Tilting sheaves for real groups and Koszul duality

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ABSTRACT

For a certain class of real analytic varieties with the real Lie group action we define a t-structure on the category of equivariant-monodromic sheaves and develop the theory of tilting sheaves. In case of a quasi-split real form of an algebraic group acting on the flag variety we construct an analog of a Soergel functor, which fully-faithfully embeds the subcategory of tilting objects to the category of coherent sheaves on a block variety. We apply the results to give a new, purely geometric, proof of the Soergel's conjecture for quasi-split groups. The thesis is based on a joint work with Zhiwei Yun.

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1. Introduction

The present thesis is based on the joint work with Zhiwei Yun.

The formalism of Koszul duality in representation theory were introduced in [21] and [5] and were further developed in subsequent works. Since then it was one of the most important concepts and tools. The classical results establish that the category $\mathcal{O}$ of representations is controlled by a quadratic Koszul self-dual algebra. The category $\mathcal{O}$ of representations is well known to be equivalent to the category of Harish-Chandra bimodules as well as the certain category of perverse sheaves or D-module over the flag variety of the corresponding algebraic group. Such duality sends irreducible objects of the category to projective objects.

The central step of this theory is constructing the aforementioned Koszul algebra. The key ingredient of the original approaches is the category of the Soergel bimodules, which forms a full subcategory of the category of coherent sheaves on the so called block variety $\mathfrak{t}^* \times_{\mathfrak{t}^*/W} \mathfrak{t}^*$, where $\mathfrak{t}$ is the abstract Cartan subalgebra of the Lie algebra of the algebraic group and W is its Weyl group. There is a Soergel functor $\mathbb{V}$ between category $\mathcal{O}$ and the Soergel bimodules category. Its essential property that allows the construction of the model algebra is that it is fully faithful on the additive subcategory of projective objects.

Another important duality namely the Ringel duality were introduced in [19]. In the context of the highest weight categories this is an equivalence of categories, which sends standard objects to costandard objects and projective objects to tilting objects. In the geometric setting the theory of tilting perverse sheaves were developed in [3]. In particular, it is proved that the long intertwining functor provide a Ringel self-duality of the category of the perverse sheaves on the flag variety of the algebraic group. The composition of these Koszul duality and Ringel duality were first studied in [4]. It sends irreducible objects to the tilting objects and, moreover, preserves the convolution monoidal structure on the category. The last property also stands for the composition of the Soergel functor $\mathbb{V}$ with the Ringel duality.

The category of free-monodromic perverse sheaves was considered in [9] to establish the dualities for Kac-Moody groups and the techniques were further developed in [7]. In these papers the authors work with the tilting version of Koszul duality mentioned above.

In [22] W. Soergel conjectured the Koszul duality for real groups. In [8] it was verified for quasi-split groups. Namely, let $G_{\mathbb{R}}$ be a quasi-split real semisimple algebraic group and let $\mathcal{M}$ be a block of its representations. Let $\check{G}$ be the Langlands dual group of the complexification of $G_{\mathbb{R}}$. Passing through Vogan's duality ([25]) one associates to $\mathcal{M}$ a full subcategory $\check{\mathcal{D}}$ in the $\check{K}$-equivariant derived category of the flag variety of $\check{G}$ for a certain $\check{K} \subset \check{G}$.

Theorem 1.1. *(*[8, Theorem 5.1]*). There is a Koszul duality equivalence of triangulated categories $D^b(\mathcal{M}) \xrightarrow{\sim} \check{\mathcal{D}}$.*

Once again an important role is played by the analog of the Soergel functor. In [8] it is constructed in representation theoretic terms via the translation functor to the singular central character and the real block variety is $\mathfrak{a}^*/W'_{\mathcal{M}} \times_{\mathfrak{t}^*/W} \mathfrak{t}^*$, where $\mathfrak{a}$ is the complexification of the Lie algebra of the maximal split torus of $G_{\mathbb{R}}$ and $W'_{\mathcal{M}} \subset W$ is a certain subgroup of the Weyl group associated to $\mathcal{M}$.

The last (but not the least) result that we bring in is the Matsuki correspondence for sheaves proved in [18]. The result states that in the above notations we have an equivalence between the derived categories of K-equivariant and $G_{\mathbb{R}}$-equivariant constructible sheaves on the flag variety of G. The same holds for the categories of monodromic constructible sheaves. Notably, the equivalence takes perverse standard objects to costandard objects up to shift and in the special case of $G_{\mathbb{R}}$ being a complex Lie group coincides with the long intertwining endofunctor.

1.2. **In the present paper.** We put together the ideas above in the following way. For a certain class of spaces X with a left action of a real group $G_{\mathbb{R}}$ and a right action of a torus T we define a t-structure on the category $\mathcal{M}_{G_{\mathbb{R}},X} := \widehat{D}^b_{G_{\mathbb{R}}}(X)_{T-\mathrm{mon}}$ of $G_{\mathbb{R}}$-equivariant, T-free-monodromic sheaves constructible with respect to $G_{\mathbb{R}}$-orbit stratification. It is defined by a perversity function, which in a way generalizes the middle perversity function defined only for complex stratifications, but takes into account not only the dimension of the stratum but also the size of its fundamental group. We then develop the theory of tilting sheaves for the heart of this t-structure.

We consider the special case associated to the real form $G_{\mathbb{R}}$ of a connected semisimple complex Lie group G. Let $T, U \subset G$ be a maximal torus and a maximal unipotent subgroup inside the same Borel subgroup. We put $X = G/U$ for the enhanced flag variety of G. We have a right $G_{\mathbb{R}}$ and left T-action on X, for which we check that it fits into our general setting. The verification involves the results and constructions of [18]. Moreover, we observe that

Theorem 1.3. *(Theorem 5.6) The Matsuki correspondence for sheaves of* [18] *is a Ringel duality between the category of perverse K-equivariant T-free-monodromic sheaves on the enhanced flag variety X and the heart of the t-structure on $\mathcal{M}_{G_{\mathbb{R}},X}$.*

We then study the convolution properties of the tilting sheaves in $\mathcal{M}_{G_{\mathbb{R}},X}$ under the action of the free-monodromic Hecke category.

It is worth noting that the common approach to the representation theory of $G_{\mathbb{R}}$ passes to the Harish-Chandra $(\mathfrak{g}, K)$-modules, which could be studied by algebraic methods. Similarly, in geometry it is more common to look at the orbits of K rather then orbits of $G_{\mathbb{R}}$, partially because the perverse t-structure is a well-developed tool in the K-setting, while in $G_{\mathbb{R}}$-setting the t-structure was missing until now. Our approach works with the real group more directly and provides a more direct geometric flavour to the representation theory of $G_{\mathbb{R}}$. The category $\mathcal{M}_{G_{\mathbb{R}},X}$ is related to the category of $G_{\mathbb{R}}$-representations by globalization functors of [16] (see also [23]), so the developed theory could be transferred to the representation theoretic setting. The constructed t-structure for the real group orbit stratifications is also of independent geometric interested and should be further investigated as we hope it should be applicable in various other situations.

Assume further that $G_{\mathbb{R}}$ is quasi-split. In this case we define a real Soergel functor $\mathbb{V}_{\mathbb{R}}$ as a generic vanishing cycles functor to the closed $G_{\mathbb{R}}$-orbit. We check the compatibility between the convolution action of the Hecke category and the functors $\mathbb{V}_{\mathbb{R}}$ and the classical Soergel functor $\mathbb{V}$. This allows us to prove a generalization of Struktursatz and Endomorphismensatz of [21]. Namely, let $\mathcal{B}$ be the algebra of functions on the union of completions at the origin of block varieties $\mathfrak{a}^*/W'_{\mathcal{M}} \times_{\mathfrak{t}^*/W} \mathfrak{t}^*$ for all different blocks. For simplicity here we give formulation of our result only with the additional assumption of G being adjoint.

Theorem 1.4. *(Theorem 9.2,Theorem 9.17). 1) The algebra of endomorphisms of the functor $\mathbb{V}_{\mathbb{R}}$ restricted to the full additive subcategory of tilting sheaves $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}},X})$ is isomorphic to $\mathcal{B}$.*

2) The functor $\mathbb{V}_{\mathbb{R}} \colon \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}},X}) \to \mathcal{B}-\mathrm{mod}$ is fully-faithful.

This theorem then allows us to reprove Theorem 1.1. We hope that our techniques and results could be further generalized in order to prove Soergel's conjecture in the general case, with the quasi-splitness assumption dropped. The key tool used in the proof of Theorem 1.4 is the technique of the localization of the free-monodromic categories, which we develop. It could be viewed as parallel to the equivariant localization (see [12]).

In Section 2 we recall some background on complex Lie groups with antiholomorphic involution. In Section 3 we recall some background on completed monodromic categories with focus on real analytic setting. In Section 4 we study real equivariant-monodromic sheaves in general case, define the t-structure on them and develop the tilting theory. In Section 5 we study the tilting sheaves on flag varieties with the real orbit stratifications. In Section 6 we investigate the relationships with the Hecke action. In Section 7 we develop the technique of the localization of the free-monodromic categories. In Section 8 in the case of a quasi-split real form we define the real Soergel functor. In Section 9 we proof our version of Struktursatz and Endomorphismensatz. In Section 10 for a quasi-split real form we construct an explicit dg-model for the category $\mathcal{M}_{G_\mathbb{R},X}$, which allows for the proof the Soergel conjecture.

1.5. **Conventions and notations.** Let $\mathbf{k}$ be an algebraically closed field. All sheaves will be assumed to be the sheaves of $\mathbf{k}$-vector spaces. All (co)homology groups are taken with $\mathbf{k}$ coefficients unless otherwise stated.

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2. Complex Lie groups with an antiholomorphic involution

2.1. **Abstract Cartan and abstract Weyl group.** Let G be a connected reductive complex Lie group. Let X be the flag variety of G. Let $\mathbf{W}$ be the abstract Weyl group of G. As a set, it is defined as the set of G-orbits on $X \times X$. For $w \in \mathbf{W}$ let $X^2_w \subset X \times X$ be the corresponding G-orbit. Simple reflections in $\mathbf{W}$ are those $s \in \mathbf{W}$ such that $\dim X^2_w = \dim X + 1$. When $(B, B') \in X^2_w$, we write $pos(B, B') = w$.

Consider the space Y of pairs $T \subset B$ where T is a maximal torus of G and B is a Borel subgroup containing it. Let $\mathscr{T}$ be the space of maximal tori in G. If we choose a maximal torus $T \subset G$, we may identify Y with G/T and $\mathscr{T}$ with $G/N_G(T)$. We get the following diagram where the maps are forgetting T or B:

$$X \xleftarrow{\beta} Y \xrightarrow{\gamma} \mathscr{T} \tag{2.1}$$

Both maps β, γ are G-equivariant. For each $(T \subset B) \in Y$ and $w \in \mathbf{W}$, there is a unique $(T \subset B^w) \in Y$ such that $pos(B, B^w) = w$. This defines a group structure on $\mathbf{W}$ so that γ becomes a G-equivariant $\mathbf{W}$-torsor.

For different choices of Borel subgroups B and B' of G, their reductive quotients are canonically identified, which we call the abstract Cartan $\mathbf{T}$ of G. There is a canonical right action of $\mathbf{W}$ characterized as follows: for any $(T \subset B) \in Y$, and $w \in \mathbf{W}$, the following diagram is commutative

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d, equal] & B \arrow[r, "can"] & \mathbf{T} \arrow[d, "(-)w"] \\ T \arrow[r, hook] & B^w \arrow[r, "can"] & \mathbf{T} \end{tikzcd} \tag{2.2}$$

where the maps "can" are the canonical quotients.

For each $(T \subset B) \in Y$ we have the based root system $\Phi(G, B, T)$ where positive roots are those appearing in B. For different choices of $(T \subset B) \in Y$ these based root systems are canonically identified with one another. We denote the resulting canonical based root system by Φ. It can be viewed as a based root system for the abstract Cartan $\mathbf{T}$ with Weyl group $\mathbf{W}$. Let $\underline{\Phi}$ be the underlying root system of Φ (i.e., without the basis).

2.2. **Real form.** Let $\sigma \colon G \to G$ be an antiholomorphic involution on G compatible with the group structure. We put $G_{\mathbb{R}} = G^\sigma$ for the corresponding real form. Put $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}(G_{\mathbb{R}}) = \mathfrak{g}^\sigma$.

Lemma 2.3.
(1) *Each Borel subgroup $B \subset G$ contains a σ-stable maximal torus T.*
(2) *Any two σ-stable maximal tori in B are conjugate under $G_{\mathbb{R}} \cap B$.*

Proof. (1) The subgroup $H := B \cap \sigma(B)$ of G is stable under σ, hence is the complexification of a real group $H_{\mathbb{R}} \subset G_{\mathbb{R}}$. Note that H is solvable and contains a maximal torus of G. By [10, Proposition 7.10], H contains a maximal torus T defined over $\mathbb{R}$, then $T \subset H \subset B$ is σ-stable and is a maximal torus of G.

(2) If T, T' are two σ-stable maximal tori in B, then they are both in H, hence they are conjugate by some $u \in H^u$ (the unipotent radical of H). This implies $u^{-1}\sigma(u) \in N_H(T) \cap H^u = \{1\}$ hence $u \in H^u_{\mathbb{R}} \subset G_{\mathbb{R}} \cap B$. $\square$

2.4. **Real orbits on the flag variety.** Let I be the set of $G_{\mathbb{R}}$-orbits on X by left translation. For $\lambda \in I$, we denote the corresponding orbit by $O^{\mathbb{R}}_{\lambda}$, so that

$$X = \bigcup_{\lambda \in I} O^{\mathbb{R}}_{\lambda}. \tag{2.3}$$

Let $\mathscr{T}^{\sigma} \subset \mathscr{T}$ be the set of σ-stable maximal tori in G. Let $Y_{\sigma} \subset Y = \gamma^{-1}(\mathscr{T}^{\sigma})$ whose points are pairs $(T \subset B) \in Y$ where T is σ-stable. Both $\mathscr{T}^{\sigma}$ and Y_{σ} carry left actions of $G_{\mathbb{R}}$ by conjugation. We have the following diagram where the maps are forgetting T or B:

$$X \xleftarrow{\beta_{\sigma}} Y_{\sigma} \xrightarrow{\gamma_{\sigma}} \mathscr{T}^{\sigma} \tag{2.4}$$

Both maps $\beta_{\sigma}, \gamma_{\sigma}$ are $G_{\mathbb{R}}$-equivariant. The map γ_{σ} is a $G_{\mathbb{R}}$-equivariant $\mathbf{W}$-torsor.

Lemma 2.5. (1) *The map $\beta_{\sigma} : Y_{\sigma} \to X$ is surjective and it induces a bijection on $G_{\mathbb{R}}$-orbits $\underline{\beta}_{\sigma} : G_{\mathbb{R}} \backslash Y_{\sigma} \leftrightarrow G_{\mathbb{R}} \backslash X$.*

(2) *The map $\gamma_{\sigma} : Y_{\sigma} \to \mathscr{T}^{\sigma}$ is a $\mathbf{W}$-torsor. It induces a surjective map $\underline{\gamma_{\sigma}} : G_{\mathbb{R}} \backslash Y_{\sigma} \leftrightarrow G_{\mathbb{R}} \backslash \mathscr{T}^{\sigma}$ whose fibers are $\mathbf{W}$-orbits.*

(3) *The right $\mathbf{W}$ action on $G_{\mathbb{R}} \backslash Y_{\sigma}$ defines a right $\mathbf{W}$-action on $I = G_{\mathbb{R}} \backslash X$ via the bijection $\underline{\beta_{\sigma}}$, and the composition $\underline{\gamma_{\sigma}} \circ \underline{\beta_{\sigma}}^{-1} : I \to G_{\mathbb{R}} \backslash \mathscr{T}^{\sigma}$ is the quotient map by $\mathbf{W}$. We denote the right $\mathbf{W}$-action on I by $\lambda \mapsto \lambda \cdot w$ ($\lambda \in I$, $w \in W$).*

Proof. (1) follows from Lemma 2.3. (2) and (3) are clear. $\square$

Lemma 2.6. *Let $\lambda \in I, B \in O^{\mathbb{R}}_{\lambda}$ and $T \subset B$ be a σ-stable maximal torus. Consider the isomorphism of tori*

$$\iota_B : T \subset B \twoheadrightarrow \mathbf{T}. \tag{2.5}$$

We have:

(1) *The real structure $\sigma|_T$ induces via ι_B a real structure $\sigma_{T \subset B}$ (anti-holomorphic involution) on $\mathbf{T}$. Then $\sigma_{T \subset B}$ depends only on the orbit λ. We denote it by σ_{λ}.*

(2) *The real points $\mathbf{T}^{\sigma_{\lambda}}$ under σ_{λ} is the image of the canonical projection $G_{\mathbb{R}} \cap B \subset B \twoheadrightarrow \mathbf{T}$ (which is then independent of $B \in O^{\mathbb{R}}_{\lambda}$).*

Proof. (1) By Lemma 2.5(1), any two such $(T \subset B)$ (with $B \in O_\lambda^{\mathbb{R}}$) are $G_{\mathbb{R}}$-conjugate. For $g \in G_{\mathbb{R}}$, we have a commutative diagram

$$\begin{tikzcd} T \arrow[r, "\iota_B"] \arrow[d, "\mathrm{Ad}(g)"] & \mathbf{T} \arrow[d, equal, "\mathrm{id}"] \\ \mathrm{Ad}(g)T \arrow[r, "\iota_{\mathrm{Ad}(g)B}"] & \mathbf{T} \end{tikzcd} \tag{2.6}$$

From this we conclude that $\sigma_{T\subset B}$ is the same as $\sigma_{\mathrm{Ad}(g)T\subset \mathrm{Ad}(g)B}$.

(2) We use the notation H from the proof of part (1) of Lemma 2.3. We have $G_{\mathbb{R}} \cap B = H_{\mathbb{R}}$. Moreover, $T_{\mathbb{R}}$ is a maximal torus in the solvable real group $H_{\mathbb{R}}$. Since the kernel of the projection $H_{\mathbb{R}} \to \mathbf{T}$ is unipotent, its image is the reductive quotient of $H_{\mathbb{R}}$. Therefore $T_{\mathbb{R}}$ maps isomorphically to the image of $H_{\mathbb{R}} \to \mathbf{T}$. By definition, $T_{\mathbb{R}}$ also maps isomorphically via ι_B to $\mathbf{T}^{\sigma_\lambda}$. The statement follows. $\square$

Definition 2.7. *Let T be a σ-stable maximal torus of G with real points $T_{\mathbb{R}}$. We say that an orbit $O_\lambda^{\mathbb{R}}$ is* attached *to T (or $T_{\mathbb{R}}$) if $\underline{\gamma_\sigma} \circ \underline{\beta_\sigma}$ maps λ to the $G_{\mathbb{R}}$-orbit of T. In other words, $O_\lambda^{\mathbb{R}}$ is attached to T if there exists a T-fixed point in $O_\lambda^{\mathbb{R}}$.*

2.8. **Roots.** Fix a σ-stable maximal torus $T \subset G$. Let $\Phi(G,T)$ be the set of roots of G with respect to T.

Note that σ atcs on the set of roots $\Phi(G,T)$: if $\alpha \in \Phi(G,T)$ viewed as a homomorphism $T \to \mathbb{G}_m$ over $\mathbb{C}$, then $\sigma\alpha : T \to \mathbb{G}_m$ is defined as $t \mapsto \overline{\alpha(\sigma t)}$.

Definition 2.9. *A root $\alpha \in \Phi(G,T)$ is called*

1. *complex if $\sigma\alpha \neq \pm\alpha$;*
2. *real if $\sigma\alpha = \alpha$;*
3. *compact imaginary if $\sigma\alpha = -\alpha$ and for nonzero $x \in \mathfrak{g}_\alpha$ the Cartan-Killing pairing between x and $\sigma(x)$ is negative.*
4. *noncompact imaginary if $\sigma\alpha = -\alpha$ and for nonzero $x \in \mathfrak{g}_\alpha$ the Cartan-Killing pairing between x and $\sigma(x)$ is positive.*

Remark 2.10. This definition is compatible with the definition for a root system invariant under the corresponding Cartan involution (see, for example, [25, Section 2].

2.11. **Based root system attached to a real orbit.** Recall we have the absrtact based root system Φ on $\mathbf{T}$ with Weyl group $\mathbf{W}$. To each point $(T \subset B) \in Y_\sigma$ we have a canonical isomorphism of based root systems $\Phi(G,B,T) \cong \Phi$, under the isomorphism $T \subset B \twoheadrightarrow \mathbf{T}$. Since T is σ-stable, σ acts on the root system $\Phi(G,T)$ without necessarily preserving the positive roots. In particular, we get an involution σ on the underlying root system $\underline{\Phi}$ of Φ. The assignment $Y_\sigma \ni (T \subset B) \mapsto \mathrm{Inv}(\underline{\Phi})$

(the set of involutions on $\underline{\Phi}$) is $G_{\mathbb{R}}$-invariant, hence it induces a map $G_{\mathbb{R}}\backslash Y_\sigma \to \mathrm{Inv}(\underline{\Phi})$. Using the bijection $\underline{\beta}_\sigma$ in Lemma 2.5, we get a map $G_{\mathbb{R}}\backslash X = I \to \mathrm{Inv}(\underline{\Phi})$. For $\lambda \in I$, we denote by Φ_λ the based root system Φ equipped with the involution constructed above on $\underline{\Phi}$.

For $\alpha \in \Phi_\lambda$, we can talk about whether it is real, complex or imaginary according to Definition 2.9.

For a simple root $\alpha \in \Phi$, let X_α be the partial flag variety parametrizing parabolic subgroups conjugate to P_α (generated by a Borel B and root subgroup of $-\alpha$). Let $\pi_\alpha \colon X \to X_\alpha$ be the projection which is a $\mathbb{P}^1$-fibration.

There is a partial order on I: $\mu \le \lambda$ if and only if $O^{\mathbb{R}}_\mu \subset \overline{O^{\mathbb{R}}_\lambda}$.

The following statement is analogous to the results of [24, Lemma 5.1] and [20, Sections 2.2, 2.3].

Lemma 2.12. *Let $\lambda \in I$. Let $\alpha \in \Phi_\lambda$ be a simple root and $s_\alpha \in \mathbf{W}$ be the corresponding simple reflection.*

(1) *If α is a complex root, then $\lambda \cdot s_\alpha \neq \lambda$, and $\pi_\alpha^{-1}\pi_\alpha(O^{\mathbb{R}}_\lambda) = O^{\mathbb{R}}_\lambda \cup O^{\mathbb{R}}_{\lambda\cdot s_\alpha}$.*
   - *If $\sigma\alpha > 0$, then $\lambda < \lambda \cdot s_\alpha$ and $\pi_\alpha | O^{\mathbb{R}}_\lambda$ is an isomorphism onto its image.*
   - *If $\sigma\alpha < 0$, then $\lambda \cdot s_\alpha < \lambda$ and $\pi_\alpha | O^{\mathbb{R}}_\lambda$ is an $\mathbb{A}^1$-fibration over its image.*

(2) *If α is a real root, then $\lambda\cdot s_\alpha = \lambda$. Moreover, one of the following happens:*
   - *Type I: there are two orbits $\mu^+, \mu^- > \lambda$ such that $\mu^- = \mu^+ \cdot s_\alpha$, and $\pi_\alpha^{-1}\pi_\alpha(O^{\mathbb{R}}_\lambda) = O^{\mathbb{R}}_\lambda \cup O^{\mathbb{R}}_{\mu^+} \cup O^{\mathbb{R}}_{\mu^-}$. Moreover, $\pi_\alpha | O^{\mathbb{R}}_\lambda$ is an S^1-fibration, and $\pi_\alpha | O^{\mathbb{R}}_{\mu^+}$ and $\pi_\alpha | O^{\mathbb{R}}_{\mu^-}$ are D^2-fibrations (D^2 is an open real 2-dimensional disk).*
   - *Type II: there is $\mu > \lambda$ such that $\mu \cdot s_\alpha = \mu$ and $\pi_\alpha^{-1}\pi_\alpha(O^{\mathbb{R}}_\lambda) = O^{\mathbb{R}}_\lambda \cup O^{\mathbb{R}}_\mu$, and $\pi_\alpha | O^{\mathbb{R}}_\lambda$ is an S^1-fibration over its image.*

(3) *If α is compact imaginary, then $\lambda\cdot s_\alpha = \lambda$ and $\pi_\alpha^{-1}\pi_\alpha(O^{\mathbb{R}}_\lambda) = O^{\mathbb{R}}_\lambda$.*

(4) *If α is noncompact imaginary, then there is a unique $\mu < \lambda$ with $\mu \cdot s_\alpha = \mu$ such that one of the following happens:*
   - *Type I: $\lambda \cdot s_\alpha \neq \lambda$, $\mu < \lambda \cdot s_\alpha$, and $\pi_\alpha^{-1}\pi_\alpha(O^{\mathbb{R}}_\lambda) = O^{\mathbb{R}}_\lambda \cup O^{\mathbb{R}}_{\lambda\cdot s_\alpha} \cup O^{\mathbb{R}}_\mu$. Moreover, $\pi_\alpha | O^{\mathbb{R}}_\mu$ is an S^1-fibration, and $\pi_\alpha | O^{\mathbb{R}}_\lambda$ and $\pi_\alpha | O^{\mathbb{R}}_{\lambda\cdot s_\alpha}$ are D^2-fibrations (D^2 is an open real 2-dimensional disk).*
   - *Type II: $\lambda \cdot s_\alpha = \lambda$, $\pi_\alpha^{-1}\pi_\alpha(O^{\mathbb{R}}_\lambda) = O^{\mathbb{R}}_\lambda \cup O^{\mathbb{R}}_\mu$, and $\pi_\alpha | O^{\mathbb{R}}_\mu$ is an S^1-fibration over its image.*

Proof. See *loc.cit.* ∎

2.13. **Real Weyl groups.** Let T be a σ-stable maximal torus. Let $T_{\mathbb{R}} = T^{\sigma}$. We denote the Weyl group of T by W, which carries an action of σ. Let $W_{\mathbb{R}} = W^{\sigma} \subset W$ be the fixed point subgroup of σ. On the other hand we have the Weyl group $W(G_{\mathbb{R}}, T_{\mathbb{R}}) = N_{G_{\mathbb{R}}}(T_{\mathbb{R}})/T_{\mathbb{R}}$. Clearly we have $W(G_{\mathbb{R}}, T_{\mathbb{R}}) \subset W_{\mathbb{R}}$.

Lemma 2.14. (1) *We have* $W_{\mathbb{R}} = \mathrm{Stab}_W(T_{\mathbb{R}})$.

(2) *Suppose T is a σ-stable maximal torus that is maximally split, then the inclusion $W(G_{\mathbb{R}}, T_{\mathbb{R}}) \subset W_{\mathbb{R}}$ is an equality.*

Proof. (1) The inclusion $W_{\mathbb{R}} \subset \mathrm{Stab}_W(T_{\mathbb{R}})$ is clear. Note that W commutes with the real involution σ_c, whose fixed points are the maximal compact subgroup. Then $W_{\mathbb{R}}$ also commute with the Cartan involution $\theta = \sigma\sigma_c$ and we have $\mathrm{Stab}_W(T^{\theta}) = \mathrm{Stab}_W(T_{\mathbb{R}})$. For the opposite inclusion it is sufficient to check that $\mathrm{Stab}_W(T^{\theta})$ preserves the eigenspace decomposition of θ on $\mathfrak{t}$. By definition it preserves the $+1$-eigenspace. It is also compatible with the Cartan-Killing form and, hence, also preserves the -1-eigenspace as it equals to the orthogonal of $\mathfrak{t}^{\theta}$.

(2) follows from [25, Propositions 3.12 and 4.16] as there are no noncompact imaginary roots for maximally split torus. $\square$

If $T \in \mathscr{T}^{\sigma}$ and B is a Borel subgroup containing T, we get a canonical identification $T \cong \mathbf{T}$ and $W \cong \mathbf{W}$ compatible with the actions.

Lemma 2.15. *In the above situation, suppose $B \in O_{\lambda}^{\mathbb{R}}$. Then under the isomorphism $W \cong \mathbf{W}$ (induced by B), $W(G_{\mathbb{R}}, T_{\mathbb{R}})$ is identified with the stabilizer $\mathbf{W}_{\lambda}$ of λ under the right action of $\mathbf{W}$ on I.*

Proof. Let $\iota : W \xrightarrow{\sim} \mathbf{W}$ be the canonical isomorphism. If $\dot{w} \in N_{G_{\mathbb{R}}}(T_{\mathbb{R}})$ has image $w \in W$, then B and $\mathrm{Ad}(\dot{w})B$ are both in $O_{\lambda}^{\mathbb{R}}$ and both contain T. By definition $\iota(w) = pos(B, \mathrm{Ad}(\dot{w})B)$. By the definition of the $\mathbf{W}$-action, we see that $\lambda \cdot \iota(w) = \lambda$. This proves $\iota(W(G_{\mathbb{R}}, T_{\mathbb{R}})) \subset \mathbf{W}_{\lambda}$.

Conversely, suppose $v \in \mathbf{W}$ is such that $\lambda \cdot v = \lambda$, then B^v (the unique Borel containing T such that $pos(B, B^v) = v$) lies in $O_{\lambda}^{\mathbb{R}}$. Therefore there exists $g \in G_{\mathbb{R}}$ such that $B^v = \mathrm{Ad}(g)B$. Now T and $\mathrm{Ad}(g)T$ are both σ-stable maximal tori in B^v, by Lemma 2.3, there exists $h \in G_{\mathbb{R}} \cap B^v$ such that $\mathrm{Ad}(hg)T = T$, i.e., $hg \in G_{\mathbb{R}} \cap N_G(T) = N_{G_{\mathbb{R}}}(T_{\mathbb{R}})$. Since $\mathrm{Ad}(hg)B = \mathrm{Ad}(h)(B^v) = B^v$, we see that the image w of hg in W satisfies $\iota(w) = v$. Therefore $v \in \iota(W(G_{\mathbb{R}}, T_{\mathbb{R}}))$. This finishes the proof. $\square$

3. Completed monodromic category

3.1. **The setup.** Let X be a real analytic variety (for example the real points of a scheme of finite type over $\mathbb{R}$). Let T^c be a *compact* torus. Let $\pi : \widetilde{X} \to X$ be a principal right T^c-bundle.

Let H be a Lie group with an analytic action on $\widetilde{X}$ from the left commuting with T^c. Then there is an induced H-action on X such that π is H-equivariant.

In [9, Appendix A], a completed monodromic category is introduced in the context of $\overline{\mathbb{Q}}_\ell$-sheaves over stratified schemes over a field. In [7], this construction has been adapted to the topological contex allowing an arbitrary coefficient field $\mathbf{k}$. The rough idea in both cases is to take certain pro-objects in $D^b(\widetilde{X}, \mathbf{k})$ that include local systems with unipotent monodromy with infinite Jordan block along the fibers of π.

For the purpose of this paper we need to extend the known construction of completion to the case of the equivariant derived category $D^b_H(\widetilde{X}, \mathbf{k})$. The new issue here is that the "size" of the monodromy will vary on different strata. We will show that the completion construction still gives a well-behaved category of sheaves.

Remark 3.2. In application, we consider $X = G/B$ to be the flag manifold of a complex reductive group G. Let $Y = G/U$ (where U is the unipotent radical of B), then $Y \to X$ is a T-torsor, where $T = B/U$ is the abstract Cartan of G. The canonical decomposition $\mathbb{C}^\times = \mathbb{R}^{>0} \times S^1$ gives a canonical decomposition $T = T^{>0} \times T^c$ where $T^{>0} = \mathbb{R}^{>0} \otimes_{\mathbb{Z}} \mathbb{X}_*(T)$ and $T^c = S^1 \otimes_{\mathbb{Z}} \mathbb{X}_*(T)$, a compact torus. We then let $\widetilde{X} = Y/T^{>0}$. Since $T^{>0}$ is contractible, the pullback functor $D^b(\widetilde{X}) \to D^b(Y)$ is fully faithful, so if we are interested only in sheaves on Y monodromic under the right T-action, we may equivalently consider sheaves on $\widetilde{X}$ monodromic under the right A-action.

3.3. **Completion.** Consider the adjoint functors

$$D^b_H(\widetilde{X}) \underset{\pi^!}{\overset{\pi_!}{\rightleftarrows}} D^b_H(X) \tag{3.1}$$

Let $D^b_H(\widetilde{X})_{T^c-\mathrm{mon}} \subset D^b_H(\widetilde{X})$ be the full subcategory generated by the image of $\pi^!$.

Let $\mathcal{R}$ be the completion of the group algebra $\mathbf{k}[\pi_1(T^c)]$ at the augmentation ideal. Equip $\mathcal{R}$ with the adic topology coming from the augmentation ideal. Then $\mathcal{R}$ is a complete regular local ring with residue field $\mathbf{k}$ and its cotangent space canonically isomorphic to $H_1(T^c, \mathbf{k}) = \pi_1(T^c) \otimes_{\mathbb{Z}} \mathbf{k}$. The monodromy action along the fibers of π gives a $\mathcal{R}$-linear structure on $D^b_H(\widetilde{X})_{T^c-\mathrm{mon}}$, namely $\mathcal{R}$ acts on the identity functor of $D^b_H(\widetilde{X})_{T^c-\mathrm{mon}}$.

Following [9, A.3], let $\widehat{D}^b_H(\widetilde{X})_{T^c-\mathrm{mon}}$ be the category of pro-objects in $D^b_H(\widetilde{X})_{T^c-\mathrm{mon}}$ indexed by positive integers. Let

$\widetilde{D}^b_H(\widetilde{X})_{T^c-\mathrm{mon}} \subset \mathrm{pro}D^b_H(\widetilde{X})_{T^c-\mathrm{mon}}$ be the full subcategory consisting of pro-objects $(\mathcal{F}_n)_{n\geq 0}$ that satisfy two conditions:

(1) (π-constancy) The pro-object $(\pi_!\mathcal{F}_n)_n \in \mathrm{pro}D^b_H(X)$ lies in the essential image of the natural functor $D^b_H(X) \to \mathrm{pro}D^b_H(X)$ consisting of constant pro-objects.

(2) (uniform boundedness) $(\mathcal{F}_n)_n$ is isomorphic to a pro-object $(\mathcal{F}'_n)_n$ in $\mathrm{pro}D^b_H(\widetilde{X})_{T^c-\mathrm{mon}}$ where each $\mathcal{F}'_n$ has perverse degrees in $[-N, N]$ for some $N > 0$ independent of n.

It is proved in [9, Theorem A.3.2] that $\widehat{D}^b_H(\widetilde{X})_{T^c-\mathrm{mon}}$ is an $\mathcal{R}$-linear triangulated category. By the π-constancy of objects in $\widehat{D}^b_H(\widetilde{X})_{T^c-\mathrm{mon}}$, we have an adjunction induced from the adjunction $(\pi_!, \pi^!)$

$$\widehat{D}^b_H(\widetilde{X})_{T^c-\mathrm{mon}} \underset{\widehat{\pi}^!}{\overset{\widehat{\pi}_!}{\rightleftarrows}} D^b_H(X) \tag{3.2}$$

It will be convenient to introduce adjunctions $(\pi_\dagger, \pi^\dagger)$ between the same categories:

$$\pi_\dagger := \widehat{\pi}_![\dim T^c], \quad \pi^\dagger := \widehat{\pi}_![-\dim T^c] \cong \widehat{\pi}^*. \tag{3.3}$$

Moreover, given an H-equivariant map $f : X \to Y$ that lifts to an H-equivariant map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ of T^c-torsors over X and Y, under the assumption that H has finitely many orbits on X and Y, the functors $\widetilde{f}^*, \widetilde{f}_*, \widetilde{f}_!, \widetilde{f}^!$ and their adjunctions induce functors $\widehat{f}^*, \widehat{f}_*, \widehat{f}_!, \widehat{f}^!$ between the completed categories $\widehat{D}_H(\widetilde{X})_{T^c-\mathrm{mon}}$ and $\widehat{D}_H(\widetilde{Y})_{T^c-\mathrm{mon}}$.

3.4. **Situation over a point.** Consider the special case where $X = \mathrm{pt}$, $\widetilde{X} = T^c$, and $H = T^c_1$ is a closed subgroup of T^c acting on $\widetilde{X}$ by left translation. We shall give an algebraic description of the completed category $\widehat{D}^b_{T^c_1}(T^c)_{T^c-\mathrm{mon}}$.

Let $(T^c_1)^\circ$ be the neutral component of T^c_1, and let $\overline{T}^c = T^c/(T^c_1)^\circ$, the quotient torus. Then $\pi_0(T^c_1) = T^c_1/(T^c_1)^\circ$ is a finite subgroup of $\overline{T}^c$.

For each character $\chi : \pi_0(T^c_1) \to \mathbf{k}^\times$, let $\mathbf{k}_\chi$ be the one-dimensional $\mathbf{k}$-vector space with an T^c_1-equivariant structure via χ. Let $D^b_{T^c_1}(\mathrm{pt})_\chi$ be the full subcategory of $D^b_{T^c_1}(\mathrm{pt})$ consisting of objects $\mathcal{F}$ such that the action of T^c_1 on $H^i\mathcal{F}$ (an $\mathbf{k}$-vector space) is via χ (pulled back to T^c_1). Then we have a decomposition

$$D^b_{T^c_1}(\mathrm{pt}) = \bigoplus_{\chi:\pi_0(T^c_1)\to\mathbf{k}^\times} D^b_{T^c_1}(\mathrm{pt})_\chi. \tag{3.4}$$

Tensoring with $\mathcal{L}_\chi$ gives an equivalence $D^b_{T^c_1}(\mathrm{pt})_1 \xrightarrow{\sim} D^b_{T^c_1}(\mathrm{pt})_\chi$.

Similarly, let $D^b_{T^c_1}(T^c)_{T^c-\mathrm{mon},\chi}$ be the full triangulated subcategory generated by the image of $D_{T^c_1}(\mathrm{pt})_\chi$ under the pullback $\pi^! : D_{T^c_1}(\mathrm{pt}) \to D^b_{T^c_1}(T^c)_{T^c-\mathrm{mon}}$. Again we have a decomposition

$$D^b_{T^c_1}(T^c)_{T^c-\mathrm{mon}} \cong \bigoplus_{\chi:\pi_0(T^c_1)\to\mathbf{k}^\times} D^b_{T^c_1}(T^c)_{T^c-\mathrm{mon},\chi}. \tag{3.5}$$

Let $\underline{\mathbf{k}}_\chi$ be the rank one T^c_1-equivariant local system on T^c with monodromy action given by χ (whose underlying local system is trivial).

Lemma 3.5. (1) *The forgetful functor $D^b_{T^c_1}(T^c)_{T^c-\mathrm{mon}} \to D^b_{(T^c)^\circ_1}(T^c)_{T^c-\mathrm{mon}}$ induces an equivalence*

$$D^b_{T^c_1}(T^c)_{T^c-\mathrm{mon},\chi} \xrightarrow{\sim} D^b_{(T^c)^\circ_1}(T^c)_{T^c-\mathrm{mon}}. \tag{3.6}$$

(2) *Let $\sigma : T^c \to \overline{T}^c$ be the projection. Then σ^* induces an equivalence of categories*

$$\sigma^* : D^b_{(T^c_1)^\circ}(T^c)_{T^c-\mathrm{mon}} \cong D^b(\overline{T}^c)_{\overline{T}^c-\mathrm{mon}}. \tag{3.7}$$

(3) *Let $\overline{\mathcal{R}}$ be the completion of $\mathbf{k}[\pi_1(\overline{T}^c)]$ at the augmentation ideal. We have an equivalence*

$$D^b(\overline{\mathcal{R}}\text{-}\mathrm{mod}_{\mathrm{nil}}) \cong D^b(\overline{T}^c)_{\overline{T}^c-\mathrm{mon}}. \tag{3.8}$$

Here $\overline{\mathcal{R}}$-$\mathrm{mod}_{\mathrm{nil}}$ is the category of continuous $\overline{\mathcal{R}}$-modules of finite dimension over $\mathbf{k}$. It sends $M \in \overline{\mathcal{R}}$-$\mathrm{mod}_{\mathrm{nil}}$ to the local system $\mathcal{L}_M$ on $\overline{T}^c$ whose stalk at $1 \in \overline{T}^c$ is M and the monodromy representation of $\pi_1(\overline{T}^c)$ is given by the $\overline{\mathcal{R}}$-module structure on M.

(4) *Combining (1)(2)(3) we get an equivalence*

$$D^b_{T^c_1}(T^c)_{T^c-\mathrm{mon}} \cong \bigoplus_{\chi:\pi_0(T^c_1)\to\mathbf{k}^\times} D^b(\overline{\mathcal{R}}\text{-}\mathrm{mod}_{\mathrm{nil}}). \tag{3.9}$$

For a collection of finite-dimensional continuous $\overline{\mathcal{R}}$-modules $(M_\chi)_\chi$ indexed by characters $\chi : \pi_0(T^c_1) \to \mathbf{k}^\times$, the equivalence (3.9) *sends $\oplus M_\chi$ on the right side to $\oplus(\sigma^*\mathcal{L}_{M_\chi} \otimes \underline{\mathbf{k}}_\chi) \in D^b_{T^c_1}(T^c)_{T^c-\mathrm{mon}}$.*

Proof. For (3) see [9, Corollary A.4.7(1)] or [7, Lemma 4.1]. The rest of the lemma is clear. □

We also have a description of $D^b_{T^c_1}(\mathrm{pt})$ following [12] as modules over the homology algebra of the torus $(T^c_1)^\circ$. Let

$$\Lambda_\bullet = H_*((T^c_1)^\circ, \mathbf{k}) \tag{3.10}$$

as a graded algebra in degrees $0, -1, \cdots, -\dim T^c_1$.

