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EQUIVARIANT COHOMOLOGY AND SMOOTH P-TORAL ACTIONS

by

JEANNE DUFLOT

B.A., University of Texas
at Austin
(1974)

SUBMITTED IN PARTIAL
FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE
OF

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1980

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May 6, 1980

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ABSTRACT

Let G be a compact Lie group and let X be a space on which G acts continuously. Choose a classifying bundle $PG \to BG$ for principal G-bundles. G acts freely on the contractible space PG, and there is a diagonal action of G on $PG \times X$. Let $PG \times^G X$ denote the orbit space of this diagonal action. Let p be a prime integer.

The mod-p equivariant cohomology ring of the G-space X is defined by the formula

$$H_G^*(X, Z/pZ) = H^*(X \times^G PG, Z/pZ).$$

One result of this thesis gives a lower bound on the depth of $H_G^*(X, Z/pZ)$.

Theorem: The depth of $H_G^*(X, Z/pZ)$ is greater than or equal to the maximum rank of a central p-torus acting trivially on X.

The second result of the thesis concerns the differentiable action of a p-torus A on a manifold M. We define a filtration on $H_A^*(M, Z/pZ)$ and identify the successive quotients of this filtration as the equivariant cohomology rings associated to certain subsets of M. As a consequence of this, we obtain an equation that expresses the Poincare series of the graded ring $H_A^*(M, Z/pZ)$ in terms of the Poincare series of the cohomology rings of these subsets.

Thesis Supervisor: Daniel G. Quillen

Title: Professor of Mathematics

Dedication

To my parents

Rosemary Collins Duflot and Leo S.M. Duflot

and to my sisters and brother

Rene, Carol, Merrie and Joe .

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Introduction

Let G be a compact Lie group and let X be a space on which G acts continuously. Choose a classifying bundle $PG \to BG$ for principal G-bundles. The group G acts freely on the contractible space PG, and there is a diagonal action of G on $PG \times X$. Let $PG \times^G X$ denote the orbit space of this diagonal action.

Let p be a prime integer. The mod-p equivariant cohomology ring of the G-space X is defined by the formula

$$H_G^*(X, Z/pZ) = H^*(PG \times^G X, Z/pZ).$$

In a series of papers [Q1, Q2] Daniel Quillen investigated the algebraic structure of this ring. For example, suppose X has finite-dimensional mod-p cohomology. In this case Quillen proves the following

<u>Theorem:</u> (Theorem 7.7 of [Q1]) The Krull dimension of the commutative ring

$$H_G(X, Z/pZ) = \begin{cases} H_G^{ev}(X, Z/pZ) & p \text{ odd} \\ H_G^*(X, Z/pZ) & p = 2 \end{cases}$$

is equal to the maximum rank of a p-torus A of G such that $X^A \neq \emptyset$.

Here a <u>p-torus</u> is a direct product of cyclic groups of order p, and the <u>rank</u> of a p-torus A is the number of cyclic factors of A.

Chapter Two of this thesis contains a result in the same spirit as the above theorem of Quillen's. This result gives a lower bound on the depth of $H_G^*(X, Z/pZ)$. The main theorem of Chapter Two is

<u>Theorem 2.1</u>: The depth of $H_G^*(X, Z/pZ)$ is greater than or equal to the maximum rank of a central p-torus acting trivially on X.

The second result of the thesis, contained in Chapter Three, concerns the differentiable action of a p-torus A on manifold M. In this section, we allow $p = 0$, and interpret a 0-torus as an ordinary torus, i.e. a product of circles. Also, we do not consider the case $p = 2$. The main theorem, Theorem 3.13, defines a filtration

$$0 = F_0 \leq F_1 \leq \cdots \leq F_{m-1} \leq F_m = H_A^*(M, k)$$

on $H_A^*(M, k)$ (where k is a field of characteristic p), and identifies the successive quotients of this filtration as the equivariant cohomology groups associated to certain subsets of M. As a consequence of this theorem we obtain an equation (Theorem 3.14) that expresses the Poincaré series of the graded ring $H_A^*(M,Z/pZ)$ in terms of the Poincaré series of the cohomology rings of these subsets.

In case M is totally non-homologous to zero in the fibration $PA \times^A M = M_A \to BA$, we recover (Corollary 3.17) a result of Borel's [Bl]; namely,

$$\dim_k H^*(M, k) = \dim_k H^*(M^A, k).$$

In addition, we obtain in this corollary equations relating the k-Euler characteristics of M and M^A.

Chapter One

The purpose of this preliminary chapter is to state some basic definitions and results, and to set notation.

Let G be a compact Lie group. There is a classifying bundle $PG \to BG$ for principal G-bundles with paracompact base. The spaces PG and BG may be assumed to be (paracompact) CW complexes. This bundle is characterized up to homotopy equivalence as the orbit projection $PG \to PG/G$ of the free action of G on a contractible space PG (e.g., see [H]).

Suppose that G acts continuously on a topological space X. We define

$$X_G = PG \times^G X$$

to be the orbit space of the diagonal action of G on $PG \times X$. We assume that the space X is such that X_G is a paracompact, locally contractible Hausdorff space. For example, take X to be locally compact, paracompact, locally contractible and Hausdorff.

If R is a commutative ring, define the equivariant cohomology ring of the G-space X with coefficients in R to be

$$H_G^*(X, R) = H^*(X_G, R),$$

where the right hand side of the equation is ordinary singular cohomology with coefficients in R. The restrictions on X enable us to say that this definition of equivariant cohomology agrees with that of Borel [B1] and Quillen [Q1] (they use sheaf cohomology); so we may use some results of their work.

We will make use of the following properties.

1.1) [Q1, sect. 1] $H_G^*(X, R)$ is independent of the choice of classifying bundle for G.

1.2) [Q1, (1.5)] Functoriality: If $u:G \to G'$ is a homomorphism of compact Lie groups and $f:X \to X'$ is u-equivariant, then there is a homomorphism $(u,f)^*:H_{G'}^*(X', R) \to H_G^*(X, R)$. If f and u are inclusions, this homomorphism will be denoted "res".

1.3) If $X = pt$ is a point, then $H_G^*(pt, R) = H^*(PG \times^G X, R) = H^*(BG, R)$. So, if G is finite, $H_G^* = H_G^*(pt, R)$ is classical group cohomology with coefficients in the trivial G-module R.

We will continue the list of properties after introducing some notation.

Let $x \varepsilon X$ be a point of X. Then

$Gx = \{gx \mid g \varepsilon G\}$ is the <u>orbit</u> of x

and

$G_x = \{g \varepsilon G \mid gx = x\}$ is the <u>isotropy group</u> at x.

The orbit Gx is homeomorphic to the homogeneous space G/G_x. Denote the orbit space of the G-action on X by X/G.

1.4) If $x \varepsilon X$, $H_G^*(Gx, R) = H^*(BG_x, R)$. This is because $Gx \simeq G/G_x$, and $PG/G_x \simeq PG \times^G (G/G_x)$ is a classifying space for G_x.

1.5) [Q1, (1.10), (1.11)] Consider the two maps

$$BG \leftarrow X_G \rightarrow X/G \quad .$$

Each of these maps has an associated spectral sequence.

a) There is the Serre spectral sequence of the fibration $X_G \to BG$:

$$H^*(BG, \{H^*(X, R)\}) \Rightarrow H^*(X_G, R) \ .$$

Here, $\{\cdot\cdot\}$ denotes local coefficients.

b) For the map $X_G \xrightarrow{q} X/G$ we have the Leray spectral sequence:

$$H^*(X/G, \mathcal{H}^*q) \Rightarrow H^*(X_G, R) \ .$$

The cohomology on the left is sheaf cohomology with coefficients in the sheaf associated to the presheaf

$$U \mapsto H_G^*(q^{-1}(U), R)$$

on X/G. The stalk of this sheaf at $x \varepsilon X$ is $H^*(BG_x, R)$ [Q1, p. 553].

1.6) If G acts freely on X, then

$$H^*(X/G, R) \xrightarrow{\cong} H_G^*(X, R) \ .$$

(We assume that X is paracompact, locally contractible, and Hausdorff; since the action of G on X is free, X/G is also paracompact, locally contractible and Hausdorff.) This follows from 1.5b) (see [Q1, (1.12)]).

1.7) If G acts with finite isotropy groups on X , and R is a field of characteristic zero, then

$$H^*(X/G, R) \xrightarrow{\cong} H_G^*(X, R) \ .$$

The Leray spectral sequence gives the isomorphism (e.g., see [Q1]) because $H^i(BG_x, R) = 0$ if $i>0$ since R has characteristic zero, and G_x is finite [C-E]. Again,

$_{\mathcal{S}}H^*(X/G, R)$ denotes sheaf cohomology with constant coefficients. The only place we use 1.5) - 1.7) is in Chapter Three, and there we do not denote the distinction between $_{\mathcal{S}}H^*(X/G, R)$ and $H^*(X/G, R)$.

1.8) If G acts trivially on X , and R is a field, then $H^*(X_G, R) \simeq H^*(BG, R) \otimes_R H^*(X, R)$. This is the Kunneth isomorphism for $X_G = BG \times X$.

Next, we discuss orientability, G-vector bundles, and characteristic classes.

A real vector bundle (or a disk bundle) $\nu : E \to X$ of constant fibre dimension n is R-orientable if there is a class $U \varepsilon H^n(E, E_0, R)$ ($E_0 = E - X$, where X is considered as the 0-section of ν) such that for each $x \varepsilon X$, the image of U under

$$\text{res}: H^n(E, E_0, R) \to H^n(\nu(x), \nu(x) - \{0\}, R) \simeq R$$

is a generator of R . The cohomology class U is called an orientation class for ν . Of course, if ν has different fibre dimensions over different components of X , we will say that ν is R-orientable if and only if the restriction of ν to each component is R-orientable.

