MIT

OPTIMAL IRRIGATION CONTROL USING STOCHASTIC CLUSTER POINT PROCESSES FOR RAINFALL MODELLING AND FORECASTING

by
Jorge Alberto Ramirez Rodriguez
and
Rafael L. Bras

RALPH M. PARSONS LABORATORY

HYDROLOGY AND WATER RESOURCE SYSTEMS

DEPARTMENT
OF
CIVIL
ENGINEERING

SCHOOL OF ENGINEERING
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Cambridge, Massachusetts 02139

Report Number 275

Prepared with the support of the
National Science Foundation

MAY 1982

R82-26

OPTIMAL IRRIGATION CONTROL USING STOCHASTIC CLUSTER POINT PROCESSES FOR RAINFALL MODELLING AND FORECASTING

by

Jorge Alberto Ramirez Rodriguez

and

Rafael L. Bras

RALPH M. PARSONS LABORATORY

HYDROLOGY AND WATER RESOURCE SYSTEMS

Report Number 275

Prepared with the support of the
National Science Foundation.

MAY 1982

ABSTRACT

Optimal irrigation control is performed. The control accounts for the intraseasonal variation of the crop water requirements and for the dynamics of soil moisture depletion process. The clustering dependence structure of rainfall occurrences is explicitly accounted for. Stochastic rainfall inputs to the soil-plant system are characterized by storm intensities, storm durations, interarrival times, and number of storms in a given period of time. Precipitation occurrences are modelled as a Neyman-Scott cluster process; and using Palm-Khinchin theory conditional distributions of the time to the next rainfall events are derived. These distributions are conditional on part of the immediate history of storm arrivals. The derived distributions are seen to possess characteristics desired for short term forecasting of rainfall occurrences. Particularly, they exhibit the ability to detect short term trends in precipitation occurrences.

The probabilistic description of precipitation is coupled with a probabilistic description of cumulative infiltration from storms and a Markov chain approach to the dynamics of soil moisture throughout the growing season. Conditional probabilities of soil moisture are derived and used within a Stochastic Dynamic Programming algorithm to obtain irrigation decisions. The control is obtained in the form of decision functions which yield the optimal irrigation depth as a function of soil moisture content at the root zone, volume of irrigation water available, and number of days since the last rainfall occurrence.

Case study results confirm the existence of a clustering dependence structure in rainfall occurrences as well as the goodness of the Neyman-Scott process in its modelling. However, there appears to be no significant difference in expected maximum net benefits when comparing results obtained with the control model under the homogeneous Poisson assumption and under the conditional Neyman-Scott model. Furthermore, slightly lower expected benefits are obtained with the conditional Neyman-Scott model than with the non-homogeneous Poisson model.

ACKNOWLEDGEMENTS

This report is based upon work supported by the National Science Foundation under NSF Grant No. 78-20245 ENG. It represents many months of effort which would not have been possible without the help and support of many people.

TABLE OF CONTENTS

Chapter | Page

Chapter		Page

LIST OF FIGURES

LIST OF TABLES

Chapter 1

INTRODUCTION

1.1 General Description of the Problem

Irrigated agriculture is one of the largest consumptive users of water in the world (Córdova and Bras, 1981). Increasing water scarcity produced by ever increasing alternative demands, and a steady rise in irrigation costs, such as water costs, and labor and energy costs, require that greater attention be paid to developing more efficient methods of irrigation water management. Optimal management of irrigation water could eventually lead to significant water conservation, to lower or steadier water costs, and to more reliable food supply in a world faced with forecasts of severe world-wide food shortages.

The general problem to be addressed is the optimal allocation of a finite amount of irrigation water throughout the growing season; specifically the problem is determining the timing of applications and the amount of water to be applied to a particular crop so as to optimize a given measure of performance; this is commonly known as the irrigation scheduling problem.

Considerable effort has been devoted to the study of the complex interactions of the main factors affecting the irrigation scheduling problem such as the characteristics of the climate-soil-plant system. In this work, the main effort is devoted to the climate portion of the climate-soil-plant system; specifically, to the modelling of the precipitation process.

Stochastic rainfall inputs to the soil-plant system are characterized by rainfall intensities, rainfall durations, inter-arrival times and number of storms within a given period of time. The random input to the soil, defined as the cumulative infiltration from a given rainstorm, is determined not only by the dynamics of the soil moisture depletion process, but also by the characteristics of the precipitation process. Consequently, the modelling of the rainfall is of primary importance in the achievement of efficient water use; especially in regions where water is a limited resource but where rainfall plays an important role as a water supply source.

In recent years, the mathematical theory of Point Processes has played an important role in describing the precipitation process and other processes driven by rainfall inputs (Gupta and Waymire, 1981). Kavvas and Delleur (1981) have shown the ability of the so-called cluster processes in modelling the statistical dependence of rainfall occurrences in the time domain. In particular, these authors apply the Neyman-Scott cluster process to model rainfall occurrences in Indiana. The main appeal of the cluster models is not only their ability to represent and preserve the statistical dependence in the occurrence of rainfall but also their ability to represent mathematically some recognizable basic physical structure of precipitation; namely, the clustering of rainfall events in time and space.

Storm arrivals have been often modelled as a Poisson process. This assumes that the number of storms within disjoint time intervals are independent. Within the context of the irrigation scheduling problem, Córdova and Bras (1979) used the Poisson model to obtain optimal irrigation control. However, the probabilistic independence of the Poisson assumption implies that the history of past rainfall occurrences contains no valuable information about the future of the process.

The existence of a dependence structure in the rainfall occurrence process in different regions of the world has been acknowledged by several authors in past years (Gabriel and Neumann, 1957, 1962; Smith and Schreiber, 1973; Kavvas and Delleur, 1975; and others). More recently, Kavvas and Delleur (1975), Gupta and Waymire (1981) and others have recognized that this dependence in precipitation is caused by the clustering of the rainfalls in time and space. Consequently, in this work, the Neyman-Scott cluster process is used to model the occurrence of rainfall in the time domain. Doing so it is possible to include into the decision process the conditional information contained in the history of storm arrivals as the growing season progresses. In this way, as opposed to the work of Córdova and Bras (1981) and Bras and Córdova (1981), the precipitation model becomes dynamic, changing throughout the growing season, according to the immediate history of storm arrivals.

To summarize, the problem is to allocate a finite amount of irrigation water during the growing season taking into account the seasonal variability of the crop response to soil moisture stresses, the dynamics of the soil moisture depletion process, and the randomness of the precipitation process. The allocation is carried out optimally so as to maximize net benefits. The solution is obtained in the framework of the general irrigation scheduling model presented by Córdova and Bras (1981) and Bras and Córdova (1981). Storm arrivals are modelled using the Neyman-Scott cluster model to account for the clustering dependence of the rainfall occurrences. Conditional information, contained in the rainfall occurrence process, is encoded in the model by deriving conditional distributions of the time to the next rainfall event. Finally, using Stochastic Dynamic Programming (SDP), a solution to the problem is obtained as optimal decision functions which yield the optimal amount of water to be applied at each decision stage as a function of the soil moisture content at the root zone, the volume of irrigation water available and the immediate history of storm arrivals.

1.2 Literature Review

Recent trends in hydrologic research indicate that the modelling of hydrologic processes starts by first recognizing their basic physical structure and then representing it mathematically. This has been permitted by a better understanding of the physical processes involved as well as by the use of mathematical tools adequate for modelling the recognized physical structure (Gupta and Waymire, 1981). In the following sections a brief literature review on precipitation modelling and irrigation scheduling is presented. For more detailed reviews, the reader is referred to Gupta and Waymire (1981), and Kavvas and Delleur (1975) on precipitation modelling; and to Córdova and Bras (1979), and Rhenals and Bras (1981) on irrigation scheduling.

1.2.1 On Rainfall Occurrences Modelling

Point Processes in general, and Counting Processes in particular, are naturally suited to describe the occurrence of rainfall events in time. Rainfall occurrences are modelled by counting the number of storm events in a given period of time. Depending on the definition of a rainfall event, the counting is carried out either in discrete time or in continuous time. However, it should be pointed out that even though the definition of a storm event becomes obscure (since storm events do not occur instantaneously), most descriptions of the storm arrival process are carried out in continuous time. Among the counting process descriptions, three different types of models can be identified in the literature. First, there are the models that assume that the sequence of rainy days and non-rainy days has no dependence structure. This assumption leads to the well known counting processes with independent increments. Second, there are the models that assume that the dependence structure of the sequence of rainy days and non-rainy days is Markovian. Finally, there are the models that recognize that clustering is the basic kinematic structure of space-time rainfall, and that this structure produces, in general, a non-Markovian dependence in the rainfall counts.

The sequence of rainy and non-rainy days can be represented by the binary sequence $\{W_i\}$, where W_i is equal to one if day i is rainy; and equal to zero otherwise. Models in the first of the above categories assume that the W_i's conform a sequence of independent and identically distributed random variables, with $\Pr[W_i=1] = p$ and $\Pr[W_i=0] = 1-p$. In discrete time, this assumption gives rise to the Binomial model for the random variable counting the number of storms in a given time interval. In continuous time, the above assumption leads to the well known and widely used Poisson models. Several authors have used the Poisson assumption to model rainfall occurrences. The complete spectrum of the Poisson models has been used; from the simple homogeneous model, to the compound and inhomogeneous forms of the Poisson model (Grant, 1938; Thom, 1959; Shane, 1964; Todorovic and Yevjevich, 1969; Duckstein, et al., 1972; Eagleson, 1978). The widespread use of the models with independent increments stems out from their simplicity, their manageability, and the ease with which their complete stochastic description is achieved as a simple product of marginal distributions. In the context of the irrigation scheduling problem, Córdova and Bras (1979) used the inhomogeneous Poisson model to describe the process of storm arrivals.

Acknowledging the inherent drawback of the independence assumption, some authors have resorted to assuming that the sequence of rainy days and non-rainy days has a Markovian dependence structure. First order, as well as higher order Markov models have been assumed to describe rainfall occurrences. As in the case of the Poisson model, both the homogeneous and the inhomogeneous forms of the models have been used (Gabriel and Neumann, 1957; Gabriel, 1959; Gabriel and Neumann, 1962; Caskey, 1963; Wiser 1965).

One of the main drawbacks of the above modelling schemes is their strongly localized applicability. In fact, for both the Poisson models and the Markov models, there exists evidence in the literature to support the fact that even though these models may describe, reasonably well, certain sets of data, they fail to do so when tested using different data (Wiser, 1965; Smith and Schreiber, 1973). With respect to the Poisson model, even the definition of a storm event has a bearing on how well rainfall occurrences can be described with the model in a given region (Todorovic and Yevjevich, 1969). Furthermore, some authors first assume the process to be Poisson, and then define a storm event to fit the assumption (Restrepo and Eagleson, 1979).

A stochastic process can be completely described by defining all of its finite dimensional probability distribution functions. In the case of the Poisson model it is easy to do so, since the independence assumption allows one to obtain the above distribution functions as products of simple marginal distributions. However, the independence assumption constitutes the main drawback of the model. In the Markovian case, only the marginal description of the counting process seems possible.

Both the Poisson models and the Markov models have been shown many times to be poor models for the rainfall occurrences (Wiser, 1965; Smith and Schreiber, 1973; Kavvas and Delleur, 1975). The former because of the independence assumption and the latter because the Markovian dependence fails to account for the observed clustering in the storm arrivals. Finally, both types of models can be considered as black box models in the sense that they are fitted to particular sets of data. Thus, their components and parameters lack physical meaning, and the models, as a whole, fail to represent any physical structure of the dynamics of space-time rainfall.

In the last category of models are those that account for the clustering dependence of the rainfall occurrences in time and space. The identification of certain physical features common to storm events has been possible from systematic observations of diverse types of storms (Petterssen, 1956; Houze, 1969; Austin and Houze, 1972). Gupta and Waymire (1979) and Waymire and Gupta (1981) provide an excellent description of the main characteristics of space-time rainfall. With respect to the clustering dependence, Kavvas and Delleur (1975) use the Neyman-Scott cluster model to describe storm arrivals in time. Rainfall occurrences in the form of clusters are assumed to be triggered by some rainfall generating mechanism (RGM) (cyclone belts, fronts, thunderstorm clouds, etc.). These RGM's constitute the primary level of the rainfall occurrence process. The actual occurrence of storms, triggered by the RGM's, constitutes the secondary level of the process. The observed dependence in the rainfall occurrences is explained by the superposition of storms triggered by different RGM's, or by the persistence of a certain type of RGM, over a given area. They applied the model, successfully, to describe rainfall occurrences in Indiana. Cluster models have also been shown to be adequate for modelling the space-time evolution of precipitation (Gupta and Waymire, 1979; Waymire and Gupta, 1981).

The advantages of the cluster models can be summarized as follows. First, they permit the definition of the complete stochastic structure of the process, a characteristic that is highly desirable for any model. This is easily obtained by using the concept of probability generating functionals (p.g.fl.). Second, their dependence structure is general enough to render the model generally applicable. Third, they are models physically based, in the sense that some physical meaning can be assigned to the model components and to their parameters. And last, but most important, they account for the observed clustering dependence of the rainfall occurrences.

1.2.2 On Irrigation Scheduling Problem

The irrigation scheduling problem can be viewed as a finite horizon, stochastic, multistage decision process. The ultimate objective in solving the scheduling problem is to find a sequence of irrigation decisions that optimizes a pre-specified measure of performance, under a given set of constraints and initial conditions. This objective can only be achieved after a description of the complex interactions taking place in the climate-soil-plant system. Several solutions to the irrigation scheduling problem can be found in the literature; all of them differing according to how the authors chose to model each subsystem of the climate-soil-plant system; and according to the solution algorithms employed.

Systems analysis techniques, such as simulation, linear programming, and dynamic programming have all been used to determine optimal policies. Simulation has been used to derive transition matrices for the soil moisture content within decision stages. Linear programming has been used to obtain optimal cropping patterns, as well as optimal irrigation scheduling when the irrigation applications are on fixed dates (Blank, 1975; Matanga and Mariño, 1977; Matanga and Mariño, 1979). Stochastic dynamic programming has also been widely used, especially when the irrigation applications are on variable dates (De Lucia, 1969; Hall and Dracup, 1970; Dudley, et al., 1971; Matanga and Mariño, 1979; Córdova and Bras, 1979; Rhenals and Bras, 1981).

The climate subsystem of the climate-soil-plant system is encoded in the models by describing potential evapotranspiration and precipitation. Potential evapotranspiration is often assumed deterministic (De Lucia, 1969; Hall and Dracup, 1970). It has also been considered as deterministic but varying throughout the growing season (Córdova and Bras, 1979) or considered stochastic and modelled as a first order

Markov process (Rhenals and Bras, 1981). The same comments can be made about the modelling of precipitation. Some authors ignore it all together (Rhenals and Bras, 1981), while some others consider it stochastic (De Lucia, 1969; Dudley et al., 1971; Córdova and Bras, 1979).

In general, most authors describe yield as a function of actual evapotranspiration. However, there exist discrepancies related to the form of this relationship. Some authors prefer a multiplicative form (Jensen, 1968; Minhas, et al., 1974; Hanks, 1974), others an additive formulation (Hiller and Clark, 1971; Stewart, 1974; Blank, 1975; Córdova and Bras, 1979). Finally, the soil system is generally described in terms of the soil moisture content at the root zone. A water balance in a conceptual soil column defines this state variable. The water balance is carried out either analytically (Córdova and Bras, 1979) or by simulation (Matanga and Mariño, 1979).

1.3 Thesis Outline

Chapter 2 reviews some elemental but important concepts from the theory of Point Processes. Probability Generating Functions (PGF), probability generating functionals (p.g.fl) and cluster processes are introduced.

The development of the Neyman-Scott cluster process is presented in Chapter 3. First and Second order moments are presented. Finally, the conditional distributions of the time to the next rainfall occurrence are derived.

The general irrigation scheduling model is then described in Chapter 4. The conditional distributions derived in Chapter 3 are then included in the model and transition matrices for the soil moisture state are derived. Finally, the irrigation scheduling problem is formulated as a multistage decision process and solved by stochastic dynamic programming (SDP).

Case study results and model calibration issues are presented and discussed in Chapter 5. Finally, Chapter 6 gives a brief summary of the work. Conclusions are presented and recommendations for future research are made.

Chapter 2

BRIEF REVIEW OF SOME FUNDAMENTAL CONCEPTS OF THE THEORY OF STOCHASTIC POINT PROCESSES

2.1 Introduction

This chapter presents a very brief and quick review of some of the fundamental concepts of the Theory of Point Processes. It relies heavily on the works of Neyman and Scott (1952), Jowett and Vere-Jones (1971), Daley and Vere-Jones (1971), and Waymire and Gupta (1981). The review is intended to be neither complete, nor mathematically rigorous. Instead, its purpose is to provide an understandable working basis, so that the reader may get a quick glance at the theory, its computational tools, and its potential applications. Especially, the emphasis is on those concepts which play a major role in the development of this study. For more detailed, more complete, and more mathematically rigorous treatments, the reader is encouraged to study the above papers and the references given therein.

As was presented in Chapter 1, the main concern of this work is the modelling of rainfall occurrences in time. The final objective is the incorporation of the observed clustering dependence structure of space-time rainfall occurrences into a decision model to optimize irrigation decisions. To do so in the time domain, rainfall occurrences in time are conceptualized as a point process. In the following sections, some elemental concepts from the Theory of Point Processes are presented. These concepts are needed in order to understand the description of rainfall occurrences as a point process and in the development of the precipitation model.

Specifically, the first section defines a point process and establishes the duality between its counting properties and its interval properties. In the next section, probability generating functions and functionals are introduced as a means to completely define a point process in terms of its joint finite dimensional multivariate distribution functions. Also, first and second-order moments are introduced. Following, cluster processes are introduced very briefly. In the last section, some concepts from the Palm-Khinchin theory are presented. They are used in a later chapter to derive conditional distributions from the precipitation model.

2.2 Definition and Basic Properties

A stochastic point process is a mathematical abstraction which arises when considering point occurrences of certain random phenomena; for example, rainfall occurrences in time or in space-time, equipment failure in time, customers arriving at a queueing facility, earthquake occurrences, etc. Thus, to define a point process, a state space over which the random phenomenon evolves, and a sequence of points in that space representing a possible realization of the phenomenon, are needed. Two basic characteristics can be defined for a point process: first, the counting properties which relate to the number of points falling within specified subsets of the state space; and second, the interval properties which relate to the relative spacings between points. For example, in the case of rainfall occurrences in time, the counting properties refer to the number of rainfall occurrences in a given period of time, while the interval properties refer to the relative times between the occurrences. Both properties serve to uniquely define a point process, and in that sense, both are equivalent. However, the relationship between the counting properties and the interval properties is not simple. This problem of expressing the ones in terms of the others is addressed in Section 2.5.

The definition of a stochastic point process is most conveniently given in terms of its counting properties. To do so, a counting measure which counts the number of occurrences within sub-regions of a given space is defined, and then the probability distribution of these counts is studied.

Let X be the state space over which the random phenomenon occurs. X can be taken as the real line R_1^+, or in general as any finite dimensional Euclidean space. A point process is, then, a collection of non-negative integer valued random variables N(A), parameterized (indexed) by subsets A of the state space X, and for which the following conditions hold true:

$$N(\emptyset) = 0 \tag{2.1}$$

where $\emptyset$ represents the empty set;

$$N(A) < \infty \tag{2.2}$$

with probability one for bounded sets A; and

$$N\left[\bigcup_{n=1}^{\infty} A_n\right] = \sum_{n=1}^{\infty} N(A_n) \tag{2.3}$$

with probability one for mutually disjoint sets $A_1, A_2, \ldots$. Thus, the non-negative integer valued random variable N(A) represents the number of occurrences of some random phenomenon within the interval or region $A \subset X$.

Consider now Ω as the family of all countable sequences of points in X:

$$w = \{x_n\} \qquad n \in Z_+ \equiv \{1, 2, \ldots\} \tag{2.4}$$

where w represents a possible realization of point occurrences in X. A fundamental theorem in the Theory of Point Processes states that every point process, N(·), induces a unique probability measure, Pr, on Ω, and conversely (Waymire and Gupta, 1981; also see Moyal, 1962 for proof of theorem). The counting measure for the point process, N(·), is defined for each possible realization $w \in \Omega$ as:

$$N(A,w) = \text{card}\{n: x_n \in w \cap A\}, \; w \in \Omega \; , \; A \subset X \tag{2.5}$$

The counting measure of equation 2.5, counts the number of occurrences of some random phenomenon in a region $A \subset X$, which were produced by a particular realization $w \in \Omega$. Observe that in this sense, the counting measure description of a point process, N(·), is a function of each possible realization w, of the random phenomenon. Thus, it should be clear that there is a one-to-one correspondence between the probability space associated with the sample space, Ω, and the space of non-negative integer valued counting measures, N(·). Consequently, a stochastic point process can be defined as a mapping from the former space into the latter. With this in mind, and assuming that $w \in \Omega$ is mapped into N(·), with value N(A,w) on $A \subset X$, the conditions 2.1 through 2.3 can be expressed as:

a) Let $\{A_n\}$, $n \in Z_+$, be a decreasing sequence of bounded sets A_n, such that as $n \to \infty$, $A_n \to \emptyset$; then:

$$\Pr\{w: N(A_n, w) \to 0\} = 1 \tag{2.6}$$

b) Let A be any bounded set, then:

$$\Pr\{w: N(A, w) < \infty\} = 1 \tag{2.7}$$

c) For every pair of disjoint sets A_1, A_2,

$$\Pr\{w: N(A_1 \cup A_2, w) = N(A_1, w) + N(A_2, w)\} = 1 \tag{2.8}$$

where $w = \{x_n\} \in \Omega$ is a possible realization, and $A \subset X$ represents subsets of the state space. $N(\cdot,\cdot)$ is as defined in equation 2.5.

According to equation 2.5, the counting measure, $N(\cdot)$, of a point process is a function of w. However, for notational convenience and unless otherwise stated, $N(\cdot)$ and $N(\cdot,\cdot)$ are used interchangeably in the sequel. But it should be stressed again that $N(\cdot) \equiv N(\cdot,\cdot)$ is well defined only as a function of a realization of the random phenomenon.

The complete mathematical definition of a stochastic point process, as introduced above, is obtained by specifying its complete finite dimensional structure. This can be accomplished by specifying all of its joint finite dimensional multivariate probability distribution functions (Daley and Vere-Jones, 1971). In the next section, this complete mathematical description is obtained straightforwardly after the introduction of the probability generating functionals (p.g.fl.).

2.3 Moments and Probability Generating Functionals

The moments of a point process to be discussed in this section, refer to the counting measure description of the point process.

The first moment or expectation measure:

$$M_1(A) \equiv E\{N(A)\} \tag{2.9}$$

is said to exist when it is finite for all bounded sets A. In equation 2.8, $E\{\cdot\}$ stands for the expectation operator. In general, higher order moments can be defined as:

$$M_r(A_1, \ldots, A_r) \equiv E\{N(A_1) \ldots N(A_r)\} \tag{2.10}$$

It should be stressed that the second-order moment measure $M_2(\cdot) \equiv E\{N(A_1)\ N(A_2)\}$ can indicate dependence between $N(A_1) \equiv N(A_1,w)$ and $N(A_2) \equiv N(A_2,w)$.

The probability generating functional (p.g.fl.), introduced below, is a natural generalization of the probability generating function (PGF), from a non-negative integer valued random variable to an infinite family $N(\cdot)$ of non-negative integer valued random variables. Let $A_1, \ldots A_k$, be fixed but arbitrary subregions of X. The joint distribution of the random vector $[N(A_1), \ldots, N(A_k)]$ can be uniquely described in terms of its multivariate PGF (Feller, 1968) as:

$$g(t_1, \ldots, t_k) = E\left\{\prod_{n=1}^{k} t_n^{N(A_n)}\right\} , \; 0 \leq t_n \leq 1 \; , \; 1 \leq n \leq k \qquad (2.11)$$

or more explicitly:

$$g(t_1,\ldots,t_k) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_k=0}^{\infty} \prod_{n=1}^{k} t_n^{j_n} \Pr\{N(A_1)=j_1, N(A_2)=j_2 \ldots N(A_k)=j_k\}$$

$$0 \leq t_n \leq 1 \; , \; 1 \leq n \leq k \qquad (2.12)$$

A PGF so defined is indefinitely differentiable with respect to all of its arguments for $0 \leq t_n \leq 1$ and $1 \leq n \leq k$. The partial derivatives of the PGF are related to the multivariate joint distribution of the random vector $[N(A_1), \ldots, N(A_k)]$, by:

$$\Pr\{N(A_1)=j_1,\ldots,N(A_k)=j_k\} = \left. \frac{\partial^{j_1+j_2+\ldots+j_k} g(t_1,\ldots t_k)}{\partial t_1^{j_1} \, \partial t_2^{j_2} \ldots \partial t_k^{j_k}} \right|_{t_1=t_2=\ldots t_k=0} \prod_{n=1}^{k} \frac{1}{j_n!} \qquad (2.13)$$

Moments of any order are also related to the partial derivatives of the PGF. In particular, the first and second-order moments can be expressed as follows. The expectation of the number of occurrences of A is (Appendix A):

$$M_1(A) \equiv E\{N(A)\} = \left.\frac{\partial g(t)}{\partial t}\right|_{t=1} \tag{2.14}$$

The variance of N(A) is (Appendix A):

$$M_2(A,A) \equiv \mathrm{var}[N(A)] = \left.\frac{\partial g(t)}{\partial t} + \frac{\partial^2 g(t)}{\partial t^2} - \left[\frac{\partial g(t)}{\partial t}\right]^2\right|_{t=1} \tag{2.15}$$

Finally, the covariance between $N(A_1)$ and $N(A_2)$ is (Appendix A):

$$M_2(A_1,A_2) \equiv \mathrm{cov}[N(A_1),N(A_2)] = \left.\frac{\partial^2 g(t_1,t_2)}{\partial t_1\, \partial t_2}\right|_{t_1=t_2=1}$$

$$- \left[\left.\frac{\partial g(t_1,t_2)}{\partial t_1}\right|_{t_1=t_2=1} \cdot \left.\frac{\partial g(t_1,t_2)}{\partial t_2}\right|_{t_2=t_1=1}\right] \tag{2.16}$$

Now, in order to define the p.g.fl., first define the stochastic integral of a real valued function f on X, with respect to the point process N(•) as:

$$\int_X f(x)\, dN(x,w) = \sum_n f(x_n) \qquad w = \{x_n\} \tag{2.17}$$

$$n \in Z_+ \equiv \{1, 2, \ldots\}$$

where N(•,•) is as defined in equation 2.4.

Equation 2.10 can be rewritten as:

$$g(t_1,\ldots, t_k) = E\left\{\exp\left[\log \prod_{n=1}^{k} t_n^{N(A_n)}\right]\right\}$$

$$g(t_1,\ldots, t_k) = E\left\{\exp\left[\sum_{n=1}^{k} N(A_n) \log(t_n)\right]\right\} \qquad (2.18)$$

Keeping in mind definition 2.17, let $\xi(\cdot)$ be the function on X given by:

$$\xi(x) = \begin{cases} t_n , & x \varepsilon A_n \qquad 1 \leq n \leq k \\ 1 , & x \notin \bigcup_{n=1}^{k} A_n \end{cases} \qquad (2.19)$$

then, using the stochastic integral defined in equation 2.17:

$$\int_X \log\xi(x) \, dN(x) = \sum_{n=1}^{k} N(A_n)\log t_n \qquad (2.20)$$

Substituting equation 2.20 in equation 2.18:

$$g(t_1,\ldots,t_k) = E\{\exp[\int_X \log (x) \, dN(x)]\} \qquad (2.21)$$

From equation 2.21, it is clear that in order to represent the entire finite dimensional structure of a point process $N(\cdot)$, and consequently its complete probabilistic structure, it is only necessary to consider the following functional (Waymire and Gupta, 1981):

$$G(\xi) = E\{\exp[\int_X \log\xi(x)\ dN(x)]\} \qquad (2.22)$$

for arbitrary real valued functions $\xi(\cdot)$ on X such as those defined in equation 2.19. In general, the function $\xi(\cdot)$ has to satisfy only the following conditions:

$$0 \leq \xi(x) \leq 1\ ,\ x \in X \qquad (2.23)$$

and, for a given x outside a defined subset of X:

$$\xi(x) \equiv 1 \qquad (2.24)$$

The functional defined by equation 2.22 is referred to as the probability generating functional of the point process $N(\cdot)$. The p.g.fl. is a powerful, but simple, transform technique due to Moyal (1962). Its development and use have been expanded by Vere-Jones (1968, 1970), and Westcott (1972). More recently, Waymire and Gupta (1981) have also presented the p.g.fl. and used it in the context of modelling physical random phenomena. The p.g.fl. is absolutely necessary if a complete stochastic description of a point process is needed. In a later chapter, it will be shown how, with suitable choices of the function $\xi(x)$ defined by conditions 2.23 and 2.24, all the joint finite dimensional multivariate probability distribution functions of a point process can be obtained from its p.g.fl., thus obtaining a complete stochastic description of the point process.

2.4 Cluster Processes

One of the most common types of dependence structure encountered in natural phenomena is the so-called clustering dependence. In Chapter 1, it was stressed that this kind of dependence has been determined to exist in the occurrence of rainfall in time and space. In fact, authors like Le Cam (1961), Kavvas and Delleur (1975, 1981), and Waymire and Gupta (1981) have acknowledged this cluster structure in the context of precipitation modelling. The spatial distribution of galaxies has also been hypothesized to occur in clusters and modelled as such by Neyman and Scott (1952). Cluster dependence structure has also been observed in the occurrence of earthquakes and their after-shocks. Vere-Jones (1970) used cluster processes to model earthquake occurrence. In fact, the literature is full of examples of various natural phenomena exhibiting clustering behavior. In the hydrologic literature, this is true, especially during recent years, when works by Kavvas and Delleur (1975, 1980), Kavvas (1982), and Waymire and Gupta (1979, 1981) have popularized the use of cluster models.

One very important class of cluster processes is the so-called Moyal Cluster Processes. They are the superposition of two different point processes, and can be constructed in the following manner. Define on X, a point process $N_1(\cdot)$ as the process of cluster centers. Also, for each $x \in X$, define $N_2(\cdot/x)$ as the point process of cluster members. Then, the cluster process is given by:

$$N(\cdot) = \sum_{x_n \varepsilon N_1(\cdot)} N_2(\cdot/x_n) \tag{2.25}$$

The p.g.fl. of the cluster process defined in equation 2.25 is (Waymire and Gupta, 1981):

$$G(\xi) = G_1\{G_2(\xi/x)\} \tag{2.26}$$

where $G_1(\cdot)$ is the p.g.fl. of $N_1(\cdot)$ and $G_2(\cdot/x)$ is the p.g.fl. of $N_2(\cdot/x)$.

Among the members of the class of cluster processes defined by equations 2.25, the one used in this work is the so-called Neyman-Scott cluster process. This model was introduced by Neyman and Scott (1952) to model the spatial distribution of the galaxies. The cluster process of equation 2.25 becomes a Neyman-Scott cluster process whenever the process of cluster centers $N_1(\cdot)$ is Poisson and whenever a random number of points $N_2(\cdot/x)$ are independently distributed about the cluster centers according to a common distance distribution. When cluster models of the kind presented above are used for modelling precipitation occurrence in time, one effectively assumes that rainfall occurrence is a two-level process. In the primary level, or parent process, there is the occurrence of rainfall generating mechanisms (RGM's), $N_1(\cdot)$, or cluster centers. In the secondary level, for each cluster center at x, there is the actual associated number of rainfall events, $N_2(\cdot/x)$, or cluster members. The RGM's can be, for example, cyclonic belts persisting over a region, or fronts sweeping over a given area. The next chapter will deal more explicitly with the modelling of precipitation occurrence in time as a cluster process. Before closing this section, it should be made clear that the cluster processes are just the result of operating on a simple point process as defined in equations 2.1 through 2.8. Just as a given point process can be operated on to transform it into a compound point process by replacing each point in the original process $N(\cdot)$ with an associated random variable, it is possible to obtain a cluster process by replacing each point in the original process $N(\cdot)$ with a cluster of points (Daley and Vere-Jones, 1970).

The closing section of this chapter introduces some basic concepts from the Palm-Khinchin theory. These concepts will allow the derivation of the conditional probability distributions of storm occurrence, that will later be used to obtain optimal irrigation decisions throughout the growing season.

2.5 Some Concepts from the Palm-Khinchin Theory

Given that the precipitation occurrence process exhibits cluster dependence and that, in general, this cluster behavior invalidates the common assumption of independence between the rainfall counts by introducing a non-negligible correlation in the occurrence process, it is highly desirable to be able to use this additional information in the forecasting of rainfall occurrence. To do so, conditional probability distribution functions are needed. The approach taken in the next chapter to obtain conditional information is to derive the conditional distribution functions of the time to the next rainfall event, conditional on the immediate history of storm arrivals. As can be inferred, now it is necessary to have either a description of the point process in terms of its interval properties, or a way of obtaining interval properties from a counting measure description of the given process.

Restricting the discussion to the real line or one-dimensional Euclidean space, where the notion of an interval is more easily comprehensible, and keeping in mind that the work to be carried out in later chapters is in the real line, the interval description of a point process is presented as follows.

Again, as in equation 2.4, consider the family Ω of sequences $w = \{x_n\}$. Equation 2.5, defined on subsets of X, allowed the description of the point process $N(\cdot)$ for each possible $w \in \Omega$. All that is needed now is to obtain the inverse description of equation 2.5. To do so, for each $w = \{x_n\} \in \Omega$ define:

$$x_n = y > 0 \quad \text{such that} \quad N[0,y) \leq n < N[0,y] \;, \; n = 0, 1, \ldots \tag{2.27}$$

In equation 2.27, the argument of the set function $N(\cdot) \equiv N(\cdot,\cdot)$ defined in equation 2.5 is an interval. However, instead of writing, for example, $N([a,b))$, here and in the sequel, the outer parentheses are omitted for notational convenience. Consequently, $N[0,y)$ counts the number of occurrences in the semi-closed interval $[0,y)$ and $N[0,y]$ counts the number of occurrences in the closed interval $[0,y]$. It is easy to observe that equation 2.27 is the inverse of equation 2.5. To construct a sample realization of a point process, it is required to specify the points $\{x_n\}$. According to the duality between equation 2.5 and equation 2.27, an equivalent construction is the specification of such quantities as $\{\eta_n\} \equiv \{x_n - x_{n-1}\}$, representing the sequence of times (intervals) between successive events. Denoting by H the space of all such sequences, by virtue of equation 2.5 and equation 2.27, there exists a one-to-one correspondence between Ω and H.

Having defined the duality between equation 2.5 and equation 2.27, the so-called Palm Functions are now introduced. These functions relate probability distributions Pr on Ω (counting properties) to probability distributions on H (interval properties). The Palm Functions are defined as:

$$q_k(y) = \lim_{x \to 0} Pr\{N(0,y] = k/N(-x,0] > 0\} \quad y > 0 \ , \ k \in Z_+ \tag{2.28}$$

When the intensity of the process defined as:

$$\lambda = \lim_{x \to 0} Pr\{N(0,x] > 0\}/x \tag{2.29}$$

is finite, these limits are shown to exist, for stationary Pr, in the works of Khinchin (1955), and Daley and Vere-Jones (1970). Observe that equation 2.28 simply yields the limit of the probability that k events occur in an interval (0,y], given that at least one event has occurred in an arbitrary, immediately preceding interval, (-x,0], as the length of this interval goes to zero. Also, observe that equation 2.28 expresses a relationship between a function of an interval measure y, and a function of a counting measure, N(•).