Lemma 3.6. (1) *The forgetful functor* $D^b_{T^c_1}(\mathrm{pt}) \to D^b_{(T^c)^\circ_1}(\mathrm{pt})$ *restricts to an equivalence for each* χ*:*

$$D^b_{T^c_1}(\mathrm{pt})_\chi \xrightarrow{\sim} D^b_{(T^c_1)^\circ}(\mathrm{pt}). \tag{3.11}$$

(2) *Assume* $ch(\mathbf{k}) = 0$*. Taking the stalk induces an equivalence*

$$\mathrm{R}\Gamma(\mathrm{pt}, -) : D^b_{(T^c_1)^\circ}(\mathrm{pt}) \cong D^f_+(\Lambda_\bullet\text{-mod}). \tag{3.12}$$

Here $D^f_+(\Lambda_\bullet\text{-mod})$ *denotes the full subcategory of the bounded below derived category of dg* $\Lambda_\bullet$*-modules with cohomology finitely generated over* $\Lambda_\bullet$*.*

(3) *Assume* $ch(\mathbf{k}) = 0$*. Combining (1)(2), there is an equivalence*

$$D^b_{T^c_1}(\mathrm{pt}) \cong \bigoplus_{\chi:\pi_0(T^c_1)\to\mathbf{k}^\times} D^b(\Lambda_\bullet\text{-mod}). \tag{3.13}$$

Proof. (2) follows from [12, Theorem 11.2]. The rest of the statements are clear. □

Lemma 3.7. (1) *The equivalence* (3.9) *extends to a canonical equivalence*

$$\widehat{D}^b_{T^c_1}(T^c)_{T^c-\mathrm{mon}} \cong \bigoplus_{\chi:\pi_0(T^c_1)\to\mathbf{k}^\times} D^f(\overline{\mathcal{R}}\text{-mod}) \tag{3.14}$$

where $D^f(\overline{\mathcal{R}}\text{-mod})$ *is the bounded derived category of finitely generated continuous* $\overline{\mathcal{R}}$*-modules.*

(2) *When* $ch(\mathbf{k}) = 0$*, under the equivalences* (3.14) *and* (3.13)*, the adjunction* $(\pi_\dagger, \pi^\dagger)$ *(see* (3.3)*) can be identified with the composition of adjunctions*

$$\bigoplus_\chi D^f(\overline{\mathcal{R}}\text{-mod}) \underset{\text{forg}}{\overset{\mathbf{k}\overset{\mathbb{L}}{\otimes}_{\overline{\mathcal{R}}}(-)}{\rightleftarrows}} \bigoplus_\chi D^f(\mathbf{k}\text{-mod}) \underset{\text{forg}}{\overset{\Lambda_\bullet\otimes_{\mathbf{k}}(-)}{\rightleftarrows}} \bigoplus_\chi D^f_+(\Lambda_\bullet\text{-mod}) \tag{3.15}$$

Here both right adjoints are the forgetful functors for the ring homomorphisms $\overline{\mathcal{R}} \to \mathbf{k}$ *(augmentation) and* $\mathbf{k} \to \Lambda_\bullet$*.*

Proof. (1) Let $\sigma : T^c \to \overline{T}^c$ be the projection. Combining the equivalences in Lemma 3.5(1)(2) we have an equivalence

$$\Phi : \bigoplus_\chi D^b(\overline{T}^c)_{\overline{T}^c-\mathrm{mon}} \xrightarrow{\sim} D^b_{T^c_1}(T^c)_{T^c-\mathrm{mon}} \tag{3.16}$$

given by sending $(\mathcal{F}_\chi)_\chi$ to $\oplus\sigma^*\mathcal{F}_\chi\otimes\underline{\mathbf{k}}_\chi$. Passing to pro-objects we get an equivalence pro(Φ). We claim that pro(Φ) restricts to an equivalence of full subcategories

$$\widehat{\Phi}:\bigoplus_{\chi}\widehat{D}^b(\overline{T}^c)_{\overline{T}^c-\mathrm{mon}}\xrightarrow{\sim}\widehat{D}^b_{T^c_1}(T^c)_{T^c-\mathrm{mon}}. \tag{3.17}$$

Note here the completions on the two sides are with respect to different torus actions. Let $\mathcal{F}_\chi=(\mathcal{F}_{\chi,n})_{n\geq 0}\in \mathrm{pro}D^b(\overline{T}^c)_{\overline{T}^c-\mathrm{mon}}$. We need to show that each $\mathcal{F}_\chi$ satisfies the two conditions defining $\widehat{D}^b(\overline{T}^c)_{\overline{T}^c-\mathrm{mon}}$ if and only if the pro-object $\oplus_\chi \underline{\mathbf{k}}_\chi\otimes\sigma^*\mathcal{F}_{\chi,n}$ satisfies the two conditions defining $\widehat{D}^b_{T^c_1}(T^c)_{T^c-\mathrm{mon}}$. This easily reduces to check the same statement for $\chi=1$: i.e., $(\mathcal{F}_n)_n$ satisfies $\overline{\pi}$-constancy (where $\overline{\pi}:\overline{T}^c\to\mathrm{pt}$) and if and only if $(\sigma^*\mathcal{F}_n)_n$ satisfies π-constancy and uniform boundedness. Since σ^* is t-exact up to a shift, the equivalence of uniform boundedness is clear. Now

$$\pi_!\sigma^*\mathcal{F}_n=\overline{\pi}_!\sigma_!\sigma^*\mathcal{F}_n\cong(\overline{\pi}_!\mathcal{F}_n)\otimes H^*_c((T^c_1)^\circ,\mathbf{k}). \tag{3.18}$$

Therefore, $(\pi_!\sigma^*\mathcal{F}_n)_n$ is isomorphic to a constant object in $\mathrm{pro}D^b_{T^c_1}(\mathrm{pt})$ if and only if $(\overline{\pi}_!\mathcal{F}_n)_n$ is isomorphic to a constant object in $\mathrm{pro}D^b(\mathrm{pt})$.

By [9, Corollary A.4.7(2)] or [7, Corollary 4.6] , we have $D^b(\overline{T}^c)_{\overline{T}^c-\mathrm{mon}}\cong D^f(\overline{\mathcal{R}}\text{-mod})$. Combining with $\widehat{\Phi}$ it gives the equivalence (3.14).

(2) By tensoring with $\underline{\mathbf{k}}_\chi$ the case for general χ reduced to the case of trivial character χ. In this case we need to describe the functor

$$\pi_\dagger=\widehat{\pi}_![\dim T^c]:\widehat{D}^b_{(T^c_1)^\circ}(T^c)\to\widehat{D}^b_{(T^c_1)^\circ}(\mathrm{pt}). \tag{3.19}$$

By Lemma 3.5, for $M\in D^f(\overline{\mathcal{R}}\text{-mod})$, the corresponding object in $\widehat{D}^b_{(T^c_1)^\circ}(T^c)$ is $\sigma^*\mathcal{L}_M$. Then (3.18) implies

$$\begin{aligned}\pi_\dagger\sigma^*\mathcal{L}_M &\cong \widehat{\pi}_!\sigma^*\mathcal{L}_M[\dim T^c]\\ &= (\widehat{\overline{\pi}}_!\mathcal{L}_M)[\dim\overline{T}^c]\otimes H^*_c((T^c_1)^\circ,\mathbf{k})[\dim T^c_1]\\ &\cong \overline{\pi}_\dagger\mathcal{L}_M\otimes\Lambda_\bullet.\end{aligned}$$

Here $\overline{\pi}_\dagger\mathcal{L}_M\in D^b(\mathrm{pt})\cong D^f(\mathbf{k}\text{-mod})$. It is well-known that $\overline{\pi}_\dagger\mathcal{L}_M\cong\mathbf{k}\overset{\mathbb{L}}{\otimes}_{\overline{\mathcal{R}}}M$. Therefore

$$\pi_\dagger\sigma^*\mathcal{L}_M\cong(\mathbf{k}\overset{\mathbb{L}}{\otimes}_{\overline{\mathcal{R}}}M)\otimes\Lambda_\bullet. \tag{3.20}$$

□

4. Real Tilting Excercises

4.1. **The setting.** We are back to the setup of Section 3.1. Namely, let X be a real analytic variety. Let T^c be a *compact* torus. Let $\pi:\widetilde{X}\to X$ be a principal right T^c-bundle. Let H be a Lie group with an analytic action on $\widetilde{X}$ from the left commuting with T^c. We assume additionally that the action of H on X has finitely many orbits $\{X_\lambda\}_{\lambda\in I}$, which then gives a Whitney stratification of X. We put $\widetilde{X}_\lambda = \pi^{-1}(X_\lambda)$. Let $i_\lambda\colon X_\lambda \to X$ and $\widetilde{i}_\lambda\colon \widetilde{X}_\lambda \to \widetilde{X}$ be the inclusions. We say $\lambda \leq \mu$ if $X_\lambda \subset \overline{X}_\mu$. Put $d_\lambda := \dim X_\lambda$.

Let $X_\lambda \subset X$ be an H-orbit, and $\widetilde{X}_\lambda = \pi^{-1}(X_\lambda)$. We choose a point $x_\lambda \in X_\lambda$. Let H_{x_λ} be its stabilizer in H. Then H acts on the fiber $\pi^{-1}(x_\lambda)$ commuting with the right T^c-action. This defines a homomorphism $\varphi_{x_\lambda} : H_{x_\lambda} \to T^c$ such that the action of $h \in H$ on $\pi^{-1}(x_\lambda)$ is by right translation by $\varphi_{x_\lambda}(h)^{-1}$. Let $T^c_{x_\lambda} \subset T^c$ be the image of φ_{x_λ}. Changing the choice of x_λ changes φ_{x_λ} by H-conjugation. Since T^c is abelian, $T^c_{x_\lambda}$ stays the same. Therefore $T^c_{x_\lambda}$ is independent of the choice of x_λ, and we denote it by T^c_λ.

The irreducible H-equivariant local systems on X_λ with $\mathbf{k}$-coefficients are in bijection with characters $\pi_0(T^c_\lambda) \to \mathbf{k}^\times$. Let

$$\widetilde{I} = \{(\lambda,\chi) | \lambda \in I, \chi\colon \pi_0(T^c_\lambda) \to \mathbf{k}^\times \text{ is a character}\}.$$

For $(\lambda,\chi) \in \widetilde{I}$, let $\underline{\mathbf{k}}_{\lambda,\chi}$ be the corresponding rank one local system on X_λ.

We assume:

(4.1) The subgroup $T^c_\lambda \subset T^c$ is closed, and $\ker(H_{x_\lambda} \to T^c_\lambda)$ is contractible.

In particular, the identity component $(T^c_\lambda)^\circ$ of T^c_λ is a compact torus. We put

$$\overline{T}^c_\lambda = T^c_\lambda/(T^c_\lambda)^\circ, \quad d_\lambda = \dim X_\lambda, \quad n_\lambda = \dim T^c - \dim T^c_\lambda = \dim \overline{T}^c_\lambda, \quad \lambda \in I.$$

We impose the following parity condition:

(4.2) The parity of the numbers $d_\lambda + n_\lambda$ is the same for all $\lambda \in I$.

For $\lambda < \mu \in I$ and $x_\lambda \in X_\lambda$, let $L^\mu_{x_\lambda}$ be the link of X_λ in X_μ at x_λ. More precisely, let D_{x_λ} be a sufficiently small transversal slice to X_λ at x_λ, and take $L^\mu_{x_\lambda} = D_{x_\lambda} \cap X_\mu$. Then $L^\mu_{x_\lambda}$ is a smooth manifold of dimension $d_\mu - d_\lambda$, well-defined up to diffeomorphism.

We assume there is a weakly increasing function $p : I \to \mathbb{Z}$, denoted $\lambda \mapsto p_\lambda$, such that for any $(\mu,\chi) \in \widetilde{I}$ and any $\lambda < \mu$, we have

$$H^i(L^\mu_{x_\lambda}, \underline{\mathbf{k}}_{\mu,\chi}) = 0 \text{ for } i > \frac{1}{2}(d_\mu + n_\mu - d_\lambda - n_\lambda). \tag{4.3}$$

Here we use $\underline{\mathbf{k}}_{\mu,\chi}$ to denote the restriction of $\underline{\mathbf{k}}_{\mu,\chi}$ to $L^\mu_{x_\lambda}$.

Since $L^\mu_{x_\lambda}$ is a smooth manifold of dimension $d_\mu - d_\lambda$, by Poincaré duality (4.3) is equivalent to the following bound for all $\chi : \pi_0(T^c_\mu) \to \mathbf{k}^\times$

$$H^i_c(L^\mu_{x_\lambda}, \underline{\mathbf{k}}_{\mu,\chi}) = 0 \text{ for } i < \frac{1}{2}(d_\mu - n_\mu - d_\lambda + n_\lambda). \tag{4.4}$$

Remark 4.2. If $n_\lambda = n_\mu$, a typical situation where the bounds (4.3) and (4.4) hold is when $L^\mu_{x_\lambda}$ is diffeomorphic to a Stein manifold (e.g. smooth affine complex algebraic variety) of complex dimension $p_\mu - p_\lambda$. If $n_\mu > n_\lambda$, then there is a possible overlap of length $n_\mu - n_\lambda$ for the nonvanishing degrees of $H^*_c(L^\mu_{x_\lambda}, \underline{\mathbf{k}}_{\mu,\chi})$ and $H^*(L^\mu_{x_\lambda}, \underline{\mathbf{k}}_{\mu,\chi})$, which can happen if $L^\mu_{x_\lambda}$ fibers over a Stein manifold of complex dimension $p_\mu - p_\lambda - n_\mu + n_\lambda$ with fibers compact manifolds of real dimension $n_\mu - n_\lambda$ (e.g. compact torus fibration). On the other hand, if $n_\mu < n_\lambda$, then there is a gap of length at least $n_\lambda - n_\mu$ between the lowest nonvanishing degree of $H^*_c(L^\mu_{x_\lambda}, \underline{\mathbf{k}}_{\mu,\chi})$ and the highest nonvanishing degree of $H^*(L^\mu_{x_\lambda}, \underline{\mathbf{k}}_{\mu,\chi})$, which can happen if $L^\mu_{x_\lambda}$ admits a fiber bundle whose total space is diffeomorphic to a Stein manifold complex dimension $p_\mu - p_\lambda - n_\mu + n_\lambda$ and whose fibers are compact manifolds of real dimension $n_\lambda - n_\mu$. In our applications, the cohomological bounds hold essentially for these reasons.

4.3. **Standard and costandard sheaves.** We define a perversity function $p : I \to \mathbb{Z}$ by

$$p_\lambda = \lfloor \frac{1}{2}(d_\lambda + n_\lambda) \rfloor. \tag{4.5}$$

As in [2, Section 2.1], it defines a t-structure on $D^b_H(X)$ using the perversity function p, whose heart we denote by ${}^pP_H(X)$.

For $(\lambda, \chi) \in \widetilde{I}$, let $\Delta_{\lambda,\chi}$ and $\nabla_{\lambda,\chi} \in D^b_H(X)$ be the ! and $*$ extensions of the local system $\underline{\mathbf{k}}_{\lambda,\chi}[p_\lambda]$ on X_λ.

Lemma 4.4. *For $(\mu, \chi) \in \widetilde{I}$ and $\lambda \leq \mu$, $i^!_\lambda \Delta_{\mu,\chi}$ lies in degrees $\geq -p_\lambda + n_\lambda - n_\mu$, and $i^*_\lambda \nabla_{\mu,\chi}$ lies in degrees $\leq -p_\lambda$. In particular $\nabla_{\mu,\chi}$ lies in the heart of the t-structure ${}^pP_H(X)$.*

Proof. We first show the statement about $\nabla_{\mu,\chi}$. The stalk of $i^*_\lambda \nabla_{\mu,\chi}$ at $x_\lambda \in X_\lambda$ is $H^*(L^\mu_{x_\lambda}, \underline{\mathbf{k}}_{\mu,\chi})[p_\mu]$. By the cohomological bound (4.3), it is concentrated in degrees $\leq -p_\mu + p_\mu - p_\lambda = -p_\lambda$. Since $\nabla_{\mu,\chi}$ has vanishing costalks, it lies in the heart of the t-structure ${}^pP_H(X)$.

For $i^!_\lambda \Delta_{\mu,\chi}$, we note that $\Delta_{\mu,\chi}$ is Verdier dual to $\nabla_{\mu,\chi'}[d_\mu - 2p_\mu]$ for some χ'. Therefore $i^!_\lambda \Delta_{\mu,\chi}$ is Verdier dual to $i^*_\lambda \nabla_{\mu,\chi'}[d_\mu - 2p_\mu]$. Since $i^*_\lambda \nabla_{\mu,\chi'}[d_\mu - 2p_\mu]$ lies in degrees $\leq -p_\lambda - d_\mu + 2p_\mu$, $i^!_\lambda \Delta_{\mu,\chi}$ lies in degrees $\geq -d_\lambda + p_\lambda + d_\mu - 2p_\mu = -p_\lambda + n_\lambda - n_\mu$.

□

4.5. **Free-monodromic local systems on an orbit.** We apply results from Section 3.4 to the situation of a single orbit $H\backslash X_\lambda$. We have adjoint functors

$$\widehat{D}^b_H(\widetilde{X}_\lambda)_{T^c-\mathrm{mon}} \underset{\pi_\lambda^\dagger}{\overset{\pi_{\lambda\dagger}}{\rightleftarrows}} D^b_H(X_\lambda) \tag{4.6}$$

Let $\overline{\mathcal{R}}_\lambda$ be the completion of the group algebra $\mathbf{k}[\pi_1(\overline{T}^c_\lambda)]$ at the augmentation ideal. Let $\Lambda_{\lambda,\bullet} = H_*((T^c_\lambda)^\circ, \mathbf{k})$ be the homology of the torus $(T^c_\lambda)^\circ$, viewed as a graded algebra in degrees $0, -1, \cdots, -\dim T^c_\lambda$.

Corollary 4.6. *For $\lambda \in I$, we have canonical equivalences*

$$\widetilde{\Phi}_\lambda : \widehat{D}^b_H(\widetilde{X}_\lambda)_{T^c-\mathrm{mon}} \cong \bigoplus_{\chi:\pi_0(T^c_\lambda)\to\mathbf{k}^\times} D^f(\overline{\mathcal{R}}_\lambda\text{-mod}), \tag{4.7}$$

$$\Phi_\lambda : D^b_H(X_\lambda) \cong \bigoplus_{\chi:\pi_0(T^c_\lambda)\to\mathbf{k}^\times} D^f_+(\Lambda_{\lambda,\bullet}\text{-mod}). \tag{4.8}$$

Under these equivalences, the adjunction $(\pi_{\lambda\dagger}, \pi_\lambda^\dagger)$ takes the form
(4.9)

$$\bigoplus_\chi D^f(\overline{\mathcal{R}}_\lambda\text{-mod}) \underset{\mathrm{forg}}{\overset{\mathbf{k}\overset{\mathbb{L}}{\otimes}_{\overline{\mathcal{R}}_\lambda}(-)}{\rightleftarrows}} \bigoplus_\chi D^f(\mathbf{k}\text{-mod}) \underset{\mathrm{forg}}{\overset{\Lambda_{\lambda,\bullet}\otimes_{\mathbf{k}}(-)}{\rightleftarrows}} \bigoplus_\chi D^f_+(\Lambda_{\lambda,\bullet}\text{-mod})$$

Proof. Let $\widetilde{x}_\lambda \in \widetilde{X}_\lambda$ with image $x_\lambda \in X_\lambda$. Then $\pi^{-1}(x_\lambda) \cong T^c$ given by the base point $\widetilde{x}_\lambda$. Restricting to $\pi^{-1}(x_\lambda)$ gives an equivalence $i^* : D^b_H(X_\lambda) \overset{\sim}{\to} D^b_{H_{x_\lambda}}(T^c)_{T^c-\mathrm{mon}}$, which extends to the completions $\widehat{i^*} : \widehat{D}^b_H(X_\alpha) \overset{\sim}{\to} \widehat{D}^b_{H_{x_\lambda}}(T^c)_{T^c-\mathrm{mon}}$. Since $\ker(\varphi_{x_\lambda})$ is contractible, we have $D^b_{H_{x_\lambda}}(T^c)_{T^c-\mathrm{mon}} \cong D^b_{T^c_\lambda}(T^c)_{T^c-\mathrm{mon}}$ which also extends to completions. Similarly, $D^b_H(X_\lambda) \cong D^b_{T^c_\lambda}(T^c)$. It remains to apply Lemma 3.7. It is easy to check that the equivalence thus defined is independent of the choice of $\widetilde{x}_\lambda$. □

In the situation of Corollary 4.6, for each $(\lambda, \chi) \in \widetilde{I}$, we have a *free-monodromic* local system $\mathcal{L}_{\lambda,\chi} \in \widehat{D}^b_H(\widetilde{X}_\lambda)_{T^c-\mathrm{mon}}$ that corresponds to the free rank one $\overline{\mathcal{R}}_\lambda$-module $\overline{\mathcal{R}}_\lambda$ placed in the χ-summand on the right side of (4.7).

4.7. **Free-monodromic sheaves.** For the triangulated category of free-monodromic sheaves we put $\mathcal{M}_{H,X} := \widehat{D}^b_H(\widetilde{X})_{T^c-\mathrm{mon}}$ for short, whenever it does not provide an ambiguity. Define the a t-structure on $\mathcal{M}_{H,X}$ using the same perversity function p as in (**??**), whose heart we denote by $\mathcal{P}_{H,X}$.

For $\lambda \in I$ let $\mathcal{M}_\lambda := \widehat{D}_H(\widetilde{X}_\lambda)_{T^c-\mathrm{mon}}$. We have adjunctions

$$\mathcal{M}_\lambda \underset{\widetilde{i}_\lambda^!}{\overset{\widetilde{i}_{\lambda!}}{\rightleftarrows}} \mathcal{M}_{H,X} \qquad \mathcal{M}_\lambda \underset{\widetilde{i}_{\lambda*}}{\overset{\widetilde{i}_\lambda^*}{\leftrightarrows}} \mathcal{M}_{H,X} \tag{4.10}$$

For $(\lambda, \chi) \in \widetilde{I}$, we have the free-monodromic local system $\mathcal{L}_{\lambda,\chi} \in \mathcal{M}_\lambda$ as defined in Section 4.5. We define standard and costandard objects $\widetilde{\Delta}_{\lambda,\chi}$ and $\widetilde{\nabla}_{\lambda,\chi}$ of $\mathcal{M}_{H,X}$ as, respectively, the !- and $*$-extensions under $\widetilde{i}_\lambda$ of $\mathcal{L}_{\lambda,\chi}[p_\lambda]$.

Lemma 4.8. *For any $(\lambda, \chi) \in \widetilde{I}$, we have*

$$\pi_\dagger \widetilde{\Delta}_{\lambda,\chi} \cong \Lambda_{\lambda,\bullet} \otimes \Delta_{\lambda,\chi}, \quad \pi_\dagger \widetilde{\nabla}_{\lambda,\chi} \cong \Lambda_{\lambda,\bullet} \otimes \nabla_{\lambda,\chi}. \tag{4.11}$$

Proof. By Corollary 4.6 we have

$$\pi_{\lambda\dagger}\mathcal{L}_{\lambda,\chi} \cong \Lambda_{\lambda,\bullet} \otimes \underline{\mathbf{k}}_{\lambda,\chi}. \tag{4.12}$$

Shift by p_λ and apply $i_{\lambda!}$ to the above we get

$$\pi_\dagger \widetilde{\Delta}_{\lambda,\chi} = i_{\lambda!}\pi_{\lambda\dagger}\mathcal{L}_{\lambda,\chi}[p_\lambda] \cong i_{\lambda!}(\Lambda_{\lambda,\bullet} \otimes \underline{\mathbf{k}}_{\lambda,\chi}[p_\lambda]) = \Lambda_{\lambda,\bullet} \otimes \Delta_{\lambda,\chi}. \tag{4.13}$$

The argument for the costandard sheaf is the same, using that $\pi_\dagger \widetilde{i}_{\lambda*} \cong i_{\lambda*}\pi_{\lambda\dagger}$ because $\widehat{\pi}_! = \widehat{\pi}_*$ (since π is proper). □

Lemma 4.9. *Let $(\mu, \chi) \in \widetilde{I}$ and $\lambda < \mu$.*

1. *The restriction $\widetilde{i}_\lambda^* \widetilde{\nabla}_{\mu,\chi}$ lies in degrees $\le -p_\lambda$.*
2. *Under the equivalence Φ_λ, the corestriction $\widetilde{i}_\lambda^! \widetilde{\Delta}_{\mu,\chi}$ corresponds to a collection $M_{\chi'} \in D^f(\overline{\mathcal{R}}_\lambda\text{-mod})$ (where $\chi' : \pi_0(T_\lambda^c) \to \mathbf{k}^\times$) where each $M_{\chi'}$ can be represented by a complex of free $\overline{\mathcal{R}}_\lambda$ in degrees $\ge -p_\lambda$.*
3. *The standard and costandard objects $\widetilde{\Delta}_{\mu,\chi}$ and $\widetilde{\nabla}_{\mu,\chi}$ lie in the heart of the t-structure $\mathcal{P}_{H,X}$.*

Proof. (1) By Lemma 4.8, $\pi_{\lambda\dagger}\widetilde{i}_\lambda^*\widetilde{\nabla}_{\mu,\chi} \cong i_\lambda^*\pi_\dagger\widetilde{\nabla}_{\mu,\chi} \cong \Lambda_{\mu,\bullet} \otimes i_\lambda^*\nabla_{\mu,\chi}$. By Lemma 4.4, $i_\lambda^*\nabla_{\mu,\chi}$ lies in degrees $\le -p_\lambda$, hence $\pi_{\lambda\dagger}\widetilde{i}_\lambda^*\widetilde{\nabla}_\mu$ lies in degrees $\le -p_\lambda$ (note $\Lambda_{\mu,\bullet}$ is in non-positive degrees). From the description of $\pi_{\lambda\dagger}$ given in Corollary 4.6 we see that $\widetilde{i}_\lambda^*\widetilde{\nabla}_\mu$ is in degrees $\le -p_\lambda$.

(2) Note the statement is stronger than saying that $M_{\chi'}$ lies in cohomological degrees $- \ge p_\lambda$, but saying that it admits a free resolution (as $\overline{\mathcal{R}}_\lambda$-modules) in degrees $\ge -p_\lambda$.

By Lemma 4.8, we have $\pi_{\lambda\dagger}\widetilde{i}_\lambda^!\widetilde{\Delta}_{\mu,\chi} \cong i_\lambda^!\pi_\dagger\widetilde{\Delta}_{\mu,\chi} \cong \Lambda_{\mu,\bullet} \otimes i_\lambda^!\Delta_{\mu,\chi}$. By Lemma 4.4, $i_\lambda^!\Delta_{\mu,\chi}$ lies in degrees $\ge -p_\lambda + n_\lambda - n_\mu$. Therefore $\pi_{\lambda\dagger}\widetilde{i}_\lambda^!\widetilde{\Delta}_{\mu,\chi} \cong \Lambda_{\mu,\bullet} \otimes i_\lambda^!\Delta_{\mu,\chi}$ lies in degrees $\ge -p_\lambda + n_\lambda - n_\mu - \dim T_\mu^c = -p_\lambda + n_\lambda - \dim T^c = -p_\lambda - \dim T_\lambda^c$.

By Corollary 4.6, $\pi_{\lambda\dagger}\widetilde{i}^!_\lambda\widetilde{\Delta}_{\mu,\chi}$ corresponds to $\oplus_{\chi'}\Lambda_{\lambda,\bullet}\otimes(\mathbf{k}\overset{\mathbb{L}}{\otimes}_{\overline{\mathcal{R}}_\lambda} M_{\chi'})$, hence $\mathbf{k}\overset{\mathbb{L}}{\otimes}_{\overline{\mathcal{R}}_\lambda} M_{\chi'}$ lies in degrees $-p_\lambda - \dim T^c_\lambda + \dim T^c_\lambda = -p_\lambda$ (the lowest degree of $\Lambda_{\lambda,\bullet}$ is $-\dim T^c_\lambda$). This implies that $M_{\chi'}$ admits a free resolution (as $\overline{\mathcal{R}}_\lambda$-modules) in degrees $\geq -p_\lambda$.

(3) The statement follows from (1)(2) and the observation that $\widetilde{i}^*_\lambda\widetilde{\Delta}_{\mu,\chi}=0$ and $\widetilde{i}^!_\lambda\widetilde{\nabla}_{\mu,\chi}=0$.

□

4.10. **Tilting sheaves.** We are in the situation of Section 4.1.

Definition 4.11. *An object $\mathcal{T}$ of $\mathcal{M}_{H,X}$ is called a* free-monodromic tilting sheaf, *if for each $\lambda\in I$, both complexes $\widetilde{i}^*_\lambda\mathcal{T}$ and $\widetilde{i}^!_\lambda\mathcal{T}$ are free-monodromic local systems in degree $-p_\lambda$.*

From the definition, we see that an object $\mathcal{T}$ of $\mathcal{M}_{H,X}$ is a free-monodromic tilting sheaf if and only if $\mathcal{T}\in\mathcal{P}_{H,X}$ and $\mathcal{T}$ has a $\widetilde{\Delta}$-flag and $\widetilde{\nabla}$-flag, i.e. it is both a successive extension of $\widetilde{\Delta}_{\lambda,\chi}$'s and a successive extension of $\widetilde{\nabla}_{\lambda,\chi}$'s in $\mathcal{P}_{H,X}$.

We will denote by $\mathrm{Tilt}(\mathcal{M}_{H,X})\subset\mathcal{M}_{H,X}$ the full additive subcategory free-monodromic tilting sheaves.

Proposition 4.12. *1) For each $(\lambda,\chi)\in\widetilde{I}$ there exists an indecomposable free-monodromic tilting sheaf $\mathcal{T}_{\lambda,\chi}$ whose restriction to $\widetilde{X}_\lambda$ is $\mathcal{L}_{\lambda,\chi}$ and whose support is the closure of $\widetilde{X}_\lambda$.*

2) If $\mathcal{T}\in\mathrm{Tilt}(\mathcal{M}_{H,X})$ is supported on the closure of $\widetilde{X}_\lambda$ and the restriction $\widetilde{i}^*_\lambda\mathcal{T}$ is a decomposable local system, then $\mathcal{T}$ is decomposable.

Proof. 1) Proceeding by the descending induction on strata we may assume that $Z=X_\mu$ is a minimal stratum of X and on the preimage $\widetilde{U}=\pi^{-1}(U)$ of its complement $U=X-Z$ there is a free-monodromic tilting sheaf $\mathcal{T}_U$ satisfying the required conditions. Let $\widetilde{j}\colon\widetilde{U}\to\widetilde{X}$ and $\widetilde{i}\colon\widetilde{Z}=\pi^{-1}(Z)\to\widetilde{X}$ be the inclusions.

Let $\mathcal{C}=\widetilde{i}^*\widetilde{j}_*\mathcal{T}_U$ and $(M_{\chi'})_{\chi'}=\Phi_\mu(\mathcal{C})\in D^f(\overline{\mathcal{R}}_\mu\text{-mod})$. Since $\mathcal{T}_U$ has a $\widetilde{\nabla}$-flag, by Lemma 4.9, $M_{\chi'}$ is in degrees $\leq -p_\mu$. Since $\mathcal{T}_U$ has a $\widetilde{\Delta}$-flag and $\mathcal{C}[-1]\cong\widetilde{i}^!\widetilde{j}_!\mathcal{T}_U$, then by Lemma 4.9, each $M_{\chi'}[-1]$ can be represented by a complex of free $\overline{\mathcal{R}}_\mu$-modules in degrees $\geq -p_\mu$, therefore $M_{\chi'}$ can be represented by a complex of $\overline{\mathcal{R}}_\lambda$-modules in degrees $\geq -p_\mu-1$. Combining these we get that $\mathcal{C}$ can be represented by a two-step complex $[\mathcal{A}\xrightarrow{\varphi}\widetilde{\mathcal{B}}]$ of free-monodromic local systems on Z in degrees $-p_\mu-1$ and $-p_\mu$ respectively.

As in [3] we now put $\mathcal{T} \in \mathcal{P}_{H,X}$ to be the extension

$$0 \to \widetilde{i}_*\mathcal{A}[p_\mu] \to \mathcal{T} \to \widetilde{j}_*\mathcal{T}_U \to 0 \tag{4.14}$$

defined by the map $\widetilde{j}_*\mathcal{T}_U \to \widetilde{i}_*\mathcal{C} \to \widetilde{i}_*\mathcal{A}[p_\mu + 1]$. From this we see that

$$\widetilde{i}^!\mathcal{T} \cong \mathcal{A}[p_\mu]. \tag{4.15}$$

Applying $\widetilde{i}^*$ to the exact sequence (4.14) we see that $\mathcal{A}[p_\mu] \to \widetilde{i}^*\mathcal{T} \to \mathcal{C}$ is a distinguished triangle hence

$$\widetilde{i}^*\mathcal{T} \cong \mathcal{B}[p_\mu]. \tag{4.16}$$

Therefore we also have an exact sequence

$$0 \to \widetilde{j}_!\mathcal{T}_U \to \mathcal{T} \to \widetilde{i}_*\mathcal{B} \to 0 \tag{4.17}$$

in $\mathcal{P}_{H,X}$. From (4.15) and (4.16) and the fact that $\mathcal{T}_U$ is free-monodromic tilting on U we conclude that $\mathcal{T}$ is a free-monodromic tilting sheaf on X.

2) Proof repeats the argument of [3, 1.2 and 1.4]. □

We next prove the functoriality of free-monodromic tilting sheaves under proper pushforward. Let $\widetilde{X} \to X$ and $\widetilde{Y} \to Y$ be H-equivariant T^c-torsors satisfying the conditions of Section 4.1 [1].

Proposition 4.13. *In the above situation, assume $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ is an $H \times T^c$-equivariant proper map. Then for any free-monodromic tilting sheaf $\mathcal{T} \in \mathcal{M}_{H,X}$, $\widetilde{f}_*\mathcal{T} \in \mathcal{M}_{H,Y}$ is also a free-monodromic tilting sheaf.*

Proof. Since $\widetilde{f}$ is proper, $\widetilde{f}_*$ commutes with restriction and corestriction to $H \times T^c$-orbits, it suffices to assume that both X and Y has a single stratum, so that $X = H/H_x, Y = H/H_y$ and $y = f(x)$, $H_x \subset H_y$. Let $T^c_x \subset T^c_y \subset T^c$ be the images of H_x and H_y.

Let $n_X = \dim T^c/T^c_x$, $n_Y = \dim T^c/T^c_y$, $d_X = \dim X$, $d_Y = \dim Y$, $p_X = \lfloor \frac{d_X + n_X}{2} \rfloor$ and $p_Y = \lfloor \frac{d_Y + n_Y}{2} \rfloor$.

We claim that

$$p_X - n_X = p_Y - n_Y. \tag{4.18}$$

Indeed, let $U_x = \ker(H_x \to T^c_x)$ and $U_y = \ker(H_x \to T^c_x)$, which are contractible Lie groups by assumption. Since $f : X \to Y$ is proper, H_y/H_x is compact. On the other hand, H_x/H_y is a fibration over T^c_y/T^c_x with contractible fiber U_y/U_x. This implies $U_x = U_y$, and $H_y/H_x \cong T^c_y/T^c_x$. Hence $d_X - d_Y = \dim H_y - \dim H_x = \dim T^c_y - \dim T^c_x = n_X - n_Y$. Therefore $d_X - n_X = d_Y - n_Y$, which implies (4.18).

[1] In fact we don't need to assume the cohomological bounds for links in X and Y.

Restricting to the fibers over x and y respectively we have a commutative diagram

$$\begin{tikzcd} \mathcal{M}_{H,X} \arrow[r, "\widetilde{i}_x^*", "\sim"'] \arrow[d, "\widetilde{f}_*"] & \widehat{D}^b_{T^c_x}(T^c)_{T^c-\mathrm{mon}} \arrow[d, "\varphi_*"] \\ \mathcal{M}_Y \arrow[r, "\widetilde{i}_y^*", "\sim"'] & \widehat{D}^b_{T^c_y}(T^c)_{T^c-\mathrm{mon}} \end{tikzcd} \tag{4.19}$$

where φ_* is the induction functor for the inclusion $T^c_x \subset T^c_y$. In this situation, we may assume $\widetilde{i}_x^*\mathcal{T} \in D^b_{T^c_x}(T^c)_{T^c-\mathrm{mon}}$ is the shifted free-monodromic local system $\mathcal{L}_\chi[p_X]$ for some $\chi : \pi_0(T^c_x) \to \mathbf{k}^\times$. The fiber of the quotient map $\overline{\varphi} : T^c_x\backslash T^c \to T^c_y\backslash T^c$ is isomorphic to $T^c_x\backslash T^c_y$, a compact Lie group whose neutral component is a torus of dimension $n_X - n_Y$. Therefore $\varphi_*\mathcal{L}_\chi$ is a direct sum of free-monodromic local systems in degree $n_X - n_Y$. This implies that $\widetilde{f}_*\mathcal{T}$ is a direct sum of free-monodromic local systems in degree $-p_X + n_X - n_Y$. By (4.18), $-p_X+n_X-n_Y = -p_Y$, therefore $\widetilde{f}_*\mathcal{T}$ is a free-monodromic tilting sheaf on $\widetilde{Y}$. $\square$

5. Matsuki correspondence and real tilting sheaves on flag variety

5.1. **Setup.** Let G be a connected semisimple complex Lie group together with the antiholomorphic involution σ. We put $G_\mathbb{R} = G^\sigma \subset G$ be the corresponding real form and $T \subset B \subset G$ for a maximal torus and a Borel subgroup. Put $U \subset B$ for the unipotent radical. Let $K_\mathbb{R} \subset G_\mathbb{R}$ be a maximal compact subgroup of $G_\mathbb{R}$ and $K \subset G$ be the complexification of $K_\mathbb{R}$.

Let $\mathbf{T} = \mathbf{T}^{>0} \times \mathbf{T}^c$ be the decomposition of the $\mathbb{C}$-points of the abstract Cartan $\mathbf{T}$ into the neutral component $\mathbf{T}^{>0}$ and the maximal compact subgroup $\mathbf{T}^c$.

Consider the flag variety X of G. Choose a Borel subgroup $B \subset G$ with unipotent radical U, and consider the $\mathbf{T}^c$-torsor $\pi\colon \widetilde{X} = (G/U)/\mathbf{T}^{>0} \to X$. When G is adjoint, we can define $\widetilde{X}$ as the space of $(B, \{x_\alpha\})$ where B is a Borel subgroup and for each simple root $\alpha \in \Phi$, x_α is basis of the α-weight space of $U/[U,U]$ under the $\mathbf{T}$-action. For general G, we need to choose a base point $B \in X$ in order to define $\widetilde{X}$. We will consider the left action of $H = G_\mathbb{R}$ on $\widetilde{X}$.

5.2. **The cross action of W on $\widetilde{I}$.** Recall from Lemma 2.6(1) the real form σ_λ on $\mathbf{T}$ for $\lambda \in I$. Let $\mathbf{T}^c_\lambda \subset \mathbf{T}^c$ be the image of the real points $\mathbf{T}^{\sigma_\lambda}$ under the projection $\mathbf{T} \to \mathbf{T}^c$. By Lemma 2.6(2), $\mathbf{T}^c_\lambda$ is the image of $G_{\mathbb{R}} \cap B \to \mathbf{T} \twoheadrightarrow \mathbf{T}^c$ for any $B \in O_\lambda^{\mathbb{R}}$, therefore this notation is consistent with that of Section 4.1.

If T is a σ-stable maximal torus and $O_\lambda^{\mathbb{R}}$ is attached to T, then by Lemma 2.6, via a choice of $B \in (O_\lambda^{\mathbb{R}})^T$, $\mathbf{T}_\lambda^c$ can be identified with the compact part of $T_{\mathbb{R}}$. In particular, we have an isomorphism

$$\iota_B : \pi_0(T_{\mathbb{R}}) \xrightarrow{\sim} \pi_0(\mathbf{T}_\lambda^c). \tag{5.1}$$

Recall the right action of $\mathbf{W}$ on $\mathbf{T}$, which induces a right action on $\mathbf{T}^c$. From the commutative diagram (2.2) we see that

$$\mathbf{T}_\lambda^c \cdot w = \mathbf{T}_{\lambda \cdot w}^c, \quad \forall \lambda \in I, w \in \mathbf{W} \tag{5.2}$$

as subgroups of $\mathbf{T}^c$.

In [25, Definition 4.1], Vogan defines a *cross action* of $\mathbf{W}$ on $\widetilde{I}$ that lifts the action of $\mathbf{W}$ on I from Lemma 2.5. We will turn the cross action $w \times (-)$ into a right action and denote it by

$$(\lambda, \chi) \cdot w := w \times (\lambda, \chi), \quad \forall w \in \mathbf{W}, (\lambda, \chi) \in \widetilde{I}. \tag{5.3}$$

By [24, Definition 6.3], for a simple reflection $s \in \mathbf{W}$, its action on $(\lambda, \chi) \in \widetilde{I}$ is as follows:

(1) If α_s is a complex root for $O_\lambda^{\mathbb{R}}$, then there is a canonical isomorphism $\pi_0(\mathbf{T}_\lambda^c) \cong \pi_0(\mathbf{T}_{\lambda \cdot s}^c)$ (both are identified with the $G_{\mathbb{R}}$-equivariant fundamental group of the image of $O_\lambda^{\mathbb{R}}$ in the partial flag variety X_s. Under this isomorphism, we have $(\lambda, \chi) \cdot s = (\lambda \cdot s, \chi)$.

(2) If α_s is type I noncompact imaginary, then there is a canonical isomorphism $\pi_0(\mathbf{T}_\lambda^c) \cong \pi_0(\mathbf{T}_{\lambda \cdot s}^c)$ for the same reason as above. Under this isomorphism, we have $(\lambda, \chi) \cdot s = (\lambda \cdot s, \chi)$.

