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A MINKOWSKI INEQUALITY FOR HOROWITZ-MYERS GEON

AGHIL ALAEE AND PEI-KEN HUNG

Abstract. We prove a sharp inequality for toroidal hypersurfaces in three and four dimensional Horowitz-Myers geon. This extend previous results on Minkowski inequality in the static spacetime to toroidal surfaces in asymptotically hyperbolic manifold with flat toroidal conformal infinity.

1. Introduction

The classical Minkowski inequality for a closed convex hypersurface Σ with induced metric γ in $\mathbb{R}^n$ reads

$$\int_\Sigma H \,\mathrm{dvol}_\gamma \geq |\mathbb{S}^{n-1}|^{\frac{1}{n-1}} |\Sigma|^{\frac{n-2}{n-1}}. \tag{1.1}$$

Here H stands for the mean curvature of Σ with respect to the outward normal. Moreover, the rigidity holds if and only if Σ is isometric to the $(n-1)$-sphere. The inequality (1.1) has been improved to include star-shaped surfaces [7, 8]. One can replace the star-shaped condition by an outward minimizing condition using the weak inverse mean curvature flow of Huisken and Ilmanen [11]. A further extension of (1.1) was proved in [1], where the only condition on Σ is that it encloses a bounded region. This inequality has been generalized to different settings. In [5], Brendle, Wang and the second author extended this inequality to convex, star-shaped hypersurface Σ with induced metric γ in the anti-de-Sitter Schwarzschild space with horizon Σ_H and static potential ϕ

$$\int_\Sigma \phi H \,\mathrm{dvol}_\gamma - n(n-1)\int_\Omega \phi \,\mathrm{dvol}_g \geq (n-1)|\mathbb{S}^{n-1}|^{\frac{1}{n-1}} \left(|\Sigma|^{\frac{n-2}{n-1}} - |\Sigma_H|^{\frac{n-2}{n-1}} \right), \tag{1.2}$$

where Ω is a bounded region with boundary $\partial\Omega = \Sigma \cup \Sigma_H$. Moreover, the equality holds if and only if Σ is a coordinate sphere. For bounded region with outward minimizing boundary Σ in the Schwarzschild space with mass m, Wei [13] proved that

$$\frac{1}{(n-1)|\mathbb{S}^{n-1}|}\int_\Sigma \phi H \,\mathrm{dvol}_\gamma \geq \left(\frac{|\Sigma|}{|\mathbb{S}^{n-1}|} \right)^{\frac{n-2}{n-1}} - 2m, \tag{1.3}$$

with rigidity for coordinate spheres. Recently, McCormick [10] showed that the inequality (1.3) holds for asymptotically flat static spacetimes of dimension $3 \leq n \leq 7$. Note that all of these inequalities are for static manifolds with spherical infinity. Recently, Minkowski-type inequalities were investigated in [12] for general warped product spaces which allows non-spherical infinity.

In 1999, Horowitz and Myers [9] discovered a complete static spacetime $(\mathbf{M}^{n+1}, -\phi^2 dt^2 + g_{\mathrm{HM}})$ with negative cosmological constant $-n$, that is a solution of the following static equations

$$g_{\mathrm{HM}}\Delta\phi + \phi \mathrm{Ric} - \mathrm{Hess}\phi = 0, \qquad R = -n(n-1), \tag{1.4}$$

where Ric, R, Δ, and Hess are the Ricci curvature, scalar curvature, Laplace-Beltrami operator, and the Hessian with respect to metric g_{HM}, respectively. The time constant slice of this spacetime

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is an asymptotically hyperbolic manifold with flat toroidal conformal infinity and it is called *the Horowitz-Myers geon* $(M^n_{\mathrm{HM}}, g_{\mathrm{HM}}, \phi)$ [14]. The manifold M_{HM} is diffeomorphic to $[0,\infty)\times T^{n-1}$ and the metric takes the form

$$g_{\mathrm{HM}} = ds^2 + \left(\frac{d\phi}{ds}\right)^2 d\xi^2 + \phi^2 \sum_{i=3}^{n} (d\theta^i)^2, \qquad \phi(s) := \cosh^{\frac{2}{n}}\left(\frac{ns}{2}\right), \tag{1.5}$$

where $s \in [0,\infty)$ is the geodesic coordinate from central torus, $\xi \in [0, 4\pi/n]$, and $\theta^i \in [0, a_i]$ for $0 < a_3 \leq \cdots \leq a_n$. The period of ξ is chosen such that, after a coordinate change, the metric is smooth up to the central torus $\{s = 0\}$. The function $\phi(s)$ satisfies the equation

$$\frac{d\phi}{ds} = \phi(1-\phi^{-n})^{1/2}. \tag{1.6}$$

The total mass of the Horowitz-Myers geon is

$$m = -\frac{4\pi}{n}\prod_{i=3}^{n} a_i\,. \tag{1.7}$$

They postulated that this may be the ground state for a conjectural nonsupersymmetric AdS/CFT correspondence [9]. More precisely, let (M,g) be an asymptotically hyperbolic manifold with flat toroidal conformal infinity and scalar curvature $R(g) \geq -n(n-1)$. Then the total mass is at least equal to the mass of the Horowitz-Myers geon. Moreover, the rigidity holds if and only if (M,g) is isometric to the Horowitz-Myers geon. The Horowitz-Myers conjecture follows from Conjecture 3 of [9]. Progress on the Horowitz-Myers conjecture has been very limited. For small perturbations of Horowitz-Myers the conjecture is true by Constable and Myers [6] and Woolgar proved the rigidity of this conjecture in 3 dimensions [14]. Recently, in [2], there is more supporting evidence in the validity of this conjecture.

In this paper, we investigate a Minkowski-type inequality for toroidal hypersurfaces Σ with induced metric γ in the Horowitz-Myers geon $(M^n_{\mathrm{HM}}, g_{\mathrm{HM}}, \phi)$. We consider a Minkowski-type quantity

$$Q(\Sigma) := n(n-1)\int_\Omega \phi \,\mathrm{dvol}_{g_{\mathrm{HM}}} - \int_\Sigma \phi H \,\mathrm{dvol}_\gamma. \tag{1.8}$$

Here H is the mean curvature of Σ in M and Ω is the bounded region enclosed by Σ. Applying the Divergence Theorem and the static equation (1.4), $Q(\Sigma)$ can be rewritten as

$$Q(\Sigma) = \int_\Sigma (n-1) g_{\mathrm{HM}}(\nabla\phi, \nu) - \phi H \,\mathrm{dvol}_\gamma. \tag{1.9}$$

We aim to prove that under certain convexity assumption,

$$Q(\Sigma) \leq -2^{-1} n m, \tag{1.10}$$

where m is the mass defined in (1.7). The main results are

Theorem 1.1. *Let $n = 3$, $\Sigma \subset M^3_{HM}$ be a graph over T^2 and $\tilde{h}_{ab}$ be the second fundamental of Σ with respect to the conformal metric $\tilde{g} = \phi^2 g_{\mathrm{HM}}$. Suppose that*

$$\tilde{h}_{ab} \text{ is non-negative definite} \tag{1.11}$$

and that

$$\min_\Sigma \phi > 1. \tag{1.12}$$

Then (1.10) *holds and the equality is achieved if and only if Σ is given by a coordinate torus $s \equiv$* constant.

Theorem 1.2. *Let $n = 4$, $\Sigma \subset M^4_{HM}$ be a graph over T^3 and $\bar{h}_{ab}$ be the second fundamental form of Σ with respect to the conformal metric $\bar{g} = \left(\phi \frac{d\phi}{ds}\right)^2 g_{\mathrm{HM}}$. Assume Σ is symmetric along the ξ direction. Suppose that*

$$\bar{h}_{ab} \text{ is non-negative definite} \tag{1.13}$$

and that

$$\min_{\Sigma} \phi^4 \geq 1 + \frac{2}{\sqrt{3}}. \tag{1.14}$$

Then (1.10) *holds and the equality is achieved if and only if Σ is given by a coordinate torus $s \equiv$ constant.*

Let us comment on the proofs of Theorem 1.1 and Theorem 1.2. We are following a similar strategy as in [5, 8]. That is, we consider a flow for which we are able to prove a monotonicity formula for $Q(\Sigma_t)$. Then we show that under certain assumptions, the flow exists for all time and the limit sup of $Q(\Sigma_t)$ is bounded from above by $-2n^{-1}m$. In the 3-dimensional case, these are the content of Lemma 3.1 and Lemma 3.2; for the 4-dimensional case, Lemma 4.1 and Lemma 4.2 are analogous statements. However, the flow we use is different from the one in the previous literature, which we explain in the next paragraph.

We use certain unit normal flows, (3.1) and (4.1), instead of the inverse mean curvature flow (IMCF) which was applied in [8, 5]. The main reason is that (3.1) and (4.1) give the desired monotonicity. See Lemma 2.1. In contrast, it seems difficult to show monotonicity under the IMCF. The main reason is that coordinate tori are not umbilic. This implies even if we start an IMCF from a coordinate torus, there is no clear way to deal with the second fundamental form in the evolution equation.

Next, we discuss assumptions in our main theorems. For the 3-dimensional case, the assumption (1.12) in Theorem 1.1 is to avoid the coordinate singularity at $\{s \equiv 0\}$. The conformal metric $\tilde{g} = \phi^2 g_{\mathrm{HM}}$ is defined in the way that (3.1) is the unit normal flow with respect to $\tilde{g}$. The convexity assumption (1.11), together with the fact that $\tilde{g} = \phi^2 g_{\mathrm{HM}}$ is negatively curved, ensures the long time existence of the unit normal flow. See Lemma 3.2.

