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THE EFFECTS OF ANNUAL STORAGE AND RANDOM POTENTIAL EVAPOTRANSPIRATION ON THE ONE-DIMENSIONAL ANNUAL WATER BALANCE

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RALPH M. PARSONS LABORATORY
FOR
WATER RESOURCES AND HYDRODYNAMICS

Report No. 251

Prepared under support of the
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PREFACE

This report is a part of a series of publications which describe various activities and studies undertaken in the Technology Adaptation Program at the Massachusetts Institute of Technology.

In 1971, the United States Department of State, through the Agency for International Development, awarded the Massachusetts Institute of Technology a grant, the purpose of which was to provide support for the development at MIT, in conjunction with institutions in selected developing countries, of capabilities useful in the adaptation of technologies and problem-solving techniques to the needs and conditions of those countries. At MIT, the Technology Adaptation Program provides the means by which the long-term objective for which the AID grant was made, can be achieved.

In the process of making this TAP-supported study, some insight has been gained into how appropriate technologies can be identified and adapted to the needs of developing countries per se, and it is expected that recommendations developed will serve as a guide to other developing countries for the solution of similar problems which may be encountered there.

Fred Moavenzadeh
Program Director

February 1980
Cambridge, Massachusetts

THE EFFECTS OF ANNUAL STORAGE AND RANDOM POTENTIAL EVAPOTRANSPIRATION ON THE ONE-DIMENSIONAL ANNUAL WATER BALANCE

ABSTRACT

An analysis is presented leading to the incorporation of storage terms into an existing first-order dynamic water balance.

Annual change in storage in the unsaturated zone of an idealized soil column is included through the addition of one characteristic vegetal parameter, the estimated depth of the root zone. This defines the storage volume in the unsaturated zone.

Annual change in storage in the saturated zone of the soil column is accounted for by assuming the dynamic linkage between percolation to the groundwater table and discharge from the groundwater reservoir to behave as a linear reservoir. The storage coefficient of this reservoir must be determined from streamflow data.

The effect on the frequency of annual basin yield of annual change in storage is tested for two contrasting climates. In both test cases, the model is found to reduce the unexplained variance of the basic model without storage mechanisms.

A simplified analysis is conducted to determine the effect on the frequency of the annual basin yield of a randomly varying rate of annual average potential evaporation. A modified Penman equation is used to derive an approximate relationship for the annual average rate of potential evaporation. A cdf is derived for the annual basin yield from

a Gamma distribution for annual point precipitation and a double exponential distribution for the annual average rate of potential evaporation.

A linearized version of the water balance model indicates for two contrasting climates that a random rate of potential evaporation has little effect on the variance of the annual basin yield. This is interpreted as a justification for considering the rate of potential evaporation to be constant when modeling the water balance on a seasonal basis.

ACKNOWLEDGEMENTS

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Particular thanks go to the German Academic Exchange Service, which provided additional funds through a governmental fellowship for Mr. Metzger.

Mr. Tobin Tellers, a research engineer at MIT, assisted in modifying one of his computer programs to the special use in this study. Ms. Anne Clee did an exceptionally conscientious job in typing the manuscript.

The material contained in this report was submitted by Mr. Metzger in partial fulfillment of the requirements for the degree of Master of Science in Civil Engineering at MIT.

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a_k	constants related to annual average rate of potential evaporation
A_k	coefficients of linearized water balance equation
A_o	gravitational infiltration rate as modified by capillary rise from water table
A	coefficient related to frequency of annual basin yield
$\bar{A}$	shortwave albedo of soil surface
B	coefficient related to frequency of annual basin yield
$\bar{B}$	annual average turbulent transfer coefficient
c	pore disconnectedness index
C_k	coefficients of linearized annual change in groundwater storage
d	diffusivity index
D_k	coefficients of annual change in the unsaturated zone
D	soil moisture diffusivity
D_e	desorption diffusivity
D_i	sorption diffusivity
$\bar{e}_p$	annual average rate of potential evaporation
e_T	evapotranspiration rate
e_s	saturated vapor pressure
e_{s_2}	saturated vapor pressure at elevation 2
e_z	vapor pressure at elevation 2
E	exfiltration parameter
E_k	coefficients of linearized evapotranspiration function

E_{p_A}	annual average potential evapotranspiration
$E^*_{p_A}$	annual average potential soil moisture evapotranspiration
E_r	surface retention loss
E_{r_s}	surface retention loss from bare soil fraction
E_{r_v}	surface retention loss from vegetated fraction
E_{s_j}	interstorm bare soil evaporation
E_{T_j}	interstorm evapotranspiration
E_{T_A}	annual evapotranspiration
$E^*_{T_A}$	annual evapotranspiration from soil moisture
f_e	exfiltration rate
f^*_e	exfiltration capacity
f_i	infiltration rate
f^*_i	infiltration capacity
G	gravitational infiltration parameter
h	storm depth
h_s	piezometric head in aquifer
h_o	surface retention capacity
i	precipitation rate
I	inflow into groundwater reservoir
I_j	storm infiltration
I_A	annual infiltration
j	counting variable
$k(1)$	saturated effective intrinsic permeability
k_v	effective transpiring leaf area per unit of land surface
$\overline{K}$	annual average fraction of cloudless-sky insolation

$K(1)$	saturated hydraulic conductivity
L	leakage rate into soil column
L_e	latent heat of vaporization
m	pore size distribution index
m_H	mean storm depth
m_i	mean storm intensity
m_{P_A}	average annual precipitation
m_{t_b}	mean time between storms
m_{t_r}	mean storm duration
m_{t_a}	mean storm interarrival time
m_τ	mean length of rainy season
$m_{r_{g_o}}$	mean initial groundwater runoff
m_ν	mean number of storms per year
n	effective medium porosity
N	mass transfer coefficient
$\overline{N}$	annual average cloud cover
$\overline{\overline{N}}$	long-term average cloud cover
O	outflow from groundwater reservoir
p_N	net percolation rate to the groundwater table
P_A	annual precipitation
P_{N_A}	annual net percolation to the groundwater table
q_{xy}	specific discharge vector of flow in x-y plane of aquifer
q_a	rate of advection of energy by precipitation, surface runoff, etc.

q_b	rate of longwave back radiation
q_i	rate of receipt of shortwave radiation from the sun
q_r	rate of reflection of shortwave radiation at surface
q_s	rate of energy storage in soil
r_g	groundwater runoff rate
r_s	surface runoff rate
r_u	subsurface runoff rate
r_{g_o}	initial groundwater runoff rate
R_{g_A}	annual groundwater runoff
R_{s_A}	annual surface runoff
$R^*_{s_A}$	annual rainfall excess
R_{s_j}	storm surface runoff
$R^*_{s_j}$	storm rainfall excess
R	Bowen ratio
R(x)	region of integration
s	degree of effective medium saturation, which equals volume of active soil moisture divided by effective volume of voids
$\bar{s}$	annual spatial average effective soil moisture concentration in surface boundary layer
s_o	time and spatial average effective soil moisture concentration
ΔS_{s_A}	annual change in surface storage
ΔS_{g_A}	annual change in groundwater storage
ΔS_{u_A}	annual change in storage in unsaturated zone (i.e., surface boundary layer)
S_t	groundwater storage at time t

$\overline{S}$	annual average saturation ratio
$\overline{\overline{S}}$	long-term average saturation ratio
t	time variable
t_a	storm interarrival time
t_b	time between storms
t_r	storm duration
T	one year
$\overline{T}_A$	annual average atmospheric temperature
$\overline{\overline{T}}_A$	long-term average atmospheric temperature
T_z	temperature of atmosphere at elevation z
u_z	wind speed at elevation z
v_{ss}	rate of surface storage
v_{sg}	rate of groundwater storage
v_{su}	rate of storage in unsaturated zone
V_{sg}	groundwater storage
V_{ss}	surface storage
V_{su}	storage in unsaturated zone
V_x	coefficient of variation of x
w	capillary rise from water table
$\overline{w}$	annual average capillary rise
w_o	long-term average capillary rise
y	yield rate
Y_A	annual basin yield
z	value of dimensionless water balance term

Z	elevation of water table
Z_o	long-term average elevation of water table
Z_b	elevation of impervious layer
Z_r	depth of the root zone
α	reciprocal of average rainstorm intensity
β	reciprocal of average time between storms
γ	sensible heat transfer coefficient
γ_ω	specific weight of water
Δ	slope of vapor pressure-temperature curve
Δt	arbitrary small time interval
δ	reciprocal of average storm duration
η	reciprocal of mean storm depth
θ	number of storms in wet season
κ	parameter of Gamma distribution of storm depth
κ_s	groundwater reservoir coefficient
λ	parameter of Gamma distribution of storm depth, equal to κ/m_H
λ_T	parameter of double exponential distribution of fluctuations of annual average atmospheric temperature
μ_ω	dynamic viscosity of water
∇_{xy}	Nabla operator for two components
ν	counting variable of number of storms
ρ_e	mass density of evaporating liquid
σ	capillary infiltration parameter
σ_x	standard deviation of x
σ_w	surface tension of pore liquid

τ	length of rainy season
Φ	pore shape parameter, latitude
ϕ_e	exfiltration diffusivity function
ϕ_i	infiltration diffusivity function
$\Psi(1)$	saturated soil matrix potential
ω	average arrival rate of storms
COV[a, b]	covariance of [a, b]
E[ ]	expected value of [ ]
f( )	probability density function of ( ), functional notation
$f^{(n)}(\ )$	n^{th} derivative of ( )
G( )	Gamma distribution
I( )	integral
J( )	evapotranspiration function
P[,]	Pearson's incomplete Gamma function
VAR[ ]	variance of [ ]
$\gamma[a, x]$	incomplete Gamma function

Chapter I

INTRODUCTION

Planning for water resource development requires, among many other things, estimates of the average return interval of extreme annual events, such as water yield. These are conventionally achieved by extrapolation of assumed probability distributions which have been fitted to the available set of observations. This method bears a number of shortcomings, however. The limited length of streamflow records available in developing countries generally prevents a reasonable estimate of the parameters of a fitted distribution. Or, even if the records of data are long enough, they might belong to a period in which the hydrologic system of interest was undergoing physical changes. Examples of such nonstationary behavior are urbanization, deforestation, drainage and irrigation and surface water storage.

Insight into hydrologic processes and mechanisms can be gained only through identification and investigation of the underlying physical determinisms. Maximum understanding of hydrologic variability would be achieved with a model that incorporated the geophysical dynamics through which those atmospheric disturbances, which produce precipitation and temperature, are generated and propagated.

Lack of scientific knowledge, however, makes it impossible for us to formulate the detailed physics of atmospheric processes. Thus we must isolate the hydrologic system from the global system (Figure 1.1)

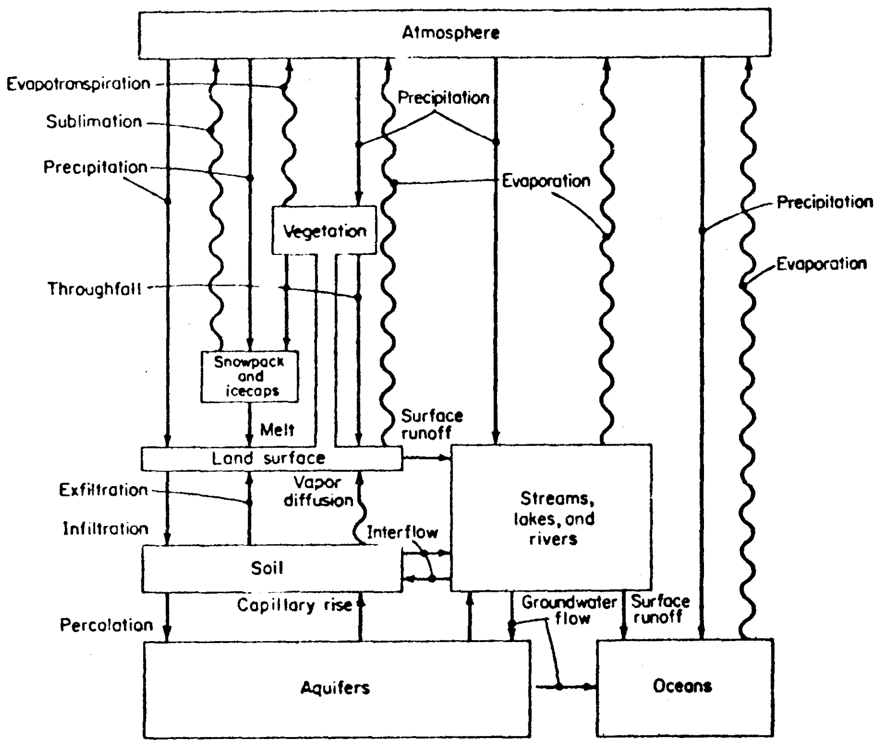


Figure 1.1

GLOBAL HYDROLOGIC CYCLE

(from Eagleson, 1970)

without being able to state the proper boundary conditions. Uncertainty must be introduced. Probability distributions of atmospheric system variables such as precipitation and temperature become the substitute for a set of equations which would define the generating processes of those variables.

This stochastic sub-unit (atmosphere) of the hydrologic system now must be linked with the soil-vegetation sub-system (catchment). It is well known that there is a dynamic coupling of these elements through the physical processes which produce the transport of thermal energy and water across the land surface. These processes depend very much upon the physical properties of the soil and vegetation as well as upon the local weather conditions. In the long term, the quantities of water and heat are each conserved.

Empirical studies, due to their weak physical basis, lack both the generality and the parametric incorporation of climate, soil and vegetal properties that are necessary for understanding the interactions of atmosphere, soil and vegetation. To assess the system response to a change in the system properties (parameters), it is necessary to develop a model which accounts for all those interactions. Moreover, that model should have a broad analytical formulation in order to provide generalizations concerning system behavior.

A conceptual model has been developed by Eagleson (1978a, b, c, d, e, f, g), which satisfies closely the requirements stated above. It is reviewed and summarized in Chapter III of this work, which attempts to elaborate on some of the components of that model.

Chapter II

PROBLEM STATEMENT AND OBJECTIVES

II.1 Background

The mathematical model this research is based on [Eagleson, 1978a, b, c, d, e, f, g] is a statistical dynamic formulation of the water budget of an arbitrary hydrologic system as schematized in Figure 2.1.

It is a dimensionless analytical representation of the one-dimensional annual water balance based on simplified models of the various interacting hydrologic subprocesses.

In the following, a notation is chosen that is as consistent as possible with the basic literature. For the sake of completeness, the original [Eagleson, 1978a] principal assumptions and simplifications are quoted in order to define the analytical framework for the first-order water balance.

1. General
    a. One-dimensional analysis (only vertical processes) is used
    b. No consideration is given to snow or ice.
    c. All processes are stationary in the long-term average.
2. Precipitation
    a. Storm series is represented by Poisson arrivals of independent and identically distributed rectangular pulses.
    b. Average interstorm period is much greater than average storm duration.

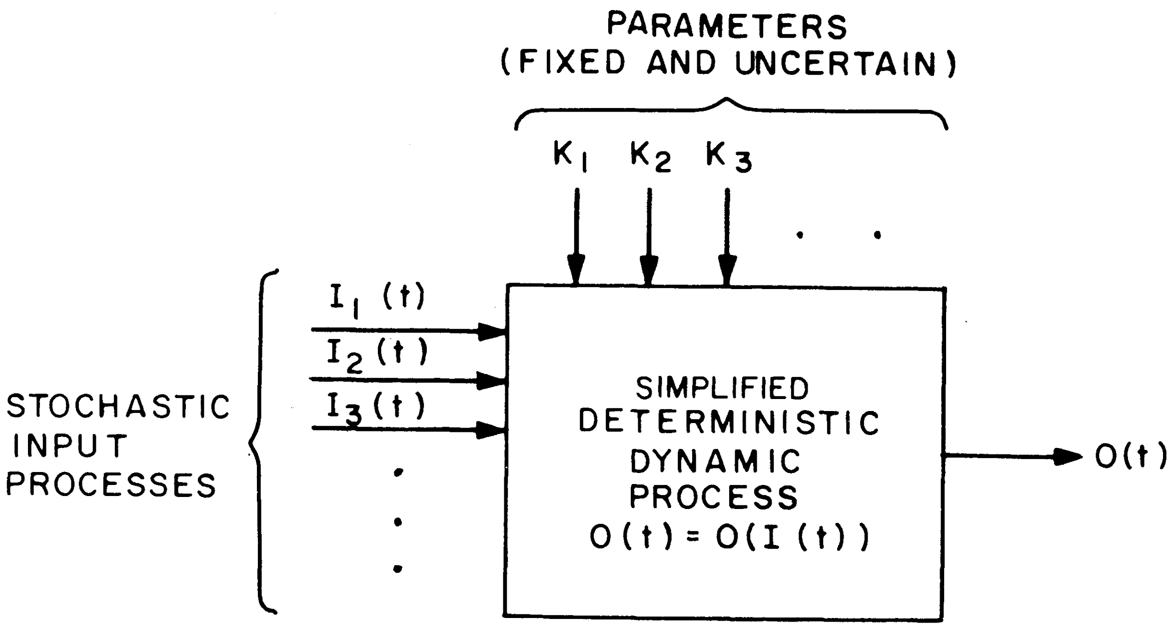


Figure 2.1

A SIMPLE STATISTICAL DYNAMIC PROCESS
(from Eagleson, 1978a)

c. Interstorm period and storm duration are statistically independent.

3. Soil

a. Soils are homogeneous.

b. Movement of water vapor is not considered.

c. Column is effectively semiinfinite as far as surface processes are concerned.

d. Infiltration, exfiltration, percolation, and capillary rise from water table are formulated separately and their fluxes are linearly superimposed.

e. Carryover moisture storage (or deficit) from storm to interstorm period (and vice versa) is neglected with internal moisture at the start of every period being s_o the space and time average in the surface boundary layer.

4. Vegetation (natural systems only)

a. Transpiration occurs at the potential rate.

b. Rate of soil moisture extraction by the root system is a constant throughout the soil volume above the maximum root depth.

c. Canopy density seeks a short-term equilibrium state at which soil moisture is a maximum.

d. In water-limited systems, species evolve in the long term toward maximum water use.

5. Infiltration and surface runoff

a. No surface inflows from outside the region are considered.

b. Storm intensity and duration are statistically independent.

6. Evapotranspiration

   a. Vegetation transpires at the potential rate.

   b. Potential rate of evaporation averaged over the interstorm period has a negligible coefficient of variation during the rainy season.

7. Percolation to water table

   a. Percolation is steady throughout rainy season at a rate determined by the average soil moisture s_o.

   b. Percolation is zero during dry season.

8. Capillary rise from the water table

   a. Potential rate of evaporation is much greater than rate of capillary rise from water table.

   b. Dry surface matrix potential is much greater than saturated matrix potential.

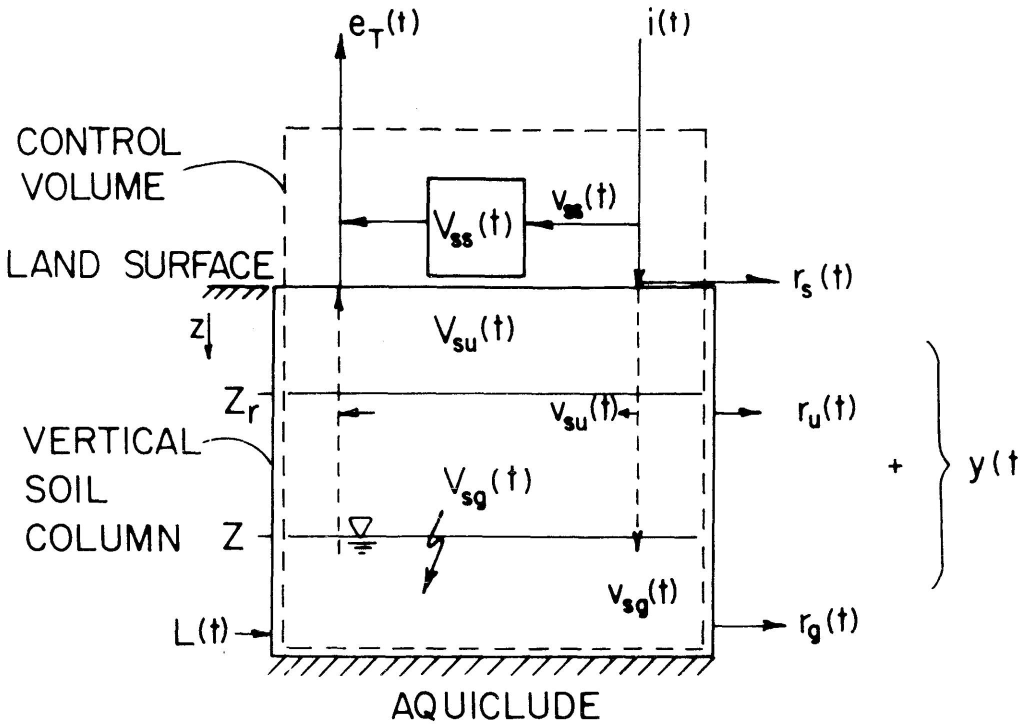
9. Miscellaneous

   a. Water table is constant (no carryover groundwater storage from year-to-year).

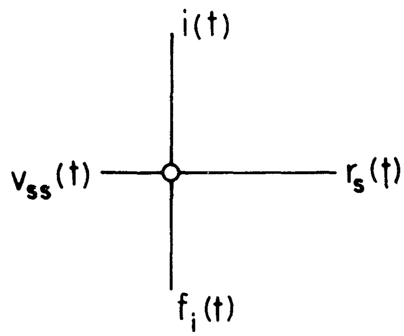
   b. Relation among annual water balance components is given to the first order by the relation among the average annual quantities.

A volumetric water balance per unit of surface area over time t (Figure 2.2) can be given by

$$\int_0^t \left\{ i(t) + L(t) - e_T(t) - \frac{\partial}{\partial t} [V_{ss}(t) + V_{su}(t) + V_{sg}(t)] \right\} dt$$
$$= \int_0^t [r_s(t) + r_u(t) + r_g(t)] \, dt = \int_0^t y(t) \, dt \qquad (2.1)$$



a. CONTROL VOLUME FOR WATER BALANCE



b. PARTITION OF PRECIPITATION AT SURFACE

Figure 2.2

INSTANTANEOUS WATER BALANCE

where

$i(t)$ = precipitation intensity

$L(t)$ = leakage rate into the soil column

$e_T(t)$ = evapotranspiration rate

$V_{ss}(t)$ = surface storage

$V_{su}(t)$ = storage in the unsaturated zone

$V_{sg}(t)$ = storage in the saturated zone

$r_s(t)$ = surface runoff rate

$r_u(t)$ = subsurface runoff rate

$r_g(t)$ = groundwater runoff rate

$y(t)$ = yield rate

Since subsurface runoff is hard to deal with both analytically as well as experimentally, this component of the water balance is commonly neglected and included in the remaining runoff rates. Likewise, leakage into or out of a catchment is an elusive term. It depends on nongeneral properties of individual catchments such as geological formations, etc. According to the goal of formulating a general model, we neglect leakage.

Finally, it is very obvious that surface storage is due to surface nonuniformities. Large surface depressions cause long-term storage in the form of lakes and ponds having annual fluctuations in volume. To include such storage again exceeds the scope of a general model. From now on, therefore, the term $V_{ss}(t)$ refers only to the retention in precipitation by small depressions and plants from which it will be completely evaporated in the interstorm periods.

$$\int_0^{1\text{ year}} \frac{\partial}{\partial t} V_{ss}(t)\, dt \equiv E_{r_A} = \int_0^{1\text{ year}} v_{ss}(t)\, dt \qquad (2.2)$$

where

E_{r_A} = annual total evaporative loss from surface retention

v_{ss} = rate of capture of precipitation in surface storage

If the hydrologic system is assumed stationary in the mean, and if the integration interval of Eq. (2.1) is taken to be a very large number of full years, the average annual water balance is obtained. All storage terms disappear, giving

$$E[P_A] - E[E_{T_A}] = E[R_{s_A}] + E[R_{g_A}] = E[Y_A] \qquad (2.3)$$

where

P_A = annual (seasonal) total precipitation

E_{T_A} = annual (seasonal) total evapotranspiration

R_{s_A} = annual (seasonal) total surface runoff

R_{g_A} = annual (seasonal) total groundwater runoff

Y_A = annual (seasonal) total yield

E[ ] = expected value of [ ]

As can be seen from Figure 2.3, the above equation (2.3) represents the balance of fluxes external to the indicated control volume. The balance of fluxes internal to the control volume is given by Eq. (2.4).

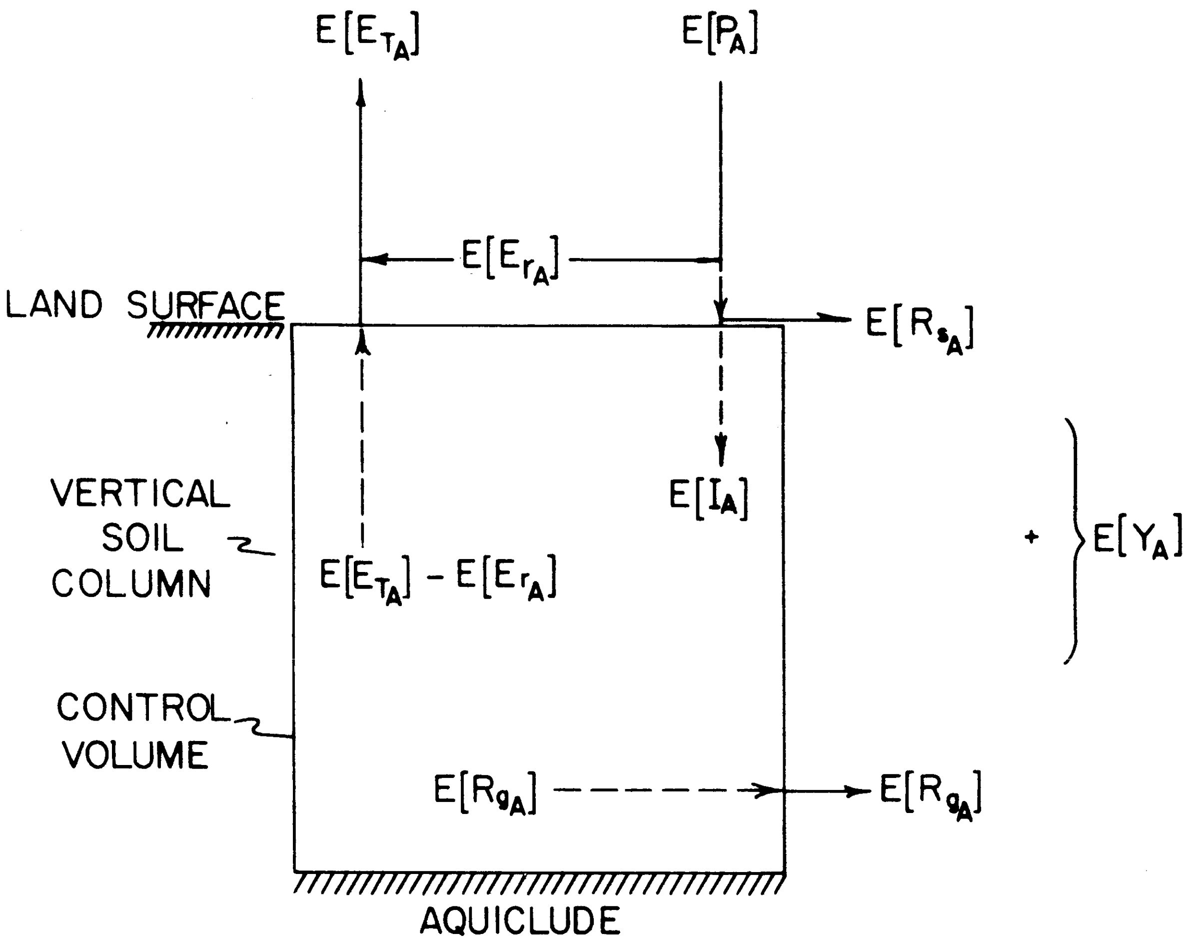


Figure 2.3

AVERAGE ANNUAL WATER BALANCE
(from Eagleson, 1978a)

$$E[I_A] = E[E_{T_A}] - E[E_{r_A}] + E[R_{g_A}] \tag{2.4}$$

This equation also defines the long-term time and spatial average of soil moisture content, s_o, in terms of which each term is defined. The soil moisture will be the subject of a special analysis in Chapters III, IV and V of this research.

II.2 Formulation of Objectives

From the literature [Eagleson, 1978a], we learn:

> "In the true three-dimensional system, the lateral properties such as surface physiography and medium transmissivity will provide a coupling between the groundwater flow and the water table elevation. A similar feedback results from the evaporation and infiltration of surface runoff as it is conveyed away from its point of generation. These feedback links are shown by the dashed lines in Figure 2.4 but are not included in the present model."

a) Storage Components of the Annual Water Balance

It is one of the objectives of this work to add another state variable to the first-order model as stated in the literature [Eagleson, 1978a, b, c, d, e, f, g]. This state variable, $\bar{s}$, represents the annual (seasonal) average of the soil moisture content with the long-term mean of s_o.

$$E[\bar{s}] \equiv s_o \tag{2.5}$$

$$\bar{s} = f(s_o, P_A, E_{T_A}, R_{g_A}, E_{r_A}, \text{parameters}) \tag{2.6}$$

Chapters IV and V will deal mainly with an analysis designed to find the relationship presented as Equation (2.6). Moreover, those chapters will deal with the consequences of introducing this new independent variable.

One of those consequences is the generation of annually varying soil moisture contents, and thus of annually varying volumes of water stored in the unsaturated zone. This has to be accounted for by an additional component in the annual water balance. This term is

$$\int_{0}^{1\ \text{year}} \frac{\partial}{\partial t} V_{su}(t)\ dt \equiv \Delta S_{u_A} \qquad (2.7)$$

where

ΔS_{u_A} = annual total change in storage in the unsaturated zone which is defined by the depth of the root zone, Z_r (Figure 2.2), centimeters

Flowing out of the fact of annually differing soil moisture content is an annually varying total percolation down to the saturated zone. Thus, another storage term has to be included in Equation (2.3). This one is

$$\int_{0}^{1\ \text{year}} \frac{\partial}{\partial t} V_{sg}(t) \equiv \Delta S_{g_A} \qquad (2.8)$$

where

ΔS_{g_A} = annual total change in storage in the saturated zone which is defined by the water table and the impervious bottom layer of the soil column, centimeters

Since the fluctuations in piezometric head must necessarily be reflected in some way or other in the groundwater runoff, an analysis must be concluded to model the linkage between storage and runoff from the saturated zone.

Hence, the lateral properties or feedback in the ground will now be included in a "second-order water balance" (Figure 2.4). The assumption of a constant water table elevation is removed.

The relation among the components of the annual water balance was given to the first-order (Eq. 2.9) by the relation of their average annual quantities (Eq. 2.3 without E[ ] operators). The relation among the components of the annual water balance is now given to the second order (Eq. 2.10) by the relation of the annual quantities of the previous components supplemented by the two storage terms:

$$P_A - E_{T_A} = Y_A \tag{2.9}$$

$$P_A - E_{T_A} - \Delta S_{g_A} - \Delta S_{u_A} = Y_A \tag{2.10}$$

By converting the water balance from its first order approximation, Eq. (2.9), into a second-order approximation, Eq. (2.10) a more realistic description of the system behavior is achieved.

b) Random Annual Average Ambient Temperature, $\overline{T}_A$

In the first-order model, the atmospheric temperature is assumed to have a small coefficient of variation in its annual average value and hence is replaced by its long-term mean, $\overline{\overline{T}}_A$.

According to the analytical relationship used (modified Penman equation) for determining the annual average rate of potential

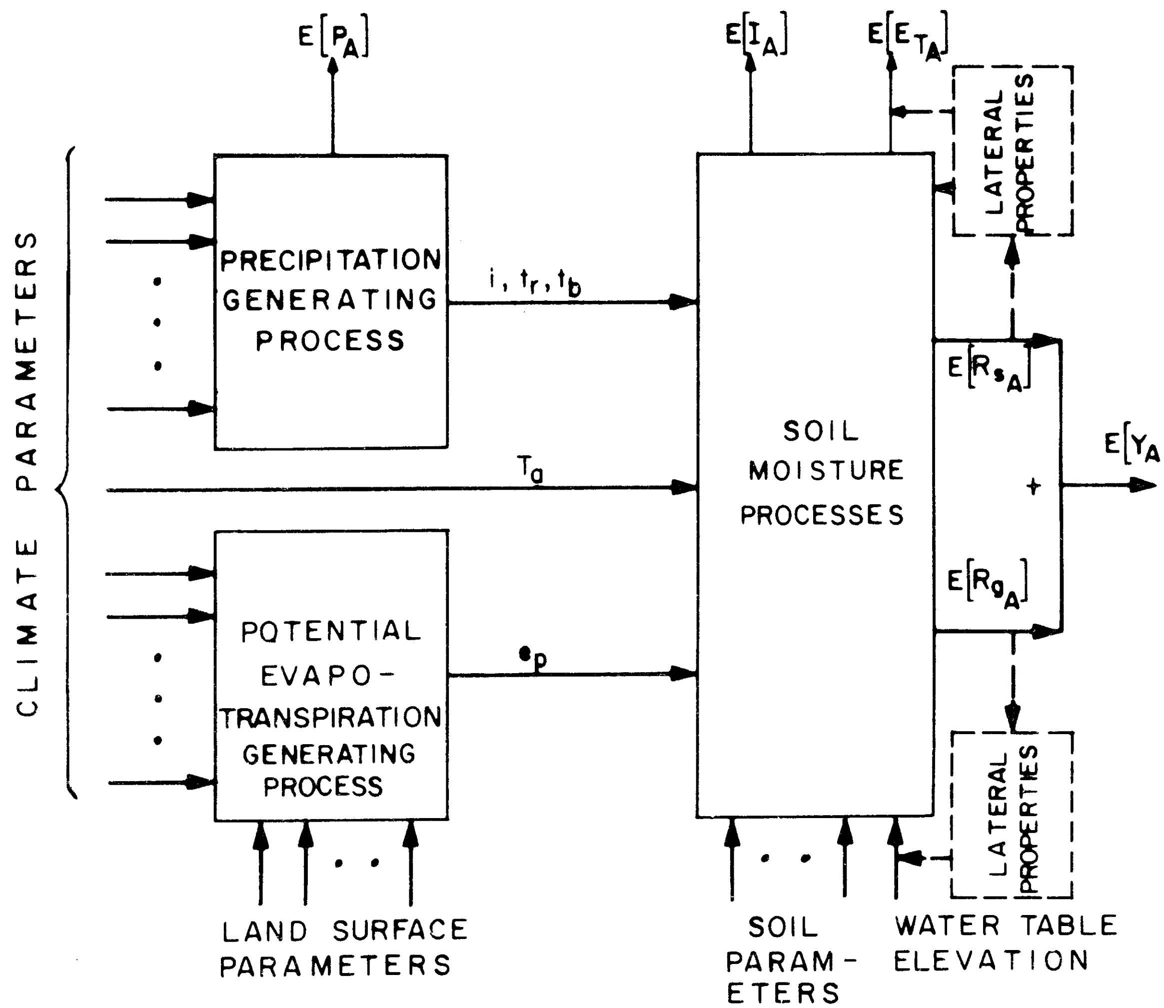


Figure 2.4

COMPUTATIONAL FLOW CHART FOR ANNUAL WATER BALANCE
(from Eagleson, 1978a)

evaporation, the annual average temperature $\overline{T}_A$ turns out to be its principal random variable. Since the coefficient of variation of this temperature is small compared to that of annual precipitation, the coefficient of variation of the annual average rate of potential evaporation is considered small as well.

The second objective of this work is to analyze the effect of a randomly varying atmospheric temperature (in its annual mean) on the annual water balance. Another assumption, that of a constant annual average rate of potential evapotranspiration, $\overline{\overline{e}}_p$, is thus being relaxed.

Chapters V and VI will deal with the statistical incorporation of a second independent random variable into the water balance model.

Together with the newly defined storage terms, a randomly-varying annual average rate of potential evapotranspiration will help to further reduce the unexplained variance of the annual basin yield, Y_A.

c) Verification of the Hypotheses

Two cases will reveal the validity of the various assumptions made to define the storage terms. A comparison of first- and second-order models may prove or reject the utility of those additional terms in the annual water balance. Moreover, the case study will show whether temperature can indeed be neglected a variable and thus can be dealt with as an atmospheric parameter.

A subhumid climate, Clinton, Mass., and a semi-arid climate, Santa Paula, Ca., will serve as test cases.

Simplifications will be made throughout the necessary analyses whenever possible in order to maintain analytical tractability of the problem. Accuracy in detail is traded off against overall utility of the model.

Chapter III

REVIEW OF THE LITERATURE

The water balance dynamics upon which this work is based are analyzed by Eagleson (1978a, b, c, d, e, f, g). A summary of this work follows.

Various physical sub-processes combine to produce a cycle of transported water mass called the water balance. Physical considerations lead to analytical relations for the separate components of this water balance.

a) Infiltration Depth during the j^{th} Storm

$$I_j = g_1(i,\ t_r:\ s_o,\ n,\ k(1),\ c,\ Z) \tag{3.1}$$

where

t_r = duration of j^{th} storm

n = effective medium porosity

$k(1)$ = saturated effective intrinsic permeability of soil

c = pore disconnectedness index

Z = depth to water table

b) Evapotranspiration Depth during the j^{th} Interstorm Period

$$E_{T_j} = g_2(t_b,\ h,\ e_p:\ s_o,\ n,\ k(1),\ c,\ M,\ k_v,\ h_o,\ Z) \tag{3.2}$$

where

t_b = j^{th} interstorm period

$h = j^{th}$ storm depth

e_p = potential evaporation rate

M = canopy density

k_v = plant coefficient = $\frac{\text{potential transpiration rate}}{\text{potential bare soil evaporation rate}}$

h_o = surface retention capacity

c) Evaporation from Surface Retention during the j^{th} Interstorm Period

$$E_{r_j} = g_3(h, t_b, M, h_o) \tag{3.3}$$

d) Uniform Flow Rate to Water Table

$$r_g = K(1)\ s_o^c - w[n, k(1), Z] \tag{3.4}$$

where

K(1) = saturated effective hydraulic conductivity

Since all of these sub-processes depend on a number of random atmospheric independent variables (Fig. 2.4), expected values for the quantities of transported mass are derived. Mutual independence of the random independent variables and analytical marginal distributions is assumed as a mathematical expediency. Integration of the instantaneous volumetric water balance, Eq. (2.1), is performed by summation over the expected number of events occurring in the period of interest.

Precipitation is chosen as the only independent random variable for evaluating the annual water balance of a given climate-soil-vegetation system (2.9). Other random variables, such as season length, τ, and annual potential evapotranspiration, E_{p_A}, have been left expressed by their expected values. Their coefficient of variation is small in most climates when compared to that of annual precipitation, P_A.

III.1 Distribution of Annual Precipitation

Precipitation is modeled according to a Poisson distribution of individual storm arrivals (point precipitation). This method is most suitable among the various types of statistical distributions since it places emphasis on important physical features of precipitation as a random time series of discrete storm events.

