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Nonlinear Targeted Energy Transfer: State-of-the-Art and New Perspectives

Dedicated to the memory of Leonid Isakovich Manevitch: Scholar, teacher, mentor, colleague

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Abstract

Following a brief review of current progress in the field of nonlinear targeted energy transfer (TET), we discuss some general ideas and methods in this field, and describe certain possible future venues for further developments; these go beyond the current paradigm of implementing TET by means of nonlinear energy sinks (NESs). Four such emerging research fields are discussed, namely, (i) the new and promising concept of intermodal targeted energy transfer (IMTET), (ii) the implementation of TET in nonlinear acoustic metamaterials, (iii) the break of classical reciprocity in elastodynamics in the context of TET, and (iv) the role of TET on the bandwidth of general classes of nonlinear resonators. Our aim is to describe the main ideas, summarize recent developments, outline possible directions for future work, and possibly trigger further research in in the discussed topics and also in other possible TET-related topics not discussed herein.

Keywords: Targeted energy transfer, nonlinear energy sink, acoustic metamaterials, non-reciprocity, bandwidth

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1. Overview

Arguably, modern exact science started from Newton's anagram in his letter to Leibnitz: *Data aequatione quotcunque fluentes quantitates involvente, fluxiones invenire; et vice versa*. A somewhat free English translation reads: "Given an equation involving any number of fluent quantities to find the fluxions, and vice versa". In even simpler language, as V.I. Arnold put it very concisely: "To find forces from trajectories, and trajectories from forces, it is useful to solve differential equations."

However, not all differential equations are created equal, as some are easier to solve and analyze than others. This realization was emphasized even more by Poincaré's discovery that some (*nonlinear*) differential equations being chaotic do not admit analytical solutions at all, and so will never be solved! Nevertheless, it is not an exaggeration to say that many major scientific advances of the past three centuries occurred because a multitude of important physical systems could be accurately described by *linear* differential equations. Then, one can apply a powerful analytical machinery that includes superposition, fundamental solutions, harmonic analysis, and integral transforms, if not to solve them, at least to describe the general structure of their solutions. At the same time, for many applications the accuracy of linear models is not sufficient, but the nonlinearity can be considered small in a certain sense. This idea led to major advances in perturbation and asymptotic methods, i.e., in building analytical approximations to the solutions based on linear or known generating functions – typically (but not always) harmonic.

Moreover, in many applications (e.g., the three-body problem with gravitational interactions, to mention the most famous), it turns out that the governing nonlinear equations are, in general, neither exactly solvable, nor tractable as quasilinear. Then, one encounters major progress in analysis and understanding of genuinely nonlinear systems based on ideas of qualitative analysis, symmetry and integrability (exact and approximate). If the global dynamical behavior of a system is beyond comprehension, one can learn a lot from representative partial solutions, e.g., nonlinear normal modes (NNMs). In addition, the discovery of deterministic chaos in nonlinear dynamical systems revealed an intrinsic relationship between dynamical and statistical approaches in describing natural phenomena. Numerical analysis is crucially important in all these developments. Still, systems at least partially comprised of elements that exhibit substantially nonlinear phenomena are often perceived as frightening and unwelcome, especially by the engineering community which is well-versed in linear theoretical settings.

This work addresses an important shift of paradigm that occurred mainly over the last 25 years. Our main claim is that *intentional* use of substantially nonlinear elements in dynamical systems and engineering structures can lead to significant improvement of their performance, opening new possibilities for design that simply would not be available in linear settings. Specifically, in this article we will discuss phenomena related to *nonlinear targeted energy transfer (TET)*. This term is somewhat loosely defined as a dynamical process wherein energy (in some form) is directed or guided from a source (donor) to a receiver (recipient) in a nearly one-way, irreversible fashion. This definition already points to two central characteristics of TET that pose distinct challenges for analysis, that is, often the requirement for essential (i.e., non-linearizable) or nearly essential nonlinearity, and the transient or non-stationary character of the process. However, one should be optimistic since to continue with English classics, “Difficulties mastered are opportunities won” (W. Churchill).

Here we do not intend to provide a detailed review on the general topic of TET – many past works have appeared that address this aim. Instead, we will outline the general *ideas* and *methods* related to nonlinear TET and describe what we consider to be some interesting venues for future developments. To this end, Section 2 is devoted to the current state-of-the-art. After a general introduction into the broader field of non-stationary nonlinear dynamics and acoustics, we dwell in some detail into an example of the TET concept – dynamics of systems with *nonlinear energy sinks (NESs)*. This choice is dictated by rapid recent development of this topic, as well as by professional experience of the authors. Again, there is no intention here to replace existing references and review papers; rather, we concentrate on physical understanding of the TET phenomenon in systems with NESs, and relate this understanding to introspection into the analytical framework and techniques. Thus, our goal is to outline the relationship between physical and engineering aspects of the problem and explore the connection between the TET concept and NES design. Section 3 is devoted to further developments and a forward look into nonlinear TET beyond the NES paradigm. Due to limited space, only four such emerging research fields have been selected by us, namely. (i) intermodal targeted energy transfer (IMTET), (ii) TET in nonlinear acoustic metamaterials, (iii) nonlinear acoustic non-reciprocity in the context of TET, and (iv) the role of TET on nonlinear bandwidth of resonators. In all these topics, we will describe the main ideas, summarize recent developments, outline possible directions for future work, and hopefully trigger further research by the interested reader into these topics, but also in other TET-related topics not discussed here.

This article is devoted to the memory of Professor Leonid Isakovich Manevitch (1938-2020), a prominent scientist in the broader field of nonlinear dynamics and acoustics, and their applications in physics, chemistry and engineering. Many of the results and ideas presented here are due to his inspiration and collaboration; in a certain sense all the authors feel as his students. He was an exemplar of broad and deep (sometimes frightening) knowledge, insatiable curiosity, unmatched love and devotion to research, positive and supportive attitude towards colleagues and his students, and human and professional dignity. His modesty and passion for science and engineering, so uncommon in our age and times, will be a lasting memory and inspiration for us. To the very end a Gentleman and a Scholar, he was working on research up to his last few days. Let us remind ourselves of one of his favorite sayings: “Any reasonable approximation can be rigorously justified”. We cannot be sure that in what follows we meet this superior standard of his.

2. State-of-the-art

2.1 Nonlinear non-stationary dynamics and acoustics

In most dynamics and acoustics applications the operation regime is linear or weakly nonlinear. This is perfectly reasonable since such regimes are deeply studied and assessable by common and widely known methods of analysis, e.g., asymptotic techniques such as the method of averaging, and, more generally, different forms of multiple-scale expansions. Still, considerable, and even essential nonlinearities occur in mechanical systems due to, e.g. clearances, impacts, friction, material, geometric or kinematic effects, external fields, scale effects and plasticity (Fidlin, 2006, Babitsky, 1998, Pilipchuk, 2010). In many cases such behavior is dictated and important for operational purposes (Fidlin, 2006, Babitsky, 1998). In other cases, the opposite is true – although the strong nonlinearity is completely unwelcome, it still reveals itself profoundly. One simple example are cracks in continuous structures (Babitsky and Hiwarkar, 2014, Hiwarkar et al, 2012, Andreaus et al. 2007).

Moreover, analysis of the dynamics and acoustics of essentially nonlinear systems constitutes a major mathematical challenge. A full analysis is available only for extremely rare completely integrable cases, yet multi-dimensional nonlinear systems are, generically, non-integrable. For other particular systems, it is sometimes possible to derive (explicitly or at least numerically) exact periodic solutions – examples are nonlinear normal modes - NNMs (Rosenberg, 1962, Rand, 1974, Vakakis et al., 1996, Haller and Ponsioen, 2016). Another interesting example of periodic solutions are discrete breathers in lattice models (Flach and Gorbach, 2008, Ablowitz and Ladik, 1976, Ovchinnikov and Flach, 1999, Gendelman, 2013). The nonstationary and transient processes in dynamics and acoustics pose even more difficult challenges for analysis and understanding, but precisely such processes are most interesting when one studies certain aspects of the physics such as nonlinear energy transfers.

Theoretical study of energy transport in essentially nonlinear systems made substantial progress, since it was perceived that most efficient transport usually occurs under conditions of *resonance*. This observation allows one to consider the system in the vicinity of a resonance manifold (RM) with reduced dimension, thus restricting the consideration to averaged equations of motion (sometimes referred to as the reduced *slow-flow system*). This crucial simplification sometimes yields even approximate conservation laws, absent in the complete system beyond the resonance manifold. In the specific case of a conservative system with two degrees of freedom (two-DOF), the existence of an additional integral of motion in the isolated-resonance approximation leads to an even stronger outcome – the complete integrability. This classical result was first formulated in quasilinear settings and dates back to Birkhoff's theory of normal forms (Birkhoff, 1927, Moser, 1973, Verhulst, 1979). In additional woks (Augusteijn and Breitenberger, 1980, Breitenberger and Mueller, 1981) this approach has been invoked to explore beating in a spring pendulum under condition of 1:2 resonance. More recent applications (Ianets and Shiff, 2018) considered propagation of asymmetric Gaussian beams in nonlinear waveguides.

The approach of isolated resonance in its initial setting is quasilinear but, as it is demonstrated above, is often used formally far beyond such regimes. Probably, the first well-known example of this approach is harmonic balance with slowly varying amplitudes (Hayashi, 2014). A rigorous mathematical justification for this extension does not exist, but still the results are often very accurate and so are widely used in engineering. Mathematically, it is convenient to formulate the slow-flow dynamics in terms of complex variables. Examples include models with self-trapping (Eilbeck et al., 1985), as well as rotating-wave approximations (Flach and Gorbach, 2008) widely used in lattice dynamics.

L.I. Manevitch with his collaborators expanded this approach and used it for far-reaching investigations of nonstationary processes in a multitude of physical contexts. In recent papers the method is commonly referred to as *complexification-averaging (CxA)* (Manevitch, 1999, 2001, 2007, 2014, Manevitch and Gendelman, 2011, Manevitch et al., 2018). The CxA approach has been applied in many problems and diverse applications, but two of its main contributions are summarized below.

The first contribution concerns the *description of slow-flow systems in terms of complex variables.* This feature should be viewed as a mathematical convenience, since from a purely mathematical point of view the slow-flow equations are equivalent to those obtained through classical harmonic balance with slowly varying amplitudes. However, the simplification is substantial and by no means incidental, as we shall see below.

The second important contribution of the CxA approach is its use to study the *topology and bifurcations of special phase trajectories on RMs.* The complete integrability of the slow flow of a general two-DOF Hamiltonian system under condition of an isolated resonance, in fact reduces it to a single-DOF Hamiltonian system. The latter can be conveniently analyzed on the phase plane. Commonly, planar phase portraits are characterized by their special invariant submanifolds, such as (in the considered Hamiltonian case) fixed points and separatrices – homoclinic/heteroclinic orbits. However, to explore non-stationary processes, one should also consider other individual orbits on the RM that (approximately) correspond to certain initial conditions of the original problem. For the interesting special case of initial excitation of a single element of the system, L.I. Manevitch coined the term *Limiting Phase Trajectory (LPT)*; such an orbit approximately describes the evolution of an initially, completely localized excitation in the system. Substantial modification of the response for given initial conditions (for instance, the transition from localization to beating) corresponds to passage of the LPT through a saddle point, in which case the LPT coincides with a (part of the) separatrix in the system for a given set of parameters. From a viewpoint of dynamical systems theory, such event is not considered as a genuine global bifurcation since the overall topological structure of the phase space does not change; still, it is clear that such transitions lead to qualitative modification of the global system response.

To illustrate these important contribution of the CxA approach for the case of isolated resonance, we consider an example with a two DOF system with its Hamiltonian expressed in terms of action – angle variables, I_1, I_2 and θ_1, θ_2, respectively,

$$H = H(I_1, I_2, \theta_1, \theta_2) \tag{1}$$

where the 2π -periodicity of the angle variables enables the representation of the Hamiltonian in terms of Fourier series (Chirikov, 1979):

$$H(I_1, I_2, \theta_1, \theta_2) = \sum_{m,n} V_{m,n}(I_1, I_2)\exp\left[j(m\theta_1 - n\theta_2)\right], \quad V_{m,n} = V^*_{-m,-n}, \quad j = (-1)^{1/2} \tag{2}$$

with $(\cdot)^*$ denoting complex conjugate. Averaging in such systems is possible due to existence of “fast angles,” whereas inapplicability of averaging (and, hence, resonance) is due to the appearance of certain “slow phases.” Formally, we assume that such a slow phase, θ, corresponds to a particular resonance and is defined by a relation such as, $\theta = m_0\theta_1 - n_0\theta_2,\ m_0, n_0 \in \mathbb{Z}$. When one assumes that the given resonance is *isolated*, it means that the terms with in (2) with all other fast angles (e.g., $m\theta_1 - n\theta_2,\ m \neq m_0,\ n \neq n_0$) can be averaged out, except for the (resonance) term associated with the slow phase θ. Thus, one ends with the following *averaged Hamiltonian*,

$$\bar{H}(J_1, J_2, \theta) = \sum_l V_{m_0 l, n_0 l}(J_1, J_2)\ \exp\left[jl(m_0\theta_1 - n_o\theta_2)\right] = \sum_l V_{m_0 l, n_0 l}(J_1, J_2)\ \exp(jl\theta)$$
$$J_k = \langle I_k \rangle,\ k = 1, 2,\ l \in \mathbb{Z} \tag{3}$$

where $\langle \bullet \rangle$ denotes the averaging operation. One can prove (Gendelman and Sapsis, 2017) that the averaged dynamics according to the averaged Hamiltonian (3) obeys an additional conservation law (besides the conservation of the Hamiltonian itself), and, thus, is completely integrable. This additional integral has the following form:

$$n_0 J_1 + m_0 J_2 = N^2 = \text{const} \tag{4}$$

Performing the trigonometric change of variables $J_1 = N^2 n_0^{-1} \sin^2(\gamma/2)$, and $J_2 = N^2 m_0^{-1} \cos^2(\gamma/2)$, one arrives to the following conservation law for the averaged system:

$$h(\gamma, \theta) = \bar{H}\left(N^2 n_0^{-1} \sin^2(\gamma/2),\ N^2 m_0^{-1} \cos^2(\gamma/2),\ \theta\right) = \text{const},\ \ \theta \in [0, 2\pi),\ \ \gamma \in [0, \pi] \tag{5}$$

Due to the conservation law (5), the dynamical system, when reduced on a sphere – where N is the radius, γ the polar angle, and θ the azimuth angle, is completely integrable. Then, various dynamical regimes are revealed by varying the parameters and initial conditions in (5).