The assumptions in Theorem 1.2 are similar to the ones in Theorem 1.1. The flow (4.1) is the unit normal flow with respect to the conformal metric $\bar{g} = \left(\phi \frac{d\phi}{ds}\right)^2 g_{\mathrm{HM}}$. Also, we require the convexity assumption (1.13). The difference is that $\bar{g}$ is only negatively curved in the region given by (1.14). See Lemma 4.2 for more details. The symmetry assumption in Theorem 1.2 is used in proving the monotonicity Lemma 2.1 in order to apply the Gauss-Bonnet Theorem. We believe the symmetry assumption is pure technical and could be replaced by more general conditions. In general we claim the following conjecture is true.

Conjecture 1.3. *Let $\Sigma \subset M^n_{HM}$ be a graph over T^{n-1} for $n \geq 5$. Under certain convexity assumptions on Σ, the inequality* (1.10) *holds and the equality is achieved if and only if Σ is given by a coordinate torus $s \equiv$ constant.*

The structure of this article is as follows. In Section 2 we discuss the geometry of graphs and prove the monotinicity of the Q in (1.8) under a weighted normal flow. In Section 3, we prove the global existence of the flow in dimension 3 and prove Theorem 1.1. Finally, in Section 4, we prove the global existence of the flow in dimension 4 for axially symmetric graphs and prove Theorem 1.2.

2. GEOMETRY OF GRAPHS

In this section, we investigate the geometry of a graph Σ in $(M^n_{\mathrm{HM}}, g_{\mathrm{HM}}, \phi)$. We start by considering conformal metrics that will be important to us. Recall that g_{HM} given in (1.5) is the metric of the Horowitz-Myers geon. We define conformal metrics

$$\tilde{g} = \phi^2 g_{\mathrm{HM}}, \qquad \bar{g} = \left(\phi \frac{d\phi}{ds}\right)^2 g_{\mathrm{HM}}. \tag{2.1}$$

The reason is that flows we consider, (3.1) and (4.1), are unit normal flows in $\tilde{g}$ and $\bar{g}$ respectively. We need to compute curvatures and second fundamental forms with respect to g, $\tilde{g}$ and $\bar{g}$. To unify the calculations, we define the function

$$q(s) := \int_0^s \frac{ds'}{\phi(s')}. \tag{2.2}$$

Using q as a coordinate, the conformal metric $g' := \phi^{-2} g_{\mathrm{HM}}$ is asymptotic to a flat metric on $\mathbb{R}\times T^{n-1}$

$$g' = dq^2 + \left(\phi^{-1}\frac{d\phi}{ds}\right)^2 d\xi^2 + \sum_{i=3}^n (d\theta^i)^2. \tag{2.3}$$

In Appendix A, we calculate the curvatures and the second fundamental forms with respect to the metric

$$dq^2 + \Psi(q)^2 d\xi^2 + \sum_{i=3}^n (d\theta^i)^2,$$

for a general positive function $\Psi(q)$. The metric g' corresponds to $\Psi = \phi^{-1}\frac{d\phi}{ds}$, which is actually the only case we will use. We further record the curvatures and the second fundamental forms under a conformal transformation $\breve{g} = e^{2\psi} g'$. The metrics g_{HM}, $\tilde{g}$ and $\bar{g}$ correspond to $\psi = \log\phi$, $\psi = 2\log\phi$ and $\psi = 2\log\phi + \log\frac{d\phi}{ds}$ respectively.

We use ∇ and $R_{\alpha\beta\lambda\mu}$ to denote the Levi-Civita connection and the Riemannian curvature tensor of g_{HM} respectively. For a hypersurface Σ, we use $\{x^a\}, a = 1, 2, \ldots, n-1$ as a local coordinate. The induced metric and the second fundamental form are denoted by γ_{ab} and h_{ab} respectively. We write D for the Levi-Civita connection of γ. We use $\mathrm{dvol}_{g_{\mathrm{HM}}}$ and $d\mathrm{vol}_\gamma$ to denote the volume form of g_{HM} and γ respectively. The tensors and connections induced by the conformal metrics $g', \tilde{g}$ or $\bar{g}$ will carry the corresponding accents. For instance, the second fundamental form with respect to g' is written as h'_{ab}. Also, the index in Weingarten tensor is raised accordingly. For example, $\tilde{h}^b_a = \tilde{\gamma}^{bc}\tilde{h}_{ca}$.

Let Σ be a graph given by $s = v(\xi, \theta^i)$ for some smooth function v defined on T^{n-1}. Define the height function in the q coordinate as

$$u(\xi, \theta^i) = q(v(\xi, \theta^i)).$$

The function $u(\xi, \theta^i)$ is introduced to simply some of the geometric quantities. In particular, the slope of Σ is given by

$$\rho^2 = 1 + \left(\phi \middle/ \frac{d\phi}{ds}\right)^2 \left(\frac{\partial u}{\partial \xi}\right)^2 + \delta^{ij}\frac{\partial u}{\partial\theta^i}\frac{\partial u}{\partial\theta^j}. \tag{2.4}$$

Let H be the mean curvature of Σ with respect to g_{HM}. Let Δ' be the Laplace-Beltrami operator of γ', the induced metric on Σ with respect to g' defined in (2.3). A direct computation using (A.11) with $\Psi = \phi^{-1}\frac{d\phi}{ds}$, $\psi = \log\phi$ and (1.6) shows

$$H = -\rho\phi^{-1}\Delta' u + (2^{-1}n)\rho^{-1}\phi^{1-n}\left(\frac{d\phi}{ds}\right)^{-1}\delta^{ij}\frac{\partial u}{\partial\theta^i}\frac{\partial u}{\partial\theta^j} + (n-1)\rho^{-1}\phi^{-1}\frac{d\phi}{ds} + (2^{-1}n)\rho^{-1}\phi^{1-n}\left(\frac{d\phi}{ds}\right)^{-1}.$$

Together with

$$d\mathrm{vol}_\gamma = \rho\phi^{n-2}\frac{d\phi}{ds}\, d\xi\wedge d\theta^3\wedge\cdots\wedge d\theta^n,$$

we derive from (1.9) that

$$Q(\Sigma) + 2^{-1}nm = \int_{T^{n-1}} \rho^2\phi^{n-2}\frac{d\phi}{ds}\Delta' u - (2^{-1}n)\delta^{ij}\frac{\partial u}{\partial\theta^i}\frac{\partial u}{\partial\theta^j}\, d\xi\wedge d\theta^3\wedge\cdots\wedge d\theta^n. \tag{2.5}$$

Let p be a function defined on M^n_{HM} to be determined. Let $F:[0,T_0)\times T^{n-1}\to M^n_{\mathrm{HM}}$ be a family of embeddings that satisfies

$$\frac{\partial F}{\partial t} = p\nu, \tag{2.6}$$

where ν is the unit outward normal. Denote by Σ_t the image of $F(t,\cdot)$ and by γ_t the induced metric.

Lemma 2.1.

- *For $n=3$, by taking $p=\phi^{-1}$, $Q(\Sigma_t)$ is monotone non-decreasing along the flow* (2.6).
- *For $n=4$, assume Σ_0 is symmetric along the ξ direction. Then by taking $p = (\phi\frac{d\phi}{ds})^{-1}$, $Q(\Sigma_t)$ is monotone non-decreasing along the flow* (2.6).

Furthermore, in the two cases above, $Q(\Sigma_t)$ is strictly increasing unless Σ_0 is a coordinate torus.

Proof. We compute

$$\frac{d}{dt}Q(\Sigma_t) = n(n-1)\int_{\Sigma_t} p\phi\, d\mathrm{vol}_{\gamma_t} - \int_{\Sigma_t}\left(\frac{\partial\phi}{\partial t}H + p\phi H^2 + \phi\frac{\partial H}{\partial t}\right) d\mathrm{vol}_{\gamma_t}. \tag{2.7}$$

Using the first variation of area and the Gauss equation,

$$\frac{\partial H}{\partial t} = -\Delta_{\gamma_t}p - p|h|^2 - p\mathrm{Ric}(\nu,\nu), \tag{2.8}$$

$$n(n-1) - H^2 + |h|^2 = -R_{\gamma_t} - 2\mathrm{Ric}(\nu,\nu). \tag{2.9}$$

Hence

$$\frac{d}{dt}Q(\Sigma_t) = \int_{\Sigma_t} p\phi\left[-R_{\gamma_t} - 2\mathrm{Ric}(\nu,\nu)\right] - pHg_{\mathrm{HM}}(\nabla\phi,\nu) + p\phi\mathrm{Ric}(\nu,\nu) + \phi\Delta_{\gamma_t}p\, d\mathrm{vol}_{\gamma_t}. \tag{2.10}$$

Using the static equation (1.4),

$$\Delta_{\gamma_t}\phi = \Delta\phi - \nabla^2\phi(\nu,\nu) - Hg_{\mathrm{HM}}(\nabla\phi,\nu) = -\phi\mathrm{Ric}(\nu,\nu) - Hg_{\mathrm{HM}}(\nabla\phi,\nu), \tag{2.11}$$

we obtain

$$\begin{aligned}\frac{d}{dt}Q(\Sigma_t) &= \int_{\Sigma_t} -p\phi R_{\gamma_t} + p\Delta_{\gamma_t}\phi + \phi\Delta_{\gamma_t}p\, d\mathrm{vol}_{\gamma_t}\\ &= \int_{\Sigma_t} -p\phi R_{\gamma_t} - 2\gamma_t(D\phi, Dp)\, d\mathrm{vol}_{\gamma_t}.\end{aligned} \tag{2.12}$$

For $n=3$, by choosing $p=\phi^{-1}$, we obtain from the Gauss-Bonnet Theorem that

$$\frac{d}{dt}Q(\Sigma_t) = \int_{\Sigma_t} -R_{\gamma_t} + 2\phi^{-2}\gamma_t(D\phi, D\phi)\, d\mathrm{vol}_{\gamma_t} \geq 0. \tag{2.13}$$

Moreover, $\frac{d}{dt}Q(\Sigma_t) = 0$ if and only if ϕ is a constant on Σ_t. This implies Σ_t is a coordinate torus. By reversing the flow (2.6), Σ_0 is also a coordinate torus.