(3) If α_s is type II real, and the local system $\mathbf{k}_\chi$ on $O_\lambda^{\mathbb{R}}$ extends to $\pi_s^{-1}\pi_s(O_\lambda^{\mathbb{R}})$. Let $\mu > \lambda$ be as in Lemma 2.12. Then $\mathbf{T}_\mu^c \cap \mathbf{T}_\lambda^c \subset \mathbf{T}_\lambda^c$ has index 2, which induces a sign character

$$\mathrm{sgn}_s : \pi_0(\mathbf{T}_\lambda^c) \twoheadrightarrow \mathbf{T}_\lambda^c / \mathbf{T}_\lambda^c \cap \mathbf{T}_\mu^c \cong \{\pm 1\} \subset \mathbf{k}^\times. \tag{5.4}$$

Then $(\lambda, \chi) \cdot s = (\lambda, \chi \otimes \mathrm{sgn}_s)$.

(4) In all other cases, $(\lambda, \chi) \cdot s = (\lambda, \chi)$.

5.3. **Matsuki correspondence.** Let us recall some of the results and constructions of [18].

The Matsuki correspondence is a canonical order-reversing bijection between the $G_{\mathbb{R}}$-orbits and K-orbits on X. This bijection is realized by a $K_{\mathbb{R}}$-invariant flow $\Phi_t \colon X \to X$ ($t \in \mathbb{R}$), such that

(1) The fixed point set of Φ_t is a finite union of $K_{\mathbb{R}}$-orbits $\{C_\lambda\}_{\lambda \in I}$ indexed a finite set I.

(2) For any $\lambda \in I$ set $O_\lambda^{\mathbb{R}}$ (resp. O_λ^K) to be the $G_{\mathbb{R}}$-orbit (resp. K-orbit) of X containing C_λ. Then we have $O_\lambda^{\mathbb{R}} = \{x \in X | \lim_{t\to+\infty} \Phi_t(x) \in C_\lambda\}$ and $O_\lambda^K = \{x \in X | \lim_{t\to-\infty} \Phi_t(x) \in C_\lambda\}$. The bijection $O_\lambda^{\mathbb{R}} \leftrightarrow O_\lambda^K$ gives an order reversing bijection between the $G_{\mathbb{R}}$-orbits and K-orbits on X which is called the *Matsuki correspondence for orbits*.

(3) The orbits $\{O_\lambda^{\mathbb{R}}\}$ and $\{O_\lambda^K\}$ intersect pairwise transversally. The natural projections $O_\lambda^{\mathbb{R}} \to C_\lambda$ and $O_\lambda^K \to C_\lambda$ given by the limits of the flow Φ_t are fibrations with contractible fibres.

For $\lambda \in I$, let $\widetilde{O}_\lambda^{\mathbb{R}}$ be the preimage of $O_\lambda^{\mathbb{R}}$ in $\widetilde{X}$.

Lemma 5.4. *Let $x \in C_\lambda$ and $K_{\mathbb{R},x}$ be the stabilizer of x under C_λ. Then the projection $K_{\mathbb{R},x} \to T^c$ is injective and its image is T_λ^c. Moreover, the $G_{\mathbb{R}}$-action on X satisfies the condition* (4.1).

Proof. Since $K_{\mathbb{R},x}$ is compact and solvable (as an algebraic group over $\mathbb{R}$), its neutral component is a compact torus. The projection $K_{\mathbb{R},x} \to T^c$ is injective with closed image.

By definition, T_λ^c is the image of the projection $\gamma_x : G_{\mathbb{R},x} \to T \to T^c$. The square of the projection $T \to T^c$ is real algebraic, hence $\gamma_x^2 : G_{\mathbb{R},x} \to T^c$ is real algebraic, and its image is therefore a real algebraic subgroup of T^c, hence closed with finitely many components. This implies that the image T_λ^c is a closed subgroup of T^c with finitely many components. The kernel $\ker(\gamma_x)$ is an extension of a closed subgroup of $T^{>0}$ and the unipotent real algebraic group $\ker(\widetilde{\gamma}_x)$, hence $\ker(\gamma_x)$ is contractible, and $G_{\mathbb{R},x} \to T_\lambda^c$ is a homotopy equivalence.

Since x lies in the critical $K_{\mathbb{R}}$-orbit C_λ, which is homotopy equivalent to $O_\lambda^{\mathbb{R}}$, the inclusion $K_{\mathbb{R},x} \hookrightarrow G_{\mathbb{R},x}$ is a homotopy equivalence. Therefore $K_{\mathbb{R},x} \hookrightarrow T_\lambda^c$ is also a homotopy equivalence. Now $T_\lambda^c/K_{\mathbb{R},x}$ is both a compact manifold and contractible, it must be a point. We conclude that $K_{\mathbb{R},x}$ maps isomorphically to $T_\lambda^c \subset T^c$.

□

We will now observe that

Proposition 5.5. *The T^c-torsor $\pi\colon \widetilde{X} \to X$ with the action of $H = G_{\mathbb{R}}$ satisfies the conditions of Section 4.1.*

Proof. (4.1) is already checked in Lemma 5.4.

We check the parity condition (4.2). From the transversality of $O_\lambda^{\mathbb{R}}$ and O_λ^K we get

$$d_\lambda + 2\dim_{\mathbb{C}} O_\lambda^K = 2\dim_{\mathbb{C}} X + \dim C_\lambda. \tag{5.5}$$

By Lemma 5.4,

$$\dim C_\lambda = \dim K_{\mathbb{R}} - \dim T^c_\lambda = \dim K_{\mathbb{R}} - \dim T^c + n_\lambda. \tag{5.6}$$

These imply

$$d_\lambda - n_\lambda = 2\operatorname{codim}_{\mathbb{C}} O^K_\lambda + \dim K_{\mathbb{R}} - \dim T^c. \tag{5.7}$$

Since the right side is independent of λ, (4.2) holds.

We check the cohomological bound of links (4.3). Suppose $\lambda < \mu$ and consider now the intersection $O^{\mathbb{R}}_\mu \cap O^K_\lambda$. The limit maps $\lim_{t\to\pm\infty} \Phi_t(p)$ provide a diagram

$$\begin{tikzcd} & Y := O^{\mathbb{R}}_\mu \cap O^K_\lambda \arrow[dl, "h_\lambda"'] \arrow[dr, "h_\mu"] & \\ C_\lambda & & C_\mu \end{tikzcd}$$

with the maps being the fibrations. For $x_\lambda \in C_\lambda$, the fiber of $O^K_\lambda \to C_\lambda$ over x_λ is a transversal slice to $O^{\mathbb{R}}_\lambda$, therefore the fiber $h_\lambda^{-1}(x_\lambda)$ is diffeomorphic to the link $L^\mu_{x_\lambda}$, which we shall denote by ${}^{\mathbb{R}}L^\mu_{x_\lambda}$ to emphasize it is the link for $G_{\mathbb{R}}$-orbits. Similarly, for $x_\mu \in C_\mu$, $h_\mu^{-1}(x_\mu)$ is diffeomorphic to the link ${}^K L^\lambda_{x_\mu}$ for the K-orbit O^K_μ in O^K_λ. Let

$$\mathcal{F}_\chi = Rh_{\lambda*}h_\mu^*\underline{\mathbf{k}}_{\mu,\chi}, \quad \mathcal{F}^i_\chi := R^i h_{\lambda*}h_\mu^*\underline{\mathbf{k}}_{\mu,\chi}. \tag{5.8}$$

Since h_λ is $K_{\mathbb{R}}$-equivariant, $\mathcal{F}^i_\chi$ is a $K_{\mathbb{R}}$-equivariant local system on C_λ. We need to show that

$$\mathcal{F}^i_\chi = 0, \quad i > \Delta := \frac{1}{2}(d_\mu + n_\mu - d_\lambda - n_\lambda). \tag{5.9}$$

As a $K_{\mathbb{R}}$-equivariant local system on C_λ, $\mathcal{F}^i_\chi$ is determined by its stalk at x_λ and the monodromy action of $\pi_0(K_{\mathbb{R},x_\lambda}) = \pi_0(T^c_\lambda)$ (by Lemma 5.4) on $\mathcal{F}^i_\chi|_{x_\lambda}$. Then $\mathcal{F}^i_\chi = 0$ if and only if $H^{\dim C_\lambda}(C_\lambda, \mathcal{F}^i_\chi \otimes \underline{\mathbf{k}}_{\lambda,\theta}) = 0$ for any character $\theta : \pi_0(T^c_\lambda) \to \mathbf{k}^\times$.

Now we introduce $K_{\mathbb{R}}$-equivariant complexes and local systems on C_μ for any character $\theta : \pi_0(T^c_\lambda) \to \mathbf{k}^\times$

$$\mathcal{G}_\theta = Rh_{\mu*}h_\lambda^*\underline{\mathbf{k}}_{\lambda,\theta}, \quad \mathcal{G}^i_\theta = R^i h_{\mu*}h_\lambda^*\underline{\mathbf{k}}_{\lambda,\theta}. \tag{5.10}$$

Note that

$$H^*(C_\lambda, \mathcal{F}_\chi \otimes \underline{\mathbf{k}}_{\lambda,\theta}) \cong H^*(Y, {}_\theta\underline{\mathbf{k}}_\chi) \cong H^*(C_\mu, \mathcal{G}_\theta \otimes \underline{\mathbf{k}}_{\mu,\chi}). \tag{5.11}$$

Here ${}_\theta\underline{\mathbf{k}}_\chi)$ is the local system $h_\lambda^*\underline{\mathbf{k}}_{\lambda,\theta} \otimes h_\mu^*\underline{\mathbf{k}}_{\mu,\chi}$ on Y.

Let N be the largest number such that $\mathcal{F}^N_\chi \neq 0$. We need to show $N \leq \Delta$. By the first isomorphism in (5.11) and the Leray spectral sequence, $H^{\dim C_\lambda+N}(Y, {}_\theta\underline{\mathbf{k}}_\chi)) \cong H^{\dim C_\lambda}(C_\lambda, \mathcal{F}^N_\chi \otimes \underline{\mathbf{k}}_{\lambda,\theta})$. Therefore it suffices to show that

$$(5.12) \qquad H^{\dim C_\lambda+N}(Y, {}_\theta\underline{\mathbf{k}}_\chi)), \quad \text{for } i > \dim C_\lambda + \Delta, \forall(\theta, \chi).$$

Reversing the argument using the second equality in (5.11), we see that (5.12) holds if and only if

$$(5.13) \qquad \mathcal{G}^i_\theta = 0, \quad i > \dim C_\lambda + \Delta - \dim C_\mu, \forall\theta.$$

Since $\mathcal{G}^i_\theta$ is a local system whose stalks calculate link cohomology for K-orbits, (5.13) holds if and only if

$$(5.14) \qquad H^i({}^K L^\lambda_{x_\mu}, \underline{\mathbf{k}}_{\lambda,\theta}) = 0, \quad i > \dim C_\lambda + \Delta - \dim C_\mu, \forall\theta.$$

By (5.6) and (5.7), we have

$$(5.15) \qquad \dim C_\lambda - \frac{1}{2}(d_\lambda + n_\lambda) = \frac{1}{2}(\dim K_\mathbb{R} - \dim T^c) - \operatorname{codim}_\mathbb{C} O^K_\lambda.$$

Therefore

$$\begin{aligned} (5.16) \qquad & \dim C_\lambda + \Delta - \dim C_\mu \\ = & \left(\dim C_\lambda - \frac{1}{2}(d_\lambda + n_\lambda)\right) - \left(\dim C_\mu - \frac{1}{2}(d_\mu + n_\mu)\right) \\ = & \operatorname{codim}_\mathbb{C} O^K_\mu - \operatorname{codim}_\mathbb{C} O^K_\lambda = \dim_\mathbb{C} O^K_\lambda - \dim_\mathbb{C} O^K_\mu \end{aligned}$$

Let $i^K_\lambda : O^K_\lambda \hookrightarrow X$ be the inclusion of the K-orbits. Then $H^*({}^K L^\lambda_{x_\mu}, \underline{\mathbf{k}}_{\lambda,\theta})$ is the stalk of $i^K_{\lambda*}\underline{\mathbf{k}}_{\lambda,\theta}$ at x_μ. By [13, Proposition 4.1] (a result due to Beilinson and Bernstein), i^K_λ is an affine map. Therefore $i^K_{\lambda*}\underline{\mathbf{k}}_{\lambda,\theta}[\dim_\mathbb{C} O^K_\lambda]$ is perverse, and the stalk of $i^K_{\lambda*}\underline{\mathbf{k}}_{\lambda,\theta}$ vanishes in degrees $> \dim_\mathbb{C} O^K_\lambda - \dim_\mathbb{C} O^K_\mu = \dim C_\lambda + \Delta - \dim C_\mu$ (by (5.16)). This proves (5.14) and confirms (4.3). □

In this present setting, we further contract the notation by putting $\mathcal{M}_{G_\mathbb{R}} := \mathcal{M}_{G_\mathbb{R},X}$. We denote the t-structure on $\mathcal{M}_{G_\mathbb{R}}$ given by the perversity function p by $(\mathcal{M}^{\leq 0}_{G_\mathbb{R}}, \mathcal{M}^{\geq 0}_{G_\mathbb{R}})$, and denote its heart by $\mathcal{P}_{G_\mathbb{R}}$.

The Matsuki equivalence functors of [18, Theorem 6.6 (2)] are compatible with passing to the completion. By the general argument of 2.3 in [3] the following result is now also a formal consequence of [18, Theorems 5.7 and 6.6].

Theorem 5.6. *The Matsuki correspondence for sheaves is a Ringel duality between the category of perverse K-equivariant B-free-monodromic sheaves on G and $\mathcal{P}_{G_\mathbb{R}}$, i.e. it is an equivalence of derived categories, which sends standard objects to costandard. In particular, it takes projective perverse sheaves to the tilting sheaves in $\mathcal{P}_{G_\mathbb{R}}$.*

5.7. **Quasisplit, split rank 1 case.** The following three cases together with the complex group cases $G = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}), G_{\mathbb{R}} = \mathrm{SL}_2(\mathbb{C})$ and $G = \mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C}), G_{\mathbb{R}} = \mathrm{PGL}_2(\mathbb{C})$ are the only quasisplit real forms, whose maximally split torus has rank 1. The latter complex case is well understood and fits into the classical theory for complex groups. We will omit its detailed discussion.

1) Let $G = \mathrm{SL}_2(\mathbb{C})$ and $G_{\mathbb{R}} = \mathrm{SL}_2(\mathbb{R})$. Respectively, we have $K = \mathrm{SO}_2(\mathbb{C})$ and $K_{\mathbb{R}} = \mathrm{SO}_2(\mathbb{R})$. The flag variety is $X = \mathbb{P}^1 = \mathbb{P}^1(\mathbb{C}) = S^2$ and $\widetilde{X} = S^3 \subset \mathbb{C}^2 - 0 = G/U$.

There are three $G_{\mathbb{R}}$-orbits on $\mathbb{P}^1$: upper and lower half planes $\mathbb{H}_+, \mathbb{H}_-$ and the real flag variety $\mathbb{P}^1(\mathbb{R}) = S^1$. We will label them by $I = \{+, -, 0\}$. The stabilizer of a point in $\mathbb{H}_+$ or $\mathbb{H}_-$ is conjugated to $\mathrm{SO}_2(\mathbb{R})$ and $n_+ = n_- = 0$. The constant local systems are the only $G_{\mathbb{R}}$-equivariant free-monodromic local systems on $\pi^{-1}(\mathbb{H}_+)$ and $\pi^{-1}(\mathbb{H}_-)$. The stablizer of a point in $\mathbb{P}^1(\mathbb{R})$ is conjugated to the group of real points of the Borel subgroup $B(\mathbb{R}) := \{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} | a \in \mathbb{R}^\times, b \in \mathbb{R}\}$. Since $\pi_0(B(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(B(\mathbb{R})) = 0$, we have $n_0 = 1$ and there are two indecomposable $G_{\mathbb{R}}$-equivariant free-monodromic local systems $\mathcal{L}_{0,\mathrm{triv}}$ and $\mathcal{L}_{0,\mathrm{sgn}}$ on the closed orbit, corresponding to the trivial and sign characters of $\pi_0(B(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$.

We see that $p_+ = p_- = p_0 = 1$ and the abelian category $\mathcal{P}_{G_{\mathbb{R}}}$ is (up to a shift) the category of constructible free-monodromic sheaves in the orbit stratification.

The category $\mathcal{M}_{G_{\mathbb{R}}}$ is the direct sum of two blocks: $\mathcal{M}^{\circ}_{G_{\mathbb{R}}} \oplus \mathcal{M}^{\mathrm{sgn}}_{G_{\mathbb{R}}}$. The block $\mathcal{M}^{\mathrm{sgn}}_{G_{\mathbb{R}}}$ is generated by the standard object $\widetilde{\Delta}_{0,\mathrm{sgn}}$ (extension by zero of $\mathcal{L}_{0,\mathrm{sgn}}$), and the other standard objects $\widetilde{\Delta}_+, \widetilde{\Delta}_-$ and $\widetilde{\Delta}_{0,\mathrm{triv}}$ generate the other block $\mathcal{M}^{\circ}_{G_{\mathbb{R}}}$. Correspondingly $\mathcal{P}_{G_{\mathbb{R}}} = \mathcal{P}^{\circ}_{G_{\mathbb{R}}} \oplus \mathcal{P}^{\mathrm{sgn}}_{G_{\mathbb{R}}}$.

Below we focus on the block $\mathcal{P}^{\circ}_{G_{\mathbb{R}}}$. We represent an object in $\mathcal{P}^{\circ}_{G_{\mathbb{R}}}$ by the following diagram

$$\begin{tikzcd} V_+ & V_0 \arrow[l, "s_+"'] \arrow[r, "s_-"] \arrow[loop above, "m"] & V_- \end{tikzcd},$$

where V_+, V_0 and V_- are the vector spaces of stalks on $\pi^{-1}(\mathbb{H}_+), \pi^{-1}(\mathbb{P}^1(\mathbb{R}))$ and $\pi^{-1}(\mathbb{H}_+)$, $m \colon V_0 \to V_0$ is the prounipotent monodromy operator along the fibers of π, and $s_\pm \colon \mathrm{Coker}(m-1) \to V_\pm$ are the cospecialisation maps. In these terms we have

$$\widetilde{\Delta}_+ = \left(\begin{tikzcd} \mathbf{k} & 0 \arrow[l] \arrow[r] \arrow[loop above] & 0 \end{tikzcd} \right), \quad \widetilde{\Delta}_- = \left(\begin{tikzcd} 0 & 0 \arrow[l] \arrow[r] \arrow[loop above] & \mathbf{k} \end{tikzcd} \right),$$

$$\widetilde{\nabla}_+ = \left(\begin{tikzcd} \mathbf{k} & \mathbf{k} \arrow[l, "\mathrm{id}"'] \arrow[r] \arrow[loop above, "\mathrm{id}"] & 0 \end{tikzcd} \right), \quad \widetilde{\Delta}_+ = \left(\begin{tikzcd} 0 & \mathbf{k} \arrow[l] \arrow[r, "\mathrm{id}"] \arrow[loop above, "\mathrm{id}"] & \mathbf{k} \end{tikzcd} \right),$$

$$\widetilde{\Delta}_0 = \widetilde{\nabla}_0 = \mathcal{T}_0 = \left(\begin{tikzcd} 0 & \mathbf{k}[[x]] \arrow[l] \arrow[r] \arrow[loop above, "(1+x)\cdot"] & 0 \end{tikzcd} \right).$$

We see that the following are the tilting extensions of the local systems on the open orbits

$$\mathcal{T}_+ = \left(\begin{tikzcd} \mathbf{k} & \mathbf{k}[[x]] \arrow[l, "\mathrm{id}"'] \arrow[r] \arrow[loop above, "(1+x)\cdot"] & 0 \end{tikzcd} \right), \quad \mathcal{T}_- = \left(\begin{tikzcd} 0 & \mathbf{k}[[x]] \arrow[l] \arrow[r, "\mathrm{id}"] \arrow[loop above, "(1+x)\cdot"] & \mathbf{k} \end{tikzcd} \right).$$

2) Let $G = \mathrm{PGL}_2(\mathbb{C})$ and $G_{\mathbb{R}} = \mathrm{PGL}_2(\mathbb{R})$. Respectively, we have $K = \mathrm{O}_2(\mathbb{C})/\{\pm I_2\}$ and $K_{\mathbb{R}} = \mathrm{O}_2(\mathbb{R})/\{\pm I_2\}$. The flag variety is $X = \mathbb{P}^1 = \mathbb{P}^1(\mathbb{C}) = S^2$ and $\widetilde{X} = \mathbb{P}^3(\mathbb{R}) = (\mathbb{C}^2 - 0)/\mathbb{R}^\times = G/U$.

There are two $G_{\mathbb{R}}$-orbits on $\mathbb{P}^1$: the real flag variety $\mathbb{P}^1(\mathbb{R})$ and its complement $\mathbb{H}_\pm = \mathbb{H}_+ \coprod \mathbb{H}_-$. We label them by $I = \{0, h\}$. The stablizer of a point in $\mathbb{P}^1(\mathbb{R})$ is conjugated to the group of real points of the Borel subgroup $B(\mathbb{R}) := \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, c \in \mathbb{R}^\times, b \in \mathbb{R}\}/\{\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} | d \in \mathbb{R}^\times\}$. Since $\pi_0(B(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(B(\mathbb{R})) = 0$, we have $n_0 = 1$, there are two indecomposable $G_{\mathbb{R}}$-equivariant free-monodromic local systems $\mathcal{L}_{0,\mathrm{triv}}$ and $\mathcal{L}_{0,\mathrm{sgn}}$ on the closed orbit corresponding to the trivial and sign characters of $\pi_0(B(\mathbb{R}))$. The stabilizer of a point in $\mathbb{H}$ is conjugated to $\mathrm{SO}_2(\mathbb{R})/\{\pm 1\}$ and $n_h = 0$.

We see that $p_h = p_0 = 1$ and the abelian category $\mathcal{P}_{G_{\mathbb{R}}}$ is (up to a shift) the category of constructible free-monodromic sheaves in the orbit stratification. We represent an object of this category by the following diagram

$$\begin{tikzcd} V_{\mathrm{triv}} \arrow[r, "s_{\mathrm{triv}}"] \arrow[loop above, "m_{\mathrm{triv}}"] & V_h & V_{\mathrm{sgn}} \arrow[l, "s_{\mathrm{sgn}}"'] \arrow[loop above, "m_{\mathrm{sgn}}"] \end{tikzcd},$$

where V_h is the stalk on $\pi^{-1}(\mathbb{H}_\pm)$, V_{triv} and V_{sgn} are eigenspaces of the stalk at a point of $\pi^{-1}(\mathbb{P}^1(\mathbb{R}))$ corresponding, respectively, to the trivial and sign characters of $\pi_0(B(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$. The maps $m_{\mathrm{triv}} : V_{\mathrm{triv}} \to V_{\mathrm{triv}}$

and $m_{\mathrm{sgn}} : V_{\mathrm{sgn}} \to V_{\mathrm{sgn}}$ are the prounipotent monodromy automorphisms along the fibers of π, and $s_{\mathrm{triv}}\colon \mathrm{Coker}(m_{\mathrm{triv}} - 1) \to V_h$, $s_{\mathrm{sgn}}\colon \mathrm{Coker}(m_{\mathrm{sgn}} - 1) \to V_h$ are the cospecialization maps. In these terms we have

$$\widetilde{\Delta}_{0,\mathrm{triv}} = \widetilde{\nabla}_{0,\mathrm{triv}} = \mathfrak{T}_{0,\mathrm{triv}} = \left(\begin{tikzcd} \mathbf{k}[[x]] \arrow[loop above, "(1+x)\cdot"] \arrow[r] & 0 & 0 \arrow[l] \arrow[loop above] \end{tikzcd} \right),$$

$$\widetilde{\Delta}_{0,\mathrm{sgn}} = \widetilde{\nabla}_{0,\mathrm{sgn}} = \mathfrak{T}_{0,\mathrm{sgn}} = \left(\begin{tikzcd} 0 \arrow[loop above] \arrow[r] & 0 & \mathbf{k}[[x]] \arrow[l] \arrow[loop above, "(1+x)\cdot"] \end{tikzcd} \right),$$

$$\widetilde{\Delta}_h = \left(\begin{tikzcd} 0 \arrow[loop above] \arrow[r] & \mathbf{k} & 0 \arrow[l] \arrow[loop above] \end{tikzcd} \right), \widetilde{\nabla}_h = \left(\begin{tikzcd} \mathbf{k} \arrow[loop above, "\mathrm{id}"] \arrow[r, "\mathrm{id}"] & \mathbf{k} & \mathbf{k} \arrow[l, "\mathrm{id}"'] \arrow[loop above, "\mathrm{id}"] \end{tikzcd} \right).$$

We see that the following is the tilting extensions of the local system on the open orbit

$$\mathfrak{T}_h = \left(\begin{tikzcd} \mathbf{k}[[x]] \arrow[loop above, "(1+x)\cdot"] \arrow[r, "\mathrm{id}"] & \mathbf{k} & \mathbf{k}[[x]] \arrow[l, "\mathrm{id}"'] \arrow[loop above, "(1+x)\cdot"] \end{tikzcd} \right).$$

3) Let $G = \mathrm{SL}_3(\mathbb{C})$ and $G_{\mathbb{R}} = \mathrm{SU}(2,1)$ or $G = \mathrm{PGL}_3(\mathbb{C})$ and $G_{\mathbb{R}} = \mathrm{PU}(2,1)$. These cases have an identical geometry and, so we will discuss only the first one.

The six orbits are defined by the signature of the hermitian form on each vector space in the flag. Namely, we put $I = \{0| + 0, 0| + -, +| + 0, -| + -, +| + -, +| + +\}$, where to the left of $|$ we have a signature of the hermitian form restricted to V_1 and to the right restricted on V_2 for a given flag $V_1 \subset V_2$. The poset of the orbits is

$$\begin{tikzcd}
 & & 0\mid+0 \arrow[dl] \arrow[dr] & & \\
 & 0\mid+- \arrow[dl] \arrow[dr] & & +\mid+0 \arrow[dl] \arrow[dr] & \\
-\mid+- & & +\mid+- & & +\mid++.
\end{tikzcd}$$

We have

$$n_{0|+0} = n_{0|+-} = n_{+|+0} = 1, n_{-|+-} = n_{+|+-} = n_{+|++} = 0$$

and

$$d_{0|+0} = 3, d_{0|+-} = d_{+|+0} = 5, d_{-|+-} = d_{+|+-} = d_{+|++} = 6.$$

The stabilizer of a point in each orbit is connected.

Closures $\overline{O}^{\mathbb{R}}_{0|+-}$ and $\overline{O}^{\mathbb{R}}_{+|+0}$ are smooth and are tangent to each other along the closed orbit $O^{\mathbb{R}}_{0|+0}$. We conclude that the stalks of $\widetilde{\Delta}_{-|+-}, \widetilde{\Delta}_{+|++}$ vanish on the nonopen orbits, the stalks of $\widetilde{\nabla}_{-|+-}, \widetilde{\nabla}_{+|++}$ are one dimensional at the point of orbits in the closure of the corresponding open orbit with the stalks at the points of the closed orbit being at perverse degree -1 and other being at perverse degree 0. The stalks of $\widetilde{\nabla}_{+|+-}$ are one dimensional at perverse degree 0 at points of $O^{\mathbb{R}}_{+|+-}, O^{\mathbb{R}}_{0|+-}, O^{\mathbb{R}}_{+|+0}$ and at points of $O^{\mathbb{R}}_{0|+0}$ the stalks are one dimensional at degree 0 and one dimensional at degree -1. The stalks of $\widetilde{\Delta}_{+|+-}$ vanish at the points of $O^{\mathbb{R}}_{0|+-}, O^{\mathbb{R}}_{+|+0}$ and at points of $O^{\mathbb{R}}_{0|+0}$ the stalks are one dimensional at perverse degree 0.

For $\lambda \in I$ put $\mathcal{C}_\lambda := \mathrm{Cone}(\mathcal{T}_\lambda \to \widetilde{i}_{0|+0,*}\widetilde{i}^*_{0|+0}\mathcal{T}_\lambda)$. It follows that we have the exact triangles:

$$\widetilde{\Delta}_{-|+-} \to \mathcal{C}_{-|+-} \to \widetilde{\Delta}_{0|+-} \to, \tag{5.17}$$

$$\widetilde{\Delta}_{+|++} \to \mathcal{C}_{+|++} \to \widetilde{\Delta}_{+|+0} \to, \tag{5.18}$$

$$\widetilde{\Delta}_{+|+-} \to \mathcal{C}_{+|+-} \to \widetilde{\Delta}_{0|+-} \oplus \widetilde{\Delta}_{+|+0} \to . \tag{5.19}$$

6. Hecke action

6.1. **Hecke category.** If we view the complex group G as a real group, i.e., $R_{\mathbb{C}/\mathbb{R}}G$, its complexification $(R_{\mathbb{C}/\mathbb{R}}G)_{\mathbb{C}}$ is $G \times G'$, where G' is G

equipped with the opposite complex structure. Similarly, the flag variety of $(R_{\mathbb{C}/\mathbb{R}}G)_{\mathbb{C}}$ is $X \times X'$ (where X' is X equipped with the opposite complex structure). We can identify $\mathcal{M}_{R_{\mathbb{C}/\mathbb{R}}G}$ with the (free-monodromic) *Hecke category*

$$\mathcal{H}_G = \widehat{D}^b_G(\widetilde{X} \times \widetilde{X})_{T^c \times T^c-\mathrm{mon}}. \tag{6.1}$$

It is equivalent to the triangulated category $\widehat{D}^b_U(\widetilde{X})_{T^c-\mathrm{mon}}$, which was studied in [9] and [7].

Note that in this case we have $\widetilde{I} \xrightarrow{\sim} I = W$, where W is the Weyl group of G, which we consider equipped with the Bruhat order. For $w \in W$ let $X^2_w \subset X^2$ be the corresponding G-orbit, and $\widetilde{X}^2_w$ be its oreimage in $\widetilde{X}^2$. In particular, for $w = e$ (identity in W), $\widetilde{X}^2_e$ is the preimage of the diagonal $\Delta(X) \subset X^2$ in $\widetilde{X}^2$.

For any $w \in W$, we have that $T^c_w \subset T^c \times T^c$ is the graph of the w-action on T^c. In particular, $n_w = r = \dim T^c$. Also $\dim_{\mathbb{R}} X^2_w = 2(\dim_{\mathbb{C}} X + \ell(w))$. The perversity function is $p_w = \dim_{\mathbb{C}} X + \ell(w) + \lfloor r/2 \rfloor$. We will use a slightly different perversity function $p' : W \to \mathbb{Z}$

$$p'_w = \ell(w) + r \tag{6.2}$$

to define a t-structure on $\mathcal{H}_G$. Denote its heart by $\mathcal{H}_G^{\heartsuit}$.

We have the free-monodromic standard orbjects and costandard objects $\widetilde{\Delta}_w, \widetilde{\nabla}_w \in \mathcal{H}_G^{\heartsuit}$ being the !- and $*$- extensions of the free-monodromic G-equivariant local system on $\widetilde{X}^2_w$ placed in degree $-p'_w$. In particular, we denote $\widetilde{\Delta}_e = \widetilde{\nabla}_e$ by $\widetilde{\delta}$. This is a free-monodromic local system on the closed stratum $\widetilde{X}^2_e$ placed in degree $-r$. If we identify $\widetilde{X}^2_e$ with $\widetilde{X} \times T^c$ such that $(\widetilde{x}, t) \in \widetilde{X} \times T^c$ corresponds to $(\widetilde{x}, \widetilde{x}t)$, then $\widetilde{\delta}$ is the extension by zero of $\mathbf{k} \boxtimes \mathcal{L}[r]$ on $\widetilde{X} \times T^c$.

We define a monoidal structure on $\mathcal{H}_G$ by convolution. Let $\mathrm{pr}_{ij} : \widetilde{X}^3 \to \widetilde{X}^2$ be the projection to the (i,j)-factors. The convolution product on $\mathcal{H}_G$ is defined using

$$\mathcal{K}_1 \star \mathcal{K}_2 := \mathrm{pr}_{13*}(\mathrm{pr}_{12}^* \mathcal{K}_1 \otimes \mathrm{pr}_{23}^* \mathcal{K}_2) \tag{6.3}$$

for $\mathcal{K}_1, \mathcal{K}_2 \in \mathcal{H}_G$. This equips $\mathcal{H}_G$ with a monoidal structure with monoidal unit $\widetilde{\delta}$. If we identify $\mathcal{H}_G$ with $\widehat{D}^b_U(\widetilde{X})_{T^c-\mathrm{mon}}$, this monoidal structure is the same as the one defined in [9, Section 4.3] and [7, Section 7].

6.2. **Hecke action.** We define a right action of $\mathcal{H}_G$ on $\mathcal{M}_{G_{\mathbb{R}}}$ as follows.

For a $G_{\mathbb{R}}$-equivariant, T^c-monodromic sheaf $\mathcal{F}_1$ on $\widetilde{X}$ and an U-equivariant T^c-monodromic sheaf $\mathcal{F}_2$ on $\widetilde{X}$ we have a $G_{\mathbb{R}}$-equivariant,

T^c-monodromic sheaf $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ on the fibered product $G \times^{T^{>0}U} \widetilde{X}$. We push it forward with respect to the action map $G \times^{T^{>0}U} \widetilde{X} \to \widetilde{X}$ to define a $G_{\mathbb{R}}$-equivariant, T^c-monodromic sheaf $\mathcal{F}_1 \star \mathcal{F}_2$ on $\widetilde{X}$. As in Lemma 4.3.1 in [9] or Lemma 7.4 in [7] the operation is compatible with passing to the limit and we have an action of the monoidal category $\mathcal{H}_G$ on $\mathcal{M}_{G_{\mathbb{R}}}$:

$$\star \colon \mathcal{M}_{G_{\mathbb{R}}} \times \mathcal{H}_G \to \mathcal{M}_{G_{\mathbb{R}}}.$$

Let $\mathcal{F} \in \mathcal{M}_{G_{\mathbb{R}}}$ and $\mathcal{K} \in \mathcal{H}_G$. Consider the two projections

$$\mathrm{pr}_1, \mathrm{pr}_2 : \widetilde{X} \times \widetilde{X} \to \widetilde{X}. \tag{6.4}$$

Define

$$\mathcal{F} \star \mathcal{K} = \mathrm{pr}_{2*}(\mathrm{pr}_1^* \mathcal{F} \otimes \mathcal{K}). \tag{6.5}$$

More precisely, we start with the above definition for the uncompleted monodromic categories and pass to the limit as in [9, Lemma 4.3.1] or [7, Lemma 7.4]. This defines a right action of the Hecke category $\mathcal{H}_G$ on $\mathcal{M}_{G_{\mathbb{R}}}$:

$$\star \colon \mathcal{M}_{G_{\mathbb{R}}} \times \mathcal{H}_G \to \mathcal{M}_{G_{\mathbb{R}}}.$$

We now compute the action of some (co)standard objects in $\mathcal{H}_G$ on some (co)standard objects of $\mathcal{M}_{G_{\mathbb{R}}}$. The calculation is parallel to [17, Lemma 3.5] (which is for K-orbits on X).

Lemma 6.3. *Let $(\lambda, \chi) \in \widetilde{I}$ and $s \in \mathbf{W}$ be a simple reflection. Let α_s be the corresponding simple root in the based root system Φ_λ. We will use the notations from Lemma 2.12 and Section 5.2.*

Suppose $O_\lambda^{\mathbb{R}}$ is closed inside $\pi_s^{-1}\pi_s(O_\lambda^{\mathbb{R}})$. Then we have the following cases:

(1) *If α_s is a complex root and $\sigma\alpha > 0$. Let $\mu = \lambda \cdot s > \lambda$ be such that $O_\mu^{\mathbb{R}} = \pi_s^{-1}\pi_s(O_\lambda^{\mathbb{R}}) - O_\lambda^{\mathbb{R}}$. Then there is a canonical isomorphism $\pi_0(\mathbf{T}_\lambda^c) \cong \pi_0(\mathbf{T}_\mu^c)$, through which we may view χ as a character of $\pi_0(\mathbf{T}_\mu^c)$. We then have*

$$\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s \cong \widetilde{\Delta}_{\mu,\chi}, \quad \widetilde{\nabla}_{\lambda,\chi} \star \widetilde{\nabla}_s \cong \widetilde{\nabla}_{\mu,\chi},$$

$$\widetilde{\Delta}_{\mu,\chi} \star \widetilde{\nabla}_s \cong \widetilde{\Delta}_{\lambda,\chi}, \quad \widetilde{\nabla}_{\mu,\chi} \star \widetilde{\Delta}_s \cong \widetilde{\nabla}_{\lambda,\chi}.$$

(2) *If α_s is a compact imaginary root. Then*

$$\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s \cong \widetilde{\Delta}_{\lambda,\chi}[-1], \quad \widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\nabla}_s \cong \widetilde{\Delta}_{\lambda,\chi}[1],$$

$$\widetilde{\nabla}_{\lambda,\chi} \star \widetilde{\Delta}_s \cong \widetilde{\nabla}_{\lambda,\chi}[-1], \quad \widetilde{\nabla}_{\lambda,\chi} \star \widetilde{\nabla}_s \cong \widetilde{\nabla}_{\lambda,\chi}[1].$$

(3) *If α_s is a real root. Then exactly one of the following options holds:*

- *The local system* $\underline{\mathbf{k}}_{\lambda,\chi}$ *on* $O_\lambda^{\mathbb{R}}$ *does not extend to* $\pi_s^{-1}\pi_s(O_\lambda^{\mathbb{R}})$. [2] *Then*
$$\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s \cong \widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\nabla}_s \cong \widetilde{\Delta}_{\lambda,\chi},$$
$$\widetilde{\nabla}_{\lambda,\chi} \star \widetilde{\nabla}_s \cong \widetilde{\nabla}_{\lambda,\chi} \star \widetilde{\Delta}_s \cong \widetilde{\nabla}_{\lambda,\chi}.$$
- *The local system* $\underline{\mathbf{k}}_{\lambda,\chi}$ *on* $O_\lambda^{\mathbb{R}}$ *extends (uniquely) to a* $G_{\mathbb{R}}$*-equivariant local system* $\widetilde{\underline{\mathbf{k}}_{\lambda,\chi}}$ *on* $\pi_s^{-1}\pi_s(O_\lambda^{\mathbb{R}})$*, and* α_s *is type II real.*
Let $\mu > \lambda$ *be such that* $O_\mu^{\mathbb{R}} = \pi_s^{-1}\pi_s(O_\lambda^{\mathbb{R}}) - O_\lambda^{\mathbb{R}}$. *Let* $\underline{\mathbf{k}}_{\mu,\psi}$ *be the restrictions of* $\widetilde{\underline{\mathbf{k}}_{\lambda,\chi}}$ *to* $O_\mu^{\mathbb{R}}$. *There are distinguished triangles*

$$\widetilde{\Delta}_{\mu,\psi} \to \widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s \to \widetilde{\Delta}_{(\lambda,\chi)\cdot s} \to, \tag{6.6}$$
$$\widetilde{\nabla}_{(\lambda,\chi)\cdot s} \to \widetilde{\nabla}_{\lambda,\chi} \star \widetilde{\nabla}_s \to \widetilde{\nabla}_{\mu,\psi} \to, \tag{6.7}$$
$$\widetilde{\Delta}_{\mu,\psi}[-1] \to \widetilde{\Delta}_{\mu,\psi} \star \widetilde{\nabla}_s \to \widetilde{\Delta}_{\lambda,\chi} \oplus \widetilde{\Delta}_{(\lambda,\chi)\cdot s} \to, \tag{6.8}$$
$$\widetilde{\nabla}_{\lambda,\chi} \oplus \widetilde{\nabla}_{(\lambda,\chi)\cdot s} \to \widetilde{\nabla}_{\mu,\psi} \star \widetilde{\Delta}_s \to \widetilde{\nabla}_{\mu,\psi}[1] \to . \tag{6.9}$$

- *The local system* $\underline{\mathbf{k}}_{\lambda,\chi}$ *on* $O_\lambda^{\mathbb{R}}$ *extends (uniquely) to a* $G_{\mathbb{R}}$*-equivariant local system* $\widetilde{\underline{\mathbf{k}}_{\lambda,\chi}}$ *on* $\pi_s^{-1}\pi_s(O_\lambda^{\mathbb{R}})$*, and* α_s *is type I real.*
Let $\mu^+, \mu^- > \lambda$ *be such that* $O_{\mu^+}^{\mathbb{R}} \coprod O_{\mu^-}^{\mathbb{R}} = \pi_s^{-1}\pi_s(O_\lambda^{\mathbb{R}}) - O_\lambda^{\mathbb{R}}$. *Let* $\underline{\mathbf{k}}_{\mu^+,\psi^+}$ *and* $\underline{\mathbf{k}}_{\mu^-,\psi^-}$ *be the restrictions of* $\widetilde{\underline{\mathbf{k}}_{\lambda,\chi}}$ *to* $O_{\mu^+}^{\mathbb{R}}$ *and* $O_{\mu^-}^{\mathbb{R}}$. *There are distinguished triangles*

$$\widetilde{\Delta}_{\mu^+,\psi^+} \oplus \widetilde{\Delta}_{\mu^-\psi^-} \to \widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s \to \widetilde{\Delta}_{\lambda,\chi} \to, \tag{6.10}$$
$$\widetilde{\Delta}_{\mu^\mp,\psi^\mp}[-1] \to \widetilde{\Delta}_{\mu^\pm,\psi^\pm} \star \widetilde{\nabla}_s \to \widetilde{\Delta}_{\lambda,\chi} \to, \tag{6.11}$$
$$\widetilde{\nabla}_{\lambda,\chi} \to \widetilde{\nabla}_{\lambda,\chi} \star \widetilde{\nabla}_s \to \widetilde{\nabla}_{\mu^+,\psi^+} \oplus \widetilde{\nabla}_{\mu^-,\psi^-} \to, \tag{6.12}$$
$$\widetilde{\nabla}_{\lambda,\chi} \to \widetilde{\nabla}_{\mu^\pm,\psi^\pm} \star \widetilde{\Delta}_s \to \widetilde{\nabla}_{\mu^\mp,\psi^\mp}[1] \to \tag{6.13}$$

Proof. In the proof we will compute the non-monodromic versions $\Delta_{\lambda,\chi} \overline{\star} \Delta_s$. Here $\Delta_w \in D_G(X \times X)$ is the !-extension of $\underline{\mathbf{k}}[\ell(w)]$ on the G-orbit X_w^2, and for $\mathcal{F} \in D_{G_{\mathbb{R}}}(X)$, $\mathcal{K} \in D_G(X \times X)$, $\mathcal{F} \overline{\star} \mathcal{K} := \mathrm{pr}_{2*}(\mathrm{pr}_1^* \mathcal{F} \otimes \mathcal{K})$. Note the degree shift for Δ_w differs from the one for $\widetilde{\Delta}_w$ by $\rho = \dim T^c$. Using Lemma 4.8 we have

$$\pi_\dagger(\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s) \cong \pi_\dagger(\widetilde{\Delta}_{\lambda,\chi}) \overline{\star} \Delta_s \cong \Lambda_{\lambda,\bullet} \otimes (\Delta_{\lambda,\chi} \overline{\star} \Delta_s), \quad \forall (\lambda,\chi) \in \widetilde{I}. \tag{6.14}$$

[2] This happens if and only if the composition $\{\pm 1\} = \pi_0(\mathbb{R}^\times) \xrightarrow{\alpha_s^\vee} \pi_0(T_{\mathbb{R}}) \xrightarrow{\sim} \pi_0(\mathbf{T}_\lambda^c) \xrightarrow{\chi} \mathbf{k}^\times$ is nontrivial, where $O_\lambda^{\mathbb{R}}$ is attached to $T_{\mathbb{R}}$.

