Essays on Corporate Finance Theory and Dynamic Games

by

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Submitted to the Department of Management
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Management

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 2022

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Abstract

This dissertation consists of three chapters.

In Chapter 1, I study the optimal algorithmic disclosure in a lending market where lenders use a predictive algorithm to mitigate adverse selection. The predictive algorithm is unobservable to borrowers and uses a manipulable borrower feature as input. A regulator maximizes market efficiency by disclosing information about the statistical properties of variables embedded in the predictive algorithm to borrowers. Under the optimal disclosure policy, the posterior belief consists of two disjoint regions in which the borrower feature is more relevant and less relevant in predicting borrower quality, respectively. The optimal disclosure policy differentiates posterior lending market equilibria by the equilibrium data manipulation levels. Equilibria with more data manipulation hurt market efficiency, but also discourage lenders' use of the borrower feature. Equilibria with less data manipulation benefit from that and generate more efficient market outcomes. Unconditionally, the borrower feature is used less intensively under optimal disclosure.

In Chapter 2, joint work with Mehmet Ekmekci, Leandro Gorno, Lucas Maestri and Dong Wei, we study a dynamic stopping game between a principal and an agent. The agent is privately informed about his type. The principal learns about the agent's type from a noisy performance measure, which can be manipulated by the agent via a costly and hidden action. We fully characterize the unique Markov equilibrium of this game. We find that terminations/market crashes are often preceded by a spike in (expected) performance. Our model also predicts that, due to endogenous signal manipulation, too much transparency can inhibit learning. As the players get arbitrarily patient, the principal elicits no useful information from the observed signal.

In Chapter 3, joint work with Dan Luo, we study SPACs in a continuous-time delegated investment model. The sponsor has an increasing incentive to propose unprofitable projects to the investor over time; in response, the investor exerts more stringent screening based on her information. The screening helps curb the sponsor's moral hazard, but also dampens the disciplining effect of partial alignment in incentives. When the investor's information is sufficiently noisy, the second effect dominates, so giving the investor the control over investment approval reduces everyone's welfare.

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Acknowledgments

I am deeply grateful to my dissertation chair Hui Chen, for his invaluable help, support, and encouragement along this journey. Hui not only helped me with every aspect of my research papers but also helped me go through every challenging moment during my Ph.D. The completion of this dissertation could not have been possible without his guidance. A debt of gratitude is also owed to Andrey Malenko for inspiring my interest in theoretical research, and for his suggestions and feedback on my research ideas from my first day at MIT. And I would like to express my sincere thanks to Haoxiang Zhu, who is not only a great advisor but also the best coauthor. I benefited tremendously from every discussion we had, the insightful comments, and his way of thinking about theoretical research.

I also want to thank the entire MIT Sloan finance group and finance Ph.D. program for every class I took, every meeting we had, every question raised during seminars, and every chat we had in elevators. All of these make me a better researcher. I want to extend my special thanks to Andrew Lo for his tremendous support and help during the most challenging time.

It's fortunate to have met so many great classmates and friends at MIT. I want to express special thanks to Yupeng Wang, Olivia Kim, Peter Hansen, and Jiaheng Yu for making this journey enjoyable. We share excitement, frustration, confusion, and hope all the time. I also want to thank Matthew Rothman for being a great friend and mentor. I miss our every Monday night chat on the 6th floor in E62. His insightful thoughts and unique perspectives helped me rethink this journey and encouraged me all the time. My sincere thanks also go to Yixin Chen and Fangzhou Lu for their invaluable support since my first day at MIT.

I won't be here without the help of my coauthors. It's a great honor and pleasure to work with many exceptional researchers. Besides my advisor Haoxiang Zhu, I also want to thank Mehmet Ekmekci, Leandro Gorno, Dan Luo, Lucas Maestri, Gideon Saar, Dong Wei, and Ron Yang. I've learned from them how to frame a research idea, write a good introduction, and unleash the full potential of a project. I want to express special thanks to Dan Luo and Dong Wei; they have been not just my coauthors but also friends for many years. They have the sharpest mind, and it's a great pleasure to work with them.

I want to thank my parents, Pijian Sun and Yuejing Li, for their love, support, and sacrifice. Most importantly, I want to thank my wife Jouan Yu. From 20 to 30, from

Beijing to Toulouse, to Boston, she shared all my excitement and happiness as well as all my frustration, failure, depression, and disappointment. Her bravery, optimism, and courage make me stronger and wiser. Meeting her makes me the luckiest person in the world.

Jian Sun, April 28, 2022 Cambridge, MA

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Chapter 1

Algorithmic Transparency

1.1 Introduction

Predictive algorithms have been widely used to mitigate adverse selection in various decision making processes, including hiring, college admission, and lending[1]. In these settings, decision makers use predictive models that establish links between variables that are relevant in their decision making problems. For example, employers score resumes to predict capability, schools use results of standardized tests to predict academic potential, and FinTech lenders use alternative data to predict credit quality. In these examples, the exact relationship between input and output is opaque to the public, and economic agents (such as job candidates, students, and borrowers) have little information about that. With the development of big data and data processing technology, predictive algorithms have become more complex and nonintuitive, involving variables that have no obvious relationship with each other, and thus become even more opaque. A popular argument to justify this opaque nature of predictive algorithms is the extent to which they can be manipulated by "gaming the system", that is, when economic agents know more about the predictive model, they are more likely to change their behavior strategically, which hurts the informativeness of the input. Despite the importance of this question, the effects of algorithmic transparency/opacity on market outcomes is still underexplored in academic research. Although some of the recent regulations start to consider this issue[2], the motivation usually comes from concerns about

[1] See Bogen and Rieke (2018) for algorithmic hiring, Kizilcec and Lee (2020) for algorithmic fairness in education, Bruckner (2018) and Di Maggio et al. (2021) for algorithmic lending.

[2] "...company using algorithmic decision-making must know what data is used in its model and how that data is used to arrive at a decision and explain that to the consumer."—Federal Trade Commission;

fairness, and largely ignores the effects on market efficiency. Moreover, due to the limited understanding of the consequences of algorithmic transparency, there is still uncertainty about future regulation[3], which may add another layer of inefficiency.

To better understand this question, this paper studies the optimal disclosure of a predictive algorithm that maximizes market efficiency in a FinTech lending setup. There are three types of players in this model: borrowers, lenders, and a regulator. Borrowers have private types, which is either good or bad. There is a borrower feature, such as phone usage behavior or social media activities, that can be observed by lenders but can also be manipulated by borrowers privately and at cost. Borrower feature is perfectly correlated with borrower type if not manipulated. Each borrower owns a borrower-specific project that needs to be financed by lenders. The required initial investment is the same for all borrowers, and the project payoffs are independent random variables that depend on borrower type. A predictive algorithm reveals the statistical properties of fundamental random variables in an economic environment. In this paper, the predictive algorithm is a mapping from borrower type to payoff distribution[4]. Specifically, when projects are financed, bad type borrowers always receive zero payoff and thus always default, and good type borrowers will receive i.i.d. nonnegative random payoffs with the same cumulative distribution function indexed by a one-dimensional parameter: *relevance*, denoted by ρ. When the relevance is higher(lower), the expected value of the random payoff from good type borrowers is higher (lower)[5]. Borrowers do not observe the true value of the relevance but share a common belief about it. Lenders observe the exact value of it, which partially determines how to use borrower data in lending decisions. This two-sided private information, i.e., one side (lenders) privately observes the statistical properties of fundamental random variables in the economic environment, and the other side (borrowers) privately manipulates their data, is novel in the literature and is the key feature of this model.

"Whenever personal data is subject to automated decision making, people have ....the right to an explanation"— General Data Protection Regulation

[3]For example, in June 2021, NCRC, Affirm, Lending Club, Oportun, PayPal Holdings Inc, Square and Varo Bank asked the Consumer Financial Protection Bureau (CFPB) to provide guidance on how it will apply disparate impact rules to any systems that use artificial intelligence (AI), machine learning (ML), algorithms, or alternative data to make lending decisions.

[4]In practice, a predictive algorithm usually refers to a mapping from the observed data to the output but not the unobserved type to the output. In Section 1.3.7, I provide a discussion on the equivalence of these two views when algorithmic disclosure only plays an informational role but not serves as a commitment device, which is the feature of this model.

[5]In this paper, I use *relevance* and *relevance of the feature* interchangeably, because ρ measures how useful the feature is in lending decisions if there is no manipulation.

The lending market equilibrium consists of the manipulation behavior of borrowers and the use of borrower data by lenders in lending decisions. The regulator can establish a disclosure rule and ask lenders to disclose any possible information about the true state of relevance to borrowers. It is clear that the lending market equilibrium is determined by the updated public belief in the relevance, so the market outcome depends on the choice of disclosure policy. In this paper, I consider the optimal design of a disclosure policy that maximizes market efficiency.

In this model, I focus on the informational role of algorithmic disclosure, but not the role as a commitment device. From the disclosure, borrowers receive new information about the true value of relevance, which updates their belief about the usefulness of their data. Lenders, on the other hand, cannot commit to how to use borrower data in their lending decisions. This lack of commitment problem turns out to be the source of the inefficiency in this model. Because lenders always make the most efficient use of borrower data, this ex post efficient use of borrower data gives borrowers excessive ex ante manipulation incentives, which in turn makes the feature noisier and also makes market outcome less efficient from the unconditional perspective. The optimal disclosure policy mitigates this problem and generates lower levels of data manipulation unconditionally.

