UNIQUE CONTINUATION THEOREMS for the DIRAC OPERATOR and the LAPLACE OPERATOR

by

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Unique Continuation Theorems for the Dirac Operator and the Laplace Operator

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ABSTRACT

Let u be a solution of the following differential operator L in some non-empty connected open subset in R^n:

$$L = \Delta + \sum a_j \partial/\partial x_j + b, \quad where$$

$a_j \in L^r_{loc}(R^n), b \in L^s_{loc}(R^n)$ for some suitable r, s. L is said to satisfy the strong unique continuation property if solutions to Lu=0 that vanish at a point to infinite order, vanish identically. One way of proving this property is to prove Carleman type inequalities. The first inequality we are going to consider is the following.

$$||e^{t\phi}\nabla f||_{L^2(U\setminus\{0\},dx)} \leq C||e^{t\phi}\Delta f||_{L^p(U\setminus\{0\},dx)} \quad for \ all \ \ f \in C_0^\infty(U\setminus\{0\};C^m),$$

where ϕ is a suitable weight function, which is a radially decreasing convex function. And U is an open neighborhood of the origin. The second type of inequality we are going to consider is about the Laplacian.

$$||e^{t\phi}u||_{L^q(U\setminus\{0\},dx)} \leq C||e^{t\phi}\Delta f||_{L^p(U\setminus\{0\},dx)} \quad for \ all \ \ f \in C_0^\infty(U\setminus\{0\},C^m)$$

We are proving this for one special p.

Introduction

We are going to consider unique continuation theorems for the following differential operator L:

$$L = \Delta + \sum a_j \partial/\partial x_j + b \tag{1}$$

or

$$L = D + V \tag{1'}$$

Here Δ is the Laplace operator and D is the Dirac operator, $a_j \in L^r_{loc}(R^n), b \in L^s_{loc}(R^n)$ for some suitable r, s. V is a $m \times m$ matrix- valued potential with entries acting on functions with values in C^m

L is said to satisfy the strong unique continuation property if solutions to Lu =0 that vanish at a point to infinite order, vanish identically.

On the other hand, L is said to satisfy the unique continuation property if solutions to Lu =0 that vanish in a non-empty, open subset of a connected set, vanish identically.

Also L is said to satisfy the weak unique continuation property if solutions to Lu = 0 that are compactly supported vanish identically. One way of proving this property is to prove Carleman type estimates.

When n = 2, $a_j = 0$, for all j, and b is bounded, Carleman [8] proved a unique continuation theorem, and all subsequent work follows his basic idea.

The first inequality we are going to consider is the following.

$$||e^{t\phi} f||_{L^2(U,dx)} \leq C||e^{t\phi} Df||_{L^p(U,dx)} \quad for\ all\ f \in C_0^\infty(U; C^m) \tag{2}$$

for U an open neighborhood of the origin, where ϕ is a suitable weight function.

For instance, weight function $\phi(x_n)$ implies weak unique continuation. If ϕ peaks on a hypersurface, we get unique continuation. On the otherhand, if ϕ peaks at a single point we get strong unique continuation.

Then (2) implies an inequality concerns to the gradient.

$$||e^{t\phi} \nabla f||_{L^2(U,dx)} \leq C||e^{t\phi} \Delta f||_{L^p(U,dx)} \quad for\ all\ f \in C_0^\infty(U; C^m) \tag{2'}$$

The second type of inequality we are going to consider is about the Laplacian.

$$||e^{t\phi} u||_{L^q(U,dx)} \leq C||e^{t\phi} \Delta u||_{L^p(U,dx)} \tag{3}$$

The above inequality (2) implies strong unique continuation result for L= D + V;

Let Ω be a connected, oped subset of R^n.

If $V \in L^\gamma(\Omega; M(m, C))$ and u satisfies (D +V)u =0 in Ω, $Du \in L^2(\Omega; C^m)$.

If u vanishes in a non-empty open subset of Ω, then $u \equiv 0$ in Ω. On the other hand the two inequalities (2'), (3) imply strong unique continuation result for the Laplacian, which put together first and zero order perturbations;

Our main result is the following

Theorem
Let $0 \in U \subset R^n$ be an connected open set, $u \in H^{2,p}_{loc}$ for some p, $a_j \in L^r_{loc}(R^n)$, *for all j*, $b \in L^s_{loc}(R^n)$,
$r = (3n-2)/2$, $s > n/2$ and u satisfies the equation

$$\Delta u(x) + \sum a_j(x)\partial u/\partial x_j + b(x)u(x) = 0$$

Then if u vanishes to infinite order at the origin, u must be zero on U.

Historically, (2') was proved by Hörmander [1] with a C^∞ weight function, peaking on a smooth hypersurface. This gives unique continuation not strong unique continuation.
Our main contribution is to extend his result to strong unique continuation. This question is explicitly posed in [1](p.).

Later, Jerison [2] proved the best possible Carleman-type inequalities for (2) and obtained unique continuation property for L=D +V, $V \in L^\gamma_{loc}(R^n)$ $for\ \gamma = \frac{3n-2}{2}$.
Also Jerison and Kenig [3] proved strong unique continuation for solution of (1) with $a_j = 0$, $b \in L^{n/2}$. On the other hand, when p, q are not dual exponents, Barcelo, Ruiz, Kenig, Sogge [4] proved weak unique continuation with exactly the hypothesis on the coefficients in our theorem.

Now we want to make some comments on the weight function.
First, for the weak unique continuation, it is enough to have ϕ as a function of one variable e.g.
$\phi(x) = \phi(x_n) = x_n + x_n^2/2$, $x = (x', x_n) \in R^n$ and the solution u (Lu =0) vanishes on the upper half space i.e. on $x_n > 0$. Then $\phi(x_n)$ has peaks on the hyperplane $x_n = 0$.
For the unique continuation, it is enough to have a radial function i.e. $\phi(r)$ for $0 < a < b$ and u has support on a compact interval i.e. $u \in C_0^\infty([a, b])$. Then ϕ has peaks on some sphere.
Finally, for the strong unique continuation, ϕ is also a radial function and support of u is non-compact i.e.$u \in C_0^\infty(0 < r < b)$ and as $r \to 0$, ϕ is allowed to blow up at a single point.

Statements of Results

Theorem1
Let $n \geq 3$, let $\phi(x)$ be a smooth real valued convex radial function, which does not vanish identically.

There is a constant C depending only on n such that for all $t \in R$, and for all $h \in C_0^\infty((-\infty, 0) \times S)$

$$t||e^{t\phi}h||_{L^2((-\infty,0)\times S, dydw)} \leq C||e^{t\phi}Dh||_{L^2((-\infty,0)\times S, dydw)} \quad (2)$$

Corollary 1
Let $U \ni 0$ be a connected, open subset of R^n.
Suppose we have a solution of a Schrödinger operator (D+V)u=0 in U, $V \in L^\infty$ and $\int_{|x|<\epsilon} |u(x)|^2 dx = 0(\epsilon^N)$ for any N, then $u \equiv 0$ on U.
Theorem 2
Let $n \geq 3$, $\phi(x)$ same as before.

p =(6n-4)/(3n+2), i.e., $1/p - 1/2 = 1/\gamma$, with $\gamma = (3n-2)/2$
There is a constant C depending only on n such that for all $t \in R$

$$||e^{t\phi} f||_{L^2((-\infty,0)\times S, dy d\omega)} \le C||e^{t\phi} Df||_{L^p((-\infty,0)\times S, dy d\omega)} \ for \ all \ f \in C_0^\infty((-\infty,0)\times S) \quad (2')$$

Corollary 2
Let Ω be a connected, open subset of R^n, $n \ge 3$.
If $V \in L^\gamma(\Omega; M(m,C))$ and u satisfies $Du \in L^2(\Omega; C^m)$, $(D+V)u = 0$ *in* Ω.
If u vanishes in a non-empty open subset of Ω, then u is identically zero in Ω.
Corollary 2'
Let the conditions be the same as above. The fact that $D^2 = -\Delta$ implies the following inequality.

$$||e^{t\phi}\nabla f||_{L^2((-\infty,0)\times S, dy d\omega)} \le C||e^{t\phi}\Delta f||_{L^p((-\infty,0)\times S, dy d\omega)} \quad (2'')$$

Theorem 3
Let $n \ge 3$ and $\phi(x)$ as before.
Then there is a constant C depending only on n such that

$$||e^{t\phi} u||_{L^q((-\infty,0)\times S, dy d\omega)} \le C||e^{t\phi}\Delta u||_{L^p((-\infty,0)\times S, dy d\omega)} \ for \ all \ f \in C_0^\infty(R^n \setminus 0)$$

and for (1/q, 1/p) in the open triangle ABC (fig.1) with vertices A(1/2, 1/2), $B(n/(2n-2), 1/q_b), C((n^2+2n-4)/(2n-2), 1/q_c)$ *where*

$$\frac{n}{2(n-1)} - \frac{1}{q_b} = \frac{n^2+2n+4}{2(n-1)} - \frac{1}{q_c} = \frac{2}{n}.$$

Corollary 3
Let $0 \in U \subset R^n$ be an open set, $u \in H^{2,p}_{loc}(U)$, p=(6n-4)/(3n+2) , u satisfies the equation

$$\Delta u(x) + v(x)\nabla u(x) + w(x)u(x) = 0 \quad (3')$$

where $v \in L^r_{loc}(R^n)$, $w \in L^s_{loc}(R^n)$, for r=(3n-2)/2, $s > n/2$ and

$$\int_{\frac{r}{2}<|x|<r} |u(x)|^p dx = 0(r^N) \ for \ any \ N,$$

$$\int_{\frac{r}{2}<|x|<r} |\nabla u(x)|^p dx = 0(r^N) \ for \ any \ N, \quad then \ u \equiv 0 \ in \ U.$$

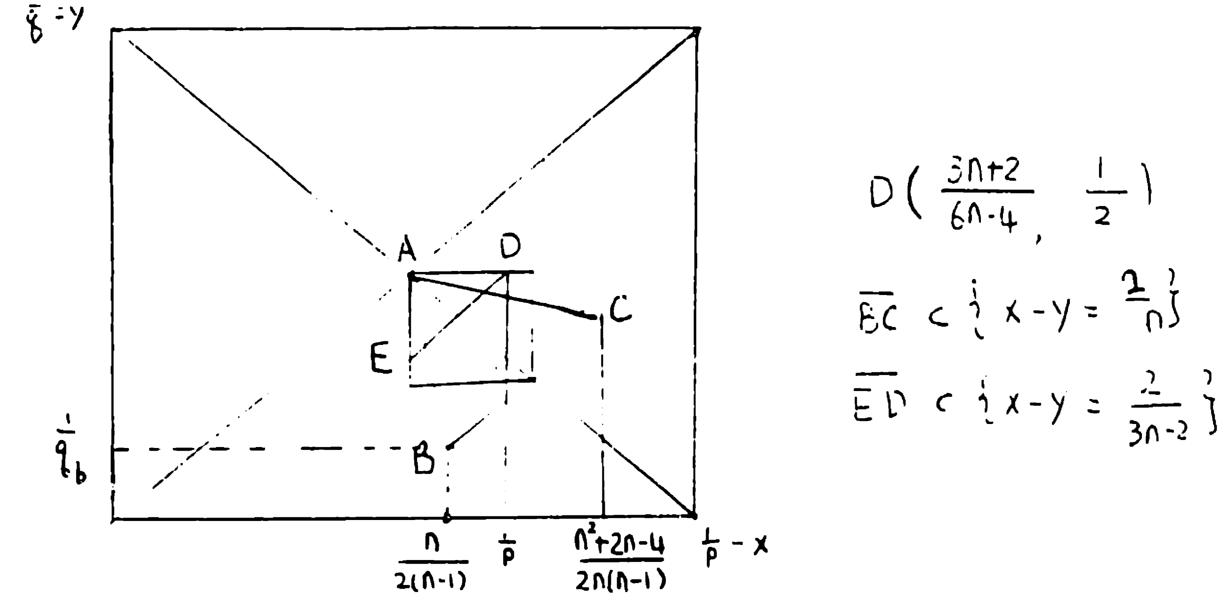

Figure 1

We will first prove these inequalities on $I \times S$, I is some fixed interval in R^-. Later on we can extend the results to $(-\infty, 0) \times S$ using partition of unities. The key ingredient is a theorem of C.Sogge [5] giving L^p to L^q estimates for the projection operator onto spherical harmonics of a given degree. Jerison [3] used a discrete restriction theorem for the Fourier transform and we will follow same methods in this paper. And for the choice of weight function we will adopt the same one which Baouendi and Alinhac [5] used first, and Hörmander [1] used later for the convexity reason.

First, we want to set up some notations and elementary results, following Jerison [2].

Polar coordinates

Let S denote the unit sphere in R^n.
For $y \in R$, and $w \in S, x = e^y w$ gives polar coordinates on R^n, i.e., $y = log|x|$ and$w = x/|x|$.
The operator
$L = \sum_{j<k} \alpha_j \alpha_k (x_j \partial/\partial x_k - x_k \partial/\partial x_j)$ acts only in the w-variables-$[L, \partial/\partial y] = 0$.
We will view L as an operator on the sphere S. Let

$$\hat{\alpha} = \sum_{j=1}^{n} \alpha_j x_j/|x|, \quad then$$

$$\hat{\alpha} D = e^{-y}(\partial/\partial y - L);$$

and since $\hat{\alpha}^2 = -1$,

$$e^y D = \hat{\alpha}(\partial/\partial y - L) \tag{4}$$

Note that $\hat{\alpha}$ is unitary and $L^* = L$. If we recall that

$$\Delta = e^{-2y}(\partial^2/\partial y^2 + (n-2)\partial/\partial y + \Delta_S), \tag{5}$$

where Δ_S denotes the Laplace-Beltrami operator of the sphere.
It follows from $D^* = D, \quad D^2 = -\Delta$ that

$$L(L+n-2) = -\Delta_S \tag{6}$$

Since $\phi(x) \sim -log|x| = -y$, we want to give a rigorous definition i.e.,
Define $\psi \in C^\infty(R)$ implicitly as a new weight function given by Hörmander as $y = -\psi(y) + e^{-\epsilon\psi(y)}$.

In general if $\psi \in C^\infty(R)$, then (4) implies that in polar coordinates $x = e^y w$,

$$e^{t\psi(y)} e^y D e^{-t\psi(y)} h = \hat{\alpha} A_t h \tag{7}$$

where $A_t = \partial/\partial y - (t\psi'(y) + L)$.

Proof of Theorem 1

To prove the inequality, it suffices to compute $A_t^* A_t$ and show $A_t^* A_t \geq ct\psi''(y)$. Since this means,

$$(A_t^* A_t h, h) \geq (t\psi''(y)h, h) = t||\sqrt{\psi''(y)}h||^2_{L^2(dx)}$$

and

$$(A_t^* A_t h, h) = ||A_t h||^2_{L^2(dx)}.$$

Combining these two implies that theorem 1 holds. But

$$A_t^* A_t = (-\partial/\partial y - n - (t\psi'(y) + L))(\partial/\partial y - (t\psi'(y) + L))$$

$$= -(\partial/\partial y + n/2)^2 + (t\psi'(y) + L + n/2)^2 + t\psi''(y) \geq t\psi''(y).$$

We had the relation $y = -\psi(y) + e^{-\epsilon\psi(y)}$. From this, we get

$$\psi'(y) = -1/1 + \epsilon e^{-\epsilon\psi(y)} < 0,$$

which means $\psi(y)$ is a decreasing function of y. We also find

$$\psi''(y) = \epsilon^2 e^{-\epsilon\psi(y)}/(1 + \epsilon e^{-\epsilon\psi(y)})^3 \geq e^{\epsilon y}$$

So the claim is true. Since $\psi(y)$ goes to ∞ as y goes to $-\infty$, $\psi''(y)$ vanishes as y goes to $-\infty$.
Now we want to show how such an inequality implies unique continuation property.

Proof of Corollary 1
First, introduce a cut off function $\eta \in C_0^\infty(R)$ such that $\eta = 1$ for $(-\infty, -T_0)$ for sufficiently large $T_0 \in R$, *and* $\eta = 0$ *for* $y > 0$.
Then by limiting process, (2) is equivalent to

$$t\|e^{t\psi}(\eta u)\|^2_{L^2((-\infty,0)\times S,dx)} \le C\|e^{t\psi}D(\eta u)\|^2_{L^2((-\infty,0)\times S,dx)} \quad since$$

$$D(\eta u) = (D\eta)u + \eta(Du) = (D\eta)u + \eta(-Vu),$$

this means that the right hand side

$$\|e^{t\psi}D(\eta u)\|^2_{L^2((-\infty,0)\times S,dx)} \le C'\|e^{t\psi}u\|^2_{L^2((-\infty,0)\times S,dx)} \quad since\ \|V\|_\infty \le B.$$

This implies that as t goes to ∞ u is identically zero on the set where η is 1.

Now we want to prove theorem 2.
proof
Since $A_t = e^{t\psi(y)}e^{y}De^{-t\psi(y)}$, where $A_t = \partial/\partial y - (t\psi(y) + L)$,
it suffices to prove the following inequality first and show later (2') by duality.

$$\|f\|_{L^q(e^{ny}dydw, R^- \times S)} \le C\|A_tf\|_{L^2(e^{ny}dydw, R^- \times S)} \quad for\ f \in C_0^\infty(U)$$

Let π_k denote the projection of $L^2(S; C^m)$ onto E_k. We can rewrite

$$A_tf = \sum_k(\partial/\partial y - (t\psi'(y) + k))\pi_kf.$$

If we have an operator of type $\partial/\partial y - ay + b$ for a, b constant coefficients, and $a > 0$ then we can find a left inverse operator for $\partial/\partial y - ay + b$ easily.
So first consider an operator
$\Omega = d/dy - y$.
Jerison [2] exhibited the following exact formula for the symbol of a left inverse of Ω:
There is a unique operator B on R satisfying $B\Omega = I$ and $B(e^{-y^2/2}) = 0$ given by

$$Bf(y) = (1/2\pi)\int F_0(y,\eta)e^{iy\eta}\hat{f}(\eta)d\eta, where$$

$$F_0(y,\eta) = \sqrt{2}\int_0^\infty e^{-s^2-2sy}ds\,e^{-iy\eta-(y^2+\eta^2)/2} - \int_0^\infty e^{-s^2-s(y-i\eta)}ds. \qquad (8)$$

Now if we have an operator $\partial/\partial y - ay + b$ then

$$\sigma(y,\eta;a,b) = \frac{1}{\sqrt{a}}F_0(\sqrt{a}y - \frac{b}{\sqrt{a}}, \frac{\eta}{\sqrt{a}})$$

is the symbol of the left inverse of $\partial/\partial y - ay + b$
Then by the method of freezing coefficient, we get an approximate symbol for the inverse of $\partial/\partial y - (t\psi'(y) + k)$. Namely

$$F(y,\eta) = \sigma(y,\eta; t\psi''(y), -t\psi'(y) + t\psi''(y)y - k).$$

Also the following symbol estimate is true.

$$|(\partial/\partial y)^j(\partial/\partial)^l F_o(y,\eta)| \leq C_{j,l}(1 + |y + i\eta|)^{-1-j-l} \quad j,l = 0,1,\ldots \tag{9}$$

Now we want to have similar estimates for our symbol $F(y,\eta)$ using (9).