In order to recover stalks of $\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s$ from that of $\Delta_{\lambda,\chi}\overline{\star}\Delta_s$, we will use the following simple observation which follows from Corollary 4.6. Suppose $(\mu,\psi) \in \widetilde{I}$, $\mathcal{F} \in \mathcal{M}_{G_{\mathbb{R}}}$:

$$\text{If } i_\mu^* \pi_\dagger \mathcal{F} \cong \Lambda_{\mu,\bullet} \otimes \underline{\mathbf{k}}_{\mu,\psi} \otimes V \text{ as a free } \Lambda_{\mu,\bullet}\text{-module (where } V \text{ is a complex of } \mathbf{k}\text{-vector spaces), then } \widetilde{i}_\mu^* \mathcal{F} \cong \mathcal{L}_{\mu,\psi} \otimes V. \tag{6.15}$$

Choose a point $x \in O_\lambda^{\mathbb{R}}$ corresponding to a Borel B_x, and let $\mathbb{P}_s^1 = \pi_s^{-1}\pi_s(x)$. Let P_x be the parabolic containing B_x with negative root $-\alpha_s$. Note that $G_{\mathbb{R}} \cap P_x$ acts on $\mathbb{P}_s^1$. The stalk of $\Delta_{\lambda,\chi}\overline{\star}\Delta_s$ at $y \in \mathbb{P}_s^1$ is

$$i_y^*(\Delta_{\lambda,\chi}\overline{\star}\Delta_s) \cong \begin{cases} H_c^*(O_\lambda^{\mathbb{R}} \cap \mathbb{P}_s^1 - \{y\}, \underline{\mathbf{k}}_{\lambda,\chi})[p_\lambda + 1] & y \in O_\lambda^{\mathbb{R}}; \\ H^*(O_\lambda^{\mathbb{R}} \cap \mathbb{P}_s^1, \underline{\mathbf{k}}_{\lambda,\chi})[p_\lambda + 1] & y \notin O_\lambda^{\mathbb{R}}. \end{cases} \tag{6.16}$$

Moreover, in the case $y \notin O_\lambda^{\mathbb{R}}$ (corresponding to a Borel B_y), the inclusion $B_y \hookrightarrow P_x$ induces an isomorphism $\pi_0(G_{\mathbb{R}} \cap B_y) \xrightarrow{\sim} \pi_0(G_{\mathbb{R}} \cap P_x)$, and the isomorphism above is equivariant under the actions of $\pi_0(G_{\mathbb{R}} \cap B_y)$ on the left and $\pi_0(G_{\mathbb{R}} \cap P_x)$ on the right via this isomorphism.

Moreover, we have the triangule $\delta \to \Delta_s \to \mathrm{IC}_s \to$ in $D_G(X \times X)$ gives a distinguished triangle

$$\Delta_{\lambda,\chi} \to \Delta_{\lambda,\chi}\overline{\star}\Delta_s \to \pi_s^*\pi_{s*}\Delta_{\lambda,\chi}[1] \to \tag{6.17}$$

(1) In this case, $O_\lambda^{\mathbb{R}} \cap \mathbb{P}_s^1 = \{x\}$. By (6.16), $\Delta_{\lambda,\chi}\overline{\star}\Delta_s$ has nonzero (1-dimemsional) stalk only at $y \in \mathbb{P}_s^1 - \{x\}$, and $\pi_0(G_{\mathbb{R}} \cap B_y)$ acts via χ via the natural isomorphisms $\pi_0(G_{\mathbb{R}} \cap B_y) \cong \pi_0(G_{\mathbb{R}} \cap P_x) \cong \pi_0(G_{\mathbb{R}} \cap B_x) = \pi_0(\mathbf{T}_\lambda^c)$. We conclude that

$$\Delta_{\lambda,\chi} \star \Delta_s \cong \Delta_{\mu,\chi} \tag{6.18}$$

By (6.15), this implies $\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s \cong \widetilde{\Delta}_{\mu,\chi}$. The formula $\widetilde{\nabla}_{\lambda,\chi} \star \widetilde{\nabla}_s \cong \widetilde{\nabla}_{\mu,\chi}$ follows from the same argument from $\nabla_{\lambda,\chi} \star \nabla_s \cong \nabla_{\mu,\chi}$, which follows from (6.18) by Verdier duality. The third and the forth isomorphsims now follow from the fact that

$$\widetilde{\Delta}_s \star \widetilde{\nabla}_s \cong \widetilde{\nabla}_s \star \widetilde{\Delta}_s \cong \widetilde{\delta} \in \mathcal{H}_G \tag{6.19}$$

(see [7, Lemma 7.7 (1)]).

(2) In this case $\mathbb{P}_s^1 \subset O_\lambda^{\mathbb{R}}$. From (6.16) we see that $\Delta_{\lambda,\chi}\overline{\star}\Delta_s$ has stalk $H_c^*(\mathbb{P}_s^1 \backslash \{y\}, \mathbf{k})[p_\lambda + 1] \cong \mathbf{k}[p_\lambda - 1]$ at every point $y \in \mathbb{P}_s^1$. Moreover, the first map in (6.17) induces an isomorphism on degree $1 - p_\lambda$ stalks, and the monodromy on $\pi_s^*\pi_{s*}\Delta_{\lambda,\chi}$ is given by the same character χ, we conclude that the same is true for the monodromy of $\Delta_{\lambda,\chi}\overline{\star}\Delta_s$, i.e., $\Delta_{\lambda,\chi}\overline{\star}\Delta_s \cong \Delta_{\lambda,\chi}[-1]$. By (6.15), this implies $\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s \cong \widetilde{\Delta}_{\mu,\chi}[-1]$. The other formulae are proved similarly.

(3) Fix a σ-stable maximal torus $T \subset B_x$. Let L be the Levi subgroup of G generated by T and root spaces $\pm\alpha_s$. Then L is σ-stable with real form $L_{\mathbb{R}} = L^\sigma$. We have an L-equivariant isomorphism $\widetilde{X}_L \cong \pi^{-1}(\mathbb{P}^1_s)$. Therefore we may assume $G = L$. Note that the derived group of $L_{\mathbb{R}}$ is either $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{PGL}_2(\mathbb{R})$. The flag variety $X_L \cong \mathbb{P}^1$ is defined over $\mathbb{R}$ with real points the real projective line $\mathbb{P}^1(\mathbb{R}) = O^{\mathbb{R}}_\lambda$.

In the first option, $\underline{\mathbf{k}}_{\lambda,\chi}|_{\mathbb{P}^1(\mathbb{R})}$ has nontrivial monodromy, hence $\pi_{s*}\Delta_{\lambda,\chi} = 0$. By (6.17) we have an isomorphism $\Delta_{\lambda,\chi} \xrightarrow{\sim} \Delta_{\lambda,\chi}\overline{\star}\Delta_s$. By (6.14), we conclude that $\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s \cong \widetilde{\Delta}_{\lambda,\chi}$. The other isomorphisms follow from this one by Verdier duality and (6.19).

In the second option, we have $O^{\mathbb{R}}_\mu = \mathbb{P}^1_s - \mathbb{P}^1(\mathbb{R})$. We first compute $\widetilde{i}^*_\lambda(\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s)$. Using (6.16), the stalk of $\Delta_{\lambda,\chi}\overline{\star}\Delta_s$ at a point $y \in \mathbb{P}^1(\mathbb{R})$ is $H^*_c(\mathbb{P}^1(\mathbb{R}) - \{y\}, \underline{\mathbf{k}}_{\lambda,\chi})[p_\lambda + 1] \cong \mathbf{k}[p_\lambda]$. Moreover, the action of $\pi_0(G_{\mathbb{R}} \cap B_y)$ on $H^*_c(\mathbb{P}^1(\mathbb{R}) - \{y\}, \underline{\mathbf{k}}_{\lambda,\chi})$ is χ twisted by the sign character sgn_s (see (5.4)) through which $\pi_0(G_{\mathbb{R}} \cap B_y)$ acts on $H^1_c(\mathbb{P}^1(\mathbb{R}) - \{y\}) \cong H^1(\mathbb{P}^1(\mathbb{R}))$. We conclude that

$$i^*_\lambda(\Delta_{\lambda,\chi}\overline{\star}\Delta_s) \cong \underline{\mathbf{k}}_{\lambda,\chi\otimes\mathrm{sgn}_s}[p_\lambda] \tag{6.20}$$

By (6.15) we conclude that

$$\widetilde{i}^*_\lambda(\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s) \cong \mathcal{L}_{\lambda,\chi\otimes\mathrm{sgn}_s}[p_\lambda] = \mathcal{L}_{(\lambda,\chi)\cdot s}[p_\lambda]. \tag{6.21}$$

Next we show that

$$\widetilde{i}^*_\mu(\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s) \cong \mathcal{L}_{\mu,\psi}[p_\mu]. \tag{6.22}$$

For this, computing $i^*_\mu(\Delta_{\lambda,\chi}\overline{\star}\Delta_s)$ as a local system is not enough, since we need to keep track of the $\Lambda_{\mu,\bullet}$-action in order to recover $\widetilde{i}^*_\mu(\widetilde{\Delta}_{\lambda,\chi}\star\widetilde{\Delta}_s)$. Nevertheless we first use (6.16) to see the stalk of $\Delta_{\lambda,\chi}\overline{\star}\Delta_s$ at a point $y \in \mathbb{P}^1_s - \mathbb{P}^1(\mathbb{R})$ is equal to $H^*(\mathbb{P}^1(\mathbb{R}), \underline{\mathbf{k}}_{\lambda,\chi})[p_\lambda + 1]$. Using the long exact sequence attached to the triangle $j_!\underline{\mathbf{k}}_{\mu,\psi} \to \widetilde{\underline{\mathbf{k}}}_{\lambda,\chi} \to i_!\underline{\mathbf{k}}_{\lambda,\chi} \to$ on $\pi_s^{-1}\pi_s(O^{\mathbb{R}}_\lambda)$, we see that the action of $\pi_0(G_{\mathbb{R}} \cap B_y) \cong \pi_0(\mathbf{T}^c_\mu)$ on the stalk $i^*_y(\Delta_{\lambda,\chi}\overline{\star}\Delta_s)$ is via ψ. To show (6.22), it remains to show that

$$\widetilde{i}^*_\mu(\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s) \text{ is free-monodromic of rank 1 in degree } -p_\mu = -p_\lambda. \tag{6.23}$$

Let $A = (ZG)^\circ$, then we have an isogeny $A \times \mathrm{SL}_2 \to G$ defined over $\mathbb{R}$ with kernel either trivial or of order two. It is then sufficient to prove (6.23) for $A \times \mathrm{SL}_2$ (by pullback from $\widetilde{X}$), and eventually for $G_{\mathbb{R}} = \mathrm{SL}_2(\mathbb{R})$ (just for the statement (6.23)). In the rest of this paragraph we assume $G = \mathrm{SL}_2$. In this case, the statement (6.23) becomes

$$\text{For } \widetilde{y} \in \pi^{-1}(\mathbb{P}^1 - \mathbb{P}^1(\mathbb{R})),\ i^*_{\widetilde{y}}(\widetilde{\Delta}_{\lambda,\mathrm{triv}} \star \widetilde{\Delta}_s) \cong \mathbf{k}[1]. \tag{6.24}$$

We identify $G/U \cong \mathbb{A}^2\backslash\{0\}$ hence $G/UT^{>0} \cong S^3$ (unit sphere in $\mathbb{C}^2$). The preimage $\widetilde{O}_\lambda^{\mathbb{R}} \subset S^3$ consists of $(z_1, z_2) \in \mathbb{C}^2$ with $|z_1|^2 + |z_2|^2 = 1$ and $z_1/z_2 \in \mathbb{R} \cup \{\infty\}$. It is easy to see that $\widetilde{O}_\lambda^{\mathbb{R}}$ is a 2-dimensional torus. Let $\widetilde{y} = (y_1, y_2) \in S^3$. Then

$$i_{\widetilde{y}}^*(\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s) \cong H^*(\widetilde{O}_\lambda^{\mathbb{R}}, \mathcal{L}_{\lambda,\chi}[1] \otimes \epsilon^*\mathcal{L}[2]). \tag{6.25}$$

Here $\mathcal{L}$ is the rank one free-monodromic local system on S^1, and $\epsilon : \widetilde{O}_\lambda^{\mathbb{R}} \to S^1$ sends (z_1, z_2) to $(z_1y_2 - z_2y_1)/|z_1y_2 - z_2y_1|$, and $\epsilon^*\mathcal{L}[2]$ is the contribution of $\widetilde{\Delta}_s$ to the the fiber of the convolution. Consider the $G(\mathbb{R})$-equivariant embedding $\theta : S^1 \to \widetilde{O}_\lambda^{\mathbb{R}}$ sending $u + iv \mapsto (u, v)$. Then $\theta^*\mathcal{L}_{\lambda,\chi}$ is trivial since $G(\mathbb{R})$ acts transitively on S^1. However, calculation shows that $\epsilon \circ \theta : S^1 \to S^1$ is a homeomorphism. These facts combined imply that $\mathcal{L}_{\lambda,\chi} \otimes \epsilon^*\mathcal{L}$ is a rank one free-monodromic local system on the 2-dimensional torus $\widetilde{O}_\lambda^{\mathbb{R}}$. Using (6.25) we see that (6.24) holds. Now (6.22) is proved.

Combining (6.21) and (6.22) we get the distinguished triangle (6.6).

The same argument for showing (6.21) shows

$$\widetilde{i}_\lambda^*(\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\nabla}_s) \cong \mathcal{L}_{\lambda,\chi}[p_\lambda + 1]. \tag{6.26}$$

Here we are using that for $y \in \mathbb{P}^1(\mathbb{R})$, $i_y^*(\Delta_{\lambda,\chi} \overline{\star} \nabla_s) \cong H^*(\mathbb{P}^1(\mathbb{R}) - \{y\}, \underline{\mathbf{k}}_{\lambda,\chi})[p_\lambda + 1] \cong \mathbf{k}[p_\lambda + 1]$. On the other hand, we have

$$\widetilde{i}_\mu^*(\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s) \cong \widetilde{i}_\mu^*(\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\nabla}_s), \tag{6.27}$$

which is isomorphic to $\mathcal{L}_{\mu,\psi}[p_\mu]$ by (6.22). This together with (6.26) imply a distinguished triangle

$$\widetilde{\Delta}_{\mu,\psi} \to \widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\nabla}_s \to \widetilde{\Delta}_{\lambda,\chi}[1] \to \tag{6.28}$$

Replacing (λ, χ) by $(\lambda, \chi) \cdot s$, and shift by $[-1]$, we get

$$\widetilde{\Delta}_{\mu,\psi}[-1] \to \widetilde{\Delta}_{(\lambda,\chi)\cdot s} \star \widetilde{\nabla}_s[-1] \to \widetilde{\Delta}_{(\lambda,\chi)\cdot s} \to \tag{6.29}$$

Now we convolve (6.6) with $\widetilde{\nabla}_s$, rotating it and using (6.19) we obtain a distinguished triangle

$$\widetilde{\Delta}_{(\lambda,\chi)\cdot s} \star \widetilde{\nabla}_s[-1] \to \widetilde{\Delta}_{\mu,\psi} \star \widetilde{\nabla}_s \to \widetilde{\Delta}_{(\lambda,\chi)\cdot s} \to .$$

Combined with (6.29), observing that there is nontrivial nonzero extension between $\widetilde{\Delta}_{\lambda,\chi}$ and $\widetilde{\Delta}_{(\lambda,\chi)\cdot s}$, we get the asserted distinguished triangle (6.8).

To prove (6.7), we take the Verdier dual of (6.20) to get $i_\lambda^!(\nabla_{\lambda,\chi} \overline{\star} \nabla_s) \cong \underline{\mathbf{k}}_{(\lambda,\chi)\cdot s}[p_\lambda]$. Using (6.15) we get $\widetilde{i}_\lambda^!(\widetilde{\nabla}_{\lambda,\chi} \star \widetilde{\nabla}_s) \cong \mathcal{L}_{(\lambda,\chi)\cdot s}[p_\lambda]$. The calculation of $\widetilde{i}_\mu^!(\widetilde{\nabla}_{\lambda,\chi} \star \widetilde{\nabla}_s)$

boils down to the same thing as $\widetilde{i}^*_\mu(\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s)$, and we have $\widetilde{i}^!_\mu(\widetilde{\nabla}_{\lambda,\chi} \star \widetilde{\nabla}_s) \cong \mathcal{L}_{\mu,\psi}[p_\mu]$. Combining these facts we get the distinguished triangle (6.7). Then the same deduction from (6.6) to (6.8) allows us to deduce (6.9) from (6.7).

Finally consider the third option. In this case $O^{\mathbb{R}}_{\mu^+}$ and $O^{\mathbb{R}}_{\mu^+}$ are the two hemispheres of $\mathbb{P}^1 - \mathbb{P}^1(\mathbb{R})$, which we denote by $\mathbb{H}^+$ and $\mathbb{H}^-$. The calculations made for (6.6) shows (6.10); the argument for (6.8) gives a distinguished triangle

(6.30)
$$\widetilde{\Delta}_{\mu^+,\psi^+}[-1] \oplus \widetilde{\Delta}_{\mu^-,\psi^-}[-1] \to \widetilde{\Delta}_{\mu^+,\psi^+} \star \widetilde{\nabla}_s \oplus \widetilde{\Delta}_{\mu^-,\psi^-} \star \widetilde{\nabla}_s \to \widetilde{\Delta}^{\oplus 2}_{\lambda,\chi} \to .$$

To deduce (6.11), it remains to show that $\widetilde{i}^*_{\mu^+}(\widetilde{\Delta}_{\mu^+,\psi^+} \star \widetilde{\nabla}_s) = 0$, or $i^*_{\mu^+}(\Delta_{\mu^+,\psi^+} \overline{\star} \nabla_s) = 0$. For $y \in \mathbb{H}^+$, $i^*_y(\Delta_{\mu^+,\psi^+} \overline{\star} \nabla_s) \cong H^*_c(\mathbb{H}^+, j_* \underline{\mathbf{k}})[2]$ where $j : \mathbb{H}^+ - \{y\} \hookrightarrow \mathbb{H}^+$ is the inclusion. The vanishing of $H^*_c(\mathbb{H}^+, j_* \underline{\mathbf{k}})$ is clear.

The proofs of (6.12) and (6.13) are similar to those of (6.7) and (6.9). We omit details. □

Let $\mathrm{Tilt}(\mathcal{H}_G) \subset \mathcal{H}_G$ be the additive subcategory of free-monodromic tilting objects. The category $\mathrm{Tilt}(\mathcal{H}_G)$ is closed under convolution. See [9, Proposition 4.3.4] and [7, Lemma 7.8, Remark 7.9]. Hence $\mathrm{Tilt}(\mathcal{H}_G)$ (as an additive category) inherits a monoidal structure from $\mathcal{H}_G$.

Proposition 6.4. *For $\mathcal{T}_1$ in $\mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}})$ and $\mathcal{T}_2$ in $\mathrm{Tilt}(\mathcal{H}_G)$ the convolution product $\mathcal{T}_1 \star \mathcal{T}_2$ is in $\mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}})$. In other words, $\mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}})$ is a module category for $\mathrm{Tilt}(\mathcal{H}_G)$ under convolution.*

Proof. Every object $\mathcal{T} \in \mathrm{Tilt}(\mathcal{H}_G)$ is a direct summand of successive convolutions of $\mathcal{T}_s$ for simple reflections $s \in \mathbf{W}$ (see [9, Corollary 5.2.3] and [7, Remark 7.9]). Hence, it is sufficient to assume that $\mathcal{T}_2 = \mathcal{T}_s$. We start by checking that for any $(\lambda,\chi) \in \widetilde{I}$, the convolution product $\widetilde{\Delta}_{\lambda,\chi} \star \mathcal{T}_s$ is a successive extension of the objects of the form $\widetilde{\Delta}_{\mu,\psi}[-n]$ with $(\mu,\psi) \in \widetilde{I}$ and $n \geq 0$.

If λ and s are in position of one of the options of Lemma 6.3 we use the exact traingle $\tilde{\delta} \to \mathcal{T}_s \to \widetilde{\Delta}_{s_\alpha} \to$ and the calculation of $\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s$ in Lemma 6.3 to obtain needed filtration. Otherwise, we consider the exact triangle $\widetilde{\nabla}_s \to \mathcal{T}_s \to \tilde{\delta} \to$ and use the calculation of $\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\nabla}_s$ in Lemma 6.3 to conclude.

It follows that $\widetilde{\Delta}_{\lambda,\chi} \star \mathcal{T}_s$ is a successive extension of the objects of the form $\widetilde{\Delta}_{\mu,\psi}[-n]$, $n \geq 0$. By considering a standard filtration on $\mathcal{T}_1$, we see that the same is true for $\mathcal{T}_1 \star \mathcal{T}_s$. In particular,

$$\text{(6.31)} \qquad \begin{array}{l}\text{For each } \mu \in I, \text{ the restriction } \widetilde{i}_\mu^*(\mathcal{T}_1 \star \mathcal{T}_s) \text{ is a bounded} \\ \text{complex of free-monodromic local systems concentrated} \\ \text{in degrees} \geq -p_\mu.\end{array}$$

Similarly, using the calculations in Lemma 6.3, the convolution product $\widetilde{\nabla}_{\lambda,\chi} \star \mathcal{T}_s$ is a successive extension of the objects of the form $\widetilde{\nabla}_{\mu,\psi}[\geq 0]$ and $(\mu,\psi) \in \widetilde{I}$. By considering the costandrad filtration on $\mathcal{T}_1$, we see that the same is true for $\mathcal{T}_1 \star \mathcal{T}_s$. In particular,

$$\text{(6.32)} \qquad \text{For each } \mu \in I, \ \widetilde{i}_\mu^!(\mathcal{T}_1 \star \mathcal{T}_s) \text{ is in degrees } \leq -p_\mu.$$

By Lemma 4.9(1), $\widetilde{i}_\mu^* \widetilde{\nabla}_{\mu',\psi}$ lies in degrees $\leq -p_\mu$. Since $\mathcal{T}_1 \star \mathcal{T}_s$ is a successive extension of $\widetilde{\nabla}_{\mu',\psi}[\geq 0]$, $\widetilde{i}_\mu^*(\mathcal{T}_1 \star \mathcal{T}_s)$ also lies in degrees $\leq -p_\mu$. Combined with (6.31), we conclude that $\widetilde{i}_\mu^*(\mathcal{T}_1 \star \mathcal{T}_s)$ is concentrated in degree $-p_\mu$ and is free-monodromic.

By Lemma 4.9(2), $\widetilde{i}_\mu^! \widetilde{\Delta}_{\mu',\psi}$ is a complex of free-monodromic local systems in degrees $\geq -p_\mu$. Since $\mathcal{T}_1 \star \mathcal{T}_s$ is a successive extension of $\widetilde{\Delta}_{\mu',\psi}[\leq 0]$, $\widetilde{i}_\mu^!(\mathcal{T}_1 \star \mathcal{T}_s)$ is also a complex of free-monodromic local systems in degrees $\geq -p_\mu$. Combined with (6.32), we conclude that $\widetilde{i}_\mu^!(\mathcal{T}_1 \star \mathcal{T}_s)$ is concentrated in degree $-p_\mu$ and is free-monodromic.

Combining the last two paragraphs, we conclude that $\mathcal{T}_1 \star \mathcal{T}_s$ is a free-monodromic tilting sheaf.

□

Lemma 6.5. *Recall the assumptions and notations of Lemma 6.3. Then*

1. *If α_s is complex and $\sigma\alpha_s > 0$, then $\mathcal{T}_{\lambda,\chi} \star \mathcal{T}_s$ contains $\mathcal{T}_{\lambda\cdot s,\chi}$ as a direct summand with multiplicity one.*
2. *If α_s is real and $\underline{\mathbf{k}}_{\lambda,\chi}$ extends to a $G_{\mathbb{R}}$-equivariant local system on $\pi_s^{-1}\pi_s(O_\lambda^{\mathbb{R}})$. Then*
   - *If α_s is type II real, then $\mathcal{T}_{\lambda,\chi} \star \mathcal{T}_s$ contains $\mathcal{T}_{\mu,\psi}$ as a direct summand with multiplicity one.*
   - *If α_s is type I real, then $\mathcal{T}_{\lambda,\chi} \star \mathcal{T}_s$ contains $\mathcal{T}_{\mu^+,\psi^+} \oplus \mathcal{T}_{\mu^-,\psi^-}$ as a direct summand with multiplicity one.*

Proof. Let $O_\mu^{\mathbb{R}} = \pi_s^{-1}\pi_s(O_\lambda^{\mathbb{R}}) - O_\lambda^{\mathbb{R}}$ (which is a union of two orbits in type I real case), and let $\widetilde{O}_\mu^{\mathbb{R}}$ be its preimage under π. Let $\widetilde{i}_\mu : \widetilde{O}_\mu^{\mathbb{R}} \hookrightarrow \widetilde{X}$ be the inclusion. Then $\widetilde{O}_\mu^{\mathbb{R}}$ is the open stratum in the support of $\mathcal{T}_{\lambda,\chi} \star \mathcal{T}_s$. By support considerations, we have

$$\text{(6.33)} \qquad \widetilde{i}_\mu^*(\mathcal{T}_{\lambda,\chi} \star \mathcal{T}_s) \cong \widetilde{i}_\mu^*(\Delta_{\lambda,\chi} \star \widetilde{\Delta}_s).$$

By Lemma 6.3, the above is $\mathcal{L}_{\mu,\chi}[p_\mu]$ in case (1), $\mathcal{L}_{\mu,\psi}[p_\mu]$ in case (2) type II, and $\mathcal{L}_{\mu^+,\psi^+}[p_{\mu^+}]\oplus\mathcal{L}_{\mu^-,\psi^-}[p_{\mu^-}]$ in case (2) type I. The conclusion follows. $\square$

The Hecke action on $\mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}})$ enjoys the following self-adjunction property.

Proposition 6.6. *Let $s \in W$ be a simple reflection and $\mathcal{T}_s$ the corresponding tilting object of $\mathcal{H}_G$. Then the convolution endo-functor $-\star\mathcal{T}_s$ on $\mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}})$ is self-adjoint. Namely for $\mathcal{T}_1, \mathcal{T}_2$ in $\mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}})$ we have a canonical isomorphism functorial in $\mathcal{T}_1$ and $\mathcal{T}_2$*

$$\mathrm{RHom}_{\mathcal{M}_{G_\mathbb{R}}}(\mathcal{T}_1\star\mathcal{T}_s, \mathcal{T}_2) \cong \mathrm{RHom}_{\mathcal{M}_{G_\mathbb{R}}}(\mathcal{T}_1, \mathcal{T}_2\star\mathcal{T}_s).$$

Proof. To construct a unit and counit of the adjunction it suffices to construct maps $u\colon \tilde{\delta} \to \mathcal{T}_s\star\mathcal{T}_s$ and $c\colon \mathcal{T}_s\star\mathcal{T}_s \to \tilde{\delta}$ in $\mathcal{H}_G$, such that both compositions

$$\mathcal{T}_s \xrightarrow{u\star\mathrm{id}} \mathcal{T}_s\star\mathcal{T}_s\star\mathcal{T}_s \xrightarrow{\mathrm{id}\star c} \mathcal{T}_s \tag{6.34}$$

$$\mathcal{T}_s \xrightarrow{\mathrm{id}\star u} \mathcal{T}_s\star\mathcal{T}_s\star\mathcal{T}_s \xrightarrow{c\star\mathrm{id}} \mathcal{T}_s$$

are equality to the identity of $\mathcal{T}_s$.

It is known that the Soergel's functor $\mathbb{V}$ (recalled in Section 8.6) provides a fully-faithful embedding of $\mathrm{Tilt}(\mathcal{H}_G)$ into the category of Soergel bimodules (i.e. [7, Proposition 11.2] in the present setting). It is thus sufficient to construct u and c in the category of Soergel bimodules, where we have $\mathbb{V}(\mathcal{T}_s) = \mathcal{R}\otimes_{\mathcal{R}^s}\mathcal{R}$. Note that $\mathcal{R}^{s=-1}$, as an $\mathcal{R}^s$-bimodule, is isomorphic to the regular bimodule. Let us fix an element $x_s \in \mathcal{R}^{s=-1}$ such that multiplication by x_s gives an isomorphism $\mathcal{R}^s \xrightarrow{\sim} \mathcal{R}^{s=-1}$ of $\mathcal{R}^s$-bimodules. We get a splitting $\mathcal{R} = \mathcal{R}^s \oplus x_s\mathcal{R}^s$ as $\mathcal{R}^s$-bimodules.

We then have a splitting

$$\mathbb{V}(\mathcal{T}_s\star\mathcal{T}_s) = \mathcal{R}\otimes_{\mathcal{R}^s}\mathcal{R}\otimes_{\mathcal{R}^s}\mathcal{R} = \mathcal{R}\otimes_{\mathcal{R}^s}(\mathcal{R}^s\oplus x_s\mathcal{R}^s)\otimes_{\mathcal{R}^s}\mathcal{R} =$$

$$= \mathcal{R}\otimes_{\mathcal{R}^s}\mathcal{R}^s\otimes_{\mathcal{R}^s}\mathcal{R} \oplus \mathcal{R}\otimes_{\mathcal{R}^s} x_s\mathcal{R}^s\otimes_{\mathcal{R}^s}\mathcal{R}.$$

We put $c\colon \mathcal{R}\otimes_{\mathcal{R}^s}\mathcal{R}\otimes_{\mathcal{R}^s}\mathcal{R}\to\mathcal{R}$ for a composition of the projection onto the second summand $\mathcal{R}\otimes_{\mathcal{R}^s}x_s\mathcal{R}^s\otimes_{\mathcal{R}^s}\mathcal{R}\simeq\mathcal{R}\otimes_{\mathcal{R}^s}\mathcal{R}$ and the multiplication map $\mathcal{R}\otimes_{\mathcal{R}^s}\mathcal{R}\to\mathcal{R}$. We put $u\colon \mathcal{R}\to\mathcal{R}\otimes_{\mathcal{R}^s}\mathcal{R}\otimes_{\mathcal{R}^s}\mathcal{R}$ for the composition of the map $\mathcal{R}\to\mathcal{R}\otimes_{\mathcal{R}^s}\mathcal{R}$ given by $1\mapsto x_s\otimes 1 + 1\otimes x_s$ and the inclusion of the first summand $\mathcal{R}\otimes_{\mathcal{R}^s}\mathcal{R}^s\otimes_{\mathcal{R}^s}\mathcal{R}\simeq\mathcal{R}\otimes_{\mathcal{R}^s}\mathcal{R}$. The verification of the adjunction identities is now straightforward.

$\square$

The following generation property for the $\mathrm{Tilt}(\mathcal{H}_G)$-module $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$ will play an important role later.

Recall there is a unique closed $G_{\mathbb{R}}$-orbit $O^{\mathbb{R}}_{\lambda_0}$ in X and its preimage $\widetilde{O}^{\mathbb{R}}_{\lambda_0} := \pi^{-1}(O^{\mathbb{R}}_{\lambda})$ in $\widetilde{X}$.

Proposition 6.7. *The category* $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$ *is generated under taking direct sums and summands and applying the convolution action of* $\mathrm{Tilt}(\mathcal{H}_G)$ *by the free-monodromic local systems supported on* $\widetilde{O}^{\mathbb{R}}_{\lambda_0}$.

Proof. Let $\mathrm{Tilt}' \subset \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$ be the full subcategory generated under taking direct sums and summands and the convolution action of $\mathrm{Tilt}(\mathcal{H}_G)$ by the free-monodromic local systems supported on $\widetilde{X}_{\lambda_0}$. Suppose for purpose of contradiction that Tilt' is not the whole $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$, let $(\lambda, \chi) \in \widetilde{I}$ be such that $\mathcal{T}_{\lambda,\chi} \notin \mathrm{Tilt}'$ and that $\dim O^{\mathbb{R}}_{\lambda}$ is minimal among such (λ, χ).

Let B be a Borel corresponding to a point in $O^{\mathbb{R}}_{\lambda}$ and let $\mathfrak{b}$ be its Lie algebra and $\mathfrak{t} \subset \mathfrak{b}$ its σ-invariant subtorus. Let α be a simple root of $\mathfrak{t}$ in $\mathfrak{g}$ positive with respect to $\mathfrak{b}$. Lemma 6.5 implies that if α is noncompact imaginary or α is complex and $\sigma\alpha$ is negative, there is $(\mu, \psi) \in \widetilde{I}$ with $\mu < \lambda$ such that $\mathcal{T}_{\lambda,\chi}$ is a direct summand in $\mathcal{T}_{\mu,\psi} \star \mathcal{T}_{s_\alpha}$. This would contradict the minimality assumption.

From the above we conclude that any simple root α of $(\mathfrak{b}, \mathfrak{t})$ either satisfies $\sigma\alpha > 0$, or α is compact imaginary. The next lemma shows that in this situation $O^{\mathbb{R}}_{\lambda}$ is closed and finishes the proof.

□

Lemma 6.8. *Let $\lambda \in I$. The orbit $O^{\mathbb{R}}_{\lambda}$ is closed if and only if any simple root $\alpha \in \Phi_\lambda$ is either complex and $\sigma\alpha > 0$ or compact imaginary.*

Proof. The "only if" part follows from Lemma 2.12 as the intesections with the α-lines has to be closed subvarieties of $\mathbb{P}^1$.

Let us proof the "if" part. Under the Matsuki correspondence, the closed $G_{\mathbb{R}}$-orbit corresponds to the unique open K-orbit in X. Consider now the corresponding K-orbit O^K_λ under the Matsuki correspondence. Let $\mathfrak{k}$ be the Lie algebra of K and let θ be the corresponding Cartan involution. We choose $B \in C_\lambda = O^K_\lambda \cap O^{\mathbb{R}}_\lambda$. We would like to prove that $\mathfrak{b} + \mathfrak{k} = \mathfrak{g}$, which then implies that O^K_λ is open and $O^{\mathbb{R}}_\lambda$ is closed.

Translating the constraints on the roots in terms of θ, we see that for each simple root α of $(\mathfrak{b}, \mathfrak{t})$, either $\theta\alpha < 0$, or α is compact imaginary (i.e., $\theta\alpha = \alpha$ and $\mathfrak{g}_{\pm\alpha} \subset \mathfrak{k}$).

We want to show that for each root α we have $\mathfrak{g}_\alpha \subset \mathfrak{k}+\mathfrak{b}$. We proceed by downward induction on the height of α. If $ht(\alpha) > 0$ then $\alpha > 0$ and $\mathfrak{g}_\alpha \subset \mathfrak{b}$. If $ht(\alpha) < 0$ and suppose $\mathfrak{g}_\beta \subset \mathfrak{k} + \mathfrak{b}$ for all $ht(\beta) > ht(\alpha)$.

Write α as a sum of simple roots α_i, and each α_i either satisfies $\theta\alpha_i < 0$ or $\theta\alpha_i = \alpha_i$. Therefore $ht(\theta\alpha) \leq ht(\alpha)$.

If $ht(\theta\alpha) < ht(\alpha)$ then by inductive hypothesis we have $\mathfrak{g}_{\theta\alpha} \subset \mathfrak{k} + \mathfrak{b}$. On the other hand, $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\theta\alpha})^\theta \subset \mathfrak{k}$. Therefore $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\theta\alpha} + (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\theta\alpha})^\theta \subset \mathfrak{b} + \mathfrak{k}$.

If $ht(\theta\alpha) = ht(\alpha)$, then this can happen only when α is compact imaginary. In this case, $\mathfrak{g}_\alpha \subset \mathfrak{k}$. This completes the induction.

□

7. Localizations of free-monodromic categories

7.1. **Preliminary on invariant theory.** Let $V = \operatorname{Spec} \mathcal{R}$. The formal version $\operatorname{Spf} \mathcal{R}$ of V can be identified with the formal completion of the dual torus $T^\vee$ (defined over $\mathbf{k}$) at the identity element. We have a natural W-action on V.

For any $\lambda \in I$ we have the subscheme $V_\lambda := \operatorname{Spec} \overline{\mathcal{R}}_\lambda \subset V$. Again the formal version $\operatorname{Spf} \overline{\mathcal{R}}_\lambda$ of V_λ can be identified with the formal completion of a subtorus of $T^\vee$ at the identity element.

Let $\mathcal{S} = \overline{\mathcal{R}}_{\lambda_0}$ be the completion of the group ring $\mathbf{k}[\pi_1(\overline{T}^c_{\lambda_0})]$ at the augmentation ideal (recall $\lambda_0 \in I$ indexes the closed $G_\mathbb{R}$-orbit). We have $V_{\lambda_0} = \operatorname{Spec} \mathcal{S} \subset V$. We consider the projection map

$$\mathrm{pr}_V : V_{\lambda_0} \times_{V /\!\!/ W} V \to V. \tag{7.1}$$

Let V_r be the scheme-theoretic image of pr_V; this is a W-stable closed subscheme of V. It is easy to see that the reduced structure of V_r is the union

$$V_r^{\mathrm{red}} = \bigcup_{w \in W} w V_{\lambda_0} \subset V. \tag{7.2}$$

Equivalently, V_r^{red} is the union of V_λ for those λ attached to a fixed maximally split σ-stable torus T_0.

Let $A \subset T_0$ be the maximal split subtorus. Consider the completion $\mathcal{S} = \mathbf{k}\widehat{[\pi_1(A)]}$. Note that $W_\mathbb{R} := W(G, T_0)^\sigma \subset W(G, T_0)$ fixes A and so acts on $\mathcal{S}$. Let $\mathcal{K}$ be the field of fractions of $\mathcal{S}$. We also put $\mathcal{Q} := \mathcal{K}^{W_\mathbb{R}}$ for the field of fractions of $\mathcal{S}^{W_\mathbb{R}}$. Consider the natural composition $\mathcal{R}^W \to \mathcal{S}^{W_\mathbb{R}} \to \mathcal{K}$ [3]. Note that $\mathcal{S} \otimes_{\mathcal{R}^W} \mathcal{Q} \cong \mathcal{K}$.

The category $\mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}})$ is linear over $\mathcal{R}$, i.e., $\mathcal{R}$ acts on the identity functor $\mathrm{id}_{\mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}})}$.

---

[3] As was proved in [14] the first map is surjective outside 4 exceptional cases for G of type E.

Lemma 7.2. *The action of $\mathcal{R}$ on $\mathrm{id}_{\mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}})}$ factors through the quotient $\mathcal{O}(V_r)$ (regular functions on V_r). The action of $\mathcal{R}^W$ on $\mathrm{id}_{\mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}})}$ factors through $\mathcal{O}(V_r)^W$, which coincides with the image of $\mathcal{R}^W$ in $\mathcal{S}^{W_\mathbb{R}}$.*

Proof. By Proposition 6.7, it suffices to check that the action of $\mathcal{R}$ on $\mathcal{T} \star \mathcal{T}'$ (via right monodromy on $\mathcal{T}'$) factors through $\mathcal{O}(V_r)$, for $\mathcal{T} \in \mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}})$ supported on $\widetilde{X}_{\lambda_0}$, and $\mathcal{T}' \in \mathrm{Tilt}(\mathcal{H}_G)$. The action of $\mathcal{R} \otimes \mathcal{R}$ on $\mathcal{T}'$ by the left and right monodromy factors through $\mathcal{R} \otimes_{\mathcal{R}^W} \mathcal{R}$ by Proposition 8.7, and the first copy of $\mathcal{R}$-action is the same as the right monodromy on $\mathcal{T}$. For $\mathcal{T} \in \mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}})$ supported on the closed orbit, $\mathcal{R}$ acts on $\mathcal{T}$ through the quotient $\mathcal{S}$, hence the $\mathcal{R} \otimes_{\mathcal{R}^W} \mathcal{R}$-action on $\mathcal{T} \star \mathcal{T}'$ factors through $\mathcal{S} \otimes_{\mathcal{R}^W} \mathcal{R}$. Therefore the second copy of $\mathcal{R}$-action factors through the image of $\mathcal{R} \to \mathcal{S} \otimes_{\mathcal{R}^W} \mathcal{R}$ ($r \mapsto 1 \otimes r$), which is $\mathcal{O}(V_r)$ by definition.

