THREE ESSAYS IN OPTIMAL CONSUMPTION

by

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ABSTRACT

The three papers that comprise this dissertation deal with the arbitrage free pricing of contingent claims and with the modelling of optimal consumption and portfolio choice. They all exploit the martingale connection of an arbitrage free price system introduced by Harrison and Kreps in 1979 and the approach developed later by Cox and Huang to solve a dynamic optimization problem with the tools of the martingale representation.

The paper that comprises chapter one, "Optimal Policies when Markets are Incomplete," develops a model of continuous trading where the incompleteness of the markets stems from an insufficient number of risky securities. The main object is to elicit the structure of the consumption space according to alternative definitions of the admissible trading strategies. The model is then specialized to solve the optimal consumption and portfolio choice problem when there is no stochastic endowment.

Returning to the complete markets setup, chapter two, "Arbitrage and Optimal Policies with an Infinite Horizon," solves some of the problems created upon letting the agents trade over an infinite horizon. Sufficient conditions are given to preclude all arbitrage opportunities. The optimal consumption portfolio problem is also taken care of by an appropriate change of time which allows one to fall back on the finite horizon case. It is shown that a solution exists whenever the subjective rate of time preference is sufficiently high. Explicit examples are given for utility functions displaying constant risk aversion and for the family of HARA utility functions.

Chapter three, "Consumption of an Endowment," reintroduces risks in the form of a stochastic endowment, while it is assumed that agents cannot carry over negative non human wealth at any point in time. Sufficient conditions for the existence of a solution are derived, which leads to a characterization of the optimal consumption path in terms of an implicit shadow price. The shadow price has two components, one depending only

on the markets model, the other on the investor's preferences and endowment. This model may be potentially useful to reconcile the life cycle hypothesis with the issues of lifetime consumption smoothing.

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Contents

Chapter 1
Optimal Policies
when Markets are Incomplete

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Abstract. This chapter develops a model of continuous trading when markets are incomplete. We first define a set of feasible consumption, the set of "marketed" contingent claims, obtained when trade in the risky assets is quantatively constrained. This set is independent of the choice of the risk neutral probabilities of Cox and Ross and has a reasonable economic property: it corresponds to claims whose "state prices" (i.e., prices consistent with the absence of arbitrage) are uniquely determined. We then consider another consumption space obtained when strategies are constrained to produce non negative wealth at all times. We show that the latter have a "fair" price in terms of the initial cost of the strategies which manufacture them. Finally, we use the model to solve the optimal consumption and portfolio choice problem when there are no uninsurable risks.

1. Introduction

In this chapter, we develop a model of incomplete markets, in which the price processes are continuous and exogenously specified. An important feature of the model is that there are fewer risky securities than there are Brownian motions required for the description of uncertainty. We offer two examples of application. First we compare two consumption sets: one is obtained with an integral restriction on trading and the other when wealth is constrained to be non negative. The second example deals with a classical issue in financial economics, the intertemporal consumption portfolio problem. Both applications exploit the martingale connection of an arbitrage free price system as examplified by Cox and Ross [**8**] and Harrison and Kreps [**17**].

Recent advances in financial theory have shown how arbitrage arguments underlie the valuation of contingent claims. In the models of Black and Scholes[**3**] and Merton[**27**], the pricing of contingent claims is based on a replication argument. One tries to synthesize the returns stream of a given contingent claim by a controlled portfolio of the basic securities. Barring arbitrage (with an integral constraint on trading), one evaluates the claim at the initial cost of the strategy which replicates it. By this means one can create "synthetic" contingent claims from the existing traded securities. The claims obtained in this manner form what we call the *marketed* space.

As a first application of the model, we compare the "marketed" space with the space consisting of claims manufactured by trading strategies satisfying only a non negative

wealth constraint. Related work is that of Dybvig [11] and Dybvig and Huang [12]. The second application focuses on the optimal consumption and portfolio choice problem. In our version, investors are not endowed with any uncertain income. Consumption of an endowment is deferred to chapter 3, within a complete markets framework. Our model allows us to extend this approach without assuming the effects of complete markets.

The main results of the paper are outlined in more details below. In section 2, we characterize the trading strategies that are available to agents. We take as primitive a probability space and an n-dimensional Brownian motion defined on it. There are m risky securities, where m can be less than n, whose price processes are governed by a system of stochastic differential equations

$$dS_t = b(t, S_t)\, dt + \sigma(t, S_t)\, dw(t),$$

where the $m \times n$ matrix σ has full rank. (Here we have neglected dividends.) The 0-th security is a bond, whose instantaneous rate of return is riskless. Since there is a single physical good allocated to consumption in this economy, all prices are initially expressed in terms of units of this good. A *strategy* is a decision regarding the continuous trading of the existing securities from time 0 to time 1, where 1 stands for the finite horizon of the model. Since there is no endowment, the proceeds are entirely generated by the capital gains and dividends from the investment in assets and can be either consumed or reinvested. For expositional reasons we rule out bequests. This implies, in particular, that the value of strategies at the final date is zero.

We then proceed to define a set of trading strategies which does not allow for arbitrage opportunities. An example of such arbitrage is given by the *doubling strategies* of Harrison and Kreps [17], named after the strategy of doubling one's bet at roulette until one eventually wins. To quote Dybvig and Huang [12]:

> Presence of the doubling strategies strikes at the core of the continuous time model, rendering it potentially vacuous. Having arbitrage opportunities precludes having a solution to the optimal investment problem (for strictly monotone preferences) and, of course, invalidates option pricing theory based on the *assumption* that there is no arbitrage opportunity.

Harrison and Kreps were the first to tackle this problem in a continuous time model. They remove arbitrage opportunities by restricting trading in the basic securities to occur at a finite number of prespecified times. They call these strategies *simple trading strategies* and show that barring arbitrage, one finds a reassignment of probabilities under which all assets have the same expected rate of returns. Such a reassignment is referred to as an *equivalent martingale measure.* While the example of simple strategies gives the continuous time analysis a rigorous foundation, it is too restrictive for our purpose, since strategies arising from either dynamic optimization or option pricing are generally not simple.

Harrison and Pliska [**18**], Cox and Huang [**6**] and Duffie and Huang [**10**] set out to define a larger class of trading strategies that depends on a fixed martingale measure (the "reference measure"). When markets are incomplete, this is unsatisfactory. On the one hand the martingale measure is some kind of mathematical artifact and it is not clear why the feasible set should be related to it in any specific way. On the other hand, the multiplicity of the martingale measures makes the dependence on any particular one arbitrary, for the set of admissible strategies is not invariant to a change in the reference measure.

To avoid these problems, we impose integral restrictions on trading à la Harrison and Pliska, but directly with respect to the *original* probability beliefs. This yields a first definition of the feasible consumption set which is independent of the martingale measures. In our setup, feasible consumption corresponds, when markets are complete, to the positive orthant of the whole consumption space and, when markets are incomplete, to the set of non negative claims whose expected discounted value is invariant to a change in the martingale measures. This means, roughly, that for each of these claims the price consistent with the absence of arbitrage is uniquely determined. Of course, marketed claims do not have to be in the filtration generated by the basic securities. This is a reflection of the fact that agents are able to use the whole information generated by the underlying Brownian motions, rather than the information conveyed by the stock prices alone.

Recently Dybvig and Huang [**12**], capitalizing on a conjecture made by Harrison and Kreps [**17**], demonstrated that strategies that generate non negative wealth at all points in time also preclude arbitrage opportunities. Doubling strategies are ruled out, since they require unbounded negative wealth with positive probability, and so are all arbitrage

opportunities. The question that arises is whether the integral constraint on trading can be replaced by the simpler requirement of a non negative wealth. Dybvig and Huang show that when markets are incomplete the consumption set under non negative wealth is larger than that with restriction on trading, and that the two sets are the same "up to closure".

In section 3 we elaborate on these issues by invoking new results by Stricker [**30**]. When wealth is constrained to be non negative, it is known that the gains process is a supermartingale (with respect to any martingale measure) and thus can only decrease in expectations. Hence the initial investment required can be no smaller than the expected discounted value of the claim, and will generally be higher. It is shown that there is a "best" strategy in terms of a least initial cost, though this strategy is generally not unique. This least initial cost defines what we call the "fair price" of the contingent claim.

In section 4 of the paper we take up the issues of existence and characterization in the problem of optimal consumption and portfolio choice. The traditional approach, as in the pioneering work of Merton [**26**], is that of *dynamic programming*. This approach has some limitations; cf. Cox and Huang [**6**], [**7**]. Recently, an alternative approach has been put forward in place of the dynamic programming: notably, Cox and Huang [**6**], [**7**] and Pliska [**28**] in portfolio theory, or Chamberlain [**4**] and Huang [**19**] in general equilibrium. Our solution draws heavily upon the results of Cox and Huang. Their approach is as follows. First, map the original dynamic optimization problem into a static variational problem that can be solved with the standard tools of Lagrangian theory. Second, implement the solution of this static proplem via a martingale representation theorem which uncovers the underlying dynamic strategy.

Their results can in fact be generalized to the incomplete markets setting at quite little cost. The reason is that the solution to the static problem is measurable with respect to the securities prices. Under those circumstances, the method of Cox and Huang is conducive to a martingale representation technology which is similar but does not rely on completeness of the markets. So we can derive existence and characterization of optimal policies in a situation which is virtually that of complete markets. In a private communication Mr. Duffie observed that one could solve for the optimal strategies using the very tools of Cox and Huang, after an elementary transformation of the data of the problem. This is correct, but

the approach we take in this section is, we believe, not more complicated than that of these authors.

The intuitive (and obvious) result is that even when markets are incomplete, the optimal consumption plan can be chosen in the space generated by the risky securities. Any other choice would in fact increase the volatility of consumption, rendering it less desirable. So when investors determine their optimal policies, they only take into account the information conveyed by the securities prices.

Section 5 provides a simple example carried out for the class of constant relative risk aversion utility functions. Section 6 concludes.

2. A MODEL OF TRADING IN INCOMPLETE MARKETS

In this section we develop a dynamic model of securities trading with the important qualification that markets are incomplete. Intuitively, this means that the information traders have at any point in time is more than just the past realizations of the securities price processes. The price system cannot suffice in itself to give a complete description of the exogenous uncertain environment. As a result, traders will not be able to synthetize the payoff of every contingent claim by a controlled portfolio of the basic securities.

Certainly, one would like to include among investment possibilities as many types of trading strategies as we can, for this implies more "synthetic" securities in addition to the traded securities. Yet, that class of trading strategies should not be so large that it allows arbitrage opportunities to exist. To get a satisfactory model of dynamic trading, it seems reasonable to expect that the space of synthetic securities will include the following contingent claims: when markets are complete, *every* non negative contingent claim in the commodity space, and, when markets are incomplete, *every* non negative contingent claim whose price is uniquely determined. We now proceed to define a class of trading strategies which fulfills this hope, when the prices of the underlying securities can be assumed to be as they are specified in this model.

Our commidity space will be a standard L^2 space. There would be no difficulty in extending our results to an arbitrary L^p space, provided $p > 1$. However, this would be at the cost of expositional ease. The horizon of the model is confined in the time interval $[0, 1]$. An infinite horizon is considered in chapter 2, but within a complete markets framework.

It is convenient to describe the economy in terms of its information structure, its price system and its dynamic strategies. Then we provide a mathematical characterization of the "marketed" space. Finally, we show that a contingent claim is marketed if and only if its expected discounted value is invariant to a change in the martingale measures.

2.1. The Information Structure.

Taken as a primitive is a probability space $(\Omega, \mathcal{F}, P)$. The set Ω gives a complete description of the states of the world. Hence, if one knows $\omega \in \Omega$, one knows everything that happened in this economy up until time 1. The σ-field $\mathcal{F}$ is a collection of distinguishable events, i.e., events that anyone can tell whether they did or did not happen. Finally the probability P stands for the common *beliefs* of the agents. We may regard P as an objective probability assessment of events in $\mathcal{F}$. We require for technical reasons that $\mathcal{F}$ be augmented by all subsets of events of zero probability. (The σ-field $\mathcal{F}$ is said to be complete.)

Next we want to say how the information is revealed over time. Agents know nothing at time 0, and their uncertainty about the future is gradually resolved over time. At time 1 they learn the true state of nature. This is modeled by means of an increasing family of sub-σ-fields or *filtration* $\mathbf{F} = \{\mathcal{F}_t; t \in [0,1]\}$. The fact that this family is increasing means that events are never forgotten. We may think of $\mathcal{F}_t$ as the collection of events that can occur up until time t. Let w be an n-dimensional standard Brownian motion defined on the probability space, and take $\mathcal{F}_t$ to be the right continuous completion of the σ-algebra $\sigma\{w(s): 0 \leq s \leq t\}$ generated by the paths of the Brownian motion between 0 and t. In other terms, the distinguishable events of $\mathcal{F}_t$ consist entirely of sample paths of w in restriction to the time interval $[0,t]$. Since we do not want any residual uncertainty at the final date, we take $\mathcal{F} = \mathcal{F}_1$. Symetrically, since w starts from zero with probability 1, the tribe $\mathcal{F}_0$ is generated by Ω and the probability zero sets. (It is said to be almost trivial.)

For economic as well as mathematical consistency, all the stochastic processes we shall consider are *measurable* and *adapted* to $\mathbf{F}$; see, e.g., Chung and Williams [5]. In words, a process is adapted to a filtration if its values at time t are non anticipative, i.e., if they depend only upon the information at that time and not on the future realizations of w. In fact we will also need to work with processes that are not only adapted but also measurable

with respect to some suitable σ-field on $[0,1] \times \Omega$. In the case of a filtration generated by Brownian motions, it turns out that the natural notion is that of *progressively measurable* processes.[1] We use PM to denote the σ-field generated by the progressive sets of $[0,1] \times \Omega$.[2] There is no distinction in our framework between measurable adapted and progressively measurable processes. It is clear that if a process is progressively measurable with respect to $\mathcal{F}_t$, it is also adapted to $\mathcal{F}_t$. In a continuous information structure the converse is also true. The following lemma addresses this rather pedantic measurability question.

LEMMA 2 1. *Let $X(\cdot,\omega)$ be right continuous for each $\omega \in \Omega$ and let $\mathcal{F}_t = \sigma\{ X_s : 0 \leq s \leq t \}$. If a process is adapted to $\mathcal{F}_t$, it is also progressively measurable.*

PROOF: See Stroock and Varadhan[**31**, exercise 1.5.6.] ∎

Since $\mathcal{F}_t$ is generated by the continuous Brownian motion w_t, the lemma applies. Hence, PM is the natural σ-field on $[0,1] \times \Omega$ associated with the adapted processes. We now turn to the study of the price system.

2.2. The Securities Price System.

We consider a securities markets model with $m+1$ long lived securities traded, indexed by $i = 0, 1, \ldots, m$. Since there is only one good available for consumption in this economy, all prices can be initially numerated in units of this commodity. Security $i = 1, 2, \ldots, m$ is risky, pays dividends at rate $\delta_i(t)$ and sells at time t for $S_i(t)$ ex-dividends. We assume that $\delta_i(t)$ can be written as $\delta_i(t, S_t)$. Security 0—the bond—is locally riskless, pays no dividend and continually compounds interest at the positive rate $r(t, S_t)$, having price

$$B(t) = \exp \int_0^t r(s, S_s)\, ds.$$

Note that B is bounded below by 1, and that for convenience we have taken $B_0 = 1$. Both $\delta_i(x,t) : R^m \times [0,1] \to R_+$ and $r(x,t) : R^m \times [0,1] \to R_+$ are taken to be continuous.

Agents in this economy are assumed to have rational expectations in the sense that they all agree on the law of the securities price processes. In effect $S : [0,1] \times \Omega \to R^m$ is given

[1] A process $Z : R_+ \times \Omega \to R^N$ is said to be *$\mathcal{F}_t$-progressively measurable* if for each t the restriction of Z to $[0,t] \times \Omega$ is $\mathcal{B}_t \times \mathcal{F}_t$ measurable, where $\mathcal{B}_t$ is the Borel σ-field of $[0,t]$.
[2] A subset of $[0,1] \times \Omega$ is progressive if its indicator function is itself progressively measurable.

as a progressively measurable, $(a.s., P)$ continuous process satisfying, in vector notation,

$$(1) \qquad S_t + \int_0^t \delta(s, S_s)\, ds = S_0 + \int_0^t b(s, S_s)\, ds + \int_0^t \sigma(s, S_s)\, dw(s)$$

for all $t \in [0,1]$, $(a.s., P)$. The vector δ is formed by stacking δ_i, with $i = 1, 2, \ldots, m$, and S_0, a vector of constants, gives the prices of the risky securities at time 0. The stochastic integral on the right hand side is defined in the sense of Itô. The functions $b(t,x)$: $[0,1] \times R^m \to R^m$ and $\sigma(t,x)$: $[0,1] \times R^m \to R^{m \times n}$ are continuous and satisfy, along with δ and r, a growth condition and a local Lipschitz condition.[3] The first condition implies that the diffusion S will not "explode" until time 1, while the second guarantees pathwise uniqueness of the solution; cf. Friedman [**15**, chapter 5]. Finally, the matrix σ is assumed to have full rank for all values of t and x. We repeat that the main feature of our model is that $m \leq n$, i.e., that there are fewer securities than Brownian motions required to generate uncertainty. Hence, the rank of σ is m.

Because $B_t \geq 1$, one can rewrite (1) in terms of units of the bond. Let $S_t^* = S_t / B_t$. The *gains process* in discounted units associated with the risky securities is defined by

$$G_t = S_t^* + \int_0^t \frac{\delta_s}{B_s}\, ds.$$

Since $G_0 = S_0$, the difference $G_t - S_0$ represents the sum of the accumulated capital gains and accumulated dividends, in units of the bond. Itô's formula implies that

$$(2) \qquad G_t = S_0 + \int_0^t \frac{b(s, S_s) - r(s, S_s) S_s}{B_s}\, ds + \int_0^t \frac{\sigma(s, S_s)}{B_s}\, dw_s$$

for all $t \in [0,1]$, $(a.s., P)$.

To be a reasonable model of securities markets, the price system should not allow one to create something out of nothing or to create *free lunches*. In words, a free lunch is a

[3] More precisely, the growth condition states that

$$|\sigma(t,x)| + |b(t,x)| + |\delta(t,x)| + |r(t,x)| \leq K(1 + |x|)$$

and the local Lipschitz condition states that for any $n > 0$ there is a positive constant K_n such that

$$|\sigma(t,x) - \sigma(t,y)| \leq K_n |x - y|,$$

and similarly for b, δ and r, for $|x|, |y| \leq n$, and $0 \leq t \leq 1$. We have used the following notation: if σ is a matrix, then $|\sigma|$ denotes the Hilbert-Schmidt norm $(\text{trace}\, \sigma\sigma^{\mathsf{T}})^{1/2}$.

consumption plan wich is non negative, non zero, and is financed by a trading strategy with zero initial cost. Obviously, one needs to specify which types of trading strategies can be used by investors. Harrison and Kreps considered a particularly simple markets model, where trading in the basic securities is only possible at a finite, though arbitrarily large, number of dates specified in advance. The corresponding trading strategies are said to be *simple*.[4] Since simple strategies can easily be implemented in real life, they certainly represent a minimal set of investment strategies one would wish to allow. Harrison and Kreps [17] and Huang [19] demonstrated that to preclude free lunches for that class of simple strategies or their L^2 limits, it is necessary and sufficient that there exists a reassignment of probabilities under which all assets have the same expected rate of return. They refer to this reassignment as an *equivalent martingale measure*. Formally, an equivalent martingale measure Q is a probability measure on $(\Omega, \mathcal{F})$ equivalent[5] to P so that the density dQ/dP is square integrable under P and the gains process $(G_t, \mathcal{F}_t, Q)$ is a martingale.[6] For internal consistency we thus have to ascertain whether such a probability reassignment exists in the special case of the price process given by (1). This will be ensured by an additional regularity condition on the parameters of the price processes. We thus make the following assumption, which will be maintained throughout.

ASSUMPTION 2.2. *Let*

$$\kappa(t,x) = \sigma^{\top}(t,x)(\sigma(t,x)\sigma^{\top}(t,x))^{-1}(b(t,x) - r(t,x)x).$$

There exists a positive constant $K < \infty$ *such that* $|\kappa(t,x)| \leq K$ *for all* (t,x) *in* $[0,1] \times R^m$.

We note that this assumption is in particular satisfied in the models originally considered by Samuelson [29] and Merton [26]. Indeed, in the geometric Brownian motion case with $b(t,x) = \mathbf{b}x$ and $\sigma(t,x) = \boldsymbol{\sigma}x$, where $\mathbf{b}$ and $\boldsymbol{\sigma}$ are respectively a vector and a matrix of constants, and $r$is a scalar, one finds $\kappa = \boldsymbol{\sigma}^{\top}(\boldsymbol{\sigma}\boldsymbol{\sigma}^{\top})^{-1}(\mathbf{b} - r1)$, where 1 is a vector of

[4]For a formal definition of simple strategies, see section 2.3.

[5]Two probability measures P and Q are said to be equivalent if they have the same sets of probability zero. For this it is necessary and sufficient that the density dQ/dP be strictly positive. If P and Q are equivalent, all statements that are true with respect to one probability are also true with respect to the other. This is in particular the case for the almost sure probability statements.

[6]Given a function Z on $[0,1] \times \mathcal{F}$ into R^n, one says that $(Z_t, \mathcal{F}_t, Q)$ is a martingale on $[0,1]$ if Z is a progressively measurable, almost surely right continuous function such that $Z_t = Z(t,\cdot)$ is Q integrable for all $t \in [0,1]$ and $\mathrm{E}^Q[Z_s \mid \mathcal{F}_t] = Z_t$ almost surely, for $0 \leq t \leq s \leq 1$.

m ones. We will now show that under assumption 2.2, there is an abundance of equivalent martingale measures. A random function η: $[0,1] \times \Omega \to R^n$ is said to be in the *null space* of σ if $\sigma_t \eta_t = 0$ for all t in $[0,1]$, $(a.s., P)$.

PROPOSITION 2.3. *A probability $\tilde{Q}$ on $(\Omega, \mathcal{F})$ is an equivalent martingale measure if and only if its density $d\tilde{Q}/dP$ is strictly positive, P square integrable and factors through Q as $d\tilde{Q}/dP = (d\tilde{Q}/dQ)(dQ/dP)$ where*

$$\frac{dQ}{dP} = \exp\left\{-\int_0^1 \kappa_t \cdot dw_t - \frac{1}{2}\int_0^1 |\kappa_t|^2\, dt\right\}$$

and

$$\frac{d\tilde{Q}}{dQ} = \exp\left\{\int_0^1 \eta_t \cdot dw_t - \frac{1}{2}\int_0^1 |\eta_t|^2\, dt\right\},$$

for some η in the null space of σ. In particular, Q is itself an equivalent martingale measure.

We first fix some notation. We denote by ν the product measure generated by the Lebesgue measure and P, and by $L^2(\nu)$ the space $L^2([0,1] \times \Omega, PM, \nu)$; cf. section 2.1.

PROOF OF THE PROPOSITION: Let $R = d\tilde{Q}/dP$ be the Radon-Nikodym derivative of $\tilde{Q}$ with respect to P. Then $R_t = \mathrm{E}(R|\mathcal{F}_t)$ is a martingale under P which can be taken to be continuous, $(a.s., P)$. By Itô's martingale representation theorem, there exists $\alpha \in L^2(\nu)$ such that

$$R_t = 1 + \int_0^t \alpha_s \cdot dw_s. \tag{3}$$

Now $(G_t, \mathcal{F}_t, Q)$ is a martingale if and only if $(R_t G_t, \mathcal{F}_t, P)$ is a martingale (cf. Dellacherie and Meyer [**9**, lemma VII.48]). By Itô's formula, we have from (2) and (3)

$$d\,R_t G_t = R_t \frac{b_t - r_t S_t}{B_t}\, dt + \frac{R_t \sigma_t}{B_t}\, dw_t + G_t \alpha_t \cdot dw_t + \frac{\sigma_t \alpha_t}{B_t}\, dt.$$

But an Itô integral is a martingale only if it has zero drift; cf. Liptser and Shiryayev [**25**]. Therefore $R_t(b_t - r_t S_t) + \sigma_t \alpha_t = 0$, or by the definition of κ, $\sigma_t(\alpha_t + R_t \kappa_t) = 0$. Solving for α yields $\alpha_t = -R_t \kappa_t + \eta'_t$ for some η' in the null space of σ. Substituting the above relation for α in equation (3), one finds

$$\begin{aligned} dR_t = \alpha_t \cdot dw_t &= (-R_t \kappa_t + \eta'_t) \cdot dw_t \\ &= R_t(-\kappa_t + \eta_t) \cdot dw_t, \qquad \eta = \eta'/R. \end{aligned}$$

Hence

$$R_t = \exp\left\{\int_0^t (-\kappa_s + \eta_s)\cdot dw_s - \frac{1}{2}\int_0^t |-\kappa_s+\eta_s|^2\, ds\right\}$$
$$= \exp\left\{-\int_0^t \kappa_s\cdot dw_s - \frac{1}{2}\int_0^t |\kappa_s|^2\, ds\right\}\exp\left\{\int_0^t \eta_s\cdot dw_s - \frac{1}{2}\int_0^t |\eta_s|^2\, ds\right\},$$

where we have used the fact that κ and η lie in orthogonal subspaces of R^n for all t. This proves the first part of the proposition. To prove the second part, we have to verify that $\xi = dQ/dP$ is square integrable and strictly positive almost surely. Square integrability follows from lemma 2.4 below. So it remains to demonstrate that

$$-\int_0^1 \kappa_t\cdot dw_t - \frac{1}{2}\int_0^1 |\kappa_t|^2\, dt > -\infty, \quad (a.s., P).$$

Since κ is bounded, one has $\int_0^1 |\kappa_t|^2\, dt < \infty$, and this implies $\left|\int_0^1 \kappa_t\cdot dw_t\right| < \infty$; cf. Liptser and Shiryayev [**25**, theorem 7.1]. Hence $\xi = dQ/dP > 0$, $(a.s., P)$, as desired. ∎

We have asserted in the proof of the proposition that $\xi \equiv dQ/dP$ is square integrable. In fact, a stronger result holds under the boundedness assumption of κ. Since this property is used repeatedly in the sequel, we state it in a separate lemma.

