Jünger Can't Borrow: Demographic Imbalances and Currency Risk Premia

by

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Abstract

Empirically, countries with relatively old populations have significantly lower interest rates and currency returns. As a first step towards explaining this fact, I develop a two-country overlapping generations model to study the relationship between the global wealth distribution and currency risk premia. Relatively wealthy countries in the model have low currency risk premia because their bonds insure wealthy households against increases in the price of their own consumption basket. I discuss how the model can be extended to incorporate demographic heterogeneity across countries. Given observed household savings patterns over the life cycle, differences in population age across countries can potentially generate large differences in financial wealth and currency risk premia.

Thesis Supervisor: Adrien Verdelhan
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Contents

List of Figures

List of Tables

Chapter 1

Introduction

The empirical failure of uncovered interest parity is one of the most important and puzzling facts documented by the field of international finance. Short-term interest rates exhibit large and persistent differences across countries. These interest rate differentials do not systematically predict exchange rate movements, which implies that international investors can earn large average excess returns by borrowing in low interest rate currencies and saving in high interest rate currencies.[1] From a traditional finance perspective, these observed differences in average returns across currencies must correspond to differences in their exposure to systematic risks: low interest rate currencies must appreciate at times of high marginal utility for global investors in order to justify their low average returns, while high interest rate currencies must depreciate at these times. This line of reasoning has inspired a search for persistent country characteristics that are correlated with both interest rates and excess currency returns, along with an explanation of why these characteristics shape currencies' exposures to systematic risks.

In this paper, I contribute to this search by documenting a strong empirical relationship between population age and currency risk premia across countries. As a first step towards building a complete explanation of this fact, I incorporate overlapping generations into a standard two-country general equilibrium asset pricing model, and discuss how this new framework can be extended to model demographic heterogeneity across countries.

Empirically, I document a strong negative relationship between population age and short-term interest rates across advanced economies. A one-year difference in the average age of an individual across countries is associated with large differences in one-year interest rates of roughly 0.5-0.6

[1] See, for example: Lustig, Roussanov, and Verdelhan (2011) and Hassan and Mano (2019).

percentage points. These cross-country differences in interest rates translate almost one-for-one to differences in excess currency returns. This relationship between population age, interest rates, and excess currency returns remains after controlling for other persistent country characteristics that have previously been proposed to explain differences in currency risk premia, such as the economy's share of global GDP (Hassan, 2013) and trade network centrality (Richmond, 2019). Further results based on sorting currencies into portfolios by population age at the start of each year confirm that global investors earn lower average returns investing in the currencies of old countries relative to those of young countries. The strong cross-country relationship between demographics and currency risk premia also arises using an alternative summary statistic for each country's age distribution, based on predicted household savings behavior over the life cycle.

To work towards a theoretical explanation of the observed relationship between demographics and currency risk premia, I develop a two-country overlapping generations model featuring aggregate risk. The model builds on previous work by Gârleanu and Panageas (2015) in a closed-economy setting. By incorporating stochastic prices of each country's consumption bundle, opportunities for international risk sharing arise, and households trade in international financial markets in order to stabilize their own consumption in the presence of fluctuating local goods prices. The key feature of the model that I explore is the relationship between the global wealth distribution and currency risk premia. When households in one country own a large share of global wealth, their desire to hedge against increases in the price of their own country's consumption bundle leads to a large asymmetry in the pricing of currency risks across countries. Under reasonable restrictions on risk aversion satisfied by almost all common asset pricing models, this hedging demand leads to a large increase in real bond prices in the wealthier country, and consequently a large decrease in expected real bond returns. The opposite is true for real bonds of the country that owns a smaller share of global financial wealth, since wealthy foreign investors require large currency risk premia in order to bear currency risk.

By developing and solving a two-country model featuring overlapping generations, I make progress towards building a complete general equilibrium model incorporating demographic heterogeneity across countries. Under standard life cycle savings patterns documented by Gourinchas and Parker (2002) and others, countries with relatively old populations would be expected to accumulate greater household savings and thus represent a larger share of the global wealth distribution. Applying the logic of my model, these large differences in financial wealth across young and old countries could potentially lead to large differences in currency risk premia, consistent with the empirical evidence that I document. Developing a model that incorporates this demographic heterogeneity across countries is beyond the scope of my current paper. However, I view my contribution as an important first step towards building such a model. In influential work that inspired the title of this paper, Constantinides, Donaldson, and Mehra (2002) study a closed economy with a similar overlapping generations structure, in which the concentration of stock market risk among a small share of wealthy middle-age households generates a large equity premium. I attempt to develop a similar argument applied *across* countries, in order to understand how demographic heterogeneity across countries determines the relative pricing of assets denominated in different currencies (i.e. deviations from uncovered interest parity).

Several previous papers have developed risk-based explanations of currency risk premia centered around persistent, fundamental characteristics of countries. For brevity, I cite only a few of these papers here. Hassan (2013) links currency risk premia with the size of countries' economies: real bonds of larger economies provide insurance against declines in non-tradable goods consumption for a larger group of global consumers, and these bonds accordingly earn a low risk premium. Ready, Roussanov, and Ward (2017) focus on the composition of countries' exports, arguing that the currencies of commodity-exporting countries should earn higher risk premia (relative to currencies of final goods-exporting countries). Richmond (2019) argues that countries occupying central locations within global trade networks earn low currency risk premia relative to peripheral countries. Colacito, Croce, Gavazzoni, and Ready (2018) develop a multi-country endowment economy model with heterogenous country-level exposure to global growth shocks, and show that the model generates a carry trade risk premium. However, they do not take a stance on the country characteristics that determine these heterogenous risk exposures, and demographic factors may be a natural candidate. In this paper, I demonstrate empirically that population age has explanatory value for cross-country differences in interest rates and currency risk premia above and beyond several other fundamental country characteristics listed here, and lay the foundations of a theoretical framework to help understand how country demographics determine interest rates and currency risk premia in general equilibrium.

Similar results on the cross-country relationship between demographics and interest rates have been documented by Carvalho, Ferrero, and Nechio (2016), Auclert, Malmberg, Martenet, and Rognlie (2020), and others. I show that these observed differences in interest rates across young and old countries translate almost directly into differences in average excess currency returns across countries. Other papers have developed models relating demographics and capital flows, such as

Backus, Cooley, and Henriksen (2014) and Barany and Coeurdacier (2019). I start with a similar overlapping generations model and incorporate both aggregate risk and nontradable goods, so that the model can be used to study risk premia and the stochastic properties of exchange rates. Moreover, my attempt to link demographics and currency risk premia can be viewed as a bridge between these previous papers linking demographics and capital flows, and the work of Della Corte, Riddiough, and Sarno (2016) linking external imbalances and currency risk premia. External imbalances are an endogenous market outcome, and demographics are one potential primitive characteristic of countries that can shape both these quantity outcomes and currency risk premia.

The rest of the paper proceeds as follows. Section 2 presents the two-country overlapping generations model and explores a numerically-solved version of the model. Section 3 presents empirical results and documents the cross-country relationship between population age and currency risk premia. Section 4 concludes.

Chapter 2

Theory

In this section, I build a two-country overlapping generations model featuring exchange rate risk. I model risk sharing between households across two countries who transact in complete international financial markets. This set of households is assumed to be “small” in the sense that local goods prices are determined exogenously. At the same time, these households are assumed to be the sole participants in international financial markets, and the distribution of wealth across the two countries will determine asset prices. Therefore, I take exchange rate dynamics as exogenous in order to focus on the determination of asset prices.

The model builds on the overlapping generations model developed by Gârleanu and Panageas (2015), which also features heterogenous types of households. As in that model as well as other international macro-finance models (e.g. Dou and Verdelhan, 2015; Sauzet, 2021), the distribution of wealth across types of agents - here households located in the home and foreign countries - emerges as a state variable determining asset prices. I numerically solve the model and explore the relationship between the wealth distribution and asset prices, in particular the price of exchange rate risk that determines currency risk premia in both countries.

2.1 Model Setup

The model is set in continuous time. There are two countries, which I will refer to as the “home country” and “foreign country”; star ($\star$) superscripts denote prices and quantities corresponding to the foreign country.

Each country contains a unit mass of households who trade in international financial markets. I model demographics following the perpetual youth framework of Blanchard (1985): individual households in each country randomly perish at a rate $\pi > 0$, and are replaced by an equal mass of new households that arrive in each country at the same rate π. The mass of existing households at time t that were born before time $s \leq t$ is then given by $\exp(-\pi(t-s))$, and the average household age in each country is $\frac{1}{\pi}$.

Households consume a country-specific consumption bundle. Let $c_{u,t}$ denote the flow consumption at time u of a household born on date $t \leq u$. Given a consumption plan $\{c_{u,t}\}_{u \geq t}$ specifying consumption at each future date that the household remains alive, lifetime utility takes the constant relative risk aversion form

$$U_t\left(\{c_{u,t}\}_{u \geq t}\right) \equiv \mathbb{E}_t\left[\int_t^{+\infty} \exp(-(\rho+\pi)(u-t))\frac{c_{u,t}^{1-\gamma}}{1-\gamma}du\right] \tag{2.1}$$

where $\gamma > 0$ denotes relative risk aversion. $\rho > 0$ is the household's subjective discount rate, which is lower than the household's effective discount rate $\rho + \pi$ due to mortality risk.

Let $P_{C,t}$ denote the price of the home country consumption bundle, and $P^{\star}_{C,t}$ denote the price of the foreign country's consumption bundle. Both prices are relative prices quoted in terms of a tradable numeraire good consumed by households in both countries. For parsimony, I assume that the prices $P_{C,t}$ and $P^{\star}_{C,t}$ are determined exogenously, rather than being endogenously determined by the demand of households in the model. This assumption can be interpreted as imposing that the consumption expenditure of households that participate in international financial markets is small relative to the expenditure of households that do not participate, so that changes in the distribution of financial wealth between countries have negligible effects on the real exchange rate. This allows me to focus on the determination of asset prices and risk premia, taking exchange rate dynamics as given.

Uncertainty arises from fluctuations in the price of each country's consumption basket. I assume that the log of the home price index $\widehat{P}_{C,t} \equiv \log P_{C,t}$ follows an Ornstein-Uhlenbeck process driven by a standard Brownian motion B_t:

$$d\widehat{P}_{C,t} = -\beta\widehat{P}_{C,t}dt + \sigma_{PC}dB_t \tag{2.2}$$

I assume that the foreign price index is equal to the reciprocal of the home price index, so that $P^{\star}_{C,t} = \frac{1}{P_{C,t}}$ in levels, or equivalently $\widehat{P}^{\star}_{C,t} \equiv \log P^{\star}_{C,t} = -\widehat{P}_{C,t}$ in logs. The log price index $\widehat{P}^{\star}_{C,t}$ in the foreign country then evolves according to an Ornstein-Uhlenbeck process whose innovations are perfectly negatively correlated with the innovations to $d\widehat{P}_{C,t}$:

$$d\widehat{P}^{\star}_{C,t} = -\beta \widehat{P}^{\star}_{C,t} dt - \sigma_{PC} dB_t \tag{2.3}$$

In levels, the price indices thus evolve according

$$\frac{dP_{C,t}}{P_{C,t}} = \left(-\beta \widehat{P}_{C,t} + \frac{\sigma^2_{PC}}{2}\right) dt + \sigma_{PC} dB_t \equiv \overline{\mu}_{PC,t} dt + \overline{\sigma}_{PC} dB_t \tag{2.4}$$

$$\frac{dP^{\star}_{C,t}}{P^{\star}_{C,t}} = \left(-\beta \widehat{P}^{\star}_{C,t} + \frac{\sigma^2_{PC}}{2}\right) dt - \sigma_{PC} dB_t \equiv \overline{\mu}^{\star}_{PC,t} dt + \overline{\sigma}^{\star}_{PC} dB_t \tag{2.5}$$

The inverse relationship between consumption basket prices across countries creates opportunities for international risk sharing. Households in either country can potentially insure themselves against fluctuations in the price of their own consumption basket by trading in international financial markets with their counterparts abroad, who face negatively correlated shocks to the price of their own consumption basket and are thus ideally positioned to provide this insurance.

In both countries, households born on date t receive a newly-created claim to income $y_{u,t} = \overline{y}\exp(-\delta(u-t))$ in *all* future dates $u \geq t$ (regardless of whether the household is still alive at this time). This claim can be thought of as a stream of pledgeable income from a business created at birth, where $\delta > 0$ denotes the rate at which the income stream depreciates. I assume this claim can be frictionlessly traded in financial markets; let $\overline{y}P_{\delta,t}$ denote its market price. The steady state level $\overline{Y}$ of households' aggregate income is such that depreciation $\delta\overline{Y}$ of existing income streams is exactly equal to the inflow $2\pi\overline{y}$ created by newly-entering households across both countries, so that $\overline{Y} = \frac{2\pi}{\delta}$.