A complex vector bundle is R-orientable for any ring R, and any vector bundle is Z/2Z-orientable [M]. Also, a vector bundle is R-orientable if and only if its associated disk bundle is R-orientable.

For an R-orientable real vector bundle $\nu : E \to X$ of constant fibre dimension n , there is an Euler class, $e(\nu) \ \varepsilon \ H^n(X, R)$ [M,S]. Also, there is the Thom isomorphism for U [M, S]:

$$\underbrace{H^k(X, R) \overset{\cong}{\underset{p}{\to}} H^k(E, R) \overset{\cong}{\underset{\cdot U}{\to}} H^{k+n}(E, E_0, R)}_{\tau} \quad .$$

The composition of τ and $\mathrm{res}: H^{k+n}(E, E_0, R) \to H^{k+n}(X, R)$ is multiplication by $e(\nu) = (p\ )^{-1}(U)$ [M,S].

If $\nu: E \to X$ is a complex vector bundle of constant fibre dimension n, there are Chern classes (e.g., see [H]) $c_i(\nu) \in H^{2i}(X, Z)$ of ν. Via the map $Z \to R$, we consider $c_i(\nu) \in H^{2i}(X, R)$. Regard ν as a real vector bundle of dimension $2n$. It is R-orientable, so we have an Euler class $e(\nu) \in H^{2n}(X, R)$. The Euler class $e(\nu)$ is equal to $c_n(\nu)$, the top Chern class of ν [H, M].

We define real or complex G-vector bundles over the G-space X as in Atiyah [A1] and Atiyah and Segal [A-S]. Namely, a real (or complex) G-vector bundle $\xi: E \to X$ over X consists of a G-space E and an equivariant map $E \to X$ such that $\xi: E \to X$ is a real (or complex) vector bundle over X, and for each $g \in G$, the map $\xi(x) \to \xi(gx)$ is a vector space map. Here, $\xi(x)$ is the fibre of $\xi: E \to X$ over $x \in X$.

If $\xi: E \to X$ is a G-vector bundle over X, then

$$\xi_G: E_G = PG \times^G E \to PG \times^G X = X_G$$

is a vector bundle [A-S] with the same fibres as ξ; i.e., $\xi_G([p,x]) = \xi(x)$ for $[p,x] \in PG \times^G X$. As usual, we have chosen a classifying bundle $PG \to BG$ for G.

The assignment $\xi \mapsto \xi_G$ has at least the following two properties:

$$1)\quad (\nu \oplus \nu')_G \simeq \nu_G \oplus \nu'_G$$

and

2) If $Y \leq X$ is G-invariant, then

$$(\nu|_Y)_G = \nu_G|_{Y_G} .$$

Chapter Two

The main theorem of this chapter is Theorem 2.1. In the proof of Theorem 2.1, we construct a regular sequence of length n in $H_G^*(X, Z/pZ)$, where n is the rank of a central p-torus acting trivially on X.

Section One: A Regular Sequence in $H_G^*(X, Z/pZ)$

Let G be a finite group acting on a space X. In this section, the cohomology groups have coefficients in Z/pZ, where p is a fixed prime, unless otherwise indicated.

Let

$$H = \begin{cases} H_G^{ev} = \bigoplus_{i\geq 0} H_G^{2i}(pt) & p \text{ odd} \\ H_G^* = \bigoplus_{i\geq 0} H_G^{i}(pt) & p = 2 \end{cases}$$

H is a commutative ring. The graded group $H_G^*(X)$ may be considered as an H-module via the map $X \to pt$. An H-sequence on $M = H_G^*(X)$ (or on any H-module M) may be defined in the following way [K].

A sequence of elements $x_1, x_2, \ldots, x_n, \ldots$ of positive degree in H is said to be an <u>H-sequence on M</u> (or a regular sequence on M) if

x_1 is not a zero divisor on M

and if for each $i>1$,

x_i is not a zero divisor on $M/(x_1,\ldots,x_{i-1})M$.

Let $n > 0$. For each i such that $1 \leq i < n$, the sequence of elements $x_1, \ldots, x_n$ of H is an H-sequence on $H_G^*(X)$ if and only if $x_1, \ldots, x_i$ is an H-sequence on $H_G^*(X)$ and $x_{i+1}, \ldots, x_n$ is an H-sequence on $H_G^*(X)/(x_1, \ldots, x_i)H_G^*(X)$.

Theorems of Evens [E] and Venkov ([V], see also [Q1]) show that H is Noetherian and that $H_G^*(X)$ is a finitely generated H-module if $H^*(X)$ is finite dimensional over Z/pZ. In this case, any two maximal H-sequences have the same length (e.g., see [K]; in Theorem 121, take the ideal I to be the positive degree elements of H). This common length we call the <u>depth</u> of $H_G^*(X)$.

Here is the main theorem.

<u>Theorem 2.1</u> Let A be a p-torus that is contained in the center of G . Suppose also that A acts trivially on X . Then there is a regular sequence on $H_G^*(X)$ of length greater than or equal to rank(A). Thus if $H^*(X)$ is finite dimensional over Z/pZ , then

$$\operatorname{depth} H_G^*(X) \geq \operatorname{rank}(A) .$$

This theorem will be proved by induction on the rank of A .

Proof:

<u>Case One: Rank(A) = 1 .</u>

Let A be a cyclic group of order p contained in the center of G , such that $X^A = X$. Let $\ell = N/p$ be the index of A in G , where N is the order of G.

Consider the representation $\rho : A \to \mathbb{C}^*$ given by $\rho(a) = e^{2\pi i/p}$ where a is a fixed generator for A. Corresponding to this one-dimensional representation of A there is the ℓ-dimensional induced representation, $\mathrm{ind}(\rho)$, of G.

The representation ρ of A gives an A-action on $\mathbb{C}$. Using this action, we may define a one-dimensional vector bundle, also called ρ, over the classifying space BA for A:

$$PA \times^A \mathbb{C} \xrightarrow{\rho} BA \qquad [A2].$$

Similarly, there is an ℓ-dimensional complex vector bundle, $\mathrm{ind}(\rho)$, over the classifying space BG for G:

$$PG \times^G \mathbb{C} \xrightarrow[\mathrm{ind}(\rho)]{} BG \quad .$$

There are Chern classes for these vector bundles, $c_i(\rho) \varepsilon H_A^{2i}(pt, Z) = H^{2i}(BA, Z)$ and $c_i(\rho)\ \varepsilon\ H^{2i}(BG, Z)$ [A2]. Via the homomorphism $Z \to Z/pZ$, we obtain mod-p Chern classes $c_i(\rho)\ \varepsilon\ H^{2i}(BA) = H_A^{2i}$ and $c_i(\mathrm{ind}(\rho))\ \varepsilon\ H^{2i}(BG) = H_G^{2i}$.

In Corollary 2.4 we will prove that $e = c_\ell(\mathrm{ind}(\rho))$ is a non-zero-divisor on $H_G^*(X)$. The key result used to prove this Corollary is Lemma 2.2. Before stating Lemma 2.2, we must state some results of Section Two of this chapter.

Since A acts trivially on X, we have the spectral sequence of Section Two:

$$E_2^{**} = H^*(X_{G/A}, \{H_A^*\}) \Rightarrow H^*(X_G) \quad .$$

We also show in Section Two that because A is central in G, and coefficients are in a field, that

$$E_2^{p,q} = H^p(X_{G/A}, H_A^q) \simeq H^p(X_{G/A}) \otimes_{Z/pZ} H_A^q .$$

Now, it is well known that

$$H_A^* = \begin{cases} Z/pZ[c_1(\rho)] \otimes_{Z/pZ} \Lambda[x] & p \text{ odd} \\ \quad \text{(a polynomial algebra on } c_1(\rho) \text{ tensored with an exterior algebra on } x = \beta(c_1(\rho)) \text{, the Bockstein of } c_1(\rho)) & \\ Z/2Z[y] & p = 2 \\ \quad \text{(a polynomial algebra on } y \text{, where } y\cdot y = c_1(\rho)) & \end{cases}$$

(see, e.g. [Q1]).

Lemma 2.2 (Evens [E]) Let $\alpha \varepsilon H_G^{2M}(X)$ be any cohomology class such that $H_G^{2M}(X) \underset{res}{\rightarrow} H_A^{2M} = E_2^{0,2M}$ takes α to $c_1(\rho)^M$ where $M>0$. Then

a) $c_1(\rho)^M \;\varepsilon\; E_r^{0,2M}$ for every $r \geq 2$,

b) Multiplication by $c_1(\rho)^M$ induces an isomorphism $E_r^{*,j} \overset{\rightarrow}{\simeq} E_r^{*,j+2M}$ for every $r \geq 2$ and $j \geq 0$ $(E_r^{*,j} = \bigoplus_{i \geq 0} E_r^{i,j})$,

and c) $E_{2M+2}^{**} = E_\infty^{**}$.

Proof of Lemma 2.2:

a) We have the fibration (see Section Two!)

$$BA \rightarrow PG \times^G X \rightarrow P(G/A) \times^{G/A} X$$

giving rise to the spectral sequence. It is enough to note that that $c_1(\rho)^M$ is the restriction to the cohomology of the fibre of the class α in the cohomology of the total space. Thus, for every $r \geq 2$, $d_r(c_1(\rho)^M) = 0$, where d_r is the r-th differential of the spectral sequence.

b) Multiplication by $c_1(\rho)^M$ is an isomorphism

$$H^*(X_{G/A}) \otimes_{Z/pZ} H_A^j \xrightarrow{\cong} H^*(X_{G/A}) \otimes_{Z/pZ} H_A^{j+2M}$$

for every $j \geq 0$, since $c_1(\rho)$ is a polynomial generator of H_A^*. So, for $r=2$, b) is true.

