ANALYSIS OF A TWO-SENSOR TANDEM DISTRIBUTED DETECTION NETWORK

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ABSTRACT

The following distributed detection problem is formulated. We consider a team that comprises of two decision makers, who are referred to as Decision Maker A (DMA) and Decision Maker B (DMB). Both decision makers receive uncertain measurements or observations, and the goal of the team is to make a decision with the objective of trying to minimize the probability of making an incorrect decision. DMA processes his measurement first and communicates to DMB one of K messages, M_1, M_2, ..., M_K. Based on this message and his own observation, DMB makes the final decision of the team.

The goal is to analyze the performance of the above scheme (using values of K greater than two) and compare it with the well known case where only two messages, M_1 and M_2, are used by DMA. It is interesting to see how the performance of the team approaches that of the centralized version of the problem (i.e., two independent observations available to a single decision maker) with increasing values of K. Furthermore, results for the General K case have been presented and can be used to evaluate the performance of a team where K messages (K taking on any value) are used for communication between DMA and DMB.

Thesis Supervisor : Dr. Michael Athans
Professor of Systems Science and Engineering

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Chapter 1

INTRODUCTION

This thesis addresses the design and performance evaluation of a special class of team decision problems, where each member of a team of decision makers receives conditionally independent observations about some underlying binary hypothesis. The objective of the team is to make an optimal decision. We study the effects of increasing the communication between the team members.

1.1 Problem Definition

The problem can be formulated as one of target detection. The decision team consists of two decision makers, namely Decision Maker A (DMA) and Decision Maker B (DMB). Each decision maker receives conditionally independent observations and the goal of the team is to make an optimal decision to choose between two hypotheses, H_0 and H_1. In a target detection context, H_0 may mean no target, while H_1 denotes the presence of a target.

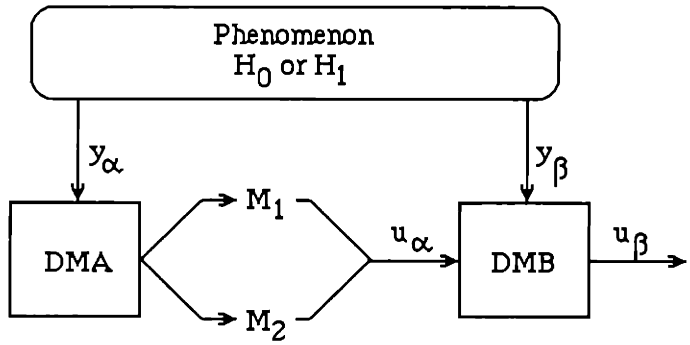
The operation of the team is described as follows. DMA processes the measurement from his sensor and arrives at a decision. This preliminary decision, which can be one of

K messages, M_1, M_2, ..., M_K (K = 2, 3, ...), is passed on to DMB. Using his own measurement and the message from DMA, DMB arrives at the final decision of the team. In other words, DMB has the final responsibility for declaring the presence or absence of a target.

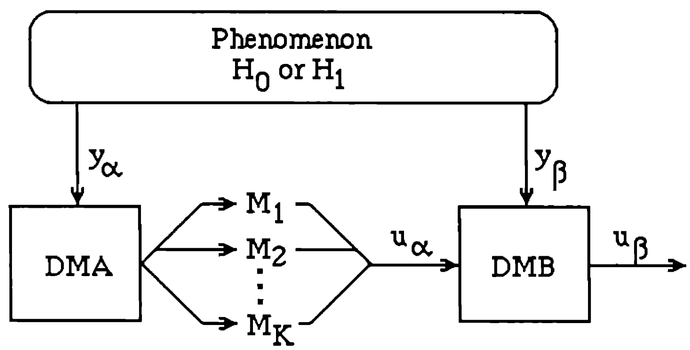
The objective of the team is to make an optimal decision, i.e., one that minimizes both the probability of missed detection and the probability of false alarm. In this context, as the number (K) of messages increases, we are providing DMB with more accurate information about DMA's observation. We study the implications of this on the probabilities of missed detection and false alarm of the final team decision.

The theme of this research is to evaluate the performance of the K-message (using values of K greater than two) Distributed Detection Network shown in Figure 1b, and compare it with that of the two-message (K = 2) Distributed Detection Network shown in Figure 1a. In order to accomplish this, the linear Gaussian case is considered (i.e., where the observations are linear and their probability distributions are Gaussian). In addition, we perform sensitivity analyses for the linear Gaussian case to gain some insight into the behavior of the team. Varying key parameters such as the quality of the observations that the decision makers receive and the a priori probabilities of the hypotheses, we see the effect this has on the performance of the team.

Finally, the results for the General K case are presented, where the number of messages used by DMA (i.e., K) can take on any value. This provides the framework to evaluate the performance of a team that uses K messages for communication between DMA and DMB. Increasing the communication between the team members in this manner results in the performance of the team approaching that of the centralized version of the problem, where a single decision maker receives two independent observations.



(a) Two-message (K=2) Tandem Distributed Detection Network, ($u_\alpha = \{M_1, M_2\}$)



(b) K-message Tandem Distributed Detection Network, ($u_\alpha = \{M_1, M_2, ..., M_K\}$)

Figure 1 Problem Formulation

1.2 Motivation for this Research

Hypothesis-testing problems in the field of Command and Control are one of the many areas where this research can be applied. More specifically, we discuss the target detection problem where the two hypotheses, H_0 and H_1, are defined as before. Two geographically distributed sensors (or DM's) receive independent noisy measurements. Based on these measurements, the team has to declare the presence or the absence of a target. The final decision of the team, however, is made by the downstream DM and there is a one-way communication between the DM's (from DMA to DMB). It is possible to allocate different costs to the probabilities of false alarm and missed detection and attempt to minimize this cost, which in turn minimizes the probability of error of the detection team. It is evident that if the downstream DM relied only on the measurement from his sensor, we have a classical centralized detection problem. Furthermore, if the upstream DM were to communicate his entire observation to the downstream DM, we once again are dealing with a classical centralized detection problem where the decisive DM has two measurements with which he can make the final decision of the team. In the latter case, however, the communication of raw data is involved which could be expensive from a channel bandwidth point of view. In addition, communication of this type could easily be intercepted by the enemy.

To minimize the cost and the potential risks involved, the communication mechanism described in the previous section is employed. As a consequence, communication is cheap and is more likely to escape enemy interception.

The need for communicating with a few bits rather than with raw data can be appreciated if we consider detecting an enemy airplane using radar as depicted in Figure 2. We associate DMB with the control room in the building and DMA with our surveillance airplane (both having means through which they can receive measurements). In this

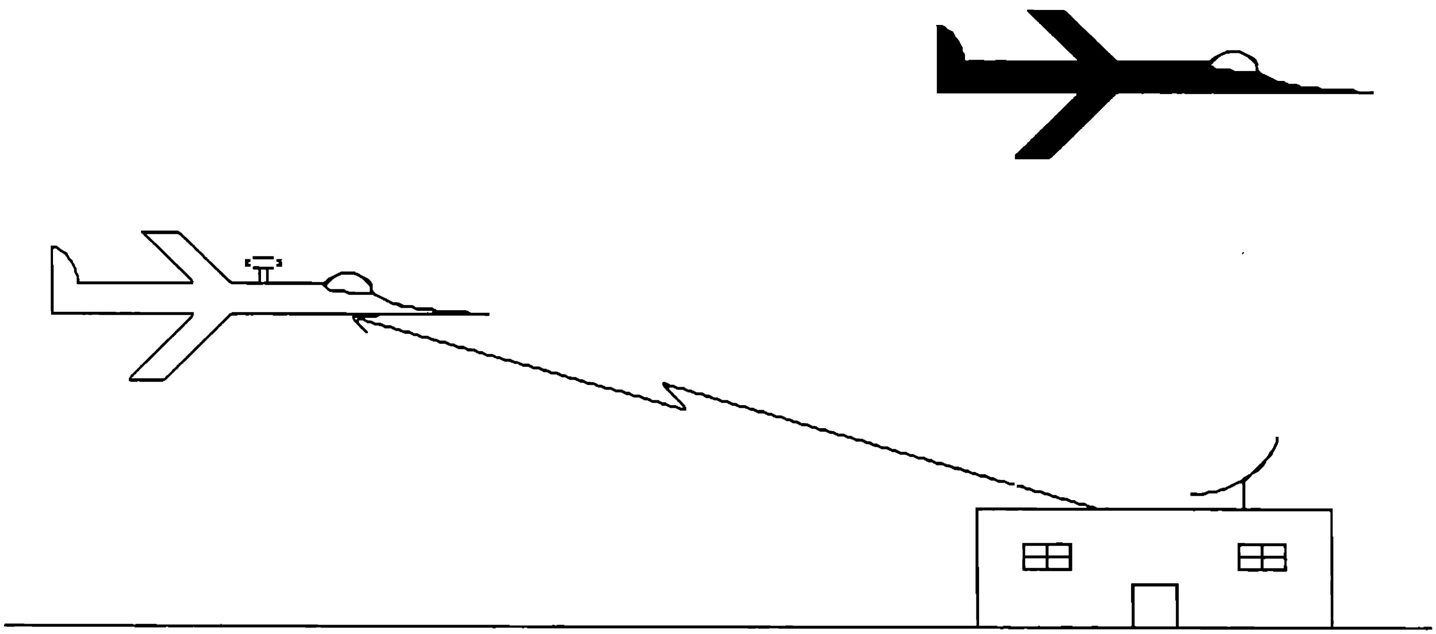


Figure 2 Target Detection Example

situation, a short communication message can be used to transmit the surveillance aircraft message to the ground center, who in turn makes the final decision of the team.

Although the primary motivation for this research stemmed from military surveillance problems, distributed detection problems are also prevelant in civilian air traffic control systems and many other areas.

1.3 Literature Survey

The area of distributed detection is relatively new and there are only a few papers that deal with solutions to such problems. One of the first papers to consider the distributed detection problem was the paper by Tenney and Sandell [5]. They demonstrated the difficulties encountered in solving a two-sensor problem, and showed that the optimal decision rule for each sensor is deterministic and is a likelihood ratio test. The decision thresholds were, in general, coupled. Eckhian [1] analyzed the decision rules for Distributed Detection Networks and investigated organizational issues in such networks. He derived the solution to the two-sensor problem where DMA uses two messages (K = 2) to communicate with DMB. The problem where DMA uses more than two messages (K = 3, 4, ...), however, has not been solved as yet. He also considered problems in which more than two sensors made a team. Eckhian and Tenney [2] pointed out that individual DM's make decisions based on their own observations and decisions passed down by the "upstream" DM's. In Papastavrou [3] and Papastavrou and Athans [4], the notion of communication cost was introduced. A two-sensor problem was considered where the primary decision maker could solicit the opinion of the consulting decision maker at a cost. The final decision, however, was made by the primary decision maker. The optimal decision rules were shown to be tightly coupled and numerical sensitivity analyses provided valuable insight on the overall behavior of the team. Tsitsiklis [6] and Tsitsiklis and Athans [7] dealt with distributed hypothesis-testing problems and showed that they were NP-complete; they demonstrated the computational complexity of solving optimal distributed decision problems while showing that the solutions to the centralized versions were trivial. Kushner and Pacut [9] introduced a delay cost (somewhat similar to the communications cost in [3] and [4]) in the case where the observations have exponential distributions. They performed a simulation study and presented the results. In Chair and

Varshney [10], the results of [5] have been extended to more generalized settings. Boettcher [11] and Boettcher and Tenney [12], have shown how to modify the normative solutions in [2] to reflect human limitation constraints, and arrive at a normative/descriptive model that captures the constraints of human implementation in the presence of decision deadlines and increasing human workload; experiments using human subjects showed close agreement with the prediction of their normative/descriptive model. In Reibman and Nolte [13], using the minimal global cost criterion, it is shown that the optimal structure of the local processor in a general distributed detection network is a likelihood ratio test when the input observations are statistically independent. In addition, it is shown that the local thresholds and the network performance can be expressed as a function of the receiver operating characteristics (ROCs) of the local processors. The performance of five distributed nerworks are compared numerically using local ROCs from the conic ROC family. Vishwanathan, Thomopoulos and Tumuluri [14] consider a serial distributed decision scheme (also called a tandem network) and carry out a performance analysis of such a scheme to compare it to the performance of a parallel decision scheme. Finally, Polychronopoulos [15] dealt with the decentralized detection problem in which a large number of identical sensors transmit a finite-valued function of their observations to a fusion center which then decides which one of the M hypotheses is true. An asymptotically optimal solution to the problem is presented for the case where observations are generated from one of a simple set of discrete symmetric distributions.

1.4 Contributions of this Research

Through this research, we see the degree of improvement in the performance of the detection team when DMB (the decision maker making the final decision of the team) is given more information than if it were operating in isolation. When operating in isolation, DMB has its own observation y_β, with which to make a decision. We now provide a message from DMA to assist DMB in making a more accurate decision. The optimal decision thresholds with the DM's operating as a team are different from those of DMB operating in isolation. In particular, these decision thresholds are tightly coupled. Hence, DMA and DMB are operating as team members. We see the difference in the performance of the team if DMA can use three messages, as opposed to two messages, to assist DMB. As expected, the three message case provides more information to DMB (regarding DMA's observation), and the performance is better.

On the other hand, we compare the overall distributed team performance to that of the centralized version of the problem, in which DMB has access to both sets of observations, y_α and y_β. This shows the performance degradation due to enforcing the distributed decision making process. If the number of messages used by DMA is increased to four, we see that the performance of the team gets closer to that of its centralized counterpart. If, for instance, the number of messages used by DMA is allowed to approach infinity, it is expected that the performance of such a scheme would approach that of the centralized version. However, we see that there is not much room for improvement in performance beyond the use of two bits of information.

The overall behavior of the team is studied by varying the quality of the observations of the decision makers and the a priori probabilities of the hypotheses. This study is very informative and answers questions regarding the positioning of the DM's (i.e., upstream or downstream), and the allocation of communication capability to the DM's (depending on whether the decision maker is "smart" or not). It also shows how the team members operate intuitively and in the best interest of the team.

Chapter 2

THE GENERAL PROBLEM

In this chapter we extend the well-known two-message case to one where three messages are used by DMA to communicate with DMB. In addition to summarizing the results of the three-message case, we also include the results of the two-message case for the sake of completeness. Once we have understood the three-message case completely, in Chapter 4 we will further extend the problem to one where four or even K messages (K taking on any value) are used for communication by a detection team.

2.1 Notation

The problem studied is one of hypothesis testing, where the detection team has to choose between two hypotheses, H_0 and H_1, with a priori probabilities

$$P(H_0) \qquad \text{and} \qquad P(H_1)$$

Each of the two decision makers, DMA and DMB, receives an uncertain measurement

y_α and y_β respectively, distributed with known joint probability density functions

$$P(y_\alpha, y_\beta \mid H_i) \qquad \text{for } i = 0, 1$$

The objective of the decision strategies is to minimize the expected cost incurred, where the minimization is done over the decision rules of the two decision makers. The cost function is written as $J(u_\beta, H_i)$, and is the cost incurred by the detection team choosing u_β, when H_i is the true hypothesis.

2.2 Assumptions

Assumption 1 : $J(1,H_0) > J(0,H_0)$; $J(0,H_0) > J(1,H_1)$ (1)

or the cost incurred for making an error is greater than that for being correct.

This logical assumption is made in order to motivate the team members to avoid making errors and in order to enable us to put the optimal decisions in the form of likelihood ratio tests.

Assumption 2 : $P(y_\alpha \mid y_\beta, H_i) = P(y_\beta \mid H_i)$; $P(y_\beta \mid y_\alpha, H_i) = P(y_\beta \mid H_i)$ for $i = 0, 1$ (2)

or the observations y_α and y_β are conditionally independent.

This implies that one observation is not dependent on the other and enables us to write the optimal decision rules as likelihood ratio tests with constant thresholds.

Assumption 3 : Without loss of generality we assume that

$$\frac{P(u_\alpha = M_1 \mid H_0)}{P(u_\alpha = M_1 \mid H_1)} \geq \frac{P(u_\alpha = M_2 \mid H_0)}{P(u_\alpha = M_2 \mid H_1)} \geq \frac{P(u_\alpha = M_3 \mid H_0)}{P(u_\alpha = M_3 \mid H_1)} \qquad (3)$$

This assumption is made in order to distinguish between the messages of DMA.

2.3 Two-message Case

Given $P(H_0)$, $P(H_1)$, the distributions $P(y_\alpha, y_\beta \mid H_i)$ for $i = 0, 1$ with $y_\alpha \in Y_\alpha$, $y_\beta \in Y_\beta$, and the cost function $J(u_\beta, H_i)$, the optimal decision rules (one that minimizes the expected cost) of DMA and DMB are derived. DMA can use one of two messages, M_1 or M_2, to communicate his decision to DMB. DMB has his own observation and the message from DMA to make the final decision of the team.

The proofs of the theorems of the two-message case appear in *Optimal Design of Distributed Detection Networks*, which is listed in the References section. Hence, they will not be repeated here.

Theorem 1

Given the decision u_α of DMA, the optimal decision rule of DMB is a deterministic function

$$\gamma_\beta : Y_\beta \times \{M_1, M_2\} \rightarrow \{0, 1\}$$

defined by the following likelihood ratio tests:

$$\gamma_\beta(y_\beta, u_\alpha) = \begin{cases} 0, & \text{if } \Lambda_\beta(y_\beta) \geq \beta_i \\ 1, & \text{otherwise} \end{cases} \qquad (4)$$

where

$$\Lambda_\beta(y_\beta) = \frac{P(H_0)\, P(y_\beta \mid H_0)}{P(H_1)\, P(y_\beta \mid H_1)} \tag{5}$$

and

$$\beta_i = \frac{P(u_\alpha \mid H_1)\, [\, J(0,H_1) - J(1,H_1)\,]}{P(u_\alpha \mid H_0)\, [\, J(1,H_0) - J(0,H_0)\,]} \qquad \text{for } i = 0, 1 \tag{6}$$

Theorem 2

Given the decision $u_\beta \in \{0, 1\}$ of DMB, the decision rule of DMA is a deterministic function

$$\gamma_\alpha : Y_\alpha \rightarrow \{ M_1, M_2 \} = \{ 0, 1 \}$$

defined by the following likelihood ratio tests:

$$\gamma_\alpha(y_\alpha) = \begin{cases} M_1, & \text{if } \Lambda_\alpha(y_\alpha) \geq \alpha^* \\ M_2, & \text{otherwise} \end{cases} \tag{7}$$

where

$$\Lambda_\alpha(y_\alpha) = \frac{P(H_0)\, P(y_\alpha \mid H_0)}{P(H_1)\, P(y_\alpha \mid H_1)} \tag{8}$$

and

$$\alpha^* = \frac{\sum_{u_\beta} J(u_\beta, H_1)\,[\,P(u_\beta \mid u_\alpha = 0, H_1) - P(u_\beta \mid u_\alpha = 1, H_1)\,]}{\sum_{u_\beta} J(u_\beta, H_0)\,[\,P(u_\beta \mid u_\alpha = 1, H_0) - P(u_\beta \mid u_\alpha = 0, H_0)\,]} \qquad (9)$$

Although the structure of the decision rules above seem simple, the computation of the thresholds is quite complex. It is necessary to solve a system of non-linear simultaneous equations since the threshold equations are coupled. This can be done iteratively using a computer algorithm. In addition, the computation of the thresholds require certain quantities that are not easy to determine. For example, the computation of β_0 requires the calculation of the conditional density $p(u_\alpha = 0 \mid H_0)$, which requires the calculation of $p(u_\alpha = 0 \mid y_\alpha)$ as shown below :

$$p(u_\alpha = 0 \mid H_0) = \int p(u_\alpha = 0 \mid y_\alpha)\, p(y_\alpha \mid H_0)\, dy_\alpha \qquad (10)$$

2.4 Three-message case

Given $P(H_0)$, $P(H_1)$, the distributions $P(y_\alpha, y_\beta \mid H_i)$ for i = 0, 1 with $y_\alpha \in Y_\alpha$, $y_\beta \in Y_\beta$, and the cost function $J(u_\beta, H_i)$, the optimal decision rules of DMA and DMB are derived. DMA can use one of three messages, M_1, M_2, or M_3, to communicate his decision to DMA. DMB has his own observation and the message from DMA to make the final decision of the team.

The detailed proofs of the theorems for the three-message case appear in the appendix.

Theorem 3

Given the decision u_α of DMA, the optimal decision rule of DMB is a deterministic function

$$\gamma_\beta : Y_\beta \times \{ M_1, M_2, M_3 \} \rightarrow \{ 0, 1 \}$$

defined by the following likelihood ratio tests :

$$\gamma_\beta(y_\beta, u_\alpha) = \begin{cases} 0, & \text{if } \Lambda_\beta(y_\beta) \geq \beta_i \\ 1, & \text{otherwise} \end{cases} \tag{11}$$

where

$$\Lambda_\beta(y_\beta) = \frac{P(H_0)\, P(y_\beta \mid H_0)}{P(H_1)\, P(y_\beta \mid H_1)} \tag{12}$$

and

$$\beta_i = \frac{P(u_\alpha = i \mid H_1)\, [\, J(0, H_1) - J(1, H_1)\,]}{P(u_\alpha = i \mid H_0)\, [\, J(1, H_0) - J(0, H_0)\,]} \qquad \text{for } i = M_1, M_2, M_3 \tag{13}$$

Theorem 4

Given the decision $u_\beta \in \{0, 1\}$ of DMB, the decision rule of DMA is a deterministic function

$$\gamma_\alpha : Y_\alpha \rightarrow \{ M_1, M_2, M_3 \}$$

defined by the following likelihood ratio tests:

$$\gamma_\alpha(y_\alpha) = \begin{cases} M_1, & \text{if } \Lambda_\alpha(y_\alpha) \geq \alpha_1 \\ M_2, & \text{if } \alpha_3 \leq \Lambda_\alpha(y_\alpha) < \alpha_1 \\ M_3, & \text{if } \Lambda_\alpha(y_\alpha) < \alpha_3 \end{cases} \tag{14}$$

where

$$\Lambda_\alpha(y_\alpha) = \frac{P(H_0)\, P(y_\alpha \mid H_0)}{P(H_1)\, P(y_\alpha \mid H_1)} \tag{15}$$

and

$$\alpha_1 = \frac{\sum_{u_\beta} J(u_\beta, H_1)\, [\, P(u_\beta \mid u_\alpha = M_2, H_1) - P(u_\beta \mid u_\alpha = M_1, H_1)\,]}{\sum_{u_\beta} J(u_\beta, H_0)\, [\, P(u_\beta \mid u_\alpha = M_1, H_0) - P(u_\beta \mid u_\alpha = M_2, H_0)\,]} \tag{16}$$

$$\alpha_3 = \frac{\sum_{u_\beta} J(u_\beta, H_1)\, [\, P(u_\beta \mid u_\alpha = M_3, H_1) - P(u_\beta \mid u_\alpha = M_2, H_1)\,]}{\sum_{u_\beta} J(u_\beta, H_0)\, [\, P(u_\beta \mid u_\alpha = M_2, H_0) - P(u_\beta \mid u_\alpha = M_3, H_0)\,]} \tag{17}$$

Once again, the nature of the above equations neccessitates an iterative solution.

