Metallic states beyond the Tomonaga-Luttinger liquid in one dimension

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In this paper, we propose strongly correlated gapless states (or critical states) of spin-1/2 electrons in 1+1 dimensions, such as the doped ferromagnetic and antiferromagnetic spin-1/2 Ising chains. We find that the metallic phases in the doped ferromagnetic and antiferromagnetic Ising chain are different strongly correlated gapless phases, despite the fact that the two phases have the same symmetry. The doped antiferromagnetic Ising chain has a finite energy gap for all charge-1 fermionic excitations even without pairing caused by the attractive interaction, resembling the pseudogap phase of underdoped high T_c superconductors. Applying a transverse field to the ferromagnetic and antiferromagnetic metallic phase can restore the Z_2 symmetry, which gives rise to two distinct critical points despite the fact that the two transitions have exactly the same symmetry-breaking pattern. We also propose chiral metallic states. Some of these gapless states are strongly correlated in the sense that they do not belong to the usual Tomonaga-Luttinger phase of fermions, i.e., they cannot be smoothly deformed into noninteracting fermion systems with the same symmetry. Our nonperturbative results are obtained by noting that gapless quantum systems have emergent *categorical symmetries* (i.e., noninvertible gravitational anomalies), which are described by multicomponent partition functions that are modular covariant. This allows us to calculate the scaling dimensions and quantum numbers of all the low-energy operators for those strongly correlated gapless states. This demonstrates an application of emergent categorical symmetries in determining low-energy properties of strongly correlated gapless states, which are hard to obtain otherwise.

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I. INTRODUCTION

The simplest one-dimensional (1D) metallic states are Fermi liquids of noninteracting electrons, the low-energy properties of which are described by noninteracting fermionic quasiparticles. In the low-energy limit, Fermi liquids are described by several decoupled sectors and each sector contains a few modes. In this paper, we will try to develop a general understanding of gapless states by viewing the gapless states as formed by several decoupled sectors, and using the notion of *categorical symmetry* [1] (i.e., modular covariance of a noninvertible gravitational anomaly [2]).

Readers who are just interested in 1D strongly interacting metallic states can directly go to Sec. III. Section II contains some general discussions.

If a strongly interacting metallic state is stable against all symmetry preserving perturbations, then it will represent a stable phase of quantum matter. However, most strongly interacting metallic states are not stable against certain symmetry preserving perturbations. Those metallic states will correspond to critical states (or multicritical points) that describe continuous phase transitions between different phases of quantum matter. Thus the constructions discussed in this paper can be viewed as a systematic way to discover 1D gapless quantum phases, as well as 1D (multi)critical points. In this paper, we will use "1D" ("2D," etc.) to refer to one-dimensional (two-dimensional, etc.) space and "1+1D" ("2+1D," etc.) to refer to 1+1-dimensional (2+1-dimensional, etc.) space-time.

II. GENERAL PICTURE FOR GAPLESS QUANTUM STATES

With the developments of the last 30 years, we have begun to have a comprehensive understanding of all gapped quantum states in one-dimensional, two-dimensional (2D), and three-dimensional (3D) spaces, in terms of spontaneous symmetry-breaking [3,4], group cohomology [5,6], and braided fusion (higher) categories [7–17]. In fact, we have classified (or proposed to classify) all 1D [18–21], 2D [12–14,22], and 3D [15–17] gapped liquid [23,24] states of boson/fermion systems with any finite on-site symmetry. The classification is achieved via the realization that gapped quantum phases are described by symmetry-breaking orders, topological orders [7,25,26], and/or symmetry protected trivial (SPT) orders [27,28].

Such a systematic understanding of topological orders [7,25,26] and SPT orders [27] (including topological insulators and superconductors [29–40]) leads to a deeper understanding of gauge and gravitational anomalies, in terms of the boundaries of topological order or SPT order in one-higher-dimensional lattice models [41–44]. This results in a generalization of anomalies including noninvertible anomalies [2,43,45–47]. Those generalized anomalies (including perturbative and global gauge/gravity anomalies) are classified in terms of topological orders and SPT orders in one higher dimension [42,43]. Such an understanding of anomalies also leads to a solution to the long-standing chiral fermion problem [48,49].

In comparison, there is a lack of comprehensive understanding of gapless quantum states of matter, despite the fact that we know many examples of them, such as superfluid, antiferromagnets, nodal d-wave superconductors, graphene, and Weyl semimetals. But in one dimension, thanks to Belavin-Polyakov-Zamolodchikov, we do have a good understanding of gapless quantum states with linear velocities via conformal field theories (CFTs) [50–52]. In particular, we can use the modular invariant partition function, which is parametrized by a complex number τ describing the shape of the space-time torus,

$$Z(\tau) = Z(\tau + 1) = Z(-1/\tau), \tag{1}$$

to systematically study 1D gapless states. In this paper, we will try to develop a systematic point of view of gapless quantum matter based on a gauge/gravity anomaly, hoping this may lead to a more general understanding of gapless states in higher dimensions.

First, the low-energy part of a gapless state may become several decoupled sectors, where the interactions between different sectors approach zero in the infrared limit under renormalization-group flow. Consequently, in the low-energy limit, there are often emergent symmetries. For example, the original UV symmetry G (the lattice symmetry) may be enlarged at low energies, $G \to G \times G \times \cdots$, with one copy for each decoupled sector. Since each decoupled low-energy sector is not a full system, each sector by itself is often anomalous. Thus there are also emergent anomalies (i.e., the low-energy effective theory is anomalous).

Recently, it was pointed out that, when restricted to the symmetric sub-Hilbert space, a symmetry can be fully characterized [1] by a noninvertible gravitational anomaly [2,43,45–47]. So we can treat the emergent symmetries and emergent anomalies in a unified way by restricting to the symmetric sub-Hilbert space. In this case, we only have an emergent noninvertible gravitational anomaly. To stress this close connection between a noninvertible gravitational anomaly and the symmetry, we refer to a noninvertible gravitational anomaly as a categorical symmetry [2]. This point of view is very general. Not only can emergent zero-symmetries (i.e., the usual global symmetries) be viewed as emergent noninvertible gravitational anomalies, but emergent higher symmetries and even more general emergent higher algebraic symmetries can also be viewed as emergent noninvertible gravitational anomalies (i.e., emergent categorical symmetries) [53,54].

For example, the 1+1D gapless state with on-site symmetry G in the original lattice system also has a dual algebraic symmetry denoted by $\widetilde{G}$ [2]. The total symmetry is the categorical symmetry denoted by $G \vee \widetilde{G}$ [2]. Note that a categorical symmetry is nothing but a generalized gravitational anomaly (which can be a noninvertible gravitational anomaly). Also note that a generalized gravitational anomaly is nothing but a topological order in one higher dimension [42,43,45–47]. The topological order in one higher dimension that describes the categorical symmetry $G \vee \widetilde{G}$ is the topological order described by G gauge theory. The 1+1D gapless state corresponds to the minimal gapless boundary of the 2+1-dimensional (2+1D) G gauge theory [2], that has neither condensation of gauge charge nor gauge flux.

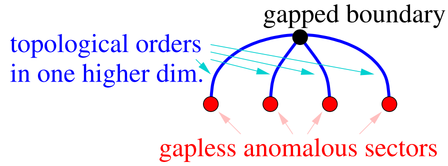

FIG. 1. A general picture of a gapless quantum state, which is formed by decoupled anomalous gapless sectors restricted to the symmetric sub-Hilbert spaces (the red dots). The emergent symmetry and emergent anomalies are described by noninvertible gravitational anomalies (i.e., the topological orders in one higher dimension). Thus, the anomalous sectors are the boundaries of corresponding topological orders in one higher dimension.

To have more information describing a gapless state, we want to decompose the gapless state into smallest decoupled sectors. This allows us to see the maximal emergent symmetry and emergent anomalies. In other words, this allows us to obtain the maximal categorical symmetry [1]. It may be possible that the maximal categorical symmetry fully characterizes the gapless state. This may be a way to systematically understand strongly correlated gapless states.

Since each decoupled sector has a generalized gravitational anomaly, it can be viewed as a boundary of topological order in one higher dimension (see Fig. 1). For example, the right-moving sector of a 1+1D gapless state has a perturbative gravitational anomaly characterized by its central charge c_R. Similarly, the left-moving sector also has a gravitational anomaly characterized by its central charge c_L. The right-moving sector is a boundary of a 2+1D chiral topological order. The left-moving sector is also a boundary of a 2+1D chiral topological order. The two chiral topological orders allow us to describe the 1+1D gapless state.

For a system with a generalized gravitational anomaly (i.e., a noninvertible gravitational anomaly), its partition function has multiple components. This multicomponent partition function transforms covariantly under mapping-class-group transformations of the space-time [2,55]. So the multicomponent partition function forms a representation of the mapping class group. Such a representation turns out to be the representation that describes the topological order in one higher dimension. Since

$$\begin{aligned}&\text{topological order in one higher dimension}\\&\quad= \text{noninvertible gravitational anomaly}\\&\quad= \text{categorical symmetry},\end{aligned} \tag{2}$$

we see that the categorical symmetry determines the representation of the mapping class group formed by the multicomponent partition function, which in turn determines the dynamical properties (such as scaling dimensions) of the 1+1D gapless state. This is how emergent maximal categorical symmetry systematically describes a strongly correlated gapless state.

In this paper, we will use this line of thinking, i.e., use multicomponent partition functions and their modular covariance, to study strongly correlated metals. This approach is beyond perturbation.

III. SUMMARY OF RESULTS

A. Ising strongly correlated metal with ferromagnetic and antiferromagnetic correlation

In Sec. IV, we consider a spin-$\frac{1}{2}$ electron chain close to one electron per site with strong on-site repulsive interaction and (anti)ferromagnetic Ising spin interaction. The model has Z_2 spin-flip symmetry, $S^z \to -S^z$ with $U(1)$ electron conservation symmetry, as well as translation symmetry, $U(1) \times Z_2^s \times \mathbb{Z}$.

We note that for the insulating Ising chain there are two Z_2 symmetry-breaking phases, one when the interaction is ferromagnetic and the other when the interaction is antiferromagnetic. The phase in the antiferromagnetic case breaks the translation symmetry.

After doping, there are also two Z_2 symmetry-breaking phases in the metallic states for ferromagnetic and antiferromagnetic Ising interactions. However, the antiferromagnetic metallic phase does not break the translation symmetry. We will show that despite the two metallic phases having the same symmetry they are two distinct phases separated by phase transitions, if we do not explicitly break the symmetry. In particular, the fermionic charge-1 excitation is gapless in the ferromagnetic metallic phase, and is gapped in the antiferromagnetic metallic phase. Thus the ferromagnetic and antiferromagnetic metallic phases provide examples of symmetry protected gapless phases [56,57].

The Z_2 spin-flip symmetry breaking in the two metallic phases can be restored if we add a strong transverse magnetic field, which will change the two metallic phases into the same metallic phase of polarized spins. We find that the critical theories of the transition points are different for the ferromagnetic and antiferromagnetic cases. The ferromagnetic critical point is described by a CFT,

$$u1 \oplus \text{Is} \oplus \overline{u1} \oplus \overline{\text{Is}}, \tag{3}$$

while the antiferromagnetic critical point is described by a different CFT. We see that even the same symmetry-breaking pattern can have distinct critical theories [58].

B. Spin-rotation symmetric strongly correlated metal

In this paper, we also construct some 1D chiral gapless states. One way to do so is to start with a 2D fractional quantum Hall (FQH) stripe. On one side of the stripe, we have a gapless edge state (the bottom part of Fig. 1), and on the other side of the stripe (the top part of Fig. 1) we have a fully gapped edge (assuming the FQH state supports gapped edges). This way, we can obtain a strongly interacting gapless state. In Appendix A, we show that, if we start with an Abelian FQH state and consider only $U(1)$ symmetry of electron number conservation, the above construction actually will always give us a Tomonaga-Luttinger (TL) liquid, not a new gapless phase. Thus, in this paper, we consider electron systems with more than just the $U(1)$ symmetry. As an application, in Sec. V, we start with a 2D integer quantum Hall stripe with $SO(3)$ spin rotation symmetry, and obtain a chiral metallic state of spin-1/2 charge-1 electrons, where the right-moving and left-moving gapless fermions carry different spins. This chiral metallic state is beyond the TL liquids of spin-1/2 electrons.

Furthermore, we consider an electron system with $U(1)$ charge, $SU(2)$ spin, and $\mathbb{Z}$ lattice translation symmetries. The lattice fermions carry charge 1 and spin 1/2. Such an electron system can realize a chiral metallic phase (see Sec. VI). In this chiral metallic state, the low-energy excitations are described by the CFT

$$su2_2 \oplus u1 \oplus \text{Is} \oplus \overline{su2_1} \oplus \overline{su2_1} \oplus \overline{u1}. \tag{4}$$

Note that the right movers and left movers are described by different CFTs (i.e., different chiral algebras), and those different sectors may have different velocities. We see that the single lattice $SU(2)$ spin rotation symmetry is enlarged to $SU(2) \times SU(2) \times SU(2)$ symmetry at low energies. The single lattice $U(1)$ charge conservation symmetry is enlarged to $U(1) \times U(1)$ symmetry at low energies. In the clean limit, the chiral metallic state has a quantized two-terminal thermal conductance $\kappa = c\frac{\pi}{6}\frac{k_B^2 T}{\hbar}$, where $c = \frac{3}{2} + 1 + \frac{1}{2} = 3$ is the total central charge for right movers (or left movers). Since the spin S_z is conserved, we can treat it as a conserved charge where each electron carries $\pm\hbar/2 S_z$ charge. The corresponding two-terminal S_z conductance is also quantized:

$$\sigma_{S_z} = \nu_s \frac{(\hbar/2)^2}{h} = \nu_s \frac{\hbar}{8\pi}, \tag{5}$$

with $\nu_s = 4$. For TL liquids of spin-1/2 electrons, c and ν_s are always integers, and they are always the same:

$$c = \nu_s. \tag{6}$$

For the chiral metallic state (4), $c = 3$ and $\nu_s = 4$. Thus the constructed chiral metallic state (4) is beyond the TL liquid. Note that the central charges of some sectors are fractional. Thus the chiral metallic state is a chiral "non-Abelian" metallic state.

IV. ISING PHASE TRANSITIONS IN THE METALLIC STATE OF A SPIN-$\frac{1}{2}$ ELECTRON CHAIN

In this section, we consider a spin-$\frac{1}{2}$ electron chain with ferromagnetic or antiferromagnetic S^z-spin interactions. The system has a symmetry $U(1) \times Z_2^s \times \mathbb{Z}$. We show that the Z_2^s symmetry-breaking transitions for the two cases are described by different CFTs in the metallic state, despite the fact that the two transitions cause the identical symmetry change, i.e., reduce the symmetry group of the ground state from $U(1) \times Z_2^s \times \mathbb{Z}$ to $U(1) \times \mathbb{Z}$.

A. Model

Let us first consider a spin-$\frac{1}{2}$ chain with Ising interaction

$$H = -J \sum_i \sigma_i^z \sigma_{i+1}^z - B \sum_i \sigma_i^x, \tag{7}$$

where B is the external magnetic field. We then add some doping to obtain a metallic state of a spin-$\frac{1}{2}$ electron chain. In this paper, we will mainly consider the case when Fermi energy of the dropped electrons is much less than $|J|$, $|B|$. In this case, the system is in the Z_2 symmetry-breaking phase when $B = 0$, with $\sigma^z = \pm 1$ (i.e., all the electrons either have $\sigma^z = +1$ or -1). In the Z_2 symmetry-breaking phase, the charge degree of freedom remains gapless. The phase is described by the $U(1)$ CFT with central charge $c = \bar{c} = 1$. The Z_2 symmetry-breaking state has central charge $c = \bar{c} = 1$. In the large B limit, the system is in a Z_2 symmetric phase where all the electrons have $\sigma^x = +1$. The Z_2 symmetric state has central charge $c = \bar{c} = 1$. We would like to consider the critical point of the Z_2 symmetry-breaking transition. The critical point of pure Z_2 symmetry breaking has central charge $c = \bar{c} = \frac{1}{2}$. With the contribution from the conserved $U(1)$ charge fluctuations, the critical point is expected to have a total central charge $c = \bar{c} = \frac{3}{2}$.

The symmetry of a fermion system is described by a pair of groups: (Z_2^f, G_f). Here G_f is the full symmetry group and Z_2^f is generated by fermion number parity, which is a central subgroup of G_f. For our spin-1/2 chain $G_f = U(1) \times Z_2^s$ and Z_2^f is the subgroup of $U(1)$.

In general, to fully describe a critical theory with a global symmetry G, we can consider the partition function twisted under the symmetry. More specifically, a twisted partition function defined on a Euclidean space-time torus parametrized by a complex number τ is indexed by a pair of elements g and h of G_f:

$$Z_{g,h}(\tau), \quad gh = hg, \quad g, h \in G_f. \tag{8}$$

It records all low-energy excitations ϕ that satisfy twisted boundary conditions along spatial and temporal directions, $\phi(x + L, -it) = g\phi(x, -it)$ and $\phi(x, -it + T) = h\phi(x, -it)$, where $-it$ denotes the imaginary time.

If the symmetry G_f is nonanomalous, the partition functions twisted under the symmetry satisfy the following relations:

$$\begin{aligned} Z_{h^{-1},g}(-1/\tau) &= Z_{g,h}(\tau), \\ Z_{g,hg}(\tau + 1) &= Z_{g,h}(\tau), \\ Z_{ugu^{-1},uhu^{-1}}(\tau) &= Z_{g,h}(\tau). \end{aligned} \tag{9}$$

For example, for a fermionic system with only fermion-number-parity symmetry, $G_f = Z_2^f$, the partition function depends on the boundary conditions along temporal and spatial directions. To put it in plain words, we consider the four-component partition function indexed by $g, h \in \{P, A\}$: $Z_{AP}(\tau)$, $Z_{AA}(\tau)$, $Z_{PA}(\tau)$, and $Z_{PP}(\tau)$, where P and A represent the periodic and antiperiodic boundary conditions of a local fermion.

B. Partition functions

Thus, for a CFT of a fermionic system, there are at least four sectors of partition functions defined as

$$\begin{aligned} Z_{PE^f}(\tau) &= \mathrm{Tr}E e^{-\mathrm{Im}(\tau)H_P - i\mathrm{Re}(\tau)K_P}, \\ Z_{PO^f}(\tau) &= \mathrm{Tr}O e^{-\mathrm{Im}(\tau)H_P - i\mathrm{Re}(\tau)K_P}, \\ Z_{AE^f}(\tau) &= \mathrm{Tr}E e^{-\mathrm{Im}(\tau)H_A - i\mathrm{Re}(\tau)K_A}, \\ Z_{AO^f}(\tau) &= \mathrm{Tr}O e^{-\mathrm{Im}(\tau)H_A - i\mathrm{Re}(\tau)K_A}, \end{aligned} \tag{10}$$

where TrE is the trace over the states with even (total) numbers of fermions and TrO is the trace over the states with odd numbers of fermions. H_P (H_A) is the Hamiltonian for a system where fermion fields satisfy a(n) (anti)periodic boundary condition in the x direction. Similarly, K_P (K_A) is the total momentum operator of the systems where fermion fields satisfy a(n) (anti)periodic boundary condition in the x direction.

Alternatively, we may define the torus partition functions for fermion systems through the space-time path integral, which also include four types, $Z_{PP}(\tau)$, $Z_{PA}(\tau)$, $Z_{AP}(\tau)$, and $Z_{AA}(\tau)$. Here the first and second subscription P and A correspond to the periodic and antiperiodic boundary conditions for fermions in x and t directions, respectively. The two sets of partition functions are related:

$$\begin{aligned} Z_{PE^f} &= \tfrac{1}{2}(Z_{PP} + Z_{PA}), \quad Z_{PO^f} = -\tfrac{1}{2}(Z_{PP} - Z_{PA}), \\ Z_{AE^f} &= \tfrac{1}{2}(Z_{AP} + Z_{AA}), \quad Z_{AO^f} = -\tfrac{1}{2}(Z_{AP} - Z_{AA}). \end{aligned} \tag{11}$$

Each partition function can be expanded as

$$Z(\tau) = q^{-\frac{c}{24}}(q^*)^{-\frac{\bar{c}}{24}} \sum_{(h,\bar{h})} N_{h,\bar{h}} q^h (q^*)^{\bar{h}}, \tag{12}$$

where c and $\bar{c}$ are the central charges for right and left movers:

$$q = e^{-i\tau\frac{2\pi}{L}}, \tag{13}$$

where L is the size of the 1D system. The summation $\sum_{(h,\bar{h})}$ is over a set of pairs $(h, \bar{h})$, which gives rise to the spectrum of scaling dimensions of local operators. In particular, the expansion coefficients $N_{h,\bar{h}}$ must be positive integers for each of $Z_{PE^f}(\tau)$, $Z_{PO^f}(\tau)$, $Z_{AE^f}(\tau)$, and $Z_{AO^f}(\tau)$.

Unlike CFTs from bosonic lattice systems that have a modular invariant partition function Eq. (1), for a CFT realizable by a fermionic lattice model, the above four types of partition functions transform covariantly under modular transformations. More explicitly, under $S : \tau \to -\frac{1}{\tau}$,

$$\begin{aligned} Z_{PP}\left(-\frac{1}{\tau}\right) &= Z_{PP}(\tau), \quad Z_{AA}\left(-\frac{1}{\tau}\right) = Z_{AA}(\tau), \\ Z_{AP}\left(-\frac{1}{\tau}\right) &= Z_{PA}(\tau), \quad Z_{PA}\left(-\frac{1}{\tau}\right) = Z_{AP}(\tau), \end{aligned} \tag{14}$$

and, under $T : \tau \to \tau + 1$,

$$\begin{aligned} Z_{PP}(\tau + 1) &= Z_{PP}(\tau), \quad Z_{AA}(\tau + 1) = Z_{AP}(\tau), \\ Z_{AP}(\tau + 1) &= Z_{AA}(\tau), \quad Z_{PA}(\tau + 1) = Z_{PA}(\tau). \end{aligned} \tag{15}$$

In the basis $(Z_{AE^f}, Z_{PO^f}, Z_{PE^f}, Z_{AO^f})$, the partition function transforms as [2]

$$Z_I(\tau + 1) = T_{IJ}^{Z_2^f} Z_J(\tau), \quad Z_I(-1/\tau) = S_{IJ}^{Z_2^f} Z_J(\tau), \tag{16}$$

where $I, J = AE^f, PO^f, PE^f, AO^f$ and

$$T^{Z_2^f} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad S^{Z_2^f} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{17}$$

As a warm-up example, we consider the 1D charge-1 spinless noninteracting fermions. The four-component partition functions for a charge-1 spinless fermion satisfy Eq. (16) and are given by the characters of a $u1_4$ CFT for right movers near k_F, and by the characters of a $\overline{u1}_4$ CFT for left movers near

$-k_F$ (see Appendix B 3). As a result, the four-component partition functions for a charge-1 spinless noninteracting fermion can be constructed from the characters of a $u1_4 \oplus \overline{u1}_4$ CFT:

$$\begin{aligned} Z_{AE^f} &= |\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2, \\ Z_{PO^f} &= \chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}, \\ Z_{PE^f} &= |\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2, \\ Z_{AO^f} &= \chi_0^{u1_4}\bar{\chi}_2^{u1_4} + \chi_2^{u1_4}\bar{\chi}_0^{u1_4}. \end{aligned} \tag{18}$$

Here, the primary field corresponding to $\chi_2^{u1_4}$ ($\bar{\chi}_2^{u1_4}$) is the charge-1 right(left)-moving fermion. We also note that the right (left) mover near k_F $(-k_F)$ can be viewed as the edge state for the integer quantum Hall state with filling fraction $\nu = 1$ $(\nu = -1)$.