One final result obtained by Khinchin allows the Palm Functions of equation 2.28 to be obtained as derivatives of the probability distribution Pr of N(•) on Ω as follows. Khinchin (1955) proves that for a stationary point process with single occurrences and finite intensity λ:

$$Pr\{N(0,x] \leq k\} = 1 - \lambda \int_0^x q_k(u)\ du = \lambda \int_x^\infty q_k(u)\ du \tag{2.30}$$

$$x > 0 \ , \ k \in Z_+$$

Denoting $\Pr\{N(0,x] \leq k\} \equiv P_k(x)$, equation 2.30 can be expressed in differential form as:

$$D\,P_k(x) = -\lambda\, q_k(x) \tag{2.31}$$

Generalization of equation 2.30 leads ultimately to the highly desirable one-to-one relation between the description of a stochastic point process in terms of its counting properties and that in terms of its interval properties (Daley and Vere-Jones, 1970). For the purposes of this work, equations 2.27 through 2.31 are sufficient.

2.6 Summary

Some of the basic concepts from the Theory of Point Processes have been reviewed. The duality between the counting measure description and interval measure description of a point process has also been introduced. The p.g.fl. has been defined and presented as a powerful tool in the definition of the complete stochastic structure of a point process. Furthermore, cluster processes were also defined as a way to model natural clustering behavior. Finally, some basic functions establishing the relationship between the counting description and the interval description of a point process were defined. The entire chapter, instead of pretending to be a deep mathematical treatise of the theory, is intended to provide a quick reference of the elemental concepts used in the following chapter in developing the precipitation occurrence model.

Chapter 3

PRECIPITATION MODEL: FORECASTING RAINFALL OCCURRENCES USING THE NEYMAN-SCOTT CLUSTER MODEL

3.1 Introduction

The main goal of this work is the incorporation of the conditional information contained in the immediate history of rainfall occurrences, into a model of optimal irrigation control. In recent years, several authors (see Chapter 1) have argued that the observed dependence structure in the precipitation process is due to the clustering of the rainfall occurrences in time. It is then highly desirable that this cluster dependence structure be correctly modelled. This would allow, in some cases, the derivation of conditional probability distribution functions (CDF's) useful, for instance, in obtaining optimal irrigation control.

In this chapter, the process of rainfall arrivals is modelled as a Neyman-Scott (N-S) cluster process. Using basic concepts from the theory of point processes, presented in the previous chapter, the complete mathematical stochastic structure of the N-S model is obtained. First and second-order moments are also derived. Finally, using the Palm-Khinchin theory, conditional distributions of the time to the next rainfall occurrence, conditional on all or part of the immediate history of the process, are derived. These CDF's will then be included into the general formulation of the irrigation scheduling problem in Chapter 4. The probabilistic description of the occurrence of precipitation effectively accounts for the observed clustering dependence in the rainfall counts, and through the CDF's, yields a dynamic model that changes according to the immediate history of the process. It is then possible to use this information to predict future behavior.

3.2 The Neyman-Scott Cluster Model

The N-S cluster model is a particular member of the more general class of Moyal cluster processes introduced in Section 2.4. It is a two-level process in which the process of cluster centers is Poisson, and in which a random number of cluster members are identically distributed about each cluster center. The N-S cluster process was originally introduced and derived by Neyman and Scott (1952), and used in modelling the spatial distribution of galaxies. More recently, and in the context of precipitation, Kavvas and Delleur (1975, 1981), used it to model rainfall occurrences in the time domain. The N-S cluster model proposed by Kavvas and Delleur (1981) has the RGM's in its primary level (parent process), and the actual occurrence of the rainfalls generated by each RGM in its secondary level.

Le Cam (1961) used the Neyman-Scott cluster process for modelling the areal clustering of precipitation. The basic element in Le Cam's model is the shower cell. These occur in clusters which correspond to fronts; and the fronts also occur in clusters called storms. Vere-Jones (1970) applied the N-S cluster process in the time domain to model earthquake occurrences.

In the work of Kavvas and Delleur (1975, 1981), the RGM's correspond to the fronts of Le Cam. In their work, they hypothesize that, for the case of rainfall occurrences in Indiana, the clustering dependence in the rainfall counts is caused, not only by the persistence of a given RGM over a region, but also by the actual superposition of two or more RGM's. The clusters in their model are made up of the rainfalls generated by the same RGM. The clustering behavior of the rainfall arrival process in Indiana is tested and determined by observing the behavior of the estimated variance-time and log-survivor functions. The convexity of the respective estimated functions indicates an overdispersion and clustering of the rainfall occurrences. In Chapter 6, these functions will be analyzed for the particular case study under consideration.

3.2.1 Assumptions and Definitions

The fundamental assumption in constructing the N-S model for precipitation is that the rainfall occurrence process exhibits a cluster dependence structure in time. Effectively, the N-S cluster model assumes that precipitation events occur in clusters in the time domain. It also assumes that the occurrence of rainfall events in any given period of time is not only caused by RGM's which occurred in the given period, but may also be caused by RGM's which occurred previously. Following the original work of Neyman and Scott (1952), the following so-called "structural postulates" are essential to the N-S model:

a) Precipitation events occur in clusters in the time domain.

b) Cluster centers are determined by the times of occurrence of the RGM's. It is assumed that these cluster centers are randomly distributed according to a Poisson model.

c) To each cluster center, there exists an associated group of rainfalls forming the cluster. Each cluster is characterized by the number of rainfalls within the cluster (cluster sizes), and by their time of occurrence with respect to the cluster center.

d) The cluster sizes are mutually independent and identically distributed and also independent of all other variables in the process.

e) For any given cluster, the times of occurrence of events within the cluster are independent, identically distributed random variables.

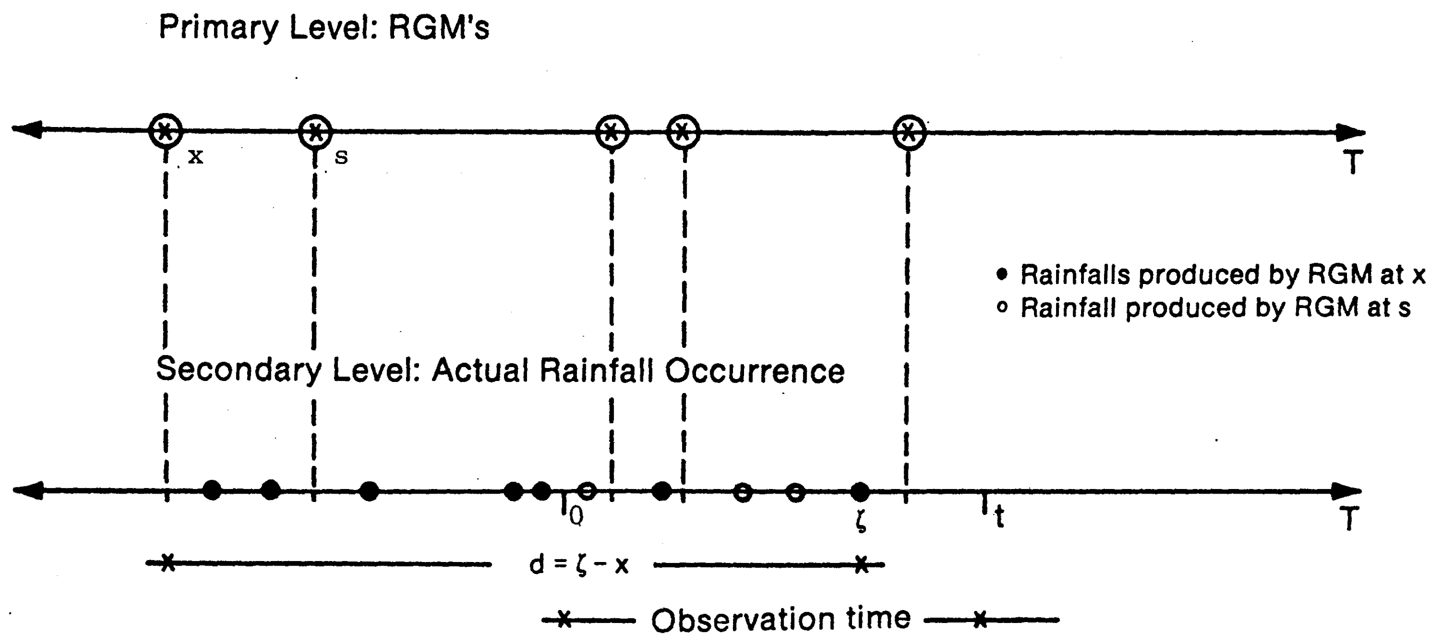
Finally, the main assumption upon which the complete model rests is simply that rainfall occurrences in time can be modelled as a Point Process. Accepting the above assumptions, the N-S cluster model can be constructed in terms of the following elemental random variables (see Figure 3.1).

Let N(0,t) be the counting random variable, counting the number of rainfall events in the interval (0,t). N(0,t) is as defined throughout Chapter 2, and describes the following Point Process:

$$\{N(0,t) \ , \ t > 0 \ , \ t \in T\} \ , \qquad (3.1)$$

which is now to be modelled as a N-S cluster process.

Let $N_1(a,b)$ be the counting random variable, counting the number of RGM's in time interval (a,b). As in Section 2.4, $N_1(\cdot)$ defines the process of cluster centers:



$N(o,t)$	Number of Rainfall Occurrences in (o,t).
$N_{1(-\infty,t)}$	Number of RGM Occurrences previous to t.
$N_2(x)$	Number of Rainfalls produced by RGM at x.
Υ	Time positions of rainfalls within cluster.

Figure 3.1

TWO LEVEL SCHEMATIC DESCRIPTION OF RAINFALL OCCURRENCES

$$\{N_1(a,b)\ ,\ -\infty < a < b < \infty\ ,\ \ a,b \in T\}\ . \tag{3.2}$$

According to structural postulate b), $N_1(\cdot)$ is Poisson with parameter μ.

Also, let $N_2(x)$ be the random variable representing the number of cluster members (rainfall events) in a cluster (RGM) centered at time x; let its PGF be $g_{N_2}(z) = \sum_{n=1}^{\infty} z^n \Pr[N_2(x) = n]$.

Finally, let $f_T(\zeta - x)$ denote the probability density function (p.d.f.) of the time positions of the cluster members within each cluster. In this way, $d = \zeta - x$ represents the time distance between the occurrence of the RGM at x, and the actual occurrence of the rainfall being considered, at ζ.

According to these definitions and assumptions, the scheme considered here to model precipitation arrivals is the same as introduced by Kavvas and Delleur (1981), in which RGM's constitute the primary level of the process and in which each RGM generates a cluster of rainfalls. The N-S cluster model is such that the random variable $N(\cdot)$ is constructed in terms of $N_1(\cdot)$, $N_2(\cdot)$, and T.

3.2.2 Complete Stochastic Structure of the Neyman-Scott Model

As was stated in Chapter 2, in order to completely define a stochastic point process, it is necessary to determine all of its joint finite dimensional multivariate probability distribution functions (PDF). Moyal (1962), and later Vere-Jones (1968), developed a powerful technique, the probability generating functional, that can be used to obtain the above definition of a stochastic point process. In principle, and as defined in Section 2.3, the p.g.fl. can be used to obtain the distributions of all random variables determined by the point process. This is so since, in heuristic terms, the p.g.fl. is an extension of the multivariate PGF, and it is well known that the latter uniquely determines the distribution of a given random vector. However, in practice, the main disadvantage of the p.g.fl. is that it may be impossible to obtain in closed form. Fortunately, for the case of cluster processes derived from the Poisson process, such as the N-S cluster process, this closed form of the p.g.fl. exists and consequently, these processes can be handled very nicely in terms of their p.g.fl.'s.

Neyman and Scott (1952) first obtained the characterization of the N-S cluster process in terms of its bivariate PGF; at this time the technique used was somewhat primitive. Later, Kavvas and Delleur (1975), using the same technique and following the derivation of Neyman and Scott (1952), obtained again the univariate and bivariate PGF for the N-S cluster process. In these two instances, only a partial definition of the process was obtained. The first to have suggested the use of the p.g.fl. in the context of the N-S cluster process appears to have been Moyal (1962). Later, Vere-Jones (1970) derived its p.g.fl. in the context of modelling earthquake occurrences in time.

In general, the p.g.fl. of a point process, $N(\cdot)$, was defined as (Section 2.3):

$$G[\xi] \quad = \quad E\{\exp[\int \log\ (t)\ dN(t)]\} \qquad (3.3)$$

for a given class of functions $\xi(\cdot)$. In equation 3.3, as well as throughout the rest of this chapter, and unless otherwise stated, all integrations are to be taken over a doubly infinite set.

The p.g.fl. for the general Moyal cluster process:

$$N(\cdot) \quad = \quad \sum_{t_n \,\varepsilon\, N_1(\cdot)} N_2(\cdot\,|\,t_n) \qquad (3.4)$$

was also introduced in Chapter 2 as:

$$G[\xi] \quad = \quad G_1[G_2(\xi\,|\,t)] \qquad (3.5)$$

Now, according to the assumptions and definitions of Section 3.2.1, the process of RGM's, $N_1(\cdot)$, is Poisson with parameter μ. The p.g.fl. of the Poisson process is (Vere-Jones, 1970):

$$G_1[\xi] \quad = \quad \exp\{\mu\int[\xi(t) - 1]dt\} \qquad (3.6)$$

The p.g.fl. of the process of cluster members, $N_2(\cdot)$, is (Vere-Jones, 1970):

$$G_2[\xi|t] = g_{N_2}\{\int \xi(t+\zeta)\, f_T(\zeta)d\zeta\} \qquad (3.7)$$

where $g_{N_2}(\cdot)$ is the PGF of the cluster member process, $N_2(\cdot)$.

Substituting equations 3.6 and 3.7 into equation 3.5, the p.g.fl. for the N-S cluster process is obtained as:

$$G[\xi] = \exp\left\{\mu \int [g_{N_2}\{\int \xi(t+\zeta)\, f_T(\zeta)d\zeta\} - 1]dt\right\} \qquad (3.8)$$

Under suitable choices for the function $\xi(\cdot)$, equation 3.8 yields the complete finite dimensional structure of the N-S cluster process. For example, to obtain the univariate representation of the N-S cluster process, and assuming that the PGF of the number of rainfall occurrences $N(0,t_1)$ is desired, define:

$$\xi(x) = 1 - (1 - z)\, I_{(0,t_1)}(x) \;,\; |z| < 1 \qquad (3.9)$$

where $I(\cdot)$ is the indicator function such that:

$$I_{(0,t_1)}(s) = \begin{cases} 1 & \text{if } s \in (0,t_1) \\ 0 & \text{otherwise} \end{cases} \qquad (3.10)$$

Clearly, the function defined by equation 3.9 belongs to the class of functions characterized by equations 2.23 and 2.24, and required in the definition of p.g.fl.. In general, and substituting equation 3.9

into the general definition of p.g.fl. of equation 3.3, the univariate PGF of the random variable $N(0,t_1)$ is immediately obtained. Consequently, substituting equation 3.9 into equation 3.8 yields:

$$g(z) = \exp\left[\mu \int_{-\infty}^{t_1} \{g_{N_2}[1 - (1 - z)\, p(t)] - 1\}dt\right] \tag{3.11a}$$

where:

$$p(t) = \int_0^{t_1} f_T(\zeta - t)d\zeta \tag{3.11b}$$

Equation 3.11a is the univariate PGF of the N-S cluster process. Kavvas and Delleur (1975) obtained the same result using more primitive techniques.

Finally, to obtain the complete finite dimensional stochastic structure of the N-S cluster process, and assuming that the multivariate PGF of the random vector:

$$[N(0,t_1),\ N(t_1,t_2),\ldots,\ N(t_{k-1},t_k)]$$

is desired, where $N(\cdot)$ counts the number of rainfall occurrences in the indicated non-overlapping intervals, select:

$$\xi(x) = 1 - \sum_{i=1}^{k} (1 - z_i)\, I_{[t_{i-1},t_i)}(x) \tag{3.12}$$

which, after substituting in equation 3.8 yields:

$$g(z_1,\ldots,z_k) = \exp\left[\mu \int_{-\infty}^{t_k} \{g_{N_2}[1 - \sum_{i=1}^{k} (1 - z_i)\, p_i(t)] - 1\}dt\right] \tag{3.13a}$$

where:

$$p_i(t) = \int_{t_i-1}^{t_i} f_T(\zeta - t)d\zeta \tag{3.13b}$$

Equation 3.13a represents the multivariate PGF for the above random vector when the random variable N(•) is distributed according to the N-S cluster model. In equations 3.11b and 3.13b, above, p(t) and $p_i(t)$ represent the probability that a rainfall whose RGM occurred at time t, falls within the time intervals indicated by the upper and lower limits of the integral. From this definition, it is clear that the nature of the p.d.f., $f_T(\zeta)$, determines the memory of the rainfall process.

By defining the p.g.fl. of the N-S cluster process in equation 3.8, it has been possible to obtain its complete finite dimensional structure, in equations 3.11 and 3.13. However, in this case, and as opposed to the Poisson process in which a complete stochastic description is obtained at the expense of the independence assumption, the N-S cluster model renders a general cluster dependence structure. In fact, the model expressed in equations 3.8, 3.11, and 3.13 has the Poisson process as one of its particular cases. This is easily seen by realizing that the independent increments assumption establishes a rainfall process with zero memory. For this case, the p.d.f, $f_T(\zeta - u)$ becomes a delta function, so that:

$$p(u) = \int_0^{t_1} f_T(\zeta - u)d\zeta = \begin{cases} 1 & \text{if } \zeta = u \\ 0 & \text{otherwise} \end{cases} \tag{3.14}$$

Substituting equation 3.14 in equation 3.11a yields:

$$g(z) = \exp\{\mu\ t_1 [g_{N_2}(z) - 1]\} \tag{3.15}$$

which is the PGF of the generalized Poisson process (Parzen, 1967).

Thus, a stochastic model has been devised, with a dependence structure general enough to render it widely applicable. In addition, the model acknowledges and represents the observed clustering behavior of the rainfall occurrences in time. Ideally, its components and parameters will have some physical meaning.

Up to now, all expressions presented for the N-S cluster process are general in the sense that forms for the distributions of the cluster sizes, $N_2(\cdot)$, and of the times of occurrence of the cluster members within each cluster, T, have not yet been specified. Explicit forms of these expressions are presented in Section 3.2.4, for a particular choice of distributions.

3.2.3 General First and Second Order Moments

General expressions for the first and second-order moments of the N-S cluster model are now introduced. Their derivation is carried out in Appendix B, using the general expressions of Section 2.3 and very simple algebraic manipulations. For more sophisticated derivations, using the p.g.fl. and its relation to the factorial moment measures and the factorial cumulant measures, the reader is referred to Vere-Jones (1970).

The first moment of the number of rainfall events, $N(0,t_1)$, in time interval $(0,t_1)$, is given in terms of the mean-time function (Appendix B):

$$E[N(0,t_1)] = \mu \cdot E[N_2(\cdot)] \cdot t_1 \tag{3.16}$$

The mean-time function of equation 3.16 is a function of the length of the interval, but not of its origin. From equation 3.16, the rate of rainfall occurrence under the N-S cluster process is easily obtained as:

$$\lambda(t_1) = \frac{d}{dt_1} E[N(0,t_1)] = \mu \cdot E[N_2(\cdot)] \tag{3.17}$$

The variance of the number of rainfall events in interval $(0,t_1)$, $N(0,t_1)$, is given in terms of the variance-time function (Appendix B):

$$\mathrm{var}[N(0,t_1)] = \mu \cdot E[N_2(t)] \cdot t_1 + \mu \cdot E[N_2^2(t) - N_2(t)] \cdot \int_{-\infty}^{t_1} p^2(t)dt \tag{3.18}$$

and where p(t) is as defined in equation 3.11b.

The covariance of the number of rainfall events in two disjoint time intervals, ζ time units apart is introduced in terms of the covariance function (Appendix B):

$$\mathrm{cov}[N(0,t_1)\ ,\ N(t_1+\zeta\ ,t_2)] = \mu \cdot E[N_2^2(t) - N_2(t)] \int_{-\infty}^{t_2} p_1(u) \cdot p_2(u)du \tag{3.19}$$

where:

$$p_1(u) = \int_0^{t_1} f_T(x-u)dx \tag{3.20}$$

and

$$p_2(u) = \int_{t_1+\zeta}^{t_2} f_T(x-u)dx \tag{3.21}$$

The most important feature of these first and second-order moments is exhibited by the covariance function. In fact, as can be observed from equation 3.19, as long as the functions $p_1(u)$ and $p_2(u)$ are different from zero, there exists correlation between the counts in disjoint time intervals. It is in equation 3.19 where the influence of the p.d.f., $f_T(\cdot)$, on the dependence structure of the N-S cluster model, is more easily grasped.

Now, define a differential process as $\Delta N_t = \lim_{\Delta t \to 0} N(0,t+\Delta t) - N(0,t)$.

Also, define the covariance density of the process ΔN_t as:

$$c(u) = \lim_{\Delta t \to 0} \frac{\text{cov}\left[\Delta N_t, \Delta N_{t+u}\right]}{(\Delta t)^2} \tag{3.22}$$

The spectrum of counts is defined as the Fourier transform of the covariance density, $c(u)$, of the differential process, ΔN_t, and is given as (Vere-Jones, 1970):

$$g(w) = \frac{1}{2\pi}\left[\lambda + \int_{-\infty}^{\infty} \exp(-iwu)\ c(u)du\right] \tag{3.23}$$

where λ is the intensity of the process.

For the N-S cluster process, the spectrum of counts is given as (Vere-Jones, 1970):

$$g_+(w) = \frac{1}{\pi}\left\{\mu \cdot E[N_2(t)] + \mu \cdot E[N_2^2(t) - N_2(t)]\left|\int_{-\infty}^{\zeta} f_T(\zeta-u)e^{iw(\zeta-u)}du\right|^2\right\}_{w>0} \tag{3.24}$$

The spectrum of counts will be used in Chapter 6 to calibrate the model. Since the interest is to preserve the dependence structure of the rainfall arrivals, nothing is more appropriate to fit the data to than the spectrum of counts, which is the covariance density after a change of basis.

3.2.4 Explicit Forms of the Neyman-Scott Cluster Model

In the above development of the N-S cluster model, the forms of the distributions for $N_2(\cdot)$ and T have been left undefined. The only requirement is that they agree with the structural postulates of Section 3.2.1. To obtain explicit forms for the equation defining the model, it is then necessary to hypothesize particular forms for the above two distributions. It is noteworthy to observe that in order to completely specify at least the first and second-order moments of Section 3.2.3, it is only necessary to specify the p.d.f. of T. In fact, in equations 3.16 through 3.24, the random variable representing the cluster sizes, $N_2(t)$, enters only through its first and second moments. It is also important to observe again, that the specific form of the p.d.f. for T determines the memory of the cluster model and the structure of the clusters as can be seen from equations 3.18 and 3.19.

In this work, as in Kavvas and Delleur (1981), it is assumed that the random variable characterizing the number of rainfalls in a given cluster follows a geometric distribution with parameter, p. It is well known that the PGF of $N_2(t)$ is then (Parzen, 1962):

$$g_{N_2}(z) = \frac{pz}{1 - (1 - p)z} \quad , \ |z| < 1 \tag{3.25}$$

The distribution of the time positions of the individual rainfalls within their respective clusters, T, is assumed to be negative exponential with parameter α, so that:

$$f_T(\zeta - t) = \begin{cases} \alpha \exp[-\alpha(\zeta - t)] & \text{if } \zeta - t > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.26}$$

The assumption of an exponential distribution for T, implies that no matter how far in the future from the time of occurrence of an RGM, there always exists a positive probability that a rainfall occurring at that time has been generated by the RGM which occurred in the infinite past.

Once the p.d.f.'s for $N_2(\cdot)$ and T have been chosen as the geometric distribution and the exponential distribution, the following expressions for the variance time function and the spectrum of counts are obtained (Kavvas and Delleur, 1981):

$$\mathrm{var}[N(0,t_1)] = \mu \cdot E[N_2^2(t)] \cdot t_1 + \frac{\mu}{\alpha} E[N_2^2(t) - N_2(t)][e^{-\alpha t_1} - 1] \tag{3.27}$$

and

$$g_+(w) = \frac{1}{\pi}\left[\mu \cdot E[N_2(t)] + \mu \cdot E[N_2^2(t) - N_2(t)] \frac{\alpha^2}{\alpha^2 + w^2}\right], \quad w > 0 \tag{3.28}$$

where: $E[N_2(t)] = 1|p$ and $E[N_2^2(t)] = (2 - p)|p^2$

Explicit forms for the N-S cluster model p.g.fl. result by substituting equations 3.25 and 3.26 in equation 3.8. For the particular case of $N(0,t_1)$ the substitution is done in equations 3.11.

Finally, for the random vector $N(0,t_1]$, $N(t_1,t_2],\ldots,$ $N(t_{k-1},t_k]$, the substitution is done in equations 3.13.

3.3 Forecasting and Conditional Probability Distributions

In Section 3.2 of this chapter, the Neyman-Scott cluster model has been developed. Its general dependence structure has been acknowledged, and in particular, the covariance function between the counts in disjoint time intervals has been shown to be non-zero in the general case. It would be very desirable then if it were possible to obtain conditional distributions of some sort, so that the model can be used to forecast the future of the process. For the particular kind of N-S cluster model being considered in this work, it turns out that explicit forms of these CDF's can be obtained. In order to do so, the few concepts from the Theory of Palm functions presented in Chapter 2 are used.

3.3.1 Assumptions and Definitions

Just as a point process can be described in two different ways, either in terms of its counting properties, or in terms of its interval properties, forecasting problems for point processes are also of two different kinds. The first kind corresponds to the problem of finding the distributions of the number of events in future time intervals. The second kind corresponds to the problem of finding the distributions of the time to and between various configurations of events. It is this last kind of problem that is dealt with in this chapter. As can be easily inferred, the second kind of problem leads naturally to a description of the process in terms of its interval properties. The function under consideration is the CDF of the time to the next rainfall event, given some or all of the past of the process. The immediate past, in particular the time which has elapsed since the preceding rainfall event, is of utmost importance.

In order to use the Palm-Khinchin theory introduced in Section 2.4, the following assumptions are in order:

a) That the process being considered is stationary.

b) That the process being considered has single occurrences.

c) That the process being considered has finite rate of occurrence.

d) That the complete stochastic structure of the process is known.

All four assumptions hold true in the case of the N-S cluster model developed in previous section. Before proceeding, denote the finite dimensional distributions of the point process $N(\cdot)$ as:

$$P_{i,j,k}\ldots(x,y,z,\ldots)$$

$$= \Pr\{N(0,x) = i,\ N(x,x+y) = j,\ N(x+y,x+y+z) = k,\ldots\} \qquad (3.29)$$

which are assumed to be known from the p.g.fl. of the N-S model.

Also, define X as the quantity being forecasted and described as the time to the next occurring rainfall event. Its distribution is most conveniently characterized in terms of the survivor function, defined as one minus the corresponding distribution function. Finally, the information on which the survivor function depends, is taken in this work as the time elapsed since the occurrence of the last rainfall event. With this in mind, denote the survivor function as:

$$S(x;H) = Pr[X > x|H] = 1 - Pr[X \leq x|H] \qquad (3.30)$$

which gives the probability that the time X to the next rainfall event is greater than x conditional on the history H of the process.

3.3.2 Conditional Distribution Functions and Palm-Khinchin Theory

In order to arrive at the desired CDF's, consider the worst situation or simplest case, namely the situation when no information about the past of the process is available. According to this, equation 3.30 reduces to the unconditional probability:

$$S(x) = Pr[X > x] \qquad (3.31)$$

Using the duality between the counting properties and the interval properties of a point process, equation 3.31 can be rewritten as:

$$S(x) = Pr[X > x] = Pr[N(0,x) = 0] = P_0(x) \qquad (3.32)$$

Now, consider the case when the available information about the past of the process is that a period of time ζ has elapsed without any rainfall events occurring. For this case, equation 3.30 becomes:

$$\Pr[X > x | N(0,\zeta) = 0] = \frac{\Pr[N(0,\zeta) = 0,\ N(\zeta,\zeta+x) = 0]}{\Pr[N(0,\zeta) = 0]} \tag{3.33}$$

where use of the fact $P(A|B) = P(A \cap B)|P(B)$ has been made. Equation 3.33 can also be written as:

$$\Pr[X > x | N(0,\zeta) = 0] = \frac{\Pr[N(0,\zeta+x) = 0]}{\Pr[N(0,\zeta) = 0]} \tag{3.34}$$

Substituting equation 3.32 in equation 3.34 yields:

$$S_0(x;\zeta) = \Pr[X > x | N(0,\zeta) = 0] = \frac{S(x+\zeta)}{S(\zeta)} = \frac{P_0(x+\zeta)}{P_0(\zeta)} \tag{3.35}$$

Equation 3.35 yields the CDF of the time to the next rainfall event, conditional on having observed the process for a period of time ζ, without any rainfall events occurring. The subscript 0 indicates that no events were observed during the interval ζ.

When actual rainfall occurrences start to appear in the history of the process, the situation complicates, since by the assumption of point occurrences, the survivor function would now be conditioned on probabilities over infinitesimal intervals. To solve this problem it is necessary to obtain the multivariate distributions of the time intervals between successive events (Jowett and Vere-Jones, 1972). The theory of Palm Functions, sketched briefly in Section 2.4, asserts that a well defined set of multivariate distributions for the intervals exists.

Assume that the available information about the past of the process is restricted to the time ζ since the occurrence of the last rainfall event (an actual rainfall occurrence was observed at the beginning of interval ζ); the desired CDF is:

$$S_{10}(x;\zeta) = \Pr[N(\zeta,\zeta+x) = 0 | \zeta \text{ time units have elapsed since last event}] \tag{3.36}$$

It is clear that equation 3.36 can be rewritten as:

$$S_{10}(x;\zeta) = \lim_{h\to 0} \frac{P_{10}(h,\zeta+x)}{P_{10}(h,\zeta)} \tag{3.37}$$

From the definition of Palm Functions presented in equation 2.28, and for k = 0:

$$q_0(x) = \lim_{h\to 0} \Pr\{N(0,x] = 0 | N(-h,0] > 0\} \ , \ x > 0 \tag{3.38}$$

Now, observe that equation 3.38 can be expressed as:

$$q_0(x) = \lim_{h\to 0} \frac{\Pr\{N(-h,0] > 0, \ N(0,x] = 0\}}{\Pr\{N(-h,0] > 0\}} \tag{3.39}$$

but, since $\Pr\{N(-h,0] > 0\} = 1 - \Pr\{N(-h,0] = 0\}$, equation 3.39 reduces to:

$$q_0(x) = \lim_{h\to 0} \frac{P_{10}(h,x)}{1 - P_0(h)} \tag{3.40}$$

Here, using the Palm-Khinchin theory and recalling equation 2.31, equation 3.40 becomes:

$$q_0(x) = \lim_{h \to 0} \frac{P_{10}(h,x)}{1 - P_0(h)} = -\frac{1}{\lambda} \cdot \frac{dP_0(x)}{dx} \tag{3.41}$$

Finally, using equation 3.41 in equation 3.37, it is easy to obtain for the desired CDF:

$$S_{10}(x;\zeta) = \frac{q_0(x+\zeta)}{q_0(\zeta)} = \left[\frac{dP_0(x+\zeta)}{dx} \middle| \frac{dP_0(\zeta)}{dx}\right] \tag{3.42}$$

All the desired CDF's have now been obtained in terms of the survivor functions $S_0(x;\zeta)$ and $S_{10}(x;\zeta)$ given in equations 3.35 and 3.42, respectively. By an extension of the development presented above, it is possible to define Palm-type functions of higher order in which more complex situations for the past of the process can be accounted for.

3.3.3 Explicit Form of CDF's for the N-S Cluster Model

In this concluding section, explicit expressions for equations 3.35 and 3.42 are obtained for the particular case of the N-S cluster process in which the cluster center process is Poisson with parameter μ, the distribution of the cluster size is geometric with parameter p, and the distribution of the time positions about cluster centers is exponential with parameter α.

To do so, it is needed to have the following probability $P_0(x) = Pr[N(0,x) = 0]$. This can be derived using equation 2.13 and equation 3.11 as follows:

$$P_0(x) = Pr[N(0,x) = 0] = g(z)\Big|_{z=0} \tag{3.43}$$

Substitution in equation 3.11 of the PGF for $N_2(\cdot)$ given in equation 3.25 and of the p.d.f. of T given in equation 3.26, and carrying out equation 3.43 yields (Appendix C):

$$P_0(x) = \left[\frac{p}{1 - qe^{-\alpha x}}\right]^{\frac{\mu}{\alpha}} \cdot e^{-\mu x} \tag{3.44}$$

where $q = 1 - p$.

For the first case, equation 3.35 is:

$$S_0(x;\zeta) = \frac{S(x+\zeta)}{S(\zeta)} \tag{3.45}$$

where $S(\cdot) = P_0(\cdot)$; so that substituting equation 3.44 in equation 3.45 yields:

$$S_0(x;\zeta) = \left[\frac{1 - qe^{-\alpha\zeta}}{1 - qe^{-\alpha(\zeta+x)}}\right]^{\frac{\mu}{\alpha}} \cdot e^{-\mu x} \tag{3.46}$$

$S_0(x;\zeta)$ is the conditional probability that the time to the next rainfall X is greater than x, conditioned on the fact that the process has evolved for a period of time ζ without any events occurring, and that storm arrivals follow the N-S model developed.

When one rainfall event occurs at the beginning of the interval ζ, equation 3.42 yields:

$$S_{10}(x;\zeta) = \frac{q_0(x+\zeta)}{q_0(\zeta)} = \frac{\dfrac{dP_0(x+\zeta)}{dx}}{\dfrac{dP_0(\zeta)}{dx}} \qquad (3.47)$$

Differentiating equation 3.44:

$$\frac{dP_0(x)}{dx} = -\mu \cdot e^{-\mu x} \cdot p^{\mu|\alpha} \left[1 - qe^{-\alpha x}\right]^{-(1+\frac{\mu}{\alpha})} \qquad (3.48)$$

Now, substitution of equation 3.48 in equation 3.47 after evaluating equation 3.48 at $(x+\zeta)$ and ζ, produces:

$$S_{10}(x;\zeta) = \left[\frac{1 - qe^{-\alpha\zeta}}{1 - qe^{-\alpha(x+\zeta)}}\right]^{(1+\frac{\mu}{\alpha})} \cdot e^{-\mu x} \qquad (3.49)$$

$S_{10}(x;\zeta)$ is the conditional probability that the time to the next rainfall X is greater than x, conditioned on the fact that ζ time units have elapsed since the last rainfall occurrence, and that storm arrivals follow the N-S cluster model proposed.

Observe that for both equation 3.46 and equation 3.49, as the time ζ increases (time without rain or time since last rainfall) the probabilities $S_0(x;\zeta)$ and $S_{10}(x;\zeta)$ approach those of a Poisson model with parameter μ, meaning that in fact, the farther in the future an event is considered, the more it starts to look as independent of the present or past of the process. From equation 3.46:

$$\lim_{\zeta \to \infty} S_0(x;\zeta) = e^{-\mu x} \tag{3.50a}$$

and from equation 3.49:

$$\lim_{\zeta \to \infty} S_{10}(x;\zeta) = e^{-\mu x} \tag{3.50b}$$

With the CDF's $S_0(x;\zeta)$ and $S_{10}(x;\zeta)$, the forecasting of future events is now possible. Thus, conditional information contained in the immediate history of the process can be used, for instance, in obtaining optimal irrigation control in regions of deficit irrigation. This will be accomplished in the next chapter.

3.4 Summary

This chapter has completely introduced the N-S cluster process. For a particular choice of distributions, explicit expressions for the model and its first and second-order moments have also been presented. Finally, with the use of the Palm-Khinchin theory, CDF's of the time to the next rainfall event conditional on the immediate history of arrivals were derived. The stage is now set for the introduction of these CDF's into the general irrigation scheduling problem.

Chapter 4

IRRIGATION SCHEDULING MODEL

4.1 Introduction

In this chapter, the general irrigation scheduling problem, presented by Córdova and Bras (1979), is reformulated to include the precipitation model described in the previous chapters. The framework in which this reformulation is carried out is a general conceptual soil column with only vertical flows considered. The probabilistic behavior of the storm arrival process, encoded in the conditional distribution functions, $S_0(x;\zeta)$ and $S_{10}(x;\zeta)$, is now coupled with the probabilistic description of the cumulative infiltration volume from a given rainstorm. This coupling provides the means to describe the random behavior of the soil moisture content at the root zone.