For $n = 4$, we assume that Σ_0 is symmetric along the ξ direction. Hence Σ_t preserves the same symmetry. Denote by $\mathring{\gamma}_t$ the induced metric on $\mathring{\Sigma}_t = \Sigma_t \cap \{\xi \equiv \text{constant}\}$. Then

$$\gamma_t = \mathring{\gamma}_t + \left(\frac{d\phi}{ds}\right)^2 d\xi^2.$$

The scalar curvature of γ_t and $\mathring{\gamma}_t$ are related through

$$R_{\gamma_t} = R_{\mathring{\gamma}_t} - 2\left(\frac{d\phi}{ds}\right)^{-1} \Delta_{\mathring{\gamma}_t}\left(\frac{d\phi}{ds}\right).$$

Assuming p is also symmetric along the ξ direction, we get

$$\frac{d}{dt}Q(\Sigma_t) = \frac{4\pi}{n}\int_{\mathring{\Sigma}_t} -p\phi\frac{d\phi}{ds}R_{\mathring{\gamma}_t} + 2p\phi\Delta_{\mathring{\gamma}_t}\left(\frac{d\phi}{ds}\right) - 2\frac{d\phi}{ds}\mathring{\gamma}_t(D\phi, Dp)\,d\mathrm{vol}_{\mathring{\gamma}_t}.$$

Taking $p = \left(\phi\frac{d\phi}{ds}\right)^{-1}$, we deduce from the Gauss-Bonnet Theorem that

$$\begin{aligned}\frac{d}{dt}Q(\Sigma_t) =& \frac{4\pi}{n}\int_{\mathring{\Sigma}_t} -R_{\mathring{\gamma}_t} + 2\left(\frac{d\phi}{ds}\right)^{-1}\Delta_{\mathring{\gamma}_t}\left(\frac{d\phi}{ds}\right) - 2\frac{d\phi}{ds}\mathring{\gamma}_t\left(D\phi, D\left(\phi\frac{d\phi}{ds}\right)^{-1}\right) d\mathrm{vol}_{\mathring{\gamma}_t}\\ =& \frac{4\pi}{n}\int_{\mathring{\Sigma}_t}\left(-2\frac{d}{ds}\left(\frac{d\phi}{ds}\right)^{-1}\frac{d^2\phi}{ds^2} - 2\left(\frac{d\phi}{ds}\right)^2\frac{d}{ds}\left(\phi\frac{d\phi}{ds}\right)^{-1}\right)\cdot\mathring{\gamma}_t(Dv_t, Dv_t)\,d\mathrm{vol}_{\mathring{\gamma}_t}.\end{aligned}$$

From (1.6) and $n = 4$, we derive

$$-2\frac{d}{ds}\left(\frac{d\phi}{ds}\right)^{-1}\frac{d^2\phi}{ds^2} - 2\left(\frac{d\phi}{ds}\right)^2\frac{d}{ds}\left(\phi\frac{d\phi}{ds}\right)^{-1} = (1-\phi^{-4})^{-1}(6+2\phi^{-8}) \geq 0.$$

Thus $Q(\Sigma_t)$ is monotone non-decreasing along the flow. Using the same argument for the 3 dimensional case, $\frac{d}{dt}Q(\Sigma_t) = 0$ if and only if Σ_0 is a coordinate torus. □

3. Three Dimensional Case

In this section, we fix $n = 3$ and prove Theorem 1.1. Recall that $F : [0, T_0) \times T^{n-1} \to M^3_{\mathrm{HM}}$ solves

$$\frac{\partial F}{\partial t} = \phi^{-1}\nu. \tag{3.1}$$

Here we take $T_0 \in (0, \infty]$ to be the largest number such that (3.1) has a smooth graphical solution in $t \in [0, T_0)$.

Equation (3.1) can be viewed as the unit normal flow with respect to the conformal metric $\tilde{g} = \phi^2 g_{\mathrm{HM}}$. Recall that $\tilde{h}_{ab}$ is the the second fundamental form with respect to $\tilde{g}$. Our main assumption is

$$\tilde{h}_{ab} \text{ is non-negative definite for } \Sigma_0 \text{ and } \min_{\Sigma_0}\phi > 1. \tag{3.2}$$

We list two lemmas which allow us to prove Theorem 1.1.

Lemma 3.1. *Suppose $T_0 = \infty$. Then*

$$\limsup_{t\to\infty} Q(\Sigma_t) \leq -\frac{3m}{2}. \tag{3.3}$$

Lemma 3.2. *Under the assumption* (3.2), *we have $T_0 = \infty$.*

Proof of Theorem 1.1. Let Σ_t be the solution of (3.1) starting from $\Sigma_0 = \Sigma$. Combining Lemmas 2.1, 3.2 and 3.1,

$$Q(\Sigma) \leq \limsup_{t\to\infty} Q(\Sigma_t) \leq -\frac{3m}{2}.$$

Furthermore, the equality implies $Q(\Sigma_t)$ is a constant along the flow. And Lemma 2.1 implies Σ is a coordinate torus. □

We prove Lemmas 3.1 and Lemma 3.2 in the rest of this section. We adapt the convention that C denotes a large constant depending on Σ_0. The value of C may change from line to line.

Denote by $v_t(\xi, \theta)$ the height function of Σ_t and $u_t = q(v_t)$. Then v_t solves the equation

$$\frac{\partial v_t}{\partial t} = \rho\phi(v_t)^{-1}. \tag{3.4}$$

Here ρ is the slope of Σ_t defined in (2.4). We start with the C^0 and C^1 estimates for v_t and u_t.

Lemma 3.3. *There exists a constant C depending on Σ_0 such that for all $t \in [0, T_0)$,*

$$|\phi(v_t) - t| \leq C,\ \ \phi(v_t) \geq 1 + C^{-1}(t+1). \tag{3.5}$$

Proof. At the minimum point of v_t, we have $\rho = 1$. Together with (1.6),

$$\frac{d}{dt}\min_{T^2}\phi(v_t) \geq \left(1 - (\min_{T^2}\phi(v_t))^{-3}\right)^{1/2}.$$

Let $\Phi(t, a)$ be the solution to the ODE

$$\frac{\partial \Phi}{\partial t} = (1 - \Phi^{-3})^{1/2},\ \ \Phi(0, a) = a.$$

Through the ODE comparison,

$$\min_{T^2}\phi(v_t) \geq \Phi(t, \min_{T^2}\phi(v_0)).$$

From now on, we take $a = \min_{T^2}\phi(v_0)$. By the assumption (3.2), $a > 1$. Because $\Phi(t, a)$ is increasing in t, we have $\frac{\partial\Phi}{\partial t} \geq (1 - a^{-3})^{1/2}$. Through integration, $\Phi(t, a) \geq (1 - a^{-3})^{1/2}t + a$. This implies $\phi(v_t) \geq 1 + C^{-1}(t+1)$ for C large enough. Furthermore, using the equation of Φ again,

$$\left|\frac{\partial\Phi}{\partial t} - 1\right| \leq \frac{C}{(1+t)^3}.$$

Thus $|\Phi(t, a) - t| \leq C$ and $\min_{T^2}\phi(v_t) \geq t - C$. With a similar argument, we can also derive

$$\max_{T^2}\phi(v_t) \leq t + C.$$

The proof is finished. □

For the notation simplicity, we later denote $\phi(v_t)$ and $\frac{d\phi}{ds}(v_t)$ by ϕ and $\frac{d\phi}{ds}$ respectively. Recall that

$$\rho^2 = 1 + \left(\frac{\partial u_t}{\partial \theta}\right)^2 + \left(\phi \Big/ \frac{d\phi}{ds}\right)^2 \left(\frac{\partial u_t}{\partial \xi}\right)^2.$$

Lemma 3.4. *There exists a constant C depending on Σ_0 such that for $t \in [0, T_0)$,*

$$\max_{\Sigma_t}(\rho^2 - 1) \le (C^{-1}t + 1)^{-4} \max_{\Sigma_0}(\rho^2 - 1). \tag{3.6}$$

Proof. Using (1.6) and (3.4), we have

$$\frac{\partial^2 u_t}{\partial x^a \partial t} = \phi^{-2}\frac{\partial \rho}{\partial x^a} - 2\rho\phi^{-1}(1-\phi^{-3})^{1/2}\frac{\partial u_t}{\partial x^a}$$

and

$$\frac{\partial}{\partial t}\left(\phi \Big/ \frac{d\phi}{ds}\right)^2 = -\,3\rho\phi^{-4}(1-\phi^{-3})^{-3/2}.$$

It follows that

$$\begin{aligned}\frac{\partial}{\partial t}(\rho^2 - 1) =& 2\phi^{-2}\left(\frac{\partial u_t}{\partial \theta}\frac{\partial \rho}{\partial \theta} + \left(\phi \Big/ \frac{d\phi}{ds}\right)^2 \frac{\partial u_t}{\partial \xi}\frac{\partial \rho}{\partial \xi}\right)\\ &- 3\rho\phi^{-4}(1-\phi^{-3})^{3/2}\left(\frac{\partial u_t}{\partial \xi}\right)^2 - 4\rho\phi^{-1}(1-\phi^{-3})^{1/2}(\rho^2-1).\end{aligned}$$

At the maximum point of ρ^2, $d\rho = 0$ and

$$\frac{\partial}{\partial t}(\rho^2 - 1) \le -\,4\rho\phi^{-1}(1-\phi^{-3})^{1/2}(\rho^2 - 1).$$

Using (3.5), there exists a constant C such that for all $t \in [0, T_0)$,

$$4\rho\phi^{-1}(1-\phi^{-3})^{1/2} \ge 4\phi^{-1}(1-\phi^{-3})^{1/2} \ge 4(t+C)^{-1}.$$

Therefore,

$$\frac{d}{dt}\max_{\Sigma_t}(\rho^2 - 1) \le -\,4(t+C)^{-1}\max_{\Sigma_t}(\rho^2 - 1).$$

By the ODE comparison,

$$\max_{\Sigma_t}(\rho^2 - 1) \le (C^{-1}t + 1)^{-4}\max_{\Sigma_0}(\rho^2 - 1).$$

The proof is finished. □

Recall that $g' = \phi^{-2} g_{\mathrm{HM}}$ is a conformal metric which asymptotic to the flat metric on $\mathbb{R} \times T^2$. Let γ'_t be the induced metric of g' on Σ_t and D' be the Levi-Civita connection of γ'_t. Lemmas 3.3 and 3.4 imply, provided $T_0 = \infty$, γ'_t converges the the flat metric $d\xi^2 + d\theta^2$ in C^0. Next, we use the second fundamental form h'_{ab} to bound the Hessian of u_t from below.