I model this optimal disclosure problem as a Bayesian persuasion problem (Kamenica and Gentzkow (2011)), and characterize the optimal public disclosure of the relevance. First, I show that it is suboptimal to disclose nothing. In this no disclosure equilibrium, borrowers' manipulation behavior and lenders' lending decisions are jointly determined by the public prior belief about the relevance of the feature. Since lenders always make efficient lending decisions ex post using all information available, there must exist scenarios where the surplus from using the feature in lending decisions is small but positive, and the lenders choose to use it ex post because it is efficient to do so. However, this possibility gives borrowers extra incentives to manipulate their features ex ante, and hurts efficiency in other scenarios. This cross-state externality through data manipulation makes lenders use the feature too intensively in their lending decisions. Second, it is also suboptimal to disclose everything, and I show that this full transparency policy leads to the worst outcome. This result relies on our assumption that there are sufficiently many bad type borrowers, so the adverse section problem is severe when there is no borrower data available. The intuition is that when borrowers know exactly how relevant their feature is, they will choose their manipulation behavior such that in equilibrium lenders are indifferent between using the feature or not, and thus there is zero surplus from lending market, resulting in the worst market outcome.

The optimal disclosure policy features partial disclosure, and differentiates the posterior lending market equilibria by their equilibrium data manipulation levels. The regulator can implement the optimal disclosure policy by assigning a score to the feature based on its relevance. Notably, the score function is not monotone in relevance, that is, features with higher relevance may be assigned with lower scores. Under each score realization, the posterior belief about the relevance consists of two disjoint intervals, denoted as approval region and rejection region, respectively. The true relevance is either in the approval region in which manipulating feature can help to get financing, or in a rejection region in which lenders do not use the feature at all and therefore manipulation is useless. The relative fraction of these two regions in the posterior belief determines all the equilibrium outcomes, including the data manipulation level and loan approval rate. Specifically, for all the posterior equilibria, when the feature is used in lending decisions with higher probability, there will be more data manipulation, and lenders will use the feature only when the relevance is high enough.

Several economic implications follow accordingly. First, unconditionally, the use of the feature in lending decisions is monotone in relevance, i.e., there exists a cutoff such that lenders will use the feature in their lending decisions if and only if the true relevance is above the cutoff. This property is obviously true for any posterior equilibrium (no matter what the posterior belief is), but I show that it also holds unconditionally under the optimal disclosure policy. Second, under the optimal policy, the unconditional probability that the lenders use the feature in their lending decisions is strictly lower than that under the no disclosure equilibrium. This confirms our intuition that lenders use the feature too intensively without disclosure. Thirdly, compared to the no disclosure equilibrium, the "worst" posterior equilibrium under the optimal policy induces more manipulation, while the "best" posterior equilibrium induces less manipulation. This uncovers the intuition of why the optimal disclosure policy improves efficiency: although the worse equilibria induce more manipulation and hurt market efficiency, they also force lenders to have a higher standard for the use of the feature in their lending decisions, and in turn deter borrowers' manipulation incentives. Better equilibria benefit from this and induce less data manipulation. Unconditionally, there is less data manipulation, and the negative cross-state externality is mitigated.

With the general structure of the optimal policy, I also provide a closed-form characterization by imposing a mild distributional assumption on the borrowers' manipulation cost. In this case, the optimal score function consists of a discrete part, which induces the equilibrium with lowest level of data manipulation, and a continuous part with data manipulation level continuously moving from the lowest level to the highest level. Furthermore, for any posterior belief, the highest relevance in the rejection region is exactly the level at which the lenders break even. With this result, I can simplify the optimal disclosure problem to a one-dimensional optimization problem, and the optimal score function is solved by an ordinary differential equation (ODE).

I also consider an extension with costly verification. In practice, lenders can verify the types of borrowers by manually reviewing their profiles, conducting interviews, and using various fraud detection techniques. In this extension, lenders can reveal the true type of any borrower with a linear cost function. I explore how costly verification interacts with algorithmic disclosure under the optimal policy. It turns out that in the optimal joint design, these two channels work as substitutes: verification is used when the relevance of the feature is higher than a threshold and disclosure becomes irrelevant in this case; otherwise, disclosure will be used and there is no verification in equilibrium. The optimal joint design can be implemented by two steps: The regulator first reveals if the relevance of the feature is above or below the threshold. If it is above the threshold, all lenders will verify all applicants with a positive probability; otherwise, the regulator reveals additional information under the updated belief and no verification will be used.

The rest of this paper is organized as follows. In this section, I continue to discuss related literature. Section 2 provides a simple model, and Section 3 introduces the general model. In Section 4, I discuss the source of friction and the intuition of improving market outcome from the no disclosure equilibrium. In Section 5, I discuss the results of the general model. Section 6 studies an extension where costly verification is available, and Section 7 concludes.

Related Literature

This paper mainly contributes to three strands of the literature. First, there is a nascent but growing literature on the impact and regulation of algorithmic decision-making. Most of the existing research mainly focuses on fairness, bias, or discrimination (e.g. Bartlett et al. (2021), Milone (2019), Gillis and Spiess (2019),Raghavan et al. (2020),Coston et al. (2021)).

This paper contributes to the literature by considering the regulation on algorithmic disclosure from the perspective of market efficiency. A closely related paper that asks a similar question and also considers strategic manipulation of data is Wang et al. (2020). They consider both the correlational and causal observables in their model and only consider the full transparency and no disclosure policies. Compared to their work, my model only considers correlational features as input in the predictive algorithm, and thus the fundamental frictions are different. Besides, I consider flexible disclosure policies, the results on the optimal design of algorithmic disclosure in my paper do not have a counterpart in their paper. Another theoretical paper that also focuses on algorithmic transparency and governance is Blattner et al. (2021). They consider the trade-off between model complexity and transparency and the role of algorithmic audit in regulating algorithms, which is different from my focus. Björkegren et al. (2020) examine the interplay between strategic manipulation of data and algorithmic transparency. Although the question in their paper is quite different from mine, the results in their field experiment empirically verify the existence of data manipulation when people know more about the algorithms. There is also a growing literature in computer science about algorithmic explainability or explainable AI (Bhatt et al. (2020), Carvalho et al. (2019), Lundberg and Lee (2017), Murdoch et al. (2019)). But the computer science literature usually focuses on algorithmic audit and explainability which mainly consider the "black box" nature of machine learning algorithms, while my paper simplifies this "black box" nature and considers an information design question in a finance setting.

I also add to a growing literature on Bayesian persuasion (Kamenica (2019) and Bergemann and Morris (2019) provide excellent surveys) and its applications in finance. The way I model information structure follows Kamenica and Gentzkow (2011), and I consider a persuasion problem with continuous state as in Dworczak and Martini (2019). Methodologically, Bayesian persuasion problems with continuous states are in general not tractable, except for some special cases (for example, Gentzkow and Kamenica (2016), Dworczak and Martini (2019), Goldstein and Leitner (2018)). In my model, the information designer's objective function depends on the entire distribution of posterior beliefs, and this question does not fit into any existing tractable framework. I obtain my theoretical results using a novel "guess and verify" method. There are many applications of Bayesian persuasion in finance literature, including shareholder voting (Malenko et al. (2021)), security design (Szydlowski (2021)), bank stress test (Goldstein and Leitner (2018), Goldstein and Leitner

(2020) Inostroza (2019), Inostroza and Pavan (2021), Leitner and Williams (2020)) and financial network (Huang (2020)). This paper contributes to this literature by considering a new question (algorithmic disclosure) and provides a novel optimal signal structure.

Lastly, this project is related to the literature on strategic manipulation of data (Frankel and Kartik (2019a), Frankel and Kartik (2019b), Ball (2019)), or more broadly, the signaling models. The way I model private information on the borrower side is similar to Frankel and Kartik (2019b). Ball (2019) considers a problem with multi-dimensional features, and shows that the optimal scoring rule underweights some features to deter data manipulation. All of these papers focus mainly on how committing to certain decision rules will improve efficiency. Relative to these work, I consider an information design question, and focus on how commitment on information structure (disclosure policy) will improve efficiency.

1.2 A Simple Model

To fix ideas, let's consider a simple model. There is a competitive lending market with many identical lenders and a unit mass of borrowers. Each borrower i is endowed with zero initial wealth and a borrower-specific project which requires an initial investment I at time 0. The project generates a positive cash flow V if it succeeds, and zero if it fails. The probability of success is a random variable, and its distribution is formally introduced later. All the borrowers can get private benefit b if their own borrower-specific project is successfully financed regardless of the outcome.

There is a manipulable feature for each borrower, which takes two possible levels: High or Low. Borrowers who are born with High (Low) feature make up μ $(1-\mu)$ of the entire population, and they are called good (bad) type borrowers. Manipulating feature is possible for bad borrowers[6], and they can privately change their feature to High by paying a cost c. The manipulation cost c follows a uniform distribution on $[0, 1]$ among the bad borrowers. A key assumption here is that manipulation behavior does not change borrower type. In this lending market, the only data that lenders can collect and observe is the the borrower feature after potential manipulation.