Time between storms, t_b, duration of the storms, t_r, and storm depth, h, are of specific interest for modeling water balance processes (Fig. 3.1). The outstanding advantage of the presented precipitation model is the fact that a considerably better estimate of the variance of the distribution of the normalized annual point precipitation is achieved by utilizing the additional information contained in the storm observations (where available!) than is obtained by working with annual totals only. This is of enormous importance if only a few years of data exist (Fig. 3.2).

Independence of successive events is assumed in a Poisson model. Simple but realistic distributions are adopted for the significant times, t_a, t_r, t_b, again assuming mutual independence as pointed out before.

A Gamma distribution is chosen to represent the probability density of storm depths, h. If each storm depth is similarly Gamma

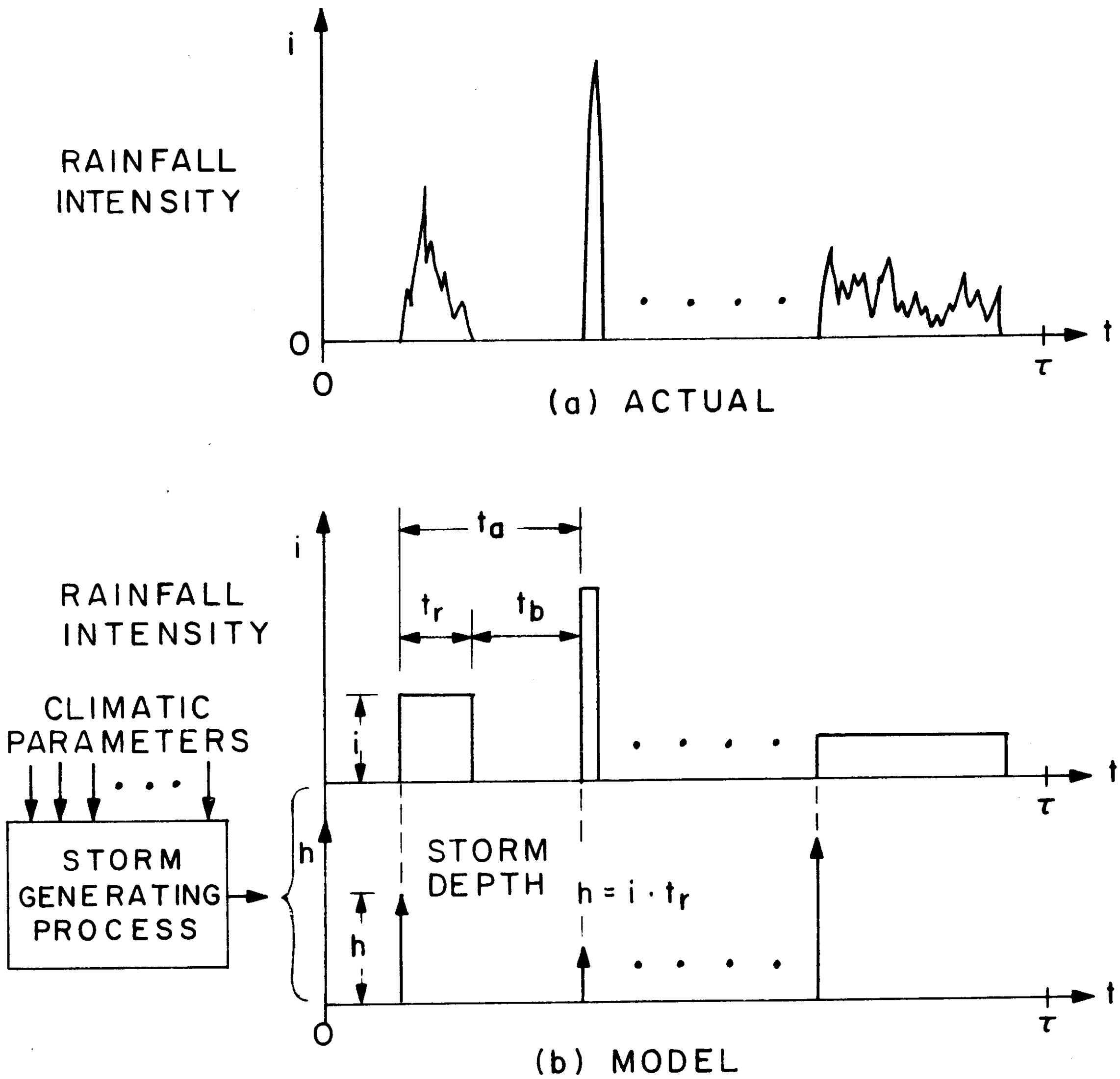


Figure 3.1

MODEL OF PRECIPITATION EVENT SERIES
(from Eagleson, 1978b)

distributed, then the sum over a number of events can be modeled by a Gamma distribution provided the parameters, κ, λ, are unique (Eq. 3.5).

The 'PDF' of total precipitation from ν storm events is given by:

$$f_{p(\nu)}(y) = G(\nu\kappa, \lambda) = \frac{\lambda(\lambda y)^{\nu\kappa-1} e^{-\lambda y}}{\Gamma(\nu\kappa)} \qquad (3.5)$$

where

κ = order of Gamma distribution

$m_H = \kappa/\lambda$ = mean of the distribution of h of a single storm event

$m_{p(\nu)} = \nu\kappa/\lambda$ mean of the distribution of h of the sum of ν events

$\sigma_H^2 = \kappa/(\lambda)^2$ = variance for single event

$\sigma_{p(\nu)}^2 = \nu\kappa/(\lambda)^2$ = variance for sum of ν events

The PDF of cumulative point precipitation can then be derived as:

$$f_p(y) = \sum_{\nu=1}^{\infty} \frac{\eta\kappa(\eta\kappa y)^{\nu\kappa-1}\, e^{-\eta\kappa y}}{\Gamma(\nu\kappa)} \frac{(\omega\tau)^{\nu}\, e^{-\omega m_\tau}}{\nu!} \qquad y > 0 \qquad (3.6)$$

$$P_\theta(0) = e^{-\omega m_\tau} \qquad y = 0$$

where

m_τ = mean value for length of rainy season

$\eta = m_H^{-1}$ = inverse of mean storm depth

$\omega = m_{t_a}^{-1}$ = inverse of mean interarrival time of individual storms

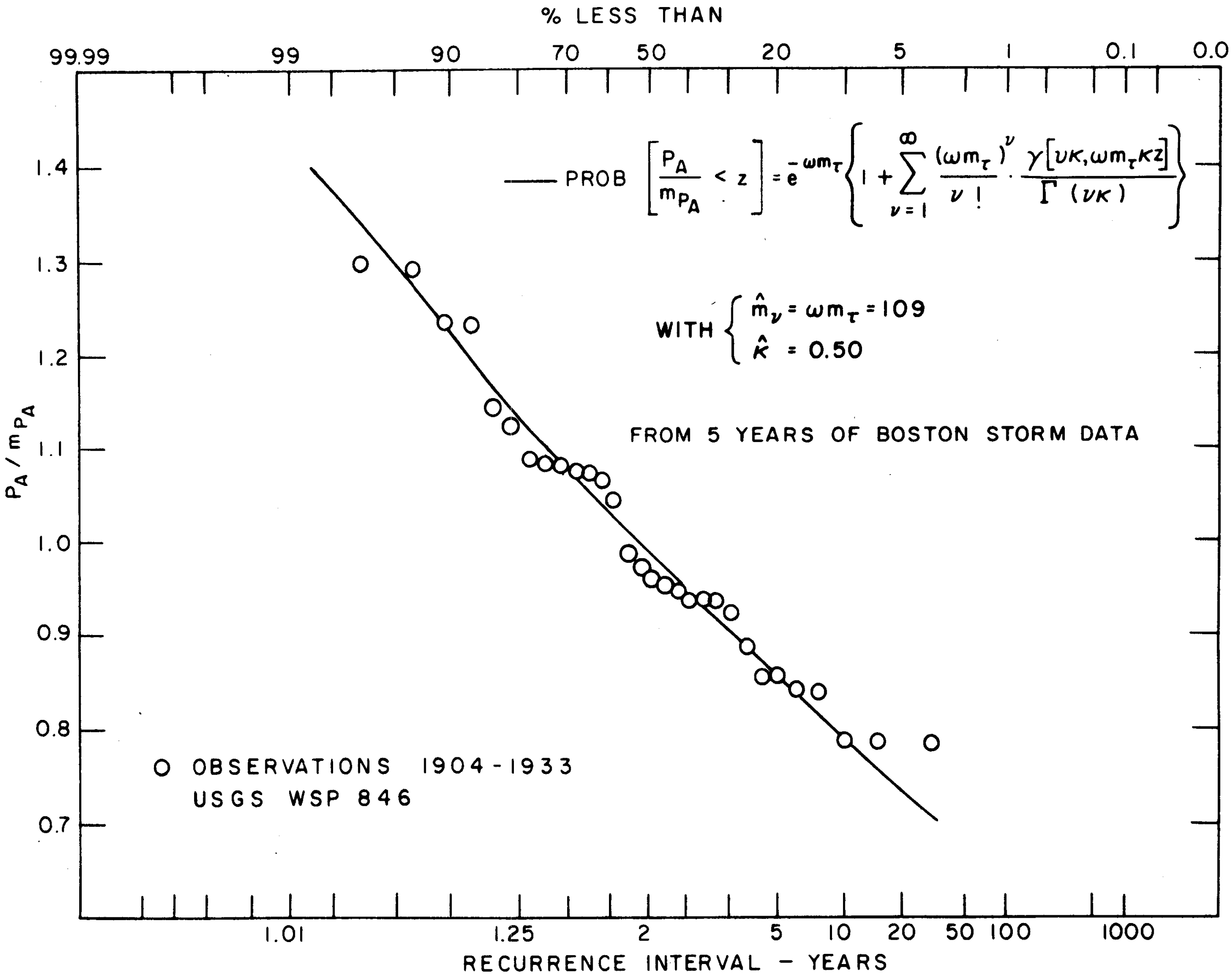


Figure 3.2
ANNUAL POINT PRECIPITATION AT BOSTON, MASS.
(from Eagleson, 1978b)

Since the expected value of annual point precipitation is given by:

$$E[P_A] \equiv m_{P_A} = m_\nu m_H \tag{3.7}$$

where

$$E[\theta|m_\tau] \equiv m_\nu = \omega m_\tau \tag{3.8}$$

the dimensionless CDF of the annual point precipitation can be readily integrated to

$$\text{Prob}[\frac{P_A}{m_{P_A}} < z] = e^{-\omega m_\tau} \left\{1 + \sum_{\nu=1}^{\infty} \frac{(\omega m_\tau)^\nu}{\nu!} p[\nu\kappa, \omega m_\tau \kappa z] \right\} \tag{3.9}$$

where

$$z = \frac{Y}{m_{P_A}} = \text{dimensionless point precipitation}$$

$$P[a, x] \equiv \gamma[a, x]/\Gamma(a) = \text{Pearson's incomplete Gamma function}$$

It is seen that this distribution is specified by two parameters only, κ and m_ν.

Figure 3.2 exhibits the powerful technique of using information of short period observations of storm characteristics in generating a frequency curve of annual point precipitation.

III.2 Model of Soil Moisture Movement

A distinction is made among the different kinds of one-dimensional (vertical) soil moisture movements. Four different processes

processes are analyzed and their effects superimposed linearly. Thus, the possibility of analytical treatment is retained [Eagleson, 1978c].

3.2.1 Soil Parameters

In order to analyze the different soil moisture movements, some preliminary remarks must be made on the properties of soil. The following parameters must be defined:

$$K(\theta) = K(1)\ s^c = k(1)\ \frac{\gamma_\omega}{\mu}\ s^c \tag{3.10}$$

where

s = effective degree of medium saturation

γ_ω = specific weight of liquid

μ = dynamic viscosity of liquid

$K(\theta)$ = effective hydraulic conductivity

$$c = (2 + 3m)/m \tag{3.11}$$

where

m = pore size disconnectedness index

$$\psi(1) = \psi(\theta)\ s^{1/m} \tag{3.12}$$

where

$\psi(1)$ = matrix potential of effective saturation

An empirical relation is given for the pore shape parameter as

$$\Phi = 10^{.66 + .55/m + .14/m^2} \tag{3.13}$$

which leads to a different definition of the matrix potential

$$\psi(1) = \frac{\sigma_\omega}{\gamma_\omega} \left[\frac{n}{k(1)\Phi}\right]^{1/2} \tag{3.14}$$

where

σ_ω = surface tension.

Thus, only three independent soil parameters, n, k(1) and c, characterize the soil's behavior.

3.2.2 Infiltration and Exfiltration

The analysis of the infiltration and exfiltration processes, Eq. (3.1) and Eq. (3.2), are based on a one-dimensional concentration-dependent diffusion equation [Philip, 1960] which has to be supplemented by a sink term for the effect of plant roots on extraction of water from the soil.

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial\theta}{\partial z}\right] - \frac{\partial K(\theta)}{\partial z} - g_r(z, \theta) \tag{3.15}$$

where

θ = effective volumetric moisture content

$D(\theta)$ = diffusivity

$\psi(\theta)$ = soil matrix potential

A rather complicated analytical derivation for the sorption and desorption processes going on during consecutive storm and interstorm periods finally leads to the following dimensionless diffusivities:

$$\frac{3mn\, D_i}{5K(1)\psi(1)} \equiv \phi_i(d, s_o) = (1 - s_o)^{-5/3} \int_{s_o}^{1} s^d [s - s_o]^{2/3}\, ds \tag{3.16}$$

where

$$D = D_i = \text{sorption diffusivity}$$

and similarly

$$\frac{mn\ D_e}{K(1)\ \psi(1)} \equiv s_o^d\ \phi_e(d) = 1.85\ s_o^{-1.85} \int_o^{s_o} s^d[s_o - s]^{.85}\ ds \qquad (3.17)$$

where

$$D = D_e = \text{desorption diffusivity}$$

For sorption, the boundary condition at the surface of the soil column is assumed $s_1 = 1$. For desorption, it is $s_1 = 0$. These equations are displayed in graphical form in Figures 3.3 and 3.4.

For infiltration, the diffusion equation (3.15) can be solved approximately to give:

$$\frac{f_i^*(t,s_o)}{K(1)} = (1 - s_o) \left[\frac{5n\ \psi(1)\ \phi_i(d,\ s_o)}{3\pi mt\ K(1)}\right]^{1/2} + \frac{1}{2}\ [1 + s_o^c] \quad (3.18)$$

where

$f_i^*(t,\ s_o)$ = infiltration capacity

t = duration of infiltration

Similarly, for exfiltration, one can get:

$$\frac{f_e^*(t,\ s_o)}{K(1)} = s_o^{1+d/2} \left[\frac{n\ \psi(1)\ \phi_e(d)}{\pi mt\ K(1)}\right]^{1/2} - \frac{M_v}{K(1)} \qquad (3.19)$$

where

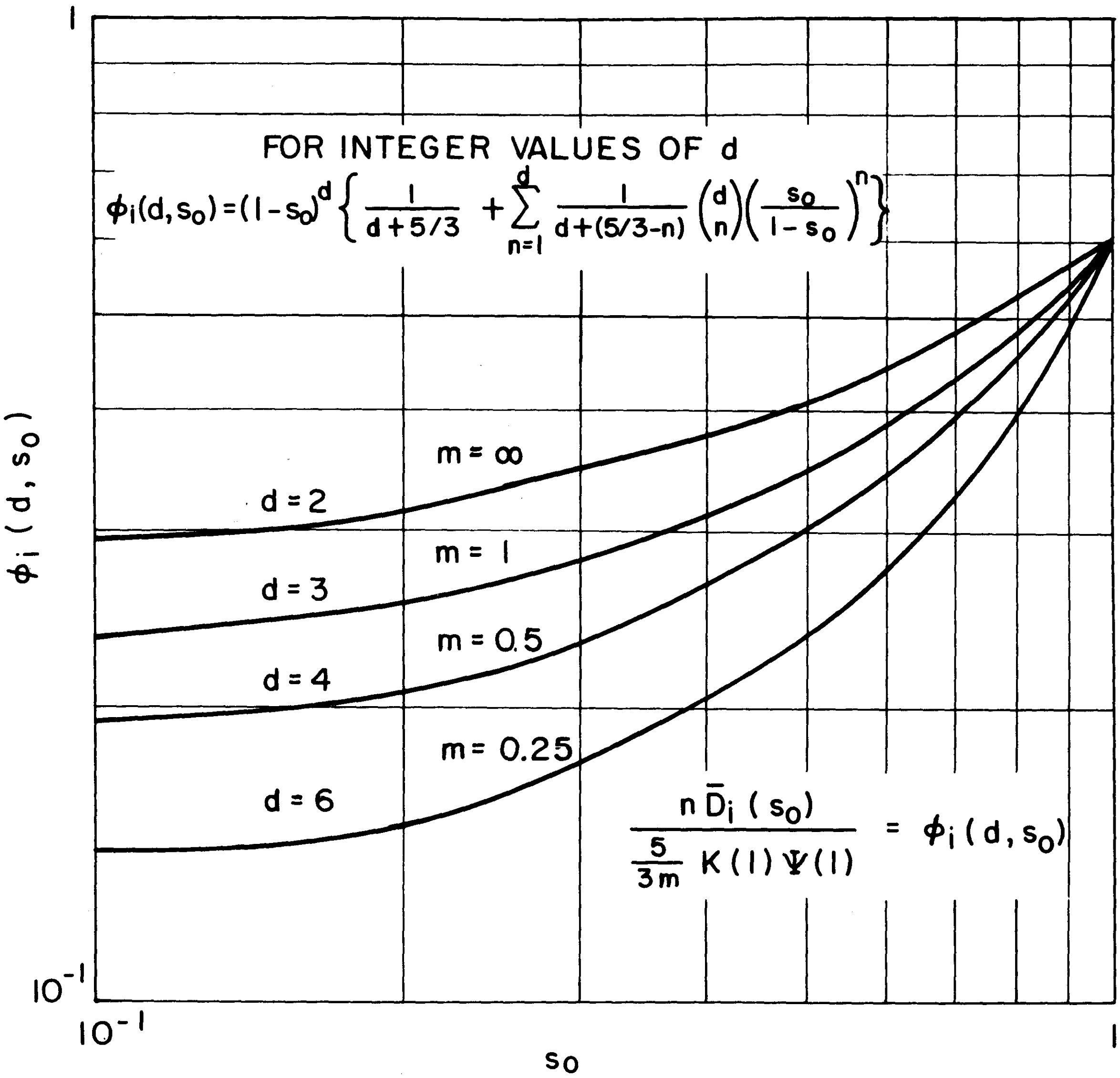
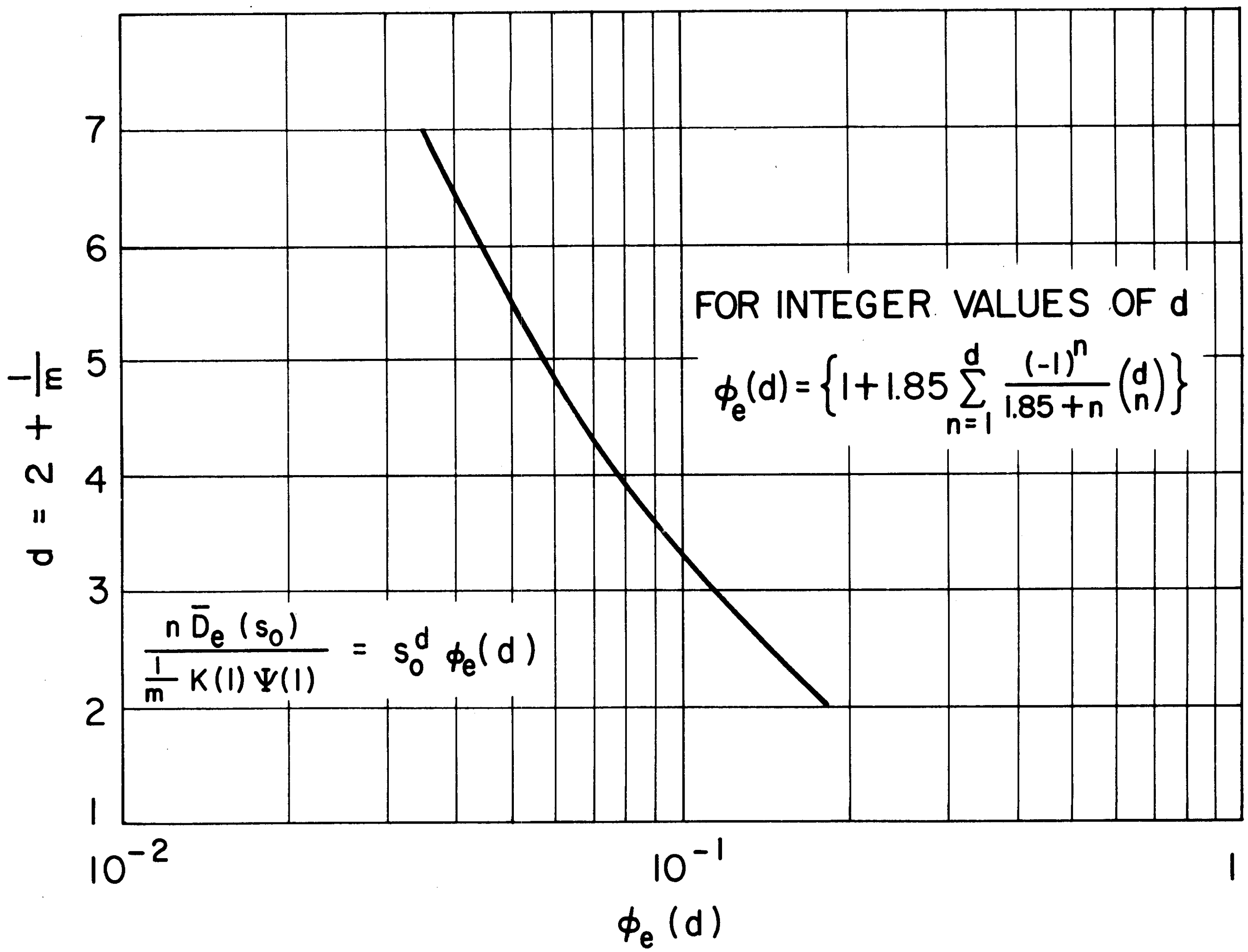


Figure 3.3

DIMENSIONLESS INFILTRATION DIFFUSIVITY
(from Eagleson, 1978c)

Figure 3.4

DIMENSIONLESS EXFILTRATION DIFFUSIVITY (from Eagleson, 1978c)



$f_e^*(t, s_o)$ = exfiltration capacity

e_v = rate of transpiration by vegetation

t = duration of exfiltration

The initial value for the soil moisture, s_o, is assumed to be, to the 'zeroth order', the space-time average soil moisture. Hence, Eq. (3.18) and Eq. (3.19) represent the relationships (dimensionless apparent velocities) necessary to find expressions for the integrated rates of infiltration, Eq. (3.1), and exfiltration, Eq. (3.2).

3.2.3 Percolation

The soil column is subdivided into three different regions according to Figure 3.5. Soil moisture moves out of the unsaturated zone, the thickness of which can be estimated by the penetration depth of infiltration and exfiltration being of the order of vegetal root depth. This soil moisture then percolates down through the intermediate zone reaching the groundwater table. The apparent percolation velocity, υ, due to gravity is given by

$$\frac{\upsilon(s_o)}{K(1)} = s_o^c \qquad (3.20)$$

During the dry season, s_o is assumed zero and therefore $\upsilon = 0$.

3.2.4 Capillary Rise from Water Table

Using a simplified governing diffusion equation, steady capillary rise from the water table at elevation $z = Z$ to the surface

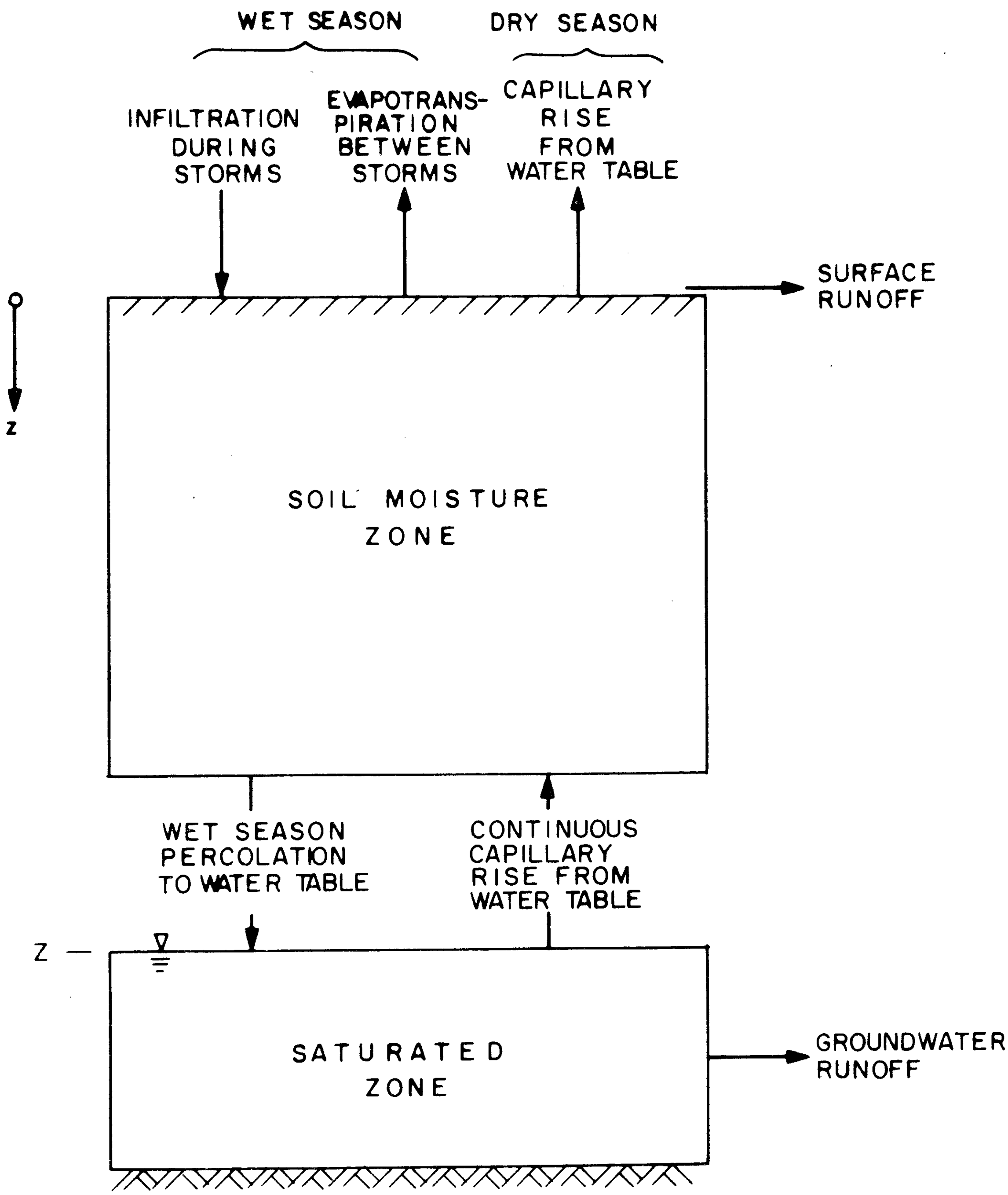


Figure 3.5

SCHEMATIC REPRESENTATION OF SOIL COLUMN
(from Eagleson, 1978c)

has been studied by various researchers. The solution, as given by Gardner (1958), is

$$\frac{w}{K(1)} = [1 + \frac{3/2}{mc-1}][\frac{\psi(1)}{Z}]^{mc} , \qquad w/e_p << 1 \qquad (3.21)$$

Capillary rise is the only operative process during the dry season. During the rainy season, Eq. (3.21) can be added to Eq. (3.19), whereas it must be subtracted from Eq. (3.18) and Eq. (3.20) (Figure 3.5). If $Z = \infty$, w must be set equal to zero.

III.3 Expected Value of Annual Evapotranspiration

The evapotranspiration process is subdivided into three separate components (Eagleson, 1978d)

1. Surface retention loss, E_r, is the depth of water left on all surfaces after cessation of precipitation and runoff. It will be removed by evaporation at the surface potential rate, e_p, from the vegetation.

2. Bare soil evaporation, E_s, is the depth of soil moisture evaporated from the bare soil fraction of the surface. This exfiltration takes place at the rate f_e.

3. Transpiration, E_v, is the depth of soil moisture evaporated by plants from the vegetated fraction of the surface. This process takes place at rate e_v.

Considering a homogeneous mixture of vegetation and bare soil, the total evapotranspiration from a unit land surface may be proportioned according to

$$E_T = (1 - M)\ E_s + ME_v \qquad (3.22)$$

where E_s and E_v include surface retention losses.

To calculate the volume, E_{T_j}, of evapotranspiration during the j^{th} interstorm period, Eq. (3.2), evaporation from bare soil is treated separately from transpiration, E_{v_j}.

Expressions are given in Eagleson (1978d) for both the expected values of E_{s_j} and E_{v_j}. The interested reader is referred to the original text.

The desired interstorm evapotranspiration is obtained by weighting $E[E_{s_j}]$ and $E[E_{v_j}]$ according to the canopy density, M. The expected value is given by

$$E[E_{T_j}] = (1 - M)\ E[E_{s_j}] + ME[E_{v_j}] \qquad (3.23)$$

which is the expected value of the function g_2 in Eq. (3.2).

By summing over the mean number of storm events, the expected annual evapotranspiration is obtained as

$$E[E_{T_A}] = E[\sum_{j=1}^{\nu} E_{T_j}] = E[\nu]\ E[E_{T_j}] = m_\nu\ E[E_{T_j}] \qquad (3.24)$$

The weighted average potential evapotranspiration rate is given by

$$\bar{e}_p^* = (1 - M)\ \bar{e}_p + M\ \bar{e}_{p_v} = [1 - M(1 - k_v)]\ \bar{e}_p \qquad (3.25)$$

The weighted mean annual (seasonal) potential evapotranspiration then is

$$E[E_{p_A}] = m_\nu \, m_{t_b} \, \bar{e}_p^* \qquad (3.26)$$

Finally, by dividing (3.24) by (3.26), the so-called evapotranspiration function $J(E, M, k_v, h_o)$ is obtained

$$J(E, M, k_v, h_o) = \frac{E[E_{T_A}]}{E[E_{p_A}]} = \frac{(1 - M)\, E[E_{s_j}] + ME[E_{v_j}]}{m_{t_b} \, \bar{e}_p^*} \qquad (3.27)$$

This function simplifies for bare soil, $M = 0$, no surface retention, h_o, and negligible capillary rise, w, to

$$J[E] = 1 - [1 + \sqrt{2}\, E]\, e^{-E} + (2E)^{1/2}\, \Gamma(3/2, E) \qquad (3.28)$$

where

$$E = [2\beta n K(1)\, \psi(1)/\pi m\, \bar{e}_p^2]\, \phi_e(d)\, s_o^{d+2} \qquad (3.29)$$

$$\beta = m_{t_b}^{-1} = \text{inverse average interstorm time} \qquad (3.30)$$

$$d = c - (1/m) - 1 \qquad (3.31)$$

Equation (3.28) is plotted in Figure 3.6. As precipitation increases, parameter E increases and the evapotranspiration reaches its potential limit.

$$\lim_{E \to \infty} E[E_{T_A}]/E[E_{p_A}] = 1 \qquad (3.32)$$

Thus, the actual evaporation is controlled primarily by the climate which in our decoupled model provides the independent variable, $\bar{e}_p$.

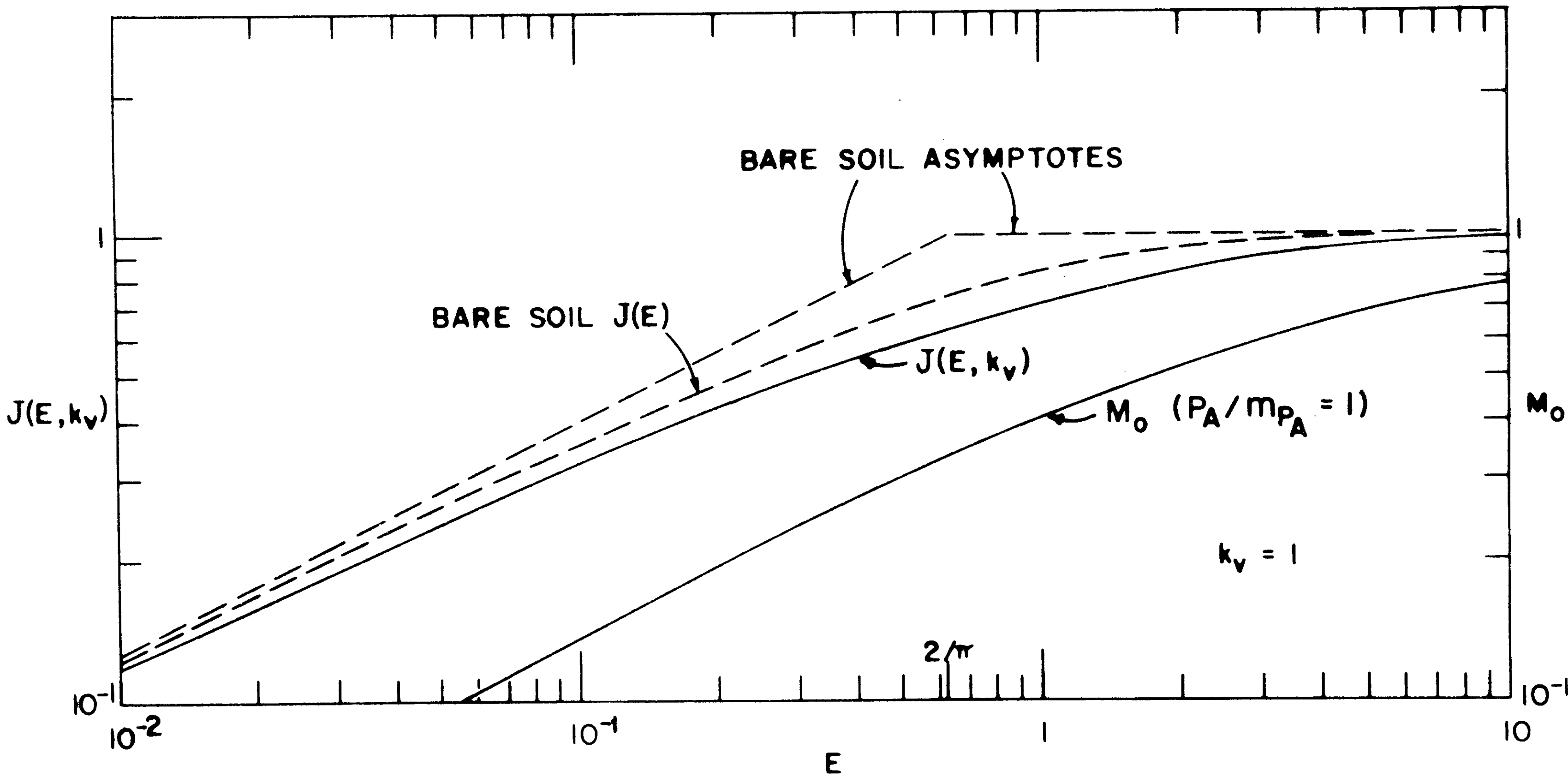


Figure 3.6

BARE SOIL EVAPORATION FUNCTION ($w/\bar{e}_p << 1$)
(from Eagleson, 1978d)

As precipitation events occur less and the potential evaporation rate is high, parameter E goes to zero.

$$\lim_{E \to 0} E[E_{T_A}]/E[E_{p_A}] = [\pi E/2]^{1/2} \qquad (3.33)$$

This is typical for arid conditions where evaporation is controlled primarily by the soil.

Surface retention capacity, h_o, is estimated to be $h_o = 0(1)$ mm.

Surface retention makes a difference in J only for arid climates where E is small. The evapotranspiration from surface retention is

$$\frac{E[E_{r_A}]}{E[E_{p_A}]} = \frac{(1 - M)\, E[E_{r_{s_j}}] + ME[E_{r_{v_j}}]}{m_{t_b}\, \bar{e}_p^*} \qquad (3.34)$$

where

$E_{r_{s_j}}$ = surface retention loss from bare soil

$E_{r_{v_j}}$ = surface retention loss from vegetation

III.4 Infiltration and Surface Runoff

From the known distributions of the independent climatic variables, i, t_r, a distribution for the surface runoff is derived [Eagleson, 1978e] applying the dynamics of the infiltration process as summarized earlier.

From Figure 2.2, we find the instantaneous partition of precipitation at the soil surface. Integrated over the duration of a storm, one obtains:

$$\int_o^{t_r} [i(t) - f_i(t)]\ dt = \int_o^{t_r} [r_s(t) + v_{ss}(t)]\ dt \tag{3.35}$$

Since

$$R_{s_j} = \int_o^{t_r} r_s(t)\ dt \tag{3.36}$$

and

$$E_r = \int_o^{t_r} v_{ss}(t)\ dt \tag{3.37}$$

Equation (3.35) can now be written

$$\int_o^{t_r} [i(t) - f_i(t)]\ dt = R_{s_j} + E_r \tag{3.38}$$

Since point precipitation is assumed to occur in rectangular pulses

$$i(t) = i = \text{const.} \qquad 0 \leq t \leq t_r \tag{3.39}$$

Integration of the infiltration rate over the duration of the storm according to the different time periods in Figure 3.7 gives

$$R_{s_j}(i,\ t_r,\ h_o,\ s_o) \simeq (i - A_o)\ t_r - S_i(t_r/2)^{1/2} - E_r \tag{3.40}$$

where

$$A = \frac{1}{2}\ K(1)(1 + s_o^c) - w \tag{3.41}$$

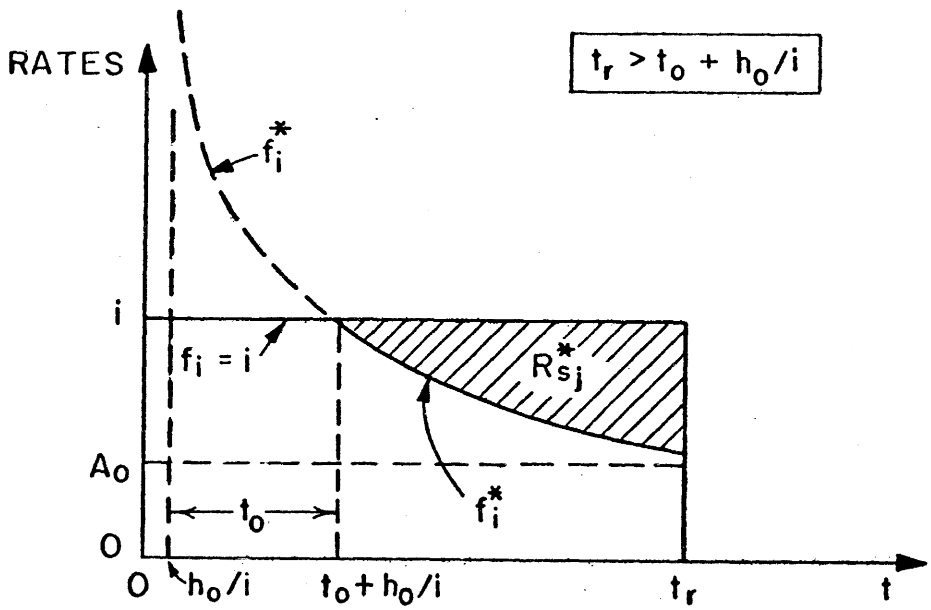


Figure 3.7

SURFACE RUNOFF GENERATION DURING TYPICAL STORM
(from Eagleson, 1978e)

$$S_i = 2(1 - s_o)[(5n\ K(1)\ \psi(1)\ \phi_i(d,\ s_o))/3m\pi]^{1/2} \tag{3.42}$$

The rainfall excess is

$$R^*_{s_j} = R_{s_j} + E[E_r] \tag{3.43}$$

Independence of i and t_r is assumed so that the CDF of rainfall excess $R^*_{s_j}$ can be found by integrating the joint PDF of i and t_r over an integration region which is defined by $t_r = t_o$ and $R^*_{s_j}$.