The second assertion follows immediately from the first. $\square$

7.3. **Localization.** For any orbit $\lambda \in I$, the universal Cartan $\mathbf{T}$ carries a real form (anti-holomorphic involution) σ_λ from Lemma 2.6(1). Let θ_λ be the Cartan involution on $\mathbf{T}$ corresponding to the real form σ_λ (so θ_λ is the composition of σ_λ with the compact real form σ_c).

Let $\mathbf{T}^\vee = \mathsf{Hom}(\mathbb{X}_*(\mathbf{T}), \mathbb{G}_{m,\mathbf{k}}) = \mathsf{Hom}(\pi_1(\mathbf{T}^c), \mathbb{G}_{m,\mathbf{k}})$ be the dual torus of $\mathbf{T}$ over $\mathbf{k}$. Then V can be canonically identified with the formal completion of $\mathbf{T}^\vee$ at the identity element. The Cartan involution θ_λ acts on $\mathbf{T}^\vee$ hence on V. Write $-\theta_\lambda$ to be θ_λ composed with inversion. We may identify V_λ with the fixed point formal scheme $V^{-\theta_\lambda}$.

Let $\mathbf{A} \subset \mathbf{T}$ be the neutral component of $\mathbf{T}^{-\theta_{\lambda_0}}$. Then V_{λ_0} is canonically identified with the formal completion of the dual torus $\mathbf{A}^\vee$ at the identity.

We fix a Borel $B_0 \in O^\mathbb{R}_{\lambda_0}$ and a σ-stable maximal torus $T_0 \subset B_0$. Let $P_0 \supset B_0$ be the minimal σ-stable parabolic subgroup containing B_0. Let $A_0 \subset T_0$ be the subtorus corresponding to the split part of $T_{0,\mathbb{R}}$. Via the isomorphism $\iota_{B_0} : T_0 \subset B_0 \twoheadrightarrow \mathbf{T}$, A_0 gets identified with $\mathbf{A}$.

Consider the restricted root system $\Phi(G, A_0)$, with basis given by B_0. Via the isomorphism $\iota_{B_0} : A_0 \xrightarrow{\sim} \mathbf{A}$, we view $\alpha \in \Phi(G, A_0)$ as characters on $\mathbf{A}$, and the corresponding coroot $\alpha^\vee$ as characters on $\mathbf{A}^\vee$. Let J be a subset of simple roots in $\Phi(G, A_0)$. Let $\mathbf{A}_J \subset \mathbf{A}$ and $A_J \subset A_0$ be the neutral component of $\cap_{j \in J} \ker(\alpha_j^\vee)$. Similarly, let $\mathbf{A}_J^\vee \subset \mathbf{A}^\vee$ be the neutral component of $\cap_{j \in J} \ker(\alpha_j^\vee)$. Let $V_J \subset V_{\lambda_0}$ be the formal completion of $\mathbf{A}_J^\vee$ at the identity.

Let $L := L_J = C_G(A_J) \subset G$ and $P := P_J \supset P_0$ be the unique parabolic subgroup of G containing P_0 with L as a Levi subgroup. Then L and P are also σ-stable hence defined over $\mathbb{R}$.

Let X_L be the flag variety of L_J and let $\widetilde{X}_L$ be the $\mathbf{T}^c$-torsor over X_L. We have the monodromic category $\mathcal{M}_{L_\mathbb{R}}$ and the subcategory of free-monodromic tilting sheaves $\mathrm{Tilt}(\mathcal{M}_{L_\mathbb{R}})$.

Lemma 7.4. *The map $L_\mathbb{R} \to L_\mathbb{R}^{ad}$ is surjective.*

Proof. Consider the roots of T in L. They are permuted by σ. The center $Z(L)$ inside T is defined by vanishing of all other roots, which are also permuted by σ. We conclude that $Z(L_\mathbb{R})$ is the product of the copies of $\mathbb{C}^\times$, corresponding to nontrivial orbits of σ and $\mathbb{R}^\times$ corresponding to the fixed points of σ. □

Let $p_L \colon P \to L$ be the projection. We have a closed embedding

$$i_P : X_L \hookrightarrow X \tag{7.3}$$

sending $B' \in X_L$ to $p_L^{-1}(B') \in X$. The image of i_P is the set of Borel subgroups of G contained in P. Similarly we have a closed embedding of enhanced flag varieties

$$\widetilde{i}_P : \widetilde{X}_L \hookrightarrow \widetilde{X} \tag{7.4}$$

covering i_P.

Lemma 7.5. *Recall J is a subset of simple roots of $\Phi(G, A_0)$ that cut out $A_J, \mathbf{A}_J, \mathbf{A}_J^\vee$ and V_J. For a fixed $\lambda \in I$ we have $V_J \subset V_\lambda$ if and only if the intersection $O_\lambda^\mathbb{R} \cap i_P(X_L)$ is nonempty, i.e. if and only if there exists a Borel subgroup $B' \subset P \subset G$ contained in $O_\lambda^\mathbb{R}$ as a point of X.*

Proof. Note that the following are equivalent:

$$V_J \subset V_\lambda \iff \mathbf{A}_J^\vee \subset \mathbf{T}^{\vee, -\theta_\lambda} \iff \mathbf{A}_J \subset \mathbf{T}^{-\theta_\lambda}. \tag{7.5}$$

Suppose $B \subset P$ is a Borel subgroup and $B_L = B \cap L$. Let $T \subset B_L$ be a σ-stable torus. Note that A_J is the split center of L, hecne $A_J \subset T$. Consider the image of A_J under $\iota_B : T \subset B \twoheadrightarrow \mathbf{T}$. On the one hand, ι_B and ι_{B_0} restricts to the same map $A_J \to \mathbf{T}$ (because they differ by $\mathbf{W}_L$), hence $\iota_B(A_J) = \mathbf{A}_J$. On the other hand, A_J is contained in complexification of the split part of $T_\mathbb{R}$, hence $\iota_B(A_J) \subset \mathbf{T}^{-\theta_\lambda}$. Therefore $\mathbf{A}_J \subset \mathbf{T}^{-\theta_\lambda}$, and $V_J \subset V_\lambda$ by (7.5).

Conversely, suppose $V_J \subset V_\lambda$, hence $\mathbf{A}_J \subset \mathbf{T}^{-\theta_\lambda}$ by (7.5). Let $B \in O_\lambda^\mathbb{R}$ and $T \subset B$ be a σ-stable maximal torus. Let $A_1 \subset T$ be the complexification of the split part of $T_\mathbb{R}$. Changing (T, B) by $G_\mathbb{R}$-conjugacy, we may assume $A_1 \subset A_0$. Under $\iota_B : T \subset B \twoheadrightarrow \mathbf{T}$, we have $\iota_B(A_1) = \mathbf{T}^{-\theta_\lambda, \circ} \supset \mathbf{A}_J$. Let $A'_J = \iota^{-1}(\mathbf{A}_J) \subset A_1$.

Now we have two subtori A_J, A'_J of A_0, which is in turn in T_0. Via ι_{B_0} and ι_B respectively, they map isomorphically to $\mathbf{A}_J$. Let $a'_J \in A'_J$ be a generic element, and let $a_J \in A_J$ such that $\iota_B(a'_J) = \iota_{B_0}(a_J)$. Both a'_J

and a_J are in T_0 and they are in the same conjugacy class of G, there exists $w \in W(G, T_0)$ such that $w(a'_J) = a_J$. Since a'_J is generic in A'_J, w restricts to the isomorphism $A'_J \xrightarrow{\iota_B} \mathbf{A}_J \xrightarrow{\iota_{B_0}^{-1}} A_J$. Since $a_J, a'_J \in A_0$ are the same $W(G, T_0)$-orbit, they are also in the same $W(G, A_0)$-orbit. Let $u \in W(G, A_0)$ be such that $u(a'_J) = a_J$, then $u|_{A'_J} = w|_{A'_J}$. Since $W(G, A_0) = W(G_{\mathbb{R}}, T_{0,\mathbb{R}})$, we can lift u to $\dot{u} \in G_{\mathbb{R}}$ normalizing T_0. We have a commutative diagram

$$
\begin{tikzcd}
A_J \arrow[rr, bend left, equal] \arrow[dr, "\iota_{\mathrm{Ad}(\dot{u})B}"'] & A'_J \arrow[l, "u"'] \arrow[r, "w"] \arrow[d, "\iota_B"] & A_J \arrow[dl, "\iota_{B_0}"] \\
& \mathbf{T} &
\end{tikzcd}
\tag{7.6}
$$

By Lemma 7.6 below, B_0 and $\mathrm{Ad}(\dot{u})B$ are in the same $L = C_G(A_J)$-orbit of X. Now $B_0 \in i_P(X_L)$, which is the L-orbit through B_0, we have $\mathrm{Ad}(\dot{u})B \in i_P(X_L) \cap O_\lambda^{\mathbb{R}}$. $\square$

Lemma 7.6. *Let $A \subset G$ be a torus. Consider the map $\kappa : X^A \to \mathsf{Hom}(A, \mathbf{T})$ sending $B \in X^A$ (i.e., a Borel subgroup B containing A) to the map $\iota_B : A \subset B \twoheadrightarrow \mathbf{T}$. Then each non-empty fiber of κ is stable under the Levi subgroup $C_G(A)$ of G, and is $C_G(A)$-equivariantly isomorphic to the flag variety of $C_G(A)$.*

Proof. The map κ is equivariant under $N_G(A)$, therefore each fiber has an action by $C_G(A)$.

Let $B_1, B_2 \in X^A$ with the same image under κ. Let $T_i \subset B_i$ be a maximal torus containing A, $i = 1, 2$. Let $\iota_i : T_i \subset B_i \twoheadrightarrow \mathbf{T}$ be the isomorphisms induced by B_i. The fact that $\kappa(B_1) = \kappa(B_2)$ implies $\iota_1|_A = \iota_2|_A$. Let $g \in G$ be such that $\mathrm{Ad}(B_1) = B_2$ and $\mathrm{Ad}(T_1) = T_2$. Then $\iota_2 \circ \mathrm{Ad}(g) = \iota_1 \in \mathsf{Hom}(T_1, \mathbf{T})$. Restricting to A we get that $\iota_2(\mathrm{Ad}(g)a) = \iota_1(a)$, which is $\iota_2(a)$ for all $a \in A$. Therefore $\mathrm{Ad}(g)a = a$ hence $g \in C_G(A)$. This shows that each non-empty fiber of κ is a homogeneous space for $C_G(A)$.

The stabilizers of $C_G(A)$ on X^A are clearly Borel subgroups of $C_G(A)$. Hence each non-empty fiber of κ is isomorphically to the flag variety of $C_G(A)$. $\square$

Let $\mathcal{K}_J$ be the localization of V at the generic point of V_J. The category $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$ is linear over $\mathcal{R}$. We define the localization of $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$ at the generic point of V_J as the $\mathcal{K}_J$-linear additive category $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}}) \otimes_{\mathcal{R}} \mathcal{K}_J$ whose objects are the same as those in $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$, and the morphisms are defined as

$$\mathsf{Hom}_{\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}}) \otimes_{\mathcal{R}} \mathcal{K}_J}(\mathcal{T}_1, \mathcal{T}_2) := \mathsf{Hom}_{\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})}(\mathcal{T}_1, \mathcal{T}_2) \otimes_{\mathcal{R}} \mathcal{K}_J.$$

Applying Lemma 7.2 to $L_{\mathbb{R}}$ we see that the action of $\mathcal{R}$ on $\mathrm{id}_{\mathrm{Tilt}(\mathcal{M}_{L_{\mathbb{R}}})}$ also factors through $\mathcal{O}(V_r)$ (in fact we can replace V_r by a further subscheme, which is the image of $V_{\lambda_0} \times_{V /\!\!/ W_L} V \to V$). Therefore it makes sense to define the localization $\mathrm{Tilt}(\mathcal{M}_{L_{\mathbb{R}}}) \otimes_{\mathcal{R}} \mathcal{K}_J$.

Proposition 7.7. *Under the above notations, The restriction map along $\widetilde{i}_P : \widetilde{X}_L \hookrightarrow \widetilde{X}$ induces an equivalence.*

$$\widetilde{i}_P^* : \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}}) \otimes_{\mathcal{R}} \mathcal{K}_J \xrightarrow{\sim} \mathrm{Tilt}(\mathcal{M}_{L_{\mathbb{R}}}) \otimes_{\mathcal{R}} \mathcal{K}_J. \tag{7.7}$$

Moreover the equivalence is equivariant under the actions of $\mathrm{Tilt}(\mathcal{H}_L)$.

Proof. We claim that the inverse functor is given by the composition $\mathrm{Av}_{G_{\mathbb{R}}} \circ \widetilde{i}_{P,*}$, where $\mathrm{Av}_{G_{\mathbb{R}}}$ is the averaging functor with respect to $G_{\mathbb{R}}$-action. It is clear that $\widetilde{i}_P^* \circ \mathrm{Av}_{G_{\mathbb{R}}} \circ \widetilde{i}_{P,*}$ is an identity (even before the localization) and we need to verify that $\mathrm{Av}_{G_{\mathbb{R}}} \circ \widetilde{i}_{P,*} \circ \widetilde{i}_P^*$ is an identity on the localised categories. By Lemma 7.5 the orbits intersecting the image of i_P are exactly the $O_\lambda^{\mathbb{R}}$ such that $V_1 \subset V_\lambda$. These are exactly the orbits that have the local systems not vanishing after the localization and the statement follows.

The convolution product commutes with $\mathrm{Av}_{G_{\mathbb{R}}}$, which implies the compatibility with $\mathrm{Tilt}(\mathcal{H}_L)$-action.

□

7.7.1. *Localization of the Hecke category.* We keep working in the above setting. Consider the map

$$\mathrm{pr}_J \colon V_J \times_{V /\!\!/ W} V \to V$$

and let $V_{J,r}$ be its scheme-theoretic image. Let $\mathcal{R}_J$ be the localization of V at the generic points of the irreducible componenent of $V_{J,r}$. It is isomorphic to the product of the copies of $\mathcal{K}_J$ numbered by the W-orbit of V_J as a subspace of V.

Recall that the Hecke category is a $\mathcal{R} \otimes_{\mathcal{R}^W} \mathcal{R}$-linear category. We define the localization of the Hecke category $\mathcal{R}_J \otimes_{\mathcal{R}} \mathrm{Tilt}(\mathcal{H}_G) \otimes_{\mathcal{R}} \mathcal{R}_J$ to be a $\mathcal{R}_J \otimes_{\mathcal{R}^W} \mathcal{R}_J$-linear category whose objects are the same as the objects of $\mathrm{Tilt}(\mathcal{H}_G)$ and the morphisms are given by

$$\mathsf{Hom}_{\mathcal{R}_J \otimes_{\mathcal{R}} \mathrm{Tilt}(\mathcal{H}_G) \otimes_{\mathcal{R}} \mathcal{R}_J}(\mathcal{T}_1, \mathcal{T}_2) := \mathcal{R}_J \otimes_{\mathcal{R}} \mathsf{Hom}_{\mathrm{Tilt}(\mathcal{H}_G)}(\mathcal{T}_1, \mathcal{T}_2) \otimes_{\mathcal{R}} \mathcal{R}_J.$$

Note that $\mathcal{R}_J \otimes_{\mathcal{R}} \mathrm{Tilt}(\mathcal{H}_G) \otimes_{\mathcal{R}} \mathcal{R}_J$ splits into the direct product of categories numbered by a pair of elements in the W-orbit of V_J. The $\star$-action of $\mathrm{Tilt}(\mathcal{H}_G)$ on $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$ passes to a functor

$$\star \colon \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}}) \otimes_{\mathcal{R}} \mathcal{R}_J \times \mathcal{R}_J \otimes_{\mathcal{R}} \mathrm{Tilt}(\mathcal{H}_G) \otimes_{\mathcal{R}} \mathcal{R}_J \to \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}}) \otimes_{\mathcal{R}} \mathcal{R}_J. \tag{7.8}$$

The action is compatible with the direct product decompositions.

7.7.2. *Localization at codimension* 0. Consider the localisation $\mathcal{M}_{G_\mathbb{R}} \otimes_{\mathcal{R}^W} \mathcal{Q}$ of the category $\mathcal{M}_{G_\mathbb{R}}$ over the monodromy action of $\mathcal{R}^W \subset \mathcal{R}$, whose objects are the same as the objects of $\mathcal{M}_{G_\mathbb{R}}$ and the morphisms are given by

$$\mathsf{Hom}_{\mathcal{M}_{G_\mathbb{R}} \otimes_{\mathcal{R}^W} \mathcal{Q}}(\mathcal{F}_1, \mathcal{F}_2) := \mathsf{Hom}_{\mathcal{M}_{G_\mathbb{R}}}(\mathcal{F}_1, \mathcal{F}_2) \otimes_{\mathcal{R}^W} \mathcal{Q}.$$

We have $\mathcal{R}_\mathcal{Q} := \mathcal{R} \otimes_{\mathcal{R}^\mathbf{W}} \mathcal{Q} \cong \prod_{\lambda \in I_0} \mathcal{K}_\lambda$ and, therefore,

$$\mathcal{M}_{G_\mathbb{R}} \otimes_{\mathcal{R}^W} \mathcal{Q} \cong \prod_{\lambda \in I_0} \mathcal{M}_{G_\mathbb{R}} \otimes_\mathcal{R} \mathcal{K}_\lambda$$

We say that an orbit $O_\lambda^\mathbb{R}$ is *attached to* T if it contains a T-fixed point. Note that for an orbit attached to T we have $n_\lambda = \dim A$ and for each orbit not attached to T we have $n_\lambda < \dim A$. Let $I_0 \subset I$ be the set of orbits attached to T, and $\widetilde{I}_0$ be the preimage of I_0 in $\widetilde{I}$. For $\lambda \in I_0$, let $\mathcal{K}_\lambda = \mathrm{Frac}(\overline{\mathcal{R}}_\lambda)$. Then $\mathrm{Spec}\,\overline{\mathcal{R}}_\lambda = w(\mathrm{Spec}\,\mathcal{S}) \subset \mathrm{Spec}\,\mathcal{R}$ for some $w \in W$. We have an isomorphism $\mathcal{K} \cong \mathcal{K}_\lambda$ unique up to precomposing with the action of $W_\mathbb{R}$. We have $\overline{\mathcal{R}}_\lambda \otimes_{\mathcal{R}^W} \mathcal{Q} = \mathcal{K}_\lambda$.

Proposition 7.8. (1) *If* $(\lambda, \chi) \in \widetilde{I} - \widetilde{I}_0$, *then* $\widetilde{\Delta}_{\lambda,\chi}$ *and* $\widetilde{\nabla}_{\lambda,\chi}$ *are zero in* $\mathcal{M}_{G_\mathbb{R}} \otimes_{\mathcal{R}^W} \mathcal{Q}$.

(2) *For* $(\lambda, \chi) \in \widetilde{I}_0$, *we have* $\mathsf{End}_{\mathcal{M}_{G_\mathbb{R}} \otimes_{\mathcal{R}^W} \mathcal{Q}}(\widetilde{\Delta}_{\lambda,\chi}, \widetilde{\Delta}_{\lambda,\chi}) \cong \mathcal{K}_\lambda$.

(3) *The functor*

$$\bigoplus_{(\lambda,\chi) \in \widetilde{I}_0} D^b(\mathcal{K}_\lambda\text{-mod}) \to \mathcal{M}_{G_\mathbb{R}} \otimes_{\mathcal{R}^W} \mathcal{Q} \tag{7.9}$$

sending $(M_{\lambda,\chi})_{(\lambda,\chi) \in \widetilde{I}_0}$ *to* $\bigoplus_{(\lambda,\chi) \in \widetilde{I}_0} M_{\lambda,\chi} \otimes_{\mathcal{K}_\lambda} \widetilde{\Delta}_{\lambda,\chi}$ *is an equivalence.*

Proof. (1) If $\lambda \notin I_0$, the action of $\mathcal{R}$ on $\widetilde{\Delta}_{\lambda,\chi}$ and $\widetilde{\nabla}_{\lambda,\chi}$ factor through $\mathcal{R}_\lambda$ but $\mathrm{Spec}\,\overline{\mathcal{R}}_\lambda$ has dimension n_λ which is smaller than the dimension of $\mathrm{Spec}\,\mathcal{S}^{W_\mathbb{R}}$. Therefore the actions of $\mathcal{R}^W$ on $\widetilde{\Delta}_{\lambda,\chi}$ and $\widetilde{\nabla}_{\lambda,\chi}$ also factor through a quotient with smaller dimension than $\mathrm{Spec}\,\mathcal{S}^{W_\mathbb{R}}$, hence localizing to the generic point of $\mathrm{Spec}\,\mathcal{S}^{W_\mathbb{R}}$ kills $\widetilde{\Delta}_{\lambda,\chi}$ and $\widetilde{\nabla}_{\lambda,\chi}$.

(2) If (λ, χ) and $(\lambda, \psi) \in \widetilde{I}_0$, then by Lemma 3.7,

$$\mathrm{RHom}_{\mathcal{M}_{G_\mathbb{R}}}(\widetilde{\Delta}_{\lambda,\chi}, \widetilde{\Delta}_{\lambda,\psi}) = \begin{cases} \overline{\mathcal{R}}_\lambda, & \chi = \psi, \\ 0, & \chi \neq \psi. \end{cases}$$

Tensoring with $\mathcal{Q}$ we get

$$\mathrm{RHom}_{\mathcal{M}_{G_\mathbb{R}} \otimes_{\mathcal{R}^W} \mathcal{Q}}(\widetilde{\Delta}_{\lambda,\chi}, \widetilde{\Delta}_{\lambda,\psi}) = \begin{cases} \mathcal{K}_\lambda, & \chi = \psi, \\ 0, & \chi \neq \psi. \end{cases}$$

(3) Since the $\{\widetilde{\Delta}_{\lambda,\chi}\}_{(\lambda,\chi)\in\widetilde{I}}$ generate $\mathcal{M}_{G_\mathbb{R}} \otimes_{\mathcal{R}^W} \mathcal{Q}$, in view of part (1) and (2), it remains to show that $\mathrm{RHom}_{\mathcal{M}_{G_\mathbb{R}}\otimes_{\mathcal{R}^W}\mathcal{Q}}(\widetilde{\Delta}_{\lambda,\chi}, \widetilde{\Delta}_{\mu,\psi}) = 0$ for distinct orbits $\lambda, \mu \in I_0$. Note that the support of $\mathrm{RHom}_{\mathcal{M}_{G_\mathbb{R}}}(\widetilde{\Delta}_{\lambda,\chi}, \widetilde{\Delta}_{\mu,\psi})$ as an $\mathcal{R}$-module is contained in $\mathrm{Spec}\,\overline{\mathcal{R}}_\lambda \cap \mathrm{Spec}\,\overline{\mathcal{R}}_\mu$. By Lemma 2.14 and [1, Proposition 12.9 and 12.14], that $(T^c_\lambda)^\circ \neq (T^c_\mu)^\circ$ as subtori of T^c, hence the intersection $\mathrm{Spec}\,\overline{\mathcal{R}}_\lambda \cap \mathrm{Spec}\,\overline{\mathcal{R}}_\mu$ has smaller dimension than $\mathfrak{a}$, which is the dimension of $\mathrm{Spec}\,\mathcal{S}^{W_\mathbb{R}}$. The same holds for the support of $\mathrm{RHom}_{\mathcal{M}_{G_\mathbb{R}}}(\widetilde{\Delta}_{\lambda,\chi}, \widetilde{\Delta}_{\mu,\psi})$ as an $\mathcal{R}^W$-module. Therefore tensoring with $\mathcal{Q}$ kills $\mathrm{RHom}_{\mathcal{M}_{G_\mathbb{R}}}(\widetilde{\Delta}_{\lambda,\chi}, \widetilde{\Delta}_{\mu,\psi})$.

□

By Proposition 7.8, indecomposable objects in the localized category $\mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}})_\mathcal{Q} := \mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}}) \otimes_{\mathcal{R}^\mathbf{W}} \mathcal{Q}$ are of the form $\widetilde{\Delta}_{\lambda,\chi}$ where $(\lambda,\chi) \in \widetilde{I}_0$ (these are objects in the localized tilting category because they are direct summands of $\mathcal{T}_{\lambda,\chi}$ after localization). On the other hand, for the base-changed Hecke category $\mathrm{Tilt}(\mathcal{H}_G)_\mathcal{Q} := \mathrm{Tilt}(\mathcal{H}_G) \otimes_{\mathcal{R}^\mathbf{W}} \mathcal{Q}$, the assignment $w \mapsto \widetilde{\Delta}_w$ (now $\widetilde{\Delta}_w$ is a direct summand of $\mathcal{T}_w$ after base change to $\mathcal{Q}$) gives a monoidal functor from $\mathbf{W}$ (viewed as a category with objects $\mathbf{W}$ and only identity morphisms) to $\mathrm{Tilt}(\mathcal{H}_G)_\mathcal{Q}$. In particular, the $\mathrm{Tilt}(\mathcal{H}_G)_\mathcal{Q}$-action on $\mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}})_\mathcal{Q}$ induces a right $\mathbf{W}$ action on the set of isomorphisms classes of indecomposable objects in $\mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}})_\mathcal{Q}$, and hence induces a right action of $\mathbf{W}$ on $\widetilde{I}_0$.

Lemma 7.9. *Assume that $G_\mathbb{R}$ is quasi-split. The action of $\mathbf{W}$ on $\widetilde{I}_0$ defined above is the same as the restriction of the cross action. In other words, in $\mathcal{M}_{G_\mathbb{R},\mathcal{Q}}$ we have an isomorphism $\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_w \cong \widetilde{\Delta}_{(\lambda,\chi)\cdot w}$ for $(\lambda,\chi) \in \widetilde{I}_0$ and $w \in \mathbf{W}$.*

Proof. When $G_\mathbb{R}$ is quasi-split and $\lambda \in I_0$, all simple roots in Φ_λ are in case (1) or (3) in Lemma 6.3. For $(\lambda,\chi) \in \widetilde{I}_0$, by Lemma 6.3 we have $\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_s \cong \widetilde{\Delta}_{(\lambda,\chi)\cdot s}$ in $\mathcal{M}_{G_\mathbb{R},\mathcal{Q}}$ (the contribution of orbits μ or $\mu^\pm$ become zero after localization). Writing any $w \in \mathbf{W}$ as a product of simple reflections, we see that $\widetilde{\Delta}_{\lambda,\chi} \star \widetilde{\Delta}_w$ and $\widetilde{\Delta}_{(\lambda,\chi)\cdot w}$ become isomorphic in $\mathcal{M}_{G_\mathbb{R},\mathcal{Q}}$. This implies the lemma. □

8. Real Soergel Functor

In this section we assume that $G_\mathbb{R}$ is quasi-split. Let B be the Borel defined over $\mathbb{R}$ and let $A \subset T \subset B$ be a maximally split with respect to σ torus inside a σ-fixed torus. Note that $W_\mathbb{R}$ acts on A.

8.1. **Regular covector.** Let $\mathcal{N}^* \subset \mathfrak{g}^*$ be the nilcone. Also we have a decomposition $\mathfrak{g}^* = \mathfrak{g}^*_{\mathbb{R}} \oplus i\mathfrak{g}^*_{\mathbb{R}}$. Let

$$i\mathcal{N}^*_{\mathbb{R}} := \mathcal{N}^* \cap i\mathfrak{g}^*_{\mathbb{R}}. \tag{8.1}$$

Let $\mathcal{O}_{\mathrm{reg}} \subset \mathcal{N}^*$ be the regular nilpotent orbit.

Lemma 8.2. (1) *We have $i\mathcal{N}^*_{\mathbb{R}} \cap \mathcal{O}_{\mathrm{reg}}$ if and only if $G_{\mathbb{R}}$ is quasi-split.*

(2) *Suppose $\xi \in i\mathcal{N}^*_{\mathbb{R}} \cap \mathcal{O}_{\mathrm{reg}}$, then the Springer fiber $\mathcal{B}_\xi$ is a single point, and is contained in the closed $G_{\mathbb{R}}$-orbit of X.*

Proof. Suppose $\xi \in i\mathcal{N}^*_{\mathbb{R}} \cap \mathcal{O}_{\mathrm{reg}}$, then the Springer fiber $\mathcal{B}_\xi \subset X$ is a single point, hence a real point of the flag variety X since ξ is pure imaginary. In particular, the point $\mathcal{B}_\xi$ gives a Borel subgroup of G defined over $\mathbb{R}$. Since the closed $G_{\mathbb{R}}-$-orbit in X parametrizes Borel subgroups that are defined over $\mathbb{R}$, $\mathcal{B}_\xi$ is contained in the closed $G_{\mathbb{R}}$-orbit of X.

Conversely, assume $G_{\mathbb{R}}$ is quasi-split and $B_{\mathbb{R}} \subset G_{\mathbb{R}}$ is a Borel subgroup defined over $\mathbb{R}$. Let $\mathfrak{n}_{\mathbb{R}}$ be the nilpotent radical of Lie$B_{\mathbb{R}}$. Then a generic element $\xi \in i\mathfrak{n}_{\mathbb{R}}$ (i.e., its projection to each simple root space is nonzero), viewed as an element in $i\mathfrak{g}^*_{\mathbb{R}}$ using the Killing form, is regular. □

8.3. **Generic vanishing cycles.** In the rest of the section we assume $G_{\mathbb{R}}$ is quasi-split. In this case, the closed $G_{\mathbb{R}}$ orbit $O^{\mathbb{R}}_{\lambda_0} \subset X$ is the set of real points of X.

Consider the moment map of $\widetilde{X} = G/UT^{>0}$ for the left action of G:

$$\mu : T^*\widetilde{X} \to \mathfrak{g}^*. \tag{8.2}$$

For T^c-monodromic sheaves $\mathcal{F} \in D^b(\widetilde{X})_{T^c-\mathrm{mon}}$, its singular support $SS(\mathcal{F})$ is contained in the image of the pullback $T^*X \times_X \widetilde{X} \to T^*\widetilde{X}$, which is equal to $\mu^{-1}(\mathcal{N}^*)$ (here $\mathcal{N}^*$ is the nilcone in $\mathfrak{g}^*$). On the other hand, if $\mathcal{F} \in D^b_{G_{\mathbb{R}}}(\widetilde{X})$, then $SS(\mathcal{F}) \subset \mu^{-1}(i\mathfrak{g}^*_{\mathbb{R}})$ (note). Let

$$\Lambda_{\mathbb{R}} = \mu^{-1}(i\mathcal{N}^*_{\mathbb{R}}) \subset T^*\widetilde{X}. \tag{8.3}$$

Then the above discussion shows that for $\mathcal{F} \in D^b_{G_{\mathbb{R}}}(\widetilde{X})_{T^c-\mathrm{mon}}$, $SS(\mathcal{F}) \subset \Lambda_{\mathbb{R}}$. Also, since $G_{\mathbb{R}} \times T^c$ has finitely many orbits on $\widetilde{X}$, $\Lambda_{\mathbb{R}}$ is the union of conormals of the orbits $\{\widetilde{O}^{\mathbb{R}}_\lambda\}$:

$$\Lambda_{\mathbb{R}} = \bigcup_{\lambda \in I} T^*_{\widetilde{O}^{\mathbb{R}}_\lambda}\widetilde{X}. \tag{8.4}$$

Let $\xi \in i\mathcal{N}^*_{\mathbb{R}} \cap \mathcal{O}_{\mathrm{reg}}$ and let

$$\Xi := \mu^{-1}(\xi) \subset T^*\widetilde{X}. \tag{8.5}$$

Then by Lemma 8.2, the Springer fiber $\mathcal{B}_\xi$ is a single point x contained in the closed $G_\mathbb{R}$-orbit $O^\mathbb{R}_{\lambda_0} \subset X$ that is a real form of X. Hence projection to $\widetilde{X}$ is an isomorphism $\Xi \xrightarrow{\sim} \pi^{-1}(x) \subset \widetilde{X}$ (a single T^c-orbit). Moreover, $\Xi \subset T^*_{\widetilde{O}^\mathbb{R}_{\lambda_0}} \widetilde{X}$ but is disjoint from the conormals of other $\widetilde{O}^\mathbb{R}_\lambda$.

For a manifold M and a submanifold $N \subset M$, and $\mathcal{F} \in D^b(M)$, we use $\mu_N \mathcal{F} \in D^b(T^*_N M)$ to denote the microlocalization of $\mathcal{F}$ along N. See [15, Definition 4.3.1].

For $\mathcal{F} \in D^b_{G_\mathbb{R}}(\widetilde{X})_{T^c-\mathrm{mon}}$, denote the microlocalization $\mu_{\widetilde{O}^\mathbb{R}_{\lambda_0}}(\mathcal{F}) \in D^b(T^*_{\widetilde{O}^\mathbb{R}_{\lambda_0}} \widetilde{X})_{T^c-\mathrm{mon}}$ along $\widetilde{O}^\mathbb{R}_{\lambda_0}$ simply by $\mu_{\lambda_0(\mathcal{F})}$; it is locally constant on the generic part of $T^*_{\widetilde{O}^\mathbb{R}_{\lambda_0}}$. Restricting to Ξ gives a functor

(8.6)
$$D^b_{G_\mathbb{R}}(\widetilde{X})_{T^c-\mathrm{mon}} \xrightarrow{\mu_{\lambda_0}} D^b(T^*_{\widetilde{O}^\mathbb{R}_{\lambda_0}} \widetilde{X})_{T^c-\mathrm{mon}} \to D^b(\Xi)_{T^c-\mathrm{mon}} \cong D^b(\pi^{-1}(x))_{T^c-\mathrm{mon}}$$

Passing to completions, we get a functor

$$\mathbb{V}_{\mathbb{R},\xi} : \mathcal{M}_{G_\mathbb{R}} = \widehat{D}^b_{G_\mathbb{R}}(\widetilde{X})_{T^c-\mathrm{mon}} \to \widehat{D}^b(\Xi)_{T^c-\mathrm{mon}} \tag{8.7}$$
[4]

If we choose a point $\widetilde{x} \in \pi^{-1}(x)$ and let $\widetilde{\xi} = (\widetilde{x}, \xi) \in \Xi$, the base point $\widetilde{\xi}$ trivializes Ξ as a T^c-torsor, and gives an equivalence $\widehat{D}^b(\Xi)_{T^c-\mathrm{mon}} \cong D^b(\text{mod-}\mathcal{R})$. Then $\mathbb{V}_{\mathbb{R},\xi}$ induces a functor

$$\mathbb{V}_{\mathbb{R},\widetilde{\xi}} : \mathcal{M}_{G_\mathbb{R}} \to D^b(\text{mod-}\mathcal{R}) \tag{8.8}$$

that depends on the choice of $\widetilde{\xi} \in T^*\widetilde{X}$ over the regular ξ.

When $\widetilde{\xi}$ is fixed, we also write the functor as $\mathbb{V}_\mathbb{R}$.

Proposition 8.4. *Let $(\lambda, \chi) \in \widetilde{I}$.*

(1) $\mathbb{V}_\mathbb{R}(\nabla_{\lambda,\chi})$ *is concentrated in degree* 0
(2) $\mathbb{V}_\mathbb{R}(\Delta_{\lambda,\chi})$ *is concentrated in degree* $n_{\lambda_0} - n_\lambda$.

Proof. (1) We argue by induction on $d_\lambda = \dim_\mathbb{R} O^\mathbb{R}_\lambda$. If $\lambda = \lambda_0$ corresponds to the closed orbit, then $\mathbb{V}_\mathbb{R}(\nabla_{\lambda,\chi})$ is the stalk of $\mathcal{L}_{\lambda,\chi}$ along $\widetilde{X}_{\lambda_0}$ (up to a shift), and it is normalized to be in degree 0. Otherwise, choose a point $x \in O^\mathbb{R}_\lambda$ (corresponding to a Borel x) such that B contains a σ-stable maximal torus T, and we can talk about the based root system $\Phi(G, T)$ with positive roots defined by B. Since $O^\mathbb{R}_\lambda$ is

[4] There is an action of $C_{G_\mathbb{R}}(\xi)$ on $\Theta_{\mathbb{R},\xi}$. This gives lifts $\mathbb{V}_{\mathbb{R},\xi}$ to take values in $\widehat{D}^b_{Z^c_\mathbb{R}}(T^c)_{T^c-\mathrm{mon}}$, where $Z^c_\mathbb{R}$ is the image of $Z_\mathbb{R} = Z(G_\mathbb{R}) \to T^c$. Hence there is a decomposition of this enhanced $\mathbb{V}_{\mathbb{R},\xi}$ according to characters of $\pi_0(Z^c_\mathbb{R}) \cong \pi_0(Z_\mathbb{R})$.

not closed, there is a simple root $\alpha \in \Phi(G, T)$ and $\mu < \lambda$, such that $\pi_\alpha(O^{\mathbb{R}}_\lambda) = \pi_\alpha(O^{\mathbb{R}}_\mu)$ for the projection $\pi_\alpha \colon X \to X_\alpha = G/P_\alpha$. Since $O^{\mathbb{R}}_\lambda$ is not closed, we have the following cases according to Lemma 6.8:

- α is complex and $\sigma\alpha < 0$. In this case, we have $p_\lambda - p_\mu = 1$, and an exact triangle $\pi_\alpha^*\pi_{\alpha*}\nabla_{\lambda,\chi} \to \nabla_{\lambda,\chi} \to \nabla_{\mu,\psi} \to$ for some $\psi : \pi_0(T^c_\mu) \to \mathbf{k}^\times$.
- α is noncompact imaginary and $\pi_\alpha^{-1}\pi_\alpha(O^{\mathbb{R}}_\lambda) = O^{\mathbb{R}}_\lambda \cup O^{\mathbb{R}}_\mu$. We have $p_\lambda = p_\mu$, and an exact triangle $\pi_\alpha^*\pi_{\alpha*}\nabla_{\lambda,\chi} \to \nabla_{\lambda,\chi} \to \nabla_{\mu,\psi} \to$ for some $\psi : \pi_0(T^c_\mu) \to \mathbf{k}^\times$.
- α is noncompact imaginary and $\pi_\alpha^{-1}\pi_\alpha(O^{\mathbb{R}}_\lambda) = O^{\mathbb{R}}_\lambda \cup O^{\mathbb{R}}_{\lambda'} \cup O^{\mathbb{R}}_\mu$, with $\mu < \lambda$ and $\mu < \lambda'$. Then $p_\lambda = p_{\lambda'} = p_\mu$, and we have an exact triangle $\pi_\alpha^*\pi_{\alpha*}\nabla_{\lambda,\chi} \to \nabla_{\lambda,\chi} \oplus \nabla_{\lambda',\chi'} \to \nabla_{\mu,\psi} \to$ for some $\chi' : \pi_0(T^c_{\lambda'}) \to \mathbf{k}^\times$ and $\psi : \pi_0(T^c_\mu) \to \mathbf{k}^\times$.

We have $\mathbb{V}_{\mathbb{R}}(\pi_\alpha^*(\mathcal{G})) = 0$ for any $\mathcal{G} \in D^b_{G_{\mathbb{R}}}(X_\alpha)$, as the covector ξ does not lie in the image of the pullback of cotangent bundles $d\pi_\alpha : (T^*X_\alpha) \times_{X_\alpha} X \to T^*X$. From the exact triangles above we see that $\mathbb{V}_{\mathbb{R}}(\nabla_{\lambda,\chi})$ is a direct summand of $\mathbb{V}_{\mathbb{R}}(\nabla_{\mu,\psi})$. Since $d_\mu < d_\lambda$, by inductive hypothesis $\mathbb{V}_{\mathbb{R}}(\nabla_{\mu,\psi})$ is concentrated in degree 0, therefore the same is true for $\mathbb{V}_{\mathbb{R}}(\nabla_{\lambda,\chi})$.

(2) The argument is similar to the costandard case. In the induction step, we have the following cases

- α is complex and $\sigma\alpha < 0$. In this case, we have $p_\lambda - p_\mu = 1$, and an exact triangle $\Delta_{\mu,\psi} \to \Delta_{\lambda,\chi} \to \pi_\alpha^!\pi_{\alpha!}\Delta_{\lambda,\chi} \to$. We conclude that $\mathbb{V}_{\mathbb{R}}(\Delta_{\lambda,\chi}) \cong \mathbb{V}_{\mathbb{R}}(\Delta_{\mu,\psi})$. Note that $n_\mu = n_\lambda$ in this case.
- α is noncompact imaginary and $\pi_\alpha^{-1}\pi_\alpha(O^{\mathbb{R}}_\lambda) = O^{\mathbb{R}}_\lambda \cup O^{\mathbb{R}}_\mu$. We have $p_\lambda = p_\mu$, and an exact triangle $\Delta_{\lambda,\chi} \to \pi_\alpha^!\pi_{\alpha!}\Delta_{\lambda,\chi} \to \Delta_{\mu,\psi} \to$. We conclude that $\mathbb{V}_{\mathbb{R}}(\Delta_{\lambda,\chi}) \cong \mathbb{V}_{\mathbb{R}}(\Delta_{\mu,\psi})[-1]$. Note that $n_\mu - n_\lambda = 1$ in this case.
- α is noncompact imaginary and $\pi_\alpha^{-1}\pi_\alpha(O^{\mathbb{R}}_\lambda) = O^{\mathbb{R}}_\lambda \cup O^{\mathbb{R}}_{\lambda'} \cup O^{\mathbb{R}}_\mu$, with $\mu < \lambda$ and $\mu < \lambda'$. Then $p_\lambda = p_{\lambda'} = p_\mu$, and we have an exact triangle $\Delta_{\lambda,\chi} \oplus \Delta_{\lambda',\chi'} \to \pi_\alpha^!\pi_{\alpha!}\Delta_{\lambda,\chi} \to \Delta_{\mu,\psi} \to$. We conclude that $\mathbb{V}_{\mathbb{R}}(\Delta_{\lambda,\chi})$ is a summand of $\mathbb{V}_{\mathbb{R}}(\Delta_{\mu,\psi})[-1]$. Again $n_\mu - n_\lambda = 1$ in this case.