Lemma 3.5. *Suppose $T_0 = \infty$. Then there exists a constant C depending on Σ_0 such that for $t \ge C + 1$,*

$$D'_b(D')^a u_t \ge -\left(C(t-C)^{-2}\log t\right)\delta^a_b.$$

Proof. The flow (3.1) can be rewritten as

$$\frac{\partial F}{\partial t} = \phi^{-2}\nu'.$$

Here $\nu' = \phi\nu$ is the unit normal vector with respect to g'. The evolution equation of $(h')^a_b$ is given by

$$\frac{\partial}{\partial t}(h')^a_b = -\phi^{-2}(h')^c_b(h')^a_c - D'_b(D')^a\phi^{-2} - \phi^{-2}(dx^a)_\alpha(\nu')^\beta\left(\frac{\partial}{\partial x^b}\right)^\lambda(\nu')^\mu(R')^\alpha{}_{\beta\lambda\mu}.$$

Here $(R')_{\alpha\beta\lambda\mu}$ is the Riemannian curvature tensor of g'. In the estimates below, the norms are computed with respect to γ'_t. From (A.3) and (1.6), the only non-zero component of R' is

$$R'_{q\xi q\xi} = 3\phi^{-1}(1-\phi^{-3}).$$

Together with (3.5), for t large enough,

$$\left|-\phi^{-2}(dx^a)_\alpha(\nu')^\beta\left(\frac{\partial}{\partial x^b}\right)^\lambda(\nu')^\mu(R')^\alpha{}_{\beta\lambda\mu}\right| \le C(t-C)^{-3}.$$

Using (1.6) and (2.2),

$$-D'_b(D')^a\phi^{-2} = 2\phi^{-1}(1-\phi^{-3})^{1/2}D'_b(D')^a u_t + (-2+5\phi^{-3})D'_b u_t(D')^a u_t.$$

The relation between h'_{ab} and $D'_bD'_au_t$ is given in Corollary A.3 with $\Psi = \phi^{-1}\frac{d\phi}{ds}$. Using (1.6), we derive

$$D'_b(D')_a u_t = -\rho^{-1}h'_{ab} + \frac{3}{2}\phi^{-2}(1-\phi^{-3})^{1/2}D'_b\xi D'_a\xi. \tag{3.7}$$

Hence

$$\begin{aligned}-D'_b(D')^a\phi^{-2} =& -2\rho^{-1}\phi^{-1}(1-\phi^{-3})^{1/2}(h')^a_b + (-2+5\phi^{-3})D'_bu_t(D')^a u_t\\ &+ 3\phi^{-3}(1-\phi^{-3})D'_b\xi(D')^a\xi.\end{aligned}$$

From (3.5) and (3.6), for t large enough, we have

$$|D'_bu_t(D')^au_t| \le C(t-C)^{-4}$$

and

$$|3\phi^{-3}(1-\phi^{-3})D'_b\xi(D')^a\xi| \le (t-C)^{-3}.$$

Putting the above together, we get

$$\left|\frac{\partial}{\partial t}(h')^a_b + 2\rho^{-1}\phi^{-1}(1-\phi^{-3})^{1/2}(h')^a_b + \phi^{-2}(h')^c_b(h')^a_c\right| \le C(t-C)^{-3}.$$

Let w_t be the maximum eigenvalue of $(h')^b_a$. Using again (3.5) and (3.6), for t large enough, we have $2\rho^{-1}\phi^{-1}(1-\phi^{-3})^{1/2} \ge 2(t+C)^{-1}$. Therefore,

$$\frac{dw_t}{dt} \le -2(t+C)^{-1}w_t + C(t-C)^{-3}.$$

From the ODE comparison, $w_t \le C(t-C)^{-2}\log t$ for t large enough. The assertion then follows in view of (3.7). □

Proof of Lemma 3.1. From (2.5),

$$Q(\Sigma_t) + \frac{3m}{2} = \int_{T^2} \rho^2 \phi \frac{d\phi}{ds} \Delta' u_t - \frac{3}{2}\left(\frac{\partial u_t}{\partial \theta}\right)^2 d\xi \wedge d\theta.$$

From Lemma 3.4, $\left(\frac{\partial u_t}{\partial \theta}\right)^2 \le C(t-C)^{-4}$. It remains to prove

$$\limsup_{t\to\infty} \int_{T^2} \rho^2 \phi \frac{d\phi}{ds} \Delta' u_t \, d\xi \wedge d\theta \le 0.$$

Recall that γ'_t is the induced metric of Σ from g'. Because

$$d\mathrm{vol}_{\gamma'_t} = \rho\phi^{-1} \frac{d\phi}{ds}\, d\xi \wedge d\theta,$$

$$\int_{T^2} \rho^2 \phi \frac{d\phi}{ds} \Delta' u_t \, d\xi \wedge d\theta = \int_{\Sigma_t} \rho \phi^2 \Delta' u_t \, d\mathrm{vol}_{\gamma'_t} = \underbrace{\int_{\Sigma_t} (\rho - 1)\phi^2 \Delta' u_t \, d\mathrm{vol}_{\gamma'_t}}_{\mathrm{I}} + \underbrace{\int_{\Sigma_t} \phi^2 \Delta' u_t \, d\mathrm{vol}_{\gamma'_t}}_{\mathrm{II}}.$$

From Lemmas 3.3, 3.4 and 3.5, $\phi^2 \le C(t+C)^2$, $|\rho - 1| \le C(t-C)^{-4}$ and

$$\int_{\Sigma_t} |\Delta' u_t| \, d\mathrm{vol}_{\gamma'_t} = 2 \int_{\Sigma_t} (\Delta' u_t)_- \, d\mathrm{vol}_{\gamma'_t} \le C(t-C)^{-2} \log t.$$

Therefore the first term I goes to zero. Through integration by parts,

$$\mathrm{II} = -\int_{\Sigma_t} 2\phi \frac{d\phi}{ds} |D' u_t|^2_{\gamma'_t} \, d\mathrm{vol}_{\gamma'_t} \le 0.$$

Thus the assertion (3.3) follows. □

Lastly, we prove Lemma 3.2. Recall that $\tilde{g} = \phi^2 g_{\mathrm{HM}}$ and that $\tilde{h}^a_b$ is the Weingarten tensor with respect to $\tilde{g}$. The flow (3.1) is a unit normal flow with respect to $\tilde{g}$. The evolution equation of $\tilde{h}^a_b$ is given by

$$\frac{\partial}{\partial t} \tilde{h}^a_b = -\tilde{h}^c_b \tilde{h}^a_c - (dx^a)_\alpha \tilde{\nu}^\beta \left(\frac{\partial}{\partial x^b}\right)^\lambda \tilde{\nu}^\mu \tilde{R}^\alpha{}_{\beta\lambda\mu}. \tag{3.8}$$

Lemma 3.6. *Suppose the assumption* (3.2) *holds. Then there exists a continuous function* $C(t)$ *defined on* $[0,\infty)$ *such that for all* $t \in [0, T_0)$,

$$|\tilde{h}^a_b|_{\tilde{\gamma}_t} \le |\tilde{h}^a_b|_{\tilde{\gamma}_0} + C(t).$$

Proof. Using (A.9) with $\Psi = \phi^{-1}\frac{d\phi}{ds}$, $\psi = 2\log\phi$ and (1.6), we derive that non-zero components of $\widetilde{R}_{\alpha\beta\lambda\mu}$ are

$$\begin{aligned}
\widetilde{R}_{q\xi q\xi} &= -\phi^6(2+\phi^{-3})(1-\phi^{-3}),\\
\widetilde{R}_{q\theta q\theta} &= -\phi^6(2+\phi^{-3}),\\
\widetilde{R}_{\xi\theta\xi\theta} &= -\phi^6(4-\phi^{-3})(1-\phi^{-3}).
\end{aligned}$$

In particular, the sectional curvature of $\tilde{g}$ is non-positive. In view of (3.8), $\tilde{h}^a_b$ remains non-negative definite under the assumption (3.2). From (3.5), there exists a continuous function $C(t)$ such that for any $t \in [0, T_0)$,

$$\left| (dx^a)_\alpha \tilde{\nu}^\beta \left(\frac{\partial}{\partial x^b}\right)^\lambda \tilde{\nu}^\mu \tilde{R}^\alpha{}_{\beta\lambda\mu} \right| \le C(t).$$

Applying the ODE comparison to (3.8), we have an upper bound for the maximum eigenvalue of $\tilde{h}^a_b$ in any finite time. The proof is finished. $\square$

Proof of Lemma 3.2. Recall that v_t is the height function of Σ_t in the s coordinate and $u_t = q(v_t)$. It suffices to show that for any $T < \infty$, there exists constants $C_j(T)$, $j = 0, 1, \ldots$ such that for all $0 \le t < T$

$$\left| \frac{\partial^j u_t}{\partial x^{a_1} \partial x^{a_2} \ldots \partial x^{a_j}} \right| \le C_j(T).$$

Lemma 3.3 and Lemma 3.4 provide such bounds for the case $j = 0$ and $j = 1$ respectively. In view of (3.7) and Lemma A.5, Lemma 3.6 shows the case $j = 2$. Furthermore, by differentiating (3.7), it suffices to show that for all $j \ge 1$,

$$|\widetilde{D}^j \tilde{h}|_{\tilde{\gamma}} \le C_{j+2}(T). \tag{3.9}$$

Here $\widetilde{D}$ is the Levi-Civita connection of $\tilde{\gamma}$. We derive (3.9) by computing the evolution equation of $\tilde{\gamma}(\widetilde{D}^j \tilde{h}, \widetilde{D}^j \tilde{h})$. To present the calculation in a systematic manner, we adapt the notation that $A * B$ stands for a term obtained from contracting some of indices in A and B through $\tilde{\gamma}$.