The resulting irrigation scheduling model makes use of the conditional information contained in the immediate history of rainfall occurrences as the growing season progresses. In this sense, the new model is improved with respect to the model used by Córdova and Bras (1979), who, by assuming independence in the rainfall arrival process, neglect the observed statistical dependence in precipitation. Finally, using SDP, the irrigation scheduling problem is solved. The solution is presented in the form of optimal decision functions which yield the optimal amount of irrigation water to be applied at each decision stage, as a function of the soil moisture content at the root zone, the volume of water available for irrigation, and the number of days that have elapsed since the last rainfall occurrence.

4.2 Climate-Soil-Plant Model

The general irrigation scheduling model has three main components which represent each of the three subsystems in the climate-soil-plant system. The climate subsystem is modelled in terms of the precipitation process and of the potential evapotranspiration. The former determines the contribution of nature to the soil moisture content at the root zone as well as how this contribution is distributed throughout the growing season. The climate controlled potential evapotranspiration is an upper bound to the rate at which moisture can be extracted from the soil by evaporation and plant transpiration. It determines the stress status of given vegetal species depending on whether the actual evapotranspiration rate is equal to or less than the potential.

The precipitation process can be completely described by describing storm intensities, storm durations and number of storms in a given time interval. Storm intensities and durations coupled with the dynamics of the infiltration process determine the amount of water contributed by each storm to the soil moisture content. In this chapter, the modelling of the precipitation process is completed. Probability distribution functions for storm intensities and durations are hypothesized, and using a derived distribution approach, the probability distribution function of cumulative infiltration from a given storm is obtained.

The soil subsystem is represented in terms of a conceptual soil column in which the components of the soil water balance interact to determine the soil moisture content at the root zone. Infiltration replenishes soil moisture, while evapotranspiration and percolation out of the root zone deplete it. The rates of infiltration, evapotranspiration and percolation are all expressed as functions of the soil moisture content.

Finally, the plant subsystem is described in terms of a crop function which relates actual evapotranspiration to actual yield.

The interface between the climate subsystem and the soil-plant system is provided by the volume of water infiltrated and by the actual evapotranspiration rate.

4.2.1 Components of Soil Water Balance

The processes which govern the soil water balance are infiltration, evapotranspiration and percolation. Their description is of primary concern in defining the dynamics of the soil moisture storage process.

4.2.1.1 Infiltration and Surface Runoff

Córdova and Bras (1981), using results obtained by Philip (1957) and Eagleson (1978), derived an expression for the cumulative volume of water infiltrated from a given rainstorm, under a given set of initial conditions.

Assuming a rainfall of constant intensity i and duration t_r, the cumulative volume of water infiltrated, $V(i,t_r)$, can be expressed as:

$$V(i,t_r) = i\,t_r - R_g(i,t_r) \tag{4.1}$$

where $R_g(i,t_r)$ represents the amount of surface runoff produced by the given rainfall.

Using Philip's infiltration equation (Philip, 1957) and assuming constant rainfall intensity, Eagleson (1978) derived the following expression for the volume of surface runoff produced by a given rainfall:

$$R_g(i,t_r) = \begin{cases} 0 & \text{if } t_r \leq t_0 \\ (i-A)t_r - S\left(\frac{t_r}{2}\right)^{\frac{1}{2}} & \text{if } t_r > t_0 \end{cases} \tag{4.2}$$

where S is the soil sorptivity, A is gravitational infiltration rate, and t_0 is the time from the beginning of the storm at which the soil surface becomes saturated. They are expressed as (Eagleson, 1978):

$$S = 2\left(1 - \frac{\theta_0}{\theta_s}\right) \left|\frac{5n\ K(1)\ \psi(1)\ \phi(d,\theta_0/\theta_s)}{3m\pi}\right|^{\frac{1}{2}} \tag{4.3}$$

$$A = \frac{1}{2} K(1)\,[1 + (\theta_0/\theta_s)^c] - w \tag{4.4}$$

$$t_0 = \frac{S^2}{2(i - A)^2} \qquad (4.5)$$

where:

n	=	porosity
$K(1)$	=	saturated hydraulic conductivity
$\psi(1)$	=	saturated soil matric potential
$\phi(d,\theta_0/\theta_s)$	=	infiltration diffusivity function
w	=	capillary rise from the water table
m	=	pore size distribution index
d	=	diffusivity index
c	=	pore connectivity index
θ_0	=	initial soil moisture content (mm)
θ_s	=	soil moisture content at saturation (mm)

Substituting equation 4.2 into equation 4.1, the volume of water infiltrated from a storm of constant intensity i and duration t_r is:

$$V(i,t_r) = \begin{cases} it_r & \text{if } t_r \leq t_0 \\ At_r + S\left(\frac{t_r}{2}\right)^{\frac{1}{2}} & \text{if } t_r > t_0 \end{cases} \qquad (4.6)$$

Assuming the water table elevation constant throughout the growing season, the values of the parameters S and A depend only on the soil properties and on the initial soil moisture content, θ_0. Thus, for a given soil, the volume of water infiltrated from a given storm is a function of the soil moisture content at the beginning of the storm. Using a derived distribution approach, the probability distribution function (PDF) of the random input to the soil system $V(\cdot,\cdot)$ can be obtained from the PDF of the random variables storm intensity and storm duration (Córdova and Bras, 1979).

4.2.1.2 Actual Evapotranspiration

The actual evapotranspiration rate represents the combined rate at which water is being extracted from the soil by plant transpiration and by evaporation of exfiltrated water. For the transpiration process it has been experimentally corroborated that for a given potential transpiration rate, there exists a threshold average soil moisture content below which the actual rate of transpiration is less than the potential (Denmead and Shaw, 1962; Minhas, 1974; Hanson, 1976). Considering the evaporation process to behave similarly, a threshold soil moisture can also be defined below which the actual rate of evaporation is less than the potential. In the general case, these two threshold soil moisture contents have different values.

To avoid the problem of defining both processes separately, Córdova and Bras (1979), following an approach proposed by Gardner, et al. (1975) combine both processes defining a single soil moisture threshold θ^*. The actual evapotranspiration rate $E_a(\theta)$, can then be written as (see Figure 4.1a):

$$E_a(\theta) = \begin{cases} ET_p & \text{if } \theta^* < \theta \\ a\theta^b & \text{if } 0 \leq \theta \leq \theta^* \end{cases} \quad (4.7)$$

where:

$$a = \frac{K_c \cdot E_0}{(\theta^*)^b}$$

ET_p = $K_c \cdot E_0$

K_c = crop coefficient

E_0 = potential evaporative flux

b = coefficient

Throughout this work, the soil moisture content θ is measured with respect to permanent wilting point (PWP). The parameters a, b, K_c, E_0, θ^* and PWP depend on the growth stage of the crop as well as on the soil, crop, and climate characteristics.

4.2.1.3 Percolation

Eagleson (1978) relates the percolation rate $P(\theta)$ to the soil moisture content by the following expression in which capillary rise from the water table has been included (Figure 4.1a):

$$P(\theta) = d\theta^c - w \quad (4.8)$$

where $d = K(1)/n^c$; n is the soil porosity; c is the pore connectivity index and w is the rate of capillary rise.

It should be emphasized that the processes of infiltration, evapotranspiration, and percolation are all dependent on the soil moisture content at the root zone. In this way, the soil moisture content becomes the state variable representing the response of the soil system.

4.2.2 Conceptual Soil Column Model: Moisture Depletion Process

Córdova and Bras (1979) consider the climate-soil-plant system in terms of a conceptual soil column model (Figure 4.1b). The inputs to the system are the climate controlled potential evapotranspiration, the amount of water infiltrated from a given storm and the irrigation applications. Actual evapotranspiration and percolation out of the root zone constitute the system outputs. The state variable describing and controlling the response of the system is the soil moisture content at the root zone. The evolution of the state of the system can then be described by:

$$\frac{d\theta}{dt} = I_t + f_t - P - E_a \qquad (4.9)$$

where I_t and f_t are the irrigation and infiltration rates, respectively. In equation 4.9 only vertical flows are considered.

Considering the storm duration t_r only through the volume of water infiltrated $V_t(i,t_r)$ and assuming that the storms occur instantaneously in time, during which no evapotranspiration, percolation or irrigation occurs, the soil moisture content after the occurrence of a storm event can be expressed as:

$$\theta_t = \begin{cases} \theta_s & \text{if } \theta_t + V_t(i,t_r) > \theta_s \\ \theta_t + V_t(i,t_r) & \text{otherwise} \end{cases} \tag{4.10}$$

Equation 4.10 also holds for the irrigation applications if these are assumed to be instantaneous and producing no runoff. In this case $V_t(\cdot,\cdot)$ becomes the volume of irrigation water applied.

The soil moisture storage process is then characterized by moisture replenishment from irrigation and infiltration from storm events and by moisture depletion from evapotranspiration and percolation. The former are assumed to occur instantaneously. Thus, the soil moisture depletion process can be described by defining the evaluation of the state of the system during the interstorm period. This can be expressed as:

$$\frac{d\theta}{dt} = -\begin{cases} a\theta^b + d\theta^c - w & 0 \leq \theta \leq \theta^* \\ ET_p + d\theta^c - w & \theta^* \leq \theta \leq \theta_s \end{cases} \tag{4.11}$$

The terminal soil moisture content at time t, θ_t, and the total actual evapotranspiration during the time interval (0,t], $ET_a(t)$, can be obtained by integrating equation 4.11.

Depending on the initial conditions, several cases have to be considered (see Figure 4.1c). If at time $t_0 = 0$ $\theta = \theta_0 \leq \theta^*$, then θ at time t is the solution of the following equation:

$$\int_{\theta_t}^{\theta_0} \frac{\partial\theta}{a\theta^b + d\theta^c - w} = t \tag{4.12}$$

The total actual evapotranspiration during the time interval (0,t] is:

$$ET_a(t) = \int_{\theta_t}^{\theta_0} \frac{a\theta^b \, \partial\theta}{a\theta^b + d\theta^c - w} \tag{4.13}$$

Now, if the initial soil moisture content θ_0 at $t_0 = 0$ is greater than θ^*, $\theta_0 > \theta^*$, the time required to deplete the soil moisture from θ_0 to θ^* has to be defined:

$$t^* = \int_{\theta_*}^{\theta_0} \frac{\partial\theta}{ET_p + d\theta^c - w} \tag{4.14}$$

where t^* is the time required to bring the soil moisture to the value θ^*. Once t^* is defined, there exist two possible cases. If the time interval under consideration is smaller than t^*, then the terminal soil moisture content θ_t is obtained as the solution of:

$$\int_{\theta_t}^{\theta_0} \frac{\partial\theta}{ET_p + d\theta^c - w} = t \tag{4.15}$$

and by definition:

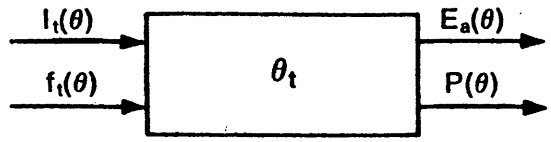


Figure 4.1a

CONCEPTUAL SOIL COLUMN MODEL

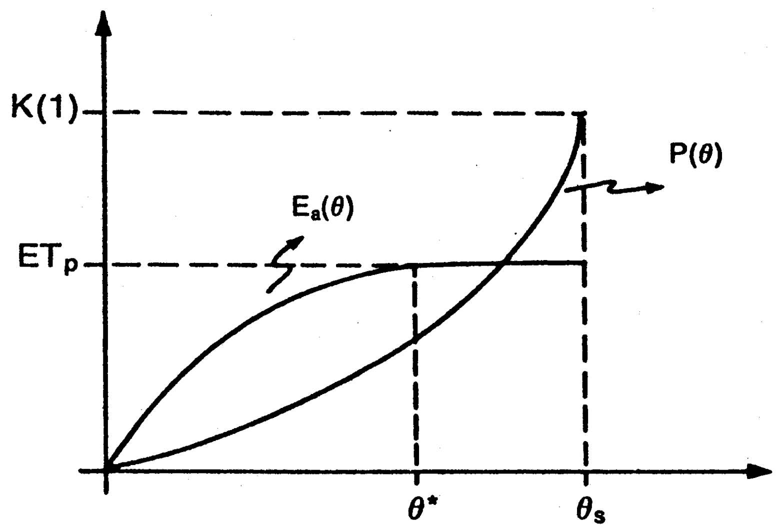


Figure 4.1b

EVAPOTRANSPIRATION AND PERCOLATION FUNCTIONS

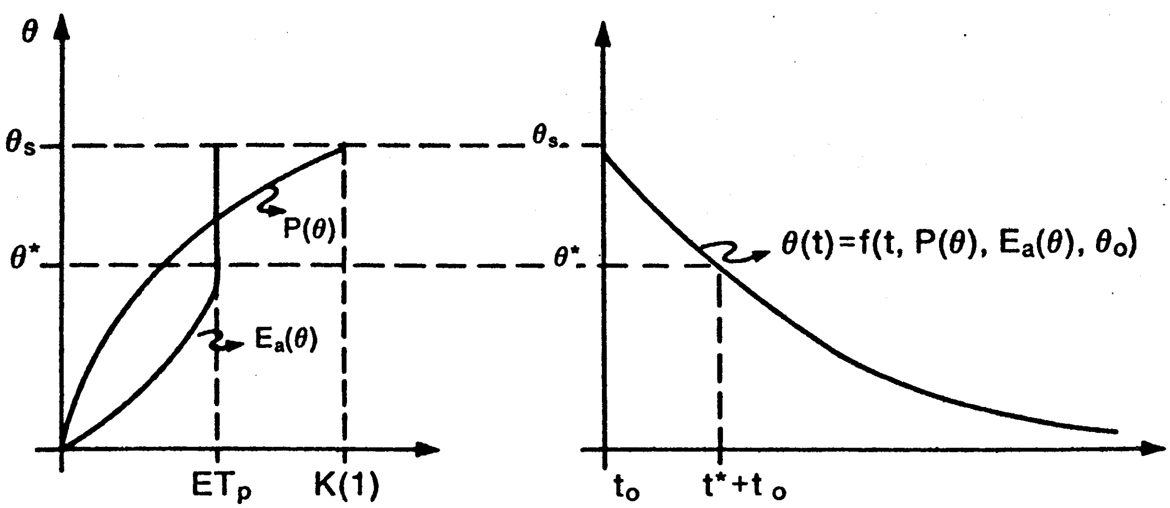


Figure 4.1c

SOIL MOISTURE DEPLETION CURVES

$$ET_a(t) = t \cdot ET_p \tag{4.15}$$

If the time interval (0,t] is longer than t*, then the terminal soil moisture content θ_t is obtained as the solution of:

$$\int_{\theta_t}^{\theta^*} \frac{\partial\theta}{a\theta^b + d\theta^c - w} = t - t^* \tag{4.17}$$

and the total actual evapotranspiration is:

$$ET_a(t) = t^* \cdot ET_p + \int_{\theta_t}^{\theta^*} \frac{a\theta^b \, \partial\theta}{a\theta^b + d\theta^c - w} \tag{4.18}$$

4.2.3 Crop Model

Real time control of irrigation systems requires knowledge of the crop response to water applications throughout the growing season. The total yield and economic return from a given crop are a function of the history of the distribution of water throughout the growing season and not only of the total volume of water applied. Stewart and Hagan (1974) emphasize this point, acknowledging the existence of critical growth stages for many crops, during which adverse responses to moisture stresses are greater. Knowing the crop response function for the different growth stages allows optimal control of the irrigation water. This is performed by considering the temporal variability of the random precipitation inputs together with the temporal crop response to moisture stresses and the dynamics of the soil moisture storage process (Córdova and Bras, 1979). The surrogate variable that best describes the effect of moisture stresses on actual yield is the evapotranspiration ratio, defined by E_a/ET_p, (Stewart, et al., 1974; Morey, et al., 1975; Stewart, et al., 1977).

Córdova and Bras (1979) consider the effect of moisture stresses on actual crop yield to be additive. The relationship between crop growth and total actual evapotranspiration can then be expressed as (Blank, 1975):

$$Y = Y_M \sum_{z=1}^{NP} A_z \frac{ET_z}{ETP_z} \tag{4.19}$$

where Y is the actual crop yield, Y_M is the species dependent maximum yield, NP is the number of growing periods, and A_z is the moisture stress sensitivity parameter of the crop. Finally, the stress factor is defined as the ratio of the total actual evapotranspiration ET_z to the total potential evapotranspiration ETP_z during each growing period z.

4.2.4 Stochastic Representation of Soil Moisture

The soil moisture storage process is a dynamic process. Figure 4.2 illustrates the variability of soil moisture over time, as different storms arrive to the site. During the duration of the storms, t_r, soil moisture gradually increases due to infiltration. On the other hand, during the interstorm time, t_b, soil moisture gradually decreases due to the action of the soil moisture depletion

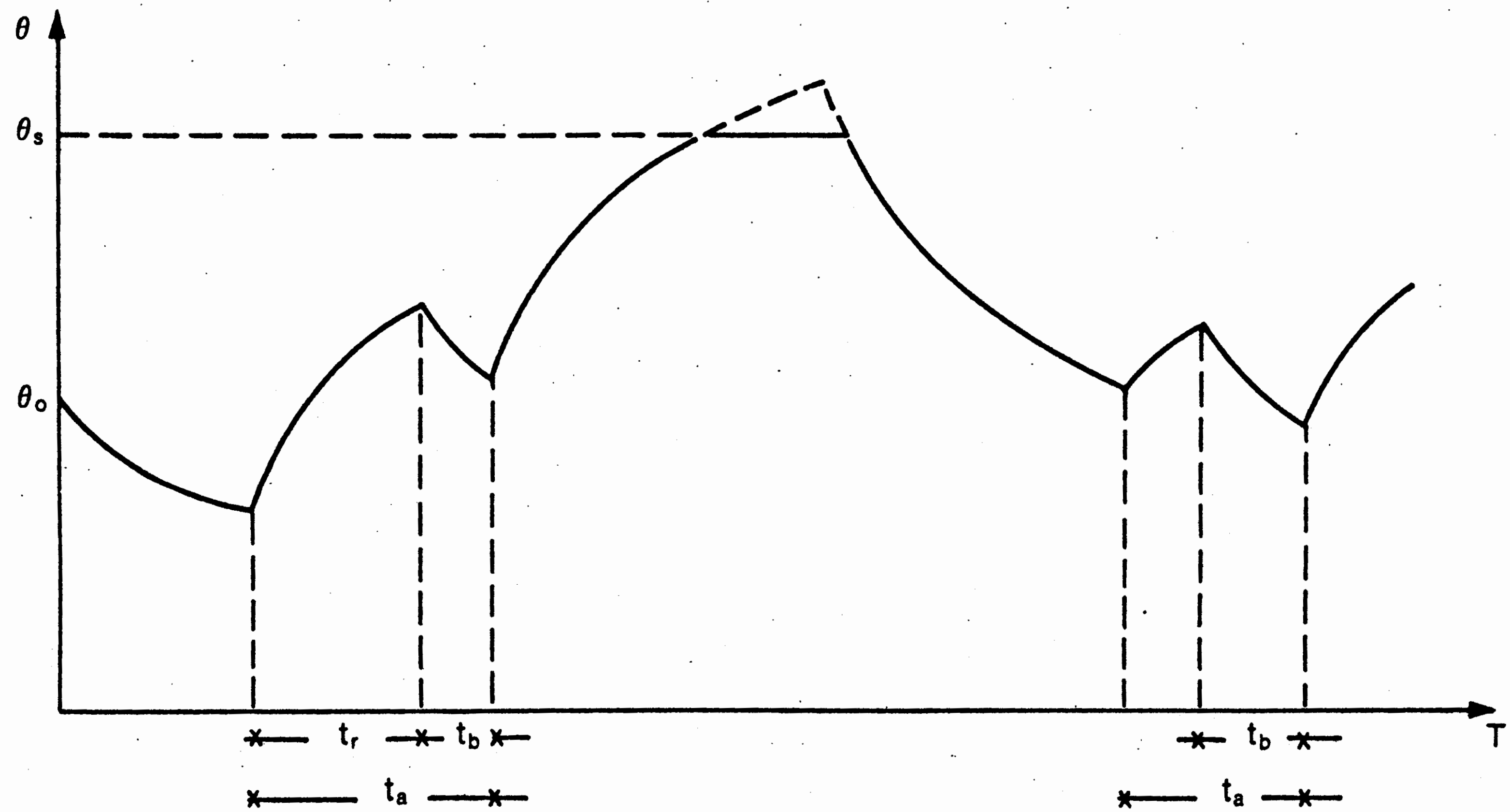


Figure 4.2
DYNAMICS OF SOIL MOISTURE

processes, namely, evapotranspiration and percolation.

The random input to the soil system, the volume of water infiltrated from a given rainstorm, depends on the soil characteristics, the initial soil moisture content, the topographic conditions, and the storm intensity and duration. Thus, it is evident that the total amount of water infiltrated during a given time interval depends both on the number, depth, duration and timing of the storm events, and on the dynamics of the soil moisture depletion process.

In making irrigation decisions throughout the growing season, it is necessary to consider all the possible future soil moisture states, especially if the amount of available irrigation water is limited. However, since the input to the soil system over a time interval is random, future soil moisture states become uncertain. Consequently, making optimal irrigation decisions requires a probabilistic description of the general terminal soil moisture content. This can be obtained by coupling the dynamics of the soil moisture storage process with the probabilistic description of precipitation in terms of the process of storm arrivals and the storm characteristics, intensity and duration. The process of storm arrivals has already been presented in previous chapters. For the storm characteristics, intensity and duration, Grayman and Eagleson (1969) observed that they could be closely fitted by exponential distributions. Furthermore, data analysis for two locations in the continental U.S., provided by Eagleson (1978), corroborate the goodness of fit obtained with the exponential distribution. For storm intensity, they found:

$$f_I(i) = \alpha e^{-\alpha i} \qquad i \geq 0 \tag{4.20}$$

and for storm duration:

$$f_{T_r}(t_r) = \delta e^{-\delta t_r} \qquad t_r > 0 \tag{4.21}$$

where $f_I(\cdot)$ and $f_{T_r}(\cdot)$ are the probability density functions (p.d.f.) for storm intensity and duration respectively; $1/\alpha$ is the average storm intensity and $1/\delta$ is the average storm duration.

When surface runoff can be considered negligible with respect to total storm depth, the p.d.f. of infiltrated volume is simply the p.d.f. of storm depth. However, in areas where surface runoff is important, the p.d.f. of infiltrated volume has to be derived from the p.d.f.'s of storm intensity and duration. Assuming that the random variables i and t_r are independent and using a derived distribution approach, Córdova and Bras (1979) obtained the probability distribution function (PDF) of the volume of water infiltrated from a given rainstorm $V(i,t_r)$ as:

$$F_V(v) = \Pr[V \leq v] = 1 - \exp[-\alpha i^* - \delta t_0^*] - \alpha \int_0^{i^*} \exp[-\alpha i - \frac{\delta v}{i}]\, di \tag{4.22}$$

where:

$$t_0^* = \frac{S^2}{2(i^* - A)^2} \tag{4.23}$$

and

$$i^* = \frac{4vA + S^2 + S\sqrt{8vA + S^2}}{4v} \qquad (4.24)$$

It is easy to see from equations 4.22 through 4.24 that when $v \to 0$, then $i^* \to \infty$ and $F_V(0) = 0$. Also, when $v \to \infty$, $i^* \to A$ and $F_V(\infty) = 1$.

In deriving equation 4.22 it is assumed that storm intensity and duration are exponentially distributed according to equations 4.20 and 4.21. It is also assumed that the volume of water infiltrated is related to storm intensity and duration as expressed in equation 4.6.

This completes the description of the general components of the irrigation scheduling model. In the following sections the general water balance elements are integrated with the probabilistic description of storm arrivals given in Chapter 3 and the probabilistic description of infiltration from storms presented above. The resulting system description of the irrigation scheduling problem acknowledges its nature as a finite horizon, multistage, stochastic decision problem. Once this nature of the problem has been recognized, its solution by stochastic dynamic programming follows straightforwardly.

4.3 Systems Description of Irrigation Scheduling Problem

In Chapter 3 the process of storm arrivals has been defined. Rainfall occurrences in time have been modelled as a Neyman-Scott cluster process. Consequently, the observed statistical dependence in the rainfall occurrences in time, caused by the clustering of storms has been accounted for. This allowed the derivation of the conditional distribution functions $S_{10}(x;\zeta)$ and $S_0(x;\zeta)$ which give the conditional probability of the time to the next rainfall event, conditioned on the immediate history of occurrences. Now, the goal is to take advantage of the information contained in the history of immediate past rainfall occurrences, encoded in the conditional PDF's $S_{10}(x;\zeta)$ and $S_0(x;\zeta)$, to obtain optimal irrigation control. The system description of the irrigation scheduling problem, given in the following sections, incorporates this conditional information to define the evolution of the state of the system throughout the growing season. In general, the system is composed of the following elements: a time scale T, a state space S, a control space C, an output space Y, a state transition function F, a stochastic law of motion P, and an output function G.

4.3.1 Time Scale, T

The time horizon of the irrigation scheduling problem is the length of the irrigation season. At the beginning of the season, the farmer has a limited but known volume of irrigation water. Thus, the objective is to manage this limited amount of water optimally, so that net profits at the end of the season are maximized. The time scale can be expressed as:

$$T = \{k : k = 1, 2, \ldots, N + 1\} \tag{4.25}$$

where k represents days (decision stages) and N is the number of days in the growing season.

4.3.2 State Space, S

A state space representation of a given system requires that the state variable chosen contains all the necessary information, besides the inputs, to determine future states of the system. Let $\overline{X}_k$ be the state vector, such that:

$$\overline{X}_k \in S \ , \ k \in T$$

and (4.26)

$$\overline{X}_k = (\theta_k, \zeta_k, r_k)$$

where:

θ_k = soil moisture content at the root zone at beginning of decision stage (day) k

ζ_k = number of days, at decision stage k, since the occurrence of the last rainfall

r_k = volume of irrigation water available at beginning of decision stage (day) k

The element ζ_k of the vector state $\overline{X}_k$, as defined by equation 4.26, represents the knowledge the decision-maker has about the immediate history of the precipitation process. Due to the dependence in the rainfall occurrences, this history contains valuable information about the future of storm arrivals. This information is encoded in the conditional distributions $S_{10}(x;\zeta)$ and $S_0(x;\zeta)$, derived in Chapter 3, and that are to be included in the general irrigation model.

4.3.3 Control Space, C

At each decision stage (days, weeks) during the irrigation season, the farmer (decision-maker) has to decide whether or not to irrigate, and how much. It is by making irrigation decisions that the farmer can control the soil moisture states. Thus, the control variable is the amount of irrigation water U_k applied at decision stage k, such that:

$$U_k \in C \text{ , } k \in T \quad \text{and} \quad 0 \leq U_k \leq r_k \quad \forall_k \in T \tag{4.27}$$

4.3.4 Output Space, Y

As has been stated before, the objective of the decision-maker (farmer) is to maximize net profits at the end of the season. Net profits are a function of crop yield. Thus, the output variable chosen is the contribution of decision stage k to total actual crop yield, Y_k.

4.3.5 State Transition Function, F

The unforced or free motion of the system, expressed in terms of the dynamics of the vector state $\overline{X}_k$, is given by the state transition function:

$$\overline{X}_{k+1} = F(\overline{X}_k) \ , \ k \in T \tag{4.28}$$

The state transition function is a composite function of three different operators or dynamic equations, one for each element of the state vector $\overline{X}_k$.

First, the soil moisture depletion operator, which yields the soil moisture content at the root zone at the end of the current decision stage, θ_{k+1} is:

$$\theta_{k+1} = g(\theta_k, \Delta t) + V_k(\theta_k, \Delta t) \tag{4.29}$$

where V_k is the volume of water infiltrated from storm events during decision stage k; Δt is the duration of decision stage k, taken as one day; and θ_{k+1} is the soil moisture content at the end of the stage. In equation 4.29, it is assumed that an irrigation decision has been taken at the beginning of the interval. Consequently, θ_k in 4.29, is the soil moisture content after the respective irrigation application.

The first right-hand member of equation 4.29, $g(\theta_k, \Delta t)$, represents the deterministic part of the soil moisture state transition from decision stage to decision stage. It is given by equations 4.11, 4.12, 4.14, 4.15 and 4.17, and describes the dynamics of the soil moisture depletion process. The second member of the right-hand side of the equation, $V_k(\theta_k, \Delta t)$, represents the stochastic volume of water infiltrated from storm events during time interval Δt. Consequently, θ_{k+1} is random. Its probabilistic description is presented later as part of the stochastic law of motion, P.

It is necessary to emphasize that infiltration, either from storm events or irrigation applications, is assumed to occur instantaneously. Thus, the dynamics of the soil moisture storage process is as presented in Figure 4.3.

Second, the elapsed time operator, which yields the value of the variable ζ_{k+1} at the beginning of the next decision stage as a function of its present value, ζ_k, and of a random disturbance, W_k, as follows:

$$\zeta_{k+1} = h(\zeta_k, W_k) = \begin{cases} \zeta_k + 1 & \text{if } W_k = 0 \text{ , } k \in T \\ W_k & \text{if } W_k = 1 \text{ , } k \in T \end{cases} \tag{4.30}$$

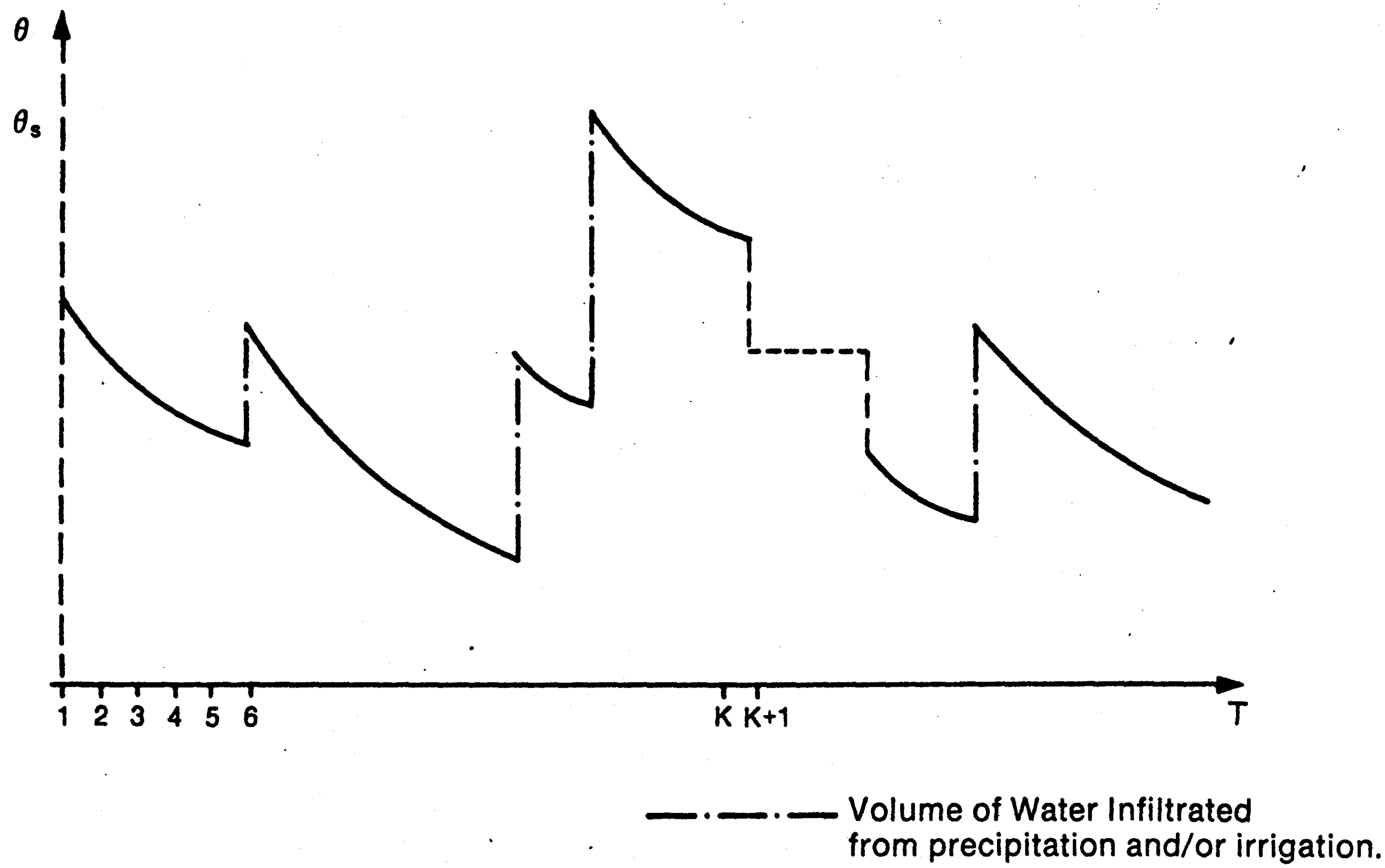


Figure 4.3

CONCEPTUAL DYNAMICS OF SOIL MOISTURE

where W_k is a binary random variable that is equal to one if it rains during decision stage k, or equal to zero otherwise.

Finally, the irrigation water operator given as a simple mass balance equation to guarantee that the volume of water used for irrigation during each decision stage k, is less than or equal to the available volume of irrigation water:

$$r_{k+1} = \ell(r_k, U_k) , k \in T$$

such that:

$$r_{k+1} = r_k - U_k$$

and

$$r_{k+1} \geq 0 \tag{4.31}$$

4.3.6 Stochastic Law of Motion, P

The stochastic law of motion P, is a family of conditional distribution functions $P_k(\overline{X}_{k+1}/\overline{X}_k, U_k)$, which for each decision stage k, yield the conditional distribution function of the state vector $\overline{X}_{k+1}$, conditional on its present value and the irrigation decision. Only two elements of the state transition vector $\overline{X}_{k+1}$ are stochastic, namely the soil moisture content θ_{k+1} and the time in days since the occurrence of the last rainfall ζ_{k+1}. The state transition function for the available irrigation water, $\ell(\cdot,\cdot)$, is totally deterministic.

Consequently, the stochastic law of motion involves only the description of the joint stochastic behavior of θ_{k+1} and ζ_{k+1}. To do so, the probability distribution function (PDF) of the terminal soil moisture content, as well as the PDF of the time since the last rainfall event have to be defined.