To find modular covariant partition functions for the Ising critical point in the spin-1/2 electron system, we use the characters of a $u1_4$ CFT, $\chi_m^{u1_4}$, and the characters of an Ising CFT, χ_h^{Is}, to construct the four-component partition functions that satisfy Eq. (16) (details shown in Appendix B 3):

$$\begin{aligned} &\chi_m^{u1_M}(\tau), \quad 0 \leqslant m < M = 4, \\ &\chi_h^{\mathrm{Is}}(\tau), \quad h = 0, \tfrac{1}{2}, \tfrac{1}{16}. \end{aligned} \tag{19}$$

Here the $u1$ CFT describes the gapless $U(1)$ charge fluctuations. Also the Ising CFT describes the gapless spin fluctuations at the Ising transition point. Equation (16) can have many solutions. For example, the following four-component partition functions represent a solution satisfying Eq. (16):

$$\begin{aligned} Z_{AE} &= \left(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\right)\left(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\right) \\ &\quad + \left(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2, \\ Z_{PO} &= \left(\chi_0^{u1_4}\bar{\chi}_2^{u1_4} + \chi_2^{u1_4}\bar{\chi}_0^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\bar{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\bar{\chi}_0^{\mathrm{Is}}\right) \\ &\quad + \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2, \\ Z_{PE} &= \left(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2 \\ &\quad + \left(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\right)\left(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\right), \\ Z_{AO} &= \left(\chi_0^{u1_4}\bar{\chi}_2^{u1_4} + \chi_2^{u1_4}\bar{\chi}_0^{u1_4}\right)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2 \\ &\quad + \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\bar{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\bar{\chi}_0^{\mathrm{Is}}\right). \end{aligned} \tag{20}$$

In the above four-component partition function, we have considered the symmetry twist and the quantum number of Z_2^f. To obtain more information, let us also consider the partition functions for the spin symmetry twist Z_2^s—$Z_{PP}(\tau)$, $Z_{PA}(\tau)$, $Z_{AP}(\tau)$, and $Z_{AA}(\tau)$, which also satisfy Eqs. (15) and (22). We introduce $Z_{PE^s}(\tau)$, $Z_{PO^s}(\tau)$, $Z_{AE^s}(\tau)$, and $Z_{AO^s}(\tau)$ in a similar but slightly different way:

$$\begin{aligned} Z_{PE^s} &= \tfrac{1}{2}(Z_{PP} + Z_{PA}), \quad Z_{PO^s} = \tfrac{1}{2}(Z_{PP} - Z_{PA}), \\ Z_{AE^s} &= \tfrac{1}{2}(Z_{AP} + Z_{AA}), \quad Z_{AO^s} = \tfrac{1}{2}(Z_{AP} - Z_{AA}), \end{aligned} \tag{21}$$

where Z_{PE^s} is the partition function in the Z_2^s even sector and Z_{PO^s} is the partition function in the Z_2^s odd sector. Similarly, Z_{AE^s} is the partition function in the Z_2^s even sector, and Z_{AO^s} is the partition function in the Z_2^s odd sector, but now there is a Z_2^s symmetry twist in the spatial direction. In the basis $(Z_{PE^s}, Z_{PO^s}, Z_{AE^s}, Z_{AO^s})$, the partition function transforms as

$$Z_I(\tau+1) = T_{IJ}^{Z_2^s} Z_J(\tau), \quad Z_I(-1/\tau) = S_{IJ}^{Z_2^s} Z_J(\tau), \tag{22}$$

where $I, J = PE^s, PO^s, AE^s, AO^s$ and

$$T^{Z_2^s} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad S^{Z_2^s} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \tag{23}$$

which is the same as Eq. (17).

For example, the four-component partition functions for the critical point of a 1D Ising model (7) satisfy Eq. (22) and are given by the characters of the Ising CFT (see Appendix B 3):

$$\begin{aligned} Z_{PE^s} &= |\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2, \\ Z_{PO^s} &= |\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2, \\ Z_{AE^s} &= |\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2, \\ Z_{AO^s} &= \chi_0^{\mathrm{Is}}\bar{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\bar{\chi}_0^{\mathrm{Is}}. \end{aligned} \tag{24}$$

Now we would like to include symmetry twists and the quantum numbers for both Z_2^f and Z_2^s, which gives us the 16-component partition functions $Z_{II'}(\tau)$, where $I = AE^f, PO^f, PE^f, AO^f$ and $I' = PE^s, PO^s, AE^s, AO^s$. $Z_{II'}(\tau)$ satisfy the modular covariant condition (see Ref. [2])

$$\begin{aligned} Z_{II'}(\tau+1) &= T_{II',JJ'}^{Z_2^f\times Z_2^s} Z_{JJ'}(\tau), \\ Z_{II'}(-1/\tau) &= S_{II',JJ'}^{Z_2^f\times Z_2^s} Z_{JJ'}(\tau), \end{aligned} \tag{25}$$

where

$$\begin{aligned} T^{Z_2^f\times Z_2^s} &= T^{Z_2^f} \otimes T^{Z_2^s}, \\ S^{Z_2^f\times Z_2^s} &= S^{Z_2^f} \otimes S^{Z_2^s}. \end{aligned} \tag{26}$$

Equation (25) has many solutions. The list of 36 solutions are given in Appendix G. But which one of the partition functions describes the Ising transition of spin-1/2 electrons?

If the electron spins have a ferromagnetic interaction [i.e., $J < 0$ in Eq. (7)], then we can view the doped holes as spinless fermions. Thus, in this case, we can view the Ising transition point as the decoupled critical point of the Ising chain and the metallic state of spinless fermions. Therefore, the ferromagnetic Ising transition point of spin-1/2 electrons is described by the following 16-component partition functions [see Eq. (G1)]:

$$\begin{aligned} Z_{AE^f,PE^s} &= \left(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\right)\left(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\right), \\ Z_{PO^f,PE^s} &= \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)\left(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\right), \\ Z_{PE^f,PE^s} &= \left(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\right)\left(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\right), \\ Z_{AO^f,PE^s} &= \left(\chi_0^{u1_4}\bar{\chi}_2^{u1_4} + \chi_2^{u1_4}\bar{\chi}_0^{u1_4}\right)\left(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\right); \end{aligned} \tag{27}$$

$$
\begin{aligned}
Z_{AE^f,PO^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{PO^f,PO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{PE^f,PO^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{AO^f,PO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2; \qquad (28)\\
Z_{AE^f,AE^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{PO^f,AE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{PE^f,AE^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{AO^f,AE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2; \qquad (29)\\
Z_{AE^f,AO^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)\left(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\right),\\
Z_{PO^f,AO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\right),\\
Z_{PE^f,AO^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)\left(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\right),\\
Z_{AO^f,AO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\right). \qquad (30)
\end{aligned}
$$

The above 16-component partition function is the multicomponent partition function mentioned in Sec. III, which is a reflection of the noninvertible gravitational anomaly if we restrict to the symmetric sub-Hilbert space of the $Z_2^f\times Z_2^s$ symmetry. The modular covariance of the above multicomponent partition function can help us to determine many properties of the strongly correlated gapless state. We remark that the above 16-component partition function only describes part of the emergent noninvertible gravitational anomaly (i.e., part of the emergent categorical symmetry), which is not the maximal categorical symmetry.

We also note that the 16-component partition function reduces to the following four-component partition function if we only consider the Z_2^f symmetry twist:

$$
\begin{aligned}
Z_{AE^f} &= Z_{AE^f,PE^s}+Z_{AE^f,PO^s} = \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)Z_{\rm Is},\\
Z_{PO^f} &= Z_{PO^f,PE^s}+Z_{PO^f,PO^s} = \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)Z_{\rm Is},\\
Z_{PE^f} &= Z_{PE^f,PE^s}+Z_{PE^f,PO^s} = \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)Z_{\rm Is},\\
Z_{AO^f} &= Z_{AO^f,PE^s}+Z_{AO^f,PO^s} = \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)Z_{\rm Is}, \qquad (31)
\end{aligned}
$$

where

$$
Z_{\rm Is} = |\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2+|\chi_{\frac{1}{16}}^{\rm Is}|^2. \qquad (32)
$$

When the electron spins have an antiferromagnetic interaction [i.e., $J>0$ in Eq. (7)], the Ising transition point will be described by a different CFT. This is because when there is an odd number of electrons on the ring the spins carried by the electrons will behave like those in a spin chain with a Z_2^s symmetry twist. In other words, a state with an odd number of fermions is like a Neel ordered Ising spin configuration with an odd number of spins, thus satisfying the antiperiodic boundary condition.

This means that in the partition functions the first label of which is AO^f or PO^f (i.e., with an odd number of electrons) the (untwisted) spin part of the excitations (the sectors labeled by PO^s and PE^s) is given by the Z_2^s twisted sector of the Ising CFT. Specifically, if the second label is PO^s, the spin part is described by Ising character $\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}$ [which is Z_{AO^s} shown in Eq. (24)]; if the second label is PE^s, it is described by Ising character $|\chi_{\frac{1}{16}}^{\rm Is}|^2$ [which is Z_{AE^s} shown in Eq. (24)]. Still, in the partition functions the first label of which is AO^f or PO^f, the $\mathbb{Z}_2$ twisted spin part is given by the Z_2^s untwisted sector of the Ising CFT. In summary, the partition functions the first label of which is AO^f or PO^f are as follows:

$$
\begin{aligned}
Z^{\rm Is}_{AO^f/PO^f,PE^s} &= |\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z^{\rm Is}_{AO^f/PO^f,PO^s} &= \chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is},\\
Z^{\rm Is}_{AO^f/PO^f,AE^s} &= |\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2,\\
Z^{\rm Is}_{AO^f/PO^f,AO^s} &= |\chi_{\frac{1}{16}}^{\rm Is}|^2. \qquad (33)
\end{aligned}
$$

Also, in the partition functions with first label AE^f or PE^f (i.e., with an even number of electrons), the (untwisted) spin part is given by the untwisted sector of the Ising CFT, and the $\mathbb{Z}_2$ twisted spin part is given by the Z_2^s twisted sector of the Ising CFT:

$$
\begin{aligned}
Z^{\rm Is}_{AE^f/PE^f,PE^s} &= |\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2,\\
Z^{\rm Is}_{AE^f/PE^f,PO^s} &= |\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z^{\rm Is}_{AE^f/PE^f,AE^s} &= |\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z^{\rm Is}_{AE^f/PE^f,AO^s} &= \chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}. \qquad (34)
\end{aligned}
$$

Furthermore, since a fermion always carries an odd number of the $U(1)$ charge, the partition functions labeled by AO^f and PO^f (i.e., with an odd number of electrons) must be described by $u1$ character $\chi_m^{u1_4}\overline{\chi}_n^{u1_4}$ with $m-n=2 \mod 4$. We find the partition functions Eq. (G13) satisfy the above conditions. Thus, the antiferromagnetic Ising transition point of spin-1/2 electrons is described by the following 16-component partition functions [see Eq. (G13)]:

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)\left(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\right),\\
Z_{PO^f,PE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2, [3pt]\\
Z_{PE^f,PE^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)\left(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\right),\\
Z_{AO^f,PE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2; \qquad (35)\\
Z_{AE^f,PO^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{PO^f,PO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\right),\\
Z_{PE^f,PO^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{AO^f,PO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\right); \qquad (36)
\end{aligned}
$$

TABLE I. Quantum numbers of local and nonlocal operators in the critical point of the ferromagnetic Ising model Eq. (7). Here σ^x is the Z_2 spin quantum number, k is the crystal momentum, and $(h, \bar{h})$ are the right and left scaling dimensions. ψ and σ are the Is CFT primary fields associated with the Ising character $\chi^{\text{Is}}_{\frac{1}{2}}$ and $\chi^{\text{Is}}_{\frac{1}{16}}$, which have scaling dimensions $\frac{1}{2}$ and $\frac{1}{16}$, respectively. Similarly, $\bar{\psi}$ and $\bar{\sigma}$ are the $\overline{\text{Is}}$ CFT fields.

Operators	σ^x	k	$h, \bar{h}$
$\psi\bar{\psi}$	1	0	$\frac{1}{2}, \frac{1}{2}$
$\sigma\bar{\sigma}$	−1	0	$\frac{1}{16}, \frac{1}{16}$
ψ	−1	0	$\frac{1}{2}, 0$
$\bar{\psi}$	−1	0	$0, \frac{1}{2}$
$\sigma\bar{\sigma}\psi \sim \sigma\bar{\sigma}\bar{\psi}$	1	0	$\frac{1}{16}, \frac{1}{16}$

$$
\begin{aligned}
Z_{AE^f,AE^s} &= \left(\left|\chi^{u1_4}_0\right|^2 + \left|\chi^{u1_4}_2\right|^2\right)\left|\chi^{\text{Is}}_{\frac{1}{16}}\right|^2,\\
Z_{PO^f,AE^s} &= \left(\chi^{u1_4}_1\bar{\chi}^{u1_4}_{-1} + \chi^{u1_4}_{-1}\bar{\chi}^{u1_4}_1\right)\left(\left|\chi^{\text{Is}}_0\right|^2 + \left|\chi^{\text{Is}}_{\frac{1}{2}}\right|^2\right),\\
Z_{PE^f,AE^s} &= \left(\left|\chi^{u1_4}_1\right|^2 + \left|\chi^{u1_4}_{-1}\right|^2\right)\left|\chi^{\text{Is}}_{\frac{1}{16}}\right|^2,\\
Z_{AO^f,AE^s} &= \left(\chi^{u1_4}_0\bar{\chi}^{u1_4}_2 + \chi^{u1_4}_2\bar{\chi}^{u1_4}_0\right)\left(\left|\chi^{\text{Is}}_0\right|^2 + \left|\chi^{\text{Is}}_{\frac{1}{2}}\right|^2\right);
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
Z_{AE^f,AO^s} &= \left(\left|\chi^{u1_4}_1\right|^2 + \left|\chi^{u1_4}_{-1}\right|^2\right)\left(\chi^{\text{Is}}_0\bar{\chi}^{\text{Is}}_{\frac{1}{2}} + \chi^{\text{Is}}_{\frac{1}{2}}\bar{\chi}^{\text{Is}}_0\right),\\
Z_{PO^f,AO^s} &= \left(\chi^{u1_4}_0\bar{\chi}^{u1_4}_2 + \chi^{u1_4}_2\bar{\chi}^{u1_4}_0\right)\left|\chi^{\text{Is}}_{\frac{1}{16}}\right|^2,\\
Z_{PE^f,AO^s} &= \left(\left|\chi^{u1_4}_0\right|^2 + \left|\chi^{u1_4}_2\right|^2\right)\left(\chi^{\text{Is}}_0\bar{\chi}^{\text{Is}}_{\frac{1}{2}} + \chi^{\text{Is}}_{\frac{1}{2}}\bar{\chi}^{\text{Is}}_0\right),\\
Z_{AO^f,AO^s} &= \left(\chi^{u1_4}_1\bar{\chi}^{u1_4}_{-1} + \chi^{u1_4}_{-1}\bar{\chi}^{u1_4}_1\right)\left|\chi^{\text{Is}}_{\frac{1}{16}}\right|^2.
\end{aligned}
\tag{38}
$$

The above 16-component partition functions reduce to the four-component partition functions given in Eq. (20), if we only consider the Z_2^f symmetry twist.

C. Scaling operators and their quantum numbers

Let us first consider the scaling operators and their quantum numbers of the critical point (24) of the Ising model Eq. (7) without doping. The partition functions Eq. (24) tell us the Z_2^s quantum numbers. For ferromagnetic spin coupling [$J < 0$ in Eq. (7)], the low-energy states all carry crystal momentum near zero. The states described by $|\chi^{\text{Is}}_{\frac{1}{2}}|^2$ in Z_{PE^s} are created by local operator $\psi\bar{\psi}$ from the ground state in $|\chi^{\text{Is}}_0|^2$. Thus the operator $\psi\bar{\psi}$ carries Z_2^s quantum number $\sigma^x = 1$. The states described by $|\chi^{\text{Is}}_{\frac{1}{16}}|^2$ in Z_{PO^s} are created by local operator $\sigma\bar{\sigma}$ from the ground state. Thus the operator $\sigma\bar{\sigma}$ carries Z_2^s quantum number $\sigma^x = -1$. The states described by $\chi^{\text{Is}}_{\frac{1}{2}}\bar{\chi}^{\text{Is}}_0$ in Z_{AO^s} are created by nonlocal operator ψ from the ground state. Thus the nonlocal operator ψ carries Z_2^s quantum number $\sigma^x = 1$. Similarly, the nonlocal operator $\bar{\psi}$ also carries Z_2^s quantum number $\sigma^x = 1$. The states described by $|\chi^{\text{Is}}_{\frac{1}{16}}|^2$ in Z_{AE^s} are created by nonlocal operator $\sigma\bar{\sigma}\psi \sim \sigma\bar{\sigma}\bar{\psi}$ from the ground state. Thus the operator $\sigma\bar{\sigma}\psi \sim \sigma\bar{\sigma}\bar{\psi}$ carries Z_2^s quantum number $\sigma^x = 1$. The above results are summarized in Table I.

TABLE II. Quantum numbers of local and nonlocal operators in the critical point of the antiferromagnetic Ising model Eq. (7).

Operators	σ^x	k	$h, \bar{h}$
$\psi\bar{\psi}$	1	0	$\frac{1}{2}, \frac{1}{2}$
$\sigma\bar{\sigma}$	−1	$\pm\frac{\pi}{a}$	$\frac{1}{16}, \frac{1}{16}$
ψ	−1	$\pm\frac{\pi}{a}$	$\frac{1}{2}, 0$
$\bar{\psi}$	−1	$\pm\frac{\pi}{a}$	$0, \frac{1}{2}$
$\sigma\bar{\sigma}\psi \sim \sigma\bar{\sigma}\bar{\psi}$	1	0	$\frac{1}{16}, \frac{1}{16}$

However, for antiferromagnetic spin coupling [$J > 0$ in Eq. (7)], the low-energy states carry crystal momentum near $k = \pm\frac{\pi}{a}$ if the Z_2^s quantum number $\sigma^x = -1$ (and carry crystal momentum near zero if the Z_2^s quantum number $\sigma^x = 1$). The scaling operators and their quantum numbers for the antiferromagnetic Ising critical point are summarized in Table II.

Now let us consider the scaling operators and their quantum numbers for the spin-1/2 electrons at the Ising transition point. The partition functions Eqs. (27)–(30) and (35)–(38) tell us the Z_2^f and Z_2^s quantum numbers. In the following, we will discuss the $U(1)$ and momentum quantum numbers.

Let us first consider the ferromagnetic Ising transition point described by Eqs. (27)–(30). The $u1_4$ character $\chi^{u1_4}_m$ describes states with $U(1)$ charge $q = \frac{m}{2}$ mod 2, and momentum $k = \frac{k_F}{2}$ mod $2k_F$. Here $k_F = \pi n_F$, where n_F is the fermion number per site. The $\overline{u1}_4$ character $\bar{\chi}^{u1_4}_m$ describes states with $U(1)$ charge $q = -\frac{m}{2}$ mod 2, and momentum $k = \frac{k_F}{2}$ mod $2k_F$. For such $U(1)$ charge assignment, we see that the states described by the partition function $Z_{AE^f,\dots}$ ($Z_{AO^f,\dots}$) carry even (odd) $U(1)$ charges. The states described by the Ising character do not carry any $U(1)$ charge or momentum.

The states described by the partition function $Z_{AE^f,\dots}$ ($Z_{AO^f,\dots}$) are created by local gapless bosonic (fermionic) operators from the ground state in the sector $|\chi^{u1_4}_0|^2|\chi^{\text{Is}}_0|^2$. So the above discussion gives us a list of scaling operators, as well as their quantum numbers and scaling dimensions. The results are summarized in Table III. For example [see Eq. (27)], the bosonic operator $e^{\pm i(\varphi\pm\bar{\varphi})}$ creates the states in $|\chi^{u1_4}_2|^2|\chi^{\text{Is}}_0|^2$. The local fermionic operator $e^{\pm i\varphi}$ creates the states in $\chi^{u1_4}_2\bar{\chi}^{u1_4}_0|\chi^{\text{Is}}_0|^2$.

From Table III, we see that there is only one relevant operator that carries a trivial quantum number, $\bar{\psi}\psi$, with total scaling dimension $h + \bar{h} = 1$. This is the operator that drives the ferromagnetic Ising transition.

Next, let us consider the antiferromagnetic Ising transition point described by Eqs. (35)–(38). The $u1_4$ character $\chi^{u1_4}_m$ still describes states with $U(1)$ charge $q = \frac{m}{2}$ mod 2, and momentum $k = \frac{k_F}{2}$ mod $2k_F$. The $\overline{u1}_4$ character $\bar{\chi}^{u1_4}_m$ still describes states with $U(1)$ charge $q = -\frac{m}{2}$ mod 2, and momentum $k = \frac{k_F}{2}$ mod $2k_F$. For such $U(1)$ charge assignment, again the states described by the partition function $Z_{AE^f,\dots}$ ($Z_{AO^f,\dots}$) carry even (odd) $U(1)$ charges. The states described by the Ising character do not carry any $U(1)$ charge. But they can carry momentum $\pm k_F$ if $\sigma_x = -1$. The results are summarized in Table IV. For example [see Eq. (35)], the local gapless bosonic operator $e^{\pm i(\varphi\pm\bar{\varphi})}$ creates the states in $|\chi^{u1_4}_2|^2|\chi^{\text{Is}}_0|^2$.

TABLE III. Quantum numbers of local gapless bosonic and fermionic operators in the ferromagnetic Ising transition point of the strongly interacting spin-1/2 electron system (the doped ferromagnetic Ising model). Here σ^x is the Z_2 spin quantum number, q is the $U(1)$ charge, k is the crystal momentum, and $(h, \overline{h})$ are the right and left scaling dimensions [the values in brackets are for $\theta = 0$, see Eq. (44)]. φ is the bosonic field to describe $u1_4$ CFT, where φ is normalized such that $e^{i\varphi}$ has a scaling dimension $\frac{1}{2}$. ψ and σ are the Is CFT fields with scaling dimension $\frac{1}{2}$ and $\frac{1}{16}$, respectively. Similarly, $\overline{\varphi}$ is the bosonic field to describe $\overline{u1}_4$ CFT and $\overline{\psi}$ and $\overline{\sigma}$ are the $\overline{\text{Is}}$ CFT fields.