LEMMA 2.4. *Let $\xi \equiv dQ/dP$ be as in proposition 2.3. Then ξ and ξ^{-1} are in $L^q(P)$ for all finite $q \geq 1$.*

PROOF: Let $q \geq 1$. Then

$$\xi^q = \exp\left\{-\int_0^1 q\kappa_t\cdot dw_t - \frac{1}{2}\int_0^1 q|\kappa_t|^2\, dt\right\}$$
$$= \underbrace{\exp\left\{-\int_0^1 q\kappa_t\cdot dw_t - \frac{q^2}{2}\int_0^1 |\kappa_t|^2\, dt\right\}}_{\text{a martingale under } P}\exp\left\{\frac{q(q-1)}{2}\int_0^1 |\kappa_t|^2\, dt\right\}.$$

Since $|\kappa_t| < K$ by assumption 2.2, we get $\mathrm{E}[\xi^q] \leq \exp(q(q-1)K^2/2)$, and so $\xi \in L^q(P)$. A similar argument works for ξ^{-1}. ∎

The existence of an equivalent martingale measure ensures that there are no free lunches for simple strategies. In this case, the density of the martingale measure can be interpreted as the shadow price of consumption in units of the bond per unit of probability P. We will rely heavily on the probability Q evinced in proposition 2.3. This probability has some

interesting properties of its own. For example, one can prove that among all martingale measures, it is the one with the smallest entropy with respect to P. In this respect, it is the least "different" from the original probability beliefs. One can also show that its density is measurable with respect to the filtration generated by prices. (In fact, it is the only one with that property, but we will not use this fact later.)

LEMMA 2.5. *Let $\xi = dQ/dP$ be as in proposition 2.3, and ξ_t be an almost surely continuous version of the martingale $\mathrm{E}[\xi|\mathcal{F}_t]$. Then ξ_t is measurable with respect to the filtration generated by prices.*

PROOF: By proposition 2.3,

$$\log \xi_t = -\int_0^t \kappa(s, S_s) \cdot dw - (1/2) \int_0^t |\kappa(s, S_s)|^2 \, ds.$$

The second integral of the right hand side is price measurable. (It can be approximated as the sum of price measurable functions.) For the stochastic integral, observe that

$$\begin{aligned} \kappa(t, S_t) \cdot dw(t) &= \big(\sigma(t, S_t)\sigma^{\top}(t, S_t)\big)^{-1} \cdot \sigma_t \, dw_t \\ &= \big(\sigma(t, S_t)\sigma^{\top}(t, S_t)\big)^{-1} \cdot \big(dS_t - b(t, S_t)\, dt - \delta(t, S_t)\, dt\big). \end{aligned}$$

This shows that ξ is price measurable. ∎

Henceforth, we fix $\xi = dQ/dP$ as in proposition 2.3. We already know that $(G_t, \mathcal{F}_t, Q)$ is a martingale. It is interesting to note that if we substitute for P the martingale measure Q, we change the drift term of the gains process but the instantaneous standard deviation remains unaffected. This is the subject to which we now turn.

LEMMA 2.6. *The gains process is a driftless Itô integral under Q. In fact, there exists an n-dimensional standard Brownian motion $\bar{w}$ under Q such that*

$$G_t = S_0 + \int_0^t \frac{\sigma(s, S_s)}{B_s} \, d\bar{w}_s. \tag{4}$$

PROOF: Let $\bar{w}_t = w_t + \int_0^t \kappa_s \, ds$. The conclusion follows directly from (2), Girsanov's theorem and the definition of κ; cf. Liptser and Shiryayev [**25**, chapter 6]. ∎

We conclude this section with a technical result concerning the filtrations generated by w and $\bar{w}$ which is needed in the sequel. Incidentally, the proof shows that under the martingale

measure Q, the undiscounted gains process $S_t + \int_0^t \delta_s\, ds$ has an instantaneous expected rate of return equal to the riskless rate. We thus find that Q also corresponds to the *risk neutral probability* of Cox and Ross [8].

LEMMA 2.7. *Let $\bar{w}$ be as in proposition 2.6. Up to completion, the filtrations generated by w and $\bar{w}$ are the same.*

PROOF: Let $\mathcal{F}_t^{\bar{w}}$ be the completion of $\sigma\{\, \bar{w}_s : s \leq t \,\}$. Since $\bar{w}_t$ is $\mathcal{F}_t$ measurable and $\mathcal{F}_t$ is already complete, the inclusion $\mathcal{F}_t^{\bar{w}} \subseteq \mathcal{F}_t$ is obvious. On the other hand consider the vector diffusion S which is governed by the stochastic differential equation

$$dS_t = (r(t, S_t)S_t - \delta(t, S_t))\, dt + \sigma(t, S_t)\, d\bar{w}_t.$$

Since P and Q are equivalent, this relation holds also under Q. Under our assumptions, one can show that the processes r, δ and σ satisfy the usual growth and local Lipschitz conditions. So S is a strong solution and is adapted to $\mathcal{F}_t^{\bar{w}}$, and so is $w_t = \bar{w}_t - \int_0^t \kappa(s, S_s)\, ds$. This proves $\mathcal{F}_t \subseteq \mathcal{F}_t^{\bar{w}}$. ∎

After these preliminaries, we turn to the task of defining the set of admissible trading strategies.

2.3. The Trading Strategies.

In our economy, an individual's object of choice is a consumption rate process generically denoted by c, where c_t is the random flow of consumption at time t. We take the consumption space to be the positive orthant of $L^2(\nu)$ which, we repeat, is the space $L^2([0,1] \times \Omega, PM, \nu)$; cf. remark following proposition 2.3. Hence a consumption plan is simply a progressively measurable function which specifies the distribution of the single commodity across all states of nature and at all times. It would be possible to include final wealth in the consumption set of agents. Here we are content to assume that agents leave no bequests.

In this model, trading is frictionless. There are no such things as constraints on short selling or transaction costs like brokerage fees for example. A strategy is a decision regarding the trading of the existing securities from time 0 to time 1. More precisely, a trading strategy is completely specified by an $m+1$-vector process $(\alpha, \theta) = \left\{\, (\alpha_t, \theta_t^i) : i = 1, \ldots, m \,\right\}$, where

α_t and θ_t^i are the number of shares of the 0-th and the i-th security, respectively, held at time t. Thus, the value of the strategy (α, θ) at time t in units of the riskless bond is

$$W_t \stackrel{\text{def}}{=} \alpha_t + \theta_t \cdot S^*(t).$$

A strategy is said to *finance* the consumption rate process $c \in L_+^2(\nu)$ if, with this definition of W,

$$W_t + \int_0^t \frac{c_s}{B_s}\, ds = W_0 + \int_0^t \theta_s \cdot dG_s \tag{5}$$

and

$$W_1 = 0, \tag{6}$$

and if the stochastic integral on the right hand side of (5) is well defined. We will specify the set of admissible strategies more fully later on. For now, we interpret conditions (5) and (6). Condition (5) is just the natural budget constraint expressed in units of the bond. The left hand side represents the market value of the portfolio held at time t plus the accumulated flow of consumption from time 0 to time t. The right hand side consists of the initial value of the portfolio plus the accumulated capital gains or capital losses plus the dividends received from securities holding up until time t. (Recall that the gains process is defined by $G_t = S_t^* + \int_0^t (\delta_s/B_s)\, ds$.) So consumption plans are financed by continuous withdrawals from the portfolio during the interval $[0,1]$. Note that condition (5) in fact defines implicitly α for any choice of the "controls" (c, θ). Hence one may view the investor as choosing first his most preferred consumption plan and holdings in the risky securities, in which case the number of shares held in the bond is determined through the budget constraint. Condition (6) says that final wealth must be equal to zero. It is not possible to borrow to finance consumption without paying back at the final date.

A well known example of strategy satisfying, alluded to in section 2.2, is given by the *simple* trading strategies. A trading strategy (α, θ) is simple if there exist a finite sequence of dates $0 = t_0 < t_1 < \cdots < t_n = 1$, along with bounded $\mathcal{F}_{t_j}$-measurable functions θ_i^j and constants $\theta_i^0 \in R$ such that

$$\theta_t^i = \theta_i^0 I_{\{0\}}(t) + \sum_{j=0}^{n-1} \theta_i^j I_{(t_j, t_{j+1}]}(t),$$

with $i = 1, 2, \ldots, m$. An investor employing a simple trading strategy can change his portfolio immediately after observing prices at t_j. Such portfolio policies are close to being observed in the real world. Any reasonable model of securities markets should include them among the set of admissible trading strategies.

If all strategies were simple, the existence of an equivalent martingale measure as defined in proposition 2.3 would suffice to rule out all arbitrage opportunities. When more complex strategies are involved, additional constraints must be imposed. A simple example of arbitrage opportunity is provided by the *doubling trading strategies.* They involve shorting more and more the riskless bond and buying arbitrarily large amounts of the risky securities, until one eventually wins. Such strategies can be implemented in a model with continuous trading, because agents are allowed to do in any finite time interval what would otherwise require an infinite sequence of trades. We first introduce an integral constraint, as in Harrison and Pliska[18].

In the proofs to follow we will use many different norms. It will be convenient to adopt uniformly the following notation. For a progressively measurable process $\rho\colon [0,1]\times\Omega \to R^n$, we note

$$\|\rho\|_{p,P} = \left(\mathrm{E}\left[\left(\int_0^1 |\rho_t|^2\, dt\right)^{p/2}\right]\right)^{1/p}.$$

Accordingly let

$$\mathcal{L}^2(G) = \left\{\theta_t \text{ adapted}\colon \|\sigma^\top\theta/B\|_{2,P} < \infty\right\}. \tag{7}$$

Note that $\mathcal{L}^2(G)$ is defined entirely in terms of the probability P. The notation $\mathcal{L}^2(G)$ is used because, as we now show, the stochastic integral $\int_0^1 \theta_t \cdot dG_t$ is well defined and square integrable under P.

LEMMA 2.8. *Suppose that $\theta \in \mathcal{L}^2(G)$. Then the integral $\int_0^t \theta_t \cdot dG_t$ is an $L^q(Q)$ martingale on $[0,1]$ for all $q \in [1,2)$, and is square integrable under P.*

PROOF: From (2) one has

$$\int_0^1 \theta_t \cdot dG_t = \int_0^1 \theta_t \cdot \frac{b(t,S_t) - r(t,S_t)S_t}{B_t}\, dt + \int_0^1 \frac{\sigma^\top(t,S_t)\theta_t}{B_t} \cdot dw_t.$$

Note that $\|\sigma^\top\theta/B\|_{2,P} < \infty$ by assumption, and so

$$\left\| \int_0^1 \frac{\sigma^\top(t,S_t)\theta_t}{B_t} \cdot dw_t \right\|_{2,P} = \|\sigma^\top\theta/B\|_{2,P} < \infty.$$

This proves that the second integral on the right hand side is well defined and square integrable under P. On the other hand by assumption 2.2, there exists a constant $K < \infty$ such that $|\kappa| \leq K$. By the definition of κ, we have

$$\left|\theta_t \cdot \frac{b_t - r_t S_t}{B_t}\right| = \left|\theta_t \cdot \frac{\sigma_t \kappa_t}{B_t}\right| = \left|\frac{\sigma_t^\top \theta_t}{B_t} \cdot \kappa_t\right|,$$

and so

$$\left|\theta_t \cdot \frac{b_t - r_t S_t}{B_t}\right| \leq K \left|\frac{\sigma_t^\top \theta_t}{B_t}\right|.$$

Hence, by Cauchy Schwarz inequality, the $L^2(P)$ norm of the first integral on the right hand side is majorized by $K\|\sigma^\top\theta/B\|_{2,P} < \infty$. This proves that $\int_0^1 \theta_t \cdot dG_t$ is square integrable under P. Hence it is also q integrable under Q whenever $q < 2$. (Use Hölder's inequality and the fact that $\xi = dQ/dP$ raised to any power is integrable; cf. lemma 2.4.) So the martingale $(\int_0^t \theta_s \cdot dG_s, \mathcal{F}_t, Q)$ is in fact an $L^q(Q)$-martingale, when $1 \leq q < 2$. ∎

We can then define the *admissible trading strategies* as follows.

DEFINITION 2.9. *A pair (α,θ) is an admissible trading strategy if there exist $c \in L^2_+(\nu)$ and $W_0 \in R$ such that, putting $W_t \stackrel{\text{def}}{=} \alpha_t + \theta_t \cdot S_t^*$,*

(i) $\theta \in \mathcal{L}^2(G)$;

(ii) $W_t + \int_0^t \frac{c_s}{B_s}\, ds = W_0 + \int_0^t \theta_s \cdot dG_s$, *for all* $t \in [0,1]$;

(iii) $W_1 = 0$.

In words, a strategy is admissible if it finances some $c \in L^2_+(\nu)$ and if it satisfies the integrability condition (7). The set of all contingent claims in $L^2_+(\nu)$ which can be attained by admissible trading strategies is called the *marketed space* and is denoted by $\mathcal{C}$. A contingent claim which is marketed represents a feasible consumption plan.

Let us use the notation $\tilde{\mathrm{E}}$ for expectation under the probability Q. It develops from the definition and lemma 2.8 that, if $c \in L^2_+(\nu)$ is financed by an admissible trading strategy,

$$\tilde{\mathrm{E}}\left[\int_0^1 \frac{c_s}{B_s}\, ds\right] = \tilde{\mathrm{E}}\left[W_0 + \int_0^1 \theta_s \cdot dG_s\right] = W_0.$$

Since W_0 is the initial endowment that finances the consumption plan, we see that $\widetilde{\mathrm{E}}$ serves us a present value operator. In particular if c is different from 0, we have $W_0 > 0$. One cannot find a positive consumption plan financed by $W_0 \leq 0$ at time 0: there is no free lunch.

We try now to identify $\mathcal{C}$ in $L^2_+(\nu)$. To this aim, we introduce

$$\mathcal{M} = \left\{ M \in L^2(P) \colon M = W_0 + \int_0^1 \theta(t) \cdot dG(t), \text{ for some } W_0 \in R \text{ and } \theta \in \mathcal{L}^2(G) \right\}.$$

The following lemma gives the relation between $\mathcal{M}$ and $\mathcal{C}$.

LEMMA 2.10. *Let $c \in L^2(\nu)$. Then $c \in \mathcal{C}$ if and only if c is positive and $\int_0^1 (c_t/B_t)\,dt$ is in $\mathcal{M}$.*

PROOF: We prove necessity first. By definition 2.9, (2) and (3),

$$\int_0^1 \frac{c_t}{B_t}\,dt = W_0 + \int_0^1 \theta_t \cdot dG_t, \tag{8}$$

where $\theta \in \mathcal{L}^2(G)$. So $\int_0^1 (c_t/B_t)\,dt \in \mathcal{M}$.

Sufficiency. Suppose $c \geq 0$ satisfies $\int_0^1 (c_t/B_t)\,dt \in \mathcal{M}$. There exist $W_0 \in R$ and $\theta \in \mathcal{L}^2(G)$ such that (8) holds. By lemma 2.8, the gains process $\int_0^t \theta_s \cdot dG_s$ is a martingale under Q. Define the residual value of c at time t as

$$W_t = \widetilde{\mathrm{E}}\left[\int_t^1 \frac{c_s}{B_s}\,ds \,\middle|\, \mathcal{F}_t \right].$$

We have

$$W_t + \int_0^t \frac{c_s}{B_s}\,ds = \widetilde{\mathrm{E}}\left[\int_0^1 \frac{c_s}{B_s}\,ds \,\middle|\, \mathcal{F}_t \right] = W_0 + \int_0^t \theta_s \cdot dG_s.$$

So in definition 2.9, (ii) and (iii) are satisfied. The strategy (α, θ), where α is implicitly defined by $W_t = \alpha_t + \theta_t \cdot S_t^*$, is admissible and finances c, as desired. ∎

Proposition 2.13 below gives a characterization of $\mathcal{M}$ based on the null space of σ. I am thankful to Pr. Stroock for the proofs of lemma 2.11 to proposition 2.13. Below the space $L^2(\nu; R^n)$ is the space of progressively measurable, R^n-valued processes whose components belong to $L^2(\nu)$.

LEMMA 2.11. *For each $X \in L^2(P)$, there is a unique $X_0 \in R$ and a unique $\rho_X \in L^2(\nu; R^n)$ such that*

$$X = X_0 + \int_0^1 \rho_X(t) \cdot d\bar{w}_t, \quad (a.s., P).$$

In fact $X_0 = \tilde{\mathrm{E}}[X]$ and there is a $B < \infty$ such that

$$\|\rho_X\|_{2,P} \leq B \|X\|_{2,P}.$$

PROOF: Suppose that $X_0 \in R$ and $\rho \in L^2(\nu; R^n)$ are such that $X_0 + \int_0^1 \rho_t \cdot d\bar{w}_t = 0$. Since by the Cauchy Schwarz inequality $\|\rho\|_{1,Q} \leq \|\xi\|_{2,P} \|\rho\|_{2,P}$, the process $(X_0 + \int_0^{t\wedge 1} \rho_t \cdot d\bar{w}_t, \mathcal{F}_t, Q)$ is a continuous martingale and so $X_0 = 0 = \int_0^{t\wedge 1} \rho(t) \cdot d\bar{w}(t)$. Hence $\|\rho\|_{1,Q} = 0$ and so $\rho = 0$, $(a.s., \nu)$. This proves the uniqueness.

To prove existence, note that by Hölder's inequality and lemma 2.8, $X \in L^q(Q)$ for every $q \in [1,2)$. Choose $q \in (1,2)$. By the martingale representation theorem for Q, which is legitimate since $\bar{w}_t$ generates $\mathcal{F}_t$ (cf. lemma 2.7),

$$X = X_0 + \int_0^1 \rho_X(t) \cdot d\bar{w}(t), \quad X_0 = \tilde{\mathrm{E}}[X],$$

where $\rho_X \in L^q(\nu; R^n)$. Now define $X_t = \tilde{\mathrm{E}}[X \mid \mathcal{F}_t]$ and choose K with $|\kappa| \leq K$ as in assumption 2.2. We have

$$X_{t+\delta} - X_t = \int_t^{t+\delta} \rho_X \cdot d\bar{w} = \int_t^{t+\delta} \rho_X \cdot dw + \int_t^{t+\delta} \rho_X \cdot \kappa \, ds$$

and therefore

$$\begin{aligned}
\left\| \left(\int_t^{t+\delta} \rho_X^2 \, ds \right)^{1/2} \right\|_{2,P} &= \left\| \int_t^{t+\delta} \rho_X \cdot dw \right\|_{2,P} \\
&\leq \left\| X_{t+\delta} - X_t \right\|_{2,P} + K \left\| \int_t^{t+\delta} |\rho_X| \, ds \right\|_{2,P} \\
&\leq \left\| X_{t+\delta} - X_t \right\|_{2,P} + K\delta^{1/2} \left\| \left(\int_t^{t+\delta} |\rho_X|^2 \, ds \right)^{1/2} \right\|_{2,P},
\end{aligned}$$

by Jensen's inequality. Choose $\delta > 0$ such that $K\delta^{1/2} < 1/2$. Then

$$\left\| \left(\int_t^{t+\delta} |\rho_X|^2 \, ds \right)^{1/2} \right\|_{2,P} \leq 2 \left\| X_{t+\delta} - X_t \right\|_{2,P} \leq 4e^K \|X\|_{2,P},$$

by lemma 2.12 below. Finally, choose $n \geq 1$ so that $1/n < \delta$. Then

$$\|\rho_X\|_{2,P}^2 = \sum_0^{n-1} \left\| \left(\int_{k/n}^{(k+1)/n} \rho_X^2 \, ds \right)^{1/2} \right\|_{2,P}^2 \leq 16ne^{2K} \|X\|_{2,P}^2.$$

This completes the proof. ∎

LEMMA 2.12. *Let* $X_t = \widetilde{\mathrm{E}}[X \mid \mathcal{F}_t]$ *and* K *majorize* κ, *as before. Then* $\|X_t\|_{2,P} \leq e^K \|X\|_{2,P}$.

PROOF: We have

$$\widetilde{\mathrm{E}}[X \mid \mathcal{F}_t] = \frac{\mathrm{E}[\xi X \mid \mathcal{F}_t]}{\xi_t} = \mathrm{E}[(\xi/\xi_t)X \mid \mathcal{F}_t]$$

and so

$$\begin{aligned} |X_t| &\leq \mathrm{E}[(\xi/\xi_t)^2 \mid \mathcal{F}_t]^{1/2}\, \mathrm{E}[X^2 \mid \mathcal{F}_t]^{1/2} \\ &\leq e^K\, \mathrm{E}[X^2 \mid \mathcal{F}_t]^{1/2} \end{aligned}$$

by lemma 2.4 with $q = 2$. Therefore $\|X_t\|_{2,P} \leq e^K \|X\|_{2,P}$. ∎

Here is the main result of this section.

PROPOSITION 2.13. $\mathcal{M}$ *is a closed linear subspace in* $L^2(P)$ *and* $X \in \mathcal{M}$ *if and only if* ρ_X *is orthogonal to the null space of* σ. *Finally, if* $X \in \mathcal{M}$, *then*

$$\theta_t = B_t\left(\sigma_t \sigma_t^\top\right)^{-1} \sigma_t \rho_X(t)$$

is the unique $\theta \in \mathcal{L}^2(G)$ *such that* $X = X_0 + \int_0^1 \theta_t \cdot dG_t$.

PROOF: Say $X \in \mathcal{M}$. Then by definition and from proposition 2.6,

$$X = X_0 + \int_0^1 \theta_t \cdot dG_t = X_0 + \int_0^1 \frac{\sigma_t^\top \theta_t}{B_t} \cdot d\bar{w}_t \tag{9}$$

with $X_0 = \widetilde{\mathrm{E}}\, X$ and so $\rho_X(t) = \sigma_t^\top \theta_t / B_t \perp \mathrm{Null}(\sigma_t)$ for all $t \in [0,1]$.

Conversely, suppose $\rho_X(t) \perp \mathrm{Null}(\sigma_t)$, $(a.s., P)$, for all $t \in [0,1]$. Define θ_t as in the proposition. Then $\sigma_t^\top \theta_t / B_t = \rho_X(t)$ and so (9) holds. The unicity of θ follows from the fact that for all t, σ_t is onto, and thus $\sigma_t^\top$ is one-to-one. Finally,

$$\|\!|\sigma^\top \theta / B|\!\|_{2,P} = \|\!|\rho_X|\!\|_{2,P} \leq B\|X\|_{2,P}$$

by lemma 2.11, and so $\theta \in \mathcal{L}^2(G)$, as desired. It remains to show that $\mathcal{M}$ is closed. But this is immediate for if X^n is a sequence in $\mathcal{M}$ which converges to X in $L^2(P)$, then

$$\|\!|\rho_X - \rho_X^n|\!\|_{2,P} \leq B\|X - X^n\|_{2,P} \to 0,$$

and so $\rho_X(t) \perp \mathrm{Null}(\sigma_t)$. ∎

COROLLARY 2.14. *$\mathcal{C}$ is a closed subspace of $L^2(\nu)$. If, in addition, markets are complete ($m = n$), then $\mathcal{C} = L^2_+(\nu)$.*

PROOF: The proof of the first assertion is obvious from lemma 2.10 and proposition 2.13. Moreover if $m = n$, then $\text{Null}(\sigma_t) = \{0\}$ for all t in $[0,1]$, and so $\mathcal{M} = L^2(P)$. ∎

We have seen above that $\tilde{\mathrm{E}}$ can serve us a present value operator. To conclude this section, we will show that if a claim is marketed, the expectation under any equivalent martingale measure (cf. proposition 2.3) can serve the same purpose.