Chapter 3

THE GAUSSIAN CASE

This chapter contains the detailed threshold equations for the cases where the observations of DMA and DMB are linear and their probability distributions are Gaussian. The Gaussian distribution, despite its cumbersome algebraic formulae, is chosen due to its generality.

3.1 The Gaussian Problem

The distribution of the observations of the decision makers are Gaussian and are given by

$$y_\alpha \sim N(\mu, \sigma_\alpha^2) \quad ; \quad y_\beta \sim N(\mu, \sigma_\beta^2) \qquad (18)$$

The hypotheses, namely H_0 and H_1, are described by

$$H_0 : \ \mu = \mu_0 \quad ; \quad H_1 : \ \mu = \mu_1 \qquad (19)$$

It is assumed that $\mu_0 < \mu_1$ (with no loss of generality).

It can be shown that the decision thresholds for the Gaussian case are on the observation axes. These thresholds appear in Figure 3. For the two-message case, the thresholds are referred to as $Y_\alpha{}^*$, $Y_\beta{}^{M_1}$ and $Y_\beta{}^{M_2}$ (where '*' refers to the only threshold of DMA). For the three-message case, the thresholds are referred to as $Y_\alpha{}^l$, $Y_\alpha{}^u$, $Y_\beta{}^{M_1}$, $Y_\beta{}^{M_2}$ and $Y_\beta{}^{M_3}$ (where 'l' and 'u' refer to the lower and upper thresholds of DMA respectively). The threshold equations are written in terms of the error function which is given by

$$\Phi_i^j(k) = \int_{-\infty}^{\frac{Y_i^j - \mu_k}{\sigma_i}} (2\pi)^{-0.5} \exp(-0.5x^2)\, dx \tag{20}$$

where $i = \alpha, \beta$

$j = *, M_1, M_2$ for $K = 2$

$= l, u, M_1, M_2, M_3$ for $K = 3$

$k = 0, 1$

It is useful to note that the centralized maximum likelihood estimations of the thresholds are given by

$$\text{DMA}: \quad Y_\alpha^{ML} = \frac{\mu_0 + \mu_1}{2} + \frac{\sigma_\alpha^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] \tag{21}$$

$$\text{DMB}: \quad Y_\beta^{ML} = \frac{\mu_0 + \mu_1}{2} + \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] \tag{22}$$

These are the estimations of the thresholds in an isolated setting (i.e., DMA or DMB in isolation rather than as team members).

3.2 Two-message Case

The decision rules that follow are applicable to the case where DMA uses two messages, M_1 and M_2, to communicate with DMB.

A discussion for the corollaries of this case appears in the appendix.

Corollary 1

Given the decision u_α by DMA, the optimal decision rule of DMB is a deterministic function defined by

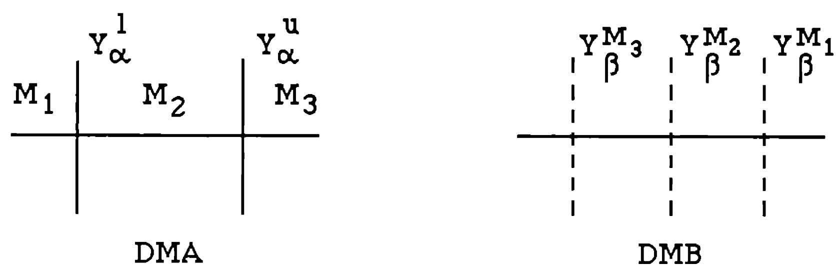
$$\gamma_\beta(y_\beta, u_\alpha) = \begin{cases} 0, & \text{if } y_\beta \leq Y_\beta^{M_i} \\ 1, & \text{if } y_\beta > Y_\beta^{M_i} \end{cases} \qquad \text{for } i = 1, 2 \tag{23}$$

where

$$Y_\beta^{M_1} = \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\alpha^*(0)}{\Phi_\alpha^*(1)}\right] + \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \tag{24}$$

$$Y_\beta^{M_2} = \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{1 - \Phi_\alpha^*(0)}{1 - \Phi_\alpha^*(1)}\right] + \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \tag{25}$$

(a) Two-message case



(b) Three-message case

Figure 3 The Gaussian Example

Corollary 2

Given the decision u_β of DMB, the optimal decision rule of DMA is a deterministic function defined by

$$\gamma_\alpha(y_\alpha) = \begin{cases} M_1 , & \text{if } y_\alpha \le Y_\alpha^* \\ M_2 , & \text{if } y_\alpha > Y_\alpha^* \end{cases} \tag{26}$$

where

$$Y_\alpha^* = \frac{\sigma_\alpha^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\beta^{M_1}(0) - \Phi_\beta^{M_2}(0)}{\Phi_\beta^{M_1}(1) - \Phi_\beta^{M_2}(1)}\right] + \frac{\sigma_\alpha^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \tag{27}$$

The coupling of the equations appearing in Corollary 1 and Corollary 2 can be seen by writing out the error functions using the definition in Section 3.1.

$$\Phi_\alpha^* (k) = \int_{-\infty}^{\frac{Y_\alpha^* - \mu_k}{\sigma_\alpha}} (2\pi)^{-0.5} \exp(-0.5\, x^2)\, dx \qquad \text{for } k = 0, 1 \tag{28}$$

$$\Phi_\beta^{M_1} (k) = \int_{-\infty}^{\frac{Y_\beta^{M_1} - \mu_k}{\sigma_\beta}} (2\pi)^{-0.5} \exp(-0.5\, x^2)\, dx \qquad \text{for } k = 0, 1 \tag{29}$$

$$\Phi_\beta^{M_2} (k) = \int_{-\infty}^{\frac{Y_\beta^{M_2} - \mu_k}{\sigma_\beta}} (2\pi)^{-0.5} \exp(-0.5\, x^2)\, dx \qquad \text{for } k = 0, 1 \tag{30}$$

Hence, we see that both $Y_\beta^{M_1}$ and $Y_\beta^{M_2}$ are functions of Y_α^*, which in turn is a function of both $Y_\beta^{M_1}$ and $Y_\beta^{M_2}$. The equations are, therefore, tightly coupled.

3.3 Three-message Case

The decision rules that follow are applicable to the case where DMA uses three messages, M_1, M_2 and M_3, to communicate with DMB.

The corollaries of this case have been proven to illustrate how the Gaussian equations are derived. These proofs appear in the appendix.

Corollary 3

Given the decision u_α of DMA, the optimal decision rule of DMB is a deterministic function defined by

$$\gamma_\beta(y_\beta, u_\alpha) = \begin{cases} 0, & \text{if } y_\beta \le Y_\beta^{M_i} \\ 1, & \text{if } y_\beta > Y_\beta^{M_i} \end{cases} \quad \text{for } i = 1, 2, 3 \tag{31}$$

where

$$Y_\beta^{M_1} = \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\alpha^l(0)}{\Phi_\alpha^l(1)}\right] + \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \tag{32}$$

$$Y_\beta^{M_2} = \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\alpha^u(0) - \Phi_\alpha^l(0)}{\Phi_\alpha^u(1) - \Phi_\alpha^l(1)}\right] + \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \tag{33}$$

$$Y_{\beta}^{M_3} = \frac{\sigma_{\beta}^2}{\mu_1 - \mu_0} \ln\left[\frac{1 - \Phi_{\alpha}^{u}(0)}{1 - \Phi_{\alpha}^{u}(1)}\right] + \frac{\sigma_{\beta}^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \quad (34)$$

Corollary 4

Given the decision u_β of DMB, the optimal decision rule of DMA is a deterministic function defined by

$$\gamma_\alpha(y_\alpha) = \begin{cases} M_1, & \text{if } y_\alpha \le Y_\alpha^l \\ M_2, & \text{if } Y_\alpha^l < y_\alpha \le Y_\alpha^u \\ M_3, & \text{if } y_\alpha > Y_\alpha^u \end{cases} \quad (35)$$

where

$$Y_{\alpha}^{l} = \frac{\sigma_{\alpha}^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_{\beta}^{M_2}(0) - \Phi_{\beta}^{M_1}(0)}{\Phi_{\beta}^{M_2}(1) - \Phi_{\beta}^{M_1}(1)}\right] + \frac{\sigma_{\alpha}^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \quad (36)$$

$$Y_{\alpha}^{u} = \frac{\sigma_{\alpha}^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_{\beta}^{M_3}(0) - \Phi_{\beta}^{M_2}(0)}{\Phi_{\beta}^{M_3}(1) - \Phi_{\beta}^{M_2}(1)}\right] + \frac{\sigma_{\alpha}^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \quad (37)$$

Having listed the error functions in detail for the two-message case, we can see by inspection that the equations for the three-message case are also coupled in the similar manner.

3.3 Computation of Thresholds

It is clear that the threshold equations for both the two-message and three-message cases are coupled. However, the equations can be solved numerically using an iterative method. This iterative method is implemented using a computer algorithm that receives as input initial estimates of the thresholds of the downstream decision maker, and yields the optimum thresholds for both decision makers. In other words, for the three-message case, the algorithm is given initial estimates of $Y_\beta^{M_1}$, $Y_\beta^{M_2}$ and $Y_\beta^{M_3}$. It then computes Y_α^l and Y_α^u using these estimates and completes one iteration. Given Y_α^l and Y_α^u, the thresholds of the downstream decision maker are computed and this iterative process is carried on until the threshold values converge (i.e., the values from the Nth iteration have not changed by more than a specified percentage from those of the N-1st iteration). The solutions have found to be unique (i.e., they are not merely locally optimal).

The algorithm was coded using Microsoft Basic. Due to the nature of the program it is essential that the initial estimates of the downstream decision maker be unequal. Equal estimates would result in the program terminating in one iteration. Although the number of iterations required for convergence depends on the initial estimates provided to the program, a typical computation usually takes about 15-20 iterations.

The two-message case entails solving a system of three equations (i.e., (25)-(26) and (28)) while the three-message case involves solving five equations (i.e., (33)-(35) and (37)-(38)).

In order to get numerical results, the following baseline parameter values were used for the problem :

$$\text{Under } H_0: \ \mu = \mu_0 = 0 \qquad \sigma_\alpha^2 = 100$$

$$\text{Under } H_1: \ \mu = \mu_1 = 10 \qquad \sigma_\beta^2 = 100$$

$$P(H_0) = 0.5$$

Using the values on the previous page, the thresholds for the two-message and three-message cases are computed.

Two-message Case :

$$Y_{\alpha}^{*} = 5.0$$

$$Y_{\beta}^{M_1} = 13.0697 \qquad Y_{\beta}^{M_2} = -3.0697$$

Three-message Case :

$$Y_{\alpha}^{l} = -0.1922 \qquad Y_{\alpha}^{u} = 10.1922$$

$$Y_{\beta}^{M_1} = 16.6188 \qquad Y_{\beta}^{M_2} = 5.0 \qquad Y_{\beta}^{M_3} = -6.6188$$

Chapter 4

THE EXTENDED PROBLEM

The general problem is extended to cases where the number of messages used by the upstream decision maker is greater than three (i.e., $K > 3$). More specifically, in this chapter we consider the four message case. Both the general problem and its Gaussian version have been studied. Results for the case where K can take on any value are also presented.

4.1 The Four-message Case

Given $P(H_0)$, $P(H_1)$, the distributions $P(y_\alpha, y_\beta \mid H_i)$ for $i = 0, 1$ with $y_\alpha \in Y_\alpha$, $y_\beta \in Y_\beta$, and the cost function $J(u_\beta, H_i)$, the optimal decision rules (one that minimizes the expected cost) of DMA and DMB are derived. DMA can use one of four messages, M_1, M_2, M_3 or M_4, to communicate his decision to DMB. DMB has his own observation and the message from DMA to make the final decision of the team.

The assumptions for the four-message case are the same as those for the three-message case (a four-message version of Assumption 3 is used). The proofs of the theorems in this section have not been included in the appendix since they are similar to those for the three-message case.

Theorem 5

Given the decision u_α of DMA, the optimal decision rule of DMB is a deterministic function

$$\gamma_\beta : Y_\beta \times \{ M_1, M_2, M_3, M_4 \} \rightarrow \{ 0, 1 \}$$

defined by the following likelihood ratio tests :

$$\gamma_\beta(y_\beta, u_\alpha) = \begin{cases} 0 , & \text{if } \Lambda_\beta(y_\beta) \geq \beta_i \\ 1 , & \text{otherwise} \end{cases} \tag{38}$$

where

$$\Lambda_\beta(y_\beta) = \frac{P(H_0)\, P(y_\beta \mid H_0)}{P(H_1)\, P(y_\beta \mid H_1)} \tag{39}$$

and

$$\beta_i = \frac{P(u_\alpha = i \mid H_1)\, [\, J(0, H_1) - J(1, H_1)\,]}{P(u_\alpha = i \mid H_0)\, [\, J(1, H_0) - J(0, H_0)\,]} \qquad \text{for } i = M_1, M_2, M_3, M_4 \tag{40}$$

Theorem 6

Given the decision $u_\beta \in \{0, 1\}$ of DMB, the decision rule of DMA is a deterministic function

$$\gamma_\alpha : Y_\alpha \rightarrow \{ M_1, M_2, M_3, M_4 \}$$

defined by the following likelihood ratio tests:

$$\gamma_\alpha(y_\alpha) = \begin{cases} M_1, & \text{if } \Lambda_\alpha(y_\alpha) \geq \alpha_1 \\ M_2, & \text{if } \alpha_4 \leq \Lambda_\alpha(y_\alpha) < \alpha_1 \\ M_3, & \text{if } \alpha_6 \leq \Lambda_\alpha(y_\alpha) < \alpha_4 \\ M_4, & \text{if } \Lambda_\alpha(y_\alpha) < \alpha_6 \end{cases} \tag{41}$$

where

$$\Lambda_\alpha(y_\alpha) = \frac{P(H_0)\, P(y_\alpha | H_0)}{P(H_1)\, P(y_\alpha | H_1)} \tag{42}$$

and

$$\alpha_1 = \frac{\sum_{u_\beta} J(u_\beta, H_1)\, [\, P(u_\beta | u_\alpha = M_2, H_1) - P(u_\beta | u_\alpha = M_1, H_1)\,]}{\sum_{u_\beta} J(u_\beta, H_0)\, [\, P(u_\beta | u_\alpha = M_1, H_0) - P(u_\beta | u_\alpha = M_2, H_0)\,]} \tag{43}$$

$$\alpha_4 = \frac{\sum_{u_\beta} J(u_\beta, H_1)\,[\,P(u_\beta | u_\alpha = M_3, H_1) - P(u_\beta | u_\alpha = M_2, H_1)\,]}{\sum_{u_\beta} J(u_\beta, H_0)\,[\,P(u_\beta | u_\alpha = M_2, H_0) - P(u_\beta | u_\alpha = M_3, H_0)\,]} \tag{44}$$

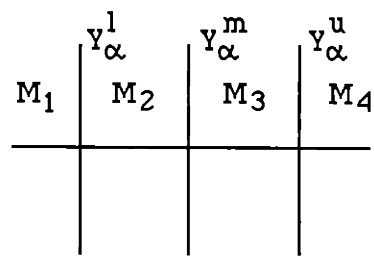
$$\alpha_6 = \frac{\sum_{u_\beta} J(u_\beta, H_1)\,[\,J(u_\beta | u_\alpha = M_4, H_1) - P(u_\beta | u_\alpha = M_3, H_1)\,]}{\sum_{u_\beta} J(u_\beta, H_0)\,[\,P(u_\beta | u_\alpha = M_3, H_0) - P(u_\beta | u_\alpha = M_4, H_0)\,]} \tag{45}$$

4.2 Gaussian Version for K = 4

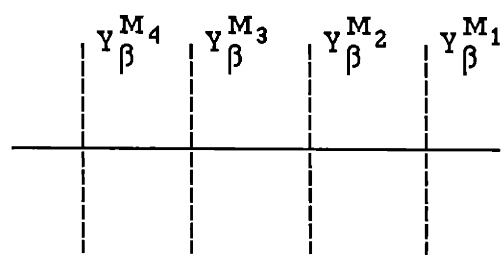
The decision rules that follow are applicable to the case where DMA uses four messages, M_1, M_2, M_3 and M_4, to communicate with DMB.

The decision thresholds for the Gaussian case are once again on the observation axes and can be seen in Figure 4. For the four-message case, the thresholds are referred to as Y_α^l, Y_α^m, Y_α^u, $Y_\beta^{M_1}$, $Y_\beta^{M_2}$, $Y_\beta^{M_3}$ and $Y_\beta^{M_4}$ (where 'l', 'm' and 'u' refer to the lower, middle and upper thresolds of DMA respectively). The threshold equations are written in terms of the same error function

$$\Phi_i^j(k) = \int_{-\infty}^{\frac{Y_i^j - \mu_k}{\sigma_i}} (2\pi)^{-0.5} \exp(-0.5x^2)\, dx \tag{46}$$



(a) Decision Thresholds for DMA



(b) Decision Thresholds for DMB

Figure 4 Gaussian Example for K = 4

where $i = \alpha, \beta$

$j = l, m, u, M_1, M_2, M_3, M_4$

$k = 0, 1$

A discussion for the corollaries of this case appears in the appendix.

Corollary 5

Given the decision u_α of DMA, the optimal decision rule of DMB is a deterministic function defined by

$$\gamma_\beta(y_\beta, u_\alpha) = \begin{cases} 0, & \text{if } y_\beta \leq Y_\beta^{M_i} \\ 1, & \text{if } y_\beta > Y_\beta^{M_i} \end{cases} \qquad \text{for } i = 1, 2, 3, 4 \tag{47}$$

where

$$Y_\beta^{M_1} = \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\alpha^l(0)}{\Phi_\alpha^l(1)}\right] + \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \tag{48}$$

$$Y_\beta^{M_2} = \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\alpha^m(0) - \Phi_\alpha^l(0)}{\Phi_\alpha^m(1) - \Phi_\alpha^l(1)}\right] + \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \tag{49}$$

$$Y_\beta^{M_3} = \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\alpha^u(0) - \Phi_\alpha^m(0)}{\Phi_\alpha^u(1) - \Phi_\alpha^m(1)}\right] + \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \tag{50}$$

$$Y_\beta^{M_4} = \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{1 - \Phi_\alpha^u(0)}{1 - \Phi_\alpha^u(1)}\right] + \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \quad (51)$$

Corollary 6

Given the decision u_β of DMB, the optimal decision rule of DMA is a deterministic function defined by

$$\gamma_\alpha(y_\alpha) = \begin{cases} M_1, & \text{if } y_\alpha \le Y_\alpha^l \\ M_2, & \text{if } Y_\alpha^l < y_\alpha \le Y_\alpha^m \\ M_3, & \text{if } Y_\alpha^m < y_\alpha \le Y_\alpha^u \\ M_4, & \text{if } y_\alpha > Y_\alpha^u \end{cases} \quad (52)$$

where

$$Y_\alpha^l = \frac{\sigma_\alpha^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\beta^{M_2}(0) - \Phi_\beta^{M_1}(0)}{\Phi_\beta^{M_2}(1) - \Phi_\beta^{M_1}(1)}\right] + \frac{\sigma_\alpha^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \quad (53)$$

$$Y_\alpha^m = \frac{\sigma_\alpha^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\beta^{M_3}(0) - \Phi_\beta^{M_2}(0)}{\Phi_\beta^{M_3}(1) - \Phi_\beta^{M_2}(1)}\right] + \frac{\sigma_\alpha^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \quad (54)$$

$$Y_\alpha^u = \frac{\sigma_\alpha^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\beta^{M_4}(0) - \Phi_\beta^{M_3}(0)}{\Phi_\beta^{M_4}(1) - \Phi_\beta^{M_3}(1)}\right] + \frac{\sigma_\alpha^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \quad (55)$$

4.3 Computation of Thresholds

The threshold equations for this case are also coupled. They are solved using the iterative method described in the previous chapter. The four-message case involves solving a system of seven non-linear simultaneous equations.

In order to get numerical results, the parameters of the problem are assigned values that appear in Section 3.3. The thresholds for the four-message case are computed. The results are shown below.

Four-message Case :

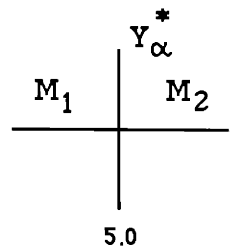
$$Y^{l}_{\alpha} = -3.15 \qquad Y^{m}_{\alpha} = 5.0 \qquad Y^{u}_{\alpha} = 13.15$$

$$Y^{M_1}_{\beta} = 18.8454 \qquad Y^{M_2}_{\beta} = 8.8548 \qquad Y^{M_3}_{\beta} = 1.1442 \qquad Y^{M_4}_{\beta} = -8.8462$$

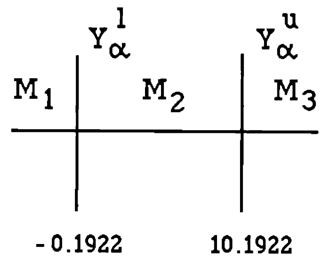
A comparison of the thresholds of DMA for the two-message, three-message and four-message cases appears in Figure 5. The following observations regarding the behavior of the thresholds are made :

(i) The thresholds for the four-message case consist of one threshold placed on either side of, and equidistant from, the stationary threshold, Y_{α}^{*}, for the two-message case.