Operators	σ^x	q	k	$h, \overline{h}$	$(\theta = 0)$
$e^{\pm i(\varphi+\overline{\varphi})}$	1	0	$\pm 2k_F$	$\frac{(\text{ch}\theta-\text{sh}\theta)^2}{2}, \frac{(\text{ch}\theta-\text{sh}\theta)^2}{2}$	$(\frac{1}{2}, \frac{1}{2})$
$e^{\pm i(\varphi-\overline{\varphi})}$	1	± 2	0	$\frac{(\text{ch}\theta+\text{sh}\theta)^2}{2}, \frac{(\text{ch}\theta+\text{sh}\theta)^2}{2}$	$(\frac{1}{2}, \frac{1}{2})$
$\psi\overline{\psi}$	1	0	0	$\frac{1}{2}, \frac{1}{2}$	$(\frac{1}{2}, \frac{1}{2})$
$\sigma\overline{\sigma}$	-1	0	0	$\frac{1}{16}, \frac{1}{16}$	$(\frac{1}{16}, \frac{1}{16})$
$e^{\pm i\varphi}$	1	± 1	$\pm k_F$	$\frac{\text{ch}^2\theta}{2}, \frac{\text{sh}^2\theta}{2}$	$(\frac{1}{2}, 0)$
$e^{\pm i\overline{\varphi}}$	1	∓ 1	$\pm k_F$	$\frac{\text{sh}^2\theta}{2}, \frac{\text{ch}^2\theta}{2}$	$(0, \frac{1}{2})$
$e^{\pm i\varphi}\sigma\overline{\sigma}$	-1	± 1	$\pm k_F$	$\frac{\text{ch}^2\theta}{2} + \frac{1}{16}, \frac{\text{sh}^2\theta}{2} + \frac{1}{16}$	$(\frac{7}{16}, \frac{1}{16})$
$e^{\pm i\overline{\varphi}}\sigma\overline{\sigma}$	-1	∓ 1	$\pm k_F$	$\frac{\text{sh}^2\theta}{2} + \frac{1}{16}, \frac{\text{ch}^2\theta}{2} + \frac{1}{16}$	$(\frac{1}{16}, \frac{7}{16})$

The local fermionic operator $e^{\pm i\varphi}\sigma\overline{\sigma}\psi \sim e^{\pm i\varphi}\sigma\overline{\sigma}\overline{\psi}$ creates the states in $\chi_2^{u1_4}\overline{\chi}_0^{u1_4}|\chi^{\text{Is}}_{\frac{1}{16}}|^2$.

From Table IV, we see that there is only one relevant operator that carries trivial quantum numbers, $\overline{\psi}\psi$, with total scaling dimension $h + \overline{h} = 1$. This is the operator that drives the antiferromagnetic Ising transition.

D. Low-energy effective theory

Let us further compare the ferromagnetic and antiferromagnetic Ising transition for the spin-1/2 electrons when there is interaction. Both the ferromagnetic and antiferromagnetic Ising transition points are described by the same low-energy effective field theory:

$$\mathcal{L} = \frac{1}{4\pi}(\partial_x\varphi\partial_t\varphi - \partial_x\varphi\partial_x\varphi - \partial_x\overline{\varphi}\partial_t\overline{\varphi} - \partial_x\overline{\varphi}\partial_x\overline{\varphi}) - \frac{1}{2\pi}V\partial_x\varphi\partial_x\overline{\varphi} + \psi(\partial_t - \partial_x)\psi + \overline{\psi}(\partial_t + \partial_x)\overline{\psi}. \quad (39)$$

However, the sets of local operators are different for the two Ising transition points. For the ferromagnetic Ising transition point, the local operators are given in Table III, while for the antiferromagnetic Ising transition point the local operators are given in Table IV. In the tables, the $\sigma(x)$ [$\overline{\sigma}(x)$] operator is the operator that creates the sign flip at x in the ψ [$\overline{\psi}$] field.

In the last section, we study the case with $V = 0$. And the $U(1)$ charge fluctuations are described by the $u1_4 \oplus \overline{u1}_4$ CFT. Here we will consider the effect of V on the scaling dimensions h and $\overline{h}$. Let us introduce

$$\begin{pmatrix}\phi\\ \overline{\phi}\end{pmatrix} = \begin{pmatrix}\text{ch}\theta & \text{sh}\theta\\ \text{sh}\theta & \text{ch}\theta\end{pmatrix}\begin{pmatrix}\varphi\\ \overline{\varphi}\end{pmatrix},$$
$$\begin{pmatrix}\varphi\\ \overline{\varphi}\end{pmatrix} = \begin{pmatrix}\text{ch}\theta & -\text{sh}\theta\\ -\text{sh}\theta & \text{ch}\theta\end{pmatrix}\begin{pmatrix}\phi\\ \overline{\phi}\end{pmatrix}, \quad (40)$$

with θ satisfying

$$V = \frac{2\text{ch}\theta\text{sh}\theta}{\text{ch}^2\theta + \text{sh}^2\theta}. \quad (41)$$

The Lagrangian for ϕ and $\overline{\phi}$ is diagonal:

$$\mathcal{L} = \frac{1}{4\pi}(\partial_x\phi\partial_t\phi - v\partial_x\phi\partial_x\phi - \partial_x\overline{\phi}\partial_t\overline{\phi} - v\partial_x\overline{\phi}\partial_x\overline{\phi}). \quad (42)$$

Thus the scaling dimensions h and $\overline{h}$ for operator

$$e^{i(m\varphi+\overline{m}\,\overline{\varphi})} = e^{i[m(\text{ch}\theta\,\phi-\text{sh}\theta\,\overline{\phi})+\overline{m}(\text{ch}\theta\,\overline{\phi}-\text{sh}\theta\,\phi)]} \quad (43)$$

are given by

$$h(\theta) = \frac{(m\,\text{ch}\theta - \overline{m}\,\text{sh}\theta)^2}{2}, \quad \overline{h}(\theta) = \frac{(\overline{m}\,\text{ch}\theta - m\,\text{sh}\theta)^2}{2}. \quad (44)$$

From Tables III and IV, we see that the charge neutral operator with $\sigma^x = -1$ has scaling dimensions $\frac{1}{16}, \frac{1}{16}$ and $\frac{2(\text{ch}\theta-\text{sh}\theta)^2+1}{16}, \frac{2(\text{ch}\theta-\text{sh}\theta)^2+1}{16}$ for the ferromagnetic and the antiferromagnetic critical points, respectively. The scaling dimensions for the antiferromagnetic critical points are always larger than $\frac{1}{16}, \frac{1}{16}$. So the ferromagnetic and the antiferromagnetic critical points are really distinct critical points, despite the fact that they describe identical symmetry-breaking patterns.

Next we compare the ferromagnetic and antiferromagnetic Ising transitions for the spin-1/2 electrons by considering the total scaling dimension $h_+(\theta) = h + \overline{h}$ for the electron operator with Z_2^s quantum number $\sigma^x = 1$ and $h_-(\theta) = h + \overline{h}$ for the electron operator with Z_2^s quantum number $\sigma^x = -1$. As a function of interaction θ, $h_+(\theta)$ and $h_-(\theta)$ have different relations for the ferromagnetic and antiferromagnetic Ising transitions, as shown in Fig. 2. For example, in the ferromagnetic transition, the Z_2^s even and odd electron operators can be $e^{\pm i\varphi}$ and $e^{\pm i\varphi}\sigma\overline{\sigma}$, respectively. And in the antiferromagnetic transition they can be $e^{\pm i\varphi}\sigma\overline{\sigma}\psi$ and $e^{\pm i\frac{\varphi-\overline{\varphi}}{2}}\psi$, respectively. From Fig. 2, we see that in the ferromagnetic case the scaling

TABLE IV. Quantum numbers of local gapless bosonic and fermionic operators in the antiferromagnetic Ising transition point of the strongly interacting spin-1/2 electron system (the doped antiferromagnetic Ising model).

Operators	σ^x	q	k	$h, \overline{h}$
$e^{\pm i(\varphi+\overline{\varphi})}$	1	0	$\pm 2k_F$	$\frac{(\text{ch}\theta-\text{sh}\theta)^2}{2}, \frac{(\text{ch}\theta-\text{sh}\theta)^2}{2}$
$e^{\pm i(\varphi-\overline{\varphi})}$	1	± 2	0	$\frac{(\text{ch}\theta+\text{sh}\theta)^2}{2}, \frac{(\text{ch}\theta+\text{sh}\theta)^2}{2}$
$\psi\overline{\psi}$	1	0	0	$\frac{1}{2}, \frac{1}{2}$
$e^{\pm i\frac{\varphi+\overline{\varphi}}{2}}\sigma\overline{\sigma}$	-1	0	$\pm k_F$	$\frac{2(\text{ch}\theta-\text{sh}\theta)^2+1}{16}, \frac{2(\text{ch}\theta-\text{sh}\theta)^2+1}{16}$
$e^{\pm i\varphi}\sigma\overline{\sigma}\psi$	1	± 1	$\pm k_F$	$\frac{\text{ch}^2\theta}{2} + \frac{1}{16}, \frac{\text{sh}^2\theta}{2} + \frac{1}{16}$
$e^{\pm i\overline{\varphi}}\sigma\overline{\sigma}\psi$	1	∓ 1	$\pm k_F$	$\frac{\text{sh}^2\theta}{2} + \frac{1}{16}, \frac{\text{ch}^2\theta}{2} + \frac{1}{16}$
$e^{\pm i\frac{\varphi-\overline{\varphi}}{2}}\psi$	-1	± 1	$\pm k_F$	$\frac{(\text{ch}\theta+\text{sh}\theta)^2+4}{8}, \frac{(\text{ch}\theta+\text{sh}\theta)^2}{8}$
$e^{\pm i\frac{\varphi-\overline{\varphi}}{2}}\overline{\psi}$	-1	± 1	$\pm k_F$	$\frac{(\text{ch}\theta+\text{sh}\theta)^2}{8}, \frac{(\text{ch}\theta+\text{sh}\theta)^2+4}{8}$

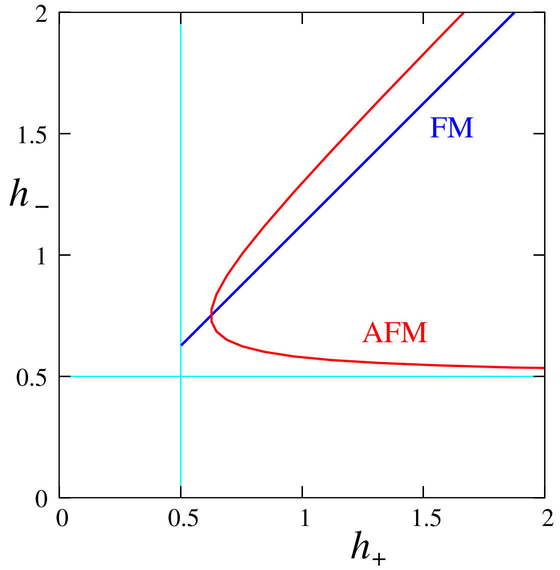

FIG. 2. The relations between the total scaling dimensions h_+ and h_- of the electron operators with Z_2^s quantum number $\sigma^x = 1$ and -1, respectively, for the ferromagnetic (FM) and antiferromagnetic (AFM) Ising transitions.

dimension of the $\mathbb{Z}_2^s$ odd electron operator is always larger than that of the $\mathbb{Z}_2^s$ one by $\frac{1}{8}$, which is independent of interaction. However, in the antiferromagnetic case, the difference in the scaling dimension of the Z_2^s odd and Z_2^s even operator increases with the attractive interacting strength ($\theta > 0$), and decreases with the repulsive interacting strength ($\theta < 0$), comparing to the noninteracting case $\theta = 0$.

E. Two metallic phases of a spin-$\frac{1}{2}$ electron chain with the same symmetry

Let us consider the spin-$\frac{1}{2}$ Ising chain Eq. (7) with $B > 0$. As we change J from $0 \to +\infty$, the Ising chain goes into a state that breaks the Z_2^s spin-flip symmetry. If we change J from $0 \to -\infty$, the Ising chain goes into a state that breaks both the Z_2^s spin-flip and translation symmetries.

However, for a doped Ising chain which is a metallic state, both the $J \to +\infty$ and the $J \to -\infty$ cases have the same symmetry: the Z_2^s spin-flip symmetry is spontaneously broken while the translation symmetry is not broken. Despite the two large $|J|$ metallic phases having the same symmetry, our previous discussions indicate that the transitions from the $J = 0$ metallic phase to $J = \pm\infty$ metallic phases are described by two distinct critical points. Thus, even the transitions that have identical spontaneous symmetry-breaking patterns can be described by different critical points.

The two distinct critical points also suggest that $J = \pm\infty$ metallic phases are two distinct metallic phases despite the fact that they have the same symmetry. Thus, they are examples of symmetry protected gapless phases, i.e., distinct gapless phases with the same symmetry. To understand these two distinct metallic phases, we consider modular covariant partition functions with $U(1) \times Z_2^f$ symmetry. We will consider the 16-component partition functions with $Z_2^f \times Z_2^s$ symmetry twists. Since Z_2^s symmetry is spontaneously broken,

TABLE V. Quantum numbers of local gapless bosonic and fermionic operators in the metallic phase of spin-1/2 electrons with strong ferromagnetic Ising interaction. Here, q is the $U(1)$ charge, k is the crystal momentum, and $(h, \bar{h})$ are the right and left scaling dimensions [the values in brackets are for $\theta = 0$, see Eq. (44)]. φ is the bosonic field to describe $u1_4$ CFT. Similarly, $\bar{\varphi}$ is the bosonic field to describe $\overline{u1}_4$ CFT.

Operators	q	k	$h, \bar{h}$	$(\theta = 0)$
$e^{\pm i(\varphi+\bar{\varphi})}$	0	$\pm 2k_F$	$\frac{(\text{ch}\theta-\text{sh}\theta)^2}{2}, \frac{(\text{ch}\theta-\text{sh}\theta)^2}{2}$	$(\frac{1}{2}, \frac{1}{2})$
$e^{\pm i(\varphi-\bar{\varphi})}$	± 2	0	$\frac{(\text{ch}\theta+\text{sh}\theta)^2}{2}, \frac{(\text{ch}\theta+\text{sh}\theta)^2}{2}$	$(\frac{1}{2}, \frac{1}{2})$
$e^{\pm i\varphi}$	± 1	$\pm k_F$	$\frac{\text{ch}^2\theta}{2}, \frac{\text{sh}^2\theta}{2}$	$(\frac{1}{2}, 0)$
$e^{\pm i\bar{\varphi}}$	∓ 1	$\pm k_F$	$\frac{\text{sh}^2\theta}{2}, \frac{\text{ch}^2\theta}{2}$	$(0, \frac{1}{2})$

the partition functions with nontrivial Z_2^s symmetry twist vanish. Using the $u1_4$ CFT characters to construct the modular covariant partition functions, we identify the following two sets of partition functions to describe the $J = \pm\infty$ metallic phases.

For the $J = +\infty$ metallic phase (ferromagnetic Ising interaction), we have

$$\begin{aligned}
Z_{AE^f,PE^s} &= \left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2,\\
Z_{PO^f,PE^s} &= \chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4},\\
Z_{PE^f,PE^s} &= \left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2,\\
Z_{AO^f,PE^s} &= \chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4};
\end{aligned} \tag{45}$$

$$\begin{aligned}
Z_{AE^f,PO^s} &= \left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2,\\
Z_{PO^f,PO^s} &= \chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4},\\
Z_{PE^f,PO^s} &= \left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2,\\
Z_{AO^f,PO^s} &= \chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4};
\end{aligned} \tag{46}$$

$$\begin{aligned}
Z_{AE^f,AE^s} &= 0,\\
Z_{PO^f,AE^s} &= 0,\\
Z_{PE^f,AE^s} &= 0,\\
Z_{AO^f,AE^s} &= 0;
\end{aligned} \tag{47}$$

$$\begin{aligned}
Z_{AE^f,AO^s} &= 0,\\
Z_{PO^f,AO^s} &= 0,\\
Z_{PE^f,AO^s} &= 0,\\
Z_{AO^f,AO^s} &= 0.
\end{aligned} \tag{48}$$

The corresponding primary fields (i.e., gapless operators) and their quantum numbers are listed in Table V.

For the $J = -\infty$ metallic phase (antiferromagnetic Ising interaction), we have

$$\begin{aligned}
Z_{AE^f,PE^s} &= \left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2,\\
Z_{PO^f,PE^s} &= 0,\\
Z_{PE^f,PE^s} &= \left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2,\\
Z_{AO^f,PE^s} &= 0;
\end{aligned} \tag{49}$$

TABLE VI. Quantum numbers of local gapless bosonic operators in the metallic phase of spin-1/2 electrons with strong antiferromagnetic Ising interaction. Local fermionic operators (i.e., odd-charge operators) are all gapped.

Operators	q	k	$h, \bar{h}$	$(\theta = 0)$
$e^{\pm i(\varphi+\overline{\varphi})}$	0	$\pm 2k_F$	$\frac{(\text{ch}\theta-\text{sh}\theta)^2}{2}, \frac{(\text{ch}\theta-\text{sh}\theta)^2}{2}$	$(\frac{1}{2}, \frac{1}{2})$
$e^{\pm i(\varphi-\overline{\varphi})}$	± 2	0	$\frac{(\text{ch}\theta+\text{sh}\theta)^2}{2}, \frac{(\text{ch}\theta+\text{sh}\theta)^2}{2}$	$(\frac{1}{2}, \frac{1}{2})$

$$\begin{aligned} Z_{AE^f,PO^s} &= \left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2, \\ Z_{PO^f,PO^s} &= 0, \\ Z_{PE^f,PO^s} &= \left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2, \\ Z_{AO^f,PO^s} &= 0; \end{aligned} \tag{50}$$

$$\begin{aligned} Z_{AE^f,AE^s} &= 0, \\ Z_{PO^f,AE^s} &= \chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}, \\ Z_{PE^f,AE^s} &= 0, \\ Z_{AO^f,AE^s} &= \chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}; \end{aligned} \tag{51}$$

$$\begin{aligned} Z_{AE^f,AO^s} &= 0, \\ Z_{PO^f,AO^s} &= \chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}, \\ Z_{PE^f,AO^s} &= 0, \\ Z_{AO^f,AO^s} &= \chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}. \end{aligned} \tag{52}$$

The corresponding primary fields (i.e., gapless operators) and their quantum numbers are listed in Table VI.

In particular, from the above partition function, we can read that in the antiferromagnetic metallic phase the single electron excitations are all gapped. For example, in Eq. (49), we see that $Z_{AE^f,PE^s} \neq 0$, which means a sector with even fermions and integer S_z spins is gapless. If we add an electron, we obtain a sector with odd fermions and half-integer S_z spins described by Z_{AO^f,PO^s} in Eq. (50). $Z_{AO^f,PO^s} = 0$ means the sector has an energy gap.

This is in contrast to the ferromagnetic metallic phase. $Z_{AE^f,PE^s} \neq 0$ in Eq. (46) and $Z_{AO^f,PO^s} \neq 0$ in Eq. (47) imply that the sectors differ by an electron and are both gapless. Thus the single electron excitations are gapless.

To understand this result, we note that the spins of electrons have a Neel-like $\uparrow\downarrow\uparrow\downarrow\cdots$ pattern. As a result, for an even number of electrons, the partition function is nonzero only when there is no Z_2^s symmetry twist. For an odd number of electrons, the partition function is nonzero only when there is a Z_2^s symmetry twist. Since the fermion number and the Z_2^s symmetry twist are locked, the fermion operators (i.e., odd-charge operators) are all gapped. We can also see the gapping of charge-1 fermions by noticing that applying a charge-1 fermion operator to states in the sector Z_{AE^f,PE^s} gives us states in the sectors Z_{AO^f,PE^s} and Z_{AO^f,PO^s}, where $AE^f \to AO^f$ (adding a fermion) and $PE^s \to PE^s, PO^s$ (the Z_2^s symmetry twist cannot be changed). Since $Z_{AO^f,PE^s} = Z_{AO^f,PO^s} = 0$, meaning the two sectors are gapped, the charge-1 fermionic excitations are all gapped.

TABLE VII. Strongly interacting spin-1/2 electrons can form chiral metallic states, whose low energy excitations are right-moving and left-moving fermionic quasiparticles. Here two possible chiral metallic states are listed in terms of spins carried by the right-moving and left-moving quasiparticles.

	Right movers	Left movers	$\nu_s = 4\text{Tr}(S_z^R)^2$	c
Spin	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{9}{2}$	$\frac{7}{2}, \frac{7}{2}$	336	16
Spin	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}$	$\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}$	80	16

V. CHIRAL METALLIC PHASES OF SPIN-$\frac{1}{2}$ ELECTRONS

Following the ideas in Ref. [59], we can also construct a strongly interacting metallic phase of spin-1/2 electrons where the left movers and right movers have very different behavior. We will call such metallic phases chiral metallic phases.

In the first example, the left movers and right movers have the same emergent symmetry at low energy. However, they carry different representations under the symmetries. Specifically, one such chiral metallic phase has $SU(2)$-spin and $U(1)$-charge symmetries with symmetry group $[SU(2) \times U(1)]/Z_2$. At low energies, the chiral metallic phase has n left-moving and n right-moving fermions, which are noninteracting. Those noninteracting fermions all carry charge 1. But the left-moving and right-moving fermions form different $SU(2)$ representations. Let S_z^R be the $n \times n$ Hermitian matrix for the S_z spin of the right-moving fermions, and let S_z^L be the $n \times n$ Hermitian matrix for the left-moving fermions. For the low-energy fermions to be free from perturbative $SU(2)$ anomaly, the $SU(2)$ representations must satisfy

$$\text{Tr}\big(S_z^R\big)^2 = \text{Tr}\big(S_z^L\big)^2. \tag{53}$$

Then combining the results in Refs. [48,59], we find that such a chiral metallic phase is free of all $U(1) \times SU(2)$ anomalies, and can be realized by interacting fermions on a 1D lattice.

Equation (53) has solutions only when $n \geqslant 16$, if we require all the fermions to have half-integer spins. At $n = 16$, we only have the two solutions in Table VII. All the fermions in Table VII carry charge 1. The spin-1/2 fermions correspond to the spin-1/2 electrons. The fermions with higher spins can be viewed as bound states of several spin-$\frac{1}{2}$ electrons and spin-$\frac{1}{2}$ holes.

The chiral metallic phases of charge-1 fermions carrying the list of spins in Table VII can be realized by interacting electrons on a 1D lattice, according to the argument presented in Refs. [48,59]. However, such chiral metallic states cannot be smoothly deformed into the noninteracting spin-1/2 electron systems since ν_s [defined in Eq. (5)] and the chiral central charge c are not equal.

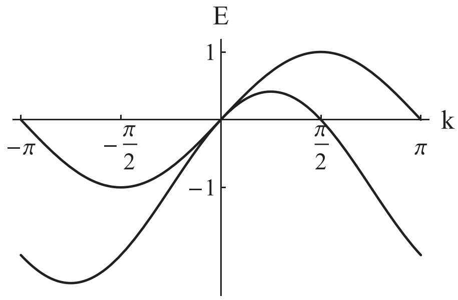

FIG. 3. The band structure of free fermions to construct the chiral metallic phase. The two bands are for two flavors. Each band is doubly degenerate corresponding to two spin-$\frac{1}{2}$ states. Velocities $v(k)$ at $k=0$ are the same, and likewise $v(\frac{\pi}{2}) = v(\pi)$.