Probability of success depends on borrower type (good or bad). Specifically, bad bor-

[6] Here we only allow bad borrowers to manipulate their features for simplicity of exposition. But this assumption is not necessary. Even if we assume that good type borrowers can costly manipulate their features, they will never do this in equilibrium.

rowers always fail. For good borrowers, the probability of success ρ is drawn from a uniform distribution on $[0,1]$. The true value of ρ is only observable to lenders, and all borrowers share the common prior about the distribution of ρ. In this example, we call ρ the *relevance* of the borrower feature, because it represents how relevant the borrower feature is in lending decisions when there is no manipulation. Besides, from the perspective of borrowers, the probability of success ρ is the only uncertain element in the mapping from borrower type to payoff distribution. In this example, let's impose the following assumption[7].

Assumption 1.2.1. *(Severe Adverse Selection)* $b \geq 1$ *and* $\mu V \leq I$.

Note that the manipulation cost follows $c \sim U[0,1]$, so $b \geq 1$ implies that if lenders lend to borrowers with feature High with probability 1, the private benefit always dominates the manipulation cost for all bad borrowers, and thus all of them will choose to manipulate their features. And the second condition $\mu V \leq I$ implies that lenders will never lend to any borrower if all bad type borrowers choose to manipulate their features. These two conditions jointly imply that in any equilibrium, not all bad borrowers choose to manipulate their features.

No Disclosure On The Relevance ρ

First we consider the equilibrium when no additional information about ρ is disclosed to borrowers. Lenders can make their lending decisions based on the observed feature. The lending market is competitive, so lenders always make zero profit in equilibrium. In this case, it can be shown that there is a unique equilibrium which consists of two cutoffs $\bar{c}_N$ and ϱ_N, such that

- all bad type borrowers with manipulation cost lower than $\bar{c}_N$ choose to manipulate their features to High;
- lenders will lend to borrowers with feature High if $\rho > \varrho_N$.

For bad type borrowers with manipulation cost $\bar{c}_N$, the indifference condition is

$$\text{Prob}\left(\rho > \varrho_N\right) \cdot b = \bar{c}_N,$$

[7] We'll have a similar assumption in the main model

where $\text{Prob}\,(\rho > \varrho_N)$ is the probability that the loan application is approved for borrowers with feature High. For lenders, the total surplus from lending is zero when $\rho = \varrho_N$, implying

$$\mu\varrho_N V = (\mu + (1-\mu)\text{Prob}\,(c \le \bar{c}_N))\, I.$$

Based on our distributional assumptions, the unique solution of the equilibrium is

$$\left(\varrho_N = \frac{I}{\mu V + (1-\mu)I}, \bar{c}_N = b \cdot \frac{\mu(V-I)}{\mu V + (1-\mu)I}\right).$$

Let

$$k_N = (\mu + (1-\mu)\text{Prob}\,(c \le \bar{c}_N))\, I$$

be the effective financing cost, then the lending market surplus (measured by the net value of all projects financed) is

$$W_N = \int_{\varrho_N}^{1} (\mu\rho V - k_N)\, d\rho.$$

For simplicity, let's take the following parameters:

$$I = 3, V = 10, b = 1, \mu = 3/10,$$

then the equilibrium variables are

$$(\varrho_N = 0.59, \bar{c}_N = 0.41, W_N = 0.25)\,. \tag{1.1}$$

Figure 1-1 summarizes the above equilibrium. The green triangle in Figure 1-1 is the surplus W_N. In equilibrium, the expected payoff outweighs the cost only when $\rho > 0.59$, and borrowers with High feature are financed only when ρ is in this region. The green line on the horizontal axis represents the support of posterior belief. In this no disclosure equilibrium, the posterior belief is the same as the prior belief, and thus the support of the posterior belief is the interval $[0, 1]$.

Full Transparency

Another natural disclosure policy is full transparency which reveals the true state of ρ perfectly to borrowers. It turns out that the surplus equals to zero in this case, which leads to the worst market outcome. To see this, suppose the true relevance ρ satisfies $\rho < \frac{I}{V}$,

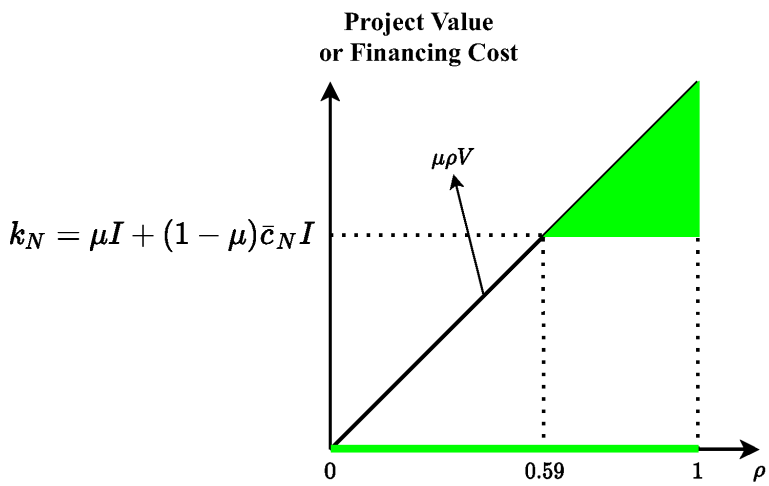


Figure 1-1: Equilibrium–No Disclosure

then even only lending to good type borrowers is inefficient, and thus there will be no financing and the market outcome must be zero. For any $\rho \geq \frac{I}{V}$, we know in equilibrium, the probability that lenders lend to borrowers with feature High must be less than 1, which means that they must be indifferent between lending and not lending. Then the surplus also must be zero for any $\rho \geq \frac{I}{V}$.

A Binary Color Signal

Our question is, can a regulator achieve a strictly higher outcome by designing a signal about ρ and disclosing it to the market? The answer is yes. The definition of disclosure policy is formally introduced in Section 1.3.4, here let's take the numbers from the no disclosure example and consider the following specific score function which consists of two levels $R(ed)$ and $B(lue)$:

$$s = \begin{cases} R(ed) & \text{if } \rho \in [0, 0.54) \cup (0.64, 0.91) \\ B(lue) & \text{if } \rho \in [0.54, 0.64] \cup [0.91, 1] \end{cases}. \tag{1.2}$$

This score function assigns colors to state of ρ, which is represented by Figure 1-2. It only reveals which region (Red or Blue) that the true state of ρ belongs to, and induces two possible posterior equilibria.

Specifically, if the signal realization is R, then the posterior belief about ρ is a uniform distribution on two disjoint intervals $[0, 0.54) \cup (0.64, 0.91)$, and it can be shown that the

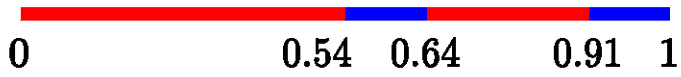


Figure 1-2: The Color Signal

equilibrium outcomes are

$$(\bar{c}_R = 0.34, \underline{\rho}_R = 0.54, W_R = 0.19). \tag{1.3}$$

Similarly, if the signal realization is B, the posterior belief about ρ is a uniform distribution on $[0.54, 0.64] \cup [0.91, 1]$, and the equilibrium outcomes are

$$(\bar{c}_B = 0.48, \underline{\rho}_B = 0.64, W_B = 0.09). \tag{1.4}$$

Figure 1-3 summarizes the surpluses of these two equilibria. The red trapezoid in the left graph represents the surplus on observing R, and the two red intervals on the horizontal line represent the support of the posterior belief. In this equilibrium, lenders will lend to borrowers with feature High only when $\rho \in (0.64, 0.91)$. Similarly, the right graph in Figure 1-3 shows the surplus on observing B. The total surplus with this color signal (1.2) is

$$W_s = W_R + W_B = 0.19 + 0.09 = 0.28 > 0.25 = W_N.$$

So the surplus improves.

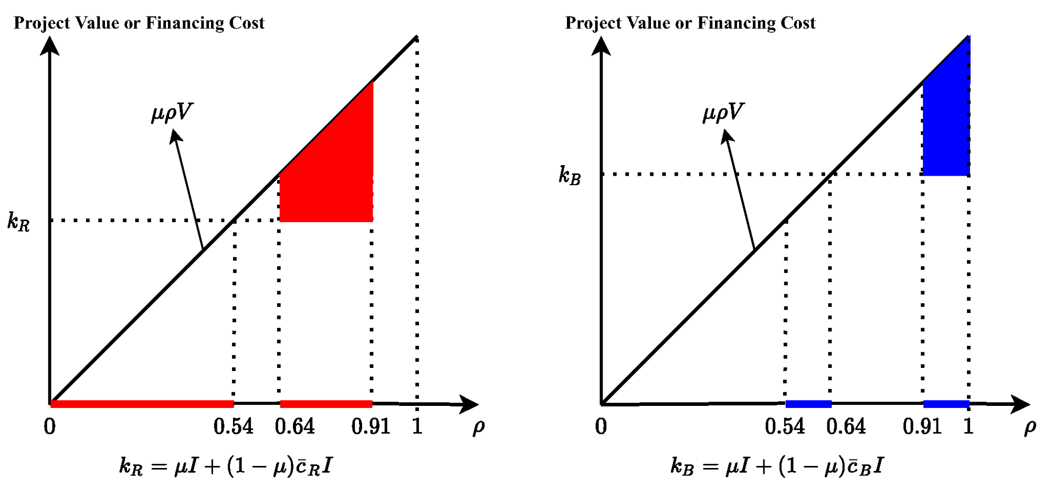


Figure 1-3: Equilibrium–Score R and B

Our analysis shows that the binary color signal dominates both the no disclosure policy and full transparency policy. But what is the intuition behind this result? The result that full transparency policy is dominated is clear: when the exact information about the relevance of the feature is disclosed to the market, bad type borrowers will manipulate their features such that in equilibrium the surplus from using the borrower data in lending decisions is always zero, and all lenders are indifferent between using and not using the borrower data in their lending decisions. The inefficiency embedded in the no disclosure equilibrium is the lenders' lack of commitment problem, that is, lenders always make the most efficient use of borrower data ex post in their lending decisions. To see this, suppose in the no disclosure equilibrium, the regulator is able to "force" the lenders to use the feature in their lending decisions only when ρ is greater than an exogenous cutoff $\varrho_x = \varrho_N + x$, where $x \ll 1$. Then the bad type borrowers with manipulation cost