Claim

$$|(\partial/\partial y)^l(\partial/\partial\eta)^j F(y,\eta)| \leq C_{j,l}(\sqrt{a} + |t\psi'(y) + k - i\eta|)^{-1-j-l}(a + |t\psi'(y) + k|)^l \tag{10}$$

Let's denote $A_j = \partial^j y'/\partial y^j, B_j = \partial^j \eta'/\partial y^j$. Then,

$$\begin{aligned} |(\partial/\partial\eta)^j(\partial/\partial y)^l F(y,\eta)| &= |(\partial/\partial\eta)^j(\partial/\partial y)^l 1/\sqrt{a}F_0(y',\eta')| \\ &= \sum_{k=0}^{l}(1/\sqrt{a})^j(\partial/\partial\eta')^j[(\partial/\partial y)^k 1/\sqrt{a}][(\partial/\partial y)^{l-k}F_0(y',\eta')], \end{aligned}$$

since $\eta' = \eta/\sqrt{a}$.
Now we have to compute $(\partial/\partial y)^k(1/\sqrt{a})$, and $(\partial/\partial y)^{l-k}F_0(y',\eta')$. First,

$$\partial^k/\partial y^k F_0(y',\eta') = \sum_{j+i+s+t=k} C_{i,j}A_1^{\tau_1}\ldots A_{q_s}^{\tau_{q_s}}B_1^{l_1}\ldots B_{r_t}^{l_{r_t}}(\partial/\partial y')^j(\partial/\partial\eta')^i F_0 \quad \textit{for}$$

$$1 + 2 + \ldots + q_s = s, \; 1 + 2 + \ldots + r_t = t \quad \textit{and}$$

$$\tau_1 + \ldots + \tau_{q_s} = j, \; l_1 + \ldots + l_{r_t} = i$$

Now from the relation $y = -\psi(y) + e^{-\epsilon\psi(y)}, \epsilon > 0$ we get

$$\psi'(y) = \frac{1}{1 + \epsilon e^{-\epsilon\psi(y)}} \quad \psi''(y) = \frac{\epsilon^2 e^{-\epsilon\psi(y)}}{(1 + \epsilon e^{-\epsilon\psi(y)})^3}$$

(*)

$$|\psi^{(k)}(y)| \leq C_k e^{-\epsilon\psi(y)} \quad k \geq 2$$

Now since

$$a = t\psi''(y) \sim te^{-\epsilon\psi(y)}, \quad 1/\sqrt{a} \sim t^{-1/2}e^{\frac{\epsilon}{2}\psi(y)}. \; \textit{Then,}$$

(**)

$$|(\partial/\partial y)^k \frac{1}{\sqrt{a}}| \le C_k t^{-1/2}\epsilon^k e^{\frac{\epsilon}{2}\psi(y)} \le C'_k \epsilon^k \frac{1}{\sqrt{a}}.$$

We can use this estimate to get one for $\frac{\partial^k y'}{\partial y^k}$.

$$\begin{aligned}
(\alpha)\quad \partial^k y'/\partial y^k &\sim \sum_{l=o}^{k-1} \partial^l/\partial y^l (1/\sqrt{a})(t\psi^{k-l+1}(y)) + \partial^k/\partial y^k (1/\sqrt{a})(t\psi'(y)+k) \\
&\le \sum_{l=0}^{k-1} C_l t^{-1/2}\epsilon^{l-1} e^{\frac{\epsilon}{2}\psi(y)}(t\epsilon^{k-l+1}e^{-\epsilon\psi(y)}) + C_k\epsilon^k(1/\sqrt{a})(t\psi'(y)+k) \\
&\le C'_k t^{1/2}\epsilon^k e^{-\frac{\epsilon}{2}\psi(y)} + C_k\epsilon^k(1/\sqrt{a})(t\psi'(y)+k) \\
&\le C''_k(\sqrt{a} + \frac{1}{\sqrt{a}}|t\psi'(y)+k|)
\end{aligned}$$

Using (α) we can deduce that each

$$|A_j| \le C(\sqrt{a} + \frac{1}{\sqrt{a}}|(t\psi'(y)+k)|) \quad \textit{for all } j$$

which implies

$$\begin{aligned}
(\beta)\quad |A_1^{r_1}A_2^{r_2}\ldots A_{q_0}^{r_{q_0}}| &\le C|\frac{1}{\sqrt{a}}(t\psi'(y)+k)|^{r_1+r_2+\ldots+r_{q_0}} \\
&\le C(\frac{1}{\sqrt{a}}(t\psi'(y)+k))^i \textit{ for } i = r_1+\ldots+r_{q_0}.
\end{aligned}$$

$$|B_1^{l_1}B_2^{l_2}\ldots B_{r_t}^{l_{r_t}}| \le C(\frac{1}{\sqrt{a}}\eta)^{l_1+\ldots+l_{r_t}} = C(\frac{1}{\sqrt{a}}\eta)^j \textit{ for } j = l_1+\ldots+l_{r_t}$$

the last inequality holds since

$$(\gamma)\qquad \frac{\partial^k\eta'}{\partial y^k} = \frac{\partial^k}{\partial y^k}(1/\sqrt{a})\eta \le \frac{c}{\sqrt{a}}\eta \qquad \textit{by } (**)$$

If we combine all the terms,

$$\begin{aligned}
|(\partial/\partial\eta)^j(\partial/\partial y)^l\frac{1}{\sqrt{a}}F_0(y',\eta')| &= \sum_{k=0}^{l}(1/\sqrt{a})^j(\partial/\partial\eta)^j[(\partial/\partial y)^k(1/\sqrt{a})][(\partial/\partial y)^{l-k}F_0(y',\eta')] \\
&= \sum_{k=0}^{l}(1/\sqrt{a})^j[(\partial/\partial y)^k(1/\sqrt{a})](\partial/\partial\eta')^j[(\partial/\partial y)^{l-k}F_0(y',\eta')] \\
&\simeq \sum_{k=0}^{l} C(1/\sqrt{a})^{j+1}(\partial/\partial\eta')^j|(\partial/\partial y)^{l-k}F_0(y',\eta')|.
\end{aligned}$$

From (**) and (β) the above is bounded by

$$C(1/\sqrt{a})^{j+1}(\partial/\partial\eta')^j \sum_{0\le k\le l} \sum_{\alpha+\beta\le l-k} C_{\alpha,\beta}|\frac{1}{\sqrt{a}}(t\psi'(y)+k)|^\alpha|\frac{1}{\sqrt{a}}\eta|^\beta|(\partial/\partial y')^\alpha(\partial/\partial\eta')^\beta F(y',\eta')|\}$$

Using (9) and other estimates above this is bounded by

$$C_j(1/\sqrt{a})^{j+1} \sum_{\alpha+\beta\le l-k, 0\le k\le l} C_\alpha|\frac{1}{\sqrt{a}}(t\psi'(y)+k)|^\alpha|\frac{1}{\sqrt{a}}\eta|^\beta(1+\frac{1}{\sqrt{a}}|t\psi'(y)+k-i\eta|)^{-1-j-\alpha-\beta}\}$$

$$\le C'_{j,l}(\sqrt{a}+|t\psi'(y)+k-i\eta|)^{-1-j-l}(a+|t\psi'(y)+k|)^l$$

by cancellations.

The main tool in the prooof is the spherical restriction theorem of C.Sogge.

Theorem
Let ξ_k denote the projection operator from $L^2(S)$ to the space of spherical harmonics o degree k. Then there is a constant c such that

$$||\xi_k g||_{L^{p'}(S)} \le ck^{1-2/n}||g||_{L^p(s)} \quad where$$
$$p = 2n/(n+2), p' = 2n/(n-2).$$

Formula (6) implies that

$$(L+(n-2)/2)^2 = -\Delta_s + (n-2)^2/4.$$

Hence

$$T = sgn(L+(n-2)/2) = (L+(n-2)/2)(-\Delta_s+(n-2)^2/4)^{-1/2}$$

is a classical pseudodifferential operator on S. Thus T is bounded from $L^q(S;C^m)$ to $L^q(S;C$ for all q,$1 < q < \infty$. Moreover,

$$\pi_k = \frac{1}{2}(1+T)\xi_k, \quad k = 0,1,2...$$

$$\pi_k = \frac{1}{2}(1-T)\xi_k, \quad k = 1-n, -n, -n-1, ...$$

Therefore, Sogge's theorem implies that

$$||\pi_k g||_{L^{p'}(S;C^m)} \le Ck^{1-2/n}||g||_{L^p(S;C^m)}.$$

Define $\pi_{M,N}$ by

$$\pi_k\pi_{M,N}g = \{\pi_k g \ \ if M \le k \le N, \ \ 0 \ otherwise.\}$$

The triangle inequality implies

$$||\pi_{M,N}g||_{L^{p'}(S;C^m)} \leq CN^{1-2/n}(N-M+1)||g||_{L^p(S;C^m)}$$

Next use a device due to P.Tomas:

$$||\pi_{M,N}g||_{L^2}^2 = \int_S < \pi_{M,N}g, g > \leq ||\pi_{M,N}g||_{L^{p'}}||g||_{L^p}$$
$$\leq CN^{1-2/n}(N-M+1)||g||_{L^p}^2.$$

We conclude that

$$||\pi_{M,N}G||_{L^2(S;C^m)} \leq CN^{1/p'}(N-M+1)^{1/2}||g||_{L^p(S;C^m)}$$

and by duality

$$||\pi_{M,N}g||_{L^{p'}(S;C^m)} \leq CN^{1/p'}(N-M+1)^{1/2}||g||_{L^2(S;C^m)}$$

If we interpolate with the trivial estimate

$$||\pi_{M,N}g||_{L^2(S;C^m)} \leq ||g||_{L^2(S;C^m)}$$

we find that

$$||\pi_{M,N}g||_{L^q(S;C^m)} \leq C(N^{\frac{n-2}{2}}(N-M+1)^{n/2})^{1/2-1/q}||g||_{L^2(S;C^m)} \tag{11}$$

for $2 \leq q \leq p' = 2n/n-2$.

Let N be the integer satisfying $2^{N-1} \leq 10e^{\epsilon j/2}t^{1/2} \leq 2^N$.
Consider a partition of unity $\{\phi_\beta\}_{\beta=0}^N$ of the positive real axis satisfying

$$\sum_{\beta=0}^{N} \phi_\beta(r) = 1 \qquad all\ r > 0$$
$$supp\phi_\beta \subset \{r : 2^{\beta-2} \leq r \leq 2^\beta\}, \beta = 1, 2, ...N-1.$$
$$supp\phi_0 \subset \{r : r \leq 1\}, \quad supp\phi_N \subset \{r : r \geq s/400\}, \tag{12}$$
$$|(\partial/\partial r)^l \phi_\beta(r)| \leq C_l 2^{-\beta l}, \quad l = 0, 1....$$

Define $F_t^\beta(y,\eta,k)=\phi_\beta(\frac{1}{\sqrt{a}}|t\psi'(y)+k-i\eta|)F_t(y,\eta,k)$ Then F_t satisfies

$$|(\partial/\partial\eta)^j(\partial/\partial y)^l F_t(y,\eta,k)|\le C_{j,l}(\sqrt{a}+|t\psi'(y)+k-i\eta|)^{-1-j-l}(a+|t\psi'(y)+k|)^l \quad (13)$$

$$|(\partial/\partial\eta)^j(F_t(y,\eta,k)-F_t(y,\eta,k+1))|\le C_j(\sqrt{a}+|t\psi'(y)+k-i\eta|)^{-2-j}$$

From (13) and the property of $|(\partial/\partial r)^l\phi_\beta(r)|\le 2^{-\beta l}$, we deduce the following inequallities hold uniformly for $y\in I=I_l=(-l,-l+1)$.

$$|(\partial/\partial\eta)^j F_t^\beta(y,\eta,k)|\le C_j(2^\beta\sqrt{a})^{-1-j} \quad (14$$

$$|(\partial/\partial\eta)^j(F_t^\beta(y,\eta,k)-F_t^\beta(y,\eta,k+1))|\le C_j(2^\beta\sqrt{a})^{-2-j}$$

Define $(F_t^\beta f)(y,w)=\sum_k\frac{1}{2\pi}\int F_t^\beta(y,\eta,k)\pi_k\tilde{f}(\eta,.)(w)e^{i\eta m}d\eta$. Then $F_t=\sum_{\beta=0}^N F_t^\beta$. W begin by estimating F_t^N.
In the case $\beta=N$, we need different estimates. By the choice of N such that $2^N\sim 10e^{cj}\surd$ we have the following.
Since F_t^N is supported where

$$|t\psi'(y)+k-i\eta|\ge 2^N\sqrt{a}\sim 10t,$$

we have

$$|t\psi'(y)+k-i\eta|>c(1+|\eta|+|k|)\ \ uniformly\ for\ y<0.$$

Hence

$$|(\partial/\partial\eta)^j(\partial/\partial y)^m F_t^N(y,\eta,k)|\le C_{j,m}(1+|\eta|+|k|)^{-1-j}j=0,1,.. \quad (1$$

F_t^N is controlled by standard pseudodifferential operators.
In fact,

$$e^yDF_t^Nf=\sum_k\frac{1}{2\pi}\int r_t(y,\eta,k)\pi_k\tilde{f}(\eta,.)(w)e^{i\eta k}d\eta,\quad where$$

$$r_t(y,\eta,k)=\hat{\alpha}((i\eta-k)F_t^N+\partial/\partial yF_t^N).$$

It follows from (15) that

$$|(\partial/\partial\eta)^j(\partial/\partial y)^m r_t(y,\eta,k)| \leq C_{j,m}(1+|k|+|\eta|)^{-j-m} \quad (16).$$

In other words r_t is a classical symbol in the (y,η) variables and a bounded multiplier in the k variable, and thus

$$||e^y DF_t^N f||_{L^2(I\times S,dx)} \leq C||f||_{L^2(I\times S,dx)} \qquad I = I_l.$$

Now by the Sobolev inequality, we get

$$||f||_{L^{p'}(I\times S, e^{ny}dydw)} \leq ||rDf||_{L^2(I\times S,e^{ny}dydw)}, \quad for\ all\ \ f\in C_0^\infty(I_{\cdot}\times S)$$

and p'=2n/(n-2). here $r \sim e^{-l}$ *in* I_l. So we have

$$\begin{aligned} ||F_t^N f||_{L^q(I\times s,dx)} &\leq C||e^{-j}DF_t^N f||_{L^2(I\times S,dx)} \\ &\leq C'||e^y DF_t^N f||_{L^2(I\times S,dx)} \\ &\leq C''||f||_{L^2(I\times S,dx)} \quad forall\ \ f\in C_0^\infty(I\times S;C^m) \end{aligned}$$

for $1\leq q\leq p'$. In particula this holds for q=(6n-4)/(3n-6).
Let $M = [-t\psi'(y) - 2^\beta\sqrt{a}]$, $M' = [M + 22^\beta\sqrt{a}] + 1$. Denote

$$T_t^\beta(y,\eta)g(w) = \sum_K F_t^\beta(y,\eta,k)\pi_k g(w)$$

Here $F_t^\beta(y,\eta,k) = 0$ unless $M\leq k\leq M'$. A summation by parts gives

$$T_t^\beta(y,\eta) = \sum_M^{M'}(F_t^\beta(y,\eta,k) - F_t^\beta(y,\eta,k+1))\pi_{M,k} \ \ forM \leq k \leq M'$$

Now (11) implies

$$||\pi_{M,k}g||_{L^q(S;C^m)} \leq C(t^{\frac{n-2}{2}}(2^\beta\sqrt{a})^{n/2})^{1/2-1/q}||g||_{L^2(S;C^m)} \ \ for\ \beta\leq N-1$$

Therefore, by (14)

$$||(\partial/\partial\eta)^j T_t^\beta(y,\eta)\pi_{M,k}g||_{L^q(S;C^m)} \leq C_j(2^\beta\sqrt{a})^{-1-j}(t^{\frac{n-2}{2}}(2^\beta\sqrt{a})^{n/2})^{1/2-1/q}||g||_{L^2(S;C^m)}$$

uniformly for $y\in I$.
Define

$$\begin{aligned} K_t^\beta(y,z) &= \frac{1}{2\pi}\int T_t^\beta(y,\eta)e^{iz\eta}d\eta \\ &= \frac{1}{2\pi}\int(\partial/\partial\eta)^j T_t^\beta(y,\eta)1/(iz)^j e^{iz\eta}d\eta \end{aligned}$$

and since the length of the interval in η where T_t^β is non-zero is less than $2 \times 2^\beta \sqrt{a}$,

$$||K_t^\beta(y,z)g||_{L^q(S;C^m)} \leq C_j |2^\beta \sqrt{a} z|^{-j} (t^{\frac{n-2}{2}} (2^\beta \sqrt{a})^{n/2})^{1/2-1/q} ||g||_{L^2(S;C^m)}$$

Furthermore,

$$||K_t^\beta(y,z)g||_{L^q(S;C^m)} \leq C(1 + |2^\beta \sqrt{a} z|)^{-10} (t^{\frac{n-2}{2}} (2^\beta \sqrt{a})^{n/2})^{1/2-1/q} ||g||_{L^2(S;C^m)}$$

Note that

$$F_t^\beta f(y,w) = \int K_t^\beta(y, y-y') f(y',.)(w) dy'$$

Lemma
Let H(y, y') be a bounded operator from $L^p(S)$ to $L^q(S)$ of operator norm $\leq h(y-y')$ for each $y, y' \in R$. Suppose that $h \in L^r(R)$ for
1/r +1/p =1+1/q . Then

$$Tf(y,w) = \int H(y,y') f(y',.)(w) dy' \quad \textit{satisfies}$$
$$||Tf||_{L^q(R\times S)} \leq ||h||_{L^r(R)} ||f||_{L^p(R\times S)}.$$

Note that

$$||(1 + |2^\beta \sqrt{a} z|)^{-10}||_{L^r(dz)} \leq C(2^\beta \sqrt{a})^{-1/r}$$

we see that the Lemma implies that for $\beta \leq N-1$,

$$\begin{aligned} ||F_t^\beta f||_{L^q(I_j\times S,dz)} &\leq C(2^\beta \sqrt{a})^{-1/r} (t^{3n-4/4} 2^{\beta n/2} \sqrt{\psi''(y)}^{n/2})^{1/2-1/q} ||g||_{L^2(I_j\times S,dz)} \\ &= C 2^{-(n-2)\beta/\gamma} t^{-\frac{1}{2r}} (t^{\frac{3n-4}{4}})^{\frac{1}{\gamma}} (\sqrt{\psi''(y)})^{-1/r} (\sqrt{\psi''(y)})^{n/2\times 1/\gamma} ||g||_{L^2(I_j\times S,dz)} \\ &= C 2^{-(n-2)\beta/\gamma} (\psi''(y))^{-1/2r+n/4\gamma} ||g||_{L^2(I_j\times S,dz)} \end{aligned}$$

with $1/2 - 1/q = 1/\gamma$, $1/r + 1/2 = 1 + 1/q$, $\gamma = (3n-2)/2$,
$r = (3n-2)/(3n-4)$ and $m = -1/2r + n/4\gamma = -2n + 4/2(3n-2) > -1/3$.
So the above is bounded by

$$C 2^{-(n-2)\beta/\gamma} (e^{cy})^m ||g||_{L^2(I_j\times S,dz)}$$

If we sum the series in β and add the final term $\beta = N$, we get

$$||\sum_\beta F_t^\beta f||_{L^q(I_j\times S, e^{ny}dydw)} \leq C' e^{cj/3} ||f||_{L^2(I_j\times S, e^{ny}dydw)} \tag{17}$$

Since we have an estimate only for the main term, we will work on the remainder term from now on. As a reminder, we defined

$$
\begin{aligned}
F_t f(y) &= \sum_k \int F_t(y,\eta,k) e^{iy\eta} \pi_k \hat{f}(\eta) d\eta \\
&= \int\int F_t(y,\eta,k) e^{i(y-y')\eta} f(y',.) dy' d\eta \pi_k \quad and
\end{aligned}
$$

$$B_t(y',\eta,k) = \partial/\partial y' - a(y)y' + b(y)$$

is the operator whose inverse has symbol $F_t(y,\eta,k)$ i.e.

$$F_t B_t f(y) = \sum_k \int\int F_t(y,\eta,k) e^{i(y-y')\eta} (\partial/\partial y' - a(y)y' + b(y)) f(y') dy' d\eta$$

$$= \int \delta(y-y') e^{i(y-y')\eta} f(y') dy' = f(y)$$

Then

$$
\begin{aligned}
R_t f(y) &= F_t A_t f(y) - Id f(y) = F_t A_t f(y) - F_t B_t f(y) \\
&= \sum_k \int\int F_t(y,\eta,k) \{(\partial/\partial y' - (t\psi'(y) + k)) - (\partial/\partial y' - a(y)y' + b(y))\} \\
&\quad \times \; e^{i(y-y')\eta} f(y',.)(w) dy' d\eta \pi_k \\
&= \int\int F_t(y,\eta,k) t(y-y')^2 g(y,y') e^{i(y-y')\eta} f(y',.)(w) dy' d\eta \pi_k \quad for
\end{aligned}
$$

$$g(y,y') = \int_0^1 (1-s)\psi'''(y)(y+s(y'-y)) ds$$

Since we want similar type of inequality, i.e.

$$||R_t f||_{L^q(I_l \times S, dx)} \le C_l ||f||_{L^2(I_l \times S, dx)} \quad in \; I_l = (-l, -l+1)$$

we will adopt same techniques as in the main step by introducing $\{\phi_\beta\}_{\beta=0}^N$ as partition of unity for R^+ and T_t^β, K_t^β having the same definitions as before. Then

$$R_t^\beta f(y) = \sum \int\int F_t^\beta(y,\eta,k) t(y-y')^2 e^{i\eta(y-y')} g(y,y') f(y',.)(w) dy' d\eta \pi_k$$

The kernel K_t^β was defined as

$$K_t^\beta(y,z) = \frac{1}{2\pi} \int T_t^\beta(y,\eta) e^{iz\eta} d\eta.$$

Using this definition, we define

$$\tilde{K}_t^\beta(y,y') = \frac{1}{2\pi}\int T_t^\beta(y,\eta)t(y-y')^2 e^{i(y-y')\eta} g(y,y')d\eta.$$

We know already when z=y-y',

$$||K_t^\beta(y,z)f||_{L^q(S;C^m)} \leq C_j|2^\beta\sqrt{a}z|^{-j}(t^{n-2/2}(2^\beta\sqrt{a})^{n/2})^{1/2-1/q}||f||_{L^2(S;C^m)}$$

From this we get

$$||\tilde{K}_t^\beta(y,y')f||_{L^q(S;C^m)} \leq C_j|2^\beta\sqrt{a}z|^{-j}t|z|^2||g||_\infty(t^{n-2/2}(2^\beta\sqrt{a})^{n/2})^{1/2-1/q}||f||_{L^2(S;C^m)}.$$

$$= C_jt(2^\beta\sqrt{a})^{-2}||g||_\infty|2^\beta\sqrt{a}z|^{-j+2}(t^{n-2/2}(2^\beta\sqrt{a})^{n/2})^{1/2-1/q}||f||_{L^2(S;C^m)}$$

Note that

$$R_t^\beta f(y,w) = \int \tilde{K}_t^\beta(y,y')f(y',.)(w)dy'.$$

As we know $a = t\psi''(y) \sim te^{-cl}$ uniformly for $y \in I_l$, and $||g||_\infty \leq e^{-cl}$ since $y,y' \in I_l$. As a result, $||t(2^\beta\sqrt{a})^{-2}g||_\infty \leq C'$. Then using the same method it gives us

$$||R_t^\beta f||_{L^q(I_l\times S,dx)} \leq C'2^{-\beta(n-2/\gamma)}e^{cl/3}||f||_{L^2(I_l\times S,dx)} \quad when \quad \gamma = (3n-2)/2 \qquad (17')$$

Now consider the case $\beta = N$. From the definition,

$$(R_t^N f)(y,w) = \sum_k \frac{1}{2\pi}\int\int F_t^N(y,\eta,k)\pi_k f(y',.)(w)(y-y')^2 g(y,y')e^{i\eta(y-y')}dy'd\eta$$

$$= \sum_k \frac{1}{2\pi}\int\int \tilde{F}_t^N(y,y',\eta,k)\pi_k f(y',.)(w)e^{i\eta(y-y')}dy'dw$$

when $\tilde{F}_t^N(y,y',\eta,k) = (y-y')^2 g(y,y')F_t^N$

Now we can claim this operator is controlled by standard pseudodifferential operators. In fact,

$$e^y D\tilde{F}_t^N f = \sum_k \frac{1}{2\pi}\int \tilde{r}_t(y,y',\eta,k)\pi_k\hat{f}(\eta,.)(w)e^{i\eta y}d\eta,$$

where

$$\tilde{r}_t(y,y',\eta,k) = (y-y')^2 g(y,y')r_t(y,,\eta,k)$$

We already have the estimate for $r_t(y,\eta,k)$ i.e.

$$|(\partial/\partial\eta)^j(\partial/\partial y)^m r_t(y,\eta,k)| \le C_{j,m}(1+|\eta|+|k|)^{-j-m}.$$

Since $y, y' \in I_l$,

$$(y-y')^2 g(y,y') \sim (y-y')^2 e^{-cl}$$

and $|(\partial/\partial y)^{m_1}(\partial/\partial y')^{m_2}(y-y')^2 g(y,y')| \le B$. So we get

$$|(\partial/\partial\eta)^j(\partial/\partial y)^{m_1}(\partial/\partial y')^{m_2}\tilde{r}_t(y,y',\eta,k)| \le C_{j,m_1,m_2}(1+|k|+|\eta|)^{-j-m_1}.$$

In other words $\tilde{r}_t$ is a classical symbol in the (y,y',η) variables and a bounded multiplier in the k variable. Now following the same method as in the main step and we can deduce

$$||\tilde{F}_t^N f||_{L^q(I_l\times S,dx)} \le C||f||_{L^2(I_l\times S,dx)}.$$

If we sum the series in β and add the final term $\beta = N$, we get

$$||R_t f||_{L^q(I_l\times S,dx)} \le Ce^{\epsilon j/3}||f||_{L^2(I_l\times S,dx)} \qquad (15'')$$

Since we have an estimate on the unit anulus i.e. $I_l \times S$, we want to extend it to the whole ball , in this case $R^- \times S$.