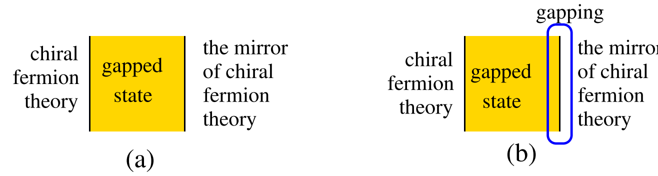

FIG. 4. (a) Chiral fermions and the mirror of chiral fermions can appear on the boundary of a 4+1D slab of a gapped state. (b) Sometimes [such as in the $SO(10)$ case] the boundary mirror chiral fermions can be gapped by interaction, which leads to a solution of the chiral fermion problem.

VI. CHIRAL NON-ABELIAN METALLIC PHASES

A. Construction

In this section, we are going to construct another stable chiral metallic phase which is also non-Abelian. We start with a noninteracting 1D electron system described by

$$\mathcal{H}_0 = \psi^\dagger_{\alpha a}(x) i v_0 \partial_x \psi_{\alpha a}(x) - \overline{\psi}^\dagger_{\alpha a}(x) i v_0 \partial_x \overline{\psi}_{\alpha a}(x), \tag{54}$$

where α and a are the spin $SU(2)$ and flavor $SU(2)$ labels, and v_0 is the Fermi velocity. At low energy, the model has an emergent $[SU_s(2)\times SU_f(2)\times U(1)]_R \times [SU_s(2)\times SU_f(2)\times U(1)]_L$ symmetry for right movers and left movers. The right movers of the above system are described by the CFT

$$su2^s_2 \oplus su2^f_2 \oplus u1^c, \tag{55}$$

where the excitations in $su2^s_2$ carry $SU_s(2)$ spin quantum numbers, the excitations in $su2^f_2$ carry $SU_f(2)$ flavor quantum numbers, and the excitations in $u1$ carry the $U(1)$ charges. Similarly, the left movers of the above system are described by the CFT

$$\overline{su2^s_2} \oplus \overline{su2^f_2} \oplus \overline{u1^c}. \tag{56}$$

In the above, suN_k denotes both the level-k $su(N)$ Kac-Moody algebra and the CFT built from it. The CFT has central charge

$$c = \frac{k(N^2-1)}{k+N}. \tag{57}$$

Likewise $u1_M$ denotes the $U(1)$ current algebra, and the central charge of the associated CFT is $c=1$. For details, see Appendix B.

In Eq. (54) the fermions also carry crystal momenta. In particular, $\psi_{\alpha a}$ carry crystal momentum $k_F=0$, $\overline{\psi}_{\alpha 1}$ carry crystal momentum $\overline{k}_{F1}=\pi$, and $\overline{\psi}_{\alpha 2}$ carry crystal momentum $\overline{k}_{F2}=\pi/2$. Such a free fermion model is easily realized, for example, with a band structure shown in Fig. 3. In particular, the low-energy fermion operator $\psi_{\alpha a,k-k_F}$ can be represented in terms of lattice fermion operator $c_{\alpha a,k}$ as follows. For $k\sim k_F$,

$$C_{\alpha a,k} = \left(\sum_{\mu=0}^{3} e^{i\mu(k-k_F)}\right) c_{\alpha a,k} \sim \psi_{\alpha a,k-k_F}, \tag{58}$$

and, for $k \sim \overline{k}_{Fa}$,

$$\overline{C}_{\alpha a,k} = \left(\sum_{\mu=0}^{3} e^{i\mu(k-\overline{k}_{Fa})}\right) c_{\alpha a,k} \sim \overline{\psi}_{\alpha a,k-\overline{k}_{Fa}}. \tag{59}$$

Additionally, in real space,

$$\psi_{\alpha a,i} = \sum_{\mu=0}^{3} e^{-i\mu k_F a} c_{\alpha a,i+\mu},\ \ \overline{\psi}_{\alpha a,i} = \sum_{\mu=0}^{3} e^{-i\mu \overline{k}_{Fa}} c_{\alpha a,i+\mu}. \tag{60}$$

$C_{\alpha a,k}$ reach the maximum at $k=0$ and vanish at $k=\frac{\pi}{2},\pi$; $\overline{C}_{\alpha 1,k}$ reach the maximum at $k=\pi$ and vanish at $k=0,\frac{\pi}{2}$; $\overline{C}_{\alpha 2,k}$ reach the maximum at $k=\frac{\pi}{2}$ and vanish at $k=0,\pi$.

To obtain the chiral metallic phase from the above free fermion model, we add interactions that respect the spin $SU(2)$, charge $U(1)$, and translation symmetry. We will add interactions in three steps, which finally lead to the Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + \delta\mathcal{H} + \overline{\delta\mathcal{H}} + \delta\mathcal{H}'. \tag{61}$$

It is crucial here that the interactions are different for left movers and right movers. For the right movers, we add interaction

$$\delta\mathcal{H} = g_s \mathbf{J}_s(x)\cdot\mathbf{J}_s(x) + g_c J_c(x) J_c(x), \tag{62}$$

where

$$J_c(x) = \psi^\dagger_{\alpha a}(x)\psi_{\alpha a}(x),$$
$$\mathbf{J}_s(x) = \tfrac{1}{2}\psi^\dagger_{\alpha a}(x)\sigma_{\alpha\beta}\psi_{\beta a}(x) \tag{63}$$

are the $U(1)$ charge and $SU(2)$ spin current (or density), and σ are the Pauli matrices. As a current-current interaction, when the coupling constant g's are not too large, the above interaction term [with scaling dimension $(h,\overline{h})=(2,0)$] is always *exactly marginal*. It does not open up any energy gap, but only modifies the velocities in the corresponding sector. With the interactions, the right movers have $SU(2)\times SU(2)\times U(1)$ symmetry, and are described by the CFT

$$su2^f_2 \oplus su2^s_2 \oplus u1^c. \tag{64}$$

The three sectors, each containing the flavor, spin, and charge degrees of freedom, respectively, can have separate velocities, while the excitations within each sector have the same velocity.

For left movers we add interactions

$$\overline{\delta\mathcal{H}} = \overline{g}_{s1}\overline{\mathbf{J}}_{s1}(x)\cdot\overline{\mathbf{J}}_{s1}(x) + \overline{g}_{s2}\overline{\mathbf{J}}_{s2}(x)\cdot\overline{\mathbf{J}}_{s2}(x) + \overline{g}_{c1}\overline{J}_{c1}(x)\overline{J}_{c1}(x) + \overline{g}_{c2}\overline{J}_{c2}(x)\overline{J}_{c2}(x). \tag{65}$$

Here,

$$\bar{J}_{c1}(x) = \bar{\psi}^\dagger_{\alpha1}(x)\bar{\psi}_{\alpha1}(x),$$
$$\bar{J}_{c2}(x) = \bar{\psi}^\dagger_{\alpha2}(x)\bar{\psi}_{\alpha2}(x),$$
$$\bar{\mathbf{J}}_{s1}(x) = \tfrac{1}{2}\bar{\psi}^\dagger_{\alpha1}(x)\sigma_{\alpha\beta}\bar{\psi}_{\beta1}(x),$$
$$\bar{\mathbf{J}}_{s2}(x) = \tfrac{1}{2}\bar{\psi}^\dagger_{\alpha2}(x)\sigma_{\alpha\beta}\bar{\psi}_{\beta2}(x). \tag{66}$$

Such current-current interactions also do not open up gaps, but modify the velocity in the corresponding sector. With the interaction, the left movers have the symmetry $SU(2) \times U(1) \times SU(2) \times U(1)$, and are described by the CFT

$$\overline{su2^s_1} \oplus \overline{u1^{c1}} \oplus \overline{su2^s_1} \oplus \overline{u1^{c2}}. \tag{67}$$

The chiral metallic phase that we have constructed so far [see Eqs. (64) and (67)] can be smoothly connected to the TL liquid (i.e., interacting 1D Fermi liquids) [60,61], as we reduce g's to zero. To construct a chiral chiral metallic phase that is not connected to the TL liquid, we add an additional interaction term:

$$\delta\mathcal{H}' = g(\psi^\dagger_{\uparrow1}\psi^\dagger_{\downarrow1} - \psi^\dagger_{\downarrow1}\psi^\dagger_{\uparrow1})(\bar{\psi}_{\uparrow1}\bar{\psi}_{\downarrow1} - \bar{\psi}_{\downarrow1}\bar{\psi}_{\uparrow1}) + \text{H.c.} \tag{68}$$

We note that the above operator carries a crystal momentum $k = 0 + 0 + \pi + \pi = 0 \bmod 2\pi$. Thus the term respects the translation symmetry. Such a term is not a current-current interaction and can induce energy gaps for some excitations and drive the system into a new phase.

To understand the new phase, note that the above operator respects the spin $SU(2)$, the diagonal charge $U(1)$, and the translation symmetry (since the crystal momentum carried by the operator vanishes). Such an operator only causes interaction within the sector $su2^f_2 \oplus u1^c \oplus \overline{u1^{c1}}$. If the interaction g is strong enough, it will gap out the $\overline{u1^{c1}}$ and part of the $su2^f_2 \oplus u1^c$ sector, which reduces $su2^f_2 \oplus u1^c \oplus \overline{u1^{c1}}$ down to $\text{Is} \oplus u1^{cf}$, where Is denotes the Ising CFT. In this way, we obtain a chiral metallic phase described by the CFT

$$su2_2 \oplus u1^{cf} \oplus \text{Is} \oplus \overline{su2_1} \oplus \overline{su2_1} \oplus \overline{u1}, \tag{69}$$

which is beyond the TL liquid.

B. Gapping process

We would like to show the gapping process of the interaction (68) more explicitly. This is accomplished by using CFT and current algebras. Furthermore, we can derive the physical properties such as local operators, correlation functions, and partition functions, which will be done in the next subsection.

We start with $\mathcal{H}_0$ in Eq. (54) plus the interactions Eqs. (64) and (67). The resulting low-energy theory has the following emergent symmetry:

$$\text{right movers,} \quad U(1) \times SU_f(2) \times SU_s(2);$$
$$\text{left movers,} \quad U_1(1) \times SU_s(2) \times U_2(1) \times SU_s(2). \tag{70}$$

$\psi_{\alpha a}$ carry the $U(1)$ charge 1 and transform as doublets of both the flavor and spin $SU(2)$. In contrast, $\bar{\psi}_{\alpha1}$ carry the charge 1 for $U_1(1)$ and form a doublet of the first $SU_s(2)$, and $\bar{\psi}_{\alpha2}$ carry the charge 1 for $U_1(1)$ and form a doublet of the second $SU(2)_s$. The low-energy excitations are described by the following current algebras:

$$\text{right movers,} \quad u1^c \oplus su2^s_2 \oplus su2^f_2;$$
$$\text{left movers,} \quad \overline{u1^1} \oplus \overline{su2^1_1} \oplus \overline{u1^2} \oplus \overline{su2^2_1}. \tag{71}$$

The theory is free from any gravitational anomaly, since the left central charge $c = 1 + \frac{3}{2} + \frac{3}{2} = 4$ is equal to the right central charge $\bar{c} = 1 + 1 + 1 + 1$.

The local operators of our theory are powers of the fermion operators $\psi_{\alpha a}$ and $\bar{\psi}_{\alpha a}$. The fermion operators can be represented in terms of the primary fields of the above CFT. In particular, they can be written in terms of simple free boson fields and free Majorana fermion fields in $u1$, $su2_2$, and $su2_1$ CFTs (see Appendix B 1):

$$\psi_{\alpha a} = e^{i\varphi_c/2}\sigma_s e^{\pm i\frac{\phi_s}{2}}\sigma_f e^{\pm i\frac{\phi_f}{2}} = e^{i\frac{\varphi_c}{2}} V^{su2^s_2}_{\frac12,\pm\frac12} V^{su2^f_2}_{\frac12,\pm\frac12},$$
$$\bar{\psi}_{\alpha1}(\bar{z}) = e^{i\frac{\bar{\varphi}_1}{\sqrt{2}}} e^{\pm i\frac{\bar{\phi}_1}{\sqrt{2}}} = e^{i\frac{\bar{\varphi}_1}{\sqrt{2}}} \bar{V}^{su2^1_1}_{\frac12,\pm\frac12}, \tag{72}$$
$$\bar{\psi}_{\alpha2} = e^{i\frac{\bar{\varphi}_2}{\sqrt{2}}} e^{\pm i\frac{\bar{\phi}_2}{\sqrt{2}}} = e^{i\frac{\bar{\varphi}_1}{\sqrt{2}}} \bar{V}^{su2^2_1}_{\frac12,\pm\frac12}.$$

Here, for right movers, (1) φ_c is the bosonic field to describe $u1^c$; (2) $\eta_s, \sigma_s,$ and ϕ_s are the Ising CFT fields and the bosonic field to describe $su2^s_2$; and (3) $\eta_f, \sigma_f,$ and ϕ_f are the Ising CFT fields and the bosonic field to describe $su2^f_2$. Similarly, for left movers, (1) $\bar{\varphi}_1$ is the bosonic field to describe $\overline{u1^1}$, (2) $\bar{\varphi}_2$ is the bosonic field to describe $\overline{u1^2}$, (3) $\bar{\phi}_1$ is the bosonic field to describe $\overline{su2^1_1}$, and (4) $\bar{\phi}_2$ is the bosonic field to describe $\overline{su2^2_1}$.

We adopt the convention that the correlation function of all bosonic fields is

$$\langle\phi(z_1)\phi(z_2)\rangle \sim -\ln(z_1 - z_2), \tag{73}$$

where $z_i = \tau_i + ix_i$ is the complex coordinate. The scaling dimensions of operators in Eq. (72) all equal $\frac{1}{2}$, a necessary condition for chiral fermion operators. This fixes the $u1$ parts of the fermion operators in Eq. (72).

Now the gapping term (68) can be rewritten as [via operator product expansion (OPE); see Appendix B 2]

$$\delta\mathcal{H}' \sim -g\cos(\varphi_c + \phi_f - \sqrt{2}\bar{\varphi}_1). \tag{74}$$

When $g > 0$ is large, the sectors generated by $\varphi_c + \phi_f$ and $\bar{\varphi}_1$ are fully gapped. Other sectors are not affected. Consequently, the gapless excitations in the new phase are described by the following CFT:

$$\text{right movers,} \quad u1^{cf} \oplus su2^s_2 \oplus \text{Is}^f;$$
$$\text{left movers,} \quad \overline{su2^1_1} \oplus \overline{u1^2} \oplus \overline{su2^2_1}. \tag{75}$$

Here $u1^{cf}$ is the $u1$ CFT represented by the field $\varphi_c - \phi_f$, the conjugate field of $\varphi_c + \phi_f$ that remains gapless. And $\text{Is}^f = \frac{su2^f_2}{u1^f}$ is the Ising CFT with primary fields $1, \sigma_f$, and η_f. Primary fields of the above CFTs are summarized in Appendix B 1.

The scaling dimension of $e^{i(\varphi_c+\phi_f-\sqrt{2}\bar{\varphi}_1)}$ is $(h, \bar{h}) = (1, 1)$. We emphasize that, even though the local interaction term gaps out the left mode and right mode in equal numbers, it selects an inequivalent combination of right modes, comparing to the left modes, since the operator is in the CFT

$$u1_4^c \oplus u1_4^{fz} \oplus \overline{u1_2^1} \subset u1_4^c \oplus su2_2^f \oplus \overline{u1_2^1}. \tag{76}$$

The resulting phase is anomaly free and has a lattice realization.

C. Local operators

To compute the physical properties of the chiral metallic phase, we first identify local operators in the above CFT. In the chiral metallic phase, the fermion operators $\psi_{\alpha 1}$ and $\overline{\psi}_{\alpha 1}$ are gapped (i.e., their imaginary-time correlations have exponential decay), since they contain either $e^{i\frac{\varphi_c+\varphi_f}{2}}$ or $e^{i\frac{\overline{\phi}_1}{\sqrt{2}}}$. They are local operators but do not appear in the low-energy CFT. All other fermion operators,

$$\psi_{\alpha 2} = e^{i\frac{\varphi_c-\phi_f}{2}}\sigma_s e^{\pm i\phi_s/2}\sigma_f = e^{i\frac{\varphi_c-\phi_f}{2}} V^{su2_2^s}_{\frac{1}{2},\pm\frac{1}{2}}\sigma_f,$$
$$\overline{\psi}_{\alpha 2} = e^{i\frac{\overline{\varphi}_2}{\sqrt{2}}} e^{\pm i\frac{\overline{\phi}_2}{\sqrt{2}}} = e^{i\frac{\overline{\varphi}_2}{\sqrt{2}}}\overline{V}^{su2_1^2}_{\frac{1}{2},\pm\frac{1}{2}}, \tag{77}$$

are still gapless and, therefore, are local operators in the CFT. Operators generated from the OPEs of $\psi_{\alpha 2}$ and $\overline{\psi}_{\alpha 2}$ are also local operators.

The above local operators are purely chiral with either only right movers or only left movers. Another type of local operators containing both right movers and left movers is

$$\psi_{\alpha 1}^\dagger \overline{\psi}_{\beta 1} = e^{-i\frac{\varphi_c+\phi_f-\sqrt{2}\overline{\phi}_1}{2}}\sigma_s e^{\pm i\phi_s/2}\sigma_f e^{\pm i\frac{\overline{\phi}_1}{\sqrt{2}}}$$
$$\sim \sigma_s e^{\pm i\phi_s/2}\sigma_f e^{\pm i\frac{\overline{\phi}_1}{\sqrt{2}}} = \sigma_f V^{su2_2^s}_{\frac{1}{2},\pm\frac{1}{2}}\overline{V}^{su2_1^1}_{\frac{1}{2},\pm\frac{1}{2}}, \tag{78}$$

where we have used the knowledge that in the chiral metallic phase the cos term in Eq. (74) is frozen to the maximum value, i.e., $e^{-i\frac{\varphi_c+\phi_f-\sqrt{2}\overline{\phi}_1}{2}} \sim 1$. Therefore $\psi_{\alpha 1}^\dagger\overline{\psi}_{\beta 1}$'s are also low-energy local operators in the chiral metallic phase. The above results are summarized in Table VIII.

D. Partition functions

To find modular covariant partition functions [see Eq. (16)], we use the CFT characters for $u1_M$, Is $\cong \frac{su2_2}{u1}$, $su2_1$, and $su2_2$ (details shown in Appendix B 3):

$$\begin{aligned} &\chi_m^{u1_M}, && 0 \leqslant m < M,\\ &\chi_\mu^{\mathrm{Is}}, && \mu = 0, \eta, \sigma,\\ &\chi_\rho^{su2_1}, && \rho = 0, \tfrac{1}{2},\\ &\chi_\nu^{su2_2}, && \nu = 0, \tfrac{1}{2}, 1. \end{aligned} \tag{79}$$

The scaling dimensions of the $U(1)$ part in $\psi_{\alpha 2}$ and $\overline{\psi}_{\alpha 2}$ in Eq. (77) are both $\frac{1}{4}$, and thus correspond to $u1$ primary fields with $R = \sqrt{2}$. This determines the level of $u1_M$ CFT to be $M = 2$ [see Eq. (B9)].

We find that the simplest solutions of covariant partition functions that contain local operators (77) and (78) are the following.

(1) The first solution is the antiperiodic boundary condition with an even number of fermions:

$$\begin{aligned} Z_{AE} = {} & \chi_0^{u1_2}\big(\chi_0^{su2_2}\chi_0^{\mathrm{Is}} + \chi_1^{su2_2}\chi_\eta^{\mathrm{Is}}\big)\overline{\chi}_0^{u1_2}\overline{\chi}_0^{su2_1^1}\overline{\chi}_0^{su2_1^2}\\ & + \chi_1^{u1_2}\big(\chi_1^{su2_2}\chi_0^{\mathrm{Is}} + \chi_0^{su2_2}\chi_\eta^{\mathrm{Is}}\big)\overline{\chi}_1^{u1_2}\overline{\chi}_{1/2}^{su2_1^1}\overline{\chi}_{1/2}^{su2_1^2}\\ & + \chi_0^{u1_2}\chi_{1/2}^{su2_2}\chi_\sigma^{\mathrm{Is}}\overline{\chi}_0^{u1_2}\overline{\chi}_{1/2}^{su2_1^1}\overline{\chi}_0^{su2_1^2}\\ & + \chi_1^{u1_2}\chi_{1/2}^{su2_2}\chi_\sigma^{\mathrm{Is}}\overline{\chi}_1^{u1_2}\overline{\chi}_0^{su2_1^1}\overline{\chi}_{1/2}^{su2_1^2}. \end{aligned} \tag{80}$$

The primary field corresponding to each term of characters in Z_{AE} is bosonic with integral spin $h - \overline{h} \in \mathbb{Z}$. We list the scaling dimensions of all primary fields in Appendix E.

(2) The second solution is the antiperiodic boundary condition with an odd number of fermions:

$$\begin{aligned} Z_{AO} = {} & \chi_0^{u1_2}\big(\chi_0^{su2_2}\chi_0^{\mathrm{Is}} + \chi_1^{su2_2}\chi_\eta^{\mathrm{Is}}\big)\overline{\chi}_1^{u1_2}\overline{\chi}_0^{su2_1^1}\overline{\chi}_{1/2}^{su2_1^2}\\ & + \chi_1^{u1_2}\big(\chi_1^{su2_2}\chi_0^{\mathrm{Is}} + \chi_0^{su2_2}\chi_\eta^{\mathrm{Is}}\big)\overline{\chi}_0^{u1_2}\overline{\chi}_{1/2}^{su2_1^1}\overline{\chi}_0^{su2_1^2}\\ & + \chi_0^{u1_2}\chi_{1/2}^{su2_2}\chi_\sigma^{\mathrm{Is}}\overline{\chi}_1^{u1_2}\overline{\chi}_{1/2}^{su2_1^1}\overline{\chi}_{1/2}^{su2_1^2}\\ & + \chi_1^{u1_2}\chi_{1/2}^{su2_2}\chi_\sigma^{\mathrm{Is}}\overline{\chi}_0^{u1_2}\overline{\chi}_0^{su2_1^1}\overline{\chi}_0^{su2_1^2}. \end{aligned} \tag{81}$$

The primary field corresponding to each term of characters in Z_{AE} is fermionic with half-integral spin $h - \overline{h} \in \mathbb{Z} + \frac{1}{2}$. We list the scaling dimensions of all primary fields in Appendix E.