$$c \leq \bar{c}_x = b \cdot \text{Prob}\left(\rho \geq \varrho_x\right)$$

will choose to manipulate their features, and the total surplus is a function of the exogenous cutoff ρ_x:

$$W\left(x\right) = \int_{\varrho_x}^{1} \left[\mu\rho V - \left(\mu + (1-\mu)(1-\varrho_x)\right) I\right] d\rho. \tag{1.5}$$

Note $W\left(0\right) = W_N$, then

$$\begin{aligned} \left.\frac{dW\left(x\right)}{dx}\right|_{x=0} &= -\left[\mu\varrho_N V - \left(\mu + \left(1-\mu\right)\left(1-\varrho_N\right)\right) I\right] + \int_{\varrho_N}^{1} (1-\mu) I d\rho \\ &= \int_{\varrho_N}^{1} (1-\mu) I d\rho > 0. \end{aligned} \tag{1.6}$$

This is because the lender's equilibrium condition in the no disclosure equlibrium is

$$\left[\mu\varrho_N V - \left(\mu + \left(1-\mu\right)\left(1-\varrho_N\right)\right) I\right] = 0. \tag{1.7}$$

The result in (1.6) shows that the equilibrium cutoff ϱ_N is inefficiently low from the ex ante perspective. So the probability that the borrower data is used in lending decisions, $(1-\varrho_N)$, is inefficiently high. This result is based on condition (1.7),which is the ex post efficient use of borrower data in lending decisions. In equilibrium, when lenders use borrower data more often in some states ex post, more bad borrowers will choose to manipulate their features ex ante, and the effective financing cost will increase for all other states from the ex ante perspective. This cross-state externality makes no disclosure equilibrium inefficient. In the first order derivative (1.6), when lenders increase their lending cutoff by x, the approval probability decreases by

$$\text{Prob}\left(\rho > \varrho_N\right) - \text{Prob}\left(\rho > \varrho_x\right) = x$$

from the perspective of bad type borrowers, then the fraction of bad type borrowers who would like to manipulate decreases by

$$\text{Prob}\left(c \leq \bar{c}_N\right) - \text{Prob}\left(c \leq \bar{c}_x\right) = x,$$

implying that the effective financing cost decreases by

$$\left(1 - \mu\right) I \cdot x.$$

Then the total cost saving from all states $\rho > \varrho_N$ is

$$\int_{\varrho_N}^{1} (1 - \mu) I d\rho \cdot x$$

which corresponds to the last term in (1.6).

To mitigate the excess manipulation, the signal (1.2) defers lenders' use of borrower data by differentiating the two lending market equilibria by data manipulation levels. To see this, upon observing B, in equilibrium we have

$$\bar{c}_B = 0.48 > \bar{c}_N = 0.41,$$

which means there are more bad type borrowers manipulating their features compared to the no disclosure equilibrium. As a result, lenders have a more stringent lending standard, and lend to borrowers with feature High when $\rho > \varrho_B = 0.64$, which is greater than the cutoff in the no disclosure equilibrium ($\varrho_N = 0.59$). Then the lenders will not use borrower data in their lending decisions when $\rho \in [0.59, 0.64]$ under signal B. But note this is the region when lenders lend to borrowers with feature High in the no disclosure equilibrium. For $s = R$, there is less manipulation compared to the no disclosure equilibrium because $\bar{c}_R = 0.34 < 0.41 = \bar{c}_N$, and borrower data is used by lenders only when $\rho \in (0.64, 0.91)$.

Unconditionally, with the binary color signal, the feature is used when $\rho > 0.64$, while it is $\rho > \rho_N = 0.59$ in the no disclosure equilibrium. So the feature is used less frequently with the binary color signal. Intuitively, by differentiating the two equilibria by data manipulation levels, the "worse" equilibrium ($s = B$) effectively guarantees that the feature will not be used in cases when it was indeed used in no disclosure equilibrium, and the "better" equilibrium ($s = R$) has lower level of data manipulation and generates more efficient outcome.

Actually this binary color signal is optimal among all binary signals. In the main model, I'll consider a general space of disclosure policies. But this binary color signal has several notable properties that are still robust in the optimal disclosure policy in the main model. First, there exists a threshold ($\underline{\rho}^\star = 0.64$), such that unconditionally, the feature is used in lending decisions if and only if the true state is above the threshold. It is clear that this cutoff property always holds for any posterior equilibria, and here I show it also holds unconditionally. The intuition is clear: the relevance ρ represents how useful borrower feature is in lending decisions. When ρ is higher, borrowers with feature High are of better qualities and will have higher probability of success. Then it is efficient to lend to borrowers with feature High when the true relevance ρ is higher. Second, for any score realization (R or B), the support of posterior belief is always a union of two disjoint intervals. These two intervals correspond to lenders' equilibrium lending decisions. The interval below $\underline{\rho}^\star$ represents the rejection region, and the lenders will reject all borrowers when the true relevance is in this region; while the interval above $\underline{\rho}^\star$ represents the approval region and lenders will lend to all borrowers with feature High when the true relevance is in this region. Thirdly, the unconditional probability of using the feature in lending decisions is less than that in the no disclosure equilibrium, implying that the feature is used less intensively with optimal disclosure. Lastly, the binary color signal induces two posterior equilibria, with one equilibrium (B) having a higher data manipulation level than the no disclosure equilibrium, and the other one (R) having a lower data manipulation level. All of these properties still hold in the optimal disclosure policy in the main model.

1.3 The Main Model

The main model is a generalization of the simple model.

1.3.1 Players

There are three types of players in this model: a unit mass of borrowers, $N(>1)$ lenders, and a regulator. All players are risk neutral. We model borrowers in a similar way as the agents in Frankel and Kartik (2019b). Borrowers have two-dimensional private information: **quality** type $\theta \in \{G(ood), B(ad)\}$, and (manipulation) **cost** type c. For the joint distribution of (θ, c), I assume the unconditional probability of good type borrowers in the population is

$$\text{Prob}\,(\theta = G) = \mu > 0.$$

And the conditional probability $c|\theta$ is

$$c|\theta \begin{cases} \equiv \infty & \text{if } \theta = G \\ \sim F_c(\cdot) & \text{if } \theta = B \end{cases}, \tag{1.8}$$

where $F_c\,(\cdot)$ is the cumulative distribution function for a continuous random variable defined on $[0, \bar{c}]$[8]. Assume $F_c'\,(x) > 0$ and $F_c''\,(x)$ is bounded for all $x \in [0, \bar{c}]$

All lenders are identical and operate in a competitive lending market. At time 0, each borrower i receives a borrower-specific project (project i) and has zero initial wealth. Each project i requires an initial investment I; otherwise it fails, and is liquidated with zero liquidation value. If project i is financed at time 0, it will generate a nonnegative random payoff $\tilde{V}_i$[9] at $t = 1$, and the realization of the random payoff is publicly observable. Besides, borrower i also receives a constant nontransferable private benefit b if the project is successfully financed. Any project can be financed by at most one lender.

If the project i is financed by lender j with debt face value D_j^i, then when the payoff $\tilde{V}_i$

[8] Here I assume $c|\theta = G \equiv \infty$ for simplicity of exposition. Actually $c|\theta = G$ is irrelevant for all of my results. For example, we can assume $c|\theta \sim F_c\,(\cdot)$ for both $\theta \in \{G, B\}$, and all the results will be the same.

[9] For expositional convenience, we sometimes use $\tilde{V}$ to represent the random payoff for an arbitrary borrower.

is realized, borrower i receives

$$\max\left\{\tilde{V}_i - D_j^i, 0\right\} + b,$$

lender j receives

$$\min\left\{\tilde{V}_i, D_j^i\right\} - I,$$

and the regulator's payoff (surplus) is the total outcome of lending market, which is[10]

$$\tilde{V}_i - I.$$

1.3.2 Predictive Algorithm

For each borrower i, his quality type θ_i is informative about his random payoff $\tilde{V}_i$. Specifically, when $\theta_i = B$, $\tilde{V}_i \equiv 0$, i.e., bad type borrowers always fail. When $\theta_i = G$, $\tilde{V}_i$ is a nonnegative, continuous random variable on $\left[0, \bar{V}\right]$, with cumulative distribution function $F(\cdot)$. The key feature of our model is that $F(\cdot)$ is drawn from a family of distribution functions $\{F_\rho(\cdot)\}_{\rho\in\mathcal{P}}$, where $\mathcal{P}$ is a subset of R. Intuitively, since bad type borrowers always fail, ρ effectively measures how the quality type θ can be used to predict payoff distribution. Throughout the paper, I call ρ the *relevance*. In practice, machine learning algorithms adopted by FinTech lenders are hard to explain and interpret and can rarely be summarized by a one-dimensional parameter. In this paper, since I focus on disclosure instead of explainability (which is the primary focus of the computer science literature, see Lundberg and Lee (2017)), I abstract away the "black box" feature of the predictive algorithms and assume them to be summarized by a one-dimensional parameter ρ.