As simple example we consider the dynamics in a system of two coupled trigonometric oscillators (Gendelman and Sapsis, 2017), with the following Hamiltonian:

$$H = \frac{p_1^2}{2} + \frac{1}{2}\tan^2 q_1 + \frac{p_2^2}{2} + \frac{1}{2}\tan^2 q_2 + \frac{\varepsilon}{2}\left(\sin q_1 - \sin q_2\right)^2 \tag{6}$$

or in terms of action-angle variables:

$$H = \frac{I_1^2}{2} + I_1 + \frac{I_2^2}{2} + I_2 + \frac{\varepsilon}{2}\left(\frac{\sqrt{I_1^2 + 2I_1}}{I_1 + 1}\sin\theta_1 - \frac{\sqrt{I_2^2 + 2I_2}}{I_2 + 1}\sin\theta_2\right)^2 \tag{7}$$

For the case of *1:1 resonance*, one sets $n_0 = m_0 = 1$ and defines the slow phase as $\theta = \theta_1 - \theta_2$. Then, according to (5), we obtain the following second independent integral of motion for the averaged Hamiltonian system:

$$h(\gamma, \theta) = -N^2 \sin^2(\gamma/2)\ \cos^2(\gamma/2) + (\varepsilon/4)\left(S_1 + S_2 - S_3\right) = \text{const},$$
$$S_1 = \frac{N^2 \sin^4(\gamma/2) + 2\sin^2(\gamma/2)}{\left[1 + N^2 \sin^2(\gamma/2)\right]^2},\ \ S_2 = \frac{N^2 \cos^4(\gamma/2) + 2\cos^2(\gamma/2)}{\left[1 + N^2 \cos^2(\gamma/2)\right]^2}, \tag{8}$$
$$S_3 = 2\cos\theta\ \frac{\sqrt{N^2 \sin^4(\gamma/2) + 2\sin^2(\gamma/2)}\sqrt{N^2 \cos^4(\gamma/2) + 2\cos^2(\gamma/2)}}{\left[1 + N^2 \sin^2(\gamma/2)\right]\left[1 + N^2 \cos^2(\gamma/2)\right]}$$

The typical evolution of the phase portrait of the averaged Hamiltonian system on the sphere $N = const,\ \theta \in [0, 2\pi),\ \gamma \in [0, \pi]$ is presented in Fig. 1 for increasing values of N. Among the initial conditions explored on the phase portrait are the ones corresponding to $(\gamma,\theta)=(0,0)$, i.e., the initial conditions corresponding to the LPT of the averaged dynamics.

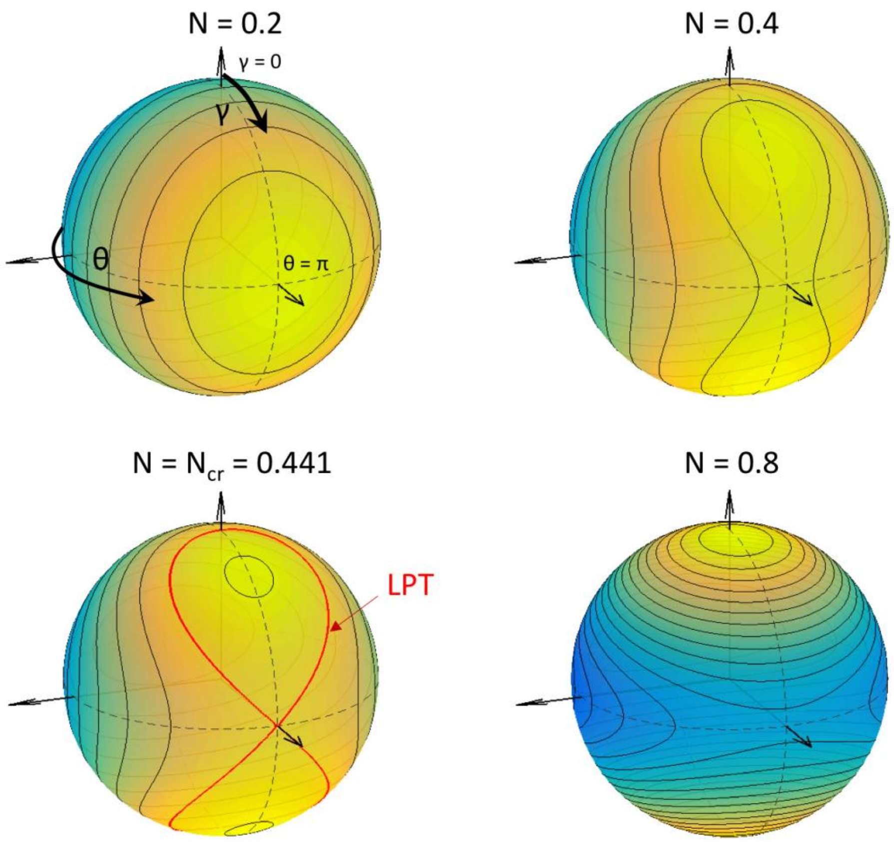

Figure 1. Phase portrait showing the orbits of the averaged Hamiltonian system on the sphere for $\varepsilon=0.1$ and increasing values of N; the trajectory in red for $N = N_{cr} \approx 0.441$ corresponds to the transition of the LPT through the saddle and represents a global bifurcation of the dynamics.

From the plots of Fig. 1 one observes that the LPT, which always starts on the north pole, for low values of *N* ends up to the south pole. This regime corresponds to nonlinear beat phenomena in the original Hamiltonian system, that is, the energy is exchanged repeatedly between the two oscillators. For high values of *N*, however, the LPT remains in the vicinity of the north pole and this results in motion localization. The transition between these two regimes occurs through a pitchfork bifurcation of the localized states and passage of the LPT through the saddle point $\theta=\pi,\ \gamma=\pi/2$ for the critical value of $N=N_{cr}\approx 0.441$. The conservation law (8) dictates that for this value of *N* the following equality should hold:

$$h(0,\pi)=h(\pi/2,\pi)\Rightarrow \varepsilon=\frac{N_{cr}^2(N_{cr}^2+1)^2(N_{cr}^2+2)^2}{3N_{cr}^6+18N_{cr}^4+24N_{cr}^2+8} \tag{9}$$

Direct numerical simulations demonstrate that the predictions obtained from Equation (9) are very accurate, especially for relatively small coupling values of ε.

Generalizing to a $P-$ DOF dynamical system, under certain conditions an alternative CxA approach can be applied starting from the original nonlinear equations of motion,

$$\ddot{u}_k = F_k(u_1,...,u_P,\dot{u}_1,...,\dot{u}_P,t),\ \ k=1,...,P \tag{10}$$

and introducing the following new complex variables,

$$\psi_k = \dot{u}_k + j\Omega u_k \tag{11}$$

where $u_k, \dot{u}_k,\ k=1,...,P$ are the physical displacements and velocities, respectively, and Ω the single fast frequency assumed to exist in the regime of the dynamics considered; in quasilinear systems Ω is set equal to the linearized natural frequency. From (10,11) one obtains the following complex representations:

$$u_k = \frac{-j}{2\Omega}\left(\psi_k - \psi_k^*\right),\ \ \dot{u}_k = \frac{1}{2}\left(\psi_k - \psi_k^*\right),\ \ \ddot{u}_k = \dot{\psi}_k - \frac{j}{2}\left(\psi_k + \psi_k^*\right) \tag{12}$$

Substituting (12) into (10), one obtains an alternative complexified set of P nonlinear equations of first order which equivalent to (10):

$$\dot{\psi}_k = G_k(\psi_1,\psi_1^*...,\psi_P,\psi_P^*,t),\ \ k=1,...,P \tag{13}$$

Then, one assumes that the dynamics admit a *fast-slow decomposition*, so that the new complex variables can be expressed as $\psi_k = \varphi_k \exp(j\Omega t)$, where φ_k is a slowly-varying complex amplitude (envelope) that modulates the fast-varying oscillation $\exp(j\Omega t)$. Finally, averaging out the complexified equations with respect to the fast frequency Ω (and its higher harmonics) we obtain the *slow-flow system*, i.e., the set of P complex equations governing the slow evolutions of the amplitudes φ_k:

$$\dot{\varphi}_k = Q_k(\varphi_1,\varphi_1^*...,\varphi_P,\varphi_P^*),\ \ \ k=1,...,P \tag{14}$$

Solving these equations, we obtain analytical approximations of even strongly nonlinear transient responses of the dynamical system (10).

As an example, we consider a system of two coupled Duffing oscillators in 1:1 internal resonance, and apply the CxA method with $\Omega=1$:

$$\ddot{u}_k = -u_k - u_k^3 - \varepsilon(u_k - u_{3-k}),\ k=1,2\ \Rightarrow\ \dot{\varphi}_k = \frac{3j}{8}\left|\varphi_k\right|^2\varphi_k + \frac{j\varepsilon}{2}(\varphi_k - \varphi_{3-k}),\ k=1,2 \tag{15}$$

Apart from the (obvious) first integral corresponding to conservation of total energy, the averaged slow-flow system (15) possesses the additional integral of motion $\left|\varphi_1\right|^2 + \left|\varphi_2\right|^2 = L^2 = \text{const}$. After the change of variables $\varphi_1 = L\sin(\gamma/2)\exp(i\delta_1),\ \varphi_2 = L\cos(\gamma/2)\exp(i\delta_2)$ one obtains the following integrable system on the sphere:

$$h_1 = -(3/32)L^4\sin^2\gamma - (\varepsilon/2)L^2\sin\gamma\cos\theta = \text{const} \tag{16}$$

where $\theta = \delta_1 - \delta_2$. A detailed analysis of a system equivalent to (16) is presented in (Manevitch, 2014). The transition from nonlinear beat phenomena to motion localization occurs through a scenario analogous to the one presented in Fig. 1, and expression (16) is similar to (5). This similarity is by no means occasional.

Indeed, a linear oscillator can be expressed in terms of action-angle variables:

$$H = \frac{p^2}{2} + \frac{\Omega^2 q^2}{2}, \;\; q = \sqrt{\frac{2I}{\Omega}} \sin\phi, \;\; p = \sqrt{2I\Omega} \cos\phi \tag{17}$$

Then, it is easy to elucidate the meaning of the complex variables in the CxA when one considers the relations:

$$q = u_k, \; p = \dot{u}_k, \; \psi_k = \dot{u}_k + j\Omega u_k = \sqrt{2I\Omega} \cos\phi + j\sqrt{2I\Omega} \sin\phi = \sqrt{2I\Omega} \exp(j\phi) \tag{18}$$

It follows that the CxA transformation is in fact a special case of a canonical transformation to action-angle variables, and the subsequent averaging is a particular case of the general approach of the isolated resonance (Gendelman and Sapsis, 2017). The CxA method generally leads to much simpler and more analytically tractable expressions compared to the generic action-angle transformation and allows for a comprehensive analysis of transient and nonstationary dynamical responses with good accuracy in many dynamical settings. At the same time, for some problems involving strongly nonlinear transient responses, e.g., dynamics of the forced escape from a potential well, the complete action-angle transformation should be used (Gendelman, 2018; Gendelman and Karmi, 2019).

One should note that the idea of complexification appeared first and was instrumental in the exploration of TET in systems with NESs. After that, this idea has been invoked by L.I. Manecitch and his collaborators in extremely diverse dynamical and acoustical settings, including small-size oscillatory systems (Manevitch, 1999, 2001, Manevitch et al., 2018, Manevitch and Kovaleva, 2013), systems behaving as local or nonlocal sonic vacua (Manevitch and Gendelman 2011, Manevitch and Vakakis 2014), one-dimensional (1D) and quasi-1D nonlinear lattices (Manevitch and Smirnov 2010a,b), systems with self-excitation (Kovaleva et al., 2013, Manevitch et al., 2013), and auto-resonances (Kovaleva and Manevitch 2013a,b), or systems exhibiting the Landau-Zener tunneling effect (Kosevich et al., 2010, Manevitch et al., 2011). More recent contributions include dynamical transitions in strongly nonlinear forced pendula (Manevitch et al., 2018, Manevitch et al., 2016a,b), Frenkel-Kontorova models (Smirnov and Manevitch, 2017) and nonlinear vibrations of carbon nanotubes (Smirnov et al., 2014, Manevitch et al., 2017).

2.2 TET and NESs

The notion of TET first appeared in the context of energy transport between "donor" and "acceptor" systems via discrete breathers (Kopidakis et al., 2001). The mechanism for TET in mechanical systems was first studied by investigating an impulsively forced, linear, grounded, dissipative oscillator (referred to as the primary system) weakly coupled to an essentially nonlinear (that is, with non-linearizable or almost non-linearizable nonlinearity), grounded dissipative oscillator (referred to as the NES). It was shown that, under condition of 1:1 transient resonance capture resonance (TRC) (Arnold, 1988; Gourdon and Lamarque, 2006) energy could be irreversibly transferred from the primary system to the NES (Gendelman, 2001; Gendelman et al., 2001; Vakakis and Gendelman, 2001). The main conclusion was that, for TET to occur, a single- or multi-mode linear primary system should be augmented by a single or a set of NESs with both essential or near-essential nonlinearity and some form of dissipation. The fact that the oscillation frequency of the NES is highly tunable with energy enables it to engage in single TRC or cascades of TRCs with modes of the primary system, absorbing and locally dissipating energy from them. The phenomenon of TET is realized through the excitation of subsets of NNMs of the combined primary-NES system, which themselves are generated by internal resonance and have no counterparts in linear theory. Later studies demonstrated that a similar TET phenomenon can be achieved with ungrounded NESs possessing significantly smaller mass, but with essentially nonlinear coupling to the primary system (Gendelman et al., 2005; Gourdon et al., 2007; Gourdon and Lamarque, 2006). That NES configuration is of particular interest due to its practicality, since it implies that, with the addition of lightweight, strongly nonlinear, *local* attachments, one could gain the capacity to alter drastically the *global* dynamics of the primary system, by inducing nonlinear modal interactions through TRCs, thus achieving broadband passive TET from different modes of the primary system to the NESs. Moreover, the achieved TET, being a strongly nonlinear phenomenon, is energy-dependent, so it yields passively adaptive dynamics (or acoustics) that are fully tunable with energy. The NESs then act, in essence, as passive, self-adaptive boundary vibration controllers.

