THEORETICAL & EXPERIMENTAL RESEARCH [229]

On the force of torsion, and on the elasticity of metal wires: Application of this theory to the use of metals in the Arts and in various physics experiments: Construction of different kinds of torsion balances, for measuring the smallest force levels. Observations on the laws of elasticity and of coherence.

By M. Coulomb

I.[1]

This Memoir has two objectives: the first, to determine the elastic force of torsion of filaments of iron and of brass as a function of their length, their thickness, and their degree of tension. I have already had need, in a Memoir on magnetized needles printed in the *neuvieme volume des Savans etrangers*, to determine the force of torsion of hair and of silk; but I have never occupied myself with filaments of metal, because the nature of my research led me to choose the most flexible suspensions for the same force, and I have found that the filaments of silk had incomparably more flexibility than filaments of metal. The second objective of this Memoir is to evaluate the imperfection of the elastic reaction [inelastic behavior] of filaments of metal, and to examine the consequences that one can deduce about the laws of coherence and elasticity of bodies.

II. [230]

The method to determine the force of torsion, via experiment, consists of suspending a cylindrical weight by a filament of metal in a manner such that its axis is vertical, in the direction of the filament of suspension. As long as the filament of suspension is not twisted, the weight will remain at rest; but if one turns the weight about its axis, the filament twists, and will attempt to re-establish itself in its natural situation; if one lets go of the weight, it will oscillate for a longer or shorter length of time, accordingly as the elastic reaction in torsion is more or less perfect. If in this type of test, one carefully observes the duration of a fixed number of oscillations, it will be easy to determine, from the formulae of oscillatory movement, the force of reaction of torsion which produces these oscillations. Thus, in varying the weight (pesanteur) of the suspended weight (poids), the length and the thickness of the filament of suspension, one can expect to determine the laws of reaction of torsion with respect to the tension, the length, the thickness, and the nature of the filaments.

III.

If the filament of metal be perfectly elastic, and the resistance of the air does not alter the amplitude of oscillations, the weight supported by the filament of metal, once set in motion, will oscillate until one [forcibly] stops it. The diminution of the amplitudes of oscillations can be attributed to air resistance and to the imperfection of the elasticity of torsion; thus, in observing the successive diminution of the amplitude of each oscillation,

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1. Mémoires de l'Académie royale des sciences (vol. 87, 229-269) imprimé en 1787
Translated by Louis L. Bucciarelli, Emeritus Professor of Engineering and Technology Studies (MIT), MIT, 2000 .

and in taking out the part of the alteration that it is necessary to attribute to air resistance, one could, by means of the formulas of oscillatory movement, applied to these tests, determine according to which laws this force of elasticity of torsion is altered. [231]

IV.

This Memoir is divided into two sections; in the first, we will determine the law of the force of torsion, in supposing the forces of torsion are proportional to the angle of twist, a supposition conforming to experience when one does not give too great an amplitude to the angle of twist: we will give several applications of this theory to practice.

In the second section, we will explore, by experiment, how the laws of elastic force of torsion is altered in large oscillations: we will make use of this research to determine the laws of coherence and of elasticity of metals and of all solid bodies.

V.

FIRST SECTION

Formulas of oscillatory movement, in supposing the reaction of the force of torsion proportional to the angle of twist, or altered by a very small term.

A cylindrical body *B (fig. 1, n.^{o} 1)* is supported by a filament *RC*, in a manner such that the axis of the cylinder is vertical, in ligne with the prolongation of the filament of suspension; we turn this cylinder about its axis, without disturbing this axis from the vertical; it is necessary to determine, in assuming the force of torsion proportional to the angle of twist, the formulas of oscillatory motion.

no. 2, fig. I, shows a horizontal section of the cylinder; all the elements of the cylinder are projected on this circular section at π, π', π'', etc we assume that the starting angle of twist is *ACM = A*, and that after time *t*, this angle is *ACm*, or that it is diminished by the angle *MCm = S*, so that *ACm = (A - S)*.

Since we suppose the force of torsion is proportional to the angle of twist, the moment [momentum] of this force will be represented by *n(A -S)*, *n* being a constant coefficient, whose value will depend on the nature of the filament of metal, on its length and on its thickness. If we call *v* the velocity of any point π, after a time *t*, when the angle of twist is *ACm*, we will have, by the principles of Dynamics, [232]

$$n(A-S)\cdot dt = \int \pi r\, dv$$

where *r* is the distance $C\pi$ from the point π to the axes of rotation *G [C]*.

But if the radius *CA'* of the cylindrical weight = *a*, and the velocity of the point *A*' on the circumference of the cylinder, be at the end of time *t*, represented by *u*, we will have

$v = \dfrac{ru}{a}$; from which it results

$$n(A-S)\cdot dt = du\int\frac{\pi r^2}{a};$$

and as $dt = \frac{adS}{u}$, we will have for the integrated equation

$$n(2AS-SS) = uu\int\frac{\pi r^2}{a^2},$$

from which we draw

$$dt = \frac{dS\int(\pi r^2)^{1/2}}{\sqrt{(n)}\cdot\sqrt{(2AS-SS)}}$$

But $\frac{dS}{\sqrt{(2AS-SS)}}$ represents an angle of which A is the radius and S the verse sine, which vanishes when $S=0$, and which becomes equal to 90 degrees when $S = A$.

Thus the time of an complete oscillation will be [Note: A complete oscillation here is but one half of what we today call a full cycle.] [233]

$$T = \left(\int\frac{\pi r^2}{n}\right)^{1/2}\cdot 180^d$$

VI.

In order to compare the force of torsion with the force of gravity in a pendulum, it is necessary to remember that in the pendulum the time T of a complete oscillation

$$= \left(\frac{\lambda}{g}\right)^{1/2}180^d$$

where λ is the length of the pendulum and g the force of gravity. Thus a pendulum which is isochronous to the oscillations of the cylinder gives

$$\int\frac{\pi r^2}{n} = \frac{\lambda}{g}$$

From this formula we will easily draw the value of n from the experiment, since the dimensions of the cylinder or of the weight are given, and so too the time of one oscillation, which determines the value of λ.

If we wish then to search for a weight Q which, acting at the extremity of the lever b, would have a *moment (momentum)* equal to the moment of the force of torsion, when the angle of twist is $(A - S)$, it requires setting $Qb = n(A - S)$.

VII.

It necessary now to search for a cylinder such that the value of $\int \pi r^2$, we will find equal to $\frac{\varphi\delta L \cdot a^4}{4}$, where ϕ is the ratio of the circumference to the radius [2π], δ is the density of the cylinder and a its radius. But as the mass M of the cylinder is $\frac{\varphi\delta L \cdot a^2}{2}$, we have $\int \pi r^2 = \frac{Ma^2}{2}$, and consequently $T = \left(\frac{Ma^2}{2n}\right)^{1/2} 180^d$: in comparing this, as in the preceding article, with the isochronous pendulum, there results $\frac{\lambda}{g} = \frac{Ma^2}{2n}$, and as gM is the weight P of the cylinder, we will have $n = \frac{Pa^2}{2\lambda}$; which gives a very simple formula for determining n from the experiment. [234]

VIII.

If the force of torsion, which we have taken equal to *n(A - S)*, be altered by a quantity *R*, the formula of oscillatory motion would give as a law

$$[n(A-S)-R] \cdot \partial t = \partial u \int \frac{\pi r^2}{a}$$

and putting as before, in place of ∂t, its value $\frac{adS}{u}$, we will have for the integration

$$n(2AS-SS) - 2\int R dS = uu\int \frac{\pi r^2}{aa}$$

If we wish to extend this integration to a complete oscillation, it requires dividing it into two parts, the first from *M* until *A*, where the force of torsion accelerates the velocity *u*, while the force of retardation diminishes [the velocity]; the second from *A* unto *M*, [This should be M’] where all the forces together retard the motion.

EXAMPLE I. Suppose $R = \mu(A-S)^m$, we will have, for the state of movement in the first portion *MA*,

$$n(2AS-SS) + \frac{2\mu(A-S)^{m+1}}{m+1} - \frac{2\mu A^{m+1}}{m+1} = uu\int \frac{\pi r^2}{aa}$$

thus when the angle of twist will be null, or that *(A - S) = 0*, we will have

$$nA^2 - \frac{2\mu A^{m+1}}{m+1} = UU\int \frac{\pi r^2}{aa}$$

Let us consider now the other part of the movement from *A* to *M'*, and suppose the angle *ACm' = S'*, we will find, in calling *U* the velocity of point *A*;

$$\frac{nS'^2}{2} + \frac{\mu S'^{m+1}}{m+1} = \frac{UU - uu}{2}\int\frac{\pi r^2}{aa}$$

Substituting in place of U^2 its value [235]

$$\left(\frac{nA^2 - \frac{2\mu A^{m+1}}{m+1}}{\int\frac{\pi r^2}{aa}}\right)$$

we will have for the total integral, when the velocity becomes null, or when the oscillation will be completed,

$$(A - S') = \frac{2\mu}{n(m+1)}\frac{(A^{m+1} + S'^{m+1})}{A + S'}$$

and if the retarding forces are such that at each oscillation, the amplitude be a little bit reduced, we will have approximately for the value of *(A - S)*

$$(A - S') = \frac{2\mu A^m}{n(m+1)}$$

and if this quantity *(A - S')* be so small so that it can be treated as an ordinary differential, we would have then, for a number *Z* of oscillation,

$$\frac{2\mu}{n(m+1)}Z = \frac{1}{m-1}\left(\frac{1}{S^{m-1}} - \frac{1}{A^{m-1}}\right)$$

where *S* represents this that becomes *A* after a number of oscillations Z. Thus we will have

$$S = \frac{1}{\left[\frac{2\mu \cdot m-1}{n(m+1)}Z + \frac{1}{A^{m-1}}\right]\frac{1}{m-1}}$$

which determines the value S, after any number of oscillations Z.