The relevance ρ is drawn from a continuous distribution with cumulative (probability) distribution function $\Pi_0(\rho)$ $(\pi_0(\rho))$. Without loss of generality, we assume ρ is drawn from a uniform distribution in $[0, 1]$[11], so $\mathcal{P} = [0, 1]$. The key assumption of our model is that ρ is only observable to all lenders but not borrowers, and we assume all borrowers share the common prior belief about the distribution of ρ.

[10]Note that the private benefit is not included in the regulator's utility, but this assumption is not crucial. Actually the key result, that partial disclosure is optimal, is still robust even if we include private benefit in the regulator's payoff function.

[11]Note that for any continuous random variable x with cumulative distribution function $T(\cdot)$, the new variable $y = T(x)$ always follows a uniform distribution on $[0, 1]$.

Let

$$m(\rho) = E\left(\tilde{V}|G,\rho\right) = \int_0^{\bar{V}} v \cdot dF_\rho(v)$$

be the expected payoff from any good type borrower if he is successfully financed. Then we impose the following assumptions on $m(\rho)$:

Assumption 1.3.1. *$m(\rho)$ satisfies the following conditions:*

1. *$m(\rho)$ is continuous and strictly increasing;*

2. *$m(0) = 0$ and $m(1) > I$;*

3. *$\mu m(1) \leq I$.*

The first assumption is mainly for expositional convenience; relaxing this assumption does not affect our main results. In the second assumption, $m(0) = 0$ is also mainly for expositional convenience so we can relax it without changing the main results. $m(1) > I$ is to make sure that when $\rho = 1$, it is efficient to lend to good type borrowers, otherwise it is always efficient to reject any borrower and the equilibrium becomes trivial. The last assumption means that the adverse selection in the market is severe and it is inefficient to lend to all borrowers, this assumption helps to establish a clear benchmark, but my main results do not rely on this specific assumption.

1.3.3 Feature and Manipulation

Although the quality type is informative about borrowers' riskness, it is the private information of borrowers, and thus can not be directly used by lenders in their lending decisions. There is a feature $\hat{\theta} \in \left\{\hat{G}, \hat{B}\right\}$ for each borrower and can be publicly observed by lenders. If borrowers do not manipulate their features, $\hat{\theta} = \hat{G}(\hat{B})$ if $\theta = G(B)$, i.e., borrower feature $\hat{\theta}$ can perfectly reveal borrower type θ. However, each borrower can change his feature to the other value by privately paying the non-pecuniary manipulation cost c. The cost structure is introduced in (1.8). Intuitively, good type borrowers are not able to manipulate their features, while bad type borrowers can manipulate their features by paying cost c, which follows a continuous distribution on $[0, \bar{c}]$ with cumulative distribution function $F_c(\cdot)$. The assumption that good type borrowers are not able to manipulate their features is actually redundant. We can show that good type borrowers will never manipulate in equilibrium even if they have finite manipulation cost (see Appendix A.1).

In equilibrium, lenders use the feature $\hat{\theta}$ to assess borrowers' riskness, but the informativeness of the feature $\hat{\theta}$ is determined by bad type borrowers' manipulation behavior. Lenders' lending decisions and bad type borrowers' data manipulation levels are jointly determined in equilibrium.

A key assumption in this model is that manipulation behavior does not change borrower type, i.e., the distribution of $\tilde{V}$ is not influenced by the manipulation behavior, so feature $\hat{\theta}$ only plays an informational role. This assumption is motivated by the "gaming the system" concern in the algorithmic transparency debate. For example, lenders find variables that can predict default risk using historical training data and machine learning algorithms, which focus more on correlation but not causation between input and output. If borrowers strategically change their behavior, their true riskness does not change but the predictive algorithm may become less effective.

I impose the following assumption which also shows up in Assumption 1.2.1 in the simple model.

Assumption 1.3.2. *$b \geq \bar{c}$.*

This assumption implies that if lenders lend to $\hat{G}$ borrowers for sure, then all of the bad type borrowers will choose to manipulate and the borrower feature becomes useless. This assumption, together with the condition $\mu m(1) \leq I$ in Assumption 1.3.1, jointly imply that in any equilibrium not all bad type borrowers choose to manipulate their features. This result that helps to characterize the optimal policy, but my main results can easily be extended to the case when this condition is violated.

1.3.4 Disclosure Policy

This project primarily focuses on the public disclosure of the relevance ρ. Although its realization is unobservable to borrowers, we consider the scenario in which the regulator can publicly reveal some information about the true state of ρ to all borrowers before they choose their manipulation behavior. Below is the formal definition of a disclosure policy.

Definition 1.3.1. *A disclosure policy $(\mathcal{S}, \tilde{\sigma})$ consists of a signal space $\mathcal{S}$ and a mapping $\tilde{\sigma}$ from the realization $\rho \in \mathcal{P} = [0, 1]$ to a distribution over signal space $\mathcal{S}$:*

$$\tilde{\sigma}(s|\rho) : [0, 1] \rightarrow \Delta(\mathcal{S}).$$

So $\tilde{\sigma}(s|\rho)$ is the (generalized) probability distribution function[12] of s conditional on state ρ. The regulator publicly announces the disclosure rule and then draws a realization of s based on it. After observing the realization s, all borrowers can update their beliefs on the distribution of ρ by Bayesian updating and then choose their manipulation strategies.

A special case of policies defined in Definition 1.3.1 is the deterministic policy. For these policies, the signal realization conditional on any state ρ is deterministic, so the conditional probability can be summarized by a deterministic function. Below is the definition of a deterministic policy. For notational simplicity, let's denote $\delta(x)$ as the Dirac function[13].

Definition 1.3.2. *A disclosure policy $(\mathcal{S}, \tilde{\sigma})$ is deterministic if for any $\rho \in [0,1]$, the signal realization is deterministic, i.e., there exists a* ***message function***

$$\sigma : [0,1] \to \mathcal{S},$$

such that

$$\tilde{\sigma}(s|\rho) = \delta(s - \sigma(\rho)).$$

Throughout this paper, when there is no confusion, we use $(\mathcal{S}, \sigma)$ to represent a deterministic disclosure policy with signal space $\mathcal{S}$ and message function σ. To gain more intuitions on how disclosure policies work, note that the full transparency can be implemented by a deterministic policy with signal space $\mathcal{S} = [0,1]$, and the message function σ is

$$\sigma(\rho) = \rho.$$

In this case, the regulator assigns a unique signal $s = \rho$ to each state ρ. When borrowers observe a realization s, the public belief will be updated and it is sure that the true state of relevance ρ is $\rho = s$. This disclosure policy effectively reveals all information about the true state of ρ. Another example is the no disclosure policy, i.e., the regulator does not reveal any information. It can also be implemented by a deterministic policy with only one element in the signal space. Then borrowers will always observe the same realization no

[12] See Ziółkowski (2009) for discussion on generalized probability distribution function.

[13] A Dirac function $\delta(x)$ is defined as

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases},$$

and $\int_{-\infty}^{+\infty} \delta(x)\, dx = 1$.

matter what the true state of relevance ρ is, and thus they will learn nothing from the signal and no information is revealed by this disclosure policy.

A more complex but also commonly used signal structure is the cutoff disclosure, i.e, the regulator only reveals whether the true state of relevance ρ is above a threshold $\hat{\rho}$ or not. In this case, the disclosure can be implemented by a deterministic policy with two elements in the signal space $\mathcal{S} = \{s_L, s_H\}$, and the message function $\sigma(\rho)$ is

$$\sigma(\rho) = \begin{cases} s_L & \text{if } \rho \in [0, \hat{\rho}] \\ s_H & \text{if } \rho \in (\hat{\rho}, 1] \end{cases}.$$

So the regulator discloses s_H if $\rho > \hat{\rho}$ and s_L otherwise. Then borrowers can only learn if the true state of relevance ρ is above the threshold $\hat{\rho}$ or not.

The main advantage of modeling information disclosure in this way is the flexibility. Intuitively, the information structure defined in Definition 1.3.1 summarizes all possible ways of disclosing information, which also sheds light on the boundary of the pure information channel on mitigating manipulation in this problem.