Compute

$$\frac{\partial}{\partial t} \tilde{\gamma}(\widetilde{D}^j \tilde{h}, \widetilde{D}^j \tilde{h}) = \underbrace{\left(\frac{\partial}{\partial t} \tilde{\gamma} \right) (\widetilde{D}^j \tilde{h}, \widetilde{D}^j \tilde{h})}_{\mathrm{I}} + \underbrace{\tilde{\gamma} \left(\left[\frac{\partial}{\partial t}, \widetilde{D}^j \right] \tilde{h}, \widetilde{D}^j \tilde{h} \right)}_{\mathrm{II}} + \underbrace{\tilde{\gamma} \left(\widetilde{D}^j \frac{\partial}{\partial t} \tilde{h}, \widetilde{D}^j \tilde{h} \right)}_{\mathrm{III}}.$$

Because $\frac{\partial}{\partial t} \tilde{\gamma}_{ab} = 2 \tilde{h}_{ab}$,

$$\mathrm{I} = \widetilde{D}^j \tilde{h} * \widetilde{D}^j \tilde{h} * \tilde{h}.$$

For any tensor N, we have the commutation relation $[\frac{\partial}{\partial t}, \widetilde{D}^j] N = \sum_{i=0}^{j-1} \widetilde{D}^{j-i} \tilde{h} * \widetilde{D}^i N$. Hence

$$\mathrm{II} = \sum_{i=0}^{j} \widetilde{D}^j \tilde{h} * \widetilde{D}^{j-i} \tilde{h} * \widetilde{D}^i \tilde{h}.$$

Finally, from (3.8),

$$\mathrm{III} = \sum_{i=0}^{j} \widetilde{D}^j \tilde{h} * \widetilde{D}^{j-i} \tilde{h} * \widetilde{D}^i \tilde{h} + \widetilde{D}^j K * \widetilde{D}^j \tilde{h}.$$

Here

$$K_{ab} = \left(\frac{\partial}{\partial x^a} \right)^{\alpha} \tilde{\nu}^{\beta} \left(\frac{\partial}{\partial x^b} \right)^{\lambda} \tilde{\nu}^{\mu} \tilde{R}_{\alpha\beta\lambda\mu}.$$

Let

$$\omega_j(t) = \max_{\Sigma_t} |\widetilde{D}^j \tilde{h}|^2_{\tilde{\gamma}}.$$

We use induction and assume $\omega_i(t) \le C_i(T)$ for $0 \le i \le j-1$ and $t \in [0, T)$. By Lemma B.2, $|\widetilde{D}^i K|_{\tilde{\gamma}}$ is bounded for $0 \le i \le j$. We then deduce

$$\frac{d}{dt} \omega_j \le C \omega_j + C.$$

From the ODE comparison, ω_j remains bounded in $t \in [0, T)$. Thus (3.9) follows. $\square$

4. Four-dimensional with symmetry in ξ

In this section, we fix $n = 4$ and prove Theorem 1.2. Recall that $F : [0, T_0) \times T^{n-1} \to M^4_{\mathrm{HM}}$ solves

$$\frac{\partial F}{\partial t} = \left(\phi \frac{d\phi}{ds}\right)^{-1} \nu. \tag{4.1}$$

Here we take $T_0 \in (0, \infty]$ to be the largest number such that (4.1) has a smooth graphical solution in $t \in [0, T_0)$.

Equation (4.1) can be viewed as the unit normal flow with respect to the conformal metric $\bar{g} = \left(\phi \frac{d\phi}{ds}\right)^2 g_{\mathrm{HM}}$. Recall that $\bar{h}_{ab}$ is the the second fundamental form with respect to $\bar{g}$. Our main assumption is

$$\bar{h}_{ab} \text{ is non-negative definite for } \Sigma_0 \text{ and } \min_{\Sigma_0} \phi^4 \geq 1 + \frac{2}{\sqrt{3}}. \tag{4.2}$$

We list two lemmas which allow us to prove Theorem 1.2.

Lemma 4.1. *Suppose $T_0 = \infty$. Then*

$$\limsup_{t\to\infty} Q(\Sigma_t) \leq -2m. \tag{4.3}$$

Lemma 4.2. *Under the assumption* (4.2) *, we have $T_0 = \infty$.*

Proof of Theorem 1.2. Let Σ_t be the solution of (4.1) starting from $\Sigma_0 = \Sigma$. Combining Lemmas 2.1, 4.2 and 4.1,

$$Q(\Sigma) \leq \limsup_{t\to\infty} Q(\Sigma_t) \leq -2m.$$

Furthermore, the equality implies $Q(\Sigma_t)$ is a constant along the flow. And Lemma 2.1 implies Σ is a coordinate torus. $\square$

We prove Lemmas 4.1 and Lemma 4.2 in the rest of this section. We again adapt the convention that C denotes a large constant depending on Σ_0. The value of C may change from line to line.

Denote by $v_t(\xi, \theta)$ the height function of Σ_t and $u_t = q(v_t)$. Then v_t solves the equation

$$\frac{\partial v_t}{\partial t} = \rho \left(\phi(v_t) \frac{d\phi}{ds}(v_t)\right)^{-1}. \tag{4.4}$$

Here ρ is the slope of Σ_t defined in (2.4). Because Σ_t are symmetric along the ξ direction, the slope ρ is given by

$$\rho^2 = 1 + \delta^{ij} \frac{\partial u_t}{\partial \theta^i} \frac{\partial u_t}{\partial \theta^j}.$$

We start with the C^0 and C^1 estimates for v_t and u_t.

Lemma 4.3. *There exists a constant C depending on Σ_0 such that for all $t \in [0, T_0)$,*

$$|\phi^2(v_t) - 2t| \leq C. \tag{4.5}$$

Proof. At the maximum point of v_t, we have $\rho = 1$. Together with (1.6),

$$\frac{d}{dt}\max_{T^2}\phi^2(v_t) \le 2.$$

Similarly,

$$\frac{d}{dt}\min_{T^2}\phi^2(v_t) \ge 2.$$

Thus the assertion follows by taking $C = \max_{T^2}\phi^2(v_0)$. $\square$

Lemma 4.4. *There exists constant C depending on Σ_0 such that for $t \in [0, T_0)$,*

$$\max_{\Sigma_t}(\rho^2 - 1) \le (C^{-1}t + 1)^{-3}\max_{\Sigma_0}(\rho^2 - 1). \tag{4.6}$$

Proof. From

$$\frac{\partial^2 u_t}{\partial x^a \partial t} = \phi^{-2}\left(\frac{d\phi}{ds}\right)^{-1}\frac{\partial \rho}{\partial x^a} + \rho\phi^{-2}(-3 + \phi^{-4})(1 - \phi^{-4})^{-1}\frac{\partial u_t}{\partial x^a},$$

we deduce

$$\frac{\partial}{\partial t}(\rho^2 - 1) = 2\phi^{-2}\left(\frac{d\phi}{ds}\right)^{-1}\delta^{ij}\frac{\partial u_t}{\partial \theta^i}\frac{\partial \rho}{\partial \theta^j} + \rho\phi^{-2}(-3 + \phi^{-4})(1 - \phi^{-4})^{-1}(\rho^2 - 1).$$

At the maximum point of $\rho^2 - 1$, $d\rho = 0$ and

$$\frac{\partial}{\partial t}(\rho^2 - 1) \le \rho\phi^{-2}(-3 + \phi^{-4})(1 - \phi^{-4})^{-1}(\rho^2 - 1).$$

Using (4.5), there exists a constant C such that for all $t \in [0, T_0)$,

$$\rho\phi^{-2}(-3 + \phi^{-4})(1 - \phi^{-4})^{-1} \ge \phi^{-2}(-3 + \phi^{-4})(1 - \phi^{-4})^{-1} \ge -3(t + C)^{-1}.$$

Hence

$$\frac{d}{dt}\max_{\Sigma_t}(\rho^2 - 1) \le -3(t + C)^{-1}\max_{\Sigma_t}(\rho^2 - 1).$$

By the ODE comparison,

$$\max_{\Sigma_t}(\rho^2 - 1) \le (C^{-1}t + 1)^{-3}\max_{\Sigma_0}(\rho^2 - 1).$$

The proof is finished. $\square$

Recall that $g' = \phi^{-2}g_{\mathrm{HM}}$ is a conformal metric which asymptotic to the flat metric on $\mathbb{R} \times T^3$. Let γ'_t be the induced metric of g' on Σ_t and D' be the Levi-Civita connection of γ'_t. Lemmas 4.3 and 4.4 imply, provided $T_0 = \infty$, γ'_t converges the the flat metric $d\xi^2 + d\theta^2$ in C^0. Next, we use the second fundamental form h'_{ab} to bound the Hessian of u_t from below.