PROPOSITION 2.15. *A contingent claim c belongs to $\mathcal{C}$ if and only if its expected discounted value does not depend on the choice of the equivalent martingale measure.*

PROOF: We will prove first that if ρ is orthogonal to the null space of σ, with $\|\rho\|_{2,P} < \infty$, then

$$\mathrm{E}^{\tilde{Q}} \int_0^1 \rho_t \cdot d\bar{w}_t = 0$$

for any equivalent martingale measure $\tilde{Q}$. By proposition 2.3, the density of $\tilde{Q}$ with respect to Q in restriction to $\mathcal{F}_t$ can be taken to be the continuous Q- martingale

$$Z_t = \exp\left\{\int_0^t \eta_s \cdot d\bar{w}_s - \frac{1}{2}\int_0^t |\eta_s|^2\, ds\right\} = \int_0^t Z_s \eta_s \cdot d\bar{w}_s,$$

where η is in the null space of σ. An application of Girsanov's theorem shows then that $w^*_t = \bar{w}_t - \int_0^t \eta_s\, ds$ is a Brownian motion under $\tilde{Q}$. Since

$$\|\rho\|_{1,\tilde{Q}} \le \left\|\frac{d\tilde{Q}}{dP}\right\|_{2,P} \|\rho\|_{2,P} < \infty$$

by the Cauchy Schwarz inequality, the stochastic integral $\int_0^t \rho_s \cdot dw^*_s$ is a uniformly integrable martingale under $\tilde{Q}$. Hence

$$\mathrm{E}^{\tilde{Q}} \int_0^1 \rho_t \cdot d\bar{w}_t = \mathrm{E}^{\tilde{Q}} \int_0^1 \rho_t \cdot dw^*_t + \mathrm{E}^{\tilde{Q}} \int_0^1 \rho_t \cdot \eta_t\, dt = 0$$

since ρ and η lie in orthogonal subspaces. This proves the claim.

Now consider necessity. Let $c \in \mathcal{C}$ and let $\tilde{Q}$ be some equivalent martingale measure. Then $X = \int_0^1 (c_t/B_t)\, dt \in \mathcal{M}$ and so

$$X = X_0 + \int_0^1 \rho_X(t) \cdot d\bar{w}_t, \qquad \tilde{\mathrm{E}}\, X = X_0,$$

where ρ_X is orthogonal to the null space of σ; cf. lemma 2.11. Applying the claim, we see that $\mathrm{E}^{\tilde{Q}} X = X_0$.

To prove sufficiency suppose that $c \in L^2_+(\nu)$ and define as above $X = \int_0^1 (c_t/B_t)\,dt$. Note that

$$\mathrm{E}\, X^2 \le \mathrm{E}\left(\int_0^1 c_t\,dt\right)^2 \le \mathrm{E}\left(\int_0^1 c_t^2\,dt\right) < \infty,$$

and so we can write

$$(\#) \qquad X = X_0 + \int_0^1 \rho_X(t)\cdot d\bar{w}_t + \int_0^1 \rho_X^{\perp}(t)\cdot d\bar{w}_t,$$

where for all t, ρ_X is in the range of $\sigma^{\top}$ and $\rho_X^{\perp}$ is orthogonal to it; cf. lemma 2.11. In view of proposition 2.13, we want to prove that $\rho_X^{\perp}(t) = 0$, $(a.s., P)$. We will use a stopping time argument. Define the stopping time

$$\tau_n = \inf\left\{t : \int_0^t |\rho_X^{\perp}(s)|^2\,ds \ge n\right\},$$

and $\tau_n = 1$ if the above set is empty. Since $\|\rho_X + \rho_X^{\perp}\|_{2,P} \le B\|X\|_{2,P}$, the integral $\int_0^1 |\rho_X^{\perp}(t)|^2\,dt$ is finite, $(a.s., P)$, and so $\tau_n \to 1$, $(a.s., P)$. Let $\bar{Q}_n$ be the probability with density

$$\frac{d\bar{Q}_n}{dQ} = \eta^n = \exp\left\{\int_0^1 \rho_X^{\perp,n}(t)\cdot d\bar{w}_t - \frac{1}{2}\int_0^1 |\rho_X^{\perp,n}(t)|^2\,dt\right\},$$

where $\rho_X^{\perp,n}(t) = \rho_X^{\perp}(t)1_{[0,\tau_n]}$. Since by construction

$$\tilde{\mathrm{E}}\exp\left\{(1/2)\int_0^1 |\rho_X^{\perp,n}(t)|^2\,dt\right\} \le e^{n/2},$$

$\bar{Q}_n$ is a well defined probability measure equivalent to Q. Note that by proposition 2.3, $\bar{Q}_n$ is an equivalent martingale measure. Now we take the expectation of $(\#)$ under $\bar{Q}_n$ and apply again our claim to see that if X has constant expectation then it must be that

$$\mathrm{E}^{\bar{Q}_n}\left[\int_0^1 \rho_X^{\perp}(t)\cdot d\bar{w}_t\right] = 0.$$

But $\eta^n = \int_0^1 \eta_t^n \rho_X^{\perp,n}(t)\cdot d\bar{w}_t$ and we can rewrite this equation as

$$\tilde{\mathrm{E}}\left[\left(\int_0^1 \eta_t^n \rho_X^{\perp,n}(t)\cdot d\bar{w}_t\right)\left(\int_0^1 \rho_X^{\perp}(t)\cdot d\bar{w}_t\right)\right] = \tilde{\mathrm{E}}\left[\int_0^1 \eta_t^n |\rho_X^{\perp,n}(t)|^2\,dt\right] = 0$$

and so $\rho_X^{\perp}(t) = 0$ on $[\![0, \tau_n]\!]$. Since $\tau_n \to 1$ as $n \to \infty$, we get $\rho_X^{\perp}(t) = 0$, $(a.s., P)$, as desired.

■

To summarize, one has constructed a space of feasible consumption which satisfies the following properties:

(i) It depends only on the original probability beliefs P, not on the equivalent martingale measures, which are not unique when markets are incomplete;

(ii) It is a closed subspace of the consumption space, and the two sets are identical when markets are complete;

(iii) A claim is marketed if and only if its present value as given by any equivalent martingale measure is uniquely determined.

We now turn to some applications of the preceding model.

3. Admissible strategies and non negative wealth.

We consider in this section another class of strategies. We replace the condition $\theta \in \mathcal{L}^2(G)$ by the weaker assumption

$$\int_0^1 \left| \frac{\sigma_t^{\top} \theta_t}{B_t} \right|^2 < \infty, \quad (a.s., P); \tag{10}$$

cf. Liptser and Shiryayev [**25**, chapter 4]. On the other hand we add the condition $W_t \geq 0$ for all $t \in [0,1]$. We have the following definition.

Definition 3.1. *A strategy (α, θ) satisfies the non negative wealth constraint if there exists $c \in L^2(\nu)$ and $W_0 \in R$ such that, putting $W_t \stackrel{\text{def}}{=} \alpha_t + \theta_t \cdot S_t^*$,*

(i) *θ satisfies* (10),

(ii) $\displaystyle\int_0^t \frac{c_s}{B_s}\, ds + W_t = W_0 + \int_0^t \theta_s \cdot dG_s$, *for all* $t \in [0,1]$;

(iii) $W_t \geq 0$, *for all* $t \in [0,1]$.

The set of contingent claims $c \in L_+^2(\nu)$ which are attained by such strategies is noted C'. It is known that there can be no free lunch under the non negative wealth constraint; cf. Dybvig and Huang [**12**]. For the sake of completeness we repeat the argument.

Lemma 3.2. *When strategies satisfy the non negative wealth constraint, no arbitrage opportunities exist.*

PROOF: Suppose $c \in C'$ is non zero. Let (X, Q) be the continuous local martingale

$$X_t = W_0 + \int_0^t \theta_s \cdot dG_s,$$

and let τ_n, $\tau_n \to 1$, be a localizing sequence for X. Since

$$X(\tau_n) = \int_0^{\tau_n} \frac{c_s}{B_s}\, ds + W(\tau_n) \geq 0,$$

Fatou's lemma implies that $\tilde{\mathrm{E}}\, X_1 \leq \liminf_n \tilde{\mathrm{E}}\, X_{\tau_n} = W_0$. But $X_1 = \int_0^1 (c_t/B_t)\, dt$ and c is non zero and non negative (under both P and Q, since they are equivalent), so $\tilde{\mathrm{E}}\, X_1 > 0$. Thus $W_0 \geq \tilde{\mathrm{E}}\, X_1 > 0$ and there is no arbitrage, as announced. ∎

Since C' has no arbitrage opportunities, it is natural to ask whether one may replace the constraint $\theta \in \mathcal{L}^2(G)$ with the more easily interpretable non negative wealth constraint. In fact, C is strictly included in C', since there are strategies which "throw money away" such as running a doubling strategy in reverse; cf. Harrison and Pliska[**17**]. We conclude this section by showing what could be the "fair price" of a consumption good $c \in C'$. Since the initial cost $W_0 \geq \tilde{\mathrm{E}}\left[\int_0^1 (c_t/B_t)\, dt\right]$, it is in the interest of the investor to find a strategy which minimizes the cost of manufacturing c.

DEFINITION 3.4. *Let $c \in C'$. The fair price at $t = 0$ for c is the number*

$$\alpha \stackrel{\text{def}}{=} \inf \left\{ W_0 : \begin{array}{l} W_0 \text{ is the initial cost of a strategy financing } c \\ \text{under the non negative wealth constraint} \end{array} \right\}.$$

PROPOSITION 3.5. *Let $c \in C'$. Then there exists a strategy satisfying the non negative wealth constraint which manufactures c and whose initial cost W_0 is the fair price of the claim. This strategy is not unique in general. Moreover if*

$$(\#) \qquad W_0 = \tilde{\mathrm{E}}\left[\int_0^1 \frac{c_s}{B_s}\, ds\right],$$

then the consumption plan c is in fact marketed.

PROOF: The first two assertions follow from the proposition proved in [**30**, section 5]. Now suppose $c \in C'$ satisfies (#). In the budget constraint

$$\int_0^t \frac{c_s}{B_s}\, ds + W_t = W_0 + \int_0^t \theta_s \cdot dG_s,$$

we want to prove that $\theta \in \mathcal{L}^2(G)$. By lemma 7.10 of Jacod [21], the positive supermartingale $X_t = W_0 + \int_0^t \theta_s \cdot dG_s$ is a uniformly integrable martingale under Q and so $X_t = \tilde{\mathrm{E}}[X_1 \mid \mathcal{F}_t]$. Since $X_1 \in L^2(P)$, there exists by lemma 2.11 a process ρ, with $\|\rho\|_{2,P} < \infty$, such that

$$X_t = X_0 + \int_0^t \rho_s \cdot d\bar{w}_s.$$

Define the stopping time

$$\tau_n = \inf\left\{ t \leq 1 : \int_0^t \left| \frac{\sigma_s^\top \theta_s}{B_s} \right|^2 ds \geq n \right\},$$

and $\tau_n = 1$ if the set above is empty. But θ satisfies (10), so $\tau_n \to 1$ as $n \to \infty$. Moreover, $X_t^n = X_{t \wedge \tau_n}$ is a continuous $L^2(Q)$ martingale (in fact, bounded) and this implies $\sigma^\top \theta / B = \rho$ on $[\![0, \tau_n]\!]$. Sending n to ∞, we have $\sigma^\top \theta / B = \rho$, $(a.s., P)$ and so

$$\|\sigma^\top \theta / B\|_{2,P} = \|\rho\|_{2,P} < \infty.$$

This proves $\theta \in \mathcal{L}^2(G)$, as desired. ∎

4. Optimization under incomplete markets

In our second application we consider a version of the optimal consumption and portfolio choice problem. An agent is endowed with a time additive utility function on consumption rate processes $u\colon R_+ \times [0,1] \to R$ and an initial wealth $W_0 > 0$. We assume that $u(c,t)$ is continuous, increasing and concave in c for each $t \in [0,1]$, and possibly unbounded from below at $c = 0$. The agent wants to solve the following problem

$$(\mathcal{P}_1) \qquad \sup_{c \in L_+^2(\nu)} \mathrm{E} \int_0^1 u(c_t, t)\, dt$$

$$\text{s.t.} \quad \begin{cases} c \text{ is financed by an admissible trading} \\ \text{strategy with initial investment } W_0. \end{cases}$$

The budget constraint has the form

$$\int_0^t \frac{c_s}{B_s}\, ds + W_t = W_0 + \int_0^t \theta_s \cdot dG_s,$$

where we have put $W_t = \alpha_t + \theta_t \cdot S_t^*$ and $\theta \in \mathcal{L}^2(G)$. Our task is to provide sufficient conditions for the existence of a solution and to derive an optimal policy when it exists. This is an extension of Cox and Huang [7], [8].

To solve the dynamic problem, we map $(\mathcal{P}_1)$ into the following static variational problem:

$$(\mathcal{P}_2) \qquad \begin{aligned} &\sup_{c \in L^2_+(\nu)} \mathrm{E} \int_0^1 u(c_t, t)\, dt \\ &\text{s.t.} \quad \widetilde{\mathrm{E}} \left[\int_0^1 \frac{c_t}{B_t}\, dt \right] = W_0. \end{aligned}$$

It is clear at this stage that the budget constraint of $(\mathcal{P}_2)$ comes from $(\mathcal{P}_1)$ by taking expectations under Q. It is convenient to transform slightly $(\mathcal{P}_2)$ by use of the following lemma.

LEMMA 4.1. *Let $\xi = dQ/dP$ and ξ_t denote a continuous version of $\mathrm{E}[\xi|\mathcal{F}_t]$; cf. section 2. Then*

$$\widetilde{\mathrm{E}} \left[\int_0^1 \frac{c_t}{B_t}\, dt \right] = \mathrm{E} \left[\int_0^1 \frac{c_t \xi_t}{B_t}\, dt \right].$$

PROOF: Let A be the increasing, adapted process $\int_0^{\cdot} (c_s/B_s)\, ds$. We want to prove that

$$\mathrm{E}[\xi_1 A_1] = \mathrm{E} \left[\int_0^1 \xi_t dA_t \right].$$

This follows, for example, from Elliott [**13**, corollary 7.16]. ∎

Hence the static variational program reduces to

$$\begin{aligned} &\sup_{c \in L^2_+(\nu)} \mathrm{E} \int_0^1 u(c_t, t)\, dt \\ &\text{s.t.} \quad \mathrm{E} \left[\int_0^1 \frac{c_t \xi_t}{B_t}\, dt \right] = W_0. \end{aligned}$$

Sufficient conditions for a solution to that program to exist are well known; cf. Aumann and Perles (1), Berliocchi and Lasry (2), and Cox and Huang (6). For future reference, we appeal to two theorems given by Cox and Huang, and apply them in a way consistent with our assumptions. The theorems cover utility functions that are either bounded from below or differentiable. Before that a technical lemma is recorded.[7]

[7] A function $f\colon R_+ \times \Omega \times [0,1] \to R_+$ measurable with respect to the product σ-algebra $\mathcal{B}(R_+) \times \mathcal{F} \times \mathcal{B}([0,1])$ is said to be $o(x)$ integrably in $L^q(\nu)$ as $x \to \infty$, denoted $f \underset{q,\nu}{\ll} x$, if for each $\epsilon > 0$, there exists $\eta \in L^q_+(\nu)$ such that

$$x \geq \eta \Longrightarrow f(x, \omega, t) \leq \epsilon x.$$

LEMMA 4.2. *Suppose* $u(x,t)$: $R_+ \times [0,1] \to R_+$ *exhibits discounting in that*

$$u(x,t) \leq u(x,0), \quad \text{for all } t \in [0,1]$$

and that there exist $b \in (0,1]$ *and* A, $B > 0$ *such that for all* $x \geq 0$

$$u(x,0) \leq A + Bx^{1-b}.$$

Then $u(x,t)/\zeta(\omega,t)$ *is* $o(x)$ *integrably in* $L^q(\nu)$ *if* $\zeta^{-1} \in L^{q/b}(\nu)$.

PROOF: By the assumption of the lemma, $u(x,t)\zeta^{-1} \leq (A + Bx^{1-b})\zeta^{-1}$. Since $\zeta^{-1} \in L^{q/b}(\nu) \subseteq L^q(\nu)$ implies $\zeta^{-1} \underset{q,\nu}{\ll} x$ (immediate verification), all that has to be proved is that $x^{1-b}\zeta^{-1}(\omega,t) \underset{q,\nu}{\ll} x$. Choose k so large that $k^{-b} < \epsilon$ and let $\eta = k\zeta^{-1/b}$. Then $\|\eta\|_{q,\nu} = k\|\zeta^{-1}\|^{1/b}_{q/b,\nu}$, which is bounded by assumption. On the other hand $x \geq \eta$ implies $x\zeta^{1/b}/k \geq 1$, so we have $(x\zeta^{1/b}/k)^{1-b} \leq x\zeta^{1/b}/k$, which in turn implies $x^{1-b}/\zeta \leq k^{-b}x < \epsilon x$, as desired. ∎

PROPOSITION 4.3. *Suppose* $u(x,t)$: $R_+ \times [0,1] \to R_+$ *is Borel measurable and is continuous, strictly increasing, and concave in* x *for all* $t \in [0,1]$. *Suppose also that* u *exhibits discounting in that*

$$u(x,t) \leq u(x,0), \quad \text{for all } t \in [0,1],$$

and satisfies the growth condition of lemma 4.2. Then there exists a solution to $\mathcal{P}_2$ *if* $(\xi/B)^{-1} \in L^{2/b}(\nu)$.

PROOF: This follows directly from Cox and Huang [**6**, theorem 3.7] and lemma 4.2 above, with $\zeta = \xi/B$ and $q = 2$. ∎

Here is the second result covering utility functions that are possibly unbounded from below at 0.

PROPOSITION 4.4. *Suppose that* $u(x,t) = e^{-\beta_t t}u(x)$ *is a separable utility function, where* $\beta_t \in R_+$ *is the discount rate at time* t. *Suppose also that* $u(x)$ *is non trivial, increasing, concave and differentiable, and is strictly concave on any subset of its domain where* u' *is strictly positive. In the case that* u *is strictly increasing, assume further that there exists*

$b > 0$ and $A > 0$ such that $u'(x) \leq Ax^{-b}$. Then there exists a solution to $\mathcal{P}_2$ if $(\xi/B)^{-1} \in L^{2/b}(\nu)$.

PROOF: This is a specialization of Cox and Huang [6, theorem 3.10]. A slight modification of proposition 3.2 (op. cit.) shows that one can take $\delta = 0$ in the theorem. ∎

The integrability condition on the inverse of ξ/B is related to how fast the discounted commodity prices converge to zero. If prices asymptote to zero fast enough and the utility function grows rapidly at infinity, it can be "worthwhile" to concentrate all consumption on a time interval close to 1, i.e., to choose c arbitrarily high for $t = 1$ and zero elsewhere. In such a case, there would be no optimum. Sufficient conditions for $(\xi/B)^{-1} \in L^{2/b}(\nu)$ are readily available. For instance, Cox and Huang show that this obtains if the parameters of a system of stochastic differential equations completely derived from the price processes satisfy a local Lipschitz and a growth condition. In the present case the system to consider is that formed by the pair $(S, \xi/B)$. Let us assume that for some $\epsilon > 0$,

$$(\sigma_t \sigma_t^{\mathsf{T}}) \geq \epsilon I_m,$$

for all (t, x) in $[0,1] \times R^m$. It is then easily verified that the local Lipschitz and growth condition satisfied by the price system itself, together with assumption 2.2, ensure that the system of stochastic differential equations governing $(S, \xi/B)$ has all the required regularity properties. Hence we have the following corollary.

COROLLARY 4.5. *Under the assumptions of section 2, and if the conditions on the utility function of either proposition 4.3 or 4.4 are satisfied, there exists a solution to the static variational problem $(\mathcal{P}_2)$.*

Finally, we turn to the implementation of the solution of $(\mathcal{P}_2)$ through dynamic trading. Our approach differs from that of Cox and Huang in that markets are not complete. This implies that the "martingale measure" is not unique. However, a martingale representation technique is still available because the probability law of the price process is itself uniquely determined. A relevant concept is the martingale formulation of Stroock and Varadhan.[8] The following lemma is instrumental for the proof of proposition 4.6.

[8]Given locally bounded measurable functions $a\colon [0,1] \times R^d \to S_+(R^d)$ and $b\colon [0,1] \times R^d \to R^d$, where

LEMMA 4.6. *Consider the system of stochastic differential equations compactly written as*

$$d\begin{bmatrix} S_t^* \\ B_t \end{bmatrix} = \begin{bmatrix} \sigma(t, B_t S_t^*)/B_t \\ 0 \end{bmatrix} d\bar{w}_t + \begin{bmatrix} -\delta(t, B_t S_t^*)/B_t \\ B_t r(t, B_t S_t^*) \end{bmatrix} dt,$$

and let $\mathcal{L}_t$ be the differential generator of (S^*, B) under Q.[9] *Then the martingale problem for $\mathcal{L}_t$ starting from $(S_0, 1)$ at zero has a unique solution, and any $L^q(Q)$-bounded $(q > 1)$ martingale X adapted to the filtration generated by the price processes can be written as*

$$X_t = \tilde{\mathrm{E}}\, X + \int_0^t \theta_s \cdot dG_s, \quad \text{for some } \theta \in \mathcal{L}^2(G).$$

PROOF: Since the drift term and the diffusion matrix are locally Lipschitz continuous, the diffusion (S^*, B) exists uniquely. Hence the martingale problem has a unique solution; cf. Stroock and Varadhan [**31**, chapter 8]. This proves the first part. For $\phi \in C_0^\infty(R^{m+1})$ (cf. footnote 8) define

$$X_\phi(t) = \phi(S_t^*, B_t) - \phi(S_0, 1) - \int_0^t (\mathcal{L}_u \phi)(S_u^*, B_u)\, du,$$

and let $\mathcal{F}_t^S = \sigma\{S(s) : 0 \le s \le t\}$ be the filtration generated by S. Since the probability law of the diffusion $Q \circ (S^*, B)^{-1}(\cdot)$ is the unique solution to the martingale problem, Q is the one and only probability such that $(X_\phi(t), \mathcal{F}_t^S, Q)$ is a martingale for all $\phi \in C_0^\infty(R^{m+1})$.

$S_+(R^d)$ denotes the space of positive definite symmetric matrices, define the operator

$$\mathcal{L}_t = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,\cdot)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^j(t,\cdot)\frac{\partial}{\partial x^i}.$$

One says that the probability measure P on $C([0,1]; R^d)$ and its Borel σ- field solves the *martingale problem* for L_t, starting from x at zero if $P(x(0) = x) = 1$ and

$$f(x(t)) - \int_0 (L_u f)(x(u))\, du$$

is a martingale under P for all functions f in $C_0^\infty(R^d)$, the space of compactly supported, infinitely differentiable functions; cf. Stroock and Varadhan.[**31**]

[9]The differential generator of (S^*, B) is the differential operator given in the last footnote with $a(t,x,y)$: $[0,1] \times R^m \times R \to S_+^{m+1}$ and $b(t,x,y)$: $[0,1] \times R^m \times R \to R^{m+1}$ defined as

$$a(t,x,y) = \begin{bmatrix} \sigma(t,yx)/y \\ 0 \end{bmatrix}\begin{bmatrix} \sigma(t,yx)/y \\ 0 \end{bmatrix}^{\mathsf{T}}$$
$$b(t,x,y) = \begin{bmatrix} -\delta(t,yx)/y \\ yr(t,yx) \end{bmatrix}.$$

Note that here the matrix a is degenerate.

Hence by Jacod's theorem, the space H^q, $(q > 1)$, of $L^q(Q)$-bounded martingales adapted to $\mathcal{F}^S$ is the stable subspace generated by the family $\{X_\phi(t)\}$ and the constant functions; cf. Jacod [**21**, theorem 11.2 and corollary 11.4]. But by Itô's formula,

$$X_\phi(t) = \int_0^1 \theta_s \cdot dG_s,$$

where

$$\theta_s = 1_{[0,t]}(s)\frac{\partial \phi}{\partial S}(S_s^*, B_s).$$

Hence every element of $H^q(Q, \mathcal{F}^S)$ is of the form

$$X_t = c + \int_0^t \theta_s \cdot dG_s, \quad c = \tilde{\mathrm{E}}\, X_1,$$

where $c \in R$ and θ is $\mathcal{F}_t^S$-adapted and such that $\|\sigma\theta/B\|_{q,Q} < \infty$. Finally, by lemma 2.11, $\theta \in \mathcal{L}_2(G)$. This completes the proof. ∎

Here is the main proposition of this section.

PROPOSITION 4.7. *Under the assumptions of section 2, and if the conditions on the utility function of either proposition 4.3 or proposition 4.4 are satisfied, there is a solution to the dynamic problem* $(\mathcal{P}_1)$.

PROOF: . Let $c^* \in L^2(\nu)$ be a solution to $(\mathcal{P}_2)$. Note that if we define $\hat{c}_t = \tilde{\mathrm{E}}[c_t^* | \mathcal{F}_t^S]$, then $\mathrm{E}^\nu(\hat{c}/B) = \mathrm{E}^\nu(c^*/B) = W_0$ and moreover

$$\begin{aligned} \mathrm{E}^P\left[\int_0^1 u(\hat{c}_t, t)\, dt\right] &= \tilde{\mathrm{E}}\left[\int_0^1 \frac{u(\hat{c}_t, t)}{\xi_t}\, dt\right] \\ &\geq \tilde{\mathrm{E}}\left[\int_0^1 \frac{u(c_t, t)}{\xi_t}\, dt\right] = \mathrm{E}\left[\int_0^1 u(c_t, t)\, dt\right], \end{aligned}$$

by Jensen's inequality and the fact that ξ is price measurable. So we may and will assume that c^* is in the filtration generated by prices. By lemma 4.6, we have

$$\int_0^1 \frac{c_t^*}{B_t}\, dt = W_0 + \int_0^1 \theta_t \cdot dG_t,$$

for some $\theta \in \mathcal{L}^2(G)$. Hence by lemma 2.10, c^* is financed by an admissible strategy with initial investment W_0. This completes the proof. ∎

In the proposition above, we have a direct proof of the existence of a solution to the dynamic problem which does not rely on the completeness of the markets. The method is similar to that of Cox and Huang and well adapted to the definition of our marketed space. On the other hand, there are crucial assumptions without which the martingale representation technology we have used becomes unavailable. We have to assume that there are no "uninsurable risks". The presence of an uninsurable endowment is by no means the only way such risks can be introduced in the model. For instance the riskless rate could fail to be a function of the price process alone, or the securities prices themselves could depend on some additional, non marketed processes. In this case our technique would not provide insights into the structure of the optimal policies, if these optimal policies exist.