In Theorem 1, we proved

$$t||f||^2_{L^2(I\times S,e^{(n+\epsilon)y}dydw)} \le C||A_t f||^2_{L^2(I\times S,e^{ny}dydx)} \quad for\ f \in C_0^\infty(I\times S).$$

when $I = I_j$,this means

$$t\int_{I_j\times S}|f|^2 e^{(n+\epsilon)y}dydw \le C||A_t f||^2_{L^2(I_j\times S,e^{ny}dydw)}.$$

Since $e^{\epsilon y} \sim e^{-\epsilon j}$ in I_j, this is equivalent to

$$\sqrt{t}(\int_{I_j\times S}|f|^2 e^{-\epsilon j}e^{ny}dydw)^{1/2} \le C||A_t f||_{L^2(I_j\times S,e^{ny}dydx)}.$$

i.e.

$$||f||_{L^2(I_j\times S,e^{ny}dydw)} \le Ce^{\epsilon j/2}||A_t f||_{L^2(I_j\times S,e^{ny}dydw)}.$$

So (10) implies

$$||R_t f||_{L^q(I_j\times S,e^{ny}dydw)} \le Ce^{5\epsilon j/6}||A_t f||_{L^2(I_j\times S,e^{ny}dydw)}.$$

Now from the relation $f(y) = F_t A_t f(y) - R_t f(y)$, we get

$$\begin{aligned}
\|f\|_{L^q(I_j\times S, e^{nv}dydw)} &\le \|F_t A_t f\|_{L^q(I_j\times S, e^{nv}dydw)} + \|R_t F\|_{L^q(I_j\times S, e^{nv}dydw)} \\
&\le e^{\epsilon j/3}\|A_t f\|_{L^2(I_j\times S, e^{nv}dydw)} + e^{5\epsilon j/6}\|A_t f\|_{L^2(I_j\times S, e^{nv}dydw)} \\
&\le C' e^{5\epsilon j/6}\|A_t f\|_{L^2(I_j\times S, e^{nv}dydw)} \quad \textit{which is}
\end{aligned}$$

$$\int_{I_j\times S} |f|^q e^{nv} dydw \le e^{8\epsilon jq/6}\left(\int_{I_j\times S} |A_t f|^2 e^{-\epsilon j} e^{nv} dydw\right)^{q/2}.$$

i.e.

$$\int_{I_j\times S} |f|^q e^{4\epsilon qv+nv} dydw \le \left(\int_{I_j\times S} |A_t f|^2 e^{(\epsilon+n)v} dydw\right)^{q/2}$$

We get from this

$$\|f\|^q_{L^q(I_j\times S, e^{(4\epsilon q/3+n)v}dydw)} \le C\|A_t f\|^q_{L^2(I_j\times S, e^{(n+\epsilon)v}dydw)}.$$

Now choose $\{\psi_{jk}\}_{j\in N, k=1,2}$ be partition of unities such that

$$\psi_{j1} \in C_0^\infty(-j, -j+3/4), \quad \psi_{j2} \in C_0^\infty(-j+2/4, -j+5/4).$$

Then using $f = \sum \psi_j f$, (we will just call $\{\psi_{jk}\}$ as $\{\psi_j\}$)and

$$\begin{aligned}
\|A_t(\psi_j f)\|^q_{L^2(I_j\times S)} &= \|\psi_j' f + \psi_j A_t f\|^q_{L^2(I_j\times S)} \\
&\le C\|\psi_j' f\|^q_{L^2(I_j\times S)} + C\|\psi_j A_t f\|^q_{L^2(I_j\times S)}
\end{aligned}$$

We come to the final estimate.

$$\begin{aligned}
\|f\|^q_{L^q(e^{(4\epsilon q/3+n)v}dydw, R^-\times S)} &= \Big\|\sum_j \psi_j f\Big\|^q_{L^q(e^{(4\epsilon q/3+n)v}dydw, I_j\times S)} \\
&\le^{(1)} C_0 \sum_j \|\psi_j f\|^q_{L^q(e^{(4\epsilon q/3+n)v}dydw, I_j\times S)} \\
&\le C_1 \sum_j \|A_t(\psi_j f)\|^q_{L^2(e^{(n+\epsilon)v}dydw, I_j\times S)} \\
&\le C_2 \sum \|\psi_j' f\|^q_{L^2(e^{(n+\epsilon)v}dydw, I_j\times S)} + C_2 \sum_j \|\psi_j A_t f\|^q_{L^2(e^{(n+\epsilon)v}dydw, I} \\
&\le^{(2)} C'\|f\|^q_{L^2(e^{(n+\epsilon)v}dydw, R^-\times S)} + C'\|A_t f\|^q_{L^2(e^{(n+\epsilon)v}dydw, R^-\times S)} \\
&\le^{(3)} C''\|A_t f\|^q_{L^2(e^{nv}dydw, R^-\times S)}
\end{aligned}$$

Inequality (1) and (2) hold since for each $x \in R$, only finitely many ψ_j's overlap. (3) comes from L^2 estimates. So we get

$$||f||^q_{L^q(e^{(4\epsilon q/3+n)y}dydw, R^-\times S)} \leq C||A_t f||^q_{L^2(e^{ny}dydw, R^-\times S)}$$

which is equivalent to

$$||e^{t\psi} f||^q_{L^q(e^{(4\epsilon q/3+n)y}dydw, R^-\times S)} \leq C||e^{t\psi}e^y Df||^q_{L^2(e^{ny}dydw, R^-\times S)} \tag{18}$$

Now if we choose small ϵ , then $e^y \leq e^{4\epsilon q/3}$. So the above is equivalent to

$$||e^{t\psi} f||_{L^q(R^-\times S, e^{ny}dydw)} \leq C||e^{t\psi} Df||_{L^2(R^-\times S, e^{ny}dydw)} \tag{18'}$$

Now we want to find a dual version of this i.e.

$$||e^{t\psi} f||_{L^2(R^-\times S, dx)} \leq C||e^{t\psi} Df||_{L^p(R^-\times S, dx)} \quad for\ p = q',\ f \in C_0^\infty(R^- \times S)$$

To prove this we will follow a new policy i.e. instead of finding an approximate inverse for A_t, we want to find a left inverse operator B_t satisfying $f(y) = B_t A_t f(y)$. If we assume B_t has kernel K_t, then the above is equivalent to

$$\begin{aligned} \pi_k f(y) &= \int K_t(y,s)(\partial/\partial s - t\psi'(s) + k)\pi_k f(s)ds \\ &= \int(-\partial/\partial s - t\psi'(y) + k)K_t(y,s)\pi_k f(s)ds \end{aligned}$$

Let

$$K_{1,t}(y,s) = H(y-s)e^{t(\psi(y)-\psi(s))+k(y-s)} \quad then$$

$$B_1 f(y,w) = \sum_k \int_{-\infty}^{y} e^{t(\psi(y)-\psi(s))+k(y-s)}\pi_k f(s,.)(w)ds$$

This integral converges only for $k < -t\psi'(y)$.
On the other hand, the fact that $f \in C_0^\infty(R^- \times S)$ and

$$(-\partial/\partial s - t\psi'(s) - k)e^{t(\psi(y)-\psi(s))+k(y-s)} = 0 \ tells\ us$$

$$K_{2,t}(y,s) = (H(y-s) - 1)e^{t(\psi(y)-\psi(s))+k(y-s)} \quad is\ another\ kernel.$$

From this we have another expression which is

$$B_2 f(y,w) = -\sum_k \int_y^\infty g(s)e^{t(\psi(y)-\psi(s))+k(y-s)}\pi_k f(s,.)(w)ds$$

where the integral converges for $k > -t\psi'(y)$. where g(s) is a cut off function we introduce for later use such that $g \in C^\infty$ which has value 1 for $s < 0$, and equals 0 for $s > 1$. Then g(s)f(s) = f(s) for $s < 0$. Now we can write

$$\psi(s) - \psi(y) = (s-y)\psi'(y) + \frac{(s-y)^2}{2}h(y,s), \quad where$$

$$h(y,s) = \int_0^1 (1-\xi)\psi''(y+\xi(s-y))d\xi \quad \geq 0$$

After substituting $f(s) = \int \hat{f}(\eta)e^{is\eta}d\eta$, *and* $y - s = s'$, *we get*

$$B_1 f(y,w) = \sum_k \int \sigma_{1,t}(y,\eta,k)\hat{f}(\eta,.)(w)e^{iy\eta}d\eta, \quad where$$

$$\sigma_{1,t}(y,\eta,k) = \int_{-\infty}^{y} e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds \qquad (-t\psi'(y) > k)$$

Also

$$B_2 f(y,w) = \sum_k \int \sigma_{2,t}(y,\eta,k)\hat{f}(\eta,.)(w)e^{iy\eta}d\eta, \quad where$$

$$\sigma_{2,t}(y,\eta,k) = -\int_y^{\infty} g(s)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds \qquad (k > -t\psi'(y)).$$

Since $\sigma_{1,t}(y,\eta,k)$ are defined only for $-t\psi'(y) > k$, we want to extend this for $k > -t\psi'(y)$. Also we want to extend $\sigma_{2,t}(y,\eta,k)$ for $k < -t\psi'(y)$ too. For that we will follow Stein's method ([7],p182).

Lemma

There exist a continuous function ϕ defined on $[1,\infty)$ which is rapidly decreasing at ∞, that is $\phi(\lambda) = 0(\lambda^{-N})$, as $\lambda \to \infty$, for every N, and which satisfies the following properties.

$$\int_1^{\infty} \phi(\lambda)d\lambda = 0, \quad \int_1^{\infty} \lambda^l \phi(\lambda)d\lambda = 0, \quad for\ k = 1,2,...$$

With this definition, we can extend our symbol by

$$\begin{aligned} F_{1,t}(y,\eta,k) &= \sigma_{1,t}(y,\eta,k) \quad (k < -t\psi'(y)) \\ &= \int_1^{\infty} \sigma_{1,t}(y,\eta,(1-2\lambda)(k+t\psi'(y)) - t\psi'(y))\phi(\lambda)d\lambda \quad (k > -t\psi'(y)) \end{aligned}$$

$$\begin{aligned} F_{2,t}(y,\eta,k) &= \sigma_{2,t}(y,\eta,k) \ \ (k > -t\psi'(y)) \\ &= \int_1^{\infty} \sigma_{2,t}(y,\eta,(1-2\lambda)(k+t\psi'(y)) - t\psi'(y))\phi(\lambda)d\lambda \ \ (k < -t\psi'(y)) \end{aligned}$$

Then

$$lim_{k\to-(t\psi'(y)+)} F_{1,t}(y,\eta,k) = \int_1^{\infty} \sigma_{1,t}(y,\eta,k)\phi(\lambda)d\lambda$$

Also

$$\begin{aligned} lim_{k\to-(t\psi'(y)+)} d_k^l F_{1,t}(y,\eta,k) &= \int_1^{\infty} lim_{k\to-(t\psi'(y)+)}(1-2\lambda)^l d_k^l \sigma_{1,t}(\\ &\qquad y,\eta,(1-2\lambda)(k+t\psi'(y)) - t\psi'(y))\phi(\lambda)d\lambda \\ &= d_k^l \sigma_{1,t}(y,\eta,k)\int_1^{\infty}(1-2\lambda)^l \phi(\lambda)d\lambda \\ &= d_k^l \sigma_{1,t}(y,\eta,k) \end{aligned}$$

The same argument says

$$lim_{k \to -(t\psi'(y)-)} d_k^l F_{2,t}(y,\eta,k) = d_k^l \sigma_{2,t}(y,\eta,k)$$

Now we want to find the size of symbols. First we will estimate $\sigma_{1,t}(y,\eta,k)$ *and* $\sigma_{2,t}(y,\eta,k)$ Then using those estimates we can do the same for the extended symbols. We expect $\sigma_{1,t}(y,\eta,k), \sigma_{2,t}(y,\eta,k)$ give same estimates.

Claim

$$|(\frac{\partial}{\partial y})^N (\frac{\partial}{\partial \eta})^M d_k^m \sigma_{i,t}(y,\eta,k)| \leq \frac{C_{N,M}}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|)^{M+m+1}}$$

$$\times (1 + \frac{t\psi''(y)}{\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|})^N \quad i = 1,2$$

We will work on $\sigma_{1,t}(y,\eta,k)$ from now on. First, from the above expression and $\psi''(y - \xi s) \sim e^{c(y-\xi s)}$, we have

$$h(y, y-s) \sim e^{cy} \frac{e^{-s} - 1 + s}{s^2}$$

$$\frac{1}{4} e^{cy} \leq h(y, y-s) \leq \frac{1}{2} e^{cy} \quad \textit{for } 0 \leq s \leq 2$$

$$e^{cy} \frac{1}{2s} \leq h(y, y-s) \leq e^{cy} \frac{1}{s} \quad \textit{for } s \geq 2$$

Then

$$\begin{aligned} |\sigma_{1,t}(y,\eta,k)| &\leq \int_0^2 e^{-\frac{1}{4}\psi''(y)s^2 + (t\psi'(y)+k)s} ds \\ &+ \int_2^\infty e^{-t\psi''(y)s + (t\psi'(y)+k)s} ds \qquad (k < -t\psi'(y)) \\ &\leq \frac{2}{\sqrt{t\psi''(y)} + |t\psi'(y) + k|} \end{aligned}$$

On the other hand, after putting absolute value in the integral we get

$$|\sigma_{2,t}(y,\eta,k)| \leq \frac{1}{\sqrt{t\psi''(y)} + |t\psi'(y) + k|}$$

From now on for convenience let's call

$$Q(t,y,s,k,\eta) = e^{t(\psi(y) - \psi(s)) + k(y-s) + i(s-y)\eta}$$

Then $Q(t,y,y,k,\eta) = 1$, $Q(t,y,\infty,k,\eta) = 0$. Now if we integrate by parts, we have

$$\sigma_{1,t}(y,\eta,k) = \frac{1}{i\eta} Q(t,y,s,k,\eta)|_{-\infty}^{y} - \frac{1}{i\eta} \int (t\psi'(s) + k) Q(t,y,s,k,\eta) ds$$

The first term gives us $\frac{1}{i\eta}$ and if we replace

$$t\psi'(s) + k = (t\psi'(y) + k) + h(y,s)(s-y)$$

The 2nd term equals $I_1 + I_2$, *where*

$$|I_1| = \frac{1}{|i\eta|}\int (t\psi'(y) + k)Q(t,y,s,k,\eta)ds \le \frac{C}{|\eta|}$$

$$\begin{aligned}
|I_2| &= \frac{1}{|i\eta|}\int_0^\infty th(y,y-s)se^{-th(y,y-s)s^2+(t\psi'(y)+k)s}ds \\
&\le \frac{1}{|i\eta|}\int_0^2 t\psi''(y)se^{-t\psi''(y)s^2+(t\psi'(y)+k)s}ds + \frac{1}{|i\eta|}\int_2^\infty e^{-t\psi''(y)s+(t\psi'(y)+k)s}ds \\
&\le \frac{C}{|\eta|}
\end{aligned}$$

This way we get $|\sigma_{1,t}(y,\eta,k)| \le \frac{C}{|\eta|}$ and if we combine two estimates, we get

$$|\sigma_{1,t}(y,\eta,k)| \le C(|t\psi'(y) + k - i\eta|)^{-1} \quad \textit{for all } y < 0.$$

We want the same estimate for $\sigma_{2,t}(y,\eta,k)$. Since we have half of what we need we will try integration of parts.

$$\sigma_{2,t}(y,\eta,k) = \int_{-\infty}^{y} g(s)Q(t,y,s,k,\eta)ds = I_1 + I_2 + I_3$$

$$I_1 = \frac{1}{i\eta}g(s)Q|_{-\infty}^{y} = \frac{1}{i\eta}g(y)$$

Then $|I_1| \le \frac{1}{|i\eta|}$ since $|g|_\infty \le 1$

$$I_2 = -\frac{1}{i\eta}\int g'(s)Q(t,y,s,k,\eta)ds$$

Then $|I_2| \le \frac{C}{|i\eta|}$ since $g'(s)$ vanishes except (0,1).

$$I_3 = \frac{1}{i\eta}\int g(s)(t\psi'(s) + k)Q(t,y,s,k,\eta)ds$$

We can show $|I_3| \le \frac{C}{|i\eta|}$ by the same reason as in the $|\sigma_{1,t}(y,k,\eta)| \le \frac{C}{|i\eta|}$. So after getting terms together we get

$$|\sigma_{2,t}(y,\eta,k)| \le \frac{C}{|i\eta|}$$

Now we want to differentiate $\sigma_{1,t}$ with respect to η N times and get an estimate. First,

$$(\partial/\partial\eta)^N\sigma_{1,t}(y,\eta,k) = \int_{-\infty}^{y}(s-y)^N Q(t,y,s,k,\eta)ds$$

If we put the absolute sign inside the integral and change y-s=s', we have two terms $I_1 + I_2$. We will estimate them separately.

$$I_1 = \int_0^1 s^N e^{-t\psi''(y)s^2+(t\psi'(y)+k)s} ds$$

Now

$$I_1 \le \int_0^1 s^N e^{-t\psi''(y)s^2} ds$$

Let $u = t\psi''(y)s^2$. Then using $s = (\frac{u}{t\psi''(y)})^{1/2}$, $ds = \frac{u^{-1/2}}{\sqrt{t\psi''(y)}}du$ we get

$$I_1 \le (\sqrt{t\psi''(y)})^{-1-N}$$

On the other hand

$$I_1 \le \int_0^1 s^N e^{(t\psi'(y)+k)s} ds$$

In this case by changing $u = (t\psi'(y) + k)s$, we get

$$I_1 \le (t\psi'(y) + k)^{-1-N}$$

So if we combine the above two estimates, we get

$$I_1 \le \frac{1}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k|)^{N+1}}$$

Now

$$I_2 = \int_1^\infty s^N e^{-t\psi''(y)s+(t\psi'(y)+k)s} ds$$

which implies

$$|(\partial/\partial\eta)^N \sigma_{1,t}(y,\eta,k)| \le \frac{C}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k|)^{N+1}}$$

By the same argument as in I_1 case we get the estimate

$$I_2 \le \frac{1}{(\sqrt{t\psi''(y)} + |t\psi'(y) + K|)^{N+1}}$$

Now we can check that the same argument works for $(\partial/\partial\eta)^N \sigma_{2,t}$ and get

$$|(\partial/\partial\eta)^N \sigma_{2,t}(y,\eta,k)| \le \frac{C}{(\sqrt{t\psi'(y)} + |t\psi'(y) + k|)^{N+1}}$$

Claim

$$|(\partial/\partial\eta)^N \sigma_{1,t}(y,\eta,k)| \le \frac{C}{(|i\eta|)^{N+1}}$$

We first want to check when N=1.

$$(\partial/\partial\eta)\sigma_{1,t}(y,\eta,k) = \int_{-\infty}^{y}(s-y)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

Let $e^{is\eta} = dg$, $(s-y)e^{t(\psi(y)-\psi(s))+k(y-s)-iy\eta} = f$. Then after integration by parts we get terms of type $I_1 + I_2 + I_3$ where

$$I_1 = \frac{1}{i\eta}\int e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

$$I_2 = \frac{1}{i\eta}\int(s-y)(t\psi'(y)+k)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

$$I_3 = \frac{1}{i\eta}\int(s-y)(t\psi'(y)-\psi'(s)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

The same argument to show $|\sigma_{1,t}(y,\eta,k)| \leq \frac{1}{|i\eta|}$ tells us

$$|I_1| \leq \frac{1}{|i\eta|}.$$

For other terms we perform integration by parts one more time and get terms of the following type.

$$A_1 = \frac{1}{(i\eta)^2}\int_{-\infty}^{y}(t\psi'(y)+k)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

$$A_2 = \frac{1}{(i\eta)^2}\int_{-\infty}^{y}(s-y)(t\psi''(y))e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

$$A_3 = \frac{1}{(i\eta)^2}\int_{-\infty}^{y}(s-y)(t\psi'(y)+k)^2e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

$$A_4 = \frac{1}{(i\eta)^2}\int_{-\infty}^{y}(s-y)(t\psi'(y)-t\psi'(s))^2e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

$$A_5 = \frac{1}{(i\eta)^2}\int_{-\infty}^{y}(s-y)(t\psi''(y)-t\psi''(s))e^{t(\psi(y)-\psi(s))+k(y-s)_i(s-y)\eta}ds$$