The mean value of the complete distribution (spike for $R^*_{s_j} = 0$ and continuous part for $R^*_{s_j} > 0$) is then

$$E[R^*_{s_j}] = e^{-G-2\sigma}\ \Gamma(\sigma + 1)/\eta\sigma^{\sigma} \tag{3.44}$$

where

$$\eta = m_H^{-1} = (m_i \cdot m_{t_r})^{-1} \tag{3.45}$$

$$E[R_{s_j}] = e^{-G-2\sigma}\Gamma(\sigma + 1)/\eta\sigma^{\sigma} - E[E_r] \tag{3.46}$$

The average annual surface runoff then follows by multiplying Eq. (3.46) by the mean number of storms, m_ν

$$m_\nu\ E[R_{s_j}] = E[R_{s_A}] = m_H m_\nu e^{-G-2\sigma}\ \Gamma(\sigma + 1)/\sigma^{\sigma} - m_\nu\ E[E_r] \tag{3.47}$$

or

$$\frac{E[R_{s_A}]}{E[P_A]} = e^{-G-2\sigma}\ \Gamma(\sigma + 1)/\sigma^{\sigma} - \frac{E[E_r]}{m_H} \tag{3.48}$$

Equations (3.46), (3.47) and (3.48) are only defined for positive runoff.

It follows that

$$\sigma = \left[\frac{5n\eta^2 \; K(1) \; \Psi(1)(1 - s_o)^2 \; \phi_i(d, \; s_o)}{6 \; \pi \; \delta \; m}\right]^{1/3} \tag{3.49}$$

and

$$G = A_o\alpha = [\alpha \; K(1)/2][1 + s_o^c] - \alpha w \tag{3.50}$$

where

$\delta = m_{t_r}^{-1}$ = reciprocal of average storm duration

$\alpha = m_i^{-1}$ = reciprocal of average storm intensity

Equation (3.48) is plotted in Figure 3.8 ($h_o = 0$).

A relationship is found now which represents function g_1 in Eq. (3.1)

$$I_j = g_i(i, \; t_r: \; s_o, \; n, \; k(1), \; c, \; Z) = h_j - R_{s_j}^* \tag{3.51}$$

where

h_j = depth of j^{th} storm

The expected annual value is given by

$$E[I_A] = m_\nu \; E[I_j] = E[P_A] - E[R_{s_A}^*] \tag{3.52}$$

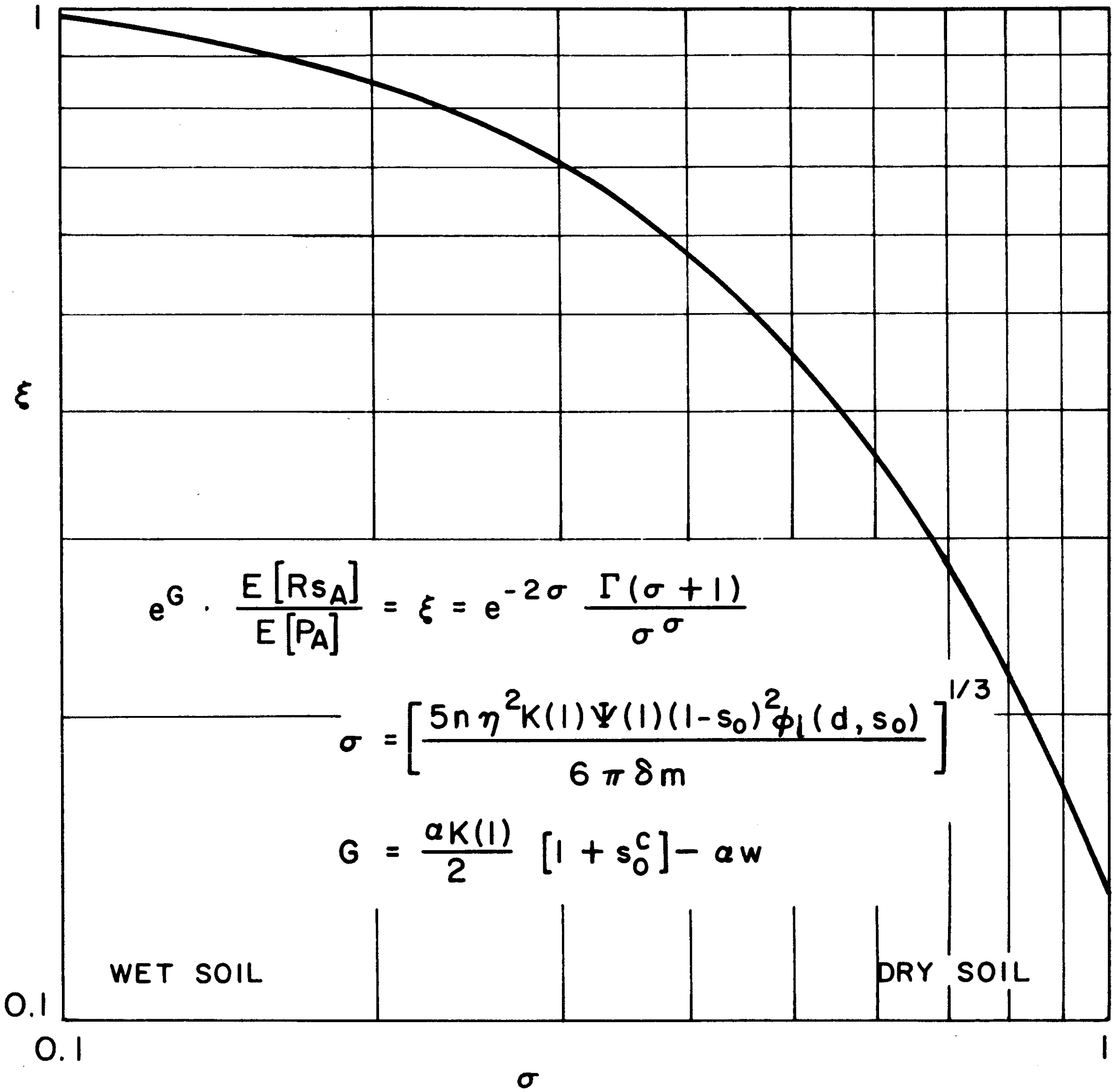


Figure 3.8

PLOT OF SURFACE RUNOFF FUNCTION ($h_o = 0$)
(from Eagleson, 1978e)

III.5 Dynamics of the Annual Water Balance

The previous subchapters demonstrate how the average annual values for the components of the annual water balance have been determined. They have been derived from the probability distributions of storm properties and from the physics of the various soil moisture fluxes.

Mass conservation on an annual basis is given by

$$P_A = R_{s_A} + R_{g_A} + E_{T_A} + \Delta S_{s_A} + \Delta S_{g_A} \tag{3.53}$$

where

P_A = annual (seasonal) precipitation

R_{s_A} = annual (seasonal) surface runoff

R_{g_A} = annual (seasonal) groundwater runoff

E_{T_A} = annual (seasonal) total evapotranspiration

ΔS_{s_A} = annual (seasonal) change in surface storage

ΔS_{g_A} = annual (seasonal) change in saturated and unsaturated zone of soil column

The average annual evapotranspiration from soil can be expressed by

$$E[E^*_{T_A}] = E[E_{T_A}] - E[E_{r_A}] \tag{3.54}$$

where

E_{r_A} = annual surface retention

By using an approximation for the exact evapotranspiration function $J(E, M, k_v, h_o)$, an expression is found for the average annual evapotranspiration from soil, $E^*_{T_A}$ [Eagleson, 1978f]

$$E[E^*_{T_A}] \simeq E[E^*_{p_A}] * J(E, M, k_v) \tag{3.55}$$

where

$E^*_{p_A}$ = potential total evapotranspiration

and

$$E[E^*_{p_A}] = E[E_{p_A}] - E[E_{r_A}] \tag{3.56}$$

and

$$J(E, M, k_v) = 1 - \left[\frac{1 - M}{1 - M + Mk_v}\right] \cdot \Big\{[1 + Mk_v + (2B)^{1/2} E]\, e^{-BE} - [Mk_v + (2C)^{1/2} E]\, e^{-CE} - (2E)^{1/2}[\gamma(3/2, CE) - \gamma(\tfrac{3}{2}, BE)]\Big\} \tag{3.57}$$

in which

$$B = \frac{1 - M}{1 + Mk_v - w/\bar{e}_p} + \frac{M^2 k_v + (1 - M)\, w/\bar{e}_p}{2(1 + Mk_v - w/\bar{e}_p)^2} \tag{3.58}$$

and

$$C = [2(Mk_v - w/\bar{e}_p)^2]^{-1} \tag{3.59}$$

To the first approximation, the average annual surface retention can be obtained by

$$E[E_{r_A}] = m_\nu \left\{(1 - M)\, E[E_{r_{s_j}}] + ME[E_{r_{v_j}}]\right\} \tag{3.60}$$

where the components of surface retention are

$$E[E_{r_{s_j}}] = m_{t_b}\, \bar{e}_p \left\{1 - e^{-\beta h_o/\bar{e}_p}\, \frac{\Gamma[\kappa, \lambda h_o]}{\Gamma(\kappa)} - \left[1 + \frac{\beta}{\lambda \bar{e}_p}\right]^{-\kappa} \frac{\gamma(\kappa, \lambda h_o - \beta h_o/\bar{e}_p]}{\Gamma(\kappa)} \right\} \tag{3.61}$$

$$E[E_{r_{v_j}}] = m_{t_b}\, \bar{e}_p k_v \left\{1 - e^{-\beta h_o/\bar{e}_p}\, \frac{\Gamma(\kappa, \lambda k_v h_o]}{\Gamma(\kappa)} - \left[1 + \frac{\beta}{\lambda k_v \bar{e}_p}\right]^{-\kappa} \frac{\gamma[\kappa, \lambda k_v h_o - \beta h_o/\bar{e}_p]}{\Gamma(\kappa)} \right\} \tag{3.62}$$

The average annual groundwater runoff results from a linear combination of wet season percolation to the water table and annual capillary rise from the water table. That is

$$E[R_{g_A}] = m_\tau\, K(1)\, s_o^c - Tw \tag{3.63}$$

To obtain a better understanding of the important role vegetation plays in a complicated climate-soil-vegetation system, a brief review of the principal assumptions may be helpful.

By including vegetal effects in the analysis of a dynamic water balance formulation, two independent parameters have to be added to the set of soil and climate parameters. These are:

k_v = potential transpiration efficiency

M = vegetal canopy density

It is hypothesized that under natural equilibrium conditions a vegetal species will operate in the unstressed state which permits approximation of the average annual rate of transpiration per unit of vegetated surface by the potential value. That is,

$$\bar{e}_{p_v} = k_v \bar{e}_p \tag{3.64}$$

where

$\bar{e}_{p_v}$ = long-term time average potential rate of transpiration

$\bar{e}_p$ = long-term time average rate of potential evaporation from a bare soil

Furthermore, it is assumed that the root system of the vegetated part, M, of the soil surface draws soil moisture uniformly from the entire soil volume above the maximum root depth Z_r. This second hypothesis resolves the conflict which is generated by introducing a distributed parameter, M, into this lumped one-dimensional representation of a catchment.

Two additional hypotheses based on Darwinian reasoning are postulated to facilitate a quantification of the parameters, k_v, M. Since the time and spatial average of soil moisture, s_o, is not uniquely defined by the water balance equation, Eq. (3.65), in the presence of vegetation, the assumption is made that those parameters can be defined in terms of the remaining climatic and soil parameters.

$$m_{P_A}[1 - e^{-G-2\sigma}\,\Gamma(\sigma + 1)/\sigma^{\sigma}] = E[E^*_{p_A}][(E, M, k_v) + E[R_{g_A}] \quad (3.65)$$

It is assumed that natural vegetal systems of given species tend toward a growth equilibrium in which soil moisture is maximized or evapotranspiration is minimized. This gives the necessary equation to determine M

$$\left.\frac{E[E^*_{T_A}]}{\partial M}\right|_{M=M_o} = 0 = J(E,M,k_v)\left.\frac{E[E^*_{p_A}]}{\partial M}\right|_{M=M_o} + E[E^*_{p_A}]\Big|_{M=M_o}\left.\frac{J(E,M,k_v)}{\partial M}\right|_{M=M_o} \quad (3.66)$$

the solution of which is $M = M_o$.

It is also assumed that equilibrium natural vegetal systems which are water limited (rather than energy limited or nutrient limited) evolve toward maximum water utilization or production of biomass $M_o k_{v_o}$, respectively. The mathematical formulation of this assumption is

$$\left.\frac{\partial(M_o k_v)}{\partial k_v}\right|_{k_v=k_{v_o}} = 0 \quad (3.67)$$

with the solution $k_v = k_{v_o}$.

Thus, the soil moisture balance is defined by the following independent parameters which can be gained by measurements and observations:

Climate: m_{P_A}, m_{t_b}, m_{t_r}, m_{τ}, $\bar{\bar{T}}_A$, $\bar{\bar{e}}_p$, κ, h_o

Soil: $k(1)$, c, n, Z

III.6 A Derived Distribution of Annual Water Yield

Solution of the average annual water balance of a climate-soil-vegetation system gives the time and space average soil moisture concentration, s_o, which can then be used to evaluate the individual water balance components such as the average annual basin yield $E[Y_A]$.

The average annual balance of soil moisture, as given by Eq. (2.4) in dimensionless form. is (for non-zero surface runoff)

$$1 - e^{-G-2\sigma}\Gamma(\sigma + 1)\sigma^{-\sigma} = \frac{E[E^*_{p_A}]}{m_{P_A}} J(E,M,k_v) + \frac{m_\tau K(1)}{m_{P_A}} s_o^c - \frac{Tw}{m_{P_A}} \quad (3.68)$$

This equation gives a unique value of the soil moisture, s_o.

$$s_o = s_o(m_{P_A}, E[E^*_{p_A}], m_\tau K(1); \text{parameters}) \quad (3.69)$$

Since s_o appears in most of the above terms in highly nonlinear fashion, thus, an iterative scheme must be employed to perform the solution.

Again, in dimensionless form, one obtains from Eq. (2.3), (3.55), (3.56) the normalized yield

$$\frac{E[Y_A]}{m_{P_A}} = 1 - \frac{E[E^*_{p_A}]}{m_{P_A}} J(E, M, k_v) - \frac{E[E_{r_A}]}{m_{P_A}} \quad (3.70)$$

Substituting the solution for soil moisture, Eq. (3.69), into the above expression eliminates the long-term average of the state variable of the system, s_o, from Eq. (3.70). This gives the solution for the average annual yield [Eagleson, 1978g]

$$E[Y_A] = g_2\{m_{P_A}, E[E^*_{p_A}], m_\tau K(1); \text{parameters}\} \tag{3.71}$$

The annual basin yield, Y_A, on the other hand, can be given by the following function the form of which is not known, however:

$$Y_A = g_1\{P_A, E^*_{p_A}, \tau K(1); \text{parameters}\} \tag{3.72}$$

Knowing the form of the above function g_1 ( ), one has the analytical basis for deriving the distribution of Y_A from known distributions of P_A, $E^*_{p_A}$ and τ.

Expanding this Eq. (3.72) about the mean of its independent random variables in a multi-dimensional Taylor series and neglecting all the terms of order higher than 1 results in a linear combination of the long-term means of those variables and the variables themselves.

Taking the expected value of this linear equation and assuming statistical independence of the three variables gives a second-order approximation of the first moment, $E[Y_A]$, of the distribution of the annual basin yield, Y_A [Benjamin and Cornell, 1970]

$$\begin{aligned} E[Y_A] \simeq\ & g_1(m_{P_A}, E[E^*_{p_A}], m_\tau K(1); \text{parameters}) \\ & + \frac{1}{2}\left\{\left.\frac{\partial^2 g_i}{\partial P_A^2}\right|_m \text{VAR}[P_A] + \left.\frac{\partial^2 g_i}{\partial E^{*2}_{p_A}}\right|_m \text{VAR}[E^*_{p_A}] + \left.\frac{\partial^2 g_i}{\partial \tau^2}\right|_m \text{VAR}[\tau]\right\} \end{aligned} \tag{3.73}$$

If, in addition, the curvatures of Eq. (3.72) and all the variances are small, their products can then be neglected giving, to the first order,

$$g_i(\ ) = g_2(\ ) \tag{3.74}$$

which means that a first-order approximation of the annual basin yield, Y_A (3.72), can be achieved by replacing the average annual quantities of the water balance components (3.71) by its annual values.

If, furthermore, the coefficients of variance of the potential evapotranspiration from soil, $E^*_{p_A}$ and of the rainy season length, τ, are small compared to that of annual point precipitation, the following simple linear function of one independent random variable is obtained:

$$Y_A = g_2(P_A;\ E[E^*_{p_A}],\ m_\tau\ K(1);\ \text{parameters}) \tag{3.75}$$

Since function g_2 is known, Eq. (3.75), the cdf of annual basin yield is readily derived:

$$\text{Prob}\left[\frac{Y_A}{m_{P_A}} < z\right] = e^{-\omega m_\tau}\left[1 + \sum_{\nu=1}^{\infty} \frac{(\omega m_\tau)^\nu}{\nu!} P[\nu\kappa,\ \omega m_\tau \kappa g_2^{-1}(z)]\right] \tag{3.76}$$

where

$$g_2^{-1}(z) = P_A \tag{3.77}$$

Up to this point, the review of a one-dimensional water balance with an adequate probability distribution for the annual basin yield has been presented. It has been derived for an idealized soil column of a unit cross-section. It is designed, however, for application to entire natural watersheds. In effect, spatial variations of parameters and variables are being averaged by a lumped model.

Case studies have been conducted which exhibited a remarkable accuracy of the model in predicting actually observed yield frequencies

(Figures 3.9 and 3.10).

Finally, it should be noted that suboptimal vegetal cases were chosen to achieve a best fit of the predicted frequencies to the observed frequencies in these case studies. The requirement of maximum biomass is relaxed, accounting for a likely disequilibrium of the vegetal systems due to limitations by nutrition, light or some other ecological factor (Figs. 3.9 and 3.10). For the semi-arid climate of Santa Paula, a variable $M_o = M_o(P_A/m_{P_A})$ was used, whereas for the sub-humid climate of Clinton, a sub-optimal plant coefficient was applied

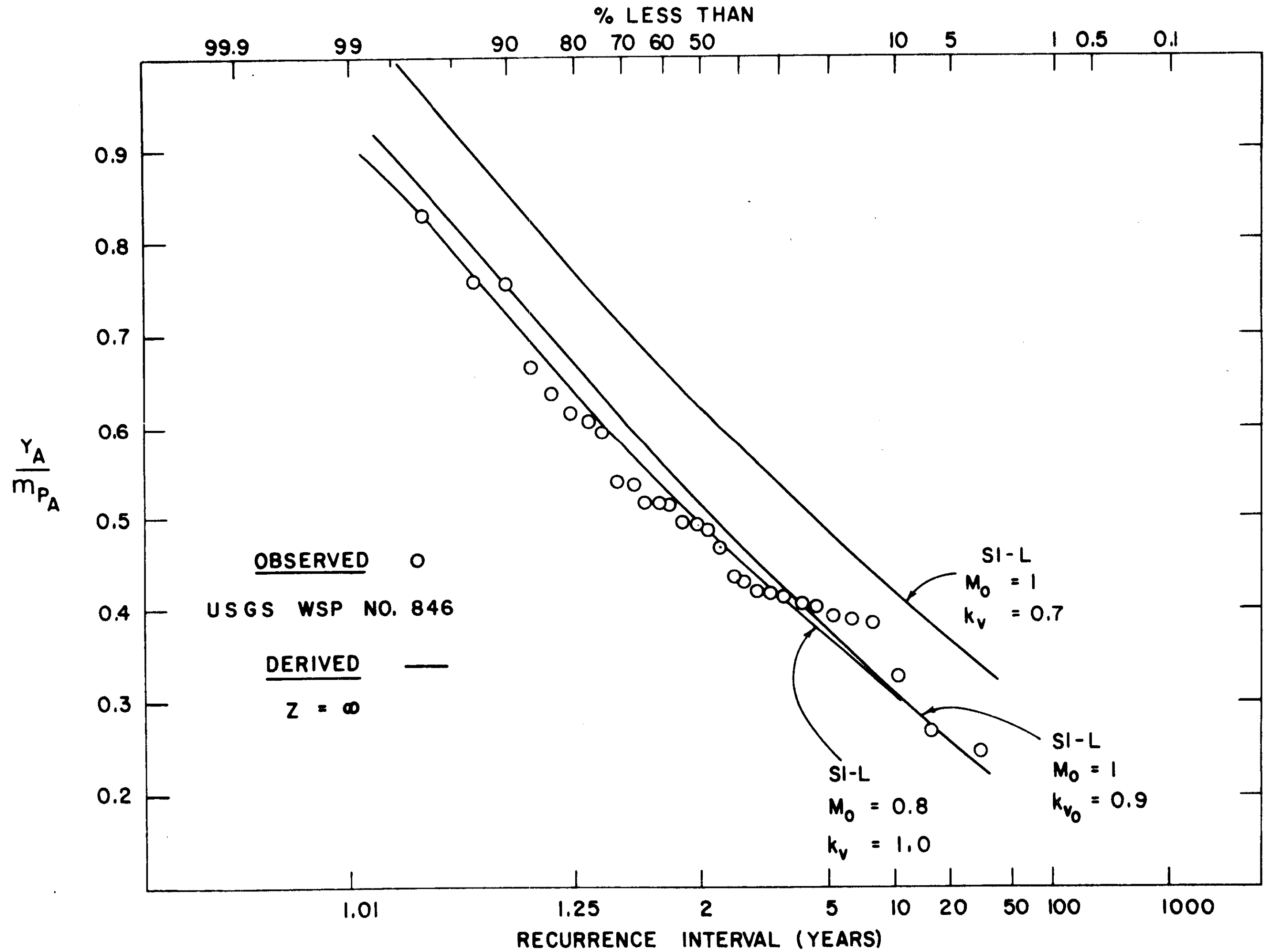


Figure 3.9

FREQUENCY OF ANNUAL BASIN YIELD WITH SUB-OPTIMAL VEGETAL COVER
(SOUTHERN BRANCH OF THE NASHUA RIVER AT CLINTON, MASS.; A = 280 km^2)
(From Eagleson, 1978g)

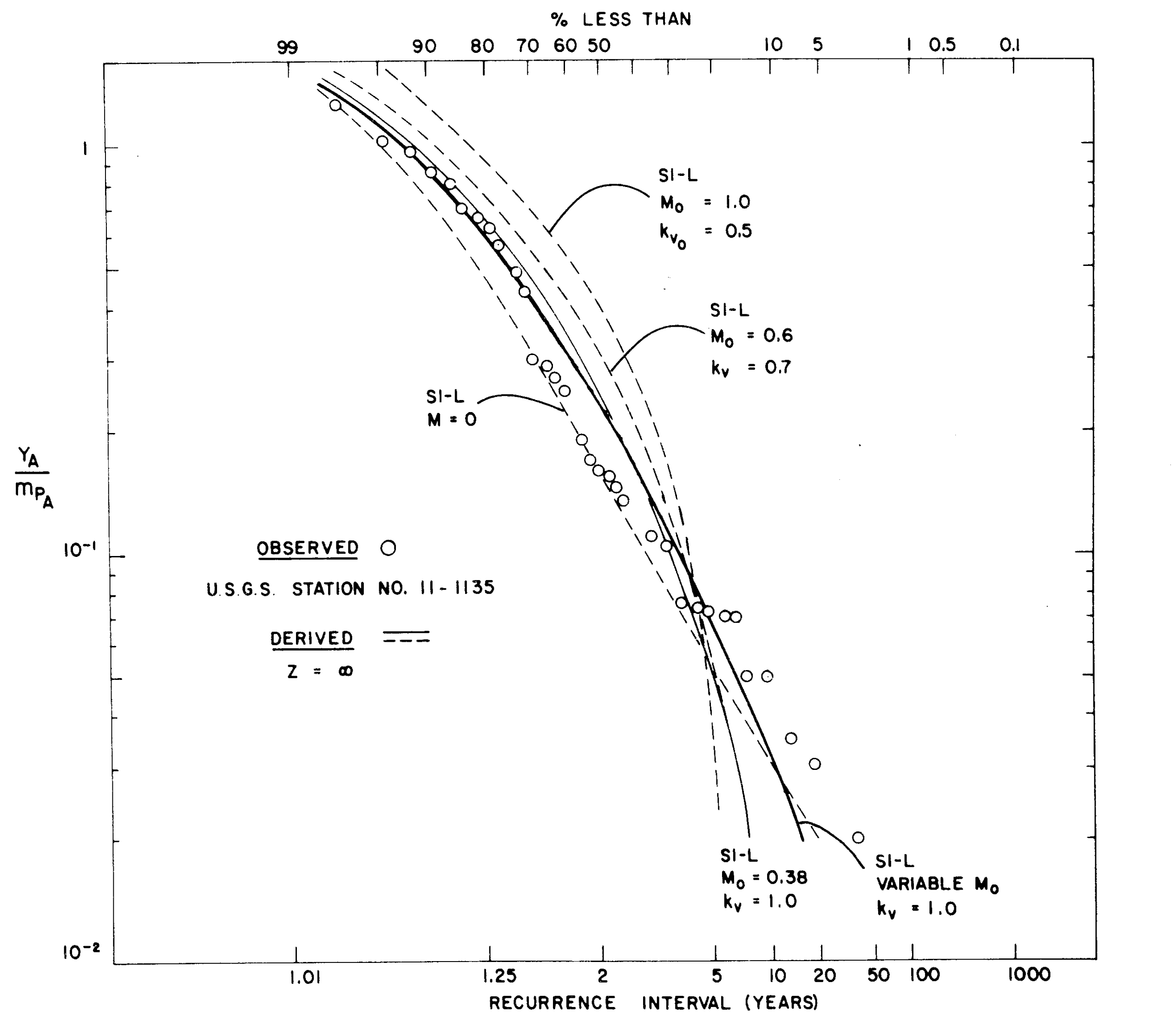


Figure 3.10

FREQUENCY OF ANNUAL YIELD WITH SUB-OPTIMAL VEGETAL COVER; $\kappa = 0.25$ (SANTA PAULA CREEK NEAR SANTA PAULA, CALIFORNIA; A = 104 km^2 (from Eagleson, 1978g)

Chapter IV

ANALYSIS OF STORAGE MECHANISMS

The one-dimensional water balance of a soil column of unit surface area is schematized in Fig. 2.2. The total control volume can be partitioned into three sub-volumes which are interconnected by appropriate boundary conditions. Each sub-volume has the capability of storing water mass. The sub-volumes act as reservoirs governed by certain physical mechanisms through which inflow is converted into change in state of the reservoir and thence into discharge.

The surface storage is dealt with in the literature already [Eagleson, 1978] as far as is possible in a general manner. Equation (2.2) defines the annual surface retention E_{r_A}, and Eq. (2.4) accounts for its average annual value $E[E_{r_A}]$. The definition of annual change in storage in the unsaturated zone of the soil column ΔS_{u_A} is given by Eq. (2.7).

Similarly, Eq. (2.8) represents the definition of the annual change in storage in the saturated zone of the soil column, ΔS_{g_A}. The latter change in storage corresponds to an annual fluctuation of the groundwater table elevation.

IV.1 Storage in the Unsaturated Zone

The instantaneous volumetric water balance for the control sub-volume for the root zone or unsaturated zone is given

$$\Delta S_{u_{\Delta t}} = nZ_r \int_{t_o}^{t_o+\Delta t} \frac{\partial s(t)}{\partial t}\, dt = \int_{t_o}^{t_o+\Delta t} \left\{ f_i(t) - e_T(t) + v_{ss}(t) - p_N(t) \right\} dt \tag{4.1}$$

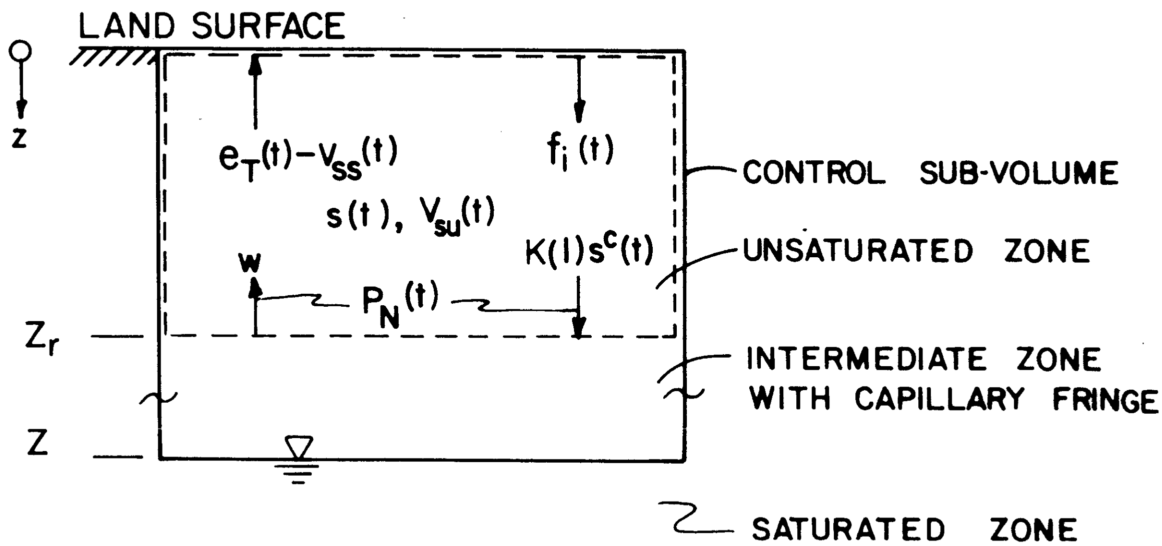


Figure 4.1

SOIL MOISTURE FLUXES

with

$$p_N(t) \equiv K(1)\ s^c(t) - w \qquad (4.2)$$

where

$p_N(t)$ = rate of net percolation to groundwater table

Since the integrand on the right hand side of Eq. (4.1) is unknown, we must seek an approximation for the time-varying function, $s(t)$, of the spatial average of soil moisture in the unsaturated zone.

In order to do so, a second state variable, $\bar{s}$, is introduced. This new state variable represents the annual (seasonal) spatial average of soil moisture in the root zone as opposed to the long-term spatial average, s_o, of the same variable as implicitly defined by Eq. (2.4).

Integrated over a period of one year (season), the rate of change of soil moisture, $\partial s(t)/\partial t$, yields the annual (seasonal) storage of soil moisture in the unsaturated zone. Since the purpose of this model is to operate with annual (seasonal) quantities, it is consistent to adopt a functional formulation for $s(t)$ that reflects its annual (seasonal) behavior. A step function is the most convenient such relationship.

$$s(t) = \bar{s} = \text{const.} \qquad (4.3)$$

This assumption refers to a period of one year (rainy season, τ), as shown in Figure 4.2, and it permits the solution of Eq. (4.1).

Equation (4.1) now gives

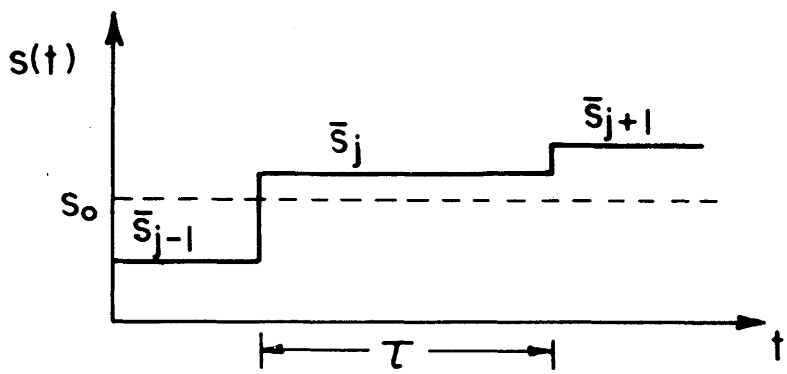


Figure 4.2

ASSUMED HYDROGRAPH OF SOIL MOISTURE

$$\Delta S_{u_A} = nZ_r(\bar{s}_j - \bar{s}_{j-1}) = V_{su_j} - V_{su_{j-1}} \tag{4.4}$$

with

$$\bar{s}_j = \frac{1}{\tau}\int_0^\tau s_i(t)\, dt \tag{4.5}$$

where

$\bar{s}_j$ = spatial average soil moisture of the j^{th} year (season)

Previous dynamic water balances [Eagleson, 1978a, b, c, d, e, f, g] have neglected annual storage entirely. Here we include it but do not account for carry-over storage effects from previous years. The initial soil moisture, s_{j-1}, is assumed to occur at the long-term spatial average quantity, s_o. Thus

$$\Delta S_{u_A} = nZ_r(\bar{s} - s_o) \tag{4.6}$$

This assumption of a long-term average initial condition may introduce considerable error as is shown in Appendix (A.2). However, it is the only way of isolating actual hydrological events from previous ones. Since the water balance finally is evaluated in a statistical manner, correlations between the consecutive events (i.e., annual quantities) have to be neglected. Suppose the water balance is to be applied deterministically in simulating a time series of the annual basin yield, Y_A, for instance, there is no reason to not account for variational initial conditions. The same holds for groundwater storage, ΔS_{g_A}.

Once the annual (seasonal) soil moisture content, $\bar{s}$, is known from evaluation of the annual (seasonal) soil moisture balance, Eq. (4.7), the annual (seasonal) change in storage in the root zone is readily determined by the preceding formula, Eq. (4.6). The annual soil moisture balance is defined by the annual quantities of the moisture fluxes internal to the sub-volume as indicated in Figure 4.1.

$$I_A = P_A - R_{s_A} - E_{r_A} = E_{T_A} - E_{r_A} + R_{g_A} + \Delta S_{g_A} + \Delta S_{u_A} \qquad (4.7)$$

or

$$P_A \left[1 - \frac{R_{s_A}}{P_A}\right] = E_{T_A} + P_{N_A} + \Delta S_{u_A} \qquad (4.8)$$

with

$$P_{N_A} = R_{g_A} + \Delta S_{g_A} = m_\tau K(1)\, \bar{s}^c - wT \qquad (4.9)$$

and

$$\frac{R_{s_A}}{P_A} = e^{-G-2\sigma}\, \Gamma(\sigma + 1)\, \sigma^{-\sigma} - \frac{E_{r_A}}{P_A} \qquad (4.10)$$

where

P_{N_A} = annual net percolation to the water table, cm

Since all the components of Eq. (4.8) are functions of the new state variable, $\bar{s}$, this variable is implicitly defined by that equation.

It must be noted that according to the assumption of constant capillary rise (i.e., constant water table elevation, which means no change in groundwater storage), there is no feedback between change in groundwater storage, ΔS_{g_A}, and storage in the unsaturated zone, ΔS_{u_A}. The new state variable thus is independent of any change in groundwater storage provided the annual fluctuations in the water table elevation, Z, are small. If the changes in groundwater storage, however, cause a considerable change in capillary rise, such a feedback has to be considered.

The change in water table elevation is

$$\Delta Z = \Delta S_{g_A}/n \tag{4.11}$$

where

ΔZ = annual change in groundwater table elevation

Substituted into Eq. (3.21), one obtains

$$\frac{\bar{w}}{K(1)} = \left[1 + \frac{3/2}{mc-1}\right]\left[\frac{\psi(1)}{Z_o - \Delta Z}\right]^{mc} \tag{4.12}$$

where

$\bar{w}$ = annual rate of capillary rise due to $Z_o - \Delta Z$

Z_o = long-term average elevation of the water table

Rearranged

$$\frac{\bar{w}}{K(1)} = \left[1 + \frac{1.5}{mc-1}\right]\left[\frac{\psi(1)}{Z_o}\right]^{mc} \left[1 - \frac{\Delta Z}{Z_o}\right]^{-mc} \tag{4.13}$$

If $\Delta Z/Z_o \ll 1$, one can simplify to

$$\frac{\bar{w}}{K(1)} = \left[1 + \frac{1.5}{mc-1}\right]\left[\frac{\psi(1)}{Z_o}\right]^{mc}\left[1 + mc\,\frac{\Delta Z}{Z_o}\right] \qquad (4.14)$$

and

$$\bar{w} \simeq \left(1 + mc\,\frac{\Delta Z}{Z_o}\right) w_o \qquad (4.15)$$

where

w_o = long-term average rate of capillary rise

This analysis becomes relevant only if the magnitude of w_o is of the order of $K(1)\ s_o^c$ or greater. Otherwise, capillary rise is being neglected entirely.

When capillary rise is not negligible, a feedback between changes in the two storages in the soil column is readily created.

$$\bar{w} = w_o\left(1 + mc\,\frac{\Delta S_{g_A}/n}{Z_o}\right) \qquad (4.16)$$

and

$$P_{N_A} = m_\tau\ K(1)\ \bar{s}^c - w_o\left(1 + mc\,\frac{\Delta S_{g_A}/n}{Z_o}\right) T \qquad (4.17)$$

which is no longer independent of ΔS_{g_A}.

IV.2 Storage in the Saturated Zone

The instantaneous volumetric water balance of the control sub-volume idealizing the saturated zone is given by

$$\Delta S_{g_{\Delta t}} = \int_{t_o}^{t_o+\Delta t} (p_N(t) - r_g(t))dt = \int_{t_o}^{t_o+\Delta t} (K(1)\ s^c(t) - w - r_g(t))\ dt \qquad (4.18)$$

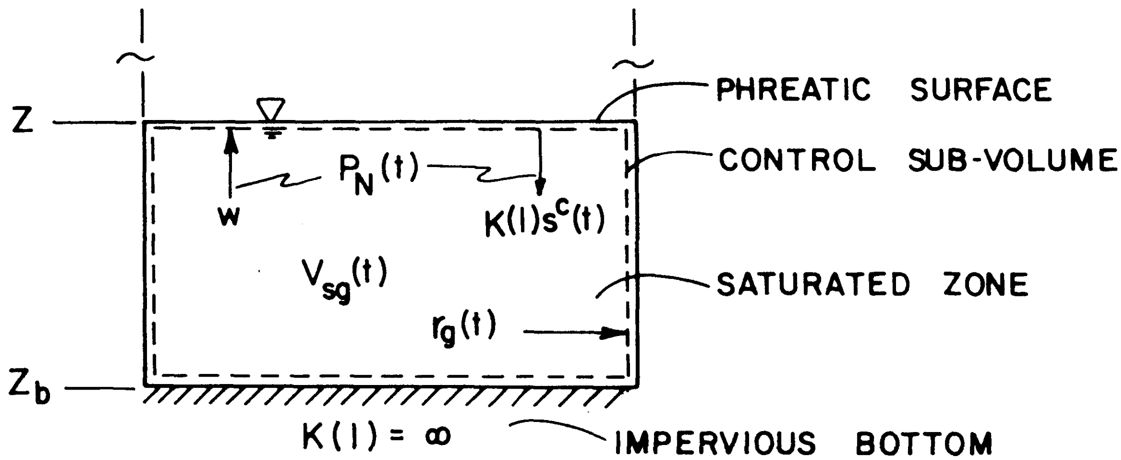


Figure 4.3

FLUXES IN SATURATED ZONE

Equation (4.18) relates the two unknowns, $r_g(t)$ and $\Delta S_{g_{\Delta t}}$, provided s(t) is a known function of time. In order to solve it uniquely, a second relationship has to be found that links the two unknowns through some lateral physical properties of the aquifer.