Soil moisture content (SMC) at the root zone is discretized, in the state space, from saturation SMC to permanent wilting point PWP. The discretization follows that of Córdova and Bras (1979) (Figure 4.4). It is a variable interval discretization whose index ranges from one at saturation to m at PWP. The length of each interval is such that, under unforced conditions (no infiltration inputs), it takes one day for the depletion processes to drive the SMC from the upper bound to the lower bound of each interval, and m days to drive it from saturation to PWP. According to this, the PDF of the terminal soil moisture content, for all soil moisture states, can be expressed as a transition matrix:

$$\Phi(k) = \{\phi_{ij}(k)\} \qquad i,j = 1, \ldots, m \tag{4.32}$$

where $\phi_{ij}(k)$ represents the probability that at the end of decision stage k (or beginning of decision stage k+1) the soil moisture content is in state j, given that at the beginning of the stage it is in state i. The generic element of matrix $\Phi(k)$ is defined as:

$$\phi_{ij}(k) = P_{ij|W_k=0}(k) \cdot \Pr[W_k=0|\zeta_k] + P_{ij|W_k=1}(k) \cdot \Pr[W_k=1|\zeta_k] \tag{4.33}$$

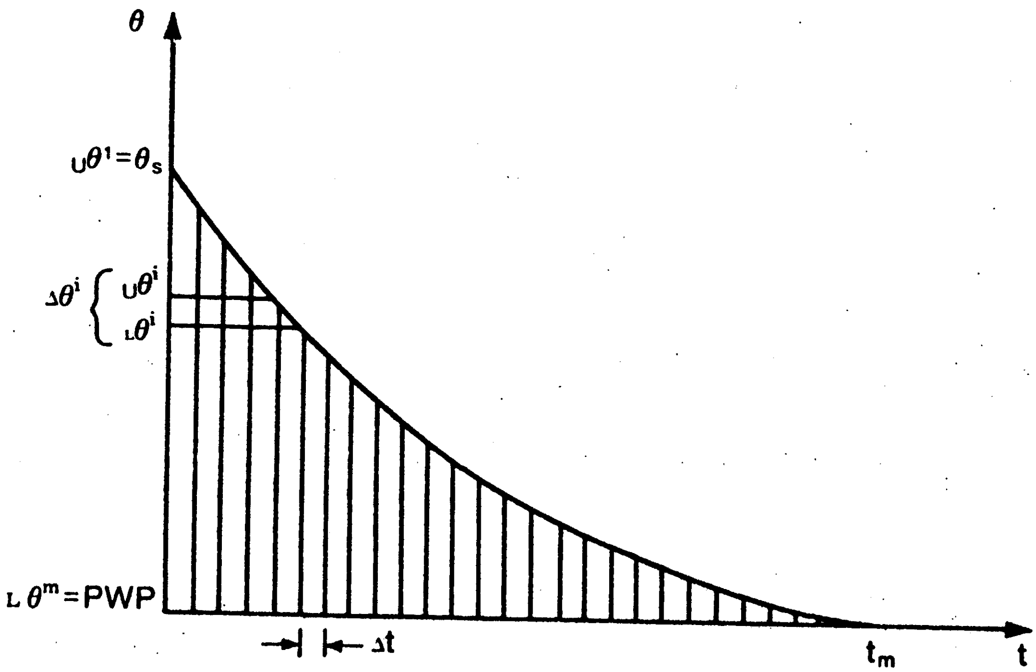


Figure 4.4
SOIL MOISTURE DISCRETIZATION

where $P_{ij|W_k=0}(k)$ is the probability that at the end of stage k, the soil moisture content is in state j, given that at the beginning of the stage it is in state i, and no precipitation occurs during the stage. $P_{ij|W_k=1}(k)$ is its analogue for the case when precipitation occurs during the stage. Finally, $Pr[W_k=0|\zeta_k]$ and $Pr[W_k=1|\zeta_k]$ are the conditional probabilities of no rain and rain during the given stage, respectively.

If there is no rain during decision stage k, then $V_k(\theta_k, \Delta t) = 0$ in equation 4.29. Thus, for Δt equal to one day and according to the above discretization:

$$\theta^j_{k+1} = g(\theta^i_k, \Delta t)$$

and (4.34)

$$j = i + 1$$

Consequently,

$$P_{ij|W_k=0}(k) = Pr\{\theta_{k+1} \varepsilon [{}_U\theta^j_{k+1}, {}_L\theta^j_{k+1}] / \theta_k \varepsilon [{}_U\theta^i_k, {}_L\theta^i_k] \text{ and } W_k=0\} = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases} \quad (4.35)$$

where the subscripts U and L stand for upper and lower bound of the given soil moisture interval.

When precipitation occurs during the time interval Δt, $(k, k+1]$, equation 4.29 can be written as:

$$\theta^{j}_{k+1} = g(\theta^{i}_{k}, \Delta t) + V_k(\theta^{i}_{k}, \Delta t) = \theta^{i+1}_{k} + V^{ij}_{k} \qquad (4.36)$$

In equation 4.36, soil moisture is first depleted from θ^{i}_{k} to θ^{i+1}_{k} and then replenished up to the value θ^{j}_{k+1}. Thus, V^{ij}_{k} represents the volume of infiltrated water (from rainfall) required to bring the SMC from θ^{i+1}_{k} to θ^{j}_{k+1}. The upper and lower bounds of the jth interval can be expressed as:

$$\begin{aligned} {}_U\theta^{j}_{k+1} &= \theta^{i+1}_{k} + {}_UV^{ij}_{k} \\ {}_L\theta^{j}_{k+1} &= \theta^{i+1}_{k} + {}_LV^{ij}_{k} \end{aligned} \qquad (4.37)$$

Consequently,

$$\begin{aligned} P_{ij|W_k=1}(k) &= \Pr\{\theta_{k+1} \varepsilon [{}_U\theta^{j}_{k}, {}_L\theta^{j}_{k}] / \theta^{i}_{k}, W_k=1\} \\ &= \Pr\{{}_LV^{ij}_{k} \le V^{ij}_{k} \le {}_UV^{ij}_{k} / \theta_k \varepsilon [{}_U\theta^{i}_{k}, {}_L\theta^{i}_{k}] \text{ and } W_k=1\} \end{aligned} \qquad (4.38)$$

Finally, and following Córdova and Bras (1979):

$$P_{ij|W_k=1}(k) = \frac{1}{\Delta\theta_k^i} \int_{_L\theta_k^i}^{_U\theta_k^i} \Pr\{_LV_k^{ij} \leq V_k^{ij} \leq {_UV_k^{ij}} / \theta_k \varepsilon [_U\theta_k^i, {_L\theta_k^i}], W_k=1\} d\theta_k^i \tag{4.39}$$

where:

$$\Delta\theta_k^i = {_U\theta_k^i} - {_L\theta_k^i} \tag{4.40}$$

Expression 4.39 can be evaluated using equations 4.36 and 4.37 to calculate $_UV_k^{ij}$ and $_LV_k^{ij}$, and using equation 4.22 which defines the PDF of the random variable V_k^{ij}, conditional on θ_k^i.

In a previous chapter, the conditional distributions $S_0(x;\zeta)$ and $S_{10}(x;\zeta)$ have been defined for a point process with single occurrences and applied to model storm arrivals. Assume that the time interval chosen, Δt equal to one day, between decision stages is sufficiently small to guarantee that:

$$\Pr[N(\Delta t) > 1] \simeq 0 \tag{4.41}$$

where $N(\Delta t)$ counts the number of rainfall occurrences in interval Δt.

By definition,

$$S_0(x;\zeta) = \Pr[X > x|\zeta]$$

and (4.42)

$$S_{10}(x;\zeta) = \Pr[X > x|\zeta]$$

where X represents the time to the next rainfall occurrence.

Also, because of the duality between the definition of a Point Process as a Counting Process or as an Interval Process, equation 4.42 can be written as:

$$\Pr[X > x] = \Pr[N(x) = 0] = 1 - \sum_{n=1}^{\infty} \Pr[N(x) = n] \tag{4.43}$$

When x equal one day, equation 4.41 holds. Consequently, substituting in equations 4.43 and 4.42:

$$\begin{aligned} S_0(x{=}1 \text{ day}; \zeta) &= \Pr[N(x) = 0|\zeta] = 1 - \Pr[N(x) = 1|\zeta] \\ S_{10}(x{=}1 \text{ day}; \zeta) &= \Pr[N(x) = 0|\zeta] = 1 - \Pr[N(x) = 1|\zeta] \end{aligned} \tag{4.44}$$

Thus, the PDF of the random variable W_k can be obtained from eqution 4.44 as:

$$\Pr[W_k{=}0|\zeta_k] = \begin{cases} S_0(x{=}1 \text{ day}; \zeta_k) \\ S_{10}(x{=}1 \text{ day}; \zeta_k) \end{cases} \tag{4.45}$$

and

$$\Pr[W_k{=}1|\zeta_k] = \begin{cases} 1 - S_0(x{=}1 \text{ day}; \zeta_k) \\ 1 - S_{10}(x{=}1 \text{ day}; \zeta_k) \end{cases} \tag{4.46}$$

Expressions to evaluate $S_0(x;\zeta)$ and $S_{10}(x;\zeta)$ are given in Chapter 3.

This completes the definition of the SMC transition matrices. It should be noted that now, as opposed to Córdova and Bras (1979), the transition matrices $\Phi(k)$ are conditional on the random variable ζ_k. Also, it should be observed that the conditional information contained in the history of the precipitation process has now been effectively accounted for.

To completely define the stochastic law of motion, it is necessary to define the PDF of the random variable ζ_{k+1}, conditional on the value of ζ_k. It has already been stated that ζ_k represents time in days since the occurrence of the last rainfall. However an upper bound on the value of ζ_k has not been defined. In order to do so, it is assumed that there exists a time lag in days beyond which the dependence in the rainfall arrival process is sufficiently weak to guarantee that:

$$S_0(x;H) \neq S_0(x;H^*) \simeq S_0(x;H_1) \qquad 0 \leq H < H^* < H_1$$

and (4.47)

$$S_{10}(x;H) \neq S_{10}(x;H^*) \simeq S_{10}(x;H_1) \qquad 0 \leq H < H^* < H_1 \, .$$

In equation 4.47, H, H* and H_1 represent the knowledge the decision-maker has about the immediate history of the storm arrival process. Thus, the previous assumption simply states that there exists an upper bound, H*, to the conditional information contained in the past of the process. According to this assumption, equation 4.30 becomes:

$$\zeta_{k+1} = h(\zeta_k, W_k) = \begin{cases} \min(\zeta_k+1, H^*) & \text{if } W_k = 0 \ , \ k \ \varepsilon \ T \\ W_k & \text{if } W_k = 1 \ , \ k \ \varepsilon \ T . \end{cases} \tag{4.48a}$$

and

$$\zeta_k \ \varepsilon \ \zeta_k^j \quad \text{if } \zeta_k = j \ , \ k \ \varepsilon \ T \ , \ j = 1, 2, \ldots, H^* \tag{4.48b}$$

The conditional PDF of the random variable ζ_{k+1}, conditional on the value of ζ_k, can be expressed as a transition matrix:

$$\Psi(k) = \{\psi_{ij}(k)\} \qquad i,j = 1, 2, \ldots, H^* \tag{4.49}$$

where $\psi_{ij}(k)$ is the probability that at decision stage k+1, ζ_{k+1} is in state j, given that at stage k, ζ_k is in state i.

The generic element $\psi_{ij}(k)$ can be expressed as:

$$\psi_{ij}(k) = q_{ij}{}_{|W_k=0}(k) \cdot \Pr[W_k=0|\zeta_k] + q_{ij}{}_{|W_k=1}(k) \cdot \Pr[W_k=1|\zeta_k] \tag{4.50}$$

where:

$$q_{ij}{}_{|W_k=0}(k) = \Pr[\zeta_{k+1} \ \varepsilon \ \zeta_{k+1}^j / \zeta_k \ \varepsilon \ \zeta_k^i \ \text{ and } \ W_k = 0] \tag{4.51}$$

and

$$q_{ij}{}_{|W_k=1}(k) = \Pr[\zeta_{k+1} \ \varepsilon \ \zeta_{k+1}^j / \zeta_k \ \varepsilon \ \zeta_k^i \ \text{ and } \ W_k = 1] \tag{4.52}$$

Expressions 4.51 and 4.52 represent the one stage transition probabilities of the random variable ζ_k, conditional on whether or not it rains during decision stage k. These probabilities are evaluated using equations 4.48a and 4.48b. They are:

$$q_{ij|W_k=0}(k) = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases} \tag{4.53}$$

and

$$q_{ij|W_k=1}(k) = \begin{cases} 1 & \text{if } j=1 \\ 0 & \text{otherwise} \end{cases} \tag{4.54}$$

The joint stochastic behavior of the random variables θ and ζ can be obtained in terms of the transition matrices $\Phi(k)$ and $\Psi(k)$. Assume that at the beginning of decision stage k, the SMC is in state i, θ_k^i, and the elapsed time is in state p, ζ_k^p; then the probability that at the beginning of stage k+1, θ_{k+1} is in state j and ζ_{k+1} is in state q can be written as:

$$\Pr\{[\theta_k^i, \zeta_k^p] \rightarrow [\theta_{k+1}^j, \zeta_{k+1}^q]\} = \phi_{ij}(k) \cdot \psi_{pq}(k) \qquad \begin{array}{l} i,j=1, \ldots, m \\ p,q=1, \ldots, H^* \end{array} \tag{4.55}$$

where $\Pr\{[\cdot,\cdot] \rightarrow [\cdot,\cdot]\}$ stands for transition probability.

Generalizing equation 4.55 for all feasible values of the indexes i, j, p, q, yields the desired stochastic law of motion.

4.3.7 Output Function, G

The contribution of decision stage k, to actual total crop yield has been defined as the output variable. According to the crop model presented in Section 4.2.3, the output function can be defined as:

$$Y_k = G_k(\overline{X}_k, \overline{X}_{k+1}, U_k) \tag{4.56}$$

such that:

$$Y_k = Y_M \cdot A_k \cdot \frac{ET_k}{ETP_k} \tag{4.57}$$

where:

Y_k = contribution of decision stage k to total crop yield

Y_M = genetically determined maximum potential crop yield

A_k = sensitivity parameter of the crop to soil moisture stress

ET_k = total actual evapotranspiration during decision stage k

ETP_k = total potential evapotranspiration during decision stage k

The systems description of the irrigation scheduling model has now been completed.

4.4 Stochastic Dynamic Programming Solution

In order to apply the Dynamic Programming algorithm to the multistage decision problem defined in the previous sections, it is necessary to define an objective function. In this study, and accounting for the stochastic nature of the variables involved, the measure of performance to be used is the maximization of the expected value of the total net profits at the end of the growing season:

$$B^* = \text{MAX } E\left[\sum_{k=1}^{N+1} R_k(\bar{X}_k, \bar{X}_{k+1}, U_k)\right] - PC \tag{4.58}$$

and

$$R_k(\bar{X}_k, \bar{X}_{k+1}, U_k) = PY_k - \beta U_k - \gamma C_k(U_k) \tag{4.59}$$

where:

P = unit price of crop yield

Y_k = contribution of decision stage k to total yield

β = unit cost of irrigation water

γ = fixed cost of irrigation (labor cost)

U_k = volume of irrigation water applied at decision stage k

$$C_k(U_k) = \begin{cases} 1 & \text{if } U_k > 0 \\ 0 & \text{if } U_k = 0 \end{cases} \tag{4.60}$$

PC = production costs different from irrigation costs

$E[\cdot]$ = expectation operator

With respect to the available irrigation water, two different cases can be defined in studying the irrigation scheduling problem: limited and unlimited water supply. The critical case is when the irrigation water supply is limited. However, from an operational point of view, the unlimited water supply case offers some advantages that make it worth solving. On one hand, it has smaller dimension since it is not necessary to account for the available volume of irrigation water. This simplifies the problem and reduces its solution cost. On the other hand, since the unlimited water problem is a particular case of the more general limited water supply problem, its solution constitutes an upper bound to the maximum net profits attainable. Furthermore, by solving the unlimited water case, it is possible to obtain a preliminary analysis of the effect of the conditional information encoded in $S_0(x;\zeta)$ and $S_{10}(x;\zeta)$ on the solution of the irrigation scheduling problem.

4.4.1 Unlimited Water Supply

When the available irrigation water, at each decision stage, is unlimited, it is not necessary to include it as a state variable. Thus, for this case:

$$\overline{X}_k = (\theta_k, \zeta_k)$$

The SDP algorithm proceeds as follows:

a) Define

$$J^*_{N+1}[\bar{X}_{N+1}] = J^*_{N+1}[\theta^i_{N+1}, \zeta^p_{N+1}] = 0 \qquad \forall i,\ i=1, \ldots, m \quad \forall p,\ p=1, \ldots, H^* \tag{4.61}$$

b) Proceed by induction as follows:

$$J^*_k[\bar{X}_k] = J_k[\bar{X}_k, U^*_k] = \underset{U_k \varepsilon Q_k}{\text{MAX}}\ E[R_k(\bar{X}_k, \bar{X}_{k+1}, U_k) + J^*_{k+1}(\bar{X}_{k+1})] \tag{4.62}$$

where Q_k is the set of feasible irrigation decisions at stage k and U^*_k is the optimal irrigation decision at stage k. More explicitly:

$$J^*_k[\theta^i_k, \zeta^p_k] = J_k[\theta^i_k, \zeta^p_k, U^*_k]$$

$$= \underset{U_k \varepsilon Q_k}{\text{MAX}}\ \{E[R_k(\theta^i_k, \zeta^p_k, U_k)] + \sum_j \sum_q \phi_{ij}(k)\ \psi_{pq}(k)\ J^*_{k+1}[\theta^j_{k+1}, \zeta^q_{k+1}]\} \tag{4.63}$$

Then, set

$$\Gamma^*_k(\theta^i_k, \zeta^p_k) = U^*_k \tag{4.64}$$

where $\Gamma^*(\cdot,\cdot)$ is the optimal decision function at stage k. It yields the optimal amount of irrigation water to be applied at decision stage k, U^*_k, as a function of the state, $\bar{X}_k$.

c) Proceeding by induction, obtain:

$$J_1^*[\theta_1^i,\zeta_1^p] = B^*[\theta_1^i,\zeta_1^p] + PC \quad (4.65)$$

4.4.2 Limited Water Supply

The SDP algorithm proceeds as follows:

a) Define

$$J_{N+1}^*[\overline{X}_{N+1}] = J_{N+1}^*[\theta_{N+1}^i,\ \zeta_{N+1}^p,\ r_{N+1}] = 0 \quad \forall i,\ i=1,\ \ldots,\ m \quad (4.67)$$

$$\forall p,\ p=1,\ \ldots,\ H^*$$

b) Proceed by induction as:

$$J_k^*[\overline{X}_k] = J_k[\overline{X}_k,U_k^*]$$

$$= \underset{U_k \varepsilon Q_k}{\text{MAX}} \{E[R_k(\overline{X}_k,U_k)] + \sum_j \sum_q \phi_{ij}(k)\ \psi_{pq}(k)\ J_{K+1}^*(\overline{X}_{K+1})\} \quad (4.68)$$

or more explicitly:

$$J_k^*[\theta_k^i,\zeta_k^p,r_k]$$

$$= \underset{U_k \varepsilon Q_k}{\text{MAX}} \{E[R_k(\theta_k^i,\zeta_k^p,U_k)] + \sum_j \sum_q \phi_{ij}(k)\ \psi_{pq}(k)\ J_{k+1}^*[\theta_{k+1}^j,\zeta_{k+1}^q,r_{k+1}]\}$$

(4.69)

and $$r_{k+1} \leq r_k - U_k \tag{4.70}$$

Then, set

$$\Gamma_k^*(\theta_k^i, \zeta_k^p, r_k) = U_k^* \tag{4.71}$$

c) Proceeding by induction, obtain:

$$J_1^*[\theta_1^i, \zeta_1^p, r_1] = B^*[\theta_1^i, \zeta_1^p, r_1] + PC \tag{4.72}$$

The SDP algorithm yields, not only the expected maximum net benefits, $J_k^*[\bar{X}_k]$, for each decision stage and state vector but also, and most importantly, optimal decision functions, $\Gamma_k^*(\bar{X}_k)$, that give the optimal amount of irrigation water to be applied as a function of the SMC at the root zone, θ_k, the time in days since the last rainfall occurrence, ζ_k, and the volume of water available for irrigation, r_k.

4.5 Summary

In this chapter, the irrigation scheduling problem has been reformulated to include the conditional information available in the history of the precipitation process as the growing season progresses. A systems description of the problem has been presented, acknowledging its nature as a finite horizon, multistage, stochastic decision process. Finally, the solution of the problem has been obtained using stochastic dynamic programming.

Chapter 5

MODEL CALIBRATION AND CASE STUDY RESULTS

5.1 Introduction

The previous four chapters have developed the theoretic setting necessary to formulate and solve the irrigation scheduling problem, considering the conditional information contained in the immediate history of rainfall occurrences. However, the main objective of this work is not so much to develop a more or less complex model of the climate-soil-plant system interactions, but to devise a model capable of being used in real world situations. In this chapter, the issues of model calibration and case study results are discussed. Issues regarding model calibration are presented in two separate sections, the first on the calibration of the plant-soil system model; the second on the calibration of the climate subsystem model. Lastly, case study results are presented.

5.2 Case Study Definition

The irrigation problem defined by Córdova and Bras (1979) constitutes the case study in this work. Córdova and Bras (1979) in turn, used parameters and data obtained by Blank (1975) from an irrigation study in Colorado.

The parameters required by the soil-plant model were obtained from field experiments conducted at Colorado State University, with an early corn variety (Northrup King PX 20) in a uniformly deep Nunn clay loam soil during the growing season of 1974.

Precipitation data was obtained from historical records at Denver, Colorado (N.O.A.A. Station Number 05-2220). Economic coefficients were obtained by Blank, from the economic study of the Fort Morgan irrigation area in Colorado, and performed by Conklin (1974). The growing season starts on May 15 and ends on September 11.

5.3 Soil-Plant System Model Calibration

The calibration of the soil-plant system is related to the estimation of the parameters defining the additive crop model adopted, as well as the elements of the soil water balance. In the definition of the above parameters, it is also implied that parameters describing the particular soil have to be defined; particularly with respect to the soil water balance. Furthermore, as required by the crop model, growth stages, and their characteristics have to be described.

5.3.1 Crop Model Parameters

For the additive crop model adopted in Chapter 4 and expressed as (equation 4.19):

$$Y = Y_M \sum_{z=1}^{NP} A_z \frac{ET_z}{ETP_z} \qquad (5.1)$$

the parameters to be estimated are the crop sensitivity parameters A_z. These parameters represent the relative sensitivity of the particular crop to water stresses in a given growth stage. Blank (1975) considered the following three growth stages:

1. Germination through vegetative growth, from May 15 to July 16.

2. Early silking, from July 16 to July 23.

3. Silking through maturity, from July 23 to September 11.

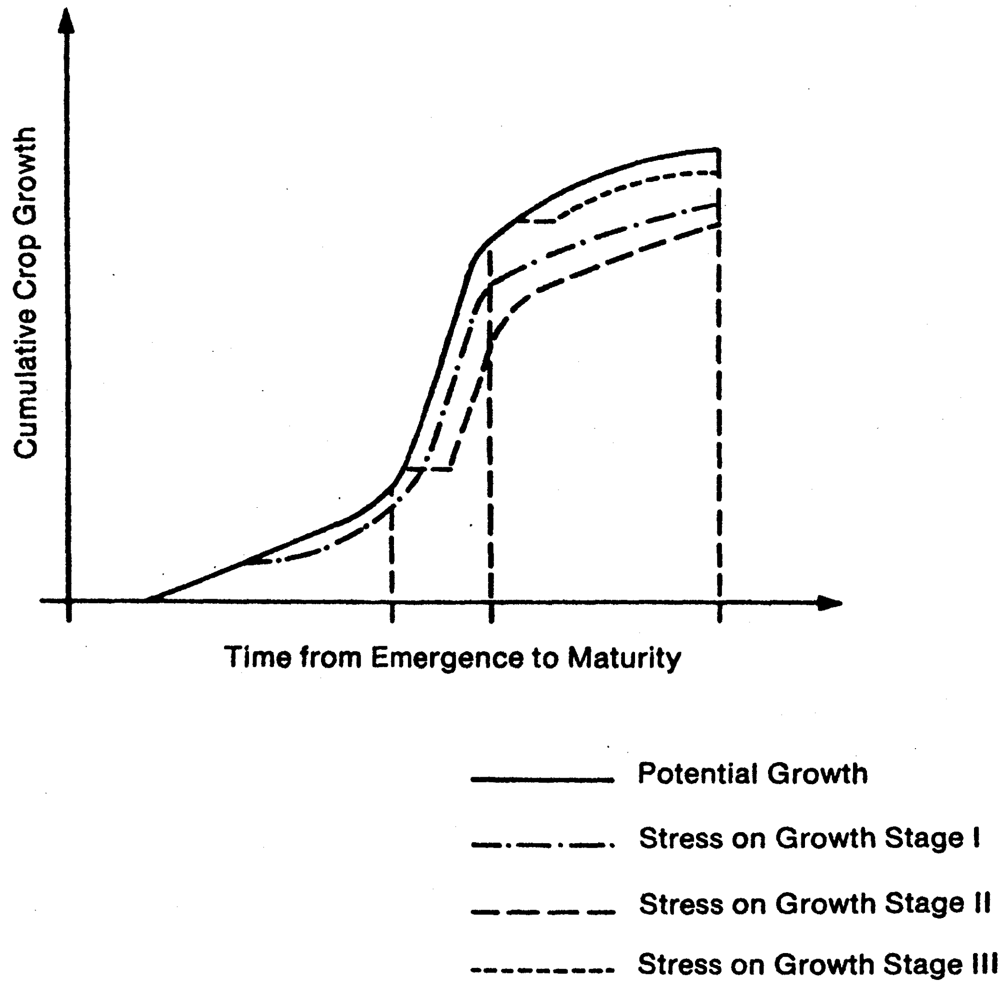
The adoption of these three particular stages takes into account the fact that the presilking period is critical for corn production. Water stresses during this growth stage induce worse adverse effects on the overall plant growth than on any of the other two growth stages (see Figure 5.1). Using a standard stepwise linear regression, Blank (1975) calibrated the model of equation 5.1 to experimental data obtained for an early maturity corn variety (Northrup King PX 20). Table 5.1 presents the values obtained for the parameter A_z in each growth stage (see Córdova and Bras, 1979).

5.3.2 Soil Water Balance Parameters

As presented in Chapter 4, the components of the soil water balance are infiltration from storm events or irrigation applications, evapotranspiration, and percolation. All of these processes are expressed as functions of the initial soil moisture content at the root zone and soil parameters.

5.3.2.1 Soil Parameters

Córdova and Bras (1979) estimated the required soil parameters necessary to account for the dynamics of the infiltration and percolation processes from the description of the soil texture. This was necessary since Blank's study neglected the processes of percolation and surface runoff and did not provide values for the soil parameters. The typical values for the parameters corresponding to a clay loam soil are taken from Eagleson (1978) and presented in Table 5.2.



RELATIVE EFFECT OF WATER STRESS ON PLANT GROWTH
(taken from Anderson and Mass (1971))

Figure 5.1

THEORETICAL CROP GROWTH CURVES

Table 5.1

CROP SENSITIVITY PARAMETERS FOR BLANK'S ADDITIVE MODEL

A_1	A_2	A_3	R^2
0.236	0.159	0.573	0.98

Note: R^2 is the square of the correlation coefficient, between measured and computed yield.

Table 5.2

SOIL PARAMETERS

Soil Type: Clay Loam

Porosity, n: 0.35

Saturated Hydraulic Conductivity, K(1): 30 mm/day

Saturated Soil Matrix Potential, $\psi(1)$: 190 mm

Diffusivity Index, d: 5.5

Pore Size Index, m: 0.286

Pore Connectivity Index, c: 10

5.3.2.2 Actual Evapotranspiration Process

For the three different growth stages defined earlier, it is necessary to determine not only the potential evapotranspiration rate, but also the corresponding crop root depth. The former is required because it determines the evaporative demand of the atmosphere or maximum water extraction rate, and the latter because it defines the depth of the conceptual soil column and consequently the total volume of water available for evapotranspiration.

Adopting the assumption made by Yaron, et al. (1973), that the actual evapotranspiration is a linear function of the available soil moisture content, equation 4.7 reduces to:

$$E_a(\theta) = \begin{cases} a\,\theta & 0 \leq \theta \leq FC \\ ET_p & FC \leq \theta \end{cases} \qquad (5.2)$$

where FC is field capacity and $a = ET_p/FC$ is called the Yaron coefficient. The values obtained by Blank for FC, ET_p, and root depth for each growth period are presented in Table 5.3.

Using the values for FC, n, and root depth presented in Tables 5.2 and 5.3, and using typical soil moisture extraction curves (Eagleson, 1978), values for the soil moisture content at saturation and permanent wilting point can be obtained (Córdova and Bras, 1979). These values for the particular case at hand are presented in Table 5.4.

Table 5.3

PARAMETERS OF EVAPOTRANSPIRATION FUNCTION

	Growth Period		
	I	II	III
ET_p (mm/day)	3.1	6.3	4.6
FC (mm)	143.1	330.0	330.0
Root Depth (m)	0.91	2.13	2.13

Table 5.4

SOIL MOISTURE CONTENT AT SATURATION AND PWP

	Growth Period		
	I	II	III
θ_s (mm)	320	747	747
PWP (mm)	74	173	173

5.3.2.3 Soil Moisture Depletion Process

In order to determine and quantify the soil moisture depletion for the case study, several assumptions are made. The evapotranspiration rate, as expressed in equation 5.2, is assumed to be a linear function of the available soil moisture content. In order to obtain a linear expression for the soil moisture depletion rate, the percolation function is also linearized. To do so, it is assumed that the percolation rate is zero for soil moisture contents below FC, and a new parameter, θ_1, $\theta^* \leq \theta_1 \leq \theta_s$ is defined. Both the linear evapotranspiration function and the linear percolation function are shown in Figure 5.2. According to this the soil moisture depletion rate can be defined as:

$$\frac{d\theta}{dt} = -\begin{cases} \alpha_1 + \beta_1\theta & \theta_1 \leq \theta \leq \theta_s \\ \alpha_2 + \beta_2\theta & \theta^* \leq \theta \leq \theta_1 \\ \beta_3\theta & 0 \leq \theta \leq \theta^* \end{cases} \tag{5.3}$$

Values for the parameters of equation 5.3 are given by Córdova and Bras (1979) (see Table 5.5). After linearization, equations 4.11 through 4.18 can be solved analytically. After solving these linearized equations, the resulting soil moisture depletion curves for the case study as well as their associated total actual evapotranspiration curves are shown in Figures 5.3 and 5.4, for each growth stage.

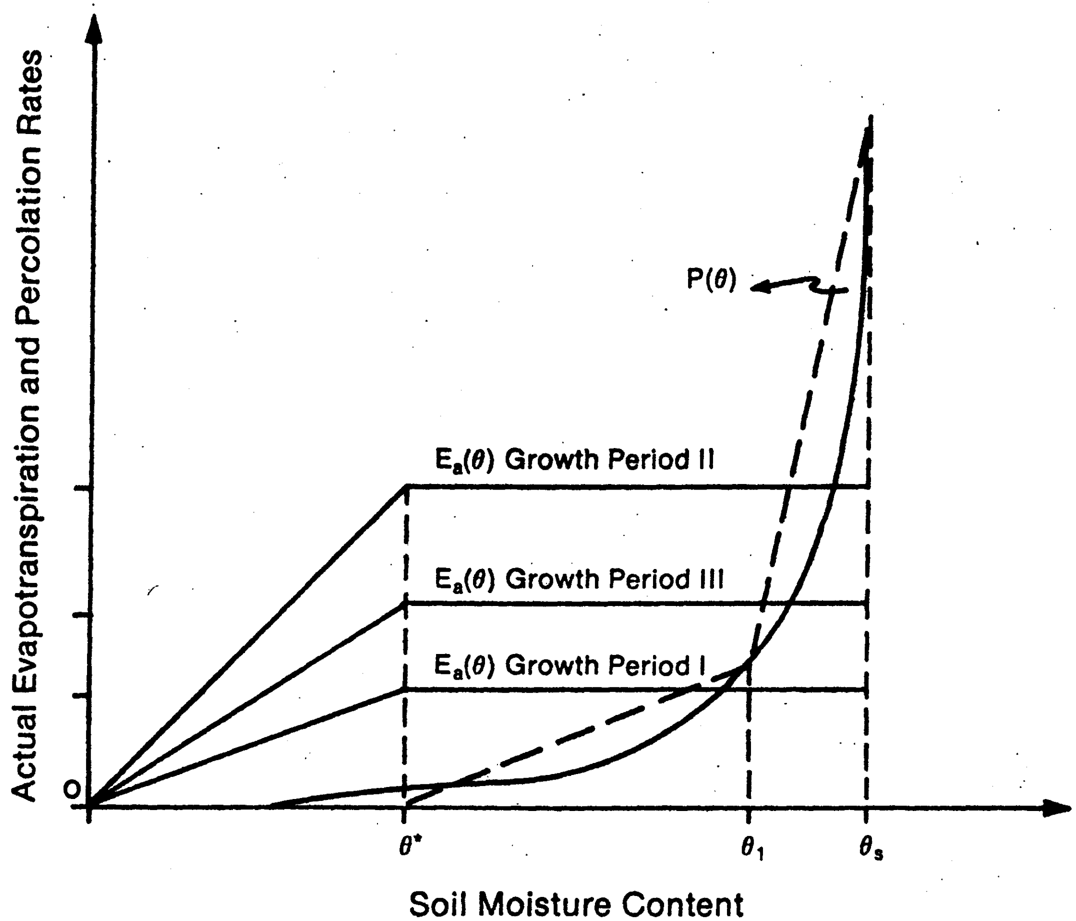


Figure 5.2

LINEARIZED EVAPOTRANSPIRATION AND PERCOLATION FUNCTIONS

Table 5.5

PARAMETERS OF LINEARIZED MOISTURE DEPLETION RATE

	Growth Period		
	I	II	III
α_1	-86.7	-89.15	-90.85
β_1	.477	.215	.215
α_2	-10.9	-6.90	-8.60
β_2	.098	.040	.040
β_3	.0217	.0191	.0139

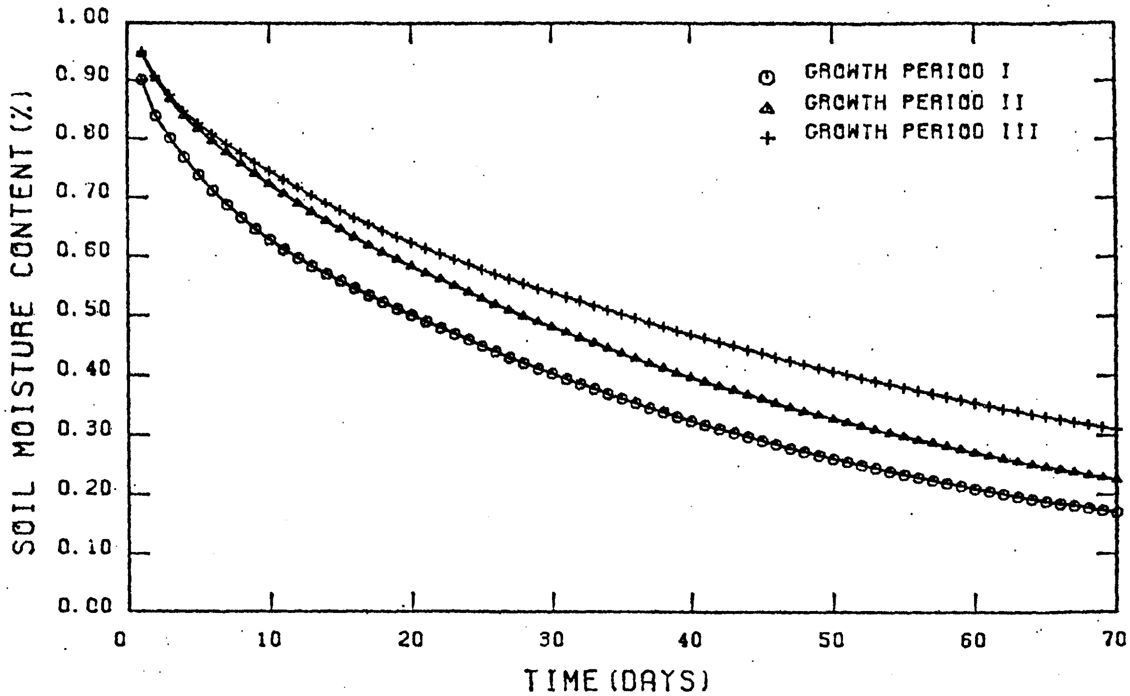


Figure 5.3

SOIL MOISTURE DEPLETION CURVES

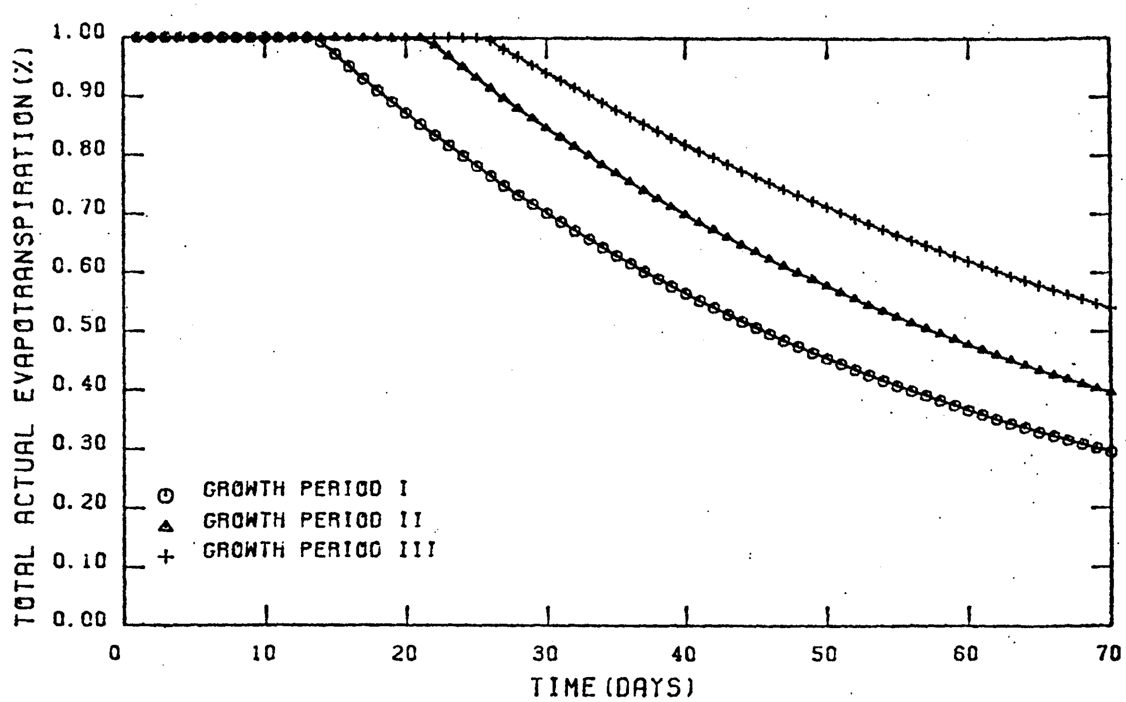


Figure 5.4

TOTAL ACTUAL EVAPOTRANSPIRATION CURVES

5.3.3 Economic Parameters

As introduced in Section 4.4, the objective is the maximization of expected net profits; this is expressed as (equations 4.58 and 4.59):

$$B^{*} = \text{MAX}\, E\left[\sum_{k=1}^{N+1} R_k(\overline{X}_k, \overline{X}_{k+1}, U_k)\right] - PC \tag{5.4}$$

$$\text{where } R_k(\overline{X}_k, \overline{X}_{k+1}, U_k) = P\, Y_k - \beta U_k - \gamma C_k(U_k). \tag{5.5}$$