Once we have a general signal structure $(\mathcal{S}, \tilde{\sigma})$, it will induce a distribution of posterior beliefs $\{f(s), \pi(\rho|s)\}_{s\in\mathcal{S}}$, where$f(s)$ is the (generalized) density function of the random variable s, and $\pi(\rho|s)$ is the public posterior belief (probability distribution function) of ρ conditional on observing the public signal realization s. To get sharp predictions on the optimal disclosure policy, we impose the following technical restriction on the posterior beliefs[14]:

Refinement 1.3.1. *We focus on disclosure policies such that for any s, and any $\rho \in (\pi(\rho|s))$, there exists a closed set $B_{s.\rho} \subset (\pi(\rho|s))$, such that $\rho \in B_{s,\rho}$ and $E\left(1_{B_{s,\rho}}(\rho)\,|s\right) > 0$, where $1_A(x)$ is the indicator function:*

$$1_A(x) = \begin{cases} 1 & \textit{if } x \in A, \\ 0 & \textit{if } x \notin A. \end{cases}$$

[14]This technical restriction is not important. In our model, any zero-measure change on the disclosure policy doesn't change the expected payoff. This restriction is to rule out some optimal policies that are almost the same as the optimal policies we characterize later.

The distribution of posteriors $\{\pi(\rho|s)\}_{s\in\mathcal{S}}$ must satisfy a necessary condition[15]:

$$\int_s \pi(\rho|s) f(s) ds = 1_{[0,1]}(\rho). \tag{1.9}$$

This is also known as the Bayes-plausible condition (Kamenica and Gentzkow (2011)). The interpretation is that the average of all posterior beliefs must be consistent with the prior belief. Then following the literature, instead of working with the signal structures directly, I work with distributions of posterior beliefs that satisfy condition (1.9) and [16]

$$\int_s f(s) ds = 1.$$

1.3.5 Lending Market

There are $N > 1$ identical lenders operating in a competitive lending market, then in equilibrium all lenders make zero profit[17]. The lending market equilibrium consists of the bad type borrowers' manipulation strategies and lenders' lending decisions. We only focus on symmetric equilibria in which all lenders choose the same strategy in equilibrium.

When the regulator commits to a disclosure policy $(\mathcal{S}, \tilde{\sigma})$, for each signal realization $s \in \mathcal{S}$, we call the lending market induced by this signal realization s the *subgame* s. Before exploring how $(\mathcal{S}, \tilde{\sigma})$ will change the market outcome, we solve the model backwards and first consider the lending market equilibrium under an arbitrary posterior belief.

Suppose updated public belief of ρ is $\pi(\rho|s)$. For any borrower i and lender j, let $\left(I_j^{s,\hat{\theta}}, D_j^{s,\hat{\theta}}\right)$ be the lender's strategy and γ_i^s be the borrower's manipulation decision where

1. $I_j^{s,\hat{\theta}} \in [0,1]$ represents the probability that lender j approves the loan application from $\hat{\theta}$ borrowers, and $D_j^{s,\hat{\theta}}$ represents the face value of the debt conditional on approval;

2. $\gamma_i^s \in \{0,1\}$ represents the probability that borrower i manipulates his feature $\hat{\theta}_i$.

It's clear that $\gamma_i^s = 0$ for all good type borrowers in any equilibrium because they have infinite manipulation cost, so the good type borrowers are passive in our model and do not

[15] The RHS of the condition represents the density of the prior belief of ρ, which is 1 under the uniform distribution on $[0,1]$.

[16] These two conditions are necessary conditions, and I'll verify the existence of the optimal policy later.

[17] This is not the key assumption of our model. Actually we can consider a model with a monopoly lender, and the results on the optimal disclosure polices are the same, as long as there is no screening by contracts. This is because in this model, the regulator wants to maximize the total surplus from financing activities, while market structure only changes the distribution of surplus but not the total surplus.

play any strategic role. For all lenders, they'll never lend to $\hat{B}$ borrowers, as those must be bad type borrowers who will default with probability 1. So we must have $I_j^{s,\hat{B}} = 0$, and the choice of $D_j^{s,\hat{B}}$ becomes irrelevant.

Let k_s be the total (effective) financing cost of lending to $\hat{G}$ borrowers, since all projects require the same initial investment, $\frac{k_s}{I}$ is the measure of $\hat{G}$ borrowers and $\left(\frac{k_s}{I} - \mu\right)$ is the measure of bad type borrowers who choose to manipulate their features. For a lender j, she lends to $\hat{G}$ borrowers only if

$$\mu m(\rho) - k_s \geq 0.$$

Then any lender j's lending decision can be summarized by[18]

$$I_j^{s,\hat{G}}(\rho) \begin{cases} = 1 & \text{if } \rho > \varrho_s \\ \in [0,1] & \text{if } \rho = \varrho_s \\ = 0 & \text{if } \rho < \varrho_s \end{cases},$$

where

$$\varrho_s = m^{-1}\left(\frac{k_s}{\mu}\right). \tag{1.10}$$

Since we only focus on symmetric equilibrium, in equilibrium we must have $E\left(I_j^{s,\hat{G}}\right) = E\left(I_k^{s,\hat{G}}\right)$ for any j, k. For simplicity, let I^s represent the equilibrium approval decision under signal s.

For a bad type borrower with private manipulation cost c_i, since he always fails, the only benefit he may get from the deal is the private benefit B, so he chooses to manipulate only if

$$E(I^s) \cdot B \geq c_i.$$

Then it's clear that the bad type borrowers' manipulation strategy can be characterized by a cutoff $\bar{c}_s$, such that all bad type borrowers with manipulation cost $c \leq \bar{c}_s$ choose to manipulate their features, where

$$\bar{c}_s = E(I^s) \cdot b. \tag{1.11}$$

[18]In equilibrium, the debt face value $D_j^{s,\hat{G}}$ must satisfy the zero-profit condition: $\mu E\left[\min\left\{\tilde{V}, D_j^{s,\hat{G}}\right\} | \theta = G\right] = k_s$. However, in our model, the debt face value only affects the distribution of surplus between lenders and borrowers and does not change the regulator's payoff (the surplus). So in this paper, we'll only focus on the approval decision.

Moreover, in equilibrium, we must have

$$k_s = [\mu + (1-\mu) F_c(\bar{c}_s)] I. \tag{1.12}$$

Then the equilibrium of the subgame s is characterized by (1.10),(1.11), and (1.12). Below is the formal definition of an equilibrium for any subgame s:

Definition 1.3.3. *An equilibrium of subgame s is a triple $(k_s, \varrho_s, \bar{c}_s)$, and a function I^s, where k_s is the total cost of financing $\hat{G}$ borrowers, ϱ_s is the cutoff in lending approval decisions, $\bar{c}_s$ is the cutoff in bad type borrowers' manipulation decisions, and I^s is the probability that $\hat{G}$ borrowers are financed, such that the following conditions are satisfied:*

1. *Lender optimization:* $I^s \begin{cases} = 1 & \text{if } \rho > \varrho_s \\ \in [0,1] & \text{if } \rho = \varrho_s \\ = 0 & \text{if } \rho < \varrho_s \end{cases}$, *where* $\varrho_s = m^{-1}\left(\frac{k_s}{\mu}\right)$;

2. *Borrower optimization:* $\bar{c}_s = E(I^s) \cdot B$;

3. *Consistency:* $k_s = [\mu + (1-\mu) F_c(\bar{c}_s)] I$.

The regulator's utility in this subgame is all the surplus generated from the financing activities, which is

$$W_s = E\left[(\mu m(\rho) - k_s)^+\right].$$

Then her unconditional expected utility is

$$W = \int_{s \in \mathcal{S}} W_s f(s)\, ds,$$

which is the expected surplus across all subgames. Then the regulator's information design problem is the following

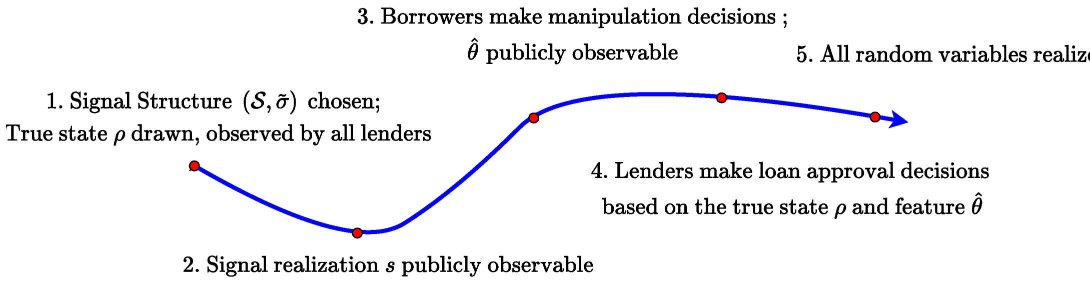


Figure 1-4: Timeline

$$\underset{\{\mathcal{S},\{f(s),\pi(\rho|s)\}_{s\in\mathcal{S}}\}}{\text{maximize}} \; W = \int_{s\in\mathcal{S}} E\left[(\mu m(\rho) - k_s)^+ |s\right] f(s)\, ds \tag{1.13}$$

$$\text{subject to} \quad \int_{s\in\mathcal{S}} f(s)\, ds = 1, \tag{1.14}$$

$$\int_{s\in\mathcal{S}} \pi(\rho|s) f(s)\, ds = 1_{\rho\in[0,1]}, \tag{1.15}$$

$$F_c\left(b \cdot \text{Prob}\left(\rho > m^{-1}\left(\frac{k_s}{\mu}\right)|s\right)\right) \leq \frac{\mu}{1-\mu}\left(\frac{k_s}{\mu I} - 1\right)$$
$$\leq F_c\left(b \cdot \text{Prob}\left(\rho \geq m^{-1}\left(\frac{k_s}{\mu}\right)|s\right)\right). \tag{1.16}$$

The solution to the regulator's problem is not unique, but in Section 1.5, we'll discuss and characterize the general properties of the optimal policies. The above regulator's problem is known as a Bayesian persuasion problem with continuous states. Bayesian persuasion models with continuous states are in general not tractable, except for some special cases (Gentzkow and Kamenica (2016), Dworczak and Martini (2019)). The regulator's problem in this paper does not fit into any existing tractable framework and I solve this model using a 'guess and verify' method.