EXAMPLE II. If

$$R = \mu(A - S)^m + \mu'(A - S)^{m'}$$

μ ' & m' have other values than μ & m, we will have, following the procedure of the last example

$$n(A - S) = \frac{2\mu}{(m+1)}\frac{(A^{m+1} + S^m + 1)}{A + S} + \frac{2\mu'}{(m'+1)}\frac{(A^{m'+1} + S^{m'} + 1)}{A + S}$$

& if the retarding force is much less than the force of torsion, we will have for the value approached, [236]

$$n(A-S) = 2\mu \frac{A^m}{m+1} + \frac{2\mu' A^{m'}}{m'+1}$$

In general, if

$$R = \mu(A-S)^m + \mu'(A-S)^{m'} + \mu''(A-S)^{m''} + \&c.$$

we will always have for an oscillation, in supposing R much smaller than the force of torsion,

$$n(A-S) = \frac{2\mu A^m}{m+1} + \frac{2\mu' A^{m'}}{m'+1} + \frac{2\mu'' A^{m''}}{m''+1} + \&c.$$

IX.

Experiments to determine the laws of the force of torsion.

Preparation

On a small, flat board *KA*, supported upon four feet, raise a post *ABD*: mount on the post *AB,* at four pieds high, the horizontal traverse DE, slid up and down on the post and fixed to it by means of a screw *E*; the cylinder or the weight *P*, carries at its top, along the prolongation of its axes, an end of a needle *b*, fixed to this cylinder. This needle is fixed by the lower part of a double clasp (pince) *a*, which is tightened by some screws; the upper part of this clasp holds the lower extremity of the filament of suspension; the lower part of this same clasp holds the extremity of the needle fixed to the cylinder. The top end of the filament of suspension is held by another clasp *g*, attached to the traverse *DE*. On the surface *AK*, which serves as a base for the apparatus, we place a circle divided into degrees, whose center *C* should be located along the prolongation of the axes of the cylinder: we attach at the bottom of the cylinder an index *eo*, whose extremity points to the divisions of the circle.

X.

Experiments on the torsion of filaments of iron. [237]

We have obtained three filaments of the clavichord (clavecin), such as one finds in commerce, rolled on bobbins, and numbered.

The no. 12 filament of iron supported, before breaking, 3 livres, 12 ounces; its six pieds of length weighed 5 grains.

The no.7 filament of iron supported, before breaking, a weight of 10 livres; its six pieds of length weighed 14 grains.

The no. 1filament of iron broke under a tension of 33 livres; its six pieds of length weighed 56 grains.

FIRST EXPERIMENT

Filament Of Iron, no. 12, The Cylinder Weighed A Half livre.

We have taken a cylinder of lead weighing a half livre, which we have suspended by the filament of iron no. 12; this cylinder had a diameter of 19 lignes and 6 1/2 lignes of height; the filament of suspension had a length of 9 lignes [pouces: This is a typographical error by the author]. We rotated the cylinder about its axes, without disturbing this axis from the vertical, and we obtained the following results:

First test. When we turned the cylinder about its axes through an angle smaller than 180 degrees, it made twenty, sensibly isochronous, oscillations in... 120”.

Second test. But in twisting three circles, the ten first oscillations have been of 2 to 3 seconds longer than the ten of the first test; and after the ten first oscillations, the amplitude of oscillations, which was at the start three circles, was reduced to five fourths of a circle.

SECOND EXPERIMENT

Filament Of Iron, No. 12, The Cylinder Weighed 2 livres.

First test. In suspending a cylinder weighing 2 livres, having the same diameter as the preceding but 26 lignes of height, from the same no. 12 filament of iron, we had, for an angle of torsion of 180 degrees or less, twenty oscillations sensibly isochronous in... 242”. [238]

THIRD EXPERIMENT

Filament Of Iron, n.^{o} 7, Cylinder Weighing One Half-livre

First test. In suspending the cylinder of a half-livre by the n.^{o} 7 string of iron, we obtained, for a torsion of 180 degrees or less, 200 oscillations sensibly isochronous in...... 42”

FOURTH EXPERIMENT

Filament of iron, n.o 7, cylinder weighing 2 livres.

Test. In suspending from the same filament a weight of 2 livres, the twenty oscillations were achieved in............85”

FIFTH EXPERIMENT

Filament Of Iron, n.^{o} 1, cylinder weighing a half-livre

Test. When we suspend a weight of a half-livre by this filament of iron of 9 pouces in length, its stiffness is so considerable that this weight is not sufficient to straighten it out; thus the oscillations are very irregular because they depend, not only on the angle of torsion, but also on the curvature that the filament of iron retains when uncoiled from the bobbin, even though it is stretched by the half-livre weight.

SIXTH EXPERIMENT

Filament Of Iron, n.^{o} 1, cylinder weighing 2 livres.

Test. But in suspending a weight of two livres from this filament of iron of 9 pouces in length, the filament is visibly straightened and one has, for an angle of torsion of 45 degrees or less, 20 oscillation sensibly isochronous in..........23”. [239]

Continuation of Experiments.

Filaments of brass (laiton).

Taking three filaments of brass, corresponding in number and approximately in thickness, to the three filaments of iron that were subject to experiment.

The *n.^{o} 12* filament of brass carries, at the moment of its rupture, 2 livres 3 ounces: its six pieds of length weighs 5 grains.

The *n.^{o} 7* filament of brass carries, at the moment of its rupture, 14 livres: its six pieds of length weighs 18 1/2 grains.

The *n.^{o} 1* filament of brass breaks under a tension of 22 livres: its six pieds of length weighs 66 grains.

SEVENTH EXPERIMENT

Brass filament no. 12, cylinder weighing one half livre.

Test. The length of the filament of suspension was 9 pouces, as in the preceding tests; we suspended a cylinder weighing a half livre from it and obtained, for an angle of twist of 360 degrees or less, twenty oscillations sensibly isochronous in...... 220”.

But with an initial angle of twist of three full circles, the first twenty oscillations took 225 seconds; and after these initial twenty oscillations, the angle of twist was still approximately two full circles.

EIGHTH EXPERIMENT

Brass filament n.o 12, cylinder weighing two livres.

Test. The filament of suspension being 9 pouces, and the cylinder weighing 2 livres, we obtained, for an angle of 360 degrees or less, twenty oscillations sensibly isochronous [240] in....442”.

With an initial angle of twist of three full circles, the first twenty oscillation took approximately 444 seconds, and the initial angle of twist was found to be reduced to two and one quarter full circles.

NINTH EXPERIMENT

Brass filament n.o 7, cylinder weighing one half livre.

Test. The length of the filament of suspension always being 9 pouces, the initial angle of twist being 360 degrees or less, one obtained twenty oscillations sensibly isochronous in....57”.

TENTH EXPERIMENT

Brass filament no.7, cylinder weighing two livres.

Test. The length of the filament of suspension again of 9 pouces, the initial angle o torsion being 360 degrees or less, one obtained twenty oscillations sensibly isochronous in........110”.

But the initial angle of twist being two full circles, it took 111 seconds for the first twenty oscillation and the initial angle of twist, originally two circumferences, was reduced to one and a half circumferences.

ELEVENTH EXPERIMENT

Brass filament n.o 1, cylinder weighing one half livre.

Test. Under a tension of one half-livre, the filament of suspension was not entirely straightened and the duration of the oscillation, depending in part on its initial curvature, is uncertain. [241]

TWELFTH EXPERIMENT

Brass filament n.o 1, cylinder weighing two livres.

Test. The length of the filament of suspension, being, as always, 9 pouces, the initial angle of torsion being 50 degrees or less, we obtained twenty oscillations sensibly isochronous in... 32”.

But the initial angle of twist being five-fourths of a circle, we observed the first twenty oscillations in 33 1/2 seconds; and at the end of these oscillations, the initial angle had been reduced to a quarter of a circle. [...reduced *by* a quarter of a circle?]

THIRTEENTH EXPERIMENT

Brass filament n.o 7, cylinder weighing two livres.

Test. The length of the filaments of suspension in all the preceding experiments being 9 pouces; needing to determine the force of torsion relative to the length of the filaments, we have given 36 pouces of length of suspension to this test and having had up to three circles of torsion or less, twenty oscillation sensibly isochronous in.......222”

XI.

Results of the preceding Experiments

The force of reaction in torsion of the filaments of metal ought to depend upon their length, their thickness, and their tension. In order to determine in general the law of this reaction, we have been obliged, in the preceding experiments, to suspend different weights from filaments of iron and brass, of different thicknesses and different lengths: Here are the results that these experiments present.

If we turn the cylinder about its axis, without disturbing this axis from the vertical, the [242]
filament twists: when we release the cylinder, the filament, by its force of reaction, will try to return to its natural situation; the cylinder will oscillate about this axis for a longer or shorter length of time accordingly as the elastic force is more or less perfect.

But we find, in all of the preceding experiments, that when the angle of twist is not very large (considerable), the period of oscillations is sensibly isochronous; thus we can regard as a first law, that for all the filaments of metal, when the angle of twist is not very great, the force of torsion is sensibly proportional to the angle of twist.