The crop model has also been defined in equation 5.1 as:

$$Y = Y_M \sum_{z=1}^{NP} A_z \frac{ET_z}{ETP_z} = \sum_{z=1}^{NP} Y_z \tag{5.6}$$

where as before, NP is the number of critical growth periods; thus, Y_z represents the contribution of growing period z to total actual crop yield. According to this, and from equation 5.6, Y_z can be expressed as:

$$Y_z = Y_M A_z \frac{ET_z}{ETP_z} \tag{5.7}$$

However, because of the additive characteristic of the crop model, Y_z can also be expressed as:

$$Y_z = \sum_{n=1}^{ND_z} Y_{n,z} = \frac{Y_M A_z}{ETP_z} \sum_{n=1}^{ND_z} ET_n \tag{5.8}$$

where ND_z represents the number of decision stages in growing period z; and $Y_{n,z}$ is the contribution of decision stage n in growing period z to total actual crop yield. Substituting equations 5.7 and 5.8 into equation 5.6 yields:

$$Y = \sum_{z=1}^{NP} \sum_{n=1}^{ND_z} Y_{n,z} = \sum_{k=1}^{N+1} Y_k \qquad (5.9)$$

where:

$$N = \sum_{z=1}^{NP} ND_z - 1 \qquad (5.10a)$$

and:

$$Y_k = Y_M \cdot A_k^* \cdot \frac{ET_k}{ETP_k^*} \qquad (5.10b)$$

is the contribution of the general decision stage k to total actual crop yield, as it appears in equations 4.59 and 5.5; A_k^* and ETP_k^* are the crop sensitivity parameter and total potential evpotranspiration for the growing period corresponding to decision stage k. Substituting equation 5.10b into equation 5.5 yields:

$$R_k(\overline{X}_k, \overline{X}_{k+1}, U_k) = P\ Y_M\ A_k^* \frac{ET_k}{ETP_k^*} - \beta U_k - \gamma C_k(U_k) \qquad (5.11)$$

Equation 5.11 can be rewritten as:

$$R_k(\overline{X}_k, \overline{X}_{k+1}, U_k) = \alpha_k\ ET_k - \beta U_k - \gamma C_k(U_k) \qquad (5.12)$$

where: $\alpha_k = P\ Y_M\ A_k^*/ETP_k^*$ represents the dollar value of one unit of actual evapotranspiration during decision stage k. In the previous equations, A_k^* and ETP_k^* are defined as follows:

$$A_k^* \equiv A_z \tag{5.13a}$$

$$ETP_k^* \equiv ETP_z \tag{5.13b}$$

and such that:

$$\sum_{n=1}^{z} ND_n \leq k \leq \sum_{n=1}^{z+1} ND_n \quad , \quad n=1,2,\ldots,NP \tag{5.14}$$

Blank (1975) estimated values for the parameters Y_M and P. These are 140 bushels/acre and 2.5 U.S.$/bushel, respectively. Using the length of each growth period and the values for the potential evapotranspiration rate, the parameter α_k was calculated. Table 5.6 shows the resulting values. The parameters PC, β, and γ were estimated by Blank from the study of the Fort Morgan irrigation area in Colorado, performed by Conklin (1974). Assuming a fifty percent efficiency in the application of water yields a value $\beta = 0.016$ U.S.$/mm, of water delivered at the root zone. Assuming that labor cost per irrigation is independent of the amount of water applied yields $\gamma = 2.5$ U.S.$/irrigation/acre. Finally, PC is estimated as 237.0 U.S.$/acre.

Table 5.6

VALUES FOR PARAMETER α_n

	Growth Period		
	I	II	III
α_n \$/mm	0.44	1.15	0.94

5.4 Climate Subsystem Model Calibration

As defined in previous chapters, the climate subsystem is modelled in terms of the potential evapotranspiration rate and the precipitation process. The potential evapotranspiration rate was assumed deterministic but varying throughout the growing season, and its values for the different growth periods were presented in Section 5.3. Precipitation is modelled in terms of the process of storm arrivals and the storm intensity and duration. As stated earlier, these parameters were estimated from twenty-seven years of precipitation data at Denver, Colorado (1949 to 1975).

The main issue to be resolved in the calibration of the complete precipitation model is the definition of a storm event. In the work of Córdova and Bras (1979), and in order to justify the use of the Poisson process for the storm arrivals, an independent storm event had to be defined. This was done by determining a minimum interstorm time that would yield a coefficient of variation equal to one in the implied one parameter exponential distribution. Neglecting seasonality in the precipitation process throughout the year, Restrepo and Eagleson (1979) determined that the minimum interstorm time was 17.7 hours for Denver, Colorado.

In this work, the objective is not to define a storm event that fits the assumptions of a prespecified model; rather, a logical storm event is defined, and then a precipitation model is fitted to the resulting time series. The only condition required is that the defined storm event effectively describes a point occurrence. This is a condition required not only by the Neyman-Scott process, but also by the Poisson process, and in general by the assumption that the precipitation arrivals constitute a point process. This is true as long as the duration of each storm event is not considered, since for a point process the occurrences should be instantaneous. Consequently, an abstraction is needed so that storms of finite duration are transformed into instantaneous occurrences in time. This transformation is obtained defining a sampling interval so that an instantaneous rainfall event is assumed to have occurred in the middle of the interval if certain conditions are met. As the size of the sampling interval decreases, each storm event starts to approximate more and more an instantaneous occurrence. This size of the interval is determined not only by the requirement that the resulting storm events should approximate as much as possible a point occurrence; but also by the computer storage requirements, computer costs, etc. Keeping this in mind, and the study of Restrepo and Eagleson mentioned above, an interval size of twenty-four hours (one day) is used in this work to define a rainfall occurrence. A storm event occurs in the middle of the day whenever the total precipitation on the given day is greater than 0.254 mm (0.01 inches).

5.4.1 Rainfall Intensity and Duration Parameters

The entire growing season (120 days) was divided into fifteen consecutive eight-day long periods to carry out the rainfall data analysis with respect to intensity and duration. The analysis was performed for each eight-day period and frequency histograms for storm intensity and duration were obtained (Tables 5.7 and 5.8): first and second-order moments for intensity and duration, as well as the number of storm events for each eight-day period were obtained.

Comparing the number of storms in each of the fifteen periods, as well as the mean and variance of the storm intensity and duration, five arbitrary statistically homogeneous precipitation periods were defined. These homogeneous precipitation periods are as shown in Table 5.9. This means, that although as stated in previous chapters, the precipitation process is considered homogeneous (stationary) with respect to the storm arrivals; it is considered inhomogeneous with respect to the storm characteristics intensity and duration. Using the method of moments, an exponential distribution was fitted to storm intensity and storm duration in each homogeneous precipitation period. Also, for the sake of comparison, a Poisson model was fitted to the process of storm arrivals. The parameters obtained for the analyzed data are as shown in Table 5.10.

5.4.2 Rainfall Arrivals Model Parameters

Only three parameters need to be calibrated when the rainfall arrivals are modelled as a Neyman-Scott cluster process of the kind considered here. Namely, these parameters are the parameter μ of the Poisson distribution for the parent process or rate of occurrence of RGM's; the parameter p of the geometric distribution for the cluster sizes; and the parameter α of the exponential distribution for the times of occurrence of the individual storms

Table 5.7

HISTOGRAMS FOR STORM INTENSITY

STORM INTENSITY (MM/H)

HISTOGRAMS FOR EACH PERIOD

INTERVAL	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0.- 1.	47.	37.	50.	47.	38.	28.	30.	41.	36.	42.	39.	32.	27.	31.	23.
1.- 2.	22.	20.	16.	17.	11.	6.	8.	13.	16.	10.	9.	10.	10.	11.	13.
2.- 3.	6.	5.	9.	4.	4.	4.	5.	5.	6.	7.	5.	8.	8.	3.	4.
3.- 4.	2.	2.	4.	6.	2.	1.	3.	3.	6.	4.	5.	5.	2.	4.	0.
4.- 5.	2.	1.	1.	1.	2.	1.	5.	3.	5.	2.	2.	1.	2.	1.	1.
5.- 6.	1.	0.	0.	4.	3.	0.	0.	2.	1.	2.	1.	0.	1.	0.	0.
6.- 7.	0.	0.	0.	0.	0.	0.	1.	0.	3.	0.	1.	0.	0.	0.	0.
7.- 8.	0.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	1.	0.	0.	2.
8.- 9.	0.	0.	0.	0.	0.	0.	2.	0.	1.	1.	0.	1.	1.	0.	1.
9.- 10.	1.	0.	0.	0.	0.	0.	0.	0.	2.	0.	0.	0.	0.	0.	0.
10.- 11.	0.	0.	0.	0.	2.	0.	1.	0.	1.	1.	0.	0.	0.	0.	0.
11.- 12.	0.	0.	0.	0.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.
12.- 13.	0.	1.	0.	1.	0.	0.	0.	0.	0.	0.	1.	0.	0.	0.	0.
13.- 14.	0.	0.	0.	0.	0.	0.	1.	0.	1.	0.	0.	0.	0.	0.	0.
14.- 15.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
15.- 16.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
16.- 17.	0.	0.	0.	0.	1.	0.	0.	0.	0.	0.	0.	1.	0.	1.	0.
17.- 18.	0.	0.	0.	0.	0.	0.	0.	0.	0.	1.	0.	0.	0.	0.	0.
18.- 19.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
19.- 20.	0.	0.	0.	0.	0.	0.	0.	0.	1.	0.	0.	0.	0.	0.	0.
20.- 21.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
21.- 22.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
22.- 23.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
23.- 24.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
24.- 25.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
25.- 26.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
26.- 27.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
27.- 28.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
28.- 29.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
29.- 30.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
30.- 31.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
31.- 32.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
32.- 33.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
33.- 34.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
> 34.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
MEAN	1.2	1.3	1.1	1.5	1.8	1.0	2.2	1.2	2.5	1.7	1.4	1.7	1.4	1.4	1.5
STD. DEV.	1.4	1.7	1.0	1.9	2.8	0.9	3.0	1.3	3.3	2.6	2.0	2.5	1.6	2.4	1.9
NUMBER OF STORMS	81	66	80	81	63	40	57	67	79	70	63	59	51	51	44

Table 5.8

HISTOGRAMS FOR STORM DURATION

STORM DURATION (H)

HISTOGRAMS FOR EACH PERIOD

INTERVAL	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0.- 1.	20.	23.	27.	26.	27.	15.	25.	29.	27.	19.	26.	32.	22.	19.	18.
1.- 2.	17.	11.	13.	19.	9.	16.	17.	17.	23.	24.	24.	13.	14.	12.	13.
2.- 3.	13.	10.	7.	10.	14.	5.	8.	10.	10.	12.	2.	5.	4.	7.	1.
3.- 4.	6.	5.	7.	5.	4.	1.	3.	6.	11.	5.	5.	6.	2.	3.	7.
4.- 5.	3.	2.	11.	5.	3.	2.	1.	2.	3.	2.	1.	1.	3.	3.	2.
5.- 6.	2.	4.	1.	5.	4.	1.	2.	1.	1.	1.	2.	0.	5.	1.	2.
6.- 7.	2.	0.	1.	6.	0.	0.	0.	2.	2.	1.	0.	1.	1.	1.	0.
7.- 8.	2.	1.	5.	1.	0.	0.	0.	0.	2.	1.	1.	0.	0.	1.	0.
8.- 9.	4.	1.	2.	1.	0.	0.	1.	0.	0.	3.	2.	1.	0.	1.	0.
9.- 10.	2.	2.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	1.	0.
10.- 11.	2.	1.	2.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	1.	0.
11.- 12.	0.	0.	3.	1.	0.	0.	0.	0.	0.	1.	0.	0.	0.	0.	1.
12.- 13.	2.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
13.- 14.	0.	1.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
14.- 15.	1.	0.	0.	0.	0.	0.	0.	0.	0.	1.	0.	0.	0.	0.	0.
15.- 16.	1.	1.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	1.	0.
16.- 17.	0.	2.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
17.- 18.	1.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
18.- 19.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
19.- 20.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
20.- 21.	1.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
21.- 22.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
22.- 23.	1.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
23.- 24.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
24.- 25.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
25.- 26.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
26.- 27.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
27.- 28.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
28.- 29.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
29.- 30.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
30.- 31.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
31.- 32.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
32.- 33.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
33.- 34.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
> 34.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
MEAN	4.9	4.3	3.8	3.4	2.7	2.0	2.1	2.2	2.5	3.0	2.3	2.0	2.4	3.1	2.5
STD. DEV.	5.1	4.9	3.5	3.4	2.7	1.2	1.5	1.5	1.7	2.7	1.9	1.6	1.8	3.0	2.1
NUMBER OF STORMS	81	66	80	81	63	40	57	67	79	70	63	59	51	51	44

Table 5.9

HOMOGENEOUS PRECIPITATION PERIODS

Precipitation Period	
I	May 15 - June 16
II	June 16 - July 18
III	July 18 - Aug. 3
IV	Aug. 3 - Aug. 19
V	Aug. 19 - Sep. 11

Table 5.10

PARAMETERS OF EXPONENTIAL DISTRIBUTION FOR STORM INTENSITY, DURATION, AND POISSON ARRIVAL RATE

	Homogeneous Precipitation Periods				
	I	II	III	IV	V
Storm Intensity Parameter	0.78	0.65	0.48	0.65	0.70
Storm Duration Parameter	0.24	0.44	0.36	0.46	0.37
Storm Arrival Rate	0.36	0.26	0.34	0.28	0.23

Poisson Model Parameter 0.294

within their respective clusters. The calibration of these three parameters is done by fitting the theoretical spectrum of counts and log-survivor function for the N-S model to their estimated counterparts. The estimation of the spectrum of counts, and of the log-survivor function is performed using the theory developed by Cox and Lewis (1966) for the statistical analysis of series of events. Before proceeding to the calibration of the N-S cluster model for the twenty-seven years of rainfall data in Denver, a partial description of the process in terms of the estimated mean rate of daily occurrence, the estimated variance-time function, the estimated log-survivor function, and the estimated spectrum of counts is obtained using the theory for the statistical analysis of series of events (Cox and Lewis, 1966).

5.4.2.1 General Description of Arrival Process

Figure 5.5 presents a plot of the cumulative number of storm events versus cumulative time from an arbitrary origin. From this plot it is possible to identify low frequency cycles or non-homogeneities in the mean rate of occurrence. The slope of the plot at any time yields the inverse mean rate of daily occurrence. However, it is not at all clear from Figure 5.5 whether the slope is increasing or decreasing or constant (or in other words, whether the function is convex, concave or both). These characteristics are important because they help to determine long-term trends or low frequency inhomogeneities in the mean rate of arrivals. In order to do so, the Cramer statistic is calculated (Cramer, 1946). The Cramer statistic is defined as:

$$U = \frac{(S - T/2)}{T/\sqrt{12n}} \tag{5.15a}$$

where:

$$S = \frac{1}{n} \sum_{i=1}^{n} t_i \tag{5.15b}$$

In equations 5.15a and 5.15b, T is the length of the period of observation, n is the number of events in T, and t_i is the time from the origin to the occurrence of the i^{th} event. Cramer showed that U is distributed N(0,1) as $n \to \infty$. If the centroid of the observed times t_i is greater than the midpoint of the period T, then U is positive, indicating an increasing mean rate of occurrence. Using the program SASE (for Statistical Analysis of Series of Events; Lewis, et al., 1969), the value of U obtained was 0.5, which indicates no trends in the rate of occurrence at the five percent level of significance. So that at this point, the assumption of a constant mean rate of occurrence seems to be called for, as it would be desired under the N-S model adopted, since as expressed in equation 3.17, its rate of occurrence is constant. However, this conclusion is not valid because both of the above tests serve only to determine low frequency trends, and high frequency cycles are filtered out.

To estimate the mean rate of occurrence of the storm arrival process, the following estimator was used:

$$\lambda_{\Delta t}(t) = \frac{N(t,\ t+\Delta t)}{\Delta t} \tag{5.16}$$

where Δt is an arbitrary interval of time.

This estimator was used by Kavvas and Delleur (1975) and proved to be unbiased for stationary processes by Cox and Lewis (1966). Figure 5.6 shows the estimated mean rate of occurrence. Since for a stationary process, the mean rate of occurrence constitutes a horizontal straight line, it is evident from Figure 5.6 that the rainfall data analyzed is not only non-homogeneous but possesses a very marked yearly periodicity. Many other significant cycles may exist in the analyzed time series, but a complete trend and cycle analysis is out of the scope of this work.

The normalized spectrum of counts was estimated using the computer program SASE by Lewis, et al. (1969). The estimated spectrum of counts for the rainfall occurrences in Denver is shown in Figure 5.7. As defined in an earlier chapter, the spectrum of counts is the Fourier transform of the covariance density. It then carries information about the dependence structure of the process. For the homogeneous Poisson process, that is, under the independence assumption, the theoretical spectrum of counts is (Kavvas and Delleur, 1975), $g_+(w) = \lambda/\pi$, $w \geq 0$, or a horizontal straight line. Figure 5.7

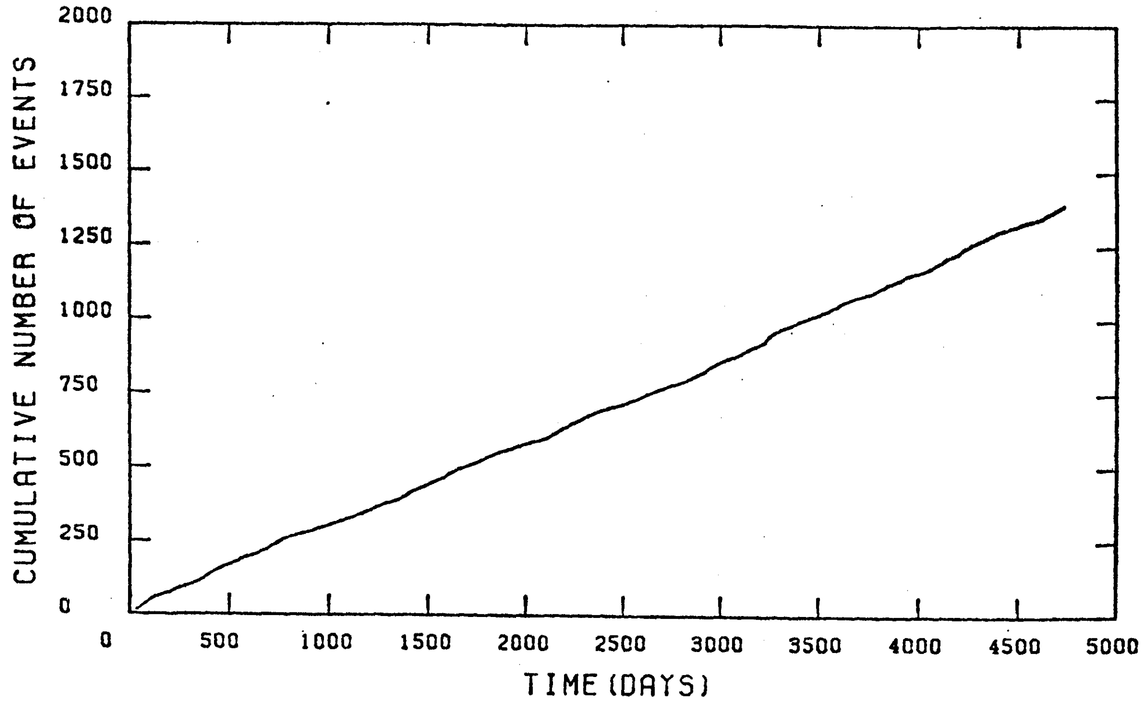


Figure 5.5

CUMULATIVE NUMBER OF STORMS VS. TIME

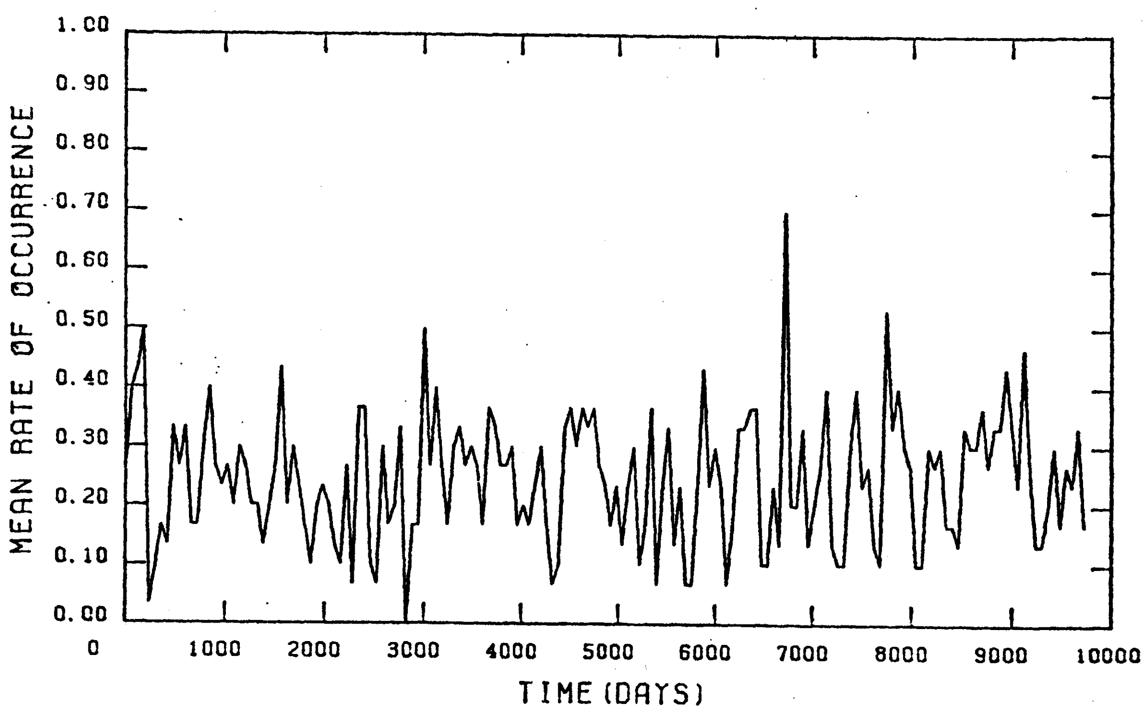


Figure 5.6

MEAN RATE OF OCCCURRENCE

also shows the normalized spectrum of counts and ninety-nine percent confidence intervals for the Poisson process when it is fitted to the data from Denver. It is evident that the estimated spectrum of counts deviates greatly from the Poisson assumption of independence, as shown by the many frequencies outside the confidence limits, indicating a definite dependence structure in the rainfall arrival process.

The variance time function, as well as the log-survivor function, was also estimated using the program SASE. For a Poisson process the variance-time function plots as a straight line with slope equal to the rate of arrival. The comparison between the estimated variance-time function and the theoretical variance-time function under the Poisson assumption is shown in Figure 5.8. It shows not only the deviation from the Poisson case of the estimated variance-time function, but also by appearing to be convex, indicates an overdispersion, and a clustering of the rainfall events (Vere-Jones, 1970); the upward deviation from the Poisson case implies a coefficient of variation greater than unity. This is all clear evidence of the clustering behavior in the occurrence of rainfall and of a significant dependence structure.

Finally, the relative frequency histograms of the interarrival times between daily occurrences, and the log-survivor function were estimated using again the above-mentioned computer program (Lewis, et al., 1969). The log-survivor function is defined as the logarithm of the survivor function of the interarrival times. For the Poisson

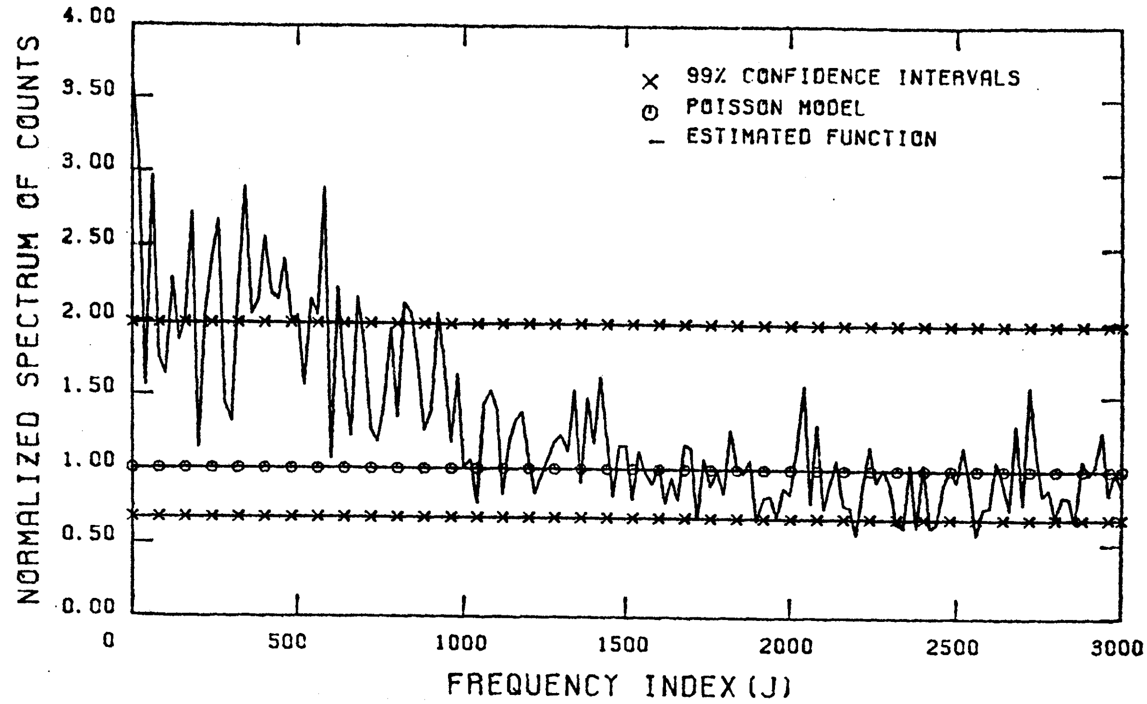


Figure 5.7

ESTIMATED NORMALIZED SPECTRUM OF COUNTS

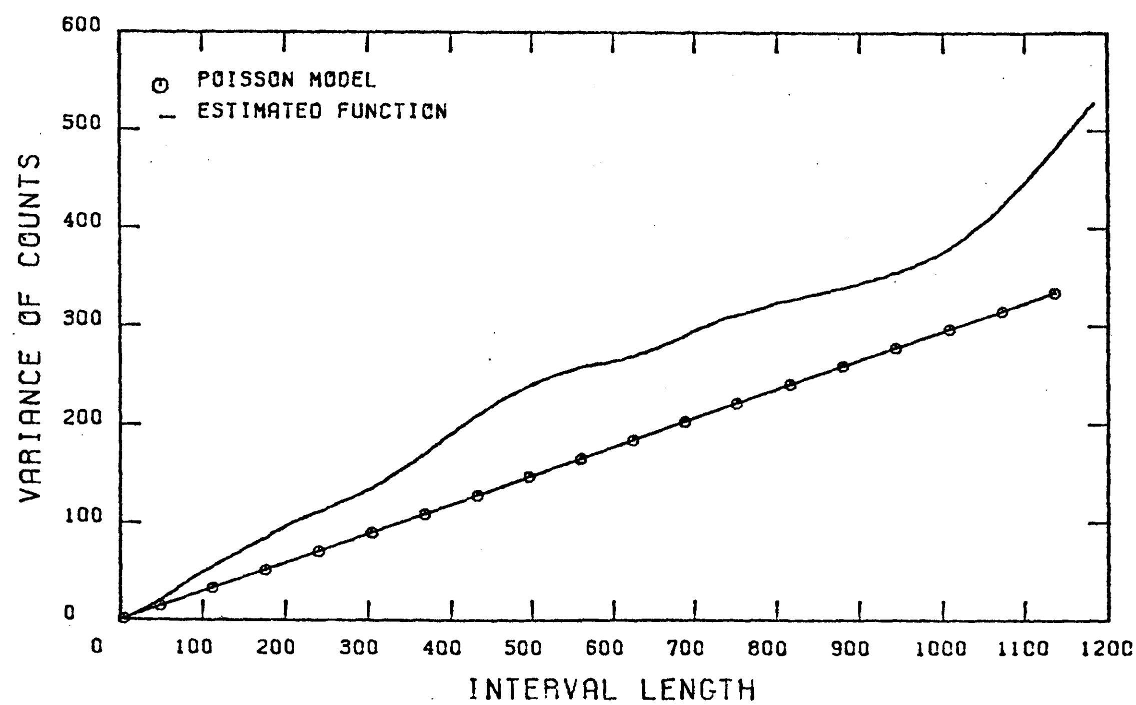


Figure 5.8

ESTIMATED VARIANCE-TIME FUNCTION

process, it is easily obtained as:

$$\ln \Pr[X > x] = -\lambda x \tag{5.17}$$

Figure 5.9 shows the estimated log-survivor function. Again, it not only deviates from the theoretical Poisson case, but its convexity indicates once more the overdispersion and clustering of the rainfall occurrences (Vere-Jones, 1970; Kavvas and Delleur, 1975). However, the deviation from the Poisson log-survivor function is not very marked, indicating that possibly, although there is some clustering, it is not sufficient to cause a very long memory in the rainfall counts, and the process at hand has a very weak dependence structure.

From the above general statistical description of the rainfall data used in the case study, three main features are of interest. The first relates to the obvious non-homogeneity of the process of rainfall occurrences. Neglecting the influence of the high frequency cycles, the only significant periodicity appears to be the marked yearly cycle. However, the N-S cluster model under consideration is stationary. Obviously then, the N-S model cannot be used to model rainfall occurrences throughout the entire year. It can only be used to model stationary sequences. For the purpose at hand, the four-month long irrigation season is a period of time short enough to make practical the assumption of stationarity. Kavvas and Delleur (1975) state that the model could be employed for small intervals where stationarity of the process can be safely assumed. Thus, for this work, although the complete process is obviously non-homogeneous, it is assumed stationary in the time interval under consideration (growing season).

The second and third features relate to the dependence structure and clustering behavior of the rainfall counts in the rainfall data from Denver. It is observed from the variance-time, spectrum of counts, and log-survivor functions that the rainfall events, in Denver, cannot be considered independent; and that the storm events have a tendency to form groups around their RGM (clustering). Thus, it has been indicated that the independence assumption of the Poisson process is not valid and that a more general model that can account for the dependence structure as well as for the clustering behavior is needed. As in Kavvas and Delleur (1975), it is here hypothesized that the N-S model is such a model.

For more details about the estimators of the variance-time function, and the spectrum of counts, as well as the computer program used in their evaluation, the reader is referred to Cox and Lewis (1966); Lewis, et al. (1969), and Kavvas and Delleur (1975).

5.4.2.2 Neyman-Scott Cluster Model Calibration

The main objective in modelling rainfall arrivals as a N-S cluster process is to be able to account for the observed dependence structure and clustering behavior of the storm counts. The statistical analysis of the previous section has confirmed once more that for Denver, Colorado, the process of storm arrivals also exhibits the cluster dependence structure found in Indiana by Kavvas and Delleur (1975). In particular, this clustering behavior and dependence structure is shown in the estimated spectrum of counts and log-survivor function. The comparison of these functions with the theoretic Poisson case invalidates the assumption of independence. The convexity of the log-survivor function has been shown to indicate an overdispersion of the rainfall events and consequently a clustering of storms. It is then only logical in order to preserve the dependence structure that the model be fitted to the estimated spectrum of counts; and in order to preserve the structure of the interarrival times and the clustering behavior, that it be fitted to the estimated log-survivor function.

The theoretical normalized spectrum of counts fitted to that estimated in the previous section is obtained from equation 3.28 as:

$$\frac{\pi g_+(w)}{\mu \cdot E[N_2(t)]} = 1 + \left[\frac{E[N_2^2(t)]}{E[N_2(t)]} - 1\right] \cdot \frac{\alpha^2}{\alpha^2 + w^2} \quad , \quad w > 0 \qquad (5.18)$$

The fit was performed using a Non-linear Adaptive Least-Squares algorithm developed at the Sloan School of Management at MIT (Dennis, et al., 1979). Letting the parameters $E[N_2^2(\cdot)]$, $E[N_2(\cdot)]$, and α free to be estimated by the algorithm, produced unrealistic values for $E[N_2^2(\cdot)]$ and $E[N_2(\cdot)]$ in the sense that they yielded negative variance. Following the approach used by Kavvas and Delleur (1975) who encountered the same problem, the ratio $E[N_2^2(\cdot)]/E[N_2(\cdot)]$ was fixed equal to 2.44 as indicated by the unrealistic fit (Table 5.11).

Equation 5.17 was fitted again producing a value of α as shown in Table 5.11.

With the ratio $E[N_2^2(\cdot)]/E[N_2(\cdot)]$ fixed at 2.44 and with the value of α obtained, the resulting fit is shown in Figure 5.10, which is quite acceptable. The value of α determined implies a rapid decay of the exponential distribution of T, indicating the anticipated short memory of the process. Since the dependence in the rainfall counts is produced in part by the superposition of RGM's, the faster the exponential distribution for T decays, the less likely it is that events from two or more RGM's superimpose, and the weaker the dependence in the rainfall counts is. This is so since the probability that a storm event occurs far in the future decreases (Figure 5.11). From Figure 5.11, H^* in equation 4.47 was taken as seven days.