A_1 is bounded by an integral of the form

$$\frac{1}{|i\eta|^2}\int_0^{\infty}ae^{as}ds \leq \frac{C}{|i\eta|^2} \quad a = t\psi'(y)+k$$

A_2 is bounded by an integral of the form

$$\frac{1}{|i\eta|^2}\int_0^{\infty}ase^{-as^2}ds \leq \frac{C}{|i\eta|^2} \quad a = t\psi''(y)$$

A_3 is bounded by an integral of the form

$$\frac{1}{|i\eta|^2}\int_0^{\infty}a^2se^{as}ds \leq \frac{C}{|i\eta|^2} \quad a = t\psi'(y)+k$$

A_4 is bounded by an integral of the form

$$\frac{1}{|i\eta|^2}\int_0^\infty a^2s^3e^{-as^2}ds \le \frac{C}{|i\eta|^2} \quad a = t\psi''(y)$$

A_5 is bounded by an integral of the form

$$\frac{1}{|i\eta|^2}\int_0^\infty ase^{-as^2}ds \le \frac{C}{|i\eta|^2} \quad a = t\psi''(y)$$

With this help we can get the general formula for $(\partial/\partial\eta)^N\sigma_{1,t}(y,\eta,k)$ i.e. after integration by parts N or N+1 times we get the following.

$$A_1 = \frac{1}{(i\eta)^N}\int e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

$$A_2 = \frac{C}{(i\eta)^{N+1}}\sum_{m=0}^{N+1}\int(s-y)^m(t\psi'(y)+k)^{m+1}e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

$$A_3 = \frac{C}{(i\eta)^{N+1}}\sum_{m=0}^{N+1}\int(s-y)^m(t\psi'(y)-\psi'(s))^{m+1}e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

Then $|A_1| \le \frac{C}{|i\eta|^{N+1}}$ and terms appearing in A_2 are bounded by

$$\frac{C}{|i\eta|^{N+1}}\int_0^\infty s^m(t\psi'(y)+k)^{m+1}e^{(t\psi'(y)+k)s}ds \le fracC|i\eta|^{N+1}$$

after change of variables i.e. putting $u = (t\psi'(y)+k)s$ gives the desired result. On the other hand, terms appearing in A_3 are bounded by

$$\frac{C}{|i\eta|^{N+1}}\int_0^1(t\psi''(y))^{m+1}s^{2m+1}e^{-t\psi''(y)s^2}ds$$
$$+\frac{C}{|i\eta|^{N+1}}\int_1^\infty(t\psi''(y))^{m+1}s^me^{-t\psi''(y)s}ds$$
$$\le \frac{C}{|i\eta|^{N+1}}$$

after change of variables i.e. putting $u = t\psi''(y)s^2$ gives the desired result.
From now on we want to find an estimate for $(\partial/\partial y)^N\sigma_{1,t}(y,\eta,k)$ with the help from the previous discussion. First, Let's look at $(\partial/\partial y)\sigma_{1,t}(y,\eta,k)$.

$$\begin{aligned}(\partial/\partial y)\sigma_{1,t}(y,\eta,k) &= 1+\int_{-\infty}^y(t\psi'(y)+k-i\eta)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds\\ &= 1+I_1+I_2\end{aligned}$$

$$I_1 = \int_{-\infty}^y(t\psi'(s)+k-i\eta)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

$$I_2 = \int_{-\infty}^{y} (t\psi'(s) - t\psi'(y)) e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds$$

Since

$$\begin{aligned} I_1 &= \int_{-\infty}^{y} (-\partial/\partial s) e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds \\ &= -e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}\Big|_{-infty}^{y} = -1 \end{aligned}$$

I_1 cancels 1 and we have only I_2. Now if we use Taylor series expansion, then I_2 equals

$$-\int_{-\infty}^{y} t\psi''(y)(s-y) e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds$$

$$+\int_{-\infty}^{y} th(y,s)(s-y)^2 e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds$$

The first term is familiar. It is the same term appearing in $(\partial/\partial\eta)\sigma_{1,t}(y,\eta,k)$ multiplied by $t\psi''(y)$. So in this case we can tell it is bounded by

$$\frac{t\psi''(y)}{|i\eta|}$$

The second term, after putting absolute sign in the integral, bounded by

$$\int_0^1 t\psi''(y) s e^{-t\psi''(y)s^2+(t\psi'(y)+k)s} ds + \int_2^{\infty} t\psi''(y) s e^{-t\psi''(y)s+(t\psi'(y)+k)s} ds$$

which is alnso bounded by

$$\frac{t\psi''(y)}{(\sqrt{t\psi''(y)} + |t\psi'(y)+k|)^2}$$

So if we combine two, we get

$$|(\partial/\partial y)\sigma_{1,t}(y,\eta,k)| \le \frac{Ct\psi''(y)}{(\sqrt{t\psi''(y)} + |t\psi'(y)+k-i\eta|)^2}$$

If we try second order differentiation, we have

$$\begin{aligned} (\partial/\partial y)^2 \sigma_{1,t}(y,\eta,k) &= \int (t\psi'(y) - t\psi'(s))^2 e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds \\ &+ \int (t\psi''(y) - t\psi''(s)) e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds \end{aligned}$$

In general Nth derivative has the following formula.

$$(\partial/\partial y)^N \sigma_{1,t}(y,\eta,k) = I_1 + I_2 + I_3 + I_4 + \ldots$$

$$I_1 = \int (t\psi'(y) - t\psi'(s))^N e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds$$

$$I_2 = \int (t\psi^{(N)}(y) - t\psi^{(N)}(s)) e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds$$

$$I_3 = \sum_{j=1}^{N} \int (t\psi'(y) - t\psi'(s))^j (t\psi^{(N-j)}(y) - t\psi^{(N-j)}(s)) e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds$$

....

$$I_m = \sum_j \int (t\psi^{(m)}(y) - t\psi^{(m)}(s))^j (t\psi^{(N-mj)}(y) - t\psi^{(N-mj)} e^{t(\psi(y)-t\psi(s))+k(y-s)+i(s-y)\eta} ds$$

I_1, after Taylor series expansion, has two components i.e.

$$\begin{aligned} I_1 &= \int (t\psi''(y))^N (s-y)^N e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds \\ &+ \int (th(y,s))^N (s-y)^{2N} e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds \end{aligned}$$

The first term is exactly $(t\psi''(y))^N (\partial/\partial\eta)^N \sigma_{1,t}(y,\eta,k)$. So the same argument says it has bound

$$\frac{(t\psi''(y))^N}{|i\eta|^{N+1}}$$

On the other hand, the second term, after putting absolute sign in the integral, is bounded by

$$\begin{aligned} &\int_0^1 (t\psi''(y))^N s^N e^{-t\psi''(y)s^2+(t\psi'(y)+k)s} ds \\ &+ \int_1^\infty (t\psi''(y))^N s^N e^{-t\psi''(y)s+(t\psi'(y)+k)s} ds \\ &\leq \frac{C(t\psi''(y))^N}{(\sqrt{t\psi''(y)} + |t\psi'(y)+k|)^{N+1}} \quad \textit{by rescalling} \end{aligned}$$

So after getting terms together we have

$$|I_1| \leq C_1 \frac{(t\psi''(y))^N}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|)^{N+1}}$$

Now for I_2, using the fact that $\psi^{(l)}(y) \leq \psi''(y)$ we can show that

$$|I_2| \leq C_2 \frac{t\psi''(y)}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|)^2}$$

Now for I_3, using Taylor series expansion, we get two terms which are

$$I_3 = \int (t\psi''(y))^j t\psi^{(N-j+1)}(y)(y-s)^{j+1} e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds$$

$$+ \int (th(y,s))^j tg(y,s)(s-y)^{j+1} e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds, \quad \textit{where}$$

$$g(y,s) = \int_0^1 (1-\xi)\psi^{(N-j+1)}(y+\xi(s-y))d\xi$$

The first term is bounded by

$$C\frac{(t\psi''(y))^{j+1}}{|i\eta|^{j+2}}.$$

On the other hand, the second term is bounded by

$$\int_0^1 (t\psi''(y))^{j+1} s^{j+1} e^{-t\psi''(y)s^2+(t\psi'(y)+k)s} ds$$

$$+\int_1^\infty (t\psi''(y))^{j+1} s^{j+1} e^{-t\psi''(y)s+(t\psi'(y)+k)s} ds$$

$$\leq \frac{C(t\psi''(y))^{j+1}}{(\sqrt{t\psi''(y)} + |t\psi'(y)+k|)^{j+2}}$$

This way we can check each of I_m separately and $(\partial/\partial y)^N \sigma_{1,t}(y,\eta,k)$ are bounded by the following.

$$|(\partial/\partial y)^N \sigma_{1,t}(y,\eta,k)| \leq \sum_{j=1}^{N} C_j \frac{(t\psi''(y))^j}{(\sqrt{t\psi''(y)} + |t\psi'(y)+k-i\eta|)^{j+1}}$$

On the other hand, if we treat k as a continuous variable and differentiate then we get (s-y) factor each time we differentiate. This is the same phenomenon as we differentiate with respect to η. so in this case we have

$$|d_k^m \sigma_{1,t}(y,\eta,k)| \leq \frac{C}{(\sqrt{t\psi''(y)} + |t\psi'(y)+k-i\eta|)^{m+1}}$$

Now if we combine all the information we have, we can tell

$$|(\frac{\partial}{\partial y})^N(\frac{\partial}{\partial \eta})^M d_k^m \sigma_{1,t}(y,\eta,k)| \leq \frac{C}{(\sqrt{t\psi''(y)} + |t\psi'(y)+k-i\eta|)^{M+m}}(1+\frac{(t\psi''(y))}{(\sqrt{t\psi''(y)} + |t\psi'(y)+k} \tag{10'}$$

Now we want to get the same estimate for $(\partial/\partial\eta)^N \sigma_{2,t}(y,\eta,k)$.
First, let N=1.

$$(\partial/\partial\eta)\sigma_{2,t}(y,\eta,k) = \int_y^\infty g(s)(s-y)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds = I_1 + I_2 + I_3$$

$$I_1 = \int \frac{1}{i\eta} \int g(s)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds$$

$$I_2 = \frac{1}{i\eta}\int g'(s)(s-y)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds$$

$$I_3 = \frac{1}{i\eta}\int g(s)(s-y)(t\psi'(s)+k)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta} ds$$

We know already that $|I_1| \leq \frac{1}{|i\eta|}$. For I_2 we perform integration by parts one more time and get terms of following.

$$I_2 = A_1 + A_2 + A_3$$

$$A_1 = \frac{1}{(i\eta)^2}\int g''(s)(s-y)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

$$A_2 = \frac{1}{(i\eta)^2}\int g'(s)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

$$A_3 = \frac{1}{(i\eta)^2}\int g'(s)(s-y)(t\psi'(y)+k)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

If we notice that derivatives of $g'(s)$ vanishes except the unit interval we can easily show that

$$|A_1| \leq \frac{C}{|i\eta|^2}$$

From I_3, after integration by parts, we get same terms as in I_2 except the following

$$\frac{1}{(i\eta)^2}\int g(s)(s-y)(t\psi'(s)+k)^2 e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

which is the same type as in $(\partial/\partial\eta)\sigma_{1,t}(y,\eta,k)$. So this is also bounded by

$$\frac{C}{|i\eta|^2}$$

In general terms appearing in the $(\partial/\partial\eta)^N\sigma_{2,t}(y,\eta,k)$ has either same type as in $(\partial/\partial\eta)^N\sigma_{1,t}(y,$ or of the following type

$$I_1 = \frac{1}{(i\eta)^{N+1}}\int g'(s)(s-y)^m(t\psi'(y)+K)^m e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds \quad or$$

$$I_2 = \frac{1}{(i\eta)^{N+1}}\int g''(s)(s-y)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

Using the fact that $g''(s)$ vanishes except the unit interval we can easily show

$$|I_i| \leq \frac{C}{|i\eta|^{N+1}} \quad i = 1,2$$

After getting terms together we can show that

$$|(\partial/\partial\eta)^N\sigma_{2,t}(y,\eta,k)| \leq \frac{C_N}{|i\eta|^{N+1}}$$

And after combining with the other half, we have

$$|(\partial/\partial\eta)^N\sigma_{2,t}(y,\eta,k)| \leq \frac{C_N}{(\sqrt{t\psi''(y)}+|t\psi'(y)+k-i\eta|)^{N+1}}$$

Now same argument as in differentiating with respect to η gives

$$|d_k^m \sigma_{2,t}(y,\eta,k)| \le \frac{C_m}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|)^{m+1}}$$

Now we want to differentiate $\sigma_{2,t}(y,\eta,k)$ with respect to y variable N times and try to get simlilar estimates. First, let N=1.

$$\begin{aligned}(\partial/\partial y)\sigma_{2,t}(y,\eta,k) &= -g(y) + \int_y^\infty g(s)(t\psi'(y) + k - i\eta)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds \\ &= -g(y) + I_1 + I_2\end{aligned}$$

$$I_2 = \int_y^\infty g(s)(-\partial/\partial s)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

$$I_3 = \int g(s)(t\psi'(y) - t\psi'(s))e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

After integration by parts

$$I_2 = g(y) + \int g'(s)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

which cancels -g(y) and the second term is the same type as $\sigma_{2,t}(y,\eta,k)$. On the other hand, I_3 is the same term appearing in $(\partial/\partial y)\sigma_{1,t}(y,\eta,k)$. So after getting terms together we get the bound

$$|(\partial/\partial y)\sigma_{1,t}(y,\eta,k)| \le \frac{1}{\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|} + C\frac{t\psi''(y)}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|)^2}$$

In general $(\partial/\partial y)^N\sigma_{2,t}(y,\eta,k)$ has same type of terms as in $(\partial/\partial y)^N\sigma_{1,t}(y,\eta,k)$ plus terms in $(\partial/\partial y)^{N-1}\sigma_{2,t}(y,\eta,k)$. So we have the bound

$$|(\partial/\partial y)^N\sigma_{2,t}| \le \sum_{j=1}^{N}\frac{(t\psi''(y))^{j+1}}{(\sqrt{t\psi''(t)} + |t\psi'(y) + k - i\eta|)^{j+1}}$$

After combining estimates with respect to y, k and η variables we finally have

$$|(\frac{\partial}{\partial y})^N(\frac{\partial}{\partial \eta})^M d_k^m\sigma_{2,t}(y,\eta,t)| \le \frac{C_{N,M,k}}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|)^{M+m}}(1+\frac{(t\psi''(y))}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k -} \tag{10'}$$

Now using (10′), we want same estimates for extended symbols. First, when $k > -t\psi'(y)$

$$|(\partial/\partial\eta)^N F_{1,t}(y,\eta,k)| = \int_1^\infty (\partial/\partial\eta)^N\sigma_{1,t}(y,\eta,(1-2\lambda)(k+t\psi'(y)) - t\psi'(y))\phi(\lambda)d\lambda|$$

$$\le \int_1^\infty \frac{C_N\phi(\lambda)d\lambda}{(\sqrt{t\psi''(y)} + |t\psi'(y) + (1-2\lambda)(k+t\psi'(y)) - t\psi'(y) - i\eta|)^{N+1}}$$

$$\leq \frac{C_N}{(\sqrt{t\psi''(y)}+|t\psi'(y)+k-i\eta|)^{N+1}}\int_1^\infty \phi(\lambda)d\lambda \quad (since\ \lambda \geq 1)$$

$$\leq \frac{C_N}{(\sqrt{t\psi''(y)}+|t\psi'(y)+k-i\eta|)^{N+1}}$$

$$|(\partial/\partial y)^M F_{1,t}(y,\eta,k)| \leq \sum_l^M \int_1^\infty (1-2\lambda)^l d_k^l (\partial/\partial y)^{M-l}(t\psi''(y))^l \sigma_{1,t}$$

$$(y,\eta,(1-2\lambda)(k+t\psi'(y))-t\psi'(y))\phi(\lambda)d\lambda$$

$$\leq \left(\frac{C_M(t\psi''(y))^M}{(\sqrt{t\psi''(y)}+|t\psi'(y)+k-i\eta|)^{M+1}}\right)\sum_l^M \int_1^\infty (1-2\lambda)^l \phi(\lambda)d\lambda$$

$$\leq C_M \frac{(t\psi''(y))^M}{(\sqrt{t\psi''(y)}+|t\psi'(y)+k-i\eta|)^{M+1}}$$

Also the same argument gives

$$|d_k^m F_{1,t}(y,\eta,k)| = |\int_1^\infty (1-2\lambda)^m d_k^m \sigma_{1,t}(y,\eta,(1-2\lambda)(k+t\psi'(y))-t\psi'(y))\phi(\lambda)d\lambda$$

$$\leq \frac{C_m}{(\sqrt{t\psi''(y)}+|t\psi'(y)+k-i\eta|)^{m+1}}$$

The same estimate is true for $F_{2,t}(y,\eta,k)$, *when* $k < -t\psi'(y)$. (From now on for the sake of simplicity we will replace $F_{1,t}(y,\eta,k)$ by $\sigma_{1,t}(y,\eta,k)$ and $F_{2,t}(y,\eta,k)$ by $\sigma_{2,t}(y,\eta,k)$. With these estimates, we can write B_t as follows.

$$B_{1,t}f(y,w) = \sum_k \int \sigma_{1,t}(y,\eta,k)\hat{f}(\eta,.)(w)e^{iy\eta}d\eta$$

$$B_{1,t}f(y,w) = \sum_k \int \sigma_{2,t}(y,\eta.k)\hat{f}(\eta,.)(w)e^{iy\eta}d\eta$$

With these definitions we want to show the following.

$$||B_{i,t}f||_{L^2(R^-\times S,dx)} \leq C||f||_{L^p(R^-\times S,dx)} \quad i=1,2$$

Now using duality at (11) we get the spherical restriction theorem we need;

$$||\pi_{M,N}g||_{L^2(S;C^m)} \leq C(N^{\frac{n-2}{2}}(N-M+1)^{n/2})^{1/2-1/q}||g||_{L^{q'}(S;C^m)} \tag{11'}$$

$$for\ q' = p = \frac{6n-4}{3n+2}, \quad q = \frac{6n-4}{3n-6}$$

With this theorem we introduce partition of unity $\{\phi_\beta\}$ satisfying (12) and define

$$\sigma_{1,t}^\beta(y,\eta,k) = \phi_\beta\left(\frac{1}{\sqrt{a}+|t\psi'(y)+k-i\eta|}\right)\sigma_{1,t}(y,\eta,k)$$

Then $\sigma_{1,t}(y,\eta,k)$ satisfies (13) and followingly we have

$$|(\partial/\partial\eta)^j\sigma_{1,t}^{\beta}(y,\eta,k)| \le C_j(2^{\beta}\sqrt{a})^{-1-j}$$

$$|(\partial/\partial\eta)^j(\sigma_{1,t}^{\beta}(y,\eta,k)-\sigma_{1,t}^{\beta}(y,\eta,k+1))| \le C_j(2^{\beta}\sqrt{a})^{-2-j} \tag{14'}$$

uniformly for $y<0$. Now define

$$F_t^{\beta}f(y,w)=\sum_k \frac{1}{2\pi}\int \sigma_{1,t}^{\beta}(y,\eta,k)\pi_k\tilde{f}(\eta,.)(w)e^{iy\eta}d\eta$$

Then $B_{1,t}=\sum_{\beta=0}^{N}F_t^{\beta}$.

We want to estimate F_t^{β} separately.

Let $M=[-t\psi'(y)-2^{\beta}\sqrt{a}]$, $M'=[M+2\times 2^{\beta}\sqrt{a}]+1$. Denote

$$T_t^{\beta}(y,\eta)g(w)=\sum_k F_t^{\beta}(y,\eta,k)\pi_k g(w)$$

Here $F_t^{\beta}=0$ unless $M\le k\le M'$. A summation by parts gives

$$T_t^{\beta}(y,\eta)=\sum_{M}^{M'}(F_t^{\beta}(y,\eta,k)-F_t^{\beta}(y,\eta,k+1))\pi_{M,k} \quad for \quad M\le k\le M'$$

Now (11') implies

$$||\pi_{M,k}g||_{L^2(S;C^m)} \le C(t^{\frac{n-2}{2}}(2^{\beta}\sqrt{a})^{n/2})^{1/2-1/q}||g||_{L^{q'}(S;C^m)} \quad for\beta \le N-1 \tag{11''}$$

Therefore by (14')

$$||(\partial/\partial\eta)^jT_t^{\beta}(y,\eta)\pi_{M,k}g||_{L^2(S,C^m)} \le C_j(2^{\beta}\sqrt{a})^{-1-j}(t^{\frac{n-2}{2}}(2^{\beta}\sqrt{a})^{n/2})^{1/2-1/q}||g||_{L^{q'}(S;C^m)}$$

uniformly for $y<0$.