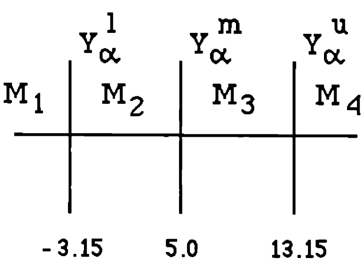
(ii) The thresholds of the three-message case are pulled apart (each by the same amount) and the stationary threshold, Y_{α}^{*}, is placed exactly in the middle to give the thresholds for the four-message case.



(a) Two-message Case



(b) Three-message Case



(c) Four-message Case

Figure 5 Threshold Comparison

(iii) We can make conjectures about the positions of the thresholds for cases where $K = 6, 8, 10$ and so on. The thresholds for the six-message case, for example, would consist of one threshold placed on either side of , and equidistant from, the three stationary thresholds for the four-message case.

(iv) We can also make conjectures about the positions of the thresholds for cases where $K = 5, 7, 9$ and so on. The thresholds for the five-message case, for example, would consist of one threshold placed on either side of , and equidistant from, the two stationary thresholds for the three-message case.

4.4 The General K Case

In this section we present results for the general problem where K messages (K can take on any value greater than two) are used by DMA to communicate his decision to DMB.

Given $P(H_0)$, $P(H_1)$, the distributions $P(y_\alpha, y_\beta \mid H_i)$ for $i = 0, 1$ with $y_\alpha \in Y_\alpha$, $y_\beta \in Y_\beta$, and the cost function $J(u_\beta, H_i)$, the optimal decision rules (one that minimizes the expected cost) of DMA and DMB are derived. DMA can use one of K messages, M_1, M_2, M_3, ..., M_K, to communicate his decision to DMB. DMB has his own observation and the message from DMA to make the final decision of the team.

The detailed proofs of the theorems for the K-message case appear in the appendix.

Theorem 7

Given the decision u_α of DMA, the optimal decision rule of DMB is a deterministic function

$$\gamma_\beta : Y_\beta \times \{ M_1, M_2, M_3, ..., M_K \} \rightarrow \{ 0, 1 \}$$

defined by the following likelihood ratio tests :

$$\gamma_\beta(y_\beta, u_\alpha) = \begin{cases} 0 , & \text{if } \Lambda_\beta(y_\beta) \geq \beta_i \\ 1 , & \text{otherwise} \end{cases} \tag{56}$$

where

$$\Lambda_\beta(y_\beta) = \frac{P(H_0)\, P(y_\beta \mid H_0)}{P(H_1)\, P(y_\beta \mid H_1)} \tag{57}$$

and

$$\beta_i = \frac{P(u_\alpha = i \mid H_1)\, [\, J(0, H_1) - J(1, H_1)\,]}{P(u_\alpha = i \mid H_0)\, [\, J(1, H_0) - J(0, H_0)\,]} \qquad \text{for } i = M_1, M_2, M_3, ..., M_K \tag{58}$$

<u>Theorem 8</u>

Given the decision $u_\beta \in \{0, 1\}$ of DMB, the decision rule of DMA is a deterministic function

$$\gamma_\alpha : Y_\alpha \rightarrow \{ M_1, M_2, M_3, ..., M_K \}$$

defined by the likelihood ratio tests on the following page.

$$\gamma_\alpha(y_\alpha) = \begin{cases} M_1\,, & \text{if } \Lambda_\alpha(y_\alpha) \geq \alpha_1 \\ M_2\,, & \text{if } \alpha_2 \leq \Lambda_\alpha(y_\alpha) < \alpha_1 \\ M_3\,, & \text{if } \alpha_3 \leq \Lambda_\alpha(y_\alpha) < \alpha_2 \\ \bullet & \\ \bullet & \\ \bullet & \\ M_{K-1}\,, & \text{if } \alpha_{K-1} \leq \Lambda_\alpha(y_\alpha) < \alpha_{K-2} \\ M_K\,, & \text{if } \Lambda_\alpha(y_\alpha) < \alpha_{K-1} \end{cases} \tag{59}$$

where

$$\Lambda_\alpha(y_\alpha) = \frac{P(H_0)\, P(y_\alpha \mid H_0)}{P(H_1)\, P(y_\alpha \mid H_1)} \tag{60}$$

and

$$\alpha_1 = \frac{\sum_{u_\beta} J(u_\beta, H_1)\, [\, P(u_\beta \mid u_\alpha = M_2, H_1) - P(u_\beta \mid u_\alpha = M_1, H_1)\,]}{\sum_{u_\beta} J(u_\beta, H_0)\, [\, P(u_\beta \mid u_\alpha = M_1, H_0) - P(u_\beta \mid u_\alpha = M_2, H_0)\,]} \tag{61}$$

$$\alpha_2 = \frac{\sum_{u_\beta} J(u_\beta, H_1)\, [\, P(u_\beta \mid u_\alpha = M_3\,, H_1) - P(u_\beta \mid u_\alpha = M_2\,, H_1)\,]}{\sum_{u_\beta} J(u_\beta, H_0)\, [\, P(u_\beta \mid u_\alpha = M_2\,, H_0) - P(u_\beta \mid u_\alpha = M_3\,, H_0)\,]} \tag{62}$$

$$\alpha_3 = \frac{\sum_{u_\beta} J(u_\beta, H_1)\,[\, J(u_\beta \mid u_\alpha = M_4, H_1) - P(u_\beta \mid u_\alpha = M_3, H_1)\,]}{\sum_{u_\beta} J(u_\beta, H_0)\,[\, P(u_\beta \mid u_\alpha = M_3, H_0) - P(u_\beta \mid u_\alpha = M_4, H_0)\,]} \qquad (63)$$

•

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•

$$\alpha_{K-1} = \frac{\sum_{u_\beta} J(u_\beta, H_1)\,[\, P(u_\beta \mid u_\alpha = M_K, H_1) - P(u_\beta \mid u_\alpha = M_{K-1}, H_1)\,]}{\sum_{u_\beta} J(u_\beta, H_0)\,[\, P(u_\beta \mid u_\alpha = M_{K-1}, H_0) - P(u_\beta \mid u_\alpha = M_K, H_0)\,]} \qquad (64)$$

4.5 Gaussian Version for Any K

The decision rules that follow are applicable to the case where DMA uses K messages (K>2), M_1, M_2, M_3, ..., M_K, to communicate with DMB.

A discussion for the corollaries of this case appears in the appendix.

Corollary 7

Given the decision u_α of DMA, the optimal decision rule of DMB is a deterministic function defined by

$$\gamma_\beta(y_\beta, u_\alpha) = \begin{cases} 0, & \text{if } y_\beta \le Y_\beta^{M_i} \\ 1, & \text{if } y_\beta > Y_\beta^{M_i} \end{cases} \qquad \text{for } i = 1, 2, 3, \ldots, K \tag{65}$$

where

$$Y_\beta^{M_1} = \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\alpha^1(0)}{\Phi_\alpha^1(1)}\right] + \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \tag{66}$$

$$Y_\beta^{M_2} = \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\alpha^2(0) - \Phi_\alpha^1(0)}{\Phi_\alpha^2(1) - \Phi_\alpha^1(1)}\right] + \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \tag{67}$$

$$Y_\beta^{M_3} = \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\alpha^3(0) - \Phi_\alpha^2(0)}{\Phi_\alpha^3(1) - \Phi_\alpha^2(1)}\right] + \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \tag{68}$$

•

•

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$$Y_\beta^{M_{K-1}} = \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\alpha^{K-1}(0) - \Phi_\alpha^{K-2}(0)}{\Phi_\alpha^{K-1}(1) - \Phi_\alpha^{K-2}(1)}\right] + \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \tag{69}$$

$$Y_\beta^{M_K} = \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{1 - \Phi_\alpha^{K-1}(0)}{1 - \Phi_\alpha^{K-1}(1)}\right] + \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \tag{70}$$

Corollary 8

Given the decision u_β of DMB, the optimal decision rule of DMA is a deterministic function defined by

$$\gamma_\alpha(y_\alpha) = \begin{cases} M_1, & \text{if } y_\alpha \le Y_\alpha^1 \\ M_2, & \text{if } Y_\alpha^1 < y_\alpha \le Y_\alpha^2 \\ M_3, & \text{if } Y_\alpha^2 < y_\alpha \le Y_\alpha^3 \\ \bullet & \\ \bullet & \\ \bullet & \\ M_{K-1}, & \text{if } Y_\alpha^{K-2} < y_\alpha \le Y_\alpha^{K-1} \\ M_K, & \text{if } y_\alpha > Y_\alpha^{K-1} \end{cases} \tag{71}$$

where

$$Y_\alpha^1 = \frac{\sigma_\alpha^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\beta^{M_2}(0) - \Phi_\beta^{M_1}(0)}{\Phi_\beta^{M_2}(1) - \Phi_\beta^{M_1}(1)}\right] + \frac{\sigma_\alpha^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \tag{72}$$

$$Y_\alpha^2 = \frac{\sigma_\alpha^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\beta^{M_3}(0) - \Phi_\beta^{M_2}(0)}{\Phi_\beta^{M_3}(1) - \Phi_\beta^{M_2}(1)}\right] + \frac{\sigma_\alpha^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2} \tag{73}$$

$$Y_{\alpha}^{3} = \frac{\sigma_{\alpha}^{2}}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_{\beta}^{M_4}(0) - \Phi_{\beta}^{M_3}(0)}{\Phi_{\beta}^{M_4}(1) - \Phi_{\beta}^{M_3}(1)} \right] + \frac{\sigma_{\alpha}^{2}}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)} \right] + \frac{\mu_0 + \mu_1}{2} \quad (74)$$

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$$Y_{\alpha}^{K-1} = \frac{\sigma_{\alpha}^{2}}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_{\beta}^{M_K}(0) - \Phi_{\beta}^{M_{K-1}}(0)}{\Phi_{\beta}^{M_K}(1) - \Phi_{\beta}^{M_{K-1}}(1)} \right] + \frac{\sigma_{\alpha}^{2}}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)} \right] + \frac{\mu_0 + \mu_1}{2} \quad (75)$$

Chapter 5

PERFORMANCE OF THE DETECTION TEAM

We can get an indication of how well a distributed detection team is doing by evaluating its performance. In this chapter, the performance is evaluated by computing the probability of error for the final team decision and generating ROC curves for the detection team. We compare the performances (by computing the probability of error) of the isolation, two-message, three-message, four-message and centralized cases to study the effect of increasing communication on the detection team. In addition, we generate ROC curves for the two-message and three-message cases to study the performance enhancement due to the increase of half a bit of information.

5.1 The Probability of Error

The computation of the probability of error of the final decision of a detection team (i.e., the probability that an incorrect decision is made) helps us evaluate the performance of the team. There are two types of errors that the team could concievably make.

I Declaring the presence of a target when in actuality there is no enemy target present. This type of error could result in firing a missile that would serve no purpose and result in a financial loss.

II Declaring the absence of a target when in actuality there is an enemy target present. This type of error could be very fatal in the event that the enemy detects you and consequently fires a missile.

The probability of error is expressed as a function of the probabilities of false alarm and miss (which is a function of the probability of detection). These probabilities are defined below.

Probability of False Alarm (P_F) *:* The probability that the team says that the target is present when it is not.

Probability of Miss (P_M) *:* The probability that the team says that the target is absent when it is present.

Probability of Detection (P_D) *:* The probability that the team says that the target is present when it is.

The Probability of Miss is a function of the Probability of Detection. The equation that relates the two quantities is given by

$$P_M = 1 - P_D \qquad (76)$$

In general, the probability of error is given by the following equation :

$$Pr(E) = P(H_0)\, P_F + P(H_1)\, P_M \qquad (77)$$

Using error functions, we have computed the probabilities of error for the two-message, three-message, four-message, isolation and centralized cases. The equations for the general K case have also been presented. The values used for the parameters of the problem are identical to those used in Section 3.3.

Two-message Case

Given that DMA can use two messages, M_1 and M_2, to communicate with DMB, the probability of error is given by

$$P(H_0)\ [\, Pr\{M_1|H_0\}\ (1 - \Phi_\beta^{M_1}(0)) + Pr\{M_2|H_0\}\ (1 - \Phi_\beta^{M_2}(0))\,]$$

$$+\ P(H_1)\ [\, Pr\{M_1|H_1\}\ (\Phi_\beta^{M_1}(1)) + Pr\{M_2|H_1\}\ (\Phi_\beta^{M_2}(1))\,] \qquad (78)$$

where

$$Pr\{M_1|H_0\} = \Phi_\alpha^*(0) \qquad \text{and} \qquad Pr\{M_1|H_1\} = \Phi_\alpha^*(1)$$

$$Pr\{M_2|H_0\} = 1 - \Phi_\alpha^*(0) \qquad \text{and} \qquad Pr\{M_2|H_1\} = 1 - \Phi_\alpha^*(1) \qquad (79)$$

Note : $Pr\{M_i|H_i\}$ is the probability that DMA says M_i given that the hypothesis H_i is the true hypothesis.

Numerically, Pr(E) = 0.25758.

Three-message Case

Given that DMA can use three messages, M_1, M_2 and M_3, to communicate with DMB, the probability of error is given by

$$\Pr(E) = P(H_0)\ [\Pr\{M_1|H_0\}\ (1 - \Phi_\beta^{M_1}(0)) + \Pr\{M_2|H_0\}\ (1 - \Phi_\beta^{M_2}(0)) + \Pr\{M_3|H_0\}\ (1 - \Phi_\beta^{M_3}(0))] + P(H_1)\ [\Pr\{M_1|H_1\}\ \Phi_\beta^{M_1}(1) + \Pr\{M_2|H_1\}\ \Phi_\beta^{M_2}(1) + \Pr\{M_3|H_1\}\ \Phi_\beta^{M_3}(1)] \tag{80}$$

where

$$\Pr\{M_1|H_0\} = \Phi_\alpha^l(0) \quad \text{and} \quad \Pr\{M_1|H_1\} = \Phi_\alpha^l(1)$$

$$\Pr\{M_2|H_0\} = \Phi_\alpha^u(0) - \Phi_\alpha^l(0) \quad \text{and} \quad \Pr\{M_2|H_1\} = \Phi_\alpha^u(1) - \Phi_\alpha^l(1)$$

$$\Pr\{M_3|H_0\} = 1 - \Phi_\alpha^u(0) \quad \text{and} \quad \Pr\{M_3|H_1\} = 1 - \Phi_\alpha^u(1) \tag{81}$$

Numerically, Pr(E) = 0.24779.

Four-message Case

Given that DMA can use four messages, M_1, M_2, M_3 and M_4, to communicate with DMB, the probability of error is given on the following page.

$$\Pr(E) = P(H_0) \; [\, \Pr\{M_1|H_0\} \; (1 - \Phi_\beta^{M_1}(0)) + \Pr\{M_2|H_0\} \; (1 - \Phi_\beta^{M_2}(0))$$

$$+ \Pr\{M_3|H_0\} \; (1 - \Phi_\beta^{M_3}(0)) + \Pr\{M_4|H_0\} \; (1 - \Phi_\beta^{M_4}(0)) \,]$$

$$+ P(H_1) \; [\, \Pr\{M_1|H_1\} \; \Phi_\beta^{M_1}(1) + \Pr\{M_2|H_1\} \; \Phi_\beta^{M_2}(1)$$

$$+ \Pr\{M_3|H_1\} \; \Phi_\beta^{M_3}(1) + \Pr\{M_4|H_1\} \; \Phi_\beta^{M_4}(1) \,] \qquad (82)$$

where

$$\Pr\{M_1|H_0\} = \Phi_\alpha^{l}(0) \quad \text{and} \quad \Pr\{M_1|H_1\} = \Phi_\alpha^{l}(1)$$

$$\Pr\{M_2|H_0\} = \Phi_\alpha^{m}(0) - \Phi_\alpha^{l}(0) \quad \text{and} \quad \Pr\{M_2|H_1\} = \Phi_\alpha^{m}(1) - \Phi_\alpha^{l}(1)$$

$$\Pr\{M_3|H_0\} = \Phi_\alpha^{u}(0) - \Phi_\alpha^{m}(0) \quad \text{and} \quad \Pr\{M_3|H_1\} = \Phi_\alpha^{u}(1) - \Phi_\alpha^{m}(1)$$

$$\Pr\{M_4|H_0\} = 1 - \Phi_\alpha^{u}(0) \quad \text{and} \quad \Pr\{M_4|H_1\} = 1 - \Phi_\alpha^{u}(1) \qquad (83)$$

Numerically, $\Pr(E) = 0.24430$.

<u>Isolated Case</u>

Given that DMB is in isolation (i.e., it receives no message from DMA), the probability of error is given by

$$P(H_0) \; [\, 1 - \Phi_\beta^{I}(0) \,] + P(H_1) \; [\, \Phi_\beta^{I}(1) \,] \qquad (84)$$

The error functions in the expression above are computed using the formula presented in Section 3.1. However, the threshold used in the computation is given by

$$Y_{\beta}^{I} = \frac{\mu_0 + \mu_1}{2} + \frac{\sigma_{\beta}^{2}}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)} \right] \tag{85}$$

Numerically, Pr(E) = 0.30854.

<u>Centralized Case</u>

Given that DMA can communicate his entire observation to DMB, the probability of error is given by

$$P(H_0) \; [1 - \Phi_{\beta}^{C}(0)] + P(H_1) \; [\Phi_{\beta}^{C}(1)] \tag{86}$$

The error functions in the expression above are computed using the formula presented in Section 3.1. However, the threshold used in the computation is given by

$$Y_{\beta}^{C} = \frac{\sqrt{2}\;(\mu_0 + \mu_1)}{2} + \frac{\sigma^{2}}{\sqrt{2}\;(\mu_1 - \mu_0)} \ln\left[\frac{P(H_0)}{1 - P(H_0)} \right] \tag{87}$$

where

$$\sigma^2 = \sigma_{\alpha}^{2} = \sigma_{\beta}^{2} \tag{88}$$

In addition, the upper limits of the integrals in the computation of the error functions are given by

$$\frac{Y_\beta^C - \sqrt{2}\ \mu_0}{\sigma} \quad \text{and} \quad \frac{Y_\beta^C - \sqrt{2}\ \mu_1}{\sigma} \quad \text{respectively.} \tag{89}$$

Numerically, Pr(E) = 0.23975.

It is evident that there is an inverse relationship between the increase in communication and the probability of error of the detection team. In particular, the isolation case has the greatest probability of error associated with it. When DMA is allowed to communicate bits of information to DMB, the probability of error of the team decreases with the increase of each half bit of information. The numerical value of the probability of error tends to approach the centralized version of the problem (which has the least probability of error associated with it).