Suppose b) has been proven for $r \geq 2$. Suppose $j \geq 0$. Consider the following diagram:

$$\begin{array}{ccccc}
E_r^{i-r,j+r-1+2M} & \xrightarrow{d_r} & E_r^{i,j+2M} & \xrightarrow{d_r} & E_r^{i+r,j-r+1+2M} \\
\uparrow \cdot c_1(\rho)^M & & \uparrow \cdot c_1(\rho)^M & & \uparrow \cdot c_1(\rho)^M \\
E_r^{i-r,j+r-1} & \xrightarrow{d_r} & E_r^{i,j} & \xrightarrow{d_r} & E_r^{i+r,j-r+1}
\end{array} ;$$

d_r is the differential of the spectral sequence. The diagram is commutative since

$$d_r(c_1(\rho)^M \cdot x) = c_1(\rho)^M \cdot d_r(x) \pm d_r(c_1(\rho)^M) \cdot x$$

by the multiplicative property of d_r, and since $d_r(c_1(\rho)^M) = 0$.

If $j-r+1 \geq 0$, then by induction all the vertical maps are isomorphisms, so the induced map on homology,

$$E_{r+1}^{i,j} \xrightarrow[\cdot c_1(\rho)^M]{} E_{r+1}^{i,j+2M} ,$$

is an isomorphism.

If $j-r+1<0$, $E_r^{i+r,j-r+1} = 0$, and a diagram chase shows that $E_{r+1}^{i,j} \xrightarrow[c_1(\rho)^M]{} E_{r+1}^{i,j+2M}$ is an isomorphism in

this case also.

c) $d_{2M+2}: E_{2M+2}^{i,j} \rightarrow E_{2M+2}^{i+2M+2,j-2M-1}$.

Suppose that $j \le 2M$, then $d_{2M+2}(E_{2M+2}^{*,j}) = 0$ since the spectral sequence is first quadrant. If $j > 2M$, there is an integer $m > 0$ such that $j = 2mM + k$, where $k \le 2M$. Let $y \in E_{2M+2}^{i,j} = E_{2M+2}^{i,2mM+k}$; then b) and induction on m show that $y = (c_1(\rho)^M)^m \cdot x$, for some $x \in E_{2M+2}^{i,k}$. So $d_{2M+2}(y) = c_1(\rho)^{mM} \cdot d_{2M+2}(x) \pm d_{2M+2}(c_1(\rho)^{mM}) \cdot x = 0 \pm 0 = 0$. So, $d_{2M+2} \equiv 0$, and $E_{2M+2}^{**} = E_{\infty}^{**}$. QED

<u>Corollary</u> 2.3 The cohomology class α of Lemma 2.2 is not a zero divisor on $H_G^*(X)$.

Proof: Lemma 2.2 shows that multiplication by $c_1(\rho)^M$ is injective on E_{∞}^{**} . So multiplication by α , which restricts to $c_1(\rho)^M$, must be injective on $H_G^*(X)$. QED

Getting back to the case at hand, we have

<u>Corollary</u> 2.4 The cohomology class $e = c_{\ell}(\mathrm{ind}(\rho))$ is a non-zero-divisor on $H_G^*(X)$.

Proof: Since A is contained in the center of G, the Mackey induction formula [Se] shows that

$$\mathrm{res}_{A\to G}(\mathrm{ind}_{A\to G}(\rho)) = \ell\rho = \underbrace{\rho \oplus \rho \oplus \ldots \oplus \rho}_{\ell \text{ times}} .$$

Thus

$$\mathrm{res}(c_{\ell}(\mathrm{ind}(\rho)) = c_{\ell}(\mathrm{res}(\mathrm{ind}(\rho))) = c_{\ell}(\ell\rho) = c_1(\rho)^{\ell}$$

by various properties of Chern classes (see, e.g., [H]) . Now use Corollary 2.3. QED

Thus, Theorem 2.1 is true in case rank(A) equals one.

To complete the proof of Theorem 2.1 we prove

Case Two:- Rank(A) $>$ 1.

Proof: Let A be a p-torus of rank n, $n>1$, contained in the center of G such that

$$X^A = \{x\varepsilon X \mid ax=x \quad \forall a\varepsilon A\} = X .$$

Let $\ell = N/p^n$ be the index of A in G. Let A_1 be a subgroup of rank 1 in A and write $A = A_1 \times B$, where B is a p-torus of rank n-1.

There is a one-dimensional representation $\bar{\rho}:A_1 \times B \to \mathbb{C}^*$ of A given by ρ on A_1 (ρ is the same representation as in Case One) and the trivial representation on B. Let $e = c_\ell(\mathrm{ind}_{A\to G}(\bar{\rho})) \varepsilon H_G^{2\ell} \leq H$ be the top Chern class of the ℓ-dimensional representation $\mathrm{ind}_{A\to G}(\bar{\rho})$ of G. If $\mathrm{res}_{A_1\to G}:H_G^* \to H_{A_1}^*$, then we have $\mathrm{res}_{A_1\to G}(e) = c_1(\rho)^\ell$. This follows from the Mackey induction formula, which implies that

$$\begin{aligned}\mathrm{res}_{A_1\to G}(\mathrm{ind}_{A\to G}(\bar{\rho})) &= \mathrm{res}_{A_1\to A}(\mathrm{res}_{A\to G}(\mathrm{ind}_{A\to G}(\bar{\rho})) \\ &= \mathrm{res}_{A_1\to A}(\ell\bar{\rho}) \\ &= \ell\rho \quad ;\end{aligned}$$

and standard properties of Chern classes.

By Corollary 2.4, e is not a zero divisor on $H_G^*(X)$.

The finite group G acts on $\mathbb{C}^{2\ell}$ via $\mathrm{ind}_{A\to G}(\bar{\rho})$ and therefore on $\mathbb{C}^{2\ell} \times X$ diagonally. So there is a vector bundle

$$\begin{array}{ccc} PG \times^G (\mathbb{C}^{2\ell} \times X) & \xrightarrow{\xi} & PG \times^G X \ , \\ \| & & \\ (PG \times X) \times^G \mathbb{C}^{2\ell} & & \end{array}$$

and the associated (orientable) sphere bundle is

$$PG \times^G (S^{2\ell-1} \times X) \xrightarrow{\xi'} PG \times^G X \quad ;$$

recall that $\mathrm{ind}_{A\to G}(\overline{\rho})$ is unitary, since $\overline{\rho}$ is unitary. Associated to this sphere bundle ξ' is a mod-p Euler class; it is the top Chern class of the vector bundle ξ. Therefore this Euler class is equal to e.

There is the exact Gysin sequence for $\xi'[S]$:

$$\cdots \to H^{j-2\ell}(X_G) \xrightarrow{\cdot e} H^j(X_G) \xrightarrow{\theta} H^j((S^{2\ell-1} \times X)_G) \to \cdots .$$

The map $H^{j-2\ell}(X_G) \to H^j(X_G)$ is multiplication by e as indicated, and since e is a non-zero-divisor, this map is injective. So there is a short exact sequence of H-modules

$$0 \to H^*(X_G) \xrightarrow{\cdot e} H^*(X_G) \xrightarrow{\theta} H^*((S^{2\ell-1} \times X)_G) \to 0 ,$$

where $H^*(X_G) = \bigoplus_{i\geq 0} H^i(X_G)$. Multiplication by e is an H-module map since e has even degree.

This short exact sequence shows that there is an H-module isomorphism

$$H^*((S^{2\ell-1} \times X)_G) \underset{\overline{\theta}}{\overset{\simeq}{\leftarrow}} H^*(X_G)/(e)H^*(X_G)$$

induced by θ.

The isomorphism $\overline{\theta}$ provides the inductive step. For, how does B act on $S^{2\ell-1} \times X$? In fact, B acts trivially.

To show this, it is enough to note that

1) $\operatorname{res}_{B\to G}(\operatorname{ind}_{A\to G}(\bar{\rho})) = \ell$

(the ℓ-dimensional trivial representation) .

(Proof: B is central since A is. So

$$\begin{aligned}\operatorname{res}_{B\to G}(\operatorname{ind}_{A\to G}(\bar{\rho})) &= \operatorname{res}_{B\to A}(\operatorname{res}_{A\to G}(\operatorname{ind}_{A\to G}(\bar{\rho})) \\ &= \operatorname{res}_{B\to A}(\ell\bar{\rho}) \\ &= \ell\cdot\operatorname{res}_{B\to A}(\bar{\rho}) \\ &= \ell\rho\ .\ )\end{aligned}$$

and

2) $X^B = X$; this follows because $X^A = X$ and $B \le A$.

Since $\operatorname{rank}(B) \le \operatorname{rank}(A)$, B is central, and $(S^{2\ell-1} \times X)^B = S^{2\ell-1} \times X$, we may use induction to obtain an H-sequence

$$e_1, e_2, \ldots, e_{n-1}$$

of length $n-1$ on $H^*((S^{2\ell-1} \times X)_G)$. Using the isomorphism $\bar{\theta}$;

$$e, e_1, e_2, \ldots, e_{n-1}$$

is an H-sequence of length n on $H^*(X_G)$.