Households can trade a complete set of contingent claims over realizations of aggregate shocks. The stochastic discount factor process ξ_t prices payoffs denominated in the numeraire good. Following Blanchard (1985), individual mortality risk is hedged through a competitive market for reverse annuities. At time u the annuity delivers a flow payout $\pi w_{u,t}$ to a household with current financial wealth $w_{u,t}$; in exchange, the household surrenders its financial wealth at the date that it perishes. Under these two standard assumptions, a household born in the home country on date t faces the lifetime budget constraint

$$\mathbb{E}_t\left[\int_t^{+\infty} \frac{\xi_u}{\xi_t} \exp(-\pi(u-t)) P_{C,u} c_{u,t} du\right] \leq \mathbb{E}_t\left[\int_t^{+\infty} \frac{\xi_u}{\xi_t} y_{u,t} du\right] \equiv \overline{y} P_{\delta,t} \tag{2.6}$$

where future consumption expenditure $P_{C,u}c_{u,t}$ is discounted at an additional rate π reflecting annuity payments. The final equality introduces

$$P_{\delta,t} \equiv \mathbb{E}_t \left[\int_t^{+\infty} \frac{\xi_u}{\xi_t} \exp(-\delta(u-t)) du \right] \tag{2.7}$$

which equals the price of a household's future income stream when current income $y_{t,s} = 1$. A household born in the foreign country faces a similar budget constraint, after substituting for the price $P^\star_{C,u}$ of the foreign country consumption bundle:

$$\mathbb{E}_t \left[\int_t^{+\infty} \frac{\xi_u}{\xi_t} \exp(-\pi(u-t)) P^\star_{C,u} c^\star_{u,t} du \right] \leq \mathbb{E}_t \left[\int_t^{+\infty} \frac{\xi_u}{\xi_t} y_{u,t} du \right] \equiv P_{\delta,t} \tag{2.8}$$

The stochastic discount factor process itself evolves according to

$$\frac{d\xi_t}{\xi_t} = -r_t dt - \kappa_t dB_t \tag{2.9}$$

where r_t denotes the instantaneous riskless interest rate and κ_t denotes the price of risk. r_t represents the interest rate for risk-free borrowing and lending denominated in the numeraire good. However, because price indices fluctuate in each country, a certain future payoff of numeraire goods delivers an uncertain number of units of each country's own consumption bundle. The relevant "safe assets" for households in each country are real bonds, which deliver a certain payout in units of each country's consumption bundle. The home country's real bond delivers a stochastic payoff in units of the numeraire good that is indexed to the change in the home country price index $dP_{C,t}/P_{C,t}$; the foreign country's real bond delivers a similar payoff in units of the numeraire good indexed to the change in its own price index, $dP^\star_{C,t}/P^\star_{C,t}$. Section 2.3 discusses the pricing of these real bonds in a numerically-solved version of the model.

Risk premia are ultimately determined by the equilibrium price of risk κ_t. From Equations (2.2) and (2.3), a positive Brownian shock $dB_t > 0$ increases the home country's price index $P_{C,t}$ while decreasing the foreign country's price index. When $\kappa_t < 0$, financial assets that deliver relatively high returns in these states where the home country experiences a real appreciation earn low expected excess returns. When $\kappa_t > 0$, these same assets earn high expected excess returns. A key determinant of the equilibrium price of risk will be the distribution of wealth between households in the home and foreign countries, which determines how the separate hedging motives of home and foreign country households are reflected in equilibrium asset prices.

2.2 Equilibrium

In defining equilibrium, it will be useful to refer to aggregate quantities. Let

$$C_t \equiv \int_{-\infty}^{t} \pi \exp(-\pi(u-t)) c_{t,s} ds \text{ and } W_t \equiv \int_{-\infty}^{t} \pi \exp(-\pi(u-t)) w_{t,s} ds \tag{2.10}$$

denote aggregate consumption and wealth for households in the home country; aggregate quantities $C_t^\star$ and $W_t^\star$ for the foreign country are defined in the same manner.

An equilibrium consists of adapted stochastic processes for household consumption $\{c_{t,s}, c_{t,s}^\star\}_{t\geq 0, s\leq t}$ and asset prices $\{r_t, \kappa_t\}_{t\geq 0}$ such that:

1. Given the stochastic discount factor process $\{\xi_t\}_{t\geq 0}$ implied by $\{r_t, \kappa_t\}_{t\geq 0}$ and the exogenous goods price process $\{P_{C,t}\}_{t\geq 0}$, the equilibrium home consumption process $\{c_{t,s}\}_{t\geq s}$ solves the utility maximization problem of home country households born on each date t:

$$\max_{\{c_{u,t}\}_{u\geq t}} U_t\left(\{c_{u,t}\}_{u\geq t}\right) \text{ subject to (2.6)} \tag{2.11}$$

2. Similarly, the equilibrium foreign consumption process $\{c_{t,s}^\star\}_{t\geq s}$ solves the utility maximization problem of foreign country households born on each date t:

$$\max_{\{c_{u,t}^\star\}_{u\geq t}} U_t\left(\{c_{u,t}^\star\}_{u\geq t}\right) \text{ subject to (2.8)} \tag{2.12}$$

3. Financial markets clear at each date t, which implies that the total consumption expenditure of home and foreign country households equals their total income $\overline{Y}$:

$$P_{C,t} C_t + P_{C,t}^\star C_t^\star = \overline{Y} \tag{2.13}$$

As in Dou and Verdelhan (2015) and Sauzet (2021), I now characterize a *stationary wealth-recursive equilibrium* in which all of the endogenous variables in the above definition can be expressed as time-invariant functions of two state variables. The first state variable is the price $P_{C,t}$ of the home country consumption bundle, which evolves exogenously according to the process in Equation

(2.2).[1] The second state variable is the home country's share of global financial wealth:

$$\chi_{W,t} \equiv \frac{W_t}{W_t + W_t^\star} \tag{2.14}$$

Moreover, I assert that in equilibrium the home wealth share $\chi_{W,t}$ follows an Ito process with diffusion and drift given by time-invariant functions of the two state variables:

$$d\chi_{W,t} = \mu_{\chi W}\left(\chi_{W,t}, \widehat{P}_{C,t}\right) dt + \sigma_{\chi W}\left(\chi_{W,t}, \widehat{P}_{C,t}\right) dB_t \tag{2.15}$$

Before stating the proposition fully characterizing such an equilibrium, I first establish some basic properties. Let $V_t(w_t)$ denote the value function of a home country household at date t with financial wealth w_t, defined by

$$V_t(w_t) \equiv \max_{\{c_u\}_{u\geq t}} U_t\left(\{c_u\}_{u\geq t}\right) \text{ subject to } \mathbb{E}_t\left[\int_t^{+\infty} \frac{\xi_u}{\xi_t} \exp(-\pi(u-t)) P_{C,u} c_u du\right] \leq w_{t,s} \tag{2.16}$$

Since the lifetime utility function $U_t\left(\{c_u\}_{u\geq t}\right)$ is homogenous of degree $1-\gamma$ and the joint distribution of $\{\xi_u, P_{C,u}\}_{u\geq t}$ is summarized by the current state variables $\left(\chi_{W,t}, \widehat{P}_{C,t}\right)$, the value function can be written as

$$V_t(w_t) = \frac{w_t^{1-\gamma}}{1-\gamma} g\left(\chi_{W,t}, \widehat{P}_{C,t}\right)^{-\gamma} \tag{2.17}$$

for some function g. Consumption can then be recovered from the envelope condition

$$u'(c_t) = c_t^{-\gamma} = \frac{\partial V_t(w_t)}{\partial w} P_{C,t} = w_t^{-\gamma} g\left(\chi_{W,t}, \widehat{P}_{C,t}\right)^{-\gamma} P_{C,t} \Leftrightarrow c_t = w_t g\left(\chi_{W,t}, \widehat{P}_{C,t}\right) P_{C,t}^{-\frac{1}{\gamma}} \tag{2.18}$$

Similarly, the value function for foreign country households is given by

$$V_t^\star(w_t^\star) = \frac{(w_t^\star)^{1-\gamma}}{1-\gamma} g^\star\left(\chi_{W,t}, \widehat{P}_{C,t}\right)^{-\gamma} \tag{2.19}$$

for some function $g^\star$. The two functions g and $g^\star$ thus characterize current consumption given the current wealth distribution. It will be analytically convenient to also define the reciprocal functions $h\left(\chi_{W,t}, \widehat{P}_{C,t}\right) \equiv \frac{1}{g\left(\chi_{W,t}, \widehat{P}_{C,t}\right)}$ and $h^\star\left(\chi_{W,t}, \widehat{P}_{C,t}\right) \equiv \frac{1}{g^\star\left(\chi_{W,t}, \widehat{P}_{C,t}\right)}$, which will be used in the statement of the main proposition below.

In order to fully characterize the equilibrium, we also need to determine the initial wealth $w_{t,t} =$

[1] Recall that the foreign country consumption bundle price is given in terms of the home country price as $P_{C,t}^\star = \frac{1}{P_{C,t}}$, so it suffices to track a single price index as a state variable.

$\overline{y}P_{\delta,t}$ of new households. Since the dynamics of the stochastic discount factor process ξ_t are fully determined by the current state variable values, we conclude from Equation (2.7) that the income stream price must also be a function of only the current state variables, so that $P_{\delta,t} = P_\delta\left(\chi_{W,t}, \widehat{P}_{C,t}\right)$ for some function P_δ.

Given the functions h, $h^\star$ and P_δ defined above, we can determine the dynamics of the endogenous state variable $\chi_{W,t}$. These three functions are in turn pinned down by a system of partial differential equations that can be solved to fully characterize the model's equilibrium. The following proposition provides further details:

Proposition 1. *Assume there exist functions $h\left(\chi_{W,t}, \widehat{P}_{C,t}\right)$, $h^\star\left(\chi_{W,t}, \widehat{P}_{C,t}\right)$ and $P_\delta\left(\chi_{W,t}, \widehat{P}_{C,t}\right)$ that solve the partial differential equations (A.37), (A.38), and (A.41). There exists a stationary wealth-recursive equilibrium where the home country wealth share $\chi_{W,t} \equiv \frac{W_t}{W_t+W_t^\star}$ evolves according to the Ito process (2.15), with drift and diffusion given by*

$$\sigma_{\chi W}\left(\chi_{W,t}, \widehat{P}_{C,t}\right) = \frac{\chi_{W,t}\left[\frac{1}{\gamma}\kappa\left(\chi_{W,t}, \widehat{P}_{C,t}\right) + \left(\frac{1}{h\left(\chi_{W,t}, \widehat{P}_{C,t}\right)}\frac{\partial h}{\partial \widehat{P}_C} - \frac{1}{P_\delta\left(\chi_{W,t}, \widehat{P}_{C,t}\right)}\frac{\partial P_\delta}{\partial \widehat{P}_C}\right)\sigma_{PC}\right]}{1 - \chi_{W,t}\left(\frac{1}{h\left(\chi_{W,t}, \widehat{P}_{C,t}\right)}\frac{\partial h}{\partial \chi_W} - \frac{1}{P_\delta\left(\chi_{W,t}, \widehat{P}_{C,t}\right)}\frac{\partial P_\delta}{\partial \chi_W}\right)} \tag{2.20}$$

$$\mu_{\chi W,t}\left(\chi_{W,t}, \widehat{P}_{C,t}\right) = \frac{\chi_{W,t}\widetilde{\mu}_{M,t} + \frac{\delta}{2}}{1 - \chi_{W,t}\left[\frac{1}{h\left(\chi_{W,t}, \widehat{P}_{C,t}\right)}\frac{\partial h}{\partial \chi_W} - \frac{1}{P_\delta\left(\chi_{W,t}, \widehat{P}_{C,t}\right)}\frac{\partial P_\delta}{\partial \chi_W}\right]} \tag{2.21}$$

where $\widetilde{\mu}_{M,t}$ is given in Equation (A.29). The price of risk $\kappa_t = \kappa\left(\chi_{W,t}, \widehat{P}_{C,t}\right)$ and the risk-free interest rate $r_t = r\left(\chi_{W,t}, \widehat{P}_{C,t}\right)$ are given by

$$\kappa\left(\chi_{W,t}, \widehat{P}_{C,t}\right) = -(\gamma - 1)\left(2\chi_{X,t} - 1\right)\overline{\sigma}_{PC} \tag{2.22}$$

$$\begin{aligned} r_t = \rho + \gamma\pi - \frac{1}{2}\left(\frac{1}{\gamma} + 1\right)\kappa_t^2 - (\gamma - 1)\left(\chi_{X,t}\overline{\mu}_{PC,t} + (1 - \chi_{X,t})\overline{\mu}^\star_{PC,t}\right) \\ - \left(1 - \frac{1}{\gamma}\right)\kappa_t\left(2\chi_{X,t} - 1\right)\sigma_{PC} + \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)\sigma_{PC}^2 \\ - \gamma\frac{\delta}{2}P_\delta\left(\chi_{W,t}, \widehat{P}_{C,t}\right)\left(P_{C,t}^{1-\frac{1}{\gamma}}g\left(\chi_{W,t}, \widehat{P}_{C,t}\right) + \left(P^\star_{C,t}\right)^{1-\frac{1}{\gamma}}g^\star\left(\chi_{W,t}, \widehat{P}_{C,t}\right)\right) \end{aligned} \tag{2.23}$$

where

$$\chi_{X,t} \equiv \frac{P_{C,t}C_t}{P_{C,t}C_t + P^\star_{C,t}C^\star_t} = \frac{\chi_{W,t}g\left(\chi_{W,t},\widehat{P}_{C,t}\right)P_{C,t}^{1-\frac{1}{\gamma}}}{\chi_{W,t}g\left(\chi_{W,t},\widehat{P}_{C,t}\right)P_{C,t}^{1-\frac{1}{\gamma}} + (1-\chi_{W,t})g^\star\left(\chi_{W,t},\widehat{P}_{C,t}\right)\left(P^\star_{C,t}\right)^{1-\frac{1}{\gamma}}} \tag{2.24}$$

denotes home households' share of total consumption expenditure.

The proof of this proposition is presented in the appendix.

The expressions for the diffusion (2.20) and drift (2.21) of the home country's wealth share are similar to those that arise in other models featuring both overlapping generations and heterogenous household types, such as Gârleanu and Panageas (2015). When $\chi_{W,t} = 0$ and home country households hold no financial wealth, the diffusion loading for the wealth share goes to zero, while the positive drift $\frac{\delta}{2}$ reflects the entry of new households in the home country. Behavior at the upper boundary $\chi_{W,t} = 1$ is similar, as will be shown in the next section: the diffusion loading also goes to zero while the drift goes to $-\frac{\delta}{2}$, reflecting the entry of new households in the foreign country.

The model's implications for currency risk premia and the pricing of exchange rate risk are determined by the equilibrium price of risk κ_t given in Equation (2.22). The sign of κ_t is determined by both household risk aversion γ and the current distribution of consumption expenditure across countries. In the numerically-solved version of the model presented in the next section, the home country's consumption expenditure share $\chi_{X,t}$ closely tracks its wealth share $\chi_{W,t}$ when price indices are equal across countries ($P_{C,t} = P^\star_{C,t} = 1$).The level of risk aversion γ determines the strength of households' motive to hedge against fluctuations in their own price index. Equations (A.15) and (A.16) in the appendix show that when $\gamma > 1$, households wish to increase overall consumption *expenditure* $P_{C,t}C_t$ in states where the price index $P_{C,t}$ is relatively high. Large declines in consumption are particularly painful for households when $\gamma > 1$, thus households attempt to keep consumption relatively stable by shifting their consumption expenditure towards states where $P_{C,t}$ is high. In the opposite case where $\gamma < 1$, the welfare loss from volatile consumption is lower, and risk-tolerant households shift consumption expenditure towards states where $P_{C,t}$ is low in order to achieve a high average level of consumption. In the knife-edge case where $\gamma = 1$ and households have log utility, such expenditure smoothing motives vanish, and the price of risk remains identically zero: $\kappa_t \equiv 0$. This latter case was studied in the influential work of Cole and Obstfeld (1991).