The General K Case

Given that DMA can use K messages, M_1, M_2, M_3, ..., M_K, to communicate with DMB, the probability of error is given by

$$\begin{aligned} Pr(E) = P(H_0)\ [\, & Pr\{M_1|H_0\}\ (1 - \Phi_\beta^{M_1}(0)) + Pr\{M_2|H_0\}\ (1 - \Phi_\beta^{M_2}(0)) \\ & + Pr\{M_3|H_0\}\ (1 - \Phi_\beta^{M_3}(0)) + \ldots + Pr\{M_K|H_0\}\ (1 - \Phi_\beta^{M_K}(0))\,] \\ & + P(H_1)\ [\, Pr\{M_1|H_1\}\ \Phi_\beta^{M_1}(1) + Pr\{M_2|H_1\}\ \Phi_\beta^{M_2}(1) \\ & + Pr\{M_3|H_1\}\ \Phi_\beta^{M_3}(1) + \ldots + Pr\{M_K|H_1\}\ \Phi_\beta^{M_K}(1)\,] \end{aligned} \tag{90}$$

where

$$\Pr\{M_1|H_0\} = \Phi^1_\alpha(0) \quad \text{and} \quad \Pr\{M_1|H_1\} = \Phi^1_\alpha(1)$$

$$\Pr\{M_2|H_0\} = \Phi^2_\alpha(0) - \Phi^1_\alpha(0) \quad \text{and} \quad \Pr\{M_2|H_1\} = \Phi^2_\alpha(1) - \Phi^1_\alpha(1)$$

$$\Pr\{M_3|H_0\} = \Phi^3_\alpha(0) - \Phi^2_\alpha(0) \quad \text{and} \quad \Pr\{M_3|H_1\} = \Phi^3_\alpha(1) - \Phi^2_\alpha(1)$$

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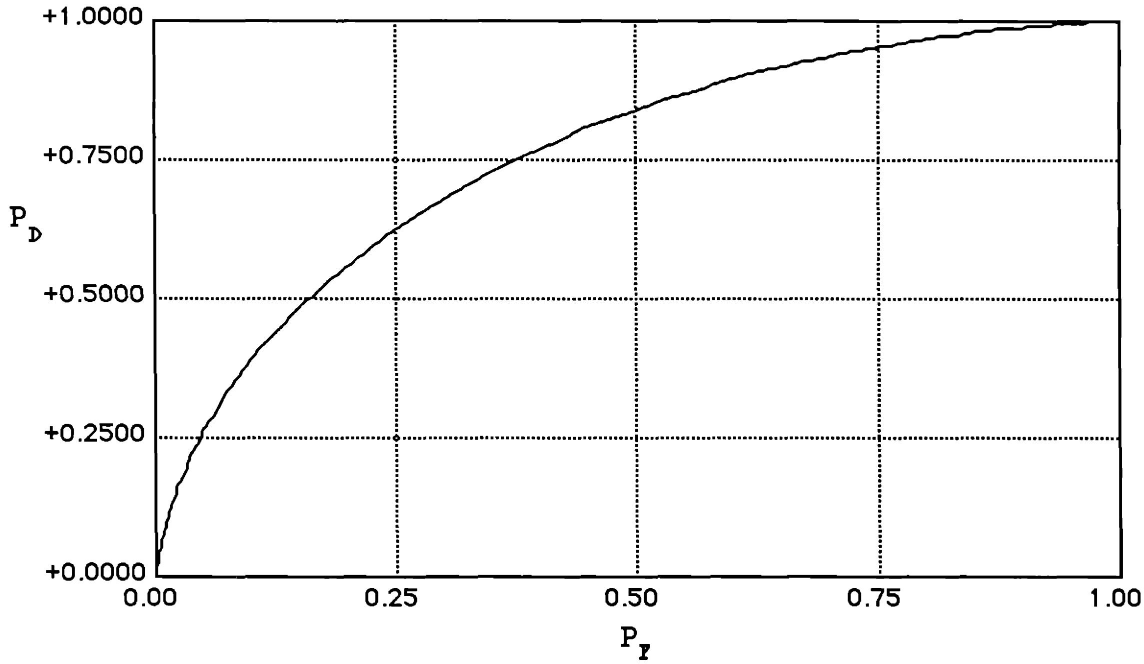
$$\Pr\{M_{K-1}|H_0\} = \Phi^{K-1}_\alpha(0) - \Phi^{K-2}_\alpha(0) \quad \text{and} \quad \Pr\{M_{K-1}|H_1\} = \Phi^{K-1}_\alpha(1) - \Phi^{K-2}_\alpha(1)$$

$$\Pr\{M_K|H_0\} = 1 - \Phi^{K-1}_\alpha(0) \quad \text{and} \quad \Pr\{M_K|H_1\} = 1 - \Phi^{K-1}_\alpha(1) \tag{91}$$

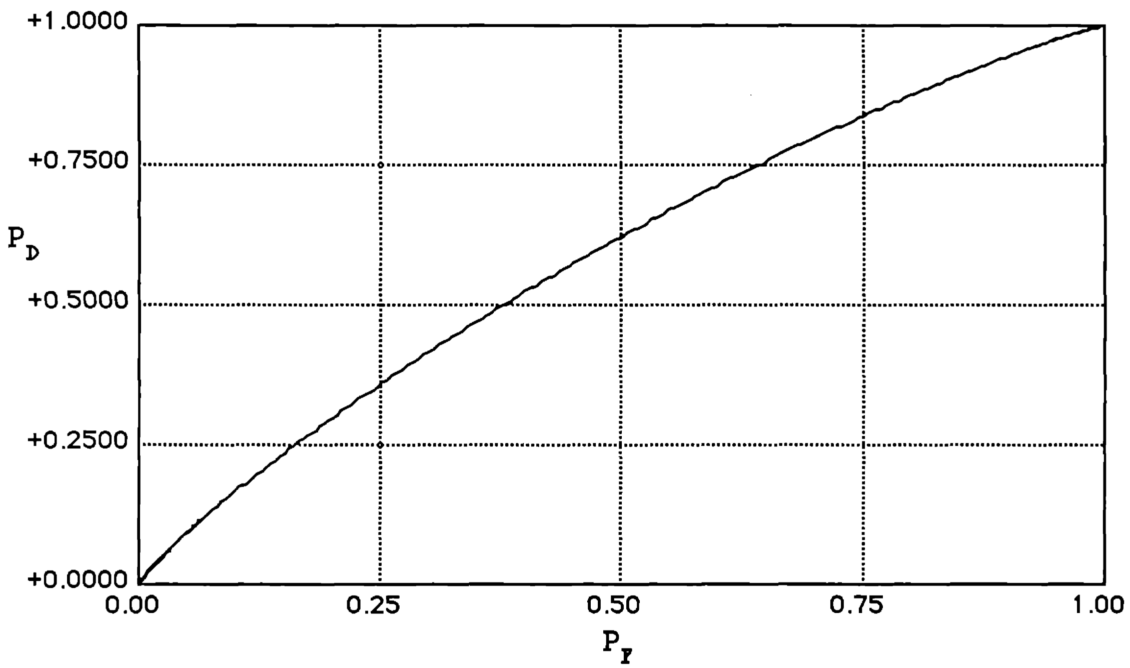
5.2 The ROC Curve

The Receiver Operating Characteristic (ROC) Curve is a plot of P_D vs. P_F. This curve gives us a good indication of the team's performance. It is derived by plotting the points (P_F, P_D) that are obtained by varying a parameter such as the decision threshold for the team. Given a specific point (P_F, P_D), it is possible to derive the numerical value of the Pr(E). However, it is usually easier to calculate the Pr(E) as described in the previous section. In general, greater concavity of the ROC curve is associated with better performance.

Before the ROC curves for the detection team are presented, we will look at the ROC curves of the individual decision makers to give an indication of how they can measure performance. In order to generate the ROC curves in Figure 5, the decision makers have



(a) ROC curve for DMA ($\sigma_\alpha^2 = 100$)

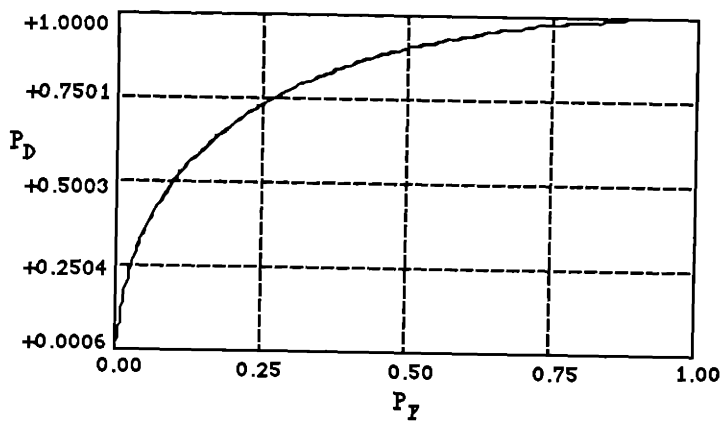


(b) ROC curve for DMB ($\sigma_\beta^2 = 1000$)

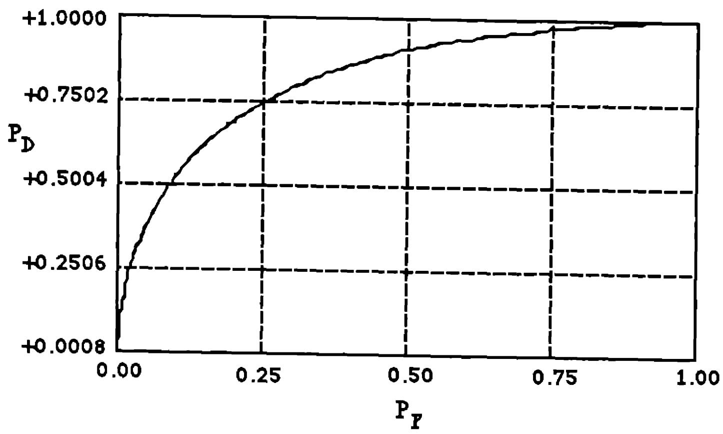
Figure 6 Individual DM's ROC Curves

been placed in isolation and their decision thresholds have been varied to obtain a set of (P_F, P_D) points. The means of the observations under the hypotheses H_0 and H_1 have been chosen to be 0 and 10 respectively. From Figure 5 we see that when a decision maker (DMA) has a better quality of observations, his performance is better than one whose observations are of poor quality (DMB). Not only is this deduced by looking at the concavity of the ROC curves, but it also makes intuitive sense.

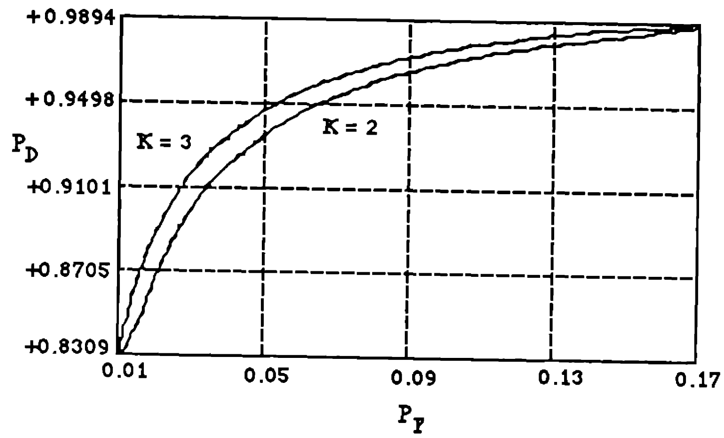
In the case of team ROC curves, we will compare the curves of the two-message and three-message cases to see if the increase in communication capability has any effect on the team performance. The points, (P_F, P_D), for these curves have been generated by varying the thresholds (which is accomplished by varying $P(H_0)$) once again, even though the expressions for P_F and P_D are significantly more complex for the team scenario. The ROC curves for the two-message and three-message cases appear in Figure 6. Since it is difficult to compare the curves when looking at them individually, we have superimposed a portion of one curve onto a portion of another. Looking at Figure 6c, we see the performance enhancement we have achieved by providing DMA with an extra half bit of communication capability.



(a) Two-message Case



(b) Three-message Case



(c) Two-message and Three-message Cases

Figure 7 Team ROC Curves

Chapter 6

NUMERICAL SENSITIVITY ANALYSES

We now investigate the behavior of the detection team by performing sensitivity studies to the solution of the linear Gaussian example. Our objective is to analyze the effects of varying the parameters of the problem on the team performance. We vary the quality of observations of each decision maker and the a priori probability of each hypotheses to see the effect on the probability of error and the thresholds of the decision makers.

The values of the parameters employed in the sensitivity analyses are the baseline parameter values given below.

$$\text{Under } H_0 : \; \mu = \mu_0 = 0 \qquad \sigma_\alpha^2 = 100$$

$$\text{Under } H_1 : \; \mu = \mu_1 = 10 \qquad \sigma_\beta^2 = 100$$

The value of $P(H_0)$ used for the analysis will be specified for each case. Also, any changes from the above values will be brought to attention.

6.1 Effects of varying parameters on the Pr(E)

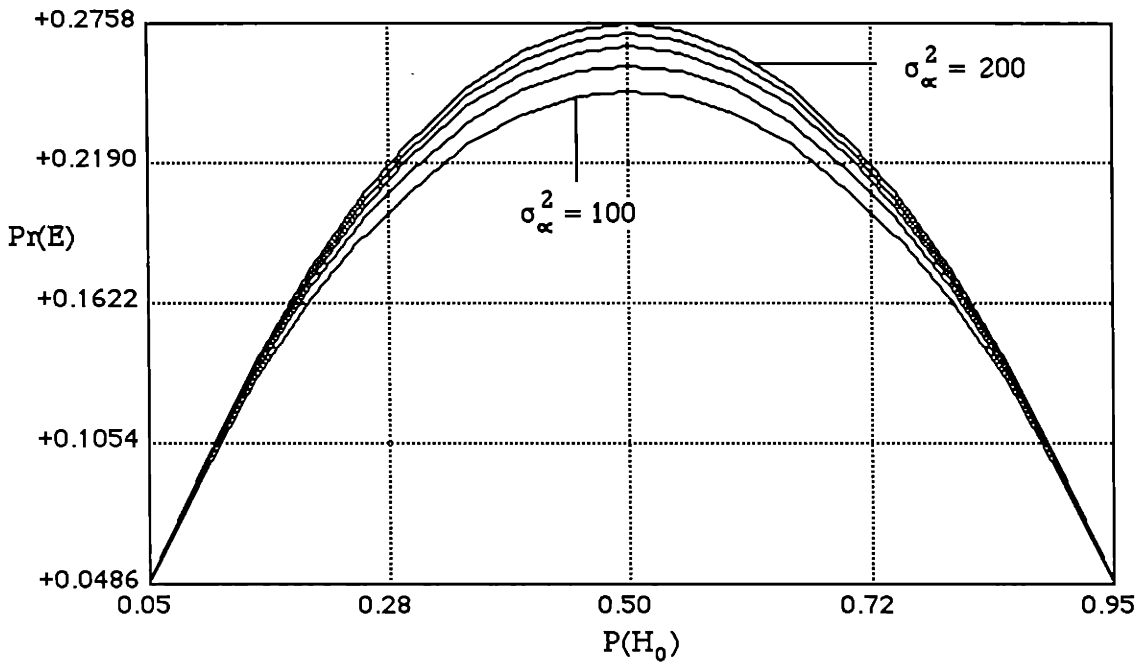
Here we vary $P(H_0)$, σ_α^2 and σ_β^2 and see the effect it has on the probability of error of the final decision of the team in the three-message case.

Effects of varying $P(H_0)$ on the Pr(E) :

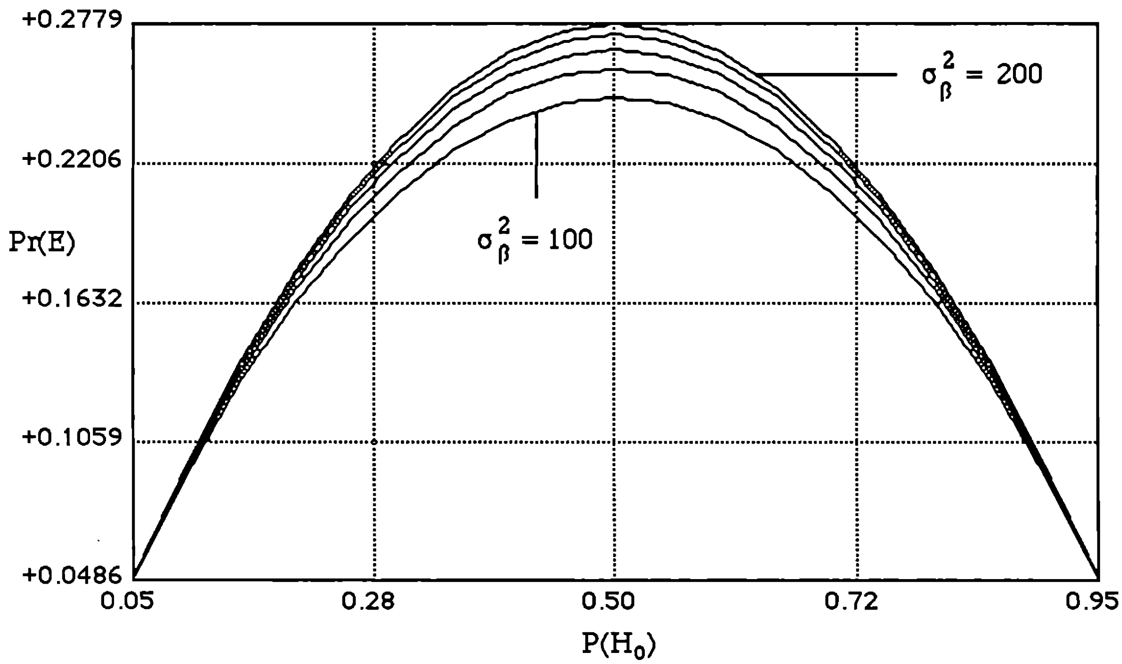
In Figure 8, families of curves (obtained by holding each of the variances of the observations of the DM's constant while varying the other) have been presented. It is clear that the Pr(E) is the largest when there is the most prior uncertainty (i.e., when $P(H_0)$ = 0.5). The curves in Figure 8a were obtained by holding σ_β^2 at 100 and varying σ_α^2 from 100 to 200 (using increments of 25). The largest value of the Pr(E) is obtained for the case where $\sigma_\alpha^2 = 200$ and $\sigma_\beta^2 = 100$. In order to get the family of curves that appear in Figure 8b, σ_α^2 is held at 100 while σ_β^2 is varied from 100 to 200 (once again using increments of 25). The performance is worst when $\sigma_\alpha^2 = 100$ and $\sigma_\beta^2 = 200$. It is interesting to note that when DMB is the "smarter" DM the worst performance is given by Pr(E) = 0.2758. However, when DMA is the smarter DM the largest error is given by Pr(E) = 0.2779. Hence, we can conclude that the smarter decision maker should be "downstream". Since the downstream DM makes the final decision of the team, this result makes intuitive sense.

Effects of varying σ_α^2 and σ_β^2 on the Pr(E) :

Figure 9a shows the results of varying σ_α^2 from 100 to 5000. There is a steep increase in the Pr(E) initially and it finally levels off with the increase in σ_α^2. In other words, there is a point ($\sigma_\alpha^2 = 5000$) beyond which the performance of the team is insensitive to the extremely poor quality of observations of DMA. The behavior of the team when σ_β^2 is

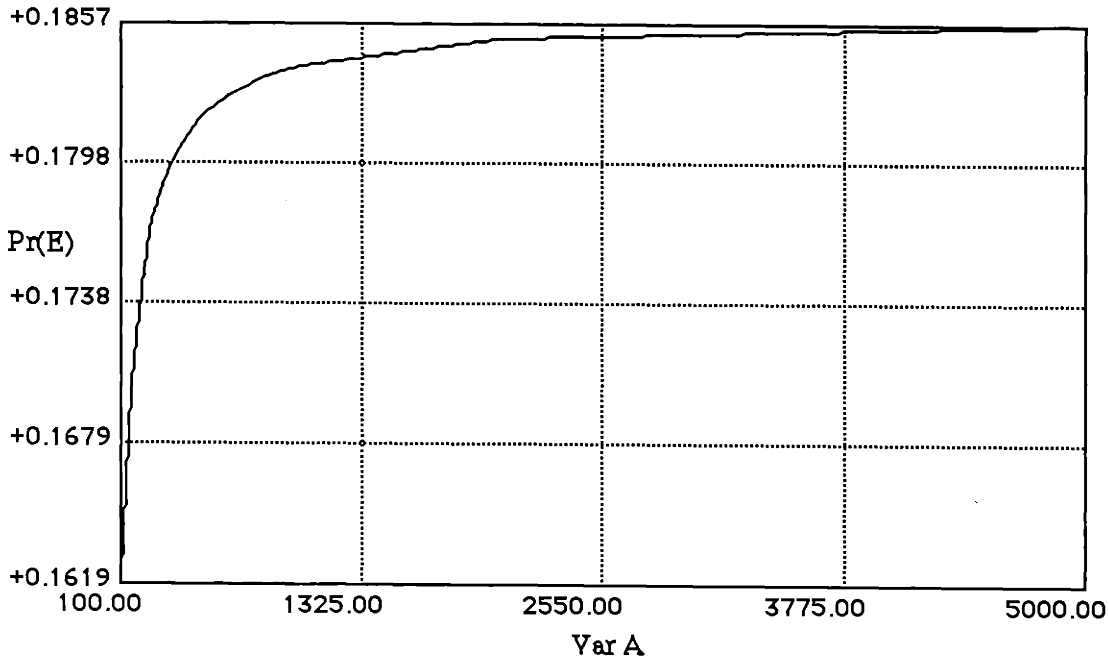


(a) $\sigma_\alpha{}^2 = 100, 125, 150, 175, 200 \quad \sigma_\beta{}^2 = 100$

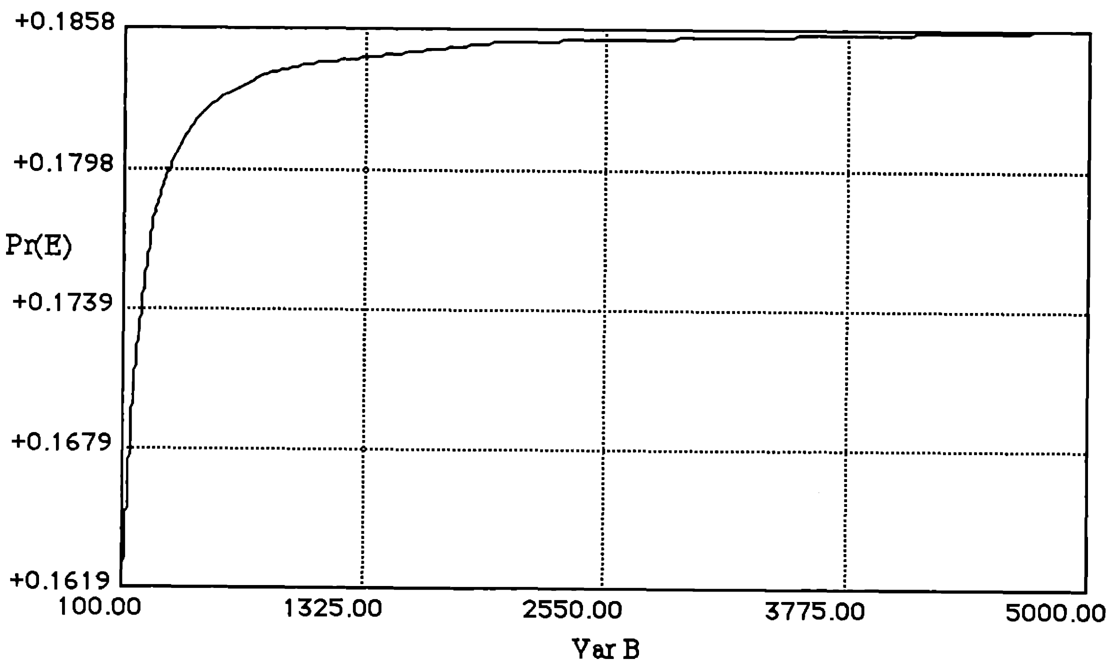


(b) $\sigma_\alpha{}^2 = 100 \quad \sigma_\beta{}^2 = 100, 125, 150, 175, 200$

Figure 8 Pr(E) vs. $P(H_0)$ (K = 3)



(a) $\sigma_\beta^2 = 100$ $P(H_0) = 0.8$



$\sigma_\alpha^2 = 100$ $P(H_0) = 0.8$

Figure 9 Pr(E) vs. σ_α^2, σ_β^2 (K = 3) varied in the same manner is similar to the above case. The results appear in Figure 9b. By comparing the two plots, we see once again that the smarter DM should be placed downstream.