As an example, we consider the following primary linear oscillator (LO) attached to a lightweight NES,

$$\begin{gathered}\ddot{x} + \lambda_1\dot{x} + \lambda_2(\dot{x} - \dot{v}) + \omega_0^2 x + C(x - v)^3 = 0 \\ \varepsilon\ddot{v} + \lambda_2(\dot{v} - \dot{x}) + C(v - x)^3 = 0\end{gathered} \tag{19}$$

where x and v are the displacements of the LO and the NES respectively, the system is viscously damped, and $0 < \varepsilon \ll 1$. Focusing on the case of *fundamental TET* in this system, we employ the CxA method and introduce the following complex variables:

$$\begin{gathered}\psi_1(t) = \dot{x}(t) + jx(t) = \varphi_1(t)e^{j\omega_0 t} \\ \psi_2(t) = \dot{v}(t) + jv(t) = \varphi_2(t)e^{j\omega t}\end{gathered} \tag{20}$$

where φ_1 and φ_2 are slowly varying complex modulations of the fast components with fast frequencies ω_0 and ω, respectively. Since fundamental TET corresponds to 1:1 transient resonance capture (TRC) between the primary linear oscillator and the NES, we assume that the transient dynamics possesses only a single dominant fast frequency, and set $\omega \approx \omega_0 = 1$. Hence, (20) can be rewritten as:

$$\begin{gathered}\psi_1(t) = \varphi_1(t)e^{jt} \\ \psi_2(t) = \varphi_2(t)e^{jt}\end{gathered} \tag{21}$$

Using the new complex variables (21) and averaging with respect to the fast frequency $\omega \approx \omega_0 = 1$ leads to the following *complex slow-flow system*,

$$\begin{gathered}\dot{\varphi}_1 - \varepsilon\lambda/2\,(\varphi_2 - \varphi_1) - 3jC/8\,|\varphi_1 - \varphi_2|^2(\varphi_1 - \varphi_2) + \varepsilon\lambda/2\,\varphi_1 = 0 \\ \dot{\varphi}_2 + j/2\,\varphi_2 + \lambda/2\,(\varphi_2 - \varphi_1) - 3jC/8\varepsilon\,|\varphi_1 - \varphi_2|^2(\varphi_2 - \varphi_1) = 0\end{gathered} \tag{22}$$

where $\lambda_1 = \lambda_2 = \lambda$ is assumed for simplicity. Furthermore, in order to solve (22) we express the complex modulation variables in polar form, $\varphi_i = a_i(t)\exp[jb_i(t)]$, $i = 1,2$. Consequently, the slow flow (22) is expressed in terms of real variables as follows,

$$\begin{aligned} &\dot{a}_1 - (\varepsilon\lambda/2)a_2\cos\phi + \varepsilon\lambda a_1 + (3C/8)(a_1^2 + a_2^2 - 2a_1a_2\cos\phi)a_2\sin\phi = 0 \\ &\dot{a}_2 + (\lambda/2)a_2 - (\lambda/2)a_1\cos\phi - (3C/8\varepsilon)(a_1^2 + a_2^2 - 2a_1a_2\cos\phi)a_1\sin\phi = 0 \\ &\dot{\phi} + (\lambda/2)[(\varepsilon a_2/a_1) + (a_1/a_2)]\sin\phi - 1/2 + (3C/8)(a_1^2 + a_2^2 - 2a_1a_2\cos\phi) \times \\ &\{(1/\varepsilon)[1 - (a_2/a_1)\cos\phi] - [1 - (a_1/a_2)\cos\phi]\} = 0 \end{aligned} \tag{23}$$

where a_1 and a_2 represent the real amplitudes of the slowly-varying envelopes of the linear and nonlinear responses, respectively, and $\phi(t) = b_1(t) - b_2(t)$ is the (slow) phase difference between the evolution of the two envelopes. Fig. 2 depicts the slow flow (23) for $\varepsilon = 0.05$, $\lambda = 0.01$, $C = 1$, $\omega_0 = 1$, and initial conditions $a_1(0) = 0.24$, $a_2(0) = 0.01$ and $\phi(0) = 0$ (this corresponds to impulsive excitation of the linear oscillator).

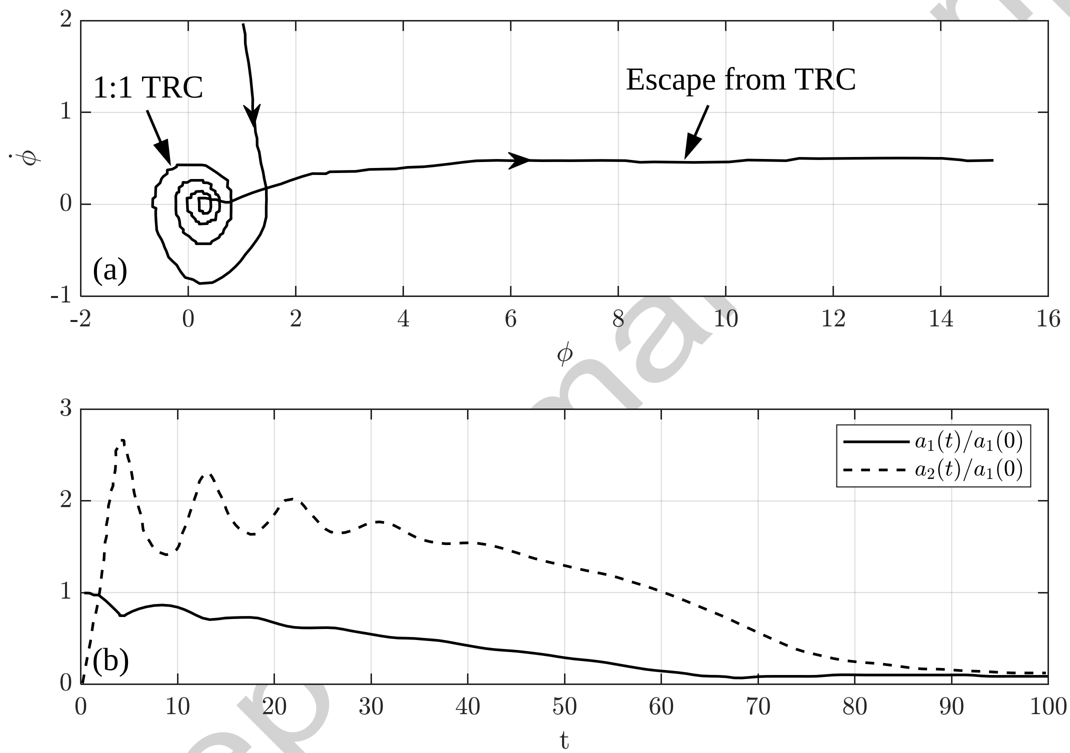

Figure 2. Dynamics of fundamental (1:1) TRC: (a) Representation in the $(\phi, \dot{\phi})$ plane, and (b) temporal evolutions of the normalized amplitude modulations of the linear oscillator and the NES.

The oscillatory (slow and not time-like) behavior of the phase variable ϕ in the neighborhood of the in-phase limit $\phi = 0$ in Fig. 2a confirms the occurrence of 1:1 TRC there, since it suggests that the phases of oscillation of the linear oscillator (LO) and the NES are "locked" in a nearly in-phase fashion. Note that away from the region of 1:1 TRC, the phase angle ϕ is not slow anymore (rather it is time-like), so the averaging operation with respect to that angle is valid in these regimes. As evidenced by the build-up of amplitude a_2 of the envelope of the NES depicted in Fig. 2b, during 1:1 TRC there occurs fundamental TET and energy flows from the LO to the NES in an irreversible way.

Later works focused on systems subject to periodic excitations (e.g., Gendelman et al., 2006). For the case of a harmonically forced single-DOF linear oscillator coupled to an NES, *strongly modulated responses (SMRs)* could be realized (Gendelman, et al., 2008); these can be regarded as sustained, periodic or quasi-periodic analogs of the transient TET that occurs for impulsive excitations (Gendelman et al., 2006, 2008). The underlying mechanism for the SMRs is a sustained series of repetitive TRCs, with each involving resonance capture, escape from resonance and recapture into resonance on a resonant manifold of the dynamics. For example, consider the system below, which is similar to (19) but with harmonic excitation applied to the linear oscillator,

$$\begin{gathered}\ddot{x} + \varepsilon\lambda(\dot{x} - \dot{v}) + x + 4/3\,\varepsilon(x - v)^3 = \varepsilon A \cos t \\ \varepsilon\ddot{v} + \varepsilon\lambda(\dot{v} - \dot{x}) + 4/3\,\varepsilon(v - x)^3 = 0\end{gathered} \tag{24}$$

where $\varepsilon\lambda$ is the weak damping coefficient, εA is the weak excitation force, and $0 < \varepsilon \ll 1$. For the set of parameters, $A = 0.225$, $\lambda = 0.2$, $\varepsilon = 0.05$, $x(0) = v(0) = 0$, $\dot{x}(0) = \dot{v}(0) = 0$, the relative steady state displacement between the LO and the NES is plotted in Fig. 3; this is a case of steady state SMR in the dynamics of (24).

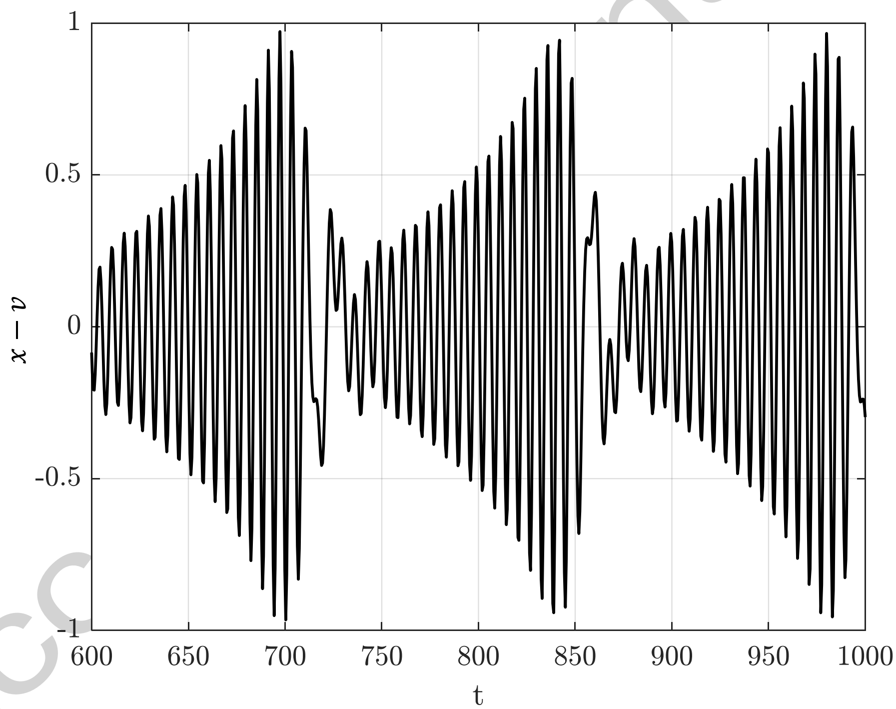

Figure 3. Strongly modulated response (SMR) of system (24).

In fact, the SMR is a repetitive series of 1:1 fundamental resonance captures, where in each cycle energy flows from the directly forced LO to the NES. Once the energy of the NES reaches a certain threshold it *saturates*, and an escape from 1:1 resonance occurs during which the oscillation frequency suddenly drops down. Once the oscillation of the system is no longer synchronized with the excitation frequency, energy is dissipated by viscous damping until it reaches a low enough level that allows for recapture into the 1:1 resonance manifold so that the cycle can be repeated.

This phenomenon is illustrated in Fig. 4 where the instantaneous frequencies of the fundamental and the second harmonic components of the relative response $x - v$ is presented. As shown in the plot of Fig. 4, the instantaneous frequency of the fundamental component of the relative response is equal to the forcing frequency, and the instantaneous frequency of the second component is three times that – due to the cubic nonlinearity of the system. Still, it should be mentioned that in the regions of fast "jump down" the frequency of the main component decreases in a rather essential manner and 1:3 resonance with the second harmonic component is also destroyed. This means that in these regions, the steady state dynamics escapes from 1:1 resonance with the external force, and is captured again into this resonance after sufficient viscous dissipation of energy (Gendelman et al., 2008).

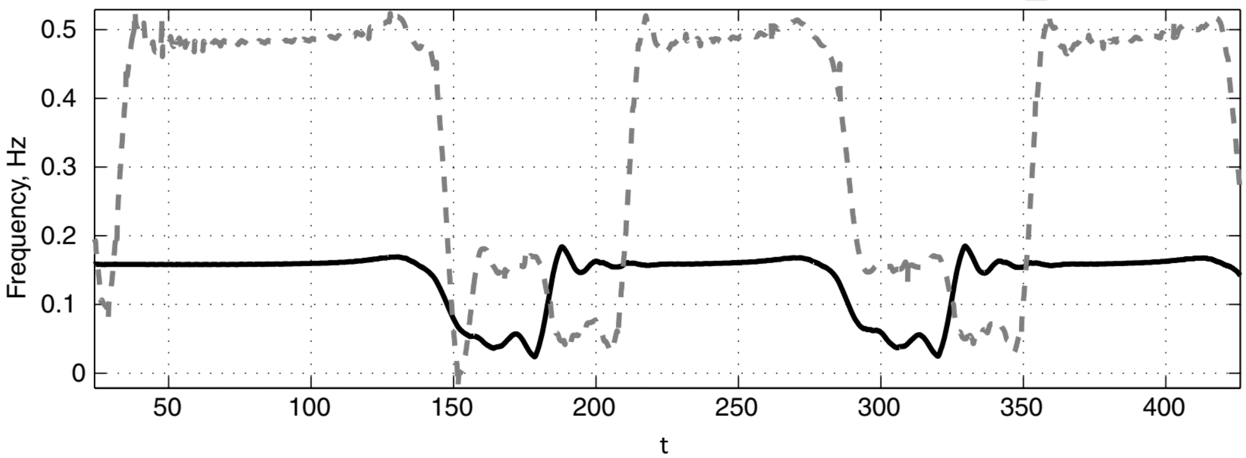

Figure 4. Instantaneous frequencies of the fundamental (solid curve) and the second (dashed curve) frequency components of the SMR of Fig. 3.

Other studies in the literature considered linear multi-DOF primary systems augmented with single- or multi-DOF NESs. The analysis of these systems is more challenging due to the higher dimensionality, and the possibility of simultaneous and more complex TRCs, and in that context the NNMs of the combined primary-NES system are of particular interest (Vakakis et al., 2003). It was shown that TET from a primary multi-DOF system and to a multi-DOF NES is realized either through isolated or cascades of TRCs yielding broadband energy transfer that significantly increase the dissipative capacity of the NES. In a later study, Gourdon and Lamarque (2005), considered a single-DOF primary system coupled to a multi-DOF NES and showed that the dissipative capacity of this system can be enhanced (compared to a single-DOF NES) due to the excitation of multiple NNMs resulting in cascades of TRCs. The common conclusion drawn from these studies was that the realization of multiple TRCs between subsets of modes of the multi-mode primary system and NNMs of the multi-mode NES is responsible for enhanced TET and increased dissipative capacity.

In parallel to these studies a considerable amount of work was devoted to modeling and understanding the complex nonlinear dynamics governing TRCs and SMRs, given the central role that these nonlinear phenomena play in transient and sustained TET, respectively. Several approaches, such as the complexification-averaging (CxA) method and singular perturbation theory were employed in these studies. In one of the first publications on transient TET, Gendelman, et al. (2001) studied transient TET by analyzing the slow-flow dynamics on an invariant 1:1 resonance manifold by CxA; the analysis verified the hypothesis that the excitation of NNMs of the *underlying Hamiltonian* (i.e., undamped and unforced) system is mainly responsible for TET in the *dissipative* system. This hypothesis enabled the reduction of the original (non-integrable) primary-NES system to an integrable slow flow averaged system which was amenable to further analytical treatment; the analysis of that system revealed that bifurcations occurring at certain energy thresholds generated (or eliminated) transient TET. Vakakis and Gendelman (2001) extended the analytical study of transient TET in the neighborhood of the 1:1 resonance manifold, by employing either action-angle transformations combined with singular perturbation theory or CxA, and determined the domain of attraction of TET in the neighborhood of a 1:1 resonance manifold.

As mentioned previously, the consideration of small-mass, ungrounded NESs was particularly convenient and practical, and paved the way for further analytical treatment of TET. The corresponding averaged slow-flow equations were in a form that enabled the application of singular perturbation theory. This led to analytical studies of SMRs leading to sustained TET in primary-NES systems subject to harmonic excitations (Gendelman et al., 2005, 2008). For example, for a single-DOF linear primary system nonlinearly coupled to a single-DOF small-mass NES subject to harmonic excitation (applied to the primary system), the existence of slow invariant manifold (SIM) was proved under condition of fundamental resonance (i.e., of 1:1 fundamental resonance between the excitation and the linearized natural frequency of the primary system). Then, it was shown that SMRs corresponded to relaxation oscillations on the SIM, that is, to periodic or quasi-periodic cycles incorporating both slow phases (on stable branches of the SIM) and fast transitions (corresponding to "jumps" between different stable branches of the SIM). Hence, sustained TET through SMRs in harmonically forced systems was interpreted in terms of relaxation oscillations on averaged SIMs of the dynamics under conditions of fundamental resonance.