Having found from experiment that the force of reaction in torsion is proportional to the angle of twist, it follows that all the oscillatory formulae that we have given, *articles IV & following,* based upon the supposition that a force of torsion proportional to the angle of twist, or altered by a very small term, can be applied to these experiments.

Thus, as we have obtained, *article VII,* by means of the formula $T = (Ma^2/2n)^{1/2}\ 180$ degrees, and that in all the preceding experiments, the cylinders of a half-livre and of 2 livres having the same diameter, it follows that *n* ought to be always proportional to (M/T^2).

Thus, if the tension in the filament, varying in magnitude, has no influence on the force of torsion, then the quantity *n* for the same filament will be the same for the case of a tension of half livre and a tension of 2 livres, and consequently we will have *T* proportional to $M^{1/2}$. Let us compare our experiments made with the two weights, one a half-livre, the
other 2 livres, of which the roots makes as 1 is to 2. [sq. root of 1/2/2 = 1/2]. [243]

First experiment. The filament of iron, *n*.º 12, stretched by the half-livre weight, makes 20 oscillation in ......120”.

Second experiment. The same filament, stretched by a weight of 2 livres, makes 20 oscillation in ......242”.

Third experiment. The filament of iron, *n*.º 7, stretched by the half-livre weight, makes 20 oscillation in ......43”.

Fourth experiment. The filament of iron, *n*.º 7, stretched by a weight of 2 livres, makes 20 oscillation in ......85”.

The *fifth experiment* can’t be compared with the *sixth*.

Seventh experiment. The filament of brass, *n*.º 12, stretched by the half-livre weight, makes 20 oscillation in ......220”.

Eighth experiment. The filament of brass, *n*.º 12, stretched by the 2 livre weight, makes 20 oscillation in ......442”.

Ninth experiment. The filament of brass, *n*.º 7, loaded with the half-livre weight, makes 20 oscillations in ......57”.

Tenth experiment. The filament of brass, *n*.º 7, loaded with the 2 livre weight, makes 20 oscillations in ......110”.

The eleventh and the twelfth experiments can’t be compared.

It thus results from all of these experiments, that for the same filament of metal, a weight of two livres makes its oscillations in a time double of this of a weight of a half livre; consequently the period of oscillations is as the root of the weights; thus the tension, of varying magnitude, has no sensible influence on the force of reaction of torsion.

However, from many tests made with very great tensions relative to the force of the metal, it appears that the large tensions diminish or alter the force of torsion a small [244]
amount. One can see in fact, that as the tension increases, the filament elongates and its diameter diminishes, which ought to reduce the period of oscillation.

We have not been able to compare the filaments of iron or of brass *n.*º *1*, under the tensions of a half livre and of two livres because, as we have said in the details of the Experiments, the tension of one half-livre is not sufficient to straighten the filament.

XII.

On the force of torsion relative to the lengths of the filaments.

We have found, in the preceding article, that the variable tension in the filaments only influences the force of torsion in a negligible way. We seek now to determine, from these same experiments, how much, for equal angles of torsion, the length of the filament of suspension increases or diminishes this force. But it is clear that to the extent that one increases the length of the filament of metal, one can make, in the same proportion, a greater number of revolutions of the cylinder, without changing the degree of torsion; thus the force of reaction of torsion ought to be, for the same number of revolutions, inversely proportional to the length of the filament. Let us see if this reasoning is in accord with experience.

The formula, of *article VII*, gives us

$$T = (Ma^2/2n)^{1/2} \ . \ 180 \text{ degrees,}$$

or for the same weight T proportional to $1/(n)^{1/2}$. Thus, if n is inversely proportional to the length, as the theory claims, T will be as the roots of the lengths of the filaments of suspension; let us compare with experience.

We find, *tenth experiment*, that the filament of brass, *n.*^{o} *7*, of 9 pouces of length, being stretched by the weight of a half-livre, makes 20 oscillations in ..... 110”.

We find, thirteenth experiment, that the same filament of brass, *n.*^{o} *7*, of 36 pouces of length, stretched by the 2 livre weight, makes 20 oscillations in..... 222”. [245]

Thus the lengths of filaments make between them:: 1 : 4, while the time of oscillations of the filaments make:: 1 : 2; thus the test proves that the times of the same number of oscillation, make, for the same filaments stretched by the same weights, as the root of the length of these filaments, in accord with the claims of theory.

We have made many tests of the same kind as the preceding, which have all very exactly confirmed this law. We have not believed it necessary to fatten this Memoir with them.

XIII.

On the force of torsion relative to the thickness of the filaments.

We have determined the law of the force of torsion relative to the tension and to the length of the filaments; it remains for us only to determine them relative to the thickness of the same filaments.

We have, in the first six experiments, three filaments of iron of different thicknesses and of the same length; and in the following six experiments, three filaments of brass of the same length and of different thicknesses: but as we have the weights of one length of 6 pieds of each of these filaments, it is easy from them to fix the ratio of their diameters. Here is our reasoning and consequent prediction; the *moment* (momentum) of the reaction of torsion ought to increase, with the thickness of the filaments, in three ways. Take for example two filaments of the same material and the same length, where the diameter of one is double that of the other, it is clear that for the one whose diameter is double, there are four times more parts stretched by the torsion, than in those which have a simple diameter; and that the mean extension of all these parts will be proportional to the diameter of

the filament, just as the mean arm of the lever relative to the axis of rotation. Thus we are led to believe, from theory, that the force of torsion of two filaments of metal of the same material and of the same length but of different thickness, is proportional to the fourth [246]
power of their diameter, or for the same length, to the square of their weights. Let us compare this with the tests.

We take here only the tests where the tension is 2 livres, in order to compare all the *n.*^{os}, the filaments of *n.*^{o}*1* not being as exactly stretched by the weights of a half-livre: we have

Filaments of iron	*Second experiment*. The filament of iron, *n.*^{o} *12*, of 6 *pieds* of length, weighing 5 *grains*, gives 20 oscillations in..242"
	Fourth experiment. The filament of iron, *n.*^{o} *7*, of 6 *pieds* of length, weighing 14 *grains*, gives 20 oscillations in..85"
	Sixth experiment. The filament of iron, *n.*^{o} *1*, of 6 *pieds* of length, weighing 56 *grains*, gives 20 oscillations in..23"
Filaments of brass	*Eighth experiment*. The filament of brass, *n.*^{o} *12*, of 6 *pieds* of length, weighing 5 *grains*, gives 20 oscillations in..442"
	Tenth experiment. The filament of brass, *n.*^{o} *7*, of 6 *pieds* of length, weighing 18 1/2 *grains*, gives 20 oscillations in...110"
	Twelvth experiment. The filament of brass, *n.*^{o} *1*, of 6 *pieds* of length, weighing 66 *grains*, gives 20 oscillations in...32"

In order to determine, from these experiments, the law of reaction of the force of torsion, relative to the diameter of the filament of suspension, let us suppose that

$$T : T' : : D^m : D'^m : : \varphi^{\frac{m}{2}} : \varphi'^{\frac{m'}{2}}$$

where one supposes that *T & T'* represent the time of a certain number of oscillations for a filament of metal, whose diameter is *D & D'*, & the weight for the same length is ϕ & ϕ'; [247]
m being the power that one searches to determine. From this proportion, we deduce

$$m = \frac{2(\log T - \log T')}{\log \varphi - \log \varphi'}$$

with which it is necessary to compare with the experiment

The *second test*, compared with the *fourth*, gives................ $m = -1.82$

The *second test*, compared with the *sixth*, ... $m = -1.95$

The *eighth test*, compared with the *tenth*, ... $m = -2.04$

The *eighth test*, compared with the *twelfth*.......................... $m = -2.02$

From which it results that

$$T : T' : : \frac{1}{D^2} : \frac{1}{D'^2} : : \frac{1}{\varphi} : \frac{1}{\varphi'}$$

But the formula of oscillatory movement

$$T = \left(\frac{Ma^2}{2n}\right)^{\frac{1}{2}} 180 \text{ degrees,}$$

gives, in the preceding experiments, because of the equality of the tensile loading, n proportional to $\frac{1}{T^2}$; thus the force of torsion, for the filaments of the same nature, of the same length, but of different thicknesses, is as the fourth power of the diameter, thus as the theory had predicted.

XIV.

General results.

It results thus from all the preceding experiments, that the *moment* (momentum) of the force of torsion is, for filaments of the same metal, proportional to the angle of twist, the fourth power of the diameter, and inversely proportional to the length of the filament; so that if we let l be the length of the filament, D its diameter, B the angle of twist, we will have for the expression which represents the torque, $\mu BD^4/l$, where μ is a constant coefficient which depends on the natural stiffness (roideur) of each metal: this quantity μ, a constant for filaments of the same metal, can be easily determined from experiment, as we see [248]
in the following article.

XV.

Effective values of the quantities n & μ.

We have seen, in *article VII*, that $n = Pa^2/2\lambda$ where P is the weight of the cylinder, a its radius, λ the length of a pendulum which is isochronous with the oscillations of the cylinder produced by the force of torsion.

Let us apply this formula to the *second experiment*, where the filament of iron, *n.º 12*, is stretched by a 2 livre weight, which has a radius of 9 1/2 lignes, and makes 20 oscillations in 242”.