□

Corollary 8.5. *For any free-monodromic tilting sheaf* $\mathcal{T} \in \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$, $\mathbb{V}_{\mathbb{R}}(\mathcal{T})$ *is concentrated in degree* 0.

8.6. **Soergel functor for the Hecke category.** Consider the Hecke category $\mathcal{H}_G = \widehat{D}^b_G(\widetilde{X} \times \widetilde{X})_{T^c \times T^c-\mathrm{mon}}$. The construction of the Soergel functor can be applied to the complex group G viewed as a real group $R_{\mathbb{C}/\mathbb{R}}G$ and giving a Soergel functor for $\mathcal{H}_G$. We spell this out.

Consider the doubled moment map $\mu^{(2)} : T^*\widetilde{X} \times T^*\widetilde{X} \to \mathfrak{g}^* \oplus \mathfrak{g}^*$. Let $\Delta^-(\mathcal{N}^*) \subset \Delta^-(\mathfrak{g}^*) \subset \mathfrak{g}^* \oplus \mathfrak{g}^*$ be the anti-diagonally embedded nilcone. Let

$$\Lambda := \mu^{(2),-1}(\Delta^-(\mathcal{N}^*)). \tag{8.9}$$

It is well-known that

$$\Lambda = \bigcup_{w \in W} T^*_{\widetilde{X}^2_w}(\widetilde{X}^2). \tag{8.10}$$

Let $\xi \in \mathcal{N}^* \cap \mathcal{O}_{\mathrm{reg}}$ and consider $(\xi, -\xi) \in \Delta^-(\mathcal{N}^*)$. Let $\Xi = \mu^{-1}(\xi), \Xi^- = \mu^{-1}(-\xi)$ and consider

$$\mu^{(2),-1}(-\xi, \xi) = \Xi^- \times \Xi \subset T^*(\widetilde{X}^2). \tag{8.11}$$

Since the Springer fiber $\mathcal{B}_\xi$ is a single point $x \in X$, $\mu^{(2),-1}(-\xi, \xi)$ projects isomorphically onto $\pi^{-1}(x)^2 \subset \widetilde{X}^2$, which is a $T^c \times T^c$-torsor. In particular, $\Xi^- \times \Xi$ is contained in the conormal bundle of $\widetilde{X}^2_e \subset \widetilde{X}^2$ and not in the closure of the conormals of $\widetilde{X}^2_w$ for $e \neq w \in W$ (i.e., it is contained in the generic part of the $T^*_{\widetilde{X}^2_e}(\widetilde{X}^2)$).

For $\mathcal{K} \in \mathcal{H}_G$, its microlocalization along $\widetilde{X}^2_e$ is locally constant on the generic part of $T^*_{\widetilde{X}^2_e}(\widetilde{X}^2)$ and $T^c \times T^c$-unipotently monodromic. Restricting to $\Xi^- \times \Xi$ gives a functor

$$\mathbb{V}_{(-\xi,\xi)} : \mathcal{H}_G \to \widehat{D}^b(\Xi^- \times \Xi)_{T^c \times T^c-\mathrm{mon}} \tag{8.12}$$

If we choose $\widetilde{x} \in \pi^{-1}(x)$ hence $\widetilde{\xi} = (\widetilde{x}, \xi) \in \Xi$ and $-\widetilde{\xi} = (\widetilde{x}, -\xi) \in \Xi^-$, we can then identify $\widehat{D}^b(\Xi^- \times \Xi)_{T^c \times T^c-\mathrm{mon}}$ with $\widehat{D}^b(T^c \times T^c)_{T^c \times T^c-\mathrm{mon}} \cong D^b(\text{mod-}\mathcal{R} \otimes \mathcal{R})$. Then $\mathbb{V}_{(-\xi,\xi)}$ induces a functor

$$\mathbb{V}_{(-\widetilde{\xi},\widetilde{\xi})} : \mathcal{H}_G \to \widehat{D}^b(T^c \times T^c)_{T^c \times T^c-\mathrm{mon}} \cong D^b(\text{mod-}\mathcal{R} \otimes \mathcal{R}). \tag{8.13}$$

[5]

It will be more convenient to turn right $\mathcal{R} \otimes \mathcal{R}$-modules into $\mathcal{R}$-bimodules. Let $\iota : \mathcal{R} \to \mathcal{R}$ be the involution given by the inversion on $\pi_1(T^c)$. We consider the equivalence

$$\tau : \text{mod-}\mathcal{R} \otimes \mathcal{R} \xrightarrow{\sim} \mathcal{R}\text{-mod-}\mathcal{R} \tag{8.14}$$

[5] There is a natural map $C_G(\xi) \to T \to T^c$ by noting $C_G(\xi) \subset {}^yB$ (the Borel corresponding to y). Then $\mathbb{V}_{(-\widetilde{\xi},\widetilde{\xi})}$ lifts to $\widehat{D}^b_{C_G(\xi)}(T^c \times T^c)_{T^c \times T^c-\mathrm{mon}}$. Note $C_G(\xi) \to T^c$ has contractible kernel and image is $Z^c = Im(Z_G \to T^c)$. Therefore $\mathbb{V}_{(-\widetilde{\xi},\widetilde{\xi})}$ lifts to $D^b_{\Delta(Z^c)}(T^c \times T^c)_{T^c \times T^c-\mathrm{mon}}$. A priori this allows us to refine $\mathbb{V}$ to a sum of functors indexed by characters of $\pi_0(Z^c)$. However, only the functor corresponding to the trivial character is nonzero, as one can check on the monoidal unit.

given by sending $M \in \text{mod-}\mathcal{R} \otimes \mathcal{R}$ to the same underlying vector space M equipped with the left and right actions of $\mathcal{R}$ defined by

$$r_1 \cdot m \cdot r_2 = m \cdot (\iota(r_1) \otimes r_2), \quad r_1, r_2 \in \mathcal{R}, m \in M. \tag{8.15}$$

Composing (8.13) with the equivalence (8.14), we get a functor

$$_{-\widetilde{\xi}}\mathbb{V}_{\widetilde{\xi}} : \mathcal{H}_G \to D^b(\mathcal{R}\text{-mod-}\mathcal{R}). \tag{8.16}$$

When $\widetilde{\xi}$ is understood from the context we simply denote $_{-\widetilde{\xi}}\mathbb{V}_{\widetilde{\xi}}$ by $\mathbb{V}$.

Recall in [9, §4.5] (in the ℓ-adic setting), a similar functor $\mathbb{V}$ was defined using the action of $\mathcal{H}_G$ on a certain Whittaker category. A topological analogue has been constructed in [7, §11.1].

We summarize the main properties of $\mathbb{V}$ in the following proposition. Most of the assertions are proved in the literature with an a priori different definition of $\mathbb{V}$.

Proposition 8.7. (1) *Let $\widetilde{\mathcal{P}}_e \in \mathcal{H}_G^{\heartsuit}$* [6] *be a projective cover of the constant sheaf on the preimage of $\Delta(X) \subset X \times X$ in $\widetilde{X} \times \widetilde{X}$. Then $\mathbb{V} \cong \mathrm{RHom}(\widetilde{\mathcal{P}}_e, -)$.*

(2) *$\widetilde{\mathcal{P}}_e$ is isomorphic to the free-monodromic indecomposable tilting sheaf $\mathcal{T}_{w_0}$ with full support. Therefore the functor $\mathbb{V} \cong \mathrm{RHom}(\mathcal{T}_{w_0}, -)$.*

(3) *We have an isomorphism of $\mathcal{R}$-bimodules $\mathsf{End}_{\mathcal{H}_G}(\mathcal{T}_{w_0}) \cong \mathcal{R} \otimes_{\mathcal{R}^W} \mathcal{R}$. In particular, if we let $(\mathcal{R}\text{-mod-}\mathcal{R})_{\mathcal{R}^W}$ be the category of $\mathcal{R}$-bimodules where the actions of $\mathcal{R}^W$ through the left and right copies of $\mathcal{R}$ coincide, then $\mathbb{V}$ takes values in $D^b((\mathcal{R}\text{-mod-}\mathcal{R})_{\mathcal{R}^W})$.*

(4) *$\mathbb{V}$ is fully faithful on free-monodromic tilting sheaves.*

(5) *For $\mathcal{T}' \in \mathrm{Tilt}(\mathcal{H}_G)$, $\mathbb{V}(\mathcal{T}')$ is a Soergel bimodule. Therefore $\mathbb{V}|_{\mathrm{Tilt}}$ (shorthand for the restriction of $\mathbb{V}$ to $\mathrm{Tilt}(\mathcal{H}_G)$) upgrades to a monoidal functor*

$$\mathbb{V}^{\sharp} : \mathrm{Tilt}(\mathcal{H}_G) \to \mathrm{SBim}. \tag{8.17}$$

Proof. (1) (this uses the $U\backslash G/U$ model). We only need to check that $\mathbb{V}(\mathrm{IC}_w) = 0$ for $w \neq e$ and $\mathbb{V}(\mathrm{IC}_e) \cong \mathbf{k}$. If $w \neq e$, then IC_w, when viewed as an object in $D^b(B\backslash G/B)$, is pulled back from a partial flag variety G/P_s, hence the generic vanishing cycle along the closed orbit vanishes. The isomorphism $\mathbb{V}(\mathrm{IC}_e) \cong \mathbf{k}$ is clear since IC_e is supported on the closed orbit. $\square$

[6] $\mathcal{H}_G^{\heartsuit}$ denotes the heart of the t-structure defined by p for the complex group, which is essentially the middle perversity.

Proposition 8.8. *The functor $\mathbb{V}$ carries a canonical monoidal structure with respect to the convolution product on $\mathcal{H}_G$ and the tensor product $(-) \otimes^{\mathbb{L}}_{\mathcal{R}} (-)$ on $D^b(\mathcal{R}\text{-mod-}\mathcal{R})$.*

Proof. First we construct a functorial isomorphism

$$c_{\mathcal{K},\mathcal{K}'} : \mathbb{V}(\mathcal{K} \star \mathcal{K}') \cong \mathbb{V}(\mathcal{K}) \otimes^{\mathbb{L}}_{\mathcal{R}} \mathbb{V}(\mathcal{K}') \tag{8.18}$$

for $\mathcal{K}, \mathcal{K}' \in \mathcal{H}_G$. Recall $\mathcal{K} \star \mathcal{K}' = \mathrm{pr}_{13*}(\mathrm{pr}_{12}^* \mathcal{K} \otimes \mathrm{pr}_{23}^* \mathcal{K}')$. Let $\mathcal{G} = \mathrm{pr}_{12}^* \mathcal{K} \otimes \mathrm{pr}_{23}^* \mathcal{K}'$. Let $\widetilde{\Delta}_{13} = \mathrm{pr}_{13}^{-1}(\widetilde{X}^2_e) \subset \widetilde{X}^3$. Let $\pi_{13} : T^*_{\widetilde{\Delta}_{13}}(\widetilde{X}^3) \to T^*_{\widetilde{X}^2_e}(\widetilde{X}^2)$ be the natural projection. By Proposition 8.11(1), we have

$$\pi_{13*}(\mu_{\widetilde{\Delta}_{13}} \mathcal{G}) \cong \mu_{\widetilde{X}^2_e} \mathrm{pr}_{13*} \mathcal{G}. \tag{8.19}$$

Recall that $\Lambda \subset T^*(\widetilde{X}^2)$ is the union of conormals of $\widetilde{X}^2_w$. Now observe that $SS(\mathcal{G}) \subset (\Lambda \times 0_{\widetilde{X}}) + (0_{\widetilde{X}} \times \Lambda)$ (here we apply [15, Proposition 5.4.14(i)], using that $(\Lambda \times 0_{\widetilde{X}}) \cap (0_{\widetilde{X}} \times \Lambda)$ is contained in the zero section). This implies that $\pi_{13}^{-1}(\Xi^- \times \Xi) \cap SS(\mathcal{G}) \subset \Xi^- \times 0_{\pi^{-1}(x)} \times \Xi \subset T^*_{\widetilde{\Delta}}(\widetilde{X}^3)$, where $\widetilde{\Delta} \subset \widetilde{X}^3$ is the preimage of the small diagonal $\Delta(X) \hookrightarrow X^3$. Therefore

$$\pi_{13,\Xi*}(\mu_{\widetilde{\Delta}}(\mathcal{G})|_{\Xi^- \times 0_{\pi^{-1}(x)} \times \Xi}) \cong (\mu_{\widetilde{X}^2_e} \mathrm{pr}_{13*} \mathcal{G})|_{\Xi^- \times \Xi} = \mathbb{V}_{(-\xi,\xi)}(\mathcal{K} \star \mathcal{K}'). \tag{8.20}$$

Here $\pi_{13,\Xi} : \Xi^- \times 0_{\pi^{-1}(x)} \times \Xi \to \Xi^- \times \Xi$ is the projection.

Let $\delta_{23} : \widetilde{X}^3 \hookrightarrow \widetilde{X}^4$ be the diagonal embedding of the middle two factors. Then $\mathcal{G} = \delta_{23}^*(\mathcal{K} \boxtimes \mathcal{K}')$. Note $\delta_{23}^{-1}(\widetilde{X}^2_e \times \widetilde{X}^2_e) = \widetilde{\Delta}$ and δ_{23} is transversal to $\widetilde{X}^2_e \times \widetilde{X}^2_e$. There is a canonical embedding induced from δ_{23} (which is a pullback of vector bundles)

$$\delta_{23}^{\natural} : T^*_{\widetilde{\Delta}}(\widetilde{X}^3) \hookrightarrow T^*_{\widetilde{X}^2_e \times \widetilde{X}^2_e}(\widetilde{X}^4) \tag{8.21}$$

By Proposition 8.11(2), we have

$$\delta_{23}^{\natural *}(\mu_{\widetilde{X}^2_e} \mathcal{K} \boxtimes \mu_{\widetilde{X}^2_e} \mathcal{K}') = \delta_{23}^{\natural *} \mu_{\widetilde{X}^2_e \times \widetilde{X}^2_e}(\mathcal{K} \boxtimes \mathcal{K}') \xrightarrow{\sim} \mu_{\widetilde{\Delta}}(\mathcal{G}). \tag{8.22}$$

Restricting both sides to $\Xi^- \times 0_{\pi^{-1}(x)} \times \Xi$ we get

$$\delta_{23,\Xi}^{\natural *}(\mathbb{V}_{(-\xi,\xi)}(\mathcal{K}) \boxtimes \mathbb{V}_{(-\xi,\xi)}(\mathcal{K}')) = \delta_{23,\Xi}^{\natural *}(\mu_{\widetilde{X}^2_e} \mathcal{K}|_{\Xi^- \times \Xi} \boxtimes \mu_{\widetilde{X}^2_e} \mathcal{K}'|_{\Xi^- \times \Xi}) \tag{8.23}$$

$$\xrightarrow{\sim} \mu_{\widetilde{\Delta}}(\mathcal{G})|_{\Xi^- \times 0_{\pi^{-1}(x)} \times \Xi}. \tag{8.24}$$

where $\delta_{23,\Xi}^{\natural} : \Xi^- \times 0_{\pi^{-1}(x)} \times \Xi \hookrightarrow \Xi^- \times \Xi \times \Xi^- \times \Xi$ is the restriction of $\delta_{23}^{\natural}$. Combined with (8.20) we get a canonical isomorphism

$$\mathbb{V}_{(-\xi,\xi)}(\mathcal{K} \star \mathcal{K}') \cong \pi_{13,\Xi*} \delta_{23,\Xi}^{\natural *}(\mathbb{V}_{(-\xi,\xi)}(\mathcal{K}) \boxtimes \mathbb{V}_{(-\xi,\xi)}(\mathcal{K}')). \tag{8.25}$$

Identifying Ξ^-, Ξ and $\pi^{-1}(x)$ with T^c using the base points $\pm\widetilde{\xi}$ and $\widetilde{x}$, (8.25) becomes

$$\mathbb{V}_{(-\widetilde{\xi},\widetilde{\xi})}(\mathcal{K}\star\mathcal{K}')\cong \dot{\mathrm{pr}}_{13*}\dot{\delta}^*_{23}(\mathbb{V}_{(-\widetilde{\xi},\widetilde{\xi})}(\mathcal{K})\boxtimes\mathbb{V}_{(-\widetilde{\xi},\widetilde{\xi})}(\mathcal{K}')). \tag{8.26}$$

where $\dot{\delta}_{23}:(T^c)^3\hookrightarrow(T^c)^4$ is the diagonal embedding of the middle factors and $\dot{\mathrm{pr}}_{13}:(T^c)^3\to(T^c)^2$ the projection. If we write $M=\mathbb{V}_{(-\widetilde{\xi},\widetilde{\xi})}(\mathcal{K})$ and $M'=\mathbb{V}_{(-\widetilde{\xi},\widetilde{\xi})}(\mathcal{K}')$ viewed as objects in $D^b(\text{mod-}\mathcal{R}\otimes\mathcal{R})$, then the right side of (8.26) becomes $(M\otimes M')\otimes^{\mathbb{L}}_{\mathcal{R}_{23}}\mathbf{k}$, where the notation $\otimes^{\mathbb{L}}_{\mathcal{R}_{23}}\mathbf{k}$ means $\otimes^{\mathbb{L}}_{\mathcal{R}}\mathbf{k}$ taken using the $\mathcal{R}$-action on $M\otimes M'$ given by $(m,m')\cdot r=(m,m')\cdot(1\otimes\delta(r)\otimes 1)$, where $\delta:\mathcal{R}\to\mathcal{R}\otimes\mathcal{R}$ is the comultiplication induced by the diagonal map $\pi_1(T^c)\to\pi_1(T^c)\times\pi_1(T^c)$. Finally note canonical isomorphism of right $\mathcal{R}\otimes\mathcal{R}$-modules (recall τ from (8.14))

$$(M\otimes M')\otimes^{\mathbb{L}}_{\mathcal{R}_{23}}\mathbf{k}\cong M\otimes^{\mathbb{L}}_{\mathcal{R}}\tau M'. \tag{8.27}$$

which induces a canonical isomorphism of right $\mathcal{R}$-bimodules

$$\tau((M\otimes M')\otimes^{\mathbb{L}}_{\mathcal{R}_{23}}\mathbf{k})\cong \tau M\otimes^{\mathbb{L}}_{\mathcal{R}}\tau M'. \tag{8.28}$$

Plugging into (8.26) we get the desired isomorphism $c_{\mathcal{K},\mathcal{K}'}$. We omit the verification that $c_{\mathcal{K},\mathcal{K}'}$ satisfies the axioms of a monoidal structure on $\mathcal{H}_G$. $\square$

8.9. **Module structure on $\mathbb{V}_{\mathbb{R}}$.** We now establish the relation between $\mathbb{V},\mathbb{V}_{\mathbb{R}}$ and the $\star$-action.

Choose $\xi\in i\mathcal{N}^*_{\mathbb{R}}\cap\mathcal{O}_{\mathrm{reg}}$, and let $x\in O^{\mathbb{R}}_{\lambda_0}$ be the unique point in the Springer fiber $\mathcal{B}_\xi$. Choose a lifting $\widetilde{x}\in\pi^{-1}(x)$ and let $\widetilde{\xi}=(\widetilde{x},\xi)\in T^*\widetilde{X}$. We use $\widetilde{\xi}$ to define the functor $\mathbb{V}_{\mathbb{R}}:=\mathbb{V}_{\mathbb{R},\widetilde{\xi}}$ as in (8.8). On the other hand, we use $(-\widetilde{\xi},\widetilde{\xi})\in T^*(\widetilde{X}^2)$ to defined the functor $\mathbb{V}:={}_{-\widetilde{\xi}}\mathbb{V}_{\widetilde{\xi}}$ as in (8.16).

Proposition 8.10. *Under the above notations, the functor $\mathbb{V}_{\mathbb{R}}$ intertwines the right $\mathcal{H}_G$-action on $\mathcal{M}_{G_{\mathbb{R}}}$ by $\star$ and the right action of $D^b(\mathcal{R}\otimes\mathcal{R}\text{-mod})$ on $D^b(\mathcal{R}\text{-mod})$ given by $(M,N)\mapsto M\otimes^{\mathbb{L}}_{\mathcal{R}}N$ (for $M\in D^b(\text{mod-}\mathcal{R})$ and $N\in D^b(\mathcal{R}\text{-mod-}\mathcal{R})$).*

Proof. We first construct a natural isomorphism

$$\alpha_{\mathcal{F},\mathcal{K}}:\mathbb{V}_{\mathbb{R}}(\mathcal{F})\otimes^{\mathbb{L}}_{\mathcal{R}}\mathbb{V}(\mathcal{K})\to\mathbb{V}_{\mathbb{R}}(\mathcal{F}\star\mathcal{K}). \tag{8.29}$$

for $\mathcal{F}\in\mathcal{M}_{G_{\mathbb{R}}}$ and $\mathcal{K}\in\mathcal{H}_G$.

Let $\mathcal{G}=\mathrm{pr}_1^*\mathcal{F}\otimes\mathcal{K}\in\widehat{D}^b(\widetilde{X}\times\widetilde{X})_{T^c\times T^c-\mathrm{mon}}$. We know that $SS(\mathcal{F})\subset\Lambda_{\mathbb{R}}$ (hence $SS(\mathrm{pr}_1^*\mathcal{F})\subset\Lambda_{\mathbb{R}}\times 0_{\widetilde{X}}\subset T^*(\widetilde{X}^2)$), and $SS(\mathcal{K})\subset\Lambda$. One checks that $(\Lambda_{\mathbb{R}}\times 0_{\widetilde{X}})\cap\Lambda$ is contained in the zero section of $T^*(\widetilde{X}^2)$.

By [15, Proposition 5.4.14(i)], $SS(\mathcal{G}) = SS(\mathrm{pr}_1^* \mathcal{F} \otimes \mathcal{K})$ is contained in the pointwise addition $\Lambda' := (\Lambda_{\mathbb{R}} \times 0_{\widetilde{X}}) + \Lambda \subset T^*(\widetilde{X}^2)$. The cone Λ' consists of $((\widetilde{x}_1, v-w), (\widetilde{x}_2, w)) \in T^*(\widetilde{X})^2$ (where $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{X}$ with image $x_1, x_2 \in X$, $v, w \in \mathfrak{g}^*$) such that $w \in {}^{x_1}\mathfrak{b}^{\perp} \cap {}^{x_2}\mathfrak{b}^{\perp}$ and $v \in i\mathcal{N}^*_{\mathbb{R}} \cap {}^{x_1}\mathfrak{b}^{\perp}$.

By definition, $\mathbb{V}_{\mathbb{R}}(\mathcal{F} \star \mathcal{K})$ is the restriction of the microlocalization $\mu_{\lambda_0}(\mathrm{pr}_{2*} \mathcal{G})$ to $\Xi = \mu^{-1}(\xi) \subset T^*_{\widetilde{O}^{\mathbb{R}}_{\lambda_0}} \widetilde{X}$. By Proposition 8.11(1), we consider

$$(8.30) \qquad T^*_{\widetilde{X} \times \widetilde{O}^{\mathbb{R}}_{\lambda_0}}(\widetilde{X}^2) \xleftarrow{\sim} 0_{\widetilde{X}} \times T^*_{\widetilde{O}^{\mathbb{R}}_{\lambda_0}}(\widetilde{X}) \xrightarrow{\pi_2} T^*_{\widetilde{O}^{\mathbb{R}}_{\lambda_0}}(\widetilde{X})$$

Let $\pi_{2,\Xi} : 0_{\widetilde{X}} \times \Xi \to \Xi$ be the second projection, then we have

$$(8.31) \qquad \mathbb{V}_{\mathbb{R},\xi}(\mathcal{F} \star \mathcal{K}) \cong (\pi_{2*}(\mu_{\widetilde{X} \times \widetilde{O}^{\mathbb{R}}_{\lambda_0}} \mathcal{G}))|_{\Xi} \cong \pi_{2,\Xi*}((\mu_{\widetilde{X} \times \widetilde{O}^{\mathbb{R}}_{\lambda_0}} \mathcal{G})|_{0_{\widetilde{X}} \times \Xi}).$$

Now $SS(\mathcal{G}) \subset \Lambda'$. We claim that

$$(8.32) \qquad \Xi' := \Lambda' \cap (0_{\widetilde{X}} \times \Xi) = \{(\widetilde{x}_1, 0), (\widetilde{x}_2, \xi)) | \widetilde{x}_1, \widetilde{x}_2 \in \pi^{-1}(x)\}.$$

Indeed, a point in Λ' takes the form $((\widetilde{x}_1, v-w), (\widetilde{x}_2, w))$. Intersecting with $0_{\widetilde{X}} \times \Xi$ means imposing $w = \xi$ and $v = w$, which also forces $x_1 = x_2 = x$ since $w = \xi \in {}^{x_1}\mathfrak{b}^{\perp} \cap {}^{x_2}\mathfrak{b}^{\perp}$. The projection to $\widetilde{X}^2$ gives an isomorphism $\Xi' \cong \pi^{-1}(x)^2$.

Now $(\mu_{\widetilde{X} \times \widetilde{O}^{\mathbb{R}}_{\lambda_0}} \mathcal{G})|_{0_{\widetilde{X}} \times \Xi}$ is supported in Ξ'. Let $\widetilde{\Delta}^{\mathbb{R}}_{\lambda_0} \subset \widetilde{X}^2$ be the preimage of the diagonal $\Delta(O^{\mathbb{R}}_{\lambda_0}) \subset X^2$ under the projection $\pi^2 : \widetilde{X}^2 \to X^2$. By (8.32) we have $\Xi' \subset T^*_{\widetilde{\Delta}^{\mathbb{R}}_{\lambda_0}}(\widetilde{X}^2)$. Therefore

$$(8.33) \qquad (\mu_{\widetilde{X} \times \widetilde{O}^{\mathbb{R}}_{\lambda_0}} \mathcal{G})|_{\Xi'} \cong (\mu_{\widetilde{\Delta}^{\mathbb{R}}_{\lambda_0}} \mathcal{G})|_{\Xi'}$$

and (8.31) gives an isomorphism

$$(8.34) \qquad \mathbb{V}_{\mathbb{R},\xi}(\mathcal{F} \star \mathcal{K}) \cong \mathrm{pr}_{2,\Xi*}((\mu_{\widetilde{\Delta}^{\mathbb{R}}_{\lambda_0}} \mathcal{G})|_{\Xi'})$$

where $\mathrm{pr}_{2,\Xi} : \Xi' \to \Xi$ is the projection.

To compute $(\mu_{\widetilde{\Delta}^{\mathbb{R}}_{\lambda_0}} \mathcal{G})|_{\Xi'}$, we consider the diagonal embedding $\delta_{12} : \widetilde{X}^2 \hookrightarrow \widetilde{X}^3$ given by $(\widetilde{x}_1, \widetilde{x}_2) \mapsto (\widetilde{x}_1, \widetilde{x}_1, \widetilde{x}_2)$. Then $\mathcal{G} = \delta_{12}^*(\mathcal{F} \boxtimes \mathcal{K})$. We have $\delta_{12}^{-1}(\widetilde{O}^{\mathbb{R}}_{\lambda_0} \times \widetilde{X}^2_e) = \widetilde{\Delta}^{\mathbb{R}}_{\lambda_0} \subset \widetilde{X}^2$, and δ_{12} is transversal with respect to $\widetilde{O}^{\mathbb{R}}_{\lambda_0} \times \widetilde{X}^2_e$. We have a canonical map induced by δ_{12} (which is an isomorphism on conormal fibers):

$$(8.35) \qquad T^*_{\widetilde{\Delta}^{\mathbb{R}}_{\lambda_0}}(\widetilde{X}^2) \xrightarrow{\delta_{12}^{\natural}} T^*_{\widetilde{O}^{\mathbb{R}}_{\lambda_0} \times \widetilde{X}^2_e}(\widetilde{X}^3)$$

By Proposition 8.11(2), we have a canonical isomorphism

$$(8.36) \qquad \delta_{12}^{\natural *}(\mu_{\lambda_0} \mathcal{F} \boxtimes \mu_{\widetilde{X}^2_e} \mathcal{K}) \cong \delta_{12}^{\natural *} \mu_{\widetilde{O}^{\mathbb{R}}_{\lambda_0} \times \widetilde{X}^2_e}(\mathcal{F} \boxtimes \mathcal{K}) \xrightarrow{\sim} \mu_{\widetilde{\Delta}^{\mathbb{R}}_{\lambda_0}} \mathcal{G}$$

Recall $\Xi^- = \mu^{-1}(-\xi)$. Then $\delta^\natural_{12}(\Xi') = (\Xi \times_{\widetilde{X}} \Xi^-) \times \Xi$, and let $\delta^\natural_{12,\Xi} : \Xi' \hookrightarrow \Xi \times \Xi^- \times \Xi$ be the induced embedding. Restricting (8.36) to Ξ' we get

$$\delta^{\natural *}_{12,\Xi}((\mu_{\lambda_0}\mathcal{F})|_\Xi \boxtimes (\mu_{\widetilde{X}^2_e}\mathcal{K})|_{\Xi^-\times\Xi}) \cong (\mu_{\widetilde{\Delta}^{\mathbb{R}}_{\lambda_0}}\mathcal{G})|_{\Xi'}. \tag{8.37}$$

Combined with (8.34) we get a canonical isomorphism in $\widehat{D}^b(\Xi)_{T^c-\mathrm{mon}}$

$$\mathbb{V}_{\mathbb{R},\xi}(\mathcal{F} \star \mathcal{K}) \cong \mathrm{pr}_{2,\Xi*}\, \delta^{\natural *}_{12,\Xi}((\mu_{\lambda_0}\mathcal{F})|_\Xi \boxtimes (\mu_{\widetilde{X}^2_e}\mathcal{K})|_{\Xi^-\times\Xi}) \tag{8.38}$$

$$= \mathrm{pr}_{2,\Xi*}\, \delta^{\natural *}_{12,\Xi}(\mathbb{V}_{\mathbb{R},\xi}(\mathcal{F}) \boxtimes \mathbb{V}_{(-\xi,\xi)}(\mathcal{K})). \tag{8.39}$$

If we identify Ξ and Ξ^- with T^c using the liftings $\widetilde{\xi} = (\widetilde{x}, \xi) \in \Xi$ and $-\widetilde{\xi} = (\widetilde{x}, -\xi) \in \Xi^-$, then $\delta^\natural_{12}$ becomes the diagonal embedding $\dot{\delta}_{12} : T^c \times T^c \hookrightarrow T^c \times T^c \times T^c$ into the first two factors, and $\mathrm{pr}_{2,\Xi}$ becomes the second projection $\dot{\mathrm{pr}}_2 : T^c \times T^c \to T^c$. Then (8.38) becomes a canonical isomorphism in $\widehat{D}^b(T^c)_{T^c-\mathrm{mon}} \cong D^b(\text{mod-}\mathcal{R})$

(8.40)

$$\mathbb{V}_{\mathbb{R},\widetilde{\xi}}(\mathcal{F}\star\mathcal{K}) \cong \dot{\mathrm{pr}}_{2*}\dot{\delta}^*_{12}(\mathbb{V}_{\mathbb{R},\widetilde{\xi}}(\mathcal{F})\boxtimes\mathbb{V}_{(-\widetilde{\xi},\widetilde{\xi})}(\mathcal{K})) \cong \dot{\mathrm{pr}}_{2*}(\dot{\mathrm{pr}}_1^*\mathbb{V}_{\mathbb{R},\widetilde{\xi}}(\mathcal{F})\otimes\mathbb{V}_{(-\widetilde{\xi},\widetilde{\xi})}(\mathcal{K})).$$

Let $M = \mathbb{V}_{\mathbb{R},\widetilde{\xi}}(\mathcal{F}) \in D^b(\text{mod-}\mathcal{R})$, $N \in \mathbb{V}_{(-\widetilde{\xi},\widetilde{\xi})}(\mathcal{K}) \in D^b(\text{mod-}\mathcal{R}\otimes\mathcal{R})$, hecne $\tau N = {}_{-\widetilde{\xi}}\mathbb{V}_{\widetilde{\xi}}(\mathcal{K}) \in D^b(\mathcal{R}\text{-mod-}\mathcal{R})$ (see (8.14) for τ). Then $\dot{\mathrm{pr}}_{2*}(\dot{\mathrm{pr}}_1^* M \otimes N') \cong (M \otimes N') \otimes^{\mathbb{L}}_{\mathcal{R}_{12}} \mathbf{k}$ where $\otimes^{\mathbb{L}}_{\mathcal{R}_{12}}\mathbf{k}$ means the functor $\otimes^{\mathbb{L}}_{\mathcal{R}}\mathbf{k}$ taken using the following right $\mathcal{R}$-module structure on $M \otimes N'$: $(m\otimes n')\cdot r = (m\otimes n)\cdot\delta(r)$, where $\delta : \mathcal{R} \to \mathcal{R}\otimes\mathcal{R}$ is the comultiplication induced by the diagonal map $\pi_1(T^c) \to \pi_1(T^c) \times \pi_1(T^c)$. Here we are using the fact that the pushforward $T^c \to \mathrm{pt}$ of monodromic sheaves corresponds to the functor $(-) \otimes^{\mathbb{L}}_{\mathcal{R}} \mathbf{k} : D^b(\text{mod-}\mathcal{R}) \to D^b(\text{mod-}\mathbf{k})$. It is easy to see there is a canonical isomorphism of right $\mathcal{R}$-modules (using the second $\mathcal{R}$-action on N and the right $\mathcal{R}$-action on τN)

$$(M \otimes N) \otimes^{\mathbb{L}}_{\mathcal{R}} \mathbf{k} \cong M \otimes^{\mathbb{L}}_{\mathcal{R}} \tau N. \tag{8.41}$$

Therefore the right side of (8.40) is canonically isomorphic to $M \otimes^{\mathbb{L}}_{\mathcal{R}} \tau N = \mathbb{V}_{\mathbb{R},\widetilde{\xi}}(\mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{R}} {}_{-\widetilde{\xi}}\mathbb{V}_{\widetilde{\xi}}(\mathcal{K})$ as a right $\mathcal{R}$-module. This completes the construction of $\alpha_{\mathcal{F},\mathcal{K}}$.

One needs to check that $\alpha_{\mathcal{F},\mathcal{K}}$ is compatible with the monoidal structure on $\mathbb{V}$ constructed in Proposition 8.8. The argument is similar and we omit it.

□

We record here the functoriality properties of microlocalization from [15] that we used in the above proof. Let $f : Y \to X$ be a map of manifolds, $M \subset X$ a closed submanifold and $N = f^{-1}(M)$. Then there is a natural correspondence between the conormals

$$T_N^* Y \xleftarrow{df_N} N \times_M (T_M^* X) \xrightarrow{f_N^\natural} T_M^* X \tag{8.42}$$

We have the microlocalization functors $\mu_M : D^b(X) \to D^b(T_M^* X)$ and $\mu_N : D^b(Y) \to D^b(T_N^* Y)$.

Proposition 8.11. *Assume that f is transversal to M (i.e. df_N above is an isomorphism).*

(1) ([15, Proposition 4.3.4]) *Let $\mathcal{G} \in D^b(Y)$ and suppose that $f|_{\mathrm{supp}(\mathcal{G})}$ is proper, then there is a canonical isomorphism*

$$\mu_M(f_* \mathcal{G}) \xrightarrow{\sim} f_{N*}^\natural df_N^* \mu_N(\mathcal{G}). \tag{8.43}$$

(2) ([15, Proposition 4.3.5]) *Let $\mathcal{F} \in D^b(X)$. Then there is a canonical isomorphism*

$$df_{N*} f_N^{\natural,*} \mu_M(\mathcal{F}) \xrightarrow{\sim} \mu_N(f^* \mathcal{F}). \tag{8.44}$$

9. Structure theorem for real tilting sheaves

9.1. **The algebras $\mathcal{A}$ and $\mathcal{A}_0$.** We choose $\xi \in i\mathcal{N}_{\mathbb{R}}^* \cap \mathcal{O}_{\mathrm{reg}}$ and $\widetilde{\xi} \in T^* \widetilde{X}$ over ξ, and define the functor $\mathbb{V}_{\mathbb{R}} = \mathbb{V}_{\mathbb{R}, \widetilde{\xi}}$.

We put $\mathcal{A} := \mathsf{End}(\mathbb{V}_{\mathbb{R}}|_{\mathrm{Tilt}})$ for the endomorphism ring of the functor $\mathbb{V}_{\mathbb{R}}|_{\mathrm{Tilt}}$ (shorthand for the restriction of $\mathbb{V}_{\mathbb{R}}$ to $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$). For an element $a \in \mathcal{A}$ and a tilting sheaf $\mathcal{T}$ we put $a_{\mathcal{T}} \in \mathsf{End}(\mathbb{V}(\mathcal{T}))$ for the action of a on $\mathbb{V}_{\mathbb{R}}(\mathcal{T})$. Then $\mathbb{V}_{\mathbb{R}}$ upgrades to an exact functor

$$\mathbb{V}_{\mathbb{R}}^\sharp : \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}}) \to \mathcal{A}\text{-mod}. \tag{9.1}$$

The goal of the rest of the section is to prove the following theorem.

Theorem 9.2. *The functor $\mathbb{V}_{\mathbb{R}}^\sharp$ is fully faithful.*

Recall $\mathcal{R} = \mathcal{R}_{T^c} := \widehat{\mathbf{k}[\pi_1(T^c)]}$ is the group algebra of the fundamental group of T^c completed at the augmentation ideal. The monodromy action along the fibers of π induces a ring map

$$\varphi_{\mathcal{R}} : \mathcal{R} \to Z(\mathcal{A}) \tag{9.2}$$

to the center of $\mathcal{A}$.

Let $\mathcal{A}_0 \subset \mathcal{A}$ be the subalgebra of endomorphisms of $\mathbb{V}_{\mathbb{R}}|_{\mathrm{Tilt}}$ commuting with the Hecke action. More precisely, $a \in \mathcal{A}_0$ if for any $\mathcal{T} \in \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$ and $\mathcal{T}' \in \mathrm{Tilt}(\mathcal{H}_G)$, the endomorphism $a_{\mathcal{T} \star \mathcal{T}'}$ of $\mathbb{V}_{\mathbb{R}}(\mathcal{T} \star \mathcal{T}') \cong \mathbb{V}_{\mathbb{R}}(\mathcal{T}) \otimes_{\mathcal{R}} \mathbb{V}(\mathcal{T}')$ is equal to $a_{\mathcal{T}} \otimes \mathrm{id}_{\mathbb{V}(\mathcal{T}')}$.

Lemma 9.3. (1) *The image of $\mathcal{R}^W$ under the map $\varphi_{\mathcal{R}}$ lies in $\mathcal{A}_0$.*

(2) *Recall the big tilting object* $\mathcal{T}_{w_0} \in \mathrm{Tilt}(\mathcal{H}_G)$. *Let* $a \in \mathcal{A}$. *Then* $a \in \mathcal{A}_0$ *if and only if for any* $\mathcal{T} \in \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$, *the endomorphism* $a_{\mathcal{T}\star\mathcal{T}_{w_0}}$ *of* $\mathbb{V}_{\mathbb{R}}(\mathcal{T}\star\mathcal{T}_{w_0}) \cong \mathbb{V}_{\mathbb{R}}(\mathcal{T})\otimes_{\mathcal{R}}\mathbb{V}(\mathcal{T}_{w_0})$ *is equal to* $a_{\mathcal{T}}\otimes\mathrm{id}_{\mathbb{V}(\mathcal{T}_{w_0})}$.

Proof. (1) Let $b \in \mathcal{R}$ and $a = \varphi_{\mathcal{R}}(b) \in Z(\mathcal{A})$. Let $\mathcal{T} \in \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$ and $\mathcal{T}' \in \mathrm{Tilt}(\mathcal{H}_G)$. Then $a_{\mathcal{T}\star\mathcal{T}'}$ is induced from the action of b on $\mathcal{T}\star\mathcal{T}'$ by right T^c-monodromy on $\mathcal{T}'$. We know that the left and right monodromy actions on $\mathcal{T}' \in \mathrm{Tilt}(\mathcal{H}_G)$ factor through $\mathcal{R}\otimes_{\mathcal{R}^W}\mathcal{R}$ (as it is so for $\mathbb{V}(\mathcal{T}')$ (Proposition 8.7(3)) and $\mathbb{V}|_{\mathrm{Tilt}}$ is fully faithful (Proposition 8.7(4))). Therefore, if $b \in \mathcal{R}^W$, then the right monodromy action of b on $\mathcal{T}'$ is the same as left monodromy, and the induced action on $\mathcal{T}\star\mathcal{T}'$ is the same as the action of b on $\mathcal{T}$ by (right) T^c-monodromy. Therefore, both $a_{\mathcal{T}\star\mathcal{T}'}$ and $a_{\mathcal{T}}\otimes\mathrm{id}_{\mathbb{V}(\mathcal{T}')}$ on $\mathbb{V}_{\mathbb{R}}(\mathcal{T}\star\mathcal{T}') \cong \mathbb{V}_{\mathbb{R}}(\mathcal{T})\otimes_{\mathcal{R}}\mathbb{V}(\mathcal{T}')$ are given by the action of $b \in \mathcal{R}^W$ on the first factor $\mathbb{V}_{\mathbb{R}}(\mathcal{T})$. This shows $a \in \mathcal{A}_0$.