Initial attempts to experimentally demonstrate TET employed cubic stiffness nonlinearities achieved through geometric effects (e.g., McFarland et al., 2005; Kerschen et al., 2007). These studies highlighted the efficacy of this type of stiffness nonlinearity in practical designs implementing TET. From an implementation perspective, however, vibro-impact and rotary nonlinearities were subsequently employed in NES designs due to their relative simplicity, even though they entailed further analytical challenges (Nucera et al., 2007; Al-Shudeifat et al., 2013, 2017; Li et al., 2017; Wierschem et al., 2017; Saeed et al., 2019, 2020). The applicability and versatility of TET by means of different types of NESs has been explored in diverse applications, including, vibration and shock mitigation of single- and multi-modal primary systems (Gourc et al., 2015; Dekemele et al., 2020; Li et al., 2021), suppression of aeroelastic instabilities (Vaurigaud et al., 2011; Hubbard et al., 2014a,b), vortex-induced vibration suppression (Tumkur et al., 2013; Blanchard et al., 2017), and vibration suppression in systems of helicopter blades, engine crankshafts, and rotor blades (Bergeot et al., 2017; Motato et al., 2017; Ebrahimzade et al., 2018; Ahmadabadi et al., 2019). For instance, Vaurigaud et al. (2011) investigated TET from a long-span bridge prone to flutter to a single-DOF NES, showing that through a 1:1:1 TRC it is possible to redirect flutter-induced vibration energy to the NES and locally dissipate it there. In another application, Blanchard et al. (2017) studied vortex-induced vibration of a sprung cylinder with an internal dissipative rotatory NES and showed that the NES can cause vortex elongation and partial wake stabilization through energy transfer from the cylinder to high-frequency fluid modes. Nowadays numerous other groups are currently working on applying the concepts of TET and NES in a diverse range of applications across scales.

3. Forward look: TET beyond NESs

Considering nonlinear TET from a broader perspective, nonlinearity may be viewed as a mechanism for scattering energy across frequencies and/or wavenumbers, and as such is prevalent in Nature. Perhaps one of the most studied examples of nonlinearity is *turbulence*, with irreversible large-to-small scale energy transfers occurring through energy cascades across multi-scale vortices (Ruelle and Takens, 1971; Frisch, 1995). Accordingly, multi-scale TET in dynamical and acoustical systems may be construed as an implementation in engineering and physics of irreversible energy cascades that occur commonly in natural systems. As described in the previous section, the traditional approach for implementing TET is by means of local NESs, inducing transient or sustained cascades of resonance captures; hence, *nonlinear resonance is a fundamental mechanism for realizing TET*. Recent work, however, has revealed that this is not the only one, since fast and intense TET can be realized also by means of *non-resonant* mechanisms, specifically by involving non-smooth nonlinearities. Hence, new paths for exciting applications of TET are now opening up, such as designing systems and structures with unprecedent features, e.g., with inherent capacity for irreversible internal energy redistributions within their modal spaces (referred to as *intermodal TET – IMTET*), or with added functionality for robustly and passively *breaking classical non-reciprocity in an energy-tunable way* through the synergy or nonlinearity and asymmetry. Moreover, new classes of nonlinear acoustic metamaterials can be conceived incorporating inherent features of targeted energy scattering across scales and controlled energy redirection – even incorporating features of *"mechanical turbulence."* Lastly, devices can be designed with capacity for irreversible nonlinear energy scattering in the frequency domain to achieve *controllable and tunable nonlinear bandwidth*. These exciting new areas, which may be considered as the next phase in the evolution of the concept of TET, will be discussed in more detail in the next sections.

3.1. Intermodal targeted energy transfer – IMTET

Inducing multi-scale TET in dynamical and acoustical systems by means of local NESs is now well studied and is the focus of intense current work. This *resonance-driven approach* enables directed transmission of broadband (shock, blast or seismic) or narrowband applied energy to the NESs where it is dissipated passively and robustly. It was recognized though (Al-Shudeifat et al., 2016), that the NESs can play an additional dual role, that is, precipitate nonlinear scattering of part of the applied energy to higher- or lower-frequency system modes, thus providing an additional means for passive energy dispersion and dissipation. Following on this finding, an alternative approach to TET was recently considered based on a *non-resonant mechanism* based on direct scattering of the applied energy in the frequency/wavenumber domain through strong local nonlinearities that eliminate altogether the need for the NES. The resulting rapid and efficient non-resonance TET mechanism is referred to as *intermodal targeted energy transfer* (*IMTET)* and yields fast-scale energy scattering within the inherent modal space of the system. Recent research (Gzal et al., 2020, 2021) indicates that this TET mechanism can be realized by imposing local non-smooth (e.g., vibro-impact) nonlinearities at certain locations of the system, but it is clear that various other types of smooth nonlinearities also could be considered for this purpose.

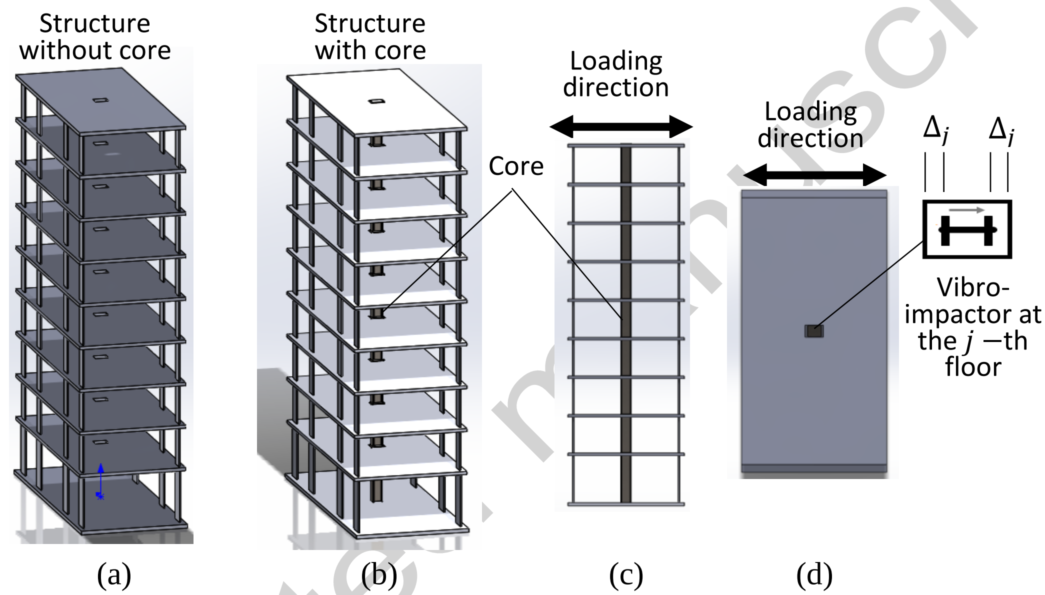

Figure 5. The nine-story structure demonstrating the implementation of IMTET: (a) Without an internal core, (b) with a core, and (c,d) side and top views indicating the loading direction of the applied blast; a detail of a vibro-impactor at the j −th floor is shown in (d).

Here IMTET will be demonstrated by means of an example in the field of blast mitigation of the model of a nine-story steel structure shown in Fig. 5, which is excited by uniform initial velocities applied to each floor simulating uniform blast loading. Typically, for blast (but also for shock or seismic) excitation the induced energy mainly excites the fundamental structural mode (and likely to a lesser extent a few lower-frequency ones) resulting in large-amplitude responses and inefficient energy dissipation by the inherent dissipative capacity of the structure, as higher-frequency structural modes remain nearly inactive. This issue is addressed by rapidly scattering the blast energy to higher-frequency modes by means of IMTET; here, this is achieved by placing an internal steel "core structure" at the center of the nine-story structure in the form of a cantilever beam with a specified clearance distribution $\Delta_j, j = 1,2, \cdots ,9$, with the nine floor slabs to inflict vibro-impact nonlinearities in the structural dynamics (cf. Fig. 5d). For simplicity in this example the core is assumed to be rigid, and the contact forces between the floor slabs and the core are modeled by an inelastic Hertzian contact law with restitution coefficient $r = 0.8$ (for comparison the elastic case with $r = 1.0$ is also considered). Details of the computations are given in (Gzal et al., 2021).

For a blast excitation with initial velocity of $0.25\, m/sec$ applied to each floor, we optimized the clearance distributions to achieve the optimal IMTET. The criterion for this optimization is the maximum scattering of the blast energy from the fundamental structural mode to higher structural modes (Gzal et al., 2021). In Fig. 6 we present the normalized energies that are eventually dissipated by the structural modes after the initial blast. In the cases of no core (linear) and purely elastic impacts (strongly nonlinear), 100% of the dissipation occurs due to the intrinsic viscous modal damping of the structure; IMTET substantially accelerates the dissipation process due to few initial strong elastic impacts. In the case of the inelastic impacts, however, only around 40% of the overall dissipation is due to the intrinsic structural viscous modal damping, and the early inelastic impacts causing IMTET are responsible for the rest. One observes that in the linear case with no core, the fundamental mode is primarily excited, and dissipates almost all of the input energy. The vibro-impact nonlinearities facilitate the energy transfer to higher modes with rapid and efficient damping. Quantitatively, one encounters a drastic decrease of the characteristic damping time, τ^*, for purely elastic Hertzian contacts, and an even stronger reduction for inelastic contacts.

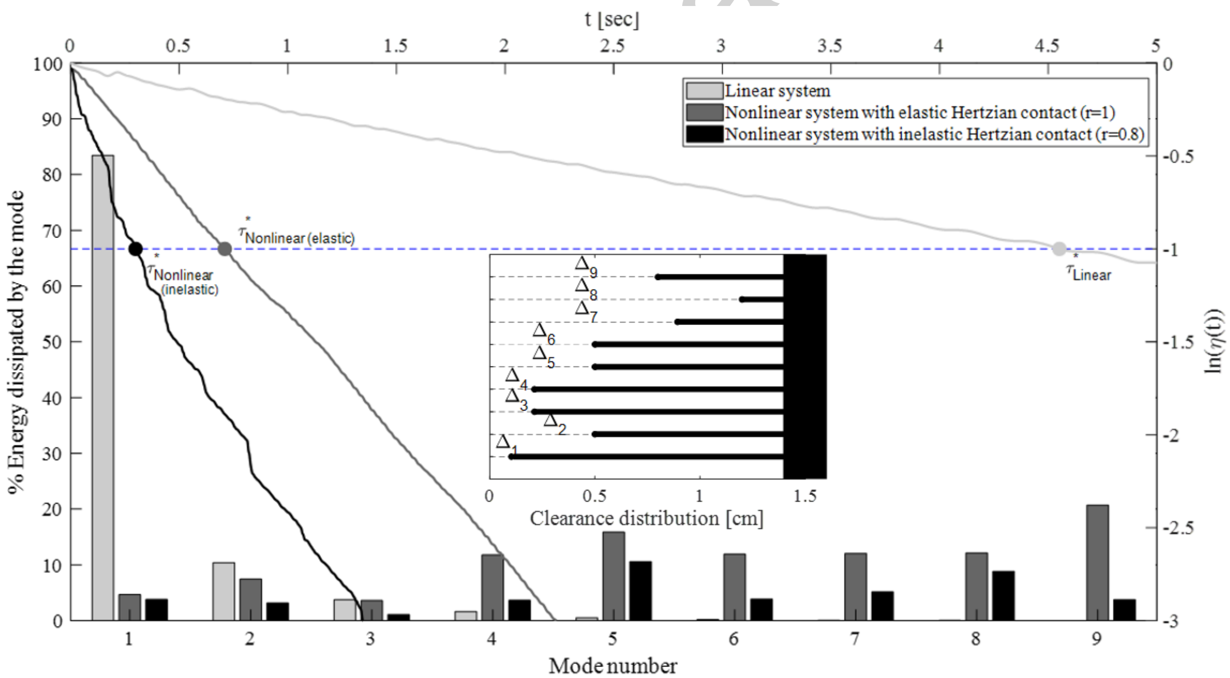

Figure 6. IMTET for purely elastic ($r = 1$) and inelastic Hertzian contacts ($r = 0.8$): Percentage of the blast energy eventually dissipated by each structural mode (left-bottom axes), superposed to the logarithm of normalized total instantaneous energy of the structure in time, $ln(\eta(t))$ (right-top axes); the inset depicts the optimized clearance distributions $\Delta_j, j = 1,2,\cdots,9$ producing the vibro-impact nonlinearities for elastic and inelastic cases (Gzal et al., 2021).

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The IMTET process based on the impacts has major advantages for the blast mitigation. First, the higher-frequency structural modes exhibit significantly smaller amplitudes compared to lower-frequency ones. Also, the overall dissipative capacity of the structure itself is utilized more effectively. Moreover, only a few strong vibro-impacts are required to achieve the energy scattering objective (Gzal et al., 2021), and these occur at the early-time, highly energetic regime of the response yielding additional strong blast energy dissipation due to inelastic vibro-impacts. It is interesting to note that IMTET is a non-resonant mechanism of scattering energy, since the vibro-impacts which are the source of the strong nonlinearity occur on a fast time scale, so resonance captures are not realizable as they occur on much slower time scales. Therefore, the low-to-high frequency nonlinear energy scattering induced by the vibro-impacts occurs in the absence of resonance which explains the extremely fast scale of energy reduction in this case.

IMTET is a new concept whereby energy induced into a system gets passively redistributed within its modal space through strong local nonlinearities. The resulting main benefit is that this enables the participation of more system modes in the response, since in the absence of IMTET the induced energy excites a limited set of modes (typically the low-frequency ones) depending on the spatial distribution of the excitation and the type of load. As a result, e.g., the overall level of system vibration is reduced not by adding extra dissipative elements but rather by redistributing energy from lower to higher frequencies (or wavenumbers) where the vibration amplitudes decrease. Moreover, the dissipative capacity of the system itself is radically enhanced since a much larger set of vibration modes (especially high-frequency ones) participate in the response, which can greatly enhance the rate of energy dissipation. Hence, IMTET provides a new approach to passive energy management, since by appropriately designing and optimizing sets of local nonlinearities (instead of NESs) one can achieve intense nonlinear energy scattering to a larger (possibly prescribed) set of system modes. The relative simplicity and passivity of the IMTET concept is emphasized at this point as, e.g., it eliminates the need to introduce NESs or active elements, which require either more involved structural design or require an external source of energy and are prone to instabilities. In addition, this nonlinear design is relatively simple, and it has modular form – it can be implemented by retrofitting existing systems with local nonlinear modules at relatively low cost.

The field of shock, blast and seismic mitigation is an obvious area where IMTET can find application, yielding new types of design approaches where instead of adding damping or stiffness, one employs local strong nonlinearities to scatter input energy to higher modes thus drastically and rapidly reducing the vibration amplitudes and enabling fast and efficient energy dissipation by the inherent dissipative capacity of the system itself. Extension to vibration isolation is obvious, since for narrowband excitations only a limited number of system modes are typically excited so one could implement IMTET at the steady state as well. In addition, implementation of TET to channel energy from a set of “unfavorable” to a set of “favorable” modes to meet specific design objectives is envisioned, leading to predictable and controllable energy management and enhanced performance with relatively low added complexity and cost. Lastly, extension of the concept of IMTET to acoustics is interesting and feasible, although this requires new concepts and

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methodologies since the concept of “vibration mode” is not applicable in the mid- to high-frequency range. Instead, one should replace it with the study of the effect of nonlinearity on the dispersion relation of the acoustic system, and how might one employ it to achieve new acoustically beneficial features. This topic is related to the TET applications in acoustics discussed in the next sections.