1.3.6 Timeline

We summarize all the key ingredients of the model in Figure 1-4. All events occur in the following order:

1. the regulator chooses a signal structure $(\mathcal{S}, \tilde{\sigma})$; and Nature chooses the realization of

ρ;

2. signal realization s is revealed, and is publicly observable to all the borrowers;

3. borrowers choose their manipulation strategies;

4. all lenders make their lending decisions simultaneously, and borrowers decide which contract to accept;

5. all random variables are realized, and all players receive their payoffs.

1.3.7 Discussion of the Assumptions

1. The notion of predictive algorithm. In practice, predictive algorithm usually refers to the mapping from observed input (which is the borrower feature after potential manipulation in this model) to the output (which is the future payoff distribution in this model). In this paper, I consider the disclosure policy from a pure informational perspective and it cannot serve as a commitment device. This means that, when lenders disclose their predictive algorithm, they are able to flexibly change their predictive algorithms privately as a response to the borrowers' manipulation behavior. Focusing on the informational role of a predictive algorithm, disclosing information about the predictive algorithm is equivalent to disclosing the fundamental statistical properties of the random variables in the economic environment, which is the mapping from the borrower type to future payoff distribution in this model. By rational expectation, the manipulation behavior and lending decision rules are known by all players in equilibrium.

2. Disclosure vs regulating decision rules. In this paper, the disclosure is about the statistical properties of variables in the economy, and the regulator is not able to monitor or regulate lenders' lending decisions directly (for example, how they use certain variables in lending decisions). This feature is motivated by the challenge of regulating algorithmic lending in practice. First, although some regulations aim at regulating lending decisions directly (for example, prohibit the use of certain variables in lending decisions), the motivation usually comes from concerns on fairness and discrimination, and thus the regulation is independent of the statistical nature of the variables and easy to implement. With the focus on market surplus, in this paper,

regulating lending decisions will depend on the statistical natural of variables, which is hard to monitor and implement. Second, regulating lending decisions by monitoring the use of certain variables may not be effective, because they can easily be deduced from other variables that correlate highly with them, known as the 'reconstruction problem' (Kleinberg et al. (2018)). Thirdly, the algorithms are dynamic and adjust over time depending on the availability of data and data processing technology, which makes it harder to monitor and regulate their decision rules directly.

3. Lending market structure. In this model, I assume all lenders are identical and the lending market is competitive. This assumption is mainly for expositional convenience. The regulator cares about the total surplus generated from all financing activities, but not the distribution of the surplus between borrowers and lenders. In this model, market structure only changes the distribution of surplus among borrowers and lenders but not the total surplus. In an extreme case with a monopoly lender, if the lender does not use differentiated contracts to screen borrowers[19], then all the results about optimal disclosure policies remain the same, and the only difference is the distribution of surplus between borrowers and lenders.

4. Bad type borrowers always fail. In the model, I assume the bad type borrowers always fail, and thus the only benefit they can receive from financing their projects is the private benefit. This is to simplify the lending market equilibrium, and make the analysis more concentrated on the disclosure side. Relaxing this assumption may make the regulator's problem messy and intractable, but our key result, that partial disclosure policy is optimal, is still robust.

5. Only bad type borrowers are able to manipulate. In the model, I assume that only bad type borrowers are able to manipulate their features. But allowing good type borrowers to manipulate their features does not change the results. The key reason is that in equilibrium, $\hat{B}$ borrowers are always viewed as a worse group than $\hat{G}$ borrowers because bad type borrowers can always be $\hat{B}$ borrowers with no cost. Then good type borrowers have no incentive to manipulate and mimic the bad type. But this result relies on the assumption that the space of $\hat{\theta}$ is binary. In a model with general feature

[19] In Appendix, I show that in our baseline model, lenders will not screen borrowers using differentiated contracts.

space, the good type borrowers may be able to signal their type by paying cost and differentiating themselves from the bad type borrowers further.

1.4 The Lack of Commitment Problem and the Inefficiency of No Disclosure

The only friction in our model is the adverse selection due to endogenous data manipulation behavior. Bad type borrowers change their manipulation behavior as a best response to the updated public belief on the relevance ρ. For the optimal policy, a natural guess would be that the regulator shouldn't disclose any information about the relevance ρ to the public and make it as opaque as possible. In this case, the lending market equilibrium is characterized by $(k_N, \varrho_N, \bar{c}_N)$, and the regulator's payoff is

$$W_N = \int_{\varrho_N}^{1} \left(\mu m\left(\rho\right) - k_N\right) d\rho.$$

However, in this scenario, the use of the feature $\hat{\theta}$ is too intensive from the regulator's perspective, and thus it creates too much manipulation unconditionally. This result comes from the lenders' lack of commitment problem: they always make the most efficient use of borrower data ex post. To see this, suppose the regulator can 'force' all lenders to choose a higher lending cutoff $\varrho_N + x$ ($x \ll 1$), so the lenders only use feature $\hat{\theta}$ in their lending decisions when the relevance $\rho > \varrho_N + x$. From the perspective of borrowers, the feature $\hat{\theta}$ will be used with lower probability, and thus discourage their manipulation incentives. The marginal change of regulator's payoff is

$$\left.\frac{dW}{dx}\right|_{x=0} = -\left(\mu m\left(\varrho_N\right) - k_N\right) + \int_{\varrho_N}^{1} \left(-\left.\frac{dk_N}{dx}\right|_{x=0}\right) d\rho. \tag{1.17}$$

In equilibrium we must have $-\left.\frac{dk_N}{dx}\right|_{x=0} < 0$, because the more stringent lending cutoff discourages borrowers' manipulation incentives, which in turn decreases the total financing cost. Besides, the ex post efficiency in the lending market equilibrium implies

$$\mu m\left(\varrho_N\right) - k_N = 0,$$

this is the lenders' break-even condition at $\rho = \varrho_N$ in the no disclosure equilibrium. The

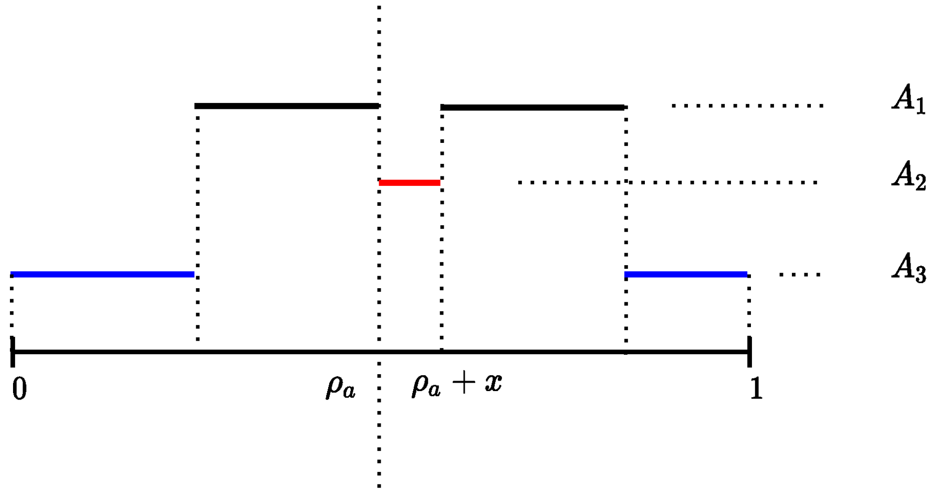


Figure 1-5: Suboptimality of No Disclosure Policy

above two observations jointly imply that

$$\left.\frac{dW}{dx}\right|_{x=0} > 0.$$

This suggests that 'forcing' lenders to use the feature less frequently improves the outcome of the lending market. Similar results show up in other economics settings where the information receivers commit to underweight some variables in decision rules to deter manipulation and improve efficiency (for example, Ball (2019)).

Although committing to the lending decisions is impossible in our model, the regulator can mitigate (average) manipulation behavior by disclosing information about the true state of relevance ρ. This leads to our first key result: the suboptimality of no disclosure equilibrium.

Proposition 1.4.1. *There exists a disclosure policy* $(\mathcal{S}, \tilde{\sigma})$ *with total surplus* W*, such that* $W > W_N$.

Proposition 1.4.1 challenges the conventional wisdom that making algorithms more transparent will always hurt efficiency because of the "gaming the system" concern. This is not true even if only correlational features are used in the predictive algorithm. The key to Proposition 1.4.1 is to find a disclosure policy under which the lenders will use feature $\hat{\theta}$ less frequently from the ex ante perspective, which will deter manipulation of the feature $\hat{\theta}$.