From groundwater hydrology, the desired set of equations consists of the conservation of mass equation (4.19) and the momentum equation (4.20) (i.e., Darcy's Law), respectively. That is

$$-\nabla_{xy}\, q_{xy} = n \frac{\partial h_s}{\partial t} \qquad (4.19)$$

and

$$q_{xy} = -k(1) \frac{\gamma_\omega}{\mu_\omega} h_s \nabla_{xy} \cdot h_s = -\frac{1}{2} K(1)\, \nabla_{xy} h_s^2 \qquad (4.20)$$

These combined give for a homogeneous soil and some vertical attrition $p_N(t)$

$$\frac{1}{2} \nabla_{xy}^2 h_s^2 + \frac{p_N(t)}{K(1)} = \frac{n}{K(1)} \frac{dh_s}{dt} \qquad (4.21)$$

where

$h_s = h_s(x, y, t)$ = elevation of the phreatic surface

Since the water balance model is based on a vertical, one-dimensional approach, Eq. (4.21) is not applicable for our purpose. Lateral dimensions don't exist by definition.

In order to circumvent this problem, the aquifer is assumed to behave like a linear reservoir. Thus, a dynamic linking between discharge, $r_g(t)$, and the state of the reservoir, V_{sg}, is established.

It is

$$V_{sg} = nh_s \equiv S = \kappa_s \, r_g(t) \qquad (4.22)$$

where

κ_s = groundwater storage reservoir coefficient

A reservoir is said to operate linearly if the discharge is linearly related to the level of storage, thus κ_s being a constant (see Appendix A.1). Combining Eq. (4.18) and Eq. (4.22) gives a linear first-order differential equation governing the groundwater reservoir

$$\frac{dS}{dt} + \frac{1}{\kappa_s} S - p_N(t) = 0 \qquad (4.23)$$

The initial condition is specified to be

$$S(0) \equiv S_o = \kappa_s r_g(0) = \kappa_s r_{g_o} \qquad (4.24)$$

Since the net percolation is

$$p_N(t) = \begin{cases} K(1)\bar{s}^c - w & \text{for} \quad 0 \leq t \leq \tau \\ \\ - w & \text{for} \quad \tau < t \leq T \end{cases} \qquad (4.25)$$

the following solution to Eq. (4.23) and Eq. (4.24) can be found [Wylie, 1975]

$$S_T = \kappa_s [(r_{g_o} + w) \, e^{-T/\kappa_s} + K(1)\bar{s}^c \, (1 - e^{-\tau/\kappa}) e^{-(T-\tau)/\kappa_s} - w] \qquad (4.26)$$

Since

$$\Delta S_{g_A} = \Delta S_{g_{o-\tau}} + \Delta S_{g_{\tau-T}} = S_\tau - S_o + S_T - S_\tau \qquad (4.27)$$

the annual change in groundwater storage is

$$\Delta S_{g_A} = \kappa_s e^{-T/\kappa_s}[K(1)\bar{s}^c\,(e^{\tau/\kappa_s} - 1) - (r_{g_o} + w)(e^{T/\kappa_s} - 1)] \qquad (4.28)$$

There are three random variables in Eq. (4.28), τ, $\bar{s}$ and r_{g_o}. An expansion of Eq. (4.28) about the mean of its variables into a Taylor series keeping only the first three terms of the expansion gives

$$\Delta S_{g_A} \simeq \Delta\bar{S}_{g_A} + (r_{g_o} - m_{r_{g_o}}) \left.\frac{\partial \Delta S_{g_A}}{\partial r_{g_o}}\right|_{m_{r_{g_o}}} + (\tau - m_\tau) \left.\frac{\partial \Delta S_{g_A}}{\partial \tau}\right|_{m_\tau}$$
$$+ (\bar{s} - s_o) \left.\frac{\partial \Delta S_{g_A}}{\partial \bar{s}}\right|_{s_o} + \frac{1}{2} (r_{g_o} - m_{r_{g_o}})^2 \left.\frac{\partial^2 \Delta S_{g_A}}{\partial r_{g_o}^2}\right|_{m_{r_{g_o}}}$$
$$+ \frac{1}{2} (\tau - m_\tau)^2 \left.\frac{\partial^2 \Delta S_{g_A}}{\partial \tau^2}\right|_{m_\tau} + \frac{1}{2} (\bar{s} - s_o)^2 \left.\frac{\partial^2 \Delta S_{g_A}}{\partial \bar{s}^2}\right|_{s_o} + \ldots \qquad (4.29)$$

where

$$\Delta\bar{S}_{g_A} = \kappa_s e^{-T/\kappa_s}[K(1)s_o^c(e^{m_\tau/\kappa_s} - 1) - (m_{r_{g_o}} + w)(e^{T/\kappa_s} - 1)] \qquad (4.30)$$

If now the expected value of this multivariate expression is taken, one obtains [Benjamin & Cornell, 1970]

$$E[\Delta S_{g_A}] = \Delta\bar{S}_{g_A} + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \left.\frac{\partial^2 \Delta S_{g_A}}{\partial x_i x_j}\right|_{mx} \mathrm{COV}[X_i, X_j] \qquad (4.31)$$

with

$$E[x - mx] = 0 \quad (4.32)$$

If we assume small variances and covariances and if the nonlinearities of Eq. (4.28) with respect to τ and $\bar{s}$ are small, the second term of Eq. (4.31) can be neglected. Some analysis of the error introduced by this assumption is conducted in Appendix (A.2.1).

We know, however, that the expected value $E[\Delta S_{g_A}]$ of the annual change in any kind of storage is, by definition, zero, if the hydrologic system is stationary in the mean. Thus,

$$E[\Delta S_{g_A}] \equiv 0 \quad (4.33)$$

Equations (4.31) and (4.33) combined give the very useful relationship which defines the average initial groundwater runoff, $m_{r_{g_o}}$:

$$(e^{T/\kappa_s} - 1)(m_{r_{g_o}} + w) \simeq K(1)\, s_o^c (e^{m_\tau/\kappa_s} - 1) \quad (4.34)$$

A sensitivity analysis (Appendix A.2.2) indicates that the variance of change in groundwater storage, ΔS_{g_A}, due to the length of rainy season, τ, can be neglected without introducing significant error. The same analysis shows that an expected initial condition, $m_{r_{g_o}}$, will cause some inaccuracy. It is an analytical expediency, however, to reduce the number of random variables in Eq. (4.28) so that

$$\Delta S_{g_A} \simeq \Delta S_{g_A}(\bar{s}, m_\tau, m_{r_{g_o}}) = \kappa_s e^{-T/\kappa_s} K(1)(e^{m_\tau/\kappa_s} - 1)(\bar{s}^c - s_o^c) \quad (4.35)$$

Substituting the value for soil moisture obtained from solving

Eq. (4.8), the annual (seasonal) change in storage in the saturated zone can be determined from Eq. (4.35).

IV.3 Analysis of the Effect of Annual Storage

Two storage components have been added to the first-order water balance as given in the literature [Eagleson, 1978]. Both of these storage terms are formulated solely in terms of the soil moisture balance, Eq. (4.8) without introducing additional independent random variables. For each additional component, just one additional independent parameter must be included:

1. The storage coefficient, κ_s, which is a characteristic of the lumped lateral features of the aquifer may be determined according to Appendix A.1.

2. The depth of the root zone, Z_r, the characteristic length of the unsaturated zone, is a vegetal parameter. Its magnitude may be found from the literature concerning the plant species which occurs in the catchment.

It should be noted that for arid climates with an expected length of the rainy season shorter than a full year there is no perennial percolation to the water table according to the assumptions of the basic model [Eagleson, 1978]. During the dry season, the soil moisture is assumed to reduce to $s(t > m_\tau) = 0$. The state variables, $\bar{s}$ and s_o, however, are defined for the length of the rainy season, Eq. (4.5). Thus, both storage terms have to be accounted for in the mass balances despite a decay of soil moisture following the rainy season.

As far as the groundwater is concerned, it is assumed that the initial runoff, r_{g_o}, occurs at its long-term average value of $m_{r_{g_o}}$ no matter when the period of integration of Eq. (4.23) begins. This means that the storage level (state of groundwater reservoir) and hence the rate of replenishment, $p_N(t)$, occur at their long-term average values too.

The loss in accuracy incurred by the above assumption is traded off against the fact that only one additional parameter is required in order to approximate the annual storage capacity of the basin aquifer.

Flowing from this is the simplification that the soil moisture in the unsaturated zone occurs at its expected concentration, $E[\bar{s}] = s_o$. A sensitivity analysis (Appendix A.2) indicates that the error incurred by assuming the expected initial state of the system to be the annual average may be significant.

Again, those simplifications appear justified by their enormous advantage. Without any major complication of the mathematics of the basic model [Eagleson, 1978], a storage concept is developed which allows for accounting for a storage effect on the annual basin yield, Y_A, at least in an approximate manner.

Due to the incorporation of this storage concept, the following change in behavior of the climate-soil-vegetation system can be anticipated:

Annual precipitation, P_A, exceeding its expected value, m_{P_A}, generates a replenishment of the two storages of the system. The annual yield, Y_A, as simulated by Eq. (2.10) occurs at a lower value than that given by the first-order model, Eq. (2.9), because a positive change (increase) in storage of the system reduces the water available for runoff. The mass conservation equations, Eq. (4.36) and Eq. (4.37) for the control sub-volumes of the idealized soil column (see Fig. 4.1 and Fig. 4.3) clearly exhibit this behavior.

Annual precipitation less than its expected value, m_{P_A}, results in a depletion of the storages, hence increasing the annual yield, Y_A, over that of the first-order model. Again, mass conservation considerations, Eq. (4.39) and Eq. (4.40) lead to the above conclusion. Yield cannot be zero for realizable values of precipitation.

Figures 4.4 and 4.5 illustrate this difference in behavior between the two models. In the first-order model, zero precipitation and zero soil moisture coincide (same horizontal axis in Fig. 4.4). In the model accounting for storage (Fig. 4.5), the annual (seasonal) average soil moisture, $\bar{s}$, never reaches zero during the rainy season by definition of the soil moisture storage, ΔS_{u_A}. Accounting for storage one finds from the two control sub-volumes:

$P_A > m_{P_A}$ or $\bar{s} > s_o$:

$$I_A - \Delta S_{u_A} = E_{T_A} + P_{N_A} \qquad (4.36)$$

and

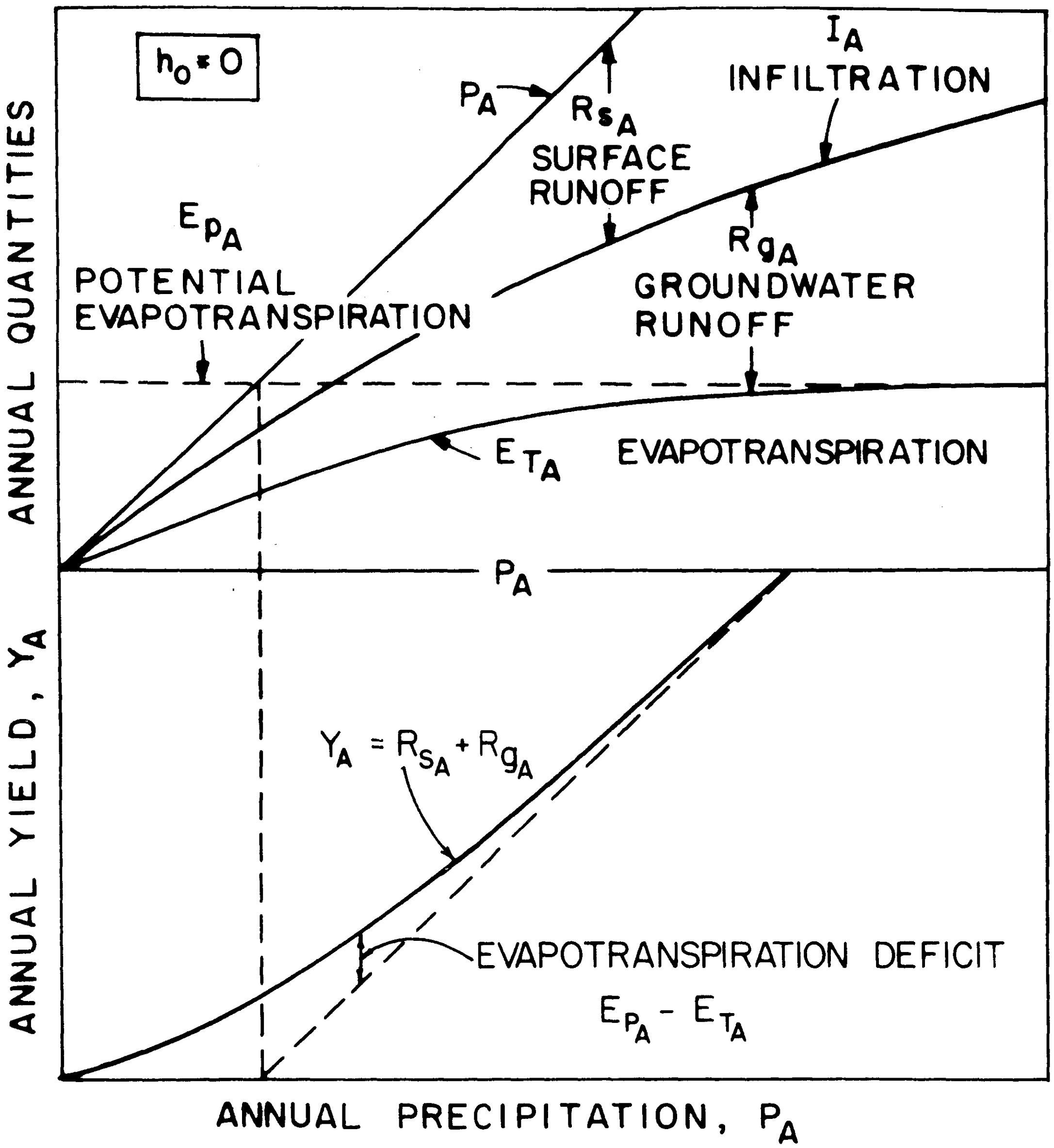


Figure 4.4

WATER BALANCE WITHOUT ANNUAL STORAGE
(from Eagleson, 1978g)

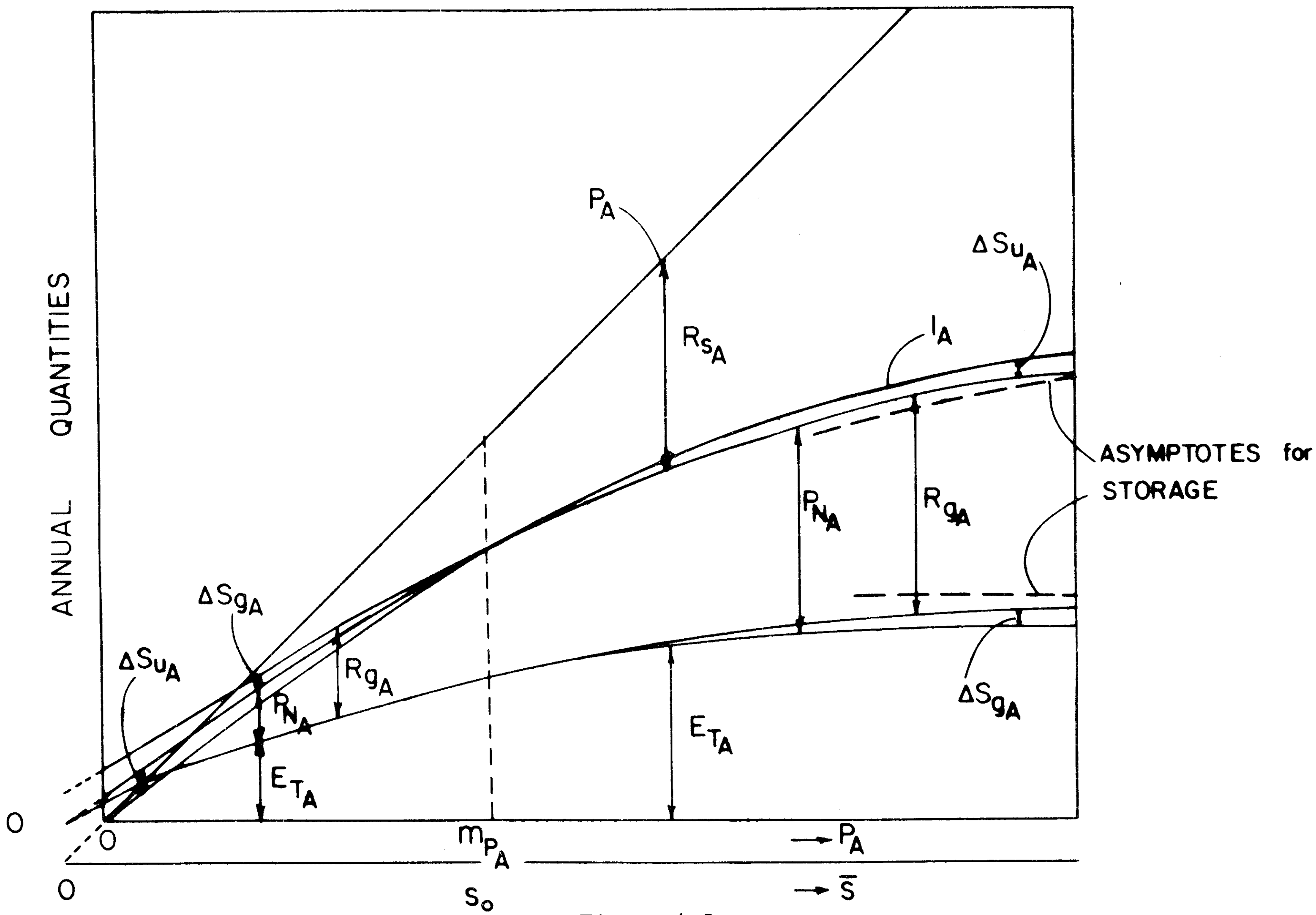


Figure 4.5
WATER BALANCE WITH ANNUAL STORAGE

$$P_{N_A} - \Delta S_{g_A} = R_{g_A} \tag{4.37}$$

Combined, these give

$$P_A - R_{s_A} - \Delta S_{u_A} = R_{g_A} + \Delta S_{g_A} + E_{T_A} \tag{4.38}$$

$P_A < m_{P_A}$ or $\bar{s} < s_o$:

$$I_A + \Delta S_{u_A} = E_{T_A} + P_{N_A} \tag{4.39}$$

and

$$P_{N_A} + \Delta S_{g_A} = R_{g_A} \tag{4.40}$$

Combined, these give

$$P_A + \Delta S_{g_A} + \Delta S_{u_A} = R_{s_A} + R_{g_A} + E_{T_A} \tag{4.41}$$

The storage terms exhibit the following asymptotic behavior:

$$\lim_{\substack{\bar{s}\to 1 \\ P_A\to\infty}} \Delta S_{u_A} = nZ_r(1 - s_o) \tag{4.42}$$

$$\lim_{\substack{\bar{s}\to 0 \\ P_A<0}} \Delta S_{u_A} = -nZ_r\, s_o \tag{4.43}$$

$$\lim_{\substack{\bar{s}\to 1 \\ P_A\to\infty}} \Delta S_{g_A} = \text{const}(1 - s_o^c) \tag{4.44}$$

$$\lim_{\substack{\bar{s} \to 0 \\ P_A < 0}} \Delta S_{g_A} = - \text{const } s_o^c \qquad (4.45)$$

where

$$\text{const} = \kappa_s e^{-T/\kappa_s} (e^{m_\tau/\kappa_s} - 1) \, K(1) \qquad (4.46)$$

A scheme is given in Appendix A.3 which displays the determination of the frequency of the annual basin yield (Table A.1).

Chapter V

LINEARIZATION OF THE SOIL MOISTURE BALANCE

The annual (seasonal) spatial average soil moisture, $\bar{s}$, is implicitly defined by the conservation of moisture internal to the control sub-volume representing the unsaturated zone. This conservation is expressed

$$P_A\left[1 - \frac{R^*_{s_A}}{P_A}\right] = E^*_{T_A} + P_{N_A} + \Delta S_{u_A} \tag{5.1}$$

If there is just one independent random variable as assumed in the 'first-order model' and as indicated in the preceding chapters, then there exists a unique relationship between that variable (P_A) and the dependent unknown soil moisture. Equation (5.1) represents a monotonic relation between precipitation, P_A, and soil moisture, $\bar{s}$.

As soon as a second independent random variable is to be accounted for (i.e., annual average potential rate of evaporation, $\bar{e}_p$), the dependent variable is no longer uniquely defined. An infinite number of combinations of the two independent variables, P_A and $\bar{e}_p$, will produce the same concentration of soil moisture. The evaluation scheme applied so far to determine the cdf of the annual basin yield, Y_A, is no longer valid.

To overcome this problem, we must first find an explicit formulation for the soil moisture, $\bar{s}$, in terms of the two independent variables and the appropriate climatic, soil and vegetal parameters.

$$\bar{s} = \bar{s}(P_A, \bar{e}_p; \text{ parameters}) \tag{5.2}$$

Equation (5.1) can be rewritten

$$P_A(1 - f_1(\bar{s})) = E^*_{P_A} f_2(\bar{s}) + f_3(\bar{s}) + f_4(\bar{s}) - wT \tag{5.3}$$

with

$$f_1(\bar{s}) = \frac{R^*_{s_A}}{P_A} = e^{-G-2\sigma[\bar{s}]} \Gamma(\sigma[\bar{s}] + 1) \sigma[\bar{s}]^{-\sigma[\bar{s}]} \tag{5.4}$$

or

$$f_1(\bar{s}) = \frac{E_{r_A}}{P_A} \tag{5.5}$$

if surface runoff, R_{s_A}, equals zero, and with

$$f_2(\bar{s}) = J(E[\bar{s}], M_o, k_{v_o}) \tag{5.6}$$

$$f_3(\bar{s}) = m_\tau K(1) \bar{s}^c \tag{5.7}$$

and

$$f_4(\bar{s}) = nZ_r(\bar{s} - s_o) \tag{5.8}$$

It is quite obvious that the soil moisture balance is highly nonlinear in terms of the soil moisture, $\bar{s}$.

An expansion into a Taylor series about the long-term average value of soil moisture, s_o, gives a rational means of simplifying the functions, $f_i(\bar{s})$. Keeping only the first three terms of the expansions gives the approximation

$$P_A[1 - f_1^{(0)}(s_o) - (\bar{s} - s_o)\, f_1^{(1)}(s_o) - \frac{1}{2!}\,(\bar{s} - s_o)^2\, f_1^{(11)}(s_o)]$$

$$= E_{P_A}^{\ *}\,[f_2^{(0)}(s_o) + (\bar{s} - s_o)\, f_2^{(1)}(s_o) + \frac{1}{2!}\,(\bar{s} - s_o)^2\, f_2^{(11)}(s_o)]$$

$$+ f_3^{(0)}(s_o) + (\bar{s} - s_o)\, f_3^{(1)}(s_o) + \frac{1}{2!}\,(\bar{s} - s_o)^2\, f_3^{(11)}(s_o) - wT$$

$$+ (\bar{s} - s_o)\, f_4^{(1)}(s_o) \qquad (5.9)$$

The error incurred in each individual function $f_i(\bar{s})$ is given by [Bronstein and Semendjajew, 1974]

$$R_{n,i} = R_{2,i} = \frac{(\bar{s} - s_o)^3}{3!}\, f_i^{(111)}(\bar{s})\Big|_{\xi_i} = c_i(\bar{s} - s_o)^3 \qquad (5.10)$$

where

$$s_o \lesseqgtr \xi \lesseqgtr \bar{s}$$

The total error involved in Eq. (5.9) adds up to

$$R_n = R_3 = \frac{(\bar{s}-s_o)^3}{3!} \sum_{i=1}^{4} f_i^{(111)}(\bar{s})\Big|_{\xi_i} \le \frac{(\bar{s}-s_o)^3}{3!} \sum_{i=1}^{4} \left| f_i^{(111)}(\bar{s})\Big|_{\xi_i} \right| \qquad (5.11)$$

which means that the summation of the absolute single errors in an upper bound for the total error. It can be anticipated, however, that these single errors cancel each other partially.

If the soil moisture, $\bar{s}$, varied over all of its theoretical range (0, 1), Equation (5.9) certainly would be a poor approximation of Eq. (5.1). This fact is indicated in Figure (5.1) where the normalized precipitation is plotted versus the normalized soil moisture

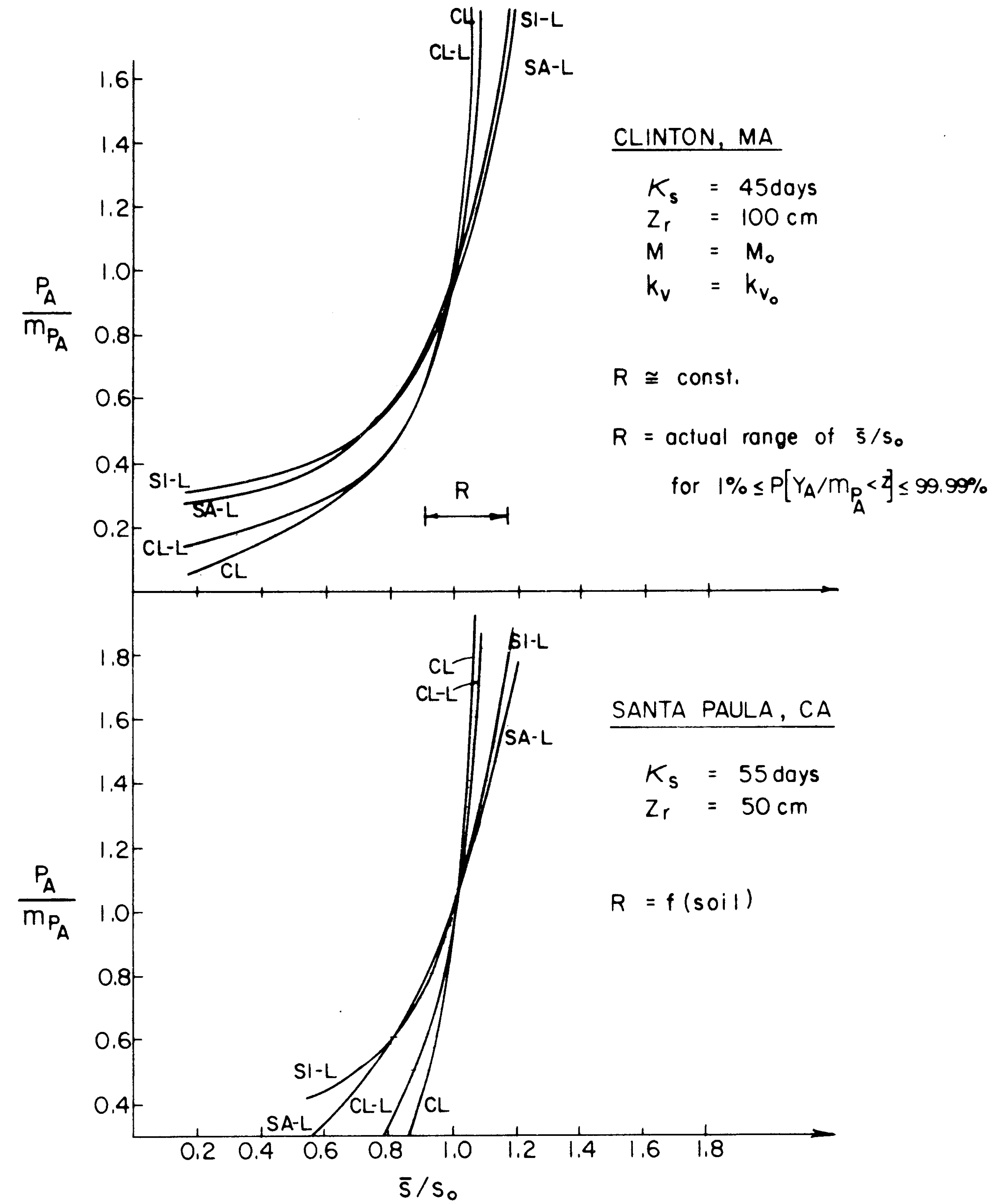


Figure 5.1

SOIL MOISTURE VARIATION WITH ANNUAL PRECIPITATION
(OPTIMAL VEGETAL COVER AND ANNUAL STORAGE)

for four characteristically different soils. The vegetal parameters, M and k_v, were kept at their optimal values [Eagleson, 1978]. The depth of the root zone, Z_r, was estimated and the storage coefficient, κ_s, was evaluated from actual records of surface waters as compiled in the USGS Water Supply papers.

It can be seen that the curvature of the graphs increases for normalized soil moisture below unity. It is in this region that higher derivatives of $f_i(\bar{s})$ contribute significantly in the Taylor series. Fortunately, the likely range of the soil moisture is quite limited.

For Clinton, this range is approximately independent of the type of soil. The limits of this range are defined by the frequency of occurrence of normalized annual yield, Y_A/m_{P_A}. For

$$1\% \leq \text{Prob}[Y_A/m_{P_A} \leq z] \leq 99.99\%$$

we have

$$.85 \leq \bar{s}/s_o \leq 1.2 \qquad (5.12)$$

For the semi-arid climate of Santa Paula, this range is dependent on the type of soil. It is widest for silty-loam and narrowest for clay. For both Clinton and Santa Paula, the curvatures of the graphs in Figure 5.1 within the common range of soil moisture (or precipitation) seem to be small enough to be explained fully by a second derivative of the functions $f_i(\bar{s})$.

Thus, from Eq. (5.9), the soil moisture can be expressed with sufficient accuracy by

$$\bar{s} - s_o \simeq a \pm \sqrt{b} \tag{5.13}$$

where the terms a and b are both functions of the independent variables P_A and $\bar{e}_p$ and the relevant climatic, soil and vegetal parameters. This quadratic formula, Eq. (5.13), may be applied whenever a highly accurate estimation of the soil moisture in terms of its independent variables is required.

This is the case when the annual basin yield, Y_A, is to be evaluated deterministically from Eq. (2.10). Elimination of the soil moisture, $\bar{s}$, in Eq. (2.10) using Eq. (5.13) quantifies the annual yield corresponding to a given pair of values of precipitation and annual average potential rate of evaporation.

The ultimate goal of this research, however, is the statistical distribution of the annual basin yield. The frequency of occurrence of that hydrologic variable is of interest rather than the quantity of the variable itself. Equations (2.10) and (5.13) combined give a very non-linear annual water balance in terms of its independent random variable precipitation. The derivation of a cdf for the annual basin yield from the known distributions for precipitation and potential evapotranspiration becomes extremely complicated. An elaborate numerical integration scheme must be employed.

Since a goal of this work is to analyze the sensitivity of the frequency of annual basin yield to a randomly varying evapotranspiration, some further expedient simplifications will be made.

An explicit formulation for soil moisture such as Eq. (5.13) which is nonlinear with respect to its independent random variable, precipitation, leads to an annual water balance (Eq. 2.10) which is also nonlinear in terms of precipitation. That is, we have an equation of the form

$$P_A = Y_A + E^*_{p_A} * J(P_A, \bar{e}_p; \text{ parameters}) + \Sigma\Delta S_A(P_A, \bar{e}_p; \text{ parameters}) \tag{5.14}$$

An explicit formulation for annual precipitation can then often not be found. Such an expression is necessary in order to specify the integration limits when determining the cdf of annual yield.

Again, Figure 5.1 indicates that linearization of the soil moisture balance with respect to soil moisture introduces an error which might be limited within the likely range of soil moisture.

The frequencies of the annual yield, Y_A, both for low and high values of Y_A become distorted slightly due to this linearization. For the purpose of investigating the effect of a random annual average evapotranspiration on the frequency of the annual basin yield, however, the linearization facilitates computations enormously at the expense of some accuracy for extreme events.

The case studies in Chapter VII of this work show that at least for Clinton the loss in accuracy by linearizing the soil moisture balance is not significant. Figures 5.2, 5.3, 5.4 and 5.5 display the error components contributed to the total error by the individual terms of the

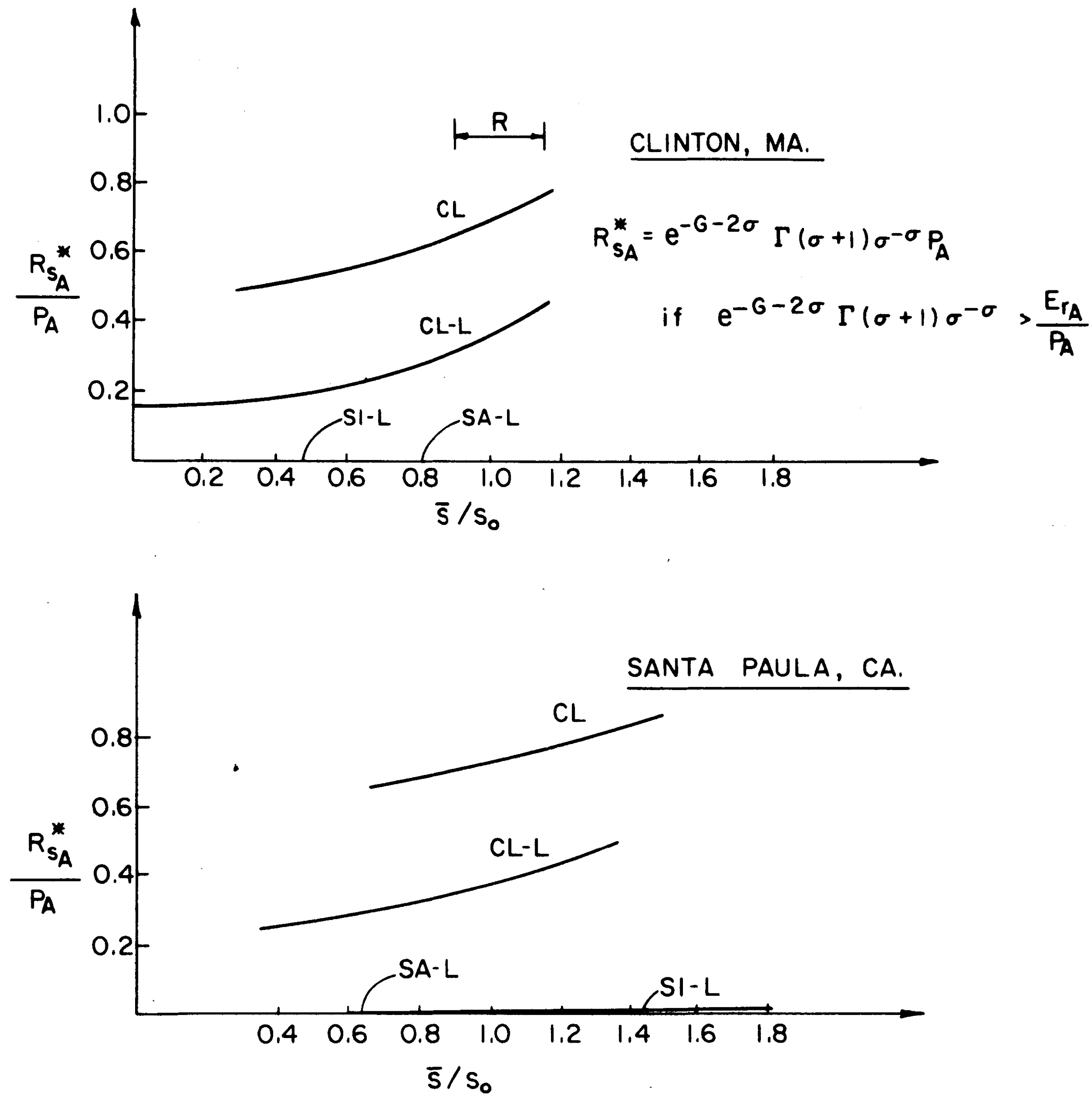


Figure 5.2

RAINFALL EXCESS DUE TO VARIATION IN SOIL MOISTURE
(OPTIMAL VEGETAL COVER AND ANNUAL STORAGE)

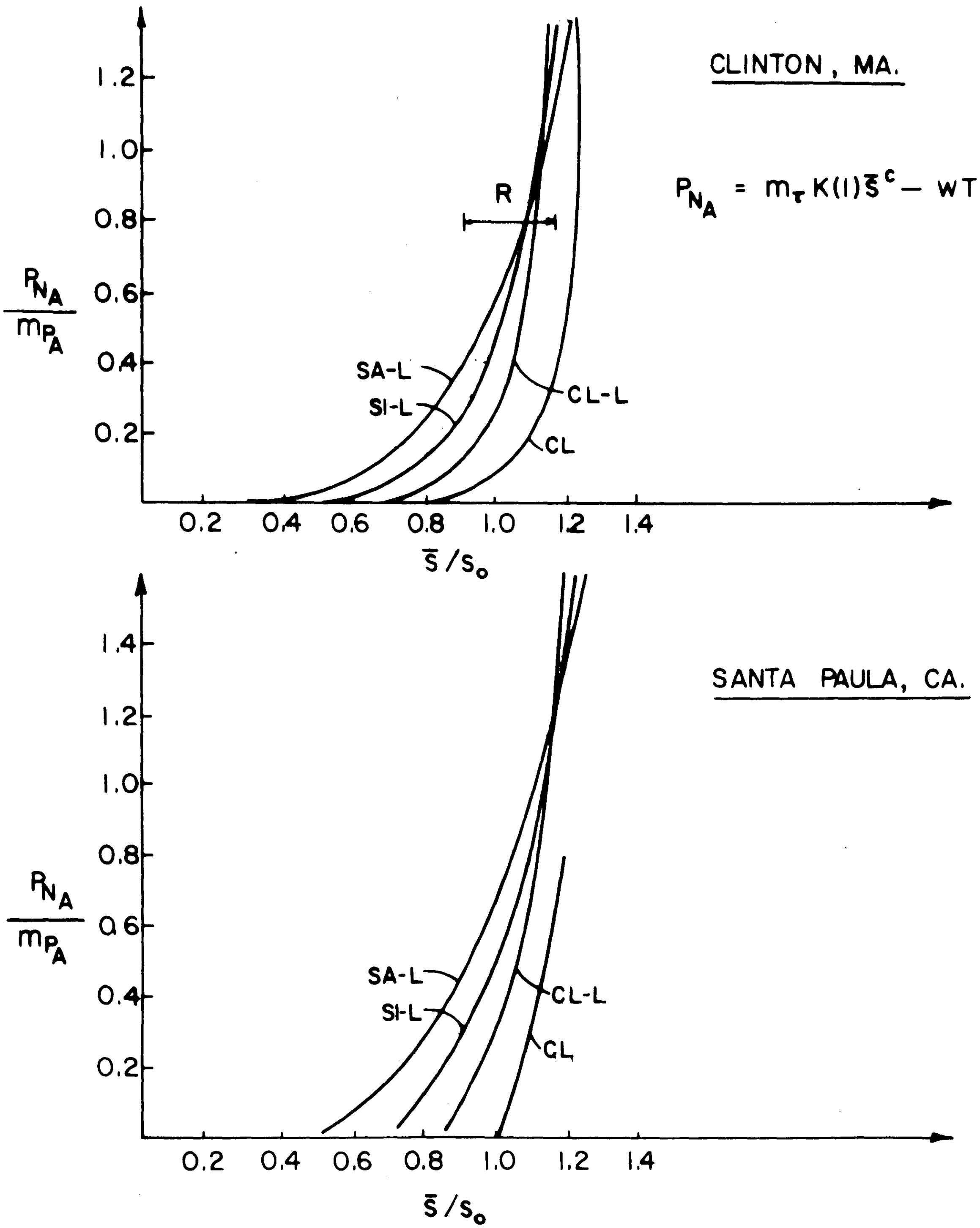


Figure 5.3

PERCOLATION TO THE GROUNDWATER TABLE DUE TO VARIATION IN SOIL MOISTURE (OPTIMAL VEGETAL COVER AND ANNUAL STORAGE)