3.2. TET in nonlinear acoustic metamaterials: Sonic vacua and “mechanical turbulence”

As stated earlier, nonlinearity-induced TET paves the way for a radically new paradigm for energy management in a broad class of dynamical and acoustical systems through predictable and controlled energy transfers across scales mimicking the directed multi-scale energy cascades that occur in Nature. The most obvious example is the often-studied energy cascades in well-developed turbulent flows. Turbulent energy transfers can be traced to the quadratic nonlinearities of the Navier-Stokes fluid flow equations and exhibit two important features: (i) They are robust and almost impossible to suppress, and (ii) their rate of energy dissipation is fast compared to laminar flows. What is the reason for this efficient dissipation in turbulent flows? In a typical turbulent flow energy is transferred through a nonlinear mechanism over a large number of fluid modes (broadband spectrum) resulting in simultaneous dissipation over a large number of scales – this is a key property for the super “dissipation efficiency” of turbulent flows. By contrast, in laminar flows energy is “trapped” in a few large-scale modes with limited energy dissipation. This leads to the natural question, if such efficient and robust TET could be induced in dynamical and acoustical systems providing them with inherent capacity for directing (channeling) energy by mimicking turbulent energy cascades, thus attaining features of *mechanical turbulence.* Such features might potentially transform the way energy management is achieved in mechanical systems and waveguides, including energy harvesting, vibration and noise control, and instability suppression. Moreover, it is clear that such aims can be achieved only through the introduction of intentional strong nonlinearity, alleviating the basic feature of linear systems to segregate energy in vibration normal modes (in dynamics) and wave modes (in acoustics), with no possibility of energy exchanges between them.

Given, however, that there can also be adverse nonlinear effects (e.g., multiple attractors, chaos or instability), it is necessary to carefully predict and warn about conditions where such effects can appear. Therefore, together with TET it is necessary to consider new concepts and approaches that enable practical realizations of the models considered. This is a challenging (but necessary) task given that in most TET problems (especially the ones involving nonlinear and nonstationary processes) no linearization nor conventional analytical quasi-linear techniques are applicable.

As the IMTET results of the previous section have demonstrated, TET from low-to-high frequency modes in dynamical systems is possible; additional work has shown that reverse TET from high-to-low frequencies is also achievable. Proceeding to acoustics, in this section the focus will be on acoustic metamaterials with inherent features of multi-scale TET and mechanical turbulence. To this end, a first objective is to study general classes of acoustical metamaterials which, due to geometric nonlinearity, are completely devoid of their linear acoustics and become nonlinear "sonic vacua." This characterization applies to highly degenerate acoustical systems with essentially nonlinear (i.e., non-linearizable) acoustics and zero speed of sound as defined in classical acoustics. Two relatively simple examples of nonlinear sonic vacua are now briefly discussed, to demonstrate that these can be realized using *linear elements* and *simple geometries*.

We start with granular metamaterials which often behave as nonlinear sonic vacua. For example, a homogeneous, initially uncompressed lattice composed of identical spherical, linearly elastic granules in Hertzian contact (Starosvetsky et al., 2017), yields the following nonlinear sonic vacuum in the long-wavelength approximation:

$$\frac{\partial^2 u}{\partial t^2} = c\left[-\frac{\partial u}{\partial x}\, H\left(-\frac{\partial u}{\partial x}\right)\right]^{\frac{1}{2}}\left(\frac{\partial^2 u}{\partial x^2}\right) + \cdots, \quad c > 0, \quad 0 < x < L \tag{25}$$

In (25) $u(x,t)$ is axial deformation, x and t spatial and temporal independent variables, c a real parameter, and $H(\cdot)$ the Heaviside function (accounting for separation between granules in the absence of compressive forces). Comparing to the classical wave equation, a (non-linearizable) "nonlinear speed of sound" is defined as $c\left[-\frac{\partial u}{\partial x}\, H\left(-\frac{\partial u}{\partial x}\right)\right]^{\frac{1}{2}}$, that is tunable with deformation (or energy). The strongly nonlinear acoustics of (25) is due to Hertzian granular interactions, whereas non-smooth effects are caused by granule separations and collisions. Hence, *the sonic vacuum (25) is due to local Hertzian interactions and is non-smooth*. Kim et al. (2015) studied computationally and experimentally wave propagation in a similar 1D diatomic granular crystal composed of "light – heavy" pairs of granules under impulsive loading. Dependent on the mass ratio of the granule pair, nonlinear resonance can be realized in this system leading to drastic attenuation of propagating velocity pulses.

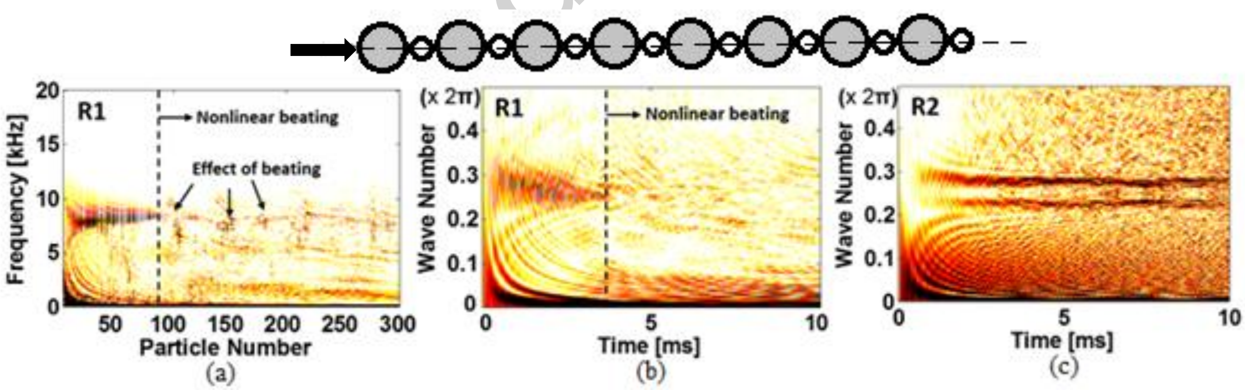


Figure 7. Impulsively loaded dimer granular metamaterial yielding a nonlinear sonic vacuum: (a) Frequency spectra of granule velocities for 1:1 nonlinear resonance (R1), and (b,c) wavenumber spectra of a the velocity of a granule for 1:1 (R1) and 1:2 (R2) nonlinear resonances, respectively, showing energy cascades and inherent multi-scale TET (Kim et al., 2015).

This is highlighted in Fig. 7a showing the frequency spectra of granule velocities during a low-frequency propagating pulse when in resonance, whereas the wavenumber spectra of specific granule velocities versus time are depicted in Figs. 7b,c for two different resonances. In Fig. 7a a strong high-frequency signal (~9 kHz) appears while the magnitude of the main low-frequency propagating pulse decreases; this low-to-high frequency energy scattering is caused by TET from the propagating pulse to an appearing wave tail up to nearly 80 granules and a nonlinear beating phenomenon after that. The resulting low-to-high frequency TET is caused by strong Hertzian interactions and vibro-impacts between granules in the wave tail and the nonlinear beat phenomenon, which appear in the wake of the low-frequency propagating pulse.

The plots of Figs. 7b,c confirm similar TET in the wavenumber domain. In the resonance of Fig. 7b the energy at the low-wavenumber propagating pulse is scattered to higher wavenumbers close to $\pi/2$ of the wave tail followed by the formation of a nonlinear beat phenomenon; this length scale corresponds to the length of two sets of heavy and light granules, suggesting the emergence of a periodic traveling wave pattern in the wave tail. In the resonance of Fig. 7c there occurs similar sustained TET from the low-wavenumber propagating pulse to a high-wavenumber wave tail, which disintegrates to spatially chaotic oscillations with eventual scattering of energy to a broad range of wavenumbers. These results indicate features of inherent mechanical turbulence in the dimer metamaterial as a result of TET across temporal and spatial scales.

Perhaps surprisingly, nonlinear phononic lattices of rather simple configuration can also behave as nonlinear sonic vacua due to strong geometric nonlinearity. As an example, consider a dissipation-less, in-plane lattice of $N-2$ particles of identical mass *m* connected by *linear* springs (with stiffness constant k) unstretched at the horizontal equilibrium position, and fixed boundary conditions (cf. Fig. 8).

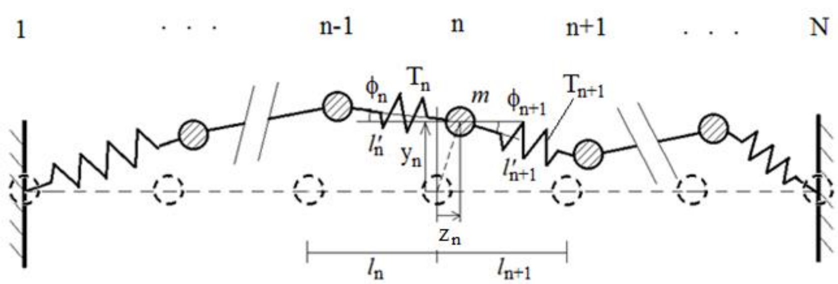


Figure 8. In-plane oscillations of a lattice giving rise to a nonlinear sonic vacuum at low energies (Manevitch and Vakakis, 2014).

For small transverse oscillations of $O(\varepsilon), |\varepsilon| \ll 1$, and even smaller axial ones of $O(\varepsilon^2)$, the dynamics is reduced approximately to the following discrete sonic vacuum (Manevitch and Vakakis, 2014),

$$y_n''(\tau) + T\left[\underline{y}(\tau)\right]\left(2y_n(\tau) - y_{n+1}(\tau) - y_{n-1}(\tau)\right) + \cdots = 0,\ n = 2, \ldots, N-1 \qquad (26)$$

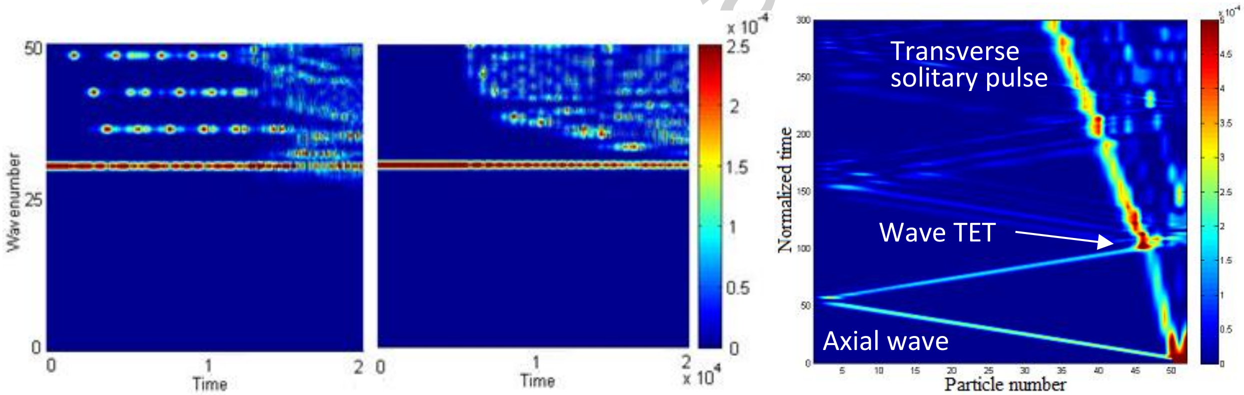
where $\underline{y}(\tau) = [y_2(\tau) \quad \ldots \quad y_{N-1}(\tau)]^T$, $y_1 = y_N \equiv 0$, and the slowly-varying tension is expressed as,

$$T\left[\underline{y}(\tau)\right] \equiv \frac{1}{2(N-1)}\left[\sum_{q=1}^{N-1}\left(y_{q+1}(\tau) - y_q(\tau)\right)^2\right] \qquad (27)$$

with $(\bullet)' \equiv d(\bullet)/d\tau$ in terms of the rescaled slow time $\tau = \varepsilon(k/m)^{1/2}t$. A long-wave approximation of (26) leads to the continuum sonic vacuum,

$$\frac{\partial^2 y(x,\tau)}{\partial \tau^2} = \left[\frac{1}{2}\int_0^1 \left(\frac{\partial y(x,\tau)}{\partial x}\right)^2 dx\right]\frac{\partial^2 y(x,\tau)}{\partial x^2} + \cdots,\ y(0,\tau) = y(1,\tau) = 0,\quad 0 \le x \le 1 \qquad (28)$$

which *contrary to (25) is generated by nonlocal geometric effects, the boundary conditions and is smooth*. It is interesting that the sonic vacuum (26) is realized only in the *finite* lattice (since the axial force generated by the boundary conditions is crucial) and disappears in the infinite limit. Moreover, in contrast to common wisdom that relates strong nonlinear effects with large energies, *this particular sonic vacuum is realized only in the limit of small energy*. This is another peculiarity of this system. The slowly-varying speed of sound of the sonic vacuum (26) is $T\left[\underline{y}(\tau)\right]$ and is tunable with energy; interestingly, this same term causes non-local coupling of all transverse displacements, despite that the original lattice having only next-neighbor (local) coupling terms! Lastly, it has exactly $(N-2)$ nonlinear normal modes (NNMs), $\underline{\phi}_p = [\sin(p\pi/N-1) \quad \ldots \quad \sin(np\pi/N-1)]^{\mathrm{T}},\ p = 1, \ldots, N-2$, which form an orthogonal basis, exactly as in the linear chain with constant tension T; all NNMs, with the exception of the highest, are unstable. The resulting nonlinear resonance modal interactions (Manevitch and Vakakis, 2014) between the NNMs produce complex dynamics and acoustics some of which are discussed below.



(a) (b) (c)

Figure 9. Low-to-high wavenumber energy cascading (TET) in the 50-particle lattice when the 30th unstable NNM is excited: Contour plots of the normalized NNM amplitudes for the (a) Full system, and (b) sonic vacuum (26) ordered by mode order – or wave number; (c) Spatiotemporal evolution of the kinetic energy of the impulsively loaded lattice of 100 particles showing nonlinear axial-to-transverse wave conversion due to wave TET (only half of the spatio-temporal domain is depicted) (Zhang et al., 2016).

The emergence of the low-energy sonic vacuum (26) provides inherent capacity for intense TET to the nonlinear lattice. As an example, in Figs. 9a,b a lattice with 50 particles is considered with its 30th (unstable) transverse NNM being harmonically excited with the initial small amplitude $A = 0.02$. TET from low-to-high transverse modes is demonstrated by plotting the wavenumber content of the response against normalized time (Zhang et al., 2016), highlighting the intense energy scattering that occurs in this system. The intense energy cascading from low-to-high wavenumbers yields chaotic responses and demonstrates capacity for mechanical turbulence. This feature is closely related to the instability of the NNMs of the lattice (except for the highest one) which enables broadband resonance exchanges across spatial and temporal scales (Manevitch and Vakakis, 2014). Recent work has revealed that this lattice may support also "energy explosions" (Zhang et al., 2018), abruptly scattering energy across a broad range of frequencies/wavenumbers. In addition, wave TET can occur in the impulsively excited lattice, resulting in wave conversions. In Fig. 9c the spatio-temporal evolution of the instantaneous kinetic energy of a lattice of 100 particles is depicted, subject to excitation of its 50th particle by the transverse impulse $F(t) = 0.1\delta(t)$. The strong geometric nonlinear coupling between axial and transverse deformations generates two distinct waves: A linear axial wave propagating at the linearized speed of sound; and a nonlinear transverse solitary pulse with energy-dependent speed and small frequency/wavenumber content. At normalized time $\tau \approx 100$ there occurs almost complete TET from the (reflected at the left fixed boundary) linear axial wave to the nonlinear transverse solitary pulse. In this case TET results in linear to nonlinear wave conversion, after which the transverse solitary pulse is transmitted through the lattice, albeit with increased speed due its increased energy.