As the length of a pendulum which completes one full swing in one second at Paris is

440 1/2 lignes, the length of a pendulum, isochronous with the oscillations of the cylinder, will be 4401/2*(242/20)^2; thus

$$n = \frac{2liv\left(9\frac{1}{2}\right)^2}{2 \cdot 440\frac{1}{2} \cdot \left(\frac{242}{20}\right)^2} = \frac{1liv}{715}$$

thus the *moment n B* of the *n.*^{o} *12* filament of iron, 9 pouces in length, is equal to 1/715 livres, multiplied by the angle of torsion B, acting at the extremity of a lever of one ligne in length.

We have seen, that for the same metal, it follows from the theory and the tests of the preceding articles that the torque is inversely proportional to the length of the filament of suspension and proportional to the fourth power of the diameter. Thus it is easy to determine the value of the torque in a filament of iron, of any length and thickness; here is the calculation. [249]

Since a cubic pied of iron weighs approximately 540 livres, the *n.*^{o} *12* filament of iron, weighing 5 grains and 6 pieds in length, has a diameter very nearly equal to a fifteenth of a ligne; thus the *moment* of torsion of a filament of iron, of a fifteenth of a ligne in diameter, is equal to 1/715 livre acting at the extremity of a lever of one ligne in length, multiplied by the angle of twist.

XVI.

Comparison of the stiffness of torsion of two different metals.

We can easily deduce, from the preceding theory and experiments, the ratio of the stiffness in torsion of two different metals, for example, iron and yellow copper: we take the *n.*^{o} *12* filament of iron to compare with the *n.*^{o} *12* filament of brass.

In the *preceding article*, we calculated the quantity *n*, for the filament of iron, which we found = 1/715 livre, multiplied by a lever of one ligne. But as the filament of brass, loaded with a weight of 2 livres, makes 20 oscillation in 442”, we will have, by the same formula for the filament of brass,

$$n' = \frac{1liv\left(9\frac{1}{2}\right)^2}{440\frac{1}{2} \cdot \left(\frac{442}{20}\right)^2} \qquad \text{thus} \qquad \frac{n}{n'} = \left(\frac{442}{242}\right)^2 = 3.34$$

thus the stiffness of the filament of iron, *n.*^{o} *12*, is to the stiffness of the filament of brass, *n.*^{o} *12* approximately in the ratio 3 1/3 : 1.

But as there is little difference between the specific weight of iron and of copper, which according to M. Musschembroek, are in the ratio 77 : 83, we can suppose that the *n.*$^{\text{o}}$ *12* filament of iron and that of copper of the same number have approximately the same diameter; thus for filaments of iron and of copper of the same diameter, every thing otherwise equal, the stiffnesses in torsion are in the ration 3 1/3 : 1, that is to say that in twisting the filament of iron one circle, one would have the same torsional reaction, in twisting the filament of copper 3 1/3 circles. [250]

If one wishes subsequently to compare the stiffness of torsion with the force of cohesion, we note that our filament of iron carries, at the instant of its rupture, 60 onces, while that of copper only carries 35 onces; thus since they are approximately the same diameter, the ratio of their force of cohesion approaches 60 : 35, while their force of torsion is found to be (in the ratio) 3 1/3 :1.

This last result, however, ought to be regarded as a special case and not as a general result. We will see, in the second section of this Memoir, that the force of metals varies following the degree of cold-working and heat treatment (d'ecroussement & de recuit), and that all the experiments which we have carried out until now aimed at determining the force of metals can only be regarded as some particular cases.

But what this last observation seems to indicate, and what practice confirms, is that if one wishes to support a moving body on a pivot point, there is an advantage to using a pivot of steel or of iron to a pivot of copper, since under the same degree of pressure the iron bends much less than the copper; thus the circle of contact formed by the pivot point, pressed by the body that it supports, will be less for iron than for copper, this which, all else be otherwise equal, reduces the *moment* of friction that it is necessary to overcome in order to rotate a body about a pivot point: We will have occasion in the following to return to this article.

From some other experiments and by means of calculations similar to the preceding, we have found that a filament of silk, formed of several strands (brins) joined by boiling water and strong enough to carry up to 60 onces (in tension), has 18 to 20 times less torsional stiffness than the filament of iron which carries the same weight at its moment of rupture. [251]

XVII.

Use of the experiments and of the preceding theory.

Using the theory which precedes, and the experiments on which it is based, we are able to measure very small forces, with a precision that ordinary means can not supply: we present an example.

XVIII.

Balance to measure the friction of fluids against solids.

The formula that expresses the resistance of fluids against a body in motion, appears composed of several terms, some of which depend on the impact of the fluids against the body, and others which are due to the friction of the fluid: among the terms due to friction, there is one which depends on adhesion, and which is believed to be constant; but this term is so small, that confounded in the experiments with the other quantities which depend on impact, it is very difficult to evaluate: one can see in the experiments that M. Newton has made in order to discover this constant quantity. *(Livre II des Principes mathematiques de la Philosophie naturelle, Scholie du vingt-cinquieme theoreme.)*

The force of torsion provides an easy means to determine the (friction due to) adhesion from experiment.

In a vase *ADBE, fig. 3*, filled with fluid of which one wishes to determine the adhesion, one suspends, by means of a filament of copper, a cylinder *abcd,* of copper or of lead; one places above the vase a circle *A'FB'*, divided in degrees; the circle is located at the level of the end *d* of an index *id* attached to the cylinder.

When one turns the cylinder about its vertical axis, without disturbing it from its verticality, one can observe, by means of the small indices, how much each oscillation is altered; and as the force of torsion of the filament which produces these oscillations, is known from the preceding experiments; thus one knows the alteration due to the imperfection of elasticity, in making the cylinder oscillate in the void or even in the air; one can expect, by this means, to find the constant quantity due to adhesion.

Example & Experiment.

We have suspended the cylinder of lead weighing two *livres*, which we used in the preceding experiments, from a filament of copper, *n.^{o} 12*, of twenty-nine *lignes* in length, in a vase filled with water: The circle *AB*, on which we observed the oscillations, had a diameter of forty-four *lignes*; we waited, before beginning our observations, until the amplitudes of oscillations diminished to the point at which the extremity *d* of the index only traveled an arc of one and one half *ligne* on the circle, corresponding to approximately $3^{d}55'$; & observing the displacement of the index through a convex [magnifying] lens (loupe), we have distinctly counted fourteen oscillations before the movement ceased.

Results of this Experiment.

If the successive diminution of each oscillation is supposed constant, & can be entirely attributed to the adhesion of the fluid to the surface of the lead cylinder, one will have, [from] *art. VIII*,

$$(A - S') = \left(\frac{2\mu}{n}\right)$$

where *(A - S')* is the diminution in each oscillation, *n(A - S')* the *moment (momentum)* of the force due to torsion, & μ the *moment* of the retarding force due to adhesion. [253]

But as, after observing the oscillations, the arc travelled diminishes one and a half *lignes* in fourteen oscillations, and given that the radius of the circle on which we observe this reduction is twenty-two *lignes*; in supposing this diminution constant, we obtain that the angle *(A-S)* by which the amplitude diminishes each oscillation $= \dfrac{3}{2 \cdot 22 \cdot 14}$.

But we found, *art. XVI*, that for a filament of brass of nine *pouces* in length, *n.º 12*,

$$n = \frac{1 \text{ livre.}\left(9\frac{1}{2}\right)^2}{440\frac{1}{2} \cdot \left(\frac{442}{20}\right)^2};$$

and as we have also found that the forces of torsion are proportional to the length of the filaments of suspension, one will have for our filament of twenty-nine *pouces* in length.

$$\mu = \frac{1}{3,155,000} \text{livre x 1 ligne,}$$

which is to say that the moment of the constant retarding force, μ, is approximately equal to three millionths of a *livre* suspended at a lever arm of one *ligne*: a quantity which would have been impossible to measure by any other means than this that we have come to employ.

In order to now deduce the value of the adhesion from this experiment, it is necessary to note that the height of the cylinder of lead, submerged in the water in the vase, is twenty-four *lignes*, and that the diameter of this cylinder is nineteen *lignes*. Thus, in taking 22/7 for the ratio of the circumference to the diameter, the surface of the submerged cylinder, is equal to $\frac{22}{7} \cdot 19 \cdot 24$; and as the movement is about the axis of the cylinder, whose radius is 9 1/2 lignes, if δ is the adhesion, the *moment* of the adhesion about the axis of rotation, will be . $\delta \frac{22}{7}(19)^2 \cdot 12$ It is then necessary to add to this quantity the *moment* of the adhesion of the circle which forms the base of the cylinder submerged in the water, of which the *moment* is [254]

$$= \delta \frac{22}{7} 19^1 \cdot \frac{19^1}{4} \cdot \frac{2}{3} \frac{19}{2}$$

so that the total *moment* of the resistance of the fluid against the cylinder will be

$$\delta \frac{22}{7}(19)^2 \cdot \left(12 + \frac{19}{12}\right) = \delta \frac{22}{7}(19)^2 \cdot \left(\frac{163}{12}\right)$$

But the experiment has shown us that this same *moment*

$$= \frac{1 \text{ livre}}{3155000} \cdot 1 \text{ ligne for a square pouce; thus}$$

$$\delta = \frac{1 \text{ livre}}{3155000} \cdot \frac{7 \cdot 12}{22 \cdot 163 \cdot (19)^2}$$

and for a square *pied* the adhesion will be

$$\delta\,(144)^2 = \frac{1 \text{ livre}}{2345000}$$

so that the constant resistance due to the adhesion of the water for a surface of 255 *pieds*, can not be more than a *grain*; thus there are few cases where this constant alteration, if it takes place, can not be neglected in the evaluation of the friction of water. We have not made any tests on other fluids.