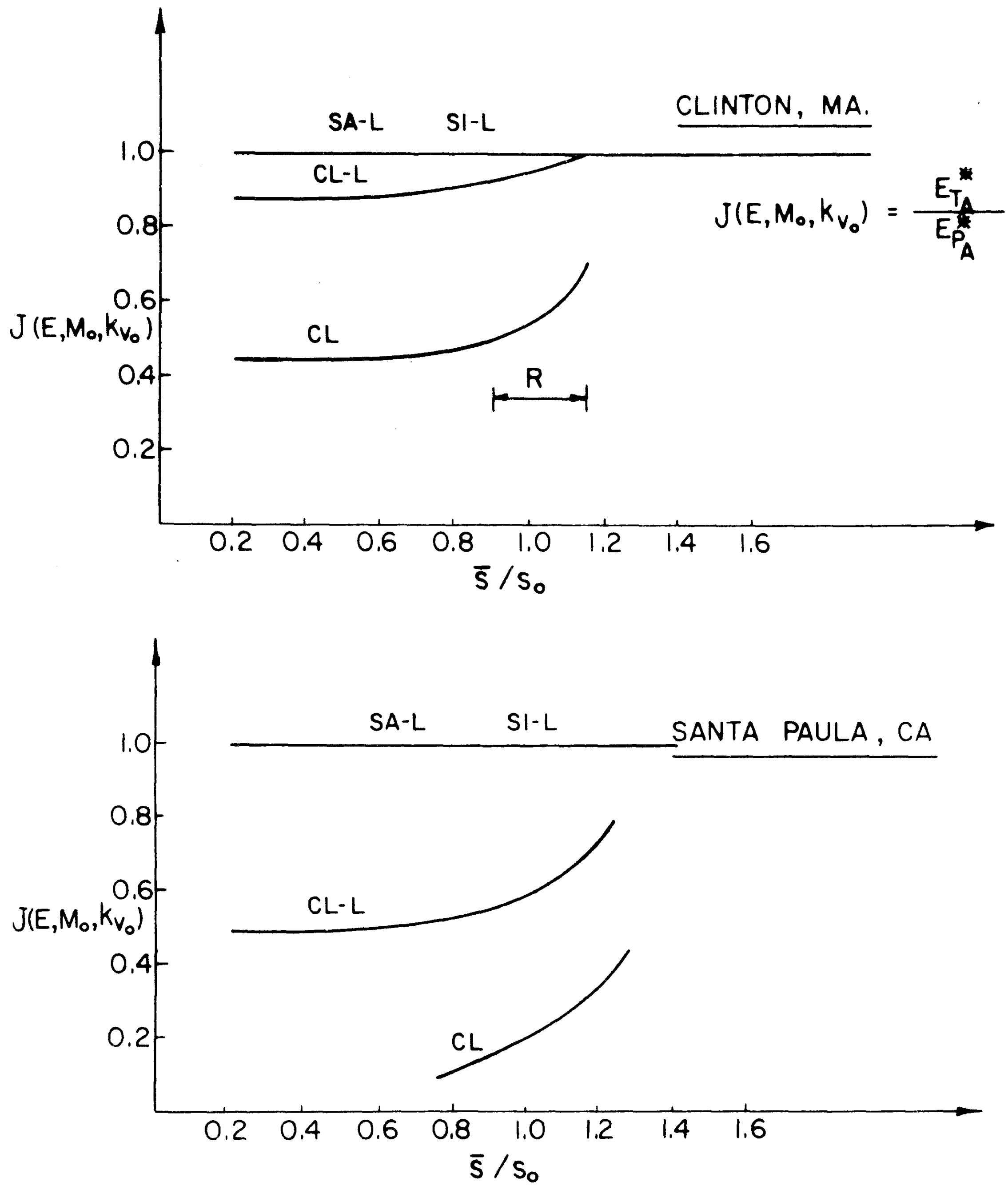
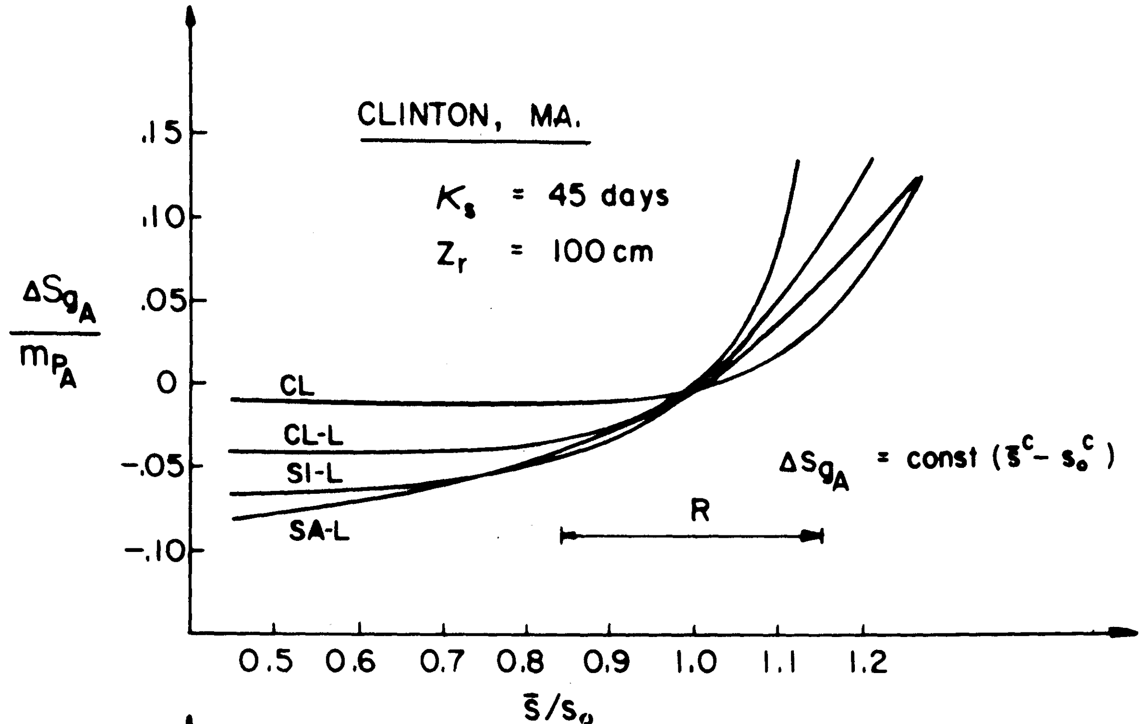


Figure 5.4

EVAPOTRANSPIRATION FUNCTION CORRESPONDING TO SOIL MOISTURE VARIATION (OPTIMAL VEGETAL COVER AND ANNUAL STORAGE)



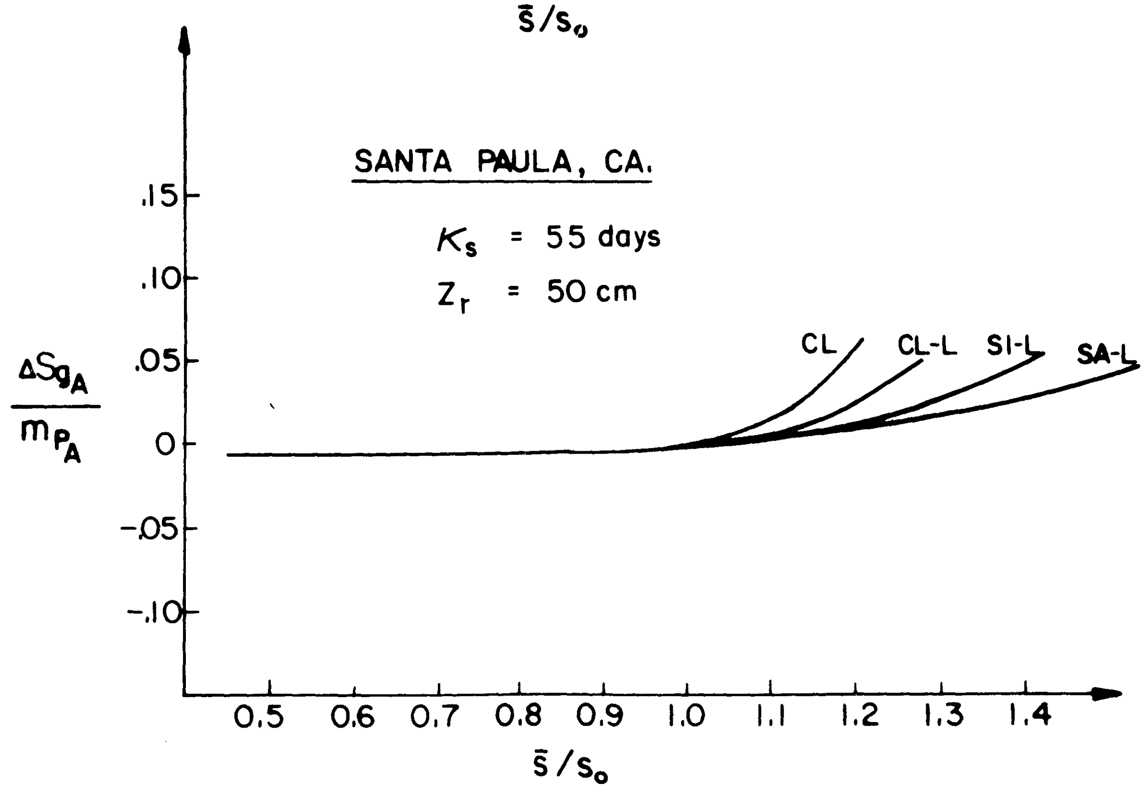


Figure 5.5

ANNUAL CHANGE IN GROUNDWATER STORAGE DUE TO VARIATION IN SOIL MOISTURE (OPTIMAL VEGETAL COVER) water balance. It can be estimated how large the discrepancy between the actual functions, $f_i(\bar{s})$ and their linearizations becomes within a practical range of soil moisture. It has to be noted that the "actual" range, R, of soil moisture for Santa Paula appears too large because of a presumably premature state of the vegetal system.

If sub-optimal vegetal equilibria are accounted for, the model's accuracy improves remarkably and the "actual" range of soil moisture shrinks. The range of annual yield, Y_A, is proportionally related to a corresponding range in soil moisture. Hence a range in annual yield which is predicted too large by considering optimal vegetal conditions generates an equivalent overestimation of the range of soil moisture. The curvatures in Figures 5.1 to 5.5 thus appear too big as far as silty-loam and sandy-loam are concerned.

Keeping in mind these circumstances, a linearization of the soil moisture balance, Eq. (5.1), with respect to precipitation seems to be accurate enough for the purpose of analyzing the effect of a random evapotranspiration. Neglecting the second-order terms in Eq. (5.9), one obtains

$$P_A[1 - f_1^{(0)}(s_o) - (\bar{s} - s_o)\, f_1^{(1)}(s_o)] = E^*_{p_A}[f_2^{(0)}(s_o) + (\bar{s} - s_o) f_2^{(1)}(s_o)] + f_3^{(0)}(s_o) + (\bar{s} - s_o)\, f_3^{(1)}(s_o) - wT + (\bar{s} - s_o)\, nZ_r \tag{5.15}$$

with error

$$R_n = R_2 \leq \frac{(\bar{s} - s_o)^2}{2!} \sum_{i=1}^{4} \left| f_i^{(11)}\bar{s}|_{\xi} \right| \tag{5.16}$$

Rearranging Eq. (5.15) gives

$$\bar{s} - s_o = \frac{P_A[1 - f_1^{(0)}(s_o)] - E^*_{p_A} f_2^{(0)}(s_o) - m_\tau K(1) s_o^c + Tw}{P_A f_1^{(1)}(s_o) + E^*_{p_A} f_2^{(1)}(s_o) + m_\tau K(1)\ c s_o^{c-1} + nZ_r} \tag{5.17}$$

This explicit formulation, Eq. (5.17) for the soil moisture, $\bar{s}$, is nonlinear with respect to both precipitation and evapotranspiration. Introduced into Equation (2.10) for annual yield, Eq. (5.17) gives an explicit quadratic-type expression for precipitation in terms of annual yield, evaporation and parameters. Equation (5.17) can be further simplified without generating major error, however.

An analysis of magnitudes indicates that the third term in the denominator of Eq. (5.17), which is related to percolation to the water table, exceeds all the other terms by at least one order of magnitude.

Figures 5.2 and 5.4 show that the gradients for rainfall excess, $f_1^{(1)}(s_o)$, and for the evapotranspiration function, $f_2^{(1)}(s_o)$, with respect to the soil moisture are extremely small in the vicinity of the average soil moisture, s_o. The gradient of percolation on the other hand is extremely steep. This means that the variables in the denominator can be replaced by their expected values without causing a significant error.

Thus, we remove that slight nonlinearity of Eq. (5.17) with respect to the independent variables. It then becomes

$$\bar{s} - s_o = \frac{P_A[1 - f_1^{(0)}(s_o)] - E^*_{p_A} \; f_2^{(0)}(s_o) - m_\tau \; K(1)s_o^c + Tw}{m_{P_A} \; f_1^{(1)}(s_o) + E[E^*_{p_A}] \; f_2^{(1)}(s_o) + m_\tau \; K(1)cs_o^{c-1} + nZ_r} \tag{5.18}$$

This equation can be written

$$\bar{s} - s_o = A_1 P_A + A_2 \bar{e}_p + A_3 \tag{5.19}$$

which demonstrates the linear relationship between soil moisture and its independent random variables.

We define

$$d \equiv m_{P_A} \; f_1^{(1)}(s_o) + E[E^*_{p_A}] \; f_2^{(1)}(s_o) + m_\tau \; K(1)cs_o^{c-1} + nZ_r \tag{5.20}$$

where

$$f_1^{(1)}(s_o) = \frac{\partial}{\partial \bar{s}} \left(e^{-G(\bar{s})-2\sigma(\bar{s})} \; \Gamma(\sigma(\bar{s}) + 1) \; \sigma(\bar{s})^{-\sigma(\bar{s})} \right) \Bigg|_{s_o} \tag{5.21}$$

with

$$\frac{\partial}{\partial x} \Gamma(x + 1) = \Gamma(x + 1) \; \psi(x + 1) \tag{5.22}$$

and

$$f_2^{(1)}(s_o) = \frac{\partial}{\partial \bar{s}} \left(J(E[\bar{s}], M_o, k_{v_o})\right) \Bigg|_{s_o} \tag{5.23}$$

The right-hand side of Eq. (5.22) is tabulated in mathematical handbooks or exists in the form of a subroutine of a mathematical statistical computer library. Then

$$A_1 \equiv d^{-1}(1 - f_1(s_o)) \tag{5.24}$$

$$A_2 \equiv -d^{-1} \; m_\nu m_{t_b}\left[(1 - M_o(1 - k_{v_o})) - \frac{E[E_r]}{\bar{\bar{e}}_p m_{t_b}}\right] f_2(s_o) \qquad (5.25)$$

$$A_3 \equiv -d^{-1}(m_\tau \; K(1)s_o^c - Tw) \qquad (5.26)$$

where

$$E[E_r] = m_\nu^{-1} \; E[E_{r_A}] \qquad (5.27)$$

We will keep the denominator constant at its long-term mean, d. This results in a very slight distortion of the tails of the frequency of the annual yield, Y_A. For a low frequency, the yield is underestimated because of an underestimated soil moisture. For high frequencies, the reverse happens. However, it must be emphasized that this distortion is almost negligible. Equation (5.19) is an extremely handy relationship allowing for a comparatively simple derivation of the cdf of annual yield, Y_A.

Flowing out of Eq. (5.19), all the terms of the water balance, Eq. (2.10), can now be expressed solely in terms of the remaining two independent random variables of our simplified hydrologic system. Linearizing the evapotranspiration term gives

$$E^*_{p_A} \; J(E, M_o, k_{v_o}) = \bar{e}_p \; m_\nu m_{t_b}\left[(1 - M_o(1 - k_{v_o})) - \frac{E[E_r]}{\bar{e}_p \; m_{t_b}}\right] * J(E,M_o,k_{v_o})$$

$$\simeq \bar{e}_p c_1[J(s_o) + (\bar{s} - s_o) \; J'(s_o)] \; m_\nu m_{t_b} \qquad (5.28)$$

where

$$c_1 = E^*_{p_A} \; / \bar{e}_p \; m_\nu m_{t_b} \qquad (5.29)$$

After substitution of Eq. (5.19), one obtains

$$E^*_{p_A} J(E, M_o, k_{v_o}) + E_{r_A} \simeq E_1 \bar{e}_p P_A + E_2 \bar{e}_p + E_3 \bar{e}_p^2 \tag{5.30}$$

where

$$E_2 \equiv m_\nu m_{t_b} [c_1 J(s_o) + \frac{E[E_r]}{\bar{e}_p m_{t_b}} + c_1 A_3 J'(s_o)] \tag{5.31}$$

$$E_1 \equiv m_\nu m_{t_b} c_1 A_1 J'(s_o) \tag{5.32}$$

and

$$E_3 \equiv m_\nu m_{t_b} c_1 A_2 J'(s_o) \tag{5.33}$$

For groundwater storage, the equivalent manipulations give

$$\Delta S_{g_A} = \kappa_s e^{-T/\kappa} K(1) (e^{m_\tau/\kappa_s} - 1)(\bar{s}^c - s_o^c) = c_2(\bar{s}^c - s_o^c)$$

$$\simeq c_2 c s_o^{c-1}(\bar{s} - s_o) \tag{5.34}$$

where

$$c_2 = \kappa_s e^{-T/\kappa} K(1) (e^{m_\tau/\kappa_s} - 1) \tag{5.35}$$

After substitution of Eq. (5.19), this becomes

$$\Delta S_{g_A} \simeq C_1 P_A + C_2 \bar{e}_p + C_3 \tag{5.36}$$

where

$$C_1 \equiv c_2 c s_o^{c-1} A_1 \tag{5.37}$$

$$C_2 = c_2 c s_o^{c-1} A_2 \tag{5.38}$$

$$C_3 = c_2 c\, s_o^{c-1}\, A_3 \tag{5.39}$$

Similarly, for storage in the unsaturated zone

$$\Delta S_{u_A} = nZ_r(\bar{s} - s_o) = D_1 P_A + D_2 \bar{e}_p + D_3 \tag{5.40}$$

where

$$D_1 \equiv nZ_r\, A_1 \tag{5.41}$$

$$D_2 \equiv nZ_r\, A_2 \tag{5.42}$$

$$D_3 \equiv nZ_r\, A_3 \tag{5.43}$$

After all these simplifications and rearrangement of terms, Equation (2.10) for the annual basin yield, Y_A, can be reformulated.

$$Y_A = P_A[1 - E_1 \bar{e}_p - C_1 - D_1] - \bar{e}_p[E_2 + C_2 + D_2] - \bar{e}_p^2\, E_3 - C_3 - D_3 \tag{5.44}$$

Due to the preceding manipulations, a very expedient approximate formulation for the basin yield is found which is linear with respect to precipitation and of second-order with respect to evapotranspiration.

If the annual average potential rate of evaporation, $\bar{e}_p$, is kept constant as is done in the 'first-order model' and in the model accounting for annual (seasonal) storage, Eq. (5.44) simplifies to a simple linear relationship between two variables.

$$Y_A = a_1 P_A - a_2 \tag{5.45}$$

where

$$a_1 \equiv 1 - E_1 \bar{\bar{e}}_p - C_1 - D_1 \tag{5.46}$$

$$a_2 \equiv \bar{\bar{e}}_p [E_2 + C_2 + D_2] + \bar{\bar{e}}_p^2 E_3 + C_3 + D_3 \tag{5.47}$$

Concluding this chapter, it may be kept in mind that the water balance equations (2.10) and (5.1) are simplified by means of some justifiable approximations. These simplified relationships are presented in Eq. (5.13) and Eq. (5.19) as far as the balance of soil moisture is concerned. The equivalent simplification for the annual basin yield is presented by Eq. (5.44) for variable evapotranspiration and by Eqs. (5.45) to (5.47) for constant evapotranspiration. For extreme events, some accuracy is traded off against a tremendous reduction in the computational burden for determining the cdf of annual yield.

Chapter VI

A RANDOM ANNUAL POTENTIAL EVAPOTRANSPIRATION

Evapotranspiration consists of the conversion to vapor and mixing with the atmosphere of the liquid (or solid) water at the earth-atmosphere boundary; this may be soil moisture, ponded water, water intercepted on surfaces, and water in plants. The potential rate of evaporation, e_p^*, which is determined by various climatic, soil and vegetal parameters of a hydrologic system provides an upper bound to the actual rate of evaporation, e_T. Time integration of this latter rate of moisture movement across the earth-atmosphere boundary gives the accumulated amount of water, $E_{T,\tau}$, 'lost' during the period of integration, τ. That is

$$E_{T,\tau} = \int_0^{\tau} e_T(t)\, dt \qquad (6.1)$$

It is an objective of this work to investigate the sensitivity of the frequency of annual basin yield, Y_A, to a randomly varying potential evapotranspiration, e_p^*. In the first-order model, the coefficient of variation of e_p^* is assumed to be small, and e_p^* is replaced by its long-term seasonal average rate, $\bar{e}_p^*$. Here we will relax this assumption and replace it by the less restrictive assumption of a seasonally constant rate.

Several simplifications are made to maintain analytical tractability while including a second random variable. No attempt is undertaken to vigorously analyze the effect on the first-order model of an instantaneously varying random variable e_p^*, which is a function of space as well. There is no doubt that averaging of the potential rate of evapotranspiration over the season length, τ, filters and hence reduces the variance of the real potential rate, $e_p^*(t)$. When interpreting the results of the following case studies, one must be mindful of this fact.

By selecting an appropriate analytical relationship for a long-term average rate of potential evapotranspiration, however, one has the means of keeping the error comparatively limited. Since all the randomness of the weighted $\bar{e}_p^*$ as defined in Eq. (3.25) comes from its bare soil component $\bar{e}_p$, an analysis of the latter atmospheric variable is conducted.

VI.1 Comparison of Different Methods of Estimating the Potential Evaporation

The evaporation process has two aspects. First, it is part of an energy balance, out of which can come quantitative estimates of water loss from a vegetated soil surface. Second, it is part of a transport process, in which the net upward flux can be estimated when relevant physical measurements over the earth-atmosphere boundary are substituted in appropriate aerodynamic equations. Thus, there are basically two different approaches to the theoretical study of evaporation from a surface.

a) Diffusion method (aerodynamic method) which involves the formulation of the mass transfer process by which vapor is removed from the surface.

b) Energy balance method which keeps account of the energy fluxes occurring across the same surface over some finite time interval.

Also several combinations of these two approaches have been used in order to take advantage of the best features of both.

One of the oldest aerodynamic methods is the one proposed by Dalton. The actual evaporation rate e_T from a free water surface is given

$$e_T = f(u_z)(e_s - e_z) \tag{6.2}$$

where

u_z = windspeed at elevation z above the surface

e_s = saturation vapor pressure corresponding to temperature of the water surface

e_z = vapor pressure of the air at elevation z

A similar relationship was tested by H. E. Jobson (1972) where the windspeed function was specified

$$e_T = N\, u_z(e_s - e_z) \tag{6.3}$$

where

N = mass transfer coefficient

Data collected during a 15-month interval at Lake Hefner near Oklahoma City were analyzed. The results indicated that errors due to averaging meteorologic variables occur which are larger than ±5% about 20% of the time for a 24 hour averaging period. It is concluded that averaging periods of 1 month should be avoided in situations similar to those at Lake Hefner.

Eagleson (1970) mentions that verification of a similar relationship for evaporation from a water surface has failed apparently because of convective instability of the atmosphere.

Tanner (1967) concludes that humidity methods like Eq. (6.2) appear least suited to general use of any methods. In addition to a necessary calibration of certain constants for the windspeed function, vapor pressure measurements are less available than other meteorological data like temperature or cloud amount.

Irrespective of possibly valuable applicability of the aerodynamic method to certain climatological conditions and to determination of evaporation from a body for short periods of time, this approach seems, therefore, to be of little use for the previously stated goals of this research.

The energy balance method applied to a water surface evaporation gives according to Eagleson (1970):

$$e_p(t) = \frac{q_i(t) - q_r(t) - q_b(t) + q_a(t) - q_s(t)}{\rho_e L_e(1 - R)} \qquad (6.4)$$

where

$q_i(t)$ = rate of receipt of short-wave radiation from the sun

$q_r(t)$ = rate of reflection of short-wave radiation

$q_b(t)$ = net rate of long-wave back radiation

$q_a(t)$ = rate of advection of energy by precipitation, surface runoff, percolation, etc.

$q_s(t)$ = rate of increase of energy storage within body

ρ_e = density of evaporated water

L_e = latent heat of vaporization

R = Bowen ratio

Again, Lake Hefner studies [Eagleson, 1970] showed that this method gives good results from book-keeping periods of more than 1 week. To use this method, there must be temperature and humidity measurements at two levels in order to determine the Bowen ratio, B. The precision needed in all the measurements, both for Eq. (6.2) and Eq. (6.4) is great. For research, with the resources of a good physics laboratory behind it, these accuracies are attainable. For general use, however, there is need for something simpler.

Penman first combined the advantages of aerodynamic aspects and energy balance considerations in order to limit the number of independent climatological variables necessary to determine the rate of potential evaporation. Again, Tanner (1967) concludes his comparative study of different methods stating that approaches such as Penman's, which are based on the energy balance, appear most valuable, and have widest applicability of all methods. It is that what makes the combination method most suitable for the scope and purpose of this research.

The "Modified Penman Equation" is, according to Eagleson (1970)

$$e_p(t) = \frac{q_i(t) - q_r(t) - q_b(t) + q_a(t) - q_s(t) + \frac{\gamma}{\Delta} L_e B(e_{s_2} - e_2)}{\rho_e L_e \left(1 + \frac{\Delta}{\gamma}\right)} \qquad (6.5)$$

with

$$\Delta = \frac{e_s - e_{s_z}}{T_s - T_z} \qquad (6.6)$$

where

γ = physical constant

B = turbulent transfer coefficient

$e_{s_2} - e_2$ = vapor-pressure deficit at elevation $z = 2$

e_s = saturation vapor-pressure at surface temperature T_s

T_z = air temperature at elevation z

Various applications of Eq. (6.5) revealed its utility for averaging periods of up to one month [Jensen & Haise, 1967; Linacre, 1967; Lane, 1964]. It was found empirically in these studies that potential pan evaporation is approximately

$$\bar{e}_{p,\tau} = \frac{1}{\tau} \int_0^\tau e_p(t)\, dt \simeq (a + b\bar{T}_{A,\tau})(\bar{q}_{i,\tau} - \bar{q}_{r,\tau}) \qquad (6.7)$$

where

$\bar{T}_{A,\tau}$ = average atmospheric temperature of the period of interest τ

a,b = empirical coefficients which vary from location to location

$q_{i,\tau} - q_{r,\tau}$ = net short-wave radiation averaged over period τ

It seems plausible that for long averaging periods the sensitivity of the averaged rate of potential evaporation to changes in energy storage in the body diminishes. If the time average of the rate of advective energy is neglected, the average potential rate of evaporation may be approximated to the first order by

$$\bar{e}_{p,\tau} = \frac{\bar{q}_{i,\tau} - \bar{q}_{r,\tau} - \bar{q}_{b,\tau} + L_e \, \bar{B}_\tau (\bar{e}_{s_{2,\tau}} - \bar{e}_{2,\tau}) \frac{\gamma}{\Delta}}{\rho_e L_e (1 + \frac{\gamma}{\Delta})} \tag{6.8}$$

where Δ/γ is evaluated at the average temperature $\bar{T}_{A,\tau}$ at elevation Z = 2.

VI.2 Development of a Simple Long-Term Relationship for Potential Evaporation

Eagleson (1977) developed a simplified relationship for the annual average rate of potential evaporation from an arbitrary surface. Making use of two empirical relationships, it is

$$\bar{e}_p \simeq (.42 + .013 \, \bar{T}_A) \frac{\bar{q}_i (1 - A) - \bar{q}_b (1 - C)}{\rho_e L_e} \tag{6.9}$$

with

$$C = .25 + \frac{1}{1-\bar{S}} \tag{6.10}$$

where

$\bar{A}$ = average value for short-wave albedo

$\overline{S}$ = average relative humidity

This Eq. (6.9) fits data fairly well. It is thus adopted as the basic model for a randomly varying annual average rate of potential evaporation. Short-wave radiation from the sun for a cloudy sky is according to Eagleson (1970):

$$\bar{q}_i = \bar{q}_{i,clear}(1 - \overline{N}(1 - \overline{K})) \qquad (6.11)$$

where

$\overline{N}$ = annual average amount of clouds

$\overline{K}$ = fraction of cloudless-sky insolation

Time average net longwave back radiation is approximately given by

$$\bar{q}_b = (1 - .8\overline{N})[.245 - .145 \cdot 10^{-10}(\overline{T}_A + 273)^4] \qquad (6.12)$$

Thus, Eq. (6.9) is formulated in terms of readily available meteorological data. The annual average rate of potential evaporation from an arbitrary surface is basically dependent upon three independent random variables.

$$\bar{e}_p \simeq \bar{e}_p(\overline{T}_A, \overline{N}, \overline{S}, \text{parameters}) \qquad (6.13)$$

A numerical analysis (Appendix B) indicates that the annual average atmospheric temperature, $\overline{T}_A$, is the principal independent random variable determining $\bar{e}_p$. It contributes most to the variance of $\bar{e}_p$. In an approximation, it is further considered the only random variable so that a cumulative distribution function (cdf) for the dependent random variable $\bar{e}_p$ can be readily derived. Although the long-wave back radiation is also dependent upon the temperature $\bar{T}_A$, a numerical analysis would show that within a realistic range of average annual temperatures the long-wave back radiation can be considered a constant. Equation (6.9) now can be recast into a highly simplified expression for $\bar{e}_p$. It is

$$\bar{e}_p \simeq (a_1 + a_2 \bar{T}_A) a_3 \qquad (6.14)$$

where

a_1, a_2 = empirical constants with general applicability for the present purpose

a_3 = constant reflecting a great many significant parameters of a catchment (Eq. 6.15)

Besides the independent random variable $\bar{T}_A$, there are five catchment parameters which are available climatologic data for most catchments

$$a_3 = f(\bar{\bar{N}}, \bar{\bar{S}}, \bar{\bar{A}}, \bar{\bar{K}}, \Phi) \qquad (6.15)$$

where

Φ = local latitude, determining clear sky insolation, $\bar{q}_i$.

Equation (6.14) strongly resembles an empirical formula for the potential rate of evaporation based on temperature measurements.

Tanner's study (1967) indicates that empirical temperature methods are best suited to monthly estimates and are not reliable for short-period estimates. They are also suited for annual estimates of $\bar{e}_p$ unlike any other method. In addition, Eq. (6.14) exploits a large set of easily attainable climatologic data. An actual calibration of the constants a_i is not necessary as opposed to purely empirical relationships.

VI.3 Probability Density Function of the Annual Average Rate of Potential Evaporation

Once a distribution is adopted for the annual average atmospheric temperature, $\bar{T}_A$, an approximate distribution for the annual average rate of potential evaporation is readily derived. The approximately linear relationship displayed in Eq. (6.14) allows for a simple derivation.

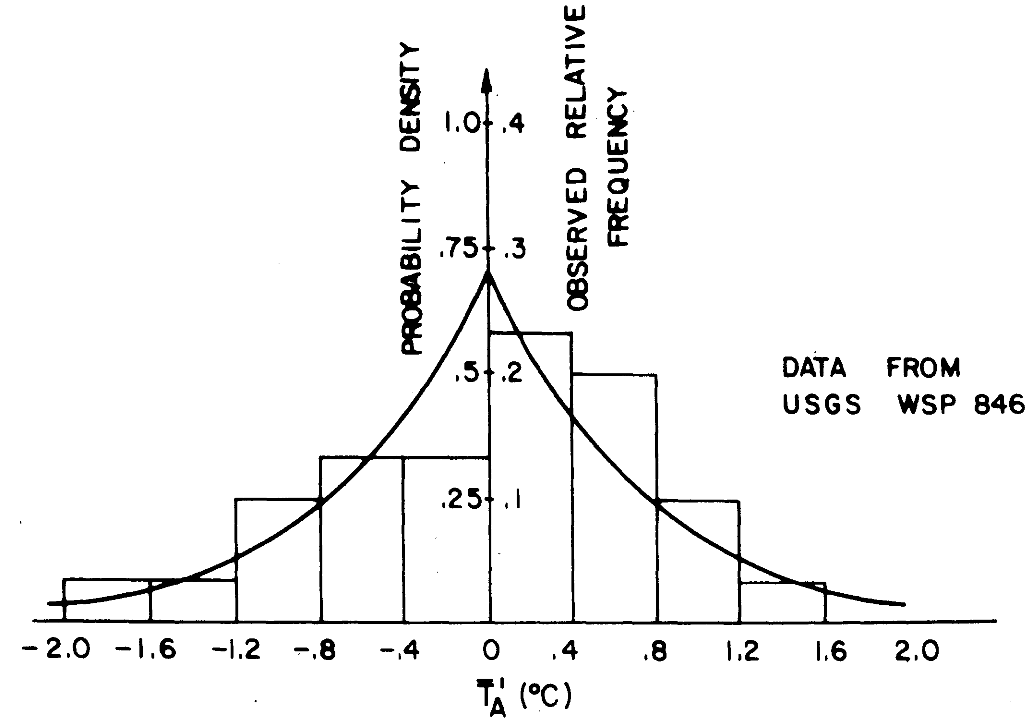
$$f_{e_p}(\bar{e}_p) = \left| \left(\frac{d\bar{e}_p}{d\bar{T}_A} \right)^{-1} \right| f_{\bar{T}_A}(g^{-1}(\bar{e}_p)) \qquad (6.16)$$

with

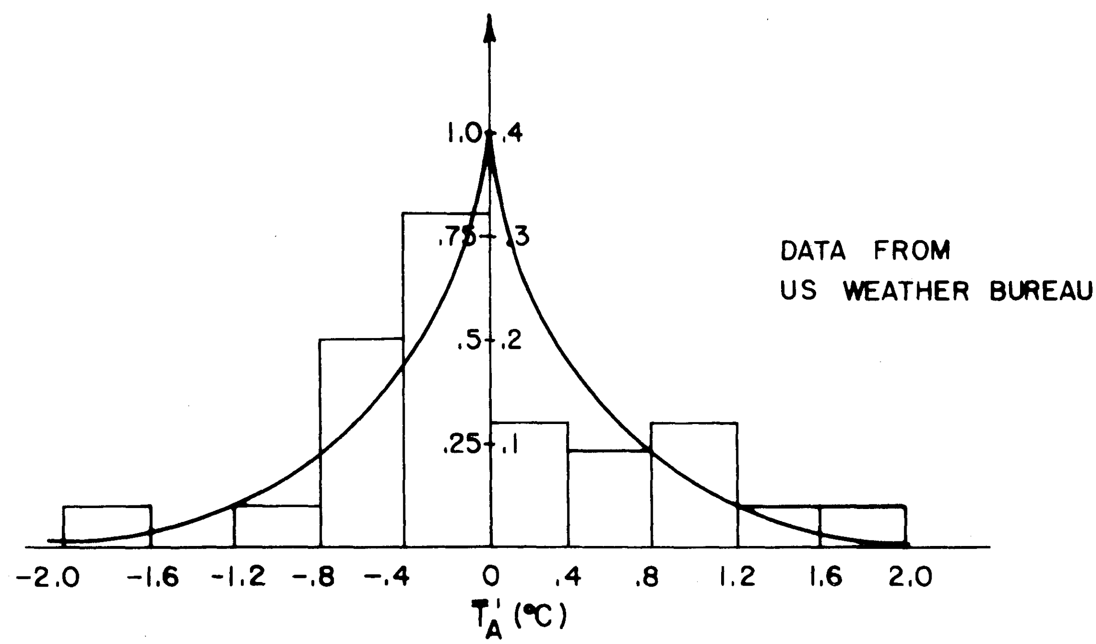
$$g^{-1}(\bar{e}_p) = \bar{T}_A = (a_3^{-1}\, \bar{e}_p - a_1)/a_2 \qquad (6.17)$$

The implication of Eq. (6.16) and Eq. (6.17) is that the approximate pdf of $\bar{e}_p$ has the same shape as the pdf of $\bar{T}_A$, only stretched and shifted.

A double exponential distribution for $\bar{T}_A$ is adopted simply as a convenient representation of the phenomenon on the basis of observed data (Figure 6.1) and the analytical tractability of the



a. CLINTON, MASSACHUSETTS, 1904 - 1933



b. SANTA BARBARA, CALIFORNIA 1946-1972

Figure 6.1
DISTRIBUTION OF FLUCTUATIONS OF ANNUAL AVERAGE TEMPERATURE AT CLINTON, MASS. (a) AND SANTA BARBARA, CALIFORNIA (b)

exponential function. Thus,

$$f_{T'_A}(\bar{T}'_A) = \frac{1}{2}\lambda_T e^{-\lambda_T|\bar{T}_A - \bar{\bar{T}}_A|} \tag{6.18}$$

where

λ_T = parameter of exponential distribution

$\bar{\bar{T}}_A$ = expected value of the annual average temperature $\bar{T}_A$

$\bar{T}'_A$ = fluctuation of annual average temperature, $\bar{T}_A - \bar{\bar{T}}_A$

Equation (6.18) combined with Eq. (6.16) gives the desired pdf for $\bar{e}_p$ in an approximate manner:

$$f_{e'_p}(\bar{e}'_p) = \frac{\lambda_T}{2a_2a_3} e^{-\frac{\lambda_T}{a_2a_3}|\bar{e}_p - \bar{\bar{e}}_p|} \tag{6.19}$$

where

$\bar{\bar{e}}_p$ = expected value of the annual average rate of potential evaporation

Since negative rates of evaporation are physically impossible, a truncation of the pdf as presented in Eq. (6.19) has to be performed. One obtains finally

$$
f_{\bar{e}_p}(\bar{e}_p) = \begin{cases} 0 & \text{as } \bar{e}_p < \bar{e}_{p,min} \\ \frac{1}{2}\, e^{-\frac{\lambda_T}{a_2a_3}\left|\bar{e}_{p,min} - \bar{\bar{e}}_p\right|}\left(1 + \frac{\lambda_T}{a_2a_3}\right) & \text{as } \bar{e}_p = \bar{e}_{p,min} \\ \frac{1}{2}\, \frac{\lambda_T}{a_2a_3}\, e^{-\frac{\lambda_T}{a_2a_3}\left|\bar{e}_p - \bar{\bar{e}}_p\right|} & \text{as } \bar{e}_{p,min} < \bar{e}_p < \bar{e}_{p,max} \\ \frac{1}{2}\, e^{-\frac{\lambda_T}{a_2a_3}\left|\bar{e}_{p,max} - \bar{\bar{e}}_p\right|}\left(1 + \frac{\lambda_T}{a_2a_3}\right) & \text{as } \bar{e}_p = \bar{e}_{p,max} \\ 0 & \text{as } \bar{e}_p > \bar{e}_{p,max} \end{cases} \tag{6.20}
$$

It has to be noted that all the preceding simplifications have been made in order to arrive at an analytically derived probability density function for the annual average rate of potential evaporation, $\bar{e}_p$. There is no doubt that a more accurate yet more complicated deterministic relationship for $\bar{e}_p$ could be found through rigorous analysis of short-term evaporative processes. Since we are interested in the approximate effect of a second random variable $\bar{e}_p$ on the frequency of the annual basin yield, Y_A, the simplifications made here appear to be justified.