So, Theorem 2.1 is proved. QED

It is nice to notice that it is possible to actually write down an H-sequence on $H^*(X_G)$. Write

$$A = A_1 \times A_2 \times \ldots \times A_n$$

as a direct product of cyclic groups of order p . For $1 \le i \le n$, let $\rho_i : A \to \mathbb{C}^*$ be the one-dimensional representation of A given by the trivial representation of A on all but the i-th factor of A and by (our usual) ρ on A_i . If $e_i = c_\ell(\operatorname{ind}_{A\to G}(\rho_i))$, the proof of Theorem

2.1 shows that $e_1, \ldots, e_n$ is an H-sequence on $H^*(X_G)$. Also,

$$H^*_G(X)/(e_1, \ldots, e_i)H^*_G(X) \simeq$$

$$H^*((\underbrace{(S^{2\ell-1} \times \ldots \times S^{2\ell-1})}_{i \text{ factors}} \times X)_G),$$

where G acts on $(S^{2\ell-1})^i$ via $\mathrm{ind}(\rho_j)$ on the j-th factor (for $1 \leq j \leq i$) and on $(S^{2\ell-1})^i \times X$ diagonally.

Section Two: A Spectral Sequence

This section constructs the spectral sequence used in the proof of Lemma 2.2 of Section One.

Let G be a compact Lie group acting on a space X. Suppose that N is a closed normal subgroup of G that acts trivially on X.

In this section we point out that there is a fibration

$$BN \rightarrow X_G \rightarrow X_{G/N}$$

giving rise to a Serre spectral sequence

$$E_2^{**} = H^*(X_{G/N}, \{H_N^*\}) \Rightarrow H^*(X_G)$$

($\{\cdot\}$ denotes local coefficients.)

We assume that cohomology has coefficients in a fixed commutative ring R, unless otherwise noted.

To get the fibration

$$BN \rightarrow X_G \rightarrow X_{G/N}$$

we start with a classifying bundle $P(G/N) \xrightarrow{\xi} B(G/N)$ for principal G/N-bundles. Then, there exists a classifying bundle $PG \to BG$ for principal G-bundles and a commutative diagram

$$\text{(A)} \qquad \begin{tikzcd} PG \arrow[r] \arrow[d] & P(G/N) \arrow[d, "\xi"] \\ BG \arrow[r, "f"'] & B(G/N) \end{tikzcd}$$

with f a fibration.

To see this, use Borel's diagram [B2]. Namely, let $P \to B$ be any classifying bundle for principal G-bundles. Form the diagram

$$\begin{tikzcd} P \arrow[d] & P \times P(G/N) \arrow[l, "pr_1"'] \arrow[r, "pr_2"] \arrow[d] & P(G/N) \arrow[d, "\xi"] \\ B & P \times^G P(G/N) \arrow[l, "\overline{pr}_1"] \arrow[r, "\overline{pr}_2"'] & B(G/N) \end{tikzcd} .$$

Here G acts on $P(G/N)$ via the homomorphism $\pi : G \to G/N$; and the space $P \times^G P(G/N)$ is, as usual, the orbit space of the diagonal action of G on $P \times P(G/N)$.

Then the right half of the above diagram

$$\begin{tikzcd} PG = P \times P(G/N) \arrow[r, "pr_2"] \arrow[d] & P(G/N) \arrow[d, "\xi"] \\ BG = P \times^G P(G/N) \arrow[r, "\overline{pr}_2"'] & B(G/N) \end{tikzcd}$$

is the desired diagram (A). We must verify

1) that $P \times P(G/N) \to P \times^G P(G/N)$ is a classifying bundle for G and

2) that $\overline{pr}_2$ is a fibration .

Now, $P \times P(G/N)$ is a contractible space on which G acts freely, and $P \times^G P(G/N)$ its orbit space. Therefore, $PG \to BG$ is a classifying bundle for G . So 1) is true.

To see 2) , we show that $P \times^G P(G/N) \underset{\overline{pr}_2}{\to} B(G/N)$ is locally a product. Since ξ is locally a product, suppose that U is an open set in $B(G/N)$ with $U \times G/N = \xi^{-1}(U)$. The action of G/N , and hence of G , is given by translation on the second factor in this product. So

$$\overline{pr}_2^{-1}(U) = P \times^G (U \times G/N) = (P \times^G G/N) \times U$$

and is locally a product. The fibre is

$$P \times^G G/N = P/N = BN ,$$

a classifying space for N .

Since N acts trivially on $P(G/N)$ there is a commutative diagram

(B)
$$\begin{tikzcd}
PG \arrow[dr] \arrow[drr, bend left] & & \\
& PG/N = P/N \times P(G/N) \arrow[r, "\bar{f}"] \arrow[d, "\xi'"] \arrow[dr, phantom, "\square"] & \bar{P}(G/N) \arrow[d, "\xi"] \\
& BG \arrow[r, "\overline{pr}_2 = f"'] & B(G/N)
\end{tikzcd}$$

and the big square $\square$ is cartesian.

Now, replace the fibres of the principal G/N-bundles ξ and ξ' by the G/N-space X , forming the commutative diagram of fibrations

C
$$\begin{tikzcd}
P/N = BN \arrow[r] \arrow[d] & PG/N \times^{G/N} X \arrow[r, "\tilde{f}"] \arrow[d] & P(G/N) \times^{G/N} X \arrow[d] \\
P \times^{G} (G/N) = BN \arrow[r] & BG \arrow[r, "f"'] & B(G/N)
\end{tikzcd}$$

The indicated square is cartesian. The only thing left to do is to notice that $PG/N \times^{G/N} X$ is homeomorphic to $PG \times^{G} X$.

So, rewriting the diagram C as C′, we have a commutative diagram of fibrations, with the indicated square cartesian:

C′
$$\begin{tikzcd}
BN \arrow[r] \arrow[d] & X_G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_{G/N} \arrow[d] \\
BN \arrow[r] & BG \arrow[r] & B(G/N)
\end{tikzcd}$$

The fibration $X_G \to X_{G/N}$ is induced from the map $BG \to B(G/N)$. Therefore, if the local coefficient system $\{H_N^*\}$ is trivial for the latter fibration, it is trivial for the former fibration [S].

From this it follows that if N is central in G, and coefficients are in a field F, then

$$E_2^{**} \simeq H^*(X_{G/N}) \otimes_F H_N^* .$$

For, $\{H_N^*\}$ is trivial for $BG \to B(G/N)$ since N is central (see, e.g. [ ]), so

$$E_2^{**} = H^*(X_{G/N}, H_N^*) \simeq H^*(X_{G/N}) \otimes_F H_N^* .$$

Chapter Three: Smooth Actions

Let A be a p-torus where p is zero, or an odd prime. We consider smooth actions of A in this chapter. The main result is Theorem 3.13 of Section Three.

Section One: Gysin Sequences

Section One constructs the Gysin sequences for the embedding of a closed invariant submanifold in a differentiable manifold. The results of this section are well known. Since they hold for smooth compact Lie group action, and not just toral actions, we let G be a compact Lie group acting smoothly on a differentiable manifold M. The manifold M has a smooth G-invariant Riemannian metric. If Y is a closed G-invariant submanifold of M, then the normal bundle $\nu : N \to Y$ is a G-vector bundle since the metric defining ν is G-invariant.

Let R be a fixed commutative ring. We assume that ν, and also

$$\nu_G : \underset{\substack{\parallel \\ N_G}}{PG \times^G N} \to \underset{\substack{\parallel \\ Y_G}}{PG \times^G Y}$$

are R-orientable vector bundles (see Chapter One). In this section, cohomology groups will have coefficients in the ring R.

<u>Proposition 3.1</u> Suppose that ν and ν_G have constant fibre dimension d over $\mathbb{R}$. Then there is an exact

equivariant Gysin sequence for the embedding $Y \to M$:

$$\cdots \to H_G^i(Y) \xrightarrow{\Phi_G} H_G^{i+d}(M) \xrightarrow{\text{res}} H_G^{i+d}(M-Y) \to H_G^{i+1}(Y) \to \cdots .$$

If $\text{res}_{M\to Y}: H_G^*(M) \to H_G^*(Y)$, then $\text{res}_{M\to Y} \circ \Phi_G$ is multiplication by the Euler class of ν_G .

Proof: To get the exact sequence, start by considering the total space D of the disk bundle associated to ν as being smoothly and equivariantly embedded as a closed G-invariant tubular neighborhood of Y [Br]. There is the exact sequence of the pair $(M_G, (M-Y)_G)$:

$$\begin{tikzcd} \cdots \arrow[r] & H_G^{i+d}(M, M-Y) \arrow[r] & H_G^{i+d}(M) \arrow[r] & H_G^{i+d}(M-Y) \arrow[dll, rounded corners] \\ & & H_G^{i+d+1}(M, M-Y) \arrow[r] & \cdots . \end{tikzcd}$$

By excision of the open set $U_G = (M-D)_G$,

$$H_G^i(M, M-Y) \xrightarrow{\cong} H_G^i(D, D-Y)$$

for every i. The space Y_G is equivariantly embedded in D_G as the zero section of the disk bundle associated to ν_G, so there is a Thom isomorphism

$$H_G^i(Y) \underset{\tau_G}{\overset{\cong}{\to}} H_G^{i+d}(D, D-Y) .$$

The composition

$$H_G^i(Y) \xrightarrow[\tau_G]{} H_G^{i+d}(D, D-Y) \xrightarrow[\text{res}]{} H_G^{i+d}(Y)$$

is multiplication by the Euler class of ν_G . The Gysin sequence is:

$$\begin{tikzcd}
\cdots \arrow[r] & H^i_G(Y) \arrow[r, "\Phi_G"] \arrow[d, "\tau_G", "\wr"'] & H^{i+d}_G(M) \arrow[r, "res"] & H^{i+d}_G(M-Y) \arrow[r] \arrow[d, "\delta"] & H^{i+1}_G(Y) \cdots \\
& H^{i+d}_G(D, D-Y) \arrow[r, "\simeq"] & H^{i+d}_G(M, M-Y) \arrow[u] & H^{i+d+1}_G(M, M-Y) \arrow[r, "\simeq"] & H^{i+d+1}_G(D, D-Y) \arrow[u, "(\tau_G)^{-1}"', "\wr"]
\end{tikzcd}$$

QED.