In giving the cylinder oscillations of two or three full circles of amplitude, and comparing the successive diminutions of amplitudes of oscillations with the formulas of changing oscillatory movement, I have believed to have seen that for very small velocities, the friction goes as the velocity, and for large velocities, as the square; but these experiments [255]
require special attention and ought to be made in different fluids.

XIX.

Since the reading of this Memoir, I have constructed, according to the theory of the reaction of torsion that I have put forward, an electric balance and a magnetic balance; but as these two instruments, as well as the results bearing on the electric and magnetic laws that they have given, will be described in the volumes following our Memoirs, I believe it suffices here to simply announce them.

XX.

SECOND SECTION

On the alteration of the elastic force in the torsion of filaments of metal. Theory of the coherence and of elasticity.

When one torques the filaments of iron or of brass, stretched, as in the preceding experiments, by a weight, one observes two things; if the angle of torsion is not so great, relative to the length of the filament of suspension, at the moment when one releases the weight, it returns approximately to the position that it had before twisting, that is to say, the filament of suspension untwists completely by the quantity by which it had been torqued; but if the angle of torsion given the suspending filament, is very large, then the filament only unwinds a certain amount, & the center of the reaction of torsion will advance the whole quantity by which the filament failed to unwind. It follows from these two considerations, that two suites of experiments are required; the first to determine, by the diminution of oscillations, how much the elastic force of torsion is altered in oscillatory movement under conditions in which the center of reaction of torsion is not displaced; the second to determine the displacement of this center of reaction, when the angle of torsion is sufficiently large for this displacement to take place. [256]

X X I.

FIRST EXPERIMENT

Filament of iron, n.^{o} 1, length, six pouces six lignes

We have taken a filament of iron of six *pouces* six *lignes* in length, that has been loaded with a weight of two *livres*, the same as has served us in the experiments in the preceding section. In turning the cylinder about its axis in order to twist the filament of suspension, we have sought to determine how many degrees the amplitude diminishes with each oscillation, & we have found:

First test, angle of torsion	90^d	loss of 10^d in	3 1/2 oscillat.
Second test	45	....	10 1/2
Third test	22 1/2	....	23
Fourth test	11 1/4	....	46

Remarks on this Experiment

The reductions in amplitudes of oscillations have been very uncertain (irregular), when the initial angle of torsion was more than 90 degrees; we have even observed that in this case, in twisting the cylinder about its axis, it did not return to its initial position, and the respective position of the constitutive parts of the filament have been altered, and and consequently, its center of reaction of torsion has remained displaced: here is what the experiments gave for this displacement.

X X I I

Follow on to the first Experiment.

In this part of the first experiment, we have searched to determine the displacement of the center of torsion, due to the degree of torsion that we have given to the filament of suspension. [257]

First test, in twisting the filament	1/2 C. {the index or the center of torsion has been displaced .....}	.... 8^d
Second test	1 C	50
Third test	2 C	310
Fourth test	3 C	....1C + 300
Fifth test	4 C	....2C + 290
Sixth test	5 C	....3C + 280
Seventh test	6 C	....4C + 260
Eighth test	10 C	....8C + 240

Ninth test. Having wished to continue to twist the filament some 15 new circles, always in the same sense, it broke at the fourteenth. After this experiment, this filament was straight and very rigid, it had separated along its length into two parts; examining it with a magnifying glass, this separation was very evident and it had exactly the shape of a cord formed of two (torons) helices.

X X I I I

Remarks concerning this Experiment.

This first experiment and its sequel appears to show that below 45 degrees, the alterations made are approximately proportional to the amplitudes of the angles of twist, as one sees from the second, third and fourth tests of the first experiment; that above 45^{d}, the alterations augment in a ratio much greater; that the center of reaction twist only begins to displace when the angle of torsion is approximately a half circumference; that this displacement increases as the torsion of the filament increases; that it is very irregular up to 1 circle 10 degrees; and that, passing this level of torsion, the reaction of torsion remains [258]
approximately the same for all the angles of twist: Thus, for example, in the fourth test, in twisting the filament three circles, the center of reaction of torsion displaces one circle + 300 degrees, so the reaction of torsion has only led the cylinder back one circle 60 degrees. In the seventh test, we see that after having already experience in the previous tests a [total] displacement of more than eight circles, that six new circles of torsion displace the center of reaction of torsion by 4C + 260 degrees, so that for more than fourteen circles of torsion, the reaction of torsion is still only one circle plus 100 degrees; thus it only differs by a tenth from the reaction of torsion for the fourth test which gave us one circle + 60 degrees: the experiments which follow clarify this remark.

X X I V

Second Experiment

Filament of iron, n.º 7, length, 6 pouces 6 lignes.

We have searched, in the first part of this experiment, how much the amplitudes of oscillations diminish at each oscillation, when the center of torsion is not yet displaced.

First test, angle of torsion	180^{d}	loss of 10^{d} in	3 1/2 oscillat.
Second test	90		12
Third test	45		27
Fourth test	22 1/2		54

Follow on to this second Experiment.

In this second part of the same experiment, we have sought the diplacement of the center of torsion. [259]

First test, in twisting the filament	3 C. {the index or the center of torsion has been displaced .....}	300^{d}
Second test	4 C	1C + 180
Third test	6 C	3 + 90
Fourth test	8 C	5 + 90
Fifth test	12 C	9 + 40
Sixth test	20 C	16 + 310
Seventh test	30 C	26 + 180
Eighth test	50 C	46 + 20
Ninth test	at seventeenth circle of torsion, the filament broke.	

X X V.

THIRD EXPERIMENT

Filament of iron, n.^{o} 12, length, 6 pouces 6 lignes.

The first part of this experiment has been made in accord with the first part of the two preceeding experiments.

First test, angle of torsion	360^{d}	loss of 10^{d} in	1 oscillat.
Second test	180	10	2
Third test	90	10	5
Fourth test	45	10	11
Fifth test	22 2/2	10	25

Follow on to this third Experiment.

Displacement of the center of torsion.

First test, in twisting the filament 4 C. {the index or the center of torsion has been displaced .....} .... 300^{d}

Second test ....6 C ....2C + 40 [260]

Third test after six other turns the filament broke.

X X V I

The preceding experiments have been continued with the filaments of brass used in the experiments of the first section.

FOURTH EXPERIMENT

Filament of brass, n.^{o} 1, length, 6 pouces 6 lignes.

First test, in twisting	189^{d}	loss of 12^{d} in	2 oscillat.
Second test	90	10	6
Third test	45	10	16
Fourth test	22 2/2	10	40
Fifth test	11 3/4	10	80

Follow on to the fourth Experiment

Displacement of the center of torsion.

First test, in twisting the filament	2 C.	{the index or the center of torsion has been displaced ....}	160^{d}
Second test	4 C		2C + 0
Third test	6 C		3 + 300
Fourth test	10 C		7 + 300
Fifth test	20 C		17 + 340
Sixth test	at the twenty-eighth circle of torsion the filament broke.		

FIFTH EXPERIMENT

Filament of brass, n.^{o} 7, length, 6 pouces 6 lignes.

First test, in twisting	360^d	loss of	10^d	in	2 1/2oscillat.
Second test	180		10		6
Third test	90		10		13
Fourth test	45		10		31
Fifth test	22 1/2		10		72

Follow on to the fifth Experiment [261]

Displacement of the center of torsion.

In twisting the filament four circles, the center is displaced 220 degrees; but in wishing to torque it six circles, the filament broke.

XXVII

In the filament employed in this last experiment, the torsion altered the oscillations, and hence the elastic force, less than in all the other experiments; it is this which occasions the great number of oscillations before the oscillatory movement dies out; it is this likewise which results in the sudden rupture of this filament, without being able to displace its center of reaction one circle. I have found in general that the filaments of brass, those available in commerce, between the n.^{os} 5 & 8, were those whose elasticity in torsion was the least imperfect: in comparing the filaments of iron & of brass with the *same numbers*, we have similarly found that the filaments of brass have an amplitude of elasticity much more extensive than the filaments of iron.

For the rest, the experiment presents many irregularities in the results: two bobbins of the same filament & and of the same number, do not always give the same displacement for the same angle of torsion, this which can only be attributed to the way in which the filaments are manufactured - to the more or less great pressure that they experience in passing under the (levre de la filiere), to the heat treatment given them in order to successively reduce the diameter from one number to the next, from large to small.

XXVIII

First Remark

Despite the uncertainty which reigns in the experiments of oscillations for the amplitudes of the (etendues), it appears that below certain limits, these alterations are approximately proportional to the amplitude of oscillation, as we have announced in the remarks on the first experiment, & as all the other experiments confirm. The resistance of the air can only alter the amplitude of oscillations very little in our experiments. I am assured of this by the following. The weight of two *livres*, which has served us in the experiments of this section, was 26 lignes in height & 19 lignes in diameter. I have formed with a very light paper, a cylindrical surface of the same diameter as this weight, but which with 70 lignes of height: I put a part of the cylinder of lead into my envelope of paper, & formed thus a cylinder of 78 lignes of height, or three times longer than the first, which should [262] have tripled, in the oscillatory movement, the alterations due to the air resistance; but I have never found that these alterations were a tenth more considerable in the second case as in the first; most often they are equal; thus the resistance of air enters into our experiments only as quantities that one can neglect.