Let us now turn to characterization. Our procedure is based on Kolmogorov's backward equation, as in Cox and Huang [7], and may be a little simpler. We will focus only on their first main theorem. Now additional assumptions must be placed on the parameters of the diffusion and on the utility function. We assume that $u(x,t)$ is continuous, increasing, strictly concave, with a right hand derivative u_{x+} verifying

$$\lim_{x\to\infty} u_{x+}(x,t) = 0.$$

Define the inverse of the marginal utility as

$$f(z,t) = \inf\{\, x \in R_+ : u_{x+}(x,t) \le z \,\}.$$

It follows from the Lagrangian theory that if c is a solution to $\mathcal{P}_2$, there exists a strictly positive real number λ such that

$$c_t = f(Z_t, t), \qquad (a.s., \nu),$$

where we have put $Z_t = \lambda\xi_t/B_t$; cf. Cox and Huang [7]. We assume that f and its first two derivatives with respect to x are continuous in (x,t) and satisfy a polynomial growth condition.[10] The stacked process (S_t, Z_t) satisfies under Q the following stochastic differential equation

$$d\begin{bmatrix} S_t \\ Z_t \end{bmatrix} = \begin{bmatrix} r(t,S_t)S_t - \delta(t,S_t) \\ Z_t\left(|\kappa(t,S_t)|^2 - r(t,S_t)\right) \end{bmatrix} dt + \begin{bmatrix} \sigma(t,S_t) \\ -Z_t\kappa(t,S_t) \end{bmatrix} d\bar{w}_t,$$

[10] We say that a function $f(x) : R^m \to R$ satisfies a polynomial growth condition if there exist strictly positive constants K and γ such that $|f(x)| \le K(1+|x|^\gamma)$ for all $x \in R^m$.

with $S(0) = S_0$ and $Z(0) = \lambda$. We suppose further that the partial derivatives with order 1 and 2 of the drift term and the diffusion coefficient of that diffusion are continuous in all variables and satisfy a polynomial growth condition.

PROPOSITION 4.8. *Let the function F be defined as*

$$F(t,x,z) = \widetilde{\mathrm{E}}\left[\int_t^1 f(Z_s,s)\exp\Big\{-\int_t^s r(u,S_u)\,du\Big\}\,ds\right]_{\left\{\begin{smallmatrix} S_t=x \\ Z_t=z \end{smallmatrix}\right\}}.$$

Then F has continuous partial derivatives up to order 2 in t, x and z and satisfies the backward Kolmogorov equation

$$\frac{\partial F}{\partial t} + \mathcal{L}F - rF + f = 0$$

with boundary conditions

$$F(1,x,z) = 0$$

$$F(0,S_0,\lambda) = W_0,$$

where $\mathcal{L}$ is the differential generator of (S,Z) under Q. The optimal consumption portfolio policy is

$$\begin{aligned}
&\theta_t = F_x(t,S_t,Z_t) - Z_t\,F_z(t,S_t,Z_t)(\sigma_t\sigma_t^{\top})^{-1}\big(b(t,S_t) - r(t,S_t)S_t\big);\\
&W_t \stackrel{\text{def}}{=} \alpha_t + \theta_t\cdot S_t^{*} = \frac{F(t,S_t,Z_t)}{B_t};\\
&c_t = f(Z_t,t).
\end{aligned}$$

PROOF: The function F arises in a natural way. It is the value of the optimal consumption plan c at time t in undiscounted units, when the price vector is $S(t) = x$ and the shadow price of consumption per unit of marginal utility is $Z(t) = \lambda\xi_t/B_t = z$. Hence

$$\frac{F(t,x,z)}{B_t} = \widetilde{\mathrm{E}}\left[\int_t^1 \frac{f(Z_s,s)}{B_s}\,ds\right] = \widetilde{\mathrm{E}}\left[\int_t^1 \frac{c_s}{B_s}\,ds\right]$$

for initial data $S_t = x$ and $Z_t = z$. The linear partial differential equation is just the Feynman Kac version of Kolmogorov's backward equation; cf. Krylov [**23**, theorem 2.9.10].

Appealing to Itô's formula we have

$$\begin{aligned}\frac{F(t,S_t,Z_t)}{B_t} &= F(0,S_0,\lambda) + \int_0^t \frac{\partial F}{\partial x}(t,S_t,Z_t)\frac{\sigma(t,S_t)}{B(t)}\,d\bar{w}(t) \\ &\quad - \int_0^t \frac{\partial F}{\partial z}(t,S_t,Z_t)\frac{Z_t\kappa(t,S_t)}{B(t)}\cdot d\bar{w}(t) \\ &\quad + \int_0^t (\mathcal{L}F + \frac{\partial F}{\partial t})(t,S_t,Z_t)B_t^{-1}\,dt - \int_0^t F(t,S_t,Z_t)\frac{r(t,S_t)}{B_t}\,dt \\ &= F(0,S_0,\lambda) + \int_0^t \frac{\partial F}{\partial x}(t,S_t,Z_t)\frac{\sigma(t,S_t)}{B_t}\,d\bar{w}(t) \\ &\quad - \int_0^t \frac{\partial F}{\partial z}(t,S_t,Z_t)\frac{Z_t\kappa(t,S_t)}{B_t}\cdot d\bar{w}(t) - \int_0^t \frac{f(Z_t,t)}{B(t)}\,dt.\end{aligned}$$

By lemma 2.10 we have

$$W_t = \tilde{\mathrm{E}}\left[\int_t^1 \frac{c_t}{B_t}\,dt \,\middle|\, \mathcal{F}_t\right] = \frac{F(t,S_t,Z_t)}{B_t}.$$

Substituting for $F(t)$ in the above equation we get

$$\int_0^t \frac{c_t}{B_t}\,dt + W_t = F(0,S_0,\lambda) + \int_0^t \frac{\sigma_t^\top}{B_t}\Big(F_x(t) - Z_tF_z(t)(\sigma_t\sigma_t^\top)^{-1}(b_t - r(t,S_t)S_t\Big)\cdot d\bar{w}_t.$$

In the last equality we have used the definition of κ. The optimal policy follows from the unicity of the martingale representation of lemma 2.11. ∎

The assumptions which were made on the parameters of the diffusion and the inverse of the marginal utility function ensure that the function F has enough derivatives for Itô's formula to apply. In some instances, one can compute F directly and verify afterwards that it has the desired derivatives. Hence there is scope for weaker regularity conditions. For instance Cox and Huang derive an explicit formula within the family of HARA utility functions. As we show with a simple example in section 6, their computations are not affected by the fact that markets are incomplete. There is also a verification procedure which is the counterpart of the verification theorem of dynamic programming. It gives an effective way of directly constructing a solution from a partial differential equation. Unlike the partial differential equation of dynamic programming, it is linear and the non negative constraint on consumption can be easily taken care of. We refer to Cox and Huang [7] for more results about the characterization of optimal policies.

5. A SPECIAL CASE

This section gives the explicit solution of the consumption portfolio problem when the investor is endowed with a utility function within the class of constant relative risk aversion. The emphasis here is on the similarity of the solutions between complete and incomplete markets, rather than on generality.

Consumption occurs only at the final date. The consumption space is $L^2_+(P)$. The interest rate is constant and set equal to zero so that $B_t = 1$. The gains process follows a multiplicative geometric Brownian motion

$$G_t = S_t + \int_0^t \delta(s, S_s)\, ds = S_0 + \int_0^t I_S(s)\mathbf{b}\, ds + \int_0^t I_S(s)\boldsymbol{\sigma}\, dw_s,$$

for all t in $[0,1]$, $(a.s., P)$. Here $\mathbf{b}$ is a m vector of constants and $\boldsymbol{\sigma}$ an $m \times n$ matrix of constants of full rank ($m \leq n$). Also $I_S(t)$ is a diagonal matrix whose i-th entry is equal to $S_i(t)$, $i = 1, \ldots, m$. In this case strategies take a particularly simple form. We have $W_t = \alpha_t + \theta_t \cdot S_t = W_0 + \int_0^t \theta_t \cdot dG_t$ and the space of marketed commodities is

$$C = \left\{ W \in L^2_+(P) \colon W = W_0 + \int_0^1 \theta_t \cdot dG_t,\ W_0 \in R_+,\ \theta \in \mathcal{L}^2(G) \right\}.$$

The utility function is of constant relative risk aversion:

$$u(x) = \begin{cases} \frac{1}{1-1/\gamma} x^{1-1/\gamma} & \text{if } x > 0 \text{ and } \gamma > 0; \\ 0, & \text{if } x = 0 \text{ and } \gamma > 1; \\ \text{not defined}, & \text{if } x = 0 \text{ and } \gamma \in (0,1). \end{cases}$$

where we have used the convention that $x^0/0 = \log x$.

The problem faced by an investor endowed with one unit of the consumption good at time 0 is

$$\max_{W \in C} \mathrm{E}^P u(W)$$
$$\text{s.t.} \quad W(0) = 1$$

The corresponding static variational program is

$$\sup_{X \in L^2(P)} \left\{ \mathrm{E}^P u(X) \colon X \geq 0 \quad \text{and} \quad \tilde{\mathrm{E}} X = 1 \right\},$$

where $dQ/dP = \xi$ and ξ_t is the lognormal variable

$$\xi_t = \exp\left\{-\boldsymbol{\kappa}\cdot w_t - \frac{1}{2}|\boldsymbol{\kappa}|^2 t\right\},$$

with $\boldsymbol{\kappa} = \boldsymbol{\sigma}^\top(\boldsymbol{\sigma}\boldsymbol{\sigma}^\top)^{-1}(\mathbf{b} - r1)$ and $1 = (1,\ldots,1)^\top$. The extremum can be found either directly or through the first order conditions. One finds

$$W = (\xi^{-1})^\gamma \,/\, \|\xi^{-1}\|^\gamma_{\gamma,Q} \in L^2_+(P).$$

As expected, the solution is price measurable, so we know there must be a dynamic strategy which implements it. Substituting for ξ in W, we get

$$W = \exp\left\{\gamma\boldsymbol{\kappa}\cdot\bar{w}_1 - \frac{1}{2}\gamma^2|\boldsymbol{\kappa}|^2\right\},$$

with $\bar{w}_t = w_t + \boldsymbol{\kappa}t$. So

$$\begin{aligned} W_t = \tilde{\mathrm{E}}[W \mid \mathcal{F}_t] &= 1 + \gamma\int_0^t W_s\boldsymbol{\kappa}\cdot dw_s \\ &= 1 + \gamma\int_0^t W_s(\boldsymbol{\sigma}\boldsymbol{\sigma}^\top)^{-1}\mathbf{b}\cdot\boldsymbol{\sigma}\,d\bar{w}_s \\ &= 1 + \gamma\int_0^t W_s I_S^{-1}(s)(\boldsymbol{\sigma}\boldsymbol{\sigma}^\top)^{-1}\mathbf{b}\cdot dG_s. \end{aligned}$$

Hence we find

$$I_S(t)\theta_t = \gamma(\boldsymbol{\sigma}\boldsymbol{\sigma}^\top)^{-1}\mathbf{b}W_t$$

so that, as expected, the shares invested in the risky assets are constant and independant of the level of wealth.

6. Conclusion

Our model sets forth a class of dynamic trading strategies independent of the "reference measures" of Harrison and Pliska, when markets are incomplete. We then consider a broader class of strategies which obtains under a non negative wealth constraint. We show that these contingent claims also have a "fair" price, which is the initial cost of one of the corresponding strategies. Finally we solve the optimal consumption and portfolio choice problem. In our model, however, there are no "uninsurable risks" in the form of an uncertain endowment. Optimization with true uninsurable risks remains one of the simplest paradigms in economics which still remains an open question.

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Chapter 2
Arbitrage and Optimal Policies with an Infinite Horizon

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Abstract. We provide sufficient conditions to preclude all arbitrage opportunities in a market model with infinite horizon. In particular, wealth is constrained to be bounded from below. We then use those conditions to solve a dynamic consumption portfolio problem when asset prices follow a diffusion process. An optimal solution can be found when the subjective rate of time preference is sufficiently high.

1. Introduction

Recent advances in financial theory have provided fresh insights into the problem of optimal intertemporal consumption and portfolio policies when the economy has a finite horizon. In the context of a market model where assets prices follow a diffusion process, Cox and Huang [3] discuss how a finite horizon model can be extended to allow for an *infinite* horizon. They show that, if one is to allow holding a constant portfolio forever (the stationary solution), the dispersion coefficient of the risky securities should go to zero when time approaches infinity, and that this rules out the case where the price process is modelled as a multidimensional geometric Brownian motion which, they argue, is the most prevalent case in applications. In this chapter, we will focus on the existence and characterization of optimal policies when the horizon is infinite, while using a model which includes the geometric Brownian price process as a special case.

We first study the question of arbitrage, for the consumption portfolio problem is well posed only if investors cannot take advantage of arbitrage opportunities. In the finite horizon case, Harrison and Kreps [10], Kreps [15] and Huang [12] showed that if no arbitrage opportunities are to prevail within a reasonably large range of trading strategies, there must exist a reassignment of probabilities under which all assets have the same expected rates of return. Such a reassignment is usually referred to as an *equivalent martingale measure*. Clearly, a necessary condition for viability under infinite horizon is that the martingale property be satisfied over any *finite* time interval. We extend the results of Harrison and Kreps and of Huang by showing that there exists a probability measure on the whole information space under which all assets have the same expected rate of return

in restriction to finite time intervals. The only difference with the findings in the earlier literature is that this measure may no longer equivalent to the original probability beliefs. In the models of Samuelson [18] and Merton [17], where prices follow geometric Brownian motions, the two are actually found to be *mutually singular*: the martingale measure is concentrated on a set to which the agents in the economy assign zero probability. This peculiarity notwithstanding, the existence of a martingale measure allows us to handle the problems of arbitrage and of optimal portfolio and consumption choice with almost as much ease as in the finite horizon case.

The main problem with infinite horizon is that one would like to allow *a priori* investors to have non trivial trades "at infinity", i.e., to hold portfolios that do not necessarily vanish when time goes to infinity. Consequently, integral restrictions on trade seem rather unappealing. Unfortunately, with no bound on their trading horizon and a rich variety of trading strategies, investors may find *arbitrage opportunities*. In our context an arbitrage opportunity is a trading strategy which is financed by nothing at time zero and yields almost surely a non negative return when liquidated at some later random time, this return being even positive with a non zero probability. As is well known from the finite horizon case, the existence of a martingale measure is not sufficient in itself to rule out arbitrage opportunities. A celebrated example is given by the *doubling strategies*, by which investors keep doubling their stakes in the risky securities, just as a gambler would do at roulette.

Given the existence of a martingale measure, a solution to eliminate arbitrage, conjectured by Harrison and Kreps [10] and investigated by Dybvig and Huang [6], is to place a lower bound on the level of wealth. This is the approach taken in this chapter. We show that if agents are denied the right to run arbitrarily large negative balances, there can exist no free lunch at any finite, even unbounded, stopping time. Hence investors have to wait "forever" for an arbitrage opportunity. It is evident that with a negative bound on wealth there can also be no "Ponzi game", where the agents roll over their debts indefinitely to finance consumption.

We then turn to the problem of optimal consumption and portfolio choice. This problem is thoroughly investigated in the finite horizon case by Cox and Huang, who give general existence results and characterize the optimal policies in feedback form, i.e., in terms of

current wealth and prices, for a wide class of utility functions and a general market model; cf. chapter 1. Here we show that, when utility is differentiable and exhibits time separability, one can reduce the infinite horizon model to the finite one by rescaling time through an appropriate *change of time*, depending on the subjective rate of time preference. In our version, the change of time is purely deterministic because we set the discount rate constant.

It is thus not surprising that our model retains many of the features contained in that of Cox and Huang. In particular, an eminent role is played by the density of the martingale measure mentioned earlier. With complete markets, which we assume in this chapter, this density in restriction to $\mathcal{F}_t$ can be interpreted as usual as the shadow price of consumption per units of probability. For an optimum to exist, we find that this price should not asymptote zero "too fast" as time approaches infinity. Otherwise, it would be worthwhile to postpone consumption forever and concentrate it "at infinity". Altogether, an optimum exists whenever the subjective rate of time preference is sufficiently large.

There are three factors that determine the minimum discount rate. One is the interest rate on the riskless bond. The higher the interest rate, the higher the discount rate has to be. The second is the riskiness of the price system. The more uncertain the price system, the lower the discount rate that can support an optimum. Conversely with an almost deterministic price system the price of taking risks is negligible, and with a low discount rate an optimum may not exist. The last effect comes from the asymptotical behavior of the utility function. When the horizon is infinite, agents consider strategies that have unbounded payoffs; cf. Cox and Huang [5]. There could be no solution if the utility function were growing too fast at infinity. At the other extreme, when the utility function exhibits satiation, the lower bound for existence on the discount rate is zero.

This chapter is organized as follows. In section 2, we start from a family of equivalent martingale measures, as in Harrison and Kreps, and construct on the sample space a new probability whose restriction to finite horizons gives back the equivalent martingale measures. A non negative constraint suffices to rule out all arbitrage opportunities. In section 3 we consider the dynamic consumption and portfolio choice problem, from the existence viewpoint. Characterization is examined in section 4. We show the connection between the optimal policies and the solution to a linear partial differential equation. In

section 5, we give two examples, with the exponential and the isoelastic utility functions. In either case the non negativity constraint on consumption is easily taken care of. Section 6 concludes.

2. Arbitrage and non negative wealth

We consider a dynamic model of securities trading with an infinite horizon. Investors have rational expectations in the sense that they all agree on the probability law of the price processes. Investors are also "small", in that their actions cannot affect the market prices. So, in this model, prices are exogenously specified. Markets are *complete* in the sense of Harrison and Pliska [11]. In our context, this amounts to equating the number of risky securities with the dimension of the Brownian motion that describes the uncertain environment. Conditions under which the completeness of the markets can be removed are developed in the first chapter.

We specify a class of trading strategies for which our model is well defined. This requires that there be no arbitrage opportunities. In particular, we want to rule out arbitrage schemes such as the *doubling strategies*. To implement a doubling strategy, one has to short sell large amounts of the riskless bond and invest the proceeds in the risky securities, and continue the strategy until one eventually wins. This implies, of course, that one gets into debt with positive probability. So we say that a strategy is *admissible* from the point of view of an investor if his wealth may never become negative. If, starting with a positive initial endowment, an investor uses a strategy which gets him into negative wealth at some point in time, bankruptcy is declared. We show that this simple non negativity constraint effectively rules out all arbitrage opportunities.

Let $(\Omega, \mathcal{F}, \mathbf{F}, P)$ be a filtered space. The triple $(\Omega, \mathcal{F}, P)$ is a conventional probability space and the filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t\in[0,\infty)}$ represents the increasing flow of information over time. The information structure is modelled by means of an n-dimensional Brownian motion w, and $\mathcal{F}_t$ is taken to be the augmentation under P of $\sigma\{w_s : 0 \leq s \leq t\}$. Since all that is of interest in this economy is what happens from time zero to infinity, we assume that $\mathcal{F}$ is generated by the family of σ-fields $\mathcal{F}_t$, $t \geq 0$. Symetrically, since w starts from zero with probability 1, $\mathcal{F}_0$ is generated by Ω and the probability zero sets, and is said to be almost trivial.

We consider a market in which $n+1$ securities are traded continuously on the infinite time horizon $[0,\infty)$. The first n assets, called stocks, are the risky securities. Their prices $S\colon [0,\infty)\times\mathcal{F}\to R^n$ are modelled by a system of stochastic differential equations

$$dS_t+\delta(t,S_t)\,dt=b(t,S_t)\,dt+\sigma(t,S_t)\,dw_t, \tag{1}$$

$S(0)=S_0$, $t\in[0,\infty)$, where $S=(S^1,\dots,S^n)^\top$ are the ex-dividend prices, and $\delta=(\delta^1,\dots,\delta^n)^\top$ the dividend rates. The stochastic integral on the right hand side is defined in the sense of Itô. The dividend $\delta(t,x)\colon [0,\infty)\times R^n\to R^n_+$, the drift term $b(t,x)\colon [0,\infty)\times R^n\to R^n$ and the dispersion matrix $\sigma(t,x)\colon [0,\infty)\times R^n\to R^{n\times n}$ are all assumed to be continuous functions, and $\sigma(t,x)$ is assumed to have full rank for all values of x and t. The remaining security, called the bond, is locally riskless. Its price is governed by the ordinary differential equation

$$dB_t=r(t,S_t)B_t\,dt,\qquad B(0)=1,\quad t\in[0,\infty),$$

where the interest rate process $r(t,x)\colon [0,\infty)\to R^+$ is also continuous. We will make enough assumptions for the solution of (1) not to explode in any finite time and be pathwise unique. More precisely, we assume that b, r and σ satisfy a linear growth condition and a local Lipschitz condition; see chapter 1, footnote 3.

As far as arbitrage is concerned, one does not really need to be as specific as we have been about the nature of the markets model. One could instead identify stock prices with some set of semimartingales, and define trading strategies appropriately. We use this model for its conventional aspects and mainly for future reference in the later sections.

We refer to the *price system* as the n-vector of normalized prices defined by $S^*_t=S_t/B_t$. Itô's formula implies that

$$S^*_t+\int_0^t\frac{\delta_s}{B_s}\,ds=S_0+\int_0^t\frac{b(s,S_s)-r(s,S_s)S_s}{B_s}\,ds+\int_0^t\frac{\sigma(s,S_s)}{B_s}\,dw_s,$$

$t\in[0,\infty)$. The process on the left hand side is called the *gains process* and here it is expressed in units of the bond. Putting

$$G_t=S^*_t+\int_0^t\frac{\delta_s}{B_s}\,ds,$$

one sees that the difference $G_t - S_0$ represents the sum of accumulated capital gains and accumulated dividends on each of the assets in units of the bond; cf. chapter 1.

Harrison and Kreps and then Huang showed that a sufficient condition to preclude arbitrage when *simple strategies*[1] are used is the existence of an equivalent probability reassignment under which all assets have the same expected rates of return. This reassignment is referred to by Harrison and Kreps as an *equivalent martingale measure*. Formally an equivalent martingale measure on $[0,T]$ is a probability measure Q on $\mathcal{F}_T$ equivalent to P with a square integrable density and under which the gains process $(G_t, \mathcal{F}_t, Q)$ is a martingale on $[0,T]$. Simple trading strategies are easy to implement in the real world. Any reasonable model of securities trading should thus satisfy the martingale measure property of Harrison and Kreps over any finite horizon. To this end we start by introducing the following assumption.

ASSUMPTION 2.1. *Let* $\kappa(t,x) = \sigma(t,x)^{-1}(b(t,x) - r(t,x)x)$. *There exists a positive constant* $K < \infty$ *such that* $|\kappa(t,x)| \leq K$ *for all* (t,x) *in* $[0,\infty) \times R^n$.

This assumption is in particular satisfied in the models originally considered by Samuelson [18] and Merton [17]; cf. chapter 1. Under assumption 2.1, the existence of a martingale measure Q_T on any finite horizon $[0,T]$ is obtained either as a specialization of proposition 2.3 of chapter 1 or from theorem 3 of Harrison and Kreps [10]. For each fixed T, we have

$$\frac{dQ_T}{dP} = \exp\left\{ -\int_0^T \kappa(t,S_t)\cdot dw_t - \frac{1}{2}\int_0^T |\kappa(t,S_t)|^2\, dt \right\},$$

where $|\ |$ denotes the Euclidian norm in R^n. For any $t \leq T$, we fix a continuous version of the P-martingale

$$\begin{aligned}\xi_t &= \mathrm{E}[\, dQ_T/dP \mid \mathcal{F}_t\,] \\ &= \exp\left\{ -\int_0^t \kappa(s,S_s)\cdot dw_s - \frac{1}{2}\int_0^t |\kappa(s,S_s)|^2\, ds \right\}.\end{aligned}$$

It follows that one can define a family of probabilities Q_n on $(\Omega, \mathcal{F}_n)$, $n \geq 0$, by $dQ_n/dP = \xi_n$, and that this family is consistent in that Q_{n+1} equals Q_n on $\mathcal{F}_n$ for each $n \geq 0$. We have the following result.

