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Review

$$\frac{du}{dt} = f(u)$$

- Forward Euler

$t_0 = 0, \quad t_1 = \Delta t, \quad t_2 = 2\Delta t \quad \ldots$

$u_0 = u(t_0) \quad \ldots \quad u_i = u(t_i)$

$$\frac{u_{i+1} - u_i}{\Delta t} = f(u_i)$$

- Midpoint rule

$$\frac{u_{i+1} - u_{i-1}}{2\Delta t} = f(u_i) \longrightarrow u_i' + \frac{1}{6} u_i''' \Delta t^2 = f(u_i) + O(\Delta t^3)$$

$$u_{i+1} = u_i + u_i' \Delta t + \frac{1}{2} u_i'' \Delta t^2 + \frac{1}{6} u_i''' \Delta t^3 + \cdots$$

$$u_{i-1} = u_i - u_i' \Delta t + \frac{1}{2} u_i'' \Delta t^2 - \frac{1}{6} u_i''' \Delta t^3 + \cdots$$

Local order: the best X scheme

$$u_{i+1} = \frac{1}{2} u'_{i+1} \Delta t + u_i + \frac{1}{2} u'_i \Delta t$$

1. Best implicit, one step scheme

$$u_{i+1} = u'_{i+1} \Delta t + u_i + u'_i \Delta t$$

2. Best explicit, two step scheme

$$u' = f(u)$$

$$u_{i+1} = -4u_i + 5u_{i-1} + \Delta t\, 4 f(u_i) + 2 \Delta t\, f(u_{i-1})$$

3. Best implicit, two step scheme

$$u_{i+1} = \frac{1}{3} u'_{i+1} \Delta t + \frac{4}{3} u'_i \Delta t + u_{i-1} + \frac{1}{3} u'_{i-1} \Delta t$$

4. Best explicit, three step scheme

$$u_{i+1} = -18 u_i + 9 u'_i \Delta t + 9 u_{i-1} + 18 u'_{i-1} \Delta t + 10 u_{i-2} + 3 u'_{i-2} \Delta t$$

$$u_{i+1} = x u_i + y u_i' + z u_{i+1}'$$

$$\begin{cases} u_{i+1} = u_i + u_i' \Delta t + \frac{u_i''}{2}\Delta t^2 + \frac{u_i'''}{6}\Delta t^3 + O(\Delta t^4) \\ u_{i+1}' = u_i' + u_i''\Delta t + \frac{u_i'''}{2}\Delta t^3 + \frac{u_i''''}{6}\Delta t^3 + O(\Delta t^4) \end{cases}$$

$$\Delta t\,\tau = u_{i+1} - x u_i - y u_i' - z u_{i+1}'$$

$$= u_i + u_i'\Delta t + u_i''\frac{\Delta t^2}{2} + u_i'''\frac{\Delta t^3}{6} + O(\Delta t^4)$$