(3) The third solution is the periodic boundary condition with an even or odd number of fermions:

$$\begin{aligned} Z_{PE} = Z_{PO} = {} & \tfrac{1}{2}\big(\chi_0^{u1_2}\chi_{1/2}^{su2_2}\chi_\sigma^{\mathrm{Is}}\overline{\chi}_0^{su2_1^1} + \chi_1^{u1_2}\chi_0^{su2_2}\chi_0^{\mathrm{Is}}\overline{\chi}_0^{su2_1^1}\big)\\ & \times\big(\overline{\chi}_1^{u1_2}\overline{\chi}_0^{su2_1^2} + \overline{\chi}_0^{u1_2}\overline{\chi}_{1/2}^{su2_1^2}\big)\\ & + \tfrac{1}{2}\big[\chi_0^{u1_2}\big(\chi_1^{su2_2}\chi_0^{\mathrm{Is}} + \chi_0^{su2_2}\chi_\eta^{\mathrm{Is}}\big)\overline{\chi}_{1/2}^{su2_1^1}\\ & + \chi_1^{u1_2}\big(\chi_1^{su2_2}\chi_\eta^{\mathrm{Is}}\overline{\chi}_0^{su2_1^1} + \chi_{1/2}^{su2_2}\chi_\sigma^{\mathrm{Is}}\overline{\chi}_{1/2}^{su2_1^1}\big)\big]\\ & \times\big(\overline{\chi}_1^{u1_2}\overline{\chi}_0^{su2_1^2} + \overline{\chi}_0^{u1_2}\overline{\chi}_{1/2}^{su2_1^2}\big), \end{aligned} \tag{82}$$

where both terms in $Z_{PE}(Z_{PO})$ are eightfold degenerate, essentially contributed from four Majorana zero modes for the periodic boundary condition, as explained in Appendix D. The primary field corresponding to each term of characters in Z_{PE} (Z_{PO}) is bosonic with integral spin $h - \overline{h} \in \mathbb{Z}$. We list the scaling dimensions of all primary fields in Appendix E.

Here we have used the fact that $Z_{PP} = 0$, since the chiral metallic phase contains free fermions $\psi_{\alpha 2}$ and $\overline{\psi}_{\alpha 2}$. The zero modes of free fermions in the space-time path integral cause $Z_{PP} = 0$.

There is a physical approach that leads to this solution of modular covariant partition functions. Consider the combination of a Heisenberg chain and a spin-1/2 Dirac fermion (referred to as the HD hybrid system). The low-energy theory for the Heisenberg chain is the $su2_1^1 \oplus \overline{su2_1^1}$ CFT. The low-energy theory of the free spin-1/2 Dirac fermion is $su2_1^2 \oplus u1_2 \oplus \overline{su2_1^2} \oplus \overline{u1_2}$. Its partition function Z_{AA}^{HD} is thus the product of the partition functions of the two CFTs,

$$\begin{aligned} Z_{AA}^{\mathrm{HD}} = {} & \big(\chi_0^{su2_1^1}\overline{\chi}_0^{su2_1^1} + \chi_{1/2}^{su2_1^1}\overline{\chi}_{1/2}^{su2_1^1}\big)\big(\chi_0^{su2_1^2}\chi_0^{u1_2} + \chi_{1/2}^{su2_1^2}\chi_1^{u1_2}\big)\\ & \times\big(\overline{\chi}_0^{su2_1^2}\overline{\chi}_0^{u1_2} + \overline{\chi}_{1/2}^{su2_1^2}\overline{\chi}_1^{u1_2}\big), \end{aligned} \tag{83}$$

TABLE VIII. Quantum numbers of local operators, where k is the crystal momentum.

Operators	Spin	Charge	k
$\psi_{\alpha 2}=e^{i\frac{\varphi_c-\phi_f}{2}}V^{su2_2^s}_{\frac12,\pm\frac12}\sigma_f$	$\frac12$	-1	0
$\overline{\psi}_{\alpha 2}=e^{i\frac{\overline{\varphi}_2}{\sqrt2}}\overline{V}^{su2_1^2}_{\frac12,\pm\frac12}$	$\frac12$	-1	$-\frac{\pi}{2a}$
$\psi^\dagger_{\alpha 1}\overline{\psi}_{\beta 1}=\sigma_f V^{su2_2^s}_{\frac12,\pm\frac12}\overline{V}^{su2_1^1}_{\frac12,\pm\frac12}$	0, 1	0	$\frac{\pi}{a}$

and can be reorganized as

$$\begin{aligned} Z^{\mathrm{HD}}_{AA} = & \big[\big(\chi_0^{su2_1^1}\chi_0^{su2_1^2}\chi_0^{u1_2}+\chi_0^{su2_1^1}\chi_{1/2}^{su2_1^2}\chi_1^{u1_2}\big)\overline{\chi}_0^{su2_1^1} \\ & +\big(\chi_{1/2}^{su2_1^1}\chi_0^{su2_1^2}\chi_0^{u1_2}+\chi_{1/2}^{su2_1^1}\chi_{1/2}^{su2_1^2}\chi_1^{u1_2}\big)\overline{\chi}_{1/2}^{su2_1^1}\big] \\ & \times\big(\overline{\chi}_0^{su2_1^2}\overline{\chi}_0^{u1_2}+\overline{\chi}_{1/2}^{su2_1^2}\overline{\chi}_1^{u1_2}\big). \end{aligned} \tag{84}$$

The interesting fact is that $su2_1^1\times su2_1^2$ characters can all be represented precisely by $su2_2\times$ Is characters. More specifically,

$$\begin{aligned} \chi_0^{su2_1^1}\chi_0^{su2_1^2} &= \chi_0^{su2_2}\chi_0^{\mathrm{Is}}+\chi_1^{su2_2}\chi_\eta^{\mathrm{Is}}, \\ \chi_0^{su2_1^1}\chi_{1/2}^{su2_1^2} &= \chi_1^{su2_1^1}\chi_0^{su2_1^2}=\chi_{1/2}^{su2_2}\chi_\sigma^{\mathrm{Is}}, \\ \chi_{1/2}^{su2_1^1}\chi_{1/2}^{su2_1^2} &= \chi_0^{su2_2}\chi_\eta^{\mathrm{Is}}+\chi_1^{su2_2}\chi_0^{\mathrm{Is}}. \end{aligned} \tag{85}$$

We find that after replacing $\chi_\mu^{su2_1^1}\chi_\nu^{su2_1^2}$ in the partition function (84) with these identities above, and rewriting in the basis with fixed fermion number parity, we reach partition functions (80)–(82).

The partition function (80) provides us a list of local gapless bosonic operators and their scaling dimensions (E1) in the chiral metallic phase. The result is summarized in Table IX.

The crystal momenta of those local gapless bosonic operators are also important quantum numbers. Note that all right movers carry zero crystal momentum. For left movers, the spin-1/2 operators in the $\overline{su_1^1}$ sector carry crystal momentum π. In the $\overline{u1_2}\oplus\overline{su_1^2}$ sector, the operator $\psi_{\alpha 2}\sim e^{\pm i\frac{\overline{\varphi}_2}{\sqrt2}}\overline{V}^{su2_1^2}_{\frac12,\pm\frac12}$ carries crystal momentum $\pm\pi/2$. From these results, we obtain the crystal momenta of the local gapless bosonic operators in Table IX.

From the $\chi_0^{u1_2}\chi_0^{su2_2}\chi_0^{\mathrm{Is}}\overline{\chi}_0^{u1_2}\overline{\chi}_0^{su2_1^1}\overline{\chi}_0^{su2_1^2}$ term in Z_{AE}, we see that there is no discrete symmetry breaking in the chiral metallic phase. If there is, say, a Z_2 symmetry breaking, $2\chi_0^{u1_2}\chi_0^{su2_2}\chi_0^{\mathrm{Is}}\overline{\chi}_0^{u1_2}\overline{\chi}_0^{su2_1^1}\overline{\chi}_0^{su2_1^2}$ will appear in Z_{AE}.

We see that all the local gapless bosonic operators carry nontrivial quantum numbers. Therefore, the chiral metallic phase is stable.

E. Phase transition from the Tomonaga-Luttinger liquid to the chiral metallic phase

This procedure signals that there can be a direct phase transition between the HD hybrid system, the low energy of which is described by Tomonaga-Luttinger liquid theory, and the chiral metallic phase, the low-energy physics of which is described by non-Abelian CFTs. The HD phase has four emergent $SU(2)$ symmetries. The chiral metallic phase has three emergent $SU(2)$ symmetries.

Indeed, the interaction operator (74) is a marginal operator with $h=\overline{h}=1$. It is a tempting indication that the zero-spin marginal perturbation can drive a transition between two stable gapless (under symmetry) phases.

VII. EXAMPLES OF STRONGLY INTERACTING GAPLESS METALLIC STATES IN HIGHER DIMENSIONS

The fact that the emergent symmetry at low energies can be anomalous plays a key role in the solution of the chiral fermion problem [48,49]. For example, in the lattice realization of $SO(10)$ chiral fermions, we start with a 4+1-dimensional (4+1D) slab, which can be viewed as a 3+1-dimensional (3+1D) system from far away. We design the gapped fermion state with $SO(10)$ on-site symmetry in the 4+1D bulk properly, such that its surface is described by 16 massless Weyl fermions, forming a 16-dimensional spinor representation of $SO(10)$. On the 4+1D slab, one 3+1D surface gives rise to 16 chiral Weyl fermions and the other 3+1D surface gives rise to 16 mirror chiral Weyl fermions [see Fig. 4(a)]. Each sector of the Weyl fermions has an emergent symmetry $U(16)$. Such an emergent $U(16)$ symmetry is anomalous for each sector [49]. In Ref. [48], the sufficient conditions are given for a sector (such as the 16 mirror chiral Weyl fermions) to be gappable via interactions without breaking the lattice and on-site symmetry [see Fig. 4(b)]. Applying

TABLE IX. Quantum numbers of local gapless bosonic operators in the chiral metallic phase $su2_2^s\times u1_2\times\mathrm{Is}\times\overline{su2_2^1}\times\overline{su2_2^2}\times\overline{u1}_2$. Here k is the crystal momentum.

Operators	Spin	Charge	k	$h,\overline{h}$
$e^{\pm i\frac{\varphi_c-\varphi_f}{2}\pm i\frac{\overline{\varphi}_2}{\sqrt2}}V^{su2_2^s}_{1,l}\overline{V}^{su2_1^1}_{\frac12,\pm\frac12}\overline{V}^{su2_1^2}_{\frac12,\pm\frac12}$	0,1,2	0, ±2	$\pi\pm\frac{\pi}{2}$	$\frac34,\frac34$
$e^{\pm i\frac{\varphi_c-\varphi_f}{2}\pm i\frac{\overline{\varphi}_2}{\sqrt2}}\eta_f\overline{V}^{su2_1^1}_{\frac12,\pm\frac12}\overline{V}^{su2_1^2}_{\frac12,\pm\frac12}$	0,1	0, ±2	$\pi\pm\frac{\pi}{2}$	$\frac34,\frac34$
$V^{su2_2^s}_{1,l}\eta_f$	1	0	0	1,0
$V^{su2_2^s}_{\frac12,\pm\frac12}\sigma_f\overline{V}^{su2_1^1}_{\frac12,\pm\frac12}$	0,1	0	π	$\frac14,\frac14$
$e^{\pm i\frac{\varphi_c-\varphi_f}{2}\pm i\frac{\overline{\varphi}_2}{\sqrt2}}V^{su2_2^s}_{\frac12,\pm\frac12}\overline{V}^{su2_1^2}_{\frac12,\pm\frac12}$	0,1	0, ±2	$\pm\frac{\pi}{2}$	$\frac12,\frac12$

to the $SO(10)$ case, we find that the 16 mirror chiral Weyl fermions can be gapped without breaking the $SO(10)$ on-site symmetry, and this solves the chiral fermion problem for the case of $SO(10)$ grant unification. We would like to stress that the gapping of 16 chiral Weyl fermions is very special, in the sense that there is no fermion mass term that can achieve such a gapping process without breaking the $SO(10)$ symmetry. It appears that the anomaly of the emergent $U(16)$ symmetry protects the 16 chiral Weyl fermions to be gapless against any small perturbations that respect the $SO(10)$ symmetry.

In the above example, each sector of 16 massless Weyl fermions is free of all anomalies. It was also pointed out in Ref. [48] that even when each sector is anomalous it is still possible that an anomalous sector can be in a topologically ordered gapped phase [62]. This offers a more general way to solve the chiral fermion problem. In general, for a gapless system, the low-energy effective theory for the gapless modes can be anomalous. Even such an anomalous low-energy effective theory can sometimes be realized by a well-defined lattice model in the same dimension, since the anomaly can be canceled by a gapped (anomalous) topological sector.

One such example is the 2D gapless theory of one single Weyl fermion with $U(1)$ (fermion number N_F) and time-reversal (T) symmetry. The time-reversal transformation satisfies $T^2 = (-)^{N_F}$. Such a single-Weyl-fermion theory has a parity anomaly (time reversal is a space-time parity transformation). It was believed (incorrectly) that there was no 2D lattice theory with on-site $U(1)$ and time-reversal symmetries that can produce a low-energy effective theory of a single Weyl fermion. Indeed, there are no *noninteracting* lattice theories with on-site $U(1)$ and time-reversal symmetries that can produce low-energy effective theory of a single Weyl fermion. However, if we include interaction, then there are *interacting* lattice theories with on-site $U(1)$ and time-reversal symmetries that can produce a low-energy effective theory of a single Weyl fermion without breaking those symmetries. One way to construct such an interacting 2D lattice model is to start with a slab of a 3D lattice model, which can be viewed as a 2D lattice model from far away. On the 3D slab we have the topological insulator with $U(1)$ symmetry and $T^2 = (-)^{N_F}$ time-reversal symmetry. The fermions do not interact near one surface of the slab, which gives rise to the low-energy effective theory of a single Weyl fermion. Near the other surface of the slab, fermions interact strongly, which gives rise to a gapped non-Abelian topologically ordered state and does not contribute to low-energy modes [see Fig. 4(b)].

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APPENDIX A: TOMONAGA-LUTTINGER LIQUID AS AN ABELIAN GAPLESS PHASE

We define the Tomonaga-Luttinger liquid (LL) as the liquid containing only excitations with integral (or bosonic) statistics. It can always be written as

$$K_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A1}$$

Thus the LL has the property that $\Gamma = \Gamma_0$. For simplicity, we assume $N_L = N_R = 1$, and there is only one left and one right mode. The essence of the proof does not depend on $N_L (= N_R)$, and can be generalized to $N_L > 1$. The task is to prove that once all excitations in the Lagrangian subgroup are condensed the partition function of the low-energy theory is the same as that of LLs, i.e., the $u1_1 \oplus \overline{u1}_1$ CFT.

We consider one kind of Abelian state, constructed from a double-layered FQH stripe, and gap sectors along only one edge (the top part of Fig. 1) while sectors from the other edge remain gapless (the bottom part of Fig. 1). The claim is that Abelian states realized by such construction are always LLs, the low-energy theory of which is the $u1_1 \oplus \overline{u1}_1$ CFT.

The edge theory of the Abelian FQH state is described by a symmetric integer matrix K. Quasiparticles created by operator $e^{il^T\phi}$ are labeled by an integer vector l. Given two quasiparticles $l, m \in \mathbb{Z}^2$, the self-statistics of the l quasiparticle and the mutual statistics of two quasiparticles are

$$\frac{\theta_l}{\pi} = l^T K^{-1} l, \quad \frac{\theta_{lm}}{2\pi} = l^T K^{-1} m. \tag{A2}$$

In particular, a local excitation is one that can be created by local operators, i.e., bosonic or fermionic operators. One set of local excitations is $\Gamma_0 = K\mathbb{Z}^2$. We see that basis vectors are columns of $K = (k_1, k_2)$. It follows that the K matrix encodes the statistics of these local operators:

$$\frac{\theta_{ij}}{2\pi} = k_i^T K^{-1} k_j = K_{ij}, \tag{A3}$$

which is integral. Another set of local operators is

$$e^{i\mathbf{l}^T(\phi+\overline{\phi})}, \quad \mathbf{l} \in \mathbb{Z}^2, \tag{A4}$$

where $\overline{\phi}$ are fields of the other edge, described by $-K$. We have the statistics

$$\frac{\theta_{\mathbf{lm}}}{2\pi} = \mathbf{l}^T K^{-1}\mathbf{m} - \mathbf{l}^T K^{-1}\mathbf{m} = 0 \tag{A5}$$

and

$$\frac{\theta_{\mathbf{l}m}}{2\pi} = \mathbf{l}^T K^{-1} m \in \mathbb{Z}^2, \tag{A6}$$

since $m \in K\mathbb{Z}^2$.

The gappable condition for a single edge is that there is a set of quasiparticles $m \in \mathbb{Z}^2$ that form a "Lagrangian subgroup" $\mathcal{M}$ [63]. One way to fully gap the edge is to add perturbation:

$$\delta\mathcal{L} = \sum_{m\in\mathcal{M}} g_n \cos(m^T\overline{\phi}). \tag{A7}$$

We see that when $g_n > 0$ are sufficiently large the quasiparticles labeled by m are condensed, i.e., $e^{im^T\overline{\phi}} \sim 1$. The question is, what is the gapless theory for the other edge that remains gapless?

When all m are condensed, the local excitations in Eq. (A4) become

$$e^{i\mathbf{m}^T(\phi+\overline{\phi})} \to e^{im^T\phi}. \tag{A8}$$

Now the lattice of local operators is

$$\Gamma = \oplus_{m\in\mathcal{M}}(m + K\mathbb{Z}^2). \tag{A9}$$

This is still a two-dimensional integral lattice, and can thus be represented by $\Gamma = U\mathbb{Z}^2$, where U is an integral matrix. Levin proved that now $P = U^T K^{-1} U$ is a symmetric integral matrix with vanishing signature, $\det P = \pm 1$. In fact, P is the effective K now:

$$P = U^T K^{-1} U. \tag{A10}$$

Next, by another linear superposition,

$$(W^T)^{-1} = UW_0, \quad W^T K W = \eta, \tag{A11}$$

which means that when the null vectors become local operators will as well. Meanwhile, we tune the interactions at the upper edge appropriately, so that $\widetilde{V}_{ij} = v\delta_{ij}$:

$$V = (W^T)^{-1}\widetilde{V}W^{-1}. \tag{A12}$$

Let us illustrate the proof with two examples. First, we consider

$$K = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}. \tag{A13}$$

Pick a null vector $\mathbf{l} = (1, 1)^T$ and k_1 to form the new basis:

$$U = \begin{pmatrix} 1 & m \\ 1 & 0 \end{pmatrix} \tag{A14}$$

with mutual statistics

$$P = \begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix}. \tag{A15}$$

By a second basis transformation W_0 we find the basis:

$$W_0 = \begin{pmatrix} \frac{1-m}{2} & \frac{1+m}{2} \\ 1 & -1 \end{pmatrix}, \quad UW_0 = \frac{1}{2}\begin{pmatrix} 1+m & 1-m \\ 1-m & 1+m \end{pmatrix}. \tag{A16}$$

With this basis, the statistics is

$$K_{\text{eff}} = K_F. \tag{A17}$$

The interaction is tuned to

$$V = \frac{v}{2}\begin{pmatrix} 1+m^2 & 1-m^2 \\ 1-m^2 & 1+m^2 \end{pmatrix}. \tag{A18}$$

In this basis, all vectors are mutually trivial:

$$e^{iu^T\phi} = e^{i\gamma^T W^{-1}\phi} = e^{i\gamma^T\widetilde{\phi}} \tag{A19}$$

where $\gamma = W^T u$ contributes

$$Z(\tau) = \sum_{\gamma\in\widetilde{\Gamma}} |\eta(q)|^{-2} q^{\frac{1}{2}\gamma_1^2}(q^*)^{\frac{1}{2}\gamma_2^2} \tag{A20}$$

to $Z(\tau)$.

γ is in the lattice

$$\widetilde{\Gamma} = W^T U\mathbb{Z}^2 = W_0^{-1}U^{-1}U\mathbb{Z}^2 = W_0^{-1}\mathbb{Z}^2. \tag{A21}$$

Since W_0 is an integer matrix with $\det W_0 = \pm 1$, so is W_0^{-1}. Then, from the theorem of lattice theory,

$$\widetilde{\Gamma} \equiv \mathbb{Z}^2. \tag{A22}$$

Therefore,

$$Z(\tau) = \sum_{\gamma\in\mathbb{Z}^2} |\eta(q)|^{-2} q^{\frac{1}{2}\gamma_1^2}(q^*)^{\frac{1}{2}\gamma_2^2} = \chi_0^{u1_1}\overline{\chi}_0^{u1_1}, \tag{A23}$$

which is the same as the partition function of the $u1_1$ CFT:

$$Z(\tau) = \sum_{m\in\mathcal{M}}\sum_{\gamma\in\Gamma_m} |\eta(q)|^{-2} q^{\frac{1}{2}\gamma_1^2}(q^*)^{\frac{1}{2}\gamma_2^2}. \tag{A24}$$

We point out that for the velocity matrix

$$V = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \tag{A25}$$

the matrix to make $K \to \eta$ and V diagonal is

$$W' = \frac{1}{\sqrt{m}}\eta, \tag{A26}$$

and in this case the partition function is

$$Z(\tau) = \sum_{m\in P\mathbb{Z}} Z_m(\tau). \tag{A27}$$

It is modular invariant, since

$$Z\left(-\frac{1}{\tau}\right) = \sum_{m,n\in P\mathbb{Z}} S_{mn}Z_n(\tau),$$
$$S_{mn} = \frac{1}{|\det P|} e^{i2\pi m^T K_{\text{eff}}^{-1} n}. \tag{A28}$$

The low-energy theory is a $u1_M \times \overline{u1}_M$ CFT, and

$$(W')^{-1}W = \frac{1}{2\sqrt{m}}\begin{pmatrix} 1+m & -1+m \\ -1+m & 1+m \end{pmatrix} \tag{A29}$$

is a boosting matrix in $SO(1, 1)$.

Second, we consider a general case with a nontrivial Lagrangian subgroup:

$$K = \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}. \tag{A30}$$

We can choose a Lagrangian subgroup $\mathcal{M} = \{(0, 0)^T, (1, 1)^T\}$, and we find

$$U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad W_0 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{A31}$$

We conclude that from the double-layered FQH and gapping one edge the gapless phase obtained is still a Tomonaga-Luttinger liquid.

APPENDIX B: CONFORMAL FIELD THEORY EXTENDED WITH CURRENT ALGEBRAS

The theory of conformal field theory with extended symmetry is well known. In this section, we summarize some defining knowledge to introduce our convention. We refer the readers to Francesco *et al.*'s textbook [64] for further details.

1. Current and primary fields

The $u1$, $su2_1$, and $su2_2$ CFTs are not only invariant under conformal symmetry, but also invariant under current algebras. Current algebras are generated by *currents*, chiral primary fields with scaling dimension 1, and denoted as $J^a(z)$. The defining OPE of the level-k current algebra g_k is

$$J^a(z)J^b(w) \sim \frac{k\delta_{ab}}{(z-w)^2} + \sum_c if_{abc}\frac{J^c(w)}{(z-w)}, \tag{B1}$$

where f_{abc} is the structure constant of the corresponding Lie algebra g. In particular, for $u1_M$ current algebra,

$$J^0(z)J^0(w) \sim \frac{M}{(z-w)^2}, \tag{B2}$$

and for $su2_k$ current algebra in the spin basis, the $su2$ generators of which satisfy

$$[J^+, J^-] = 2J^0, \quad [J^0, J^\pm] = \pm J^\pm, \tag{B3}$$

the OPEs are

$$\begin{aligned} J^0(z)J^0(w) &\sim \frac{k/2}{(z-w)^2}, \\ J^0(z)J^\pm(w) &\sim \frac{\pm J^\pm(w)}{(z-w)}, \\ J^+(z)J^-(w) &\sim \frac{k}{(z-w)^2} + \frac{2J^0(w)}{(z-w)}. \end{aligned} \tag{B4}$$

Analogous to highest weight representations of Lie algebras, the highest weight representations of the current algebras are labeled by primary fields. The defining OPE of a primary field $V_\lambda(z)$ is

$$J^a(z)V_\lambda(w) \sim \frac{t^a_\lambda V_\lambda(w)}{z-w}, \tag{B5}$$

where t^a_λ is the representation matrix for J^a of g in the representation labeled by λ.