In order to calibrate the other two parameters, namely, p and μ, and in order to preserve the cluster behavior and interarrival time structure, the theoretic log-survivor function for the N-S model is fitted to the estimated one. The theoretic unconditional survivor function can easily be obtained from equation 3.49 by letting $\zeta = 0$. This yields:

$$S_{10}(x;0) = \Pr[X > x] = \left[\frac{p}{1 - qe^{-\alpha x}}\right]^{(\frac{\mu}{\alpha}+1)} \cdot e^{-\mu x} \qquad (5.19)$$

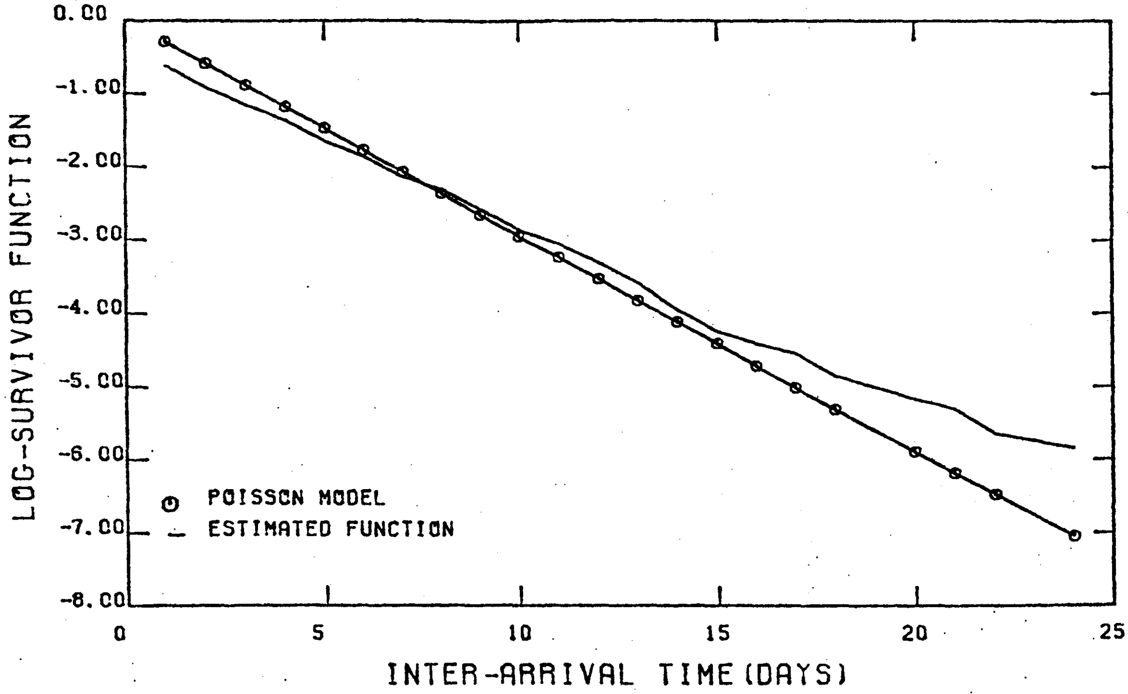


Figure 5.9

ESTIMATED LOG-SURVIVOR FUNCTION

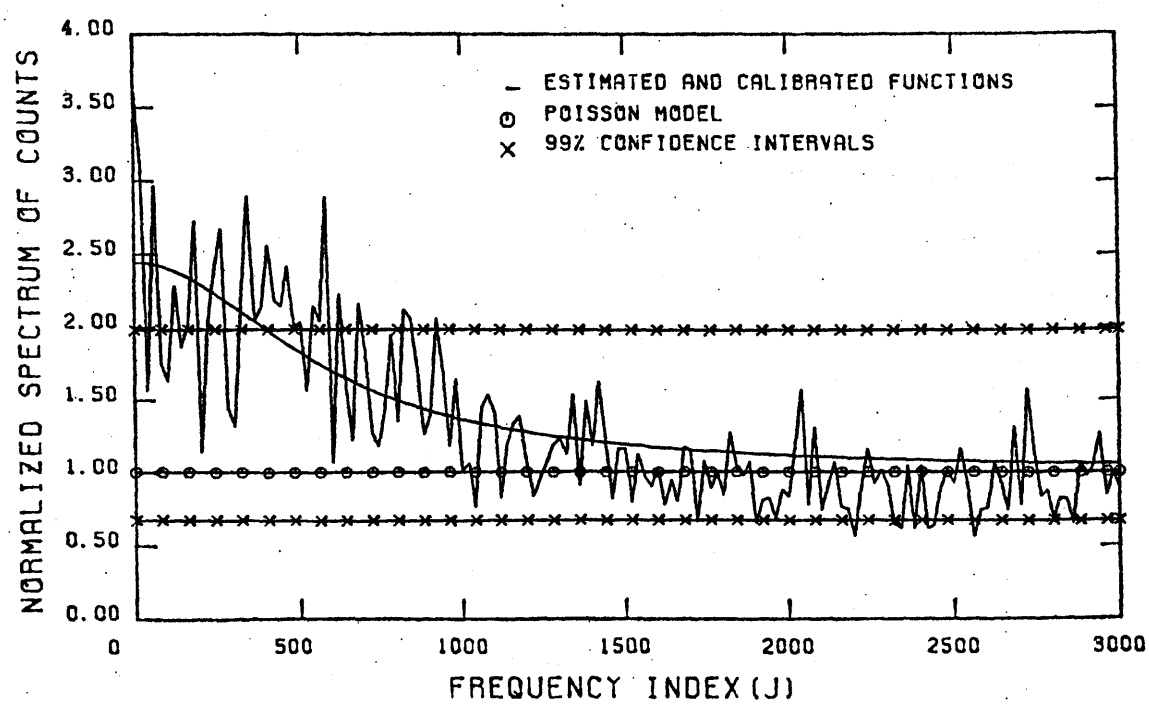


Figure 5.10

CALIBRATED NORMALIZED SPECTRUM OF COUNTS

Thus, the log-survivor function is:

$$\ln S_{10}(x;0) = \ln \Pr[X > x] = -\mu x + (\frac{\mu}{\alpha} + 1) \ln \frac{p}{1 - qe^{-\alpha x}} \qquad (5.20)$$

Using again the Non-linear Adaptive Least-Squares algorithm and the value of α previously calibrated (with ratio fixed at 2.44), values for the parameters μ and p were obtained as shown in Table 5.12. However, observe that the value of p obtained differs greatly from that implied by the fixed ratio at 2.44; namely a value of p equal to 0.58. In order to observe the behavior of the fit and expecting to obtain identical values for p in both fits, an iterative procedure was implemented. The value of the ratio $E[N_2^2(\cdot)]/E[N_2(\cdot)]$ for which identical values for p were obtained was 1.99 (Table 5.13); the associated Mean Square Errors (MSE) and coefficients of variation (CV) are also shown. Since the fit to the estimated spectrum of counts has the greater CV, it was assumed that the correct ratio was the one that yielded the smaller MSE. Thus, the ratio assumed is equal to 2.44, meaning that the parameters used are those shown in Tables 5.11 and 5.12. The fit obtained with these parameters is shown in Figure 5.12, which as before, is quite acceptable. This fit is rather good for the longer interarrival times, especially when compared to the fit that would be obtained using the Poisson assumption which implies a straight line. The N-S model is then able to preserve the clustering indicated by the convexity of the estimated log-survivor function.

Table 5.11

FIT TO SPECTRUM OF COUNTS

I)	Free Parameters:	$E[N_2^2(\cdot)]$	=	19.17
		$E[N_2(\cdot)]$	=	7.86
		α	=	0.755297
II)	Fixed Ratio	$E[N_2^2(\cdot)]/E[N_2(\cdot)]$	=	2.44
	Free Parameter:	α	=	0.755296

Table 5.12

FIT TO LOG-SURVIVOR FUNCTION

Fixed Parameter:	α	=	0.755296
Free Parameters:	p	=	0.674060
	μ	=	0.234509

Table 5.13

ITERATIVE MODEL FITTING

	Spectrum of Counts			Log-Survivor Function		
Fixed Ratio	Fitted α	Implied p	MSE	α	Fitted p	MSE
2.44	0.755296	0.5814	10.7	0.755296	0.674060	0.226
1.99	0.996443	0.668	12.1	0.996443	0.667	0.223
	CV = 2.5				CV = 0.12	

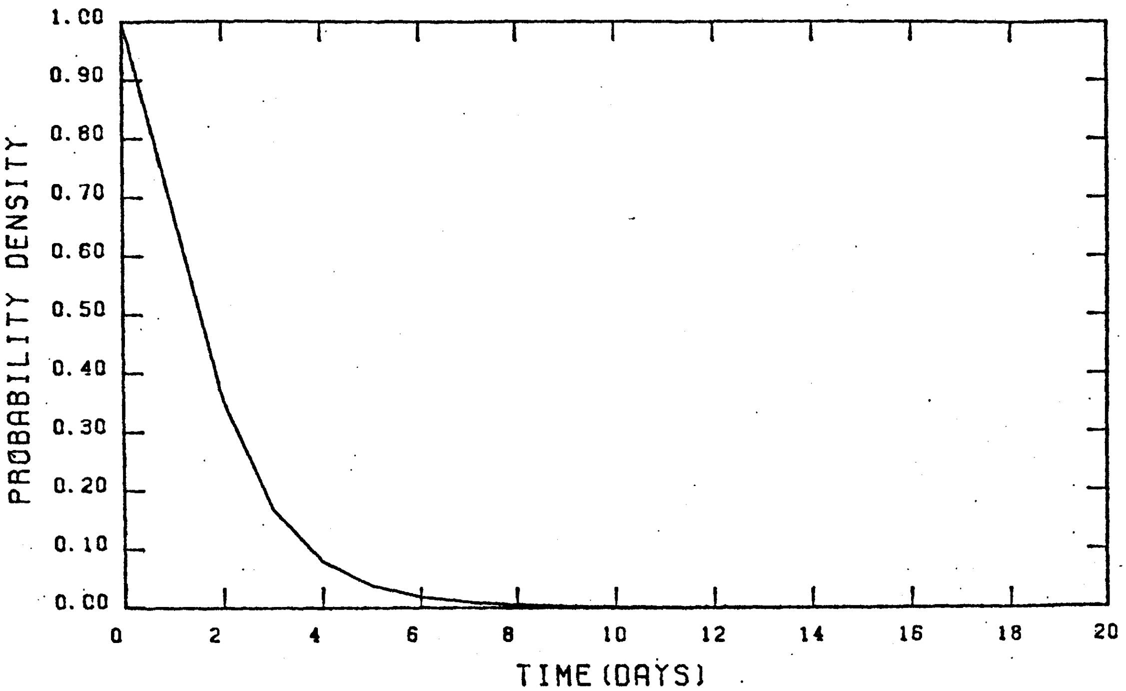


Figure 5.11

CALIBRATED PDF OF T

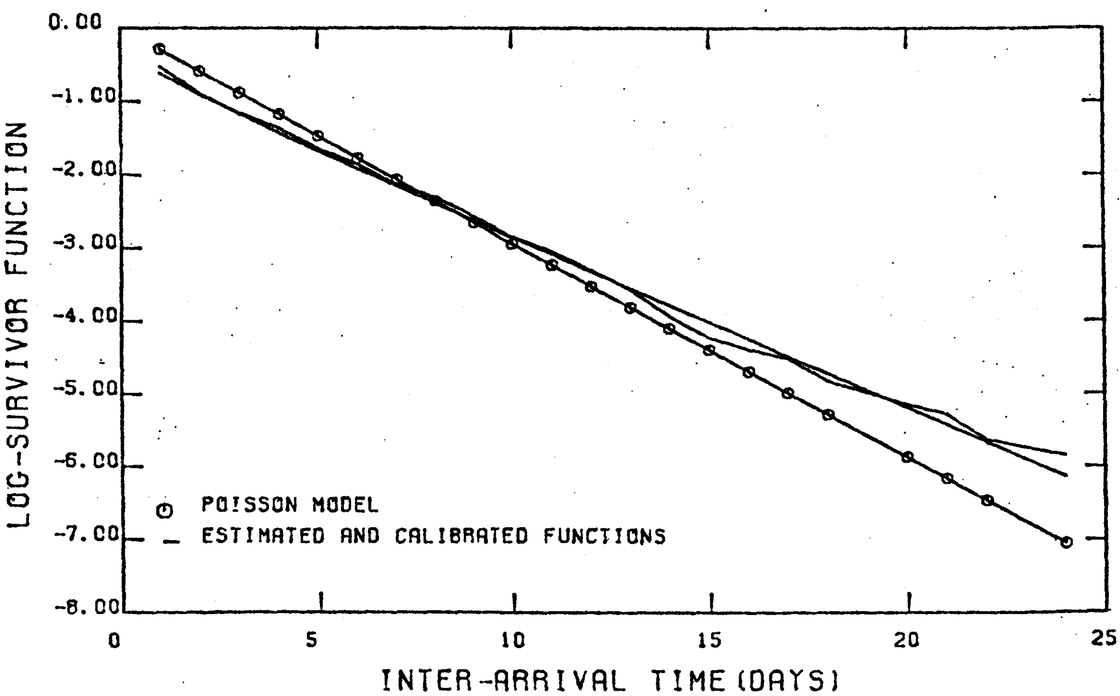


Figure 5.12

CALIBRATED LOG-SURVIVOR FUNCTION

5.4.2.3 Calibrated Conditional Distribution Functions

Any model that pretends to make use of the conditional information contained in the immediate history of rainfall arrivals should have at least the characteristic of being able to identify local short-term trends in the process. For instance, if during any given period, several days have passed without any storm occurring, the probability of rain should increase. However, when the number of days without rain keeps increasing, the model should be able to interpret this information as an indication of a drought; thus, the probability of future rain should decrease.

In the framework of the CDF's derived in Chapter 3 and expressed in equations 3.46 and 3.49, the above characteristic can be interpreted as follows. As ζ increases, that is, as the number of days since the last rainfall event increases, the model should detect a possible drought and yield a decreasing probability of rain in the future. This behavior can be observed clearly in Figure 5.13 by fixing the value of x and varying ζ. Observe that as ζ increases, $S_{10}(x;\zeta)$ also increases. On the other hand, for a fixed ζ, as x increases, $S_{10}(x;\zeta)$ decreases, meaning that the probability of rain in the future increases as we look farther and farther ahead (Figure 5.13). Thus, the simple model devised is capable of adequately modelling and forecasting short-term trends in the process. However, its simplicity neglects a great deal of valuable information by only considering the time since the last rainfall event. In spite of this, due to the short memory of the process expressed by the

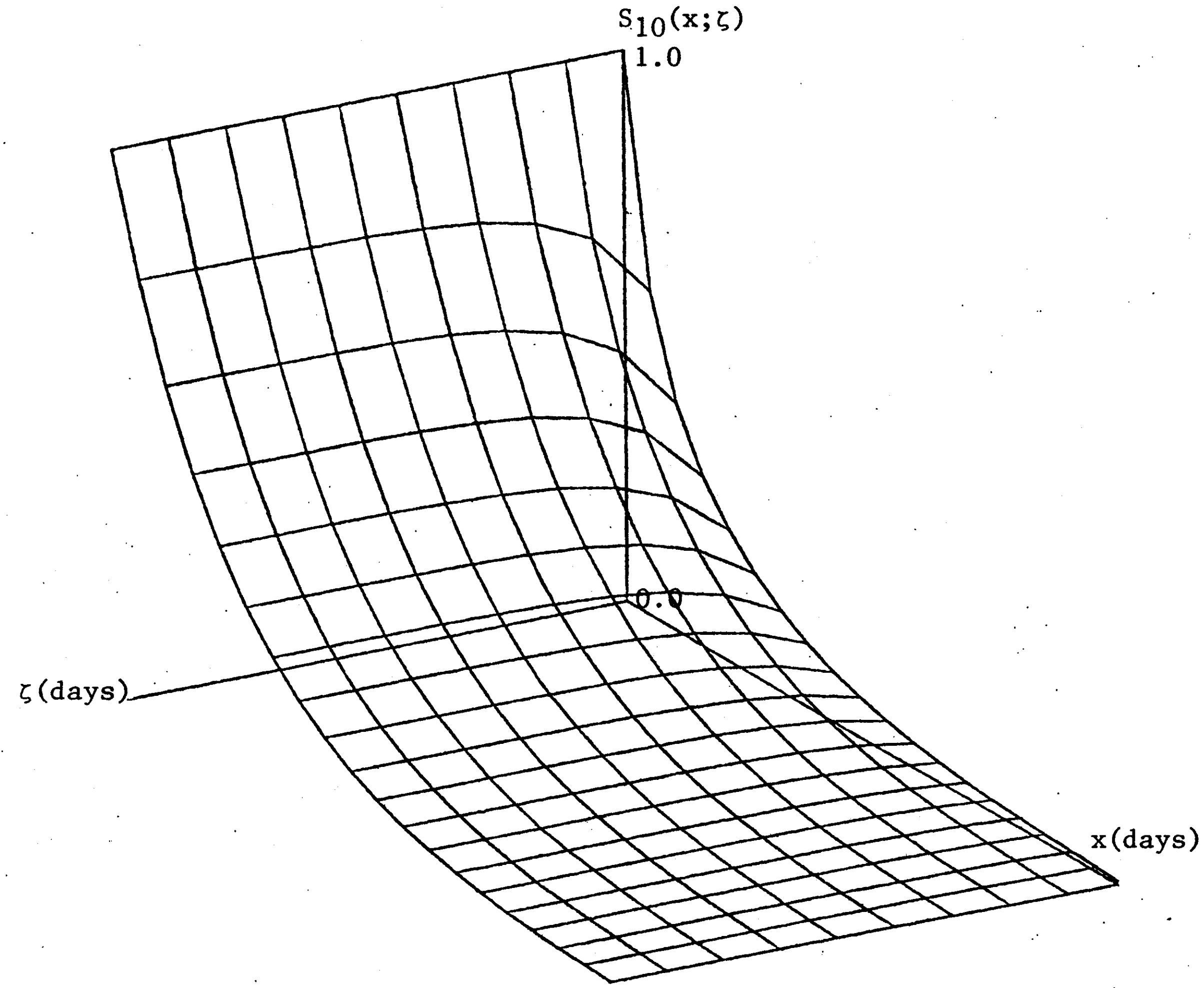


Figure 5.13

CALIBRATED CONDITIONAL DISTRIBUTION FUNCTION parameter α, the influence of the neglected information can probably be considered negligible.

5.5 Case Study Results

As in Córdova and Bras (1979), this study considers a dynamic root depth throughout the growing season. During each growth period, the water balance is carried out only in the soil column defined by the corresponding root depth. When a new period starts with a different root depth, there exists a newly available soil layer for which no water balance has been carried out, and consequently, its SMC is unknown. Assuming that percolation out of the root zone into the new layer is equal to the percolation out of the new layer (Córdova and Bras, 1979), the initial SMC in this layer at the beginning of the growing period can be assumed equal to the initial condition at the beginning of the growing season. In this study, this occurs when going from growth period I to growth period II. Following Córdova and Bras (1979) it is further assumed that the initial SMC in the new available soil layer is FC.

The irrigation scheduling problem was solved using SDP. Two different cases were considered, the unlimited irrigation water supply case, and the limited irrigation water supply case. For the unlimited water case only two feasible irrigation decisions were considered at each decision stage: not to irrigate at all or irrigate up to field capacity. For the limited water case, five feasible irrigation decisions were considered at each decision stage: irrigate up to FC, to 3/4 FC, to 1/2 FC, to 1/4 FC, or not to irrigate. The growing season was divided in 120 decision stages (daily irrigation decisions). The solution is given as optimal decision functions that yield the optimal amount of water to be applied at each decision stage as a function of the SMC and the number of days elapsed since the last rainfall occurrence. For the limited water supply case, twenty cases with different total irrigation water available during the growing season were considered. The amount of water available ranged from 475.0 mm to 0.0 mm, and as implied above, discrete intervals of 25 mm were used. In this case, the optimal decision is also a function of the available irrigation water. Typical decision functions for each case are presented in Appendix D and Appendix E.

The effect of the conditional Neyman-Scott model, encoded in the CDF's $S_0(x;\zeta)$ and $S_{10}(x;\zeta)$, on irrigation control is compared to results from four other precipitation models. Namely, the unconditional N-S model encoded in the PDF $P_0(x)$, the homogeneous and non-homogeneous Poisson models whose parameters are as presented in Tables 5.10 and 5.14, and Córdova's non-homogeneous Poisson model (Córdova and Bras, 1979) whose parameters are as presented in Table 5.14. It is necessary to make clear that the latter model was fitted by Córdova and Bras (1979) to a transformed set of data as explained earlier, so that the comparison is not valid. However, in extending Córdova's model from weekly to daily irrigation decisions results are obtained which contradict previously accepted facts, stated by Blank (1975) and Córdova and Bras (1979), about irrigation on fixed dates or variable dates. The inclusion of Córdova's model then serves the purpose of showing the effect of the particular data transformation used on the results of the control algorithm.

Table 5.14

TOTAL AMOUNT OF PRECIPITATION UNDER THE N-S AND POISSON MODELS

Poisson Model*

	Precipitation Period					
	I	II	III	IV	V	Mean
Mean Storm Intensity (mm/h)	1.28	1.54	2.27	1.52	1.47	1.62
Mean Storm Duration (h/storm)	6.67	2.33	3.13	2.33	3.70	3.63
Mean Rate of Arrival	0.262	0.228	0.308	0.266	0.184	0.250

N-S Model

	Precipitation Period					
	I	II	III	IV	V	Mean
Mean Storm Intensity (mm/h)	1.28	1.54	2.08	1.54	1.43	1.57
Mean Storm Duration (h/storm)	4.16	2.32	2.78	2.17	2.70	2.83
Mean Rate of Arrival	0.356	0.263	0.345	0.282	0.225	0.294

Rate of Occurrence under N-S Model $\mu \cdot E[N_2(\cdot)] = 0.348$

Average Precipitation for Poisson Model = 1.31 mm/day

Average Precipitation for Poisson Model* (Córdova and Bras, 1979) = 1.47 mm/day

Average Precipitaion for the N-S Model = 1.54 mm/day

*Obtained from Córdova and Bras (1979).

Figures 5.14 through 5.19 summarize the results obtained with the different models and different initial conditions. Observe that in general, all models analyzed behaved very consistently. The conditional form of the N-S model is compared first to the unconditional N-S model, and then to the Poisson model. As expected, the conditional N-S model yields greater expected net benefits at the end of the growing season than the unconditional N-S model for all the cases considered (Figures 5.14b through 5.19b). This is obviously the result of water conservation, obtained through improved irrigation water management made possible by accounting for the conditional information contained in the precipitation process.

Logical results are obtained with the unconditional N-S model; this case always yields lower expected net benefits than any of the other four models. Both the unconditional and conditional N-S models yield lower expected net benefits than any of the other models when there is no available water for irrigation; in this case, the system is being driven solely by rainfall. The behavior can be logically explained by the clustering of the precipitation occurrences encoded in the N-S process. The clustering behavior, as shown by the convexity of the log-survivor functions (Figure 5.9 and 5.12), tends to produce longer interarrival times. This is implied in Figures 5.9 and 5.12 by the fact that:

$$Pr_{N-S}[X > x] > Pr_{P}[X > x] \qquad \forall x > 0 \qquad (5.21)$$

where the subscripts N-S and P refer to the N-S and Poisson models, repectively.

When there is no available irrigation water, the longer interarrival times coupled to the soil moisture depletion processes and no possible control, lead to longer and more severe periods of water stress on the crop, thus producing reduced maximum expected net benefits as compared to those obtained with the Poisson model.

It was expected that the conditional form of the N-S model would yield greater expected net benefits than the Poisson model when irrigation was possible. As presented in Figures 5.14 through 5.19, this expectation was wrong. It is observed from these figures that the conditional N-S and the homogeneous Poisson model yield almost identical results. In fact, although the difference is not significant enough to be detected in the figures, the conditional N-S model produces slightly lower expected net benefits for higher initial SMC's and slightly greater expected net benefits for lower initial SMC's than the homogeneous Poisson model. The non-homogeneous Poisson model always yielded greater expected net benefits. Since the models were fitted to the same data set, these results can be simply explained by realizing that the clustering behavior will tend to induce greater water use and by the fact that the dependence structure present in the precipitation process is very weak as anticipated in Sections 5.4.2.1 and 5.4.2.2, and as shown in the form of the CDF $S_{10}(x;\zeta)$ (see Figure 5.13). Observe that the range of ζ (conditional information) over which there is a significant change in the value of $S_{10}(x;\zeta)$ is about two days. For regions of deficit irrigation, where the mean interstorm time tends to be long, a memory of only two days seems to be for all practical purposes negligible. However, the slightly better results obtained with respect to the homogeneous Poisson during dry initial conditions indicate that there may exist instances

