Data set contents

Title: Small spherical and projective codes

Author: Henry Cohn

List of files, types, and sizes:

File name	Type	Size
`angles.txt`	plain text	95 KB
`angles.csv`	plain text	65 KB
`coordinates.txt`	plain text	1.6 MB
`exact.txt`	plain text	323 MB
`rigor.txt`	plain text	14 KB

Originally posted in 2022. Updated in 2023 with improved N-point spherical codes in $\mathbb{R}^n$ for $(n, N) \in \{(8, 32), (9, 25), (10, 30), (11, 25)\}$ and in 2024 for $(n, N) = (4, 31)$.

Notes:

This data set describes the best spherical codes that are known (to the best of my knowledge) with at most 32 points and the best real projective codes with at most 16 lines. The *spherical code* problem asks how to arrange N points $x_1, \ldots, x_N$ in the unit sphere S^{n-1} in $\mathbb{R}^n$ so that the minimal angle between them is maximized. Equivalently, the goal is to minimize the maximal inner product $\max_{i<j}\langle x_i, x_j\rangle$, which is the cosine of this angle. The *real projective code* problem asks how to maximize the minimal angle between N lines through the origin in $\mathbb{R}^n$. It's equivalent to maximizing the minimal angle in an antipodal $2N$-point spherical code.

The data set includes all spherical codes with $n \geq 3$ and $2n + 1 \leq N \leq 32$ and all real projective codes with $n \geq 3$ and $n + 1 \leq N \leq 16$. The spherical codes with $N \leq 2n$ are omitted because the answer is known [22]: the optimal inner product is $-1/(N-1)$ for $2 \leq N \leq n+1$ and 0 for $n + 2 \leq N \leq 2n$. Similarly, the real projective codes with $N \leq n$ have optimal inner product 0, and the spherical and projective codes with $n = 2$ are simply regular N-gons.

These codes are not always unique. One simple case is when the code has *rattlers*, points with no neighbors at the minimal distance, so they can be moved freely. However, even rigid codes may not be unique; for example, two seemingly optimal spherical codes of 15 points in $\mathbb{R}^3$ are known [13].

The files provided in the data set are described below. Note that they are not intended to be especially convenient for humans to read, but rather just to record the data. For a friendlier presentation, see `https://cohn.mit.edu/spherical-codes`.

`angles.txt`

For each code in the data set, this file lists the dimension n, number N of points or lines, minimal angle in degrees, cosine of the minimal angle, number of rattlers, and whether the code is known to be optimal, as well as the minimal polynomial of the cosine in many cases and references for codes that had been found previously or proofs of optimality (the list of references is given below). The cosines are all rounded up and the angles are rounded down, so that it can be rigorously checked using interval arithmetic that a code matching these parameters exists. Whenever a minimal polynomial is listed, the file `exact.txt` gives a proof that there exists a code with this exact angle, not just the numerical approximation listed in `angles.txt`.

`angles.csv`

This file provides the same data as `angles.txt` (except for the references) as comma-separated values, intended to be parsed as easily as possible by computer programs. Each line begins with "spherical" or "projective" to indicate what sort of code it is, and then provides the remaining data in the same order as in `angles.txt`. When no minimal polynomial is available, it is listed as 0. Note also that `angles.csv` uses * for multiplication in polynomials, for convenience when using computer algebra systems, while `angles.txt` indicates multiplication by juxtaposition.

`coordinates.txt`
This file provides approximate coordinates for each code, to sufficient precision to obtain the numerical angle listed in `angles.txt`.

`exact.txt`
In each case that includes the minimal polynomial, this file provides the data needed for a straightforward existence proof using a computer algebra system. The format is designed for PARI/GP but is straightforward to adapt to other systems. The proof works by specifying the Gram matrix, i.e., the matrix of inner products. For all the non-rattlers, the Gram matrix entries will be algebraic numbers. The rattlers are a little more subtle, but they are handled using interval arithmetic.

Each line of `exact.txt` is a list $[n, G, p, a, b, q, c, d, L]$ of nine elements between square brackets and separated by commas. The first entry n is the dimension of the ambient space $\mathbb{R}^n$, and G is the Gram matrix for the non-rattlers. This Gram matrix of the M non-rattlers is given as a list of entries

$$[G_{1,1}, G_{1,2}, \ldots, G_{1,M}; G_{2,1}, G_{2,2}, \ldots, G_{2,M}; \ldots; G_{M,1}, G_{M,2}, \ldots, G_{M,M}],$$

in which rows are separated by semicolons and entries within a row are separated by commas. Note that M is determined implicitly by the format. Each entry $G_{i,j}$ is a polynomial in a variable x with exact rational coefficients, where x represents a generator over $\mathbb{Q}$ of the number field containing these entries. The third entry p in the line $[n, G, p, a, b, q, c, d, L]$ is the minimal polynomial of this generator, and a and b are rational upper and lower bounds for it. More precisely, the generator is the unique root of p in the rational interval $[a, b)$. As part of the verification of the proof, one must check that p is irreducible and that it has a unique root in this interval. The next three entries q, c, and d are the minimal polynomial q of the cosine of the minimal angle in the code and the endpoints of a rational interval $[c, d)$ containing it. In many cases $q = p$, but not always. Finally, L is a list in square brackets (possibly the empty list), where each entry of L is a list of M rational numbers in square brackets. These entries of L correspond to the rattlers, as described below.