Lemma 4.5. *Suppose $T_0 = \infty$. Then there exists a constant C depending on Σ_0 such that for $t \ge C + 1$,*

$$D'_b(D')^a u_t \ge -\left(C(t - C)^{-3/2}\log t\right)\delta^a_b.$$

Proof. The flow (4.1) can be rewritten as

$$\frac{\partial F}{\partial t} = \left(\phi^2\frac{d\phi}{ds}\right)^{-1}\nu'.$$

Here $\nu' = \phi\nu$ is the unit normal vector with respect to g'. Let $G = \phi^2 \frac{d\phi}{ds}$. Then the evolution equation of $(h')^a_b$ is given by

$$\frac{\partial}{\partial t}(h')^a_b = -G^{-1}(h')^c_b(h')^a_c - D'_b(D')^a G^{-1} - G^{-1}(dx^a)_\alpha(\nu')^\beta \left(\frac{\partial}{\partial x^b}\right)^\lambda (\nu')^\mu (R')^\alpha{}_{\beta\lambda\mu}.$$

Here $(R')_{\alpha\beta\lambda\mu}$ is the Riemannian curvature tensor of g'. In the estimates below, the norms are computed with respect to γ'_t. From (A.3) and (1.6), the only non-zero component of R' is

$$R'_{q\xi q\xi} = 6\phi^{-2}(1-\phi^{-4}).$$

Together with (4.5), for t large enough,

$$\left| -G^{-1}(dx^a)_\alpha(\nu')^\beta \left(\frac{\partial}{\partial x^b}\right)^\lambda (\nu')^\mu (R')^\alpha{}_{\beta\lambda\mu} \right| \le C(t-C)^{-5/2}.$$

Using (2.2),

$$-D'_b(D')^a G^{-1} = G^{-2}\frac{dG}{dq} D'_b(D')^a u_t + \frac{d}{dq}\left(G^{-2}\frac{dG}{dq}\right) D'_b u_t (D')^a u_t.$$

The relation between h'_{ab} and $D'_b D'_a u_t$ is given in Corollary A.3 with $\Psi = \phi^{-1}\frac{d\phi}{ds}$. Using (1.6), we derive

$$D'_b D'_a u_t = -\rho^{-1} h'_{ab} + 2\phi^{-3}(1-\phi^{-4})^{1/2} D'_b\xi D'_a\xi. \tag{4.7}$$

Together with

$$\begin{aligned} G^{-2}\frac{dG}{dq} &= \phi^{-2}(3-\phi^{-4})(1-\phi^{-4})^{-1}, \\ \frac{d}{dq}\left(G^{-2}\frac{dG}{dq}\right) &= -2\phi^{-1}(3+\phi^{-8})(1-\phi^{-4})^{-3/2}, \end{aligned}$$

we derive

$$\begin{aligned} -D'_b(D')^a G^{-1} = & -\rho^{-1}\phi^{-2}(3-\phi^{-4})(1-\phi^{-4})^{-1}(h')^a_b - 2\phi^{-1}(3+\phi^{-8})(1-\phi^{-4})^{-3/2} D'_b u_t (D')^a u_t \\ & + 2\phi^{-5}(3-\phi^{-4})(1-\phi^{-4})^{-1/2} D'_b\xi (D')^a\xi. \end{aligned}$$

From (3.5) and (3.6), for t large enough,

$$|2\phi^{-5}(3-\phi^{-4})(1-\phi^{-4})^{-1/2} D'_b\xi (D')^a\xi| \le C(t-C)^{-5/2}.$$

Putting the above together, we derive

$$\begin{aligned} \Bigg| & \frac{\partial}{\partial t}(h')^a_b + \rho^{-1}\phi^{-2}(3-\phi^{-4})(1-\phi^{-4})^{-1}(h')^a_b + G^{-1}(h')^c_b(h')^a_c \\ & + 2\phi^{-1}(3+\phi^{-8})(1-\phi^{-4})^{-3/2} D'_b u_t (D')^a u_t \Bigg| \le C(t-C)^{-5/2}. \end{aligned}$$

Let w_t be the maximum eigenvalue of $(h')^b_a$. From (3.5) and (3.6), for t large enough,

$$\rho^{-1}\phi^{-2}(3-\phi^{-4})(1-\phi^{-4}) \ge \frac{3}{2}(t+C)^{-1}.$$

We derive

$$\frac{dw_t}{dt} \le -\frac{3}{2}(t+C)^{-1} w_t + C(t-C)^{-5/2}.$$

From the ODE comparison, $w_t \le C(t-C)^{-3/2}\log t$ for t large enough. Hence the assertion follows (4.7). $\square$

Proof of Lemma 4.1. From (2.5),

$$Q(\Sigma_t) + 2m = \int_{T^3} \rho^2 \phi \frac{d\phi}{ds} \Delta' u_t - 2\delta^{ij} \frac{\partial u_t}{\partial \theta^i} \frac{\partial u_t}{\partial \theta^j}\, d\xi \wedge d\theta^3 \wedge d\theta^4.$$

From Lemma 4.4, $\delta^{ij} \frac{\partial u_t}{\partial \theta^i} \frac{\partial u_t}{\partial \theta^j} \le C(t-C)^{-3}$. It remains to prove

$$\limsup_{t\to\infty} \int_{T^3} \rho^2 \phi \frac{d\phi}{ds} \Delta' u_t\, d\xi \wedge d\theta^3 \wedge d\theta^4 \le 0.$$

Recall that γ'_t is the induced metric of Σ from g'. Because

$$d\mathrm{vol}_{\gamma'_t} = \rho \phi^{-1} \frac{d\phi}{ds} d\xi \wedge d\theta^3 \wedge d\theta^4,$$

$$\int_{T^3} \rho^2 \phi \frac{d\phi}{ds} \Delta' u_t d\xi \wedge d\theta^3 \wedge d\theta^4 = \int_{\Sigma_t} \rho \phi^2 \Delta' u_t \, d\mathrm{vol}_{\gamma'_t} = \underbrace{\int_{\Sigma_t} (\rho - 1)\phi^2 \Delta' u_t \, d\mathrm{vol}_{\gamma'_t}}_{\mathrm{I}} + \underbrace{\int_{\Sigma_t} \phi^2 \Delta' u_t \, d\mathrm{vol}_{\gamma'_t}}_{\mathrm{II}}.$$

From Lemmas 4.3, 4.4 and 4.6, $\phi^2 \le Ct$, $|\rho - 1| \le Ct^{-3}$ and

$$\int_{\Sigma_t} |\Delta' u_t| \, d\mathrm{vol}_{\gamma'} = 2 \int_{\Sigma_t} (\Delta' u_t)_- \, d\mathrm{vol}_{\gamma'} \le Ct^{-3/2} \log t.$$

Hence the first term I goes to zero. Through integration by parts,

$$\mathrm{II} = -\int_{\Sigma_t} 2\phi \frac{d\phi}{ds} |D' u_t|^2_{\gamma'_t} \, d\mathrm{vol}_{\gamma'} \le 0.$$

Thus (4.3) follows. $\square$

Lastly, we prove Lemma 4.2. Recall that $\bar g = \left(\phi \frac{d\phi}{ds}\right)^2 g_{\mathrm{HM}}$ and that $\bar h^a_b$ is the Weingarten tensor with respect to $\bar g$. The flow (4.1) is the unit normal flow with respect to $\bar g$. The evolution equation of $\bar h^a_b$ is given by

$$\frac{\partial}{\partial t} \bar h^a_b = -\bar h^c_b \bar h^a_c - (dx^a)_\alpha \bar\nu^\beta \left(\frac{\partial}{\partial x^b}\right)^\lambda \bar\nu^\mu \overline{R}^\alpha{}_{\beta\lambda\mu}. \tag{4.8}$$

Lemma 4.6. *Under the assumption* (4.2)*, there exists a continuous function* $C(t)$ *defined on* $[0,\infty)$ *such that for all* $t \in [0, T_0)$*,*

$$|\bar h^a_b|_{\bar\gamma_t} \le |\bar h^a_b|_{\bar\gamma_0} + C(t)$$

Proof. Using (A.9) with $\Psi = \phi^{-1}\frac{d\phi}{ds}$ and $\psi = 2\log\phi + \log\frac{d\phi}{ds}$, we derive that non-zero components of $\overline{R}_{\alpha\beta\lambda\mu}$ are

$$\begin{aligned}
\overline{R}_{q\xi q\xi} &= -3\phi^8(1-\phi^{-4})^3,\\
\overline{R}_{qiqi} &= -\phi^8(3 - 6\phi^{-4} - \phi^{-8}),\\
\overline{R}_{\xi i\xi i} &= -\phi^8(9-\phi^{-8})(1-\phi^{-4}),\\
\overline{R}_{ijij} &= -\phi^8(1-\phi^{-4})^2,\ i \ne j.
\end{aligned}$$

Besides $\overline{R}_{qiqi}$, other sectional curvatures are apparently non-positive. Actually, the polynomial $3x^2-6x-1$ has roots $x=1\pm\frac{2}{\sqrt{3}}$. Under the assumption (4.2), we have $\phi^4\geq 1+\frac{2}{\sqrt{3}}$ along the flow. Hence the term

$$-(dx^a)_\alpha\bar{\nu}^\beta\left(\frac{\partial}{\partial x^b}\right)^\lambda\bar{\nu}^\mu\overline{R}^\alpha{}_{\beta\lambda\mu}$$

is non-negative definite and $\bar{h}^a_b$ also remains non-negative definite . On the other hand, from (4.5), for any $t\in[0,T_0)$,

$$\left|(dx^a)_\alpha\bar{\nu}^\beta\left(\frac{\partial}{\partial x^b}\right)^\lambda\bar{\nu}^\mu\overline{R}^\alpha{}_{\beta\lambda\mu}\right|\leq C(t).$$

for some function $C(t)$. Then the assertion follows by the ODE comparison. □

With Lemma 4.6, the proof of Lemma 4.2 is similar to the one for Lemma 3.2.