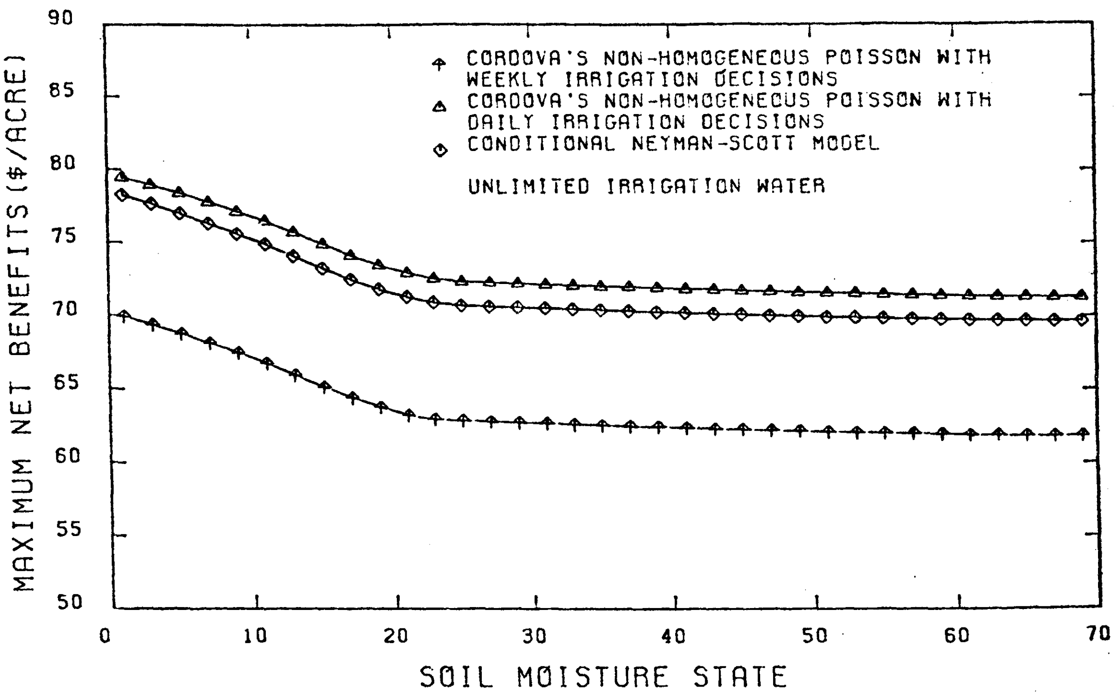


Figure 5.14a

UNLIMITED WATER CASE: MAXIMUM NET BENEFITS

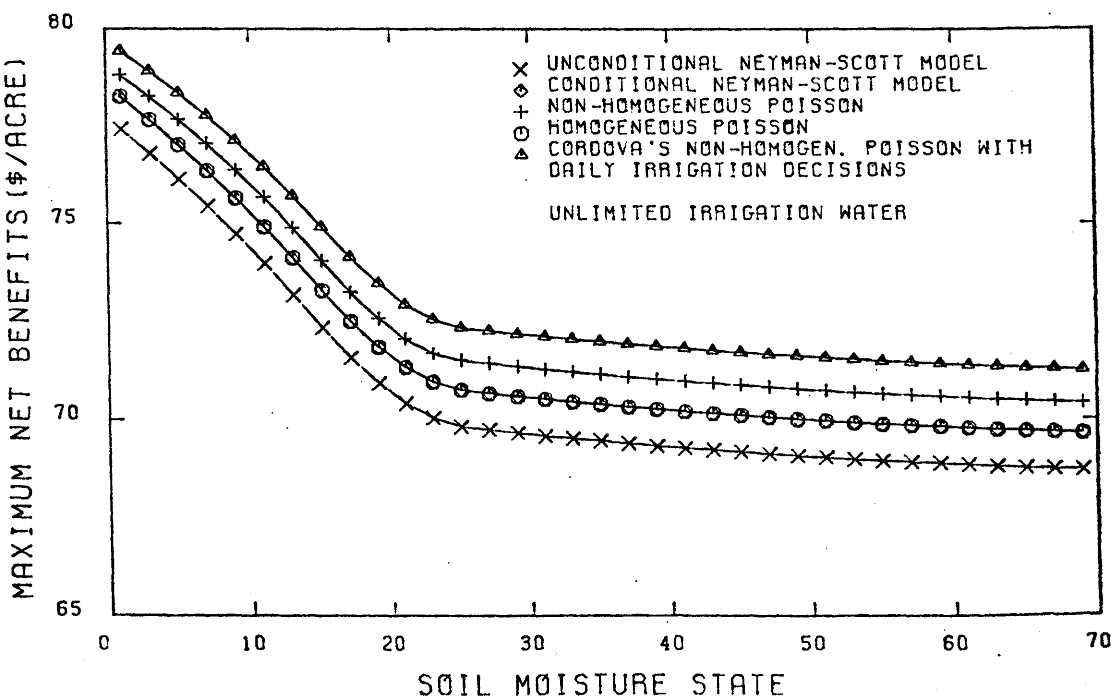


Figure 5.14b

UNLIMITED WATER CASE: MAXIMUM NET BENEFITS

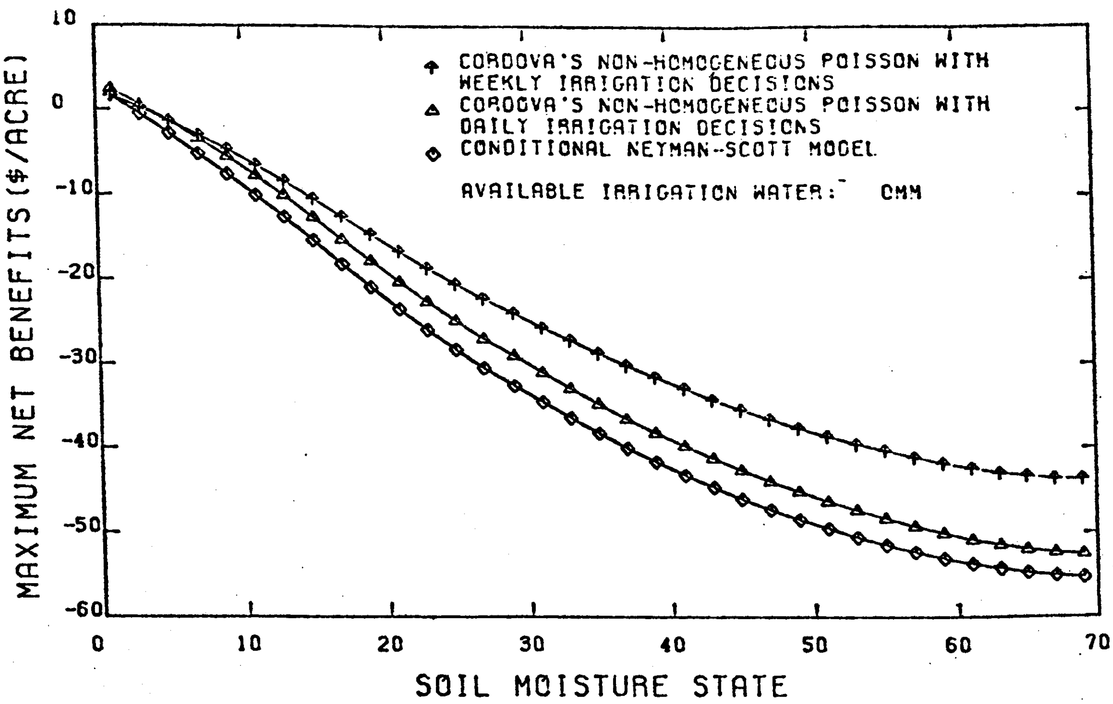


Figure 5.15a

LIMITED WATER CASE: MAXIMUM NET BENEFITS

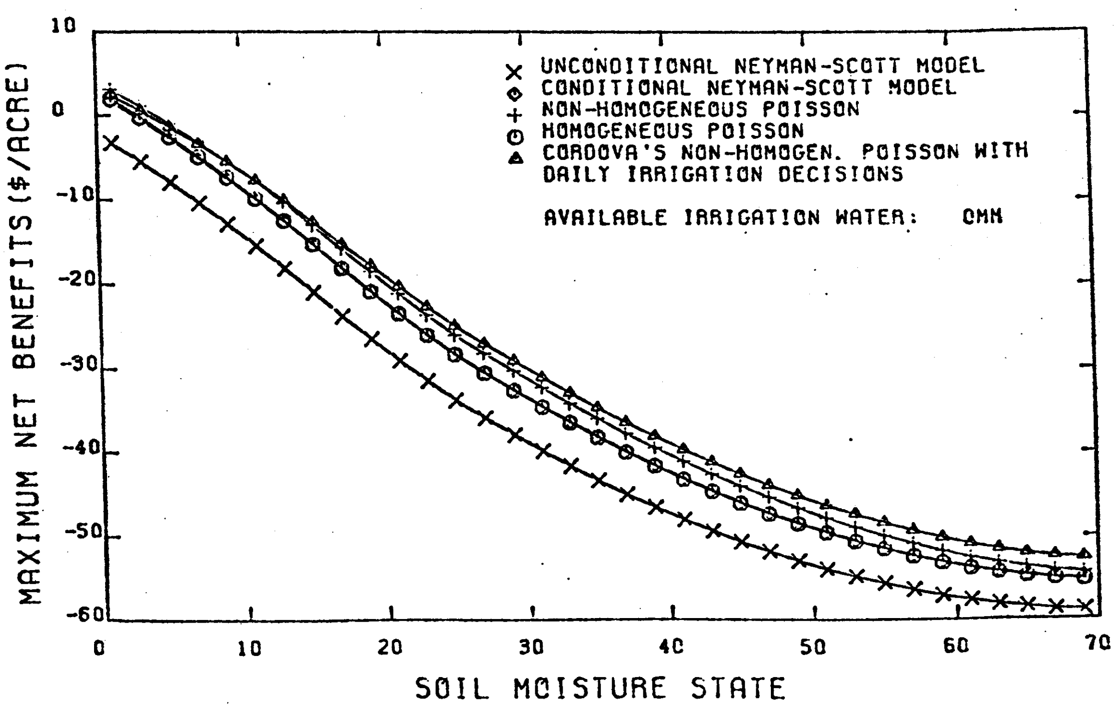


Figure 5.15b

LIMITED WATER CASE: MAXIMUM NET BENEFITS

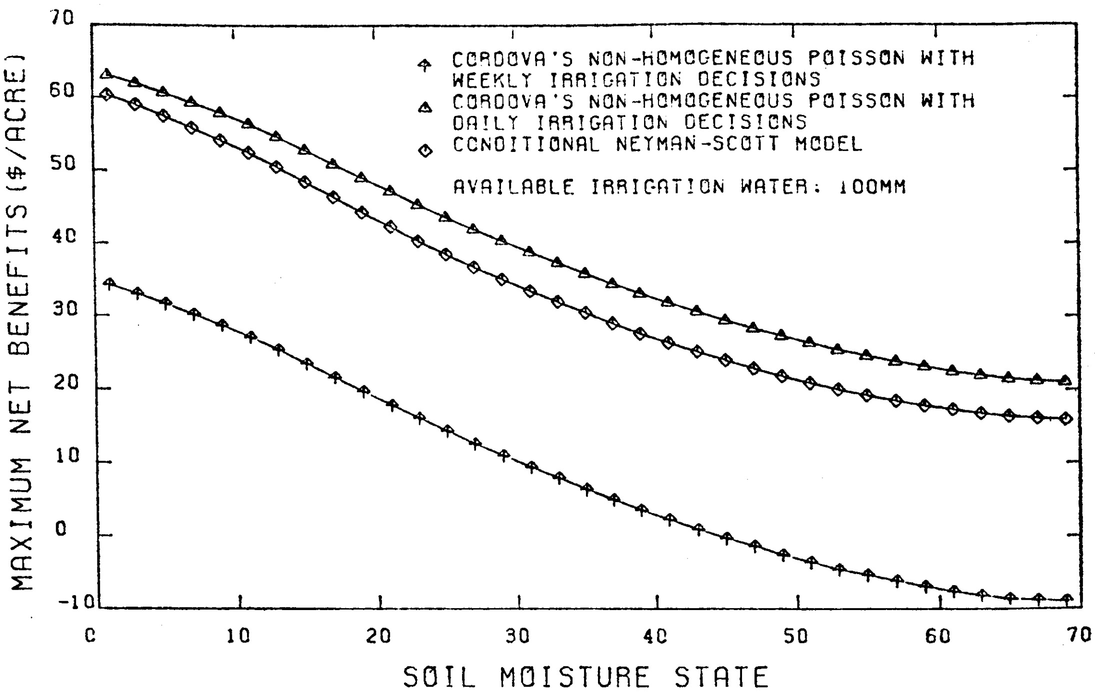


Figure 5.16a

LIMITED WATER CASE: MAXIMUM NET BENEFITS

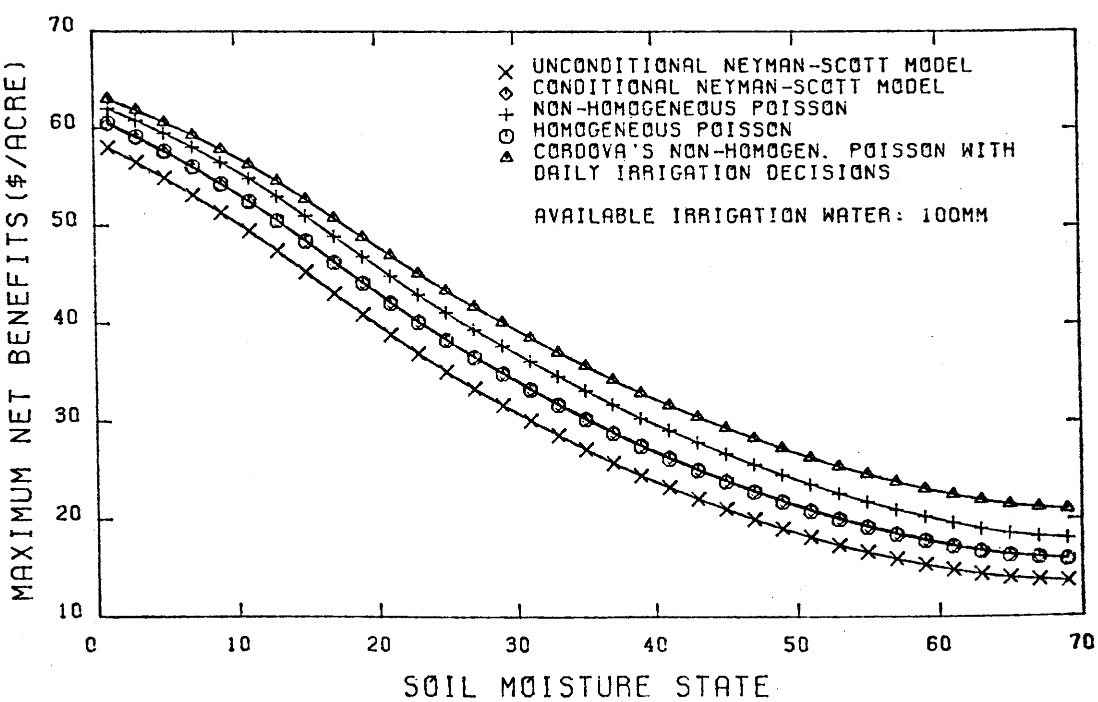


Figure 5.16b

LIMITED WATER CASE: MAXIMUM NET BENEFITS

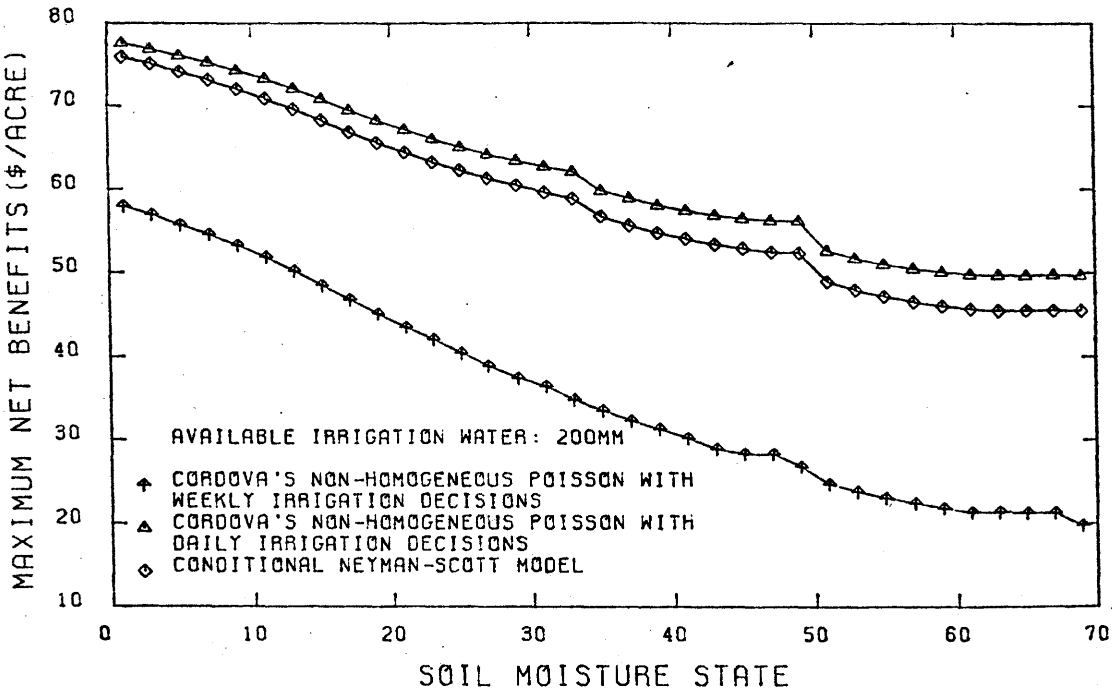


Figure 5.17a

LIMITED WATER CASE: MAXIMUM NET BENEFITS

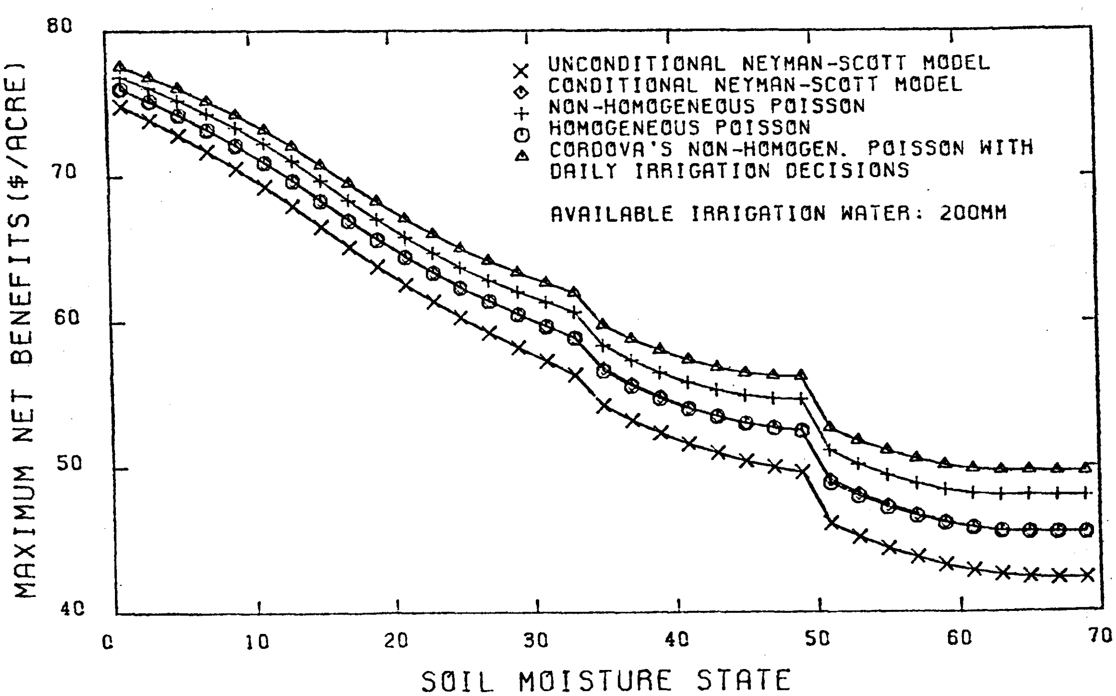


Figure 5.17b

LIMITED WATER CASE: MAXIMUM NET BENEFITS

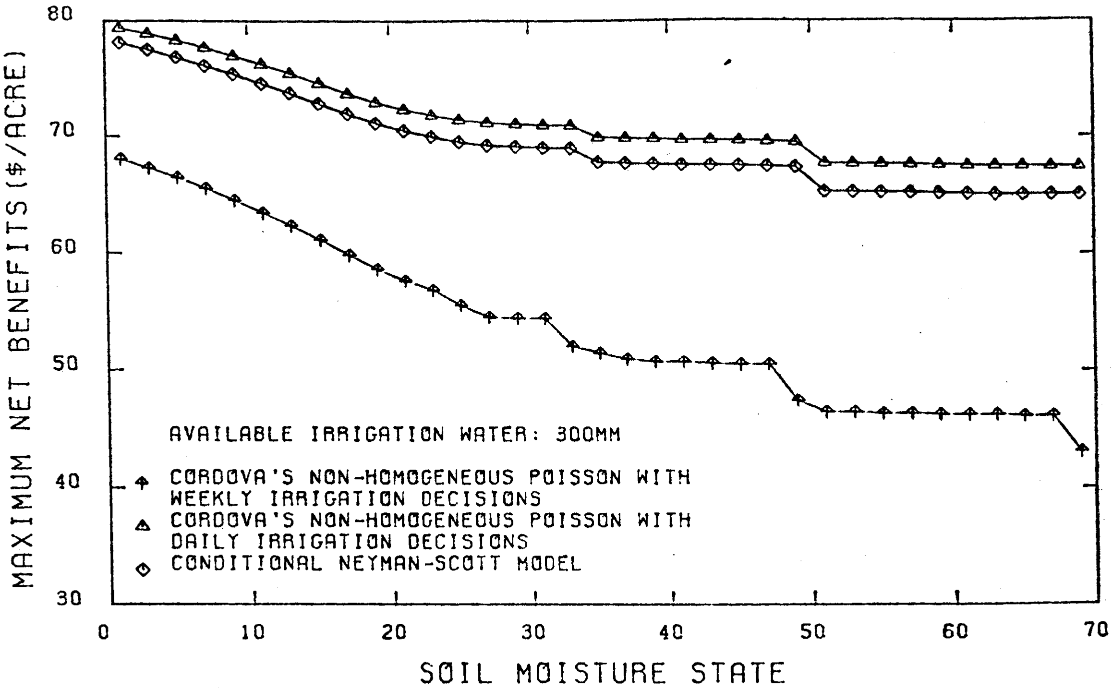


Figure 5.18a

LIMITED WATER CASE: MAXIMUM NET BENEFITS

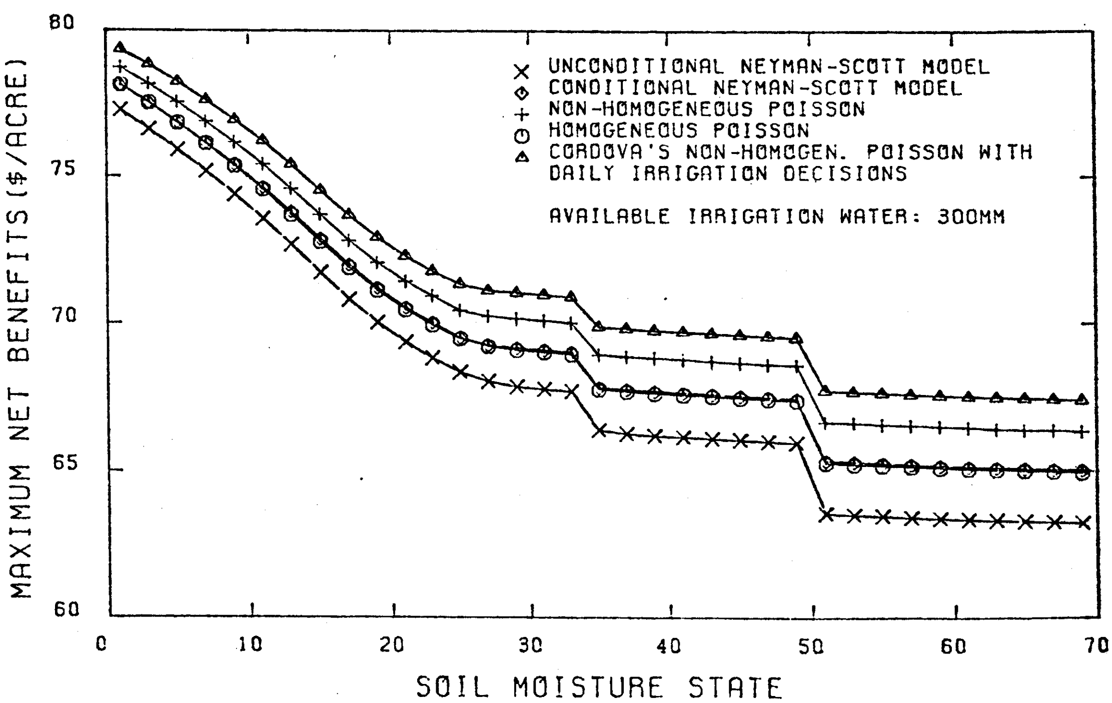


Figure 5.18b

LIMITED WATER CASE: MAXIMUM NET BENEFITS

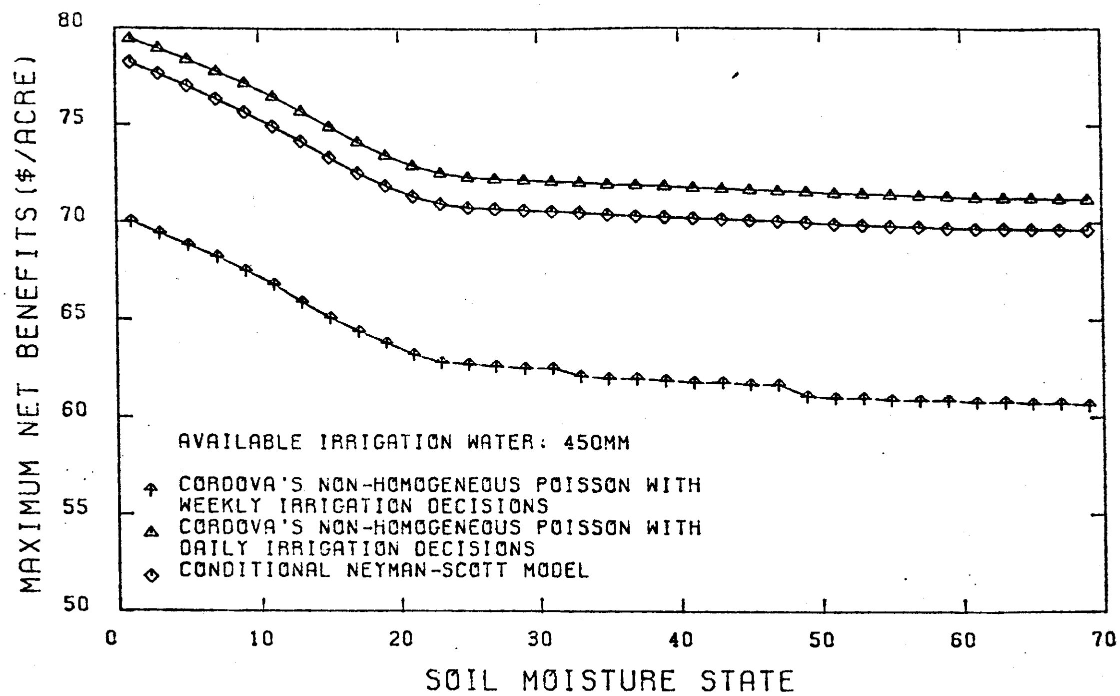


Figure 5.19a

LIMITED WATER CASE: MAXIMUM NET BENEFITS

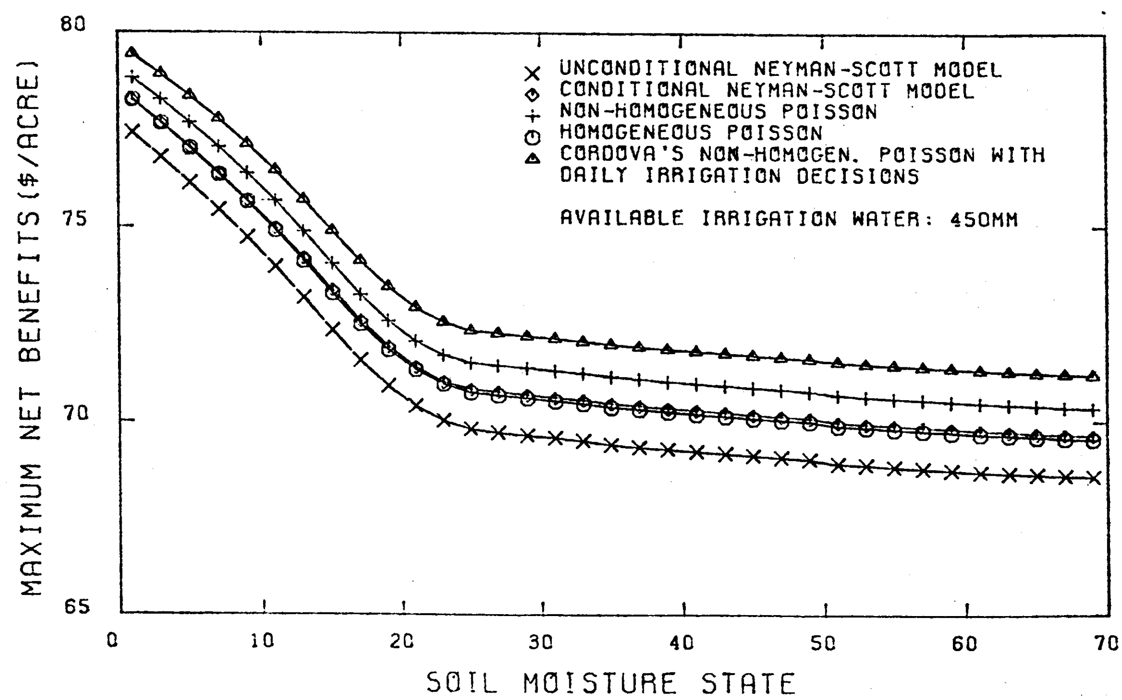


Figure 5.19b

LIMITED WATER CASE: MAXIMUM NET BENEFITS

where the use of the N-S model will improve irrigation efficiency; namely when the dependence structure of the precipitation process yields longer memory.

As presented in Table 5.14, the average precipitation expected by the N-S model is slightly greater than the average precipitation expected by the Poisson models. The longer interarrival times tend to offset this difference by producing greater water stresses on the crop, leading to more water use and lower expected net benefits. Thus, the conditional N-S model will only yield better results whenever the memory of the precipitation process is long enough to counterbalance the clustering effect.

The difference observed between the conditional N-S model and the non-homogeneous Poisson model also raises once more the question of the implied stationarity of the former model. Although it has been shown that the N-S model is a better representation of the rainfall occurrences than the homogeneous Poisson model, this has not been done with respect to the non-homogeneous Poisson model. The implied independence of the Poisson model is almost reproduced by the very short memory of the fitted N-S model, so that at the end the question remains one of determining which is the true underlying process governing precipitation occurrences. This is a question that can only be partially answered by simulating the system with the different optimal policies obtained from each model.

Some additional results are now simply stated. Observe from Figures 5.14a through 5.19a that the Poisson model fitted by Córdova and Bras (1979) always produces greater expected net benefits than the Poisson models fitted in this work, although the data used in both fits were the same. However, Córdova and Bras (1979) transformed the data by defining an independent storm event to justify the Poisson assumption. Table 5.14 compares the parameters obtained in both fits. In general, the model of Córdova and Bras expects more precipitation. Also, observe from the figures that contrary to facts stated by Blank (1975) and Córdova and Bras (1979), there is an increase in the expected net benefits when going from irrigation on fixed dates (weekly irrigations) to irrigation on variable dates (daily irrigation decisions).

Finally, it is necessary to emphasize that the results presented in this chapter are only theoretic results expressed as expected values. A simulation is required to be able to state with certainty the superiority of one model over the others.

5.6 Summary

This chapter has completely presented the calibration of the different model parameters and the case study results. The precipitation process for the case study at Denver, Colorado has been shown to exhibit a clustering dependence structure of very short memory. The homogeneous Poisson model is not a good representation of the data. After fitting the N-S model to the storm arrivals, the derived CDF's are shown to posses characteristics necessary for adequate short term forecasting of rainfall occurrence. Finally, case study results indicate that although the precipitation is better represented by the N-S process, this improvement does not produce better results for the particular data analyzed, when the models are used to obtain optimal irrigation decisions. The expected net benefits are shown to be essentially identical under the conditional N-S and homogeneous Poisson model, and lower than under the nonhomogeneous Poisson model.

Chapter 6

SUMMARY, CONCLUSIONS AND FUTURE RESEARCH

6.1 Summary and Conclusions

The objectives of this work were manyfold. It was desired to implement a real-time control model for irrigation, in regions of deficit irrigation, that accounted for the widely documented dependence and clustering behavior of the process of precipitation arrivals. At the same time, it was desired to determine whether the rainfall data from Denver also exhibited this clustering behavior and if so, to determine how well the N-S cluster process could represent the data as compared to the Poisson process.

Using the methodology proposed by Kavvas and Delleur (1975), the precipitation data from Denver was analyzed and seen to have no significant long term trends in the daily rate of occurrence. However, from the behavior of the estimated variance-time, spectrum of counts, and log-survivor functions it was concluded that the rainfall arrivals for Denver possess a definite short memory dependence structure caused by the clustering of the individual storms.

A particular form of the N-S cluster process was fitted to the data and seen to represent quite well the real precipitation time series. In order to preserve both the dependence structure and the clustering behavior, the model was fitted to the theoretical spectrum of counts and log-survivor function. The slight inconsistency encountered in the calibration can be explained by the fact that the process is not completely homogeneous as assumed.

The success obtained in modelling the precipitation data from Denver using the N-S cluster process confirms once more the ample generality and flexibility of the model; and most importantly, its ability to adequately represent the clustering dependence of the storm arrivals exhibited by the data. This last fact also confirms, once more, the widely accepted notion that precipitation events occur in clusters in the time domain. Furthermore, the fit obtained with the N-S model as compared to the one obtained with the Poisson model indicates the superiority of the former.

The use of the Palm-Khinchin theory allowed the derivation of general expressions for the conditional distribution functions of the time to the next rainfall event, conditional on part of the immediate history of storm arrivals. Explicit forms of these expressions for a particular form of the N-S model were derived. These functions are seen to possess the required characteristics for adequate short term rainfall forecasting. In fact, these functions are able to detect short term trends, particularly drought trends. The use of the N-S model and of the derived CDF's permitted a substantial increase in the accuracy of the rainfall forecasting.

The dynamic precipitation model devised, with the capability of changing as the growing season progresses and of using the conditional information to predict future behavior, was coupled with a model of the plant-soil system. This coupling allowed the implementation of a stochastic control model to obtain optimal irrigation decisions. It is shown that for the case study, the added complexity of the precipitation model results in a substantial improvement in the representation of the process, although this does not lead to significant differences in the theoretic expected maximum net benefits when compared to the Poisson model.

The results of this work do not allow any conclusion as to which model is more adequate. On one hand, it has been shown that the N-S model is a better representation of the precipitation process than the homogeneous Poisson model. But on the other hand, from the practical point of view, there seems to be no reason to favor one over the other, at least with respect to the particular case analyzed in this work. Results indicate that for Denver, the dependence structure of precipitation is very weak to allow for significant improvement in rainfall forecasting. Consequently, given that the Poisson model is not only simpler and easier to use, but also less costly to implement, the immediate conclusion is that the Poisson model should be favored over the N-S model.

6.2 Recommendations for Future Research

With respect to the precipitation model, the main issue is the assumption of homogeneity. The N-S model is stationary. Two alternatives are in order: either a homogenization scheme is implemented as suggested by Kavvas and Delleur (1975) or a non-homogeneous form of the N-S cluster model is developed. Neither alternative seems promising. First, most homogenization schemes deal only with first order moments and it is shown here that even under a homogeneous rate of occurrence, problems arise when calibrating the second order moments. Second, non-homogeneous forms of the N-S cluster model can be expected to be very complex thus invalidating their ability to be easily used. Consequently, it is necessary to study different models from the general class of Moyal cluster processes so that a simple non-homogeneous cluster model is devised.

With the use of the Palm-Khinchin theory, it has been shown how conditional distribution functions of the time to the next rainfall event could be derived. However, the situations considered here were very simple. In fact, only the time since the last rainfall event is accounted for. It is desirable to include more complex situations, when more than one event in the past is considered. In this way, the emphasis is not so much on the drought conditions but also on the wet conditions. The derivation of CDF's accounting for more complex situations seems possible by defining Palm-type functions of higher order. See Appendix F for an example. However, from the onset, the complexity of the expressions obtained, as well as the implementation of a control model able to use the added information, makes this a difficult task.

With respect to the general irrigation control model, a more detailed modelling of the different components of the soil-plant is needed. At the present level, the precipitation component of the system is much more accurate than the soil-plant component. It is highly desirable that all system components be modelled at the same level of complexity so that added accuracy in one component is not filtered out by another. In this regard, the further research proposed above to improve precipitation modelling and forecasting should be postponed in favor of further research on the dynamics of the soil-plant system. In particular, on the impact of the spatial variability of soil moisture in the root zone on plant productivity, as well as on the impact of the spatial inhomogeneity of the soil properties on the dynamics of soil moisture. Furthermore, the effect of the fluctuations of the groundwater elevation on crop yield has to be determined. Finally, and since the accuracy of the work rests on the validity of the crop response model, further research and experiments are needed to determine the form of the yield-evapotranspiration relationship as well as the sensitivity of the crop to vertical variance of soil moisture in the root zone. Lastly, since the results presented in this work are expressed in terms of theoretic expected values, the system should be simulated using the optimal operating policies determined, so that a more definite conclusion can be reached with respect to the precipitation model and its effect on the irrigation control problem.

REFERENCES

Anderson, A., and A. Maass, "Simulation of Irrigation Systems: The Effect of Water Supply and Operating Rules on Production and Income on Irrigated Farm," U.S. Department of Agriculture Technical Bulletin 1431, 1971.

Austin, P.M., and R.A. Houze, Jr., "Analysis of the Structure of Precipitation Patterns in New England," Journal of Applied Meteorology, Vol. 11, 1972.

Benjamin, J., and A. Cornell, Probability Statistics, and Decision for Civil Engineers, McGraw-Hill, New York, 1970.

Bertsekas, D., Dynamic Programming and Stochastic Control, Academic Press, New York, 1976.

Blank, H., "Optimal Irrigation Decisions with Limited Water," Ph.D. Dissertation, Civil Engineering Department, Colorado State University, Fort Collins, Colorado, 1975.

Bras, R.L., and J.R. Córdova, "Intraseasonal Water Allocation in Deficit Irrigation," Water Resources Research, Vol. 17, No. 4, 1981.

Butcher, W., "Stochastic Dynamic Programming for Optimum Reservoir Operation," Water Resources Bulletin, Vol. 7, No. 1, pp. 115-124, 1971.

Caskey, I.E., Jr., "A Markov Chain Model for the Probability Occurrence in Interval of Various Length," Monthly Weather Review, Vol. 91, pp. 289-301, 1963.

Conklin, L.R., "Farm and Irrigation Management in the South Platte Valley of Colorado," Masters Thesis, Colorado State University, Fort Collins, Colorado, August 1974.

Conklin, L.R., "Costs and Returns for Irrigated Crop Production in the Lower South Platte Valley, Colorado," Bulletin 491A, Cooperative Extension Service, Colorado State University, Fort Collins, Colorado, p. 63, July 1975.

Córdova, J.R., and R.L. Bras, "Physically-Based Probabilistic Models of Infiltration, Soil Moisture and Actual Evapotranspiration," Water Resources Research, Vol. 17, No. 1, 1981.

Córdova, J.R., and R.L. Bras, "Stochastic Control of Irrigation Systems," Technical Report No. 239, Ralph M. Parsons Laboratory for Water Resources and Hydrodynamics, Massachusetts Institute of Technology, March 1979.

Cox, D.R., and P.A.W. Lewis, The Statistical Analysis of Series of Events, London: Methuen and Barnes and Noble: New York, 1966.

Cramer, H., Mathematical Methods of Statistics, Princeton University Press, 1946.

Daley, D.J., and D. Vere-Jones, "A Survey of the Mathematical Theory of Point Processes," in Stochastic Point Processes, ed. by P.A.W. Lewis, Wiley and Sons, Incorporated, New York, 1972.

Davenport, W., Probability and Random Processes: An Introduction for Applied Scientists and Engineers, McGraw-Hill, New York, 1970.

DeLucia, R., "Operating Policies for Irrigation Systems under Stochastic Regimes," Ph.D. Thesis, Division of Engineering and Applied Physics, Harvard University, Cambridge, Massachusetts, 1969.

Dennis, J.E., Jr., D.M. Gay, and R.E. Welsch, "An Adaptive Nonlinear Least Squares Algorithm," Technical Report No. 1, Alfred P. Sloan School of Management, Massachusetts Institute of Technology, 1979.

Denmead, O.T., and R.H. Shaw, "Availability of Soil Water to Plants as Affected by Soil Moisture Content and Meteorological Conditions," Agronomy Journal, Vol. 54, pp. 385-390, 1962.

Dudley, N., D. Howell, and W. Musgreave, "Irrigation Planning 3: The Best Size of Irrigation Area for a Reservoir," Water Resources Research, Vol. 8, No. 1, 1972.

Dudley, N., "Irrigation Planning 4: Optimal Interseasonal Water Allocation," Water Resources Research, Vol. 8, No. 3, 1972.

Dudley, N., D. Howell, and W. Musgreave, "Optimal Intraseasonal Irrigation Allocation," Water Resources Research, Vol. 7, No. 4, August 1971.

Dudley, N., and O. Burt, "Stochastic Reservoir Management and System Design for Irrigation," Water Resources Research, Vol. 7, No. 4, 1971.

Dudley, N., D. Howell, and W. Musgreave, "Irrigation Planning 2: Choosing Optimal Acreage within an Irrigation Season," Water Resources Research, Vol. 7, No. 5, 1971.

Duckstein, L., M.N. Fogel, and C.C. Kisiel, "A Stochastic Model of Runoff-Producing Rainfall for Summer Type Storms," Water Resources Research, Vol. 8, No. 2, pp. 410-421, 1972.

Eagleson, P.S., "Climate, Soil, and Vegetation 1. Introduction to Water Balance Dynamics," Water Resources Research, Vol. 15, No. 5, 1978.

Eagleson, P.S., "Climate, Soil, and Vegetation 2. The Distribution of Annual Precipitation Derived from Observed Storm Sequences," Water Resources Research, Vol. 14, No. 5, 1978.

Eagleson, P.S., "Climate, Soil, and Vegetation 3. A Simplified Model of Soil Moisture Movement in the Liquid Phase," Water Resources Research, Vol. 14, No. 5, 1978.

Eagleson, P.S., "Climate, Soil, and Vegetation 4. The Expected Value of Annual Evapotranspiration," Water Resources Research, Vol. 14, No. 5, 1978.

Eagleson, P.S., "Climate, Soil, and Vegetation 5. A Derived Distribution of Storm Surface Runoff," Water Resources Research, Vol. 14, No. 5, 1978.

Feller, W., An Introduction to Probability Theory and its Applications, Vol. 1, 2nd ed., Wiley, New York, 1968.

Feller, W., An Introduction to Probability Theory and its Applications, Vol. 2, Wiley, New York, 1966.

Gabriel, K.R., and J. Neumann, "On a Distribution of Weather Cycles by Lengths," Quarterly Journal of the Royal Meteological Society, Vol. 83, pp. 375-380, 1957.

Gabriel, K.R., and J. Neumann, "A Markov Chain Model for Daily Rainfall Occurrence at Tel Aviv," Quarterly Journal of the Royal Meteological Society, Vol. 88, pp. 90-95, 1962.

Gabriel, K.R., "The Distribution of the Number of Successes in a Sequence of Dependent Trials," Biometrika, Vol. 96, pp. 454-460, 1959.

Gardner, W.R., W.A. Jury, and J. Knight, "Water Uptake by Vegetation," from Heat and Mass Transfer in the Biosphere 1. Transfer Processes in the Environment, eds., D.A. Devries and N.H. Afgan, John Wiley, New York, 1975.

Grant, E.L., "Rainfall Intensities and Frequencies," ASCE Transactions, Vol. 103, pp. 384-388, 1938.

Grayman, W.M., and P.S. Eagleson, "Streamflow Record Length for Modeling Catchment Dynamics," Report 114, Hydrodynamics Laboratory, Department of Civil Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1969.

Gupta, V.K., and E.C. Waymire, "A Stochastic Kinematic Study of Subsynoptic Space-time Rainfall," Water Resources Research, Vol. 15, No. 3, pp. 637-644, 1979.

Hall, W.A., and J.A. Dracup, Water Resources Systems Engineering, McGraw-Hill, New York, 1970.

Hall, W.A., and W. Butcher, "Optimal Timing of Irrigation," Journal of Irrigation and Drainage Division, ASCE, Vol. 94, No. IR2, 1968.

Hall, W.A., and N. Buras, "Optimum Irrigated Practice under Conditions of Deficient Water Supply," Transactions of the ASAE, Vol. 4, 1961.

Hanks, R.J., "A Model for Predicting Plant Yield as Influenced by Water Use," Agronomy Journal, Vol. 66, pp. 660-664, September 1974.

Hanson, C.L., "Model for Predicting Evapotranspiration from Native Rangelands in the Northern Great Plains," Transactions of the ASCE, Vol. 19SW, 1976.

Hiler, E.A., and R.N. Clark, "Stress Day Index to Characterize Effects of Water Stress on Crop Yields, Transactions of the ASAE, Vol. 14, pp. 757-761, 1971.

Houze, R.A., Jr., "Characteristics of Mesoscale Precipitation Areas," M.S. Thesis, Department of Meteorology, Massachusetts Institute of Technology, 1969.

Jensen, M.E., "Water Consumption by Agricultural Plants," in Water Deficits and Plant Growth, Vol. 2, ed., T.T. Kozlowski, Academic Press, New York, pp. 1-22, 1968.

Jensen, M.E., D.C.N. Robb, and C.E. Franzoy, "Scheduling Irrigations Using Climate-Crop-Soil Data," Journal of Irrigation and Drainage Division, ASCE, Vol. 96, pp. 25-38, 1970.

Jowett, J., and D. Vere-Jones, "The Prediction of Point Processes," in Stochastic Point Processes, ed. by P.A.W. Lewis, Wiley and Sons, Incorporated, New York, 1972.

Kavvas, M.L., "Stochastic Trigger Models for Flood Peaks, 1, Development of Model," Water Resources Research, Vol. 18, No. 2, 1982.

Kavvas, M.L., "Stochastic Trigger Models for Flood Peaks, 2, Application of the Model to the Flood Peaks of Goksu Karahacili," Water Resources Research, Vol. 18, No. 2, 1982.

Kavvas, M.L., and J.W. Delleur, "A Stochastic Cluster Model of Daily Rainfall Sequences," Water Resources Research, Vol. 17, No. 4, 1981.

Kavvas, M.L., and J.W. Delleur, "The Stochastic and Chronologic Structure of Rainfall Sequences; Application to Indiana," Water Resources Research Center Technical Report No. 57, Purdue University, West Lafayette, Indiana, 1975.

Khinchin, A. Ya., "Mathematical Methods in the Theory of Queueing," Trudy Mat. Inst. Steklov, Vol. 49, 1955, (English translation, 1960, Griffin, London).