In the empirically relevant case where relative risk aversion satisfies $\gamma > 1$, the price of risk is negatively related to the home country's share of total consumption expenditure $\chi_{X,t}$ (which is in turn determined by its share of total wealth $\chi_{W,t}$). This reflects the balance of offsetting exchange rate hedging motives for households in the two countries when one country currently owns a large share of global wealth. I provide a detailed discussion in the following section.

2.3 Numerical Solution and Asset Prices

In this section, I explore the features of the model through numerical simulations. These simulations are primarily intended to illustrate the qualitative properties of the model, in particular the relationship between the cross-country wealth distribution, the equilibrium price of risk κ_t, and the risk premium for each country's real bond.

I choose parameter values so that one unit of time within the model corresponds to one year of calendar time. The birth/death rate of agents is set to $\pi = 0.02$, implying an average lifespan of $\frac{1}{\pi} = 50$ years for households in the model.[2] The subjective time discount rate is set to $\rho = 0.04$, which implies an effective discount rate of $\rho + \pi = 0.06$ after adjusting for mortality risk. The depreciation rate of household income is set to $\delta = 0.07$, which implies that earnings fall by half after every period of 10 years.

As discussed in the previous section, the value of relative risk aversion γ plays a crucial role in the model. I set $\gamma = 2$, at the lower range of values used by Gourinchas, Rey, and Govillot (2010) and just above the crucial threshold of 1, so that households shift consumption expenditure towards states when the price of their consumption bundle rises. I intentionally choose a low value for risk aversion in order to illustrate that the qualitative relationship between the global wealth distribution and currency risk premia arises even for low values of household risk aversion.

The parameters β and σ_{PC} governing the persistence and volatility of the home country's consumption bundle price are set to match the empirical properties of the real exchange rate, which is defined within the model as the ratio of consumption bundle prices across countries: $Q_t \equiv P_{C,t}/P^\star_{C,t}$. Given the dynamics of log price indices $\widehat{P}_{C,t}, \widehat{P}^\star_{C,t}$ stated in Equations (2.2) and (2.3), the log real exchange rate $\widehat{Q}_t = \widehat{P}_{C,t} - \widehat{P}^\star_{C,t}$ follows an Ornstein-Uhlenbeck process with mean reversion parameter equal to β and volatility $2\sigma_{PC}$:

$$d\widehat{Q}_t = d\widehat{P}_{C,t} - d\widehat{P}^\star_{C,t} = -\beta\widehat{Q}_t dt + 2\sigma_{PC} dB_t \tag{2.25}$$

[2] Since households in the model immediately receive income and transact in financial markets at "birth", this event within the model should be interpreted as the entry of new households into the labor force. Consequently, a 50-year lifespan of households in the model corresponds to a roughly 70-year lifespan for actual households (assuming labor force entry occurs on average at age 20).

I set $\beta = 0.25$ and $\sigma_{PC} = 0.05$. These parameter values imply a half-life of real exchange rate shocks equal to $\frac{\log 2}{\beta} \approx 2.8$ years and annual real exchange rate volatility equal to $Var(\widehat{Q}_{t+1} - \widehat{Q}_t) = 10\%$, similar to the estimates used by Itskhoki and Mukhin (2021).

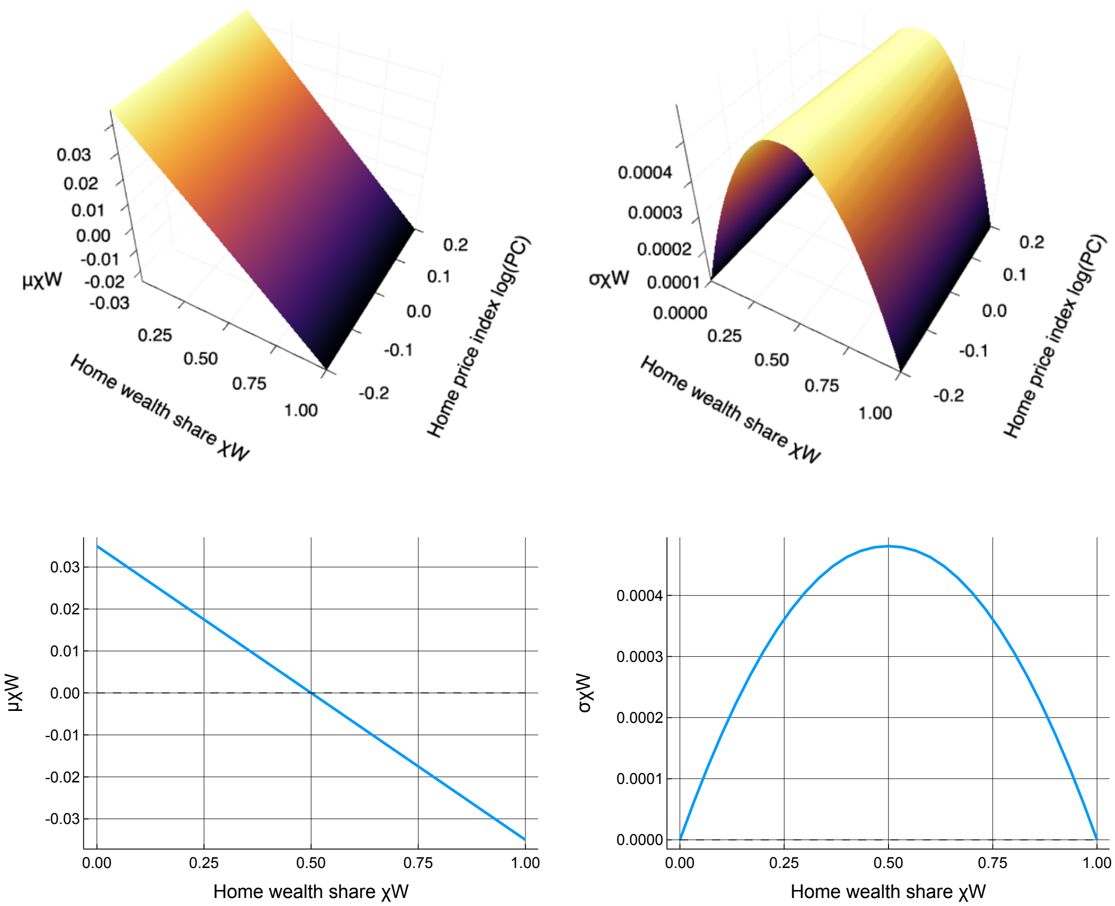
Figure 2-1 plots the drift and diffusion of the home country wealth share $\chi_{W,t}$ at each point in the state space. Since the drift and diffusion do not vary much as a function of the log price index $\widehat{P}_{C,t}$, I focus attention on the lower two panels, which fix consumption bundle prices at $P_{C,t} = P^{\star}_{C,t} = 1$ (in logs, $\widehat{P}_{C,t} = \widehat{P}^{\star}_{C,t} = 0$) while varying the home wealth share $\chi_{W,t}$.

On the left-hand side, the drift of $\chi_{W,t}$ is positive when $\chi_{W,t} < 1/2$ and the foreign country owns a greater share of financial wealth. The drift is negative when $\chi_{W,t} > 1/2$, so that the wealth share ultimately reverts back towards the steady state value $\chi_{W,t} = 1/2$ where households in each country hold equal financial wealth. At both boundaries $\chi_{W,t} = 0$ and $\chi_{W,t} = 1$, the drift is exactly equal to $+\frac{\delta}{2}$ and $-\frac{\delta}{2}$ (respectively). At these boundary points where one country's households own all financial wealth, the drift of each country's wealth share is solely determined by the entry of new households. The relative wealth of all existing households decays at rate δ as their income streams depreciate, and both countries receive equal shares of wealth created by newly-entering households (hence the scaling by $\frac{1}{2}$).

On the right-hand side, the diffusion loading of $\chi_{W,t}$ is positive everywhere for the chosen parameter values. This implies that the wealth of home country households increases when the home country experiences a real appreciation, since a positive Brownian shock $dB_t > 0$ increases the price $P_{C,t}$ of the home country's consumption bundle and decreases the price $P^{\star}_{C,t}$ of the foreign country's bundle. Since $\gamma > 1$, home country households shift their consumption expenditure towards states where $P_{C,t}$ is high. Because the real exchange rate is persistent, after an increase in $P_{C,t}$ the higher path of future consumption expenditure by home country households must be financed through a large increase in their current wealth, resulting in the positive diffusion loading for their wealth share. This real exchange rate hedging motive is at the heart of the model's implications for currency risk premia.

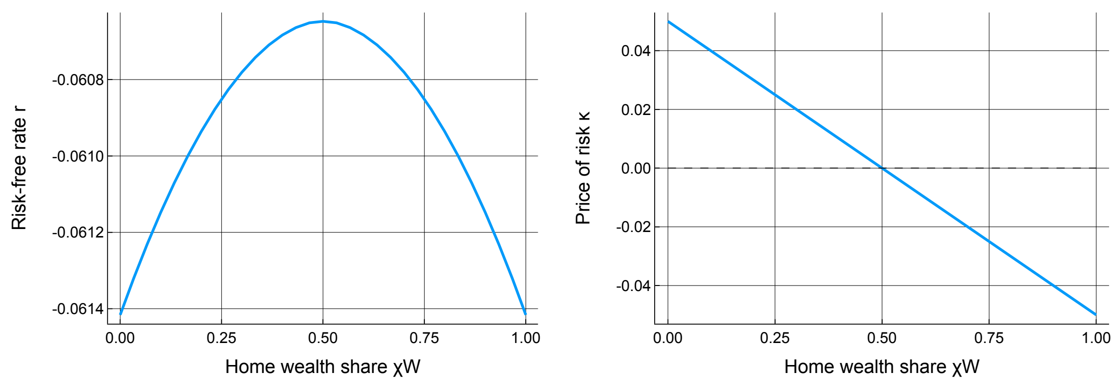
The wealth shares of the two countries are most volatile near the steady state $\chi_{W,t} = 1/2$. International risk sharing can be conducted most effectively in this case, since both households have ample wealth to insure their counterparts abroad against a real appreciation abroad. The diffusion loadings of the wealth shares decline to 0 near the boundaries as one country dominates the global wealth distribution. In the extreme case where $\chi_{W,t} = 0$ or $\chi_{W,t} = 1$, existing households in one country hold all financial wealth and thus cannot share any risk with their counterparts abroad.

Figure 2-1: Drift and diffusion for the home country's financial wealth share $\chi_{W,t}$.
The top panels depict the drift $\mu_{\chi W}\left(\chi_W, \widehat{P}_C\right)$ and diffusion $\sigma_{\chi W}\left(\chi_W, \widehat{P}_C\right)$ as a function of both state variables $\chi_{W,t}$ and $\widehat{P}_{C,t} = \log P_{C,t}$. The bottom panels depict the drift and diffusion as a function of the home wealth share $\chi_{W,t}$ only, with consumption bundle prices in both countries fixed at $P_{C,t} = P^{\star}_{C,t} = 1$.



The wealth share is locally deterministic at these points, with drift determined by the entry of new households in the relatively poor country. Since these new households are born *after* the realization of any shocks, they cannot share risk with households in the relatively wealthy country. Finally, note that because the home country wealth share has zero diffusion loading and positive (negative) drift at the boundary $\chi_{W,t} = 0$ ($\chi_{W,t} = 1$), a stationary wealth distribution exists.

Figure 2-2: Risk-free rate r and price of risk κ.
Both prices are functions of the two state variables $\left(\chi_W, \widehat{P}_C\right)$. The consumption bundle prices in both countries fixed at $P_{C,t} = P^{\star}_{C,t} = 1$. The home country's financial wealth share $\chi_{W,t}$ varies between 0 and 1 along the horizontal axes.



I now discuss the model's implications for asset prices. Figure 2-2 depict the risk-free rate r_t and price of risk κ_t as a function of the home country wealth share $\chi_{W,t}$, with consumption bundle prices again fixed at $P_{C,t} = P^{\star}_{C,t} = 1$. The risk-free interest rate (shown in the left panel) takes on large negative values, despite the relatively high effective discount rate $\rho + \pi = 0.06$. I conjecture that this is primarily due to the fact that the model features no growth in aggregate income, which remains constant at the level $\overline{Y}$. Incorporating deterministic growth in aggregate income would be relatively straightforward and could help to increase the implied risk-free interest rate to a more reasonable level.[3] At the same time, these changes would likely have at most a small effect on the price of risk.

The price of risk (shown in the right panel) is negative when $\chi_{W,t} < 1/2$ and positive when $\chi_{W,t} > 1/2$. To understand this pattern, it is helpful to consider the two boundary cases where one

[3] The hump-shaped relationship between the risk-free interest rate and the home country's wealth share is driven by a precautionary savings effect: near the boundaries $\chi_{W,t} = 0$ and $\chi_{W,t} = 1$, households in the wealthy country cannot effectively share risk with their counterparts abroad, due to the uneven distribution of global wealth. As a result, the volatility of their consumption increases and the risk-free rate declines.

country dominates the global wealth distribution. When $\chi_{W,t} = 1$, households in the home country hold all financial wealth. Because of the hedging motives discussed above, home country households would like to insure themselves against future increases in their consumption bundle price $P_{C,t}$, which occurs after a positive Brownian shock realization $dB_t > 0$. However, no international risk sharing can be carried out when $\chi_{W,t} = 1$, due to the fact that households in the foreign country do not own any claims to future income that they can pledge in states where $P_{C,t}$ increases. Households in the foreign country thus remain fully exposed to increases in $P_{C,t}$, and this asymmetric hedging demand increases the equilibrium price of insurance against such shocks. Since a positive Brownian shock $dB_t > 0$ corresponds to a real appreciation in the home country, this is reflected in a negative equilibrium price of risk $\kappa_t < 0$ for this shock.