X X I X

Second Remark

In order to make a torsion balance, it is always necessary to choose the filaments which have the least imperfect elasticity; the filaments of brass are much more preferable to those of iron: the choice of the thickness depends on the forces which one wishes to measure. I have a magnetic balance which will be described in our Memoirs, where I alternatively made use of a filament of brass of 3 pieds in length, no.s 12 & 7; the elastic force of torsion is such that in twisting the filaments eight circles, over the course of thirty hours, there is not one degree of alteration or displacement in the center of torsion.

X X X [263]

Third Remark

In all the filaments of metal, the behavior is elastic only up to a certain point: The isochrony of the oscillations teaches us that in the first degrees of torsion, the elastic force is almost perfect; but beyond the angle of torsion which serves, for thus to say, as a measure of the elastic force, the center of reaction of torsion displaces nearly the whole of all the angle of torsion which exceeds this of the elastic reaction. However, as one can note in the preceding experiments, the amplitude of the elastic reaction is not a constant quantity for all angles of twist, it increases as the torsion increases; the less the initial elasticity, in the filament subject to test, to the extent, the more this increases. (?) A filament of brass, no. 1, of 6 and one-half pouces in length, made red in a fire, in order to make it loose, by heat treatment the greatest part of its elasticity, only gives, after this operation, for the first circle of torsion, 50 degrees of reaction of elasticity; but it has acquired, after 90 circles of torsion, an elastic extension of nearly 500 degrees in this interval; From the 2nd to 3rd circle of torsion, the reaction of elasticity increases 12 degrees; from the 40 to 41st circle of torsion, the same reaction increases 6 degrees; and from 90 to 91st circle of torsion, almost a degree, such that the increase of the elastic reaction, after the center of reaction has been displaced a certain angle, is nearly inversely proportional to the angle of displacement. It is necessary to point out that after these 90 circles of torsion, I wished to twist the same filament another 50 circles, but it broke at the 49th, so this filament, before breaking, could [264]
be twisted to 140 circles. If we compare this result with that which followed from the first experiment, where the same filament, no. 1, had not been heat treated, we found that after 25 circles of torsion, the reaction of elasticity was 480 degrees and that in twisting 15 new circles, the filament fractured; this last filament can thus only take, without breaking, 40 circles of torsion. In following in this experiment the path of the elastic reaction, we deduce from it that at the point of rupture, this reaction is almost equal to that of the heat treated filament in the same point of rupture; from which it would appear that one is justified in concluding that by torsion alone one can give to a heat treated filament all the elasticity of which it is susceptible [?] and that the (ecrouissement) plastic deformation adds

nothing more to it; such that reciprocally, if in passing it through the oriface [filter, narrowing hole] or by any other means, we have been able to give to our filament of brass a cold working such that its elastic reaction had been 520 degrees, which appears to me to be this of our two filaments at the moment of rupture, for then the elastic reaction had been carried to its maximum by this first operation: There would not have been any more possible displacement in the center of the reaction of torsion; but all the time that we would have made to test this filament to a torsion of more than 520 degrees, it would break.

X XX I

Fourth Remark

From the preceding experiments, this, it appears, is how we can explain the elasticity and coherence of metals. The integral parts of the filament of iron or of brass, or of any metal, have an elasticity that one can regard as perfect, that is to say, that the forces necessary to compress or dilate these integral parts are proportional to the dilatations or compressions they experience; but they are only tied together by the coherence, a constant quantity and absolutely different from the elasticity. In the first stages of torsion, the integral parts change their shape (figure), elongating or compressing, without the points by where they adhere together changing position because the force required to produces these first stages of torsion is considerably less than the force of adhesion; but when the angle of torsion becomes such, that the force with which these parts are compressed or dilated is equal to the coherence which unites these integral parts, then they ought to separate or slide one on the other. This sliding of parts takes place in all ductile bodies but if by this sliding of parts, the ones on the others, the bodies compress, the extent of the points of contact increases and the extent of the domain of elasticity becomes greater. However as these integral parts have a determined figure, the extent of the points of contact can only [265]
increase up to a certain degree, beyond which the body breaks; it is this which explains the detailed facts of the preceding article. This which proves again that it is necessary to distinguish the cause of elasticity from the adhesion is that one can vary the coherence at will by the degree of heat treatment without altering in any way the elasticity. It is thus the case when I heated to white my no. 1 filament of copper in the preceding experiments, it lost a great part of its force of coherence: before heat treatment, it could carry up to the point of rupture 22 (livres) and after the heat treatment it only carried 12 to 14 (livres); but while the adhesion was diminished nearly by half by the heat treatment and the amplitude of elasticity was nearly diminished in the same proportion, however in all the extent (etendue) of the elastic reaction which (refloit?) to the heat treated filament, the elasticity was the same, at equal angle of torsion, as in the same filament not heat treated, since in suspending to one and the others the same weights, the time of the same number of oscillations was exactly equal in the two cases.

X X X I I

An equally interesting effect due to the deformation (rapprochement) of parts in tor- [266]
sion of filaments of metals is this which takes place when one twists a filament of iron, which by this operation alone acquires through the rapprochement of parts, the quality of taking the magnetism to a higher degree than it had before. Here is the experiment which revealed this to me; I have taken a filament of iron, such as one finds them throughout the

world of commerce, of the thickness of those which serve for the small sounding bars (sonnettes); a length of six pouces, weighing 57 grains; this filament of six pouces, magnetized (aimante) and suspended horizontally by a filament of silk, untwisted (detordu) and very fine, makes an oscillation in 18 seconds: this same filament of six pouces in length, twisted up to the point of rupture and magnetized as in the first case to saturation by the method of double touching, makes an oscillation in 6 seconds; such that the moment of the directive force for the two needles equal and similar, being as the interval of the square of the times for the same number of oscillations, the magnetic moment of the twisted needle, being nine times more considerable than that of the needle not twisted: I will have the occasion to return to this article in another Memoir.

X X X I I I

To confirm all the preceding theory regarding the coherence and elasticity, I have made the following experiments.

We have fixed, *fig. 4*, by means of a C clamp (agrafe) *CD* with a vise *V*, a lamina of steel *AB* on the edged of a very solid table; this beam being pressed and held (serree) in its part Aa, between two plates of iron *E* and *F* by the vise *V*: this lamina was 11 lignes long and one-half ligne thick form the point a to the point *B* where was suspended the weight *P*, there was seven pouces of length: we measured on the vertical rule rg, how much the [267] weight *P* made lower the lamina *AB* at its extremity *B*. Here are the details of the results which took place following the different weights with which the lamina was loaded.

We had made red the lamina to white (?) and we have given it a quenching (trempe tresroide); then we have attached at B at seven pouces from point a, different weights. The extremity B has deflected,

With a weight of a half-livre some .... 8 lignes
With a weight of one livre some 15 1/2
With a weight of one livre and a half some ....23 +

We have taken this same lamina and we have heated it until it took on a violet color and it returned to the consistency of an excellent sprint; and we have found equally, that in loading it as in the first case, the extremity B has deflected,

With 1/2 livre de .... 8 lignes
With 1 livre .... 15 1/2 +
With 1 1/2 livre ....23 +

Finally we have made red this same lamina to white and let it cool (refroidir) very slowly; and we have had, in loading the extremity B, exactly te same results as in the two preceding experiments.

It appears to us that these three experiments prove in an incontestable manner, that whatever the state of the lamina, the first degrees of its elastic force are in no way altered; since in taking account of the lever arm, which diminishes in the same(?) measure as the lamina is loaded, the same weights deflect it in the three states equally and proportionally to the load; which when one removes (otoit) the weights, it retakes exactly it original horizontal position.

I have wished to see subsequently what be the force of this lamina in these three different states; and in the case where the center of flexure would begin to displace, what would [268] be the degree of flexure where the lamina would begin to be deformed without returning to its original position. Here is the result of this experiment.

I have cut from a sheet (planche de tole) of English steel, three lamina exactly similar to this of the preceding experiment: one of these lamina have been quenched (tres-roide), the second had been returned to the consistency of an excellent spring, and the third had been heat treated to white and slowly cooled. I attached, fig. 4, a weight d at 2 pouces and 1/2 distant from the point a and I had carefully exerted a pull always perpendicular to the direction of the lamina. Here is what I observed.