6.2 Comparison of Performance

Using the plot of Pr(E) vs. $P(H_0)$ it is possible to compare the performance of the isolation, two-message, three-message, four-message and centralized cases. The goal is to see the effect increasing communication has on the team performance. In Figure 10 we have presented all the cases plotted together for the sake of comparison. It can be seen that with the increase in information, the performance of the team approaches that of the centralized version of the problem, which corresponds to an infinite number of messages. The improvement expressed as a percentage is as follows :

Isolation to the Two-message Case : 17%

Two-message to the Three-message Case : 4%

Three-message to the Four-message Case : 1.4%

Four-message to the Centralized Case : 1.8%

Hence, there is a significant improvement in performance when the team members communicate as opposed to being in isolation. However, there is no point in using more than a few bits of information to communicate from DMA to DMB since there is not much room for improvement (only 1.8%). It can also be shown that decision makers operating as a team acknowledge their team member's capability and make decisions that benefit the team. We compute the Pr(E) for three different scenarios to prove this.

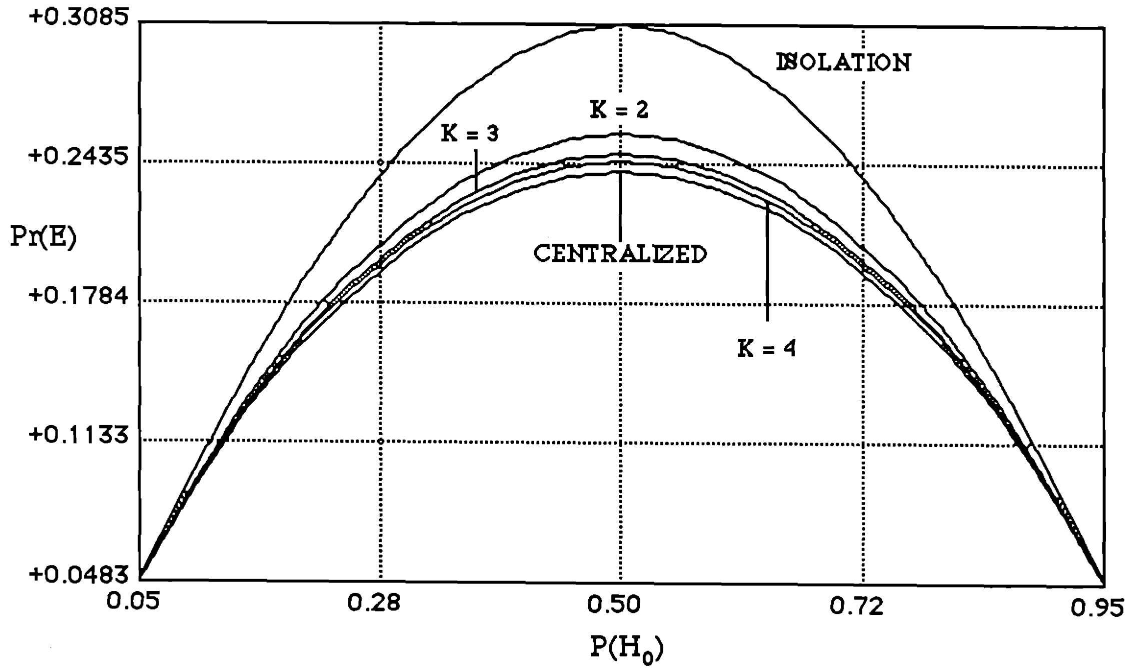


Figure 10 comparison of Performance ($\sigma_\alpha^2 = \sigma_\beta^2 = 100$)

(i) A decision maker in isolation with $\sigma_{DM}^2 = 100$.

$$Pr(E) = 0.18615$$

(ii) DMA and DMB operating as a team with DMA using three messages (K=3) to communicate with DMB. In this case, $\sigma_\alpha^2 = 20{,}000$ and $\sigma_\beta^2 = 100$.

$$Pr(E) = 0.18605$$

(iii) The centralized version of the problem where DMA passes on his entire observation to DMB. Once again, $\sigma_\alpha^2 = 20{,}000$ and $\sigma_\beta^2 = 100$.

$$Pr(E) = 0.18602$$

Thus, we see that the performance of the three cases are comparable. Comparing (i) and (ii), we see that DMB does not consider DMA's information as being of any value when the quality of DMA's observation is poor. Hence, DMB acts as though it were in isolation. Comparing (ii) and (iii), we conclude that if DMA's observations are of poor quality, the amount of communication capability given to DMA does not matter since DMB is going to ignore his observation anyway.

Finally, we compare the Pr(E) for two different scenarios to see the effect of increasing communication.

(i) The decision makers operating as a team, with DMA being the smarter one. The two-message, three-message and centralized cases are considered for comparison.

(ii) The decision makers operating as a team, with DMB being the smarter one. Again, the two-message, three-message and centralized cases are considered for comparison.

From Figure 11, we conclude that the improvement in performance from the two-message to the three-message case is greater if DMA is the smarter decision maker. Hence, we show again that there is no point in giving more communication capability to a decision maker who is not smart.

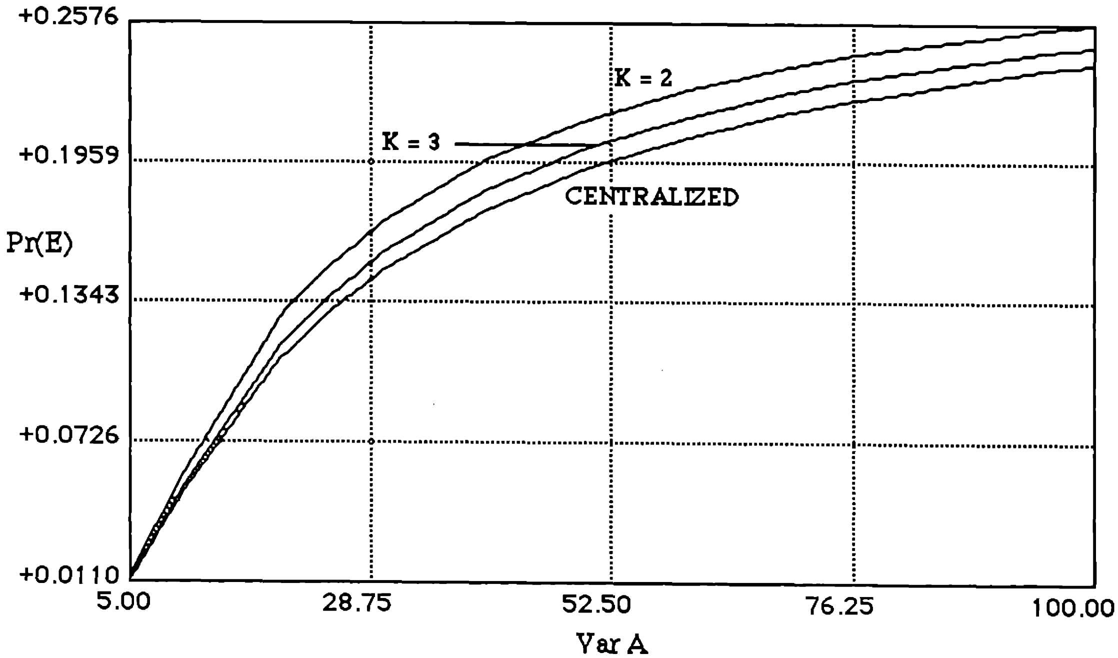
6.3 Threshold Analysis

Here we study the thresholds of the decision makers for the three-message case as we vary σ_α^2, σ_β^2 and $P(H_0)$. The thresholds of the decision makers is another way of representing probabilities of decision makers' decisions since decision regions are characterized by thresholds. Thus, performing a threshold analysis proves to be very informative.

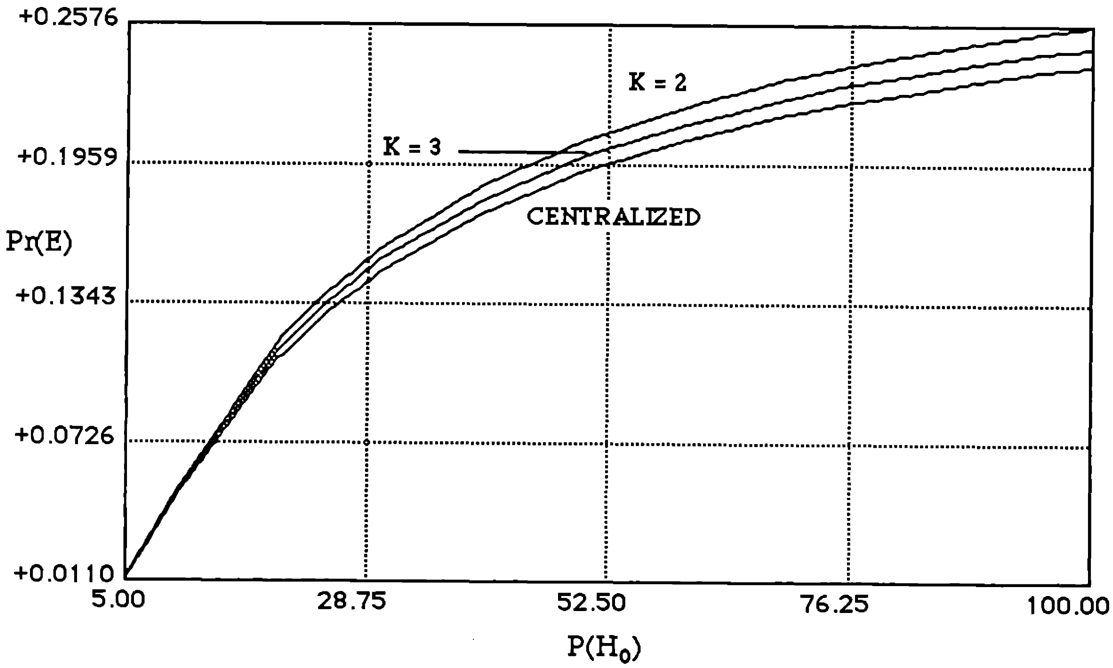
Effects of varying σ_α^2 on the thresholds of the DM's :

In Figure 12, the quality of DMA's observations (i.e., σ_α^2) has been varied to study the behavior of DMA's thresholds. Figure 11a shows the symmetric behavior of the thresholds for the case when $P(H_0) = 0.5$. We can see in Figure 11b (when $P(H_0) = 0.8$) that DMA tends to be biased towards the more likely hypothesis as σ_α^2 increases.

In Figure 13, we vary the quality of observations of DMA (i.e., σ_α^2) to study the behavior of DMB's thresholds. Again, there is symmetry in the case where $P(H_0) = 0.5$.

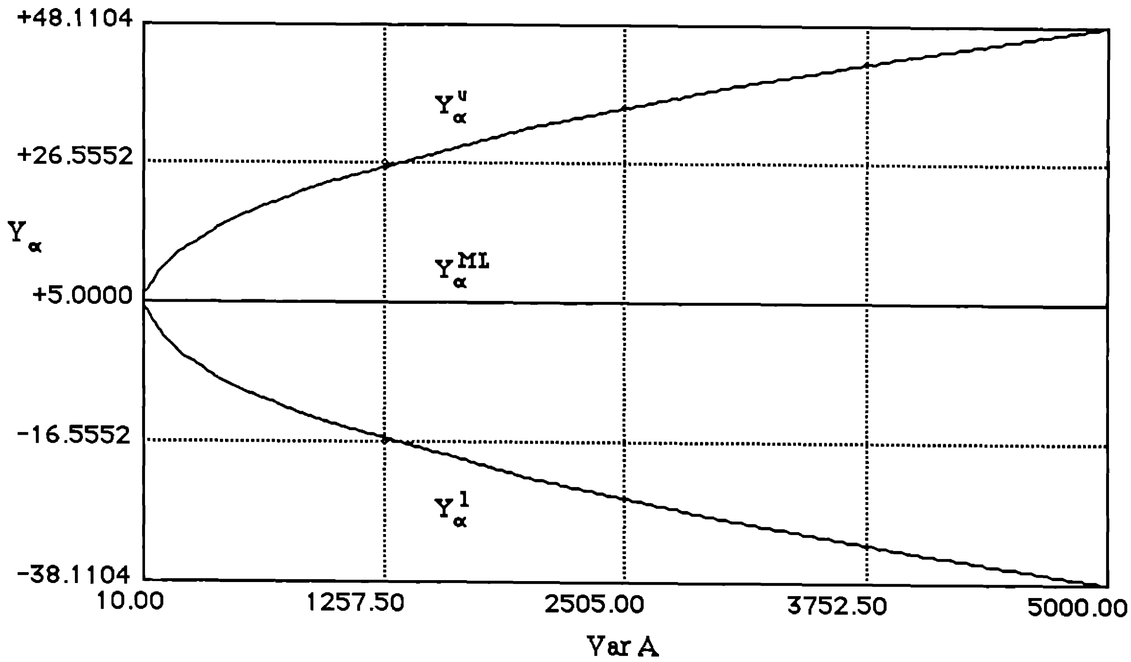


(a) DMA smarter ($\sigma_\beta^2 = 100$)

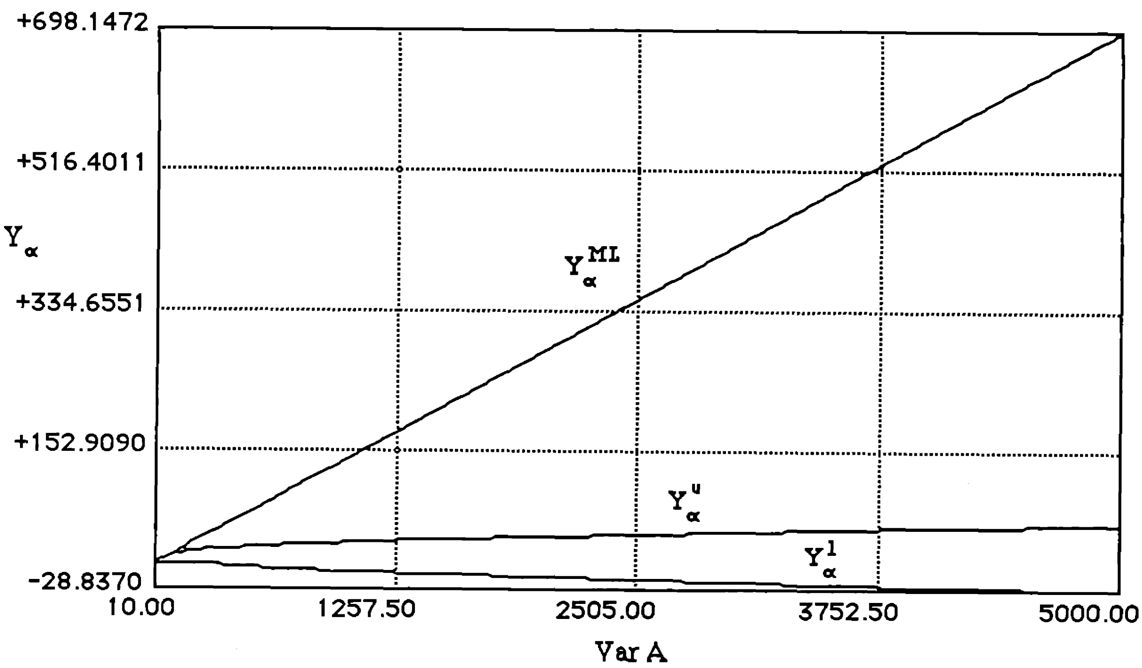


(b) DMB smarter ($\sigma_\alpha^2 = 100$)

Figure 11 Effect of Increasing Communication

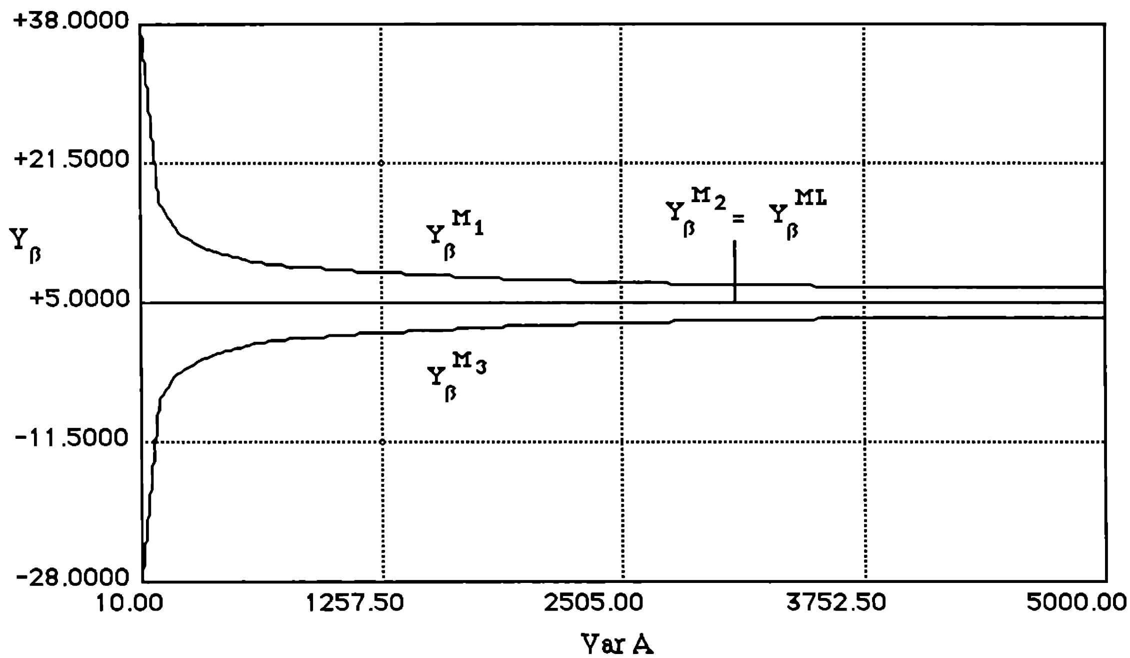


(a) $P(H_0) = 0.5 \quad \sigma_\beta^2 = 100$

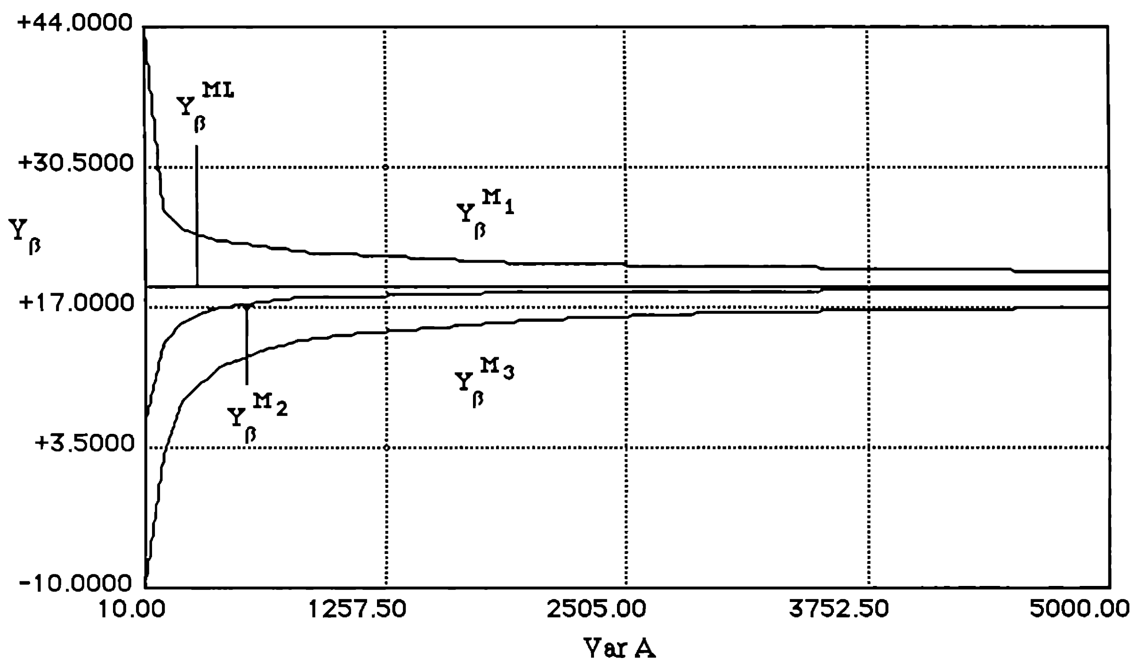


(b) $P(H_0) = 0.8 \quad \sigma_\beta^2 = 100$

Figure 12 Thresholds of DMA vs. σ_α^2 $(K = 3)$



(a) $P(H_0) = 0.5 \quad \sigma_\beta^2 = 100$



(b) $P(H_0) = 0.8 \quad \sigma_\beta^2 = 100$

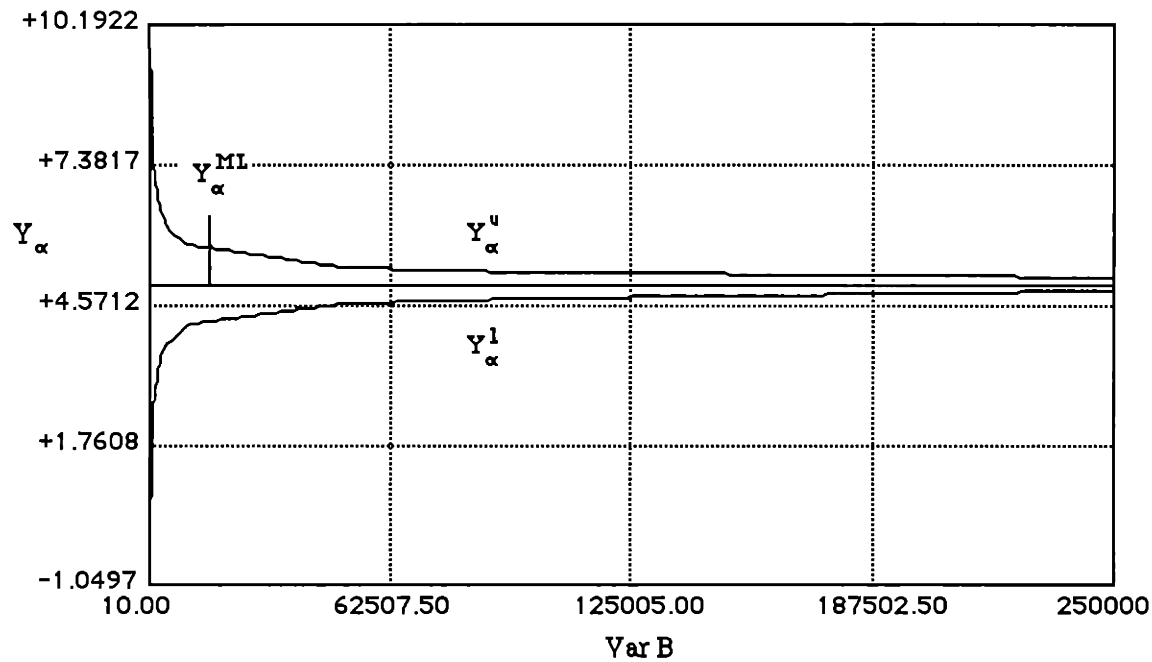
Figure 13 Thresholds of DMB vs. σ_α^2 $(K = 3)$

The threshold, $Y_\beta^{M_2}$ (when DMA declares M_2), coincides with the maximum likelihood threshold of DMB since DMA is not giving any indication regarding the absence or presence of a target to DMB. For both cases ($P(H_0) = 0.5$ and $P(H_0) = 0.8$), as σ_α^2 increases, all the thresholds approach the maximum likelihood threshold of DMB. This is due to the fact that the quality of information from DMA is very poor and DMB tends to give it less importance. Thus, DMB uses logic and intuition in making his decision.