VI.4 Derivation of a cdf for Annual Basin Yield

In Chapter V, a relationship, Eq. (5.44), for the annual basin yield, Y_A, was found which is formulated in terms of two independent random variables. Since there probably is some negative correlation between those two random variables, P_A and $\bar{e}_p$, a conditional probability density function has to be applied in order to evaluate the jointly random behavior of P_A and $\bar{e}_p$. It is

$$f_{X,Y}(x,y) = f_{X|Y}(x,y)\ f_Y(y) = f_{Y|X}(y,x)\ f_X(x) \qquad (6.21)$$

where

$f_X(x)$ = marginal probability density function of x

$f_{X|Y}(x,y)$ = conditional probability density function of x given y

$f_{X,Y}(x,y)$ = joint probability density function of x and y

Unfortunately, a conditional pdf is not available for either of the independent random variables P_A and $\bar{e}_p$. Thus, statistical independence between precipitation and potential evaporation has to be assumed in order to force any existing covariance, $COV[P_A, \bar{e}_p]$, between P_A and $\bar{e}_p$ to zero. Considering this simplification, which introduces some error in further calculations, the joint pdf can be approximated by the marginal pdf's, Eq. (3.6) and Eq. (6.20). Integration of the volume under this joint pdf finally gives the desired joint cumulative distribution function for the annual basin yield, Y_A. It is

$$\text{Prob}[Y_A \leq x] = \iint_{R(x)} f_{P_A}(P_A)\ f_{\bar{e}_p}(\bar{e}_p)\ dP_A\ d\bar{e}_p \qquad (6.22)$$

with an integration area R(x) defined by Eq. (5.44) and x equal to a certain value of Y_A. Rewritten, Eq. (6.22) becomes

$$\text{Prob}[Y_A \leq x] = \int_{\bar{e}_{p,min}}^{\bar{e}_{p,max}} \left[\int_{o}^{P_A[\bar{e}_p, x]} f_{P_A,\bar{e}_p}(P_A, \bar{e}_p)\ dP_A \right] d\bar{e}_p \qquad (6.23)$$

The integration of Eq. (6.23) is performed in Appendix B.2 together with some analysis of the behavior of the joint cdf for Y_A. The solution to Eq. (6.23) is

$$\mathrm{Prob}\left[\frac{Y_A}{m_{P_A}} < z\right] = e^{-\omega m_\tau} \left\{ 1 + A \sum_{\nu=1}^{\infty} \frac{(\omega m_\tau)^\nu}{\nu!} [P[\nu\kappa, f(z, \bar{e}_{p,max})] + P[\kappa\nu, f(z, \bar{e}_{p,min})]] + B \sum_{\nu=1}^{\infty} \frac{(\omega m_\tau)^\nu}{\nu!} * I(z, \bar{e}_p) \right\} \tag{6.24}$$

with

$$f(z, \bar{e}_p) = \frac{\eta\kappa}{m_{P_A}} P_A(x, \bar{e}_p, \text{parameter}) \tag{6.25}$$

and

$$I(z, \bar{e}_p) = \int_{\bar{e}_{p,min}}^{\bar{e}_{p,max}} e^{-\frac{\lambda_T}{a_2 a_3}|\bar{e}_p - \bar{\bar{e}}_p|} P[\nu\kappa, f(z, \bar{e}_p)]\, d\bar{e}_p \tag{6.26}$$

where

$P[a, y]$ = Pearson's incomplete Gamma function equal to $\gamma[a,y]/\Gamma(a)$

The constants A and B are given by

$$B = \frac{\lambda_T}{2a_2 a_3} \tag{6.27}$$

$$A = \frac{1}{2} e^{-2B|\bar{e}_{p,max} - \bar{\bar{e}}_p|} \tag{6.28}$$

The function $P_A(z, \bar{e}_p)$ is readily obtained from Eq. (5.44). It is

$$P_A = [z\, m_{P_A} + \bar{e}_p^2 E_3 + C_3 + D_3 + \bar{e}_p [E_2 + C_2 + D_2]]\ [1 - E_1 \bar{e}_p - C_1 - D_1]^{-1} \tag{6.29}$$

with

$$z = \frac{x}{m_{P_A}} \tag{6.30}$$

where x is equal to a certain value of Y_A for which the cumulative frequency of occurrence is to be determined. The integral $I(z, \bar{e}_p)$ has to be evaluated numerically.

Chapter VII

CASE STUDIES

The 'first-order water balance' model [Eagleson, 1978] has been verified for two catchments in contrasting climates. A sub-humid climate is represented by Clinton, Massachusetts. Storm observations of nearby Boston serve to define the cdf of annual point precipitation, P_A, for Clinton (see Figure 3.2). The derived frequency of annual basin yield, Y_A, is shown in Fig. 3.9 for a silty-loam soil and a sub-optimal vegetal cover.

A semi-arid climate is represented by Santa Paula, California. Storm observations together with a derived cdf of P_A/m_{P_A} are given in the literature [Eagleson]. The corresponding frequency of annual basin yield at Santa Paula is presented in Fig. 3.10.

Evaluation of the modifications to the above model developed in this work will now be conducted for the same two catchments. Additional streamflow data for determining the groundwater reservoir coefficient, κ_s, are analyzed as are additional climatological data for the incorporation of a random potential evapotranspiration in the model. A complete set of the independent parameters of the modified model accounting for annual storage and a random potential evapotranspiration is listed in Appendix C. Since no direct observations of the vegetal parameter, Z_r, were available estimations based upon the reported physiology of the vegetation are used. For Clinton, with its perennial vegetation, the depth of the root system is estimated to be of the order of

Z_r = 1 m. For Santa Paula with its significant percentage of annual grasses, the depth of the root zone is assumed to be Z_r = 0.5 m.

For the purpose of comparison, the first-order model is applied to four representative soils at both Clinton and Santa Paula, as was done before in the literature [Eagleson, 1978]. The resulting frequency curves of annual basin yield are shown in Figures 7.1 and 7.2, respectively. In the Santa Paula case, all derived curves in this work are for $\kappa = 0.25$ resulting from visual best fit.

In both climates, silty-loam appears to best represent the actual soil properties. It is that soil which allows the climate-soil-vegetation system to reach a maximum optimal biomass productivity, $M_o k_{v_o}$. Maximum biomass production seems to be related to maximum water use of the system through vegetal transpiration or, in other words, minimum basin yield.

In Chapter IV of this research, an analysis of storage mechanisms is conducted. It leads to the incorporation in the first-order water balance, Eq. (2.9), of the additional terms, Eq. (2.10), representing annual changes in storage in the unsaturated zone, ΔS_{u_A}, and in the saturated zone, ΔS_{g_A}, of an idealized soil column. The effect of annual storage on the frequency of annual yield is demonstrated for Clinton and Santa Paula in Figures 7.3 and 7.4, respectively. Each of the four soils responds slightly differently to this modification of the original model. The overall pattern of sensitivity of the cdf of normalized annual yield, Y_A/m_{P_A}, however, stays the same for all the soils. Due to annual change in storage, a rotation of the individual frequency curves around a

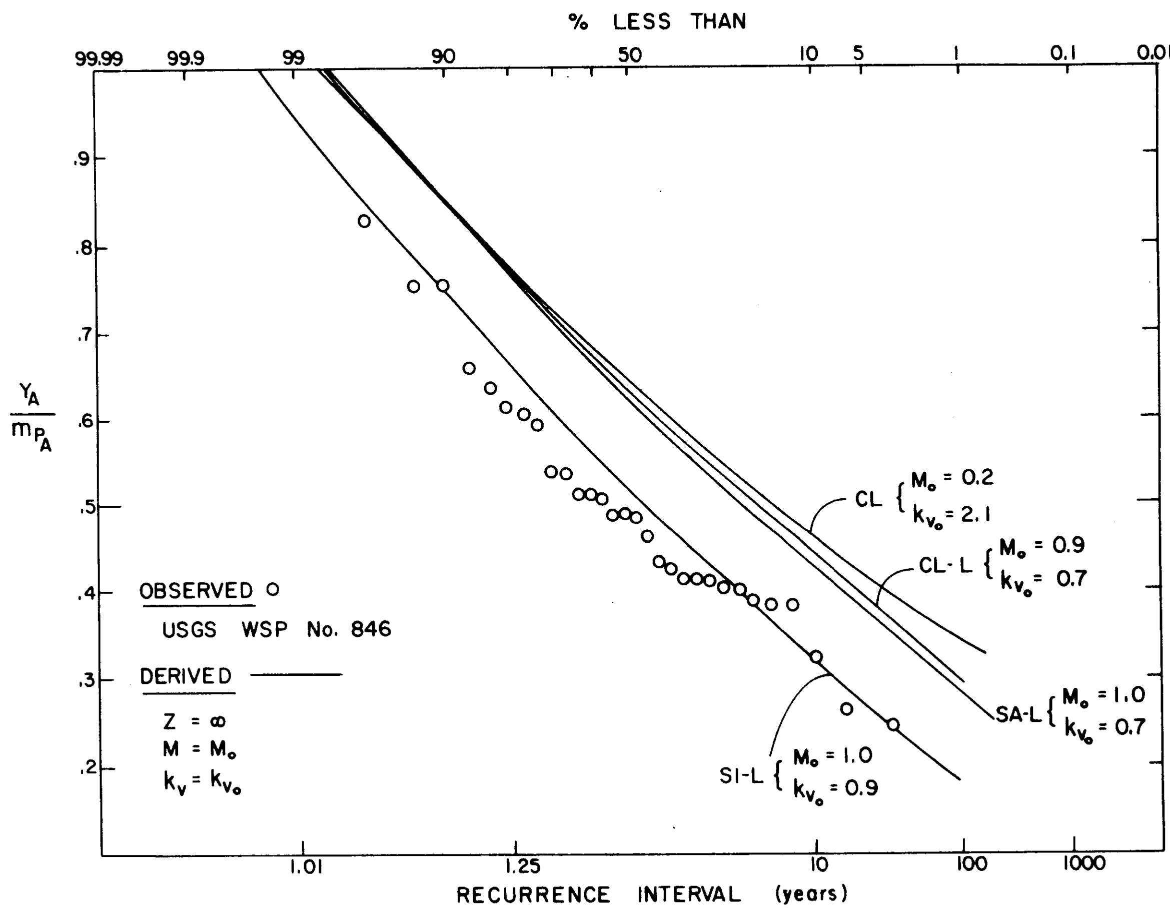


Figure 7.1

FREQUENCY OF ANNUAL BASIN YIELD WITH EQUILIBRIUM VEGETAL COVER (SOUTHERN BRANCH OF THE NASHUA RIVER AT CLINTON, MASS.; A = 280 km^2)

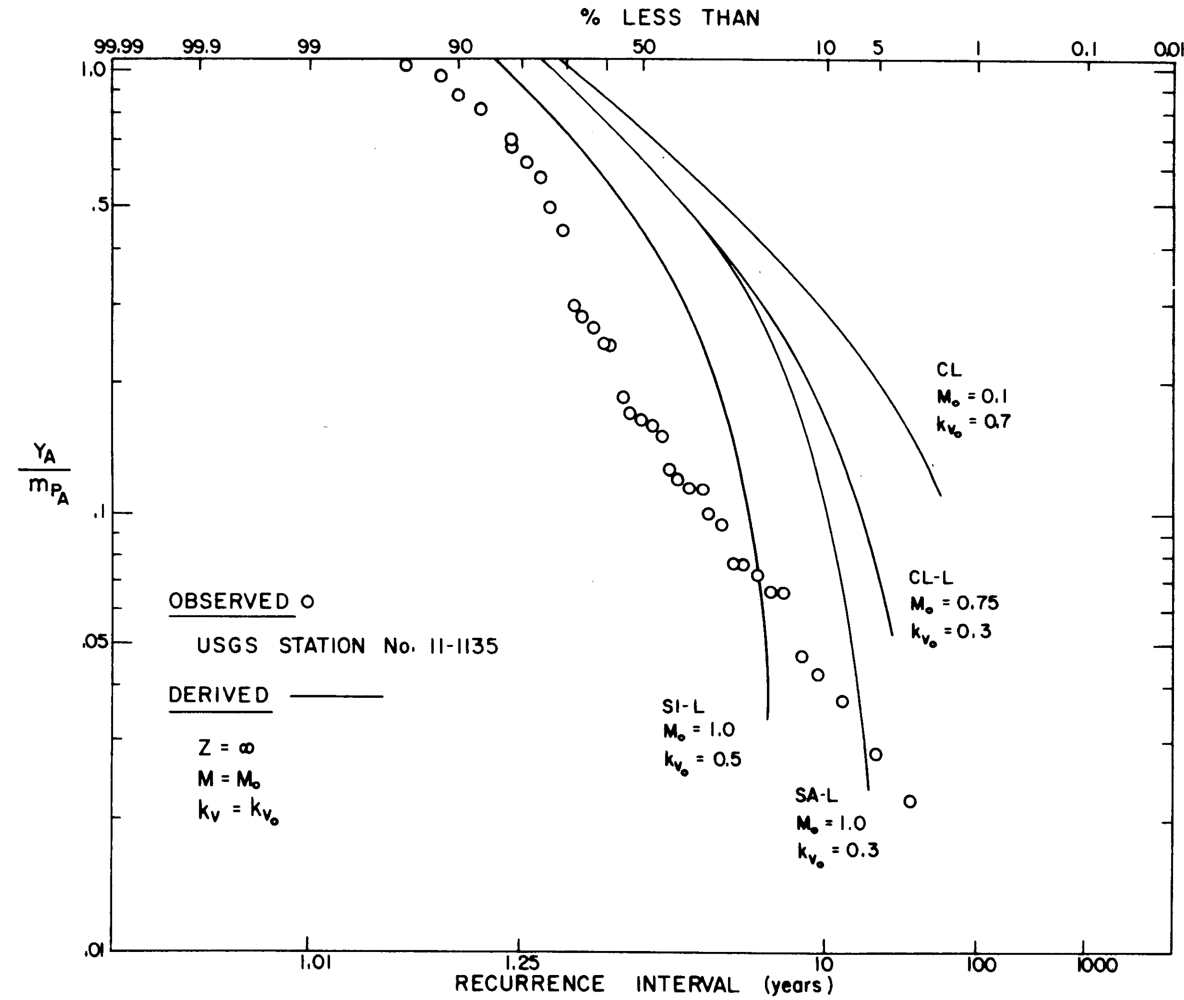


Figure 7.2

FREQUENCY OF ANNUAL BASIN YIELD WITH EQUILIBRIUM VEGETAL COVER; $\kappa = 0.25$
(SANTA PAULA CREEK NEAR SANTA PAULA, CALIFORNIA; A = 104 km^2)

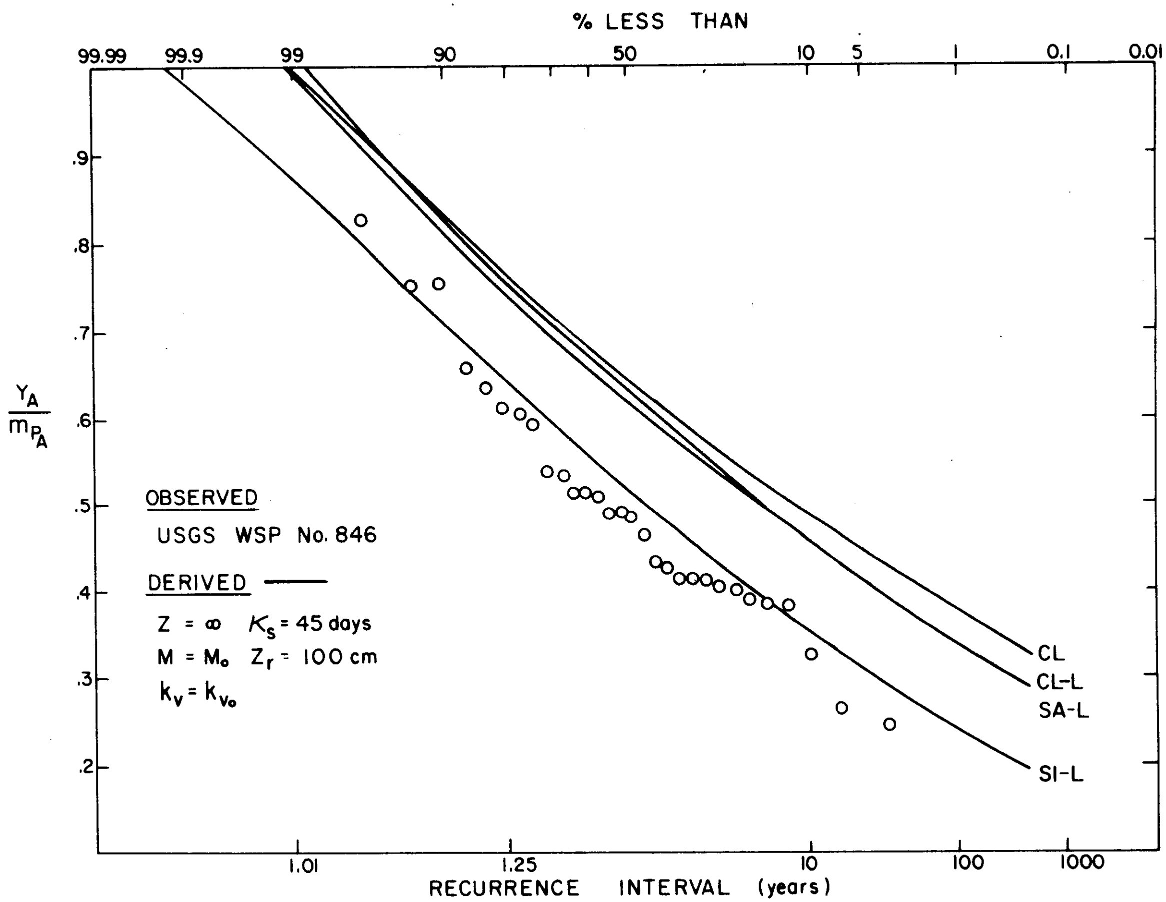


Figure 7.3

FREQUENCY OF ANNUAL BASIN YIELD WITH EQUILIBRIUM VEGETAL COVER AND ANNUAL CHANGE IN STORAGE
(CLINTON, MASS.)

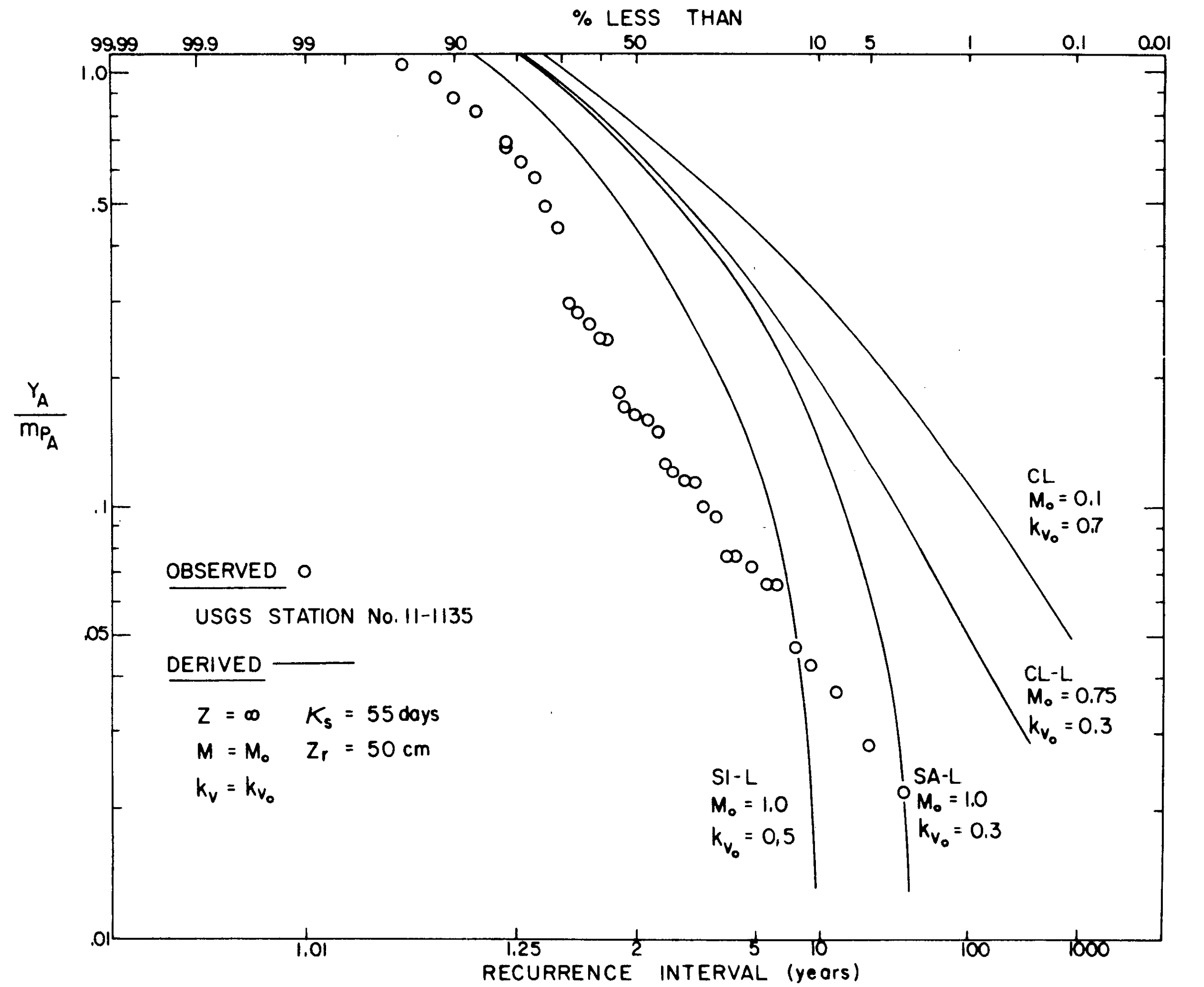


Figure 7.4

FREQUENCY OF ANNUAL BASIN YIELD WITH EQUILIBRIUM VEGETAL COVER AND ANNUAL CHANGE IN STORAGE (SANTA PAULA, CALIF.)

particular point occurs. This point, by definition of the storage mechanisms, coincides with the cumulative probability, Prob[$Y_A/m_{P_A} \leq m_{Y_A}/m_{P_A}$], of the expected normalized annual basin yield, m_{Y_A}/m_{P_A}. For Clinton, that probability is approximately 52% for m_{Y_A}/m_{P_A} depending on the set of soil properties. For Santa Paula, the centerpoint of the rotation of the frequency curves is at about 56%. This rotation of the cdf of annual yield implies that a reduction in yield variance is generated by the additional storage terms. This modification of the curves confirms the analytical argumentation concluding Chapter IV. For annual precipitation exceeding its expected value, m_{P_A}, a replenishment of the two storages of the system occurs. The high frequencies of annual yield correspond to a value of Y_A/m_{P_A}, which is lower in the case of annual storage. Accountable for this relative reduction in annual yield is a growth of both storage volumes. It can be seen from Figure 4.5 that for P_A/m_{P_A} greater than unity, the available water is partitioned into five physically different quantities. There is surface runoff, R_{s_A}, and groundwater runoff, R_{g_A}, adding up to basin yield, Y_A, and there is evapotranspiration, E_{T_A}, and a total change in storage $\Sigma\Delta S_A$. According to the first-order model, the same quantity of water is divided up into only three different components, as is shown in Figure 4.4. That is why annual yield, accounting for storage, is reduced from that of the first-order model which omits storage.

Annual precipitation less than the expected value, m_{P_A}, causes a depletion of the storages. The supply of water is not sufficient to maintain the expected level of storage in either zone. A smaller

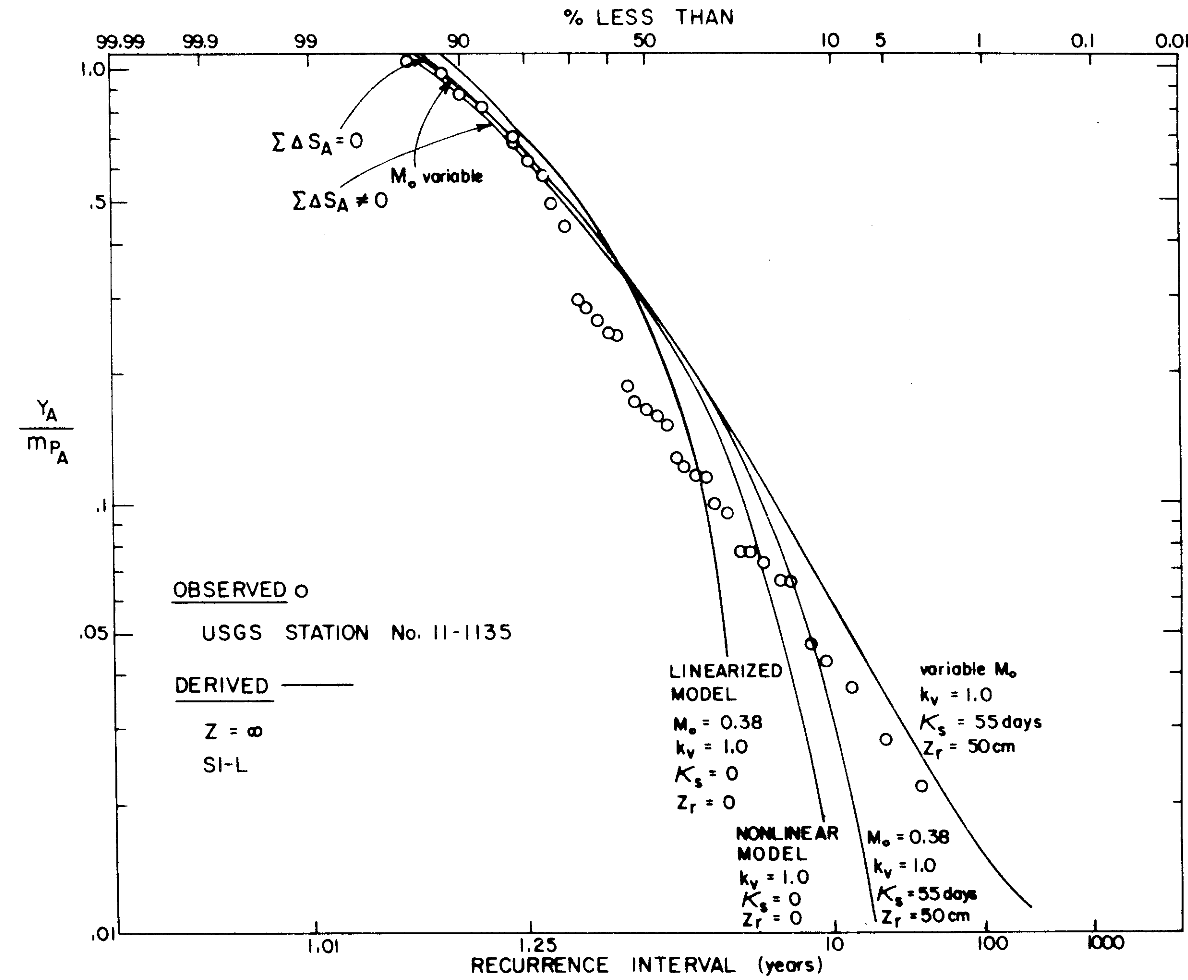


Figure 7.5

FREQUENCY OF ANNUAL BASIN YIELD WITH DIFFERENT SUB-OPTIMAL VEGETAL COVERS.
COMPARISON OF NONLINEAR WITH LINEARIZED MODEL. EFFECT OF ANNUAL CHANGE IN STORAGE.
(SANTA PAULA, CALIF.)

average concentration of soil moisture, $\bar{s}$, produces a smaller average rate of percolation down to the groundwater table and consequently the storage level drops. Depletion of the storages adds to the annual yield which occurs when storage effects are neglected. Thus, the low frequencies of annual yield, when accounting for annual change in storages, correspond to a higher value of Y_A/m_{P_A}, as compared to the first-order model. Again, this difference in behavior of the system can be visualized by comparing Figures 4.4 and 4.5.

The sensitivity of the frequency of annual basin yield to the storage parameters, κ_s and Z_r, is demonstrated in the following: First, it should be noted that in the case of Santa Paula, the vegetal system may not have reached equilibrium due to limitations by nutrients, light or some other ecological factor. Therefore, sub-optimal M-k_v combinations are tested. The literature [Eagleson, 1978] indicates that the sub-optimal condition $M_o = 0.38$ and $k_v = 1.0$ provides the best fit of any combination with a constant optimal canopy density, M_o. Two curves in Figure 7.5 represent that best sub-optimal condition both with and without annual storage. The models are in reasonable agreement with the streamflow data considering the discrepancy of the cdf's for optimal vegetal conditions in Figure 7.4.

Since in arid climates the canopy density is expected to change from year to year in response to fluctuations in precipitation, the system is assumed to approximately reach a new growth equilibrium each year. Accounting for a variable optimal canopy density, $M_o = M_o(P_A/m_{P_A})$, and incorporating annual storage gives a surprising agreement of model prediction and observation. This comparison is also shown in Figure 7.5.

Both for Clinton and Santa Paula, silty-loam evolves as the best fitting soil. Using this soil, optimal vegetal conditions for Clinton and a variable M_o for Santa Paula, the sensitivity of the water balance model to either one of the storage terms is displayed in Figures 7.6 and 7.7. The curves corresponding to no storage, $\Sigma\Delta S_A = 0$, and to two distinct storages, span a range which embraces possible combinations of κ_s and Z_r between zero and the ones chosen. It has to be emphasized that Z_r is just an estimation and thus may differ from the actual average depth of the root zone of the catchments. The Figures 7.6 and 7.7 provide a visualization of the effect of the two storage parameters on the frequency of the annual yield. The figures indicate that accounting for annual storage may improve the model's accuracy slightly.

Another objective of this work is to check the assumtpion [Eagleson, 1978] of only a minor contribution to the variance of annual basin yield from a randomly varying rate of potential evaporation, $\bar{e}_p$. It has been pointed out in Chapter V that incorporation of a second random variable (in addition to annual precipitation) in the water balance necessitates elimination, from Eq. (2.10), of the state variable, $\bar{s}$. In order to accomplish this, the highly nonlinear soil moisture balance, Eq. (4.8), must be simplified to reach an explicit formulation for the state variable, $\bar{s}$, in terms of the independent random variables, P_A and $\bar{e}_p$, and of the system parameters. Figure 5.1 indicated that a formulation, $P_A = f(\bar{s})$, which has defined second derivatives with respect to $\bar{s}$ might reflect the curvetures of the functions with sufficient

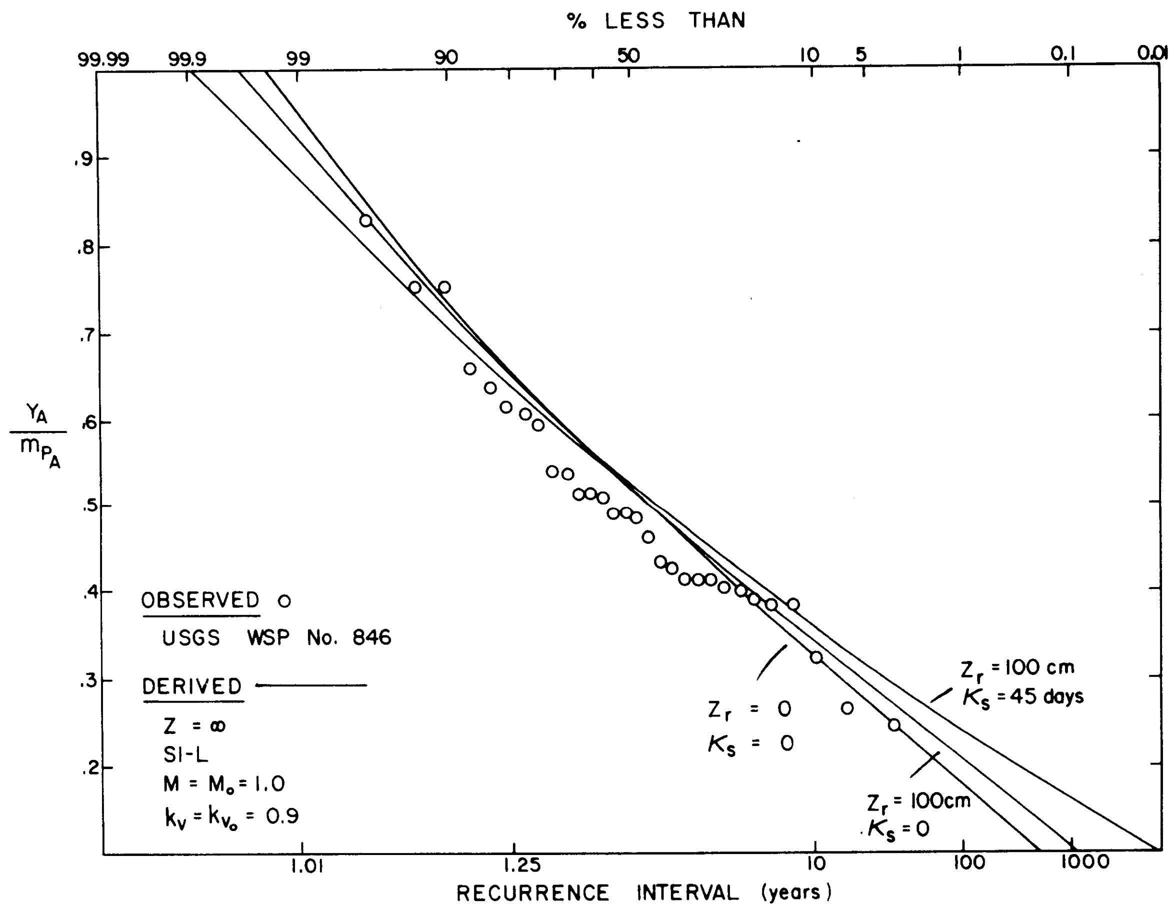


Figure 7.6

SENSITIVITY OF THE FREQUENCY OF ANNUAL BASIN YIELD (OPTIMAL VEGETAL COVER) TO THE STORAGE PARAMETERS (CLINTON, MASS.)

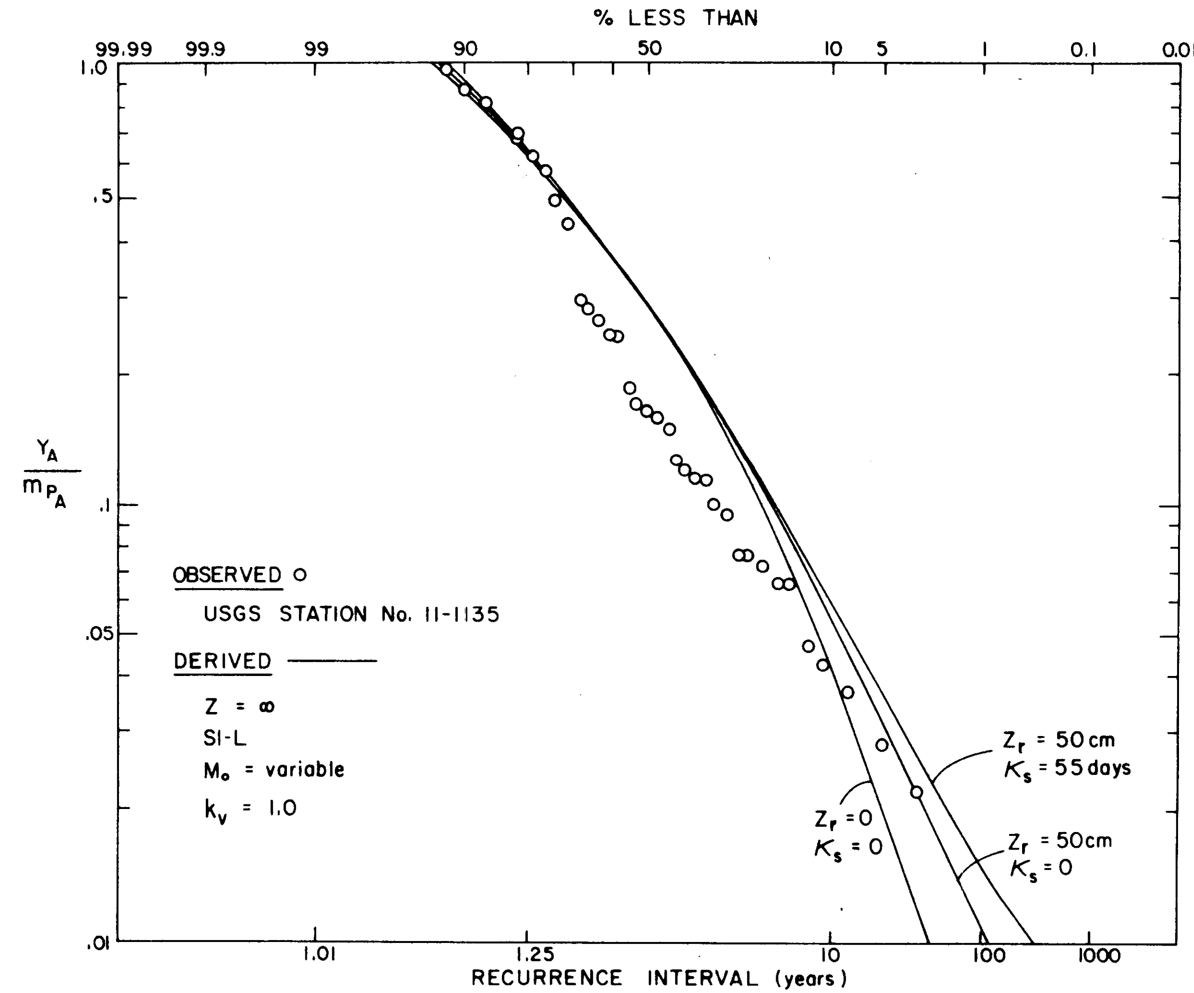


Figure 7.7

SENSITIVITY OF THE FREQUENCY OF ANNUAL BASIN YIELD (VARIABLE M_o) TO THE STORAGE PARAMETERS (SANTA PAULA, CALIF.)

accuracy, at least within the common range, R, of soil moisture, to facilitate the analytical derivation of a cdf for annual yield from distributions of two independent variables. Therefore, a linear relationship, Eq. (5.19), was developed. For Santa Paula, where the evapotranspiration, E_{T_A}, is sensitive to changes in P_A, a linearization could be expected to introduce some error in a derived cdf of annual yield.