More generally, we have

<u>Proposition 3.2</u> Suppose that $Y = \bigcup_\alpha Y_\alpha$ is the finite disjoint union of closed G-invariant submanifolds Y_α such that $\nu|_{Y_\alpha}$ has constant fibre dimension d_α. Then

1) there is an exact Gysin triangle of R-modules

$$\begin{tikzcd}
& H^*_G(M) \arrow[dr] & \\
H^*_G(Y) \arrow[ur, "\Phi_G"] & & H^*_G(M-Y) \arrow[ll]
\end{tikzcd}$$

where $H^*_G(-) = \bigoplus_{i\geq 0} H^i_G(-)$,

and

2) the composition

$$H^*_G(Y_\alpha) \to H^*_G(Y) \xrightarrow[\Phi_G]{} H^*_G(M) \xrightarrow[res]{} H^*_G(Y_\beta)$$

is zero if $\alpha \neq \beta$ and is multiplication by the Euler class of $\nu_G|_{(Y_\alpha)_G} = (\nu|_{Y_\alpha})_G$ if $\alpha = \beta$.

Here the inclusion $H^*_G(Y_\alpha) \to H^*_G(Y)$ comes from the isomorphism

$$H^*_G(Y) \xrightarrow[\simeq]{\bigoplus_\alpha res_\alpha} \bigoplus_\alpha H^*_G(Y_\alpha) \quad .$$

Proof: To see this, note that we may assume that the closed invariant tubular neighborhood D of Y is the disjoint union of invariant tubular neighborhoods D_α of Y_α. Proceeding as in Proposition 3.1, start with the exact sequence of the pair $(M_G, (M-Y)_G)$; and then use the isomorphisms

$$\begin{tikzcd}
H_G^*(Y) \arrow[r, "\cong", "\oplus_\alpha \mathrm{res}_\alpha"'] & \bigoplus_\alpha H_G^*(Y_\alpha) \arrow[r, "\cong", "\oplus_\alpha \tau_{G,\alpha}"'] & \bigoplus_\alpha H_G^*(D_\alpha, D_\alpha - Y_\alpha) \\
 & & H_G^*(D, D-Y) \arrow[u, "\wr", "\oplus_\alpha \mathrm{res}_\alpha"']
\end{tikzcd}$$

The map $\tau_{G,\alpha}$ is the Thom isomorphism for $(Y_\alpha)_G \to (D_\alpha)_G$.

It is clear that 2) holds since the D_α's are disjoint. The Gysin map Φ_G mixes degree, as does the map $H_G^*(M-Y) \to H_G^*(Y)$ in the Gysin triangle, if the d_α's are different. QED

For later use, we note that the Gysin triangle (for $G = [e]$, the identity group) implies that if any two of the three groups $H^*(M)$, $H^*(M-Y)$, or $H^*(Y)$, is finite dimensional over the coefficient <u>field</u> R, then so is the third group.

Section Two: The Decomposition of the Normal Bundle

In this section we go back to considering smooth toral actions. Let M be a differentiable manifold on which a p-torus A (p is zero, or an odd prime) acts smoothly. As in Section One, assume that M has a smooth A-invariant

Riemannian metric.

Let B be a nontrivial subtorus of A, and let Y be a smooth closed A-invariant submanifold of M on which B acts trivially. If $\nu : N \to Y$ is the normal bundle to Y, the subtorus B acts (by restriction of the A-action) on N. This B-action is an automorphism on each fibre of ν since B acts trivially on Y. Throughout this section we assume that the actions of B on the fibres of ν have no nonzero fixed vectors. Let $C = A/B$, and fix once and for all an isomorphism $A \simeq B \times C$. The p-torus C acts by restriction on N and Y making ν into a C-vector bundle.

Proposition 3.3 gives a decomposition of the normal bundle corresponding to the irreducible nontrivial real characters of B. Under "constant codimension" assumptions on Y, we then get a factorization of the Euler class of ν (Proposition 3.6). Finally, we show that this Euler class acts as a non-zero-divisor on $H^*_A(Y)$ (Propositions 3.9 and 3.10).

We begin by listing the irreducible nontrivial complex characters of B. They are one-dimensional and occur in conjugate pairs: $\{\chi_j, \overline{\chi}_j\}$. Since $p \neq 2$, the nontrivial irreducible real characters of B are two-dimensional; they are $\{\chi_j + \overline{\chi}_j\}$. Given the list $\{\chi_j, \overline{\chi}_j\}$ of irreducible complex characters, $\chi_j + \overline{\chi}_j$ is the character associated to the real representation $\rho_j : B \to \mathbf{GL}(V_j)$ given by (here, $V_j \simeq \mathbb{R}^2$):

$$\rho_j(b)(x,y) = \chi_j(b)\cdot(x+iy) = \begin{bmatrix} \operatorname{Re}\chi_j(b) & \operatorname{Im}\chi_j(b) \\ -\operatorname{Im}\chi_j(b) & \operatorname{Re}\chi_j(b) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

for each $b \in B$.

The real vector space V_j has a natural complex structure $J: V_j \to V_j$ given by $J(x,y) = (-y, x)$. Note that $b \cdot J(x,y) = \chi_j(b)\cdot i(x+iy) = i\chi_j(b)\cdot(x+iy) = J(b\cdot(x,y))$, if $b \in B$. Let γ_j be the B-vector bundle $Y \times V_j \xrightarrow{\gamma_j} Y$; the action of B is given by ρ_j on V_j . The vector bundle γ_j of course has a complex structure given by J .

<u>Proposition 3.3</u> (Atiyah, [A1]) Let Y be a smooth closed A-invariant submanifold of M on which B acts trivially, with normal bundle $\nu: N \to Y$. Then (recall that the indices j index the nontrivial irreducible real characters of B)

a) $\varepsilon_j = \operatorname{Hom}_B(\gamma_j, \nu) = [\operatorname{Hom}(\gamma_j, \nu)])^B$
is a (real) vector bundle over Y . There is an action of C on the total space of ε_j making ε_j into a C-vector bundle. Also, the vector bundle ε_j has a complex structure given by J.

b) The vector bundle $\gamma_j \otimes_{\mathbb{C}} \varepsilon_j$ has a complex structure and an A-action making it into an A-vector bundle over Y.

c) If the actions of B on the fibres of ν have no nontrivial fixed vectors, then $\nu \simeq \sum_j \gamma_j \otimes_{\mathbb{C}} \varepsilon_j$ as A-bundles. Thus, ν has a complex structure.

Proof: a) For the fact that ε_j is a vector bundle over Y, see Atiyah [A1]. If $f \varepsilon \mathrm{Hom}_B(\gamma_j(y), \nu(y))$, for $y \varepsilon Y$, and if $c \varepsilon C$, then $cf \varepsilon \mathrm{Hom}_B(\gamma_j(cy), \nu(cy))$ is given by $(cf)(cy,v) = c \cdot f(y,v)$. Since A is abelian and f is a B-homomorphism, cf is a B-homomorphism. The complex structure on ε_j is given by J; i.e., if $f \varepsilon \mathrm{Hom}_B(\gamma_j(x), \nu(x))$ then $Jf \varepsilon \mathrm{Hom}_B(\gamma_j(x), \nu(x))$ is given by $(Jf)(x,v) = f(x, Jv)$. $J^2(f) = -f$ since f is linear, and Jf is a B-homomorphism since $J(bv) = bJ(v)$ for $v \varepsilon V_j = \gamma_j(x)$.

b) It is clear that $\gamma_j \otimes_{\mathbb{C}} \varepsilon_j$ has a complex structure; the A-action on $\gamma_j \otimes_{\mathbb{C}} \varepsilon_j$ is given by the isomorphism $A = B \times C$.

c) There is a natural map

$$\sum_j \gamma_j(x) \otimes_{\mathbb{C}} \mathrm{Hom}_B(\gamma_j(x), \nu(x)) \to \nu(x)$$

for $x \varepsilon Y$. This map is an isomorphism because

1) There are no nonzero fixed vectors in the B-action on $\nu(x)$, so that $\nu(x) \simeq \sum_j n_j(x) V_j$ as B-vector spaces. (The nonnegative integers $n_j(x)$ are constant on the A-orbit (=C-orbit) of a component of Y.)

and

2) By Schur's Lemma,

$$\mathrm{Hom}_B(V_j, V_k) \underset{B}{\simeq} \begin{cases} 0 & j \neq k \\ \mathbb{R}^2 \underset{J}{=} \mathbb{C} & j = k \end{cases} .$$

Now, follow Atiyah [A1] to show that $\nu \simeq \sum_j \gamma_j \otimes_{\mathbb{C}} \varepsilon_j$ as B-vector bundles. Using the isomorphism $A = B \times C$, it is easy to see that the isomorphism is A-equivariant too. QED

Now, for each subtorus B of A, the fixed point set $M^B = \{m \in M \mid am=m \ \forall a \in B\}$ is a smooth closed submanifold of M [Br]. The fixed point set M^B is A-invariant since B is normal in A. Proposition 3.3 has two immediate corollaries that we use later on in Section Three.