These results pave the way for inducing features of multi-scale TET in acoustic metamaterials, whereby input energy is rapidly scattered to high frequencies/wavenumbers to achieve, e.g., rapid and enhanced dissipation. An interesting application of TET in that context is inducing features of "mechanical turbulence in metamaterials, that is, initiating sustained or transient large-to-small scale mechanical energy cascades similar to the ones encountered in turbulent fluids. In addition, new ways for passive wave conversion and tailoring could be achieved, as well as strong passive tunability of the system acoustics with energy. That would lead to new classes of acoustic metamaterials whose response would be passively adaptive to the applied excitations. The passive nature of these TET-based effects is emphasized, eliminating complications related to control energy inputs or instabilities.

A promising path forward would be to employ TET-based approaches to manage or guide broadband energy in acoustic metamaterials by mimicking the robust nonlinear energy transfer mechanisms occurring across scales in turbulent flows. As the previous discussion shows, this can be achieved by employing intentional strong nonlinearities leading to acoustic metamaterials that either are devoid of any linear acoustics (reaching a state of nonlinear sonic vacuum) or possess local strong nonlinearities. Considering first sonic vacua, apart from identifying and studying general classes of acoustical systems that can exhibit such characteristics, an additional general objective calls for uncertainty quantification of the chaotic dynamics resulting from a state of mechanical turbulence that is induced by the resulting multi-scale intense energy cascades. This dictates a new perspective for order-reduction of the resulting strongly nonlinear acoustics, given that turbulent systems possess broad spectra and strong interactions between scales and mode instabilities (persistent or intermittent). For example, in an energy cascade in a turbulent flow, energy flows downwards to smaller scales passing through essentially every scale of the system

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until it reaches the dissipation scale. In traditional order-reduction relying on Galerkin projections these nonlinear energy fluxes are only partially modeled or not modeled at all, so developing reduced-order models that correctly describe the acoustics in the subspaces-of-interest remains a challenge. Moreover, given that the enhanced dissipation capacity of turbulent flows is closely related to the nonlinear TET from larger to smaller spatial scales (or, equivalently, from smaller-to-larger wavenumbers) it would be of interest to study possible energy-wavenumber laws in nonlinear acoustic metamaterials that are analogous to "Kolomogorov's 5/3 law" (Frisch, 1995), expressed as, $E(k) \sim k^{-5/3}$, where $E(k)$ is the energy spectrum of a fully developed turbulent flow and k the wavenumber. The chaotic state of the acoustics due to intense targeted energy cascading is evident in the plots of Figs. 9a,b (caused by NNM instability). Accordingly, we conjecture that in this state of fully developed complexity there might be similarity laws in the acoustic metamaterial that are analogous to fully developed turbulence. Of course, the characteristics of the nonlinear energy cascading are expected to differ in a metamaterial, but such a study could reveal similarity laws for "mechanical turbulence" analogous to Kolomogorov's law for turbulent flows where dissipative effectiveness is well documented.

Taking this line of thought one step further, it would be of interest to study conditions for optimal TET and robustness of the acoustics of metamaterial systems augmented by local intentional strong nonlinearities (but not necessary reaching a final state of mechanical turbulence), and explore the possible triggering of multi-scale energy cascades in these systems that would cause energy dispersion and dissipation within the material at a fast time scale. An interesting perspective would be to couple local strong nonlinearities with an overall periodic structure of the metamaterial to yield unprecedented nonlinear acoustic filtering features, i.e., inherent capacity to transmit, localize or attenuate propagating disturbances depending on the type (frequency/wavenumber content) and rate of application (fast/slow) of the external excitation. This hints at energy-dependent nonlinear pass and stop bands in the metamaterial, and at its capacity for passive motion confinement.

Lastly, it is of interest to couple acoustic metamaterials with features of nonlinear sonic vacuum (acting, in essence, as NESs) to primary linear multi-degree-of-freedom (DOF) systems in order to explore their efficacy as passive broadband energy absorbers of drastically enhanced performance compared to the current state-of-the-art. We anticipate that features of mechanical turbulence in the attachments will enable broadband multi-scale TET from the primary systems, providing a new paradigm for an NES. The anticipated high complexity of the primary structure – sonic vacuum interaction dictates a thorough study of uncertainty quantification of the nonlinear dynamics and acoustics of such coupled systems. For example, new forms of instabilities induced by the synergistic effect of nonlinearity and high dimensionality are anticipated, which might be beneficial for energy harvesting. To this end, one needs to build a fundamental understanding for this interplay by considering new possibilities for TET due to the primary system – sonic vacuum interaction. In that regard, it is of interest to explore the capacity for targeted broadband TET for energy harvesting; e.g., one could conceive of multi-dimensional nonlinear energy harvesters operating for optimal multi-scale TET over broad frequency and energy ranges. The primary goal

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will be the optimization of the nonlinear energy harvesting system under a given spectral density of the input signal (energy source). This problem involves many different parameters, but also excitation uncertainty. To this end, one could employ suitable metrics to optimize the TET design towards specific goals, such as guiding energy to high wavenumbers (to achieve high-energy harvesting rates) or guiding energy towards or away from specific parts of the metamaterial system.

3.3. Nonlinear non-reciprocity and TET

Another interesting application of TET is related to the break of reciprocity in acoustical and dynamical systems. Reciprocity (Achenbach, 2004) is a fundamental property of linear time invariant (LTI) systems. In LTI waveguides, breaking acoustic reciprocity is only possible by breaking time reversal symmetry on the micro-level, and ways to do this include imposing odd-symmetric external biases, e.g., uni-directional static magnetic fields or uni-rotating fluid circulations (Fleury et al., 2014; Tsakmakidis et al., 2017), or time-variant modulations of system parameters through control, e.g., spatio-temporal modulations of elastic properties (Popa and Cummer, 2014). Both approaches are *active* and require a source of external energy. An alternative *passive* way for breaking reciprocity is through the synergy of nonlinearity and asymmetry (Li et al., 2019; Maldovan, 2013; Liang et al., 2010). In this section, nonlinear TET will be related to break of reciprocity in dynamical and acoustical systems.

We highlight the intricate relation of multi-scale TET and non-reciprocity with the simple example shown in Fig. 10a – namely, a linear oscillator of unit mass (referred to as the large scale – LS) coupled to a nonlinear attachment of small mass $0 < \varepsilon \ll 1$ (the small scale – SS). For nonlinear cubic coupling between LS and SS and weak damping, this system is governed by,

$$\begin{gathered} \ddot{x} + \omega_0^2 x + \varepsilon\lambda_1 \dot{x} + \varepsilon\lambda_2(\dot{x} - \dot{v}) + C(x - v)^3 = F_1(t) \\ \varepsilon\ddot{v} - \varepsilon\lambda_2(\dot{x} - \dot{v}) - C(x - v)^3 = -F_2(t) \end{gathered} \tag{29}$$

with $\omega_0 = 1,\ \varepsilon = 0.05, \lambda_1 = \lambda_2 = 0.04, C = 1$. First, we consider an impulsive excitation of the LS for zero initial conditions and $F_1(t) = 0.12\ \delta(t),\ F_2(t) = 0$ – cf. Fig. 10b. In this case most of the energy gets transferred from the impulsively excited LS to the SS. On the contrary, for SS excitation, $F_1(t) = 0,\ F_2(t) = 0.12\ \delta(t)$ (Fig. 10c), the energy mainly localizes to the SS. Key to this non-reciprocity is the fact that since the SS has no preferred resonance frequency (due to its pure cubic stiffness), its frequency is solely tunable with energy. This is confirmed by the corresponding wavelet spectrum (depicting the transient evolution of the dominant harmonics of the SS response) in Fig. 10b, revealing a 1:1 transient resonance capture (TRC) and TET in the *initial, high-energy* regime as the SS tunes its frequency to the (fixed linear) resonance frequency ω_0 of the LS, absorbing energy from it.

However, for SS excitation (Fig. 10c) its frequency is high ($> \omega_0$) in the initial high-energy regime (since its stiffening response is tunable with energy), and a *delayed* 1:1 TRC with, and TET to the LS at a reduced-energy regime occurs. It is the time delay and intensity of TET from the SS to the LS that causes the non-reciprocity in the dynamics, revealing the direct link between TET and nonlinear non-reciprocity in this system. This simple example indicates that nonlinear non-

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reciprocity through multi-scale TET can be induced in dynamical systems and acoustic waveguides with internal hierarchical structure, i.e., with multiple coupled internal scales (such as the LS and SS in the previous example). A different example of application of TET for acoustic non-reciprocity is discussed in the next example.

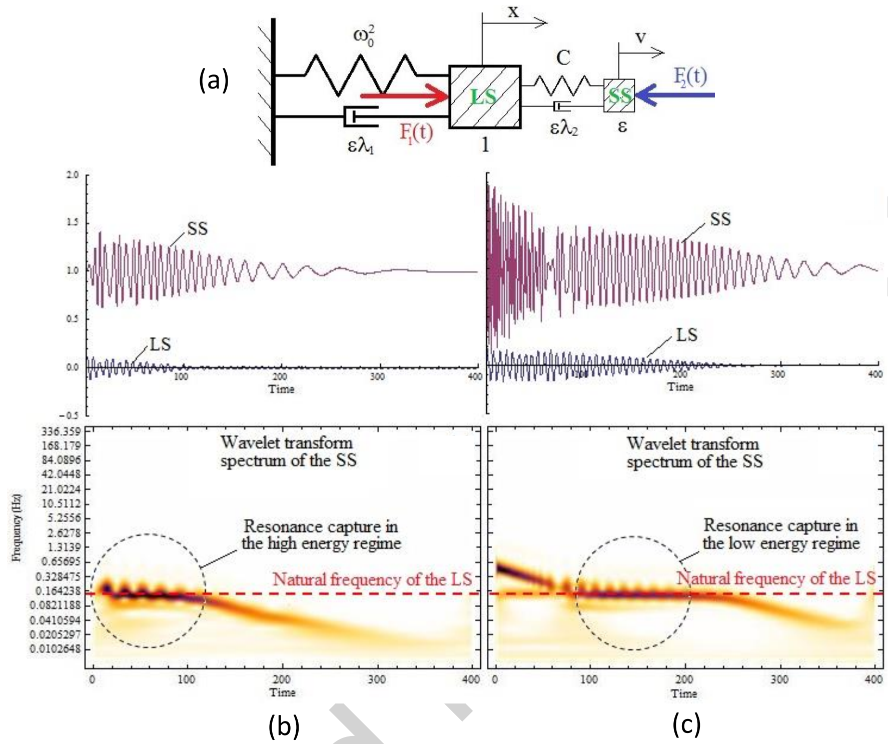

Figure 10. Oscillating system composed of a large scale (LS) nonlinearly coupled to a small scale (SS): (a) Configuration, and impulsive responses for excitation of (b) the LS – strong TET to the SS, and (c) the SS – localization to the SS; frequency axes in the wavelet spectra in (b,c) are logarithmic (Moore et al., 2018).

A second example concerns the experimental fixture of Fig. 11, composed of an "excited waveguide" (whose first unit-cell is excited by an impulsive force) – unit-cells E1-E7, and an "absorbing waveguide" – unit-cells A1-A7, which is weakly coupled to the excited one. All unit-cells are linearly grounded oscillators with non-linearizable, cubic intra-waveguide coupling, and weakly linear inter-waveguide coupling with the oscillators of the other waveguide. Unit-cells E1, and A1-A7 are grounded by softer springs compared to E2-E7, so the integrated waveguide is asymmetric. Details for this experimental fixture can be found in (Wang et al., 2020a).

In the symmetric case (all grounding springs are the same) there occur recurrent propagating breather exchanges (Mojahed et al., 2019). In the asymmetric system, however, irreversible breather redirection from the excited to the absorbing waveguide can occur, depending on the strength of the impulse excitation. This phenomenon is due to a *macroscopic spatial analogue of the Landau-Zener tunneling (LZT) quantum effect* (Wang et al., 2020b). Fig. 12 depicts the spatiotemporal evolution of the normalized instantaneous energy for an experiment and the corresponding simulation, at three different excitation levels (Wang et al., 2020a). For weak excitation (Fig. 12a), the energy remains confined in the unit-cells E1 and A1, being exchanged between them.

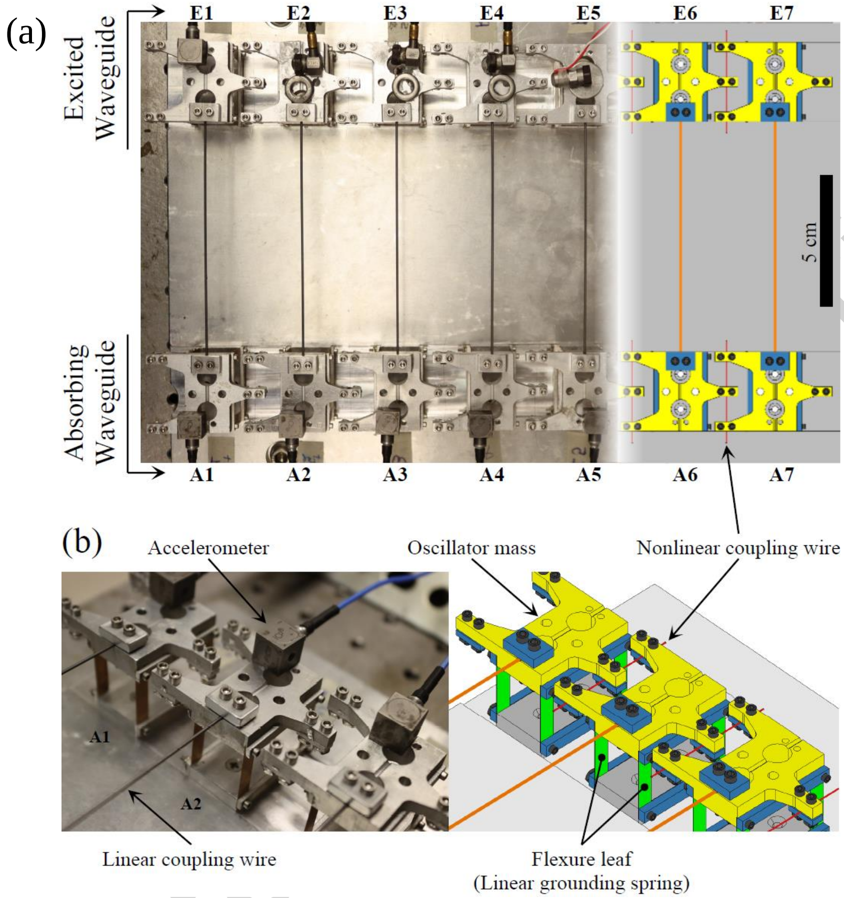

Figure 11. Experimental fixture of two nonlinear coupled waveguides of 7 unit-cells: (a) Top view with the last 4 unit-cells replaced by schematics, and (b) isometric (left) and schematic (right) views of unit-cell **A2** connected with **A1** and **A3**; each unit-cell is composed of a U-shaped aluminum block mass (yellow) grounded via a pair of flexure leaves (green), with intra-waveguide nonlinear coupling being realized by thin wires (red), and inter-waveguide linear coupling by thick wires (orange) (Kanj et al., 2021).