[1] An investor employing a simple trading strategy can change his portfolio at a finite number of time points $\{t_j\}$ immediately after observing prices at t_j. For a formal definition of simple trading strategies, cf. chapter 1, section 2.3

PROPOSITION 2.2. *There exists a probability Q on $(\Omega, \mathcal{F})$ such that $(G_t, \mathcal{F}_t, Q)$ is a martingale on any finite time interval.*

PROOF: Define a consistent family Q_n on $(\Omega, \mathcal{F}_n)$ as above. Obviously $\lim_{n\to\infty} Q_n(\{n \leq t\}) = 0$ for all t, and so there exists a unique probability measure Q on $(\Omega, \mathcal{F})$ such that Q equals Q_n on $\mathcal{F}_n$ for all $n \geq 0$; cf. Stroock and Varadhan [**20**, theorem 1.3.5]. For all n, the process $(G_t, \mathcal{F}_t, Q)$ is a martingale on $[0, n]$, as desired. ∎

The gains process G is thus a (continuous) *local* martingale under Q. Similarly, since P and Q are equivalent on any finite time interval, they are *locally* equivalent. On the other hand, they may disagree on the σ-field $\mathcal{F}$, as shown in the following lemma.

LEMMA 2.3. *The measures Q and P are mutually singular if and only if $\int_0^\infty |\kappa_t|^2\, dt = \infty$, $(a.s., Q)$.*

PROOF: By the Radon Nikodym theorem, there exists a random variable ξ_∞ with values in $[0, \infty]$ such that for all $A \in \mathcal{F}$

$$Q(A) = \mathrm{E}[\xi_\infty I_A] + Q(A \cap \{\xi = \infty\}),$$

$$P(\xi_\infty = \infty) = 0.$$

For P and Q to be mutually singular, it is necessary and sufficient that $Q(\xi_\infty < \infty) = 0$. But theorem 8.19 of Jacod [1979] shows that in fact

$$\{\xi_\infty < \infty\} = \left\{ \int_0^\infty |\kappa(t, S_t)|^2\, dt < \infty \right\}, \quad (a.s., Q),$$

and so P and Q are mutually singular if and only if $\int_0^\infty |\kappa(t, S_t)|^2\, dt = \infty$, $(a.s., Q)$, as desired. ∎

In the models of Samuelson or Merton, the process κ is constant, so by the lemma above P and Q are mutually singular. In particular, the set $A = \{\xi_\infty = \infty\}$ is such that $P(A) = 0$ and $Q(A) = 1$. Thus Q is concentrated on a set of P-measure zero. The *almost surely* statements on $\mathcal{F}$ can no longer be applied indifferently with respect to either probability, as they are on the finite horizon case. However, one can still use *almost surely* with respect to both P and Q in restriction to $\mathcal{F}_t$, for all $t > 0$.

The following lemma will be useful later.

LEMMA 2.4. *Under Q, $\bar{w}_t = w_t + \int_0^t \kappa(s, S_s)\, ds$ is a standard Brownian motion and*

$$G_t = S_0 + \int_0^t \frac{\sigma(s, S_s)}{B_s}\, d\bar{w}_s, \quad a.s.,$$

is a local martingale.

Proof. The first assertion follows from Girsanov's theorem; see, e.g., Liptser and Shiryayev [**16**, chapter 6]. The second assertion follows from substitution of $\bar{w}$ in the definition of G, which is legitimate since P and Q are locally equivalent. ∎

A trading strategy is an $n+1$ vector process $(\alpha, \theta) = \{(\alpha, \theta^i): i = 1, \ldots, n\}$, where α_t and θ_t^i are the number of shares of the bond and of asset i, respectively, owned by the investor at time t. The investor's wealth in units of the bond at time t is

$$W_t = \alpha_t + \theta \cdot S_t^*.$$

A trading strategy is said to *finance* the consumption plan $c \in L_+^2(\nu)$ if the following intertemporal budget constraint holds at all times:

$$\int_0^t \frac{c_s}{B_s}\, ds + W_t = W_0 + \int_0^t \theta_t \cdot dG_t, \qquad \forall t > 0, \quad (a.s., P), \tag{2}$$

where the stochastic integral on the right hand side is well defined. To this end we introduce the class $\mathcal{L}(G)$ of progressively measurable processes $\theta: [0, \infty) \times \Omega \to R^n$ satisfying

$$\int_0^t \left| \frac{\sigma^\top(s, S_s)\theta_s}{B_s} \right|^2 ds < \infty, \qquad \forall t \geq 0, \quad (a.s., P),$$

where $|\ |$ is the Hilbert Schmidt norm defined by $|\sigma| = \left(\text{trace}(\sigma\sigma^\top)\right)^{1/2}$.

LEMMA 2.5. *Suppose $\theta \in \mathcal{L}(G)$. Then the integral $\int_0^t \theta_s \cdot dG_s$ is an Itô process under both P and Q and is the same whether it is computed relative to $(\{\mathcal{F}_t\}, P)$ or to $(\{\mathcal{F}_t\}, Q)$.*

PROOF: From the definition of G one has

$$\int_0^t \theta_s \cdot dG_s = \int_0^t \theta_s \cdot \frac{b(s, S_s) - r(s, S_s)S_s}{B_s}\, ds + \int_0^t \frac{\sigma^\top(s, S_s)\theta_s}{B_s}\, dw_s.$$

Since $\theta \in \mathcal{L}(G)$, the stochastic integral on the right hand side is well defined under P. On the other hand by assumption 2.1, there exists $K < \infty$ such that $|\kappa| \leq K$. By the definition of κ, we have thus

$$\left| \theta_t \cdot \frac{b_t - r_t S_t}{B_t} \right| = \left| \theta_t \cdot \frac{\sigma_t \kappa_t}{B_t} \right| = \left| \frac{\sigma_t^\top \theta_t}{B_t} \cdot \kappa_t \right|,$$

and so

$$\left|\theta_t \cdot \frac{b_t - r_t S_t}{B_t}\right| \leq K \left|\frac{\sigma_t^\top \theta_t}{B_t}\right|.$$

Thus the Lebesgue integral is finite, and $\int_0^t \theta_s \cdot dG_s$ is an Itô process under P. In addition, note that the density process $(\xi_t, \mathcal{F}_t, P)$ is a continuous, strictly positive martingale satisfying $\xi_0 = 1$. The other assertions of the lemma then follow from Stroock [**19**, lemma III.4.3]. ∎

Up to now nothing was said about the existence of arbitrage opportunities. In fact with the strategies we have defined it is well known that such arbitrage opportunities do exist, even in finite time; cf. Dybvig and Huang [**6**]. Were this the case, the consumption and portfolio choice problem would not be well posed. We now show that if the wealth $W_t = \alpha_t + \theta_t \cdot G_t$ is bounded from below, no arbitrage opportunities are possible in our infinite horizon setup. We recall the following usual definition.

DEFINITION 2.6. *An arbitrage opportunity is a strategy (α, θ) with $W_0 \leq 0$, financing zero consumption, and such that there exists a finite stopping time τ for which W_τ is non negative and positive with strictly positive probability.*

(One could alternatively define an arbitrage opportunity associated with a non zero consumption plan, but this would not be more general.) For a trading strategy to be an arbitrage opportunity, we must then have

$$W_\tau = W_0 + \int_0^\tau \theta_t \cdot dG_t \geq 0,$$
$$W_0 \leq 0,$$

with $P(W_\tau > 0)$ being strictly positive.

LEMMA 2.7. *Let (α, θ) finance some $c \in L_+^2(\nu)$. Suppose $W_t \geq 0$ for all $t > 0$. Then $(\int_0^t \theta_s \cdot dG_s, \mathcal{F}_t, Q)$ is a supermartingale.*

PROOF: The right hand side of (2) is a continuous local martingale under Q for all $\theta \in \mathcal{L}(G)$, and the left hand side is positive. The result follows as in lemma 3.2 of chapter 1. ∎

COROLLARY 2.8. *Let W be bounded from below. Then there is no arbitrage.*

PROOF: Let (α, θ) be a strategy that finances $c = 0$, τ a finite stopping time, and suppose that $W_\tau \geq 0$, the inequality being strict over a set of strictly positive P probability. We have to show that $W_0 > 0$. By lemma 2.7, the local martingale $X_t = \int_0^t \theta_s \cdot dG_s$ is in fact a supermartingale under Q, and it is closed on the right by 0. The optional sampling theorem implies that X_τ is integrable and $\mathrm{E}^Q X_\tau \leq X_0 = 0$. The budget constraint implies in turn that $\mathrm{E}^Q W_\tau = \mathrm{E}^Q[W_0 + X_\tau] \leq W_0$. Hence everything comes down to showing that $Q(W_\tau > 0)$ is strictly positive. But this follows from the fact that P and Q are equivalent in restriction to $\mathcal{F}_\tau$. Indeed, suppose $A \in \mathcal{F}_\tau$. This implies that for every n, $A \cap \{\tau \leq n\} \in \mathcal{F}_n$. As τ is finite, we can write

$$P(A) = \lim_{n \to \infty} \uparrow P(A \cap \{\tau \leq n\}),$$

and so $P(A) = 0$ if and only if $P(A \cap \{\tau \leq n\}) = 0$ for all n, and similarly for Q. Hence we see that $P(A) = 0$ if and only if $Q(A) = 0$. The conclusion follows with $A = \{W_\tau > 0\}$ upon noting that $P(A) > 0$. ∎

The result of proposition 2.8 stands for arbitrary finite stopping times. Interestingly, when the non negative constraint on wealth is in force, the absence of arbitrage for *bounded* stopping times automatically extends to *finite* stopping times. I owe the following proof to Chi-fu Huang.

PROPOSITION 2.9. *Suppose that there is no free lunch for bounded stopping times and that there is a non negative wealth constraint. Then there is no free lunch for any finite stopping time.*

PROOF: Suppose not, and define

$$Y(t) = W_0 + \int_0^t \theta(s) \cdot dG(s), \quad \theta \in \mathcal{L}(G).$$

Then there exists $\tau < \infty$ such that $Y(0) = 0$, $Y(\tau)$ is almost surely positive and strictly positive with strictly positive probability. (All statements are made relative to P.) It is easily seen that there exists N such that $P(Y(\tau \geq N^{-1}) > 0$, or else $P(Y(\tau) \geq n^{-1}) = 0$

for every n implies

$$P(Y(\tau) > 0) \leq P\left(\bigcup_{n=1}^{\infty} \{Y(\tau) \geq n^{-1}\}\right)$$
$$= \sum_{n=1}^{\infty} P\left(Y(\tau) \geq n^{-1}\right) = 0.$$

Similarly, $\{\tau < \infty\} = \cup_{m=1}^{\infty} \{\tau \leq m\}$. We claim that there exists m such that

$$P\left(\{Y(\tau) \geq N^{-1}\} \cap \{\tau \leq m\}\right) > 0.$$

Suppose not. That is,

$$\bigcup_{m=1}^{\infty} \{Y(\tau) \geq N^{-1}\} \cap \{\tau \leq m\}$$

is of measure zero. This implies that

$$\{Y(\tau \geq N^{-1}\} \cap \bigcup_{m=1}^{\infty} \{\tau \leq m\} = \{Y(\tau \leq N^{-1})\} \cap \Omega$$
$$= \{Y(\tau \leq N^{-1})\}$$

is of measure zero, a contradiction. Put $A = \{\omega : Y(\tau(\omega) \geq N^{-1}\} \cap \{\tau(\omega) \leq m\}$ and

$$\widehat{\tau}(\omega) = \begin{cases} \tau(\omega), & \text{if } \omega \in A \\ m, & \text{if } \omega \notin A. \end{cases}$$

We want to show that $\widehat{\tau}$ is a stopping time. First we claim that $A \in \mathcal{F}_{\tau}$. Note that

$$A \cap \{\tau \leq t\} = \{Y(\tau \geq N^{-1}\} \cap \{\tau \leq m\} \cap \{\tau \leq t\} \in \mathcal{F}_{t \wedge m} \subset \mathcal{F}_t.$$

Thus by the definition of $\mathcal{F}_{\tau}$, we have $A \in \mathcal{F}_{\tau}$. Then

$$\tau^* = \begin{cases} \tau, & \text{on } A \\ +\infty, & \text{on } A^c \end{cases}$$

is a stopping time. It is then easily seen that $\widehat{\tau} = \tau^* \wedge m$ and $\widehat{\tau}$ is a stopping time. We have

$$Y(\widehat{\tau}) \quad \begin{cases} = Y(\tau) \geq N^{-1}, & \text{on } A \\ \geq 0, & \text{on } A^c, \end{cases}$$

where the inequality follows from the non negative wealth constraint. Thus $\widehat{\tau}$ is a free lunch, a contradiction. ∎

3. Optimal policies: existence

In this economy, an individual's object of choice is a consumption rate process denoted by c, where c_t is the random flow of consumption at time t. Consumption is constrained to be always positive, but does not have to "vanish at infinity". In particular, it could stay constant for ever. For this reason, we introduce on the space $[0,\infty) \times \Omega$ the measure ν defined by $d\nu/(\lambda \times P) = e^{-\beta t}$, where β is a discount factor and λ the Lebesgue measure on the positive real line. (The discount factor could as well be a deterministic function of time.) Then $([0,\infty) \times \Omega, PM, \nu)$ becomes a finite measure space, where PM denotes the progressively measurable σ-field associated with the filtration $\mathbf{F}$; cf. chapter 1, section 2.1. The consumption space is taken to be the positive orthant of the space $L^2([0,\infty)\times\Omega, PM, \nu)$, simply written as $L^2_+(\nu)$.

In this section we consider the optimal consumption and portfolio choice problem when the horizon is infinite. The agent's preferences can be represented by a time separable utility function on consumption rate processes $u(x,t) = u(x)e^{-\beta t}$. We assume that $u : R_+ \to R$ is concave, increasing and differentiable, and possibly unbounded from below at the origin. The strategies employed by the agent are *admissible* in the sense of section 2. Given that the investor is endowed with an initial wealth $W_0 > 0$, the budget constraint is of the form

$$\int_0^t \frac{c_s}{B_s}\, ds + W_t = W_0 + \int_0^t \theta_s \cdot dG_s, \qquad \forall t > 0,$$

where we have put $W_t = \alpha_t + \theta_t \cdot S^*_t$ with $\theta \in \mathcal{L}(G)$, $c \in L^2_+(\nu)$ and $W_t \geq 0$ for all t. We say that the strategy (α,θ) finances $c \in L^2_+(\nu)$ with initial wealth W_0. We repeat that the density of ν with respect to the product measure $\lambda \times P$ is equal to $e^{-\beta t}$. Hence, the agent's discount factor is used to define the consumption space.

The agent wants to solve the following problem:

$$(\mathcal{P}_1) \qquad \sup_{c \in L^2_+(\nu)} \mathrm{E} \int_0^\infty u(c_t)e^{-\beta t}\, dt$$
$$\text{s.t.} \quad \begin{cases} c \text{ is financed by } (\alpha,\theta), \\ W(0) = k, \text{ and } W_t \geq 0. \end{cases}$$

We provide sufficient conditions for the existence of a solution to $(\mathcal{P}_1)$ and later on characterize the optimal solution when it exists. Our technique follows Cox and Huang [3], [4]. We

first transform the budget constraint so as to map the original dynamic problem into a static variational problem whose solution is known. The solution of the second problem is then implemented with a dynamic trading strategy uncovered by a martingale representation theorem.

As in chapter one, the notation $\widetilde{\mathrm{E}}$ is used for expectation under Q.

LEMMA 3.1. *Suppose $c \in L^2_+(\nu)$ is financed by an admissible strategy (α, θ). Then*

$$\widetilde{\mathrm{E}}\left[\int_0^\infty \frac{c_t}{B_t}\,dt\right] \le k,$$

where $k = \alpha_0 + \theta_0 \cdot S_0$.

PROOF: Since the strategy is admissible, the integral $\int_0^t \theta_s \cdot dG_s$ is a supermartingale under Q; cf. lemma 2.7. So

$$\widetilde{\mathrm{E}}\left[\int_0^t \frac{c_s}{B_s}\,ds + W_t\right] = k + \widetilde{\mathrm{E}}\left[\int_0^t \theta_s \cdot dG_s\right] \le k$$

and so,

$$\widetilde{\mathrm{E}}\left[\int_0^t \frac{c_s}{B_s}\,ds\right] \le k.$$

Let then $t \to \infty$. The result follows from the monotone convergence theorem. ∎

We now consider the converse problem. Given a consumption plan $c \in L^2_+(\nu)$ that satisfies

$$\widetilde{\mathrm{E}}\left[\int_0^\infty \frac{c_t}{B_t}\,dt\right] \le k,$$

can we find an admissible trading strategy which manufactures it? In some instances we can.

PROPOSITION 3.2. *Suppose that the consumption plan $c \in L^2_+(\nu)$ is such that*

$$\int_0^\infty \frac{c_t}{B_t}\,dt \in L^1(Q).$$

Then there is an admissible strategy that manufactures it.

PROOF: Let X be an almost surely continuous version of the martingale

$$X_t = \widetilde{\mathrm{E}}\left[\int_0^\infty \frac{c_s}{B_s}\,ds \,\middle|\, \mathcal{F}_t\right].$$

The Brownian motion $\bar{w}$ defined in lemma 2.4 and the original Brownian motion w generate the same complete filtration; see chapter 1, lemma 2.7. Hence by the fundamental representation result for Brownian martingales (cf. Clark [1, theorem 3] and [2, p. 1778]), there exists a progressively measurable R^n-valued process ρ with

$$\int_0^\infty |\rho_t|^2 \, dt < \infty, \quad (a.s., Q)$$

such that

$$X_t = \tilde{\mathrm{E}}\, X + \int_0^t \rho_s \cdot d\bar{w}_s, \qquad t \geq 0, \quad (a.s., Q).$$

Let $\theta_t = B_t \rho_t \big(\sigma^\top(t, S_t)\big)^{-1}$. Since $\sigma^\top \theta / B = \rho$, then $\theta \in \mathcal{L}(G)$ and

$$(\#) \qquad X_t = \tilde{\mathrm{E}}\, X + \int_0^t \theta_s \cdot dG_s.$$

Let

$$W_t = \tilde{\mathrm{E}}\left[\int_t^\infty \frac{c_s}{B_s} \, ds \;\middle|\; \mathcal{F}_t \right]$$

and

$$\alpha_t = W_t - \theta_t \cdot S_t^*.$$

Then from (#) we get

$$\begin{aligned} W_t + \int_0^t \frac{c_s}{B_s} \, ds = \tilde{\mathrm{E}}\left[\int_0^\infty \frac{c_s}{B_s} \, ds \;\middle|\; \mathcal{F}_t \right] &= X_t \\ &= \tilde{\mathrm{E}}\, X + \int_0^t \theta_s \cdot dG_s \end{aligned}$$

which implies that the strategy (α, θ) finances c with $k = \tilde{\mathrm{E}}\, X$. This strategy is admissible since $W_t \geq 0$ for all t. ∎

Proposition 2 suggests that one may replace the original problem by the following static variational problem

$$\begin{aligned} &\max_{c \in L^2_+(\nu)} \mathrm{E} \int_0^\infty u(c_t) e^{-\beta t} dt \\ &\text{s.t.} \quad \tilde{\mathrm{E}}\left[\int_0^\infty \frac{c_t}{B_t} \, dt \right] = k. \end{aligned}$$

By lemma 3.1, any solution of $(\mathcal{P}_1)$ satisfies the constraint of the static variational problem. Thus, if a solution of the static problem verifies the condition of proposition 3.2, it must also solve $(\mathcal{P}_1)$. In fact, we will transform the constraint of the static problem slightly.

LEMMA 3.3. *Suppose $c \in L^2_+(\nu)$ is financed by an admissible strategy (α, θ). Then,*

$$\tilde{\mathrm{E}}\left[\int_0^\infty \frac{c_t}{B_t}\,dt\right] = \mathrm{E}\left[\int_0^\infty \frac{\xi_t c_t}{B_t}\,dt\right].$$

PROOF: For any finite t

$$\tilde{\mathrm{E}}\left[\int_0^t \frac{c_s}{B_s}\,ds\right] = \mathrm{E}\left[\xi_t \int_0^t \frac{c_s}{B_s}\,ds\right]$$

since ξ_t is the density of Q with respect to P in restriction to $\mathcal{F}_t$. But $\xi_{t\wedge\cdot}$ is a positive, uniformly integrable martingale. Hence

$$\mathrm{E}\left[\xi_t \int_0^t \frac{c_s}{B_s}\,ds\right] = \mathrm{E}\left[\int_0^t \frac{\xi_s c_s}{B_s}\,ds\right];$$

cf. Elliott [7, corollary 7.16]. The result follows upon letting $t \to \infty$ from the monotone convergence theorem. ∎

Therefore the static variational problem can be written as

$$(\mathcal{P}_2) \qquad \begin{aligned} &\max_{c \in L^2_+(\nu)} \mathrm{E}\int_0^\infty u(c_t)e^{-\beta t}\,dt \\ &\text{s.t. } \mathrm{E}\left[\int_0^\infty \frac{\xi_t c_t}{B_t}\,dt\right] = k. \end{aligned}$$

From Cox and Huang [3, theorem 3.6] we have immediately the following.

PROPOSITION 3.4. *Suppose $u : R_+ \to R$ is non trivial, increasing, concave, and strictly concave on any subset of its domain where u'_+ is strictly positive. In the case that u is strictly increasing, suppose further that there exists $b > 0$ and $A > 0$ such that $u'_+(x) \le Ax^{-b}$. Then there exists a solution to $(\mathcal{P}_2)$ if either*

(i) *u is not strictly increasing and $\tilde{\mathrm{E}}\left[\int_0^\infty B_t^{-1}\,dt\right] < \infty$, or*

(ii) *u is strictly increasing and $\zeta^{-1} \in L^{2/b}(\nu)$, where $\zeta_t = \xi_t e^{\beta t}/B_t$.*

PROOF: Let τ be the (deterministic) change of time defined by

$$\tau(t) = \inf\left\{ s > 0 : \int_0^s e^{-\beta u}\,du > t \right\}.$$

Then putting $\tilde{c}_t = c(\tau_t)$, we have

$$\int_0^\infty u(c_t)e^{-\beta t}\,dt = \int_0^{\beta^{-1}} u(\tilde{c}_t)\,dt;$$

cf. Elliott [**7**, theorem 7.7]. Likewise,

$$\begin{aligned} \mathrm{E}\left[\int_0^\infty \frac{c_t \xi_t}{B_t}\, dt\right] &= \mathrm{E}\left[\int_0^\infty c_t \zeta_t e^{-\beta t}\, dt\right] \\ &= \mathrm{E}\left[\int_0^{\beta^{-1}} \widetilde{c}_t \widetilde{\zeta}_t\, dt\right], \end{aligned}$$

where $\widetilde{\zeta}_t = \zeta(\tau_t)$. Hence one can rewrite $(\mathcal{P}_2)$ as

$$(\mathcal{P}_2') \qquad \begin{aligned} &\max_{\widetilde{c} \in L^2_+(\lambda \times P)} \mathrm{E} \int_0^{\beta^{-1}} u(\widetilde{c}_t)\, dt \\ &\text{s.t.} \quad \mathrm{E}\left[\int_0^{\beta^{-1}} \widetilde{c}_t \widetilde{\zeta}_t\, dt\right] = k, \end{aligned}$$

where λ is the Lebesgue measure. We take cases.

(i) u is strictly increasing. From Cox and Huang [**3**, theorem 3.6], suppose that $\widetilde{\zeta}^{-1} \in L^{2/b}(\lambda \times P)$, or equivalently, $\zeta^{-1} \in L^{2/b}(\nu)$. Then $(\mathcal{P}_2')$, and thus $(\mathcal{P}_2)$, admit a solution.

(ii) u is not strictly increasing. Define $g(y) = \inf\{z \geq 0 : u'_+(z) \leq y\}$, and let $\widetilde{c}_\mu(t) = g(\mu \widetilde{\zeta}_t)$, for some strictly positive μ. Then $\widetilde{c}_\mu(t) \leq g(0) < \infty$ since u is not strictly increasing and

$$\begin{aligned} \mathrm{E}\left[\int_0^{\beta^{-1}} \widetilde{c}_\mu(t) \widetilde{\zeta}_t\, dt\right] &\leq g(0)\, \mathrm{E}\left[\int_0^\infty \zeta_t e^{-\beta t}\, dt\right] \\ &= g(0)\, \widetilde{\mathrm{E}}\left[\int_0^\infty \frac{dt}{B_t}\right] < \infty. \end{aligned}$$

The conclusion follows as in theorem 3.6 of Cox and Huang. ∎

Remark 3.5. Program $(\mathcal{P}_2')$ is not exactly identical to that considered by Cox and Huang. The difference is that, in their setup, the process $\widetilde{\zeta}_t$ is square integrable with respect to the measure $\lambda \times P$ (or, more generally, in the dual space of the commodity space).

Remark 3.6. It follows from theorem 3.6 of Cox and Huang that if c is a solution of $(\mathcal{P}_2)$, there exists a strictly positive real number μ such that

$$u'_+(c_t) \begin{cases} = \mu \zeta_t & \text{for } c_t > 0 \ (a.s., \nu), \\ \leq \mu \zeta_t & \text{for } c_t = 0 \ (a.s., \nu). \end{cases}$$

Thus we have

$$c_t = g(\mu\zeta_t), \quad (a.s., \nu),$$

where g is the inverse of marginal utility, as defined in the proof of proposition 3.4.

In the following corollary we provide a sufficient condition for condition (ii) of proposition 3.4 to hold. It amounts to assuming that the interest rate is bounded and that the discount rate is sufficiently large.