Define for z=y-y'

$$K_t^{\beta}(y,z)=\frac{1}{2\pi}\int(\partial/\partial\eta)^jT_t^{\beta}(y,\eta)\frac{1}{(iz)^j}e^{iz\eta}d\eta$$

Then since the length of the interval in η where T_t^{β} is non-zero is less than $2\times 2^{\beta}\sqrt{a}$,

$$||K_t^{\beta}(y,z)g||_{L^2(S;C^m)} \le C_j|2^{\beta}\sqrt{a}z|^{-j}(t^{\frac{n-2}{2}}(2^{\beta}\sqrt{a})^{n/2})^{1/2-1/q}||g||_{L^{q'}(S;C^m)}$$

Now if we take the value j=1, 10 we get

$$||K_t^{\beta}(y,z)g||_{L^2(S;C^m)} \le C(1+|2^{\beta}\sqrt{a}z|)^{-10}(t^{\frac{n-2}{2}}(2^{\beta}\sqrt{a})^{n/2})^{1/2-1/q}||g||_{L^{q'}(S;C^m)}$$

Note that

$$F_t^{\beta}f(y,w)=\int K_t^{\beta}(y,y-y')f(y',.)(w)dy'$$

We can apply Young's inequality as before and we get for $\beta \leq N-1$

$$||F_t^\beta f||_{L^2(R^- \times S, dx)} \leq C||(1+|2^\beta \sqrt{a} z|)^{-10}||_{L^r(R,dz)} (t^{\frac{n-2}{2}} (2^\beta \sqrt{a})^{n/2})^{1/2-1/q} ||f||_{L^p(R^- \times S, dx)}$$

$$for \ \frac{1}{r} + \frac{1}{p} = \frac{1}{2} + 1$$

Note that

$$||(1+|2^\beta \sqrt{a} z|)^{-10}||_{L^r(dz)} \leq C(2^\beta \sqrt{a})^{-1/r}$$

Then for $\frac{1}{\gamma} = \frac{1}{2} - \frac{1}{q} = \frac{2}{3n-2}$, and $\frac{1}{r} = \frac{3n-4}{3n-2}$ we finally have

$$\begin{aligned} ||F_t^\beta f||_{L^2(R^- \times S, dx)} &\leq C(2^\beta \sqrt{a})^{-1/r} (t^{\frac{3n-4}{4}} (2^\beta)^{n/2} (\psi''(y))^{n/2})^{1/\gamma} ||f||_{L^p(R^- \times S, dx)} \\ &= C(2^\beta)^{(n/2 \times 1/\gamma) - 1/r} (\psi''(y))^{-\frac{1}{2r} + (\frac{n}{2} \times \frac{1}{\gamma})} ||f||_{L^p(R^- \times S, dx)} \end{aligned}$$

Since 2^β has negative power, if we sum the series in β we get exactly same bound as $L^2 \to L^q$ estimate, i.e.

$$||\sum_\beta^{N-1} F_t^\beta f||_{L^2(R^- \times S, e^{ny} dy dw)} \leq C||f||_{L^p(R^- \times S, e^{(n-\epsilon/3)y} dy dw)}$$

Now the case $\beta = N$, we need a different estimate for F_t^N. We had

$$F_t^N f(y,w) = \sum_k \int \sigma_{1,t}^N(y,\eta,k) \tilde{f}(\eta,.)(w) e^{iy\eta} d\eta$$

Since σ_t^N had support where

$$|t\psi'(y) + k - i\eta| \geq 2^N \sqrt{a},$$

we had

$$|t\psi'(y) + k - i\eta| > c(1+|\eta|+|k|) \ \ uniformly \ for \ y < 0.$$

Hence

$$|(\partial/\partial\eta)^j (\partial/\partial y)^m d_k^l \sigma_t^N(y,\eta,k)| \leq C_{j,m}(1+|\eta|+|k|)^{-1-j-l} \quad j = 0,1,.. \tag{15}$$

Here d_k^l is a difference operator of order l in the k variable. The above estimate means that $\sigma_{1,t}^N$ is a standard symbol in the (y,η,k) variables of order -1. So F_t^N is a standard pseudodifferential operator of order -1,[Taylor p296] and we can write F_t^N as

$$F_t^N f(y,w) = \int_{S \times R} K_t((w,y),(w',y')) f(w',y') dw' dy'$$

Then the corresponding kernel K_t has bounds

$$|K_t((w',y')(w,y))| \leq C(|w-w'| + |y-y'|)^{-n+1}$$

Hence F_t^N is a bounded operator from $L^p \to L^{\tilde{p}}$ for $\frac{1}{p} - \frac{1}{\tilde{p}} = \frac{1}{n}$ (Stein,[9],p128) i.e.

$$||F_t^N f||_{L^{\tilde{p}}(R^- \times S, dx)} \leq C||f||_{L^p(R^- \times S, dx)} \qquad (*)$$

On the otherhand, compactness of (w, w') and $n \geq 3$ gives us

$$\int |K_t((w', y'), (w, y))| dw' dy' \leq C \quad and$$

$$\int |K_t((w', y'), (w, y))| dw dy \leq C$$

Now we can apply Young's inequality to get $L^p \to L^p$ boundedness i.e.

$$||F_t^N f||_{L^p(R^- \times S, dx)} \leq C||f||_{L^p(R^- \times S, dx)} \qquad (**)$$

Now if we intrerpolate the above two estimates (*) , and (**) we get

$$||F_t^N f||_{L^q(R^- \times S, dx)} \leq C||f||_{L^p(R^- \times S, dx)}$$

for all $p \leq q \leq \tilde{p}$. And in particular this holds for q=2. So if we combine this with $\beta \leq N-1$ case we have

$$||B_{1,t} f||_{L^2(R^- \times S, e^{(n+\frac{2}{p})y} dy)} \leq C||f||_{L^p(R^- \times S, dx)}$$

Exactly same process shows us same inequality holds for $B_{2,t}$. Now define

$$P_1 f = \sum_{k<t} \pi_k f$$

$$P_2 f = \sum_{k>t} \pi_k f \qquad , then$$

$f = P_1 f + P_2 f$, and $P_1 B_1 A_t f = P_1 f$, and $P_2 B_2 A_t f = P_2 f$. Now

$$\begin{aligned}
||f||_{L^2(R^- \times S, e^{(n+\frac{2}{p})y} dy dw)} &= ||P_1 f + P_2 f||_{L^2(R^- \times S, e^{(n+\frac{2}{p})y} dy dw)} \\
&\leq ||P_1 f||_{L^2(R^- \times S, e^{(n+\frac{2}{p})y} dy dw)} + ||P_2 f||_{L^2(R^- \times S, e^{(n+\frac{2}{p})y} dy dw)} \\
&= ||P_1 B_1 A_t f||_{L^2(R^- \times S, e^{(n+\frac{2}{p})y} dy dw)} + ||P_2 B_2 A_t f||_{L^2(R^- \times S, e^{(n+\frac{2}{p})y} dy dw)} \\
&\leq ||B_1 A_t f||_{L^2(R^- \times S, e^{(n+\frac{2}{p})y} dy dw)} + ||B_2 A_2 f||_{L^2(R^- \times S, e^{(n+\frac{2}{p})y} dy dw)} \\
&\leq 2C||A_t f||_{L^p(R^- \times S, dx)}
\end{aligned}$$

Now for the future use, we would like to note the following corollary also holds.

$$||e^{i\phi} f||_{L^p(R^- \times S, dx)} \leq C||f||_{L^p(R^- \times S, dx)} \quad f \in C_0^\infty(R^- \times S)$$

We can apply exactly same argument as the $L^p \to L^2$ estimate. The idea is to prove slightly stonger estimate i.e.($L^p(R^- \times S) \to L^p(L^2(S), dy)$ estimate.) In this context we want to show the following

$$||B_i g||_{L^p(L^2(S), dy)} \leq C||g||_{L^p(R^- \times S, dx)} \quad i = 1, 2$$

First we have

$$||K_t^\beta(y,z)g||_{L^2(S)} \le C(1+|2^\beta\sqrt{a}z|)^{-10}(t^{\frac{n-2}{2}}(2^\beta\sqrt{a})^{n/2})^{\frac{1}{2}-\frac{1}{p}}||g||_{L^p(S)}$$

and

$$F_t^\beta f(y,w) = \int K_t^\beta(y, y-y')f(y',.)(w)dy'$$

Lemma
Let $H(y,y')$ be a bounded operator from $L^p(S)$ to $L^p(S)$ of operator norm $\le h(y-y')$ for each $y, y' \in R$. Suppose that $h \in L^1(R)$. Then

$$Tf(y,w) = \int H(y,y')f(y')(w)dy' \quad satisfies$$

$$||Tf||_{L^p(L^2(S),dy)} \le ||h||_{L^1(R)}||f||_{L^p(R^-\times S)}$$

Proof

$$\begin{aligned}
||Tf||^p_{L^p(L^2(S),dy)} &= \int_{-\infty}^0 \Big(\int_S \Big|\int_{-\infty}^0 H(y,y')f(y',.)(w)dy'\Big|^2 dw\Big)^{p/2} dy \\
&\le \int_{-\infty}^0 \Big(\int_{-\infty}^0 \Big(\int_S |H(y,y')f(y',.)(w)|^2 dw\Big)^{1/2} dy'\Big)^p dy \\
&\le \int_{-\infty}^0 \Big(\int_{-\infty}^0 h(y-y')||f(y',.)||_{L^p(S)}dy'\Big)^p dy \\
&\le ||h(y-y')||^p_{L^1(R)}||f||^p_{L^p(R^-\times S)}
\end{aligned}$$

Note that

$$||(1+|2^\beta\sqrt{a}z|)^{-10}||_{L^1(dz)} \le C(2^\beta\sqrt{a})^{-1}$$

and this gives

$$||\sum_\beta^{N-1} F_t^\beta f||_{L^p(L^2(S),dy)} \le C||f||_{L^p(R^-\times S, e^{(n-\epsilon/3)y}dydw)}$$

On the other hand, for $\beta = N$, we need only

$$||F_t^N f||_{L^p(R^-\times S, dz)} \le C||f||_{L^p(R^-\times S, dz)} \qquad (**)$$

After combining these two, we get

$$||B_{i,t}f||_{L^p(L^2(S),dy)} \le C||f||_{L^p(R^-\times S, e^{(n-\epsilon/3)y}dydw)} \quad i=1,2 \qquad (18'')$$

No we will prove our inequality. First, from the definition of $P_i's$ i=1,2, it is easy to check

$$||P_i f||_{L^p(L^2(S),dy)} \le ||f||_{L^p(L^2(S),dy)} \quad and$$

$$||f||_{L^p(R^-\times S, dz)} \le C||f||_{L^p(L^2(S),dy)}$$

Then as before

$$\begin{aligned}
||f||_{L^p(R^-\times S,dx)} &\leq ||P_1 f||_{L^p(R^-\times S,dx)} + ||P_2 f||_{L^p(R^-\times S,dx)} \\
&= ||P_1 B_1 A_t f||_{L^p(R^-\times S,dx)} + ||P_2 B_2 A_t f||_{L^p(R^-\times S,dx)} \\
&\leq C||P_1 B_1 A_t f||_{L^p(L^2(S),dy)} + C||P_2 B_2 A_t f||_{L^p(L^2(S),dy)} \\
&\leq C||B_1 A_t f||_{L^p(L^2(S),dy)} + C||B_2 A_t f||_{L^p(L^2(S),dy)} \\
&\leq 2C||A_t f||_{L^p(R^-\times S,dx)}
\end{aligned}$$

From now on we will use this estimate to prove Corollary 2.

Corollary 2

It suffices to prove that $u \equiv 0$ on a neighborhood of $-\infty$. Let η be a cut off function such that $\eta \equiv 1$ for $y \in (-\infty, -2\rho)$, $\eta \equiv 0$ for $y > -\rho$ for sufficirntly large ρ.
Then for $f = \eta\rho$, (18) is equivalent to the following.

$$||e^{t\psi}\eta u||_{L^q(S_\rho,dx)} \leq C||e^{t\psi}D(\eta u)||_{L^2(U,dx)}$$

where $S_\rho = (-\infty, -\rho) \times S^{n-1}$.
Since $D(\eta u) = (D\eta)u + \eta(Du)$, $\quad D\eta \equiv 0$ for $x \notin S_\rho$,

$$\begin{aligned}
||e^{t\psi}D(\eta u)||_{L^2(U,dx)} &\leq C|D\eta|_\infty ||e^{t\psi}u||_{L^2(S_\rho,dx)} \\
&+ C'||e^{t\psi}Du||_{L^2(U,dx)}
\end{aligned}$$

Furthermore, since (D+V)u =0, on U,

$$\begin{aligned}
||e^{t\psi}Du||_{L^2(U,dx)} &\leq ||e^{t\psi}Vu||_{L^2(S_\rho,dx)} \\
&+ ||e^{t\psi}Du||_{L^2(U\setminus S_\rho,dx)}
\end{aligned}$$

After getting terms together, we have

$$\begin{aligned}
||e^{t\psi}\eta u||_{L^q(S_\rho,dx)} &\leq C''(|D\eta|_\infty + ||V||_{L^r(S_\rho)})||e^{t\psi}u||_{L^q(S_\rho,dx)} \\
&+ C''||e^{t\psi}Du||_{L^2(U\setminus S_\rho,dx)}
\end{aligned}$$

For sufficiently large ρ , we can make sure

$$C''(|D\eta|_\infty + ||V||_{L^r(S_\rho)}) < 1.$$

Then we can make cancellation of the first term on the right hand side against the left hand side. Now after cancellation, we have

$$||e^{t\psi}u||_{L^q(S_\rho,dx)} \leq C||e^{t\psi}Du||_{L^2(U\setminus S_\rho)}$$

Then since $\eta \equiv 1$ for $y \in (-\infty, -2\rho) = S_{\rho/2}$,

$$||e^{t\psi}u||_{L^q(S_{\rho/2},dx)} \leq C||e^{t\psi(-\rho)}Du||_{L^2(U\setminus S_\rho)}$$

Now the fact that ψ is a decreasing function gives us

$$||e^{t(\psi(-2\rho)-\psi(-\rho))}u||_{L^q(S_{\rho/2},dx)} \leq C||Du||_{L^2(U,dx)} < \infty.$$

So as $t \to \infty$, we are forced to $u \equiv 0$ on $S_{\rho/2}$.

The Laplace operator

Proof of Theorem 3
We can decompose the right hand side of (3) i.e.

$$(e^{t\psi(y)}e^{2y}\Delta e^{-t\psi(y)})f(y,w) = \sum_k(\partial/\partial y - t\psi'(y) - k)(\partial/\partial y - t\psi'(y) + k + n - 2)\xi_k f(y,.)(w)$$

Let's denote this as $(A_t f)(y,w)$.
We are familiar with the first component of $A_t f$, i.e. the Dirac operator has the same expression. On the other hand the second component has a nice property, i.e. whose symbol $i\eta - t\psi'(y) + k + n - 2$ never vanishes. For this reason, we will first work on the main part of A_t which is

$$\sum_k(\partial/\partial y - ay + b)(\partial/\partial y - t\psi'(y) + k - n - 2)\pi_k \quad with$$

$$a = t\psi''(y), \quad b = t\psi''(y)y - t\psi'(y) - k$$

Let's denote this operator as B_t.
Now if we let $\sigma_t(y,\eta,k)$ be the symbol of the left inverse of $\partial/\partial y - ay + b$, then

$$\frac{\sigma_t(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2}$$

is a left parametrix. We want to inspect the property of $\sigma_t(y,\eta,k)$. Since

$$\sigma_t(y,\eta,k) \sim \frac{1}{\sqrt{a} + |t\psi'(y) + k - i\eta|}, \quad a = t\psi''(y) \sim te^{\epsilon y}$$

If we choose some large N_0, depending on t such that $te^{-\epsilon N_0} = \frac{1}{100}$, then for $y > -N_0$, $\sigma_t(y,\eta,k) \leq \frac{1}{\sqrt{a}} < 0(1)$.
On the other hand, for $y \leq -N_0$, $\sqrt{a}$ is vanishing fast as y goes to $-\infty$. But we can choose t such that $|t-k| > 1/4$. Also in this range $t(\psi'(y)+1) \sim te^{\epsilon y} << 1/10$.

Then $|t\psi'(y) + k - i\eta| \geq |-t + k - i\eta| - |t(\psi'(y)+1)|$ and $\sigma_t(y,\eta,k) \leq \frac{1}{|t\psi'(y)+k-i\eta|} \leq c$.
In either case,

$$\sigma_t(y,\eta,k) \sim 0(1) \; and \; \frac{\sigma_t(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2} \sim t^{-1}.$$

For $f \in C_0^\infty(R \times S)$, denote

$$F_t f(y,w) = \sum_k \int \frac{\sigma_t(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2} e^{iy\eta} \pi_k \hat{f}(\eta,.)(w) d\eta.$$

First we want to show the following inequality.

$$||F_t f||_{L^q(I\times S, dx)} \leq C e^{-ty/2} ||f||_{L^p(I\times S, dx)} \quad for\ f \in C_0^\infty(I \times S). \tag{19}$$

Because of the behavior of $\sigma_t(y,\eta,k)$, we want to proceed separately depending on the size of y.

As before we introduce partition of unity $\{\phi_\beta\}_{\beta=0}^N$ of the positive real axis satisfying (12). First, define

$$\sigma_t^\beta(y,\eta,k) = \phi_\beta(\frac{1}{\sqrt{a}}|t\psi'(y) + k - i\eta|)\sigma_t(y,\eta,k) \qquad y > -N_0.$$

$$\sigma_t^\beta(y,\eta,k) = \phi_\beta(|t\psi'(y) + k - i\eta|)\sigma_t(y,\eta,k) \quad for\ \ y < -N_0$$

From now on we will work on the first case.(i.e. $y > -N_0$) Then from (10) it is easy to show

$$|(\partial/\partial\eta)^j \frac{\sigma_t^\beta(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2}| \leq (2^\beta\sqrt{a})^{-1-j} t^{-1} \quad \beta \leq N-1 \tag{10'}$$

In the case $\beta = N$, since σ_t^N is supported where $|t\psi'(y) + k - i\eta| \geq t/400$, we have $|t\psi'(y) + k - i\eta| > c(t + |k| + |\eta|)$ uniformly for $y \in I_j$. So we get

$$|(\partial/\partial\eta)^j d_y^l \frac{\sigma_t^N(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2}| \leq C_{j,l}(1 + |k| + |\eta|)^{-2-j-l} \tag{10''}$$

Observe that since $\sigma_t^\beta(y,\eta,k)$ for $k \leq N-1$ is supported where $|t\psi'(y) + k - i\eta| \leq 2^\beta\sqrt{a}$, there are at most $22^\beta\sqrt{a}$ non-zero terms in the sum over k, and the value of k in each case is comparable to t.
Now we need Sogge's theorem for spherical harmonics.

Theorem
Let $1 \leq s \leq 2(n-1)/n, r = (n-1)s'/(n+1)$. Then

$$||\xi_k f||_{L^r(S^n)} \leq C k^{1/s'} ||f||_{L^s(S^n)}$$

Using the duality, this is equivalent to

$$||\xi_k f||_{L^{s'}(S^n)} \leq C k^{1/s'} ||f||_{L^{r'}(S^n)}$$

From the conditions of r, s we have the relation $1/s - 1/r > 2/n$. also, $1/r' - 1/s' > 2/n$. Now if we interpolate above two inequalities,we get

$$||\xi_k f||_{L^q(S^n)} \leq C k^{1/s'} ||f||_{L^p(S^n)} \quad for \tag{20}$$

$$1/q = t/r + (1-t)/s', 1/p = t/s + (1-t)/r' \quad and$$

$$1/r' = 1 - 1/r > \frac{n^2 + 2n - 4}{2n(n-1)} \; and \; 1/q - 1/p > 2/n$$

Now from the above estimate (19) and (10') we get

$$||\sum_k (\partial/\partial\eta)^j \frac{\sigma_t^\beta(y,\eta,k)}{t\psi'(y) + k - i\eta + n - 2} \xi_k f||_{L^q(S^n)} \leq (2^\beta\sqrt{a})^{-j} t^{-1} t^{1/s'} ||f||_{L^p(S^n)}.$$

Let

$$F_t^\beta f(y,w) = \sum_k \frac{1}{2\pi} \int \frac{\sigma_t^\beta(y,\eta,k)}{t\psi'(y) + k - i\eta + n - 2} \xi_k \hat{f}(\eta,.)(w) e^{iy\eta} d\eta \quad \beta = 0, 1, ...$$

Then $F_t^\beta f(y,w) = \sum_k \frac{1}{2\pi} \int K_t^\beta(y,y') f(y')(w) dy'$, where

$$\begin{aligned} K_t^\beta(z) &= \sum_k \frac{1}{2\pi} \int \frac{\sigma_t^\beta(y,\eta,k)}{t\psi'(y) + k - i\eta + n - 2} e^{iz\eta} d\eta \xi_k \\ &= \sum_k \frac{1}{2\pi} \int (\partial/\partial\eta)^j \frac{\sigma_t^\beta(y,\eta,k)}{t\psi'(y) + k - i\eta + n - 2} \frac{1}{(iz)^j} d\eta \xi_k \quad for \;\; k \leq N-1. \end{aligned}$$

the integration in η is over an interval of length $2 \times 2^\beta\sqrt{a}$. Therefore, $K_t^\beta(z)$ is a bounded operator from $L^p(S^n) \rightarrow L^q(S^n)$ whose operator norm is bounded by

$$C_j |2^\beta\sqrt{a}z|^{-j} (2^\beta\sqrt{a}) t^{-1} t^{1/s'}$$

using the values j=0, 10 we find that the norm is bounded by

$$C(2^\beta\sqrt{a}) t^{-1+1/s'} (1 + |2^\beta\sqrt{a}z|)^{-10}$$

Next, let 1/l +1/p =1/q +1, and apply Young's theorem and get

$$\begin{aligned} ||F_t^\beta f||_{L^q(I\times S,dx)} &\leq (2^\beta\sqrt{a})t^{-1+1/s'}||(1+|2^\beta\sqrt{a}z|)^{-10}||_{L^l(R,dz)}||f||_{L^p(I\times S,dx)} \\ &\leq (2^\beta\sqrt{a})t^{-1+1/s'}(2^\beta\sqrt{a})^{-1/l}||f||_{L^p(I\times S,dx)} \end{aligned}$$