The same logic applies when $\chi_{W,t} = 0$, with the roles of households reversed across the two countries. In this case, foreign households hold all financial wealth and cannot engage in risk sharing with households in the home country. However, the hedging demands of foreign households are the opposite of those for home country households: foreign country households wish to shift consumption expenditure towards states where the price $P^{\star}_{C,t}$ of their own consumption bundle is high. Given my assumption of perfectly *negatively* correlated movements in price indices across countries, these states correspond to negative realizations of the Brownian shock, $dB_t < 0$, as can be seen in Equation (2.3). As a result, the hedging demand of the foreign households who dominate the wealth distribution when $\chi_{W,t} = 0$ is reflected in a positive equilibrium price of risk $\kappa_t > 0$.

In the intermediate case where $\chi_{W,t} = 1/2$ and household wealth is equal across the home and foreign countries, the price of risk is exactly $\kappa_t = 0$. This is the case where hedging demands of households in both countries are fully balanced, and risk sharing is carried out perfectly over infinitesimal horizons. A positive Brownian shock $dB_t > 0$ increases the price index in the home country while decreasing the price index in the foreign country. Home country households increase their consumption expenditure at this time, financed by a wealth transfer from households in the foreign country. These transactions are exactly reversed in the case of a negative Brownian shock $dB_t < 0$ that causes a real appreciation in the foreign country. As a result, all households are (locally) well-insured against exchange rate fluctuations, and the equilibrium price of risk is correspondingly zero.

After determining the price of risk, I now examine the model's implications for real bond prices.[4]

[4] I focus on the pricing of short-maturity bonds. I have not yet explored the term structure of real interest rates implied by the model, and whether this is consistent with the empirical facts about holding period returns on long-maturity bonds documented by Lustig, Stathopoulos, and Verdelhan (2019).

Each country's real bond delivers a risk-free return with respect to the country's own price index. Specifically, the real bond for the home country delivers a realized instantaneous return (in units of the numeraire good) of

$$r_{B,t} + \frac{dP_{C,t}}{P_{C,t}} = \underbrace{(r_{B,t} + \overline{\mu}_{PC,t})dt}_{\text{expected return}} + \underbrace{\sigma_{PC} dB_t}_{\substack{\text{unexpected} \\ \text{price} \\ \text{appreciation}}} \tag{2.26}$$

where $r_{B,t}$ denotes the yield of the bond. Similarly, the foreign country's real bond delivers a return indexed to the change in its own price index:

$$r^{\star}_{B,t} + \frac{dP^{\star}_{C,t}}{P^{\star}_{C,t}} = (r^{\star}_{B,t} + \overline{\mu}^{\star}_{PC,t})dt - \sigma_{PC} dB_t \tag{2.27}$$

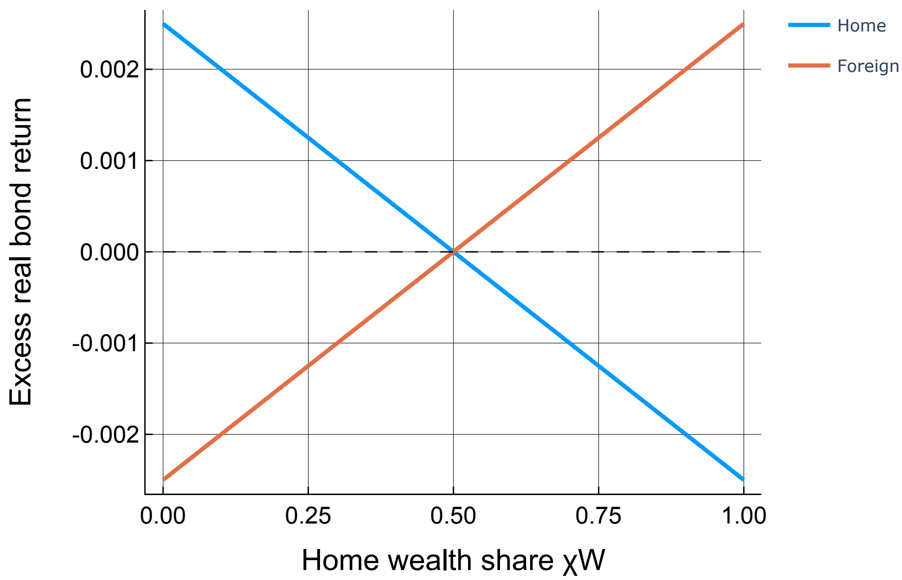
Note that since innovations to both countries' price indices are perfectly negatively correlated, the same is true for realized real bond returns across countries (as reflected by the $-\sigma_{PC} dB_t$ term in the foreign country bond return above). Since financial markets are complete, the expected instantaneous excess return on each country's real bond relative to the risk-free rate r_t (i.e. the *currency risk premium*) is determined from the risk price κ_t by standard no-arbitrage logic:

$$r_{B,t} + \overline{\mu}_{PC,t} - r_t = \kappa_t \sigma_{PC}, \quad r^{\star}_{B,t} + \overline{\mu}^{\star}_{PC,t} - r_t = -\kappa_t \sigma_{PC} \tag{2.28}$$

Since the volatility of price indices is constant over time, currency risk premia are directly determined by the price of risk κ_t and are anti-symmetric across countries: the same hedging motives that cause one country's real bond to become relatively expensive at a given point in time (as reflected by a low expected return) also cause the other country's real bond to become relatively cheap.

Figure 2-3 plots the home and foreign country currency risk premia implied by the model. This figure is the mirror image of the right panel of Figure 2-2 depicting the price of risk κ_t. When $\kappa_{W,t} > 1/2$ and home country households dominate the global wealth distribution, they bid up the price of home country real bonds that insure them against an increase in the price $P_{C,t}$ of their consumption basket. On the other hand, investing in foreign country real bonds is risky for households in the home country because these bonds deliver low returns in states where $P_{C,t}$ (and thus the marginal utility of wealth) is high; consequently, the equilibrium return on the foreign country's real bond is high. When the home country dominates the global wealth distribution, its hedging motives determine the relative prices of real bonds and thus currency risk premia. The same is true when $\kappa_{W,t} < 1/2$ and foreign country households dominate the global wealth distribution, although the implications for currency risk premia are reversed: wealthy foreign households bid up the price of their own country's real bond, which then earns a lower return than the home country's real bond.

Figure 2-3: Model-implied currency risk premia.
The expected excess returns on home and foreign country real bonds are given in Equation (2.28). The consumption bundle prices in both countries fixed at $P_{C,t} = P^{\star}_{C,t} = 1$. The home country's financial wealth share $\chi_{W,t}$ varies between 0 and 1 along the horizontal axes.



The magnitude of the currency risk premia implied by the model are relatively small. This is a common feature of many international macro-finance models, as pointed out by Hassan and Zhang (2020) in their review of the literature. The present exercise is intended to illustrate the key qualitative features of my model. A more serious quantitative exercise could incorporate standard features like disaster risk (Farhi and Gabaix, 2015) or very high values of risk aversion (possibly coupled with non-separable preferences as in Epstein and Zin, 1989) that have been used in the asset pricing literature to match the large magnitude of observed risk premia. However, I conjecture that the qualitative relationship depicted in Figure 2-3 between the distribution of financial wealth across countries and currency risk premia will still remain intact (and in fact be much stronger in magnitude) in such a model.

In summary, the model described here predicts a strong relationship between the distribution of financial wealth across countries and currency risk premia, i.e. excess returns for bonds denominated in the countries' currencies.

Chapter 3

Empirics

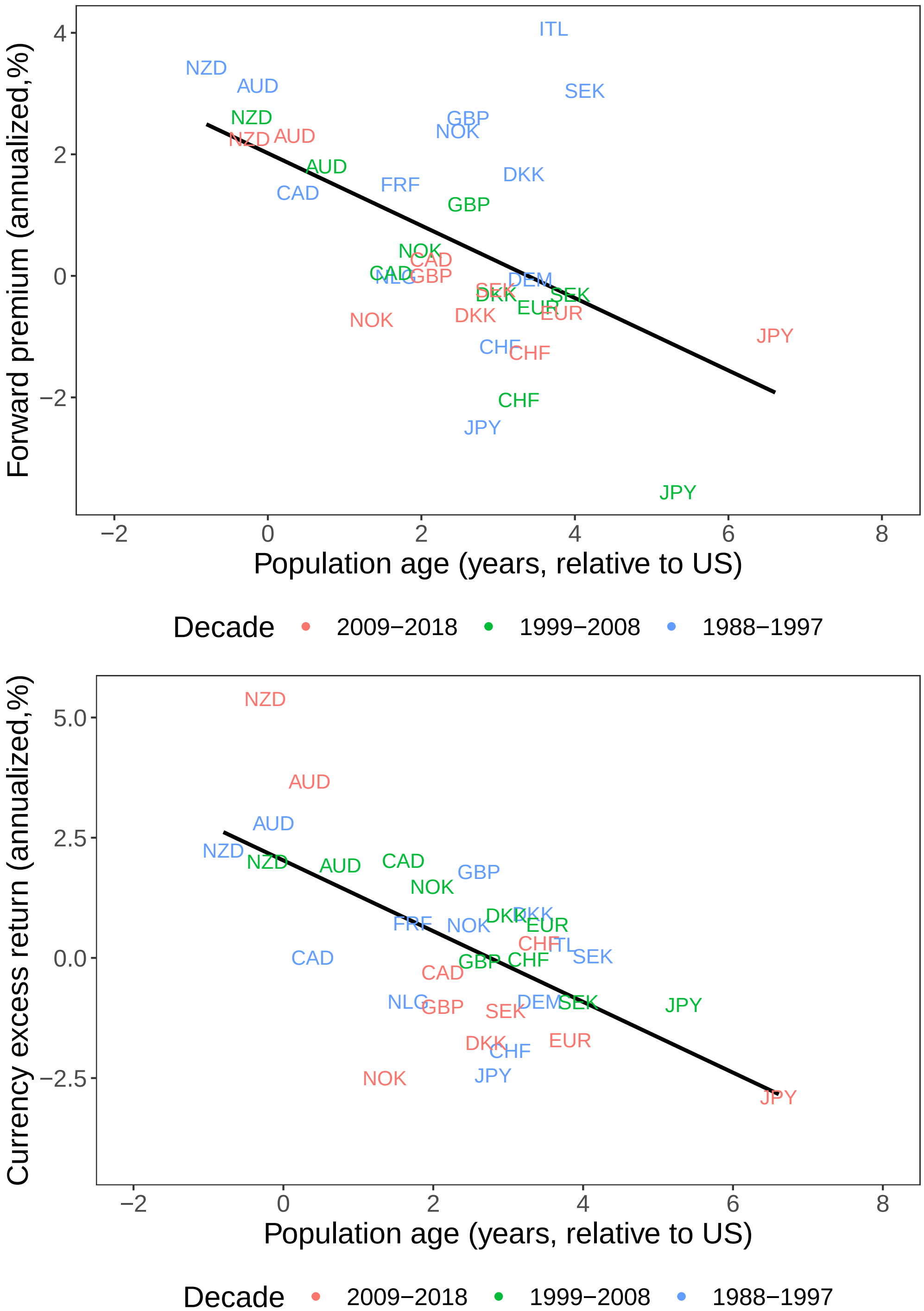
3.1 Main Results

Throughout my analysis, I use annual data (January values) on one-year currency forward premia, changes in exchange rates, and currency excess returns (all with respect to the US dollar) from Hassan and Zhang (2020). Since demographic variables are generally quite persistent, using higher-frequency data offers minimal benefits when investigating the empirical relationship between country demographics and currency returns. The benchmark sample used in my analysis consists of 17 currencies from the IMF's list of advanced economies. I obtain annual data on individual country demographics from the World Bank's World Development Indicators (WDI) dataset, which reports the fraction of the total male population in each country and year that falls within 5-year age ranges (e.g. ages 20-24, 25-29, and so on). Using these population shares, I calculate $Age_{c,t}$, the (approximate) average age of a resident in country c and year t.

Figure 3-1 plots 10-year averages of forward premia and currency excess returns (relative to the US dollar) against population age relative to the United States for my benchmark sample. Many countries with relatively high forward premia and excess returns (such as Canada, New Zealand, and Australia) have relatively young populations among countries in the sample. On the other hand, several countries and regions with persistently low forward premia and excess returns (such as Japan and the euro area) have relatively old populations.

More formally, I estimate panel regressions of log forward premia $fp_{c,t\to t+1}$, log currency depreciation $-\Delta s_{t+1}$, and log currency excess returns $rx_{c,t\to t+1}$ (all measured against the US dollar, where the excess return $rx_{c,t\to t+1} \equiv fp_{c,t\to t+1} - \Delta s_{t+1}$ is the difference between the forward premium and the depreciation of the log exchange rate s_t measured in units of foreign currency per US

Figure 3-1: Forward premia, currency excess returns, and population age.
Forward premia are measured at the one-year horizon with respect to the US dollar. Population age is computed as the difference for each country relative to the United States. Data are plotted as 10-year averages of annual observations for the periods 1988-1998, 1998-2008, and 2008-2018.



dollar) on the population age measure $Age_{c,t}$ and controls collected in the vector $X_{c,t}$:

$$
\begin{aligned}
fp_{c,t\to t+1} &= \beta_{fp} Age_{c,t} + \gamma' X_{c,t} + \varepsilon^{fp}_{c,t\to t+1} \\
-\Delta s_{c,t\to t+1} &= \beta_{\Delta s} Age_{c,t} + \gamma' X_{c,t} + \varepsilon^{fp}_{c,t\to t+1} \\
rx_{c,t\to t+1} &= \beta_{rx} Age_{c,t} + \gamma' X_{c,t} + \varepsilon^{rx}_{c,t\to t+1}
\end{aligned} \tag{3.1}
$$

The set of controls always includes either a constant or time fixed effects. Additionally, I estimate the same regressions controlling for two other primitive country characteristics that have been proposed in the literature as determinants of currency risk premia: the country's share of world GDP as proposed by Hassan (2013), and the country's global trade network centrality as proposed by Richmond (2019). Under uncovered interest parity, excess returns should not be predictable from country characteristics, which implies the restriction $\beta_{rx} = 0$ on the estimated coefficient in the excess return regression. This implies that $\beta_{fp} = \beta_{\Delta s}$ across the forward premium and currency depreciation regressions, so that any differences in forward premia across countries entirely reflect anticipated movements in the spot exchange rate rather than predictable excess returns.