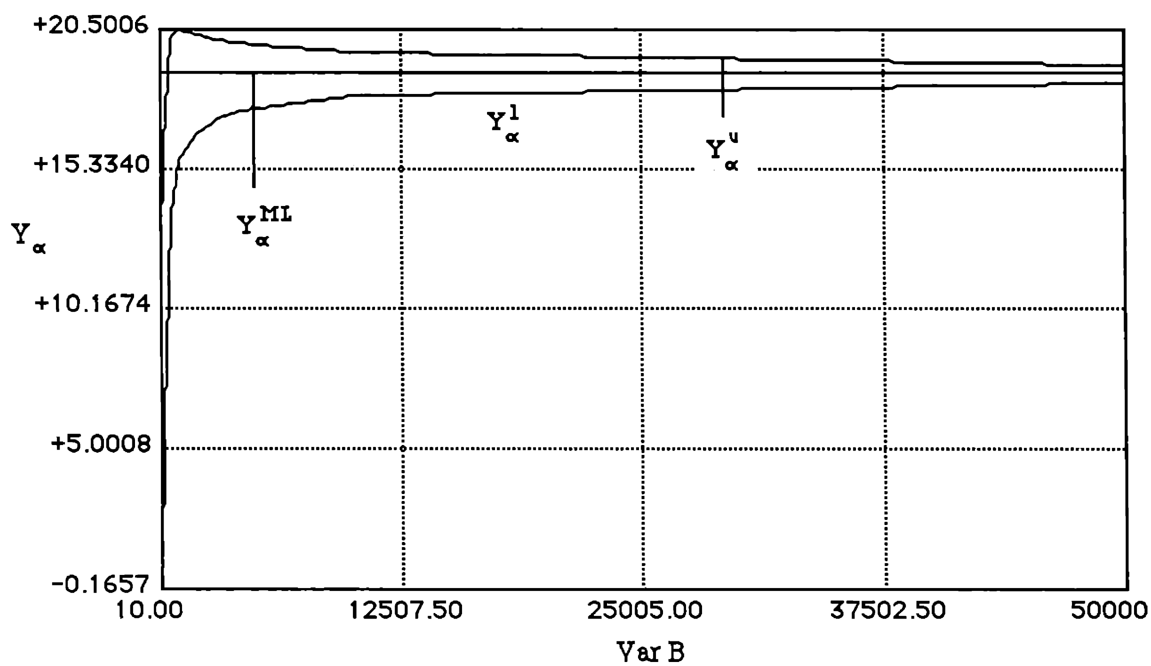
Effects of varying σ_β^2 on the thresholds of the DM's :

In Figure 14, the quality of DMB's observations (i.e., σ_β^2) has been varied to study the behavior of DMA's thresholds. Figure 14a shows symmetry due to the chosen value of $P(H_0) = 0.5$. In Figure 14b, we see that DMA tends to declare M_1 more often than in 14a since the a priori probability of the target being absent is greater (i.e., $P(H_0) = 0.8$). For both cases ($P(H_0) = 0.5$ and $P(H_0) = 0.8$), as σ_β^2 increases, DMA tends to be more decisive (by declaring M_1 or M_3) and gives a more definitive message since he knows DMB's quality of information is poor. Hence, the lower and upper thresholds of DMA tend to converge together for high σ_β^2. In Figure 14b, we see that DMA decides not to rely on DMB's observations after $\sigma_\beta^2 = 1000$ and hence the thresholds converge rather rapidly after this point. We conclude that DMA is demonstrating team behavior by acknowledging the capability of DMB.

In Figure 15, we vary the quality of observations of DMB (i.e., σ_β^2) to study the behavior of DMB's thresholds. Again, in Figure 15a we see the symmetry due to the chosen value of $P(H_0) = 0.5$. We can see the bias of DMB towards declaring the absence of the target since $P(H_0) = 0.8$ in Figure 15b. It is evident that, for high σ_β^2, DMB is biased towards what DMA declared since its own observation is of poor quality. We conclude that DMB acts in the best interest of the team.

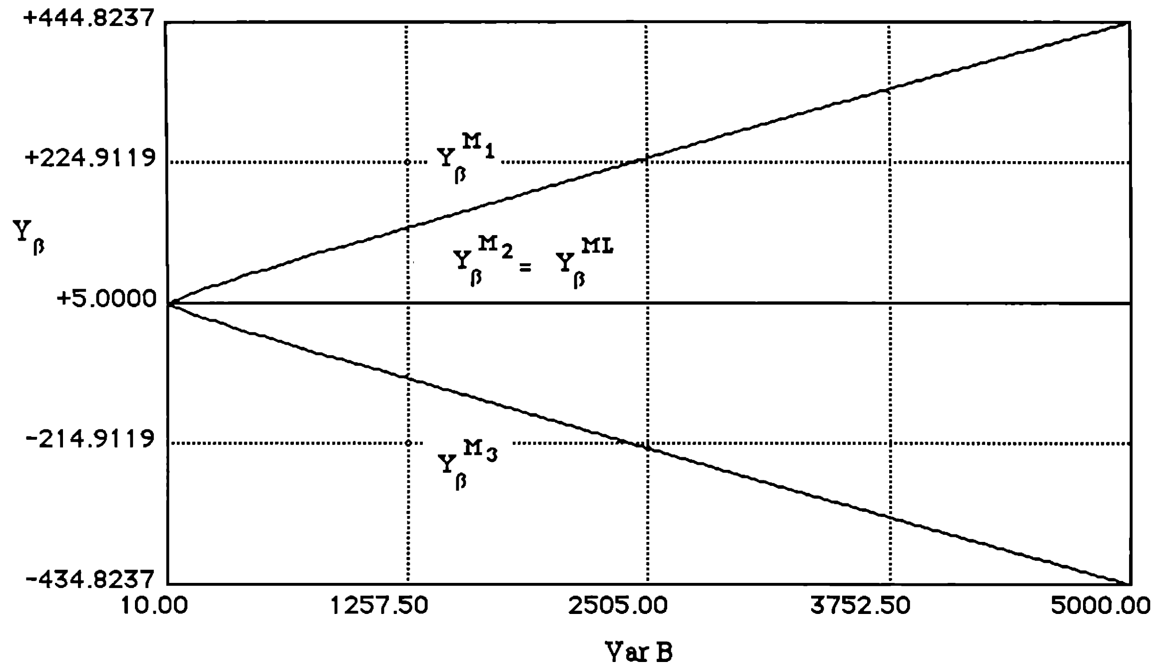


(a) $P(H_0) = 0.5 \quad \sigma_\alpha^2 = 100$

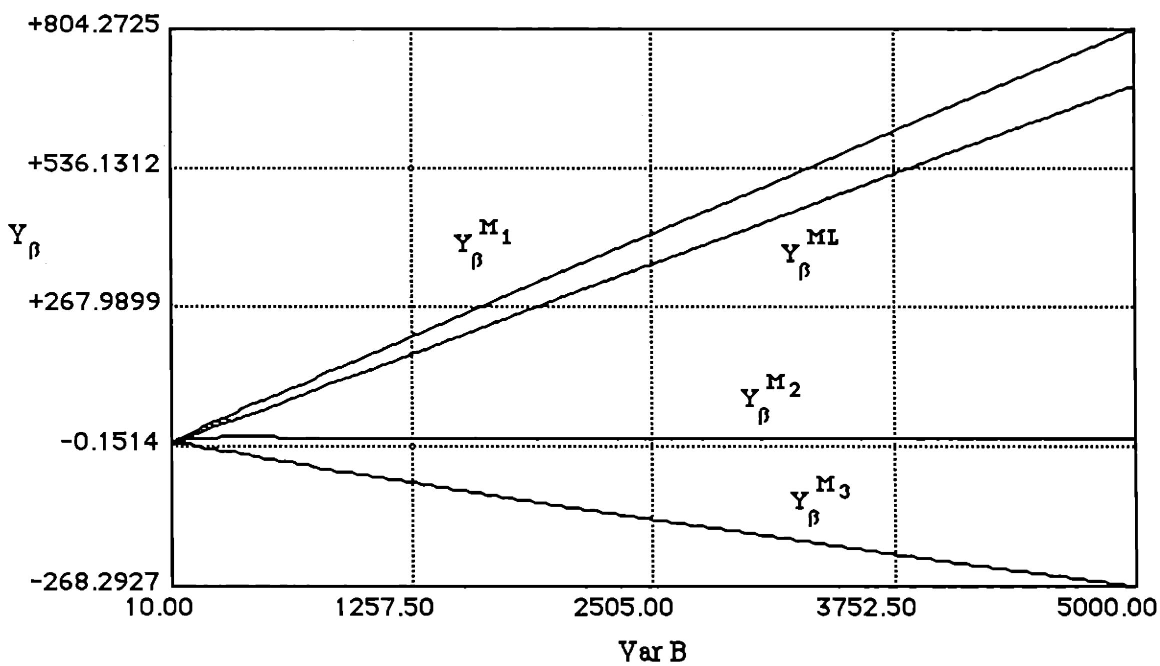


(b) $P(H_0) = 0.8 \quad \sigma_\alpha^2 = 100$

Figure 14 Thresholds of DMA vs σ_β^2 (K = 3)



(a) $P(H_0) = 0.5 \quad \sigma_\alpha^2 = 100$



(b) $P(H_0) = 0.8 \quad \sigma_\alpha^2 = 100$

Figure 15 Thresholds of DMB vs. σ_β^2 $(K = 3)$

Effects of varying $P(H_0)$ on the thresholds of the DM's :

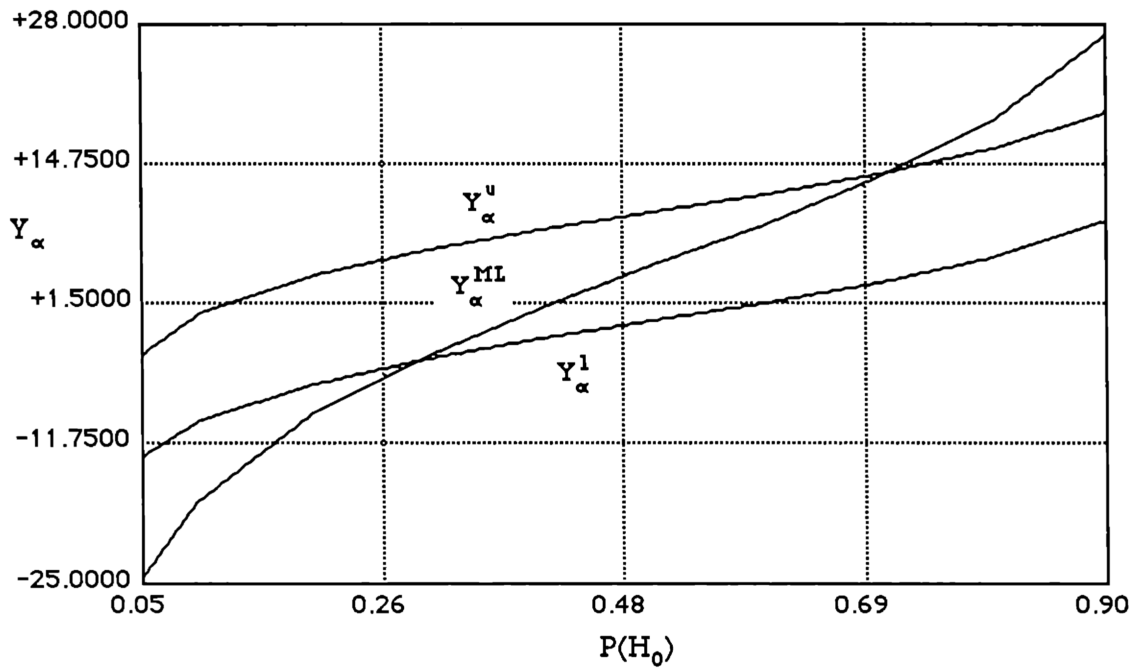
In Figure 16, the a priori probability of the hypothesis H_0, $P(H_0)$, has been varied to study the behavior of the thresholds of the decision makers. There is symmetry in the behavior of the thresholds for DMA and DMB about $P(H_0) = 0.5$. From Figure 16a, we see that DMA favors declaring M_2 around $P(H_0) = 0.5$. Towards the extreme values of $P(H_0)$ (both low and high values), DMA tends to be more decisive. In other words, for low values of $P(H_0)$ DMA is biased towards declaring M_3 and for high values of $P(H_0)$ DMA is biased towards declaring M_1. In Figure 16b, the point X represents the threshold when $P(H_0) = 0.5$ and DMA declares M_2. The point X also represents the maximum likelihood threshold for $P(H_0) = 0.5$. This makes intuitive sense and is not surprising. The point Y, in Figure 16b, shows how the decision makers operate as a team rather than in isolation. If DMB were in isolation, point Y is associated with a target being present. However, if DMA were to declare M_1 in a team setting, DMB would declare the absence of a target given point Y.

6.4 Summary

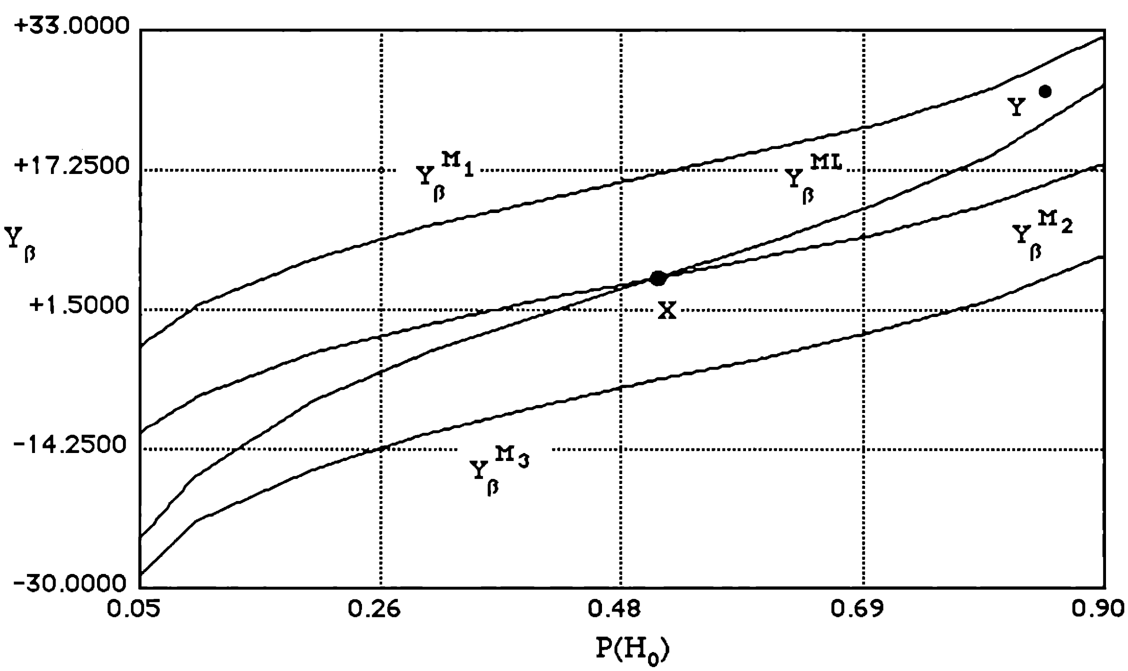
The following observations summarize the results of the numerical studies.

(i) For optimum performance, it was found that the smarter decision maker should be placed downstream.

(ii) We realized that there was no point in increasing the communication capacity of a dumb decision maker (i.e., DMA in our case).



(a) DMA Thresholds $(\sigma_\alpha{}^2 = \sigma_\beta{}^2 = 100)$



(b) DMB Thresholds $(\sigma_\alpha{}^2 = \sigma_\beta{}^2 = 100)$

Figure 16 Thresholds of DMA and DMB vs. $P(H_0)$ $(K = 3)$

(iii) The benefits of the team structure are best seen around $P(H_0) = 0.5$ (i.e., where the prior uncertainty is the greatest).

(iv) It was found that the decision makers of a team make decisions that are in total contrast of their decisions that they would make if placed in isolation. In other words, they operate as a team and make decisions that benefit the team.

Chapter 7

CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

This chapter contains the conclusions of this research and suggestions for future research.

7.1 Conclusions

A distributed detection problem, one of the simplest forms of decentralized decision making, has been studied. The detection team consists of two decision makers, DMA and DMB, each receiving an observation. There is a one-way communication between the decision makers (from DMA to DMB). The goal of the team is to make a decision regarding the presence or absence of a target, while trying to minimize the probability of error (i.e., minimizing the cost function, which depends on the team decision and the true hypothesis).

We extended the two-message case, where DMA uses two-messages, M_1 and M_2, to communicate with DMB, to one where three-messages, M_1, M_2 and M_3, are used by DMA for communication. We showed that by invoking the conditional independence assumption the optimal decision rules of both decision makers are given by deterministic functions, expressed as likelihood ratio tests with constant thresholds. The optimum decision thresholds of the two decision makers for the three-message case are coupled and cannot be expressed in closed form. In the linear Gaussian case the optimal decision rules reduce to threshold tests on the observation axes. We studied the improvement in performance achieved by increasing the communication capacity of DMA by half a bit. Furthermore, the Gaussian example was used to perform sensitivity studies to enhance our knowledge of team behavior.

We concluded that in order to optimize performance, the smarter DM (if there exists one) should be placed downstream. This result was expected since the downstream decision maker makes the final decision of the team. In addition, we found that there was no point in increasing the communication capability of a "dumb" decision maker. Regardless of the communication capability of DMA, DMB is going to give less importance to his message knowing DMA's observation is of poor quality.

Through the sensitivity analysis we learnt that the benefits of the team structure are best seen around $P(H_0) = 0.5$. This is not surprising since in an isolated setting a decision maker is most uncertain at $P(H_0) = 0.5$, and communication between decision makers of a team would be very helpful. It was also found that decision makers operate as a team and make decisions that benefit the team. For instance, a team member makes decisions which are in total contrast with the decision he would make if he were in isolation and not a member of the team.

Finally, the number of messages used by DMA to communicate with DMB is increased beyond three. The Four-message Case is solved and its performance is evaluated. In addition, the General K Case is solved and the threshold equations have been presented. The goal was to see how quickly increasing communication between the team members in this manner results in the performance of the team approaching that of the centralized version of the problem (a single decision maker receiving two observations). We conclude that on the basis of these numerical studies, there is not much potential for improvement in performance beyond the use of two bits.

7.2 Suggestions for Future Research

Using the same team structure, it would be interesting to assign a cost to each of the messages used by DMA (i.e., using four messages would be more expensive than using three messages) and find an optimum point (i.e., number of messages) without sacrificing the performance of the team. In other words, the cost and the performance of the team would have to be chosen optimally.

Another extension would be to consider the problem where the observations that the decision makers receive are n-dimensional, or rather, more complex vector-valued quantities (as opposed to mere scalars such as y_α and y_β). It would be interesting to see what implications this has on the structure of the decision rules of the members of a detection team. This generalization results in a more realistic problem and will prove to be a building block for more complex organizations.

Appendix

The detailed proofs of the various theorems and corollaries are presented in this appendix.