Since 2^β has power $1-1/l = 1/p-1/q = 2/n+\delta$, and $-1+1/s' < -n/2n-2$, we get

$$\begin{aligned} ||\sum_\beta F_t^\beta f||_{L^r(I\times S,dx)} &\leq \sum_\beta (2^\beta\sqrt{a})^{\frac{2}{n}+\delta}t^{-n/(2n-2)}||f||_{L^s(I\times S,dx)} \\ &\leq^{(1)} Ct^{(\frac{2}{n}+\delta)-\frac{n}{2n-2}}||f||_{L^p(I\times S,dx)} \\ &\leq C't^\delta||f||_{L^p(I\times S,dx)} \end{aligned}$$

(1) holds since $\sum_{\beta=0}^{N-1}2^\beta \sim \sqrt{a}$. Now we want to estimate F_t^N.
It is obvious from (10") that the operators $(\partial/\partial y)^2F_t^N$ and $\Delta_s F_t^N$(with symbols

$$-\eta^2\frac{\sigma_t^N}{i\eta - t\psi'(y)+k+n-2} \; and - k(k+n-2)\frac{\sigma_t^N(y,\eta,k)}{i\eta - t\psi'(y)+k+n-2}),$$

are standard zero order multipliers.
They are therefore bounded from L^p to L^p *for* $1<p<\infty$. Sovolev theorem [4] implies

$$||h||_{L^q(I\times S)} \leq C||(\partial/\partial y)^2h||_{L^p(I\times S)} + ||\Delta_s h||_{L^p(I\times S)} \; for \; 1/p-1/q\leq 2/n$$

for all $h\in C_0^\infty((-\infty,-N_o)\times S)$. So we find that

$$||F_t^Nf||_{L^q((-\infty,-N_0)\times S)} \leq C||f||_{L^p((-\infty,-N_0)\times S)} \tag{19}$$

Since we have an estimate for the main term we will work on the remainder term and try to get similar estimate. Since

$$\begin{aligned} &F_tB_tf(y,w) = f(y,w) \\ &= \sum_k\int\int\frac{\sigma_t^\beta(y,\eta,k)}{i\eta - t\psi'(y)+k+n-2}(\partial/\partial y' - a(y)y' + b(y)) \\ &\times(\partial/\partial y' - t\psi'(y)+k+n-2)f(y',.)(w)e^{i(y-y')\eta}dy'd\eta\xi_k \end{aligned}$$

the difference $A_t - B_t$ appears in $R_tf(y,w) = F_tA_tf(y,w) - F_tB_tf(y,w)$ as the integrand.
If we compute

$$\begin{aligned} &(\partial/\partial y' - t\psi'(y') - k)(\partial/\partial y' - t\psi'(y')+k+n-2) \\ &-(\partial/\partial y' - a(y)y' + b(y))(\partial/\partial y' - t\psi'(y)+k+n-2) \\ &= \{(\partial/\partial y' - t\psi'(y') - k) - (\partial/\partial y' - a(y)y' + b(y))\}(\partial/\partial y' - t\psi'(y')+k+n-2) \\ &+(\partial/\partial y' - a(y)y' + b(y))\{(\partial/\partial y' - t\psi'(y')+k+n-2) - (\partial/\partial y' - t\psi'(y)+k+n-2)\} \end{aligned}$$

$$= t(y-y')^2 g(y,y')(\partial/\partial y' - t\psi'(y') + k + n - 2) + (\partial/\partial y' - a(y)y' + b(y))(t\psi'(y) - t\psi'(y'))$$

$$= t(y-y')^2 g(y,y')(\partial/\partial y' - t\psi'(y) + k + n - 2) + t(y-y')^2 g(y,y')(t\psi'(y) - t\psi'(y'))$$

$$+ t(y-y')h(y,y')(\partial/\partial y' - a(y)y' + b(y)) \quad for$$

$$g(y,y') = \int_0^1 (1-s)\psi'''((1-s)y + sy')ds, \quad and$$

$$h(y,y') = \int_0^1 (1-s)\psi''((1-s)y + sy')ds.$$

Then

$$R_t f(y,w) = R_{1,t} f(y,w) + R_{2,t} f(y,w) + R_{3,t} f(y,w) \quad for$$

$R_{1,t} f(y,w)$

$$= \sum_k \int\int \frac{t(y-y')^2 \sigma_t^\beta(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2} g(y,y')(\partial/\partial y' - t\psi'(y) + k + n - 2) f(y',.)(w) e^{i(y-y')\eta} dy' d\eta \xi_k$$

$$= \sum_k \int\int t(y-y')^2 \sigma_t(y,\eta,k) g(y,y') f(y',.)(w) e^{i(y-y')\eta} dy' d\eta \xi_k$$

$R_{2,t} f(y,w)$

$$= \sum_k \int\int \frac{\sigma_t(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2} (\partial/\partial y' - a(y)y' + b(y))(t\psi'(y) - t\psi'(y')) f(y',.) e^{i(y-y')\eta} dy' d\eta \xi_k$$

$$= \sum_k \int\int \frac{t(y-y')h(y,y')}{i\eta - t\psi'(y) + k + n - 2} f(y',.)(w) e^{i(y-y')\eta} dy' d\eta \xi_k$$

$R_{3,t} f(y,y')$

$$= \sum_k \int\int \frac{\sigma_t(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2} t(y-y')^2 g(y,y') t(y-y') h(y,y') f(y',.)(w) e^{i(y-y')\eta} dy' d\,\eta \xi_k$$

We want to show

$$||R_t f||_{L^p(I\times S, dx)} \leq C ||A_t f||_{L^p(I\times S, dx)}.$$

For this we first define $F^\beta_{1,t}, F^\beta_{2,t}, F^\beta_{3,t}$ for $\beta \leq N-1$ as follows.

$$F^\beta_{1,t} f(y,w) = \sum_k \int\int \frac{\sigma_t^\beta(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2} t g(y,y')(\partial/\partial\eta)^2 \sigma_t(y,\eta,k) f(y',.)(w) e^{i(y-y')\eta} dy' d\eta$$

$$F^\beta_{2,t} f(y,w) = \sum_k \int\int \frac{\sigma_t^\beta(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2} \frac{t(y-y')h(y,y')}{i\eta - t\psi'(y) + k + n - 2} f(y',.)(w) e^{i(y-y')\eta} dy' d\eta \xi_k$$

$$F^\beta_{3,t} f(y,w) = \sum_k \int\int \frac{\phi_\beta(y,\eta,k)\sigma_t^2(y,\eta,k)}{(i\eta - t\psi'(y) + k + n - 2)^2} (y-y')^3 g(y,y') h(y,y') f(y,.)(w) e^{i(y-y')\eta} dy' d\eta \xi_k$$

With these definitions, we want to show

$$||\sum_{\beta} F^{\beta}_{i,t}f||_{L^q(I\times S,dx)} \le C||f||_{L^p(I\times S,dx)} \quad for\ \ i = 1,2,3...\ separately.$$

First, we can rewrite $F^{\beta}_{1,t}f$ as

$$F^{\beta}_{1,t}f(y,w) = \int K^{\beta}_{1,t}(y,y')f(y',.)(w)dy', \quad \beta \le N-1$$

where

$$\begin{aligned} K^{\beta}_{1,t}(y,y') &= tg(y,y')\sum_k \int \frac{\sigma^{\beta}_t(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2}(\partial/\partial\eta)^2\sigma_t(t,\eta,k)e^{i(y-y')\eta}d\eta\xi_k \\ &= tg(y,y')\sum_k \int (\partial/\partial\eta)^j\bar{\sigma}^{\beta}_{1,t}(y,\eta,k)\frac{1}{(i(y-y'))^j}e^{i(y-y')\eta}d\eta\xi_k \ \ where \end{aligned}$$

$$\bar{\sigma}^{\beta}_{1,t}(y,\eta,k) = \frac{\sigma^{\beta}_t(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2}(\partial/\partial\eta)^2\sigma_t(y,\eta,k)$$

Then using (10') we can easily check that

$$t||g||_{\infty}|(\partial/\partial\eta)^j\bar{\sigma}^{\beta}_{1,t}(y,\eta,k)| \le C_j t^{-1}(2^{\beta}\sqrt{a})^{-2-j} \tag{22}$$

Observe that since $\bar{\sigma}^{\beta}_{1,t}(y,\eta,k)$ for $\beta \le N-1$ is supported where $|i\eta - t\psi'(y) - k| \le 2^{\beta}\sqrt{a}$, there are at most $22^{\beta}\sqrt{a}$ non-zero terms in the sum over k and the value of k in each case is comparable to t. Hence by Sogge's theorem and (21)

$$||tg\sum_k(\partial/\partial\eta)^j\bar{\sigma}^{\beta}_{1,t}(y,\eta,k)\xi_k f||_{L^q(S^n)} \le C_j(2^{\beta}\sqrt{a})^{-1-j}t^{-1+1/p'}||f||_{L^p(S^n)}$$

for $\beta = 0,1,..N-1$. In the expression of $K^{\beta}_{1,t}(y,z)$ when z=y-y', we can see the integration in η is over an interval of length $\le 22^{\beta}\sqrt{a}$. Therefore, $K^{\beta}_{1,t}(y,z)$ is a bounded operator from $L^p(S^n)$ to $L^q(S^n)$ whose operator norm is bounded by

$$C_j|2^{\beta}\sqrt{a}z|^{-j}t^{-1+1/p'}.$$

Using the values j=0 and 10 we find that the norm is bounded by

$$Ct^{-1+1/p'}(1 + |2^{\beta}\sqrt{a}z|)^{-10}.$$

Next, let 1/l +1/p =1/q +1 .Then after applying Young's inequality,

$$\begin{aligned}||F^{\beta}_{1,t}f||_{L^q(I\times S,dx)} &\leq Ct^{-1+1/p'}||(1+|2^{\beta}\sqrt{a}z|)^{-10}||_{L^l(R,dz)}||f||_{L^p(I\times S,dx)}\\ &= Ct^{-1+1/p'}(2^{\beta}\sqrt{a})^{-1/l}||f||_{L^p(I\times S,dx)}\\ &\leq C't^{-1/2}||f||_{L^p(I\times S,dx)}\end{aligned}$$

On the other hand when $\beta = N$, it behaves like an ordinary fractional integral operator of order 2 and we can show

$$||F^{N}_{1,t}f||_{L^q(I\times S,dx)} \leq C||f||_{L^p(I\times S,dx)} \quad for\ 1/p - 1/q \leq 2/n \tag{21'}$$

If we sum the series in β then we get

$$||\sum_{\beta}^{N} F^{\beta}_{1,t}f||_{L^q(I\times S,dx)} \leq C'||f||_{L^p(I\times S,dx)}$$

Secondly, we want to show $||F_{2,t}f||_{L^q(I\times S,dx)} \leq C||f||_{L^p(I\times S,dx)}$.
As before, we can rewrite $F^{\beta}_{2,t}f$ as

$$F^{\beta}_{2,t}f(y,w) = \int K^{\beta}_{2,t}(y,y')f(y',.)(w)d\eta$$

where,

$$\begin{aligned}K^{\beta}_{2,t}f(y,y') &= th(y,y')\int \frac{\sigma^{\beta}_t(y,\eta,k)}{i\eta - t\psi'(y)+k+n-2}\frac{(y-y')}{i\eta - t\psi'(y)+k+n-2}e^{i(y-y')\eta}d\eta\xi_k\\ &= th(y,y')\int \frac{(\partial/\partial\eta)\sigma^{\beta}_t(y,\eta k)}{(i\eta - t\psi'(y)+k+n-2)^2}e^{i(y-y')\eta}d\eta\xi_k \quad for\ \beta \leq N-1\end{aligned}$$

Let's denote

$$\tilde{\sigma}^{\beta}_{2,t}(y,\eta,k) = \frac{(\partial/\partial\eta)\sigma^{\beta}_t(y,\eta,k)}{(i\eta - t\psi'(y)+k+n-2)^2}.$$

Then using (10'), it's easy to check that

$$t|h|_{\infty}|(\partial/\partial\eta)^j\tilde{\sigma}^{\beta}_{2,t}(y,\eta,k)| \leq C_j t^{-1}(2^{\beta}\sqrt{a})^{-2-j} \tag{22'}$$

The above and Sogge's theorem implies

$$||t|h|_{\infty}\sum_{k}(\partial/\partial\eta)^j\tilde{\sigma}^{\beta}_{2,t}(y,\eta,k)\xi_k||_{L^q(S^n)} \leq C_j t^{-1+1/p'}(2^{\beta}\sqrt{a})^{-1-j}||f||_{L^p(S^n)}.$$

So $K^{\beta}_{2,t}(y,z)$ is a bounded operator from $L^p(S^n)$ to $L^q(S^n)$, whose operator norm is bounded by

$$t^{-1+1/p'}|2^{\beta}\sqrt{a}z|^{-j}, \quad for\ z = y - y'$$

and after applying Young's inequality,we get

$$||F^{\beta}_{2,t}f||_{L^q(I\times S,dx)} \le Ct^{-1+1/p'}||(1+|2^{\beta}\sqrt{a}z|)^{-10}||_{L^1(R,dz)}||f||_{L^p(I\times S,dx)}$$

and this implies

$$||\sum_{\beta=0}^{N-1} F^{\beta}_{2,t}f||_{L^q(I\times S,dx)} \le C''t^{-1/2}||f||_{L^p(I\times S,dx)} \tag{23}$$

And as before $F^N_{2,t}$ behaves like an ordinary fractinal integral operator and we get the same estimate.

Finally, we want to show

$$||F_{3,t}F||_{L^q(I\times S,dx)} \le C||f||_{L^p(I\times S,dx)}.$$

For this we can rewrite $F^{\beta}_{3,t}f$ as

$$F^{\beta}_{3,t}f(y,w) = \int K^{\beta}_{3,t}(y,y')f(y',.)(w)dy',$$

where

$$K^{\beta}_{3,t}(y,y') = t^2 g(y,y')h(y,y')\sum_k \int \tilde{\sigma}^{\beta}_{3,t}(y,\eta,k)e^{i(y-y')\eta}d\eta\xi_k \quad and$$

$$\tilde{\sigma}^{\beta}_{3,t} = \frac{\phi_{\beta}(y,\eta,k)\sigma_t(y,\eta,k)(\partial/\partial\eta)^3\sigma_t(y,\eta,k)}{(i\eta - t\psi'(y)+k+n-2)^2}$$

Then using (10'), it is easy to check

$$t^2|g|_{\infty}|h|_{\infty}|(\partial/\partial\eta)^j\tilde{\sigma}^{\beta}_{3,t}| \le C_j t^{-2}(2^{\beta}\sqrt{a})^{-1-j} \tag{22''}$$

From this and Sogge's theorem, it follows

$$||t^2gh\sum_k(\partial/\partial\eta)^j\tilde{\sigma}^{\beta}_{3,t}(y,\eta,k)\xi_k f||_{L^q(S^N)} \le Ct^{-2+1/p'}(2^{\beta}\sqrt{a})^{-j}||f||_{L^p(S^n)}.$$

and as before when z=y-y' $K^{\beta}_{3,t}(y,z)$ is a bounded operator from $L^p(S^n)$ to $L^q(S^n)$ with norm bounded by

$$C_j t^{-2+1/p'}(2^{\beta}\sqrt{a})|2^{\beta}\sqrt{a}z|^{-j}.$$

Using the values j=0 and 10 we find that the norm is bounded by

$$C't^{-2+1/p'}(2^{\beta}\sqrt{a})(1+|2^{\beta}\sqrt{a}z|)^{-10}.$$

Then for 1/q +1 =1/l +1/p, Young's theorem gives the following.

$$\begin{aligned} \|\sum_{\beta=0}^{N-1} F_{3,t}^{\beta} f\|_{L^q(I\times S,dx)} &\leq Ct^{-2+1/p'}(2^{\beta}\sqrt{a})^{1-1/l}\|f\|_{L^p(I\times S,dx)} \\ &= Ct^{-2+1/p'}t^{\frac{2}{n}+\delta}\|f\|_{L^p(I\times S,dx)} \\ &= C'\|f\|_{L^p(I\times S,dx)} \end{aligned}$$

And as before $F_{3,t}^N$ behaves like an ordinary fractional integral operator of order 2 and we get the same estimate as $F_{1,t}^N$. i.e.

$$\|F_{3,t}^N f\|_{L^q(I\times S,dx)} \leq C\|f\|_{L^p(I\times S)}$$

Now if we denote $F_t^1 f(y,w) = \sum_{i=1}^3 F_{i,t} f(y,w)$, then from the estimate

$$\|F_t^1 f\|_{L^q(I\times S,dx)} \leq C\|f\|_{L^p(I\times S,dx)} \tag{24}$$

we get the second remainder term.i.e.

$$R_t^2 f(y,w) = F_t^1 A_t f(y,w) - R_t^1 f(y,w), \quad R_t^1 = R_t$$

Let

$$\begin{aligned} V_t(y,y',\eta,k) &= t(y-y')^2 g(y,y')\sigma_t(y,\eta,k) + \frac{t(y-y')h(y,y')}{i\eta - t\psi'(y) + k + n - 2} \\ &+ \frac{t^2(y-y')^3 g(y,y')h(y,y')\sigma_t(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2} \end{aligned}$$

Then from the following expressions

$$F_t^1 f(y,w) = \sum_k \int\int \frac{\sigma_t(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2} V(y,y',\eta,k) f(y',.)(w) e^{i(y-y')\eta} dy' d\eta \xi_k, \quad \text{and}$$

$$R_t f(y,w) = \sum_k \int\int V(y,y',\eta,k) f(y',.)(w) e^{i(y-y')\eta} dy' d\eta \xi_k$$

we get

$$\begin{aligned} R_t^2 f(y,w) &= \sum_k \int\int \frac{\sigma_t(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2} V(y,y',\eta,k)\{A_t - B_t\} f(y',.)(w) e^{i(y-y')\eta} dy' d\eta \xi \\ &= \sum_k \int\int V(y,y',\eta,k)^2 f(y,.)(w) e^{i(y-y')\eta} dy' d\eta \xi_k \end{aligned}$$

Define

$$F_t^2 f(y,w) = \sum_k \int\int \frac{\sigma_t(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2} V(y,y',\eta,k)^2 f(y',.)(w) e^{i(y-y')\eta} dy' d\eta \xi_k$$

Then using the same techniques as before we can show that

$$||F_t^2 f||_{L^q(I\times S, e^{n y}dydw)} \leq C||f||_{L^p(I\times S, e^{n y}dydw)} \tag{25}$$

In this way we have a next remainder term,which we call $R_t^3 f(y,w)$ coming from

$$R_t^3 f(y,w) = F_t^2 A_t f(y,w) - R_t^2 f(y,w)$$

Now $R_t^3 f$ has the following form.

$$R_t^3 f(y,w) = \sum_k \int\int V(y,y',\eta,k)^3 f(y',.)(w) e^{i(y-y')\eta} dy' d\eta \xi_k$$

Claim

(1) $\quad ||R_t^3 f||_{L^q(I\times S, e^{n y}dydw)} \leq c||f||_{L^q(I\times S, e^{n y}dydw)} \quad c < 1/2$

(2) $\quad ||f||_{L^q(I\times S, dx)} \leq Ce^{-ty/2}||f||_{L^p(I\times S, dx)} \quad for\ f \in C_0^\infty(I\times S)\ \ for\ 1/p - 1/q \leq 2/n$

Proof of Claim

We want to show this is true for each term separately. First, let

$$R_{1,t}^{\beta,3} f(y,w) = \sum_k \int\int \phi_\beta(y,\eta,k) t^3 (y-y')^6 g^3(y,y') \sigma_t^3(y,\eta,k) f(y',.)(w) e^{i(y-y')\eta} dy' d\eta \xi_k$$

Now let's denote $\sigma_{1,t}^{\beta,3}(y,\eta,k) = \phi_\beta(y,\eta,k)(y-y')^6 \sigma_t^3(y,\eta,k)$ then

$$R_{1,t}^{\beta,3} f(y,w) = \sum \int\int t^3 g^3(y,y') \phi_\beta(y,\eta,k) (\partial/\partial\eta)^6 \sigma_t^3(y,\eta,k) f(y',.)(w) e^{i(y-y')\eta} dy' d\eta \xi_k$$

and from the fact that $\sigma_t^3 \in S^{-3}$, and (10') we can check that

$$|t^3 |g|_\infty^3 (\partial/\partial\eta)^j \sigma_{1,t}^{\beta,3}| \leq C_j t^3 |g|_\infty^3 (2^\beta\sqrt{a})^{-9-j} \leq C_j (2^\beta\sqrt{a})^{-3-j} \tag{26}$$

which implies

$$||t^3 |g|_\infty^3 (\partial/\partial\eta)^j \sigma_{1,t}^{\beta,3}(y,\eta,k)\xi_k f||_{L^q(S)} \leq C_j (2^\beta\sqrt{a})^{-2-j} t^\alpha ||f||_{L^q(S)} \quad for\ some\ \ \alpha < 1$$

On the other hand, we can write $R_{1,t}^{\beta,3} f(y,w) = \int K_{1,t}^{\beta,3}(y,y') f(y') dy'$, where

$$K_{1,t}^{\beta,3}(y,y') = t^3 g^3(y,y') \int \sigma_{1,t}^{\beta,3}(y,\eta,k) e^{i(y-y')\eta} d\eta \xi_k$$

Then the above estimate implies $K_{1,t}^{\beta,3}(y,y')$ is a bounded operator from $L^q(S) \rightarrow L^q(S)$, whose operator norm bounded by

$$t^\alpha (2^\beta\sqrt{a})^{-1} |2^\beta\sqrt{a}x|^{-j}.$$

Then by the usual procedure(including Young's theorem) we can show that

$$\begin{aligned}||R^{\beta,3}_{1,t}f||_{L^q(I\times S,dx)} &\leq t^{\alpha}(2^{\beta}\sqrt{a})^{-1}||(1+|2^{\beta}\sqrt{a}z|)^{-10}||_{L^1(R,dz)}||f||_{L^q(I\times S,dx)} \\ &\leq t^{\alpha}(2^{\beta}\sqrt{a})^{-2}||f||_{L^q(I\times S,dx)} \\ &\leq C_j 2^{-2\beta}t^{-1+\alpha}e^{-\epsilon y}||f||_{L^q(I\times S,dx)}\end{aligned}$$