COROLLARY 3.7. *Suppose u is strictly increasing and satisfies the assumptions of proposition 3.4. If*

(i) *$r(t,x)$ is bounded by $\bar{r} < \infty$ for all (t,x),*

(ii) *$\beta > (1+b/2)^{-1}\bar{r} + K^2/b$, where K is the constant of assumption 2.1,*

then there exists a solution to $(\mathcal{P}_2)$.

PROOF: In view of proposition 3.4, we have to check that $\zeta^{-1} \in L^{2/b}(\nu)$. Let $p = 2/b$. Recall that $\zeta_t = \xi_t e^{\beta t}/B_t$ and put $\phi(t) = \mathrm{E}(\xi_t/B_t)^{-p}$. We have by Fubini's theorem

$$(\#) \qquad \mathrm{E}\int_0^\infty \zeta_t^{-p} e^{-\beta t}\, dt = \int_0^\infty \phi(t) e^{-\beta(1+p)t}\, dt,$$

provided the integral on the right hand side is finite. Note that by Itô's formula

$$\left(\frac{\xi_t}{B_t}\right)^{-p} = 1 + p\int_0^t \left(\frac{\xi_s}{B_s}\right)^{-p} \kappa_s \cdot dw_s + p\int_0^t \left(\frac{\xi_s}{B_s}\right)^{-p} (r_s + \frac{1}{2}(1+p)|\kappa_s|^2)\, ds.$$

The second term on the right hand side is in fact a martingale, because r, κ and $\mathrm{E}[\xi^{-2p}]$ are bounded; cf. lemma 2.4 in chapter 1. Putting $H = p\bar{r} + p(1+p)K^2/2$ and taking expectations under P, we get

$$\phi(t) \le 1 + H\int_0^t \phi(s)\, ds.$$

By Gronwall's inequality, (see for instance Elliott[7, lemma 14.20]), $\phi(t) \le e^{Ht}$, and since $H - \beta(1+p) = -(1+p)(\beta - (1+b/2)^{-1}\bar{r} - K^2/b) < 0$, the integral on the right hand side of $(\#)$ is finite, as desired. ∎

The process ζ can be interpreted as the shadow price of consumption per unit of probability ν. As in the finite horizon case, the integrability condition of proposition 3.4

indicates that the shadow price of consumption across time and states cannot go to zero "too fast". Otherwise the investor would postpone consumption and consume an arbitrarily large amount at infinity, and there would be no optimum. Hence, the discount rate has to be sufficiently large. Note that a reduction in uncertainty (κ large), or an increase in the riskless rate, tend to tighten the sufficient condition of corollary 3.7.

We now turn to the implementation of $(\mathcal{P}_2)$ through dynamic trading. The following proposition summarizes our earlier results.

PROPOSITION 3.8. *Suppose u is as in proposition 3.4, where in place of condition* (ii) *we assume instead* (ii'): *u is strictly increasing and $\beta > (1+b/2)^{-1}\bar{r} + K^2/b$. Then there exists a solution to the dynamic consumption and portfolio problem $(\mathcal{P}_1)$.*

PROOF: In view of proposition 3.2, 3.4 and corollary 3.7, it suffices to verify that

$$(\#) \qquad \int_0^\infty \frac{c_t}{B_t}\, dt \in L^1(Q),$$

where c is a solution of $(\mathcal{P}_2)$. Note that on $\{c_t > 0\}$, we have (by remark 3.6) $u'_+(c_t) = \mu\zeta_t$ for some strictly positive μ. We take cases.

(i) u is not strictly increasing. Then $u'_+(c_t) > 0$ implies $c_t \leq \bar{c}$, with $\bar{c} = \inf\{x \in R_+ : u'_+(x) = 0\} < \infty$. Since c is bounded, condition $(\#)$ follows from (i) of proposition 3.4.

(ii) u is strictly increasing. Then with the notation of proposition 3.4, $u'_+(c_t) = \mu\zeta_t \leq Ac_t^{-b}$, which implies

$$\begin{aligned}\frac{c_t}{B_t} &\leq \left(\frac{A}{\mu}\right)^{1/b} \frac{\zeta_t^{-1/b}}{B_t} \\ &= \left(\frac{A}{\mu}\right)^{1+p} \left(B_t^p/\xi_t^{1+p}\right) e^{-\beta(1+p)t},\end{aligned}$$

with $p = -1 + 1/b$. Let $\phi(t) = \mathrm{E}(B_t^p/\xi_t^{1+p})$. A computation similar to that of corollary 3.6 shows that $\phi(t) \leq e^{Ht}$, where we have put $H = p\bar{r} + p(1+p)K^2/2$. But

$$\begin{aligned}\beta(1+p) - H &= (1+p)(\beta - \frac{p}{1+p}\bar{r} - \frac{1}{2}pK^2) \\ &= (1+p)\left(\beta - \frac{2\bar{r}}{2+b} - \frac{K^2}{2}\right) + \frac{1+p}{2}\left(\frac{2}{b} - p\right)K^2 \\ &\geq \frac{1}{2}(1+p)\left(\frac{2}{b} - p\right)K^2 > 0\end{aligned}$$

since $-p+2/b = 1+1/b > 0$. Hence the integral defined in (#) has finite expectation under Q, and this ends the proof. ∎

The analysis above has shown that the most important decision from the part of the investor comes at time zero with the determination of the Lagrange multiplier μ. Once the initial value $\zeta(0) = \mu$ has been chosen, the the optimal policy is entirely driven by S and ζ. The evolution of ζ itself depends only on the market. More precise characterization results are provided in the following section.

4. Optimal policies: characterization

The characterization of optimal policies is virtually contained in the analysis of section 3. The optimal consumption plan is given by

$$c_t = g(\mu\zeta_t),$$

where g is the inverse of the marginal utility function; cf. remark 3.6. Moreover we know from proposition 3.2 that

$$W_t = \widetilde{\mathrm{E}}\left[\int_t^\infty \frac{c_s}{B_s}\,ds \;\middle|\; \mathcal{F}_t\right].$$

We will show that one can in general define a function F such that

$$W_t = \frac{F(t, S_t, \zeta_t)}{B_t},$$

and that when this function has appropriate derivatives, one can express the optimal policy explicitly in terms of t, S and ζ with F and its first derivative. Specializing to a market model displaying time homogeneity, we will then draw the connection between F and the solution to a degenerate elliptic (i.e., parabolic) partial differential equation. Finally, we show that the solution can be represented in explicit *feedback* form, i.e., in terms of the current value of wealth and assets prices.

Consider the vector diffusion $Z = (S, \zeta)$, defined by the system of stochastic differential equations compactly written as

$$dZ_t = \widehat{b}(t, Z_t)\,dt + \widehat{\sigma}(t, Z_t)\,d\bar{w}_t,$$

where

$$\widehat{b}(t,x,z) = \begin{bmatrix} r(t,x)x - \delta(t,x) \\ z(\beta - r(t,x) + |\kappa(t,x)|^2) \end{bmatrix},$$
$$\widehat{\sigma}(t,x,z) = \begin{bmatrix} \sigma(t,x) \\ -z\kappa(t,x) \end{bmatrix}$$

for all $(t,x,z) \in R \times R^n \times R_+^*$, where $R_+^* = R_+ \setminus \{0\}$. To guarantee that $\widetilde{b}$ and $\widetilde{\sigma}$ satisfy a linear growth condition and a local Lipschitz condition, as in section 2, we have to assume that the process κ is locally Lipschitz continuous. A sufficient condition is given by the following assumption.

ASSUMPTION 4.1. *There exists $\epsilon > 0$ such that $\sigma(t,x)\sigma^\top(t,x) \geq \epsilon I_n$ for all $(t,x) \in R_+ \times R^n$.*

This implies in particular that for all $y \in R^n$,

$$|\sigma^{-1}y| = |\sigma^\top(\sigma\sigma^\top)^{-1}y| \leq \epsilon|\sigma^\top y|.$$

This, together with the assumptions of section 2 (including the boundedness of κ), implies that κ is Lipschitz continuous. Hence the diffusion (S,ζ) has a unique strong solution on $[0,\infty)$ for all initial values $(x,z) \in R^n \times R_+^*$. Now define

$$F(t,x,z) = \mathrm{E}^{Q_{t,x,z}}\left[\int_t^\infty \exp\left\{-\int_t^s r(u,S_u)\,du\right\} g(\zeta_s)\,ds\right],$$

where the notation $Q_{t,x,z}$ indicates initial data $S_t = x$ and $\zeta_t = z$. By the markov property of solutions to stochastic differential equations and the fact that g, being positive, can be approximated by an increasing sequence of bounded functions, we have

$$\begin{aligned} F(t,S_t,\zeta_t) &= \widetilde{\mathrm{E}}\left[\int_t^\infty \exp\left\{-\int_t^s r(u,S_u)\,du\right\} g(\zeta_s)\,ds \;\middle|\; \mathcal{F}_t\right] \\ &= B_t\,\widetilde{\mathrm{E}}\left[\int_t^\infty \frac{g(\zeta_s)}{B_s}\,ds \;\middle|\; \mathcal{F}_t\right], \end{aligned}$$

and so

$$W_t = \frac{F(t,S_t,\zeta_t)}{B_t}$$

is the optimal wealth associated with the consumption plan $c_t = g(\zeta_t)$. Note that, under the assumptions of section 3, the integral defining F is finite.

Sometimes F can be computed directly. In section 5 we will give a general formula and two examples where this is so, in which the market model has constant coefficients. Sometimes, one can also uncover F by solving a partial differential equation. For simplicity, the following proposition takes the coefficients of the market model, including the interest rate, as independent of time. It is a counterpart of the verification theorem in dynamic programming.

PROPOSITION 4.2. *Suppose that under the conditions of section 2, the market model is time homogeneous, and satisfies assumption 4.1. Suppose also that the consumption portfolio problem can be characterized as in corollary 3.7. Assume that*

(i) *F is in $C^2(R^n \times R_+^*)$ and satisfies the polynomial growth condition*

$$F(x,z) \leq C(1+z^{-\gamma})$$

for all $(x,z) \in R^n \times R_+^*$ and for some strictly positive constants C and $\gamma < 1 + 2/b$;

(ii) *$\mathcal{L}F - rF + g = 0$ for all $(x,z) \in R^n \times R_+^*$, where $\mathcal{L}$ is the differential generator of (S,ζ), and g is the inverse of the marginal utility (cf. remark 3.6);*

(iii) *there exists $\mu > 0$ such that $F(S_0,\mu) = k$.*

Then there exists a solution to the optimal consumption and portfolio choice problem. The optimal wealth can be identified as

$$W_t = \frac{F(S_t,\zeta_\mu(t))}{B_t} \tag{3}$$

where $\zeta_\mu(0) = \mu$ and the optimal portfolio policy is

$$\theta_t = F_x(S_t,\zeta_\mu(t)) - \zeta_\mu(t)F_z(S_t,\zeta_\mu(t))\big(\sigma(S_t)\sigma^\top(S_t)\big)^{-1}\big(b(S_t) - r(S_t)S_t\big) \tag{4}$$

and

$$c_t = W_t - \theta_t \cdot S_t^*. \tag{5}$$

PROOF: First fix some $T > 0$. We will show that

$$F(x,z) = \mathrm{E}^{Q_{x,z}}\left[\int_0^T \frac{g(\zeta_t)}{B_t}\,dt + \frac{F(S_T,\zeta_T)}{B_T}\right]. \tag{\#}$$

For this we consider the stopping time $\tau_n = \inf\{t \le T : |Z_t| \ge n \text{or } \zeta_t \le n^{-1}\}$. Since Z is a continuous process, $\tau_n \uparrow T$, $(a.s., Q)$. By Itô's formula,

$$\frac{F(S_{\tau_n}, \zeta_{\tau_n})}{B_{\tau_n}} = F(x,z) + \int_0^{\tau_n} \frac{\mathcal{L}F(S_t, \zeta_t)}{B_t}\, dt - \int_0^{\tau_n} r(S_t)\frac{F(S_t,\zeta_t)}{B_t}\, dt$$
$$+ \int_0^{\tau_n} \frac{F_x(S_t,\zeta_t)}{B_t}\sigma(S_t)\, d\bar{w}_t - \int_0^{\tau_n} \zeta_t \frac{F_x(S_t,\zeta_t)}{B_t}\kappa(S_t)\cdot d\bar{w}_t.$$

Using assumption (ii) of the proposition and taking expectations under $Q_{x,z}$, we get

$$(\#\#) \qquad F(x,z) = \mathrm{E}^{Q_{x,z}}\left[\int_0^{\tau_n} \frac{g(\zeta_t)}{B_t}\, dt + \frac{F(S_{\tau_n},\zeta_{\tau_n})}{B_{\tau_n}}\right].$$

Now send n to ∞. Note that

$$\mathrm{E}^{Q_{x,z}}\left[\int_0^{T} \frac{g(\zeta_t)}{B_t}\, dt\right] \le \mathrm{E}^{Q_{x,z}}\left[\int_0^{\infty} \frac{g(\zeta_t)}{B_t}\, dt\right]$$

which is finite by the proof of proposition 3.8. Since the diffusion (S, ζ^{-1}) also satisfies linear growth and local Lipschitz conditions, theorem V.5.2 of Fleming and Rishel [8], together with the polynomial growth condition (i), implies that the right hand side of $(\#\#)$ tends to

$$\mathrm{E}^{Q_{x,z}}\left[\int_0^{T} \frac{g(\zeta_t)}{B_t}\, dt + \frac{F(S_T,\zeta_T)}{B_T}\right],$$

and hence $(\#)$ holds. Now let $T \to \infty$. Let us show that

$$\lim_{T\to\infty} \mathrm{E}^{Q_{x,z}} \frac{F(S_T,\zeta_T)}{B_T} = 0.$$

Since $F(S_T,\zeta_T) \le C(1+\zeta^{-\gamma})$, we have to show that

$$(\#\#\#) \qquad \lim_{T\to\infty} \mathrm{E}^{Q_{x,z}} \frac{\zeta_T^{-\gamma}}{B_T} = 0.$$

Using Itô's formula,

$$\frac{\zeta_T^{-\gamma}}{B_T} = z^{-\gamma}\exp\left\{\gamma\int_0^T \kappa_t\cdot d\bar{w}_t - \frac{\gamma^2}{2}\int_0^T |\kappa_t|^2\, dt\right\}$$
$$\exp\left\{-\gamma\int_0^T \left(\beta - \frac{\gamma-1}{\gamma}r_t - \frac{\gamma-1}{2}|\kappa_t|^2\right) dt\right\}.$$

Note that, by (ii) of corollary 3.7, $\beta > \bar{r}(\gamma-1)/\gamma + (\gamma-1)K^2/2$ whenever $\gamma \le 1 + 2/b$. Taking expectations under Q, one finds

$$\mathrm{E}^{Q_{x,z}} \frac{\zeta_T^{-\gamma}}{B_T} \le z^{\gamma} e^{-\epsilon T}$$

for some $\epsilon > 0$ and so (###) holds. Whence,

$$F(x,z) = \mathrm{E}^{Q_{x,z}}\left[\int_0^\infty \frac{g(\zeta_t)}{B_t}\,dt\right].$$

Choose $x = S_0$ and $z = \mu$. Taking assumption (iii) into account we get

$$k = \tilde{\mathrm{E}}\left[\int_0^\infty \frac{g(\zeta_\mu(t))}{B_t}\,dt\right],$$

with $\zeta_\mu(t) = \mu\xi_t e^{\beta t}/B_t$. This implies that the consumption plan $c_t = g(\zeta_\mu(t))$ has value k at time 0. By corollary 3.7, it is an optimum for $(\mathcal{P}_2)$. Moreover we have that

$$W_t = \tilde{\mathrm{E}}\left[\int_t^\infty \frac{g(\zeta_\mu(s))}{B_s}\,ds \,\middle|\, \mathcal{F}_t\right] = \frac{F(S_t,\zeta_\mu(t))}{B_t}.$$

Using Itô's formula again, we get

$$\frac{F(S_t,\zeta_\mu(t))}{B_t} + \int_0^t \frac{c_s}{B_s}\,ds = k + \int_0^t \left[F_x(t) - \zeta_\mu(t)F_z(t)(\sigma_t\sigma_t^{\mathsf{T}})^{-1}(b_t - r_tS_t)\right]\cdot dG_t.$$

A standard stopping time argument then shows that θ is as in the statement of the proposition, while the formula for α follows directly from the definition of W. This completes the proof. ∎

Remark 4.3. To prove the existence of the solution to (ii) of proposition 4.2, one has to fall back on the theory of partial differential equations. Note that it is *linear*, unlike the non linear Bellman equation of dynamic programming. On the other hand if the function F is defined as

$$F(x,z) = \mathrm{E}^{Q_{x,z}}\left[\int_0^\infty \frac{g(\zeta_t)}{B_t}\,dt\right],$$

it is always well defined with the addition of assumption 4.1. The proof of proposition 4.2 shows that in this case the optimal policy is still given by (3), (4) and (5). In particular, the vector of optimal dollar investment in the stocks is given by

$$A_t = I_{S_t}\theta_t = I_{S_t}\left(F_x(t) - \zeta_t F_z(t)(\sigma_t\sigma_t^{\mathsf{T}})^{-1}(b_t - r_tS_t)\right).$$

In formulas (3), (4) and (5) the optimal policy is given as a function of the price processes and of the process ζ of proposition 3.4. We would like the optimal controls c, α and θ to be feedback controls, i.e., to depend only upon the current values of S and the agent's optimally invested wealth. The following lemma allows us to show that in general the optimal controls are feedback controls.

LEMMA 4.4. *Suppose that the interest rate is bounded above, and that*

$$\lim_{t\to\infty} \int_0^t |\kappa(S_s)|^2\, ds = \infty, \quad a.s.$$

under either P or Q. Then the function

$$z \to F(x,z) = \mathrm{E}^{Q_{x,z}}\left[\int_0^\infty \frac{g(\zeta_t)}{B_t}\, dt\right]$$

is stictly decreasing on $(0,\infty)$. This is so, in particular, if P and Q are mutually singular.

PROOF: Let $\zeta = \xi_t e^{\beta t}/B_t$, as in proposition 3.4. Then

$$F(x,z) = \mathrm{E}^{Q_x}\left[\int_0^\infty \frac{g(z\zeta_t)}{B_t}\, dt\right].$$

Since g is non decreasing, and strictly decreasing on $(0, u'_+(0))$, and ζ is continuous, it suffices to show that the event

$$\left\{\zeta_t < u'_+(0)/z\text{: for some } t \geq 0\right\}$$

has strictly positive probability for all $z > 0$ under either P or Q, which is equivalent. But (say under P),

$$\log \zeta_t = -\int_0^t \kappa(S_s)\cdot dw_s + \int_0^t \left(\beta - r(S_s) - \frac{1}{2}|\kappa(S_s)|^2\right) ds.$$

Since $\lim_{t\to\infty}\int_0^t |\kappa(S_s)|^2\, ds = \infty$ and $M_t = \int_0^t \kappa_s \cdot dw_s$ is a continuous martingale, theorem 9.3 of Chung and Williams [1983] implies that $B_t = M_{\tau_t}$ is indistinguishable from a Brownian motion (Dubins Schwarz), where $\tau_t = \inf\left\{s > 0 : A_s = \int_0^s |\kappa_u|^2\, du > t\right\}$. But $M_t = B_{A_t}$. The result then follows from the properties of Brownian motions and the boundedness of r and κ. An identical result can be obtained under Q by expressing ζ as a stochastic integral against $\bar{w}$. The last assertion follows from lemma 2.3. ∎

From remark 4.3, we know that $W_t = F(\zeta_\mu(t), S_t)/B_t$. Assume F is continuously differentiable. Since $F_z < 0$ for all values of x and z, there exists by the implicit function theorem a function G such that $\zeta_\mu(t) = G(B_tW_t, S_t)$. Note that $B_tW_t = \alpha_t B_t + \theta_t \cdot S_t$ is

just the undiscounted wealth at time t. Hence we can write the optimal policies (3), (4) and (5) in feedback form as

$$\theta(S_t, \zeta_t) = \theta\big(S_t, G(B_t W_t, S_t)\big)$$

$$\alpha(S_t, \zeta_t) = \alpha\big(S_t, G(B_t W_t, S_t)\big)$$

$$c_t = g\big(\mu G(B_t W_t, S_t)\big).$$

5. A SPECIAL CASE

We now specialize the market model developed in the earlier sections to the model with constant coefficients considered by Merton [17] and revisited recently by Karatzas, Lehoczky, Sethi and Shreve [14]. In this case explicit formulas for the optimal consumption and portfolio policies can be computed just as in Cox and Huang [4]. The method does not use stochastic control and takes care in a natural way of the non negative constaint on consumption. We illustrate our results with two examples taken from the family of constant absolute risk aversion and HARA utility functions.

We take the model (1) of section 2 with the following specialization. The risky securities follow a geometric Brownian motion

$$dS_t + \delta(t, S_t)\, dt = I_{S_t} \mathbf{b}\, dt + I_{S_t} \boldsymbol{\sigma}\, dw_t, \quad \forall t \geq 0$$

where $\mathbf{b}$ is an n-vector of constants, $\boldsymbol{\sigma}$ an $n \times n$ non singular matrix of constants and I_{S_t} an $n \times n$ diagonal matrix having S_t^i in the (i,i)th position. We furthermore assume that r is constant and write $\mathbf{r}$ for an n-vector of r's and $\boldsymbol{\kappa}$ for the constant vector $\boldsymbol{\sigma}^{-1}(\mathbf{b} - \mathbf{r})$. Given initial data z, the process ζ of section 3 and 4 satisfies the linear stochastic differential equation

$$d\zeta_t = (\beta - r + |\boldsymbol{\kappa}|^2)\zeta_t\, dt - \zeta_t \boldsymbol{\kappa} \cdot d\bar{w}_t$$

and so

$$\zeta(t) = z \exp\left\{-\boldsymbol{\kappa} \cdot \bar{w}_t + (\beta - r + |\boldsymbol{\kappa}|^2/2)t\right\}.$$

Thus, $\log \zeta_t$ is normally distributed under Q with

$$\log \zeta_t \sim N\big(\log z + (\beta - r + \varrho^2/2)t, \varrho^2 t\big),$$

where ϱ is the square root of $|\kappa|^2$.

We will assume that whenever agents are endowed with a non strictly increasing utility function, satiation does not occur, i.e., that

$$k < \int_0^\infty g(0)e^{-rt}\,dt,$$

where g denotes the inverse of marginal utility as defined in proposition 3.4. Define F as

$$F(z) = \mathrm{E}^{Q_z}\left[\int_0^\infty e^{-rt}g(\zeta_t)\,dt\right] = \int_0^\infty \int_{-\infty}^{+\infty} e^{-rt}\frac{1}{\varrho\sqrt{t}}g(e^x) \times n\left(\frac{x - \log z - (\beta - r + \varrho^2/2)t}{\varrho\sqrt{t}}\right) dt\,dx,$$

where n stands for the standard normal density function. It follows that

$$F(z) = \int_{-\infty}^{+\infty}\frac{g(e^x)}{\varrho}\int_0^\infty \frac{e^{-rt}}{\sqrt{2\pi t}}e^{-\frac{(x-\log z-(\beta-r+\varrho^2/2)t)^2}{2\varrho^2 t}}\,dt\,dx$$
$$= \int_{-\infty}^{+\infty}\frac{g(e^x)}{\varrho\sqrt{2\pi}}e^{\frac{\widetilde{\beta}(x-\log z)}{\varrho^2}}\int_0^\infty t^{-\frac{1}{2}}e^{-\left(\frac{x-\log z}{\varrho\sqrt{2}}\right)^2\frac{1}{t}-\frac{\alpha^2 t}{2}}\,dt\,dx$$

where we have put $\widetilde{\beta} = \beta - r + \varrho^2/2$ and $\alpha^2 = 2r + \widetilde{\beta}^2/\varrho^2$,

$$= \int_{-\infty}^{+\infty}\frac{2g(e^x)}{\varrho\sqrt{2\pi}}e^{\frac{\widetilde{\beta}(x-\log z)}{\varrho^2}}\left|\frac{x-\log z}{\varrho\alpha}\right|^{1/2}K_{\frac{1}{2}}\left(\left|\frac{x-\log z}{\varrho\alpha^{-1}}\right|\right)dx,$$

where $K_{1/2}$ is associated with a Bessel function and takes here the particular form $K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}$; cf. Gradshteyn and Ryzhik [9, pp. 340 and 967]. We then evaluate the last integral to obtain the important formula:

$$F(z) = \frac{1}{\varrho\alpha}\left[\int_0^\infty g\left(ze^{-u}\right)e^{-\mu u}\,du + \int_0^\infty g\left(ze^{u}\right)e^{-\mu' u}\,du\right],$$

where

$$\mu = \frac{\widetilde{\beta}}{\varrho^2} + \frac{\alpha}{\varrho},$$
$$\mu' = -\frac{\widetilde{\beta}}{\varrho^2} + \frac{\alpha}{\varrho}$$

are both positive. Using the formula above, it is then easy to compute F for a wide range of utility functions. Note that when utility has a finite marginal utility at zero, optimal consumption may involve zero consumption. Indeed, $c_t = 0$ if and only if $u'_+(0) \le \zeta_t$, that is if and only if the nominal wealth $e^{rt}W_t$ is less than the nonstochastic time independent boundary given by

$$\underline{W} = F(u'_+(0)) = \frac{1}{\varrho\alpha}\int_0^\infty \left[g(u'_+(0)e^{-u})e^{-\mu u} + g(u'_+(0)e^{u})e^{-\mu' u}\right]du.$$

We now give examples which illustrate the simplicity of our approach.