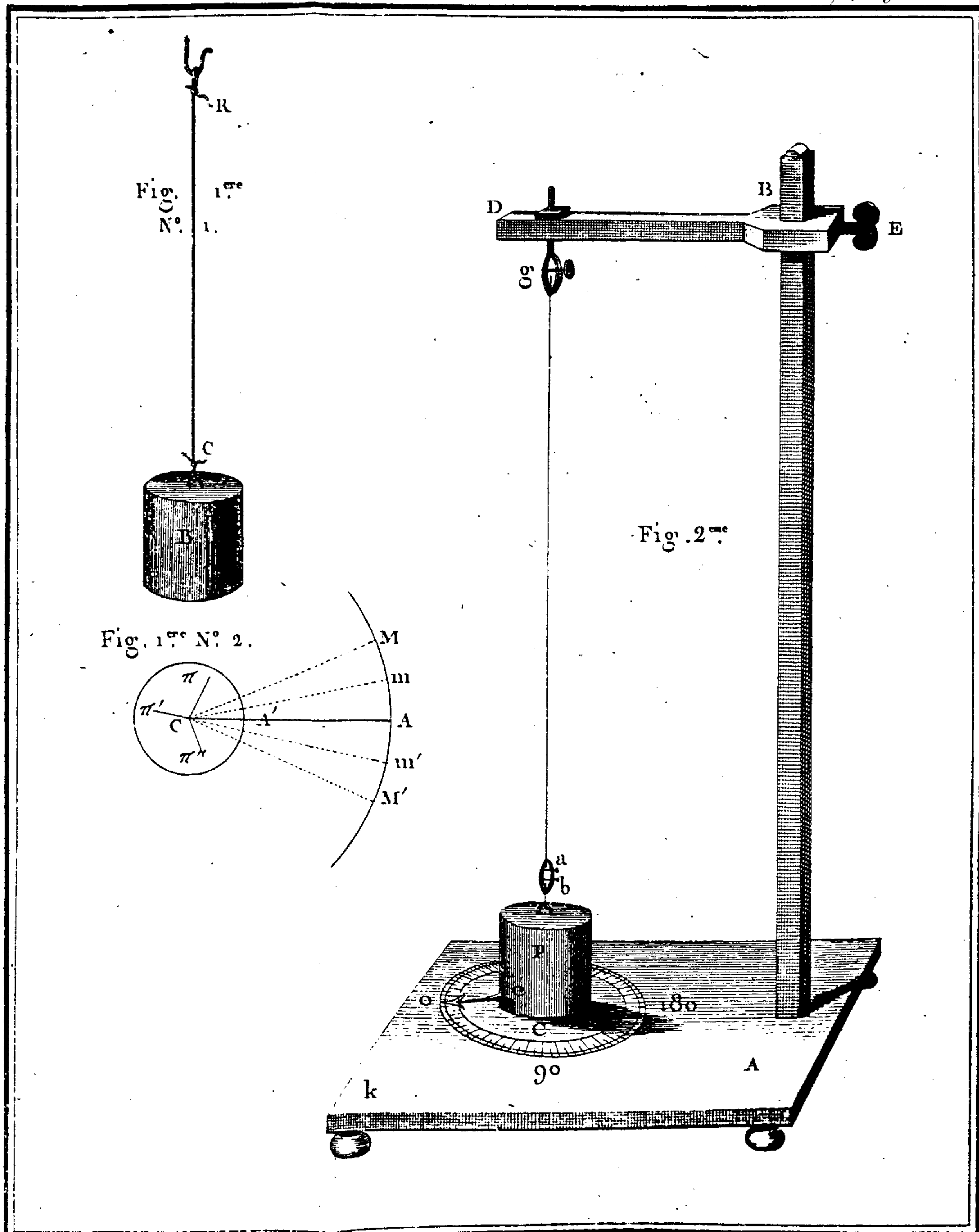
The lamina which had been rapidly quenched broke under a pull of six livres; but under whatever angle what it was deflected below this of rupture it returned exactly to its original position. The lamina returned to a violet color, forming an excellent spring, broke only under a pull of eighteen livres; it bent (se plioit) up to the point of rupture, with an angle nearly proportional to the angle of torsion(?), and under any angle that it was bent before rupture, when we freed it (la lachoit) it retook its original position. The lamina heat treated to white and slowly cooled, bent (se plioit) up to a pull of five to six livres, proportionally to this force of pull, and with an angle absolutely equal under the same force that in the state of quenched and of spring; but in pulling always subsequently perpendicular to the direction of the lamina, in order to conserve the same lever, with a force of seven livres, we have bent it under all the angles, without that it was necessary to augment this force: in letting go, it raised itself back up only by the quantity of which it had been (primitiveness) originally deflected by a pull of six livres; such that the angle of reaction of flexure, found itself changed from all the angle which we had bent it with a force greater than seven [269] livres.

These last experiment lead us back to the same results as those which went before. It is clear that in order to have an idea of what happens in the flexure of metals, it is necessary to distinguish the elastic force of the integral parts from the force of adhesion which ties these parts together: The elastic force depends, as we have already said, on the compression or dilation that the integral parts experience and is always proportional to the tractions. These integral parts are not altered, neither by the quenching nor by the heating, since we see that in theses different states, the elasticity is the same under the same degrees of flexure; but these integral parts, are only tied among themselves by a certain degree of adhesion which probably depends on their shape and on the respective portion of the different fluids with which their pores are filled, this which varies according to the quenching and the heating. In the quenched (trempe roide) steel and in the good springs the integral molecules can neither slide one on the other nor experience the least displacement without the body breaking; but in the ductile bodies, in the heat treated metals, these parts can slide one on the other and displace themselves, without the adhesion being sensibly altered.

This that we have come to explain for metals appears to be able to be applied to all bodies; their parts are always of a perfect elasticity, but the bodies made hard, mous or fluids, follow the adhesion of the integral parts. If in the hard bodies, they can slide one upon the other, without their distance being sensibly altered, the body will be ductile or malleable;

but if they can not slide one on the other, without their respective distances being sensibly altered, the bodies break when the force with which the bodies will be pulled or compressed, will be equal to the adhesion.

[*fin*]



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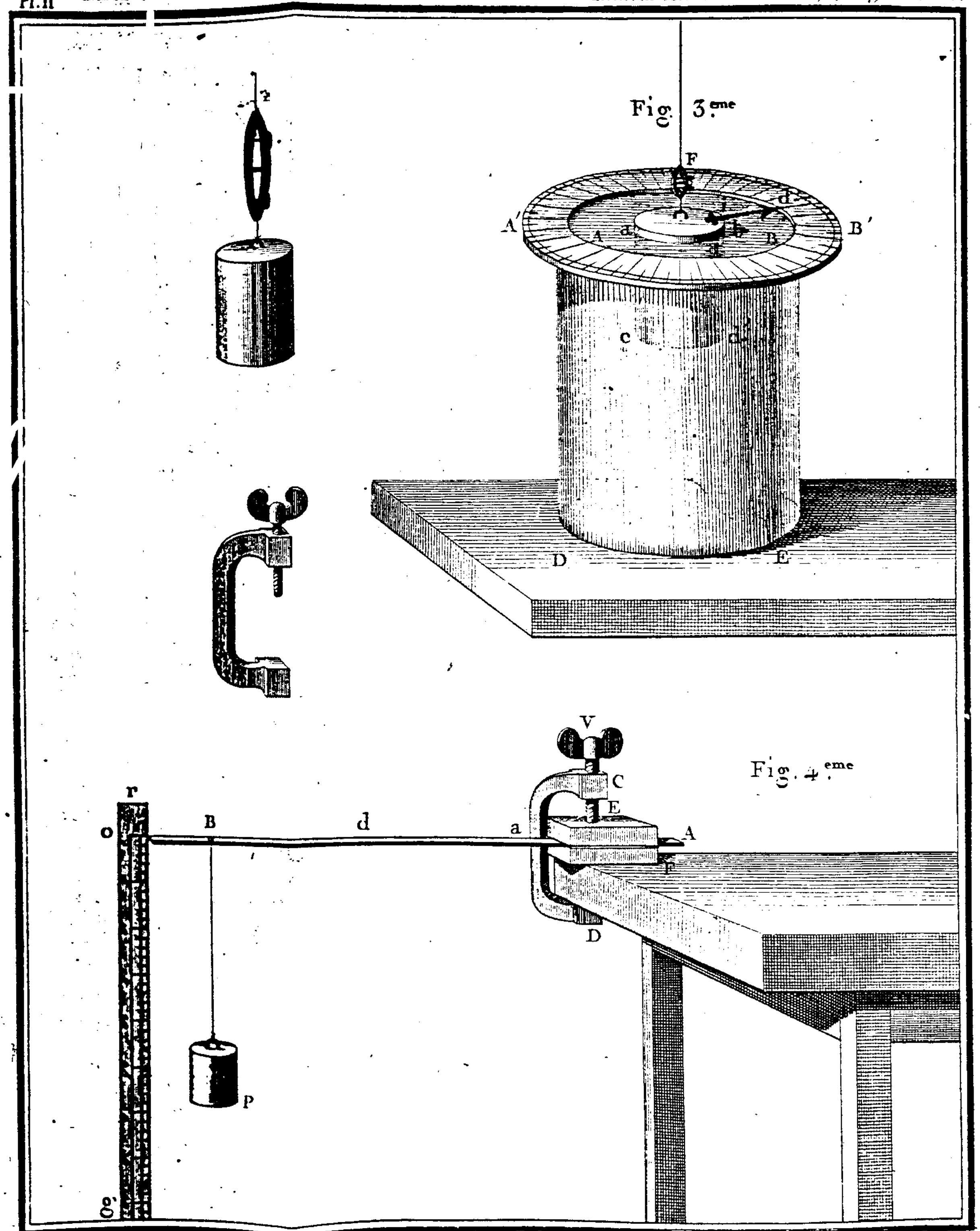
Y. le Gouaz sculp.



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OUTLINE:

COULOMB'S THEORETICAL & EXPERIMENTAL RESEARCH

On the force of torsion, and on the elasticity of metal wires:

Louis Bucciarelli MIT

I. Objectives

... to determine the elastic force of torsion of filaments of iron and brass as a function of their length, their thickness, and their degree of tension.

.... to evaluate the inelastic behavior of filaments of metals.. and consequences about the lows of coherence and elasticity of bodies.