Current algebras can be represented in terms of different quantum fields, as long as the different representations produce the same correlation functions (so-called quantum equivalence). In particular, primary fields of the above current algebras can be expressed in terms of the chiral compactified bosonic field ϕ and primary fields in the Is CFT. The representations of current fields and primary fields, and their scaling dimensions h, are listed in Table X. Table XI lists the primary fields and the scaling dimensions h.

2. Operator product expansion

The OPE of fermion operators is

$$\eta(z)\eta(w) \sim \frac{1}{z-w}. \tag{B6}$$

The OPEs of Ising primary fields are

$$\begin{aligned} \sigma(z)\sigma(w) &\sim \frac{1}{(z-w)^{\frac{1}{8}}} + C(z-w)^{\frac{3}{8}}\eta(w), \\ \eta(z)\sigma(w) &\sim \frac{1}{(z-w)^{\frac{1}{2}}}\mu(w), \\ \eta(z)\mu(w) &\sim \frac{1}{(z-w)^{\frac{1}{2}}}\sigma(w), \end{aligned} \tag{B7}$$

TABLE X. Fields of CFTs with current algebras. J^i are current fields, and others are primary fields (except the identity field with scaling dimension zero) of the current algebra. a_0 is the cutoff length scale.

CFT	Field	h
$u1_M$	$J^0 = i\sqrt{M}\partial\phi$	1
	$V_k = e^{i\frac{k}{\sqrt{M}}\phi}, k = 0, \dots, M-1$	$\frac{k^2}{2M}$
$su2_1$	$J^0 = \frac{i}{\sqrt{2}}\partial\phi$	1
	$J^\pm = e^{\pm i\sqrt{2}\phi}$	1
	$V_{\frac{1}{2},\pm\frac{1}{2}} = e^{\pm i\frac{1}{\sqrt{2}}\phi}$	$\frac{1}{4}$
$su2_2$	$J^0 = i\partial\phi$	1
	$J^\pm = \sqrt{2}\eta e^{\pm i\phi}$	1
	$V_{\frac{1}{2},\pm\frac{1}{2}} = \sigma e^{\pm i\phi/2}$	$\frac{3}{16}$
	$V_{1,\pm 1} = e^{\pm i\phi}$	$\frac{1}{2}$
	$V_{1,0} = \eta$	$\frac{1}{2}$

where μ denotes the disorder operator dual to the spin operator σ, and it has the same OPE and conformal dimensions as σ. All other OPEs can be derived from (73), (B6), and (B8).

3. Characters and modular transformations

Each primary field corresponds to a highest weight representation of the current algebra. The *character* of a highest weight representation encodes the degeneracy or *multiplicities* of states with the same quantum numbers.

a. $u1_M$ character

The $u1_M$ character $\chi^{u1_M}_m$ is given by

$$\chi^{u1_M}_m(\tau) = q^{-\frac{1}{24}}\frac{\sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(\frac{m}{R}+nR)^2}}{\prod_{n=1}^{\infty}(1-q^n)}, \quad R^2 = M, \tag{B8}$$

which contains primary fields of conformal symmetry:

$$e^{i(\frac{m}{R}+nR)\phi}. \tag{B9}$$

When M is even, under modular transformation, $\chi^{u1_M}_m$ transforms as

$$\begin{aligned} \chi^{u1_M}_i\left(-\frac{1}{\tau}\right) &= \sum_j S_{ij}\chi^{u1_M}_j(\tau), \\ S_{ij} &= \frac{1}{\sqrt{M}}e^{-i2\pi\frac{ij}{M}}, \\ \chi^{u1_M}_i(\tau+1) &= e^{i2\pi(\frac{1}{2}\frac{i^2}{M}-\frac{1}{24})}\chi^{u1_M}_i(\tau). \end{aligned} \tag{B10}$$

When M is odd, $\chi^{u1_M}_m$ corresponds to the partition function of a fermionic system.

TABLE XI. Primary fields of Ising CFT.

Primary field	h
1	0
σ	$\frac{1}{16}$
η	$\frac{1}{2}$

b. su2$_k$ character

The $\widehat{su}(2)_k$ character $\chi_j^{su2_k}(\tau)$ is

$$\chi_j^{su2_k}(q) = \frac{q^{(2j+1)^2/4(k+2)}}{[\eta(q)]^3} \times \sum_{n\in\mathbb{Z}} [2j+1+2n(k+2)]q^{n[2j+1+(k+2)n]} \tag{B11}$$

where $j \in \mathcal{P} = \{0, \frac{1}{2}, \cdots, \frac{k}{2}\}$. The modular transformations are

$$\begin{aligned} \chi_j^{su2_k}(-1/\tau) &= \sum_{l\in\mathcal{P}} S_{jl}\chi_l^{su2_k}(\tau), \\ S_{jl} &= \sqrt{\frac{2}{k+2}}\sin\left[\frac{\pi(2j+1)(2l+1)}{k+2}\right], \\ \chi_j^{su2_k}(\tau+1) &= \sum_{l\in\mathcal{P}} T_{jl}\chi_l^{su2_k}(\tau), \\ T_{jl} &= \delta_{jl}e^{i2\pi\left(\frac{j(j+1)}{k+2}-\frac{c}{24}\right)}, \end{aligned} \tag{B12}$$

and $c = \frac{3k}{k+2}$.

c. Ising characters

The Ising characters are

$$\chi_{r,s}(q) = \eta^{-1}(q)\sum_{n\in\mathbb{Z}}\left(q^{(24n+4r-3s)^2/48} - q^{(24n+4r+3s)^2/48}\right), \tag{B13}$$

where

$$\chi_{\mathbf{1}} \equiv \chi_{1,1}, \quad \chi_\sigma \equiv \chi_{1,2}, \quad \chi_\psi \equiv \chi_{2,1}. \tag{B14}$$

In the basis $(\chi_{\mathbf{1}}, \chi_\sigma, \chi_\eta)$, the S matrix is

$$S = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \tag{B15}$$

and the T operation is

$$T\chi_\mu = e^{i2\pi\left(h_\mu - \frac{1}{48}\right)}\chi_\mu, \tag{B16}$$

where $h_{\mathbf{1}} = 0$, $h_\sigma = \frac{1}{16}$, and $h_\eta = \frac{1}{2}$.

APPENDIX C: EXACTLY MARGINAL OPERATORS

Consider a perturbation $\delta S = \frac{1}{2\pi}\sum_i g_i \int d^2z\phi_i(z,\bar{z})$, where $\phi_i(z,\bar{z})$ is the marginal Virasora primary field with weights $(h_i, \bar{h}_i)$. The correction of the correlations of $O(z,\bar{z})$, a product of primary fields, is

$$\frac{\delta}{\delta g_j}\langle O\rangle = \frac{1}{2\pi}\int d^2w\langle\phi_j(w,\overline{w})O\rangle. \tag{C1}$$

In particular, by taking $O = \phi_i(z_1,\bar{z}_1)\phi_i(z_2,\bar{z}_2)$, one can show that to the first order in δg_i the correction to the weights is

$$\delta h_i = \delta\bar{h}_i = -\sum_j c_{iij}\delta g_j. \tag{C2}$$

The necessary condition for a marginal operator to be exactly marginal, i.e., preserving conformal symmetry when g_i is turned on continuously, is that $c_{iij} = 0$, for any primary field ϕ_j.

APPENDIX D: PARTITION FUNCTIONS OF FREE SPIN-$\frac{1}{2}$ DIRAC FERMIONS

The spin-$\frac{1}{2}$ Dirac fermions can be considered as the representation of the $u1_2 \oplus su2_1$ current algebra. The partition functions are

$$\begin{aligned} Z_{AE}^{\text{Dirac}} &= \chi_0^{u1_2}\chi_0^{su2_1}\overline{\chi}_0^{u1_2}\overline{\chi}_0^{su2_1} + \chi_1^{u1_2}\chi_{1/2}^{su2_1}\overline{\chi}_1^{u1_2}\overline{\chi}_{1/2}^{su2_1} \\ &= \frac{1}{2}\left(\left|\frac{\theta_3(q)}{\eta(q)}\right|^4 + \left|\frac{\theta_4(q)}{\eta(q)}\right|^4\right) \\ &= \sum_{k=0,2,4}\binom{4}{k}\left(Z_{AE}^{\text{Maj}}\right)^k\left(Z_{AO}^{\text{Maj}}\right)^{4-k}, \\ Z_{AO}^{\text{Dirac}} &= \chi_0^{u1_2}\chi_0^{su2_1}\overline{\chi}_1^{u1_2}\overline{\chi}_{1/2}^{su2_1} - \chi_1^{u1_2}\chi_{1/2}^{su2_1}\overline{\chi}_0^{u1_2}\overline{\chi}_0^{su2_1} \\ &= \frac{1}{2}\left(\left|\frac{\theta_3(q)}{\eta(q)}\right|^4 - \left|\frac{\theta_4(q)}{\eta(q)}\right|^4\right) \\ &= \sum_{k=1,3}\binom{4}{k}\left(Z_{AE}^{\text{Maj}}\right)^k\left(Z_{AO}^{\text{Maj}}\right)^{4-k}, \\ Z_{PE}^{\text{Dirac}} &= Z_{PO}^{\text{Dirac}} \\ &= \frac{1}{2}\left(\chi_0^{u1_2}\chi_{1/2}^{su2_1} + \chi_1^{u1_2}\chi_0^{su2_1}\right)\left(\overline{\chi}_0^{u1_2}\overline{\chi}_{1/2}^{su2_1} + \overline{\chi}_1^{u1_2}\overline{\chi}_0^{su2_1}\right) \\ &= \frac{1}{2}\left|\frac{\theta_2(q)}{\eta(q)}\right|^4 = 8\left(Z_{PE}^{\text{Maj}}\right)^4, \end{aligned} \tag{D1}$$

where we have used Jacobi's theta functions $\theta_a(\tau)$ to track the various identities between characters. To understand the multiplicity in $Z_{PE}^{\text{Dirac}}(Z_{PO}^{\text{Dirac}})$, we compare it with

$$Z_{PE}^{\text{Maj}} = Z_{PO}^{\text{Maj}} = \frac{1}{2}\left|\frac{\theta_2(q)}{\eta(q)}\right|. \tag{D2}$$

Since θ_2/η has twofold degeneracy, there is no degeneracy in Z_{PE}^{Maj}, but an eightfold degeneracy in Z_{PE}^{Dirac}. Physically, in n chains of a Majorana fermion with a periodic boundary condition, there is a ground-state degeneracy of 2^n due to n zero modes. For fixed fermion number parity, the degeneracy is

$$\sum_{k=0,2,\cdots,k\leqslant n}\binom{n}{k} = \sum_{k=1,3,\cdots,k\leqslant n}\binom{n}{k} = 2^{n-1}. \tag{D3}$$

APPENDIX E: SCALING DIMENSIONS OF PRIMARY FIELDS IN THE CHIRAL METALLIC PHASE

Here we list the scaling dimensions $(h, \bar{h})$ of primary fields corresponding to each term of characters Z_{AE} (80).

(1) In Z_{AE} (80),

$$(0,0),\ \left(\tfrac{3}{4},\tfrac{3}{4}\right),\ \left(\tfrac{3}{4},\tfrac{3}{4}\right),\ (1,0)\ \left(\tfrac{1}{4},\tfrac{1}{4}\right),\ \left(\tfrac{1}{2},\tfrac{1}{2}\right). \tag{E1}$$

(2) In Z_{AO} (81),

$$\left(0,\tfrac{1}{2}\right),\ \left(\tfrac{3}{4},\tfrac{1}{4}\right),\ \left(\tfrac{3}{4},\tfrac{1}{4}\right),\ \left(1,\tfrac{1}{2}\right)\ \left(\tfrac{1}{4},\tfrac{3}{4}\right),\ \left(\tfrac{1}{2},0\right). \tag{E2}$$

(3) In Z_{PE} or Z_{PO} (82),

$$\begin{aligned}&\left(\tfrac{1}{4},\tfrac{1}{4}\right),\quad \left(\tfrac{1}{4},\tfrac{1}{4}\right),\quad \left(\tfrac{1}{2},\tfrac{1}{2}\right),\quad \left(\tfrac{1}{2},\tfrac{1}{2}\right),\\ &\left(\tfrac{1}{2},\tfrac{1}{2}\right),\quad \left(\tfrac{1}{2},\tfrac{1}{2}\right),\quad \left(\tfrac{5}{4},\tfrac{1}{4}\right),\quad \left(\tfrac{5}{4},\tfrac{1}{4}\right),\\ &\left(\tfrac{1}{4},\tfrac{1}{4}\right),\quad \left(\tfrac{1}{4},\tfrac{1}{4}\right),\quad \left(\tfrac{1}{2},\tfrac{1}{2}\right),\quad \left(\tfrac{1}{2},\tfrac{1}{2}\right).\end{aligned} \tag{E3}$$

APPENDIX F: COMPUTATION OF MODULAR INVARIANT PARTITION FUNCTIONS

1. Fusion algebra in the chiral metallic phase

Fields generated from OPEs of fermion operators can be summarized in terms of fusion algebras. The primary fields in the partition functions can be organized in the following fashion. We denote the vacuum and local operators as

$$\begin{aligned}\nu_1 &= V_0^{u1_2}V_0^{su2_2}\mathbf{1}_{\rm Is}\overline{V}_0^{u1_2}\overline{V}_0^{su2_1^1}\overline{V}_0^{su2_1^2},\\ \nu_2 \equiv \overline{\psi}_{\alpha 2} &= V_0^{u1_2}V_0^{su2_2}\mathbf{1}_{\rm Is}\overline{V}_1^{u1_2}\overline{V}_0^{su2_1^1}\overline{V}_{\frac{1}{2}}^{su2_1^2},\\ \mu_1 \equiv \psi_{\alpha 2} &= V_1^{u1_2}V_{\frac{1}{2}}^{su2_2}\sigma\overline{V}_0^{u1_2}\overline{V}_0^{su2_1^1}\overline{V}_0^{su2_1^2},\\ \mu_2 \equiv \psi_{\alpha 1}^{\dagger}\overline{\psi}_{\beta 1} &= V_0^{u1_2}V_{\frac{1}{2}}^{su2_2}\sigma\overline{V}_0^{u1_2}\overline{V}_{\frac{1}{2}}^{su2_1^1}\overline{V}_0^{su2_1^2}.\end{aligned} \tag{F1}$$

All the primary fields generated from the fusion of them are

$$\begin{aligned}\mu_3 = \nu_2\times\mu_1 &= V_1^{u1_2}V_{\frac{1}{2}}^{su2_2}\sigma\overline{V}_1^{u1_2}\overline{V}_0^{su2_1^1}\overline{V}_{\frac{1}{2}}^{su2_1^2},\\ \mu_4 = \nu_2\times\mu_2 &= V_0^{u1_2}V_{\frac{1}{2}}^{su2_2}\sigma\overline{V}_1^{u1_2}\overline{V}_{\frac{1}{2}}^{su2_1^1}\overline{V}_{\frac{1}{2}}^{su2_1^2}.\end{aligned} \tag{F2}$$

The other fields are ν_{ij}, $1\leqslant i\leqslant 4, 0\leqslant j\leqslant 3$ defined as

$$\begin{aligned}\nu_{i0} &= \nu_i,\\ \nu_{i1} &= V_0^{u1_2}V_1^{su2_2}\mathbf{1}_{\rm Is}\times\nu_{i0},\\ \nu_{i2} &= V_0^{u1_2}V_0^{su2_2}\eta\times\nu_{i0},\\ \nu_{i3} &= V_0^{u1_2}V_1^{su2_2}\eta\times\nu_{i0}.\end{aligned} \tag{F3}$$

ν_1 and ν_2 have been given. ν_3 and ν_4 are determined by

$$\mu_1\times\mu_2 = \sum_{j=0}^{3}\nu_{3j},\quad \mu_1\times\mu_4 = \sum_{j=0}^{3}\nu_{4j}. \tag{F4}$$

The solution is

$$\begin{aligned}\nu_3 &= V_1^{u1_2}V_0^{su2_2}\mathbf{1}_{\rm Is}\overline{V}_0^{u1_2}\overline{V}_{\frac{1}{2}}^{su2_1^1}\overline{V}_0^{su2_1^2},\\ \nu_4 &= V_1^{u1_2}V_0^{su2_2}\mathbf{1}_{\rm Is}\overline{V}_1^{u1_2}\overline{V}_{\frac{1}{2}}^{su2_1^1}\overline{V}_{\frac{1}{2}}^{su2_1^2}.\end{aligned} \tag{F5}$$

Note that all $\nu_{i0}\equiv\nu_i$ have the partial vacuum $V_0^{su2_2}\mathbf{1}_{\rm Is}$, thus the fusion in Eq. (F3) is Abelian and trivial.

The primary fields defined above form a complete set such that the fusion algebra is closed. We denote this fusion algebra as

$$\mathcal{C} = \{\mu_l, \nu_{ij}|\ 1\leqslant l, i\leqslant 4, 0\leqslant j\leqslant 3\}. \tag{F6}$$

2. Procedure to look for modular invariant partition functions

From the fusion algebra $\mathcal{C}$, we can look for solutions of modular covariant Z.

(1) Assign non-negative integral multiplicities for the characters χ_{c_j} for $c_j\in\mathcal{C}$, and sum over them to get an initial partition function z.

(2) Generate a set of partition functions:

$$z,\quad Sz,\quad TSz,\quad STSz,\quad Tz,\quad STq. \tag{F7}$$

Since $S^2=1$ [65], $T^2=1$, and $(TS)^3=1$, they are all the partition functions generated by modular transformations. Next one checks that all the multiplicities in these vectors are non-negative integers, and the primary fields in these partition functions are either bosonic or fermionic fields.

(3) There are three basis vectors of S invariant partition functions:

$$z_1 = z+Sz,\quad z_T = Tz+STz,\quad z_{TS} = TSz+STSz. \tag{F8}$$

Therefore all vectors $Z_{AA}=\sum_{i=1}^3 a_i z_i,\ a_i\in\mathbb{Z}, a_i\geqslant 0$ are invariant under S transformation.

(4) The other sectors of partition functions can be generated by $Z_{AP}=TZ_{AA}$ and $Z_{PA}=SZ_{AP}$.

(5) Use Z_{PP} to find $\sum_{i=1}^3 a_i z_i,\ \ a_i\in\mathbb{Z}, a_i\geqslant 0$, which is purely bosonic.

The only free choice in the procedure is in the first step; the general guideline is to assign small integers to the multiplicities.

Another modular covariant partition function

There is another independent solution for modular covariant partition functions, as shown below:

$$\begin{aligned}Z'_{AE} =\ &\chi_0^{u1_2}\chi_0^{su2_2}\chi_0^{\rm Is}\overline{\chi}_0^{u1_2}\overline{\chi}_0^{su2_1^1}\overline{\chi}_0^{su2_1^2}\\ &+\chi_1^{u1_2}\chi_1^{su2_2}\chi_0^{\rm Is}\overline{\chi}_1^{u1_2}\overline{\chi}_{1/2}^{su2_1^1}\overline{\chi}_{1/2}^{su2_1^2}\\ &+\chi_1^{u1_2}\chi_0^{su2_2}\chi_\eta^{\rm Is}\overline{\chi}_1^{u1_2}\overline{\chi}_{1/2}^{su2_1^1}\overline{\chi}_{1/2}^{su2_1^2}\\ &+\chi_0^{u1_2}\chi_1^{su2_2}\chi_\eta^{\rm Is}\overline{\chi}_0^{u1_2}\overline{\chi}_0^{su2_1^1}\overline{\chi}_0^{su2_1^2}\\ &+\chi_1^{u1_2}\chi_0^{su2_2}\chi_0^{\rm Is}\overline{\chi}_0^{u1_2}\overline{\chi}_{1/2}^{su2_1^1}\overline{\chi}_0^{su2_1^2}\\ &+\chi_0^{u1_2}\chi_1^{su2_2}\chi_0^{\rm Is}\overline{\chi}_1^{u1_2}\overline{\chi}_0^{su2_1^1}\overline{\chi}_{1/2}^{su2_1^2}\\ &+\chi_0^{u1_2}\chi_0^{su2_2}\chi_\eta^{\rm Is}\overline{\chi}_1^{u1_2}\overline{\chi}_0^{su2_1^1}\overline{\chi}_{1/2}^{su2_1^2}\\ &+\chi_1^{u1_2}\chi_1^{su2_2}\chi_\eta^{\rm Is}\overline{\chi}_0^{u1_2}\overline{\chi}_{1/2}^{su2_1^1}\overline{\chi}_0^{su2_1^2}.\end{aligned} \tag{F9}$$

The scaling dimensions of the primary field corresponding to each term are

$$\begin{aligned}&(0,0),\quad \left(\tfrac{3}{4},\tfrac{3}{4}\right),\quad \left(\tfrac{3}{4},\tfrac{3}{4}\right),\quad (1,0),\\ &\left(\tfrac{1}{4},\tfrac{1}{4}\right),\quad \left(\tfrac{1}{2},\tfrac{1}{2}\right),\quad \left(\tfrac{1}{2},\tfrac{1}{2}\right),\quad \left(\tfrac{5}{4},\tfrac{1}{4}\right);\end{aligned} \tag{F10}$$

$$\begin{aligned}Z'_{AO} =\ &\chi_0^{u1_2}\chi_0^{su2_2}\chi_0^{\rm Is}\overline{\chi}_1^{u1_2}\overline{\chi}_0^{su2_1^1}\overline{\chi}_{1/2}^{su2_1^2}\\ &+\chi_1^{u1_2}\chi_1^{su2_2}\chi_0^{\rm Is}\overline{\chi}_0^{u1_2}\overline{\chi}_{1/2}^{su2_1^1}\overline{\chi}_0^{su2_1^2}\\ &+\chi_1^{u1_2}\chi_0^{su2_2}\chi_\eta^{\rm Is}\overline{\chi}_0^{u1_2}\overline{\chi}_{1/2}^{su2_1^1}\overline{\chi}_0^{su2_1^2}\\ &+\chi_0^{u1_2}\chi_1^{su2_2}\chi_\eta^{\rm Is}\overline{\chi}_1^{u1_2}\overline{\chi}_0^{su2_1^1}\overline{\chi}_{1/2}^{su2_1^2}\\ &+\chi_1^{u1_2}\chi_0^{su2_2}\chi_0^{\rm Is}\overline{\chi}_1^{u1_2}\overline{\chi}_{1/2}^{su2_1^1}\overline{\chi}_{1/2}^{su2_1^2}\end{aligned}$$

$$
\begin{aligned}
&+\chi_0^{u1_2}\chi_1^{su2_2}\chi_0^{\rm Is}\overline{\chi}_0^{u1_2}\overline{\chi}_0^{su2_1^1}\overline{\chi}_0^{su2_1^2}\\
&+\chi_0^{u1_2}\chi_0^{su2_2}\chi_\eta^{\rm Is}\overline{\chi}_0^{u1_2}\overline{\chi}_0^{su2_1^1}\overline{\chi}_0^{su2_1^2}\\
&+\chi_1^{u1_2}\chi_1^{su2_2}\chi_\eta^{\rm Is}\overline{\chi}_1^{u1_2}\overline{\chi}_{1/2}^{su2_1^1}\overline{\chi}_{1/2}^{su2_1^2}. \qquad \text{(F11)}
\end{aligned}
$$