Appendix A. Curvature and Second Fundamental Form

In this appendix, we record formulas of curvatures and second fundamental forms. Let $\Psi(q)$ be a smooth and positive function. Consider the metric

$$g'=dq^2+\Psi(q)^2d\xi^2+\sum_{i=3}^n(d\theta^i)^2. \tag{A.1}$$

The non-zero components of the Christoffel symbols are

$$(\Gamma')^\xi_{q\xi}=\Psi^{-1}\frac{d\Psi}{dq},\ (\Gamma')^q_{\xi\xi}=-\Psi\frac{d\Psi}{dq}. \tag{A.2}$$

The only non-zero component of the curvature is

$$R'_{q\xi q\xi}=-\Psi\frac{d^2\Psi}{dq^2}. \tag{A.3}$$

We now consider a hypersurface given by a graph over T^{n-1}. Let $u(\xi,\theta^i)$ be a smooth function and consider the map $F(\xi,\theta^i)=(u(\xi,\theta^i),\xi,\theta^i)$. Denote the image of F by Σ.

$$\frac{\partial F}{\partial\xi}=\frac{\partial}{\partial\xi}+\frac{\partial u}{\partial\xi}\frac{\partial}{\partial q},\ \frac{\partial F}{\partial\theta^i}=\frac{\partial}{\partial\theta^i}+\frac{\partial u}{\partial\theta^i}\frac{\partial}{\partial q}.$$

Let $\{x^a\}=\{\xi,\theta^i\}$ and define a metric on T^{n-1} as

$$\hat{\gamma}_{ab}dx^adx^b=\Psi(u)^2d\xi^2+\delta_{ij}d\theta^id\theta^j. \tag{A.4}$$

The induced metric γ'_{ab} on Σ is given by

$$\gamma'_{ab}=\hat{\gamma}_{ab}+\frac{\partial u}{\partial x^a}\frac{\partial u}{\partial x^b}. \tag{A.5}$$

The normal vector of Σ is given by

$$\nu'=\rho^{-1}\left(\frac{\partial}{\partial q}-\frac{1}{\Psi(u)^2}\frac{\partial u}{\partial\xi}\frac{\partial}{\partial\xi}-\delta^{ij}\frac{\partial u}{\partial\theta^i}\frac{\partial}{\partial\theta^j}\right), \tag{A.6}$$

where ρ is the slope given by

$$\rho^2=1+\Psi(u)^{-2}\left(\frac{\partial u}{\partial\xi}\right)^2+\delta^{ij}\frac{\partial u}{\partial\theta^i}\frac{\partial u}{\partial\theta^j}. \tag{A.7}$$

Lemma A.1. *The second fundamental form of Σ is given by*

$$\begin{aligned} h'_{\xi\xi} =& -\rho^{-1}\frac{\partial^2 u}{\partial \xi^2} + 2\rho^{-1}\Psi^{-1}\frac{d\Psi}{dq}\left(\frac{\partial u}{\partial \xi}\right)^2 + \rho^{-1}\Psi\frac{d\Psi}{dq}, \\ h'_{ij} =& -\rho^{-1}\frac{\partial^2 u}{\partial\theta^i\partial\theta^j}, \\ h'_{\xi i} =& -\rho^{-1}\frac{\partial^2 u}{\partial\xi\partial\theta^i} + \rho^{-1}\Psi^{-1}\frac{d\Psi}{dq}\frac{\partial u}{\partial \xi}\frac{\partial u}{\partial\theta^i}. \end{aligned}$$

Proof. Let ∇' be the Levi-Civita connection of g'. From (A.2),

$$\begin{aligned} \nabla'_{\frac{\partial F}{\partial \xi}}\frac{\partial F}{\partial \xi} =& \left(\frac{\partial^2 u}{\partial \xi^2} - \Psi\frac{d\Psi}{dq}\right)\frac{\partial}{\partial q} + 2\Psi^{-1}\frac{d\Psi}{dq}\frac{\partial u}{\partial\xi}\frac{\partial}{\partial\xi}, \\ \nabla'_{\frac{\partial F}{\partial \theta^i}}\frac{\partial F}{\partial \theta^j} =& \frac{\partial^2 u}{\partial\theta^i\theta^j}\frac{\partial}{\partial q}, \\ \nabla'_{\frac{\partial F}{\partial \theta^i}}\frac{\partial F}{\partial \xi} =& \frac{\partial^2 u}{\partial\xi\partial\theta^i}\frac{\partial}{\partial q} + \Psi^{-1}\frac{d\Psi}{dq}\frac{\partial u}{\partial\theta^i}\frac{\partial}{\partial\xi}. \end{aligned}$$

Taking inner product with ν' given in (A.6), the assertion follows. $\square$

We now view u as a function defined on Σ and compute its Hessian. Let D' be the Levi-Civita connection of γ'.

Lemma A.2.

$$\begin{aligned} D'_\xi D'_\xi u =& \rho^{-2}\frac{\partial \tilde{u}}{\partial \xi^2} - 2\rho^{-2}\Psi^{-1}\frac{d\Psi}{dq}\left(\frac{\partial u}{\partial \xi}\right)^2 - \rho^{-2}\Psi\frac{d\Psi}{dq} + \Psi\frac{d\Psi}{dq}, \\ D'_i D'_j u =& \rho^{-2}\frac{\partial^2 u}{\partial\theta^i\partial\theta^j}, \\ D'_\xi D'_i u =& \rho^{-2}\frac{\partial^2 u}{\partial\xi\partial\theta^i} - \rho^{-2}\Psi^{-1}\frac{d\Psi}{dq}\frac{\partial u}{\partial\xi}\frac{\partial u}{\partial\theta^i}. \end{aligned}$$

Proof. Let $\hat{\Gamma}^c_{ab}$ be the Christoffel symbols of $\hat{\gamma}_{ab}$. The non-zero components of $\hat{\Gamma}^c_{ab}$ are

$$\hat{\Gamma}^\xi_{\xi i} = \Psi^{-1}\frac{d\Psi}{dq}\frac{\partial u}{\partial\theta^i},\ \hat{\Gamma}^i_{\xi\xi} = -\Psi\frac{d\Psi}{dq}\frac{\partial u}{\partial\theta^i},\ \hat{\Gamma}^\xi_{\xi\xi} = \Psi^{-1}\frac{d\Psi}{dq}\frac{\partial u}{\partial\xi}.$$

For any smooth function $\tilde{u}$,

$$\begin{aligned} \hat{D}_\xi\hat{D}_\xi\tilde{u} =& \frac{\partial\tilde{u}}{\partial\xi^2} + \Psi\frac{d\Psi}{dq}\delta^{ij}\frac{\partial u}{\partial\theta^i}\frac{\partial\tilde{u}}{\partial\theta^j} - \Psi^{-1}\frac{d\Psi}{dq}\frac{\partial u}{\partial\xi}\frac{\partial\tilde{u}}{\partial\xi}, \\ \hat{D}_i\hat{D}_j\tilde{u} =& \frac{\partial^2\tilde{u}}{\partial\theta^i\partial\theta^j}, \\ \hat{D}_\xi\hat{D}_i\tilde{u} =& \frac{\partial^2\tilde{u}}{\partial\xi\partial\theta^i} - \Psi^{-1}\frac{d\Psi}{dq}\frac{\partial u}{\partial\theta^i}\frac{\partial\tilde{u}}{\partial\xi}. \end{aligned}$$

Here $\hat{D}$ is the Levi-Civita connection for $\hat{\gamma}$. Because $\gamma'_{ab} = \hat{\gamma}_{ab} + \frac{\partial u}{\partial x^a}\frac{\partial u}{\partial x^b}$, we have

$$D'_a D'_b u = \rho^{-2}\hat{D}_a\hat{D}_b u.$$

Combining the above, the assertion for $D'_i D'_j u$ and $D'_\xi D'_i u$ follows. For $D'_\xi D'_\xi u$, we use

$$\delta^{ij}\frac{\partial u}{\partial\theta^i}\frac{\partial u}{\partial\theta^j} = -\Psi^{-2}\left(\frac{\partial u}{\partial\xi}\right)^2 - 1 + \rho^2.$$

Then

$$\begin{aligned} D'_\xi D'_\xi u =&\rho^{-2}\hat{D}_\xi \hat{D}_\xi u = \rho^{-2}\left(\frac{\partial \tilde{u}}{\partial \xi^2} + \Psi \frac{d\Psi}{dq}\delta^{ij}\frac{\partial u}{\partial \theta^i}\frac{\partial u}{\partial \theta^j} - \Psi^{-1}\frac{d\Psi}{dq}\left(\frac{\partial u}{\partial \xi}\right)^2\right) \\ =&\rho^{-2}\left(\frac{\partial \tilde{u}}{\partial \xi^2} - 2\Psi^{-1}\frac{d\Psi}{dq}\left(\frac{\partial u}{\partial \xi}\right)^2 - \Psi\frac{d\Psi}{dq} + \rho^2\Psi\frac{d\Psi}{dq}\right) \\ =&\rho^{-2}\frac{\partial \tilde{u}}{\partial \xi^2} - 2\rho^{-2}\Psi^{-1}\frac{d\Psi}{dq}\left(\frac{\partial u}{\partial \xi}\right)^2 - \rho^{-2}\Psi\frac{d\Psi}{dq} + \Psi\frac{d\Psi}{dq}. \end{aligned}$$

□

From Lemmas A.1 and A.2, we obtain the following relation between h'_{ab} and $D'_b D'_b u$.