<u>Corollary 3.4</u> For each subtorus B of A, the normal bundle $\nu : N \to M^B$ has a complex structure.

Proof: Consider N as being equivariantly embedded as an invariant open tubular neighborhood of M^B. Since B has no fixed points on $N-M^B$ the B-action on N has no nonzero fixed vectors. Proposition 3.3 c) shows that ν has a complex structure. QED

<u>Corollary 3.5</u> 1) Every component of M^B has even codimension in M, and 2) ν is an R-orientable vector bundle for any commutative ring R.

Proof: Corollary 3.4 shows that ν has a complex structure. QED

We now look at some Euler classes. We fix a field k of characteristic p and consider cohomology with coefficients in k.

First, suppose that A acts <u>transitively</u> on the set of components of Y. Then each subbundle $\gamma_j \otimes_{\mathbb{C}} \varepsilon_j$ in the decomposition $\nu = \sum_j \gamma_j \otimes_{\mathbb{C}} \varepsilon_j$ has constant fibre dimension s_j over $\mathbb{C}$. So ν has constant fibre dimension $d = 2\sum_j s_j$ over $\mathbb{R}$.

Proposition 3.6 Suppose A acts transitively on the set of components of Y, and the s_j's are as above. Let $e_j \in H_A^{2s_j}(Y, k)$ be the top Chern class of the bundle $(\gamma_j \otimes_{\mathbb{C}} \varepsilon_j)_A$. If $e \in H_A^{2d}(Y, k)$ is the top Chern class of ν, then $e = \prod_j e_j$.

Proof: Since $\nu = \sum_j \gamma_j \otimes_{\mathbb{C}} \varepsilon_j$, we have $\nu_A = \sum_j (\gamma_j \otimes_{\mathbb{C}} \varepsilon_j)_A$. The result follows from the sum formula for Chern classes. QED

Before stating Proposition 3.7 we note that for any A-space W (no smoothness restrictions) on which B acts trivially, that there is a Kunneth isomorphism

$$(*) \qquad H_A^*(W, k) \ \underset{\simeq}{\leftarrow} \ H_B^*(pt, k) \otimes_k H_C^*(W, k).$$

To see this, suppose PB, PC are total spaces of classifying bundles for B, C respectively. Then $PB \times PC$ is the total space of a classifying bundle for A. Since there is a homeomorphism

$$PB/B \times (PC \times^C W) \ \underset{h}{\simeq} \ (PB \times PC) \times^A W$$

the isomorphism $(*)$ is a consequence of the ordinary Kunneth isomorphism.

Now we state

Proposition 3.7 Suppose A acts transitively on the set of components of Y and the s_j's and e_j's are as in Proposition 3.6. Then, using the Kunneth isomorphism

(*) ,

$$e_j = \sum_{r=0}^{s_j} c_1(\overline{\gamma}_j)^r \otimes c_{s_j-r}[(\varepsilon_j)_C]$$

for each j, where $c_{s_j-r}[(\varepsilon_j)_C]$ is the (s_j-r)-th Chern class of $(\varepsilon_j)_C$ and $c_1(\overline{\gamma}_j)$ is the first Chern class of

$$\overline{\gamma}_j : PB \times^B V_j \to PB/B .$$

(Here, B acts on V_j via the character χ_j .)

Proof: Let pr_1 and pr_2 be the projections

$$PB/B \xleftarrow{pr_1} PB/B \times Y_C \xrightarrow{pr_2} Y_C .$$

Under the homeomorphism

$$h: PB/B \times Y_C \simeq Y_A ,$$

the vector bundle $(\gamma_j \otimes_{\mathbb{C}} \varepsilon_j)_A$ over Y_A corresponds to the vector bundle $pr_1^*(\overline{\gamma}_j) \otimes_{\mathbb{C}} pr_2^*((\varepsilon_j)_C)$ over $PB/B \times Y_C$ (compare fibres). Since $\overline{\gamma}_j$ is a complex line bundle, the proposition follows from the well known

<u>Lemma</u> 3.8 If $\xi : E \to X$ is an r-dimensional complex vector bundle over X and $\gamma : L \to X$ is a line bundle, then the top Chern class $c_r(\gamma \otimes \xi) \in H^{2r}(X, k)$ of $\gamma \otimes \xi$ equals

$$\sum_{i=0}^{r} c_1(\gamma)^i \cdot c_{r-i}(\xi) ;$$

where $c_{r-i}(\xi) \in H^{2(r-i)}(X, k)$ is the $(r-i)$-th Chern class of ξ and $c_1(\gamma)$ is the first Chern class of γ.

Proof: We may use the splitting principle [H] to reduce to the case where ξ is a direct sum of line bundles. The result follows from a straightforward calculation. QED

Proposition 3.9 If A acts transitively on the set of components of Y, then the Euler class e of the normal bundle to Y is a non-zero-divisor on $H_A^*(Y, k)$.

Proof: Proposition 3.6 shows that $e = \prod_j e_j$.

We show that each e_j is a non-zero-divisor on $H^*(Y, k)$.

Fix an index j. If p is odd, we may regard $H_B^*(pt, k) = H_B^*$ as the tensor product over k of a polynomial algebra on $c_1(\overline{\gamma}_j)$ and $(\operatorname{rank}(B)) - 1$ other polynomial generators, and an exterior algebra on $(\operatorname{rank}(B))$ generators of degree one. If p is zero, H_B^* is a polynomial algebra on $c_1(\overline{\gamma}_j)$ and $(\operatorname{rank}(B)) - 1$ other polynomial generators of degree two. We write this as

$$H_B^* = \begin{cases} k[c_1(\overline{\gamma}_j), \ldots] \otimes_k \Lambda & p \text{ odd} \\ k[c_1(\overline{\gamma}_j), \ldots] & p = 0 . \end{cases}$$

(see, e.g. [Q1]).

The Kunneth isomorphism (*) gives

$$H_A^*(Y, k) \simeq H_C^*(Y, k) \otimes_k H_B^* .$$

Proposition 3.7 shows that e_j is a monic polynomial in $c_1(\overline{\gamma}_j)$, therefore it cannot be a zero-divisor in

$$H_A^*(Y, k) \simeq \begin{cases} H_C^*(Y, k) \otimes_k \Lambda \otimes_k k[c_1(\overline{\gamma}_j), \ldots] & p \text{ odd} \\ H_C^*(Y, k) \otimes_k k[c_1(\overline{\gamma}_j), \ldots] & p = 0. \end{cases}$$

QED

We generalize Proposition 3.9 by weakening the hypothesis of a transitive A-action on the set of components of Y.

<u>Proposition 3.10</u> Suppose Y has constant codimension d in M, and has only a finite number of components. Then e, the Euler class of $\nu : N \to Y$, is a non-zero-divisor on $H^*_A(Y, k)$.

Proof. We write $Y = \bigcup_\alpha Y_\alpha$ as the finite disjoint union of A-invariant closed submanifolds Y_α such that A acts transitively on the set of components of each Y_α. Then, the following diagram commutes, where e_α is the Euler class of $\nu_A|_{(Y_\alpha)_A} = (\nu|_{Y_\alpha})_A$:

$$\begin{tikzcd}
H^i_A(Y,k) \arrow[r, "\bigoplus_\alpha \mathrm{res}_\alpha", "\cong"'] \arrow[drr, "\cdot e"'] & \bigoplus_\alpha H^i_A(Y_\alpha, k) \arrow[r, "\cdot \sum_\alpha e_\alpha"] & \bigoplus_\alpha H^{i+d}_A(Y_\alpha, k) \\
 & & H^{i+d}_A(Y, k) \arrow[u, "\cong", "\bigoplus_\alpha \mathrm{res}_\alpha"']
\end{tikzcd}$$

Since each e_α is a non-zero-divisor on $H^*_A(Y_\alpha, k)$ by Proposition 3.9, $\sum_\alpha e_\alpha$ is a non-zero-divisor on $\bigoplus_\alpha H^*_A(Y_\alpha, k)$, so e is a non-zero-divisor on $H^*_A(Y, k)$. QED

<u>Section Three: A Filtration on $H^*_A(M)$</u>

Let A and M be as in Section Two. Suppose $n = \mathrm{rank}(A)$ and $r = \dim(M)$. The cohomology groups in this section have coefficients in a field k of characteristic p (p is zero, or an odd prime). In this section we define a filtration on $H^*_A(M)$; in Theorem 3.13, we identify the successive quotients of this filtration using the results of Sections One and Two. In the last part of this section we obtain an expression for the k-Poincare series of $H^*_A(M, k)$ (Theorem 3.14) and make some calculations using this series.

For each point $m \epsilon M$, define an integer, rk m , as follows. Let $A_m = \{a \epsilon A \mid am=m\}$ be the isotropy group at m . If $(A_m)_0$ denotes the connected component of the identity in A_m , define

$$\text{rk } m = \begin{cases} \text{rank}(A_m) & p \text{ odd} \\ \text{rank}((A_m)_0) & p = 0 \end{cases} .$$

For $0 \leq i \leq n+1$, let

$$M_i = \{m \epsilon M \mid \text{rk } m \geq i\} .$$

Then

$$M_0 = M \geq M_1 \geq M_2 \geq \cdots \geq M_n \geq M_{n+1} = \emptyset$$

is a decreasing filtration of M by A-invariant subsets.

<u>Proposition 3.11</u> Each M_i is closed in M .