EXAMPLE 5.1. *Utility functions of constant absolute risk aversion.*

Let the utility function be

$$u(x) = -\frac{1}{\theta}e^{-\theta x},$$

where $\theta > 0$ is the coefficient of absolute risk aversion. Condition (ii) of proposition 3.4 is met for all $b > 0$. In turn, proposition 3.8 implies that a solution exists whenever the subjective rate of time preference is strictly positive. In this case, we find

$$g(x) = \left[-\frac{1}{\theta}\log x\right]^+$$

and so

$$g(ze^{-u}) = \begin{cases} -(1/\theta)(\log z - u) & \text{if } u \geq \log z, \\ 0 & \text{otherwise,} \end{cases}$$

and similarly for $g(ze^{u})$. Straightforward computations show that if $z \geq 1$

$$F(z) = \frac{\Gamma(2, \mu \log z)}{\varrho\alpha\theta\mu^2} - \log z \frac{z^{-\mu}}{\varrho\alpha\theta\mu},$$

and that if $z \leq 1$

$$F(z) = \frac{1}{\varrho\alpha\theta\mu^2} - \frac{\log z}{\varrho\alpha\theta\mu} - \frac{\log z}{\varrho\alpha\theta\mu'}\left(1 - z^{\mu'}\right) - \frac{\gamma(2, \mu' \log z^{-1})}{\varrho\alpha\theta\mu'^2},$$

where γ and Γ are the incomplete gamma functions

$$\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1}\, dt$$
$$\Gamma(\alpha, x) = \int_x^\infty e^{-t} t^{\alpha-1}\, dt.$$

It is easily checked that F is twice piecewise continuous (F'' has a point of discontinuity at $z = 1$), and that $F \to 0$ as $z \to \infty$ and $F \to \infty$ as $z \to 0$. The vector of optimal dollar amounts invested in the stocks is given by

$$A_t = I_{S_t}\theta_t = \frac{1}{\varrho\alpha\theta\mu}\zeta_t^{-\mu}(\boldsymbol{\sigma}\boldsymbol{\sigma}^\top)(\mathbf{b} - \mathbf{r})$$

if $\zeta_t \geq 1$ and by

$$A_t = \left[\frac{1}{\varrho\alpha\theta\mu} + \frac{1}{\varrho\alpha\theta\mu'}\left(1 - \zeta^{\mu'}\right)\right](\boldsymbol{\sigma}\boldsymbol{\sigma}^\top)(\mathbf{b} - \mathbf{r})$$

if $\zeta_t \leq 1$. Feedback contols are obtained by substituting $\zeta_t = G(e^{rt}W_t)$ in the formulas for c and A, where G is the inverse of F.

EXAMPLE 5.2. *HARA utility functions.*

Let the utility function be

$$u(x) = \frac{1-\gamma}{\gamma}\left(\frac{\rho x}{1-\gamma} + \eta\right)^{\gamma}$$

with $\rho > 0$, $\eta > 0$ and $\gamma < 1$. In condition (ii) of proposition 3.4, we take $b = 1-\gamma$, and for the solution to be in $L^2(\nu)$ we will impose that the subjective rate β satisfy condition (ii) of proposition 3.8. In the HARA case we find

$$g(x) = \frac{1-\gamma}{\rho}\left[\left(\frac{x}{\rho}\right)^{-1/1-\gamma} - \eta\right]^{+}.$$

Computations whose length is the sole difficulty give

$$F(z) = \frac{1}{\varrho\alpha\mu(\mu - 1/1-\gamma)}\frac{1}{\rho}\left(\frac{z}{\rho}\right)^{-\mu}\eta^{1-\mu(1-\gamma)}$$

when $z \geq \rho\eta^{-(1-\gamma)} = u'_{+}(0)$ and

$$F(z) = \frac{1-\gamma}{\rho\widetilde{\delta}}\left(\frac{z}{\rho}\right)^{-1/1-\gamma} - \frac{1-\gamma}{\rho r}\eta + \frac{1}{\varrho\alpha\mu'(\mu' + 1/1-\gamma)}\frac{1}{\rho}\left(\frac{z}{\rho}\right)^{\mu'}\eta^{1+\mu'(1-\gamma)}$$

when $z \leq u'_{+}(0)$, where

$$\widetilde{\delta} = \frac{1}{1-\gamma}\left(\beta - \gamma\left(r + \frac{\varrho^2}{2(1-\gamma)}\right)\right).$$

The inequalities $\widetilde{\delta} > 0$, $\mu > 1/(1-\gamma)$ and $\mu' > -1/(1-\gamma)$ are all a consequence of $\beta > \gamma r + (\gamma/1-\gamma)\varrho^2/2$ which itself follows from condition (ii) of proposition 3.8. Again it is easy to verify that F is twice piecewise continuous, that $F(z) \to \infty$ as $z \to 0$ and $F(z) \to 0$ as $z \to \infty$. Hence, the inverse G of F is defined for all $W_t > 0$ and feedback formulas can easily be obtained by substituting $\zeta_t = G(e^{rt}W_t)$ for z in the formulas involving F and F'. The case $\eta = 0$ is the one that is usually taken in the approach of dynamic programming, for this is the one for which the Bellman equation can be solved quite explicitly. This yields the simpler expression

$$F(z) = \frac{1-\gamma}{\rho\widetilde{\delta}}\left(\frac{z}{\rho}\right)^{-1/1-\gamma},$$

from which one derives $c_t = \widetilde{\delta}W_t e^{rt}$. It is clear that except in the degenerate case $\eta = 0$ the optimal consumption and portfolio policies are not linear functions of wealth, because of the

non negativity constraint on consumption. When $z \geq u'_+(0)$, we have $zF'(z) = -\mu F(z)$. Hence when nominal wealth $e^{rt}W_t$ falls below the non stochastic boundary $\underline{W} = F(u'_+(0))$ consumption is zero and the optimal dollar investment in the stocks is proportional to wealth with

$$A_t = \mu e^{rt} W_t (\sigma\sigma^\top)^{-1}(\mathbf{b} - \mathbf{r}).$$

When $z \leq u'_+(0)$, i.e., when nominal wealth is above $\underline{W}$, the optimal policy is no longer linear. As z approaches zero, however, which corresponds to large values of wealth, the optimal consumption and investment policies are "almost" the linear functions of wealth given in Merton [17]

6. Conclusion

In this chapter we have extended the martingale approach to encompass a model with infinite horizon. We find a "martingale measure" independent of the horizon, under which all assets have the same expected rates of return in restriction to finite time intervals. For example, in the classical models of Samuelson or Merton, the martingale measure and the original probability beliefs are found to be mutually singular, even though they are equivalent in restriction to all finite horizons.

Fortunately, this particular feature hardly changes the main results of the finite horizon model. The non negative wealth constraint is sufficient to rule out all arbitrage opportunities at finite stopping times. Also, the dynamic consumption portfolio problem for a "small" investor can be put in correspondence with an analogous problem in finite horizon, thanks to a natural change of time depending on the subjective rate of time preference. The basic conclusion is that in a continuous time economy, one needs some qualifying assumptions for optimal policies to exist. Our condition strikes a balance between the discount rate, the interest rate, the volatility of asset prices and the asymptotical behavior of the utility function.

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Chapter 3
Consumption of an Endowment

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Abstract. A general consumption investment problem is considered for an agent endowed with an uncertain "labor" income, and whose non human wealth is constrained to be non negative. Under general conditions on the nature of the market model and on the utility function of the agent, it is shown how one can derive the existence of an optimum. When the endowment is an Itô process, the optimal consumption path is a function of the shadow price of consumption. Because of the liquidity constraint, the principle of equimarginal utility does not have to be satisfied.

1. Introduction

In the foregoing chapters, we have studied the problem of optimal consumption and portfolio choice when markets are dynamically incomplete (fewer risky securities than sources of uncertainty in the world) or when the planning horizon is sent to infinity. In this paper, we are back to the case of complete markets and finite horizon, but we assume that the agent is endowed with an uncertain "labor" income. The additional risks he faces are uninsurable in the following sense: because of a constraint on borrowing, there may be no portfolio strategy on the risky assets whose capital gains or losses can hedge completely his flow of income. Hence the model embodies some kind of market incompleteness even though the endowment process does not constitute in itself a new source of uncertainty in the economy.

In the economics literature i' is typically asked whether or not there is a "market" for labor income; cf. for example Caballero [3] or Drèze and Modigliani [8]. If such a market exists, the agent can simply "sell" his endowment at time zero. His optimal policy is then no different than that of the same agent endowed with an initial investment supplemented with the cash value of his future income. When markets are dynamically complete, the agent can sell his endowment on the financial markets, by implementing the portfolio strategy which generates the same final payoff. In that case, his behavior will present no special interest. The only difference is that his financial wealth, defined as the market value of his portfolio, will take into account the endowment yet to come and may become negative as he borrows against future labor income.

The extension of the existing literature on optimal consumption and portfolio choice policy to income uncertainty is then straightforward insofar as one allows for negative financial wealth. However, as recalled in the first two chapters, unconstrained negative wealth paves the way to arbitrage opportunities (or free lunches). Clearly one has to impose a liquidity constraint of some sort, lest the optimization problem becomes ill defined. A possibility could be that borrowing be no more than the expected discounted value of future labor income. Unfortunately, such a constraint will be difficult to enforce, especially if the law of the endowment process is not agreed upon, or is unkown. (Here of course we implicitly assume that the endowment process has no deterministic component.)

In this paper we shall pose the optimal consumption portfolio problem under the requirement that financial wealth be non negative at each point in time. For instance, one could argue that banks will not lend money against future income to a given consumer without asking for collateral requirements. They would not be able to increase those collateral requirements if wealth were already negative. Be that as it may, an important justification for the choice of our liquidity constraint relies on the assumption that there is imperfect knowledge about the distribution characteristics of income. It is clear that the non negative wealth constraint can be easily enforced without prior knowledge of the law of the consumer's endowment process. The endowment rate process is itself arbitrary, and may even become negative. Of course, not *every* lifetime income profile will give rise to a feasible consumption path. One may suspect that a consumer facing the external debt of the United States with some positive probability will not be able to satisfy the non negative wealth constraint. In the sequel, we provide necessary and sufficient conditions for this constraint to hold in all states of nature.

A model close to the one we study here is the stochastic version of the Ramsey problem, which deals with the optimal consumption of an investment. The Ramsey model examines the optimal planning of saving and consumption in a one commodity world with returns proportional to capital. It is a model of economic growth; cf. Foldes [11], Cox, Ingersoll and Ross [5] or Akian and Bensoussan [1]. Here we will talk about optimal consumption of an endowment, in the sense that the consumer has no control over the return to capital.

Our results fall under three main headings. First we outline the model, which is closely related to the models of Chapters 1 and 2, and transform the dynamic problem into a static one, in the spirit of Cox and Huang [4]. Second we provide sufficient conditions for the existence of an optimum, proved by a compactness argument found in Foldes [11]. Third we elicit conditions charaterizing an optimum, derived by methods modelled on the theory of duality. Finally, some concluding remarks deal with the martingale conditions for shadow prices and the value of final wealth. Unfortunately, we have not attempted to give explicit solutions for a class of special models.

2. The model

The only substantial modification from the model developed in chapters 1 and 2 is the introduction of an endowment process, so we will sketch the model briefly. As usual we start with a probability space $(\Omega, \mathcal{F}, P)$ on which is defined an n-dimensional Brownian motion, which is denoted by w. We let $\{\mathcal{F}_t\}$ be the filtration, satisfying the usual conditions, generated by w from time 0 to time 1, and still take $\mathcal{F}$ to be the information $\mathcal{F}_1$ known at time 1. (All uncertainty is resolved at the final date.) In the sequel, it will be made frequent use of stopping times. For a stochastic process stopped at some random time to be a well defined random variable, it is necessary to make assumptions about its measurability properties. All the processes to appear are taken to be progressively measurable. As was stressed in chapter 1, all (measurable) processes adapted to $\mathcal{F}_t$ are necessarily progressively measurable. It is also known that the optional and predictable σ-fields are, in this case, identical (though distinct from the progressive σ-field).

The definition of the markets model remains unchanged from the previous chapters, except for the assumption of completeness. Namely there are n risky securities, whose price S_t at time t is the solution to a system of stochastic differential equation

$$dS_t + \iota_t\, dt = b(t, S_t)\, dt + \sigma(t, S_t)\, dw_t. \tag{1}$$

The dividend rate ι, the drift term b and the dispersion matrix σ satisfy the same requirements as in chapters 1 and 2, the motivation being that the solution to (1) exists, is unique and does not explode on the time interval $[0,1]$. There is an additional asset, called the

bond, which is locally riskless and produces interest at a random rate r. Its price is given by

$$B_t = \exp\left\{\int_0^t r(u, S_u)\,du\right\}. \tag{2}$$

The continuous function $r : R^+ \times R^n \to R^+$ is arbitrary, but the quantity

$$\kappa(t,x) \stackrel{\text{def}}{=} \sigma^{-1}(t,x)\big(b(t,x) - r(t,x)x\big)$$

is assumed to be bounded; cf. chapter 1, assumption 2.2. Letting (θ_t, α_t) be the number of shares invested in the risky securities and the bonds, respectively, financial wealth in units of the bond at time t is defined as $W_t \stackrel{\text{def}}{=} \alpha_t + \theta_t \cdot S_t^*$, where $S^* = S/B$ denotes the normalized prices of the risky securities. The processes (α_t, θ_t) are adapted, and we will assume that

$$\int_0^1 |\sigma^\top \theta|^2\,dt < \infty, \quad (a.s., P). \tag{3}$$

The last condition ensures that capital gains or losses on the risky securities are well defined, by means of the stochastic integral $\int_0^t \theta_s \cdot dS_s$.

On the endowment front, we have an adapted process $\{Y_t; 0 \le t \le 1\}$, with $Y_0 = 0$, which stands for accumulated endowment between time 0 and time t. Note that Y must be of bounded variation, as the difference of input and output flows. The model can be taken as the continuous time analog of the classical difference equations commonly found in the literature, e.g., Caballero [3] or Sibley [25]. Often, Y may be thought of as a positive or an increasing process, but this needs not be the case. However, for mathematical reasons, it is desirable to keep some control over how large Y may be. We assume that Y is a predictable process which can be written as

$$Y_t = \int_0^t e_s\,ds,$$

where e, the income rate, is a process verifying

$$\|e\|_{2,P} = \left(\mathrm{E}\int_0^1 e_t^2\,dt\right)^{1/2} < \infty.$$

(Here we have used the notation of chapter 1.) Note that this implies that the density $e_t = dY_t/dt$ can be chosen to be predictable; cf. Jacod [14, lemma 1.36].

Accumulated consumption is also modelled as an absolutely continuous process. The rate of consumption at time t is positive and denoted by c_t, with $\int_0^1 c_t\,dt < \infty$, $(a.s., P)$. The process c is of course adapted to the information and is financed through withdrawals from the portfolio. Defining the discounted gains process as

$$G_t = S_t^* + \int_0^t \frac{\iota_s}{B_s}\,ds,$$

the budget constraint can be written as

$$\int_0^t \frac{c_s}{B_s}\,ds + W_t = k + \int_0^t \frac{e_s}{B_s}\,ds + \int_0^t \theta_s \cdot dG_s. \tag{4}$$

The interpretation is the same as in the previous chapters, except for the addition of an endowment.

We are now ready for the following definition.

DEFINITION. *Let $\pi = (\alpha, \theta)$. The pair (c, π) of a consumption and a portfolio process defines an admissible strategy for the initial endowment k if*

(i) *c is predictable, non negative, and $\int_0^1 c_s\,ds < \infty$;*

(ii) *The budget constraint (4) holds for all $t \in [0,1]$, with the qualification condition (3);*

(iii) *The liquidity constraint $W_t \geq 0$ holds for all $t \in [0,1]$.*

(Note that once θ has been found satisfying (4), the number of shares of the riskless bond α obtains from the definition of W.) As usual the objective of the investor is to find an admissible strategy for an initial capital $k > 0$ which maximizes the expected utility

$$E\int_0^1 u(c_t, t)\,dt$$

for some function $u(c,t) : R^+ \times [0,1] \to R \cup \{-\infty\}$ which is taken to be increasing, concave and upper semicontinuous in its first argument.[1] The goal of the next two lemmas is to transform the admissibility requirement into a more amenable condition. In order to do so we recall from chapters 1 and 2 that there is a unique probability Q, equivalent to P, with

[1] Note that if $u(\cdot, t)$ is finite on $(0, \infty)$ it is automatically continuous there; cf. Ekeland and Turnbull [9, corollary 3.2].

respect to which the discounted gains process G is a martingale. The density of Q with respect to P is given by the value at $t = 1$ of the exponential martingale

$$\eta_t = \exp\left\{-\int_0^t \kappa(s, S_s) \cdot dw_s - \frac{1}{2}\int_0^t |\kappa(s, S_s)|^2\, ds\right\},$$

where κ is defined as above. The economic interpretation of Q is the following. For a measurable, "discounted" contingent claim y, integration by the measure Q given by $\tilde{\mathrm{E}}[y] = E[\eta_1 y]$ is a "rational" pricing scheme—as long as y satisfies an integrability condition--and η_1 is the shadow price of consumption at time $t = 1$. In particular, the expected capital gains $\int_0^1 \theta_t \cdot dG_t$ which can be realized by bounded (more generally, integrable) trading strategies are worth zero at time 0.

Let "human capital" be the process of discounted labor income received between 0 and t, i.e., $H_t = \int_0^t (e_s/B_s)\, ds$. A measurable process $(X_t)_{t\in[0,1]}$ is said to belong to *class D* on $[0,1]$ if the collection $\{X_\tau, \tau \in \mathcal{T}\}$ is uniformly integrable, where $\mathcal{T}$ is the collection of all stopping times bounded by 1; cf. Dellacherie and Meyer [**6**]. We have the following result.

LEMMA 1. *If the density of Q with respect to P is square integrable, then H is of class D with respect to Q.*

PROOF: This follows from the fact that

$$|H_\tau| \le \int_0^1 \left|\frac{e_s}{B_s}\right| ds \le \left(\int_0^1 e_s^2\, ds\right)^{1/2}$$

and that

$$\tilde{\mathrm{E}}\left(\int_0^1 e_s^2\, ds\right)^{1/2} = \mathrm{E}\,\eta_1 \left(\int_0^1 e_s^2\, ds\right)^{1/2} \le \|\eta_1\|_{2,P} \|e\|_{2,P} < \infty.$$

Since H_τ is uniformly bounded by an integrable random variable, it is uniformly integrable.

■

LEMMA 2. *Suppose M is a continuous local martingale, and that its negative part M^- is of class D on $[0,1]$. Then M is a supermartingale on $[0,1]$.*

PROOF: We have to prove that M_t is integrable for $0 \le t \le 1$, and that $M_s \ge E[M_t|\mathcal{F}_s]$ for any $0 \le s \le t \le 1$. Let τ_n be a localizing sequence for M, with $\tau_n \uparrow 1$. By continuity of the sample paths of M, $M_t^{\tau_n} = M_{\tau_n \wedge t}$ converges a.s. to M_t as $n \to \infty$. For $A \in \mathcal{F}_s$, consider

$$\int_A M_t^{\tau_n}\, dP = \int_{A\cap\{M_t^{\tau_n} \ge -a\}} M_{t\wedge\tau_n}\, dP + \int_{A\cap\{M_t^{\tau_n} < -a\}} M_{t\wedge\tau_n}\, dP.$$

Since M^- is of class D by assumption, given $\epsilon > 0$ we can choose $a > 0$ such that $\int_{\{M_t^{\tau_n} < -a\}} |M_t^{\tau_n}|\, dP \leq \epsilon$. Then

$$\int_{A \cap \{M_t^{\tau_n} < -a\}} M_t^{\tau_n}\, dP = -\int_{A \cap \{M_t^{\tau_n} < -a\}} |M_t^{\tau_n}|\, dP \geq -\epsilon.$$

Hence

$$\begin{aligned}
\liminf_n \int_A M_t^{\tau_n}\, dP &\geq -\epsilon + \liminf_n \int_{A \cap \{M_t^{\tau_n} \geq -a\}} M_{t \wedge \tau_n}\, dP \\
&\geq -\epsilon + \int_A \liminf_n \left(1_{\{M_t^{\tau_n} \geq -a\}} M_{t \wedge \tau_n}\right) dP \\
&\geq -\epsilon + \int_A \lim_n M_{t \wedge \tau_n}\, dP \\
&= -\epsilon + \int_A M_t\, dP.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_A M_t\, dP &\leq \epsilon + \liminf_n \int_A M_t^{\tau_n}\, dP \\
&= \epsilon + \liminf_n \int_A M_s\, dP \\
&= \epsilon + \int_A M_s\, dP.
\end{aligned}$$

We have used the fact that M^{τ_n} is a martingale. Because ϵ is arbitrary, this shows that

$$E[M_t | \mathcal{F}_s] \leq M_s,$$

as desired. ∎

We now apply the foregoing two lemmas to the budget constraint (4). Since H is of class D with respect to Q and

$$M_t = k + \int_0^t \theta_s \cdot dG_s \geq -H_t,$$

M is a supermartingale with respect to Q. Doob's stopping time theorem then yields the condition

$$\tilde{\mathrm{E}}\left[\int_0^\tau \frac{c_s - e_s}{B_s}\, ds\right] \leq \tilde{\mathrm{E}}[M_\tau] \leq k$$

for all $\tau \in \mathcal{T}$, the set of $\mathcal{F}_t$ stopping times bounded by 1, and so

$$\sup_{\tau \in \mathcal{T}} \tilde{\mathrm{E}}\left[\int_0^\tau \frac{c_s - e_s}{B_s}\, ds\right] \leq k. \tag{5}$$

We now show that the necessary condition for admissibility (5) is in fact equivalent to (4), thanks to the following result (cf. also Karatzas [15]).

LEMMA 3. *Let ξ be an optional process of class D with respect to Q. If*

$$\sup_{\tau \in \mathcal{T}} \widetilde{\mathrm{E}}[\xi_\tau] \leq k,$$

then there exist two adapted processes θ and W, with $\int_0^1 |\sigma^\top \theta|^2 \, dt < \infty$ and $W_t \geq 0$, such that

$$(\#) \qquad \xi_t + W_t = k + \int_0^t \theta_s \cdot dG_s,$$

for all $0 \leq t \leq 1$.

PROOF: Let Z be the Snell envelope of ξ; cf. Dellacherie and Meyer [**6**, appendix I, 22–23]. One has

$$Z_0 = \sup_{\tau \in \mathcal{T}} \widetilde{\mathrm{E}}[\xi_\tau] \leq k$$

by assumption, $Z_1 = \xi_1$ and Z is of class D. One then use the Doob Meyer decomposition under Q

$$Z = Z_0 + M - A$$

where M is a uniformly integrable martingale and A an increasing integrable process, $A_0 = 0$; cf. Karatzas and Shreve [**17**, theorem 4.10]. The martingale representation theorem coupled with the observation made in chapter 1, lemma 2.7, allows us to write

$$M_t = \int_0^t \theta_s \cdot dG_s,$$

with $\int_0^1 |\sigma^\top \theta|^2 \, dt < \infty$. It suffices now to pose

$$\begin{aligned} W_t &\stackrel{\text{def}}{=} (k - Z_0) + (Z_t - \xi_t) + A_t \\ &= k + \int_0^t \theta_s \cdot dG_s - \xi_t \geq 0. \end{aligned}$$

One sees that $W \geq 0$ and that (#) is verified. ∎

PROPOSITION 4. *Suppose c is a predictable, positive process satisfying constraint* (5). *Then there exists an admissible strategy (c, π) for the initial investment k.*

PROOF: This follows immediately upon setting $\xi_t = \int_0^t (c_s/B_s)\, ds - \int_0^t (e_s/B_s)\, ds$ and applying the lemma above. Note that ξ is optional (here, predictable) and that $\int_0^{(\cdot)} (c_s/B_s)\, ds$ is of class D since c is positive and

$$\widetilde{\mathrm{E}} \int_0^1 \frac{c_s}{B_s}\, ds \leq k + \widetilde{\mathrm{E}} \int_0^1 \frac{e_s}{B_s}\, ds < \infty. \quad \blacksquare$$

Remark 5. If one looks at the proof of lemma 3. one sees that whenever A is a non trivial process, wealth at the final date $W_1 \geq A_1 > 0$. This might seem bothering. But the purpose of the lemma is to show that two constraints are equivalent. We will show in the concluding remarks that at the optimum, $W_1 = 0$.