$$- x u_i$$

$$- y u_i'$$

$$- z u_i' - u_i'' z\Delta t - u_i''' \frac{z\Delta t^2}{2} + zO(\Delta t^3)$$

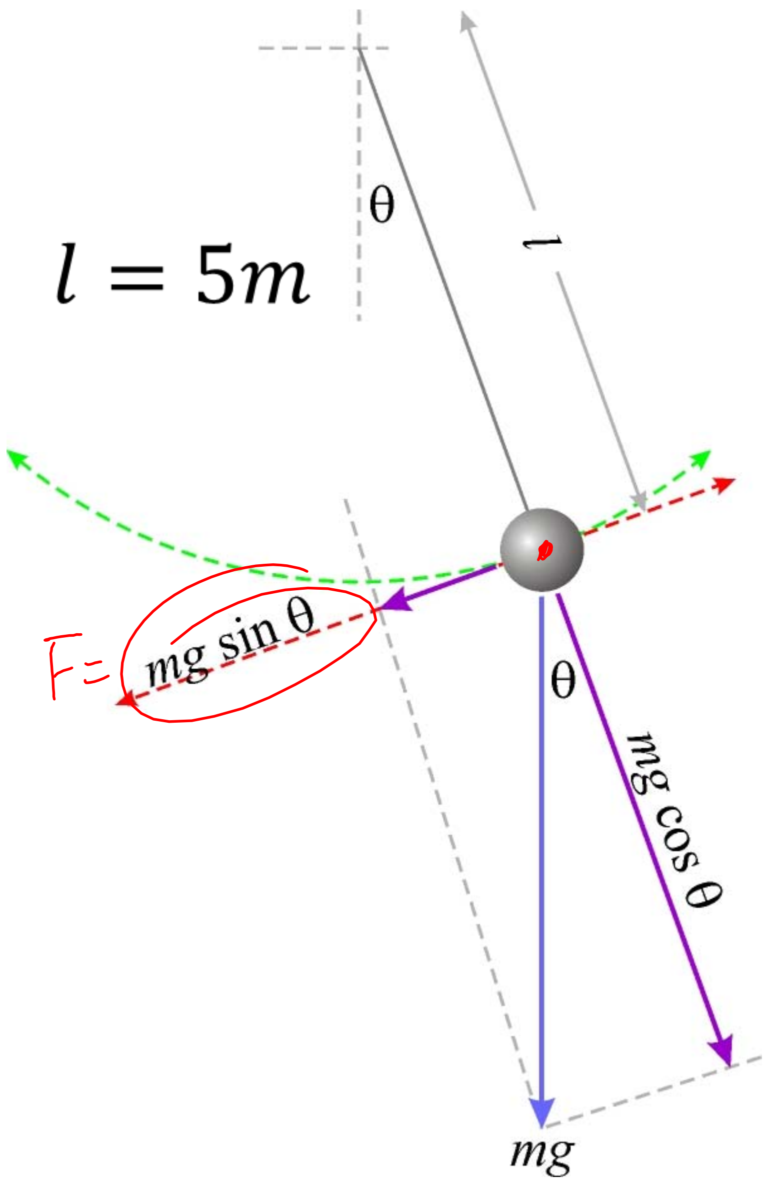
$$\begin{cases} x = 1 \\ y + z = \Delta t \\ z = \frac{\Delta t}{2} \end{cases} \qquad z = \frac{\Delta t}{2}, \quad y = \frac{\Delta t}{2}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y/\Delta t \\ z/\Delta t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

$$u_{i+1} = u_i + \frac{\Delta t}{2}u_i' + \frac{\Delta t}{2}u_{i+1}' + O(\Delta t^3)$$

$$\frac{u_i' + u_{i+1}'}{2} = \frac{u_{i+1} - u_i}{\Delta t} + O(\Delta t^2)$$

Pendulum



$$\ddot{\theta} = a/l = -g\sin\theta/l$$

$$a = \frac{F}{m} = \frac{mg\sin\theta}{m} = g\sin\theta$$

$$\sin\theta = \underbrace{\sin\theta\big|_0}_{0} + \underbrace{\theta\cdot\cos\theta\big|_0}_{\theta} + \dots$$

$$\approx \theta$$

$$\frac{d\theta}{dt} = \dot{\theta} \qquad \sin\theta$$

$$\frac{d\dot{\theta}}{dt} = \ddot{\theta} = -\frac{g\theta}{l}$$

Forward Euler

Midpoint rule

Best implicit, one step scheme

$$u_{i+1} = u_i + \frac{\Delta t}{2} u_i' + \frac{\Delta t}{2} u_{i+1}'$$

$$u_i' = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{pmatrix} u_i$$

$$u_{i+1}' = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{pmatrix} u_{i+1}$$

$$u_{i+1} = u_i + \frac{\Delta t}{2} \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{pmatrix} u_i + \frac{\Delta t}{2} \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{pmatrix} u_{i+1}$$

$$\begin{pmatrix} 1 & -\frac{\Delta t}{2} \\ \frac{\Delta t}{2}\frac{g}{l} & 1 \end{pmatrix} u_{i+1} = \begin{pmatrix} 1 & \frac{\Delta t}{2} \\ -\frac{\Delta t}{2}\frac{g}{l} & 1 \end{pmatrix} u_i$$

Best explicit, two step scheme

Best explicit, three step scheme