To prove the existence of the code, first one must check that the Gram matrix G is valid. In other words, it must be symmetric, have diagonal entries all equal to 1, be positive semidefinite, and have rank at most n. To check the rank, one can compute the characteristic polynomial of G as a polynomial of degree M in a variable t with coefficients given by polynomials in x modulo p. Checking that the rank is at most n amounts to verifying that the coefficients of $1, t, \ldots, t^{M-n-1}$ all vanish. To show that it is positive semidefinite, it suffices to check that the remaining coefficients alternate in sign. The final step in analyzing G is to determine the greatest off-diagonal entries (in absolute value in the real projective case) and check that they are described by q, c, and d. Determining the greatest entries and checking the sign alternation for the coefficients amount to verifying inequalities between polynomials in x given p, a, and b. To determine whether a given polynomial is positive or negative, one can use Sturm's theorem to check that it has no roots in the interval $[a, b)$ and then determine the sign by setting $x = (a + b)/2$. The net effect is that all these inequalities can be straightforwardly determined as long as the interval $[a, b)$ is small enough, and that is the case in this file.

The computations in the previous paragraph provide a rigorous proof that there exists a code of size M in n dimensions as described. In most cases there are no rattlers and the proof is therefore complete, but rattlers are a special case. When L is non-empty, each element of L is a list of M rational coefficients $c_1, \ldots, c_M$. If $v_1, \ldots, v_M$ are the non-rattlers specified by the above data, then the rattler determined by the coefficients is given by renormalizing $c_1 v_1 + \cdots + c_M v_M$ to have length 1. Using interval arithmetic and the entries of the Gram matrix G, one can check that the resulting non-rattlers stay far enough from $v_1, \ldots, v_M$ and each other that they are indeed rattlers. The total number of points in the code is M plus the length of the list L.

Note that this format was chosen for simplicity, not efficiency. For example, each entry of G is listed individually, and one could achieve substantial compression by taking into account that many of them are the same.

`rigor.txt`
This file contains PARI/GP code that implements the rigorous verification described above. Specifically, the functions `verifyproofspherical` and `verifyproofprojective` take a list $[n, G, p, a, b, q, c, d, L]$ as input and verify the existence of the corresponding spherical or real projective code.

References

[1] P. G. Adams, *A numerical approach to Tamme's problem in Euclidean n-space*, Master of Science thesis, Oregon State University, 1997, https://hdl.handle.net/1957/33911.

[2] C. Bachoc and F. Vallentin, *Optimality and uniqueness of the* $(4, 10, 1/6)$ *spherical code*, J. Combin. Theory Ser. A **116** (2009), 195–204, doi:10.1016/j.jcta.2008.05.001.

[3] J. Buddenhagen and D. A. Kottwitz, *Multiplicity and symmetry breaking in (conjectured) densest packings of congruent circles on a circle*, undated preprint. [I obtained a copy in 2004 and have seen it cited with a date of 2001.]

[4] B. Bukh and C. Cox, *Nearly orthogonal vectors and small antipodal spherical codes*, Israel J. Math. **238** (2020), 359–388, doi:10.1007/s11856-020-2027-7.

[5] B. W. Clare and D. L. Kepert, *The closest packing of equal circles on a sphere*, Proc. Roy. Soc. London Ser. A **405** (1986), 329–344, doi:10.1098/rspa.1986.0056.

[6] J. H. Conway, R. H. Hardin, and N. J. A. Sloane, *Packing lines, planes, etc.: packings in Grassmannian spaces*, Experiment. Math. **5** (1996), 139–159, https://projecteuclid.org/euclid.em/1047565645.

[7] L. Danzer, *Finite point-sets on* S^2 *with minimum distance as large as possible*, Discrete Math. **60** (1986), 3–66, doi:10.1016/0012-365X(86)90002-6.

[8] T. Ericson and V. Zinoviev, *Codes on Euclidean spheres*, North-Holland Mathematical Library **63**, North-Holland Publishing Co., Amsterdam, 2001.

[9] L. Fejes [Tóth], *Über eine Abschätzung des kürzesten Abstandes zweier Punkte eines auf einer Kugelfläche liegenden Punktsystems*, Jber. Deutsch. Math.-Verein. **53** (1943), 66–68.

[10] L. Fejes Tóth, *Distribution of points in the elliptic plane*, Acta Math. Acad. Sci. Hungar. **16** (1965), 437–440, doi:10.1007/BF01904849.