Le Cam, L., "A Stochastic Description of Precipitation," in IV Berkeley Symposium on Math. Stat. and Prob. Proceedings, ed. by J. Neyman, University of California Press, 1961.

Lewis, P.A.W., A.M. Katcher, and A.H. Weis, SASE IV - An Improved Program for the Analysis of Series of Events, IBM Report, RC 2365, 1969.

Lewis, P.A.W., "Remarks on the Theory, Computation and Application of the Spectral Analysis of Series of Events," Journal of Sound Vib., Vol. 12, pp. 353-375, 1970.

Matanga, G.B., and M.A. Mariño, "Irrigation Planning 2. Water Allocation for Leaching and Irrigation Purpose," Water Resources Research, Vol. 15, No. 3, June 1979.

Matanga, G.B., and M.A. Mariño, "Application of Optimization and Simulation Techniques to Irrigation Management," Report 5003, Department of Land, Air, and Water Resources, University of California, Davis, California, 1977.

Minhas, B., K. Parikm, and N. Srinivasan, "Toward the Structure of a Production Function for Wheat Yields with Dated Inputs of Irrigation Water," Water Resources Research, Vol. 10, No. 3, 1974.

Morey, R.V., J.R. Gilley, F.G. Bergsrud, and L.R. Dirkzwager, "Yield Response of Corn Related to Soil Moisture Conditions," Annual Meeting of ASAE, Davis, California, 1975.

Moyal, J.E., "The General Theory of Stochastic Population Processes," Acta Math., Vol. 108, pp. 1031, 1962.

Neyman, J., and E.L. Scott, "A Theory of the Spatial Distribution of Galaxies," Astrophysical Journal, Vol. 16, 1952.

Neyman, J., and E.L. Scott, "A Statistical Approach to Problems of Cosmology," Journal of the Royal Statistical Society, B, Vol. 20, pp. 1-43, 1958.

Palm, C., "Intensitätsschwankungen im Fernsprechverkehr," Ericsson Techniks, Vol. 44, pp. 1-189, 1943.

Parzen, E., Stochastic Processes, Holden Day, San Francisco, California, 1962.

Petterssen, S., Introduction to Meteorology, 3rd Edition, McGraw-Hill, New York, 1969.

Petterssen, S., Weather Analysis and Forecasting, Vol. 2, McGraw-Hill, New York, 1956.

Philip, J.R., "The Theory of Infiltration 1: The Infiltration Equation and Its Solution," Soil Science, Vol. 83, 1957.

Philip, J.R., "Plant Water Relations: Some Physical Aspects," Annual Review of Plant Physiology, Vol. 17, 1966.

Philip, J.R., "The Theory of Infiltration," in Advances in Hydroscience, Vol. 5, ed. by V.T. Chow, Academic Press, New York, 1969.

Restrepo, P., "Some Geomorphologic Inferences from Hydrology," Masters Thesis, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1979.

Restrepo, P., and P.S. Eagleson, "Some Geomorphologic Inferences from Hydrology," Technical Report (to be published), Ralph M. Parsons Laboratory for Water Resources and Hydrodynamics, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1979.

Rhenals, A.E., and R.L. Bras, "The Irrigation Scheduling Problem and Evapotranspiration Uncertainty," Water Resources Research, Vol. 17, No. 5, 1981.

Shane, R.M., "The Application of the Compound Poisson Distribution to the Analysis of Rainfall Records," M.S. Thesis, Cornell University, Ithaca, New York, 1964.

Smith, E.R., and H.A. Schreiber, "Point Processes of Seasonal Thunderstorm Rainfall Part 1. Distribution of Rainfall Events," Water Resources Research, Vol. 9, No. 4, pp. 841-884, 1973.

Stewart, I., R. Hagan, and W. Pruitt, "Function to Predict Optimal Irrigation Programs," Journal of Irrigation and Drainage Division, ASCE, Vol. 100, No. IR2, 1973.

Stewart, I., and R. Hagan, "Function to Predict Effects of Crop Water Deficits," Journal of Irrigation and Drainage Division, ASCE, Vol. 94, No. IR2, 1971.

Stewart, I., and R. Hagan, "Predicting Effects of Water Shortage on Crop Yield," Journal of Irrigation and Drainage Division, ASCE, Vol. 95, No. IR2, 1971.

Stewart, J.I., R.D. Misra, W.O. Pruitt, and R.M. Hagan, "Irrigating Corn and Grain Sorghum with Limited Water," Paper No. 74-2024, ASAE, St. Joseph, Michigan, 1974.

Stewart, et al., "Optimizing Crop Production through Control of Water and Salinity Levels in the Soil," Utah Water Research Laboratory, College of Engineering, Utah State University, Logan, Utah, September 1977.

Thom, H.C.S., "Time Interval Distribution for Excessive Rainfalls," Journal of the Hydraulics Division, ASCE, Vol. 85, No. HY2, 1959.

Todorovic, P., and V. Yevjevich, "Stochastic Process of Precipitation," Colorado State University Hydrology Paper No. 35, Fort Collins, Colorado, 1969.

Vere-Jones, D., "Some Applications of Probability Generating Functionals to the Study of Input/Output Streams," Journal of the Royal Statistical Society, B, Vol. 30, pp. 321-333, 1968.

Vere-Jones, D., "Stochastic Models for Earthquake Occurrences," Journal of the Royal Statistical Society, B, Vol. 32, pp. 1-62, 1970.

Waymire, E.C., and V.K. Gupta, "The Mathematical Structure of Rainfall," Parts I, II, III, University of Mississippi, 1980.

Waymire, E., and V.K. Gupta, "The Mathematical Structure of Rainfall Representations 1. A Review of the Stochastic Rainfall Models," Water Resources Research, Vol. 17, No. 5, 1981.

Waymire, E., and V.K. Gupta, "The Mathematical Structure of Rainfall Representations 2. A Review of the Theory of Point Processes," Water Resources Research, Vol. 17, No. 5, 1981.

Waymire, E., and V.K. Gupta, "The Mathematical Structure of Rainfall Representations 3. Some Applications of the Point Process Theory to Rainfall Processes," Water Resources Research, Vol. 17, No. 5, 1981.

Westcott, M., "Some Applications of the Probability Generating Functional to Point Processes," Ph.D. Thesis, Australian National University, 1970.

Westcott, M., "The Probability Generating Functional," Journal of the Australian Mathematical Society, (to appear), 1972.

Wiser, E.H., "Modified Markov Probability Models of Sequences of Precipitation Events," Monthly Weather Review, Vol. 93, pp. 511-516, 1965.

Yaron, D., "Estimation and Use of the Water Production Function in Crops," Journal of Irrigation and Drainage Division, ASCE, Vol. 97, No. IR2, 1981.

Yaron, D., H. Biebrai, J. Shalheret, and Y. Gorish, "Estimation Procedure for Response Functions of Crops to Soil Water Content and Salinity," Water Resources Research, Vol. 8, pp. 291,800, 1972.

Yaron, D., G. Straterner, D. Shimshi, and M. Weisbrod, "Wheat Response to Soil Moisture and the Optimal Irrigation Policy under Conditions of Unstable Rainfall," Water Resources Research, Vol. 9, No. 5, 1973.

Appendix A

FIRST AND SECOND ORDER MOMENTS AS FUNCTIONS OF THE PGF

A) First Moment:

By definition:

$$g(z) = \sum_{n=0}^{\infty} z^n \Pr[N(\cdot) = n] \tag{A.1}$$

Differentiating equation A.1 yields:

$$\frac{\partial g(z)}{\partial z} = \sum_{n=0}^{\infty} n\, z^{n-1} \Pr[N(\cdot) = n] \tag{A.2}$$

Evaluating equation A.2 at z = 1, yields:

$$E[N(\cdot)] = \sum_{n=0}^{\infty} n \Pr[N(\cdot) = n] = \left.\frac{\partial g(z)}{\partial z}\right|_{z=1} \tag{A.3}$$

B) Second Moments:

1) Variance:

Differentiating again equation A.2:

$$\frac{\partial^2 g(z)}{\partial z^2} = \sum_{n=0}^{\infty} n^2 z^{n-2} \Pr[N(\cdot) = n] - \sum_{n=0}^{\infty} n\, z^{n-2} \Pr[N(\cdot) = n] \tag{A.4}$$

Equation A.4, after evaluating it at $z = 1$, becomes:

$$\left.\frac{\partial^2 g(z)}{\partial z^2}\right|_{z=1} = \sum_{n=0}^{\infty} n^2 \Pr[N(\cdot) = n] - \sum_{n=0}^{\infty} n \Pr[N(\cdot) = n] \qquad (A.5)$$

and

$$\left.\frac{\partial^2 g(z)}{\partial z^2}\right|_{z=1} = E[N^2(\cdot)] - E[N(\cdot)] \qquad (A.6)$$

Finally, adding equation A.3 to equation A.6, and subtracting equation A.3 squared from equation A.6, the following is obtained:

$$\operatorname{var}[N(\cdot)] = E[N^2(\cdot)] - E^2[N(\cdot)] = \left.\frac{\partial^2 g(z)}{\partial z^2} + \frac{\partial g(z)}{\partial z} - \left[\frac{\partial g(z)}{\partial z}\right]^2\right|_{z=1}$$

(A.7)

2) Covariance:

By definition: $$g(z_1, z_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} z_1^{n_1} z_2^{n_2} \Pr[N_1(\cdot) = n_1, N_2(\cdot) = n_2]$$

(A.8)

Differentiating equation A.8 with respect to z_1 and z_2:

$$\frac{\partial g(z_1, z_2)}{\partial z_1} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} n_1 z_1^{n_1 - 1} z_2^{n_2} \Pr[N_1(\cdot) = n_1, N_2(\cdot) = n_2]$$

(A.9)

and

$$\frac{\partial^2 g(z_1,z_2)}{\partial z_1 \partial z_2} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} n_1 n_2 \; z_1^{n_1-1} \; z_2^{n_2-1} \; \Pr[N_1(\cdot) = n_1,\; N_2(\cdot) = n_2] \tag{A.10}$$

Equation A.10 evaluated as $z_1 = z_2 = 1$ yields:

$$\left.\frac{\partial^2 g(z_1,z_2)}{\partial z_1 \partial z_2}\right|_{z_1=z_2=1} = \sum_{n_1=0} \sum_{n_2=0} n_1 n_2 \; \Pr[N_1(\cdot)=n_1,\; N_2(\cdot)=n_2] = E[N_1(\cdot)N_2(\cdot)] \tag{A.11}$$

From A.9, $\left.\frac{\partial g(z_1,z_2)}{\partial z_1}\right|_{z_1=z_2=1} = E[N_1(\cdot)]$ (A.12)

and by analogy: $\left.\frac{\partial g(z_1,z_2)}{\partial z_2}\right|_{z_1=z_2=1} = E[N_2(\cdot)]$ (A.13)

Finally,

$$\begin{aligned} \operatorname{cov}[N_1(\cdot),\; N_2(\cdot)] &= E[N_1(\cdot)\; N_2(\cdot)] - E[N_1(\cdot)]E[N_2(\cdot)] \\ &= \left.\frac{\partial^2 g(z_1,z_2)}{\partial z_1 \partial z_2}\right|_{z_1=z_2=1} - \left[\left.\frac{\partial g(z_1,z_2)}{\partial z_1}\right|_{z_1=z_2=1} \cdot \left.\frac{\partial g(z_1,z_2)}{\partial z_2}\right|_{z_1=z_2=1}\right] \end{aligned} \tag{A.14}$$

Appendix B

FIRST AND SECOND ORDER MOMENTS FOR THE N-S CLUSTER MODEL

A) First Moment:

The univariate PGF of the N-S cluster model for the interval $(0,t_1)$ is (Equation 3.11):

$$g(z) = \exp[\mu \int_{-\infty}^{t_1} \{g_{N_2}[1-(1-z)p(t)] - 1\}dt] \qquad \text{(B.1)}$$

$$\text{where:} \quad p(t) = \int_0^{t_1} f_T(\zeta - t)d\zeta \qquad \text{(B.2)}$$

From equation 2.14:

$$E[N(0,t_1)] = \left.\frac{\partial g(z)}{\partial z}\right|_{z=1} = \mu \cdot g(z) \cdot \frac{\partial}{\partial z}\left.\left\{\int_{-\infty}^{t_1} g_{N_2}[1-(1-z)p(t)]dt\right\}\right|_{z=1} \qquad \text{(B.3)}$$

The derivative in equation B.3 can be written as:

$$\frac{\partial}{\partial z}\left.\left\{\int_{-\infty}^{t_1} g_{N_2}[1-(1-z)p(t)]dt\right\}\right|_{z=1} = \left.\int_{-\infty}^{t_1}\left\{\frac{\partial}{\partial z} g_{N_2}[1-(1-z)p(t)]\right\}dt\right|_{z=1} \qquad \text{(B.4)}$$

By definition:

$$g_{N_2}[1 - (1 - z)p(t)] = \sum_{n=0}^{\infty} [1 - (1 - z)p(t)]^n \Pr[N_2(t) = n] \qquad (B.5)$$

so that the derivative appearing in equation B.4 is:

$$\frac{\partial}{\partial z} g_{N_2} [1 - (1 - z)p(t)] = p(t) \sum_{n=0}^{\infty} n[1 - (1 - z)p(t)]^{n-1} \Pr[N_2(t) = n] \qquad (B.6)$$

and evaluating at z = 1 yields:

$$\frac{\partial}{\partial z} g_{N_2} [1 - (1 - z)p(t)] \Big|_{z=1} = p(t) \sum_{n=0}^{\infty} n \Pr[N_2(t) = n] = p(t) \cdot E[N_2(t)] \qquad (B.7)$$

Substituting equation B.7 in equation B.4:

$$\frac{\partial}{\partial z} \left\{ \int_{-\infty}^{t_1} g_{N_2} [1 - (1 - z)p(t)]\, dt \right\} \Bigg|_{z=1} = \int_{-\infty}^{t_1} E[N_2(t)] p(t) dt \qquad (B.8)$$

Now, by structural postulate, the $N_2(t)$'s are i.i.d. for each cluster centered at t, and consequently equation B.8 becomes:

$$\int_{-\infty}^{t_1} E[N_2(t)] p(t) dt = E[N_2(t)] \cdot \int_{-\infty}^{t_1} p(t) dt \qquad (B.9)$$

Using equation B.2 in equation B.9:

$$\int_{-\infty}^{t_1} p(t)dt = \int_{-\infty}^{t_1} \int_{0}^{t} f_T(\zeta-t)d\zeta dt = \int_{0}^{t_1} \int_{-\infty}^{\zeta} f_T(\zeta-t)dtd\zeta = t_1 \tag{B.10}$$

Finally, by definition:

$$g(z)\Big|_{z=1} = 1 \tag{B.11}$$

Substituting equation B.9, B.10, and B.11 in equation B.3 yields:

$$E[N(0,t_1)] = \mu \cdot E[N_2(t)] \cdot t_1 \tag{B.12}$$

B) Second Moments:

1) Variance:

From equation 2.15:

$$var[N(0,t_1)] = \frac{\partial^2 g(z)}{\partial z^2} + \frac{\partial g(z)}{\partial z} - \left[\frac{\partial g(z)}{\partial z}\right]^2 \Bigg|_{z=1} \tag{B.13}$$

From equation B.12:

$$\frac{\partial g(z)}{\partial z}\Bigg|_{z=1} = \mu \cdot E[N_2(\cdot)] \cdot t_1 \tag{B.14}$$

and

$$\left\{ \left. \frac{\partial g(z)}{\partial z} \right|_{z=1} \right\}^2 = \mu^2 \cdot E^2[N_2(\cdot)] \cdot t_1^{\,2} \tag{B.15}$$

Now, denoting $[1 - (1 - z)p(t)]$ by k:

$$\frac{\partial^2 g(z)}{\partial z^2} = \mu \cdot g(z) \cdot \int_{-\infty}^{t_1} \frac{\partial^2}{\partial z^2} g_{N_2}(k)dt + \mu^2 \cdot g(z) \cdot \left[\int_{-\infty}^{t_1} \frac{\partial}{\partial z} g_{N_2}(k)dt \right]^2 \tag{B.16}$$

The second-order derivative appearing in equation B.16 is:

$$\frac{\partial^2}{\partial z^2} g_{N_2}(k) = \sum_{n=0}^{\infty} n(n-1)\, p^2(t)[1 - (1 - z)p(t)]^{n-2} \;\; Pr[N_2(\cdot) = n] \tag{B.17}$$

And evaluating at $z = 1$:

$$\left. \frac{\partial^2}{\partial z^2} g_{N_2}(k) \right|_{z=1} = p^2(t) \cdot E[N_2^{\,2}(\cdot) - N_2(\cdot)] \tag{B.18}$$

Finally, after replacing equations B.14, B.15, and B.19 in equation B.13, the following expression for the variance is obtained:

$$var[N(0,t_1)] = \mu \cdot E[N_2(t)] \cdot t_1 + \mu \cdot E[N_2^{\,2}(t) - N_2(t)] \cdot \int_{-\infty}^{t_1} p^2(t)dt \tag{B.20}$$

2) Covariance:

Using equation 3.13, the PGF for the counts $N(0,t_1)$ and $N(t_1+\zeta, t_2)$ can be written as:

$$g(z_1,z_2) = \exp[\mu \int_{-\infty}^{t_2} \{g_{N_2}[1-(1-z_1)p_1(t) - (1-z_2)p_2(t)] - 1\}dt] \tag{B.21}$$

$$\text{with} \quad p_1(t) = \int_0^{t_1} f_T(x-t)dx \quad \text{and} \quad p_2(t) = \int_{t_1+\zeta}^{t_2} f_T(x-t)dx \tag{B.22}$$

From equation 2.16:

$$\mathrm{cov}[N(0,t_1),N(t_1+\zeta,t_2)] = \frac{\partial^2 g(z_1,z_2)}{\partial z_1 \partial z_2} - \left[\frac{\partial g(z_1,z_2)}{\partial z_1} \cdot \frac{\partial g(z_1,z_2)}{\partial z_1}\right]\Bigg|_{z_1=z_2=1} \tag{B.23}$$

Denoting $[1 - (1 - z_1)p_1(t) - (1 - z_2)p_2(t)]$ by m, the derivatives in equation B.23 are:

$$\frac{\partial g(z_1,z_2)}{\partial z_1} = \mu \cdot g(z_1,z_2) \cdot \int_{-\infty}^{t_2} \frac{\partial}{\partial z_1} g_{N_2}(m)dt \tag{B.24}$$

$$\frac{\partial g(z_1,z_2)}{\partial z_2} = \mu \cdot g(z_1,z_2) \cdot \int_{-\infty}^{t_2} \frac{\partial}{\partial z_2} g_{N_2}(m)dt \tag{B.25}$$

$$\frac{\partial^2 g(z_1,z_2)}{\partial z_1 \partial z_2} = \mu \cdot g(z_1,z_2) \cdot \int_{-\infty}^{t_2} \frac{\partial}{\partial z_1 \partial z_2} g_{N_2}(m)dt$$

$$+ \mu^2 \cdot g(z_1,z_2) \cdot \left[\int_{-\infty}^{t_2} \frac{\partial}{\partial z_1} g_{N_2}(m)dt\right]\left[\int_{-\infty}^{t_2} \frac{\partial}{\partial z_2} g_{N_2}(m)dt\right]$$

(B.26)

Evaluating B.24, B.25, and B.26 at $z_1=z_2=1$, yields:

$$\text{cov}[N(0,t_1),N(t_1+\zeta,t_2)] = \mu \cdot E[N_2^2(t) - N_2(t)] \cdot \int_{-\infty}^{t_2} p_1(u) \quad p_2(u)du$$

(B.27)

Appendix C

DERIVATION OF $P_0(x)$

From equation 3.11:

$$g(z) = \exp\left[\mu \int_{-\infty}^{x} \{g_{N_2}[1 - (1 - z)p(t)] - 1\}dt\right] \qquad (C.1)$$

and

$$p(t) = \int_{0}^{x} f_T(\zeta - t)d\zeta \qquad (C.2)$$

From equation 3.43:

$$P_0(x) = g(z)\Big|_{z=0} \qquad (C.3)$$

According to equation C.3:

$$P_0(x) = \exp\left[\mu \int_{-\infty}^{x} \{g_{N_2}[1 - p(t)] - 1\}dt\right] \qquad (C.4)$$

Now, from equation 3.25:

$$g_{N_2}(z) = \frac{pz}{1 - (1 - p)z}, \quad |z| < 1 \qquad (C.5)$$

and from equation 3.26:

$$f_T(\zeta-t) = \begin{cases} \alpha e^{-\alpha(\zeta-t)} & \text{if } \zeta-t > 0 \\ 0 & \text{otherwise} \end{cases} \tag{C.6}$$

According to definition C.6 and equation C.2:

$$p(t) = \int_0^x \alpha e^{-\alpha(\zeta-t)} d\zeta = e^{\alpha t}(1 - e^{-\alpha x}) \;, \; t \leq 0 \tag{C.7}$$

and

$$p(t) = \int_t^x \alpha e^{-\alpha(\zeta-t)} d\zeta = 1 - e^{-\zeta(x-t)} \;, \; t > 0 \tag{C.8}$$

Equation C.4 can then be rewritten as:

$$P_0(x) = \exp\left[\mu \int_{-\infty}^{0} \{g_{N_2}[1 - e^{\alpha t}(1 - e^{-\alpha x})] - 1\}dt\right]$$

$$+ \exp\left[\mu \int_0^x \{g_{N_2}[-e^{-\alpha(x-t)}]\}dt\right] \tag{C.9}$$

From equation C.5:

$$g_{N_2}\left[1 - e^{\alpha t}(1 - e^{-\alpha x})\right] = \frac{p\left[1 - e^{\alpha t}(1 - e^{-\alpha x})\right]}{1 - (1 - p)\left[1 - e^{\alpha t}(1 - e^{-\alpha x})\right]} \qquad \text{(C.10)}$$

and

$$g_{N_2}\left[-e^{-\alpha(x-t)}\right] = \frac{-pe^{-\alpha(x-t)}}{1 + (1 - p)e^{-\alpha(x-t)}} \qquad \text{(C.11)}$$

Now, substituting C.10 and C.11 in equation C.9 and carrying out the integrations, it is easy to obtain:

$$P_0(x) = \left[\frac{p}{1 - qe^{-\alpha x}}\right]^{\frac{\mu}{\alpha}} e^{-\mu x} \qquad \text{(C.12)}$$

Appendix D

RESULTS OF THE SDP ALGORITHM: UNLIMITED WATER CASE.

RESULTS OF THE STOCHASTIC DYNAMIC PROGRAMMING ALGORITHM

OPERATION POLICIES

OPERATION POLICY NO.	IRRIGATE UP TO SOIL MOIST.CONT. EQUALS STATE GR.PR.I	GR.PR. II	GR.PR. III
1	14	22	26
2	NO IRRIGATION	NO IRRIGATION	NO IRRIGATION

OPTIMAL IRRIGATION POLICY FOR STAGE : 1

```
------------------------------------------------------------------------------------------------------------------
DAYS                                        SOIL MOISTURE CONTENT
NO RAIN
STATES
12345678910111213141516171819202122232425262728293031323334353637383940414243444546474849505152535455565758596061626364656667686970
------------------------------------------------------------------------------------------------------------------
0222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
2222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
3222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
4222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
5222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
6222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
7222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
```

184

OPTIMAL IRRIGATION POLICY FOR STAGE : 61

```
------------------------------------------------------------------------------------------------------------------
DAYS                                        SOIL MOISTURE CONTENT
NO RAIN
STATES
12345678910111213141516171819202122232425262728293031323334353637383940414243444546474849505152535455565758596061626364656667686970
------------------------------------------------------------------------------------------------------------------
0222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
2222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
3222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
4222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
5222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
6222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
7222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
```

OPTIMAL IRRIGATION POLICY FOR STAGE : 62

SOIL MOISTURE CONTENT

DAYS NO RAIN STATES	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
0	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
6	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

OPTIMAL IRRIGATION POLICY FOR STAGE : 63

SOIL MOISTURE CONTENT

DAYS NO RAIN STATES	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
0	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
6	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

OPTIMAL IRRIGATION POLICY FOR STAGE : 118

```
-----------------------------------------------------------------------------------------------------
DAYS                                        SOIL MOISTURE CONTENT
NO RAIN
STATES
12345678910111213141516171819202122232425262728293031323334353637383940414243444546474849505152535455565758596061626364656667686970
-----------------------------------------------------------------------------------------------------
0222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
2222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
3222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
4222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
5222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
6222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
7222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
```

OPTIMAL IRRIGATION POLICY FOR STAGE : 120

```
-----------------------------------------------------------------------------------------------------
DAYS                                        SOIL MOISTURE CONTENT
NO RAIN
STATES
12345678910111213141516171819202122232425262728293031323334353637383940414243444546474849505152535455565758596061626364656667686970
-----------------------------------------------------------------------------------------------------
0222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
1222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
2222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
3222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
4222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
5222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
6222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
7222222222 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
```

Appendix E

RESULTS OF THE SDP ALGORITHM: LIMITED WATER CASE.

RESULTS OF THE STOCHASTIC DYNAMIC PROGRAMMING ALGORITHM

OPERATION POLICIES

OPERATION POLICY NO.	IRRIGATE UP TO SOIL MOIST.CONT. EQUALS STATE GR.PR.I	GR.PR. II	GR.PR. III
1	14	22	26
2	24	30	38
3	36	40	52
4	50	54	66
5	NO IRRIGATION	NO IRRIGATION	NO IRRIGATION

OPTIMAL IRRIGATION POLICY FOR STAGE 30

AVAILABLE IRRIGATION WATER AT BEGINNING OF STAGE 175.00MM

DAYS NO RAIN — SOIL MOISTURE CONTENT

STATES	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	2	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1
1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	2	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1
2	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	2	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1
3	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	2	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1
4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	2	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	2	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1
6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	2	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1
7	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	2	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1

OPTIMAL IRRIGATION POLICY FOR STAGE 30

AVAILABLE IRRIGATION WATER AT BEGINNING OF STAGE 200.00MM

DAYS NO RAIN — SOIL MOISTURE CONTENT

STATES	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	2	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1
1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	2	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1
2	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	2	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1
3	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	2	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1
4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	2	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	2	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1
6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	2	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1
7	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	2	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1

OPTIMAL IRRIGATION POLICY FOR STAGE 30

AVAILABLE IRRIGATION WATER AT BEGINNING OF STAGE 250.00MM

DAYS NO RAIN STATES — SOIL MOISTURE CONTENT

123456789	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
0555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
6555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

OPTIMAL IRRIGATION POLICY FOR STAGE 30

AVAILABLE IRRIGATION WATER AT BEGINNING OF STAGE 225.00MM

DAYS NO RAIN STATES — SOIL MOISTURE CONTENT

123456789	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
0555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	1	1	1	1	1	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	2	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	2	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	2	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	2	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	2	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	2	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
6555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	2	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	2	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

OPTIMAL IRRIGATION POLICY FOR STAGE 30

AVAILABLE IRRIGATION WATER AT BEGINNING OF STAGE 275.00MM

DAYS NO RAIN — SOIL MOISTURE CONTENT

STATES	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

191

OPTIMAL IRRIGATION POLICY FOR STAGE 30

AVAILABLE IRRIGATION WATER AT BEGINNING OF STAGE 475.00MM

DAYS NO RAIN — SOIL MOISTURE CONTENT

STATES	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

OPTIMAL IRRIGATION POLICY FOR STAGE 90

AVAILABLE IRRIGATION WATER AT BEGINNING OF STAGE 75.00MM

DAYS NO RAIN STATES — SOIL MOISTURE CONTENT

STATES	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	1	5	5	5	5	5	5	5	2	5	5	5	5	5	5	2	2	5	5	5	5	5	5	3	3	3	5
1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	1	5	5	5	5	5	5	5	2	5	5	5	5	5	5	5	2	5	5	5	5	5	5	3	3	3	5
2	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	1	5	5	5	5	5	5	5	2	5	5	5	5	5	5	5	2	5	5	5	5	5	5	3	3	3	5
3	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	1	5	5	5	5	5	5	5	2	5	5	5	5	5	5	5	2	5	5	5	5	5	5	3	3	3	5
4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	1	5	5	5	5	5	5	5	2	5	5	5	5	5	5	5	2	5	5	5	5	5	5	3	3	3	5
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	1	5	5	5	5	5	5	5	2	5	5	5	5	5	5	5	2	5	5	5	5	5	5	3	3	3	5
6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	1	5	5	5	5	5	5	5	2	5	5	5	5	5	5	5	2	5	5	5	5	5	5	3	3	3	5
7	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	1	5	5	5	5	5	5	5	2	5	5	5	5	5	5	5	2	5	5	5	5	5	5	3	3	3	5

OPTIMAL IRRIGATION POLICY FOR STAGE 90

AVAILABLE IRRIGATION WATER AT BEGINNING OF STAGE 100.00MM

DAYS NO RAIN STATES — SOIL MOISTURE CONTENT

STATES	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	1	5	5	5	5	1	1	1	1	5	5	5	5	2	2	2	2	5	2	2	2	2	2	2	2	2	2
1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	1	5	5	5	5	5	1	1	1	5	5	5	5	2	2	2	2	5	5	2	2	2	2	2	2	2	2
2	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	1	5	5	5	5	5	5	1	1	5	5	5	5	2	2	2	2	5	5	2	2	2	2	2	2	2	2
3	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	1	5	5	5	5	5	5	1	1	5	5	5	5	5	2	2	2	5	5	2	2	2	2	2	2	2	2
4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	1	5	5	5	5	5	5	1	1	5	5	5	5	5	2	2	2	5	5	5	2	2	2	2	2	2	2
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	1	5	5	5	5	5	5	1	1	5	5	5	5	5	2	2	2	5	5	5	2	2	2	2	2	2	2
6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	1	5	5	5	5	5	5	1	1	5	5	5	5	5	2	2	2	5	5	5	2	2	2	2	2	2	2
7	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	5	5	5	5	1	5	5	5	5	5	5	1	1	5	5	5	5	5	2	2	2	5	5	5	2	2	2	2	2	2	2

OPTIMAL IRRIGATION POLICY FOR STAGE 90

AVAILABLE IRRIGATION WATER AT BEGINNING OF STAGE 125.00MM

DAYS NO RAIN STATES — SOIL MOISTURE CONTENT

123456789	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
0555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	5	1	1	1	1	1	1	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	5	5	5	2	2	2	2	2	2	2
1555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	5	1	1	1	1	1	1	5	1	1	1	1	1	1	1	5	1	1	1	1	1	1	1	5	5	2	2	2	2	2	2	2	2
2555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	1	1	1	1	1	5	5	1	1	1	1	1	1	5	1	1	1	1	1	1	1	5	5	2	2	2	2	2	2	2	2
3555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	1	1	1	1	1	5	5	1	1	1	1	1	1	5	1	1	1	1	1	1	1	5	5	2	2	2	2	2	2	2	2
4555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	1	1	1	1	1	5	5	1	1	1	1	1	1	5	1	1	1	1	1	1	1	5	5	2	2	2	2	2	2	2	2
5555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	1	1	1	1	1	5	5	1	1	1	1	1	1	5	1	1	1	1	1	1	1	5	5	2	2	2	2	2	2	2	2
6555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	1	1	1	1	1	5	5	1	1	1	1	1	1	5	1	1	1	1	1	1	1	5	5	2	2	2	2	2	2	2	2
7555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	5	5	1	1	1	1	1	5	5	1	1	1	1	1	1	5	1	1	1	1	1	1	1	5	5	2	2	2	2	2	2	2	2

OPTIMAL IRRIGATION POLICY FOR STAGE 90

AVAILABLE IRRIGATION WATER AT BEGINNING OF STAGE 475.00MM

DAYS NO RAIN STATES — SOIL MOISTURE CONTENT

123456789	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
0555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
6555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7555555555	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Appendix F

DERIVATION OF A MORE COMPLEX CONDITIONAL DISTRIBUTION OF STORM ARRIVALS

Assume that the information about the past of the process is that a rainfall event has occurred during the observation interval, not at the origin. The desired conditional distribution can be written as:

$$S_{010}(x;\ \zeta_1,\ \zeta_2) = \lim_{h \to 0} \frac{P_{010}(\zeta_1,\ h,\ \zeta_2+x)}{P_{010}(\zeta_1,\ h,\ \zeta_2)} \tag{F.1}$$

where $S_{010}(x;\ \zeta_1,\ \zeta_2)$ yields the probability that the time to the next rainfall is greater than x, conditional on the fact that one storm occurred after ζ_1 time units since the beginning of the observation period, and ζ_2 time units have elapsed since the occurrence of that storm.

It is well known that:

$$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B] \tag{F.2}$$

and for mutually exclusive events:

$$Pr[A \cup B] = Pr[A] + Pr[B] \tag{F.3}$$

In the situation depicted above, the observation interval has been divided into three subintervals, a first interval of length ζ_1 with no storms, a second interval of length h with one storm, and a third interval of length ζ_2 with no storms. Since events are instantaneous, a limit is taken as $h \to 0$.

Let A now be the event (0,1,0) and B the event (0,0,1) representing the number of events in each of the above intervals. Then $A \cap B = \phi$ and $A \cup B = (0,1,1)$. Thus, using equation F.3 it is obtained:

$$P_{010}(\zeta_1, h, \zeta_2) + P_{001}(\zeta_1, h, \zeta_2) = P_{011}(\zeta_1, h, \zeta_2) \qquad \text{(F.4)}$$

And from equation F.4:

$$P_{010}(\zeta_1, h, \zeta_2) = P_{01}(\zeta_1, h+\zeta_2) - P_{01}(\zeta_1+h, \zeta_2) \qquad \text{(F.5)}$$

Now, define a Palm-type function as:

$$\psi_{010}(\zeta_1, \zeta_2) = \lim_{h \to 0} \frac{P_{010}(\zeta_1, h, \zeta_2)}{[1 - P_0(h)]} \qquad \text{(F.6)}$$

Substituting equation F.5 in F.6, and adding and subtracting $P_{01}(\zeta_1, \zeta_2)$ yields:

$$\psi_{010}(\zeta_1,\zeta_2) = \lim_{h \to 0} \frac{P_{01}(\zeta_1,\zeta_2+h) - P_{01}(\zeta_1+h,\zeta_2) + P_{01}(\zeta_1,\zeta_2) - P_{01}(\zeta_1,\zeta_2)}{[1 - P_0(h)]}$$

(F.7)

Multiplying and dividing by h:

$$\psi_{010}(\zeta_1,\zeta_2) = \lim_{h\to 0} \frac{\left[\frac{P_{010}(\zeta_1,\zeta_2+h) - P_{01}(\zeta_1,\zeta_2)}{h} - \frac{P_{01}(\zeta_1+h,\zeta_2) - P_{01}(\zeta_1,\zeta_2)}{h}\right]}{[1 - P_0(h)]/h} \tag{F.8}$$

In equation F.8:

$$1 - P_0(h) = \Pr[N(0,h) > 0]$$

so that:

$$\lim_{h\to 0} \frac{1 - P_0(h)}{h} = \lim_{h\to 0} \frac{\Pr[N(0,h) > 0]}{h} = \lambda \tag{F.9}$$

where by definition, λ is the mean rate of the process.

Finally, substituting in equation F.8:

$$\psi_{010}(\zeta_1,\zeta_2) = \frac{1}{\lambda}\left[\frac{\partial P_{01}(\zeta_1,\zeta_2)}{\partial \zeta_2} - \frac{\partial P_{01}(\zeta_1,\zeta_2)}{\partial \zeta_1}\right] \tag{F.10}$$

Using equation F.10 in equation F.1, the desired CDF is obtained as:

$$S_{010}(x;\ \zeta_1,\ \zeta_2) = \psi_{010}(\zeta_1,\ \zeta_2+x)/\psi_{010}(\zeta_1,\ \zeta_2) \tag{F.11}$$

where in general the functions $\psi_{010}(\zeta_1, \zeta_2)$ are defined in terms of the p.d.f. of the number of events in a given time interval $P_{i,j,k} \cdots (x,y,z \ldots)$ defined in Chapter 3.