Proof of Theorem 3 :

The objective is to minimize the expected value of the cost function which can explicitly be written as

$$E\{J(u_\beta, H)\} = \sum_{u_\alpha, u_\beta, H} \int_{y_\alpha, y_\beta} P(u_\alpha, u_\beta, H, y_\alpha, y_\beta)\, J(u_\beta, H)\, dy_\alpha dy_\beta$$

$$= \sum_{u_\alpha, u_\beta, H} \int_{y_\alpha, y_\beta} P(u_\beta \mid u_\alpha, H, y_\alpha, y_\beta)\, P(u_\alpha, y_\alpha, y_\beta \mid H)\, P(H)\, J(u_\beta, H)\, dy_\alpha dy_\beta$$

Invoking the appropriate independence assumptions (Assumption 2) yields

$$= \sum_{u_\alpha, u_\beta, H} \int_{y_\alpha, y_\beta} P(u_\beta \mid u_\alpha, y_\beta)\, P(u_\alpha, y_\alpha \mid H)\, P(y_\beta \mid H)\, P(H)\, J(u_\beta, H)\, dy_\alpha dy_\beta$$

Substituting $P(u_\beta = 1 \mid u_\alpha, y_\beta) = 1 - P(u_\beta = 0 \mid u_\alpha, y_\beta)$ and ignoring the constant term yields

$$= \sum_{u_\alpha, H} \int_{y_\alpha, y_\beta} P(u_\beta = 0 \mid u_\alpha, y_\beta)\, P(u_\alpha, y_\alpha \mid H)\, P(y_\beta \mid H)\; P(H)\, [\, J(0, H) - J(1, H)\,]\, dy_\alpha dy_\beta$$

$$= \sum_{u_\alpha} \int_{y_\beta} P(u_\beta = 0 \mid u_\alpha, y_\beta) \sum_{H} \int_{y_\alpha} P(u_\alpha, y_\alpha \mid H)\, P(y_\beta \mid H)\, P(H)\, [\, J(0, H) - J(1, H)\,]\, dy_\alpha dy_\beta$$

To minimize the expression on the previous page, let

$$P(u_\beta = 0 \mid u_\alpha, y_\beta) = \begin{cases} 0 & \text{if "INNERSUM"} > 0 \\ 1 & \text{otherwise} \end{cases}$$

where

$$\text{"INNERSUM"} = \sum_H \int_{y_\alpha} P(u_\alpha, y_\alpha \mid H)\, P(y_\beta \mid H)\, P(H)\, [\, J(0, H) - J(1, H) \,]\, dy_\alpha$$

The above expression can be written as

$$= \int_{y_\alpha} \sum_H P(u_\alpha \mid H)\, P(y_\beta \mid H)\, P(H)\, [\, J(0, H) - J(1, H) \,]\, dy_\alpha$$

Expanding the integrand (which must be negative) over H and invoking Assumption 1 yields

$$P(u_\alpha \mid H_0)\, P(y_\beta \mid H_0)\, P(H_0)\, [\, J(1, H_0) - J(0, H_0) \,]$$

$$\overset{u_\beta=0}{>} \; P(u_\alpha \mid H_1)\, P(y_\beta \mid H_1)\, P(H_1)\, [\, J(0, H_1) - J(1, H_1) \,]$$

Rearranging terms we get

$$\frac{P(y_\beta \mid H_0)}{P(y_\beta \mid H_1)} \overset{u_\beta=0}{>} \frac{P(H_1)\, P(u_\alpha \mid H_1)\, [\, J(0, H_1) - J(1, H_1) \,]}{P(H_0)\, P(u_\alpha \mid H_0)\, [\, J(1, H_0) - J(0, H_0) \,]} \qquad (92)$$

where the quantity on the right hand side of the inequality is referred to as β_i when $u_\alpha = i$ for $i = M_1$, M_2 and M_3.

This is the optimal decision rule for DMB appearing in Theorem 3.

Lemma 1 :

If the optimal decision rule presented in Theorem 4 is employed for u_α, then, whenever the following conditional probabilities are defined, we have

$$P(u_\beta = 0 \mid u_\alpha = M_1, H_0) \geq P(u_\beta = 0 \mid u_\alpha = M_2, H_0) \geq P(u_\beta = 0 \mid u_\alpha = M_3, H_0)$$

Proof :

Using Assumption 3 and the thresholds derived in Theorem 3 we find

$$\beta_{M_1} \leq \beta_{M_2} \leq \beta_{M_3} \tag{93}$$

From Theorem 3 we get the equality that appears below.

$$P(u_\beta = 0 \mid u_\alpha = i, H_0) = \int_{\Lambda_\beta(y_\beta) \geq \beta_i} \sum_{H} P(H)\, P(y_\beta \mid H) \tag{94}$$

Since $P(H)\, P(y_\beta \mid H)$ is always positive, (97) and (98) yield

$$P(u_\beta = 0 \mid u_\alpha = M_1, H_0) \geq P(u_\beta = 0 \mid u_\alpha = M_2, H_0) \geq P(u_\beta = 0 \mid u_\alpha = M_3, H_0)$$

This corollary has been used in the proof of Lemma 2.

Lemma 2 :

If the optimal decision rule, derived in Theorem 4, is employed for u_α, then, when the following conditional probabilities are defined, we have

$$\sum_{u_\beta} J(u_\beta, H_0)\,[\,P(u_\beta \mid u_\alpha = M_1, H_0) - P(u_\beta \mid u_\alpha = M_2, H_0)\,] \;\leq\; 0$$

$$\sum_{u_\beta} J(u_\beta, H_0)\,[\,P(u_\beta \mid u_\alpha = M_1, H_0) - P(u_\beta \mid u_\alpha = M_3\,, H_0)\,] \;\leq\; 0$$

$$\sum_{u_\beta} J(u_\beta, H_0)\,[\,P(u_\beta \mid u_\alpha = M_2\,, H_0) - P(u_\beta \mid u_\alpha = M_3\,, H_0)\,] \;\leq\; 0$$

<u>Proof</u> :

Only the following equation will be proved since the proofs of each of the three equations appearing above is similar.

$$\sum_{u_\beta} J(u_\beta, H_0)\,[\,P(u_\beta \mid u_\alpha = M_1, H_0) - P(u_\beta \mid u_\alpha = M_3\,, H_0)\,] \;\leq\; 0$$

Expanding the above we get

$$J(0, H_0)\, P(u_\beta = 0 \mid u_\alpha = M_1, H_0) + J(1, H_0)\, P(u_\beta = 1 \mid u_\alpha = M_1, H_0)$$

$$-\, J(0, H_0)\, P(u_\beta = 0 \mid u_\alpha = M_3, H_0) - J(1, H_0)\, P(u_\beta = 1 \mid u_\alpha = M_3, H_0)$$

Substituting $P(u_\beta = 1 \mid u_\alpha = i, H_0) = 1 - P(u_\beta = 0 \mid u_\alpha = i, H_0)$ for $i = M_1, M_3$ and combining terms yields

$$[\, P(u_\beta = 0 \mid u_\alpha = M_1, H_0) - P(u_\beta = 0 \mid u_\alpha = M_3, H_0) \,] \, [\, J(0, H_0) - J(1, H_0) \,]$$

Using Lemma 1 and Assumption 1 we see that the above expression is less than or equal to zero.

Proof of Theorem 4 :

Once again, the objective is to minimize the expected value of the cost function which can explicitly be written as

$$E\{J(u_\beta, H)\} = \sum_{u_\alpha, u_\beta, H} \int_{y_\alpha, y_\beta} P(u_\alpha, u_\beta, H, y_\alpha, y_\beta)\, J(u_\beta, H)\, dy_\alpha dy_\beta$$

$$= \sum_{u_\alpha, u_\beta, H} \int_{y_\alpha, y_\beta} P(u_\beta \mid u_\alpha, H, y_\alpha, y_\beta)\, P(u_\alpha, y_\alpha, y_\beta \mid H)\, P(H)\, J(u_\beta, H)\, dy_\alpha dy_\beta$$

Invoking the appropriate independence assumptions (Assumption 2) yields

$$= \sum_{u_\alpha, u_\beta, H} \int_{y_\alpha, y_\beta} P(u_\beta \mid u_\alpha, y_\beta)\, P(u_\alpha, y_\alpha \mid H)\, P(y_\beta \mid H)\, P(H)\, J(u_\beta, H)\, dy_\alpha dy_\beta$$

$$= \sum_{u_\alpha, u_\beta, H} \int_{y_\alpha, y_\beta} P(u_\beta \mid u_\alpha, y_\beta)\, P(u_\alpha \mid y_\alpha)\, P(y_\alpha \mid H)\, P(y_\beta \mid H)\, P(H)\, J(u_\beta, H)\, dy_\alpha dy_\beta$$

Explicitly summing over u_α and integrating over y_β yields

$$\int_{y_\alpha} \Bigg[P(u_\alpha = M_1 \mid y_\alpha) \sum_{u_\beta, H} P(u_\beta \mid u_\alpha = M_1, H)\, P(y_\alpha \mid H)\, P(H)\, J(u_\beta, H)$$

$$+ P(u_\alpha = M_2 \mid y_\alpha) \sum_{u_\beta, H} P(u_\beta \mid u_\alpha = M_2, H)\, P(y_\alpha \mid H)\, P(H)\, J(u_\beta, H)$$

$$+ P(u_\alpha = M_3 \mid y_\alpha) \sum_{u_\beta, H} P(u_\beta \mid u_\alpha = M_3, H)\, P(y_\alpha \mid H)\, P(H)\, J(u_\beta, H) \Bigg] dy_\alpha$$

Now set

$$P^i = \sum_{u_\beta, H} P(u_\beta \mid u_\alpha = i, H)\, P(y_\alpha \mid H)\, P(H)\, J(u_\beta, H)$$

To minimize the cost we use

$$P(u_\alpha = i \mid y_\alpha) = \begin{cases} 1 & \text{if } P^i = \min\{ P^{M_1}, P^{M_2}, P^{M_3} \} \\ 0 & \text{otherwise} \end{cases}$$

Hence, the optimal decision rule takes the form

$$u_\alpha = \gamma_\alpha(y_\alpha) = i \qquad \text{if } P(u_\alpha = i \mid y_\alpha) = 1 \quad \text{for } i = M_1, M_2, M_3$$

Finally, invoking Assumption 1, it is a matter of simple, but tedious algebraic manipulations to put the decision rule in the following form

$$\gamma_\alpha(y_\alpha) = \begin{cases} M_1, & \text{if } \Lambda_\alpha(y_\alpha) \geq \alpha_1 \text{ and } \Lambda_\alpha(y_\alpha) \geq \alpha_2 \\ M_2, & \text{if } \Lambda_\alpha(y_\alpha) < \alpha_1 \text{ and } \Lambda_\alpha(y_\alpha) \geq \alpha_3 \\ M_3, & \text{if } \Lambda_\alpha(y_\alpha) < \alpha_2 \text{ and } \Lambda_\alpha(y_\alpha) < \alpha_3 \end{cases}$$

where

$$\alpha_1 = \frac{\sum_{u_\beta} J(u_\beta, H_1)\,[\, P(u_\beta \mid u_\alpha = M_2, H_1) - P(u_\beta \mid u_\alpha = M_1, H_1)\,]}{\sum_{u_\beta} J(u_\beta, H_0)\,[\, P(u_\beta \mid u_\alpha = M_1, H_0) - P(u_\beta \mid u_\alpha = M_2, H_0)\,]} \tag{95}$$

$$\alpha_2 = \frac{\sum_{u_\beta} J(u_\beta, H_1)\,[\, P(u_\beta \mid u_\alpha = M_3, H_1) - P(u_\beta \mid u_\alpha = M_1, H_1)\,]}{\sum_{u_\beta} J(u_\beta, H_0)\,[\, P(u_\beta \mid u_\alpha = M_1, H_0) - P(u_\beta \mid u_\alpha = M_3, H_0)\,]} \tag{96}$$

$$\alpha_3 = \frac{\sum_{u_\beta} J(u_\beta, H_1)\,[\, P(u_\beta \mid u_\alpha = M_3, H_1) - P(u_\beta \mid u_\alpha = M_2, H_1)\,]}{\sum_{u_\beta} J(u_\beta, H_0)\,[\, P(u_\beta \mid u_\alpha = M_2, H_0) - P(u_\beta \mid u_\alpha = M_3, H_0)\,]} \tag{97}$$

For example, to arrive at the conditions for DMA to declare M_1 we need both (i) and (ii) below to be satisfied.

$$\text{(i)}\ P^{M_1} < P^{M_2} \qquad \text{(ii)}\ P^{M_1} < P^{M_3}$$

(i) $P^{M_1} < P^{M_2}$ can be written as

$$\sum_{u_\beta, H} P(u_\beta \mid u_\alpha = M_1, H)\, P(y_\alpha \mid H)\, P(H)\, J(u_\beta, H)$$

$$< \sum_{u_\beta, H} P(u_\beta \mid u_\alpha = M_2, H)\, P(y_\alpha \mid H)\, P(H)\, J(u_\beta, H)$$

Expanding we get

$$\sum_{u_\beta} P(u_\beta \mid u_\alpha = M_1, H_0)\, P(y_\alpha \mid H_0)\, P(H_0)\, J(u_\beta, H_0)$$

$$+ \sum_{u_\beta} P(u_\beta \mid u_\alpha = M_1, H_1)\, P(y_\alpha \mid H_1)\, P(H_1)\, J(u_\beta, H_1)$$

$$< \sum_{u_\beta} P(u_\beta \mid u_\alpha = M_2, H_0)\, P(y_\alpha \mid H_0)\, P(H_0)\, J(u_\beta, H_0)$$

$$+ \sum_{u_\beta} P(u_\beta \mid u_\alpha = M_2, H_1)\, P(y_\alpha \mid H_1)\, P(H_1)\, J(u_\beta, H_1)$$

Finally, rearranging the terms

$$\frac{P(H_0)\, P(y_\alpha \mid H_0)}{P(H_1)\, P(y_\alpha \mid H_1)} \geq \frac{\sum_{u_\beta} J(u_\beta, H_1)\, [\, P(u_\beta \mid u_\alpha = M_2, H_1) - P(u_\beta \mid u_\alpha = M_1, H_1)\,]}{\sum_{u_\beta} J(u_\beta, H_0)\, [\, P(u_\beta \mid u_\alpha = M_1, H_0) - P(u_\beta \mid u_\alpha = M_2, H_0)\,]}$$

or

$$\Lambda_\alpha(y_\alpha) \geq \alpha_1$$

(ii) $P^{M_1} < P^{M_3}$ can be written as

$$\sum_{u_\beta, H} P(u_\beta \mid u_\alpha = M_1, H)\, P(y_\alpha \mid H)\, P(H)\, J(u_\beta, H)$$

$$< \sum_{u_\beta, H} P(u_\beta \mid u_\alpha = M_3, H)\, P(y_\alpha \mid H)\, P(H)\, J(u_\beta, H)$$

Similarly, expanding and rearranging the terms we get

$$\frac{P(H_0)\, P(y_\alpha \mid H_0)}{P(H_1)\, P(y_\alpha \mid H_1)} \geq \frac{\sum_{u_\beta} J(u_\beta, H_1)\, [\, P(u_\beta \mid u_\alpha = M_3\,, H_1) - P(u_\beta \mid u_\alpha = M_1, H_1)\,]}{\sum_{u_\beta} J(u_\beta, H_0)\, [\, P(u_\beta \mid u_\alpha = M_1, H_0) - P(u_\beta \mid u_\alpha = M_3\,, H_0)\,]}$$

or

$$\Lambda_\alpha(y_\alpha) \geq \alpha_2$$

We have used Lemma 2 to write the inequalities in the manner that they appear above. Similarly, the conditions for DMA declaring M_2 and M_3 can be derived.

Finally, it is easy to show that α_2 is redundant in the above decision rule by proving that $\alpha_2 < \alpha_1$ and $\alpha_2 > \alpha_3$. We will show the proof for $\alpha_2 < \alpha_1$ only, since the proof for $\alpha_2 > \alpha_3$ is very similar.

Writing out the expressions for α_1 and α_2 we find that we have to show

$$\frac{\sum_{u_\beta} J(u_\beta, H_1)\, [\, P(u_\beta \mid u_\alpha = M_2, H_1) - P(u_\beta \mid u_\alpha = M_1, H_1)\,]}{\sum_{u_\beta} J(u_\beta, H_0)\, [\, P(u_\beta \mid u_\alpha = M_1, H_0) - P(u_\beta \mid u_\alpha = M_2, H_0)\,]}$$

$$> \frac{\sum_{u_\beta} J(u_\beta, H_1)\, [\, P(u_\beta \mid u_\alpha = M_3, H_1) - P(u_\beta \mid u_\alpha = M_1, H_1)\,]}{\sum_{u_\beta} J(u_\beta, H_0)\, [\, P(u_\beta \mid u_\alpha = M_1, H_0) - P(u_\beta \mid u_\alpha = M_3, H_0)\,]}$$

Expanding and cross-multiplying we get

$$\{ J(0, H_1) [P(u_\beta = 0 \mid u_\alpha = M_2, H_1) - P(u_\beta = 0 \mid u_\alpha = M_1, H_1)]$$

$$+ J(1, H_1) [P(u_\beta = 1 \mid u_\alpha = M_2, H_1) - P(u_\beta = 1 \mid u_\alpha = M_1, H_1)] \}$$

$$\{ J(0, H_0) [P(u_\beta = 0 \mid u_\alpha = M_1, H_0) - P(u_\beta = 0 \mid u_\alpha = M_2, H_0)]$$

$$+ J(1, H_0) [P(u_\beta = 1 \mid u_\alpha = M_1, H_0) - P(u_\beta = 1 \mid u_\alpha = M_2, H_0)] \}$$

$$>$$

$$\{ J(0, H_1) [P(u_\beta = 0 \mid u_\alpha = M_3, H_1) - P(u_\beta = 0 \mid u_\alpha = M_1, H_1)]$$

$$+ J(1, H_1) [P(u_\beta = 1 \mid u_\alpha = M_3, H_1) - P(u_\beta = 1 \mid u_\alpha = M_1, H_1)] \}$$

$$\{ J(0, H_0) [P(u_\beta = 0 \mid u_\alpha = M_1, H_0) - P(u_\beta = 0 \mid u_\alpha = M_3, H_0)]$$

$$+ J(1, H_0) [P(u_\beta = 1 \mid u_\alpha = M_1, H_0) - P(u_\beta = 1 \mid u_\alpha = M_3, H_0)] \}$$

Once again, expanding, combining terms and writing the inequality with respect to zero we write

$$J(0, H_1) J(0, H_0) [P(u_\beta = 0 \mid u_\alpha = M_2, H_1) P(u_\beta = 0 \mid u_\alpha = M_1, H_0)$$

$$- P(u_\beta = 0 \mid u_\alpha = M_1, H_1) P(u_\beta = 0 \mid u_\alpha = M_2, H_0)$$

$$+ P(u_\beta = 0 \mid u_\alpha = M_1, H_1) P(u_\beta = 0 \mid u_\alpha = M_3, H_0)$$

$$- P(u_\beta = 0 \mid u_\alpha = M_3, H_1) P(u_\beta = 0 \mid u_\alpha = M_1, H_0)$$

$$+ P(u_\beta = 0 \mid u_\alpha = M_3, H_1) P(u_\beta = 0 \mid u_\alpha = M_2, H_0)$$

$$- P(u_\beta = 0 \mid u_\alpha = M_1, H_1) P(u_\beta = 0 \mid u_\alpha = M_2, H_0)]$$

$$
\begin{aligned}
&+ J(0, H_1)\, J(1, H_0)\, [\, P(u_\beta = 0 \mid u_\alpha = M_2, H_1)\, P(u_\beta = 1 \mid u_\alpha = M_1, H_0) \\
&\qquad - P(u_\beta = 0 \mid u_\alpha = M_3, H_1)\, P(u_\beta = 1 \mid u_\alpha = M_1, H_0) \\
&\qquad + P(u_\beta = 0 \mid u_\alpha = M_1, H_1)\, P(u_\beta = 1 \mid u_\alpha = M_3, H_0) \\
&\qquad - P(u_\beta = 0 \mid u_\alpha = M_2, H_1)\, P(u_\beta = 1 \mid u_\alpha = M_3, H_0) \\
&\qquad + P(u_\beta = 0 \mid u_\alpha = M_3, H_1)\, P(u_\beta = 1 \mid u_\alpha = M_2, H_0) \\
&\qquad - P(u_\beta = 0 \mid u_\alpha = M_1, H_1)\, P(u_\beta = 1 \mid u_\alpha = M_2, H_0)\,] \\
&+ J(1, H_1)\, J(0, H_0)\, [\, P(u_\beta = 1 \mid u_\alpha = M_2, H_1)\, P(u_\beta = 0 \mid u_\alpha = M_1, H_0) \\
&\qquad - P(u_\beta = 1 \mid u_\alpha = M_2, H_1)\, P(u_\beta = 0 \mid u_\alpha = M_3, H_0) \\
&\qquad + P(u_\beta = 1 \mid u_\alpha = M_1, H_1)\, P(u_\beta = 0 \mid u_\alpha = M_3, H_0) \\
&\qquad - P(u_\beta = 1 \mid u_\alpha = M_3, H_1)\, P(u_\beta = 0 \mid u_\alpha = M_1, H_0) \\
&\qquad + P(u_\beta = 1 \mid u_\alpha = M_3, H_1)\, P(u_\beta = 0 \mid u_\alpha = M_2, H_0) \\
&\qquad - P(u_\beta = 1 \mid u_\alpha = M_1, H_1)\, P(u_\beta = 0 \mid u_\alpha = M_2, H_0)\,] \\
&+ J(1, H_1)\, J(1, H_0)\, [\, P(u_\beta = 1 \mid u_\alpha = M_2, H_1)\, P(u_\beta = 1 \mid u_\alpha = M_1, H_0) \\
&\qquad - P(u_\beta = 1 \mid u_\alpha = M_2, H_1)\, P(u_\beta = 1 \mid u_\alpha = M_3, H_0) \\
&\qquad + P(u_\beta = 1 \mid u_\alpha = M_1, H_1)\, P(u_\beta = 1 \mid u_\alpha = M_3, H_0) \\
&\qquad - P(u_\beta = 1 \mid u_\alpha = M_3, H_1)\, P(u_\beta = 1 \mid u_\alpha = M_1, H_0) \\
&\qquad + P(u_\beta = 1 \mid u_\alpha = M_3, H_1)\, P(u_\beta = 1 \mid u_\alpha = M_2, H_0) \\
&\qquad - P(u_\beta = 1 \mid u_\alpha = M_1, H_1)\, P(u_\beta = 1 \mid u_\alpha = M_2, H_0)\,] \quad > \quad 0
\end{aligned}
$$

We can see that the above inequality holds by looking at adjacent terms and using Lemma 1.

Finally, we can put the decision rule in the form that appears in Theorem 4.

Proof of Corollary 3 :

There are three cases to be derived for the decision rule of DMB.