Since $y > -N_0$, this means that

$$||\sum_{\beta} R^{\beta,3}_{1,t}f||_{L^q(I\times S,dx)} \leq c'||f||_{L^q(I\times S,dx)} \quad for\ c << 1 \tag{27}$$

Now for $\beta = N$, we need different estimate. By definition,

$$R^{N,2}_{1,t}f(y,w) = \sum \int\int t^2 g(y,y')^2\phi_\beta(y,\eta,k)(\partial/\partial\eta)^6\sigma^N_t(y,\eta,k)^3 f(y',.)(w)e^{i(y-y')\eta}dy'd\eta\xi_k$$

Then

$$|t^2(\partial/\partial\eta)^j(\partial/\partial y)^l\phi_\beta(y,\eta,k)(\partial/\partial\eta)^6\sigma^N_t(y,\eta,k)^3| \leq Ct^{-2}(1+|k|+|\eta|)^{-2-j-l} \tag{28}$$

It follows from (10') that $(\partial/\partial y)^2R^{N,3}_{1,t}$ and $\Delta_S R^{N,2}_{1,t}$ are standard zero order multipliers and therefore bounded from L^p to L^p. $1 < p < \infty$. Now Sobolev's theorem, we have

$$||h||_{L^q(I\times S,dx)} \leq C(||(\partial/\partial y)^2h||_{L^q(I\times S,dx)} + ||\Delta_S h||_{L^q(I\times S,dx)})$$

Combining above arguments, we find that

$$||R^{N,2}_{1,t}f||_{L^q(I_j\times S,dx)} \leq Ct^{-2}e^{-\epsilon j}||f||_{L^q(I_j\times S,dx)}$$

Secondly, let's define $R^{\beta,3}_{2,t}$ as

$$R^{\beta,3}_{2,t}f(y,w) = \sum_k\int\int\phi_\beta(y,\eta,k)\frac{t^3(y-y')^3h^3(y,y')}{(i\eta - t\psi'(y)+k+n-2)^3}f(y',')(w)e^{i(y-y')\eta}dy'dw\xi_k$$

If we denote

$$\sigma^{\beta,3}_{2,t}(y,\eta,k) = \frac{\phi_\beta(y-y')^3}{(i\eta - t\psi'(y)+k+n-2)^3}, \quad then$$

$$|t^3(\partial/\partial\eta)^j\sigma^{\beta,3}_{2,t}| \leq C'_j(2^{\beta}\sqrt{a})^{-3-j} \quad which\ implies \tag{29}$$

$$||\sum_k t^3(\partial/\partial\eta)^j\sigma^{\beta,3}_{2,t}(y,\eta,k)\xi_k f||_{L^q(S)} \leq C'_j t^{\alpha}(2^{\beta}\sqrt{a})^{-2-j}||f||_{L^p(S)} \quad \alpha << 1$$

On the other hand, we can write $R^{\beta,3}_{2,t}f(y,w) = \int K^{\beta,3}_{2,t}(y,y')f(y',.)dy'$, *where*

$$K^{\beta,3}_{2,t}(y,y') = h(y,y')^3\int\sigma^{\beta,3}_{2,t}(y,\eta,k)e^{i(y-y')\eta}d\eta\xi_k$$

Then the above estimate implies that when z=y-y' $K^{\beta,3}_{2,t}(y,z)$ is a bounded operator from $L^q(S)$ *to* $L^p(S)$ whose operator norm bounded by $t^{\alpha}e^{\epsilon y}(2^{\beta}\sqrt{a})^{-1}|2^{\beta}\sqrt{a}z|^{-j}$. Now by the same method as $R^{\beta,3}_{1,t}$ we can show that

$$||R^{\beta,3}_{2,t}f||_{L^q(I\times S,dx)} \leq Ct^{-1+\alpha}||f||_{L^q(I\times S,dx)} \tag{30}$$

And for $\beta = N$, we get the same estimate as $R^{N,3}_{1,t}$. All other terms appearing in $R^3_t f$ are combinations of $R^{\beta,3}_{1,t}$ and$R^{\beta,3}_{2,t}$ and they give the same estimates. So we get

$$||R^3_t f||_{L^q(I\times S,dx)} \leq c'||f||_{L^q(I\times S,dx)} \quad c' < 1/2 \tag{31}$$

Now we want to show the second half of the claim. Since we have

$$||\sum_\beta F^\beta_t f||_{L^q(I\times S)} \leq Ct^\delta ||f||_{L^p(I\times S)} \quad for\ 1/p - 1/q \geq 2/n$$

take δ small enough and use interpolation theorem coming from

$$||F^\beta_t||_{L^2(I\times S,dx)} \leq t^{-1}(2^\beta\sqrt{a}z)^{-j}||f||_{L^2(I\times S,dx)},$$

which is

$$\begin{aligned} ||\sum_\beta F^\beta_t f||_{L^2(I\times S,dx)} &\leq \sum_\beta t^{-1}(2^\beta\sqrt{a})^{-1}||f||_{L^2(I\times S,dx)} \\ &\leq Ct^{-3/2}(\psi''(y))^{-1/2}||f||_{L^2(I\times S,dx)} \\ &\leq C't^{-3/2}e^{-\epsilon y/2}||f||_{L^2(I\times S,dx)} \end{aligned}$$

And similary use the same interpolation for terms like

$$||\sum_\beta F^{i,\beta}_t f||_{L^q(I\times S)} \leq C||f||_{L^p(I\times S)} \quad i = 2,3, \ \ 1/p - 1/q \geq 2/n$$

we gain some power of t after interpolation. Now from the relation

$$f = F_t A_t f - R^1_t f, \quad and\ R^i_t f = F^{i-1}_t A_t f - R^{i-1}_t f, \ \ i = 2,3$$

we get

$$\begin{aligned} ||f||_{L^q(I\times S)} &\leq ||\sum^{N-1} F^\beta_t A_t f||_{L^q(I\times S)} + ||F^N_t A_t f||_{L^q(I\times S)} \\ &+ ||\sum^{N-1} F^{2,\beta}_t A_t f||_{L^q(I\times S)} + ||F^{2,N}_t A_t f||_{L^q(I\times S)} \\ &+ ||\sum^{N-1} F^{3,\beta}_t A_t f||_{L^q(I\times S)} + ||F^{3,N}_t A_t f||_{L^q(I\times S)} + ||R^3_t f||_{L^q(I\times S)} \\ &\leq^{(1)} Ce^{-\epsilon y/2}||A_t f||_{L^p(I\times S)} + t^{-1+\alpha}||f||_{L^q(I\times S)} \end{aligned}$$

Inequality (1) follows from (21),(21') and above interpolation argument. Now since th second term of the last inequality has negative power of t we can cancel that term again the left hand side and get the desired estimate (19) for $1/p - 1/q = 2/n - \epsilon$.

From now on we will work on the second case (i.e. $y < -N_0$).

First, introduce $\{\phi_\beta\}_0^N$, satisfying (12) and $2^N < t/10 < 2^{N+1}$. We had two symbol estimate i.e.

$$|(\partial/\partial\eta)^j \frac{\sigma_t^\beta(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2}| \leq (2^\beta)^{-1-j} t^{-1} \quad \beta \leq N-1 \tag{10}$$

In the case $\beta = N$, since $\sigma_t^N(y,\eta,k)$ is supported where $|t\psi'(y) + k - i\eta| \geq t/400$, we hav $|t\psi'(y) + k - i\eta| > c(t + |k| + |\eta|)$ uniformly for $y \in I_j$. So we get

$$|(\partial/\partial\eta)^j d_y^l \frac{\sigma_t^N(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2}| \leq C_{j,l}(1 + k + |\eta|)^{-2-j-l} \tag{10'}$$

Observe that since $\sigma_t^\beta(y,\eta,k)$ for $k \leq N-1$ is supported where $|t\psi'(y) + k - i\eta| \leq 2$ there are at most 22^β non-zero terms in the sum over k, and the value of k in each case comparable to t.

Now from the above estimates (20), (10') and Sogge's theorem, we get

$$\|\sum_k (\partial/\partial\eta)^j \frac{\sigma_t^\beta(y,\eta,k)}{t\psi'(y) + k - i\eta + n - 2} \xi_k f\|_{L^q(S^n)} \leq (2^\beta)^{-j} t^{-1} t^{1/s'} \|f\|_{L^p(S^n)}.$$

Let

$$F_t^\beta f(y,w) = \sum_k \frac{1}{2\pi} \int \frac{\sigma_t^\beta(y,\eta,k)}{t\psi'(y) + k - i\eta + n - 2} \pi_k \hat{f}(\eta,.)(w) e^{iy\eta} d\eta \quad \beta = 0, 1, \ldots.$$

Then $F_t^\beta f(y,w) = \sum_k \frac{1}{2\pi} \int K_t^\beta(y,y') f(y')(w) dy'$, where

$$\begin{aligned} K_t^\beta(y,z) &= \sum_k \frac{1}{2\pi} \int \frac{\sigma_t^\beta(y,\eta,k)}{t\psi'(y) + k - i\eta + n - 2} e^{iz\eta} d\eta \xi_k \\ &= \sum_k \frac{1}{2\pi} \int (\partial/\partial\eta)^j \frac{\sigma_t^\beta(y,\eta,k)}{t\psi'(y) + k - i\eta + n - 2} \frac{1}{(iz)^j} d\eta \xi_k \quad for \;\; k \leq N-1. \end{aligned}$$

the integration in η is over an interval of length $2 \times 2^\beta$. Therefore, $K_t^\beta(y,z)$ is a bounde operator from $L^p(S^n) \to L^q(S^n)$ whose operator norm is bounded by

$$C_j |2^\beta z|^{-j} (2^\beta) t^{-1} t^{1/s'}$$

using the values j=0, 10 we find that the norm is bounded by

$$C(2^\beta) t^{-1+1/s'} (1 + |2^\beta z|)^{-10}$$

Next, let 1/l +1/p =1/q +1, and apply Young's theorem and get

$$\begin{aligned}||F_t^\beta f||_{L^q(I\times S,dx)} &\leq (2^\beta)t^{-1+1/s'}||(1+|2^\beta z|)^{-10}||_{L^l(R,dz)}||f||_{L^p(I\times S,dx)}\\ &\leq (2^\beta)t^{-1+1/s'}(2^\beta)^{-1/l}||f||_{L^p(I\times S,dx)}\end{aligned}$$

Since 2^β has power $1-1/l=1/p-1/q=2/n+\delta$, and $-1+1/s'<\frac{-n}{(2n-2)}$, we get

$$\begin{aligned}||\sum_\beta F_t^\beta f||_{L^r(I\times S,dx)} &\leq \sum_\beta (2^\beta)^{2/n+\delta}t^{-n/(2n-2)}||f||_{L^s(I\times S,dx)}\\ &\leq^{(1)} Ct^{(\frac{2}{n}+\delta)-\frac{n}{2n-2}}||f||_{L^p(I\times S,dx)}\\ &\leq C't^\delta||f||_{L^p(I\times S,dx)}\end{aligned}$$

(1) holds since $\sum_{\beta=0}^{N-1}2^\beta\sim t$. Now we want to estimate F_t^N.
It is obvious from (10") that the operators $(\partial/\partial y)^2F_t^N$ and $\Delta_s F_t^N$(with symbols

$$-\eta^2\frac{\sigma_t^N(y,\eta,k)}{i\eta-t\psi'(y)+k+n-2}\quad and\quad -k(k+n-2)\frac{\sigma_t^N(y,\eta,k)}{i\eta-t\psi'(y)+k+n-2},\ respectively)$$

are standard zero order multipliers.
They are therefore bounded from L^p to itself for $1<p<\infty$. Sovolev theorem [4] implies

$$||h||_{L^q(I\times S)}\leq C||(\partial/\partial y)^2h||_{L^p(I\times S)}+||\Delta_s h||_{L^p(I\times S)}\ \ for\ 1/p-1/q\leq 2/n$$

for all $h\in C_0^\infty((-\infty,-N_o)\times S)$. So we find that

$$||F_t^N f||_{L^q((-\infty,-N_0)\times S)}\leq C||f||_{L^p((-\infty,-N_0)\times S)}\tag{21}$$

From now on we will work on the remainder term and try to get similar estimate a $y>-N_0$ case. Since

$$F_tB_tf(y,w)=f(y,w),$$

if we denote

$$R_tf(y,w)=F_tA_tf(y,w)-F_tB_tf(y,w)$$

as the remainder term, then as before

$$R_tf(y,w)=R_{1,t}f(y,w)+R_{2,t}f(y,w)+R_{3,t}f(y,w),\quad and$$

$$R_{1,t}f(y,w)=\sum_k\int\int t(y-y')^2g(y,y')\sigma_t(y,\eta,k)f(y',.)(w)e^{i(y-y')\eta}dy'd\eta\xi_k$$

$$R_{2,t}f(y,w)=\sum_k\int\int\frac{t(y-y')h(y,y')}{i\eta-t\psi'(y)+k+n-2}f(y',.)(w)e^{i(y-y')\eta}dy'd\eta\xi_k$$

$$R_{3,t}f(y,w) = \sum_k \int\int \frac{t^2\sigma_t(y,\eta,k)(y-y')^3}{i\eta - t\psi'(y)+k+n-2} g(y,y')h(y,y')f(y',.)(w)e^{i(y-y')\eta}dy'd\eta\xi_k$$

We want to show

$$||R_t f||_{L^q(I\times S,dx)} \leq C||A_t f||_{L^p(I\times S,dx)}.$$

For this we define $F^{\beta}_{1,t}, F^{\beta}_{2,t}, F^{\beta}_{3,t}$ for $\beta \leq N-1$ as before.

$$F^{\beta}_{1,t}f(y,w) = \sum_k \int\int \frac{tg(y,y')\phi_\beta(y,\eta,k)}{i\eta - t\psi'(y)+k+n-2}(y-y')^2\sigma_t^2(y,\eta,k)f(y',.)(w)e^{i(y-y')\eta}dy'd\eta\xi_k$$

$$F^{\beta}_{2,t}f(y,w) = \sum_k \int\int \frac{\sigma_t^{\beta}(y,\eta,k)}{i\eta - t\psi'(y)+k+n-2}\frac{t(y-y')h(y,y')}{i\eta - t\psi'(y)+k+n-2}f(y',.)(w)e^{i(y-y')\eta}dy'd\eta\xi_k$$

$$F^{\beta}_{3,t}f(y,w) = \sum_k \int\int \frac{\phi_\beta(y,\eta,k)\sigma_t^2(y,\eta,k)}{(i\eta - t\psi'(y)+k+n-2)^2}(y-y')^3 g(y,y')h(y,y')f(y,.)(w)e^{i(y-y')\eta}dy'd\eta\xi_k$$

And then we want to show

$$||\sum_\beta F^{\beta}_{i,t}f||_{L^q(I\times S,dx)} \leq C||f||_{L^p(I\times S,dx)} \ for\ i=1,2,3...\ separately.$$

First, we can rewrite $F^{\beta}_{1,t}f$ as

$$F^{\beta}_{1,t}f(y,w) = \int K^{\beta}_{1,t}(y,y')f(y',.)(w)dy', \quad \beta \leq N-1$$

where

$$\begin{aligned} K^{\beta}_{1,t}(y,y') &= tg(y,y')\sum_k \int \frac{\phi_\beta(y,\eta,k)}{i\eta - t\psi'(y)+k+n-2}(\partial/\partial\eta)^2\sigma_t^2(t,\eta,k)e^{i(y-y')\eta}d\eta\xi_k \\ &= tg(y,y')\sum_k \int (\partial/\partial\eta)^j\tilde{\sigma}^{\beta}_{1,t}(y,\eta,k)\frac{1}{(i(y-y'))^j}e^{i(y-y')\eta}d\eta\xi_k, \quad where \end{aligned}$$

$$\tilde{\sigma}^{\beta}_{1,t}(y,\eta,k) = (\partial/\partial\eta)^2\frac{\phi_\beta(y,\eta,k)\sigma_t^2(y,\eta,k)}{i\eta - t\psi'(y)+k+n-2}$$

Then using (10') we can easily check that

$$t||g||_\infty|(\partial/\partial\eta)^j\tilde{\sigma}^{\beta}_{1,t}(y,\eta,k)| \leq C_j e^{\epsilon y}(2^\beta)^{-4-j} \tag{22'}$$

Observe that since $\tilde{\sigma}^{\beta}_{1,t}(y,\eta,k)$ for $\beta \leq N-1$ is supported where $|i\eta - t\psi'(y) - k| \leq 2^\beta$, there are at most $2\times 2^\beta$ non-zero terms in the sum over k and the value of k in each case is comparable to t.

Hence by Sogge's theorem and (22')

$$||t\sum_k(\partial/\partial\eta)^j\tilde{\sigma}^{\beta}_{1,t}(y,\eta,k)\xi_k f||_{L^q(S^n)} \leq C_j(2^\beta)^{-3-j}t^{1/p'}||f||_{L^p(S^n)} \ for\ \beta=0,1,...,N-1.$$

In the expression of $K^{\beta}_{1,t}(y,z)$ when z =y - y', we can see that the integration in η is over an interval of length $\leq 22^{\beta}$. Therefore, $K^{\beta}_{1,t}(y,z)$ is a bounded operator from $L^p(S^n)$ to $L^q(S^n)$ whose operator norm is bounded by

$$C_j e^{\epsilon y}(2^{\beta})^{-2}|2^{\beta}z|^{-j}t^{1/p'}.$$

Using the values j=0 and 10 we find that the norm is bounded by

$$Ct^{1/p'}e^{\epsilon y}(2^{\beta})^{-2}(1+|2^{\beta}z|)^{-10}.$$

Next, let 1/l +1/p =1/q +1 .Then after applying Young's inequality,

$$\begin{aligned}
||F^{\beta}_{1,t}f||_{L^q(I\times S,dx)} &\leq Ct^{1/p'}(2^{\beta})^{-2}e^{\epsilon y}||(1+|2^{\beta}z|)^{-10}||_{L^l(R,dz)}||f||_{L^p(I\times S,dx)} \\
&= Ct^{1/p'}(2^{\beta})^{-2-1/l}e^{\epsilon y}||f||_{L^p(I\times S,dx)} \\
&\leq^{(1)} C'||f||_{L^p(I\times S,dx)}
\end{aligned}$$

(1) holds since $e^{\epsilon y}$ vanishes fast as y goes to $-\infty$. On the other hand when $\beta = N$, it behaves like ordinary fractional integral operator of order 2 and we can show

$$||F^{N}_{1,t}f||_{L^q(I\times S,dx)} \leq C'||f||_{L^p(I\times S,dx)} \quad 1/p-1/q \leq 2/n \tag{21''}$$

If we sum the series in β then we get

$$||\sum_{\beta}^{N} F^{\beta}_{1,t}f||_{L^q(I\times S,dx)} \leq C'||f||_{L^p(I\times S,dx)}$$

Secondly, we want to show $||F_{2,t}f||_{L^q(I\times S,dx)} \leq C||f||_{L^p(I\times S,dx)}$.
As before, we can rewrite $F^{\beta}_{2,t}f$ as

$$F^{\beta}_{2,t}f(y,w) = \int K^{\beta}_{2,t}(y,y')f(y',.)(w)d\eta \quad \textit{where},$$

$$\begin{aligned}
K^{\beta}_{2,t}f(y,y') &= th(y,y')\int \frac{\sigma^{\beta}_t(y,\eta,k)}{i\eta - t\psi'(y)+k+n-2}\frac{(y-y')}{i\eta - t\psi'(y)+k+n-2}e^{i(y-y')\eta}d\eta\xi_k \\
&= th(y,y')\int \frac{(\partial/\partial\eta)\sigma^{\beta}_t(y,\eta k)}{(i\eta - t\psi'(y)+k+n-2)^2}e^{i(y-y')\eta}d\eta\xi_k \quad \textit{for } \beta \leq N-1
\end{aligned}$$

Let's denote

$$\tilde{\sigma}^{\beta}_{2,t}(y,\eta,k) = \frac{(\partial/\partial\eta)\sigma^{\beta}_t(y,\eta,k)}{(i\eta - t\psi'(y)+k+n-2)^2}$$

Then using (10'), it's easy to check that

$$t|(\partial/\partial\eta)^j\tilde{\sigma}^{\beta}_{2,t}(y,\eta,k)| \le C_j t^{-1}(2^{\beta})^{-2-j}$$

The above and Sogge's theorem implies

$$||t\sum_k(\partial/\partial\eta)^j\tilde{\sigma}^{\beta}_{2,t}(y,\eta,k)\xi_k||_{L^q(S^n)} \le C_j t^{-1+1/p'}(2^{\beta})^{-1-j}||f||_{L^p(S^n)}.$$

So $K^{\beta}_{2,t}(y,z)$ is a bounded operator from $L^p(S^n)$ to $L^q(S^n)$, whose operator norm is bounded by $t^{-1+1/p'}e^{\epsilon y}|2^{\beta}z|^{-j}$, when z=y-y', and after applying Young's inequality,we get

$$||F^{\beta}_{2,t}f||_{L^q(I\times S,dx)} \le Ct^{-1+1/p'}e^{\epsilon y}||(1+|2^{\beta}z|)^{-10}||_{L^l(R,dz)}||f||_{L^p(I\times S,dx)}$$

and this implies

$$||\sum_{\beta=0}^{N-1}F^{\beta}_{2,t}f||_{L^q(I\times S,dx)} \le C''||f||_{L^p(I\times S,dx)} \tag{23'}$$