Proof: Any $m \epsilon M$ has an open neighborhood U such that for each $x \epsilon U$, A_x is a subgroup of A_m([Br, pg. 86], this theorem is a consequence of the existence of slices for differentiable actions). Thus if rk m < i , then rk x < i for $x \epsilon U$. So M_i is closed. QED

So, we have an increasing filtration of M by open A-invariant submanifolds:

$$\emptyset \leq M-M_1 \leq \cdots \leq M-M_n \leq M .$$

Let $M_{(i)} = \{m \epsilon M \mid \text{rk } m = i\}$ for each i such that

$0 \leq i \leq n$. We have $M_{(i)} = M_i - M_{i+1} \subseteq M - M_{i+1}$.

Let $S = \{(A_m)_0 \mid m \varepsilon M\}$.

<u>Proposition 3.12</u> Suppose that S is finite. Then for each i such that $0 \leq i \leq n$, $M_{(i)}$ is a smooth closed A-invariant submanifold of $M - M_{i+1}$.

Proof: We need only show that $M_{(i)}$ is a submanifold of $M - M_{i+1}$. We use the fact that the fixed point set of a smooth action of a compact Lie group on a differentiable manifold is a submanifold [Br] . Given this fact we proceed.

Let

$$S_i = \begin{cases} \{A_m \mid \text{rk } m = i\} & p \text{ odd} \\ \{(A_m)_0 \mid \text{rk } m = i\} & p = 0 \end{cases} .$$

The set S_i is finite for each i . We claim that

$$M_{(i)} = \bigcup_{B \varepsilon S_i} (M - M_{i+1})^B \quad ,$$

and that this union is disjoint, so that $M_{(i)}$ is a submanifold of $M - M_{i+1}$.

The equality holds because

$$x \ \varepsilon \bigcup (M - M_{i+1})^B$$

iff $B \leq A_x$ for some $B \ \varepsilon \ S_i$ and $x \ \varepsilon \ M - M_{i+1}$

iff $\text{rk } x \geq \text{rank}(B) = i$ for some $B \ \varepsilon \ S_i$ and $\text{rk } x \leq i+1$

iff $\text{rk } x = i$.

The union is disjoint, because if

$$x \ \varepsilon \ (M - M_{i+1})^B \cap (M - M_{i+1})^{B'} \text{ , then}$$

$B \leq A_x$ and $B' \leq A_x$. So, $\langle B , B' \rangle$, the subgroup generated by B and B' , is in A_x . If $B \neq B'$, then rk $\langle B, B' \rangle > i$, so rk $x > i$, contradicting $x \varepsilon M_{(i)}$. QED

Define an increasing filtration on $H_A^*(M)$:

$$F_{n+1} = 0 \leq F_n \leq F_{n-1} \leq \cdots \leq F_0 = H_A^*(M) = \bigoplus_{j \geq 0} H_A^j(M)$$

by $F_i = \ker(H_A^*(M) \xrightarrow{\text{res}} H_A^*(M - M_i))$.

From now on we assume that the set $S = \{(A_m)_0 \mid m \varepsilon M\}$ is finite. Let the sets S_i ,for i such that $0 \leq i \leq n$, be defined as in Proposition 3.12.

Theorem 3.13 If $M_{(i)}$ has only a finite number of components for every i , then

$$1) \quad F_i / F_{i+1} \simeq H_A^*(M_{(i)}) \text{ as k-modules}$$

and

$$2) \quad H_A^*(M) \xrightarrow{\text{res}} H_A^*(M - M_i) \text{ is}$$

surjective for $0 \leq i \leq n$.

Proof: Let $\mathcal{C}_i = \{c\}$ be the set of components of $M_{(i)}$. If $c \varepsilon \mathcal{C}_i$, let $\overline{c} = \underset{a \varepsilon A}{} a \cdot c$. Then $\overline{c}$ is a closed A-invariant submanifold of $M_{(i)}$, and $M_{(i)}$ is the disjoint union of $\overline{c}$'s . Note that the normal bundle to each $\overline{c}$ in $M - M_{i+1}$ has constant fibre dimension.

By Proposition 3.2 of Section One, there is an exact Gysin triangle for each i such that $0 \leq i \leq n$:

$$\begin{tikzcd} & H_A^*(M - M_{i+1}) \arrow[dr] & \\ H_A^*(M_{(i)}) \arrow[ur, "\Phi_A"] & & H_A^*(M - M_i) \arrow[ll] \end{tikzcd} .$$

We will show that Φ_A is injective. Given this, there are short exact sequences of k-modules

$$0 \to H_A^*(M_{(i)}) \xrightarrow[\Phi_A]{} H_A^*(M - M_{i+1}) \to H_A^*(M - M_i) \to 0$$

for each i such that $0 \leq i \leq n$. Since $H_A^*(M - M_{i+1}) \to H_A^*(M - M_i)$ is surjective for each i such that $0 \leq i \leq n$, by induction we see that

$$H_A^*(M) = H_A^*(M - M_{n+1}) \to H_A^*(M - M_i)$$

is surjective, yielding part 2) of the theorem. Part 1) of the theorem follows in view of the following commutative diagram of exact sequences for $0 \leq i \leq n$:

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & & \\ & 0 \arrow[r] & F_{i+1} \arrow[r] \arrow[d] & F_i \arrow[r] \arrow[d] & F_i/F_{i+1} \arrow[r] & 0 \\ & & H_A^*(M) \arrow[r, equal] \arrow[d] & H_A^*(M) \arrow[d] & & \\ 0 \arrow[r] & H_A^*(M_{(i)}) \arrow[r] & H_A^*(M - M_{i+1}) \arrow[r] \arrow[d] & H_A^*(M - M_i) \arrow[r] \arrow[d] & 0 & \\ & & 0 & 0 & & \end{tikzcd} .$$

So, it remains to show that Φ_A is injective. We use the results of Section Two.

For each d such that $0 \leq d \leq r$, let $Y_{i,d}$ be the union of the components of $M_{(i)}$ of codimension d in $M - M_{i+1}$. The set $Y_{i,d}$ is a closed A- invariant submanifold of $M - M_{i+1}$.

For each $B \varepsilon S_i$, let

$$Y_{B,d} = Y_{i,d} \cap (M - M_{i+1})^B .$$

Then

a) $Y_{B,d}$ is a closed A-invariant (smooth) submanifold of $M - M_{i+1}$,

b) B acts trivially on $Y_{B,d}$,

c) since $Y_{B,d}$ is a submanifold of $(M - M_{i+1})^B$, the normal bundle to $Y_{B,d}$ in $M - M_{i+1}$ has no nonzero fixed vectors under the action of B (Corollary 3.4),

d) $Y_{B,d}$ has a finite number of components, each of which has codimension d in $M - M_{i+1}$.

We have verified the hypotheses of Proposition 3.10 of Section Two, and may conclude that $e_{B,d}$, the Euler class of the embedding $Y_{B,d} \to M-M_{i+1}$, is a non-zero-divisor on $H^*_A(Y_{B,d})$.

We also have

e) $M_{(i)} = \bigcup_{B \varepsilon S_i} \bigcup_{d=0}^{r} Y_{B,d}$, and this union is disjoint.

Using this decomposition of $M_{(i)}$, Proposition 3.2

shows that the composition

$$\begin{tikzcd}
H_A^*(Y_{B,d}) \arrow[r] \arrow[ddddrrr] & \bigoplus_{B\in S_i}\bigoplus_{d=0}^{r} H_A^*(Y_{B,d}) & H_A^*(M_{(i)}) \arrow[l, "\cong"'] \arrow[r] & H_A^*(M - M_{i+1}) \arrow[d, "\Phi_A"] \\
 & & & H_A^*(M_{(i)}) \arrow[d, "\cong"] \\
 & & & \bigoplus_{B\in S_i}\bigoplus_{d=0}^{r} H_A^*(Y_{B,d}) \arrow[d] \\
 & & & H_A^*(Y_{B',d'})
\end{tikzcd}$$

is multiplication by $e_{B,d}$ if $(B,d) = (B', d')$; and is zero if $(B, d) \neq (B', d')$. Thus Φ_A is injective. For, identify $H_A^*(M_{(i)})$ with

$$\bigoplus_{B\varepsilon S_i} \bigoplus_{d=0}^{r} H_A^*(Y_{B,d}) \ .$$

Then

$$\begin{aligned}
(\mathrm{res} \circ \Phi_A)(\sum_{B,d} y_{B,d}) &= ((\sum_{B',d'} \mathrm{res}_{B',d'}) \circ \Phi_A)(\sum_{B,d} y_{B,d}) \\
&= \sum_{B',d'} (\mathrm{res}_{B',d'} \circ \Phi_A)(\sum_{B,d} y_{B,d}) \\
&= \sum_{B,d} e_{B,d}\, y_{B,d} \quad .
\end{aligned}$$

Since multiplication by $e_{B,d}$ is injective, $\sum_{B,d} e_{B,d}\, y_{B,d} = 0$ if and only if each $y_{B,d}$ is zero. Therefore, $\mathrm{res} \circ \Phi_A$ is injective, so Φ_A is injective. QED

The k-Poincare series of a graded k-module $H^* = \bigoplus_{i\geq 0} H^i$

that is finitely generated in each dimension is defined as

$$\text{P.S. } H^* = \sum_{i \geq 0} (\dim_k H^i)\, t^i \ .$$

If $H^*(M)$ is finite dimensional over k, then $H_A^*(M)$ is finitely generated as a ring over $k[E,V]$, so the Poincaré series of $H_A^*(M)$ is defined.

<u>Theorem 3.14</u> Suppose that $H^*(M)$ is finite dimensional over k. Using the hypotheses and notation of Theorem 3.13 we have

$$\text{P.S. } H_A^*(M) = \sum_{i=0}^{\operatorname{rank}(A)} \sum_{B \in S_i} \sum_{d=0}^{\dim M} \frac{t^d}{(1 - t^{\varepsilon})^i} \text{ P.S. } H^*(Y_{B,d}/A)$$

where $\varepsilon = \begin{cases} 1 & p \text{ odd} \\ 2 & p=0 \end{cases}$.