Corollary A.3.

$$h'_{ab} = -\rho D'_a D'_b u + \rho\Psi\frac{d\Psi}{dq}D'_a\xi D'_b\xi. \tag{A.8}$$

Next, we record the formulas under a conformal transformation. Let $\psi(q)$ be a smooth function of q. Define

$$\check{g} = e^{2\psi}g'.$$

Let $\check{R}_{\alpha\beta\lambda\mu}$ be the Riemannian curvature of $\check{g}$.

Lemma A.4. *The non-zero components of $\check{R}_{\alpha\beta\lambda\mu}$ are*

$$\begin{aligned} \check{R}_{q\xi q\xi} =& -e^{2\psi}\left(\Psi\frac{d^2\Psi}{dq^2} + \Psi^2\frac{d^2\psi}{dq^2} + \Psi\frac{d\Psi}{dq}\frac{d\psi}{dq}\right), \\ \check{R}_{qiqi} =& -e^{2\psi}\frac{d^2\psi}{dq^2}, \\ \check{R}_{\xi i\xi i} =& -e^{2\psi}\Psi\frac{d\Psi}{dq}\frac{d\psi}{dq} - e^{2\psi}\Psi^2\left(\frac{d\psi}{dq}\right)^2, \\ \check{R}_{ijij} =& -e^{2\psi}\left(\frac{d\psi}{dq}\right)^2,\ i\neq j. \end{aligned} \tag{A.9}$$

Proof. As $\check{g} = e^{2\psi}g'$, the formula of $\check{R}_{\alpha\beta\lambda\mu}$ is given by [3, page 58]

$$\check{R}_{\alpha\beta\lambda\mu} = e^{2\psi}R'_{\alpha\beta\lambda\mu} - e^{2\psi}(g'_{\alpha\lambda}T_{\beta\mu} + g'_{\beta\mu}T_{\alpha\lambda} - g'_{\alpha\mu}T_{\beta\lambda} - g'_{\beta\lambda}T_{\alpha\mu}),$$

with $T_{\alpha\beta} = \nabla'_\alpha\nabla'_\beta\psi - \nabla'_\alpha\psi\nabla'_\beta\psi + 2^{-1}|\nabla'\psi|^2_{g'}g'_{\alpha\beta}$. From (A.2), non-zero components of $T_{\alpha\beta}$ are

$$\begin{aligned} T_{qq} =&\frac{d^2\psi}{dq^2} - 2^{-1}\left(\frac{d\psi}{dq}\right)^2, \\ T_{\xi\xi} =&\Psi\frac{d\Psi}{dq}\frac{d\psi}{dq} + 2^{-1}\Psi^2\left(\frac{d\psi}{dq}\right)^2, \\ T_{ii} =&2^{-1}\left(\frac{d\psi}{dq}\right)^2. \end{aligned}$$

Combining with (A.3), the assertion follows a direct computation. □

We turn to the second fundamental form of Σ with respect to $\breve{g}$. The normal vector and the induced metric are given by $\breve{\nu} = e^{-\psi}\nu'$ and $\breve{\gamma} = e^{2\psi}\gamma'$ respectively. Let $\breve{h}_{ab}$ be the second fundamental form with respect to $\breve{g}$.

Lemma A.5.

$$\breve{h}_{ab} = e^{\psi}\left(h'_{ab} + \rho^{-1}\frac{d\psi}{dq}\gamma'_{ab}\right). \tag{A.10}$$

Proof. Let $\breve{\nabla}$ be the Levi-Civita connection of $\breve{g}$. From

$$\breve{\nabla}_X Y = \nabla'_X Y + g'(\nabla'\psi, X)Y + g'(\nabla'\psi, Y)X - g'(X,Y)\nabla'\psi,$$

$$\breve{h}_{ab} = -\breve{g}\left(\breve{\nabla}_{\frac{\partial F}{\partial x^a}}\frac{\partial F}{\partial x^b}, \breve{\nu}\right) = -e^{\psi}g'\left(\nabla'_{\frac{\partial F}{\partial x^a}}\frac{\partial F}{\partial x^b} - \gamma'_{ab}\frac{d\psi}{dq}\frac{\partial}{\partial q}, \nu\right) = e^{\psi}\left(h'_{ab} + \rho^{-1}\frac{d\psi}{dq}\gamma'_{ab}\right).$$

□

Using (A.8), (A.10), we have

$$\breve{h}_{ab} = e^{\psi}\left(-\rho D'_a D'_b u + \rho\Psi\frac{d\Psi}{dq}D'_a\xi D'_b\xi + \rho^{-1}\frac{d\psi}{dq}\gamma'_{ab}\right).$$

To get the mean curvature $\breve{H}$, we need to calculate $|D'\xi|^2_{\gamma'}$. Recall that

$$\gamma'_{ab} = \hat{\gamma}_{ab} + \frac{\partial u}{\partial x^a}\frac{\partial u}{\partial x^b}.$$

We deduce

$$(\gamma')^{ab} = \hat{\gamma}^{ab} - \rho^{-2}\hat{\gamma}^{ac}\hat{\gamma}^{bd}\frac{\partial u}{\partial x^c}\frac{\partial u}{\partial x^d}.$$

Hence

$$\begin{aligned}|D'\xi|^2_{\gamma'} =&(\gamma')^{\xi\xi} = \Psi^{-2} - \rho^{-2}\Psi^{-4}\left(\frac{\partial u}{\partial \xi}\right)^2\\ =&\rho^{-2}\Psi^{-2}\left(1 + \delta^{ij}\frac{\partial u}{\partial\theta^i}\frac{\partial u}{\partial\theta^j}\right).\end{aligned}$$

With $\breve{\gamma}^{ab} = e^{-2\psi}(\gamma')^{ab}$, we obtain

Corollary A.6. *The mean curvature of Σ with respect to $\breve{g}$ is given by*

$$\breve{H} = e^{-\psi}\left(-\rho\Delta' u + \rho^{-1}\Psi^{-1}\frac{d\Psi}{dq}\delta^{ij}\frac{\partial u}{\partial\theta^i}\frac{\partial u}{\partial\theta^j} + (n-1)\rho^{-1}\frac{d\psi}{dq} + \rho^{-1}\Psi^{-1}\frac{d\Psi}{dq}\right). \tag{A.11}$$

Here Δ' is the Laplace-Beltrami operator of γ'.

Appendix B. Projection

Let M be a n-dimensional manifold with coordinates $\{y^\alpha\}_{\alpha=1}^n$. Let $g_{\alpha\beta}$ be a Riemannian metric on M and Σ^{n-1} be a hypersurface. We use $\{x^a\}_{a=1}^n$ to denote a local coordinates of Σ. Denote by ν^α the unit normal vector of Σ, by γ_{ab} the induced metric on Σ and by h_{ab} the second fundamental form of Σ with respect to ν^α. Define the projection tensor

$$P^\alpha_\beta = \delta^\alpha_\beta - \nu_\beta\nu^\alpha. \tag{B.1}$$

Denote by ∇ and D the Levi-Civita connection of $g_{\alpha\beta}$ and γ_{ab} respectively.

Lemma B.1. *Let $\overline{F}_\alpha$ be a one-form and $F_a = P_a^\alpha \overline{F}_\alpha$ be the projection of $\overline{F}_\alpha$. Then*

$$D_a F_b = P_a^\alpha P_b^\beta \nabla_\alpha \overline{F}_\beta - \overline{F}_\alpha \nu^\alpha h_{ab}. \tag{B.2}$$

$$D_a (F_\alpha \nu^\alpha) = (P_a^\beta \nabla_\beta \overline{F}_\alpha)\nu^\alpha + F_b h_a^b. \tag{B.3}$$

Let

$$K_{ab} = P_a^\alpha \nu^\beta P_b^\lambda \nu^\mu R_{\alpha\beta\lambda\mu}. \tag{B.4}$$

Lemma B.2. *For any $m \in \mathbb{N} \cup \{0\}$ there exists a degree $m+1$ polynomial p_m such that the following holds. Suppose there exists a constant C such that for all $0 \le j \le m$,*

$$|\nabla^j Rm|_g \le C, \ |D^{j-1} h|_\gamma \le C. \tag{B.5}$$

Then

$$|D^m K|_\gamma \le p_m(C).$$

Proof. For $m, k \in \mathbb{N} \cup \{0\}$, let $L_{m,k}$ be the collection $4+m-k$ tensors on Σ obtained from projections of $\nabla^m Rm \cdot \nu^{\otimes k}$. We adapt the convention that $L_{m,-1} = 0$ and $L_{m,k} = 0$ if $k \ge m+3$. Note that $K \in L_{0,2}$. From Lemma B.1, we have

$$DL_{m,k} = L_{m+1,k} + L_{m,k-1} * h + L_{m,k+1} * h.$$

Here we write $L * h$ for any linear combination formed by contracting L and h by γ. Through induction, we have for all $m \in \mathbb{N} \cup \{0\}$,

$$D^m L_{0,2} = \sum L_{j,i} * D^{\ell_1} h * \cdots * D^{\ell_k} h,$$

where the summation goes over

$$0 \le i \le j+3, \ 0 \le \ell_i, \text{ and } j + k + \sum_{i=1}^k \ell_i = m.$$

Under the assumption (B.5), each term $L_{j,i}$ and D^{ℓ_i} above are bounded by C. Each individual summand $L_{j,i} * D^{\ell_1} h * \cdots * D^{\ell_k} h$ is bounded by C^{k+1}. Then the assertion then follows. □

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