And as before $F^N_{2,t}$ behaves like a ordinary fractinal operator and we get

$$||F^N_{2,t}f||_{L^q(I\times S,dx)} \le C||f||_{L^p(I\times S,dx)}$$

The above two inequalities imply

$$||F_{2,t}f||_{L^q(I\times S,dx)} \le C'||f||_{L^p(I\times S,dx)}$$

Finally, using the same method,we can show

$$||F_{3,t}F||_{L^q(I\times S,dx)} \le Ct^{-1/2}||f||_{L^p(I\times S,dx)}.$$

Now if we denote $F^2_t f(y,w) = \sum_{i=1}^3 F_{i,t}f(y,w)$, then from the estimate

$$||F^2_t f||_{L^q(I\times S,dx)} \le Ct^{-1/2}||f||_{L^p(I\times S,dx)} \tag{24'}$$

we get the second remainder term as before i.e.

$$R^2_t f(y,w) = F^2_t A_t f(y,w) - R^1_t f(y,w), \quad R^1_t = R_t$$

$$R^2_t f(y,w) = \sum_k \int\int V(y,y',\eta,k)^2 f(y,.)(w)e^{i(y-y')\eta}dy'd\eta\xi_k$$

Define

$$F^3_t f(y,w) = \sum_k \int\int \frac{\sigma_t(y,\eta,k)}{i\eta - t\psi'(y) + k + n - 2}V(y,y',\eta,k)^2 f(y',.)(w)e^{i(y-y')\eta}dy'd\eta\xi_k$$

Then using the same techniques as before we can show that

$$||F_t^3 f||_{L^q(I\times S, e^{n\nu}dydw)} \le C||f||_{L^p(I\times S, e^{n\nu}dydw)} \tag{25'}$$

In this way we have the next remainder term,which we call $R_t^3 f(y,w)$ coming from

$$R_t^3 f(y,w) = F_t^3 A_t f(y,w) - R_t^2 f(y,w)$$

$R_t^3 f$ has the following form.

$$R_t^3 f(y,w) = \sum_k \int\int V(y,y',\eta,k)^3 f(y',.)(w) e^{i(y-y')\eta} dy' d\eta \xi_k$$

Claim

$$(1) \quad ||R_t^3 f||_{L^q(I\times S, e^{n\nu}dydw)} \le c||f||_{L^q(I\times S, e^{n\nu}dydw)} \quad c << 1$$

$$(2) \quad ||f||_{L^q(I\times S, dx)} \le Ce^{-\epsilon y/2}||f||_{L^p(I\times S, dx)} \quad for\ 1/p - 1/q \le 2/n$$

Proof of Claim We want to show this is true for each term separately. First, let

$$R_{1,t}^{\beta,3} f(y,w) = \sum_k \int\int t^3 (y-y')^6 \phi_\beta(y,\eta,k) g^3(y,y') \sigma_t^3(y,\eta,k) f(y',.)(w) e^{i(y-y')\eta} dy' d\eta \xi_k$$

Now if we denote $\sigma_{1,t}^{\beta,3}(y,\eta k) = \phi_\beta(y,\eta,k)(y-y')^6 \sigma_t^3(y,\eta,k)$ then

$$R_{1,t}^{\beta,3} f(y,w) = \sum t^3 \int\int g^3(y,y') \phi_\beta(y,\eta,k) (\partial/\partial\eta)^6 \sigma_t^3(y,\eta,k) f(y',.)(w) e^{i(y-y')\eta} dy' d\eta \xi_k$$

and from the fact that $\sigma_t^3 \in S^{-3}$, and (10') we can check that

$$|(\partial/\partial\eta)^j \sigma_{1,t}^{\beta,3}(y,\eta,k)| \le C_j (2^\beta)^{-9-j} \tag{26'}$$

which implies

$$||\sum_k (\partial/\partial\eta)^j \sigma_{1,t}^{\beta,3}(y,\eta,k)\xi_k f||_{L^q(S)} \le C_j (2^\beta)^{-8-j} t^\alpha ||f||_{L^q(S)} \quad for\ some\ \ \alpha < 1$$

On the other hand, we can write $R_{1,t}^{\beta,3} f(y,w) = \int K_{1,t}^{\beta,3}(y,y') f(y') dy'$. where

$$K_{1,t}^{\beta,3}(y,y') = t^3 g^3(y,y') \int \sigma_{1,t}^{\beta,3}(y,\eta,k) e^{i(y-y')\eta} d\eta \xi_k$$

Then the above estimate implies $K_{1,t}^{\beta,3}(y,z)$ is a bounded operator from $L^q(S) \to L^q(S)$, whose operator norm bounded by

$$t^{\alpha+3} e^{\epsilon y} (2^\beta)^{-7} |2^\beta z|^{-j}.$$

Then by the usual procedure(including Young's theorem) we can show that

$$\begin{aligned} \|R_{1,t}^{\beta,3} f\|_{L^q(I\times S,dx)} &\leq t^{\alpha+3} e^{\epsilon y} (2^\beta)^{-1} \|(1+|2^\beta z|)^{-10}\|_{L^1(R,dz)} \|f\|_{L^q(I\times S,dx)} \\ &\leq t^{\alpha+3} e^{\epsilon y} (2^\beta)^{-2} \|f\|_{L^q(I\times S,dx)} \\ &\leq C 2^{-2\beta} t^{3+\alpha} e^{\epsilon y} \|f\|_{L^q(I\times S,dx)} \end{aligned}$$

Since $y < -N_0$, this means as y goes to $-\infty$,

$$\|\sum_{\beta}^{N-1} R_{1,t}^{\beta,3} f\|_{L^q(I\times S,dx)} \leq c'\|f\|_{L^q(I\times S,dx)}, \quad c << 1 \tag{27'}$$

and for $\beta = N$, we need different estimate. From the definition,

$$R_{1,t}^{N,3} f(y,w) = \sum_k \int\int t^3 g(y,y')^3 \phi_\beta(y,\eta,k)(\partial/\partial\eta)^6 \sigma_t^N(y,\eta,k)^3 f(y',.)(w) e^{i(y-y')\eta} dy' d\eta \xi_k$$

Then,

$$|t^2(\partial/\partial\eta)^j (\partial/\partial y)^l \phi_\beta(y,\eta,k)(\partial/\partial\eta)^6 \sigma_t^N(y,\eta,k)^3| \leq t^{-2}(1+|k|+|\eta|)^{-2-j-l} \tag{28'}$$

It follows from (28') that $(\partial/\partial y)^2 R_{1,t}^{N,3}$ and $\Delta_S R_{1,t}^{N,3}$ are standard zero order multipliers and therefore bounded from L^p to L^p for $1 < p < \infty$. Now by Sobolev's theorem we get

$$\|h\|_{L^q(I_j\times S)} \leq C(\|(\partial/\partial y)^2 h\|_{L^q(I_j\times S)} + \|\Delta_S h\|_{L^q(I_j\times S)})$$

Combining these two argument we find that

$$\|R_{1,t}^{N,3} f\|_{L^q(I_j\times S,dx)} \leq C t^{-2} e^{-\epsilon j} \|f\|_{L^q(I_j\times S)}$$

Now define

$$R_{2,t}^{\beta,3} f(y,w) = t^3 \int\int h(y,y')^3 \sum_k \frac{\phi_\beta(y,\eta,k)(y-y')^3}{(i\eta - t\psi'(y)+k+n-2)^3} f(y',.)(w) e^{i(y-y')\eta} dy' d\eta \xi_k$$

If we denote

$$\sigma_{2,t}^{\beta,3}(y,\eta,k) = \frac{(y-y')^3 \phi_\beta(y,\eta,k)}{(i\eta - t\psi'(y)+k+n-2)^3}, \quad \textit{then}$$

$$|t^3(\partial/\partial\eta)^j \sigma_{2,t}^{\beta,3}| \leq C_j (2^\beta)^{-j-3} \quad \textit{which implies} \tag{29'}$$

$$\|t^3 \sum_k (\partial/\partial\eta)^j \sigma_{2,t}^{\beta,3}(y,\eta,k)\xi_k f\|_{L^q(S)} \leq C_j (2^\beta)^{-2-j} t^\alpha \|f\|_{L^p(S)} \quad \textit{for some } \alpha < 1$$

On the other hand,we can write

$$R_{2,t}^{\beta,3} f(y,w) = \int K_{2,t}^{\beta,3}(y,y') e^{i(y-y')\eta} dy', \quad \textit{where}$$

$$K_{2,t}^{\beta,3}(y,y') = t^3 h^3(y,y') \int \sigma_{2,t}^{\beta,3}(y,\eta,k) e^{i(y-y')\eta} d\eta \xi_k$$

Then the above estimate implies when z=y-y', $K_{2,t}^{\beta,3}(y,z)$ is a bounded operator from $L^q(S)$ to $L^q(S)$, whose operator norm is bounded by

$$Ce^{\epsilon y}t^{\alpha}(2^{\beta})^{-1}|2^{\beta}z|^{-j},$$

Then we can easily show that

$$||\sum_{\beta}^{N-1} R_{2,t}^{\beta,3}f||_{L^q(I\times S,dx)} \leq c'||f||_{L^q(I\times S,dx)} \quad for\ c' << 1 \tag{30'}$$

And for $\beta = N$, we get the same estimate as $R_{1,t}^{N,3}$ since $R_{2,t}^{N,3}$ is an ordinary fractional integral operator of order 2. Now all other terms appearing in $R_t^{\beta,3}f$ are combinations of $R_{1,t}^{\beta,3}$ and $R_{2,t}^{\beta,3}$ and they give same estimates and we get

$$||R_t^3 f||_{L^q(I\times S,dx)} \leq c''||f||_{L^q(i\times S,dx)} \quad for\ c'' < 1/2 \tag{31'}$$

Now the second half of the claim comes from the same interpolation argument as $y > -N_0$ case and we have (19).

Since we have the estimate

$$||f||_{L^q(I\times S,e^{(n+2\epsilon q)y}dydw)} \leq C||A_t f||_{L^p(I\times S,dx)}, \quad I = I_j$$

we want to extend this result to the whole $R^- \times S$.
First we want to remind one result we had before.

$$||e^{t\psi}f||_{L^2(R^-\times S,e^{(n+\epsilon)y}dydw)} \leq C'||e^{t\psi}e^{y}Df||_{L^p(R^-\times S,dx)} \tag{18'}$$

Now by replacing f =Dg, the above inequality (18") implies

$$||e^{t\psi}Df||_{L^2(R^-\times S,e^{(n+\epsilon)y}dydw)} \leq C||e^{t\psi}e^{y}\Delta f||_{L^p(R^-\times S,e^{ny}dydw)}$$

and

$$||e^{t\psi}e^{y}Df||_{L^2(R^-\times S,e^{(n+\epsilon)y}dydw)} \leq C||e^{t\psi}e^{2y}\Delta f||_{L^p(R^-\times S,e^{ny}dydw)} \tag{18'*}$$

Since

$$\begin{aligned} |Df|_{L^2}^2 &= |\sum_j \alpha_j \partial f/\partial x_j|_{L^2}^2 = \sum_j |\partial f/\partial x_j|_{L^2}^2 \\ &= |\nabla f|_{L^2}^2, \quad we\ get, \end{aligned}$$

$$||e^{t\psi(y)}\nabla f||_{L^2(R^-\times S,e^{ny}dydw)} \leq C||e^{t\psi(y)}\Delta f||_{L^p(R^-\times S,e^{ny}dydw)} \tag{2'}$$

On the other hand, we had (18") i.e.

$$||e^{t\psi}f||_{L^p(R^-\times S,e^{(n+\epsilon)y}dydw)} \leq C||e^{t\psi}Df||_{L^p(R^-\times S,e^{ny}dydw)} \quad f \in C_0^{\infty}(R^- \times S) \tag{18''}$$

By replacing f=Dg, (18") becomes

$$||e^{t\psi}e^{y}Df||_{L^p(R^-\times S, e^{(n+\epsilon y)}dydw)} \leq C||e^{t\psi}e^{2y}\Delta f||_{L^p(R^-\times S, e^{ny}dydw)} \qquad (18''^{*})$$

If we combine above two inequalities, we have

$$|||e^{t\psi}f||_{L^p(R^-\times S, e^{(n+2\epsilon)y}dydw)} \leq C||e^{t\psi}e^{2y}\Delta f||_{L^p(R^-\times S, e^{ny}dydw)} \qquad (32)$$

Now with these estimates, we can continue.
We can rewrite $f = \sum_j \phi_j f$, for $\{\phi_j\}$ partition of unities, having support in $(-j - 5/4, -j + 5/4)$ and for each $x \in R^-$,
there are only finitely many ϕ_j 's such that $\phi_j(x) \neq 0$. Then,

$$\begin{aligned} ||f||_{L^q(R^-\times S, e^{(n+2\epsilon q)y}dydw)} &= ||\sum_j \phi_j f||_{L^q(R^-\times S, e^{(n+2\epsilon q)y}dydw)} \\ &\leq (\sum_j ||\phi_j f||^q_{L^q(I_j\times S, e^{(n+2\epsilon q)y})})^{1/q} \\ &\leq (\sum_j ||A_t(\phi_j f)||^q_{L^p(I_j\times S, e^{ny}dydw)})^{1/q} \qquad (33) \end{aligned}$$

Since

$$\begin{aligned} A_t(\phi_j f) &= (\phi_j'')f + \sum_k \phi_j'(\partial/\partial y - t\psi'(y) - k)\xi_k f \\ &+ \sum_k \phi_j'(\partial/\partial y - t\psi'(y) + k + n - 2)\xi_k f + \phi_j A_t f \end{aligned}$$

We want to estimate each of them separately.
First,

$$\begin{aligned} \sum_j ||\phi_j'' f||_{L^p(I_j\times S, e^{(n+2\epsilon)y}dydw)} &\leq C||f||_{L^p(R^-\times S, e^{(n+2\epsilon)y}dydw)} \\ &\leq C'||A_t f||_{L^p(R^-\times S, dx)} \ by \ (32) \end{aligned}$$

Also

$$\sum_j ||\phi_j A_t f||_{L^p(I_j\times S, e^{(n+2\epsilon)y})} \leq C||A_t f||_{L^p(R^-\times S, dx)}.$$

Claim We can get similar estimates for the intermediate terms, i.e.

$$\sum_j ||\phi_j' \sum_k (\partial/\partial y - t\psi'(y) - k)\xi_k f||_{L^p(I_j\times S, e^{(n+2\epsilon)y})} \leq C||A_t f||_{L^p(R^-\times S, e^{ny}dydw)} \qquad (*)$$

$$\sum_j ||\phi_j' \sum_k (\partial/\partial y - t\psi'(y) + k + n - 2)\xi_k f||_{L^p(I_j\times S, e^{(n+2\epsilon)y}dydw)} \leq C||A_t f||_{L^p(R^-\times S, e^{ny}dydw)} \qquad (**)$$

Then after getting terms together, and from (33) we get

$$\begin{aligned}||f||_{L^q(R^-\times S, e^{(n+2\epsilon q+2\epsilon q/p)y}dydw)} &\leq (\sum_j ||A_t(\phi_j f)||^q_{L^p(I_j\times S, e^{(n+2\epsilon)y}dydw)})^{1/q} \\ &\leq C||A_t f||_{L^p(R^-\times S, dx)} \\ &= C||e^{t\psi}e^{2y}\Delta f||_{L^p(R^-\times S, dx)}\end{aligned}$$

The first inequality is from (33), the second is from (*), (**), and the third from the fact that

$$l^q \supset l^p \; for \; q > p$$

Now if we choose small ϵ, then $e^{2py} \leq e^{2\epsilon q+2\epsilon q/p}$. So the above inequality is equivalent to

$$||e^{t\psi}f||_{L^q(R^-\times S, dx)} \leq C||e^{t\psi}\Delta f||_{L^p(R^-\times S, dx)} \tag{3}$$

Now we will prove the claim. Since

$$\begin{aligned}&\sum_j ||\phi_j' \sum_k (\partial/\partial y - t\psi'(y) - k)\xi_k f||_{L^p(I_j\times S, e^{(n+2\epsilon)y}dydw)} \\ \leq\; & C||\sum_k (\partial/\partial y - t\psi'(y) - k)\xi_k f||_{L^p(R^-\times S, e^{(n+2\epsilon)y}dydw)} \\ =\; & C||e^{t\psi}e^{y}De^{-t\psi}f||_{L^p(R^-\times S, e^{(n+2\epsilon)y}dydw)} \\ \leq\; & C'||e^{t\psi}e^{2y}\Delta e^{-t\psi}f||_{L^p(R^-\times S, e^{ny}dydw)}\end{aligned}$$

The last inequality follows from (18″*)
On the other hand, (**) is equivalent to

$$\begin{aligned}\sum_j ||\phi_j' f||_{L^p(R^-\times S, e^{(n+2\epsilon)y}dydw)} &\leq C||f||_{L^p(R^-\times S, e^{(n+2\epsilon)y}dydw)} \\ &\leq C||\sum_k (\partial/\partial y - t\psi'(y) - k)\xi_k f||_{L^p(R^-\times S, e^{ny}dydw)}\end{aligned}$$

But the second inequality is just (18″).

Strong Unique Continuation Theorem

Proof of Corollary 3
We have two Carleman type estimates.

$$||e^{t\psi}u||_{L^q(R^-\times S, e^{ny}dydw)} \leq C||e^{t\psi}\Delta u||_{L^p(R^-\times S, e^{ny}dydw)}. \tag{3}$$

$$||e^{t\psi}\nabla u||_{L^2(R^-\times S, e^{ny}dydw)} \leq C||e^{t\psi}\Delta u||_{L^p(R^-\times S, e^{ny}dydw)}. \tag{2'}$$

$for\ \ p = \frac{6n-4}{3n+2}$ and $\frac{1}{p} - \frac{1}{q} = \frac{2}{n} - \epsilon$, for small $\epsilon > 0$.
It suffices to show $u \equiv 0$ in a neighborhood of $y = -\infty$.
Now let's introduce a cut off function $\eta \in C_0^\infty(R)$ such that $\eta \equiv 1$ for $y \in (-\infty, -2\rho)$, $\eta \equiv 0$ for $y > -\rho$.
Let $f = \eta u$. Then by limiting process,(3)and (2') gives

$$||e^{t\psi}\nabla f||_{L^2(S_\rho, e^{ny}dydw)} + ||e^{t\psi} f||_{L^q(S_\rho, e^{ny}dydw)} \leq C||e^{t\psi}\Delta f||_{L^p(R^n, e^{ny}dydw)}$$

for $S_\rho = \{x \in R^n; x = e^y w, y < -4\rho\}$. Then the right side is bounded by

$$C||e^{t\psi}\Delta f||_{L^p(R^n \setminus S_\rho)} + C||e^{t\psi} v \nabla f||_{L^p(S_\rho)} + C||e^{t\psi} w f||_{L^p(S_\rho)}.$$

$$\leq C||e^{t\psi}\Delta f||_{L^p(R^n \setminus S_\rho)} + C||v||_{L^t(S_\rho)}||e^{t\psi(y)}\nabla f||_{L^2(S_\rho)} + C||w||_{L^{\frac{n}{2}+\epsilon}(S_\rho)}||e^{t\psi(y)} f||_{L^q(S_\rho)}.$$

Now if we choose ρ large enough, we can assume

$$||v||_{L^t(S_\rho)}, ||w||_{L^{\frac{n}{2}+\epsilon}(S_\rho)} < \frac{1}{2C}.$$

This means we can cancel the second and third term, i.e.

$$C||v||_{L^t(S_\rho)}||e^{t\psi(y)}\nabla f||_{L^2(S_\rho)},\ \ and\ \ C||w||_{L^{\frac{n}{2}+\epsilon}(S_\rho)}||e^{t\psi(y)} f||_{L^q(S_\rho)}$$

can be canceled against

$$||e^{t\psi(y)}\nabla f||_{L^2(S_\rho, e^{ny}dydw)},\ \ and\ \ ||e^{t\psi(y)} f||_{L^q(S_\rho, e^{ny}dydw)}$$

Then after cancellation,

$$||e^{t\psi}\nabla f||_{L^2(S_\rho, dx)} + ||e^{t\psi} f||_{L^q(S_\rho, dx)} \leq C||e^{t\psi}\Delta f||_{L^p(R^n \setminus S_\rho, dx)}.$$

Since $\psi(y)$ is a decreasing function, for $y \notin \Pi_1(supp(f) \setminus S_\rho)$ we have $\psi(y) < \psi(\rho')$ for $\rho' \in \{y \in R; y < -4\rho\}$.

This implies

$$||e^{t(\psi(y)-\psi(\rho'))}f||_{L^q(S_\rho,e^{ny}dydw)} \leq C||\Delta f||_{L^p(R^n,e^{ny}dydw)} \leq C'.$$

Since $\psi(y) - \psi(\rho') \geq 0$, as $t \to \infty$ we are forced to $f \equiv 0$ on S_ρ.

References

1. L.Hörmander, "Uniqueness theorems for second order elliptic differential equations, Comm. Partial Differential Equations 88 (1) (1983), 21-64.

2. D.Jerison, "Carleman Inequalities for the Dirac and Laplace Operators and Unique Continuation," Advances in Mathematics 62 (2) (1986), 118-134.

3. D.Jerison and C.E.Kenig, "Unique continuation and absence of positive eingenvalues for Schrödinger operators," Ann. of Math. 121 (1985).

4. Barcelo, Ruiz, Kenig, and Sogge, "Weighted Sobolev inequalities and Unique continuation," Preprint.

5. C.Sogge, "Oscillatory Integrals and Spherical Harmonics," Thesis, Princeton University, Princeton, N.J.,1985.

6. Baouendi and Alinhac, "Uniqueness for the Characteristic Cauchy problem and strong unique continuation for higher order partial differential ineqyalities," Amer.J.Math. 102 (1980), 179-217.

7. E.M.Stein, "Singular Integrals and Differentiability Properties of Functions," Princeton Univ.Press, Princeton, N.J., 1970.

8. T.Carleman, "Sur un Problèm d'unicité pour les systèmes d'équations aux dérivées partielles á deux variables independent, Ark. Mat. B26 (1939),1-9

9. Taylor, "Pseudodifferential Operators," Princeton Univ. Press, Princeton, N.J.,1981