Figures 7.8 and 7.9 for Clinton and Santa Paula, respectively, demonstrate the effect on the frequency of annual yield of a linearized water balance model. For Clinton, the result agrees remarkably with that of the original nonlinear model supplemented by annual storage. The reason for this is that under humid conditions E_{T_A} is insensitive to P_A. Thus, Eq. (2.10) gives a nearly linear function.

Under more arid conditions like those at Santa Paula, linearization of the water balance equations with respect to soil moisture causes distortions of the tails of the frequency curves. The lower tails of the curves in Figure 7.9 fall off sharply compared with those of Figure 7.4. It is the lower frequencies which correspond to significant curvatures in Figure 5.1 for ratios of $\bar{s}/s_o$ smaller than unity. High frequencies in Figure 7.9, however, are well represented because of the absence of considerable curvature in Figure 5.1 for ratios of $\bar{s}/s_o$ greater than unity.

A clearer demonstration of the comparatively poor performance of a linearization under arid conditions is offered by Figure 7.5. For the sub-optimal conditions $M_o = 0.38$ and $k_v = 1.0$, linearized and nonlinear models are compared. Storage in both cases is not accounted for.

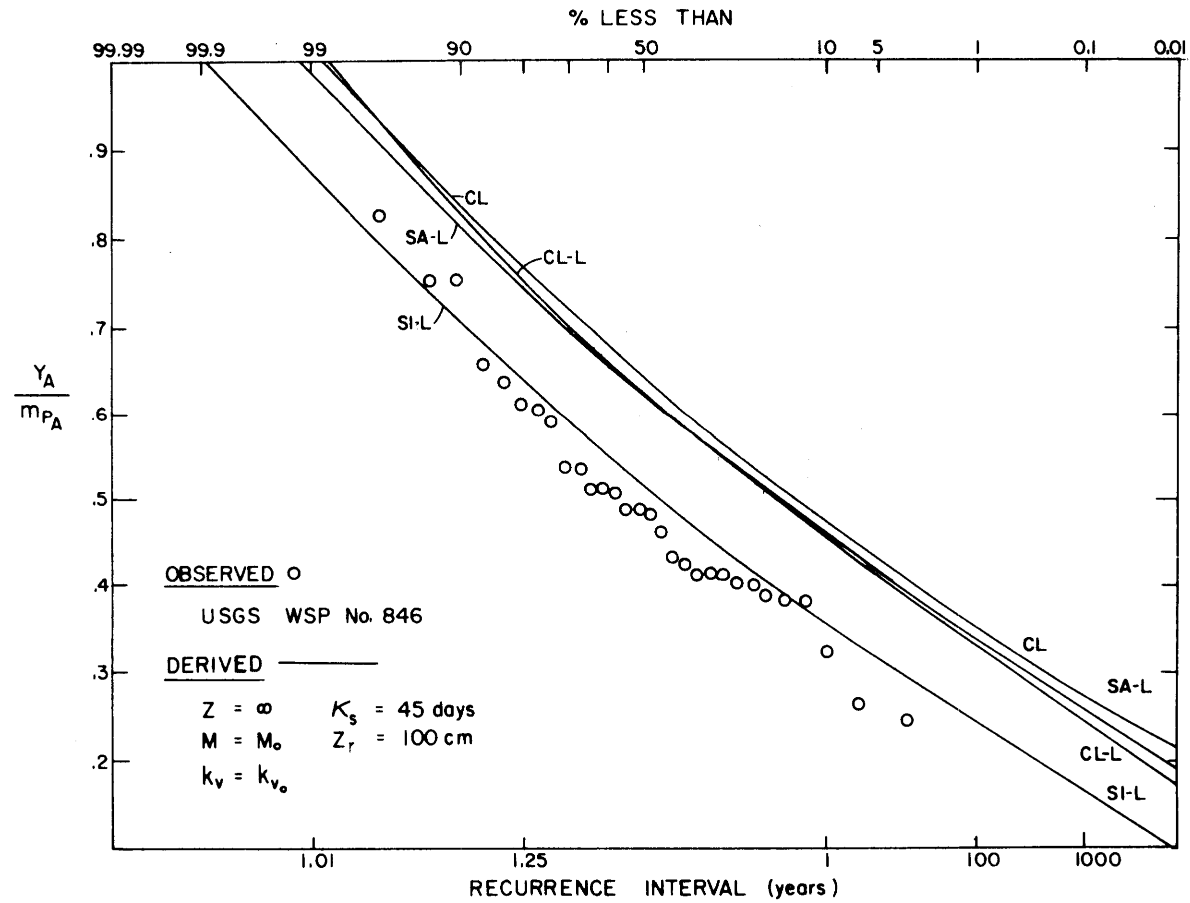


Figure 7.8

FREQUENCY OF ANNUAL BASIN YIELD WITH EQUILIBRIUM VEGETAL COVER ACCORDING TO THE LINEARIZED WATER BALANCE MODEL (CLINTON, MASS.)

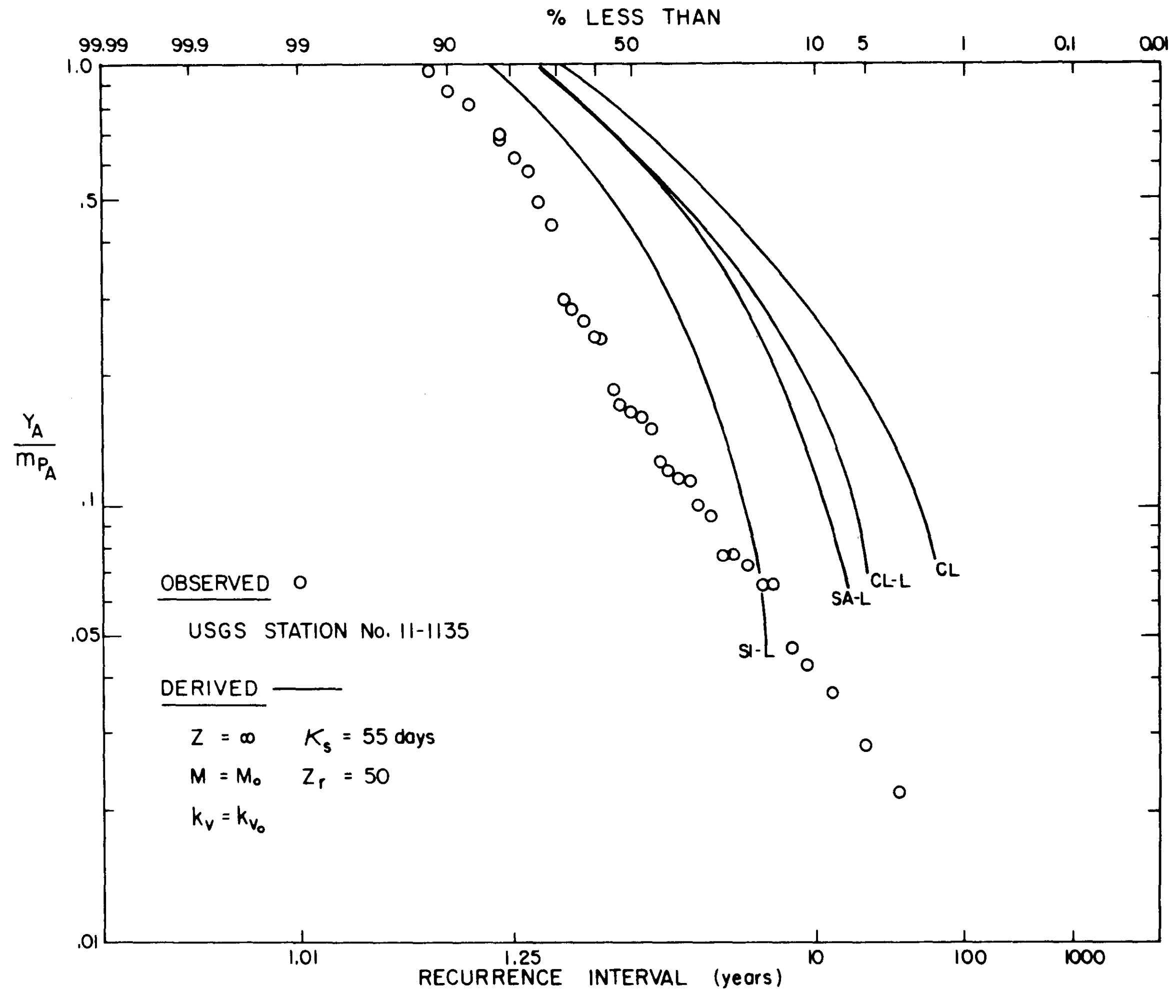


Figure 7.9

FREQUENCY OF ANNUAL BASIN YIELD WITH EQUILIBRIUM VEGETAL COVER ACCORDING TO THE LINEARIZED WATER BALANCE MODEL (SANTA PAULA, CALIF.)

It has been pointed out in Chapter IV that optimal vegetal conditions exaggerate the nonlinearity between soil moisture and precipitation (thus yield). Figure 7.10 clearly shows that the curvatures of $P_A/m_{P_A} = f(\bar{s}/s_o)$ for a variable $M_o = M_o(\bar{s})$ are considerably smaller than those for a constant optimal $M_o \neq M_o(\bar{s})$. The linear model developed is not capable, however, of handling a variable optimal canopy density. As mentioned earlier, the common range of $\bar{s}$ shrinks significantly if sub-optimal (variable M_o) conditions are accounted for.

For optimal biomass production, the corresponding range, R, as defined previously, is bounded to the left by $\bar{s}/s_o = 0.0$ (because probabilities for Y_A/m_{P_A} of less than 15% are never reached for silty-loam, see Figure 7.2) and reaches far to the right of $\bar{s}/s_o = 1.0$. For variable $M_o = M_o(\bar{s})$, the realistic range of soil moisture is rather limited. Curvatures of the graph within that range appear to be representable by a second derivative. It, therefore, can be expected that Eq. (5.13) rather than Eq. (5.19) would provide a reasonable approximation of $\bar{s} = f(P_A, \bar{e}_p, \text{parameters})$ for arid climates.

Use of this approximation for elimination of soil moisture from the expression for annual yield, Eq. (2.10), might provide a reasonably accurate formulation in the case of annual basin yield, Y_A, which is defined by two independent random variables.

Since the soil moisture must be eliminated from Eq. (2.10), if more than one independent random variable is to be accounted for, a linearization was chosen to facilitate computations. In order to analyze the sensitivity of the frequency of annual yield to a second

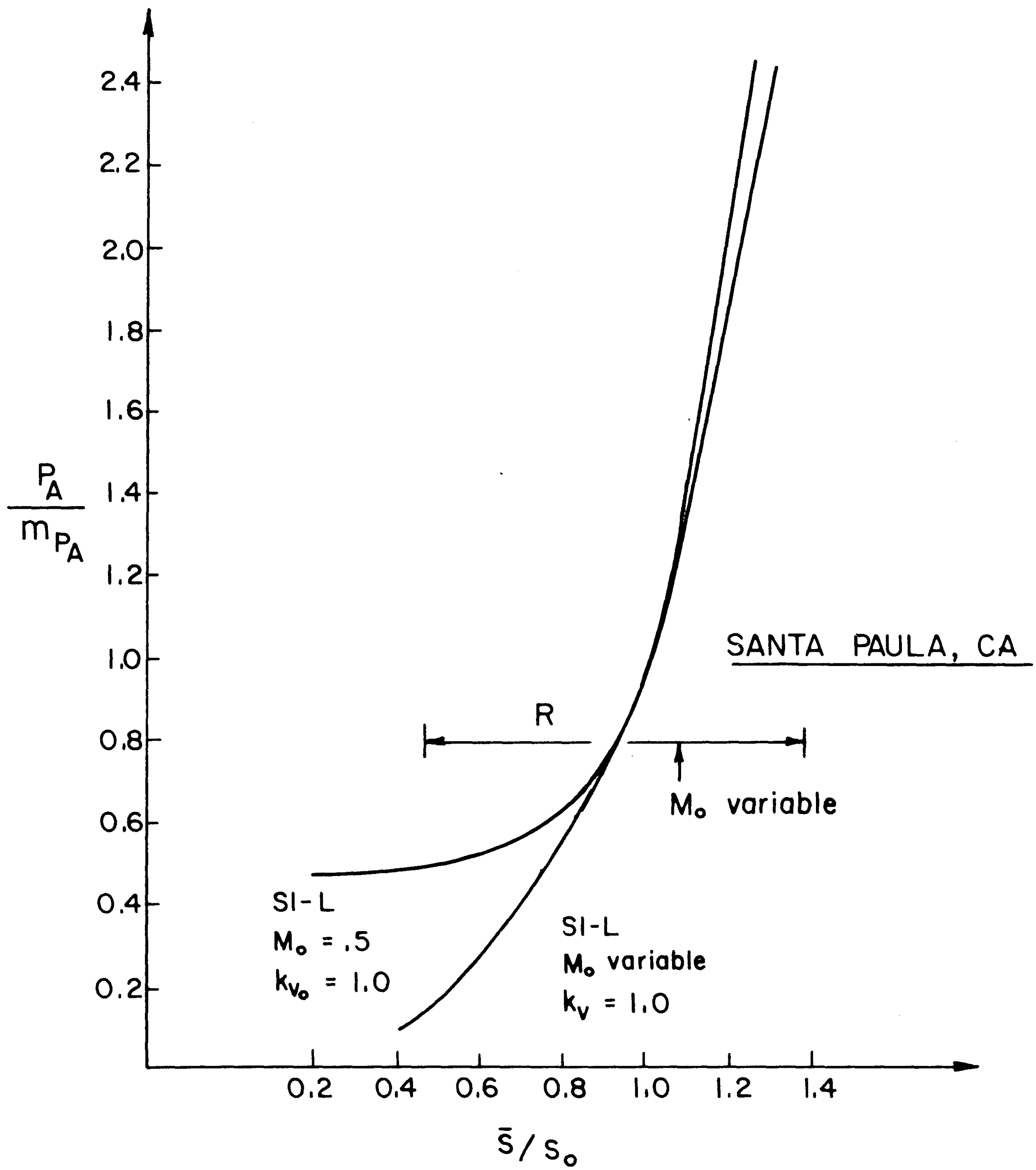


Figure 7.10

RANGE OF VARIATION OF SOIL MOISTURE WITH SUB-OPTIMAL VEGETAL COVER (SANTA PAULA, CALIF.)

independent variable, it may not matter too much how accurate the sensitivity of the same variable with respect to P_A is simulated. It is rather the difference between the cdf for a constant $\bar{e}_p$ and that for a randomly varying $\bar{e}_p$ which is of interest in order to assess the sensitivity to $\bar{e}_p$. Linearization of the water balance equations with respect to soil moisture certainly decreases the variance of Y_A due to $\bar{e}_p$ to a degree. The linearized equation for annual yield, Eq. (5.44), retains some nonlinearity with respect to $\bar{e}_p$, however.

From a computer program (Appendix D), which is set up to handle two independent random variables in a linearized water balance, a surprising result is obtained. The cdf's of Y_A for both the sub-humid climate at Clinton and the semi-arid climate at Santa Paula show no significant sensitivity whatsoever to a random variation of the annual average evaporation, $\bar{e}_p$. A comparison of the numerical values of the cumulative probabilities for both Clinton and Santa Paula is given in Table 7.1.

One would have expected that in arid climates a random evaporation would explain some of the total variance of the annual yield. The result of this analysis indicates that for both Clinton and Santa Paula the assumption of a constant annual rate of potential evaporation [Eagleson, 1978] may be excellent.

One has to keep in mind, however, that the extremely simplified Penman equation, Eq. (6.14), by no means captures the 'true' variability of the annual average rate of potential evaporation. For instance, the integration interval of 1 year filters out some of the variance of the random atmospheric temperature, $\overline{T}_A$. Similarly, a long-term constant cloud cover, $\overline{\overline{N}}$, absorbs more of the true variance of $\bar{e}_p$, which in case of an arid climate may reach the magnitude of the variance of $\overline{T}_A$.

It has been pointed out in Chapter VI of this work that an analysis of the rate of potential evaporation similar to that of point precipitation is not being undertaken.

Infiltration from storm rainfall is governed by two random variables, t_r and i, which (since they may be fitted with exponential pdf's) have coefficients of variance, CV, of unity. Thus, the randomness of both variables was considered in the analysis.

Bare soil evaporation is governed in an analogous fashion by the random variables, t_b and e_p. While t_b has an exponential pdf and thus a CV of unity, observations show (Nixon, et al., 1972, and Pruitt, et al., 1972) that $\bar{e}_p$ when averaged over an interstorm period of 10 days has a CV = $0(10^{-1})$. We can expect the CV of $\bar{e}_p$ to become smaller for averaging times longer than 10 days and vice versa.

We have assumed that <u>in all cases</u> (i.e., humid <u>as well as</u> arid climates) $CV(t_b) >> CV(\bar{e}_p)$.

Thus, considering the possibilities that some portion of the "true" variance of $\bar{e}_p$ may not be accounted for by the simplified model, Eq. (6.14), and that a linearization of the water balance may distort their contributions to the variance of Y_A, the actual sensitivity of the basin yield to a random annual average $\bar{e}_p$ seems negligible.

All calculations for the nonlinear model have been performed by means of an existing water balance program. Its use for determining the cdf of the annual basin yield follows closely that displayed in Table A.1 in Appendix A.3. A few statements have been added to the program in order to account for the two storage terms.

	Prob$[Y_A/m_{P_A} < z]$, Clinton, Ma. $\kappa_s = 45d$; $Z_r = 100$ cm; $k_v = 0.9$, $M_o = 1.0$		Prob$[Y_A/m_{P_A} < z]$, Santa Paula Ca. $\kappa_s = 55d$; $Z_r = 50$ cm, $k_v = 1.0$, $M_o = .38$	
z	Constant $\bar{e}_p$	Variable $\bar{e}_p$	Constant $\bar{e}_p$	Variable $\bar{e}_p$
0.1	0.0102	0.0112	35.6279	35.6456
0.2	0.3640	0.3773	44.6517	44.6720
0.3	3.8791	3.9326	53.2703	53.2937
0.4	17.6614	17.7308	61.1926	61.2191
0.5	43.7971	43.8046	68.2495	68.2793
0.6	71.3671	71.3140	74.3724	74.4052
0.7	89.2753	89.2232	79.5673	79.6029
0.8	97.0079	96.9830	83.8907	83.9286
0.9	99.3627	99.3561	87.4286	87.4690
1.0	99.8926	99.8924	90.2812	90.3234
1.1	99.9840	99.9854	92.5510	92.5948
1.2	99.9965	99.9981	94.3360	94.3810

Table VII.1

SENSITIVITY OF THE FREQUENCY OF ANNUAL YIELD TO A RANDOM RATE OF POTENTIAL EVAPORATION

(LINEARIZED MODEL APPROXIMATING ACTUAL VARIANCE OF e_p)

Chapter VIII

CONCLUSIONS

A. An existing dynamic water balance model [Eagleson, 1978] has been supplemented by a simple storage concept. Change of the moisture state of a hydrologic system can now be accounted for by means of two additional terms in the water balance equations.

Simple dynamic modeling of storage mechanisms operating in an idealized soil column allows for incorporating storage effects on the annual basin yield in an approximate manner.

In order to retain analytical tractability of the mathematics and to avoid inappropriate data requirements for calibration, several simplifying assumptions have been made:

1. All the relevant random variables determining the two newly-derived storage terms are assumed statistically independent.
2. The annual average soil moisture, $\bar{s}$, is considered the only independent random variable accounting for the variance of annual change in storage.
3. The initial state of the hydrologic system is assumed to occur at its expected level.

The storage concept appears to produce a slight improvement of the basic nonlinear water balance model.

It has been demonstrated that an explicit formulation of soil moisture, $\bar{s} = \bar{s}(P_A, \bar{e}_p$, parameters), becomes necessary if the water balance model [Eagleson, 1978] is to be implemented in terms of more than one independent random variable. An analysis has been performed that leads to an explicit expression for soil moisture in terms of higher order partial derivatives of the state variable with respect to its independent random variables. Numerical analyses indicate that this simplification produces a reasonable approximation of the actual nonlinearities of the water balance equations.

B. Additional simplifying assumptions have been introduced in order to analyze the validity of the assumption of a constant annual rate of potential evaporation:

1. The two random variables, precipitation, P_A, and annual average potential evaporation, $\bar{e}_p$, are statistically independent.
2. A simple linear relationship for $\bar{s} = \bar{s}(P_A, \bar{e}_p$, parameters) can be employed in order to analyze the sensitivity of the annual basin yield to a second random variable.
3. A simplified Penman equation preserves sufficient variance of the annual average rate of potential evaporation.

Case studies show that the second assumption leads to considerable distortions of the model when applying it in an arid climate, since the primary nonlinearities are at small s_o. It seems to give acceptable results for humid climates, however. Despite the failure of the linearized water balance model in terms of accurately predicting frequencies of annual basin yield for arid climates, it is believed that it still serves the purpose of qualitatively assessing the effect of a second random variable on the water balance simulations.

The linearized model indicates negligible sensitivity of the frequency of annual yield to a randomly varying annual average potential evaporation, for both humid and arid climates.

It seems advisable, however, to look more closely at a formulation for $\bar{e}_p$ which more realistically accounts for random variations of its interstorm average value.

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Appendix A

A.1 Evaluation of the Storage Coefficient from Data

A streamflow hydrograph consists of several different components. Approximate empirical procedures have been proposed to separate these flow components for the purpose of hydrograph analysis [V. T. Chow, 1964].

The recession segment represents withdrawal of water from storage after all inflow to the reservoir has ceased. Therefore, it is more or less independent of the time variation in rainfall and infiltration. From a number of recession segments of a drainage basin, an envelope curve may be developed to represent the groundwater recession curve.

If we model, using a linear reservoir, the physical process of releasing water from groundwater storage, the reservoir coefficient (recession constant) can be derived in the following fashion:

$$I - O = \frac{dS}{dt} \tag{A.1}$$

and

$$S = \kappa_s O \tag{A.2}$$

where

I = inflow to reservoir

O = outflow from reservoir

S = storage of the reservoir

κ_s = recession constant

Since there is negligible inflow, I, during a recession period as stated before, the resulting differential equation is

$$\frac{dS}{dt} + \frac{1}{\kappa_s} S = 0 \qquad (A.3)$$

with the general solution

$$S = S_o e^{-t/\kappa_s} \qquad (A.4)$$

or in logarithmic form

$$\log S - \log S_o = -t/\kappa_s \qquad (A.5)$$

The preceding equation (A.5) will plot as a straight line on semilogarithmic paper with the storage on the logarithmic scale, as shown in Fig. A.1.

The reservoir coefficient, κ_s, contains geological and soil information in lumped form.

In general, the storage coefficient is a function of various variables and parameters.

$$\kappa_s = \kappa_s(S,\ 0,\ I,\ \text{geology, soil parameters...}) \qquad (A.6)$$

Here, we postulated a linear reservoir, thus

$$\kappa_s = \kappa_s(\text{geology, soil parameters...}) \qquad (A.7)$$

Depending on this reservoir property, an arbitrary discharge, 0_o, can be generated by an innumerable number of different storage levels. For very high values of the reservoir coefficient, κ_s, the

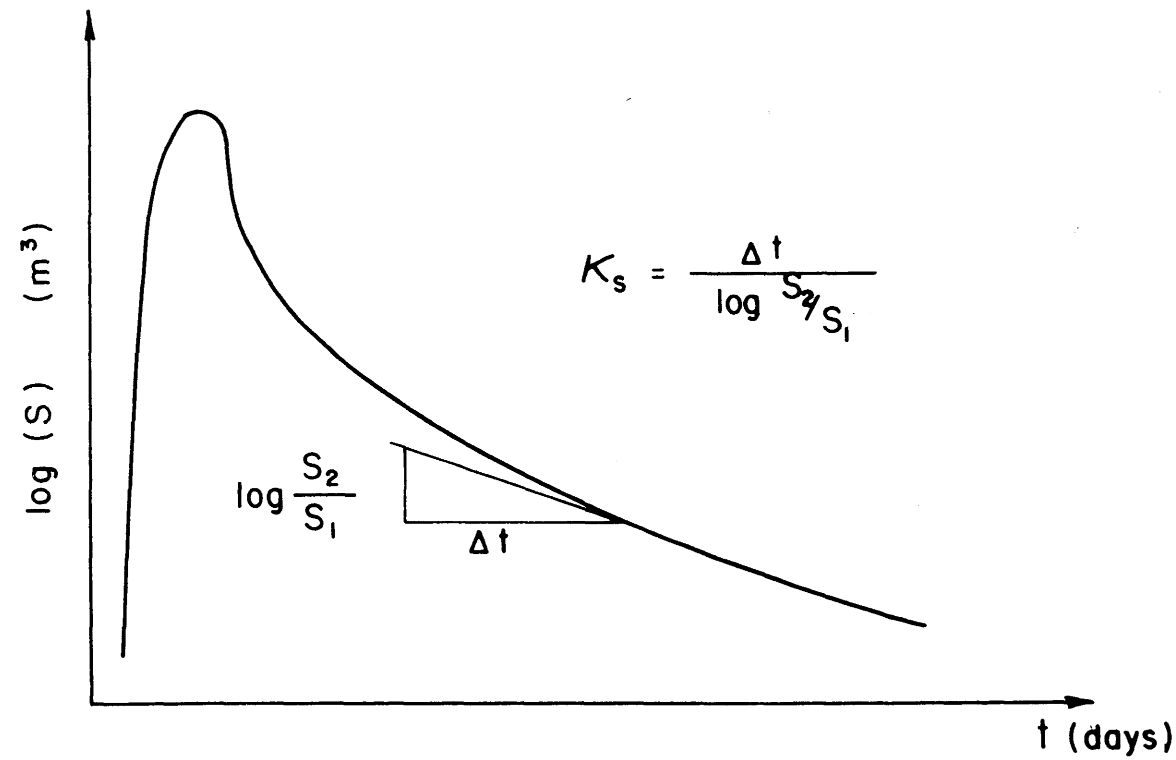

Figure A.1

RECESSION CURVE OF ANNUAL RUNOFF HYDROGRAPH discharge is almost insensitive to changes in storage. For very low values of κ_s, the discharge is extremely sensitive to any change in storage. The limit in this case is that the reservoir loses its functionality.

Equation (4.23) permits an analysis of the two cases.

$$\lim_{\kappa_s \to \infty} \left(\frac{dS}{dt} + \frac{1}{\kappa_s} S - p_N(t) \right) = \frac{dS}{dt} - p_N(t) = 0 \tag{A.8}$$

Thus

$$\frac{dS}{dt} = p_N(t) \tag{A.9}$$

and

$$\Delta S_{g_A} = \int_o^{1\ \text{year}} p_N(t)\ dt = P_{N_A} \tag{A.10}$$

Thus,

$$\int_o^{1\ \text{year}} r_g(t) = R_{g_A} = P_{N_A} - \Delta S_{g_A} = 0 \tag{A.11}$$

We see that in the case of a high value for the reservoir coefficient, the yield from the groundwater approaches zero.

In the other extreme, we find

$$\lim_{\kappa_s \to 0} (\kappa_s \frac{dS}{dt} + S - p_N(t)\ \kappa_s) = S = 0 \tag{A.12}$$

Thus,

$$S = 0 \tag{A.13}$$

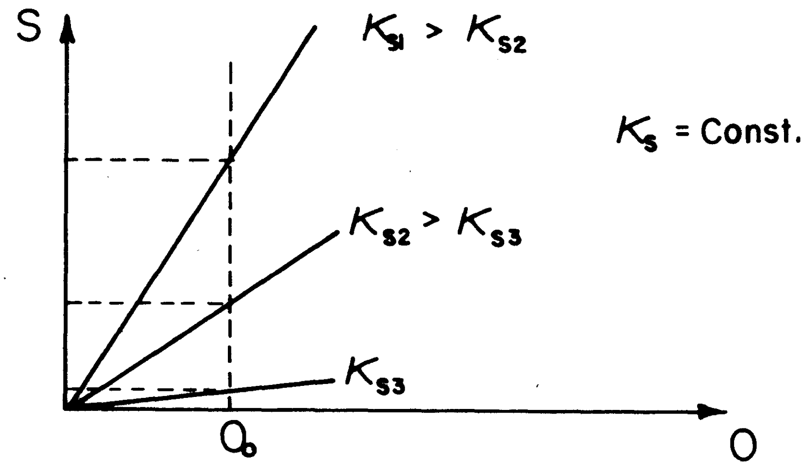

Figure A.2

GROUNDWATER RESERVOIR COEFFICIENTS

and

$$R_{g_A} = P_{N_A} = m_\tau \; K(1) \; \bar{s}^c - wT \tag{A.14}$$

Here the first-order model is approached. Storage in the saturated zone is not accounted for. The high lateral transmissivity of the aquifer doesn't allow for any storage according to a linear reservoir model.

A.2 Sensitivity Analysis for Groundwater Storage

A.2.1 Approximation of the First Moment

Eq. (4.31) contains six covariance terms. Three of those are $i \neq j$ covariances for two different random variables.

A basic assumption in formulating a change of state of a hydrologic system in a simple statistical manner is that any correlation of climatological or hydrologic events between two consecutive years are insignificant. According to this assumption, two of the $i \neq j$ covariances in Eq. (4.31) are negligible. Initial groundwater runoff, r_{g_o}, on one hand, and annual average soil moisture, $\bar{s}$, and length of rainy season, τ, on the other hand, belong, by definition, to two consecutive years.

Statistical dependence between the latter random variables, $\bar{s}$ and τ, is assumed small as well in this work.

Some analysis of the curvatures of Eq. (4.28) with respect to its independent random variables is necessary, however. From Eq. (4.28), we have

$$\left.\frac{\partial^2 \Delta S_{g_A}}{\partial \tau^2}\right|_{m_\tau, s_o} = \frac{1}{\kappa_s} K(1)\ s_o^c\ e^{-(T-m_\tau)/\kappa_s} \qquad (A.15)$$

and

$$\left.\frac{\partial^2 \Delta S_{g_A}}{\partial \bar{s}^2}\right|_{s_o, m_\tau} = \kappa_s\ K(1) \left(e^{-(T-m_\tau)/\kappa_s} - e^{-T/\kappa_s}\right) c(c-1)\ s_o^{c-2} \qquad (A.16)$$

and

$$\left.\frac{\partial^2 \Delta S_{g_A}}{\partial r_{g_o}^2}\right|_{m_{r_{g_o}}} = 0 \qquad (A.17)$$

Since the first moment of ΔS_{g_A} by definition has to be zero neglecting the three i = j covariance terms in Eq. (4.31) is only reasonable if they are of the order of zero too.

An upper bound for the magnitude of the nonlinearity of Eq. (4.28) with respect to τ is given by assuming $T = m_\tau$. Thus, Eq. (A.15) becomes

$$\left.\frac{\partial^2 \Delta S_{g_A}}{\partial \tau^2}\right|_{m_\tau, s_o} \leq \frac{1}{\kappa_s} K(1)\ s_o^c \qquad (A.18)$$

Data analyses indicate (Appendix C) that the right hand side of Eq. (A.18) evaluated for sandy-loam and the storage coefficient for Clinton, κ_s = 45 days, gives

$$\left.\frac{\partial^2 \Delta s_{g_A}}{\partial \tau^2}\right|_{m_\tau, s_o} \leq 1.8 \times 10^{-3} \left[\frac{cm}{day^2}\right] \qquad (A.19)$$

Any other combination of climate and soil parameters yields a smaller value. It is unlikely that the variance of season length, σ_τ^2, for any climate considerably exceeds $\sigma_\tau^2 = 10^3$ days^2. Taking that into account, the whole covariance term in Eq. (4.31) for τ then becomes

$$\frac{1}{2} \left.\frac{\partial^2 \Delta s_{g_A}}{\partial \tau^2}\right|_{m_\tau, s_o} \sigma_\tau^2 < 1.0 \quad [cm] \qquad (A.20)$$

An upper bound for the magnitude of the nonlinearity of Eq. (4.28) with respect to $\bar{s}$ is also given by assuming a humid climate, $T \simeq m_\tau$. Thus, Eq. (A.16) becomes

$$\left.\frac{\partial^2 \Delta s_{g_A}}{\partial \bar{s}^2}\right|_{s_o, m_\tau} \leq \kappa_s \, K(1) \, c(c-1) \, s_o^{c-2} \qquad (A.21)$$

Data analyses indicate (Appendix C) that the right hand side of Eq. (A.21) evaluated for sandy-loam and the storage coefficient for Santa Paula, $\kappa_s = 55$ days, gives

$$\left.\frac{\partial^2 \Delta s_{g_A}}{\partial \bar{s}^2}\right|_{s_o, m_\tau} \leq 600 \quad [cm] \qquad (A.22)$$

Again any other (more realistic) combination of climatic and soil parameters yields a smaller value.

As done in Chapter V of this work, an explicit relationship between soil moisture, $\bar{s}$, and a single random variable, P_A, can be gained from Eq. (4.8) by means of a Taylor series expansion. It can be found that

$$\bar{s} - s_o \simeq a_1 P_A + a_2 \qquad (A.23)$$

where

a_i = constants containing parameterized information of the climate-soil-vegetation system (Eq. 5.19)

From Eq. (A.23)

$$\sigma_{\bar{s}}^2 = VAR(\bar{s}) \simeq VAR(a_1 P_A) = a_1^2 \, \sigma_{P_A}^2 \qquad (A.24)$$

The variance of the annual point precipitation, P_A, is according to an adopted Gamma distribution

$$\sigma_{P_A}^2 = \frac{m_{P_A}^2}{m_\nu} (1 + \kappa^{-1}) \qquad (A.25)$$

The coefficient of variation, V_{P_A}, of the same variable then becomes

$$V_{P_A} = \frac{\sigma_{P_A}}{m_{P_A}} = \left[\frac{1 + \kappa^{-1}}{m_\nu}\right]^{1/2} \qquad (A.26)$$

Case studies (Chapter VII) indicate that for any soil

$$a_1^2 \leq 0(5 \times 10^{-6} \text{ cm}^{-2}) \qquad (A.27)$$

and

$$\sigma^2_{P_A} \simeq \begin{cases} 250 \text{ cm}^2, \text{ Clinton, Mass.} \\ 700 \text{ cm}^2, \text{ Santa Paula, Ca.} \end{cases} \quad \text{(A.28)}$$

Thus, from Eq. (A.24), we get

$$\sigma^2_{\bar{s}} \simeq \begin{cases} 1.25 \times 10^{-3}, \text{ Clinton} \\ 2.80 \times 10^{-3}, \text{ Santa Paula} \end{cases} \quad \text{(A.29)}$$

With this data, an upper bound for the complete i = j covariance term in Eq. (4.31) with respect to soil moisture can be determined. It is

$$\frac{1}{2} \left. \frac{\partial^2 \Delta S_{g_A}}{\partial \bar{s}^2} \right|_{s_o, m_\tau} \sigma^2_{\bar{s}} \leq 1.0 \quad \text{[cm]} \quad \text{(A.30)}$$

For realistic combinations of climate and soil, both the values for the expression Eq. (A.20) and for the term Eq. (A.30) are well below 0 (.1 cm). Considering the fact that the magnitude of ΔS_{g_A} within a common range of soil moisture (Chapter V) may well exceed values of 0 (1.0 cm) above numerical analysis may indicate that the three i = j covariances in Eq. (4.31) are of negligible magnitude. This result together with the assumption of small i ≠ j covariances may serve as a justification for the approximation of $E[\Delta S_{g_A}]$ by $\Delta \overline{S}_{g_A}$, Eq. (4.30).

A.2.2 Analysis of the Second Moment

If only the first two terms of the Taylor-series expansion of the annual change in groundwater storage, ΔS_{g_A}, are retained, a first-order approximation of the variance of ΔS_{g_A} is obtained giving [Benjamin

& Cornell, 1970]

$$\mathrm{VAR}[\Delta S_{g_A}] \approx \sum_{j=1}^{3} \sum_{i=1}^{3} \left.\frac{\partial \Delta S_{g_A}}{\partial x_i}\right|_m \left.\frac{\partial \Delta S_{g_A}}{\partial x_j}\right|_m \mathrm{COV}[x_i, x_j] \qquad (A.31)$$

Again, it is reasonable to assume statistical independence among the x_i, this being consistent with previous assumptions. Equation (A.31) then simplifies to

$$\mathrm{VAR}[\Delta S_{g_A}] \simeq \sum_{i=1}^{3} \left(\left.\frac{\partial \Delta S_{g_A}}{\partial x_i}\right|_m \right)^2 \mathrm{VAR}[x_i] \qquad (A.32)$$

This equation may be interpreted that each of the three random variables, x_i, contributes to the dispersion of ΔS_{g_A} in a manner proportional to its own variance, $\mathrm{VAR}[x_i]$, and proportional to a factor $[(\partial \Delta S_{g_A}/\partial x_i)|_m]^2$, which is related to the sensitivity of changes in ΔS_{g_A} to changes in x_i.

This formula allows one to estimate the error in the variance of ΔS_{g_A} due to the treatment of one of the independent stochastic variables as deterministic. The variance of ΔS_{g_A} has three components:

$$S_{\bar{s}} \equiv \left(\left.\frac{\partial \Delta S_{g_A}}{\partial \bar{s}}\right|_{s_o, m_\tau} \right)^2 \sigma_{\bar{s}}^2 = \left[\kappa_s \, K(1) c s_o^{c-1} \left(e^{-(T-m_\tau)/\kappa_s} - e^{-T/\kappa_s}\right) \right]^2 \sigma_{\bar{s}}^2$$

(A.33)

$$S_\tau \equiv \left(\left.\frac{\partial \Delta S_{g_A}}{\partial \tau}\right|_{m_\tau, s_o} \right)^2 \sigma_\tau^2 = [K(1) \, s_o^c \, e^{-(T-m_\tau)/\kappa_s}]^2 \, \sigma_\tau^2 \qquad (A.34)$$

$$S_{r_{g_o}} \equiv \left(\left.\frac{\partial \Delta S_{g_A}}{\partial r_{g_o}}\right|_{m_{r_{g_o}}} \right)^2 \sigma_{r_{g_o}}^2 = [-\kappa_s(1 - e^{-T/\kappa_s})]^2 \, \sigma_{r_{g_o}}^2 \qquad (A.35)$$

We wish to show that the contribution to the variance of ΔS_{g_A} from variation of the season length, τ, and from variation of the initial groundwater runoff, r_{g_o}, is negligible compared to that coming from variance of the soil moisture, $\bar{s}_o$. Accordingly, we write

$$S_\tau / S_{\bar{s}} = \frac{\sigma_\tau^2}{\sigma_{\bar{s}}^2} \frac{1}{\kappa_s^2} \left(\frac{s_o}{c}\right)^2 \left[\frac{e^{m_\tau/\kappa_s}}{e^{m_\tau/\kappa_s} - 1}\right]^2 \tag{A.36}$$

and

$$S_{r_{g_o}} / S_{\bar{s}} = \frac{\sigma_{r_{g_o}}^2}{\sigma_{\bar{s}}^2} \left[\frac{e^{T/\kappa_s} - 1}{e^{m_\tau/\kappa_s} - 1}\right]^2 \left(\frac{1}{c s_o^{c-1} K(1)}\right)^2 \tag{A.37}$$

As before (A.2.1), we assume σ_τ^2 to be smaller than 10^3 days^2 which may be valid even for extremely arid conditions. For the most unfavorable combination of climatic and soil parameters (Appendix C), one can show that the magnitude of $S_\tau / S_{\bar{s}}$ is approximately 0.5.