The scaling dimensions of the primary field corresponding to each term are

$$
\begin{aligned}
&\left(0,\tfrac{1}{2}\right),\quad \left(\tfrac{3}{4},\tfrac{1}{4}\right),\quad \left(\tfrac{3}{4},\tfrac{1}{4}\right),\quad \left(1,\tfrac{1}{2}\right),\\
&\left(\tfrac{1}{4},\tfrac{3}{4}\right),\quad \left(\tfrac{1}{2},0\right),\quad \left(\tfrac{1}{2},0\right),\quad \left(\tfrac{5}{4},\tfrac{3}{4}\right); \qquad \text{(F12)}
\end{aligned}
$$

$$
\begin{aligned}
Z'_{PE} =& Z'_{PO}\\
=&\chi_0^{u1_2}\chi_{1/2}^{su2_2}\chi_\sigma^{\rm Is}\overline{\chi}_1^{u1_2}\overline{\chi}_0^{su2_1^1}\overline{\chi}_0^{su2_1^2}\\
&+\chi_0^{u1_2}\chi_{1/2}^{su2_2}\chi_\sigma^{\rm Is}\overline{\chi}_0^{u1_2}\overline{\chi}_0^{su2_1^1}\overline{\chi}_{1/2}^{su2_1^2}\\
&+\chi_1^{u1_2}\chi_{1/2}^{su2_2}\chi_\sigma^{\rm Is}\overline{\chi}_1^{u1_2}\overline{\chi}_{1/2}^{su2_1^1}\overline{\chi}_0^{su2_1^2}\\
&+\chi_1^{u1_2}\chi_{1/2}^{su2_2}\chi_\sigma^{\rm Is}\overline{\chi}_0^{u1_2}\overline{\chi}_{1/2}^{su2_1^1}\overline{\chi}_{1/2}^{su2_1^2}. \qquad \text{(F13)}
\end{aligned}
$$

The scaling dimensions of the primary field corresponding to each term are

$$
\left(\tfrac{1}{4},\tfrac{1}{4}\right),\quad \left(\tfrac{1}{4},\tfrac{1}{4}\right),\quad \left(\tfrac{1}{2},\tfrac{1}{2}\right),\quad \left(\tfrac{1}{2},\tfrac{1}{2}\right). \qquad \text{(F14)}
$$

APPENDIX G: SOLUTIONS OF EQ. (25)

Using the characters of the $u1_4$ CFT, $\chi_m^{u1_4}$, and the characters of the Ising CFT, $\chi_h^{\rm Is}$, we can construct many solutions of Eq. (25). The following is a list of 36 solutions (the list may not be complete):

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)\left(|\chi_0^{\rm Is}|^2+|\chi_{\frac12}^{\rm Is}|^2\right),\\
Z_{PO^f,PE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(|\chi_0^{\rm Is}|^2+|\chi_{\frac12}^{\rm Is}|^2\right),\\
Z_{PE^f,PE^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)\left(|\chi_0^{\rm Is}|^2+|\chi_{\frac12}^{\rm Is}|^2\right),\\
Z_{AO^f,PE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(|\chi_0^{\rm Is}|^2+|\chi_{\frac12}^{\rm Is}|^2\right),\\
Z_{AE^f,PO^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{PO^f,PO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{PE^f,PO^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{AO^f,PO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{AE^f,AE^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{PO^f,AE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{PE^f,AE^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{AO^f,AE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{AE^f,AO^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)\left(\chi_0^{\rm Is}\overline{\chi}_{\frac12}^{\rm Is}+\chi_{\frac12}^{\rm Is}\overline{\chi}_0^{\rm Is}\right),\\
Z_{PO^f,AO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\chi_0^{\rm Is}\overline{\chi}_{\frac12}^{\rm Is}+\chi_{\frac12}^{\rm Is}\overline{\chi}_0^{\rm Is}\right),\\
Z_{PE^f,AO^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)\left(\chi_0^{\rm Is}\overline{\chi}_{\frac12}^{\rm Is}+\chi_{\frac12}^{\rm Is}\overline{\chi}_0^{\rm Is}\right),\\
Z_{AO^f,AO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\chi_0^{\rm Is}\overline{\chi}_{\frac12}^{\rm Is}+\chi_{\frac12}^{\rm Is}\overline{\chi}_0^{\rm Is}\right); \qquad \text{(G1)}
\end{aligned}
$$

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)\left(|\chi_0^{\rm Is}|^2+|\chi_{\frac12}^{\rm Is}|^2\right),\\
Z_{PO^f,PE^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)\left(|\chi_0^{\rm Is}|^2+|\chi_{\frac12}^{\rm Is}|^2\right),\\
Z_{PE^f,PE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(|\chi_0^{\rm Is}|^2+|\chi_{\frac12}^{\rm Is}|^2\right),\\
Z_{AO^f,PE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(|\chi_0^{\rm Is}|^2+|\chi_{\frac12}^{\rm Is}|^2\right),\\
Z_{AE^f,PO^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{PO^f,PO^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{PE^f,PO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{AO^f,PO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{AE^f,AE^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{PO^f,AE^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{PE^f,AE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{AO^f,AE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{AE^f,AO^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)\left(\chi_0^{\rm Is}\overline{\chi}_{\frac12}^{\rm Is}+\chi_{\frac12}^{\rm Is}\overline{\chi}_0^{\rm Is}\right),\\
Z_{PO^f,AO^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)\left(\chi_0^{\rm Is}\overline{\chi}_{\frac12}^{\rm Is}+\chi_{\frac12}^{\rm Is}\overline{\chi}_0^{\rm Is}\right),\\
Z_{PE^f,AO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\chi_0^{\rm Is}\overline{\chi}_{\frac12}^{\rm Is}+\chi_{\frac12}^{\rm Is}\overline{\chi}_0^{\rm Is}\right),\\
Z_{AO^f,AO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\chi_0^{\rm Is}\overline{\chi}_{\frac12}^{\rm Is}+\chi_{\frac12}^{\rm Is}\overline{\chi}_0^{\rm Is}\right); \qquad \text{(G2)}
\end{aligned}
$$

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)\left(|\chi_0^{\rm Is}|^2+|\chi_{\frac12}^{\rm Is}|^2\right),\\
Z_{PO^f,PE^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{PE^f,PE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{AO^f,PE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(|\chi_0^{\rm Is}|^2+|\chi_{\frac12}^{\rm Is}|^2\right),\\
Z_{AE^f,PO^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{PO^f,PO^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)\left(|\chi_0^{\rm Is}|^2+|\chi_{\frac12}^{\rm Is}|^2\right),\\
Z_{PE^f,PO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(|\chi_0^{\rm Is}|^2+|\chi_{\frac12}^{\rm Is}|^2\right),\\
Z_{AO^f,PO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{AE^f,AE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\chi_0^{\rm Is}\overline{\chi}_{\frac12}^{\rm Is}+\chi_{\frac12}^{\rm Is}\overline{\chi}_0^{\rm Is}\right),\\
Z_{PO^f,AE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{PE^f,AE^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\rm Is}|^2,\\
Z_{AO^f,AE^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)\left(\chi_0^{\rm Is}\overline{\chi}_{\frac12}^{\rm Is}+\chi_{\frac12}^{\rm Is}\overline{\chi}_0^{\rm Is}\right),
\end{aligned}
$$

$$\begin{aligned}
Z_{AE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{PO^f,AO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\big),\\
Z_{PE^f,AO^s} &= \big(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\big),\\
Z_{AO^f,AO^s} &= \big(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2;
\end{aligned}\tag{G3}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\big),\\
Z_{PO^f,PE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{PE^f,PE^s} &= \big(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{AO^f,PE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\big),\\
Z_{AE^f,PO^s} &= \big(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{PO^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\big),\\
Z_{PE^f,PO^s} &= \big(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\big),\\
Z_{AO^f,PO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{AE^f,AE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\big),\\
Z_{PO^f,AE^s} &= \big(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{PE^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{AO^f,AE^s} &= \big(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\big)\big(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\big),\\
Z_{AE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{PO^f,AO^s} &= \big(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\big),\\
Z_{PE^f,AO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\big),\\
Z_{AO^f,AO^s} &= \big(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2;
\end{aligned}\tag{G4}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\big),\\
Z_{PO^f,PE^s} &= \big(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{PE^f,PE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{AO^f,PE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\big),\\
Z_{AE^f,PO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\big),\\
Z_{PO^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{PE^f,PO^s} &= \big(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{AO^f,PO^s} &= \big(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\big)\big(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\big),\\
Z_{AE^f,AE^s} &= \big(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{PO^f,AE^s} &= \big(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\big),\\
Z_{PE^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\big),\\
Z_{AO^f,AE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{AE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{PO^f,AO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\big),\\
Z_{PE^f,AO^s} &= \big(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\big),\\
Z_{AO^f,AO^s} &= \big(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2;
\end{aligned}\tag{G5}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\big),\\
Z_{PO^f,PE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{PE^f,PE^s} &= \big(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{AO^f,PE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\big),\\
Z_{AE^f,PO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\big),\\
Z_{PO^f,PO^s} &= \big(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{PE^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{AO^f,PO^s} &= \big(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\big)\big(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\big),\\
Z_{AE^f,AE^s} &= \big(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{PO^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\big),\\
Z_{PE^f,AE^s} &= \big(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\big),\\
Z_{AO^f,AE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{AE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{PO^f,AO^s} &= \big(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\big),\\
Z_{PE^f,AO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\rm Is}\overline{\chi}_{\frac{1}{2}}^{\rm Is}+\chi_{\frac{1}{2}}^{\rm Is}\overline{\chi}_0^{\rm Is}\big),\\
Z_{AO^f,AO^s} &= \big(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2;
\end{aligned}\tag{G6}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\big),\\
Z_{PO^f,PE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{PE^f,PE^s} &= \big(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\big),\\
Z_{AO^f,PE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{AE^f,PO^s} &= \big(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{PO^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\big),\\
Z_{PE^f,PO^s} &= \big(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\big)\big|\chi_{\frac{1}{16}}^{\rm Is}\big|^2,\\
Z_{AO^f,PO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\rm Is}|^2+|\chi_{\frac{1}{2}}^{\rm Is}|^2\big),
\end{aligned}$$

$$\begin{aligned}
Z_{AE^f,AE^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{PO^f,AE^s} &= \left(\chi_0^{u1_4}\bar{\chi}_2^{u1_4}+\chi_2^{u1_4}\bar{\chi}_0^{u1_4}\right)\left(\chi_0^{\text{Is}}\bar{\chi}_{\frac{1}{2}}^{\text{Is}}+\chi_{\frac{1}{2}}^{\text{Is}}\bar{\chi}_0^{\text{Is}}\right),\\
Z_{PE^f,AE^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{AO^f,AE^s} &= \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)\left(\chi_0^{\text{Is}}\bar{\chi}_{\frac{1}{2}}^{\text{Is}}+\chi_{\frac{1}{2}}^{\text{Is}}\bar{\chi}_0^{\text{Is}}\right),\\
Z_{AE^f,AO^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)\left(\chi_0^{\text{Is}}\bar{\chi}_{\frac{1}{2}}^{\text{Is}}+\chi_{\frac{1}{2}}^{\text{Is}}\bar{\chi}_0^{\text{Is}}\right),\\
Z_{PO^f,AO^s} &= \left(\chi_0^{u1_4}\bar{\chi}_2^{u1_4}+\chi_2^{u1_4}\bar{\chi}_0^{u1_4}\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{PE^f,AO^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)\left(\chi_0^{\text{Is}}\bar{\chi}_{\frac{1}{2}}^{\text{Is}}+\chi_{\frac{1}{2}}^{\text{Is}}\bar{\chi}_0^{\text{Is}}\right),\\
Z_{AO^f,AO^s} &= \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2;
\end{aligned}\tag{G7}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)\left(|\chi_0^{\text{Is}}|^2+|\chi_{\frac{1}{2}}^{\text{Is}}|^2\right),\\
Z_{PO^f,PE^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{PE^f,PE^s} &= \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)\left(|\chi_0^{\text{Is}}|^2+|\chi_{\frac{1}{2}}^{\text{Is}}|^2\right),\\
Z_{AO^f,PE^s} &= \left(\chi_0^{u1_4}\bar{\chi}_2^{u1_4}+\chi_2^{u1_4}\bar{\chi}_0^{u1_4}\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{AE^f,PO^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{PO^f,PO^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)\left(|\chi_0^{\text{Is}}|^2+|\chi_{\frac{1}{2}}^{\text{Is}}|^2\right),\\
Z_{PE^f,PO^s} &= \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{AO^f,PO^s} &= \left(\chi_0^{u1_4}\bar{\chi}_2^{u1_4}+\chi_2^{u1_4}\bar{\chi}_0^{u1_4}\right)\left(|\chi_0^{\text{Is}}|^2+|\chi_{\frac{1}{2}}^{\text{Is}}|^2\right),\\
Z_{AE^f,AE^s} &= \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{PO^f,AE^s} &= \left(\chi_0^{u1_4}\bar{\chi}_2^{u1_4}+\chi_2^{u1_4}\bar{\chi}_0^{u1_4}\right)\left(\chi_0^{\text{Is}}\bar{\chi}_{\frac{1}{2}}^{\text{Is}}+\chi_{\frac{1}{2}}^{\text{Is}}\bar{\chi}_0^{\text{Is}}\right),\\
Z_{PE^f,AE^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{AO^f,AE^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)\left(\chi_0^{\text{Is}}\bar{\chi}_{\frac{1}{2}}^{\text{Is}}+\chi_{\frac{1}{2}}^{\text{Is}}\bar{\chi}_0^{\text{Is}}\right),\\
Z_{AE^f,AO^s} &= \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)\left(\chi_0^{\text{Is}}\bar{\chi}_{\frac{1}{2}}^{\text{Is}}+\chi_{\frac{1}{2}}^{\text{Is}}\bar{\chi}_0^{\text{Is}}\right),\\
Z_{PO^f,AO^s} &= \left(\chi_0^{u1_4}\bar{\chi}_2^{u1_4}+\chi_2^{u1_4}\bar{\chi}_0^{u1_4}\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{PE^f,AO^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)\left(\chi_0^{\text{Is}}\bar{\chi}_{\frac{1}{2}}^{\text{Is}}+\chi_{\frac{1}{2}}^{\text{Is}}\bar{\chi}_0^{\text{Is}}\right),\\
Z_{AO^f,AO^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2;
\end{aligned}\tag{G8}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)\left(|\chi_0^{\text{Is}}|^2+|\chi_{\frac{1}{2}}^{\text{Is}}|^2\right),\\
Z_{PO^f,PE^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)\left(|\chi_0^{\text{Is}}|^2+|\chi_{\frac{1}{2}}^{\text{Is}}|^2\right),\\
Z_{PE^f,PE^s} &= \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{AO^f,PE^s} &= \left(\chi_0^{u1_4}\bar{\chi}_2^{u1_4}+\chi_2^{u1_4}\bar{\chi}_0^{u1_4}\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{AE^f,PO^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{PO^f,PO^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{PE^f,PO^s} &= \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)\left(|\chi_0^{\text{Is}}|^2+|\chi_{\frac{1}{2}}^{\text{Is}}|^2\right),\\
Z_{AO^f,PO^s} &= \left(\chi_0^{u1_4}\bar{\chi}_2^{u1_4}+\chi_2^{u1_4}\bar{\chi}_0^{u1_4}\right)\left(|\chi_0^{\text{Is}}|^2+|\chi_{\frac{1}{2}}^{\text{Is}}|^2\right),\\
Z_{AE^f,AE^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{PO^f,AE^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{PE^f,AE^s} &= \left(\chi_0^{u1_4}\bar{\chi}_2^{u1_4}+\chi_2^{u1_4}\bar{\chi}_0^{u1_4}\right)\left(\chi_0^{\text{Is}}\bar{\chi}_{\frac{1}{2}}^{\text{Is}}+\chi_{\frac{1}{2}}^{\text{Is}}\bar{\chi}_0^{\text{Is}}\right),\\
Z_{AO^f,AE^s} &= \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)\left(\chi_0^{\text{Is}}\bar{\chi}_{\frac{1}{2}}^{\text{Is}}+\chi_{\frac{1}{2}}^{\text{Is}}\bar{\chi}_0^{\text{Is}}\right),\\
Z_{AE^f,AO^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)\left(\chi_0^{\text{Is}}\bar{\chi}_{\frac{1}{2}}^{\text{Is}}+\chi_{\frac{1}{2}}^{\text{Is}}\bar{\chi}_0^{\text{Is}}\right),\\
Z_{PO^f,AO^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)\left(\chi_0^{\text{Is}}\bar{\chi}_{\frac{1}{2}}^{\text{Is}}+\chi_{\frac{1}{2}}^{\text{Is}}\bar{\chi}_0^{\text{Is}}\right),\\
Z_{PE^f,AO^s} &= \left(\chi_0^{u1_4}\bar{\chi}_2^{u1_4}+\chi_2^{u1_4}\bar{\chi}_0^{u1_4}\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{AO^f,AO^s} &= \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2;
\end{aligned}\tag{G9}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)\left(|\chi_0^{\text{Is}}|^2+|\chi_{\frac{1}{2}}^{\text{Is}}|^2\right),\\
Z_{PO^f,PE^s} &= \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)\left(|\chi_0^{\text{Is}}|^2+|\chi_{\frac{1}{2}}^{\text{Is}}|^2\right),\\
Z_{PE^f,PE^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{AO^f,PE^s} &= \left(\chi_0^{u1_4}\bar{\chi}_2^{u1_4}+\chi_2^{u1_4}\bar{\chi}_0^{u1_4}\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{AE^f,PO^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{PO^f,PO^s} &= \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{PE^f,PO^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)\left(|\chi_0^{\text{Is}}|^2+|\chi_{\frac{1}{2}}^{\text{Is}}|^2\right),\\
Z_{AO^f,PO^s} &= \left(\chi_0^{u1_4}\bar{\chi}_2^{u1_4}+\chi_2^{u1_4}\bar{\chi}_0^{u1_4}\right)\left(|\chi_0^{\text{Is}}|^2+|\chi_{\frac{1}{2}}^{\text{Is}}|^2\right),\\
Z_{AE^f,AE^s} &= \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{PO^f,AE^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{PE^f,AE^s} &= \left(\chi_0^{u1_4}\bar{\chi}_2^{u1_4}+\chi_2^{u1_4}\bar{\chi}_0^{u1_4}\right)\left(\chi_0^{\text{Is}}\bar{\chi}_{\frac{1}{2}}^{\text{Is}}+\chi_{\frac{1}{2}}^{\text{Is}}\bar{\chi}_0^{\text{Is}}\right),\\
Z_{AO^f,AE^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)\left(\chi_0^{\text{Is}}\bar{\chi}_{\frac{1}{2}}^{\text{Is}}+\chi_{\frac{1}{2}}^{\text{Is}}\bar{\chi}_0^{\text{Is}}\right),\\
Z_{AE^f,AO^s} &= \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)\left(\chi_0^{\text{Is}}\bar{\chi}_{\frac{1}{2}}^{\text{Is}}+\chi_{\frac{1}{2}}^{\text{Is}}\bar{\chi}_0^{\text{Is}}\right),\\
Z_{PO^f,AO^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)\left(\chi_0^{\text{Is}}\bar{\chi}_{\frac{1}{2}}^{\text{Is}}+\chi_{\frac{1}{2}}^{\text{Is}}\bar{\chi}_0^{\text{Is}}\right),\\
Z_{PE^f,AO^s} &= \left(\chi_0^{u1_4}\bar{\chi}_2^{u1_4}+\chi_2^{u1_4}\bar{\chi}_0^{u1_4}\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{AO^f,AO^s} &= \left(|\chi_1^{u1_4}|^2+|\chi_{-1}^{u1_4}|^2\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2;
\end{aligned}\tag{G10}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)\left(|\chi_0^{\text{Is}}|^2+|\chi_{\frac{1}{2}}^{\text{Is}}|^2\right),\\
Z_{PO^f,PE^s} &= \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{PE^f,PE^s} &= \left(\chi_1^{u1_4}\bar{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\bar{\chi}_1^{u1_4}\right)|\chi_{\frac{1}{16}}^{\text{Is}}|^2,\\
Z_{AO^f,PE^s} &= \left(|\chi_0^{u1_4}|^2+|\chi_2^{u1_4}|^2\right)\left(\chi_0^{\text{Is}}\bar{\chi}_{\frac{1}{2}}^{\text{Is}}+\chi_{\frac{1}{2}}^{\text{Is}}\bar{\chi}_0^{\text{Is}}\right),
\end{aligned}$$

$$\begin{aligned}
Z_{AE^f,PO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PO^f,PO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,PO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,PO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AE^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PO^f,AE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,AE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,AO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PO^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,AO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big);
\end{aligned}\tag{G11}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PO^f,PE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,PE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,PO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PO^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,PO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AE^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PO^f,AE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,AE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,AO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PO^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,AO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big);
\end{aligned}\tag{G12}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PO^f,PE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,PE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AO^f,PE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AE^f,PO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PO^f,PO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PE^f,PO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,AE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PO^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PE^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,AE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AE^f,AO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PO^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,AO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AO^f,AO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2;
\end{aligned}\tag{G13}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PO^f,PE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,PE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AO^f,PE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AE^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PO^f,PO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PE^f,PO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,PO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,AE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PO^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PE^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,AE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AE^f,AO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PO^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,AO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AO^f,AO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2;
\end{aligned}\tag{G14}$$

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PO^f,PE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PE^f,PE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,PE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AE^f,PO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PO^f,PO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,PO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AO^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,AE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PO^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AO^f,AE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AE^f,AO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PO^f,AO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,AO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2;
\end{aligned}
\tag{G15}
$$

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PO^f,PE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PE^f,PE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,PE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AE^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PO^f,PO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,PO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AO^f,PO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,AE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PO^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AO^f,AE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AE^f,AO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PO^f,AO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,AO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2;
\end{aligned}
\tag{G16}
$$

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PO^f,PE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,PE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PO^f,PO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,PO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,AE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PO^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,AE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AE^f,AO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PO^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,AO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big);
\end{aligned}
\tag{G17}
$$

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PO^f,PE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,PE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,PO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PO^f,PO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,PO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,PO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,AE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PO^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,AE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),
\end{aligned}
$$

$$
\begin{aligned}
Z_{AE^f,AO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{PO^f,AO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PE^f,AO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AO^f,AO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2 + \left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right);
\end{aligned}
\tag{G18}
$$