3. Existence

Recall that the problem faced by the investor is to find an admissible strategy for the initial endowment $k > 0$ which maximizes

$$\mathrm{E} \int_0^1 u(c_t, t)\, dt,$$

where $u(c,t) : R^+ \times [0,1] \to R \cup \{-\infty\}$ is a concave, increasing and upper semicontinuous function of c. To convey the idea of time preference, we also assume that u is a decreasing function of t. From the section above this amounts to solving the problem

$$(\mathcal{P}) \qquad \begin{aligned} &\sup_{c \in \mathcal{G}} \mathrm{E} \int_0^1 u(c_t, t)\, dt \\ \text{s.t.} \quad &\sup_{\tau \in T} \widetilde{\mathrm{E}} \left[\int_0^\tau \frac{c_s - e_s}{B_s}\, ds \right] \leq k. \end{aligned}$$

We let $\mathcal{G}$ be the set of all predictable, non negative c satisfying the constraint of problem $(\mathcal{P})$. We assume for the present that the supremum ψ^* of $(\mathcal{P})$ is finite, and that there exists some $c \in \mathcal{G}$ such that $E \int_0^1 u(c_t, t)\, dt > -\infty$. (This assumption will be met under the conditions of proposition 10.) In this section we shall find conditions on the utility function or on "human capital" which ensure the existence of a solution to $(\mathcal{P})$. The existence question is not only of theoretical interest. It is a crucial step towards the characterization problem examined in section 4. The *extremality relations* of duality theory are valid only when a solution to the primal problem is known to exist.

We follow closely Foldes [11]. The argument runs as follows. Let $c_n \in \mathcal{G}$ be a maximizing sequence, i.e., a sequence such that $\mathrm{E} \int_0^1 u(c_n(t), t)\, dt$ converges to ψ^*. Then $u_n \stackrel{\text{def}}{=} u(c_n(t), t)$ is an element of

$$L^1 = L^1(\Omega \times [0,1], \mathcal{P}, P \times \lambda),$$

where λ stands for the Lebesgue measure and $\mathcal{P}$ denotes the predictable σ-field. If u_n has a subsequence which converges weakly in L^1, then there exists $c^* \in \mathcal{G}$ for which the supremum ψ^* is attained. This is shown in the following lemma.

LEMMA 6. *Let $c_n \in \mathcal{G}$ be a maximizing sequence, and suppose $u_n = u(c_n, t)$ has a weakly convergent subsequence in L^1. Then there exists $c^* \in \mathcal{G}$ such that*

$$E \int_0^1 u(c^*(t), t)\, dt = \psi^*.$$

PROOF: The argument is essentially contained in Foldes [11]. We sketch the proof for the sake of completeness. Let u^* be the limit of the subsequence of u_n in the weak topology. We have $\psi^* = E \int_0^1 u^*\, dt$. Now define implicitly c^* as $u^*(\omega, t) = u(c^*(\omega, t), t)$. It is easy to see that c^* is well defined, non negative and predictable. Now there exists a sequence of convex combinations of u_n which converges in norm to u^*; cf. Rudin [24, theorem 3B]. By passing to a subsequence, we can assume that it converges also almost surely. Denoting by β_{jm}, $1 \leq j \leq m$ the corresponding coefficients with $\sum_j \beta_{jm} = 1$ for $m = 1, 2, \ldots$, we define the sequence $\bar{c}_m = \sum_j \beta_{jm} c_j$. Note that $c_m \in \mathcal{G}$ since $\mathcal{G}$ is convex. The concavity of u implies that

$$\sum_{j=1}^{m} u(c_j, t) \leq u(\bar{c}_m, t)$$

and so $u^* \leq \liminf_m u(\bar{c}_m, t)$. Using the definition of c^* we get $c^*(\omega, t) \leq \liminf_m c_m(\omega, t)$. Finally we use Fatou's lemma to get

$$\tilde{\mathrm{E}} \left[\int_0^\tau \frac{c^* - e}{B}\, dt \right] \leq \tilde{\mathrm{E}} \left[\int_0^\tau \frac{\bar{c}_m - e}{B}\, dt \right] \leq k,$$

for all $\tau \in \mathcal{T}$. So $c^* \in \mathcal{G}$, as desired. ∎

Uniform integrability of the sequence u_n in L^1 (a condition equivalent to weak sequential compactness) obtains immediately if the utility function is bounded. Otherwise it may occur under the following circumstances.

COROLLARY 7. *Suppose that $u \geq 0$, and that there exists $p \in (0,1)$ such that $u(x,0) \leq a + bx^p$ for all $x \in R^+$ and some $b > 0$. Then $(\mathcal{P})$ has a solution whenever $B/\eta \in L^q(P \times \lambda)$, with $q = p/1 - p$.*

PROOF: In view of the lemma above, it suffices to check that any maximizing sequence u_n is uniformly integrable in L^1 or, equivalently, that it is uniformly bounded and uniformly continuous; cf. Ash [2, theorem 7.5.3]. Let H be a predictable set. Then

$$\int_H u_n\, dP\, dt \leq a\, P{\times}\lambda\,(H) + b \int_H c_n^p\, dP\, dt$$

and

$$\int_H c_n^p \, dP \, dt = \int_H \frac{c_n^p}{\eta} \, dQ \, dt = \int_H \left(\frac{c_n}{B}\right)^p \frac{B^p}{\eta} \, dQ \, dt$$
$$\leq \left(\int_H \frac{c_n}{B} \, dQ \, dt\right)^p \left(\int_H \left(\frac{B^p}{\eta}\right)^{1/1-p} dQ \, dt\right)^{1-p}$$
$$\leq \left(k + \tilde{\mathrm{E}}\left[\int_0^1 \frac{e}{B} \, dt\right]\right)^p \left(\int_H \left(\frac{B}{\eta}\right)^q dP \, dt\right)^{1-p}.$$

Uniform boundedness obtains upon setting $H = \Omega \times [0,1]$. Uniform continuity obtains upon letting $P \times \lambda(H)$ converge to 0. ∎

The case $u \leq 0$ can be handled separately thanks to the following lemma.

LEMMA 8. *Suppose that $u \leq 0$ and that there exists $\bar{c} \in \mathcal{G}$ such that $E \int_0^1 u(h\bar{c}_t, t) \, dt > -\infty$ for all h sufficiently small. Then the maximizing sequence u_n is uniformly integrable in L^1.*

PROOF: Foldes, pp. 53–54. The same proof applies since $\mathcal{G}$ is convex. ∎

COROLLARY 9. *Suppose that $u \leq 0$ and that $u(x,t) \geq a - bx^p$ for some $p < 0$, $b > 0$. Then $(\mathcal{P})$ has a solution whenever the discounted human capital $H = \int_0^{(\cdot)} (e/B) \, dt$ is non negative.*

PROOF: It suffices to set consumption constant and less than k. Then $E \int_0^1 u(c,t) \, dt \geq u(c,1) > -\infty$ and

$$\tilde{\mathrm{E}}\left[\int_0^\tau \frac{c-e}{B} \, dt\right] \leq c \leq k. \quad ∎$$

Summarizing those results, we conclude with the following proposition.

PROPOSITION 10. *An optimum to $(\mathcal{P})$ exists*

(i) *If the utility function u is bounded;*

(ii) *If it is positive, there exist constants a, $b > 0$, $0 < p < 1$ such that $u(x,t) \leq a + bx^p$, and $B/\eta \in L^{p/1-p}$;*

(iii) *If it is unbounded, there exist constants a, a', $b > 0$, $b' > 0$, $p < 1$ and $p' < 0$ such that*

$$a' - b'x^{p'} \leq u(x,1) \leq u(x,0) \leq a + bx^p,$$

the discounted human capital $H \geq 0$ and $B/\eta \in L^q(P \times \lambda)$, with $q = p/1-p$.

PROOF: The result for unbounded utilities results from the different cases above combined. For details see Foldes [11, pp. 55–56]. ∎

4. CHARACTERIZATION

To characterize the optimal solution of $(\mathcal{P})$, we make use of different analytical methods. We shall refer to duality theory as in Ekeland and Turnbull [9] and call $(\mathcal{P})$ the *primal problem.*

Constraint (5) prompts us to choose as commodity space the space $L^1(\nu)$, where ν is the Doléans measure associated with the increasing process $t \to \int_0^t B_s^{-1}\, ds$ and the measure Q, i.e., one has

$$\nu(X) = \tilde{\mathrm{E}} \int_0^1 \frac{X_s}{B_s}\, ds$$

for every positive and $\mathcal{P}$-measurable process X. Let h be the function $L^1(\nu) \to R$ defined as

$$h(c) = \sup_{\tau \in T} \tilde{\mathrm{E}} \left[\int_0^\tau \frac{c_s - e_s}{B_s}\, ds \right].$$

Constraint (5) may be reformulated as $h(c) \leq k$. Finally the indicator function $\delta(\cdot|K) : R \to \{0, \infty\}$ of a subset K of R is defined by

$$\delta(x|K) = \begin{cases} 0 & \text{if } x \in K; \\ +\infty & \text{if } x \notin K. \end{cases}$$

Hence problem $(\mathcal{P})$ amounts to finding the infimum of $F(c)$, $c \in L^1(\nu)$, where $F\colon L^1(\nu) \to R \cup \{\infty\}$ is given by

$$F(c) \stackrel{\text{def}}{=} -\mathrm{E}\left[\int_0^1 \bar{u}(c_t, t)\, dt \right] + \delta(h(c)|(\infty, k]).$$

Here $\bar{u}(c,t)$ is $-\infty$ if $c < 0$, identical to u if $c > 0$, and equal to $\lim u(c,t)$ as $c \downarrow 0$ if $c = 0$. Thus $\bar{u}$ penalizes negative consumption and inherits from u its increasing, concave and upper semicontinuous properties. In order to go ahead we have to check whether the function F so defined satisfies the regularity properties stated in Ekeland and Turnbull [9].

LEMMA 11. *Suppose that u satisfies one of the assumptions of proposition 10. Then the function F defined above is convex, lower semicontinuous and $F \not\equiv +\infty$.*

PROOF: First we show that the function $c \to \mathrm{E} \int_0^1 \bar{u}(c_t, t)\, dt$ from $L^1(\nu)$ to $R \cup \{-\infty\}$ is concave and upper semicontinuous. We show only upper semicontinuity. Let the sequence

$\{c_n\}$ converge to c in $L^1(\nu)$. By considering only a subsequence, we may and will assume that it converges also almost surely. Hence for almost all (ω, t), we have (dropping the argument t) $\bar{u}(c) \geq \limsup_n \bar{u}(c_n)$. Taking expectations we get

$$\mathrm{E}\int \bar{u}(c)\,dt \geq \mathrm{E}\int \limsup \bar{u}(c_n) \geq \limsup \mathrm{E}\int \bar{u}(c_n).$$

The last inequality follows from Fatou's lemma and the uniform integrability of $u(c_n)$ proved in section 3. This proves the first part. Now we consider $\delta\big(h(c) \mid (-\infty, k]\big)$. Note that h is convex (immediate verification). Moreover, h is bounded on bounded open balls of $L^1(\nu)$ since H is of class D under Q. So h is continuous; cf. Ekeland and Turnbull [9, corollary III.2]. Since δ is lower semicontinuous, the composite function $\delta \circ h$ is also lower semicontinuous. Convexity can be checked directly. Finally to find c such that $F(c) < +\infty$, it suffices to take $c = 0$ in cases (i) and (ii) of proposition 10, or a constant $c \leq k$ in case (iii). The lemma is proved. ∎

We will imbed $(\mathcal{P})$ in a family of "perturbed" problems depending upon a parameter z which will be interpreted as an increment to the rate of labor income. Since the income rate $e \in L^1(\nu)$, we will choose $L^1(\nu)$ for our parameter space. Let us define for any convex, lower semicontinuous function $\Phi : L^1(\nu)^2 \to R \cup \{+\infty\}$ such that $\Phi(c, 0) = F(c)$ the perturbed problem

$$(\mathcal{P}_z) \qquad \inf_{c \in L^1(\nu)} \Phi(c, z).$$

Let $g(z) = \inf_{c \in L^1(\nu)} \Phi(c, z)$ be the value of $(\mathcal{P}_z)$. Hence $g(z)$ is, up to a change in sign, the expected utility of the investor when his endowment e is perturbed by the small variation z. Of course $(\mathcal{P}_0) = (\mathcal{P})$. We define Φ in the natural way as

$$\Phi(c, z) = -\,\mathrm{E}\int_0^1 \bar{u}(c_t, t)\,dt + \delta\left(\sup_{\tau \in \mathcal{T}} \widetilde{\mathrm{E}}\left[\int_0^\tau \frac{c_t - e_t + z_t}{B_t}\,dt\right] \,\middle|\, (-\infty, k]\right).$$

It is easily verified that Φ inherits from F its convex and lower semicontinuous properties under the assumptions of proposition 10, and that there is a $\bar{c} \in L^1(\nu)$ such that $\Phi(\bar{c}, \cdot)$ is finite and continuous at 0 provided $\sup_\tau \widetilde{\mathrm{E}}[-H_\tau] < k$, which we will assume. (The latter condition is known as the Slater condition.)

The dual of $L^1(\nu)$ is $L^\infty(\nu)$, the space of essentially bounded, predictable processes, and the two spaces are put in duality by the continuous bilinear form

$$(z, z^*) = \tilde{\mathrm{E}} \int_0^1 \frac{z_t z_t^*}{B_t}\, dt.$$

Let g^* be the polar function of g; cf. Ekeland and Turnbull [9, §1, chapter III]. The *dual problem* of $\mathcal{P}$ with respect to Φ is

$$(\mathcal{P}^*) \qquad \sup_{z^* \in L^\infty(\nu)} -g^*(z^*).$$

From the extremality relations (cf. Proposition III.7 of Ekeland and Turnbull [9]), if c^* solves $(\mathcal{P})$, then there exists $z^* \in L^\infty(\nu)$ such that

$$(0, z^*) \in \partial\Phi(c^*, 0),$$

where $\partial\Phi(c^*, 0)$ is the subdifferential of Φ at $(c^*, 0)$. Hence for all $c \in L^1(\nu)$ and all $z \in L^1(\nu)$

$$\begin{aligned} \Phi(c, z) - \Phi(c^*, 0) &= \mathrm{E} \int_0^1 (u(c^*, t) - u(c, t))\, dt \\ &\quad + \delta\left(\sup_{\tau \in \mathcal{T}} \tilde{\mathrm{E}} \left[\int_0^\tau \frac{c_t - e_t + z_t}{B_t}\, dt \right] \,\middle|\, (-\infty, k] \right) \\ &\geq (z, z^*) = \tilde{\mathrm{E}} \int_0^1 \frac{z_t z_t^*}{B_t}\, dt. \end{aligned} \tag{6}$$

We then get the following characterization.

PROPOSITION 12. *Under one of the assumptions of proposition 10, a consumption plan c^* is a solution of $(\mathcal{P})$ if and only if there exists an essentially bounded, positive and predictable process z^* such that*

(i) *c^* maximizes $\tilde{\mathrm{E}} \int_0^1 (z^* c / B)\, dt$ in $\mathcal{G}$;*

(ii) *$z_t^*(c_t^* - c)\eta_t / B_t \leq u(c_t^*, t) - u(c, t)$ for all real $c \geq 0$.*

PROOF: Sufficiency is easy. For any $c \in \mathcal{G}$ we have

$$\mathrm{E} \int_0^1 (u(c, t) - u(c^*, t))\, dt \leq \tilde{\mathrm{E}} \int_0^1 z_t^* \frac{c - c^*}{B_t}\, dt \leq 0,$$

as desired. Consider now necessity. We show first that a.s., $z^* \geq 0$. Suppose not, and let $z_t = \mu I_{\{z_t^* < 0\}}$ for some negative number μ. Then for any $\tau \in \mathcal{T}$ and $c \in \mathcal{G}$,

$$\tilde{\mathrm{E}} \left[\int_0^\tau \frac{c - e + z}{B}\, dt \right] \leq \tilde{\mathrm{E}} \left[\int_0^\tau \frac{c - e}{B}\, dt \right] \leq k$$

and so $\Phi(c,z) - \Phi(c^*,0) < \infty$. But $(z,z^*) = \mu \int_{z^*<0} z^* \, d\nu$ converges to $+\infty$ as $\mu \to -\infty$. This contradicts (6) and shows $z^* \geq 0$. Moreover from (6) we see that for all $z \in L^1(\nu)$ such that

$$(\#) \qquad \widetilde{\mathrm{E}}\left[\int_0^\tau \frac{c - e + z}{B}\, dt\right] \leq k, \qquad \forall \tau \in \mathcal{T},$$

we have

$$\mathrm{E}\int (u(c^*,t) - u(c,t))\, dt \geq \widetilde{\mathrm{E}}\int_0^1 \frac{z_t z_t^*}{B_t}\, dt.$$

First we put $c = c^*$. This reads $\widetilde{\mathrm{E}}(z_t z_t^*/B_t)\, dt \leq 0$ for all z verifying (#). For $z = c - c^*$, with $c \in \mathcal{G}$, this implies

$$\widetilde{\mathrm{E}}\int_0^1 z_t^* \frac{c_t - c_t^*}{B_t}\, dt \leq 0.$$

Whence (i). Second put $z = c^* - c$ for any $c \in L^1(\nu)$. We obtain

$$(7) \qquad \widetilde{\mathrm{E}}\int z^* \frac{c^* - c}{B}\, dt \leq \mathrm{E}\int \big(u(c^*,t) - u(c,t)\big)\, di.$$

To deduce condition (ii) from (7), suppose that for all (ω,t) in a set S of positive measure, there exists $c \geq 0$ such that

$$z_t^* \frac{c^*(\omega,t) - c}{B_t}\eta_t > u(c^*(\omega,t),t) - u(c,t).$$

Define $c(\omega,t)$ to be such a c when it exists, and $c(t) = c^*(t)$ otherwise. Integrating, we obtain a contradiction to (7). Thus (ii) is shown, and this ends the proof. ∎

COROLLARY 13. *Suppose u is differentiable. Then condition* (ii) *of proposition 12 may be replaced with*

$$(\text{ii}') \qquad c^* = J(z_t^* \eta_t / B_t, t),$$

where $J(x,t) = \inf\{\, c \geq 0 : u_c'(c,t) \leq x \,\}$ is the "inverse" of u_c'.

PROOF: Inequality (ii) of proposition 12 implies

$$[\partial u/\partial c]_{c=c^*(t)} \leq z_t^* \eta_t / B_t$$

for all t, with equality holding whenever $c^* > 0$. In other terms $c^*(t) = J(z_t^* \eta_t / B_t, t)$, as desired. ∎

5. Concluding remarks

To interpret our results on characterization, consider the marginal utility of the dated contingent good $c(\omega, t)$. Imagine a perfect forward market at zero time in contracts, a contract being a promise to deliver one unit of the consumption good at t if the state is ω. If prices are quoted in reduced rather than natural units, the price of a promise to deliver $c(\omega, t) = c_t$ units will be proportional to

$$c_t u'_c(c_t, t) = \frac{c_t}{B_t} u'_c(c_t, t) B_t.$$

Hence, the reduced shadow price corresponding to c_t is given by the product $u'_c(c_t, t)B_t$.

Relation (ii') of corollary 13 expresses the fact that the reduced shadow price is, at the optimum, equal to the product

$$y_t = z_t^* \eta_t.$$

The process η_t is the density, in restriction to $\mathcal{F}_t$, of the measure Q mentioned earlier. It starts at $\eta_0 = 1$ and is determined solely by the market model and not by the utility function of the consumer. In chapter 1, it was interpreted as the market price of consumption in units of the bond per unit of probability P. On the second hand, the process z^* depends on the real decision on the part of the investor at time t. Unlike consumption strategies usually evinced in the economics literature, this decision is continually readjusted and depends on the investor's preferences and endowment. We interpret in turn z^* as the shadow value of the non negative wealth constraint, in other terms, as the marginal utility of wealth in the dynamic programming solution; cf. He and Pearson [**13**] for a similar treatment. Condition (i) of proposition 12 expresses the fact that an optimal consumption plan is one which is "supported" by the price z^* which it defines—in other words, one which maximizes the total value $\widetilde{\mathrm{E}} \int_0^1 z_t^*(c_t/B_t)\,dt$ calculated at these prices; cf. Foldes [**11**, theorem 4].

An important question arises of whether the principle of equimarginal utility holds, in the sense that y_t is a martingale. In other words, one would like to know if marginal utility obeys a random walk, as in Hall [**12**], when consumers maximize expected future utility. As far as I know, all explicit characterizations carried out in models of the same spirit as ours are always conducive to $z_t^* =$ constant, i.e., since η is a martingale, to the principle of

equimarginal utility. Of course, these models either do not incorporate an endowment in the budget constraint (Cox and Huang [4], [5] or Karatzas, Lehoczky and Shreve [16]) or do not have an explicit liquidity constraint of the type $W_t \geq 0$ (Hall [12], Kotlikoff and Pakes [18]). It is worth wondering what would happen to our characterization if we indexed the perturbed problems with a constant $\lambda \in R$, rather than a process $z \in L^1(\nu)$. Arguments similar in every respect to those given in the last section yield in that case the inequality

$$\mathrm{E}\int_0^1 (u(c_t^*,t) - u(c,t))\,dt \geq \lambda^* \left(k - \sup_{\tau \in T} \widetilde{\mathrm{E}}\left[\int_0^\tau \frac{c-e}{B}\,dt\right]\right)$$

for some positive λ^* and all $c \in \mathcal{G}$. Now let $H = 0$, i.e., let there be no endowment. In this case the above inequality becomes

$$\mathrm{E}\int_0^1 (u(c_t^*,t) - u(c,t))\,dt \geq \lambda^* \widetilde{\mathrm{E}}\left[\frac{c_t^* - c_t}{B_t}\,dt\right],$$

from which we obtain $z_t^* = \lambda^* = \text{constant}$.

The issue of equimarginal utility is of central importance in economic theory. It is also called the Martingale property by Lucas [20, section 8]. In the life cycle hypothesis, it leads to the tenet of consumption expenditures smoothing. In a seminal paper, Hall [12] uses the principle of equimarginal utility to distinguish the life cycle hypothesis and permanent income theory from alternative theories. In this respect excessive sensitivity to anticipated changes in income is sometimes related to the degree to which consumers are liquidity constrained. In other, more recent versions of the neoclassical models of consumption choice, equimarginal utility is used to construct tests which unambiguously reject the implications of the life cycles hypothesis; cf. Kotlikoff and Pakes [18]. Diamond and Hausman [7] show that a large fraction of households are net savers after they retire, a result quite at odds with the life cycle hypothesis.

The model suggests that when a consumer is faced with income uncertainty under a liquidity constraint of the type $W_t \geq 0$, the extremely low intertemporal substitution effects found in the empirical literature might be ascribed to the irrelevance of the law of equimarginal utility. Though it is perhaps unreasonable to expect to get a closed form solution for z^*, future research should try to give further information on the optimal paths. One obvious choice is stochastic dynamic programming. Since there exist now well

developed numerical analysis techniques to tackle non linear partial differential equations (cf. for instance the Expert system due to Quadrat [23]), at least numerical results should obtain with respect to the shape of the value function for different characteristics of labor income uncertainty.

Another issue related to the debate over the life cycle hypothesis deals with the role of intergenerational transfers. The main competing approach to the LCH takes the bequest motive as a force of major importance. In sharp contrast to Modigliani [21], [22], Kotlikoff and Summers [19] argue that wealth that can be ascribed to inheritances or gifts inter vivos accounts for up to 80% of total existing assets, wealth accumulated by the life cyclers representing only 20%. In view of our model, it is interesting to ask whether or not it is possible that, at the optimum, $W_1 \neq 0$. The answer is negative, as is shown in the following proposition.

PROPOSITION 14. *If there is no satiation, the final wealth corresponding to the optimal consumption plan is almost surely zero.*

PROOF: The optimal consumption plan satisfies

$$\sup_{\tau \in \mathcal{T}} \widetilde{\mathrm{E}} \left[\int_0^\tau \frac{c_s^* - e_s}{B_s} \, ds \right] = k,$$

where k is the initial wealth of the consumer. Let for simplicity $C_t = \int_0^t (c^*/B_s)\, ds$, $\xi_t = C_t - H_t$ and $Z_0 = \sup_{\tau \in \mathcal{T}} \widetilde{\mathrm{E}}[C_\tau - H_\tau]$. The Snell envelope of ξ can be written as $Z_t = Z_0 + M_t' - A_t$, with $\xi_1 = Z_1$, as shown in the proof of lemma 3. We now pose $W_t' = Z_t - \xi_t + k - Z_0$. Note that $W_t' \geq k - Z_0$. A simple argument shows that necessarily $Z_0 = k$ (the budget constraint is saturated). It develops that

$$W_t' + C_t + A_t = k + M_t' + H_t,$$

where by a result of El Karoui [10, theorem 2], A is absolutely continuous with respect to the Lebesgue measure. Hence if C is optimal, it must be that $A_1 = 0$. Thus $\xi_1 = Z_1 = k + M_1'$ and $\widetilde{\mathrm{E}}\, \xi_1 = k$. But the budget constraint is

$$C_1 + W_1 = k + H_1 + M_1,$$

where M is the local martingale of capital gains. Taking expectations under Q, we get $\tilde{\mathrm{E}}[\xi_1 + W_1] = k + \tilde{\mathrm{E}}\, M_1$ and so $\tilde{\mathrm{E}}\, W_1 \leq 0$. This proves $W_1 = 0$, as desired. ∎

To conclude, we would like to stress some weaknesses of this paper. The martingale representation technique requires as usual completeness of the markets. This is annoying, because we would like to deal with uncertain incomes which are not in the filtration generated by the securities price system. Second, we have not provided explicit solutions for a class of special models, as the ones presented in the foregoing chapters. The application of our model to specific simple cases awaits future research.

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