Rearrangement of Eq. (A.37) gives in combination with the approximation of Eq. (4.34)

$$S_{r_{g_o}} / S_{\bar{s}} \simeq \frac{\sigma_{r_{g_o}}^2}{\sigma_{\bar{s}}^2} \left(\frac{s_o}{c}\right)^2 \left(\frac{1}{m_{r_{g_o}} + w}\right)^2 \tag{A.38}$$

Assuming negligible capillary rise, this becomes

$$S_{r_{g_o}} / S_{\bar{s}} \simeq \frac{1}{\sigma_{\bar{s}}^2} \left(\frac{s_o}{c}\right)^2 V_{r_{g_o}}^2 \tag{A.39}$$

where

$V_{r_{g_o}}$ = coefficient of variation of the initial groundwater runoff, r_{g_o}

By definition of the storage concept developed in this work, the parameters of the distribution of groundwater runoff, r_g, and those of the distribution of the initial groundwater runoff, r_{g_o}, are identical.

To the first order (absence of groundwater storage), we can approximate the relationship between soil moisture, $\bar{s}$, and groundwater runoff, r_{g_o}, by

$$r_g = K(1)\ \bar{s}^c \tag{A.40}$$

The coefficients of variance are related by [Benjamin & Cornell, 1970]

$$V_{r_g} = \frac{\sigma_{r_{g_o}}}{m_{r_{g_o}}} \simeq \frac{\left|\left.\frac{dr_g(\bar{s})}{d\bar{s}}\right|_{s_o}\right|}{K(1) s_o^{c-1}}\ \frac{\sigma_{\bar{s}}}{s_o} \tag{A.41}$$

Rearranged this becomes

$$V_{r_{g_o}}/V_{\bar{s}} \simeq V_{r_g}/V_{\bar{s}} \simeq c \tag{A.42}$$

Combining Eq. (A.42) and Eq. (A.39) gives

$$S_{r_{g_o}}/S_{\bar{s}} \simeq \left(\frac{V_{r_{g_o}}}{V_{\bar{s}}}\right)^2 \frac{1}{c^2} \simeq 1.0 \tag{A.43}$$

This result is independent of any soil and climatic condition. It is obtained solely in accordance with the definition of the storage concept. It shows that both the initial condition for groundwater runoff, r_{g_o}, and the annual average soil moisture, $\bar{s}$, contribute to the variance of ΔS_{g_A} to the same extent.

It is obvious that significant error is introduced into Eq.

(4.31) if r_{g_o} is treated like constant. It is legitimate, however, to neglect the variability of τ.

Despite the result of the above sensitivity analysis ($S_{r_{g_o}}/S_{\bar{s}}$ = 1.0) which exactly applies to the initial condition of storage in the unsaturated zone, s_{j-1}, too (it can easily be shown that $S_{\bar{s}_{j-1}}/S_{\bar{s}_j} = 1.0$ for VAR[ΔS_{u_A}]), we must assume the initial state of the system to occur at its long-term level. This is the only way to account for storage effects without complicating the analysis to a degree which makes it analytically intractable and which creates impractical data requirements.

A.3 Determination of the CDF of Annual Basin Yield*

	E	$J(E)$	E_{T_A}	P_{N_A}	R_{s_A}/P_A	ΔS_{u_A}	P_A/m_{P_A}		$\Delta S_{g_A}/m_{P_A}$	Y_A/m_{P_A}
$\bar{s}$	$f(\bar{s})$	$f(\bar{s})$	$E_{P_A} \cdot J[E] = f(\bar{s})$	$m\,K(1)\bar{s}^{c} - wT$	$\phi_i(\bar{s})$; $\sigma(\bar{s})$; $G(\bar{s})$	$nZ_r \cdot (\bar{s}-s_o)$		$P[\frac{P_A}{m_{P_A}} < z] \times 100$	$\text{const} \cdot (\bar{s}^{c}-\bar{s}_o^{c})$	
	Eq. 3.29	Eq. 3.27	Eq. 3.27	Eq. 4.9	Eq. 4.10	Eq. 4.6	Eq. 4.8	Eq. 3.9	Eq. 4.35	Eq. 2.10
			cm	cm		cm		%		
Solution of the soil moisture balance equation, (4.8)									Sol. of Eq. (2.10)	

Table A.1

SCHEME TO EVALUATE THE ANNUAL WATER BALANCE AND THE FREQUENCY OF ANNUAL YIELD

* It must be noted that the above table is valid for the simplified case of no surface retention and no vegetation, thus, $h_o = 0$ and $M = 0$.

Appendix B

B.1 Analysis of the Variance of the Annual Average Rate of Potential Evaporation, $\bar{e}_p$

An expansion of Eq. (6.9) about the expected value of its independent random variables, $\bar{T}_A$, $\bar{N}$ and $\bar{S}$ into a multidimensional Taylor series is necessary before expectations of $\bar{e}_p$ are taken. The justification for this expansion lies in the observation that the coefficient of variation, V_x, of the independent random variables is small. Keeping only the first three terms of the expansion gives

$$\bar{e}_p \simeq \bar{e}_p(\bar{\bar{T}}_A, \bar{\bar{N}}, \bar{\bar{S}}) + (\bar{T}_A - \bar{\bar{T}}_A) \left.\frac{\partial \bar{e}_p}{\partial \bar{T}_A}\right|_{\bar{\bar{T}}_A} + (\bar{N}-\bar{\bar{N}}) \left.\frac{\partial \bar{e}_p}{\partial \bar{N}}\right|_{\bar{\bar{N}}} + (\bar{S}-\bar{\bar{S}}) \left.\frac{\partial \bar{e}_p}{\partial \bar{S}}\right|_{\bar{\bar{S}}}$$

$$+ \frac{1}{2} (\bar{T}_A - \bar{\bar{T}}_A)^2 \left.\frac{\partial^2 \bar{e}_p}{\partial \bar{T}_A^2}\right|_{\bar{\bar{T}}_A} + \frac{1}{2} (\bar{N}-\bar{\bar{N}})^2 \left.\frac{\partial^2 \bar{e}_p}{\partial \bar{N}^2}\right|_{\bar{\bar{N}}} + \frac{1}{2} (\bar{S}-\bar{\bar{S}})^2 \left.\frac{\partial^2 \bar{e}_p}{\partial \bar{S}^2}\right|_{\bar{\bar{S}}} \quad \text{(B.1)}$$

where double bars represent expectation of a variable.

Acknowledging the fact that the second-order terms in the above expression, Eq. (B.1), don't contribute significantly to the variance of $\bar{e}_p$ because of second-order derivatives, one can find [Benjamin & Cornell, 1970] to the first order

$$\text{VAR}[\bar{e}_p] \simeq \sum_{i=1}^{3} \sum_{j=1}^{3} \left.\frac{\partial \bar{e}_p}{\partial x_i}\right|_m \left.\frac{\partial \bar{e}_p}{\partial x_j}\right|_m \text{COV}[X_i, X_j] \quad \text{(B.2)}$$

where

x_i = independent random variables $\bar{T}_A$, $\bar{N}$, $\bar{S}$

Since we are interested in the relative contribution to the variance, VAR[$\bar{e}_p$], by the individual variables rather than in the exact value of VAR[$\bar{e}_p$] itself, the $i \neq j$ covariances (although considerable probably for temperature and cloud amount) can be neglected. Thus, one obtains three terms from the right hand side of Eq. (B.2), each defining the contribution of one independent variable to the total variance of $\bar{e}_p$. Relating the remaining components to the temperature component, which presumably is largest, gives the ratios

$$R_{\bar{N},\bar{T}_A} \equiv \frac{\left(\partial\bar{e}_p/\partial\bar{N}\big|_{\bar{\bar{N}}}\right)^2}{\left(\partial\bar{e}_p/\partial\bar{T}_A\big|_{\bar{\bar{T}}_A}\right)^2} \sigma^2_{\bar{N}}/\sigma^2_{\bar{T}_A} \tag{B.3}$$

and

$$R_{\bar{S},\bar{T}_A} \equiv \frac{\left(\partial\bar{e}_p/\partial\bar{S}\big|_{\bar{\bar{S}}}\right)^2}{\left(\partial\bar{e}_p/\partial\bar{T}_A\big|_{\bar{\bar{T}}_A}\right)^2} \sigma^2_{\bar{S}}/\sigma^2_{\bar{T}_A} \tag{B.4}$$

Temperature observations (Figure 6.1) for Clinton, Mass., which has a sub-humid climate and for Santa Barbara, Cal., which has a semi-arid climate, lead to the following parameters of fitted analytical probability density functions (pdf)

Clinton:

$$E[\bar{T}_A] \equiv \bar{\bar{T}}_A = 8.4°C$$

$$\sigma_{\bar{T}_A} = \lambda_T^{-1} = .714°C$$

$$V_{\bar{T}_A} = \frac{\sigma_{\bar{T}_A}}{\bar{\bar{T}}_A} = .085$$

Santa Barbara:

$$E[\bar{T}_A] = 15.8°C$$

$$\sigma_{\bar{T}_A} = .5°C$$

$$V_{\bar{T}_A} = .032$$

Santa Barbara records were chosen because of its immediate vicinity to Santa Paula for which long-term temperature observations are not available.

Observations [U.S. Weather Bureau, 1950-1978] of relative humidity, S, and cloud cover, N, have been analyzed for Boston, Mass., and Santa Maria, Ca. These stations were found closest to Clinton, Mass. and Santa Paula, Ca., among the observation stations having complete records of climatological data. Records of annual means from 1950 through 1978 revealed the following statistical parameters:

Boston:

$$E[\bar{N}] \equiv \bar{\bar{N}} = .61 \text{ tenth}$$

$$\sigma_{\bar{N}} = .029 \text{ tenth}$$

$$V_{\bar{N}} = 4.8 \times 10^{-2}$$

$$E[\bar{S}] \equiv \bar{\bar{S}} = 57.0\%$$

$$\sigma_{\bar{s}} = 2.3\%$$

$$V_{\bar{s}} = 4.0 \times 10^{-2}$$

Santa Maria:

$$E[\overline{N}] = .42 \text{ tenth}$$

$$\sigma_{\overline{N}} = .03 \text{ tenth}$$

$$V_{\overline{N}} = 7.1 \times 10^{-2}$$

$$E[\overline{S}] = 60.8\%$$

$$\sigma_{\overline{S}} = 2.6\%$$

$$V_{\overline{S}} = 4.3 \times 10^{-2}$$

Using actual numbers now for the ratios $R_{\overline{N},\overline{T}_A}$ and $R_{\overline{S},\overline{T}_A}$ and assuming that above results are transferable to Clinton and Santa Paula gives for Clinton:

$$R_{\overline{N},\overline{T}_A} \simeq \left(\frac{.069 \times .029}{.0038 \times .714}\right)^2 = .54 < 1$$

$$R_{\overline{S},\overline{T}_A} \simeq \left(\frac{8.5 \times 10^{-5} \times .023}{.0038 \times .714}\right)^2 = 5.2 \times 10^{-7} \ll 1$$

and for Santa Paula:

$$R_{\overline{N},\overline{T}_A} \simeq \left(\frac{.18 \times .03}{.0124 \times .5}\right)^2 = .76 < 1$$

$$R_{\overline{S},\overline{T}_A} \simeq \left(\frac{8.8 \times 10^{-5} \times .026}{.0124 \times .5}\right)^2 = 3.7 \times 10^{-4} \ll 1$$

This numerical analysis indicates that it is realistic to consider the annual average atmospheric temperature, $\overline{T}_A$, the primary independent variable in Eq. (6.9). Although it has to be noted that particularly in the semi-arid climate of California, the variance of $\bar{e}_p$ due to cloud cover is almost of the same magnitude as that due to temperature, $\bar{T}_A$. A study [CESS, 1976] on atmospheric feedback mechanisms appears to support the above finding of a minor impact of cloud amount on climatic variables. The author states that cloud amount, N, is not a significant climatic feedback mechanism, irrespective of how cloud amount might depend upon surface temperature, T_A. The reasons that there are compensating changes in both the solar (related to q_i) and the infrared (related to q_b) optical properties of the atmosphere. In other words, temperature variations might cause variations in cloud cover, but the latter doesn't significantly feed back into changes in potential evaporation, e_p.

Hence, it seems justifiable to assume all the variance VAR[$\bar{e}_p$] of $\bar{e}_p$ to come from temperature, $\bar{T}_A$, alone if to the first order only one independent random variable is to be accounted for in Eq. (6.9). This then leads to the simplified Eq. (6.14) where $\bar{N}$ and $\bar{S}$ are fixed at their long-term average values $\bar{\bar{N}}$ and $\bar{\bar{S}}$, respectively.

B.2 Integration of the Joint Density Function for the Annual Basin Yield

Expansion of the right hand side of Eq. (6.23) by substituting the joint pdf through the marginal pdf's for P_A, Eq. (3.6), and $\bar{e}_p$, Eq. (6.20), gives

$$\text{Prob}[Y_A \le x] = e^{-\omega m_\tau} \left\{ \left[e^{-\frac{\lambda_T}{a_2 a_3} |\bar{e}_{p,max} - \bar{\bar{e}}_p|} + \frac{\lambda_T}{a_2 a_3} \int_o^{\bar{e}_{p,max}} e^{-\frac{\lambda_T}{a_2 a_3} |\bar{e}_p - \bar{\bar{e}}_p|} d\bar{e}_p \right] \cdot 1 \right.$$

$$+ \frac{1}{2} e^{-\frac{\lambda_T}{a_2 a_3}|\bar{e}_{p,max} - \bar{\bar{e}}_p|} \sum_{\nu=1}^{\infty} \frac{(\omega m_\tau)^\nu}{\nu!} \Gamma^{-1}(\nu\kappa) \int_o^{r(x,\bar{e}_{p,max})} e^{-r} r^{\nu\kappa-1} dr$$

$$+ \frac{1}{2} e^{-\frac{\lambda_T}{a_2 a_3}|\bar{e}_{p,min} - \bar{\bar{e}}_p|} \sum_{\nu=1}^{\infty} \frac{(\omega m_\tau)^\nu}{\nu!} \Gamma^{-1}(\nu\kappa) \int_o^{r(x,\bar{e}_{p,min})} e^{-r} r^{\nu\kappa-1} dr$$

$$\left. + \frac{1}{2} \frac{\lambda_T}{a_2 a_3} \sum_{\nu=1}^{\infty} \frac{(\omega m_\tau)^\nu}{\nu!} \Gamma^{-1}(\nu\kappa) \int_{\bar{e}_{p,min}}^{\bar{e}_{p,max}} e^{-\frac{\lambda_T}{a_2 a_3}|\bar{e}_p - \bar{\bar{e}}_p|} \int_o^{r(x,\bar{e}_p)} e^{-r} r^{\nu\kappa-1} dr \right\} \tag{B.5}$$

where

$$r(x, \bar{e}_p) = \eta\kappa P_A(x, \bar{e}_p) \tag{B.6}$$

After a first integration step and normalization of $x = zm_{P_A}$, Eq. (B.5) becomes

$$\text{Prob}\left[\frac{Y_A}{m_{P_A}} < z\right] = e^{-\omega m_\tau} \left\{ 1 + A \sum_{\nu=1}^{\infty} \frac{(\omega m_\tau)^\nu}{\nu!} [P[\nu\kappa, f(z, \bar{e}_{p,max})] - P[\nu\kappa, f(z, \bar{e}_{p,min})]] + B \sum_{\nu=1}^{\infty} \frac{(\omega m_\tau)^\nu}{\nu!} \cdot I(z, \bar{e}_p) \right\} \tag{B.7}$$

where A, B and the function $f(z, \bar{e}_p)$ are given by Eq. (6.27), Eq. (6.28) and Eq. (6.25), respectively.

For the numerical evaluation of the integral $I(z, \bar{e}_p)$ as given by Eq. (6.26), the Simpson Rule [Bronstein-Semendjajew, 1974] is applied. The integral is approximated by

$$I(z, \bar{e}_p) \simeq \frac{h}{3} (y_o + \sum_{i=1}^{n-1} [3 + (-1)^{i+1}] \, y_i + y_n) \qquad \text{(B.8)}$$

where

y_i = numerical value of the integrand of $I(z, \bar{e}_p)$

n = number of nodal points where n has to be even

The interval of integration is given

$$h = \frac{\bar{e}_{p,max} - \bar{e}_{p,min}}{n} \qquad \text{(B.9)}$$

with

$$\bar{e}_{p,max} = 2\bar{\bar{e}}_p - \bar{e}_{p,min} \qquad \text{(B.10)}$$

because of a symmetrical double exponential pdf for $\bar{e}_p$, Eq. (6.20). The lower realistic limit for $\bar{e}_p$ has to be found by Eq. (6.14)

$$\bar{e}_{p,min} = (a_1 + a_2 \, \bar{T}_{A,min}) \, a_3 \qquad \text{(B.11)}$$

where $\bar{T}_{A,min}$ is picked the freezing temperature $\bar{T}_{A,min}$ = 0°C.

For large values of normalized annual basin yield, $z = Y_A/m_{P_A}$ $(z \to \infty)$, the joint cdf for Y_A has to approach unity. That Eq. (6.24) in fact approaches unity in that case can easily be shown.

Pearson's incomplete Gamma function reaches unity as the argument $f(z, \bar{e}_p)$ goes to infinity due to z going to infinity. Equation (6.24) simplifies to

$$\lim_{z \to \infty} \text{Prob}\left[\frac{Y_A}{m_{P_A}} < z\right] = e^{-\omega m_\tau} \left\{1 + 2A \sum_{\nu=1}^{\infty} \frac{(\omega m_\tau)^\nu}{\nu!} + (1 - 2A) \sum_{\nu=1}^{\infty} \frac{(\omega m_\tau)^\nu}{\nu!}\right\} = 1 \tag{B.12}$$

For small values of normalized yield ($P_A \rightarrow 0$), the lower integration limit, Eq. (6.25) reaches zero. From Eq. (6.29), one obtains by setting P_A equal to zero (see Figure B.1)

$$Y_{A,min} \equiv -\bar{e}_p^2 E_3 - \bar{e}_p[C_2 + D_2 + E_2] - C_3 - D_3 \tag{B.13}$$

The integration region R(x) for Eq. (6.22) thus displays a confined area F_o over which the integration of the joint cdf for Y_A gives a discrete probability (Figure B.1) for $Y_{A,min} = z_{min} \cdot m_{P_A}$. For $z \rightarrow \infty$, the upper boundary of the integration region $F = F_1 + F_o$ moves to infinite values of P_A.

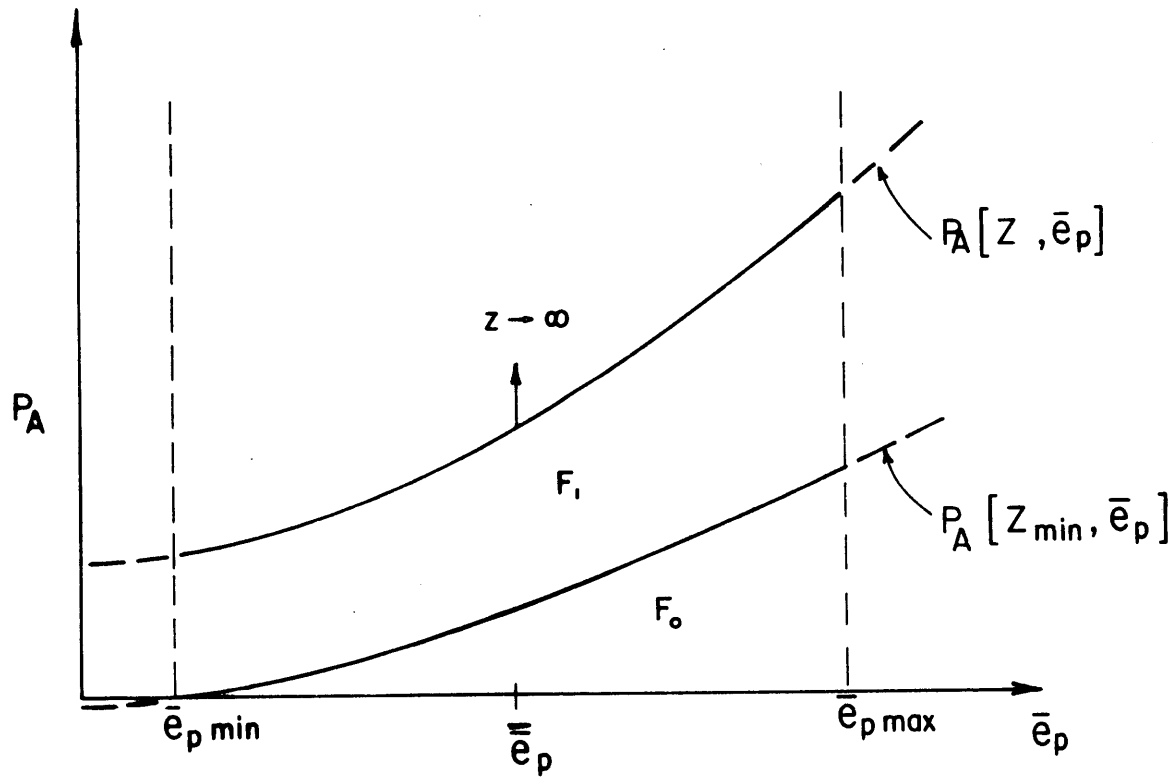

Figure B.1

INTEGRATION REGION FOR PROBABILITY OF ANNUAL BASIN YIELD

Appendix C

LIST OF PARAMETERS

Parameter	Clay		Clay Loam		Silty Loam		Sandy Loam	
k(1), cm^2	1×10^{-10}		2.8×10^{-10}		1.2×10^{-9}		2.5×10^{-9}	
n	0.45		0.35		0.35		0.25	
c	12		10		6		4	
M_o	0.2	0.1	0.9	0.75	1.0	1.0	1.0	1.0
k_{v_o}	2.1*	0.7†	0.7	0.3	0.9	0.5	0.7	0.3

Table C.1

INDEPENDENT PARAMETERS OF REPRESENTATIVE SOILS AND EQUILIBRIUM PROPERTIES OF VEGETAL COVER

* Clinton, Mass.

† Santa Paula, Calif.

Parameter	Clinton, Mass.	Santa Paula, Calif.
Φ (°N)	42.50	34.40
$\hat{m}_{P_A}$, cm (annual)	111.3	54.4
$\hat{m}_{P_A}$, cm (seasonal)	94.1	53.4
$\hat{m}_T$, d	365	212
$\hat{m}_{t_r}$, d	0.32	1.43
$\hat{m}_{t_b}$, d	3.0	10.4
κ	0.50*	0.37*, 0.25†
$\overline{\overline{T}}_A$, °C	8.4	13.8
λ_T, °C^{-1}	1.4	2.0 (Santa Barbara)
$\overline{\overline{N}}$, tenth	.61 (Boston)	.42 (Santa Maria)
$\sigma_{\overline{N}}$, tenth	.029	.03
$\overline{S}$, %	57.0	60.8
$\sigma_{\overline{S}}$, %	2.3	2.6
$\overline{\overline{N}}$, tenth	0.35	0.35
$\overline{\overline{S}}$, %	70.0	60.0
κ_s, d	45.0	55.0
Z_r, cm	100.0	50.0
$\overline{A}$	0.3	0.2

* From method of moments

† From visual best fit

Table C.2

INDEPENDENT CLIMATE AND CATCHMENT PROPERTIES

Appendix D

FORTRAN PROGRAM FOR LINEARIZED WATER BALANCE COMPUTATIONS

```
c         THIS FORTRAN PROGRAM DETERMINES THE 'CDF' OF THE ANNUAL BASIN
c         YIELD OF A CATCHMENT.IT IS A LINEARIZED VERSION OF THE FIRST-
c         ORDER MODEL [ EAGLESON,1978 ], ACCOUNTING FOR ANNUAL CHANGE IN
c         STORAGE AND FOR A RANDOMLY VARYING  ANNUAL AVERAGE RATE OF
c         POTENTIAL EVAPORATION. GIVEN A SET OF VEGETAL, CLIMATIC
c         AND SOIL-PARAMETERS THE PROGRAM EVALUATES THE CORRESPONDING
c         PROB[Ya/mpa < z] FOR SOME DISCRETE VALUES OF Ya/mpa.

          real*8    mv,mh,maxep,minep,meanep,E1,E3,cfin,afin,bfin,A,B
          real*8    kapp,znorm,probab
          real  mtr,mtau,mtb,mi,n,k1,kapr,m,J,ke,Fiso(3),j(3),e(3),JJ
          real  mpa,mo,kvo,ho,lambda,maxta,meanta,minta
          integer  pistol
c         INDEPENDENT CLIMATIC PARAMETERS FOR CLINTON, MASS.
          data  mpa/94.1/,mtr/.32/,mtau/365./,mtb/3./,lambda/1.4/
          data  ho/.1/,const/.283/,kapp/.5/,meanta/8.4/
c         INDEPENDENT CLIMATIC PARAMETERS FOR ST. PAULA, CALIF.
c         data  mpa/53.4/,mtr/1.43/,mtau/212./,mtb/10.42/,lambda/2.0/
c         data  ho/.1/,const/.457/,kapp/.25/,meanta/13.8/
          data  w/.0/,t/365./
          data  specw/7.1e4/,tens/7.5e-2/
          phi(d,os)=1./(d*(1.-os)**(1.45-.0375*d)+5./3.)

c         THE FOLLOWING INPUT PARAMETERS HAVE TO BE SPECIFIED:
c         minta = TEMPERATURE CHOSEN TO REPRESENT LOWER LIMIT OF PDF
c         kapr =  GROUNDWATER RESERVOIR COEFFICIENT, DAYS
c         rootd = DEPTH OF ROOT ZONE, CM
c         bs0ve1= SWITCH FOR BARE SOIL( 0 ), VEGETATION( 1)
c         ret10 = SWITCH FOR SURFACE RETENTION( 1 ), OTHERWISE( 0 )
          print,'input minta,kapr,rootdepth,bs0ve1,ret10'
          input,minta,kapr,rootd,bs0ve1,ret10
          pistol=1

c         INDEPENDENT SOIL-AND VEGETAL PARAMETERS, AND  LONG-TERM SO
c         so   = LONG-TERM SOIL MOISTURE CONTENT,
c         ke   = INTRINSIC SATURATED PERMEABILITY,CM**2
c         c    = PORE DISCONNECTEDNESS INDEX
c         n    = EFFECTIVE POROSITY OF THE MEDIUM
c         mo   = VEGETAL CANOPY DENSITY
c         kvo  = PLANT COEFFICIENT

  113     print,'so,ke,c,n,mo,kvo'
          input,so,ke,c,n,mo,kvo
          mo=mo*bs0ve1
          ho=ho*ret10

c         CALCULATION OF THE DEPENDENT CLIMATIC- AND SOIL PARAMETERS
c         ACCORDING TO EAGLESON(1978).
          mv=mtau/(mtr+mtb)
          mh=mpa/mv
          mi=mh/mtr
          m=2./(c-3.)
          PHI=10.**(.66+.55/m+.14/(m*m))
          k1=specw*ke*86400.
```

```
      psii=tens*(n/(ke*PHI))**.5
      d=c-1./m-1.
      soo=so-.02
      l=0
      do  333  jj=1,3
      l=l+1
      soo=soo+.01
      phii=phi(d,soo)
      phie=phe(d)
      maxta=2.*meanta - minta
      sigma=((5.*n*(1./mh)**2.*k1*psii*(1.-soo)**2.*phii)/(6.*3.14*
     &1./mtr*m))**(1./3.)
      call mgamma(sigma,gam,ier)
      G=1./mi*(.5*k1*(1.+soo**c)-w)
      Fiso(l)=sigma*gam/sigma**sigma*exp(-G-2.*sigma)
c     SIMPLIFIED PENMAN EQUATION, CHAPTER VI

      minep=(.42+.013*minta)*const
      meanep=(.42+.013*meanta)*const
      maxep=(.42+.013*maxta)*const
      e(l)=(2.*1./mtb*n*k1*psii/(3.14*m*(meanep-w)**2.))*phie*soo**(d+2.)
      epabs=mv*mtb*meanep
      EE=e(l)
c     000000000000000000000000000000000000000
      call  evtrfc(meanep,EE,w,mo,kvo,JJ)
      veg=1.-mo*(1.-kvo)
      call  era(meanep,kapp,mh,mtb,mo,kvo,ho,ret)
c     0000000000000000000000000000000000000000000
      j(l)=JJ
 333  continue
      derfis=(Fiso(3)-Fiso(1))/.02
      derJ=(j(3)-j(1))/.02
      fiso=Fiso(2)
      E=e(2)
      J=j(2)
      vegret=veg-ret
      epavr=epabs*vegret
      Era=epabs*ret/mpa
      if(fiso.le.Era)  goto  999
      goto  9999
 999  fiso=0.0
      derJ=0.0
      w=-epabs*ret/365.
 9999 denom=mpa*derfis+epavr*derJ+mtau*k1*c*so**(c-1.)+n*rootd
      A1=1./denom*(1.-fiso)
      A2=-1./denom*mv*mtb*vegret*J
      A3=-1./denom*(mtau*k1*so**c-t*w)
      w=0.0
      if(kapr.lt.1.e-6)  goto  57
      aa=exp(-t/kapr)
      ab=exp(mtau/kapr)-1.
      goto  58
 57   aa=0.0
      ab=100.
```

```
 58     rk=kapr*k1*aa*ab*c*so**(c-1.)
        C1=rk*A1
        C2=rk*A2
        C3=rk*A3
        D1=n*rootd*A1
        D2=n*rootd*A2
        D3=n*rootd*A3
        E1=mv*mtb*vegret*A1*derJ
        E2=mv*mtb*(vegret*J+ret+vegret*A3*derJ)
        E3=mv*mtb*A2*derJ*vegret
        afin=1.-C1-D1
        bfin=C2+D2+E2
        cfin=C3+D3
        B=.5*lambda/(const*.013)
        A=.5*exp(-(maxep-meanep)*2.*B)
        print,' A1      A2    minep    meanep   maxep   derfso derJ'
        print200,A1,A2,minep,meanep,maxep,derfis,derJ
 200  format('0',3x,e8.3,3x,f5.3,3x,3(f5.3,3x),f7.3,3x,f6.3)
        print,' sigma    G     fiso    E     J    epabs    denom'
        print100,sigma,G,fiso,E,J,epabs,denom
 100 format('0',3x,7(f8.3,2x))
        print,'   C1    C2    D1     D2     E1     E2'
        print300,C1,C2,D1,D2,E1,E2
 300  format('0',3x,f5.3,3x,f7.3,3x,f5.3,3x,f7.3,3x,f5.3,3x,f7.3)
        print,'  afin    bfin     A     B'
        print400,afin,bfin,A,B
 400  format('0',3x,f5.3,3x,f7.3,3x,2(e10.3,3x))
        print,'    z        Prob[Ya/mpa < z]'
        znorm=.0
        iii=0
 111    znorm=znorm+.1

c       8888888888888888888888888888888888888888888888888888888888888
        call numint(mv,mh,maxep,meanep,minep,E1,E3,cfin,afin,bfin,A,B,
 &      probab,kapp,znorm,pistol)
c       8888888888888888889888888888888888888888888888888888888888888

        print500,znorm,probab
 500    format(1x,f3.1,10x,f7.4/)
        if(iii.eq.13)  goto  112
        iii=iii+1
        pistol=2
        goto  111
 112    continue
        goto  113
        end

c       THIS SUBROUTINE DETERMINES THE CUMULATIVE FREQUENCY OF AN
c       ARBITRARY NORMALIZED ANNUAL BASIN YIELD(ZNORM), IN
c       ACCORDANCE WITH CHAPTERS V AND VI.

c       1111111111111111111111111111111111111111111111111111111111111
        subroutine numint(mv,mh,maxep,meanep,minep,E1,E3,cfin,afin,bfin,
 &      A,B,probab,kapp,znorm,pistol)
```

```
c         55555555555555555555555555555555555555555555555555555555555555

          real*8  vfac,znorm,z,kapp,k,error,mv,h,minep,maxep,meanep,ep
          real*8  fv,login,loginc,increm,maxim,mh,E1,E3,sum0n
          real*8  afin,bfin,cfin,B,sumga,A,C,cmin,cmax,n,probab
          real*8  fac(500),ga(150),integ(3),ratio(3),summ(3)
          real*4  p,maxili,minili
          integer v,vmax,pistol

c         generation of v-terms

          if(pistol.eq.2)  goto 11
          do  20  j=1,500
          vfac=.0d0
          do  10  ji=1,j
 10       vfac=vfac + dlog(dble(float(ji)))
          fac(j)=vfac
 20       continue

c         initiation of values

 11       z=znorm*mv*mh
          k=kapp
          error=1.e-5
          v=1

c         calculation of the v-th increment

          summ(1)=.0d0
          summ(2)=.0d0
          summ(3)=.0d0
          n=100.
 12       fv=float(v)
          p=k*fv
          l=1
          h=(maxep-minep)/n
          if(h.lt.error)  goto  1000
  1       ep=minep + float(l)*h
          xxy=(znorm+(E3*ep**2.+cfin+ep*bfin)/(mv*mh))/(afin-E1*ep)
          varili=mv*k*xxy

c         calc. of all discrete values of Fi(v=const)

          call mdgam(varili,p,probva,ier)
          C=exp(-abs(ep-meanep)*2.*B)
          ga(l)=C*dble(probva)
          ll= ifix(sngl(n))
          if(l-ll)  13,2,2
 13       l=l+1
          goto  1
c         numerical evaluation of the integrals  I(3),I(4)

  2       sumga=.0d0
          nn=n-1
```

```
        do  3  i=1,nn
    3   sumga=sumga + (3.+(-1.)**(i+1))*ga(i)
 1000   continue
        xx=(znorm+(E3*minep**2.+cfin+minep*bfin)/(mv*mh))/(afin-E1*minep)
        xy=(znorm+(E3*maxep**2.+cfin+maxep*bfin)/(mv*mh))/(afin-E1*maxep)
        minili=mv*k*xx
        maxili=mv*k*xy
c       minili=mv*k*znorm
c       maxili=minili
        call mdgam(minili,p,probmi,ier)
        call mdgam(maxili,p,probma,ier)
        cmin=exp(-abs(minep-meanep)*2.*B)
        cmax=exp(-abs(maxep-meanep)*2.*B)
        sum0n=cmin*dble(probmi) + cmax*dble(probma)
        integ(1)=probmi
        integ(2)=probma
        integ(3)=(sum0n + sumga)*h/3.

c       in case of constant ambient temperature we need
c       the following corrections
        if(h.lt.error)  goto  2000
        goto  3000
 2000   integ(3)=1.
        B=.0d0
 3000   continue
        if(integ(1).le..0)  goto  60

c       summation of all v-terms

        do  4  ii=1,3
        login=dlog(integ(ii))
        loginc=fv*dlog(mv) - fac(v) + login - mv
        if(loginc.le.-85.)  loginc=-85.
        increm=dexp(loginc)
        summ(ii)=summ(ii) + increm
    4   ratio(ii)=increm/summ(ii)
        maxim=dmax1(ratio(1),ratio(2),ratio(3))
        if(maxim.le.error)  goto  60
        v=v+1
        goto  12

c       end of calc. of the v-th increment

   60   vm=mv
        if(mv.gt.85.)  vm=85.
        prob=exp(-vm) + A*(summ(1)+summ(2)) + B*summ(3)
        probab=prob*100.
        return
        end

c       THIS FUNCTION SUBPROGRAM DETERMINES THE EXFILTRATIONPARAMETER

        function phe(d)
        dimension y(6)
```

```
      data  y/.18,.11,.077,.056,.044,.034/
      if(d.gt.7.)  goto  10
      x=d-1.
      i=ifix(x)
      frac=x-float(i)
      y1=alog(y(i))
      y2=alog(y(i+1))
      phe=exp((y2-y1)*frac+y1)
      return
10    phe=.034
      return
      end

c     THIS SUBROUTINE DETERMINES THE SURFACERETENTION

c     333333333333333333333333333333333333333333333333
      subroutine  era(meanep,kapp,mh,mtb,mo,kvo,ho,ret)
c     3333333333333333333333333333333333333333333333333

      real*8  meanep,kapp,mh
      real*4  mtb,mo,kvo,ho

      if(ho.lt.1e-5)  goto  1
      xlamb=kapp/mh
      xhme=ho/(mtb*meanep)
      xlah=xlamb*ho
      xlahk=xlah*kvo
      arg1=xlah+xhme
      arg2=xlahk+xhme
      quot1=1./(mtb*meanep*xlamb)
      quot2=quot1/kvo
      brack1=(1.+quot1)**(-kapp)
      brack2=(1.+quot2)**(-kapp)

      call  mdgam(xlah,kapp,trxlah,ier)
      call  mdgam(xlahk,kapp,txlahk,ier)
      call  mdgam(arg1,kapp,trarg1,ier)
      call  mdgam(arg2,kapp,trarg2,ier)

      x1=1-exp(-xhme)*(1.-trxlah)
      x2=brack1*trarg1
      x3=1.-exp(-xhme)*(1.-txlahk)
      x4=brack2*trarg2

      ret=(1.-mo)*(x1-x2) + mo*kvo*(x3-x4)
      goto  2
 1    ret=0.
 2    continue

      return
      end

c     THIS SUBROUTINE DETERMINES THE EVAPOTRANSPIRATION FUNCTION
```

```
c        77777777777777777777777777777777777777777777777777
         subroutine  evtrfc(meanep,e,w,mo,kvo,j)
c        7777777777777777777777777777777777777777777777

         real*8  meanep
         real*4  mk,mo,kvo,j

         wep=w/meanep
         mk=mo*kvo
         cc=(mk-wep)**2.
         if(cc-.005)  10,10,20
   10    c=100.
         goto  30
   20    c=(2.*cc)**(-1.)
   30    b=(1.-mo)/(1.+mk-wep)+(mk*mo+(1.-mo)*wep)/(2.*(1.+mk-wep)**2.)
         be=b*e
         ce=c*e

         call  mgamma(1.5,gamm,ier)
         call  mdgam(be,1.5,begam,ier)
         call  mdgam(ce,1.5,cegam,ier)

         if(be.gt.80.)  be=80.
         if(ce.gt.80.)  ce=80.
         gam=(cegam-begam)*gamm
         coeff=(1.-mo)/(1.-mo+mk)
         a=(1.+mk+e*(2.*b)**.5)*exp(-be)
         d=(mk+e*(2.*c)**.5)*exp(-ce)

         j=1.-coeff*(a-d-gam*(2.*e)**.5)

         return
         end
```