Table 3.1: Panel regression estimates.
This table reports estimated coefficients from panel regressions of log forward premia $fp_{c,t\to t+1}$, log currency depreciation $-\Delta s_{t+1}$, and log currency excess returns $rx_{c,t\to t+1}$ on population age $Age_{c,t}$ and controls. Data are annual (January values) for the sample of advanced economies, from 1988-2018. Standard errors are clustered by both time and currency in specifications that do not include time fixed effects, and are clustered by currency only in specifications that include time fixed effects. Significance levels: * = 10%, ** = 5%, *** = 1%.

	Dependent variable:						
	Forward premium	Forward premium	Currency depreciation	Excess return	Forward premium	Forward premium	Excess return
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Age	−0.567***	−0.519***	0.074	−0.593***	−0.422***	−0.276	−0.523***
	(0.146)	(0.163)	(0.106)	(0.149)	(0.151)	(0.184)	(0.161)
GDP share					−0.336	−0.406	−0.301**
					(0.243)	(0.295)	(0.147)
Trade network centrality					−0.288	−0.640*	−0.227
					(0.306)	(0.345)	(0.192)
Time fixed effects?		X	X	X		X	X
Observations	344	344	344	344	324	324	324
R^2	0.128	0.465	0.628	0.638	0.156	0.509	0.632

Table 3.1 presents these panel regression estimates. Column (1) reports the OLS estimate for the log forward premium. Across countries, a one-year difference in population age is associated with a roughly -0.57 percentage point lower interest rate in the older country. Column (2) reports estimates for the forward premium regression controlling for time fixed effects. The estimated coefficient is similar in magnitude and remains highly significant. Column (3) regresses currency depreciation on population age (also controlling for time fixed effects). I estimate a precise zero relationship between population age and movements in spot exchange rates. As a result, the differences in forward premia across young and old countries should correspond to differences in average excess currency returns. Column (4) confirms this: the estimated coefficient from regressing excess currency returns on population age is large and highly significant.

Columns (5) through (7) re-estimate the same three specifications, but controlling for alternative country characteristics (the country's share of global GDP and its trade network centrality) that have been linked to currency risk premia in previous work. Column (5) excludes time fixed effects and adds controls to the specification in Column (2), resulting in a large and statistically coefficient for population age and insignificant estimates for the other two characteristics. Column (6) adds time fixed effects to this regression. Only the estimated coefficient trade network centrality is significant, and then only marginally so. Column (7) regresses currency excess returns on the three characteristics. The estimated coefficient on population age is similar to the one reported in Column (4) without controls and remains highly significant. From these regressions including multiple country characteristics, we conclude that the predictive power of population age for forward premia and currency excess returns is not subsumed by other country characteristics that have been previously linked to currency risk premia.

Finally, in order to determine whether the large and statistically significant coefficient estimates reported in Table 3.1 are an artifact of the strong parametric assumptions implicit in linear regression, I sort currencies into portfolios at the beginning of each year t based on each country's value of $Age_{c,t}$ (in decreasing order, so that the first portfolio contains currencies with the highest values of $Age_{c,t}$ and the third portfolio contains currencies with the lowest values). Table 3.2 reports average excess returns and average forward premia for the three age-sorted portfolios. Both average returns and forward premia increase as we move from the portfolio of "older" to "younger" currencies. For comparison, sorting currencies into three portfolios using forward premia directly as in Lustig, Roussanov, and Verdelhan (2011) produces a 2.70 percentage point spread in average returns and a 4.06 percentage point spread in average forward premia for this sample of countries. The "demographic carry trade" strategy (that borrows in the currencies of old countries while lending in the currencies of young countries) thus produces roughly 40% of the average difference in forward premia and two thirds of the average excess return for the classic "conditional carry trade" that directly exploits the information in current forward premia.

Table 3.2: Average excess returns and forward premia for sorted currency portfolios. This table reports average log currency excess returns $rx_{c,t\rightarrow t+1}$ and average log forward premia $fp_{c,t\rightarrow t+1}$ for two sets of sorted currency portfolios. The left panel reports averages for portfolios sorted on the population age variable $Age_{c,t}$ (in decreasing order) at the start of each year, while the right panel reports averages for portfolios sorted on the one-year log forward premium $fp_{c,t\rightarrow t+1}$ at the start of each year. Data are annual (January values) from 1988-2018. Newey-West standard errors are reported in parentheses below each mean. Significance levels (for tests that each average portfolio return or forward premium is equal to zero): * = 10%, ** = 5%, *** = 1%.

	Sort on $-Age_{c,t}$				Sort on $fp_{c,t\rightarrow t+1}$			
	P1	P2	P3	P3 - P1	P1	P2	P3	P3 - P1
$\overline{rx}$	-0.432	-0.021	1.423	1.854	-1.191	0.818	1.502	2.694
(s.e.)	(2.855)	(2.819)	(4.202)	(1.238)	(3.073)	(2.644)	(4.416)	(1.885)
$\overline{fp}$	-0.201	0.447^{***}	1.516^{***}	1.717^{***}	-1.285^{***}	0.567^{**}	2.773^{***}	4.058^{***}
(s.e.)	(0.424)	(0.120)	(0.123)	(0.273)	(0.087)	(0.242)	(0.239)	(0.134)

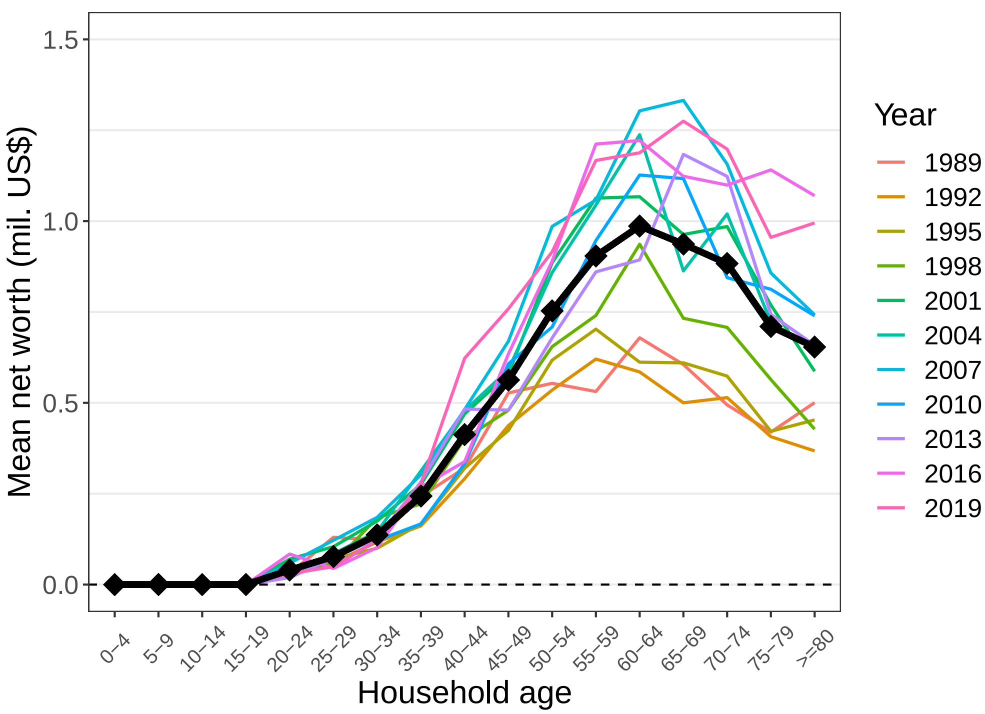
Overall, the empirical results presented here suggest a strong relationship across countries between demographics, forward discounts, and currency risk premia. Countries with relatively young populations (such as Canada, New Zealand, and Australia) have both high interest rates and currencies that earn large average excess returns against the US dollar, while the opposite is true for the currencies of countries and regions with relatively old populations (such as Japan and the euro area). In the following subsections, I further analyze this relationship between demographics and currency risk premia in a broader sample of countries, and using an alternative summary statistic for each country's age distribution based on household savings behavior over the life cycle.

3.2 An Alternative Demographic Summary Statistic

In this section I construct an alternative summary statistic for each country's age distribution using microdata on household asset accumulation over the life cycle. While population age is the simplest statistic to summarize a country's age distribution, one might wonder whether other demographic measures exhibit a similar relationship with currency risk premia across countries. Motivated by the key role of the wealth distribution across countries in determining currency risk premia within the model presented in Section 2.1, I construct a "shift-share" measure that interacts each country's age distribution with weights computed from Survey of Consumer Finances (SCF) data on U.S. household net worth.

Specifically, I first group households in each Survey of Consumer Finances wave into five-year age bins, as in the World Development Indicators data on demographics. I compute average net worth for households in each age bin, which yields the mean net worth weights depicted in Figure 3-2. I then average these mean net worth weights over all sample years; I refer to this average value for a particular five-year age bin as $\overline{NetWorth}_{[y,y+4)}$.[1] Average net worth over the life cycle displays the prominent hump-shaped pattern documented by Gourinchas and Parker (2002) and others: household net worth is initially low, then rises in middle age before peaking around retirement, at which point the average household begins to spend down its accumulated assets.

Figure 3-2: Mean household net worth over the life cycle.
Means are computed using microdata from the Survey of Consumer Finances. The black line reports averages computed over all survey years.



Using the net worth weights described above, I construct a "shift-share" summary measure of each country's age distribution by computing an estimate of its per capita savings if households

[1] I set weights for age groups below age 20 to zero.

followed the exact same asset accumulation policies as in the U.S.:

$$SCFMeasure_{c,t} \equiv \sum_{y=0,5,\ldots,80} PopShare_{[y,y+4),c,t} \overline{NetWorth}_{[y,y+4)} \tag{3.2}$$

This assumption is clearly counterfactual, since retirement savings institutions and government policies differ significantly across advanced economies. Nevertheless, I believe this measure still provides a useful summary of the potential contribution of a country's demographic structure to its share of global financial wealth.

I run an additional set of regressions using the SCF-based measure in place of population age; the specifications are otherwise identical to those reported in the first four columns of Table 3.1. Table 3.3 below reports the results. To assist with interpretation, the SCF-based demographic measure is standardized over the full panel of observations before running the regression. The results are very similar to those reported for population age in Table 3.1: a one-standard deviation in the SCF-based measure across countries is associated with a large and statistically difference in forward premia of 1.06 to 1.23 percentage points, and a 1.35 percentage point difference in average currency returns.

Table 3.3: Panel regression estimates using the SCF-based demographic variable.
This table reports estimated coefficients from panel regressions of log forward premia $fp_{c,t\to t+1}$, log currency depreciation $-\Delta s_{t+1}$, and log currency excess returns $rx_{c,t\to t+1}$ on the standardized SCF-based demographics measure and controls. Data are annual (January values) for the sample of advanced economies, from 1988-2018. Standard errors are clustered by both time and currency in specifications that do not include time fixed effects, and are clustered by currency only in specifications that include time fixed effects. Significance levels: * = 10%, ** = 5%, *** = 1%.

	Dependent variable:			
	Forward premium	Forward premium	Currency depreciation	Excess return
	(1)	(2)	(3)	(4)
SCF measure	-1.064^{***}	-1.227^{***}	0.123	-1.350^{***}
	(0.160)	(0.370)	(0.247)	(0.316)
Time fixed effects?		X	X	X
Observations	344	344	344	344
R^2	0.191	0.481	0.628	0.639

The results presented here show that the empirical relationship between demographics and currency risk premia arises using summary statistics for the full age distribution other than raw population age. Moreover, microdata on household asset accumulation over the life cycle can be used to discipline general equilibrium models building on the framework presented in Section 2.

Chapter 4

Conclusion

This paper is the first to link country demographics and currency risk premia. I present empirical evidence showing a strong negative relationship between population age and both interest rates and currency risk premia: younger countries have much higher interest rates than older countries, and global investors can earn large average excess returns by exploiting this fact. As a first step towards building a complete theoretical framework to study the relationship between demographic heterogeneity and equilibrium currency risk premia, I develop a two-country overlapping generations model in which households trade in international financial markets to hedge against fluctuating local goods prices. The model features a strong relationship between currency risk premia and the global wealth distribution: when households in one country hold a large share of global financial wealth, they drive up the equilibrium price of domestic real bonds that insure against increases in the price of their own consumption bundle. As a result, currency risk premia are negatively related to shares of global financial wealth across countries.

In future work, I plan to extend the model developed in this paper by incorporating demographic heterogeneity across countries. As Figure 3-2 illustrates, the average household accumulates assets over the course of its life cycle, eventually accumulating a large stock of financial wealth at retirement before gradually dissaving. This suggests that large differences in financial wealth can arise between young and old countries, and the model I present in Section 2 suggests that these cross-country differences in financial wealth are likely to depress currency risk premia in old countries while elevating currency risk premia in young countries. While a complete model that formalizes this logic is beyond the current scope of this paper, the work presented here lays the foundation for richer models of international risk sharing, demographic heterogeneity, and currency risk premia.

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Appendix A

Proof of Proposition 1

The proof proceeds in four steps:

1. Given the stochastic discount factor process (2.9), I determine the dynamics of aggregate consumption $C_t, C_t^\star$ and wealth $W_t, W_t^\star$.
2. Using the market clearing condition $P_{C,t}C_t + P_{C,t}^\star C_t^\star = \overline{Y}$, I determine the price of risk κ_t and risk-free interest rate r_t.
3. I then solve for the diffusion $\sigma_{\chi W}\left(\chi_{W,t}, \widehat{P}_{C,t}\right)$ and drift $\mu_{\chi W}\left(\chi_{W,t}, \widehat{P}_{C,t}\right)$ of the home country wealth share $\chi_{W,t}$.
4. Finally, I derive the partial differential equations satisfied by the functions $h\left(\chi_{W,t}, \widehat{P}_{C,t}\right)$, $h^\star\left(\chi_{W,t}, \widehat{P}_{C,t}\right)$, and $P_\delta\left(\chi_{W,t}, \widehat{P}_{C,t}\right)$.