Proof: For each i such that $0 \leq i \leq n$, there is a short exact sequence

$$0 \to \bigoplus_{B \in S_i} \bigoplus_{d=0}^{\dim M} H_A^*(Y_{B,d}) \to H_A^*(M - M_{i+1}) \to H_A^*(M - M_i) \to 0$$

using Theorem 3.13. The Gysin map Φ_A raises degree by d on the summand $H_A^*(Y_{B,d})$. Basic properties of Poincaré series and the short exact sequences above yield

$$\text{(A)} \qquad \text{P.S. } H_A^*(M) = \sum_{i=0}^{\operatorname{rank}(A)} \sum_{B \in S_i} \sum_{d=0}^{\dim M} t^d \text{ P.S.} H_A^*(Y_{B,d}) \ .$$

To prove the theorem, we need to calculate P.S. $H_A^*(Y_{B,d})$

(which is defined in view of the equation (A)) for a fixed $B \varepsilon S_i$ and d such that $0 \leq d \leq \dim M$.

For a fixed pair (B, d), $Y_{B,d}$ is an A-invariant space on which the p-torus B of rank i acts trivially. Write $A = B \times C$, where C is a p-torus of rank $n-i$. By the Kunneth formula (Equation (*) of Section Two) , $H_A^*(Y_{B,d}) \simeq H_C^*(Y_{B,d}) \otimes_k H_B^*$. If p is odd, C acts freely on $Y_{B,d}$. If $p = 0$, C acts with <u>finite</u> isotropy groups on $Y_{B,d}$. Thus by 1.6 and 1.7 of Chapter One, we have that

$$H^*(Y_{B,d}/C) \simeq H_C^*(Y_{B,d}) .$$

Since B acts trivially on $Y_{B,d}$, $Y_{B,d}/C = Y_{B,d}/A$.

So,

$$H_A^*(Y_{B,d}) \simeq H^*(Y_{B,d}/A) \otimes_k H_B^* .$$

Since $\text{P.S. } H_B^* = \dfrac{1}{(1 - t^{\varepsilon})^i}$ where $\varepsilon = \begin{cases} 1 & p \text{ odd} \\ 2 & p=0 \end{cases}$

(this follows from the structure of H_B^*; see proof of Proposition 3.9),

$$\text{P.S. } H_A^*(Y_{B,d}) = \frac{1}{(1 - t^{\varepsilon})^i} \text{ P.S. } H^*(Y_{B,d}/A) .$$

The theorem follows. QED

<u>Corollary 3.15</u> Suppose that $H^*(M)$ is finite dimensional over k , and the hypotheses of Theorem 3.13 are sat-

isfied. Then

a) $\lim_{t \to 1} (1-t)^r P.S.\ H_A^*(M) = \sum_{B \in S_r} \dim_k H^*(M^B/A)$, p odd,

b) $\lim_{t \to 1} (1-t)^r P.S. H_A^*(M) = \frac{1}{2^r} \sum_{B \in S_r} \dim_k H^*(M^B/A)$, p=0,

and c) $\lim_{t \to -1} (1+t)^r P.S. H_A^*(M) = \frac{1}{2^r} \sum_{B \in S_r} \chi(M^B/A)$, p=0.

Here, $r = \max \{\mathrm{rk}\ m \mid m \in M\}$ and

$\chi(M^B/A) = \sum_i (-1)^i \dim_k H^i(M^B/A)$ is the k-Euler characteristic of M^B/A .

Proof: For each i such that $0 \leq i \leq n$, each $B \in S_i$, and each d such that $0 \leq d \leq \dim M$, assume that $H^*(Y_{B,d})$ and $H^*(Y_{B,d}/A)$ are finite dimensional over k . (We show this in Lemma 3.16 following the proof of this corollary.)

Since $Y_{B,d} = \emptyset$ for d odd (Corollary 3.5 of Section Two), and $S_i = \emptyset$ for $i > r$, Theorem 3.14 shows that

$$\text{(B)} \quad (1-t^{\varepsilon})^r P.S. H_A^*(M) = \sum_{B \in S_r} \sum_{d \text{ even}} t^d P.S. H^*(Y_{B,d}/A)$$

$$+ \quad (1-t^{\varepsilon}) Q(t) ,$$

where $Q(t)$ is a <u>polynomial</u> in t .

For each $B \in S_r$, M^B is the disjoint union $\bigcup_{d \text{ even}} Y_{B,d}$ (Proposition 3.12) since $M = M - M_{r+1}$, so

M^B/A is the finite disjoint union $\bigcup_{d \text{ even}} Y_{B,d}/A$.

Therefore, $H^*(M^B/A) \simeq \bigoplus_d H^*(Y_{B,d}/A)$ for $B \in S_r$. Now, simply calculate the limits of the Corollary by substituting $t = 1$ (or $t = -1$, for c) if $\varepsilon = 2$) in the right hand side of equation (B) . QED

Lemma 3.16 Using the notation and assumptions of Corollary 3.15, $H^*(Y_{B,d})$ and $H^*(Y_{B,d}/A)$ are finite dimensional for each B and d . (For the characteristic zero case, this Lemma uses the fact that $S = \{(A_m)_0 \mid m \in M\}$ is finite.)

Proof: If X is an A-space with $H^*(X)$ finite dimensional then

1) $H^*(X^A)$ is finite dimensional [Q1] (this is where we need finiteness of S)

and

2) The cohomological dimension of X/A over k is finite: $cd_k(X/A) < \infty$. [Q1].

So, suppose $H^*(M)$ is finite dimensional over k . Then $M_{(n)} = M^A$ has finite dimensional cohomology (we abbreviate this to: $M_{(n)}$ has FDC) by 1) . The Gysin triangle of Proposition 3.2 shows that $M - M^A$ has FDC . Thus $M_{(n-1)} = \bigcup_{B \in S_{n-1}} (M - M^A)^B$ has FDC by 1) and so $M - M_{n-1}$ has FDC using the Gysin triangle of Proposition 3.2. Continuing in this manner, we see that $(M - M_i)^B$ and each of its finitely many components has FDC for every $B \in S_{i+1}$ and every i such that $0 \le i \le n$. Therefore $Y_{B,d}$, which is the union of components of $(M - M_i)^B$ if $B \varepsilon S_{i+1}$, has FDC for every B and d.

Now, by 2) $cd_k(Y_{B,d}/A) < \infty$. But the equation of Theorem 3.14 shows that $P.S.H^*(Y_{B,d}/A)$ is defined, so that $Y_{B,d}/A$ must have FDC. QED

Corollary 3.17 Suppose that $H^*(M)$ is finite dimensional. If M is totally non-homologous to zero in the fibration $M_A \to BA$, then

1) $\dim_k H^*(M) = \dim_k H^*(M^A)$ (Borel [Bl])

2) $\chi(M) = \chi(M^A) + \frac{2}{p}\chi(M_{(n-1)}) + \ldots + \frac{2^n}{p^n}\chi(M_{(0)})$

$$= \sum_{i=0}^{n} \frac{2^i}{p^i}\chi(M_{(n-i)}) \quad \text{if } p \text{ is odd.}$$

3) $\chi(M) = \chi(M^A)$ if p is zero.

Proof: The hypothesis that M is totally non-homologous to zero in $M_A \to BA$ implies that $H^*_A(M) \simeq H^*(M) \otimes_k H^*_A$. So,

$$P.S.\ H^*_A(M) = \frac{1}{(1 - t^{\varepsilon})^n} P.S.\ H^*(M) \text{ , where}$$

$$\varepsilon = \begin{cases} 1 & p \text{ odd} \\ 2 & p=0 \end{cases}.$$

Therefore, Theorem 3.14 implies that (n = rank A)

$$\text{(C)} \quad P.S.\ H^*(M) = \sum_{i=0}^{n} \sum_{B \varepsilon S_i} \sum_{d=0}^{\dim M} t^d (1-t^{\varepsilon})^{n-i} P.S.H^*(Y_{B,d}/A).$$

Since $P.S.\ H^*(M) \neq 0$, we must have $M^A \neq 0$ (as in Borel

[B1]).

Evaluating (C) at $t=1$, we get

$$\dim_k H^*(M) = \dim_k H^*(M^A) .$$

If p is odd, $\varepsilon = 1$. So, evaluating (C) at $t = -1$, we (recall that $Y_{B,d} = \emptyset$ if d is odd) have

$$\chi(M) = \chi(M^A) + 2\chi(M_{(n-1)}/A) + \ldots + 2^n\chi(M_{(0)}/A) .$$

Each $M_{(i)}/A$ is the finite disjoint union

$$\bigcup_{B\varepsilon S_i} \bigcup_d Y_{B,d}\big/_{A/B}$$; since each p-torus A/B

of rank $n-i$ acts freely on $Y_{B,d}$,

$$\chi(Y_{B,d}\big/_{A/B}) = \frac{1}{p^{n-i}} \chi(Y_{B,d}) ,$$

so

$$\chi(M_{(i)}/A) = \frac{1}{p^{n-i}} \chi(M_{(i)}) .$$ Equation 2)

follows.

For Equation 3), evaluate (C) at $t = -1$, and note that $\varepsilon = 2$ if $p = 0$. QED

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Biographical Note

Jeanne Duflot was born August 17, 1954, to Rosemary and Leo Duflot. She was the second of five children. She had a happy childhood, during which she was blissfully unaware of the realities of life, love, and higher mathematics.