II. Method *... to determine the force of torsion*.

III. Method ... to determine effect of dissipative forces.

IV. Contents *... two sections*

V. *First Section*

Formulas of oscillatory movement. for the period of oscillation.

$$T = \frac{\sqrt{I}}{\sqrt{(n)}} \cdot \pi = \int\left(\frac{\pi r^2}{n}\right)^{1/2} \cdot 180^d$$

VI. ...compare the force of torsion with the force of gravity in a pendulum.

Coulomb writes the period for a pendulum of length λ, $T = \left(\frac{\lambda}{g}\right)^{1/2} 180^d$

VII. Setting the periods equal, and noting that weight, P, is mass* g, he obtains *a very simple formula for determining n from the experiment.* (n is the torsional stiffness of the wire). namely $n = \frac{Pa^2}{2\lambda}$ where a is the radius of the cylinder with moment of inertia I.

VIII. The Effect of dissipative forces on the oscillatory movement.

IX. The Experiments. Preparation

X. Experiments on the torsion of filaments of iron (and brass).

XI. Results of the preceding Experiments

XII. On the force of torsion relative to the lengths of the filaments.

XIII. On the force of torsion relative to the thickness of the filaments.

XIV General results. $\mu BD^4/l$ where B is the angle of twist, so $n = \mu D^4/l$

NOTES ON:

COULOMB'S THEORETICAL & EXPERIMENTAL RESEARCH

On the force of torsion, and on the elasticity of metal wires:

Louis L Bucciarelli, MIT

In the first section of his memoir, Coulomb presents the *Formulas of oscillatory movement, in supposing the reaction of the force of torsion proportional to the angle of twist,...* for solid circular cylinder, suspended by a thin wire along its axis.

The cylinder is turned about its axis, then let free to oscillate. Coulomb *by the principles of Dynamics* writes the equation of motion as

$$n(A - S) \cdot dt = \int \pi r dv$$

The product *n(A-S)* on the left had side is the torque acting on the cylinder due to the twisting of the wire. The term *(A- S)* is the cylinder's (and wire's) angular rotation from the undisturbed, original configuration. The term *n* is what we would today call the torsional stiffness of the wire, *a constant coefficient whose value will depend on the nature of the filament of metal, on its length and on its thickness*.

The integral on the right hand side is the increment in the total angular momentum, or moment of momentum, of the cylinder (per unit time). In more modern notation we would begin with

$$\tau_{ext} = \frac{d\boldsymbol{L}}{dt}$$

where $\boldsymbol{L}$ is the angular momentum and τ_{ext} is the externally applied torque, then say that $\boldsymbol{L}$ is given by

$$\boldsymbol{L} = \int_{volume} m_i \cdot \boldsymbol{r}_i \times \boldsymbol{v}_i$$

In this we, as did Coulomb, consider the cylinder as a collection of (differential) mass points, a rigid body (the relative distances between any two points does not change) where each point is located at a radial distance $\boldsymbol{r_i}$ from the axis of rotation and has velocity $\boldsymbol{v}_{\text{i}}$, in a direction perpendicular to $\boldsymbol{r_i}$. (Bold face indicates a vector). The integral extends over all the mass points within the volume of the cylinder.

Since each mass point moves in (non-uniform) circular motion, the magnitude of the vector cross product will just be the product of the scalars r_{i} and v_{i} and its direction will lie along the axis of rotation, perpendicular to the plane of the two vectors $\boldsymbol{r_i}$ and $\boldsymbol{v}_{\text{i}}$, i.e., in the same direction as the applied torque. That is, we can write

$$\boldsymbol{L} = \int_{volume} \pi r v \cdot (\textit{unit vector in the direction of torque})$$

where I have replaced the (differential) element of mass with Coulomb's symbol pi.

Finally, since the contours of the cylinder do not change with time, nor does *r*, the position of each mass point relative to the axis of rotation, we can interchange the differentiation with respect to time and the integration over the volume, and write, noting that the velocity *does* change with time

$$\tau_{ext} dt = d\mathbf{L} \qquad \Leftrightarrow \qquad n(A-S) \cdot dt = \int \pi r dv$$

Coulomb then expresses the velocity at any point within the cylinder in terms of the velocity of a point on the circumference of the cylinder, *u*. Designating the radius of the cylinder as *a*, he writes

$$v = \frac{ru}{a}; \qquad \text{so} \qquad n(A-S) \cdot dt = du \int \frac{\pi r^2}{a};$$

Note that the differential of the velocity at the circumference of the cylinder, which is a differential change with respect to time, can be pulled out from under the integral since it is not a function of position within the cylinder. (If this strikes you as weird and suspect, recast these last few steps in more modern notation to convince yourself of their legitimacy. I would start by writing π, the mass element, as a differential element of volume).

Then Coulomb performs a trick you no doubt have seen in the past but might not recognize as such. Our author changes the independent variable from time, *t*, to angular position, *S*. and so can carry out the integration. In his words: *and as* $dt = \frac{a dS}{u}$*, we will have for the integrated equation* $n(2AS - SS) = uu \int \frac{\pi r^2}{a^2}$,

He then solves this last equation for (*a/u*) and substitutes back into the relation between the differentials of *t* and *S*. Thus... *from which we draw*

$$dt = \frac{dS \int (\pi r^2)^{1/2}}{\sqrt{(n)} \cdot \sqrt{(2AS - SS)}}$$

At this point, Coulomb extracts from this last equation, which can be re-written as an equation for the angular velocity as a function of angular position (In fact it looks like an energy integral of the rigid body equation of motion), to wit $\frac{dS}{dt} = \frac{\sqrt{(n)} \cdot \sqrt{(2AS - SS)}}{\sqrt{I}}$ - he extracts an expression for the period of oscillation as a function of the moment of inertial $I = \int (\pi r^2)$ and the torsional stiffness of the wire, *n*. In his words:

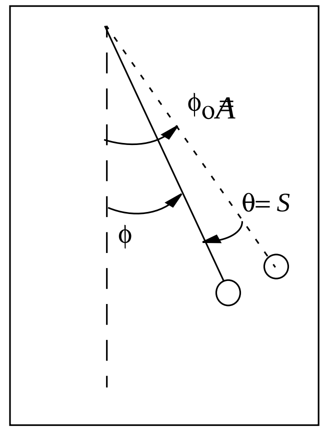
But $\frac{dS}{\sqrt{(2AS - SS)}}$ *represents an angle of which A is the radius and S the verse sine, which vanishes when S=0, and which becomes equal to 90 degrees when S = A. Thus the time of an complete oscillation will be* $T = \left(\int \frac{\pi r^2}{n}\right)^{1/2} \cdot 180^d$

[Note: A complete oscillation here is but one half of what we today call a full cycle.]

To grasp his reasoning here requires re-education in the ways simple harmonic motion was understood the last half of the 18th century. I am not sufficiently literate to translate for you. However, we can at least verify his result.

We first note that the expression $\frac{dS}{dt} = \frac{\sqrt{(n)} \cdot \sqrt{(2AS - SS)}}{\sqrt{I}}$ gives at the point of release of the cylinder, when $S = 0$, $\frac{dS}{dt} = 0$ whereas at the point when the cylinder has returned to its original, undisturbed state (but with maximum angular velocity), that is, when $S = A$, $\frac{dS}{dt} = \frac{\sqrt{(n)} \cdot A}{\sqrt{I}}$

At the right is a representation of a pendulum executing simple harmonic motion, to and fro, from right to left and back again. (The pendulum, the mass spring, and the torsional pendulum all, as you know, exhibit the same mathematical structure).



We have, assuming the pendulum is released from rest after displaced the angle A, $\phi = \phi_o \cos 2\pi f t$

and letting $\theta = \phi_o - \phi$ this becomes

$S = \theta = \phi_o(1 - \cos 2\pi f t) = A(1 - \cos 2\pi f t)$

The derivative with respect to time, gives the angular velocity, $\frac{dS}{dt} = A \cdot 2\pi f \sin 2\pi f t$

Now at $t=0$, the this is zero since we released the pendulum from rest at this point $S=0$. Whereas, it is the case that at $t = T/2$ the pendulum is vertical and $S=A$ and the angular velocity is, with $f = 1/2T$ (f is *our* cycles per unit time).

$$\left.\frac{dS}{dt}\right|_{t = T/2} = A \cdot 2\pi f \sin 2\pi f t = A \cdot \pi / T$$

NOTE: T is what Coulomb calls a period of oscillation, what we would call but a half cycle.

Comparing this expression with our manipulation of Coulomb's expression for the angular velocity at this same state gives

$$\frac{\sqrt{(n)} \cdot A}{\sqrt{I}} = A \cdot \pi / T$$

or the period (in Coulomb's sense)

$$T = \frac{\sqrt{I}}{\sqrt{(n)}} \cdot \pi = \int \left(\frac{\pi r^2}{n}\right)^{1/2} \cdot 180^d$$

Weight and Length Conversion Factors[1]

TABLE 1. Weights, Mass

	livre	*marc*	*onces*	*gros*	*deniers*	*grain*	*grams*
livre	1						489.506
marc	2	1					244.753
onces	16	8	1				30.594
gros	128	64	8	1			3.823
deniers	384	192	24	3	1		1.274
grain	9,216	4,608	576	72	24	1	.0531

TABLE 2. Lengths

	toise	*pied*	*pouce*	*ligne*	*mm*
toise	1				
pied	6	1			324.8
pouce	72	12	1		27.07
ligne	864	144	12	1	2.26

1. Kisch, R., *Scales and Weights: A Historical Outline*, New Haven: Yale University Press.