I use the notation $g_t \equiv g(\chi_{W,t}, \widehat{P}_{C,t})$, $g_t^\star \equiv g^\star(\chi_{W,t}, \widehat{P}_{C,t})$, $h_t \equiv h(\chi_{W,t}, \widehat{P}_{C,t})$, $h_t^\star \equiv h^\star(\chi_{W,t}, \widehat{P}_{C,t})$, and $P_{\delta,t} \equiv P_\delta(\chi_{W,t}, \widehat{P}_{C,t})$. Using Ito's lemma, we can determine the dynamics of each of these newly defined processes:

$$\frac{dh_t}{h_t} = \overline{\mu}_{h,t}dt + \overline{\sigma}_{h,t}dB_t, \quad \frac{dh_t^\star}{h_t^\star} = \overline{\mu}_{h,t}^\star dt + \overline{\sigma}_{h,t}^\star dB_t \quad \frac{dP_{\delta,t}}{P_{\delta,t}} = \overline{\mu}_{P\delta,t}dt + \overline{\sigma}_{P\delta,t}dB_t \tag{A.1}$$

The drifts are given by

$$
\begin{aligned}
\overline{\mu}_{h,t} &= \frac{1}{h_t}\frac{\partial h}{\partial \chi_W}\mu_{\chi W,t} + \underbrace{\frac{1}{h_t}\frac{\partial h}{\partial \widehat{P}_C}\mu_{\widehat{P}C,t} + \frac{1}{2}\frac{1}{h_t}\frac{\partial^2 h}{\partial \chi_W^2}\sigma_{\chi W,t}^2 + \frac{1}{2}\frac{1}{h_t}\frac{\partial^2 h}{\partial \widehat{P}_C^2}\sigma_{PC}^2 + \frac{1}{h_t}\frac{\partial^2 h}{\partial \chi_W \partial \widehat{P}_C}\sigma_{\chi W,t}\sigma_{PC}}_{\equiv \widetilde{\mu}_{h,t}} \\
&= \frac{1}{h_t}\frac{\partial h}{\partial \chi_W}\mu_{\chi W,t} + \widetilde{\mu}_{h,t} \\
\overline{\mu}^\star_{h,t} &= \frac{1}{h^\star_t}\frac{\partial h^\star}{\partial \chi_W}\mu_{\chi W,t} + \underbrace{\frac{1}{h^\star_t}\frac{\partial h^\star}{\partial \widehat{P}_C}\mu_{\widehat{P}C,t} + \frac{1}{2}\frac{1}{h^\star_t}\frac{\partial^2 h^\star}{\partial \chi_W^2}\sigma_{\chi W,t}^2 + \frac{1}{2}\frac{1}{h^\star_t}\frac{\partial^2 h^\star}{\partial \widehat{P}_C^2}\sigma_{PC}^2 + \frac{1}{h^\star_t}\frac{\partial^2 h^\star}{\partial \chi_W \partial \widehat{P}_C}\sigma_{\chi W,t}\sigma_{PC}}_{\equiv \widetilde{\mu}^\star_{h,t}} \\
&= \frac{1}{h^\star_t}\frac{\partial h^\star}{\partial \chi_W}\mu_{\chi W,t} + \widetilde{\mu}^\star_{h,t} \\
\overline{\mu}_{P\delta,t} &= \frac{1}{P_{\delta,t}}\frac{\partial P_\delta}{\partial \chi_W}\mu_{\chi W,t} + \underbrace{\frac{1}{P_{\delta,t}}\frac{\partial P_\delta}{\partial \widehat{P}_C}\mu_{\widehat{P}C,t} + \frac{1}{2}\frac{1}{P_{\delta,t}}\frac{\partial^2 P_\delta}{\partial \chi_W^2}\sigma_{\chi W,t}^2 + \frac{1}{2}\frac{1}{P_{\delta,t}}\frac{\partial^2 P_\delta}{\partial \widehat{P}_C^2}\sigma_{PC}^2 + \frac{1}{P_{\delta,t}}\frac{\partial^2 P_\delta}{\partial \chi_W \partial \widehat{P}_C}\sigma_{\chi W,t}\sigma_{PC}}_{\equiv \widetilde{\mu}_{P\delta,t}} \\
&= \frac{1}{P_{\delta,t}}\frac{\partial P_\delta}{\partial \chi_W}\mu_{\chi W,t} + \widetilde{\mu}_{P\delta,t}
\end{aligned}
\tag{A.2}
$$

where $\widetilde{\mu}_{h,t}, \widetilde{\mu}^\star_{h,t}, \widetilde{\mu}_{P\delta,t}$ denote the components of each drift that do not depend directly on the drift $\mu_{\chi W,t}$ of the home wealth share. The diffusions are given by

$$
\begin{aligned}
\overline{\sigma}_{h,t} &= \frac{1}{h_t}\frac{\partial h}{\partial \chi_W}\sigma_{\chi W,t} + \frac{1}{h_t}\frac{\partial h}{\partial \widehat{P}_C}\sigma_{PC} \\
\overline{\sigma}^\star_{h,t} &= \frac{1}{h^\star_t}\frac{\partial h^\star}{\partial \chi_W}\sigma_{\chi W,t} + \frac{1}{h^\star_t}\frac{\partial h^\star}{\partial \widehat{P}_C}\sigma_{PC} \\
\overline{\sigma}_{P\delta,t} &= \frac{1}{P_{\delta,t}}\frac{\partial P_\delta}{\partial \chi_W}\sigma_{\chi W,t} + \frac{1}{P_{\delta,t}}\frac{\partial P_\delta}{\partial \widehat{P}_C}\sigma_{PC}
\end{aligned}
\tag{A.3}
$$

Step 1. Using both the first-order conditions for the home household's optimization problem in (2.16) and the envelope condition (2.18), optimal consumption growth for a home country household that survives from date t to date $u > t$ can be expressed as

$$
\frac{c_{u,s}}{c_{t,s}} = \exp\left(-\frac{\rho}{\gamma}(u-t)\right)\left(\frac{\xi_u}{\xi_t}\right)^{-\frac{1}{\gamma}}\left(\frac{P_{C,u}}{P_{C,t}}\right)^{-\frac{1}{\gamma}} = \frac{w_{u,s}}{w_{t,s}}\frac{g_u}{g_t}\left(\frac{P_{C,u}}{P_{C,t}}\right)^{-\frac{1}{\gamma}} \tag{A.4}
$$

This gives a simple expression for the growth in wealth (conditional on survival):

$$
\frac{w_{u,s}}{w_{t,s}} = \exp\left(-\frac{\rho}{\gamma}(u-t)\right)\left(\frac{\xi_u}{\xi_t}\right)^{-\frac{1}{\gamma}}\left(\frac{g_u}{g_t}\right)^{-1} = \exp\left(-\frac{\rho}{\gamma}(u-t)\right)\left(\frac{\xi_u}{\xi_t}\right)^{-\frac{1}{\gamma}}\frac{h_u}{h_t} \tag{A.5}
$$

where $h_t \equiv \frac{1}{g_t}$. The total wealth W_t of home households can then be obtained by integrating over the distribution of households:

$$
\begin{aligned}
W_t &= \int_{-\infty}^{t} \pi \exp(-\pi(t-s)) w_{t,s} ds \\
&= \int_{-\infty}^{t} \pi \exp(-\pi(t-s)) \underbrace{w_{s,s}}_{=\overline{y}P_{\delta,s}} \exp\left(-\frac{\rho}{\gamma}(t-s)\right)\left(\frac{\xi_t}{\xi_s}\right)^{-\frac{1}{\gamma}} \frac{h_t}{h_s} ds
\end{aligned} \tag{A.6}
$$

Since total wealth of all home and foreign households sums to $W_t + W_t^{\star} = \overline{Y}P_{\delta,t}$, the home wealth share is given by

$$
\begin{aligned}
\chi_{W,t} &= \frac{W_t}{\overline{Y}P_{\delta,t}} \\
&= \underbrace{\frac{\pi\overline{y}}{\overline{Y}}}_{=\frac{\delta}{2}} \int_{-\infty}^{t} \exp\left(-\left(\pi + \frac{\rho}{\gamma}\right)(t-s)\right)\left(\frac{\xi_t}{\xi_s}\right)^{-\frac{1}{\gamma}} \frac{h_t}{h_s} \left(\frac{P_{\delta,t}}{P_{\delta,s}}\right)^{-1} ds
\end{aligned} \tag{A.7}
$$

Define $M_t \equiv \xi_t^{-\frac{1}{\gamma}} h_t P_{\delta,t}^{-1}$. We apply Ito's Lemma to determine the dynamics of M_t:

$$
\begin{aligned}
&\text{drift}\left(\frac{dM_t}{M_t}\right) \\
&= -\frac{1}{\gamma}(-r_t) + \overline{\mu}_{h,t} - \overline{\mu}_{P\delta,t} + \frac{1}{2}\left(-\frac{1}{\gamma}\right)\left(-\frac{1}{\gamma} - 1\right)(-\kappa_t)^2 + \frac{1}{2}(-1)(-2)\overline{\sigma}_{P\delta,t}^2 \\
&\quad + \left(-\frac{1}{\gamma}\right)(-\kappa_t)\overline{\sigma}_{h,t} + \left(-\frac{1}{\gamma}\right)(-1)(-\kappa_t)\overline{\sigma}_{P\delta,t} + (-1)\overline{\sigma}_{h,t}\overline{\sigma}_{P\delta,t} \\
&= \frac{1}{\gamma}r_t + \overline{\mu}_{h,t} - \overline{\mu}_{P\delta,t} + \frac{1}{2}\left(\frac{1}{\gamma}\right)\left(\frac{1}{\gamma} + 1\right)\kappa_t^2 + \overline{\sigma}_{P\delta,t}^2 + \frac{1}{\gamma}\kappa_t\left(\overline{\sigma}_{h,t} - \overline{\sigma}_{P\delta,t}\right) - \overline{\sigma}_{h,t}\overline{\sigma}_{P\delta,t} \\
&\equiv \overline{\mu}_{M,t}
\end{aligned} \tag{A.8}
$$

$$
\text{diffusion}\left(\frac{dM_t}{M_t}\right) = -\frac{1}{\gamma}(-\kappa_t) + \overline{\sigma}_{h,t} - \overline{\sigma}_{P\delta,t} = \frac{1}{\gamma}\kappa_t + \overline{\sigma}_{h,t} - \overline{\sigma}_{P\delta,t} \equiv \overline{\sigma}_{M,t} \tag{A.9}
$$

The dynamics of $\chi_{W,t}$ are then given by

$$
d\chi_{W,t} = \left[\left(\overline{\mu}_{M,t} - \pi - \frac{\rho}{\gamma}\right)\chi_{W,t} + \frac{\delta}{2}\right] dt + \chi_{W,t}\overline{\sigma}_{M,t} dB_t \tag{A.10}
$$

I follow a similar procedure to determine the dynamics of aggregate consumption expenditure

$P_{C,t}C_t$ for home country households. Aggregate consumption C_t is given by

$$\begin{aligned} C_t &= \int_{-\infty}^{t} \pi \exp(-\pi(t-s)) c_{t,s} ds \\ &= \int_{-\infty}^{t} \pi \exp(-\pi(t-s)) c_{s,s} \exp\left(-\frac{\rho}{\gamma}(t-s)\right) \left(\frac{\xi_t}{\xi_s}\right)^{-\frac{1}{\gamma}} \left(\frac{P_{C,t}}{P_{C,s}}\right)^{-\frac{1}{\gamma}} ds \end{aligned} \tag{A.11}$$

Total expenditure of home households is

$$P_{C,t}C_t = \int_{-\infty}^{t} \pi c_{s,s} \xi_s^{\frac{1}{\gamma}} P_{C,s}^{\frac{1}{\gamma}} \exp\left(\left(\pi + \frac{\rho}{\gamma}\right) s\right) \exp\left(-\left(\pi + \frac{\rho}{\gamma}\right) t\right) \xi_t^{-\frac{1}{\gamma}} P_{C,t}^{1-\frac{1}{\gamma}} ds \tag{A.12}$$

To determine the dynamics of total consumption expenditure, we need to determine the dynamics of $\xi_t^{-\frac{1}{\gamma}} P_{C,t}^{1-\frac{1}{\gamma}}$:

$$\begin{aligned} &\text{drift}\left(\frac{d\left[\xi_t^{-\frac{1}{\gamma}} P_{C,t}^{1-\frac{1}{\gamma}}\right]}{\xi_t^{-\frac{1}{\gamma}} P_{C,t}^{1-\frac{1}{\gamma}}}\right) \\ &= -\frac{1}{\gamma}(-r_t) + \left(1 - \frac{1}{\gamma}\right) \overline{\mu}_{PC,t} + \frac{1}{2}\left(-\frac{1}{\gamma}\right)\left(-\frac{1}{\gamma} - 1\right)(-\kappa_t)^2 + \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)\left(-\frac{1}{\gamma}\right) \overline{\sigma}_{PC,t}^2 \\ &\quad + \left(-\frac{1}{\gamma}\right)\left(1 - \frac{1}{\gamma}\right)(-\kappa_t)\overline{\sigma}_{PC,t} \\ &= \frac{1}{\gamma} r_t + \left(1 - \frac{1}{\gamma}\right) \overline{\mu}_{PC,t} + \frac{1}{2}\frac{1}{\gamma}\left(\frac{1}{\gamma} + 1\right)\kappa_t^2 - \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)\frac{1}{\gamma}\overline{\sigma}_{PC,t}^2 + \frac{1}{\gamma}\left(1 - \frac{1}{\gamma}\right)\kappa_t \overline{\sigma}_{PC,t} \end{aligned} \tag{A.13}$$

$$\text{diffusion}\left(\frac{d\left[\xi_t^{-\frac{1}{\gamma}} P_{C,t}^{1-\frac{1}{\gamma}}\right]}{\xi_t^{-\frac{1}{\gamma}} P_{C,t}^{1-\frac{1}{\gamma}}}\right) = -\frac{1}{\gamma}(-\kappa_t) + \left(1 - \frac{1}{\gamma}\right)\overline{\sigma}_{PC,t} = \frac{1}{\gamma}\kappa_t + \left(1 - \frac{1}{\gamma}\right)\overline{\sigma}_{PC,t} \tag{A.14}$$