For intermediate excitation (Fig. 12b), irreversible breather redirection occurs from unit-cell E1 to the absorbing waveguide where a propagating breather is initiated; this amounts to nonlinear TET from the excited waveguide to the absorbing one. For strong excitation (Fig. 12c), a mixed acoustic regime occurs: At an early stage (of duration ~0.2 s) the input energy remains confined into unit-cells E1 and E2 with a weak breather being released to the absorbing waveguide; after that the remaining energy is redirected to the absorbing waveguide in the form of a secondary weak propagating breather. In this case viscous damping plays an important role in the acoustics, and intermittent TET between the excited to the absorbing waveguide takes place.

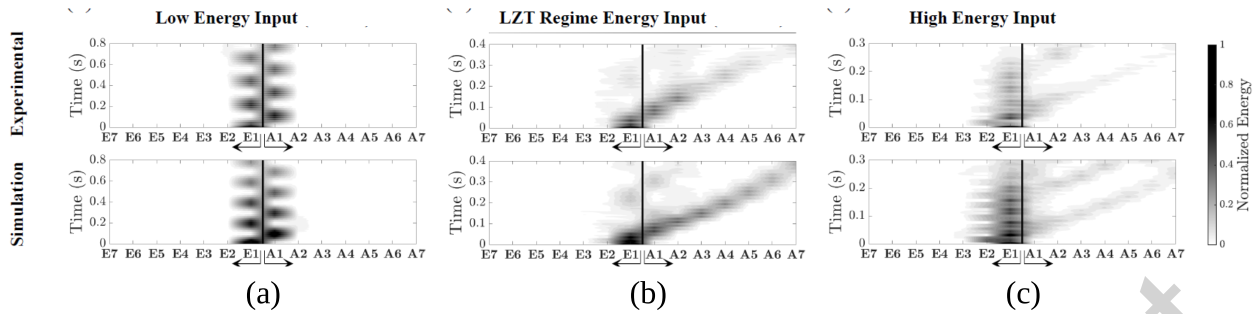

Figure 12. Spatiotemporal evolution of normalized energy in the experimental system (top) and simulated reduced-order model (bottom) when subjected to (a) low, (b) intermediate, and (c) high energy input; note LZT wave redirection and TET in (b) and (c) (Kanj et al., 2021).

Nonlinear TET (and wave redirection) in the asymmetric nonlinear coupled waveguides of Fig. 11 gives rise to non-reciprocity in the acoustics. This was confirmed in (Kanj et al., 2021) and only a brief synopsis of the findings is included here. To study non-reciprocity in the asymmetric nonlinear system of Fig. 11 we consider the waveguide response subject to same-magnitude impulsive loads (i.e., identical initial velocities v_0 since all oscillators have the same mass) to the free boundaries and adopt the non-reciprocity measures defined by Wang et al. (2021); specifically, considering pairs of boundaries consisting of E1 and any other boundary N, we define the following non-reciprocity measure,

$$\delta_{E1\,N} = \frac{\int_0^T \left[x_{E1/N}(t) - x_{N/E1}(t)\right]^2 dt}{2\int_0^T \left[{x_{E1/N}}^2(t) + {x_{N/E1}}^2(t)\right] dt}\,, \qquad N = A1, E7, A7 \qquad (30)$$

where T denotes the time domain (window) of the measured responses and $x_{E1/N}(t)$ the response at boundary E1 due to an impulsive load applied to terminal N. Note that $0 \leq \delta_{E1\,N} \leq 1$, with zero implying perfect reciprocity, 0.5 orthogonal boundary responses, and unity opposite boundary responses. Lastly, the basic assumption in defining the measure (30) is that the applied impulses at the pairs of boundaries are identical. Although this requirement could only be approximately satisfied in the experiments (performed by using a manual hammer), never the less through repeated tests we ensured that this requirement was nearly satisfied.

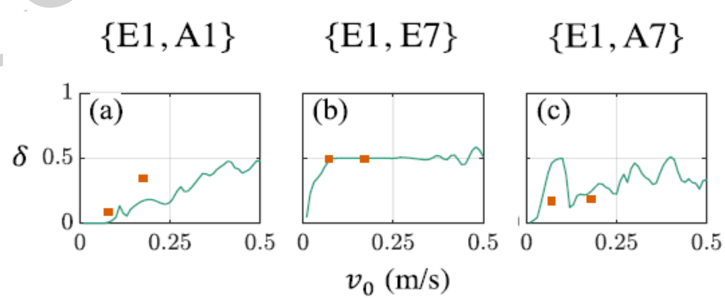

Figure 13. Non-reciprocity measure $\delta_{E1\,N}$ as function of the intensity of the applied impulse (or initial velocity v_0) when (a) N =A1, (b) N =E7, and (c) N =E7; solid curves are simulations, and square markers experimental measurements at low- (cf. Fig. 12a) and high-(cf. Fig. 12c) impulse intensities (Kanj et al., 2021).

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In Fig. 13 the experimental values of $\delta_{E1\ N}$ are compared to the theoretical predictions (with impulses at the boundaries) for $T = 0.6s$. Whereas the non-reciprocity measures are nearly zero for weak impulses, there is strong non-reciprocity for stronger impulses, indicating the influence of the intra-coupling nonlinearities. In fact, the strongest and most robust non-reciprocity ($\delta > 0.5$) occurs for {E1, E7} (Fig. 13b). This effect is due to the LZT wave redirection and TET when the impulse is applied to E1, resulting in negligible response of E7 (as most of the impulsive energy is redirected to the absorbing waveguide). Conversely, when the impulse is applied to E7, motion confinement in the excited lattice occurs, yielding finite response of E1. It follows that TET is directly responsible for the observed non-reciprocity in this case. Similar arguments can be made for other excitation-measurement pairs.

The results of this section highlight the strong relation of nonlinear TET and the break of classical reciprocity in dynamical and acoustical systems. The violation of such a fundamental property of LTI systems provides new and unprecedented technological opportunities; moreover, the passive tunability of nonlinear non-reciprocity with energy (i.e., the characteristics and intensity of the applied external stimuli) provides unique added functionality and advantage of this passive nonlinear approach compared to other linear approaches. As such, this is a promising field of novel applications of TET in the design of new classes of passive acoustical and dynamical systems with inherent capacity for non-reciprocal responses. The potential benefits are many, important and diverse. For example, inducing TET across scales can be employed towards hierarchically-structured layered media with internal microstructure capable of directing sound; that is, would enable sound transmission in preferred directions but restrict sound transmission in the opposite directions thus completely isolating sound propagation in the reverse. The applications of such potentially transformative non-reciprocal material systems in sound isolation are obvious. In addition, non-reciprocity can drastically enhance passive vibration and shock mitigation designs, especially when these are coupled with direction of shock energy in *a priori* determined paths within a system through TET; this latter feature can yield additional benefits, such as energy harvesting of enhanced efficiency and performance, capable of passively absorbing energy from narrowband or broadband environmental sources. In that context, non-reciprocity could drastically enhance energy localization and focusing in a system, a feature that renders itself immediately applicable for novel, efficient and robust energy harvesting and storage designs.

In addition, TET-induced acoustic non-reciprocity can be the basis of a new class of programmable metamaterials, e.g., with built-in capacity for acoustic logic and/or acoustic computing. For example, preliminary work by the authors has shown that acoustic AND gates, OR gates and acoustic amplifiers are feasible through irreversible breather redirection in networks of coupled nonlinear phononic lattices; this could be used for constructing “acoustic chips” embedded in material systems to provide capacity for logic operations. From a broader technological standpoint, using multi-scale TET for the creation of mechanical, non-reciprocal systems will enable new metamaterials with different responsivity to external stimuli, regulated by basic autonomous logic operations. We envision applications of these transformative reciprocity-breaking materials and acoustic systems in fields, such as medical ultrasound devices enabling new-discoveries in biology and materials science; atomic force microscope (AFM) sensing; acoustic filters; sonar; and acoustical energy management and control.

3.4. Nonlinear bandwidth in the context of TET

In this last section we provide an additional interesting demonstration of TET but showing the central role that it plays on the intrinsic unforced dynamics of a general class of nonlinear oscillators. To be more specific we show that TET between harmonic components generated by the nonlinearity is directly related to the nonlinear bandwidth of these systems. Our discussion follows two recent studies (Mojahed et al., 2021a; 2021b), and after some preliminaries we provide a specific example with a vibro-impacting oscillator.

In *linear* dynamics and acoustics, the notion of bandwidth of a single-DOF linear oscillator is related to its Q-factor, which, in turn, is proportional to the ratio of its instantaneous energy to the rate of its energy loss per cycle; hence the bandwidth is an indication of the spectral sharpness of the resonance (or localization of energy) in the frequency domain. Then, the traditional *half-power bandwidth (3 dB bandwidth)* of this resonator is computed as $\Delta\omega_{-3\text{ dB}} = \omega_0/Q$ where ω_0 is its natural frequency. Assuming weak damping and a single resonance mode, the half-power bandwidth can be approximated as $\Delta\omega_{-3\text{ dB}} \sim 2\zeta\omega_0$, where ζ its critical viscous damping ratio of the resonator.

Extending the notion of bandwidth to a general *nonlinear* resonator, one can defined it as an intrinsic measure of time locality of its energy, i.e., the capacity of the resonator to dissipate energy in time, or similarly, the measure of its energy dispersion capacity in the frequency domain. Now, from a signal processing point of view, the time-locality of a signal can be quantified by its temporal variance, σ_t^2, while its frequency dispersion can be quantified by its frequency-domain variance, σ_ω^2, with these two quantities being related through the Fourier uncertainty principle $\sigma_t^2\sigma_\omega^2 = C \geq 1/16\pi^2$, where C is a constant. In standard communication systems (Gabor, 1947) σ_ω^2 is also known as $(\Delta\omega_{rms})^2$ or *RMS bandwidth*,

$$\sigma_\omega^2 \equiv (\Delta\omega_{rms})^2 \equiv \frac{4\int_{-\infty}^{\infty}\omega^2 E^2(\omega)d\omega}{\int_{-\infty}^{\infty}E^2(\omega)d\omega} \tag{31}$$

where $E(\omega)$ is the Fourier transform of the total energy of the resonator. Based on this definition, Mojahed et al. (2021a,b) proposed the following extended bandwidth definition for a nonlinear resonator as follows:

$$\Delta\omega_{rms}^2 = \frac{4\int_0^{\infty}\omega^2|V(\omega)|^4 d\omega}{\int_0^{\infty}|V(\omega)|^4 d\omega} \tag{32}$$

where $V(\omega)$ denotes the Fourier transform of the envelope $\langle v(t)\rangle$ of the velocity time series of the resonator, given that the envelope of the kinetic energy of the resonator is approximately proportional (factored by an inertial term) to its total instantaneous energy. This extends the traditional linear bandwidth definition $\Delta\omega_{-3\text{ dB}}$, being valid for a broad class of systems including linear/nonlinear, lightly/heavily damped, single-/multi-mode and time invariant/variant systems (Mojahed et al., 2021b).

Moving one step further, the extended bandwidth (32) can be related to harmonic generation by the nonlinearity of a resonator; this is a crucial step towards establishing a direct link between the nonlinear bandwidth and TET between harmonics in the frequency domain. To this end, and truncating the velocity time series to the N leading harmonics only, the bandwidth (32) can be related to the harmonics of the measured velocity time series according to the following expression (Mojahed et al., 2021b),

$$\Delta\omega_{rms}^2 = \frac{\sum_{i=1}^{N} \alpha_i^4 E_{v_i}^2 \Delta\omega_i^2}{E_v^2} + \frac{\int_{-\infty}^{\infty} \omega^2 [V^4(\omega) - \sum_{i=1}^{N} \alpha_i^4 V_i^4(\omega)] d\omega}{E_v^2} \tag{33a}$$

where $V(\omega) = \mathcal{F}\{\langle v(t)\rangle\}$, $V_i(\omega) = \mathcal{F}\{\langle v_i(t)\rangle\}$, $E_v^2 = \int_0^{\infty} |V(\omega)|^4 d\omega$, $E_{v_i}^2 = \int_0^{\infty} |V_i(\omega)|^4 d\omega$, $\mathcal{F}\{\bullet\}$ is the Fourier transform operator and $\langle\cdot\rangle$ the envelope operator, and the envelope of the velocity time series is decomposed in terms of its harmonic components (Mojahed et al., 2021c) as follows, $\langle v(t)\rangle \approx \langle \sum_{i=1}^{N} v_i(t)\rangle \approx \sum_{i=1}^{N} \alpha_i \langle v_i(t)\rangle$ where the coefficients α_i account for the phase differences between the harmonic velocity components $v_i(t)$, with $\alpha_1 = 1$ and the rest being determined by numerical optimization. In addition, the individual bandwidth of the i-th harmonic is computed by:

$$\Delta\omega_i = 2\sqrt{\frac{\int_0^{\infty} \omega^2 |V_i(\omega)|^4 d\omega}{\int_0^{\infty} |V_i(\omega)|^4 d\omega}} \tag{33b}$$

Expressions (33a,b) directly relate the extended bandwidth (32) to nonlinear harmonic generation and TET between harmonics. The leading term represents the *individual* contributions of the leading N harmonics to the bandwidth, whereas the second term denotes the *inter-harmonic TET* contributions to the bandwidth, i.e., the influence on the bandwidth of directed energy exchanges between different harmonics.

As a demonstration we consider the pendulum depicted in Fig. 14a undergoing single-sided impacts at a rigid barrier (Mojahed et al., 2021a). For small-angle oscillations the pendulum has a natural frequency equal to $\omega_0 = 2.8247\,\mathrm{rad/s}$, normalized viscous damping coefficient $\lambda = 0.0056\,\mathrm{s}^{-1}$ and normalized friction coefficient, $\mu = 0.0056\,\mathrm{rad.\,s}^{-2}$; these parameters were extracted from system identification of an experimental fixture as discussed in (Mojahed et al., 2021a). Hence, the normalized model of the vibro-impacting pendulum is given by,

$$\ddot{\theta} + \lambda\dot{\theta} + \omega_0^2\theta + \mu\,\mathrm{sgn}(\dot{\theta}) + N(\theta) = 0 \tag{34}$$

where $N(\theta)$ is the term modeling the single-sided impacts with restitution coefficient $r \approx 0.6$, and the initial conditions are selected as $\theta(0) = 0, \dot{\theta}(0) = -3\ rad/s$. The resulting angular velocity time series and its wavelet spectrum are then presented in Figs. 14b,c, from which it is deduced that after an initial, highly energetic, strongly nonlinear regime where intense vibro-impacts occur and nonlinear scattering of energy into higher harmonics occurs, the system settles into a lower-energy, nearly linear regime (with only the friction moment persisting) where only the fundamental harmonic component close to the natural frequency remains.