(2) Suppose $a \in \mathcal{A}$ satisfies $a_{\mathcal{T}\star\mathcal{T}_{w_0}} = a_{\mathcal{T}}\otimes\mathrm{id}_{\mathbb{V}(\mathcal{T}_{w_0})}$ for all $\mathcal{T} \in \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$. Let $\mathcal{T}' \in \mathrm{Tilt}(\mathcal{H}_G)$, we want to show that $a_{\mathcal{T}\star\mathcal{T}'} = a_{\mathcal{T}}\otimes\mathrm{id}_{\mathbb{V}(\mathcal{T}')}$ as endomorphisms of $\mathbb{V}_{\mathbb{R}}(\mathcal{T}\star\mathcal{T}') \cong \mathbb{V}_{\mathbb{R}}(\mathcal{T})\otimes_{\mathcal{R}}\mathbb{V}(\mathcal{T}')$. We have a canonical map

$$\epsilon : \mathsf{Hom}(\mathcal{T}_{w_0}, \mathcal{T}')\otimes\mathbb{V}(\mathcal{T}_{w_0}) \to \mathbb{V}(\mathcal{T}') \tag{9.3}$$

sending $f\otimes v$ to the image of $v \in \mathbb{V}(\mathcal{T}_{w_0})$ under the map $\mathbb{V}(f) : \mathbb{V}(\mathcal{T}_{w_0}) \to \mathbb{V}(\mathcal{T}')$. By Proposition 8.7(2), $\mathsf{Hom}(\mathcal{T}_{w_0}, \mathcal{T}') \cong \mathbb{V}(\mathcal{T}')$ and $\mathbb{V}(\mathcal{T}_{w_0}) \cong \mathsf{End}(\mathcal{T}_{w_0})$. Take $\mathrm{id}_{\mathcal{T}_{w_0}} \in \mathsf{End}(\mathcal{T}_{w_0}) \cong \mathbb{V}(\mathcal{T}_{w_0})$, we have $\epsilon(f\otimes\mathrm{id}_{\mathcal{T}_{w_0}}) = f$ for any $f \in \mathsf{Hom}(\mathcal{T}_{w_0}, \mathcal{T}') \cong \mathbb{V}(\mathcal{T}')$. This shows that ϵ is surjective. The similarly defined map

$$\epsilon_{\mathcal{T}} : \mathsf{Hom}(\mathcal{T}_{w_0}, \mathcal{T}')\otimes\mathbb{V}_{\mathbb{R}}(\mathcal{T}\star\mathcal{T}_{w_0}) \to \mathbb{V}_{\mathbb{R}}(\mathcal{T}\star\mathcal{T}')$$

can be identified with

$$\epsilon\otimes\mathrm{id}_{\mathbb{V}_{\mathbb{R}}(\mathcal{T})} : \mathsf{Hom}(\mathcal{T}_{w_0}, \mathcal{T}')\otimes\mathbb{V}_{\mathbb{R}}(\mathcal{T})\otimes_{\mathcal{R}}\mathbb{V}(\mathcal{T}_{w_0}) \to \mathbb{V}_{\mathbb{R}}(\mathcal{T})\otimes_{\mathcal{R}}\mathbb{V}(\mathcal{T}'),$$

hence is also surjective. By the definition of $\mathcal{A}$, we have a commutative diagram

$$\begin{tikzcd} \mathsf{Hom}(\mathcal{T}_{w_0}, \mathcal{T}')\otimes\mathbb{V}_{\mathbb{R}}(\mathcal{T}\star\mathcal{T}_{w_0}) \arrow[r, "\epsilon_{\mathcal{T}}"] \arrow[d, "\mathrm{id}\otimes a_{\mathcal{T}\star\mathcal{T}_{w_0}}"] & \mathbb{V}_{\mathbb{R}}(\mathcal{T}\star\mathcal{T}') \arrow[d, "a_{\mathcal{T}\star\mathcal{T}'}"] \\ \mathsf{Hom}(\mathcal{T}_{w_0}, \mathcal{T}')\otimes\mathbb{V}_{\mathbb{R}}(\mathcal{T}\star\mathcal{T}_{w_0}) \arrow[r, "\epsilon_{\mathcal{T}}"] & \mathbb{V}_{\mathbb{R}}(\mathcal{T}\star\mathcal{T}') \end{tikzcd} \tag{9.4}$$

Rewriting $\mathbb{V}_{\mathbb{R}}(\mathcal{T} \star \mathcal{T}_{w_0})$ as $\mathbb{V}_{\mathbb{R}}(\mathcal{T}) \otimes_{\mathcal{R}} \mathbb{V}(\mathcal{T}_{w_0})$ and $\mathbb{V}_{\mathbb{R}}(\mathcal{T} \star \mathcal{T}')$ as $\mathbb{V}_{\mathbb{R}}(\mathcal{T}) \otimes_{\mathcal{R}} \mathbb{V}(\mathcal{T}')$, we similarly we have a commutative diagram

$$\begin{tikzcd} \mathsf{Hom}(\mathcal{T}_{w_0}, \mathcal{T}') \otimes \mathbb{V}_{\mathbb{R}}(\mathcal{T}) \otimes_{\mathcal{R}} \mathbb{V}(\mathcal{T}_{w_0}) \arrow[r, "\epsilon\otimes\mathrm{id}_{\mathbb{V}_{\mathbb{R}}(\mathcal{T})}"] \arrow[d, "\mathrm{id}\otimes a_{\mathcal{T}}\otimes\mathrm{id}"] & \mathbb{V}_{\mathbb{R}}(\mathcal{T}) \otimes_{\mathcal{R}} \mathbb{V}(\mathcal{T}') \arrow[d, "a_{\mathcal{T}}\otimes\mathrm{id}"] \\ \mathsf{Hom}(\mathcal{T}_{w_0}, \mathcal{T}') \otimes \mathbb{V}_{\mathbb{R}}(\mathcal{T}) \otimes_{\mathcal{R}} \mathbb{V}(\mathcal{T}_{w_0}) \arrow[r, "\epsilon\otimes\mathrm{id}_{\mathbb{V}_{\mathbb{R}}(\mathcal{T})}"] & \mathbb{V}_{\mathbb{R}}(\mathcal{T}) \otimes_{\mathcal{R}} \mathbb{V}(\mathcal{T}') \end{tikzcd} \tag{9.5}$$

Now compare (9.4) and (9.5), the left vertical maps are equal in the two diagrams by assumption. Since the horizontal maps are equal and surjective, the right vertical maps are also equal in the two diagrams, i.e., $a_{\mathcal{T}\star\mathcal{T}'} = a_{\mathcal{T}} \otimes \mathrm{id}_{\mathbb{V}(\mathcal{T}')}$. □

By Lemma 9.3, we have a ring homomorphism

$$\varphi : \mathcal{A}_0 \otimes_{\mathcal{R}^W} \mathcal{R} \to \mathcal{A} \tag{9.6}$$

where the image of $\mathcal{R}$ is central.

Proposition 9.4. *The ring map φ is an isomorphism.*

Proof. We construct a map in the other direction as follows. Let $w_0 \in W$ be the longest element, and $\mathcal{T}_{w_0} \in \mathrm{Tilt}(\mathcal{H}_G)$ be the indecomposable free-monodromic tilting object with full support. It is known (Theorem 9.1 in [7], Proposition 4.7.3 1) in [9]) that $\mathbb{V}(\mathcal{T}_{w_0}) = \mathcal{R} \otimes_{\mathcal{R}^W} \mathcal{R}$. Consider the functor $\mathbb{U}_{\mathbb{R}} : \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}}) \to \mathcal{R}\text{-mod-}\mathcal{R}$ given by $\mathbb{U}_{\mathbb{R}}(\mathcal{T}) := \mathbb{V}_{\mathbb{R}}(\mathcal{T} \star \mathcal{T}_{w_0})$, with the two $\mathcal{R}$-actions given by the left and right monodromy actions on $\mathcal{T}_{w_0}$. Using Proposition 8.7(3)

$$\mathbb{U}_{\mathbb{R}}(\mathcal{T}) \cong \mathbb{V}_{\mathbb{R}}(\mathcal{T}) \otimes_{\mathcal{R}} \mathbb{V}(\mathcal{T}_{w_0}) \cong \mathbb{V}_{\mathbb{R}}(\mathcal{T}) \otimes_{\mathcal{R}^W} \mathcal{R} \tag{9.7}$$

with the first copy of $\mathcal{R}$ acting on $\mathbb{V}_{\mathbb{R}}(\mathcal{T})$ and the second copy acting on the second tensor factor $\mathcal{R}$ by multiplication. From (9.7) we see that $\mathsf{End}(\mathbb{U}_{\mathbb{R}}) \cong \mathcal{A} \otimes_{\mathcal{R}^W} \mathcal{R}$.

We have an action of $\mathcal{A}$ on $\mathbb{U}_{\mathbb{R}}$: for $a \in \mathcal{A}$ and $\mathcal{T} \in \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$, a acts on $\mathbb{U}_{\mathbb{R}}(\mathcal{T})$ by $a_{\mathcal{T}\star\mathcal{T}_{w_0}}$. This gives a ring map

$$\psi' : \mathcal{A} \to \mathsf{End}(\mathbb{U}_{\mathbb{R}}) \cong \mathcal{A} \otimes_{\mathcal{R}^W} \mathcal{R}. \tag{9.8}$$

Claim. *The image of ψ' is contained in $\mathcal{A}_0 \otimes_{\mathcal{R}^W} \mathcal{R}$.*

Proof of Claim. Using the characterization of $\mathcal{A}_0$ given in Lemma 9.3(2), $\mathcal{A}_0 \otimes_{\mathcal{R}^W} \mathcal{R}$ consists exactly of those $b \in \mathsf{End}(\mathbb{U}_{\mathbb{R}})$ such that $b_{\mathcal{T}\star\mathcal{T}_{w_0}} = b_{\mathcal{T}} \otimes \mathrm{id}_{\mathbb{V}(\mathcal{T}_{w_0})}$. Therefore, to show $\psi'(a) \in \mathcal{A}_0 \otimes_{\mathcal{R}^W} \mathcal{R}$ for $a \in \mathcal{A}$, we need to check for any $\mathcal{T} \in \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$,

$$a_{\mathcal{T}\star\mathcal{T}'_{w_0}\star\mathcal{T}''_{w_0}} = a_{\mathcal{T}\star\mathcal{T}''_{w_0}} \otimes \mathrm{id}_{\mathbb{V}(\mathcal{T}'_{w_0})} \in \mathsf{End}(\mathbb{V}_{\mathbb{R}}(\mathcal{T} \star \mathcal{T}'_{w_0} \star \mathcal{T}''_{w_0})). \tag{9.9}$$

Here, to distinguish two copies of $\mathcal{T}_{w_0}$ we denote them by $\mathcal{T}'_{w_0}$ and $\mathcal{T}''_{w_0}$. Consider the map $\beta : \mathcal{T}''_{w_0} \to \mathcal{T}'_{w_0} \star \mathcal{T}''_{w_0}$ such that $\mathbb{V}(\beta)$ is given by

$$\mathcal{R} \otimes_{\mathcal{R}^W} \mathcal{R} \;\to\; \mathcal{R} \otimes_{\mathcal{R}^W} \mathcal{R} \otimes_{\mathcal{R}^W} \mathcal{R} \tag{9.10}$$
$$a \otimes b \;\mapsto\; a \otimes 1 \otimes b. \tag{9.11}$$

Then β induces $\mathrm{id}_{\mathcal{T}} \star \beta : \mathcal{T} \star \mathcal{T}''_{w_0} \to \mathcal{T} \star \mathcal{T}'_{w_0} \star \mathcal{T}''_{w_0}$, hence a commutative diagram

$$\begin{tikzcd} \mathbb{V}_{\mathbb{R}}(\mathcal{T} \star \mathcal{T}''_{w_0}) \arrow[r, "\mathbb{V}(\mathrm{id}_{\mathcal{T}} \star \beta)"] \arrow[d, "a_{\mathcal{T}\star\mathcal{T}''_{w_0}}"] & \mathbb{V}_{\mathbb{R}}(\mathcal{T} \star \mathcal{T}'_{w_0} \star \mathcal{T}''_{w_0}) \arrow[d, "a_{\mathcal{T}\star\mathcal{T}'_{w_0}\star\mathcal{T}''_{w_0}}"] \\ \mathbb{V}_{\mathbb{R}}(\mathcal{T} \star \mathcal{T}''_{w_0}) \arrow[r, "\mathbb{V}(\mathrm{id}_{\mathcal{T}} \star \beta)"] & \mathbb{V}_{\mathbb{R}}(\mathcal{T} \star \mathcal{T}'_{w_0} \star \mathcal{T}''_{w_0}) \end{tikzcd} \tag{9.12}$$

Using the description of $\mathbb{V}(\beta)$, we see that the above diagram can be written as

$$\begin{tikzcd} \mathbb{V}_{\mathbb{R}}(\mathcal{T}) \otimes_{\mathcal{R}^W} \mathcal{R} \arrow[r, "\mathrm{id}\otimes 1\otimes \mathrm{id}"] \arrow[d, "a_{\mathcal{T}\star\mathcal{T}''_{w_0}}"] & \mathbb{V}_{\mathbb{R}}(\mathcal{T}) \otimes_{\mathcal{R}^W} \mathcal{R} \otimes_{\mathcal{R}^W} \mathcal{R} \arrow[d, "a_{\mathcal{T}\star\mathcal{T}'_{w_0}\star\mathcal{T}''_{w_0}}"] \\ \mathbb{V}_{\mathbb{R}}(\mathcal{T}) \otimes_{\mathcal{R}^W} \mathcal{R} \arrow[r, "\mathrm{id}\otimes 1\otimes \mathrm{id}"] & \mathbb{V}_{\mathbb{R}}(\mathcal{T}) \otimes_{\mathcal{R}^W} \mathcal{R} \otimes_{\mathcal{R}^W} \mathcal{R} \end{tikzcd} \tag{9.13}$$

This shows that (9.9) holds on the subspace $\mathbb{V}_{\mathbb{R}}(\mathcal{T}) \otimes 1 \otimes \mathcal{R} \subset \mathbb{V}_{\mathbb{R}}(\mathcal{T}) \otimes_{\mathcal{R}^W} \mathcal{R} \otimes_{\mathcal{R}^W} \mathcal{R} = \mathbb{V}_{\mathbb{R}}(\mathcal{T} \star \mathcal{T}'_{w_0} \star \mathcal{T}''_{w_0})$. Since both $a_{\mathcal{T}\star\mathcal{T}'_{w_0}\star\mathcal{T}''_{w_0}}$ and $a_{\mathcal{T}\star\mathcal{T}''_{w_0}} \otimes \mathrm{id}_{\mathbb{V}(\mathcal{T}'_{w_0})}$ are linear with repsect to the three $\mathcal{R}$-actions on $\mathbb{V}_{\mathbb{R}}(\mathcal{T} \star \mathcal{T}'_{w_0} \star \mathcal{T}''_{w_0})$, we conclude that (9.9) holds. □

By the Claim, we have a map

$$\psi : \mathcal{A} \to \mathcal{A}_0 \otimes_{\mathcal{R}^W} \mathcal{R}. \tag{9.14}$$

Note that this map is $\mathcal{R}$-linear. We check that φ and ψ are inverse to each other. If $a \in \mathcal{A}_0$, then $\psi\varphi(a)$ acts on $\mathbb{U}_{\mathbb{R}}(\mathcal{T}) = \mathbb{V}_{\mathbb{R}}(\mathcal{T} \star \mathcal{T}_{w_0})$ by $a_{\mathcal{T}\star\mathcal{T}_{w_0}}$. Since $a \in \mathcal{A}_0$, $a_{\mathcal{T}\star\mathcal{T}_{w_0}} = a_{\mathcal{T}} \otimes \mathrm{id}_{\mathbb{V}(\mathcal{T}_{w_0})}$, which implies $\psi\varphi(a) = a \otimes 1 \in \mathcal{A}_0 \otimes_{\mathcal{R}^W} \mathcal{R}$. By $\mathcal{R}$-linearity, this implies $\psi\varphi = \mathrm{id}$.

On the other hand, to check $\varphi\psi = \mathrm{id}_{\mathcal{A}}$, it suffices to show that the composition

$$\mathcal{A} \xrightarrow{\psi'} \mathcal{A} \otimes_{\mathcal{R}^W} \mathcal{R} \xrightarrow{m} \mathcal{A} \tag{9.15}$$

is the identity, where m is the multiplication map. We have a canonical map $\epsilon : \mathcal{T}_{w_0} \to \tilde{\delta}$ that induces the multiplication map $\mathbb{V}(\mathcal{T}_{w_0}) = \mathcal{R} \otimes_{\mathcal{R}^W} \mathcal{R} \to \mathcal{R} = \mathbb{V}(\tilde{\delta})$. It induces a natural transformation of functors $\gamma : \mathbb{U}_{\mathbb{R}} \to \mathbb{V}_{\mathbb{R}}$ ($\mathcal{R}$-linear with respect to the second

$\mathcal{R}$-action on $\mathbb{U}_{\mathbb{R}}$). Under the isomorphism $\mathbb{U}_{\mathbb{R}} \cong \mathbb{V}_{\mathbb{R}} \otimes_{\mathcal{R}^W} \mathcal{R}$, γ corresponds to the multiplication map $\mathbb{V}_{\mathbb{R}} \otimes_{\mathcal{R}^W} \mathcal{R} \to \mathbb{V}_{\mathbb{R}}$. Therefore for any $b \in \mathsf{End}(\mathbb{U}_{\mathbb{R}}) \cong \mathcal{A} \otimes_{\mathcal{R}^W} \mathcal{R}$, we have a commutative diagram for any $\mathcal{T} \in \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$

$$\begin{tikzcd} \mathbb{U}_{\mathbb{R}}(\mathcal{T}) \arrow[r, "\gamma"] \arrow[d, "b_{\mathcal{T}}"] & \mathbb{V}_{\mathbb{R}}(\mathcal{T}) \arrow[d, "m(b)_{\mathcal{T}}"] \\ \mathbb{U}_{\mathbb{R}}(\mathcal{T}) \arrow[r, "\gamma"] & \mathbb{V}_{\mathbb{R}}(\mathcal{T}) \end{tikzcd} \tag{9.16}$$

On the other hand, by the definition of $\mathcal{A}$ we have a commutative diagram

$$\begin{tikzcd} \mathbb{V}_{\mathbb{R}}(\mathcal{T} \star \mathcal{T}_{w_0}) \arrow[r, "\mathbb{V}(\mathrm{id}_{\mathcal{T}} \star \epsilon)"] \arrow[d, "a_{\mathcal{T} \star \mathcal{T}_{w_0}}"] & \mathbb{V}_{\mathbb{R}}(\mathcal{T}) \arrow[d, "a_{\mathcal{T}}"] \\ \mathbb{U}_{\mathbb{R}}(\mathcal{T}) \arrow[r, "\mathbb{V}(\mathrm{id}_{\mathcal{T}} \star \epsilon)"] & \mathbb{V}_{\mathbb{R}}(\mathcal{T}) \end{tikzcd} \tag{9.17}$$

Now taking $b = \psi'(a)$ in (9.16), it becomes the same diagram as (9.17), from which we conclude that $m(\psi'(a)) = m(b) = a$. This implies $m\psi' = \mathrm{id}_{\mathcal{A}}$ hence $\varphi\psi = \mathrm{id}_{\mathcal{A}}$ and finishes the proof. $\square$

Now we define an action of Soergel bimodules SBim on $\mathcal{A}$-mod as follows: $M \in \mathcal{A}$-mod and $N \in$ SBim, the action of N on M is the tensor product $M \otimes_{\mathcal{R}} N$. Now $M \otimes_{\mathcal{R}} N$ is naturally a $\mathcal{A}_0 \otimes_{\mathcal{R}^W} \mathcal{R}$-module using the $\mathcal{A}_0$-action on M and the right $\mathcal{R}$-action on N. Since $\mathcal{A}_0 \otimes_{\mathcal{R}^W} \mathcal{R} \xrightarrow{\sim} \mathcal{A}$ by Proposition 9.4, we may view $M \otimes_{\mathcal{R}} N$ as an $\mathcal{A}$-module.

The following is an immediate consequence of Proposition 8.10, and the action defined above.

Corollary 9.5. *The functors $\mathbb{V}^{\sharp}_{\mathbb{R}}$ in (9.1) and $\mathbb{V}^{\sharp}$ in (8.17) intertwine the convolution action of $\mathrm{Tilt}(\mathcal{H}_G)$ on $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$ and the action of SBim on $\mathcal{A}$-mod defined above.*

The following is parallel to Proposition 6.6.

Lemma 9.6. *Let $\mathcal{B}_s = \mathcal{R} \otimes_{\mathcal{R}^s} \mathcal{R} \in$ SBim for each simple reflection $s \in W$. Then the action of $\mathcal{B}_s$ on $\mathcal{A}$-mod is self-adjoint: there is an isomorphism functorial in $M_1, M_2 \in \mathcal{A}$-mod*

$$\mathsf{Hom}_{\mathcal{A}}(M_1 \otimes_{\mathcal{R}} \mathcal{B}_s, M_2) \cong \mathsf{Hom}_{\mathcal{A}}(M_1, M_2 \otimes_{\mathcal{R}} \mathcal{B}_s) \tag{9.18}$$

Proof. As in the proof of Proposition 6.6, it suffices to give unit and counit maps $u : \mathcal{R} \to \mathcal{B}_s \otimes_{\mathcal{R}} \mathcal{B}_s$ and $\mathcal{B}_s \otimes_{\mathcal{R}} \mathcal{B}_s \to \mathcal{R}$ in SBim satisfying identities analogous to (6.34). The maps u and c are given in the proof of Proposition 6.6 because $\mathbb{V}(\mathcal{T}_s) \cong \mathcal{B}_s$. $\square$

9.7. **Real Soergel functor on localized categories.** The real Soergel functor $\mathbb{V}_{\mathbb{R}}$ is $\mathcal{R}$-linear and so we can define

$$\mathbb{V}_{\mathbb{R}} \otimes_{\mathcal{R}} \mathcal{K}_J \colon \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}}) \otimes_{\mathcal{R}} \mathcal{K}_J \to \text{mod-}\mathcal{K}_J$$

. Its endomorphism algebra is isomorphic to $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{K}_J$ and we also get the functor

$$\mathbb{V}^{\sharp}_{\mathbb{R}} \otimes_{\mathcal{R}} \mathcal{K}_J \colon \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}}) \otimes_{\mathcal{R}} \mathcal{K}_J \to \mathcal{A} \otimes_{\mathcal{R}} \mathcal{K}_J\text{-mod}.$$

They enjoy the following properties.

Lemma 9.8. *The equivalence of Proposition 7.7 intertwines the localized real Soergel functors on both sides.*

Proof. We put $\mathfrak{p} \supset \mathfrak{p}_{\mathbb{R}}$ and $\mathfrak{l} \supset \mathfrak{l}_{\mathbb{R}}$ for the relevant Lie algebras. Let $x \in X_L$ be a point in the closed orbit and let a regular nilpotent $\xi \in \mathcal{N} \cap i\mathfrak{g}^*_{\mathbb{R}}$ be the generic conormal to the closed orbit at x inside X. Since x lies in X_L, the corresponding Borel is contained in P and $\xi \in \mathcal{N} \cap \mathfrak{p}$ we conclude that ξ is orthogonal to the nilpotent radical $\mathfrak{p}_P$ of $\mathfrak{p}$ and, thus, $p_L(\xi)$ lands in $i\mathfrak{l}^*_{\mathbb{R}}$. Moreover, $p_L(\xi)$ is regular inside $i\mathfrak{l}^*_{\mathbb{R}}$. Indeed, for a nonregular element $e_0 \in \mathcal{N}_L$ the elements in $e_0 + \mathfrak{n}_P$ are also not regular. By construction we can also match the fibers $\pi^{-1}(x)$ and $\pi_L^{-1}(x)$ together with the compact part of the stabilizers of x inside G and L. We conclude that for a sheaf $\mathcal{F}$ we have an isomorphsim

$$\mu_{(x,\xi)}(\mathrm{Av}_{G_{\mathbb{R}}} \circ \widetilde{i}_{P,*}(\mathcal{F})) \cong \mu_{(x,p_L(\xi))}(\mathcal{F}) \tag{9.19}$$

of free-monodromic local systems on T^c.

□

Proposition 9.9. *Let $V_J \subset V_{\lambda_0}$. Assume that the localized Soergel functor $\mathbb{V}^{\sharp}_{\mathbb{R}} \otimes_{\mathcal{R}} \mathcal{K}_J$ is fully faithful on* $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}}) \otimes_{\mathcal{R}} \mathcal{K}_J$*. Then $\mathbb{V}^{\sharp}_{\mathbb{R}} \otimes_{\mathcal{R}} \mathcal{R}_J$ is fully faithful on* $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}}) \otimes_{\mathcal{R}} \mathcal{R}_J$*.*

Proof. By Proposition 6.7 and the assumption $V_J \subset V_{\lambda_0}$ the subcategory $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}}) \otimes_{\mathcal{R}} \mathcal{K}_J$ generates $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}}) \otimes_{\mathcal{R}} \mathcal{R}_J$ under the localized action (7.8). Therefore, by Proposition 6.6 it is sufficient to check that the map

$$\mathrm{RHom}_{\mathcal{M}_{G_{\mathbb{R}}}}(\mathcal{T}_1, \mathcal{T}_2) \otimes_{\mathcal{R}} \mathcal{R}_J \to \mathrm{RHom}_{\mathcal{A} \otimes_{\mathcal{R}} \mathcal{R}_J}(\mathbb{V}_{\mathbb{R}}(\mathcal{T}_1) \otimes_{\mathcal{R}} \mathcal{R}_J, \mathbb{V}_{\mathbb{R}}(\mathcal{T}_2) \otimes_{\mathcal{R}} \mathcal{R}_J)$$

is an isomorphism for $\mathcal{T}_2 \in \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}}) \otimes_{\mathcal{R}} \mathcal{K}_J$ and $\mathcal{T}_1 \in \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}}) \otimes_{\mathcal{R}} w(\mathcal{K}_J)$, where $w(\mathcal{K}_J) \neq \mathcal{K}_J$ is the localization of V at the generic point of $w(V_J) \neq V_J$. As an $\mathcal{R}$-module $\mathrm{RHom}_{\mathcal{M}_{G_{\mathbb{R}}}}(\mathcal{T}_1, \mathcal{T}_2)$ is supported on $V_J \cap w(V_J)$ and, therefore, vanishes after applying $- \otimes_{\mathcal{R}} \mathcal{R}_J$. Since $\mathcal{M}_{G_{\mathbb{R}}} \otimes_{\mathcal{R}} \mathcal{R}_J$ decomposes into the direct sum, so does $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{R}_J$. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ belong to the different summands of $\mathcal{M}_{G_{\mathbb{R}}} \otimes_{\mathcal{R}} \mathcal{R}_J$, the algebra $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{R}_J$ acts on $\mathbb{V}_{\mathbb{R}}(\mathcal{T}_1)$

and $\mathbb{V}_{\mathbb{R}}(\mathcal{T}_2)$ through different direct summands. It follows that $\mathrm{RHom}_{\mathcal{A}\otimes_{\mathcal{R}}\mathcal{R}_J}(\mathbb{V}_{\mathbb{R}}(\mathcal{T}_1)\otimes_{\mathcal{R}}\mathcal{R}_J, \mathbb{V}_{\mathbb{R}}(\mathcal{T}_2)\otimes_{\mathcal{R}}\mathcal{R}_J) = 0$ and we are done. $\square$

In the codimension 0 case recall that $\mathcal{R}_{\mathcal{Q}} = \mathcal{R}\otimes_{\mathcal{R}^{\mathbf{W}}}\mathcal{Q} \cong \prod_{\lambda\in I_0}\mathcal{K}_\lambda$ and consider the localized functors

$$\mathbb{V}_{\mathbb{R},\mathcal{Q}} : \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})_{\mathcal{Q}} \to \mathcal{R}_{\mathcal{Q}}\text{-mod} \tag{9.20}$$

and

$$\mathbb{V}^{\sharp}_{\mathbb{R},\mathcal{Q}} : \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})_{\mathcal{Q}} \to \mathcal{A}\otimes_{\mathcal{R}^{\mathbf{W}}}\mathcal{Q}\text{-mod} \tag{9.21}$$

We observe that

Lemma 9.10. *For any $(\lambda,\chi)\in\widetilde{I}_0$, $\mathbb{V}_{\mathbb{R},\mathcal{Q}}(\widetilde{\Delta}_{\lambda,\chi})\cong\mathcal{K}_\lambda$ as an $\mathcal{R}_{\mathcal{Q}}$-module.*

Proof. The statement is clear for $\lambda=\lambda_0$. For general $\lambda\in I_0$ we have $(\lambda,\chi)=(\lambda_0,\chi')\cdot w$ for some $w\in\mathbf{W}$. By Lemma 7.9, $\mathbb{V}_{\mathbb{R},\mathcal{Q}}(\widetilde{\Delta}_{\lambda,\chi}) \cong \mathbb{V}_{\mathbb{R},\mathcal{Q}}(\widetilde{\Delta}_{\lambda_0,\chi'}\star\widetilde{\Delta}_w)$ is the translation under w of $\mathbb{V}_{\mathbb{R},\mathcal{Q}}(\widetilde{\Delta}_{\lambda_0,\chi'})\cong\mathcal{K}$, which is $\mathcal{K}_\lambda$ as an $\mathcal{R}_{\mathcal{Q}}$-module. $\square$

We can now check the version of Theorem 9.2 for the localized categories.

Lemma 9.11. *The functor $\mathbb{V}^{\sharp}_{\mathbb{R},\mathcal{Q}}$ is fully-faithful.*

Proof. We need to check that for $\mathcal{F}_1,\mathcal{F}_2\in\mathcal{M}_{G_{\mathbb{R}}}$, the map

$$\mathrm{RHom}_{\mathcal{M}_{G_{\mathbb{R}}}}(\mathcal{F}_1,\mathcal{F}_2)\otimes_{\mathcal{R}^W}\mathcal{Q} \to \mathrm{RHom}_{\mathcal{A}\otimes_{\mathcal{R}^W}\mathcal{Q}}(\mathbb{V}_{\mathbb{R}}(\mathcal{F}_1)\otimes_{\mathcal{R}^W}\mathcal{Q}, \mathbb{V}_{\mathbb{R}}(\mathcal{F}_2)\otimes_{\mathcal{R}^W}\mathcal{Q}) \tag{9.22}$$

is an isomorphism. Since the image of $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}}) \to \mathcal{M}_{G_{\mathbb{R}}}\otimes_{\mathcal{R}}\mathcal{Q}$ generates $\mathcal{M}_{G_{\mathbb{R}}}\otimes_{\mathcal{R}}\mathcal{Q}$ by taking direct summands (for $\widetilde{\Delta}_{\lambda,\chi}$ is a direct summand of $\mathcal{T}_{\lambda,\chi}$ after localization), it suffices to check the case $\mathcal{F}_1,\mathcal{F}_2\in\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$. By Proposition 6.7, Proposition 6.6 and Corollary 9.5 we can reduce to the case where $\mathcal{F}_2$ is supported on $\widetilde{X}_{\lambda_0}$ (preimage of the closed orbit), hence we may assume $\mathcal{F}_2=\widetilde{\Delta}_{\lambda_0,\psi}$.

Now instead of assuming $\mathcal{F}_1$ is a free-monodromic tilting sheaf, by Proposition 7.8 it suffices to treat the case $\mathcal{F}_1=\widetilde{\Delta}_{\lambda,\chi}$ for some $\lambda\in I_0$. If $\lambda\neq\lambda_0$, then the left side of (9.22) is zero by Proposition 7.8(2), and the right side vanishes for the same reason: the action of $\mathcal{R}$ on $\mathbb{V}_{\mathbb{R}}(\mathcal{F}_1)$ and $\mathbb{V}_{\mathbb{R}}(\mathcal{F}_2)$ has support contained in $\mathrm{Spec}\,\overline{\mathcal{R}}_\lambda\cap\mathrm{Spec}\,\overline{\mathcal{R}}_{\lambda_0}$, which has dimension less than $\dim\mathcal{S}^{W_{\mathbb{R}}}$. If $\lambda=\lambda_0$ then $\mathbb{V}_{\mathbb{R}}(\mathcal{F}_i)$ is just the stalk of $\mathcal{F}_i$ at a point in $\widetilde{X}_{\lambda_0}$. If $\chi=\psi$, then both sides of (9.22) are isomorphically equal to $\mathcal{K}$. If $\chi\neq\psi$, then the left side is zero by Proposition 7.8. By Lemma 9.10 we have $\mathbb{V}_{\mathbb{R},\mathcal{Q}}(\widetilde{\Delta}_{\lambda_0,\chi})\cong\mathbb{V}_{\mathbb{R},\mathcal{Q}}(\widetilde{\Delta}_{\lambda_0,\psi})\cong\mathcal{K}_{\lambda_0}$. Since by Proposition 7.8 the objects $\widetilde{\Delta}_{\lambda_0,\chi}$ and $\widetilde{\Delta}_{\lambda_0,\psi}$ are two different simple objects of the semisimple category $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})_{\mathcal{Q}}$ we have an element

$a \in \mathcal{A} \otimes_{\mathcal{R}^W} \mathcal{Q}$ acting on $\mathbb{V}_{\mathbb{R},\mathcal{Q}}(\widetilde{\Delta}_{\lambda_0,\chi})$ and $\mathbb{V}_{\mathbb{R},\mathcal{Q}}(\widetilde{\Delta}_{\lambda_0,\psi})$ by different elements of $\mathcal{K}$. This implies that the right hand side also vanishes. The statement now follows.

□

9.12. **Proofs of Theorem 9.2.** The following lemma will allow us to transfer the results from the localised category to the original category.

Lemma 9.13. *For a free-monodromic tilting object $\mathcal{T} \in \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$, the $\mathcal{R}^W$-action on $\mathbb{V}_{\mathbb{R}}(\mathcal{T})$ factors through the quotient $\mathcal{S}^{W_{\mathbb{R}}}$ and $\mathbb{V}_{\mathbb{R}}(\mathcal{T})$ is torsion free as an $\mathcal{S}^{W_{\mathbb{R}}}$-module.*

Proof. By Lemma 7.2, the action of $\mathcal{R}^W$ on $\mathcal{T}$ factors through $\mathcal{S}^{W_{\mathbb{R}}}$, therefore so does its action on $\mathbb{V}_{\mathbb{R}}(\mathcal{T})$.

If $\mathcal{L}$ is a free-monodromic local system supported on the closed orbit $\widetilde{O}^{\mathbb{R}}_{\lambda_0}$ and extended by zero to $\widetilde{X}$, then $\mathbb{V}_{\mathbb{R}}(\mathcal{L})$ is the same as the stalk of $\mathcal{L}$ along $\widetilde{O}^{\mathbb{R}}_{\lambda_0}$, hence a free $\mathcal{S}$-module. Since $\mathcal{S}$ is torsion free over $\mathcal{S}^{W_{\mathbb{R}}}$, $\mathbb{V}_{\mathbb{R}}(\mathcal{L})$ is torsion free over $\mathcal{S}^{W_{\mathbb{R}}}$.

Now for any $\mathcal{T}' \in \mathrm{Tilt}(\mathcal{H}_G)$, $\mathbb{V}_{\mathbb{R}}(\mathcal{L} \star \mathcal{T}') \cong \mathbb{V}_{\mathbb{R}}(\mathcal{L}) \otimes_{\mathcal{R}} \mathbb{V}(\mathcal{T}')$. Since $\mathbb{V}(\mathcal{T}')$ is a Soergel bimodule, it is free as a left $\mathcal{R}$-module. Therefore, $\mathbb{V}_{\mathbb{R}}(\mathcal{L}) \otimes_{\mathcal{R}} \mathbb{V}(\mathcal{T}')$ is again torsion free over $\mathcal{S}^{W_{\mathbb{R}}}$. Finally, by Proposition 6.7 each free-monodromic tilting object is the direct summand of one of $\mathcal{L} \star \mathcal{T}'$, the statement holds for all free-monodromic tilting objects. □

We can now prove Theorem 9.2.

Proof of Theorem 9.2. Once again by Proposition 6.7, Proposition 6.6 and Lemma 9.6 it is sufficient to check that $\mathbb{V}_{\mathbb{R}}$ induces and isomorphism

$$\mathsf{Hom}_{\mathcal{M}_{G_{\mathbb{R}}}}(\mathcal{T}_1, \mathcal{T}_2) \cong \mathsf{Hom}_{\mathcal{A}}(\mathbb{V}_{\mathbb{R}}(\mathcal{T}_1), \mathbb{V}_{\mathbb{R}}(\mathcal{T}_2))$$

for $\mathcal{T}_2$ supported on the preimage $\widetilde{X}_{\lambda_0}$ of the closed orbit. Let $\widetilde{i} : \widetilde{X}_{\lambda_0} \hookrightarrow \widetilde{X}$ be the inclusion. We may assume $\mathcal{T}_2 = \widetilde{\nabla}_{\lambda_0,\chi} \cong \widetilde{\Delta}_{\lambda_0,\chi}$ for some character χ of $\pi_0(T^c_{\lambda_0})$.

By adjunction we have

$$\mathsf{Hom}_{\mathcal{M}_{G_{\mathbb{R}}}}(\mathcal{T}_1, \widetilde{\nabla}_{\lambda_0,\chi}) = \mathsf{Hom}_{\mathcal{M}_{G_{\mathbb{R}}}}(\widetilde{i}_*\widetilde{i}^*\mathcal{T}_1, \widetilde{\nabla}_{\lambda_0,\chi}) = \mathsf{Hom}_{\mathcal{M}_{G_{\mathbb{R}}}}(\widetilde{i}_*(\widetilde{i}^*\mathcal{T}_1)_\chi, \widetilde{\nabla}_{\lambda_0,\chi}).$$

Here $(\widetilde{i}^*\mathcal{T}_1)_\chi$ is the direct summand of $\widetilde{i}^*\mathcal{T}_1$ where $\pi_0(T^c_{\lambda_0})$ acts by χ.

We claim that the natural map

$$\mathsf{Hom}_{\mathcal{M}_{G_{\mathbb{R}}}}(\widetilde{i}_*(\widetilde{i}^*\mathcal{T}_1)_\chi, \widetilde{\nabla}_{\lambda_0,\chi}) \to \mathsf{Hom}_{\mathcal{A}}(\mathbb{V}_{\mathbb{R}}(\widetilde{i}_*(\widetilde{i}^*\mathcal{T}_1)_\chi), \mathbb{V}_{\mathbb{R}}(\widetilde{\nabla}_{\lambda_0,\chi}))$$

is an isomorphism. Indeed, it suffices to replace $\widetilde{i}_*(\widetilde{i}^*\mathcal{T}_1)_\chi$ by $\widetilde{\nabla}_{\lambda_0,\chi}$ and show $\mathsf{End}(\widetilde{\nabla}_{\lambda_0,\chi}) \cong \mathsf{Hom}_{\mathcal{A}}(\mathbb{V}_{\mathbb{R}}(\widetilde{\nabla}_{\lambda_0,\chi}))$. Now the left side is $\mathcal{S}$. As $\mathbb{V}_{\mathbb{R}}$

is the stalk functor when restricted to local systems on $\widetilde{X}_{\lambda_0}$, the right side is $\mathsf{End}_{\mathcal{A}}(\mathcal{S})$ where $\mathcal{S}$ is viewed as an $\mathcal{A}$-module via the $\mathcal{R}$-algebra homomorphism

$$\mathrm{ev}_\chi : \mathcal{A} \to \overline{\mathcal{R}}_{\lambda_0} = \mathcal{S}. \tag{9.23}$$

given by the action of $\mathcal{A}$ on $\mathbb{V}_{\mathbb{R}}(\widetilde{\nabla}_{\lambda_0,\chi}) \cong \mathcal{S}$. Since $\mathcal{R} \twoheadrightarrow \mathcal{S}$, ev_χ is surjective, hence $\mathsf{End}_{\mathcal{A}}(\mathcal{S}) = \mathcal{S}$, which coincides with the left side.

We denote $\mathbb{V}_{\mathbb{R}}(\widetilde{\nabla}_{\lambda_0,\chi})$ by $\mathcal{S}_\chi$ to emphasize that it is isomorphic to $\mathcal{S}$ as an $\mathcal{R}$-module, and $\mathcal{A}$ acts on it via ev_χ. Therefore, it remains to check that the map

$$\mathsf{Hom}_{\mathcal{A}}(\mathbb{V}_{\mathbb{R}}(\widetilde{i}_*(\widetilde{i}^*\mathcal{T}_1)_\chi), \mathcal{S}_\chi) \to \mathsf{Hom}_{\mathcal{A}}(\mathbb{V}_{\mathbb{R}}(\mathcal{T}_1), \mathcal{S}_\chi) \tag{9.24}$$

induced by the unit map $u : \mathcal{T}_1 \to \widetilde{i}_*\widetilde{i}^*\mathcal{T}_1 \to \widetilde{i}_*(\widetilde{i}^*\mathcal{T}_1)_\chi$ is an isomorphism. The left side of (9.24) is

$$\mathsf{Hom}_{\mathcal{S}}(\mathbb{V}_{\mathbb{R}}(\widetilde{i}_*(\widetilde{i}^*\mathcal{T}_1)_\chi), \mathcal{S})$$

since the action of $\mathcal{A}$ on $\mathbb{V}_{\mathbb{R}}(\widetilde{i}_*(\widetilde{i}^*\mathcal{T}_1)_\chi)$ factors through $\mathcal{S}$ via ev_χ. The right side of (9.24) can be identified with

$$\mathsf{Hom}_{\mathcal{S}}(\mathbb{V}_{\mathbb{R}}(\mathcal{T}_1) \otimes_{\mathcal{A},\mathrm{ev}_\chi} \mathcal{S}, \mathcal{S}).$$

Therefore (9.24) comes from the map of $\mathcal{S}$-modules

$$\rho : \mathbb{V}_{\mathbb{R}}(\mathcal{T}_1) \otimes_{\mathcal{A},\mathrm{ev}_\chi} \mathcal{S} \to \mathbb{V}_{\mathbb{R}}(\widetilde{i}_*(\widetilde{i}^*\mathcal{T}_1)_\chi)$$

by taking the $\mathcal{S}$-linear dual. Since $\mathcal{S}$ is regular, to show (9.24) is an isomorphism, it suffices to show

(9.25) $\ker(\rho)$ is a torsion $\mathcal{S}$-module;

(9.26) $\mathrm{coker}(\rho)$ has support of codimension ≥ 2 in $\mathrm{Spec}\,\mathcal{S} = V_{\lambda_0}$.