The dynamics of aggregate consumption expenditure $P_{C,t}C_t$ are then given by

$$
\begin{aligned}
&d\left[P_{C,t}C_t\right] \\
&= \left\{ \text{drift}\left(\frac{d\left[\xi_t^{-\frac{1}{\gamma}} P_{C,t}^{1-\frac{1}{\gamma}}\right]}{\xi_t^{-\frac{1}{\gamma}} P_{C,t}^{1-\frac{1}{\gamma}}} - \pi - \frac{\rho}{\gamma} \right) P_{C,t}C_t + \pi P_{C,t}c_{t,t} \right\} dt \\
&\qquad + \text{diffusion}\left(\frac{d\left[\xi_t^{-\frac{1}{\gamma}} P_{C,t}^{1-\frac{1}{\gamma}}\right]}{\xi_t^{-\frac{1}{\gamma}} P_{C,t}^{1-\frac{1}{\gamma}}} \right) P_{C,t}C_t dB_t \\
&= \\
&\left(\frac{1}{\gamma} r_t + \left(1 - \frac{1}{\gamma}\right) \overline{\mu}_{PC,t} + \frac{1}{2}\frac{1}{\gamma}\left(\frac{1}{\gamma} + 1\right)\kappa_t^2 - \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)\frac{1}{\gamma}\overline{\sigma}_{PC,t}^2 + \frac{1}{\gamma}\left(1 - \frac{1}{\gamma}\right)\kappa_t \overline{\sigma}_{PC,t} - \pi - \frac{\rho}{\gamma} \right) P_{C,t}C_t dt \\
&+ \pi P_{C,t}c_{t,t} dt \\
&+ \left(\frac{1}{\gamma}\kappa_t + \left(1 - \frac{1}{\gamma}\right)\overline{\sigma}_{PC,t} \right) P_{C,t}C_t dB_t
\end{aligned} \tag{A.15}
$$

Similarly, in the foreign country, aggregate consumption expenditure $P^{\star}_{C,t}C^{\star}_t$ evolves according to:

$$
\begin{aligned}
&d\left[P^{\star}_{C,t}C^{\star}_t\right] \\
&= \left\{ \text{drift}\left(\frac{d\left[\xi_t^{-\frac{1}{\gamma}} \left(P^{\star}_{C,t}\right)^{1-\frac{1}{\gamma}}\right]}{\xi_t^{-\frac{1}{\gamma}} \left(P^{\star}_{C,t}\right)^{1-\frac{1}{\gamma}}} - \pi - \frac{\rho}{\gamma} \right) P^{\star}_{C,t}C^{\star}_t + \pi P^{\star}_{C,t}c^{\star}_{t,t} \right\} dt \\
&\qquad + \text{diffusion}\left(\frac{d\left[\xi_t^{-\frac{1}{\gamma}} \left(P^{\star}_{C,t}\right)^{1-\frac{1}{\gamma}}\right]}{\xi_t^{-\frac{1}{\gamma}} \left(P^{\star}_{C,t}\right)^{1-\frac{1}{\gamma}}} \right) P^{\star}_{C,t}C^{\star}_t dB_t \\
&= \\
&\left(\frac{1}{\gamma} r_t + \left(1 - \frac{1}{\gamma}\right) \overline{\mu}^{\star}_{PC,t} + \frac{1}{2}\frac{1}{\gamma}\left(\frac{1}{\gamma} + 1\right)\kappa_t^2 - \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)\frac{1}{\gamma}\overline{\sigma}_{PC,t}^{\star 2} + \frac{1}{\gamma}\left(1 - \frac{1}{\gamma}\right)\kappa_t \overline{\sigma}^{\star}_{PC,t} - \pi - \frac{\rho}{\gamma} \right) P^{\star}_{C,t}C^{\star}_t dt \\
&+ \pi P^{\star}_{C,t}c^{\star}_{t,t} dt \\
&+ \left(\frac{1}{\gamma}\kappa_t + \left(1 - \frac{1}{\gamma}\right)\overline{\sigma}^{\star}_{PC,t} \right) P^{\star}_{C,t}C^{\star}_t dB_t
\end{aligned} \tag{A.16}
$$

Step 2. Next, we pin down the price of risk κ_t and the risk-free interest rate r_t from the market clearing condition

$$P_{C,t}C_t + P^{\star}_{C,t}C^{\star}_t = \overline{Y} \tag{A.17}$$

It will be helpful to define home households' share of total (home and foreign) consumption expenditure, which can be recovered from the state variables as follows:

$$\chi_{X,t} \equiv \frac{P_{C,t}C_t}{P_{C,t}C_t + P^\star_{C,t}C^\star_t} = \frac{\chi_{W,t} g_t P_{C,t}^{1-\frac{1}{\gamma}}}{\chi_{W,t} g_t P_{C,t}^{1-\frac{1}{\gamma}} + (1-\chi_{W,t}) g^\star_t \left(P^\star_{C,t}\right)^{1-\frac{1}{\gamma}}} \tag{A.18}$$

We set the diffusion component on the left-hand side of Equation (A.17) to 0 to pin down κ_t:

$$\begin{aligned}
&\text{diffusion}\left(P_{C,t}C_t\right) + \text{diffusion}\left(P^\star_{C,t}C^\star_t\right) \\
&= \left(\frac{1}{\gamma}\kappa_t + \left(1-\frac{1}{\gamma}\right)\overline{\sigma}_{PC,t}\right) P_{C,t}C_t + \left(\frac{1}{\gamma}\kappa_t + \left(1-\frac{1}{\gamma}\right)\overline{\sigma}^\star_{PC,t}\right) P^\star_{C,t}C^\star_t \\
&= \left(P_{C,t}C_t + P^\star_{C,t}C^\star_t\right)\left[\frac{1}{\gamma}\kappa_t + \left(1-\frac{1}{\gamma}\right)\left(\chi_{X,t}\overline{\sigma}_{PC,t} + (1-\chi_{X,t})\overline{\sigma}^\star_{PC,t}\right)\right] \\
&= 0 \\
\Leftrightarrow\ & \kappa_t = -(\gamma-1)\left[\chi_{X,t}\overline{\sigma}_{PC,t} + (1-\chi_{X,t})\overline{\sigma}^\star_{PC,t}\right]
\end{aligned} \tag{A.19}$$

Similarly, we set the drift component to 0 to pin down r_t:

$$\begin{aligned}
&\text{drift}\left(P_{C,t}C_t\right) + \text{drift}\left(P^\star_{C,t}C^\star_t\right) \\
&= \\
&\left(\frac{1}{\gamma}r_t + \left(1-\frac{1}{\gamma}\right)\overline{\mu}_{PC,t} + \frac{1}{2}\frac{1}{\gamma}\left(\frac{1}{\gamma}+1\right)\kappa_t^2 - \frac{1}{2}\left(1-\frac{1}{\gamma}\right)\frac{1}{\gamma}\overline{\sigma}^2_{PC,t} + \frac{1}{\gamma}\left(1-\frac{1}{\gamma}\right)\kappa_t\overline{\sigma}_{PC,t} - \pi - \frac{\rho}{\gamma}\right) P_{C,t}C_t \\
&+ \left(\frac{1}{\gamma}r_t + \left(1-\frac{1}{\gamma}\right)\overline{\mu}^\star_{PC,t} + \frac{1}{2}\frac{1}{\gamma}\left(\frac{1}{\gamma}+1\right)\kappa_t^2 - \frac{1}{2}\left(1-\frac{1}{\gamma}\right)\frac{1}{\gamma}\overline{\sigma}^{\star 2}_{PC,t} + \frac{1}{\gamma}\left(1-\frac{1}{\gamma}\right)\kappa_t\overline{\sigma}^\star_{PC,t} - \pi - \frac{\rho}{\gamma}\right) P^\star_{C,t}C^\star_t \\
&+ \pi\left(P_{C,t}c_{t,t} + P^\star_{C,t}c^\star_{t,t}\right) \\
&= \left(P_{C,t}C_t + P^\star_{C,t}C^\star_t\right)\left[\frac{1}{\gamma}r_t + \frac{1}{2}\frac{1}{\gamma}\left(\frac{1}{\gamma}+1\right)\kappa_t^2 - \pi - \frac{\rho}{\gamma} + \left(1-\frac{1}{\gamma}\right)\left(\chi_{X,t}\overline{\mu}_{PC,t} + (1-\chi_{X,t})\overline{\mu}^\star_{PC,t}\right)\right. \\
&\qquad + \frac{1}{\gamma}\left(1-\frac{1}{\gamma}\right)\kappa_t\left(\chi_{X,t}\overline{\sigma}_{PC,t} + (1-\chi_{X,t})\overline{\sigma}^\star_{PC,t}\right) \\
&\qquad \left. - \frac{1}{2}\left(1-\frac{1}{\gamma}\right)\frac{1}{\gamma}\left(\chi_{X,t}\overline{\sigma}^2_{PC,t} + (1-\chi_{X,t})\left(\overline{\sigma}^\star_{PC,t}\right)^2\right)\right] \\
&\quad + \pi\left(P_{C,t}c_{t,t} + P^\star_{C,t}c^\star_{t,t}\right) \\
&= 0
\end{aligned} \tag{A.20}$$

In deriving the equation for r_t, it will help to have an expression for the ratio of total consumption expenditure by *new* households at time t (the last term above) to total consumption of all home households $P_{C,t}C_t + P^\star_{C,t}C^\star_t = \overline{Y}$. Using the fact that home and foreign households start with the same initial wealth $w_{t,t} = w^\star_{t,t} = \overline{y} P_{\delta,t}$, we have

$$\begin{aligned}
\pi \frac{P_{C,t} c_{t,t} + P^\star_{C,t} c^\star_{t,t}}{\overline{Y}} &= \pi \frac{P_{C,t}^{1-\frac{1}{\gamma}} g_t w_{t,t} + \left(P^\star_{C,t}\right)^{1-\frac{1}{\gamma}} g^\star_t w^\star_{t,t}}{\overline{Y}} \\
&= \underbrace{\pi \frac{\overline{y}}{\overline{Y}}}_{=\frac{\delta}{2}} P_{\delta,t} \left(P_{C,t}^{1-\frac{1}{\gamma}} g_t + \left(P^\star_{C,t}\right)^{1-\frac{1}{\gamma}} g^\star_t \right) \\
&= \frac{\delta}{2} P_{\delta,t} \left(P_{C,t}^{1-\frac{1}{\gamma}} g_t + \left(P^\star_{C,t}\right)^{1-\frac{1}{\gamma}} g^\star_t \right)
\end{aligned} \tag{A.21}$$

We can then obtain an expression for r_t:

$$\begin{aligned}
r_t = \rho + \gamma\pi - \frac{1}{2}\left(\frac{1}{\gamma} + 1\right)\kappa_t^2 \\
- (\gamma - 1)\left(\chi_{X,t}\overline{\mu}_{PC,t} + (1-\chi_{X,t})\overline{\mu}^\star_{PC,t}\right) \\
- \left(1 - \frac{1}{\gamma}\right)\kappa_t\left(\chi_{X,t}\overline{\sigma}_{PC,t} + (1-\chi_{X,t})\overline{\sigma}^\star_{PC,t}\right) \\
+ \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)\left(\chi_{X,t}\overline{\sigma}^2_{PC,t} + (1-\chi_{X,t})\left(\overline{\sigma}^\star_{PC,t}\right)^2\right) \\
- \gamma\frac{\delta}{2} P_{\delta,t}\left(P_{C,t}^{1-\frac{1}{\gamma}} g_t + \left(P^\star_{C,t}\right)^{1-\frac{1}{\gamma}} g^\star_t\right)
\end{aligned} \tag{A.22}$$

Additionally, when solving for $\mu_{\chi W,t}$ it will be helpful to have the following expression:

$$\begin{aligned}
&\frac{1}{\gamma} r_t + \frac{1}{2}\frac{1}{\gamma}\left(\frac{1}{\gamma} + 1\right)\kappa_t^2 \\
&= \pi + \frac{\rho}{\gamma} - \left(1 - \frac{1}{\gamma}\right)\left(\chi_{X,t}\overline{\mu}_{PC,t} + (1-\chi_{X,t})\overline{\mu}^\star_{PC,t}\right) \\
&\quad - \frac{1}{\gamma}\left(1 - \frac{1}{\gamma}\right)\kappa_t\left(\chi_{X,t}\overline{\sigma}_{PC,t} + (1-\chi_{X,t})\overline{\sigma}^\star_{PC,t}\right) \\
&\quad + \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)\frac{1}{\gamma}\left(\chi_{X,t}\overline{\sigma}^2_{PC,t} + (1-\chi_{X,t})\left(\overline{\sigma}^\star_{PC,t}\right)^2\right) \\
&\quad - \frac{\delta}{2} P_{\delta,t}\left(P_{C,t}^{1-\frac{1}{\gamma}} g_t + \left(P^\star_{C,t}\right)^{1-\frac{1}{\gamma}} g^\star_t\right)
\end{aligned} \tag{A.23}$$

Step 3. Next, we solve for $\sigma_{\chi W,t}$, the diffusion of the endogenous state variable $\chi_{W,t}$. We substitute our expression for the price of risk κ_t in Equation (A.19) into Equation (A.10):

$$\begin{aligned}\sigma_{\chi W,t} &= \chi_{W,t}\overline{\sigma}_{M,t} \\ &= \chi_{W,t}\left(\frac{1}{\gamma}\kappa_t + \overline{\sigma}_{h,t} - \overline{\sigma}_{P\delta,t}\right) \\ &= \chi_{W,t}\left(-\left(1-\frac{1}{\gamma}\right)\left[\chi_{X,t}\overline{\sigma}_{PC,t} + (1-\chi_{X,t})\overline{\sigma}^{\star}_{PC,t}\right] + \overline{\sigma}_{h,t} - \overline{\sigma}_{P\delta,t}\right)\end{aligned} \tag{A.24}$$