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \left(\left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2 + \left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PO^f,PE^s} &= \left(\left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PE^f,PE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AO^f,PE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AE^f,PO^s} &= \left(\left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PO^f,PO^s} &= \left(\left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2 + \left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PE^f,PO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AO^f,PO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AE^f,AE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PO^f,AE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2 + \left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PE^f,AE^s} &= \left(\left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AO^f,AE^s} &= \left(\left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AE^f,AO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2 + \left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PO^f,AO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PE^f,AO^s} &= \left(\left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AO^f,AO^s} &= \left(\left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2;
\end{aligned}
\tag{G19}
$$

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \left(\left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2 + \left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PO^f,PE^s} &= \left(\left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PE^f,PE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AO^f,PE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AE^f,PO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PO^f,PO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2 + \left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PE^f,PO^s} &= \left(\left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AO^f,PO^s} &= \left(\left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AE^f,AE^s} &= \left(\left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PO^f,AE^s} &= \left(\left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2 + \left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PE^f,AE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AO^f,AE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AE^f,AO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2 + \left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PO^f,AO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PE^f,AO^s} &= \left(\left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AO^f,AO^s} &= \left(\left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2;
\end{aligned}
\tag{G20}
$$

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \left(\left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2 + \left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PO^f,PE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{PE^f,PE^s} &= \left(\left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AO^f,PE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AE^f,PO^s} &= \left(\left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PO^f,PO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PE^f,PO^s} &= \left(\left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2 + \left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{AO^f,PO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AE^f,AE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PO^f,AE^s} &= \left(\left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PE^f,AE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2 + \left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{AO^f,AE^s} &= \left(\left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AE^f,AO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2 + \left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PO^f,AO^s} &= \left(\left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{PE^f,AO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AO^f,AO^s} &= \left(\left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2;
\end{aligned}
\tag{G21}
$$

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \left(\left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2 + \left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PO^f,PE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{PE^f,PE^s} &= \left(\left|\chi_0^{u1_4}\right|^2 + \left|\chi_2^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AO^f,PE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AE^f,PO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PO^f,PO^s} &= \left(\left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PE^f,PO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2 + \left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{AO^f,PO^s} &= \left(\left|\chi_1^{u1_4}\right|^2 + \left|\chi_{-1}^{u1_4}\right|^2\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),
\end{aligned}
$$

$$\begin{aligned}
Z_{AE^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PO^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{AO^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}} + \chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PO^f,AO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}} + \chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,AO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2;
\end{aligned}\tag{G22}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PO^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}} + \chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PO^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}} + \chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PO^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}} + \chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PO^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}} + \chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}_0^{\mathrm{Is}}\big);
\end{aligned}\tag{G23}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PO^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}} + \chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,PO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PO^f,PO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,PO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,PO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}} + \chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PO^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}} + \chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PO^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}} + \chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}_0^{\mathrm{Is}}\big);
\end{aligned}\tag{G24}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PO^f,PE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}} + \chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PE^f,PE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{AO^f,PE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}} + \chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,PO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PO^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,PO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,PO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AE^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PO^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,AE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,AE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AE^f,AO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}} + \chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PO^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PE^f,AO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}} + \chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AO^f,AO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big);
\end{aligned}\tag{G25}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PO^f,PE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}} + \chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PE^f,PE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),
\end{aligned}$$

$$
\begin{aligned}
Z_{AO^f,PE^s} &= \big(|\chi^{u1_4}_1|^2+|\chi^{u1_4}_{-1}|^2\big)\big(\chi^{\mathrm{Is}}_0\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}}+\chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}^{\mathrm{Is}}_0\big),\\
Z_{AE^f,PO^s} &= \big(\chi^{u1_4}_1\overline{\chi}^{u1_4}_{-1}+\chi^{u1_4}_{-1}\overline{\chi}^{u1_4}_1\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PO^f,PO^s} &= \big(|\chi^{u1_4}_1|^2+|\chi^{u1_4}_{-1}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,PO^s} &= \big(|\chi^{u1_4}_0|^2+|\chi^{u1_4}_2|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,PO^s} &= \big(\chi^{u1_4}_0\overline{\chi}^{u1_4}_2+\chi^{u1_4}_2\overline{\chi}^{u1_4}_0\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AE^f,AE^s} &= \big(\chi^{u1_4}_1\overline{\chi}^{u1_4}_{-1}+\chi^{u1_4}_{-1}\overline{\chi}^{u1_4}_1\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PO^f,AE^s} &= \big(|\chi^{u1_4}_1|^2+|\chi^{u1_4}_{-1}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,AE^s} &= \big(|\chi^{u1_4}_0|^2+|\chi^{u1_4}_2|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,AE^s} &= \big(\chi^{u1_4}_0\overline{\chi}^{u1_4}_2+\chi^{u1_4}_2\overline{\chi}^{u1_4}_0\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AE^f,AO^s} &= \big(|\chi^{u1_4}_0|^2+|\chi^{u1_4}_2|^2\big)\big(\chi^{\mathrm{Is}}_0\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}}+\chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}^{\mathrm{Is}}_0\big),\\
Z_{PO^f,AO^s} &= \big(\chi^{u1_4}_0\overline{\chi}^{u1_4}_2+\chi^{u1_4}_2\overline{\chi}^{u1_4}_0\big)\big(|\chi^{\mathrm{Is}}_0|^2+|\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PE^f,AO^s} &= \big(\chi^{u1_4}_1\overline{\chi}^{u1_4}_{-1}+\chi^{u1_4}_{-1}\overline{\chi}^{u1_4}_1\big)\big(\chi^{\mathrm{Is}}_0\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}}+\chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}^{\mathrm{Is}}_0\big),\\
Z_{AO^f,AO^s} &= \big(|\chi^{u1_4}_1|^2+|\chi^{u1_4}_{-1}|^2\big)\big(|\chi^{\mathrm{Is}}_0|^2+|\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big);
\end{aligned}\tag{G26}
$$

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi^{u1_4}_0|^2+|\chi^{u1_4}_2|^2\big)\big(|\chi^{\mathrm{Is}}_0|^2+|\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PO^f,PE^s} &= \big(|\chi^{u1_4}_0|^2+|\chi^{u1_4}_2|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,PE^s} &= \big(\chi^{u1_4}_1\overline{\chi}^{u1_4}_{-1}+\chi^{u1_4}_{-1}\overline{\chi}^{u1_4}_1\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,PE^s} &= \big(\chi^{u1_4}_1\overline{\chi}^{u1_4}_{-1}+\chi^{u1_4}_{-1}\overline{\chi}^{u1_4}_1\big)\big(\chi^{\mathrm{Is}}_0\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}}+\chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}^{\mathrm{Is}}_0\big),\\
Z_{AE^f,PO^s} &= \big(|\chi^{u1_4}_1|^2+|\chi^{u1_4}_{-1}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PO^f,PO^s} &= \big(|\chi^{u1_4}_1|^2+|\chi^{u1_4}_{-1}|^2\big)\big(|\chi^{\mathrm{Is}}_0|^2+|\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PE^f,PO^s} &= \big(\chi^{u1_4}_0\overline{\chi}^{u1_4}_2+\chi^{u1_4}_2\overline{\chi}^{u1_4}_0\big)\big(\chi^{\mathrm{Is}}_0\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}}+\chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}^{\mathrm{Is}}_0\big),\\
Z_{AO^f,PO^s} &= \big(\chi^{u1_4}_0\overline{\chi}^{u1_4}_2+\chi^{u1_4}_2\overline{\chi}^{u1_4}_0\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AE^f,AE^s} &= \big(\chi^{u1_4}_1\overline{\chi}^{u1_4}_{-1}+\chi^{u1_4}_{-1}\overline{\chi}^{u1_4}_1\big)\big(|\chi^{\mathrm{Is}}_0|^2+|\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PO^f,AE^s} &= \big(\chi^{u1_4}_1\overline{\chi}^{u1_4}_{-1}+\chi^{u1_4}_{-1}\overline{\chi}^{u1_4}_1\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,AE^s} &= \big(|\chi^{u1_4}_0|^2+|\chi^{u1_4}_2|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,AE^s} &= \big(|\chi^{u1_4}_0|^2+|\chi^{u1_4}_2|^2\big)\big(\chi^{\mathrm{Is}}_0\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}}+\chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}^{\mathrm{Is}}_0\big),\\
Z_{AE^f,AO^s} &= \big(\chi^{u1_4}_0\overline{\chi}^{u1_4}_2+\chi^{u1_4}_2\overline{\chi}^{u1_4}_0\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PO^f,AO^s} &= \big(\chi^{u1_4}_0\overline{\chi}^{u1_4}_2+\chi^{u1_4}_2\overline{\chi}^{u1_4}_0\big)\big(|\chi^{\mathrm{Is}}_0|^2+|\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PE^f,AO^s} &= \big(|\chi^{u1_4}_1|^2+|\chi^{u1_4}_{-1}|^2\big)\big(\chi^{\mathrm{Is}}_0\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}}+\chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}^{\mathrm{Is}}_0\big),\\
Z_{AO^f,AO^s} &= \big(|\chi^{u1_4}_1|^2+|\chi^{u1_4}_{-1}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2;
\end{aligned}\tag{G27}
$$

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi^{u1_4}_0|^2+|\chi^{u1_4}_2|^2\big)\big(|\chi^{\mathrm{Is}}_0|^2+|\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PO^f,PE^s} &= \big(|\chi^{u1_4}_0|^2+|\chi^{u1_4}_2|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,PE^s} &= \big(|\chi^{u1_4}_1|^2+|\chi^{u1_4}_{-1}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,PE^s} &= \big(|\chi^{u1_4}_1|^2+|\chi^{u1_4}_{-1}|^2\big)\big(\chi^{\mathrm{Is}}_0\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}}+\chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}^{\mathrm{Is}}_0\big),\\
Z_{AE^f,PO^s} &= \big(\chi^{u1_4}_1\overline{\chi}^{u1_4}_{-1}+\chi^{u1_4}_{-1}\overline{\chi}^{u1_4}_1\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PO^f,PO^s} &= \big(\chi^{u1_4}_1\overline{\chi}^{u1_4}_{-1}+\chi^{u1_4}_{-1}\overline{\chi}^{u1_4}_1\big)\big(|\chi^{\mathrm{Is}}_0|^2+|\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PE^f,PO^s} &= \big(\chi^{u1_4}_0\overline{\chi}^{u1_4}_2+\chi^{u1_4}_2\overline{\chi}^{u1_4}_0\big)\big(\chi^{\mathrm{Is}}_0\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}}+\chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}^{\mathrm{Is}}_0\big),\\
Z_{AO^f,PO^s} &= \big(\chi^{u1_4}_0\overline{\chi}^{u1_4}_2+\chi^{u1_4}_2\overline{\chi}^{u1_4}_0\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AE^f,AE^s} &= \big(|\chi^{u1_4}_1|^2+|\chi^{u1_4}_{-1}|^2\big)\big(|\chi^{\mathrm{Is}}_0|^2+|\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PO^f,AE^s} &= \big(|\chi^{u1_4}_1|^2+|\chi^{u1_4}_{-1}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,AE^s} &= \big(|\chi^{u1_4}_0|^2+|\chi^{u1_4}_2|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,AE^s} &= \big(|\chi^{u1_4}_0|^2+|\chi^{u1_4}_2|^2\big)\big(\chi^{\mathrm{Is}}_0\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}}+\chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}^{\mathrm{Is}}_0\big),\\
Z_{AE^f,AO^s} &= \big(\chi^{u1_4}_0\overline{\chi}^{u1_4}_2+\chi^{u1_4}_2\overline{\chi}^{u1_4}_0\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PO^f,AO^s} &= \big(\chi^{u1_4}_0\overline{\chi}^{u1_4}_2+\chi^{u1_4}_2\overline{\chi}^{u1_4}_0\big)\big(|\chi^{\mathrm{Is}}_0|^2+|\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PE^f,AO^s} &= \big(\chi^{u1_4}_1\overline{\chi}^{u1_4}_{-1}+\chi^{u1_4}_{-1}\overline{\chi}^{u1_4}_1\big)\big(\chi^{\mathrm{Is}}_0\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}}+\chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}^{\mathrm{Is}}_0\big),\\
Z_{AO^f,AO^s} &= \big(\chi^{u1_4}_1\overline{\chi}^{u1_4}_{-1}+\chi^{u1_4}_{-1}\overline{\chi}^{u1_4}_1\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2;
\end{aligned}\tag{G28}
$$

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi^{u1_4}_0|^2+|\chi^{u1_4}_2|^2\big)\big(|\chi^{\mathrm{Is}}_0|^2+|\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PO^f,PE^s} &= \big(|\chi^{u1_4}_0|^2+|\chi^{u1_4}_2|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,PE^s} &= \big(\chi^{u1_4}_1\overline{\chi}^{u1_4}_{-1}+\chi^{u1_4}_{-1}\overline{\chi}^{u1_4}_1\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,PE^s} &= \big(\chi^{u1_4}_1\overline{\chi}^{u1_4}_{-1}+\chi^{u1_4}_{-1}\overline{\chi}^{u1_4}_1\big)\big(\chi^{\mathrm{Is}}_0\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}}+\chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}^{\mathrm{Is}}_0\big),\\
Z_{AE^f,PO^s} &= \big(\chi^{u1_4}_1\overline{\chi}^{u1_4}_{-1}+\chi^{u1_4}_{-1}\overline{\chi}^{u1_4}_1\big)\big(|\chi^{\mathrm{Is}}_0|^2+|\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PO^f,PO^s} &= \big(\chi^{u1_4}_1\overline{\chi}^{u1_4}_{-1}+\chi^{u1_4}_{-1}\overline{\chi}^{u1_4}_1\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PE^f,PO^s} &= \big(|\chi^{u1_4}_0|^2+|\chi^{u1_4}_2|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AO^f,PO^s} &= \big(|\chi^{u1_4}_0|^2+|\chi^{u1_4}_2|^2\big)\big(\chi^{\mathrm{Is}}_0\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}}+\chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}^{\mathrm{Is}}_0\big),\\
Z_{AE^f,AE^s} &= \big(|\chi^{u1_4}_1|^2+|\chi^{u1_4}_{-1}|^2\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PO^f,AE^s} &= \big(|\chi^{u1_4}_1|^2+|\chi^{u1_4}_{-1}|^2\big)\big(|\chi^{\mathrm{Is}}_0|^2+|\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),\\
Z_{PE^f,AE^s} &= \big(\chi^{u1_4}_0\overline{\chi}^{u1_4}_2+\chi^{u1_4}_2\overline{\chi}^{u1_4}_0\big)\big(\chi^{\mathrm{Is}}_0\overline{\chi}^{\mathrm{Is}}_{\frac{1}{2}}+\chi^{\mathrm{Is}}_{\frac{1}{2}}\overline{\chi}^{\mathrm{Is}}_0\big),\\
Z_{AO^f,AE^s} &= \big(\chi^{u1_4}_0\overline{\chi}^{u1_4}_2+\chi^{u1_4}_2\overline{\chi}^{u1_4}_0\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{AE^f,AO^s} &= \big(\chi^{u1_4}_0\overline{\chi}^{u1_4}_2+\chi^{u1_4}_2\overline{\chi}^{u1_4}_0\big)|\chi^{\mathrm{Is}}_{\frac{1}{16}}|^2,\\
Z_{PO^f,AO^s} &= \big(\chi^{u1_4}_0\overline{\chi}^{u1_4}_2+\chi^{u1_4}_2\overline{\chi}^{u1_4}_0\big)\big(|\chi^{\mathrm{Is}}_0|^2+|\chi^{\mathrm{Is}}_{\frac{1}{2}}|^2\big),
\end{aligned}
$$

$$
\begin{aligned}
Z_{PE^f,AO^s} &= \left(\left|\chi_1^{u1_4}\right|^2+\left|\chi_{-1}^{u1_4}\right|^2\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}}+\chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AO^f,AO^s} &= \left(\left|\chi_1^{u1_4}\right|^2+\left|\chi_{-1}^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2;
\end{aligned}
\tag{G29}
$$

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \left(\left|\chi_0^{u1_4}\right|^2+\left|\chi_2^{u1_4}\right|^2\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2+\left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PO^f,PE^s} &= \left(\left|\chi_0^{u1_4}\right|^2+\left|\chi_2^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PE^f,PE^s} &= \left(\left|\chi_1^{u1_4}\right|^2+\left|\chi_{-1}^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AO^f,PE^s} &= \left(\left|\chi_1^{u1_4}\right|^2+\left|\chi_{-1}^{u1_4}\right|^2\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}}+\chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AE^f,PO^s} &= \left(\left|\chi_1^{u1_4}\right|^2+\left|\chi_{-1}^{u1_4}\right|^2\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2+\left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PO^f,PO^s} &= \left(\left|\chi_1^{u1_4}\right|^2+\left|\chi_{-1}^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PE^f,PO^s} &= \left(\left|\chi_0^{u1_4}\right|^2+\left|\chi_2^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AO^f,PO^s} &= \left(\left|\chi_0^{u1_4}\right|^2+\left|\chi_2^{u1_4}\right|^2\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}}+\chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AE^f,AE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PO^f,AE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2+\left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PE^f,AE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}}+\chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AO^f,AE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AE^f,AO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PO^f,AO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2+\left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PE^f,AO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}}+\chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AO^f,AO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2;
\end{aligned}
\tag{G30}
$$

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \left(\left|\chi_0^{u1_4}\right|^2+\left|\chi_2^{u1_4}\right|^2\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2+\left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PO^f,PE^s} &= \left(\left|\chi_1^{u1_4}\right|^2+\left|\chi_{-1}^{u1_4}\right|^2\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2+\left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PE^f,PE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}}+\chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AO^f,PE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}}+\chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AE^f,PO^s} &= \left(\left|\chi_1^{u1_4}\right|^2+\left|\chi_{-1}^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PO^f,PO^s} &= \left(\left|\chi_0^{u1_4}\right|^2+\left|\chi_2^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PE^f,PO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AO^f,PO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AE^f,AE^s} &= \left(\left|\chi_1^{u1_4}\right|^2+\left|\chi_{-1}^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PO^f,AE^s} &= \left(\left|\chi_0^{u1_4}\right|^2+\left|\chi_2^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PE^f,AE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AO^f,AE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AE^f,AO^s} &= \left(\left|\chi_0^{u1_4}\right|^2+\left|\chi_2^{u1_4}\right|^2\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}}+\chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{PO^f,AO^s} &= \left(\left|\chi_1^{u1_4}\right|^2+\left|\chi_{-1}^{u1_4}\right|^2\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}}+\chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{PE^f,AO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2+\left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{AO^f,AO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2+\left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right);
\end{aligned}
\tag{G31}
$$

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \left(\left|\chi_0^{u1_4}\right|^2+\left|\chi_2^{u1_4}\right|^2\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2+\left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PO^f,PE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2+\left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PE^f,PE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}}+\chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AO^f,PE^s} &= \left(\left|\chi_1^{u1_4}\right|^2+\left|\chi_{-1}^{u1_4}\right|^2\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}}+\chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AE^f,PO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PO^f,PO^s} &= \left(\left|\chi_0^{u1_4}\right|^2+\left|\chi_2^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PE^f,PO^s} &= \left(\left|\chi_1^{u1_4}\right|^2+\left|\chi_{-1}^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AO^f,PO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AE^f,AE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PO^f,AE^s} &= \left(\left|\chi_0^{u1_4}\right|^2+\left|\chi_2^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PE^f,AE^s} &= \left(\left|\chi_1^{u1_4}\right|^2+\left|\chi_{-1}^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AO^f,AE^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AE^f,AO^s} &= \left(\left|\chi_0^{u1_4}\right|^2+\left|\chi_2^{u1_4}\right|^2\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}}+\chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{PO^f,AO^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}}+\chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{PE^f,AO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2+\left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{AO^f,AO^s} &= \left(\left|\chi_1^{u1_4}\right|^2+\left|\chi_{-1}^{u1_4}\right|^2\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2+\left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right);
\end{aligned}
\tag{G32}
$$

$$
\begin{aligned}
Z_{AE^f,PE^s} &= \left(\left|\chi_0^{u1_4}\right|^2+\left|\chi_2^{u1_4}\right|^2\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2+\left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{PO^f,PE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PE^f,PE^s} &= \left(\left|\chi_0^{u1_4}\right|^2+\left|\chi_2^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AO^f,PE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}}+\chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{AE^f,PO^s} &= \left(\left|\chi_1^{u1_4}\right|^2+\left|\chi_{-1}^{u1_4}\right|^2\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{PO^f,PO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}}+\chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\right),\\
Z_{PE^f,PO^s} &= \left(\left|\chi_1^{u1_4}\right|^2+\left|\chi_{-1}^{u1_4}\right|^2\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2+\left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),\\
Z_{AO^f,PO^s} &= \left(\chi_0^{u1_4}\overline{\chi}_2^{u1_4}+\chi_2^{u1_4}\overline{\chi}_0^{u1_4}\right)\left|\chi_{\frac{1}{16}}^{\mathrm{Is}}\right|^2,\\
Z_{AE^f,AE^s} &= \left(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4}+\chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\right)\left(\left|\chi_0^{\mathrm{Is}}\right|^2+\left|\chi_{\frac{1}{2}}^{\mathrm{Is}}\right|^2\right),
\end{aligned}
$$

$$\begin{aligned}
Z_{PO^f,AE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,AE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PO^f,AO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AO^f,AO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2;
\end{aligned}\tag{G33}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PO^f,PE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,PE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PO^f,PO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PE^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AO^f,PO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AE^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PO^f,AE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,AE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PO^f,AO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AO^f,AO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2;
\end{aligned}\tag{G34}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PO^f,PE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,PE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PO^f,PO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,PO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,PO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PO^f,AE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PE^f,AE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AO^f,AE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PO^f,AO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AO^f,AO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2;
\end{aligned}\tag{G35}$$

$$\begin{aligned}
Z_{AE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PO^f,PE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,PE^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,PE^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,PO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{PO^f,PO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PE^f,PO^s} &= \big(|\chi_1^{u1_4}|^2 + |\chi_{-1}^{u1_4}|^2\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AO^f,PO^s} &= \big(|\chi_0^{u1_4}|^2 + |\chi_2^{u1_4}|^2\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{AE^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PO^f,AE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PE^f,AE^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AO^f,AE^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{AE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2,\\
Z_{PO^f,AO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)\big(\chi_0^{\mathrm{Is}}\overline{\chi}_{\frac{1}{2}}^{\mathrm{Is}} + \chi_{\frac{1}{2}}^{\mathrm{Is}}\overline{\chi}_0^{\mathrm{Is}}\big),\\
Z_{PE^f,AO^s} &= \big(\chi_0^{u1_4}\overline{\chi}_2^{u1_4} + \chi_2^{u1_4}\overline{\chi}_0^{u1_4}\big)\big(|\chi_0^{\mathrm{Is}}|^2 + |\chi_{\frac{1}{2}}^{\mathrm{Is}}|^2\big),\\
Z_{AO^f,AO^s} &= \big(\chi_1^{u1_4}\overline{\chi}_{-1}^{u1_4} + \chi_{-1}^{u1_4}\overline{\chi}_1^{u1_4}\big)|\chi_{\frac{1}{16}}^{\mathrm{Is}}|^2.
\end{aligned}\tag{G36}$$

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