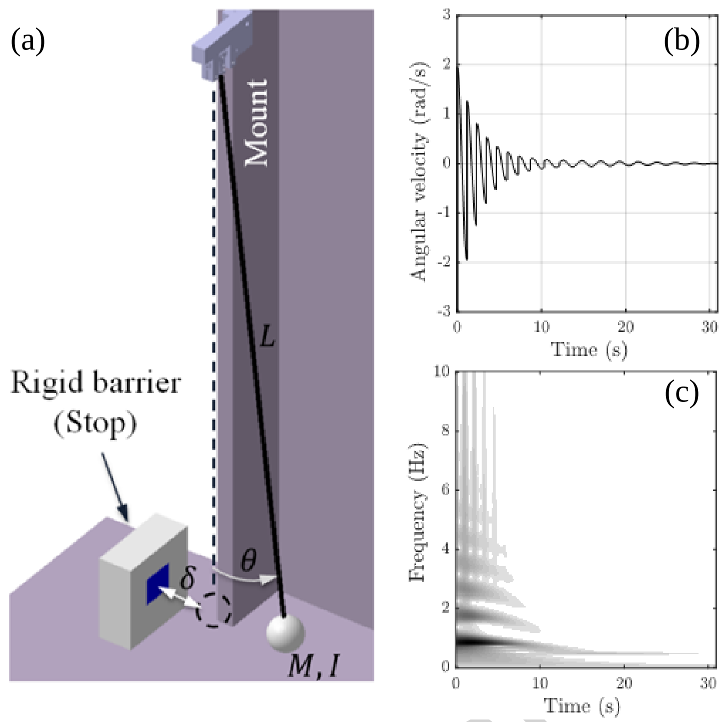

Figure 14. Pendulum undergoing single-sided impacts: (a) Schematic, (b) angular velocity time series, and (c) corresponding wavelet transform spectrum (Mojahed et al., 2021a).

Considering the wavelet spectrum of Fig. 14c we consider only the five leading harmonics (although there exist countably infinite higher harmonics, their amplitudes are rapidly decreasing with increasing frequency so they are neglected from this analysis), so we truncate the series (33a) to $N = 5$. The angular velocity time series corresponding to each of these harmonics is extracted by decomposing the wavelet spectrum of Fig. 14c through a numerical inverse continuous wavelet transform (ICWT) as developed by Mojahed et al. (2021c), yielding the harmonic decomposition in the time and wavelet domains depicted in Figure 15.

Fig. 16 depicts the energy-dependent bandwidth of the vibro-impacting pendulum (where at each input energy level the full velocity time series was needed to compute the corresponding bandwidth value), together with the contributions to the bandwidth of the five leading harmonics of the angular velocity time series shown in Figs. 14b,c, and their interactions. In this case the contributions of the higher harmonics (second and above) are negligible. In fact, the inter-harmonic interaction term denotes the overall net energy exchange between the fundamental and the other four leading harmonics. As mentioned previously, the bandwidth of a nonlinear oscillator provides a measure of its overall dissipative capacity as a function of its input energy level. The results in Fig. 16 show that the bandwidth contribution of the fundamental harmonic remains nearly constant and is smaller than the total bandwidth; this means that the fundamental harmonic loses energy at a slower rate compared to the overall rate of energy dissipation of the nonlinear resonator. At the same time, the higher harmonics do not contribute individually to the bandwidth, but they do contribute through *their nonlinear interactions with the fundamental harmonic*, i.e., through directed energy exchanges with the fundamental harmonic.

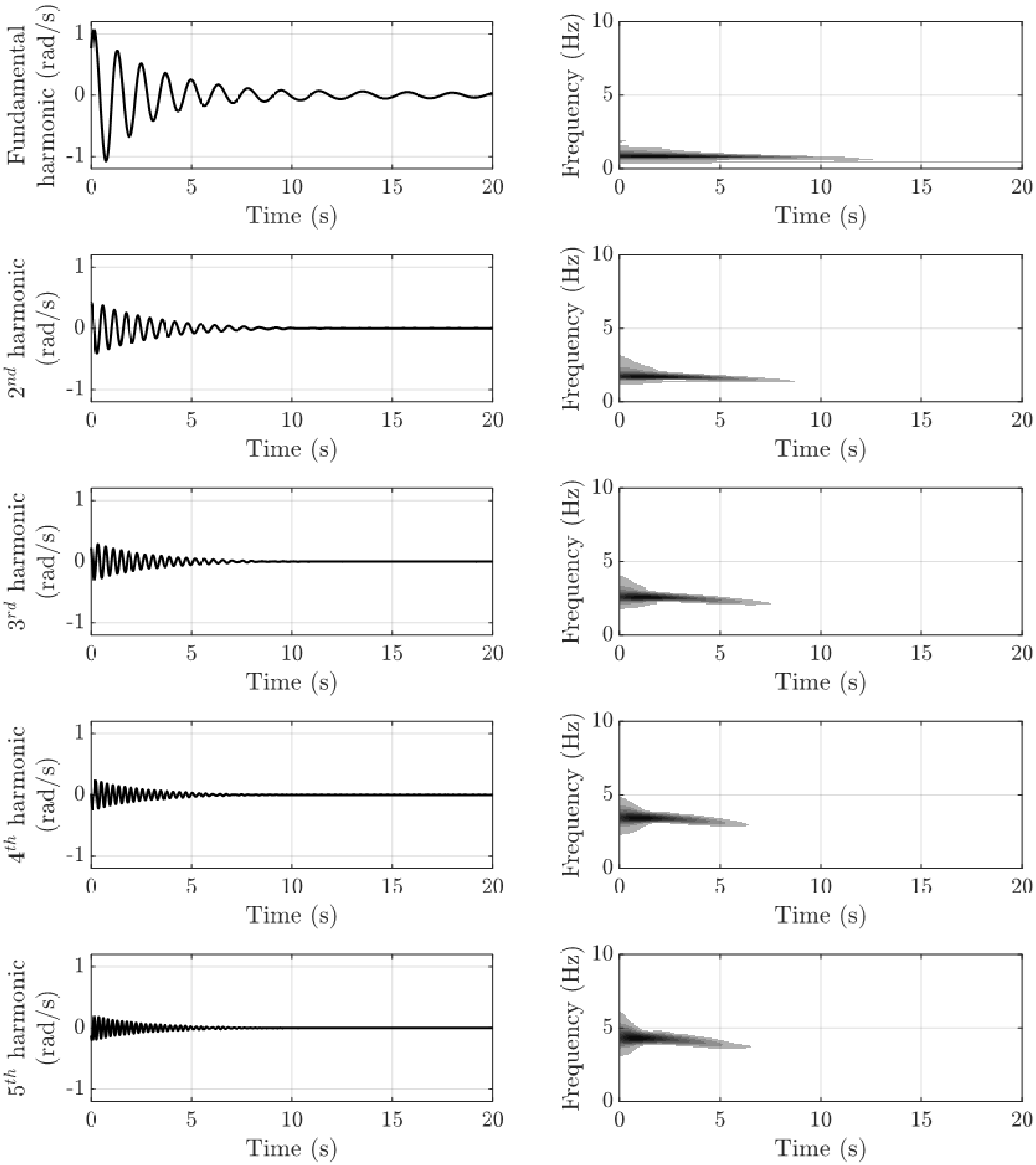

Figure 15. Harmonic decomposition of the five leading harmonics of the angular velocity time series of Figs 14b,c and their corresponding wavelet transforms.

The positive contribution to the bandwidth of the inter-harmonic interaction term in Fig. 16, indicates that the higher harmonics lose energy at a much higher rate compared to the leading harmonic; this can be explained if one considers that, in addition to dissipating energy at higher frequencies, the higher harmonics continuously lose energy to the first harmonic. This explains the fact that the dissipation rate of the fundamental harmonic remains nearly constant throughout the input energy range: Despite the hardening nonlinearity (which typically increases the energy dissipation rate due to frequency increase) the only way for the fundamental harmonic to keep a constant energy dissipation rate is for the higher harmonics to supply it with their energy through TET. Additionally, when the input energy level is below the critical threshold at which vibro-impacts are triggered, the transient dynamics becomes dominated by the friction effects and the bandwidth of the system tends to that predicted solely by the fundamental harmonic, while the contribution of the inter-harmonic interaction terms becomes zero.

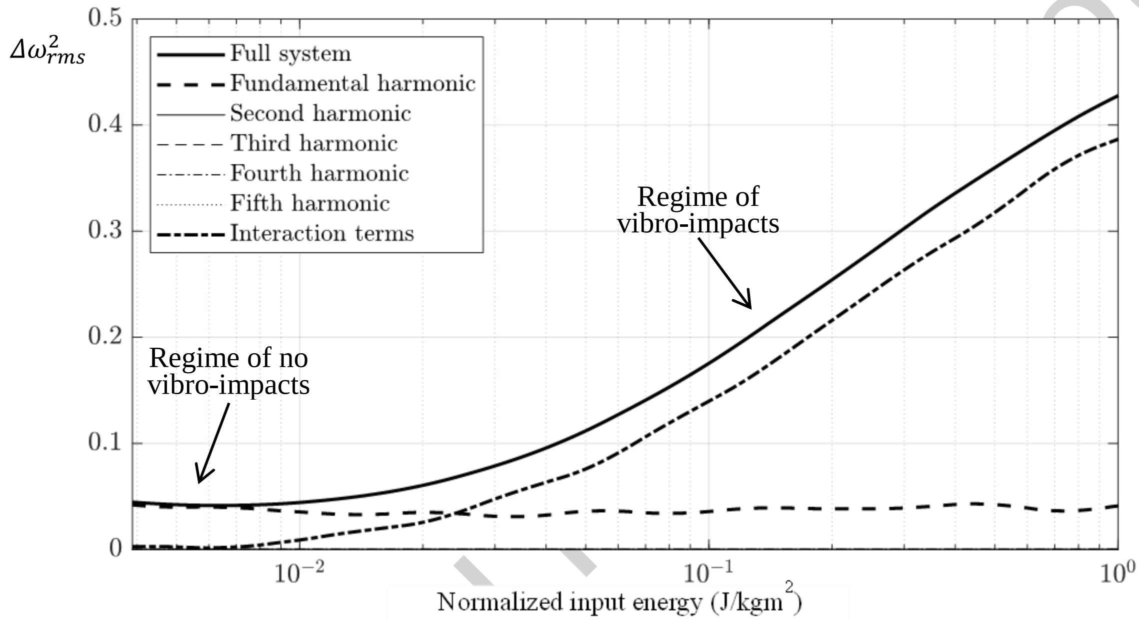

Figure 16. Nonlinear bandwidth-squared of the vibro-impacting pendulum (solid curve), and contributions to the bandwidth of the leading harmonic (dashed curve) and the inter-harmonic interactions between the five leading harmonics (dashed-dotted curve).

The previous application highlights an additional (and perhaps surprising) application of TET to the energy-dependent bandwidth of nonlinear oscillators. In fact, the outlined methodology – more details of which can be found in (Mojahed et al., 2021a,b,c) – enables the detailed quantification of the contributions to the nonlinear bandwidth of the different harmonics generated by the nonlinearity, as well as their TET inter-harmonic interactions. The individual contributions to the bandwidth of the higher harmonics are negligible, since they are generated by direct nonlinear energy scattering from the fundamental harmonic; however, once the higher harmonics are generated, targeted TET from or to the fundamental harmonic from these harmonics, as well as interactions between the higher harmonics affect significantly the nonlinear bandwidth of the resonator. Potential applications of inter-harmonic TET are broad, e.g., nonlinear systems could be designed with tailored bandwidth properties. In addition, the "energy storage" capacities of nonlinear resonators, i.e., their capacities to release their energy over fast or slow rates, could be tuned through predictive inter-harmonic TET manipulation. In general, accurate knowledge and control of TET from low-to-high harmonics or vice versa in the dynamics of nonlinear resonators can become valuable tools for a new type of tailored "nonlinear functionality" providing oscillators (or systems of oscillators) with unprecedented tunable resonance features.

4. Afterword

We attempted to briefly outline some of the prospective research venues concerning TET in dynamical and acoustical systems. One can safely say that by now this topic has reached a level of maturity and has been successfully applied in multiple interesting and important directions, with good perspectives and many promising leads for new concepts and applications.

It seems appropriate to mention here some fundamental challenges that are to a certain degree common in the prospective research venues. The first challenge is related to the "curse of dimensionality;" indeed, all applications addressed in section 3 are intrinsically one-dimensional. This choice may be well-justified and common in certain classes of metamaterials (subsections 3.2 and 3.3) but appears as an oversimplification for the general IMTET problem – after all, e.g., the civil and mechanical structures to be protected through TET-based concepts are necessarily three-dimensional. At the same time, in metamaterials the combination of high dimensionality and strong nonlinearity may give rise to new and interesting physical phenomena and design approaches. On the other hand, it might be that the observed phenomena exist in these systems just due to the confinement of the analysis to one dimension; in this unfortunate case, at higher dimensionality one may observe nothing beyond a "chaotic sea" which is deprived of any interesting, predictable and understandable structural responses. Of course, one always hopes that this is not often the case, and that the observed interesting phenomena are preserved also in higher dimensions, in which case, one will encounter a very rich and promising track for further research efforts.

The other major challenge concerns the mathematical methods used for the description of TET-related phenomena. All novel research topics described in Section 3 go far beyond the basic (and well-studied) case of isolated resonances and cannot be treated without advanced numerical methods. Any ideas that could simplify the models and their treatment, and render them at least to some extent analytically tractable, are most welcome. However, it seems highly improbable that the intrinsic mathematical difficulties related to the strong nonlinearity, large number of degrees of freedom, transient character of the responses etc. will be overcome by some type of new trick; most probably, further developments will continue to rely on advanced numerical studies, perhaps even supported by physics-based machine learning algorithms. Still, it is most important to develop the qualitative understanding of the described phenomena, as such understanding is indispensable for any practical implementations. To this end, the implementation of nonlinear system identification methods coupled to model updating (Kerschen et al., 2006) is a promising design tool for postprocessing the numerical or experimental results to yield efficacious reduced-order models capable of capturing the main features of the nonlinear phenomena studied.

But, perhaps, an even more tempting challenge would be the exploration of TET phenomena in multi-physical settings and applications. All applications presented in this article belong to the broader field of mechanics, and this was not accidental: In addition to well-known nonlinearities originating from material constitutive laws, mechanical systems offer two major additional sources of strong nonlinearity, namely, clearances/impacts, and two- and three-dimensional geometrical configurations. Still other branches of science and engineering, such as physics, e.g., optics, fluids, solid state, photonics, plasmonics, etc., chemistry, e.g. chemical oscillators and processes, biology, e.g., biological and neural processes, and multi-physical combinations of these, e.g., opto-mechanics, magneto-optics, magnetoelasticity, or bio-inspired engineering abound with nonlinear phenomena. The question then is how and to what extent one can expand and apply TET-related ideas such as the ones discussed in this article in these fields. In that context, transient and nonequilibrium processes in multi-physical, strongly nonlinear systems open even more space for imagination and exploration of TET as we move forward.

Finally, it should be clear from this article that our belief is, that (again paraphrasing V.I. Arnold) it is useful to apply the strong nonlinearity to predictably control how energy "flows" in broad classes of systems and processes. We hope that the reader will agree, at least in principle, with us, and find interest in pursuing the described forward paths, and also possibly formulate additional ones not contemplated here.

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Disclosure of potential conflicts of interest
The authors declare that they have no conflict of interest.

Data availability statement
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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