Since the diffusion loadings of price index changes $dP_{C,t}/P_{C,t}$ and $dP^{\star}_{C,t}/P^{\star}_{C,t}$ satisfy $\overline{\sigma}_{PC} = -\overline{\sigma}^{\star}_{PC}$, we have

$$\chi_{X,t}\overline{\sigma}_{PC,t} + (1-\chi_{X,t})\overline{\sigma}^{\star}_{PC,t} = (2\chi_{X,t}-1)\,\overline{\sigma}_{PC,t} \tag{A.25}$$

By substituting these diffusion terms into Equation (A.24), we obtain

$$\begin{aligned}\sigma_{\chi W,t} = {} & \chi_{W,t}\left(\frac{1}{h_t}\frac{\partial h}{\partial \chi_W} - \frac{1}{P_{\delta,t}}\frac{\partial P_\delta}{\partial \chi_W}\right)\sigma_{\chi W,t} \\ & + \chi_{W,t}\left(-\left(1-\frac{1}{\gamma}\right)(2\chi_{X,t}-1) + \frac{1}{h_t}\frac{\partial h}{\partial \widehat{P}_C} - \frac{1}{P_{\delta,t}}\frac{\partial P_\delta}{\partial \widehat{P}_C}\right)\sigma_{PC}\end{aligned} \tag{A.26}$$

Solving explicitly for $\sigma_{\chi W,t}$, we obtain

$$\sigma_{\chi W,t} = \frac{\chi_{W,t}\left(-\left(1-\frac{1}{\gamma}\right)(2\chi_{X,t}-1) + \frac{1}{h_t}\frac{\partial h}{\partial \widehat{P}_C} - \frac{1}{P_{\delta,t}}\frac{\partial P_\delta}{\partial \widehat{P}_C}\right)\sigma_{PC}}{1-\chi_{W,t}\left(\frac{1}{h_t}\frac{\partial h}{\partial \chi_W} - \frac{1}{P_{\delta,t}}\frac{\partial P_\delta}{\partial \chi_W}\right)} \tag{A.27}$$

By plugging this expression for $\sigma_{\chi W,t}$ into the expressions in (A.3), we can determine the diffusion loadings and price of risk κ_t.

Next, we solve for $\mu_{\chi W,t}$, the drift of home households' wealth share. Starting from Equation (A.8) and substituting based on Equation (A.23), we have

$$\begin{aligned}\overline{\mu}_{M,t} = {} & \frac{1}{\gamma}r_t + \frac{1}{2}\left(\frac{1}{\gamma}\right)\left(\frac{1}{\gamma}+1\right)\kappa_t^2 + \overline{\mu}_{h,t} - \overline{\mu}_{P\delta,t} + \overline{\sigma}^2_{P\delta,t} + \frac{1}{\gamma}\kappa_t\left(\overline{\sigma}_{h,t} - \overline{\sigma}_{P\delta,t}\right) - \overline{\sigma}_{h,t}\overline{\sigma}_{P\delta,t} \\ = {} & \pi + \frac{\rho}{\gamma} - \left(1-\frac{1}{\gamma}\right)\left(\chi_{X,t}\overline{\mu}_{PC,t} + (1-\chi_{X,t})\overline{\mu}^{\star}_{PC,t}\right) - \frac{1}{\gamma}\left(1-\frac{1}{\gamma}\right)\kappa_t\left(\chi_{X,t}\overline{\sigma}_{PC,t} + (1-\chi_{X,t})\overline{\sigma}^{\star}_{PC,t}\right) \\ & + \frac{1}{2}\left(1-\frac{1}{\gamma}\right)\frac{1}{\gamma}\left(\chi_{X,t}\overline{\sigma}^2_{PC,t} + (1-\chi_{X,t})\left(\overline{\sigma}^{\star}_{PC,t}\right)^2\right) - \frac{\delta}{2}P_{\delta,t}\left(P_{C,t}^{1-\frac{1}{\gamma}}g_t + \left(P^{\star}_{C,t}\right)^{1-\frac{1}{\gamma}}g^{\star}_t\right) \\ & + \overline{\mu}_{h,t} - \overline{\mu}_{P\delta,t} + \overline{\sigma}^2_{P\delta,t} + \frac{1}{\gamma}\kappa_t\left(\overline{\sigma}_{h,t} - \overline{\sigma}_{P\delta,t}\right) - \overline{\sigma}_{h,t}\overline{\sigma}_{P\delta,t}\end{aligned} \tag{A.28}$$

By substituting the expressions (A.2) for the drifts into Equation (A.28), we obtain

$$
\begin{aligned}
\overline{\mu}_{M,t} = & \left[\frac{1}{h_t}\frac{\partial h}{\partial \chi_W} - \frac{1}{P_{\delta,t}}\frac{\partial P_\delta}{\partial \chi_W}\right]\mu_{\chi W,t} + \pi + \frac{\rho}{\gamma} \\
& - \left(1-\frac{1}{\gamma}\right)\left(\chi_{X,t}\overline{\mu}_{PC,t} + (1-\chi_{X,t})\overline{\mu}^{\star}_{PC,t}\right) - \frac{1}{\gamma}\left(1-\frac{1}{\gamma}\right)\kappa_t\left(\chi_{X,t}\overline{\sigma}_{PC,t} + (1-\chi_{X,t})\overline{\sigma}^{\star}_{PC,t}\right) \\
& + \frac{1}{2}\left(1-\frac{1}{\gamma}\right)\frac{1}{\gamma}\left(\chi_{X,t}\overline{\sigma}^2_{PC,t} + (1-\chi_{X,t})\left(\overline{\sigma}^{\star}_{PC,t}\right)^2\right) - \frac{\delta}{2}P_{\delta,t}\left(P_{C,t}^{1-\frac{1}{\gamma}}g_t + \left(P^{\star}_{C,t}\right)^{1-\frac{1}{\gamma}}g^{\star}_t\right) \\
& + \widetilde{\mu}_{h,t} - \widetilde{\mu}_{P\delta,t} + \overline{\sigma}^2_{P\delta,t} + \frac{1}{\gamma}\kappa_t\left(\overline{\sigma}_{h,t} - \overline{\sigma}_{P\delta,t}\right) - \overline{\sigma}_{h,t}\overline{\sigma}_{P\delta,t} \\
\equiv & \left[\frac{1}{h_t}\frac{\partial h}{\partial \chi_W} - \frac{1}{P_{\delta,t}}\frac{\partial P_\delta}{\partial \chi_W}\right]\mu_{\chi W,t} + \pi + \frac{\rho}{\gamma} + \widetilde{\mu}_{M,t}
\end{aligned}
\tag{A.29}
$$

where $\widetilde{\mu}_{M,t}$ is defined so that terms cancel in the law of motion below. By substituting this into the drift component of Equation (A.10) describing the law of motion for $\chi_{W,t}$, we obtain

$$
\begin{aligned}
& \mu_{\chi W,t} \\
& = \left(\overline{\mu}_{M,t} - \pi - \frac{\rho}{\gamma}\right)\chi_{W,t} + \frac{\delta}{2} \\
& = \chi_{W,t}\left[\frac{1}{h_t}\frac{\partial h}{\partial \chi_W} - \frac{1}{P_{\delta,t}}\frac{\partial P_\delta}{\partial \chi_W}\right]\mu_{\chi W,t} + \chi_{W,t}\widetilde{\mu}_{M,t} + \frac{\delta}{2}
\end{aligned}
\tag{A.30}
$$

Solving for $\mu_{\chi W,t}$, we obtain

$$
\mu_{\chi W,t} = \frac{\chi_{W,t}\widetilde{\mu}_{M,t} + \frac{\delta}{2}}{1 - \chi_{W,t}\left[\frac{1}{h_t}\frac{\partial h}{\partial \chi_W} - \frac{1}{P_{\delta,t}}\frac{\partial P_\delta}{\partial \chi_W}\right]}
\tag{A.31}
$$

Step 4. Finally, we derive the partial differential equations characterizing the functions $h(\chi_{W,t}, \widehat{P}_{C,t})$, $h^{\star}(\chi_{W,t}, \widehat{P}_{C,t})$, and $P_\delta(\chi_{W,t}, \widehat{P}_{C,t})$. First, we write the time t budget constraint of a household born at time $s \leq t$, with current wealth $w_{t,s}$:

$$
w_{t,s} = E_t\left[\int_t^\infty \frac{\xi_u}{\xi_t}\exp(-\pi(u-t))P_{C,u}c_{u,s}du\right]
\tag{A.32}
$$

where $c_{u,s}$ denotes the household's consumption in period $u \geq t$ (conditional on survival). Recall that optimal consumption growth is given in Equation (A.4) by

$$
\frac{c_{u,s}}{c_{t,s}} = \exp\left(-\frac{\rho}{\gamma}(u-t)\right)\left(\frac{\xi_u}{\xi_t}\right)^{-\frac{1}{\gamma}}\left(\frac{P_{C,u}}{P_{C,t}}\right)^{-\frac{1}{\gamma}}
\tag{A.33}
$$

We then substitute using this expression to obtain

$$\begin{aligned}\frac{w_{t,s}}{c_{t,s}} &= E_t\left[\int_t^{\infty} \exp\left(-\pi(u-t)\right)\frac{\xi_u}{\xi_t} P_{C,u}\frac{c_{u,s}}{c_{t,s}} du\right] \\ &= E_t\left[\int_t^{\infty} \exp\left(-\left(\pi+\frac{\rho}{\gamma}\right)(u-t)\right)\left(\frac{\xi_u}{\xi_t}\right)^{1-\frac{1}{\gamma}} \frac{P_{C,u}^{1-\frac{1}{\gamma}}}{P_{C,t}^{-\frac{1}{\gamma}}} du\right]\end{aligned} \tag{A.34}$$

Rearranging terms, we obtain

$$\begin{aligned}\exp\left(-\left(\pi+\frac{\rho}{\gamma}\right)t\right)\underbrace{\frac{w_{t,s}}{c_{t,s}}P_{C,t}^{-\frac{1}{\gamma}}}_{=\frac{1}{g_t}=h_t}\xi_t^{1-\frac{1}{\gamma}} &= \exp\left(-\left(\pi+\frac{\rho}{\gamma}\right)t\right)h_t\xi_t^{1-\frac{1}{\gamma}} \\ &= E_t\left[\int_t^{\infty}\exp\left(-\left(\pi+\frac{\rho}{\gamma}\right)u\right)\xi_u^{1-\frac{1}{\gamma}}P_{C,u}^{1-\frac{1}{\gamma}}du\right]\end{aligned} \tag{A.35}$$

Applying Ito's lemma on both sides, we obtain

$$\begin{aligned}&\text{drift}\left[\exp\left(-\left(\pi+\frac{\rho}{\gamma}\right)t\right)h_t\xi_t^{1-\frac{1}{\gamma}}\right] \\ &= \exp\left(-\left(\pi+\frac{\rho}{\gamma}\right)t\right)h_t\xi_t^{1-\frac{1}{\gamma}}\left[-\pi-\frac{\rho}{\gamma}+\overline{\mu}_{h,t}-\left(1-\frac{1}{\gamma}\right)r_t-\frac{1}{2}\left(1-\frac{1}{\gamma}\right)\frac{1}{\gamma}\kappa_t^2-\overline{\sigma}_{h,t}\left(1-\frac{1}{\gamma}\right)\kappa_t\right] \\ &= -\exp\left(-\left(\pi+\frac{\rho}{\gamma}\right)t\right)\xi_t^{1-\frac{1}{\gamma}}P_{C,t}^{1-\frac{1}{\gamma}}\end{aligned} \tag{A.36}$$

and after cancelling terms on both sides:

$$h_t\left[-\pi-\frac{\rho}{\gamma}+\overline{\mu}_{h,t}-\left(1-\frac{1}{\gamma}\right)r_t-\frac{1}{2}\left(1-\frac{1}{\gamma}\right)\frac{1}{\gamma}\kappa_t^2-\overline{\sigma}_{h,t}\left(1-\frac{1}{\gamma}\right)\kappa_t\right] = -P_{C,t}^{1-\frac{1}{\gamma}} \tag{A.37}$$

Since $\overline{\mu}_{h,t}$ and $\overline{\sigma}_{h,t}$ depend on the partial derivatives of the function h, this gives a partial differential equation characterizing h. For the foreign country, we obtain a similar equation:

$$h_t^{\star}\left[-\pi-\frac{\rho}{\gamma}+\overline{\mu}_{h,t}^{\star}-\left(1-\frac{1}{\gamma}\right)r_t-\frac{1}{2}\left(1-\frac{1}{\gamma}\right)\frac{1}{\gamma}\kappa_t^2-\overline{\sigma}_{h,t}^{\star}\left(1-\frac{1}{\gamma}\right)\kappa_t\right] = -\left(P_{C,t}^{\star}\right)^{1-\frac{1}{\gamma}} \tag{A.38}$$

The annuity price is given by

$$\xi_t P_{\delta,t} = E_t\left[\int_t^{+\infty}\exp(-\delta(u-t))\xi_u du\right] = \exp(\delta t)E_t\left[\int_t^{+\infty}\exp(-\delta u)\xi_u du\right] \tag{A.39}$$

By equating drift terms on both sides of this equation, we obtain

$$\begin{aligned}
\text{drift}(\xi_t P_{\delta,t}) &= \xi_t P_{\delta,t}[\overline{\mu}_{P\delta,t} - r_t - \kappa_t \overline{\sigma}_{P\delta,t}] \\
&= \delta \exp(\delta t) E_t \left[\int_t^{+\infty} \exp(-\delta u) \xi_u du \right] - \xi_t \\
&= \xi_t P_{\delta,t} \left(\delta - \frac{1}{P_{\delta,t}} \right)
\end{aligned} \tag{A.40}$$

or rewritten in simpler pricing equation form:

$$\underbrace{\overline{\mu}_{P\delta,t} - \delta}_{\text{“capital gain”}} + \underbrace{\frac{1}{P_{\delta,t}}}_{\text{“dividend yield”}} - r_t = \kappa_t \overline{\sigma}_{P\delta,t} \tag{A.41}$$