[11] M. Fickus, J. Jasper, and D. G. Mixon, *Packings in real projective spaces*, SIAM J. Appl. Algebra Geom. **2** (2018), 377–409, doi:10.1137/17M1137528.

[12] T. Gosset, *On the regular and semi-regular figures in space of n dimensions*, Messenger Math. **29** (1900), 43–48.

[13] D. A. Kottwitz, *The densest packing of equal circles on a sphere*, Acta Cryst. Sect. A **47** (1991), 158–165, doi:10.1107/S0108767390011370.

[14] D. de Laat, N. Leijenhorst, and W. H. H. de Muinck Keizer, *Optimality and uniqueness of the* D_4 *root system*, preprint, 2024, https://www.arxiv.org/abs/2404.18794.

[15] D. E. Lazić, V. Šenk, and I. Šeškar, *Arrangements of points on a sphere which maximize the least distance*, Bull. Appl. Math. (Technical University Budapest) **47** (1987), 7–21.

[16] V. I. Levenšteĭn, *On bounds for packings in n-dimensional Euclidean space* (Russian), Dokl. Akad. Nauk SSSR **245** (1979), 1299–1303; English translation in Soviet Math. Dokl. **20** (1979), 417–421.

[17] J. H. van Lint and J. J. Seidel, *Equilateral point sets in elliptic geometry*, Indag. Math. **28** (1966), 335–348, https://research.tue.nl/en/publications/equilateral-point-sets-in-elliptic-geometry.

[18] A. L. Mackay, *The packing of three-dimensional spheres on the surface of a four-dimensional hypersphere*, J. Phys. A **13** (1980), 3373–3379, doi:10.1088/0305-4470/13/11/011.

[19] D. G. Mixon and H. Parshall, *Globally optimizing small codes in real projective spaces*, SIAM J. Discrete Math. **35** (2021), 234–249, doi:10.1137/19M1307895.

[20] D. G. Mixon and H. Parshall, *The optimal packing of eight points in the real projective plane*, Exp. Math. **31** (2022), 474–485, doi:10.1080/10586458.2019.1641767.

[21] O. R. Musin and A. S. Tarasov, *The strong thirteen spheres problem*, Discrete Comput. Geom. **48** (2012), 128–141, doi:10.1007/s00454-011-9392-2.

[22] O. R. Musin and A. S. Tarasov, *The Tammes problem for* $N = 14$, Exp. Math. **24** (2015), 460–468, doi:10.1080/10586458.2015.1022842.

[23] R. A. Rankin, *The closest packing of spherical caps in n dimensions*, Proc. Glasgow Math. Assoc. **2** (1955), 139–144, doi:10.1017/S2040618500033219.

[24] R. M. Robinson, *Arrangements of 24 points on a sphere*, Math. Ann. **144** (1961), 17–48, doi:10.1007/BF01396539.

[25] L. Schläfli, *Theorie der vielfachen Kontinuität* [1852], Denkschriften der Schweizerischen naturforschenden Gesellschaft, vol. 38, J. H. Graf, Bern, 1901.

[26] K. Schütte and B. L. van der Waerden, *Auf welcher Kugel haben 5, 6, 7, 8 oder 9 Punkte mit Mindestabstand Eins Platz?*, Math. Ann. **123** (1951), 96–124, doi:10.1007/BF02054944.

[27] N. J. A. Sloane, with the collaboration of R. H. Hardin, W. D. Smith, and others, *Tables of spherical codes*, 1994, https://hdl.handle.net/1721.1/147004.

[28] J. Strohmajer, *Über die Verteilung von Punkten auf der Kugel*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **6** (1963), 49–53.

[29] P. M. L. Tammes, *On the origin of number and arrangement of the places of exit on the surface of pollen-grains*, Recueil des travaux botaniques néerlandais **27** (1930), no. 1, 1–84, https://natuurtijdschriften.nl/pub/552640.

[30] T. Tarnai and Zs. Gáspár, *Improved packing of equal circles on a sphere and rigidity of its graph*, Math. Proc. Cambridge Philos. Soc. **93** (1983), 191–218, doi:10.1017/S0305004100060485.

[31] T. Tarnai and Zs. Gáspár, *Arrangement of 23 points on a sphere (on a conjecture of R. M. Robinson)*, Proc. Roy. Soc. London Ser. A **433** (1991), 257–267, doi:10.1098/rspa.1991.0046.

[32] B. L. van der Waerden, *Punkte auf der Kugel. Drei Zusätze*, Math. Ann. **125** (1952), 213–222, doi:10.1007/BF01343118.
[33] J. Wang, *Finding and investigating exact spherical codes*, Experiment. Math. **18** (2009), 249–256, `https://projecteuclid.org/euclid.em/1259158434`.
[34] L. R. Welch, *Lower bounds on the maximum cross correlation of signals*, IEEE Trans. Inform. Theory **20** (1974), 397–399, doi:10.1109/TIT.1974.1055219.