(i) DMA declares M_1 (i.e., $u_\alpha = M_1$) :

$$\frac{P(y_\beta \mid H_0)}{P(y_\beta \mid H_1)} \overset{u_\beta=0}{\geq} \frac{P(H_1)\, P(u_\alpha = M_1 \mid H_1)\, [\, J(0, H_1) - J(1, H_1)\,]}{P(H_0)\, P(u_\alpha = M_1 \mid H_0)\, [\, J(1, H_0) - J(0, H_0)\,]}$$

Substituting the Gaussian probability density functions and using the minimum error cost function (i.e., $J(0, H_0) = J(1, H_1) = 0$ and $J(0, H_1) = J(1, H_0) = 1$) we have

$$\frac{e^{-\frac{(y_\beta - \mu_0)^2}{2\sigma_\beta^2}}}{e^{-\frac{(y_\beta - \mu_1)^2}{2\sigma_\beta^2}}} \geq \frac{P(H_1)\, P(u_\alpha = M_1 \mid H_1)}{P(H_0)\, P(u_\alpha = M_1 \mid H_0)}$$

Taking natural logarithms on both sides we get

$$\frac{2y_\beta\mu_0 - \mu_0^2 - 2y_\beta\mu_1 + \mu_1^2}{2\sigma_\beta^2} \geq \ln\left[\frac{P(u_\alpha = M_1 \mid H_1)}{P(u_\alpha = M_1 \mid H_0)}\right] + \ln\left[\frac{1 - P(H_0)}{P(H_0)}\right]$$

Finally, multiplying both sides of the equation by $2\sigma_\beta^2$, using the definition of the error function and isolating y_β we get

$$y_\beta \overset{u_\beta=0}{\lessgtr} \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\alpha^l(0)}{\Phi_\alpha^l(1)}\right] + \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2}$$

or

$$y_\beta \overset{u_\beta=0}{\lessgtr} Y_\beta^{M_1}$$

(ii) DMA declares M_2 (i.e., $u_\alpha = M_2$) :

$$\frac{P(y_\beta \mid H_0)}{P(y_\beta \mid H_1)} \overset{u_\beta=0}{\gtrless} \frac{P(H_1)\, P(u_\alpha = M_2 \mid H_1)\, [\, J(0, H_1) - J(1, H_1)\,]}{P(H_0)\, P(u_\alpha = M_2 \mid H_0)\, [\, J(1, H_0) - J(0, H_0)\,]}$$

Similarly, using the Gaussian probability density functions, the minimum error cost function, taking the natural logarithm on both sides, substituting the error functions and isolating y_β we get

$$y_\beta \overset{u_\beta=0}{\lessgtr} \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\alpha^u(0) - \Phi_\alpha^l(0)}{\Phi_\alpha^u(1) - \Phi_\alpha^l(1)}\right] + \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2}$$

or

$$y_\beta \overset{u_\beta=0}{\lessgtr} Y_\beta^{M_2}$$

(iii) DMA declares M_3 (i.e., $u_\alpha = M_3$) :

$$\frac{P(y_\beta \mid H_0)}{P(y_\beta \mid H_1)} \underset{\geq}{\overset{u_\beta=0}{\leq}} \frac{P(H_1)\, P(u_\alpha = M_3 \mid H_1)\, [\, J(0, H_1) - J(1, H_1)\,]}{P(H_0)\, P(u_\alpha = M_3 \mid H_0)\, [\, J(1, H_0) - J(0, H_0)\,]}$$

Similarly, using the Gaussian probability density functions, the minimum error cost function, taking the natural logarithm on both sides, substituting the error functions and isolating y_β we get

$$y_\beta \underset{\leq}{\overset{u_\beta=0}{\geq}} \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{1 - \Phi_\alpha^u(0)}{1 - \Phi_\alpha^u(1)}\right] + \frac{\sigma_\beta^2}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2}$$

or

$$y_\beta \underset{\leq}{\overset{u_\beta=0}{\geq}} Y_\beta^{M_3}$$

This completes the proof of Corollary 3.

Proof of Corollary 4 :

This corollary can be proved by considering the cases when DMA declares M_1 and when DMA declares M_3.

When DMA declares M_1 we have

$$\frac{P(y_\alpha \mid H_0)}{P(y_\alpha \mid H_1)} \underset{\geq}{\overset{u_\alpha=M_1}{\leq}} \frac{P(H_1) \sum_{u_\beta} J(u_\beta, H_1)\, [\, P(u_\beta | u_\alpha = M_2, H_1) - P(u_\beta | u_\alpha = M_1, H_1)\,]}{P(H_0) \sum_{u_\beta} J(u_\beta, H_0)\, [\, P(u_\beta | u_\alpha = M_1, H_0) - P(u_\beta | u_\alpha = M_2, H_0)\,]}$$

Expanding the right hand side of the above equation we have

$$\frac{P(H_1)\ \{\ J(0, H_1)\ [\ P(u_\beta = 0 \mid u_\alpha = M_2, H_1) - P(u_\beta = 0 \mid u_\alpha = M_1, H_1)\ ] + J(1, H_1)\ [\ P(u_\beta = 1 \mid u_\alpha = M_2, H_1) - P(u_\beta = 1 \mid u_\alpha = M_1, H_1)\ ]\ \}}{P(H_0)\ \{\ J(0, H_0)\ [\ P(u_\beta = 0 \mid u_\alpha = M_1, H_0) - P(u_\beta = 0 \mid u_\alpha = M_2, H_0)\ ] + J(1, H_0)\ [\ P(u_\beta = 1 \mid u_\alpha = M_1, H_0) - P(u_\beta = 1 \mid u_\alpha = M_2, H_0)\ ]\ \}}$$

Substituting error functions we get

$$= \frac{J(0, H_1)\ [\ \Phi_\beta^{M_2}(1) - \Phi_\beta^{M_1}(1)\ ] + J(1, H_1)\ [\ (1 - \Phi_\beta^{M_2}(1)) - (1 - \Phi_\beta^{M_1}(1))\ ]}{J(0, H_0)\ [\ \Phi_\beta^{M_1}(0) - \Phi_\beta^{M_2}(0)\ ] + J(1, H_0)\ [\ (1 - \Phi_\beta^{M_1}(0)) - (1 - \Phi_\beta^{M_2}(0))\ ]}$$

Simplifying

$$= \frac{(\Phi_\beta^{M_2}(1) - \Phi_\beta^{M_1}(1))\ [\ J(0, H_1) - J(1, H_1)\ ]}{(\Phi_\beta^{M_2}(0) - \Phi_\beta^{M_1}(0))\ [\ J(1, H_0) - J(0, H_0)\ ]}$$

Now, considering the original inequality, we substitute the Gaussian probability density function, use the minimum error cost function (i.e., $J(0, H_0) = J(1, H_1) = 0$ and $J(0, H_1) = J(1, H_0) = 1$) and take natural logarithms on both sides to get

$$\frac{2y_\alpha \mu_0 - \mu_0^2 - 2y_\alpha \mu_1 + \mu_1^2}{2\sigma_\alpha^2} \overset{u_\alpha = M_1}{\geq} \ln\left[\frac{\Phi_\beta^{M_2}(1) - \Phi_\beta^{M_1}(1)}{\Phi_\beta^{M_2}(0) - \Phi_\beta^{M_1}(0)}\right] + \ln\left[\frac{1 - P(H_0)}{P(H_0)}\right]$$

Isolating y_α we get

$$y_\alpha \overset{u_\alpha = M_1}{\leq} \frac{\sigma^2_\alpha}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\beta^{M_2}(0) - \Phi_\beta^{M_1}(0)}{\Phi_\beta^{M_2}(1) - \Phi_\beta^{M_1}(1)}\right] + \frac{\sigma^2_\alpha}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2}$$

or

$$y_\alpha \overset{u_\alpha = M_1}{\leq} Y^l_\alpha$$

When DMA declares M_3 we have

$$\frac{P(y_\alpha \mid H_0)}{P(y_\alpha \mid H_1)} \overset{u_\alpha = M_3}{<} \frac{P(H_1) \sum_{u_\beta} J(u_\beta, H_1) \left[P(u_\beta | u_\alpha = M_3, H_1) - P(u_\beta | u_\alpha = M_2, H_1) \right]}{P(H_0) \sum_{u_\beta} J(u_\beta, H_0) \left[P(u_\beta | u_\alpha = M_2, H_0) - P(u_\beta | u_\alpha = M_3, H_0) \right]}$$

Similarly, expanding the right hand side, substituting error functions, simplifying, substituting the Gaussian probability density function, using the minimum error cost function, taking natural logarithms on both sides and isolating y_α we arrive at the inequality that appears below.

$$y_\alpha \overset{u_\alpha = M_3}{>} \frac{\sigma^2_\alpha}{\mu_1 - \mu_0} \ln\left[\frac{\Phi_\beta^{M_3}(0) - \Phi_\beta^{M_2}(0)}{\Phi_\beta^{M_3}(1) - \Phi_\beta^{M_2}(1)}\right] + \frac{\sigma^2_\alpha}{\mu_1 - \mu_0} \ln\left[\frac{P(H_0)}{1 - P(H_0)}\right] + \frac{\mu_0 + \mu_1}{2}$$

or

$$y_\alpha \overset{u_\alpha = M_3}{>} Y^u_\alpha$$

Now, to write the conditions when DMA declares M_2 we need to show that $Y_\alpha^l < Y_\alpha^u$. Proving this is equivalent to proving the Gaussian version of the proof of $\alpha_2 < \alpha_1$ appearing in the proof of Theorem 4 (which is the general problem).

Hence, we see that when DMA declares M_2 we have

$$y_\alpha > Y_\alpha^l \qquad \text{and} \qquad y_\alpha \leq Y_\alpha^u$$

This completes the proof of Corollary 4.

Proof of Theorem 7 :

The objective is to minimize the expected value of the cost function which can explicitly be written as

$$E\{J(u_\beta, H)\} = \sum_{u_\alpha, u_\beta, H} \int_{y_\alpha, y_\beta} P(u_\alpha, u_\beta, H, y_\alpha, y_\beta)\, J(u_\beta, H)\, dy_\alpha dy_\beta$$

$$= \sum_{u_\alpha, u_\beta, H} \int_{y_\alpha, y_\beta} P(u_\beta \mid u_\alpha, H, y_\alpha, y_\beta)\, P(u_\alpha, y_\alpha, y_\beta \mid H)\, P(H)\, J(u_\beta, H)\, dy_\alpha dy_\beta$$

Invoking the appropriate independence assumptions (Assumption 2) yields

$$= \sum_{u_\alpha, u_\beta, H} \int_{y_\alpha, y_\beta} P(u_\beta \mid u_\alpha, y_\beta)\, P(u_\alpha, y_\alpha \mid H)\, P(y_\beta \mid H)\, P(H)\, J(u_\beta, H)\, dy_\alpha dy_\beta$$

Substituting $P(u_\beta = 1 \mid u_\alpha, y_\beta) = 1 - P(u_\beta = 0 \mid u_\alpha, y_\beta)$ and ignoring the constant term yields the expression on the following page.

$$= \sum_{u_\alpha, H} \int_{y_\alpha, y_\beta} P(u_\beta = 0 \mid u_\alpha, y_\beta)\, P(u_\alpha, y_\alpha \mid H)\, P(y_\beta \mid H)\; P(H)\, [\, J(0, H) - J(1, H)\,]\, dy_\alpha dy_\beta$$

$$= \sum_{u_\alpha} \int_{y_\beta} P(u_\beta = 0 \mid u_\alpha, y_\beta) \sum_{H} \int_{y_\alpha} P(u_\alpha, y_\alpha \mid H)\, P(y_\beta \mid H)\, P(H)\, [\, J(0, H) - J(1, H)\,]\, dy_\alpha dy_\beta$$

To minimize the expression above, let

$$P(u_\beta = 0 \mid u_\alpha, y_\beta) = \begin{cases} 0 & \text{if "INNERSUM"} > 0 \\ 1 & \text{otherwise} \end{cases}$$

where

$$\text{"INNERSUM"} = \sum_{H} \int_{y_\alpha} P(u_\alpha, y_\alpha \mid H)\, P(y_\beta \mid H)\, P(H)\, [\, J(0, H) - J(1, H)\,]\, dy_\alpha$$

The above expression can be written as

$$= \int_{y_\alpha} \sum_{H} P(u_\alpha \mid H)\, P(y_\beta \mid H)\, P(H)\, [\, J(0, H) - J(1, H)\,]\, dy_\alpha$$

Expanding the integrand (which must be negative) over H and invoking Assumption 1 yields

$$P(u_\alpha \mid H_0)\, P(y_\beta \mid H_0)\, P(H_0)\, [\, J(1, H_0) - J(0, H_0)\,]$$

$$\overset{u_\beta = 0}{\gtrless}\; P(u_\alpha \mid H_1)\, P(y_\beta \mid H_1)\, P(H_1)\, [\, J(0, H_1) - J(1, H_1)\,]$$

Rearranging terms we get

$$\frac{P(y_\beta \mid H_0)}{P(y_\beta \mid H_1)} \overset{u_\beta=0}{>} \frac{P(H_1)\,P(u_\alpha \mid H_1)\,[\,J(0, H_1) - J(1, H_1)\,]}{P(H_0)\,P(u_\alpha \mid H_0)\,[\,J(1, H_0) - J(0, H_0)\,]}$$

where the quantity on the right hand side of the inequality is referred to as β_i when $u_\alpha = i$ for $i = M_1, M_2, M_3, \ldots, M_K$.

This is the optimal decision rule for DMB appearing in Theorem 7.

Proof of Theorem 8 :

Once again, the objective is to minimize the expected value of the cost function which can explicitly be written as

$$E\{J(u_\beta, H)\} = \sum_{u_\alpha, u_\beta, H} \int_{y_\alpha, y_\beta} P(u_\alpha, u_\beta, H, y_\alpha, y_\beta)\, J(u_\beta, H)\, dy_\alpha dy_\beta$$

$$= \sum_{u_\alpha, u_\beta, H} \int_{y_\alpha, y_\beta} P(u_\beta \mid u_\alpha, H, y_\alpha, y_\beta)\, P(u_\alpha, y_\alpha, y_\beta \mid H)\, P(H)\, J(u_\beta, H)\, dy_\alpha dy_\beta$$

Invoking the appropriate independence assumptions (Assumption 2) yields

$$= \sum_{u_\alpha, u_\beta, H} \int_{y_\alpha, y_\beta} P(u_\beta \mid u_\alpha, y_\beta)\, P(u_\alpha, y_\alpha \mid H)\, P(y_\beta \mid H)\, P(H)\, J(u_\beta, H)\, dy_\alpha dy_\beta$$

$$= \sum_{u_\alpha, u_\beta, H} \int_{y_\alpha, y_\beta} P(u_\beta \mid u_\alpha, y_\beta)\, P(u_\alpha \mid y_\alpha)\, P(y_\alpha \mid H)\, P(y_\beta \mid H)\, P(H)\, J(u_\beta, H)\, dy_\alpha dy_\beta$$

Explicitly summing over u_α and integrating over y_β yields

$$\int_{y_\alpha} \Bigg[P(u_\alpha = M_1 \mid y_\alpha) \sum_{u_\beta, H} P(u_\beta \mid u_\alpha = M_1, H)\, P(y_\alpha \mid H)\, P(H)\, J(u_\beta, H)$$

$$+ P(u_\alpha = M_2 \mid y_\alpha) \sum_{u_\beta, H} P(u_\beta \mid u_\alpha = M_2, H)\, P(y_\alpha \mid H)\, P(H)\, J(u_\beta, H)$$

$$+ P(u_\alpha = M_3 \mid y_\alpha) \sum_{u_\beta, H} P(u_\beta \mid u_\alpha = M_3, H)\, P(y_\alpha \mid H)\, P(H)\, J(u_\beta, H)$$

$$\vdots$$

$$+ P(u_\alpha = M_K \mid y_\alpha) \sum_{u_\beta, H} P(u_\beta \mid u_\alpha = M_K, H)\, P(y_\alpha \mid H)\, P(H)\, J(u_\beta, H) \Bigg] dy_\alpha$$

Now set

$$P^i = \sum_{u_\beta, H} P(u_\beta \mid u_\alpha = i, H)\, P(y_\alpha \mid H)\, P(H)\, J(u_\beta, H)$$

To minimize the cost we use

$$P(u_\alpha = i \mid y_\alpha) = \begin{cases} 1 & \text{if } P^i = \min \{ P^{M_1}, P^{M_2}, P^{M_3}, \ldots, P^{M_K} \} \\ 0 & \text{otherwise} \end{cases}$$

Hence, the optimal decision rule takes the form

$$u_\alpha = \gamma_\alpha(y_\alpha) = i \qquad \text{if } P(u_\alpha = i \mid y_\alpha) = 1 \quad \text{for } i = M_1, M_2, M_3, \ldots, M_K$$

Finally, invoking Assumption 1, it is a matter of simple, but tedious algebraic manipulations to put the decision rule in the following form

$$\gamma_\alpha(y_\alpha) = \begin{cases} M_1 , \ \text{if } \Lambda_\alpha(y_\alpha) \geq \alpha_1 \\ M_2 , \ \text{if } \alpha_2 \leq \Lambda_\alpha(y_\alpha) < \alpha_1 \\ M_3 , \ \text{if } \alpha_3 \leq \Lambda_\alpha(y_\alpha) < \alpha_2 \\ \bullet \\ \bullet \\ \bullet \\ M_{K-1} , \ \text{if } \alpha_{K-1} \leq \Lambda_\alpha(y_\alpha) < \alpha_{K-2} \\ M_K , \ \text{if } \Lambda_\alpha(y_\alpha) < \alpha_{K-1} \end{cases} \tag{98}$$

where

$$\alpha_1 = \frac{\sum_{u_\beta} J(u_\beta, H_1) \, [\, P(u_\beta | \, u_\alpha = M_2, H_1) - P(u_\beta | \, u_\alpha = M_1, H_1) \,]}{\sum_{u_\beta} J(u_\beta, H_0) \, [\, P(u_\beta | \, u_\alpha = M_1, H_0) - P(u_\beta | \, u_\alpha = M_2, H_0) \,]}$$

$$\alpha_2 = \frac{\sum_{u_\beta} J(u_\beta, H_1) \, [\, P(u_\beta | \, u_\alpha = M_3 , H_1) - P(u_\beta | \, u_\alpha = M_2 , H_1) \,]}{\sum_{u_\beta} J(u_\beta, H_0) \, [\, P(u_\beta | \, u_\alpha = M_2 , H_0) - P(u_\beta | \, u_\alpha = M_3 , H_0) \,]}$$

$$\alpha_3 = \frac{\sum_{u_\beta} J(u_\beta, H_1) \, [\, J(u_\beta | \, u_\alpha = M_4, H_1) - P(u_\beta | \, u_\alpha = M_3, H_1) \,]}{\sum_{u_\beta} J(u_\beta, H_0) \, [\, P(u_\beta | \, u_\alpha = M_3, H_0) - P(u_\beta | \, u_\alpha = M_4, H_0) \,]}$$

•

•

•

$$\alpha_{K-1} = \frac{\sum_{u_\beta} J(u_\beta, H_1) \left[P(u_\beta \mid u_\alpha = M_K, H_1) - P(u_\beta \mid u_\alpha = M_{K-1}, H_1) \right]}{\sum_{u_\beta} J(u_\beta, H_0) \left[P(u_\beta \mid u_\alpha = M_{K-1}, H_0) - P(u_\beta \mid u_\alpha = M_K, H_0) \right]}$$

The decision rule appearing in (98) has been written with all the redundant α's removed. The redundancies can be proven in the similar manner as was shown in the proof of Theorem 4 (i.e., $\alpha_1 > \alpha_2$, or more generally, in the above case, $\alpha_{K-1} > \alpha_K$).

The conditions for DMA declaring M_1, M_2, M_3, ..., M_K have been derived in the manner shown in the proof of Theorem 4. Hence, the algebraic manipulations have not been repeated here.

Proof of Corollaries 1, 2, 5, 6, 7 and 8 :

No actual proof for the Gaussian equations for these corollaries will be presented. The thresholds for the Gaussian case are obtained by substituting the Gaussian probability density functions in the threshold equations derived for the general case and solving for y_α or y_β. In other words, the proof is similar to that of the three-message case (Corollary 3 and Corollary 4).

The subscripts of the thresholds indicate the decision maker whose decision they characterize and the superscripts of the thresholds indicate the content of the decision. For the thresholds of DMA we use l (lower), m (middle) and u (upper) depending on how many messages DMA is using. For the case of K messages the superscripts are numbered 1 through K-1. For the thresholds of DMB we use M_1 through M_K as superscripts, which indicates the message of DMA that the threshold corresponds to.

This completes the discussion of Corollaries 1, 2, 5, 6, 7 and 8.

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Index of Notation

A brief explanation is presented for the symbols that appear most frequently in the text.

y_α, y_β	: observations of DMA and DMB respectively.
Y_α, Y_β	: the set of observations for DMA and DMB respectively.
u_α, u_β	: the decisions of DMA and DMB respectively.
γ_α, γ_β	: the decision rules of DMA and DMB respectively.
H_0, H_1	: the two possible hypothesis.
$P(H_0)$, $P(H_1)$	: the a priori probabilities of H_0 and H_1 respectively.
$P(y_\alpha \mid H_i)$, $P(y_\beta \mid H_i)$	: known probability density functions of the observations of DMA and DMB conditional on H_i $(i = 0, 1)$
$J(u_\beta, H_i)$	: the cost incurred by the team choosing u_β, when H_i is true.
$M_1, M_2, ..., M_K$	: the K possible messages that DMA can communicate to DMB $(K = 2, 3, ...)$.
$N(\mu, \sigma^2)$	: denotes a Gaussian distribution with mean μ and variance σ^2.
μ_0, μ_1	: the means under hypotheses H_0 and H_1 respectively.
σ_α^2, σ_β^2	: the variances of the observations of DMA and DMB respectively.
Y_α^*	: the threshold of DMA for the two-message case.
Y_α^I, Y_α^C	: the thresholds for the isolation and centralized cases.
Y_α^l, Y_α^m, Y_α^u	: the thresholds of DMA for the three-message and four-message cases.

$Y_\alpha^1, Y_\alpha^2, \ldots, Y_\alpha^{K-1}$	: the thresholds of DMA for the general K case.
$Y_\beta^{M_1}, Y_\beta^{M_2}, \ldots, Y_\beta^{M_K}$	: the thresholds of DMB for the two-message, three-message, four-message and the general K cases.
$Y_\alpha^{ML}, Y_\beta^{ML}$	: the maximum likelihood thresholds of DMA and DMB respectively.
Pr(E)	: Probability of Error.
ROC	: Receiver Operating Characteristic.
P_F, P_D, P_M	: Probabilities of False Alarm, Detection and Miss.
$\Phi_i^j(k)$	: the error function for Decision Maker i, with j being the content of the decision and k the underlying given hypothesis.