To check (9.25), we localize $\mathcal{M}_{G_{\mathbb{R}}}$ at the generic point of $V_{\lambda_0} = \mathrm{Spec}\,\mathcal{S} \subset \mathrm{Spec}\,\mathcal{R} = V$ as in Section 7.7.2 and use Lemma 9.11. We conclude that both $\ker(\rho)$ and $\mathrm{coker}(\rho)$ are torsion $\mathcal{S}$-modules.

To check (9.26), we first observe that the cokernel of ρ is supported on $\cup_{\lambda \in I_1}(V_\lambda \cap V_{\lambda_0})$ as an $\mathcal{S}$-module, where

$$I_1 = \{\lambda \in I | n_\lambda = n_{\lambda_0} - 1\}. \tag{9.27}$$

Indeed, let $\mathcal{K} \in \mathcal{M}_{G_{\mathbb{R}}}$ fit into the distinguished triangle $\mathcal{K} \to \mathcal{T}_1 \to \widetilde{i}_*(\widetilde{i}^*\mathcal{T}_1)_\chi \to \mathcal{K}[1]$. Then $\mathcal{K}$ is a successive extension of $\widetilde{\Delta}_{\lambda,\psi}$ where $(\lambda, \psi) \neq (\lambda_0, \chi)$. Taking $\mathbb{V}_{\mathbb{R}}$, using that $\mathbb{V}_{\mathbb{R}}(\mathcal{T}_1)$ is concentrated in degree 0, we get an exact sequence

$$0 \to H^0\mathbb{V}_{\mathbb{R}}(\mathcal{K}) \to \mathbb{V}_{\mathbb{R}}(\mathcal{T}_1) \to \mathbb{V}_{\mathbb{R}}(\widetilde{i}_*(\widetilde{i}^*\mathcal{T}_1)_\chi) \to H^1\mathbb{V}_{\mathbb{R}}(\mathcal{K}) \to 0.$$

We have $\operatorname{coker}(\rho) = H^1\mathbb{V}_{\mathbb{R}}(\mathcal{K})$. As an $\mathcal{R}$-module, the support of $H^1\mathbb{V}_{\mathbb{R}}(\mathcal{K})$ is contained in the union of supports of $H^1\mathbb{V}_{\mathbb{R}}(\widetilde{\Delta}_{\lambda,\psi})$, which is nonzero only when $\lambda \in I_1$ by Proposition 8.4 and supported on V_λ as an $\mathcal{R}$-module. Therefore, $\operatorname{supp}_{\mathcal{R}}(H^1\mathbb{V}_{\mathbb{R}}(\mathcal{K})) \subset \cup_{\lambda\in I_1} V_\lambda$. Since $H^1\mathbb{V}_{\mathbb{R}}(\mathcal{K})$ is also a quotient of $\mathbb{V}_{\mathbb{R}}(\widetilde{i}_*(\widetilde{i}^*\mathcal{T}_1)_\chi)$ which is supported on V_{λ_0} as an $\mathcal{R}$-module, we conclude that $\operatorname{coker}(\rho) = H^1\mathbb{V}_{\mathbb{R}}(\mathcal{K})$ is supported on $\cup_{\lambda\in I_1}(V_\lambda \cap V_{\lambda_0})$ as an $\mathcal{S}$-module.

Therefore, to show (9.26), it suffices to show that for any $\lambda \in I_1$ such that $V_\lambda \subset V_{\lambda_0}$, letting η_λ be the generic point of $V_\lambda \subset V_{\lambda_0}$ and $\widehat{\mathcal{S}}_{\eta_\lambda}$ be the completed local ring of $\mathcal{S}$ at η_λ, the map

$$\widehat{\rho}_{\eta_\lambda} = \rho \otimes_{\mathcal{S}} \mathrm{id}_{\widehat{\mathcal{S}}_{\eta_\lambda}} : \mathbb{V}_{\mathbb{R}}(\mathcal{T}_1) \otimes_{\mathcal{A},\mathrm{ev}_\chi} \widehat{\mathcal{S}}_{\eta_\lambda} \to \mathbb{V}_{\mathbb{R}}(\widetilde{i}_*(\widetilde{i}^*\mathcal{T}_1)_\chi) \otimes_{\mathcal{S}} \widehat{\mathcal{S}}_{\eta_\lambda} \tag{9.28}$$

is surjective. We start by verifying that ρ is surjective in case of $G_{\mathbb{R}}$ of split rank 1.

Lemma 9.14. *Let $G_{\mathbb{R}}$ be (adjoint) of split rank 1 then ρ is surjective.*

Proof. We should go over the cases of Section 5.7. In the complex case of $G_{\mathbb{R}} = \mathrm{PGL}_2(\mathbb{C})$ it is well known that the map is an isomorphism.

If $G_{\mathbb{R}} = \mathrm{PGL}_2(\mathbb{R})$ the functor $\mathbb{V}_{\mathbb{R}}$ is equal to $\ker(s_{\mathrm{triv}} + s_{\mathrm{sgn}})$ in the notations of Section 5.7. We should consider the case $\mathcal{T}_1 = \mathcal{T}_h$. Then we have $\widetilde{i}_{0,*}(\widetilde{i}_0^*\mathcal{T}_h) \simeq \mathcal{T}_{0,\mathrm{triv}} \oplus \mathcal{T}_{0,\mathrm{sgn}}$. For $\chi = \mathrm{triv}, \mathrm{sgn}$ we have $\mathbb{V}_{\mathbb{R}}(\mathcal{T}_{0,\chi}) = \mathbf{k}[[x]]$ and $\mathbb{V}_{\mathbb{R}}(\mathcal{T}_{0,\chi}) = \ker(\mathbf{k}[[x]] \oplus \mathbf{k}[[x]] \to \mathbf{k})$ and the map ρ is induced by projection on one of the summands. We conclude that it is indeed surjective.

Let $G_{\mathbb{R}} = \mathrm{PU}(2,1)$. We may assume that $O_\lambda^{\mathbb{R}}$ is open. Short exact sequences (5.17), (5.18), (5.19) after applying $\mathbb{V}_{\mathbb{R}}$ yield long exact sequences of cohomology. For $\lambda = -|+-, +|++$ the statement then follows form the vanishings $H^1\mathbb{V}_{\mathbb{R}}(\widetilde{\Delta}_{-|+-}) = H^1\mathbb{V}_{\mathbb{R}}(\widetilde{\Delta}_{+|++}) = 0$. In case of $\lambda = +|+-$ we get the exact sequence:

$$H^0\mathbb{V}_{\mathbb{R}}(\widetilde{\Delta}_{+|+0} \oplus \widetilde{\Delta}_{0|+-}) \to H^1\mathbb{V}_{\mathbb{R}}(\widetilde{\Delta}_{+|+-}) \to H^1\mathbb{V}_{\mathbb{R}}(\mathcal{K}) \to 0.$$

It is, therefore, sufficient to verify the surjectivity of the first map. Note that $H^1\mathbb{V}_{\mathbb{R}}(\Delta_{+|+-}) = \mathbf{k}$ and, hence, we can verify the surjectivity of the map

$$H^0\mathbb{V}_{\mathbb{R}}(\Delta_{+|+0} \oplus \Delta_{0|+-}) \to H^1\mathbb{V}_{\mathbb{R}}(\Delta_{+|+-}),$$

i.e. after pushing forward to X. The latter map coincide with the boundary homomorphism obtained by applying $\mathbb{V}_{\mathbb{R}}$ to the standard filtration of $\nabla_{+|+-}$, which we know to be surjective as $H^1\mathbb{V}_{\mathbb{R}}(\nabla_{+|+-}) = 0$. $\square$

Proposition 7.7, Lemma 9.8 and Proposition 9.9 allow to reduce the local statement to the case of Levi subgroup $L = L_J$ with J defining

$V_\lambda \subset V_{\lambda_0}$. Such Levi is quasisplit of split rank 1, but is not necessary adjoint. However, Lemma 7.4 holds, which will allow us further to reduce the statement to Lemma 9.14.

Let $Z \subset L$ be the center subgroup. We have the decomposition

$$\mathcal{M}_{L_\mathbb{R}} = \bigoplus_{\omega\colon \pi_0(Z_\mathbb{R}) \to \mathbf{k}^\times} \mathcal{M}_{L_\mathbb{R},\omega},$$

with respect to the characters of the group of real points of Z. Note that if there exists a character $\theta\colon L_\mathbb{R} \to \mathbf{k}^\times$, such that $\theta|_{Z_\mathbb{R}} = \omega$, then tensoring with the corresponding local system $\mathcal{L}_\theta$ on $\widetilde{X}_L$ provides an equivalence

$$- \otimes \mathcal{L}_\theta \colon \mathcal{M}_{L_\mathbb{R},\mathrm{triv}} \xrightarrow{\sim} \mathcal{M}_{L_\mathbb{R},\omega}$$

compatible with the real Soergel functor. In cases of $L^{ad}_\mathbb{R} = \mathrm{PU}(2,1)$ or $L^{ad}_\mathbb{R} = \mathrm{PGL}_2(\mathbb{C})$ such θ always exists.

If $L^{ad}_\mathbb{R} = \mathrm{PGL}_2(\mathbb{R})$ and such θ does not exist we claim that the sheaves in $\mathcal{M}_{L_\mathbb{R},\omega}$ are supported on the orbits with $\lambda \in I_0$ and so these summands are not relevant for our consideration. Taking into account Lemma 7.4 we have $G = (Z \times \mathrm{SL}_2(\mathbb{C}))/\mu_2$, where Z is a commutative group defined over $\mathbb{R}$ and the subgroup μ_2 restricts to $\{\pm 1\}$ inside $\mathrm{SL}_2(\mathbb{C})$ and via some fixed map $\mu_2 \to Z$ defined over $\mathbb{R}$ to the first factor. Put $L^+_\mathbb{R} \subset L_\mathbb{R}$ for the subgroup of elements preserving the hemispheres $\mathbb{H}_+$ and $\mathbb{H}_-$ on $\mathbb{P}^1$. The character ω could not be extended to θ if and only if it is nontrivial on the kernel of the map $\pi_0(Z_\mathbb{R}) \to \pi_0(L^+_\mathbb{R})$. Our block is supported on the closed orbit if and only if ω is nontrivial on the kernel of the map $\pi_0(Z_\mathbb{R}) \to \pi_0(L^+_{\mathbb{R},x})$ for a point x in the closed orbit. But the kernel of both of them is exactly the image of μ_2 in Z.

It remains to treat the case $\mathcal{M}_{L_\mathbb{R},\mathrm{triv}}$. Fixing the trivial character of $\pi_0(Z_\mathbb{R})$ gives us the functor

$$\mathcal{M}_{L^{ad}_\mathbb{R}} \to \widehat{D}^b_{L_\mathbb{R}}(\widetilde{X}_{L^{ad}})_{\mathbf{T}^{ad,c}-\mathrm{mon},\mathrm{triv}}.$$

There is also the functor

$$- \otimes \mathcal{L}_{Z_{\mathrm{split}}} \colon \widehat{D}^b_{L_\mathbb{R}}(\widetilde{X}_{L^{ad}})_{\mathbf{T}^{ad,c}-\mathrm{mon},\mathrm{triv}} \to \mathcal{M}_{L_\mathbb{R},\mathrm{triv}}$$

given by the external tensor product with the free monodromic local system $\mathcal{L}_{Z_{\mathrm{split}}}$ on the split part of the center $Z_{\mathrm{split}} \subset Z$ of L. By Lemma 7.4 $L_\mathbb{R}$-orbits on X_L are in the bijection with $L^{ad}_\mathbb{R}$-orbits on $X_{L^{ad}}$, i.e. $I = I^{ad}$. Moreover, the natural map $\widetilde{I}^{ad} \to \widetilde{I}$ is surjective as for each strata it induces in view of (5.1) the map dual to the injective map $\pi_0(T_\mathbb{R})/\pi_0(Z_\mathbb{R}) \hookrightarrow \pi_0(T^{ad}_\mathbb{R})$. We conclude that the composition of the two functors above sends indecomposable tiltings to indecomposable tiltings and is surjective on this set. Since $- \otimes \mathcal{L}_{Z_{\mathrm{split}}}$ pulls out of the real Soergel functor as $-\otimes_{\mathbf{k}} \mathbf{k}[\widehat{\pi_1(Z_{\text{split}})}]$ the surjectivity of ρ for $L_{\mathbb{R}}$ now follows from the surjectivity of ρ for $L_{\mathbb{R}}^{ad}$, which is given by Lemma 9.14.

This finishes the proof of (9.26), and completes the argument for Theorem 9.2.

□

9.15. **The algebras $\mathcal{B}$ and $\mathcal{B}_0$.** Let $\mathrm{LS}_{\lambda_0} = \{\chi : \pi_0(\mathbf{T}^c_{\lambda_0}) \to \mathbf{k}^\times\}$, the set of rank one $G_{\mathbb{R}}$-equivariant local systems on the closed orbit $O^{\mathbb{R}}_{\lambda_0}$. We have an action of $\mathbf{W}_{\lambda_0}$ on LS_{λ_0} by the cross action. The group $\mathbf{W}_{\lambda_0}$ acts on LS_{λ_0} via the cross action. It also acts on $\mathcal{S}$ compatibly with the action of $\mathbf{W}$ on $\mathcal{R}$. Define an action of $\mathbf{W}_{\lambda_0}$ on $\oplus_{\chi \in \mathrm{LS}_{\lambda_0}} \mathcal{S}$ as follows: for $f \in \mathcal{S}$ put in summand χ, denoted f_χ, $w(f_\chi) := (wf)_{\chi \cdot w^{-1}}$ for $w \in \mathbf{W}_{\lambda_0}$. Let

$$\mathcal{B}_0 = (\bigoplus_{\chi \in \mathrm{LS}_{\lambda_0}} \mathcal{S})^{\mathbf{W}_{\lambda_0}}. \tag{9.29}$$

By Lemma 2.14 and Lemma 2.15, with the choice of $B \in \mathcal{O}^{\mathbb{R}}_{\lambda_0}$ and a σ-stable maximal torus $T \subset B$, we have an isomorphism

$$W(G_{\mathbb{R}}, T_{\mathbb{R}}) \cong \mathbf{W}_{\lambda_0}. \tag{9.30}$$

By definition, $\mathcal{B}_0$ is an $\mathcal{S}^{\mathbf{W}_{\lambda_0}}$-algebra. View it as an $\mathcal{R}^{\mathbf{W}}$-algebra via the natural map $\mathcal{R}^{\mathbf{W}} \to \mathcal{S}^{\mathbf{W}_{\lambda_0}}$. Define another algebra

$$\mathcal{B} := \mathcal{B}_0 \otimes_{\mathcal{R}^{\mathbf{W}}} \mathcal{R}. \tag{9.31}$$

Remark 9.16. The algebra $\mathcal{B}$ is related to the block variety of [8] in the following way. The orbits of LS_{λ_0} under the cross action of $\mathbf{W}_{\lambda_0}$ are called blocks. This coincides with the notion of blocks for $(\mathfrak{g}, K)$-modules with a fixed regular integral infinitesimal character, see [8, Claim 2.2]. Then $\mathcal{B}$ defined above is the direct product of formal completions of the block varieties $\mathfrak{B}_{mon}$ from [8, Remark 2.3] for all blocks.

The action of $\mathcal{A}$ on $\mathbb{V}_{\mathbb{R}}(\mathcal{L}_{\lambda_0,\chi})$ for $\chi \in \mathrm{LS}_{\lambda_0}$ gives a homomorphism

$$\mathrm{act}_{\lambda_0} : \mathcal{A} \to \bigoplus_{\chi \in \mathrm{LS}_{\lambda_0}} \mathsf{End}_{\mathcal{R}}(\mathbb{V}_{\mathbb{R}}(\mathcal{L}_{\lambda_0,\chi})) = \bigoplus_{\chi \in \mathrm{LS}_{\lambda_0}} \mathcal{S}. \tag{9.32}$$

Here we use that $\mathsf{End}_{\mathcal{R}}(\mathbb{V}_{\mathbb{R}}(\mathcal{L}_{\lambda_0,\chi})) \cong \mathsf{End}_{\mathcal{R}}(\mathcal{S}) = \mathcal{S}$.

Recall from Proposition 9.4 that we have $\mathcal{A} = \mathcal{A}_0 \otimes_{\mathcal{R}^W} \mathcal{R}$, where the subalgebra $\mathcal{A}_0 \subset \mathcal{A}$ is exactly the endomorphisms commuting with the Hecke action.

Theorem 9.17. *The map* act_{λ_0} *restricts to an* $\mathcal{S}^{\mathbf{W}_{\lambda_0}}$*-algebra isomorphism* $\mathcal{A}_0 \xrightarrow{\sim} \mathcal{B}_0$*. In particular, we have an isomorphism of* $\mathcal{R}$*-algebras* $\mathcal{A} \xrightarrow{\sim} \mathcal{B}$*.*

Proof. By Proposition 6.7 the map act_{λ_0} restricted to $\mathcal{A}_0$ is injective: if $a \in \mathcal{A}_0$ acts by zero on $\mathbb{V}_{\mathbb{R}}(\mathcal{L}_{\lambda_0,\chi})$ for all $\chi \in \mathrm{LS}_{\lambda_0}$, it will act by zero on all $\mathbb{V}_{\mathbb{R}}(\mathcal{L}_{\lambda_0,\chi} \star \mathcal{T}_w)$ for all $w \in \mathbf{W}$, which contain $\mathbb{V}_{\mathbb{R}}(\mathcal{T}_{\lambda,\psi})$ as a direct summand for all $(\lambda,\psi) \in \widetilde{I}$, hence $a = 0$. In particular, $\mathcal{A}_0$ is torsion-free as an $\mathcal{S}^{\mathbf{W}_{\lambda_0}}$-module.

Let us prove that act_{λ_0} sends $\mathcal{A}_0$ to the $\mathbf{W}_{\lambda_0}$-invariants of $\oplus_{\chi\in\mathrm{LS}_{\lambda_0}}\mathcal{S}$. Since $\mathcal{A}_0$ is $\mathcal{S}^{\mathbf{W}_{\lambda_0}}$-torsion free, it suffices to show the statement after tensoring with $\mathcal{Q}$. In fact we will show that act_{λ_0} induces an isomorphism

$$\mathrm{act}_{\lambda_0,\mathcal{Q}} : \mathcal{A}_{0,\mathcal{Q}} := \mathcal{A}_0 \otimes_{\mathcal{R}^{\mathbf{W}}} \mathcal{Q} \to (\oplus_{\chi\in\mathrm{LS}_{\lambda_0}}\mathcal{Q})^{\mathbf{W}_{\lambda_0}} = \mathcal{B}_{0,\mathcal{Q}}. \tag{9.33}$$

Consider the localized category $\mathrm{Tilt}(\check{\mathcal{M}}_{G_{\mathbb{R}}})_{\mathcal{Q}}$ under the action of the base-changed Hecke category $\mathrm{Tilt}(\mathcal{H}_G)_{\mathcal{Q}}$. Then $\mathcal{A}_{\mathcal{Q}} := \mathcal{A} \otimes_{\mathcal{S}^{\mathbf{W}_{\lambda_0}}} \mathcal{Q}$ is the endomorphism ring of the functor $\mathbb{V}_{\mathbb{R},\mathcal{Q}} : \mathrm{Tilt}(\check{\mathcal{M}}_{G_{\mathbb{R}}})_{\mathcal{Q}} \to \mathcal{R}_{\mathcal{Q}}\text{-mod}$, and $\mathcal{A}_{0,\mathcal{Q}}$ is the subalgebra of $\mathcal{A}_{\mathcal{Q}}$ commuting with the Hecke action. By Lemma 9.10 we can compute $\mathcal{A}_{\mathcal{Q}}$ explicitly as

$$\mathcal{A}_{\mathcal{Q}} \xrightarrow{\sim} \bigoplus_{(\lambda,\chi)\in\widetilde{I}_0} \mathcal{K}_\lambda, \tag{9.34}$$

where the (λ,χ)-factor is the action of $\mathcal{A}_{\mathcal{Q}}$ on $\mathbb{V}_{\mathbb{R},\mathcal{Q}}(\widetilde{\Delta}_{\lambda,\chi}) \cong \mathcal{K}_\lambda$. By Lemma 7.9, the part of $\mathcal{A}$ that commutes with the Hecke action corresponds to the $\mathbf{W}$-invariants of the right side of (9.34) under the cross action. Since $\mathbf{W}$ acts transitively on I_0, we may rewrite the $\mathbf{W}$-invariants of the right side as $(\oplus_{\chi\in\mathrm{LS}_{\lambda_0}}\mathcal{K})^{\mathbf{W}_{\lambda_0}} = \mathcal{B}_{0,\mathcal{Q}}$. This proves (9.33). In particular, act_{λ_0} restricts to a ring homomorphism

$$\mathrm{act}'_{\lambda_0} : \mathcal{A}_0 \to \mathcal{B}_0 \tag{9.35}$$

that is injective and becomes an isomorphism after tensoring with $\mathcal{Q}$.

It remains to show that $\mathrm{act}'_{\lambda_0}$ is surjective. Given a collection $b = (b_\chi)_{\chi\in\mathrm{LS}_{\lambda_0}} \in \mathcal{B}_0$, we would like to construct $a \in \mathcal{A}_0$ that acts on $\mathbb{V}_{\mathbb{R}}(\mathcal{T}_{\lambda_0,\chi}) = \mathbb{V}_{\mathbb{R}}(\widetilde{\Delta}_{\lambda_0,\chi})$ by b_χ. For $\chi \in \mathrm{LS}_{\lambda_0}$ and $w \in \mathbf{W}$, define an endomorphism $a_{\chi,w}$ of $\mathbb{V}_{\mathbb{R}}(\mathcal{T}_{\lambda_0,\chi} \star \mathcal{T}_w) \cong \mathbb{V}_{\mathbb{R}}(\mathcal{T}_{\lambda_0,\chi}) \otimes_{\mathcal{R}} \mathbb{V}(\mathcal{T}_w)$ by $b_\chi \otimes \mathrm{id}_{\mathbb{V}(\mathcal{T}_w)}$. For any morphism $\varphi : \mathcal{T}_{\lambda_0,\chi} \star \mathcal{T}_w \to \mathcal{T}_{\lambda_0,\chi'} \star \mathcal{T}_{w'}$ in $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$, we claim that the following diagram is commutative

$$\begin{tikzcd}
\mathbb{V}_{\mathbb{R}}(\mathcal{T}_{\lambda_0,\chi} \star \mathcal{T}_w) \arrow[r, "a_{\chi,w}"] \arrow[d, "\mathbb{V}_{\mathbb{R}}(\varphi)"] & \mathbb{V}_{\mathbb{R}}(\mathcal{T}_{\lambda_0,\chi} \star \mathcal{T}_w) \arrow[d, "\mathbb{V}_{\mathbb{R}}(\varphi)"] \\
\mathbb{V}_{\mathbb{R}}(\mathcal{T}_{\lambda_0,\chi'} \star \mathcal{T}_{w'}) \arrow[r, "a_{\chi',w'}"] & \mathbb{V}_{\mathbb{R}}(\mathcal{T}_{\lambda_0,\chi'} \star \mathcal{T}_{w'})
\end{tikzcd} \tag{9.36}$$

Indeed, after tensoring with $\mathcal{Q}$ the above diagram is commutative since $\mathcal{A}_{0,\mathcal{Q}} \xrightarrow{\sim} \mathcal{B}_{0,\mathcal{Q}}$. Since $\mathbb{V}_{\mathbb{R}}(\mathcal{T}_{\lambda_0,\chi'} \star \mathcal{T}_{w'})$ is torsion-free as an $\mathcal{S}^{\mathbf{W}_{\lambda_0}}$-module by Lemma 9.13, the diagram is commutative. Let $\mathrm{Tilt}' \subset \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$ be the full subcategory whose objects are finite direct sums of $\mathcal{T}_{\lambda_0,\chi} \star \mathcal{T}_w$ for various $\chi \in \mathrm{LS}_{\lambda_0}$ and $w \in \mathbf{W}$. By the commutativity of (9.36), the collection $\{a_{\chi,w}\}_{\chi \in \mathrm{LS}_{\lambda_0}, w \in \mathbf{W}}$ gives an endomorphism of $\mathbb{V}_{\mathbb{R}}|_{\mathrm{Tilt}'}$. By Proposition 6.7, all objects in $\mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}})$ are direct summands of objects in Tilt', therefore the $\{a_{\chi,w}\}$ defines an endomorphism of $\mathbb{V}_{\mathbb{R}}$, i.e., an element $a \in \mathcal{A}$. By construction, a acts on $\mathbb{V}_{\mathbb{R}}(\mathcal{T}_{\lambda_0,\chi})$ by b_χ, and a commutes with the Hecke action. Therefore $a \in \mathcal{A}_0$ satisfies $\mathrm{act}_{\lambda_0}(a) = b$.

$\square$

Combining Theorem 9.2 and Theorem 9.17, we get:

Corollary 9.18. *Suppose G is adjoint and $G_{\mathbb{R}}$ is quasi-split. The enhanced real Soergel functor gives a fully faithful embedding*

$$\mathbb{V}^\sharp_{\mathbb{R}} : \mathrm{Tilt}(\mathcal{M}_{G_{\mathbb{R}}}) \to \mathcal{B}\text{-mod}. \tag{9.37}$$

9.19. **Non-adjoint group case.** Now let G be a connected reductive group over $\mathbb{C}$ with real form $G_{\mathbb{R}}$. Let $X, \widetilde{X}$ be defined in terms of G.

Let G^{ad} be the adjoint form of G, which carries a real form $G^{ad}_{\mathbb{R}}$ compatible with $G_{\mathbb{R}}$. Let $X^{ad}, \widetilde{X}^{ad}$ be defined as in Section 4.1 in terms of G^{ad}. In particular, $\pi^{ad} : \widetilde{X}^{ad} \to X^{ad}$ is a $(\mathbf{T}^{ad})^c$-torsor.

Put $G'_{\mathbb{R}}$ for the subgroup of G generated by $G_{\mathbb{R}}$ and the center of G. We put $\widetilde{G}^{ad}_{\mathbb{R}}$ for the preimage of $G^{ad}_{\mathbb{R}}$ under the projection $G \to G^{ad}$. The quotient $\widetilde{G}^{ad}_{\mathbb{R}}/G'_{\mathbb{R}} = G^{ad}_{\mathbb{R}}/\mathsf{Im}(G_{\mathbb{R}})$ is finite abelian and we denote by $\mathfrak{S}$ the abelian group dual to $G^{ad}_{\mathbb{R}}/\mathsf{Im}(G_{\mathbb{R}})$, so that

$$\widetilde{G}^{ad}_{\mathbb{R}}/G'_{\mathbb{R}} \cong G^{ad}_{\mathbb{R}}/\mathsf{Im}(G_{\mathbb{R}}) \cong \mathfrak{S}^*. \tag{9.38}$$

By definition we have an equivalence of categories

$$(\widehat{D}^b_{G'_{\mathbb{R}}}(\widetilde{X})_{\mathbf{T}^c-\mathrm{mon}})^{\mathfrak{S}^*} \simeq \widehat{D}^b_{\widetilde{G}^{ad}_{\mathbb{R}}}(\widetilde{X})_{\mathbf{T}^c-\mathrm{mon}}. \tag{9.39}$$

9.19.1. *De-equivariantization.* We recall the procedure of de-equivariantization, as explained in [11, 21 and 22]. Let Γ be a finite abelian group such that $|\Gamma|$ is prime to $\mathrm{ch}(\mathbf{k})$. Assume $\mathbf{k}$ contains enough roots of unity such that $\Gamma^* = \mathsf{Hom}(\Gamma, \mathbf{k}^\times)$ has the same cardinality as Γ. Assume Γ acts on a $\mathbf{k}$-linear idempotent complete category $\mathcal{C}$. Let $\mathcal{D} = \mathcal{C}^\Gamma$ be the category of Γ-equivariant objects in $\mathcal{C}$: an object $d \in \mathcal{D}$ is tuple $(c, \{\alpha_\gamma\}_{\gamma\in\Gamma})$ where $X \in \mathcal{C}$ and $\alpha_\gamma : \gamma(c) \xrightarrow{\sim} c$ are isomorphisms indexed by $\gamma \in \Gamma$ satisfying $\alpha_1 = \mathrm{id}_c$ and $a_{\gamma_1\gamma_2} = \alpha_{\gamma_1} \circ \gamma_1(\alpha_{\gamma_2})$. Then there is an action of the dual group Γ^* on $\mathcal{D}$ as follows: for $\chi \in \Gamma^*$ and $d = (c, \{\alpha_\gamma\}) \in \mathcal{D}$, define $\chi(d) = (c, \{\alpha'_\gamma\})$ where α'_γ is α_γ multiplied by $\langle \chi, \gamma \rangle \in \mathbf{k}^\times$. Then we can recover $\mathcal{C}$ as the category of Γ^*-equivariant objects in $\mathcal{D}$. We give the functors as follows. There is a functor $\mathrm{av}_\Gamma : \mathcal{C} \to \mathcal{D}$ sending c to the object $\mathrm{av}_\Gamma(c) = \oplus_{\gamma\in\Gamma}\gamma(c)$ with its obvious Γ-equivariant structure. Then $\mathrm{av}_\Gamma(c) \in \mathcal{D}$ in fact carries a canonical Γ^*-equivariant structure and hence lifts to an object $\mathrm{av}_\Gamma(c)^\sharp \in \mathcal{D}^{\Gamma^*}$. The assignment $c \mapsto \mathrm{av}_\Gamma(c)^\sharp$ gives a functor $\mathcal{C} \to \mathcal{D}^{\Gamma^*}$. On the other hand, let $(d, \{\alpha_\chi\}_{\chi\in\Gamma^*}) \in \mathcal{D}^{\Gamma^*}$, with $d = (c, \{\alpha_\gamma\}_{\gamma\in\Gamma}) \in \mathcal{D}$. The data of α_χ means automorphisms $\beta_\chi \in \mathsf{Aut}(c)$ that form a Γ^*-action on c compatible with $\{\alpha_\gamma\}_{\gamma\in\Gamma}$. In particular, we can extract the direct summand $c_1 = c^{\Gamma^*} \subset c$ corresponding to the trivial character of Γ^*. The assignment $(d, \{\alpha_\chi\}_{\chi\in\Gamma^*}) \mapsto c$ gives a functor $\mathcal{D}^{\Gamma^*} \to \mathcal{C}$. These two functors are inverse to each other and give an equivalence $\mathcal{C} \cong \mathcal{D}^{\Gamma^*}$.

Applying the above remarks to the $\Gamma := \mathfrak{S}^*$-action on $\widehat{D}^b_{G'_{\mathbb{R}}}(\widetilde{X})_{\mathbf{T}^c-\mathrm{mon}}$, we have an action of $\Gamma^* = \mathfrak{S}$ on $\widehat{D}^b_{\widetilde{G}^{ad}_{\mathbb{R}}}(\widetilde{X})_{\mathbf{T}^c-\mathrm{mon}}$ and a canonical equivalence

$$\widehat{D}^b_{G'_{\mathbb{R}}}(\widetilde{X})_{\mathbf{T}^c-\mathrm{mon}} \simeq (\widehat{D}^b_{\widetilde{G}^{ad}_{\mathbb{R}}}(\widetilde{X})_{\mathbf{T}^c-\mathrm{mon}})^{\mathfrak{S}}. \tag{9.40}$$

Pullback along $\widetilde{X} \to \widetilde{X}^{ad}$ gives an equivalence $\widehat{D}^b_{\widetilde{G}^{ad}_{\mathbb{R}}}(\widetilde{X})_{\mathbf{T}^c-\mathrm{mon}} \simeq \widehat{D}^b_{G^{ad}_{\mathbb{R}}}(\widetilde{X}^{ad})_{(\mathbf{T}^{ad})^c-\mathrm{mon}} = \mathcal{M}_{G^{ad}_{\mathbb{R}}}$, hence the latter also carries an action of $\mathfrak{S}$. Combining this equivalence with (9.40) we get an equivalence

$$\widehat{D}^b_{G'_{\mathbb{R}}}(\widetilde{X})_{\mathbf{T}^c-\mathrm{mon}} \simeq \mathcal{M}^{\mathfrak{S}}_{G^{ad}_{\mathbb{R}}}. \tag{9.41}$$

Let

$$\mathrm{Tilt}(\mathcal{M}_{G'_{\mathbb{R}}}) \subset \widehat{D}^b_{G'_{\mathbb{R}}}(\widetilde{X})_{\mathbf{T}^c-\mathrm{mon}} \tag{9.42}$$

be the full subcategory of free-monodromic tilting sheaves. Thus we get an equivalence

$$\mathrm{Tilt}(\mathcal{M}_{G'_{\mathbb{R}}}) \cong \mathrm{Tilt}(\mathcal{M}_{G^{ad}_{\mathbb{R}}})^{\mathfrak{S}}. \tag{9.43}$$

By Corollary 9.18, the enhanced real Soergel functor

$$\mathbb{V}^{\mathrm{ad},\sharp}_{\mathbb{R}} : \mathrm{Tilt}(\mathcal{M}_{G^{ad}_{\mathbb{R}}}) \to \mathcal{B}^{ad}\text{-mod} \tag{9.44}$$

is fully-faithful. Here $\mathcal{B}^{ad} = \mathcal{B}^{ad}_0 \otimes_{\mathcal{R}^{ad},\mathbf{W}} \mathcal{R}^{ad}$ is the algebra $\mathcal{B}$ for the group $G^{ad}_{\mathbb{R}}$.

For a point x in the closed orbit we have a short exact sequence of abelian groups

$$0 \to \pi_0(\mathrm{Stab}_{\mathsf{Im}(G_{\mathbb{R}})}(x)) \to \pi_0(\mathrm{Stab}_{G^{ad}_{\mathbb{R}}}(x)) \to \mathfrak{S}^* \to 0.$$

It follows that we have an action of $\mathfrak{S}$ on the set of the local systems on the closed $G_{\mathbb{R}}^{ad}$-orbit on X^{ad}. This yields a $\mathfrak{S}$-action on $\mathcal{B}^{ad}$, such that the functor $\mathbb{V}_{\mathbb{R}}^{ad}$ respects the $\mathfrak{S}$ action on $\widehat{D}^b_{\widetilde{G}^{ad}_{\mathbb{R}}}(\widetilde{X})_{T^c-\mathrm{mon}} \simeq \widehat{D}^b_{G^{ad}_{\mathbb{R}}}(\widetilde{X}^{ad})_{(T^{ad})^c-\mathrm{mon}}$ and $\mathcal{B}^{ad} \otimes_{\mathcal{R}^W} \mathcal{R} - \mathrm{mod}$. As a result we obtain a fully-faithful composition functor

$$\mathbb{V}_{\mathbb{R}} \colon \widehat{D}^b_{G'_{\mathbb{R}}}(\widetilde{X})_{T^c-\mathrm{mon}} \xrightarrow{\sim} (\widehat{D}^b_{\widetilde{G}^{ad}_{\mathbb{R}}}(\widetilde{X})_{T^c-\mathrm{mon}})^{\mathfrak{S}} \xrightarrow{\mathbb{V}_{\mathbb{R}}^{ad}} (\mathcal{B}^{ad} \otimes_{\mathcal{R}^W} \mathcal{R} - \mathrm{mod})^{\mathfrak{S}}.$$

The following proposition relates $\mathbb{V}_{\mathbb{R}}$ and the vanishing cycles functors for the conormals to the closed $G_{\mathbb{R}}$-orbit on X.

Proposition 9.20. *The composition of* $\mathbb{V}_{\mathbb{R}}$ *with the functor* $(\mathcal{B}^{ad} \otimes_{\mathcal{R}^W} \mathcal{R} - \mathrm{mod})^{\mathfrak{S}} \to (\mathcal{B}^{ad})^{\mathfrak{S}} \otimes_{\mathcal{R}^W} \mathcal{R} - \mathrm{mod}$ *is equivalent to the sum of vanishing cycles functor for a collection of conormals to the closed orbit, one towards each of the open* $G_{\mathbb{R}}$*-orbit.*

Proof. Note that we have $\pi_0(G_{\mathbb{R}}^{ad}) = \pi_0(\mathcal{N}_{\mathbb{R}}^{reg})$, where $\mathcal{N}_{\mathbb{R}}^{reg}$ is the regular elements of the real nilpotent cone. Its components correspond to the generic conormals to the closed orbit. Those conormals are identified under the action of $\mathsf{Im}(G_{\mathbb{R}})$ and, respectively, $\mathfrak{S}^*$ permute the classes.

On the resulting composition there is a $\mathfrak{S}$-action and, which we can turn in $\mathfrak{S}^*$-grading decomposing by characters of $\mathfrak{S}$. To define $\mathbb{V}_{\mathbb{R}}^{ad}$ we have fixed a choice of the conormal and with the $\mathfrak{S}^*$-grading at trivial character we have the vanishing cycle at the fixed conormal. We conclude that the grading is the sum over generic conormals to the closed orbit of the vanishing cycles functors.

□

10. Koszul Duality

We can now reproof the main result of [8] (Theorem 1.1). To do this we need to construct an explicit dg-model for category $\mathcal{M}_{G_{\mathbb{R}}}$.

The following constructions and results make sense in the general setting of Section 4.1.

Lemma 10.1. *For* $\mathcal{T}_1, \mathcal{T}_2$ *in* $\mathrm{Tilt}(\mathcal{M}_{H,X})$ *we have* $\mathrm{Ext}^{>0}_{\mathcal{M}_{H,X}}(\mathcal{T}_1, \mathcal{T}_2) = 0$.

Proof. Recall that $\mathcal{T}_1$ has a standard flag and $\mathcal{T}_2$ has a costandard flag. It suffices to check that $\mathrm{Ext}^{>0}_{\mathcal{M}_{H,X}}(\widetilde{\Delta}_{\lambda,\chi}, \widetilde{\nabla}_{\mu,\psi}) = 0$ for any pair of standard and costandard object. If $\lambda \neq \mu$ the vanishing follows by adjunction. If $\lambda = \mu$ the vanishing follows by adjunction and Corollary 4.6. □

Put

$$\mathcal{T}_{\oplus} := \bigoplus_{(\lambda,\chi)\in\widetilde{I}} \mathcal{T}_{\lambda,\chi}$$

for the sum of all indecomposible free-monodromic tilting sheaves in $\mathcal{M}_{H,X}$ and let $E = \mathsf{End}(\mathcal{T}_\oplus)^\oplus$. Also, put $K^b(\mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}}))$ for the homotopy category of bounded complexes in $\mathrm{Tilt}(\mathcal{M}_{G_\mathbb{R}})$.

Proposition 10.2. (1) *Let* $\mathrm{Proj}(E\text{-mod})$ *be the category of finitely generated projective E-modules. Then the functor* $\mathsf{Hom}(\mathcal{T}_\oplus, -) : \mathrm{Tilt}(\mathcal{M}_{H,X}) \to \mathrm{Proj}(E\text{-mod})$ *is an equivalence of categories.*

(2) *The natural functor* $K^b(\mathrm{Tilt}(\mathcal{M}_{H,X})) \to \mathcal{M}_{H,X}$ *is an equivalence of triangulated categories.*

(3) *Combining (1) and (2), there is a canonical equivalence of triangulated categories*

$$\mathcal{M}_{H,X} \cong \mathrm{Perf}(E\text{-mod}) := K^b(\mathrm{Proj}(E\text{-mod})) \tag{10.1}$$

under which indecomposable free-monodromic tilting sheaves correspond to indecomposable projective E-modules.

Proof. (1) The functor lands in projective E-modules because all objects in $\mathrm{Tilt}(\mathcal{M}_{H,X})$ are direct summands of $\mathcal{T}_\oplus^n$ for some n. The left adjoint of the functor is given by $M \mapsto \mathcal{T}_\oplus \otimes_E M$. One checks that these functors are inverse to each other.

(2) Follows from Proposition 4.12 and Lemma 10.1 as in [3] 1.5 (see also Proposition B.1.7 in [9]). □

We now return to $X = G/B$, $\widetilde{X} = G/UT^{>0}$ and $H = G_\mathbb{R}$. In this case we now define the graded version of category $\mathcal{M}_{G_\mathbb{R}}$. By Theorem 9.2 and Theorem 9.17 we have $M = \mathsf{End}_{\mathbf{k}[\mathfrak{S}]\ltimes(\mathcal{B}\otimes_{\mathcal{R}W}\mathcal{R})}(\mathbb{V}_\mathbb{R}(\mathcal{T}_\oplus))$. We assign the grading 2 to the generators of $\mathcal{R}$ and $\mathcal{S}$. There is a compatible grading on $\mathbb{V}_\mathbb{R}(\mathcal{T}_\oplus)$, which turns M into a graded algebra. We put $\mathcal{M}^{gr}_{G_\mathbb{R}} = \mathrm{Perf}(M - \mathrm{grmod})$.

Remark 10.3. It would be interesting to define the grading on M without appealing to Theorem 9.2 and Theorem 9.17 and in the more general setting. It would be sufficient to define a compatible gradings on the stalks and costalks of the tilting sheaves. We refer to [26] where this approach were used in the case of B-action on $X = G/B$ and raise a question of whether there is a way to define the grading on M explicitly.

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