To gain intuitions on how it works, suppose the regulator designs a deterministic disclosure policy with three elements in the signal space $\mathcal{S} = \{s_1, s_2, s_3\}$, and the message function is

$$\sigma(\rho) = s_1 1_{A_1}(\rho) + s_2 1_{A_2}(\rho) + s_3 1_{A_3}(\rho)$$

where A_1, A_2, and A_3 are (unions of) intervals shown on Figure 1-5. The above disclosure policy effectively discloses which set of A_1, A_2, and A_3 that the true state belongs to. When signal s_i is disclosed to the borrowers, updated belief $\pi(\rho|s_i)$ is a uniform distribution conditional on set A_i. The boundaries of the intervals are chosen such that:

1. the equilibrium of subgame s_1 is the same as the no disclosure equilibrium, i.e.,

$$(k_1, \varrho_1, \bar{c}_1) = (k_N, \varrho_N, \bar{c}_N);$$

2. $A_2 = [\varrho_N, \varrho_N + x]$, where $x \ll 1$;

3. $A_3 = [0, 1] - A_1 \cup A_2$.

The equilibrium of subgame s_1 is the same as the no disclosure equilibrium, so it has no effect on the change of regulator's payoff. The signal s_2 reveals that the true state is in the interval $[\varrho_N, \varrho_N + x]$. Note that the equilibrium condition

$$\mu m(\varrho_N) - k_N = 0$$

implies the surplus when $\rho \in [\varrho_N, \varrho_N + x]$ is close to zero in the no disclosure equilibrium. And in the equilibrium of subgame s_2, the surplus must be nonnegative, then the change of regulator's payoff is also negligible in this case. When s_3 is disclosed, in the equilibrium of subgame s_3, the probability of financing $\hat{G}$ borrowers is lower than that in the no disclosure equilibrium (note that $\hat{G}$ borrowers will be financed only if the true state is in the right interval of A_3), which mitigates the manipulation incentives of bad type borrowers and improves the market surplus. Then the net effect of marginally increasing lending cutoff is positive.

1.5 General Properties of Optimal Policies

In this section, I discuss the general properties of the optimal policies.

1.5.1 Structure of The Optimal Policies

We already show that no disclosure is suboptimal in Section 1.4. Another natural guess for the optimal disclosure policy is full transparency, i.e., disclosing all information about the relevance ρ to the public. We can show that full transparency leads to the worst outcome, and thus it must be suboptimal.

Lemma 1.5.1. *Suppose W_F is the regulator's payoff when she makes the true state of relevance ρ fully transparent, then $W_F = 0$.*

Note that the regulator's payoff must be nonnegative. Lemma 1.5.1 implies that disclosing all information about the true state of relevance ρ leads to the regulator's worst payoff, so it must be suboptimal. The intuition behind the result is straightforward: when bad type borrowers know perfectly about the true state of relevance ρ, then in equilibrium, the data manipulation level satisfies that there is zero surplus from financing $\hat{G}$ borrowers, and lenders are indifferent between using and not using borrower data in lending decisions. This result is consistent with the popular argument that disclosing too much information about the predictive model hurts efficiency.

Proposition 1.4.1 and Lemma 1.5.1 jointly imply that the optimal disclosure policy must feature partial disclosure. Before exploring the properties of the optimal policy, we show all the subgame equilibria are ranked by equilibrium variables.

Lemma 1.5.2. *For any disclosure policy $(\mathcal{S}, \tilde{\sigma})$, and any two signal realizations s_1 and s_2, we must have*

$$k_{s_1} k_{s_2} \Longleftrightarrow \bar{c}_{s_1} \bar{c}_{s_2} \Longleftrightarrow \underline{\rho}_{s_1} \underline{\rho}_{s_2},$$

where $\left(k_{s_1}, \underline{\rho}_{s_1}, \bar{c}_{s_1}\right)$ and $\left(k_{s_2}, \underline{\rho}_{s_2}, \bar{c}_{s_2}\right)$ are defined in Definition 1.3.3.

When more bad type borrowers manipulate their features, adverse selection is more severe in the pool of $\hat{G}$ borrowers, and the quality type θ needs to be a more relevant variable (higher ρ) in identifying borrowers with better quality in lending decisions. Another observation related to Lemma 1.5.2 is that if there exist two signal realizations s_1, s_2 such that

$$\left(k_{s_1},\varrho_{s_1},\bar{c}_{s_1}\right)=\left(k_{s_2},\varrho_{s_2},\bar{c}_{s_2}\right),$$

then "combining" these two signal realizations together does not change the equilibrium outcome. The following lemma is a formal statement of this result.

Lemma 1.5.3. *For an optimal signal structure $(\mathcal{S},\tilde{\sigma})$ with distribution of posterior beliefs $\{f(s),\pi(\rho|s)\}_{s\in\mathcal{S}}$, if there exist two distinct realizations $s_1,s_2\in\mathcal{S}$, such that*

$$\left(k_{s_1},\varrho_{s_1},\bar{c}_{s_1}\right)=\left(k_{s_2},\varrho_{s_2},\bar{c}_{s_2}\right),$$

then the signal structure $(\mathcal{S}',\tilde{\sigma}')$ is also optimal, where $\{s_0'\}\notin\mathcal{S}$ and $(\mathcal{S}',\tilde{\sigma}')$ is defined by

$$\mathcal{S}'=\left\{s_0'\right\}\cup\mathcal{S}\backslash\left\{s_1,s_2\right\}$$

and

$$\tilde{\sigma}'(s|\rho)=\tilde{\sigma}(s|\rho)\,1_{\mathcal{S}\backslash\{s_1,s_2\}}(s)+\left(\tilde{\sigma}(s_1|\rho)+\tilde{\sigma}(s_2|\rho)\right)1_{\left\{s_0'\right\}}(s)$$

for all $\rho\in[0,1]$ and $s\in\mathcal{S}'$.

Lemma 1.5.3 is very intuitive. When there are two signal realizations s_1 and s_2 that lead to the equivalent equilibria, then instead of disclosing these two signal realizations separately, we can simply disclose that "the realization is either s_1 or s_2", and the equilibrium outcome will be unchanged. Based on this observation, without loss of generality, we impose the following restriction on optimal policies: We focus on policies $(\mathcal{S},\tilde{\sigma})$ such that for any $s_1,s_2\in\mathcal{S}$ and $s_1\neq s_2$, the lending market equilibria satisfy $\left(k_{s_1},\varrho_{s_1},\bar{c}_{s_1}\right)\neq\left(k_{s_2},\varrho_{s_2},\bar{c}_{s_2}\right)$. Based on the above criterion and the suboptimality of no disclosure equilibrium, the optimal policy must differentiate the subgame equilibria by the data manipulation levels (and other equilibrium variables), which is measured by $\bar{c}_s$. The next lemma shows that data manipulation exists in all subgame equilibria, so there is no first best outcome for any subgame. And an implication of the lemma is that it is never optimal to confirm that a feature is not used in lending decisions for sure.

Lemma 1.5.4. *(Manipulation in all states) Suppose $(\mathcal{S},\tilde{\sigma})$ is an optimal policy. Then for* *almost all $s \in \mathcal{S}$, we must have*

$$k_s > I, \underline{\rho}_s > m^{-1}(I), \bar{c}_s > 0.$$

Lemma 1.5.4 rules out some disclosure policies. For example, suppose the regulator chooses a disclosure policy that reveals whether the relevance ρ is below $m^{-1}(I)$ or not. Note that for any $\rho < m^{-1}(I)$, it is inefficient to finance any borrowers, then there will be no loan approved and no manipulation. On the other hand, if $\rho > m^{-1}(I)$ is revealed, $\hat{G}$ borrowers will be financed and the unique lending market equilibrium is determined by conditions in Definition 1.3.3. This disclosure policy violates the result in Lemma 1.5.4, and thus it is inefficient. This is because compared to the no disclosure equilibrium, the regulator does not gain anything from states $\rho \leq m^{-1}(I)$ as she still only receives zero payoff, but more people will choose to manipulate in states $\rho > m^{-1}(I)$, as signal $\rho > m^{-1}(I)$ confirms the high relevance of the feature $\hat{\theta}$ and incentivizes more manipulation. This cutoff policy is dominated by the no disclosure policy, which effectively pools these two signals together. Actually, as we will discuss later, in the optimal policy, we want to mix low states (where relevance ρ is low) with high states (where relevance ρ is high) and preserve uncertainty of the true state of relevance ρ in all posterior equilibria.

The second necessary condition of optimal policy features ex ante cutoff of lending decisions. Note that under any subgame s, the loan applications from $\hat{G}$ borrowers will be approved if the relevance ρ is high enough, i.e., when $\rho > \underline{\rho}_s$. This means that the lending decision is always a cutoff decision ex post, and this is a natural result in equilibrium: the feature $\hat{\theta}$ is more useful when ρ is higher. It turns out that this condition is also satisfied ex ante under the optimal disclosure policy. The following lemma states this result.

Lemma 1.5.5. *(Ex ante lending cutoff) Suppose that $(\mathcal{S}, \tilde{\sigma})$ is an optimal policy, with induced distribution of posteriors $\{f(s), \pi(\rho|s)\}_{s\in\mathcal{S}}$, then there must exist a constant $\underline{\rho}^{\star} \in (0,1)$, such that for almost all $s \in \mathcal{S}$, $\hat{G}$ borrowers are financed if and only if*

$$\rho \in (\underline{\rho}^{\star}, 1] \cap (\pi(\rho|s)).$$

Figure 1-6 explains Lemma 1.5.5 by showing three specific signal realizations s_1, s_2,

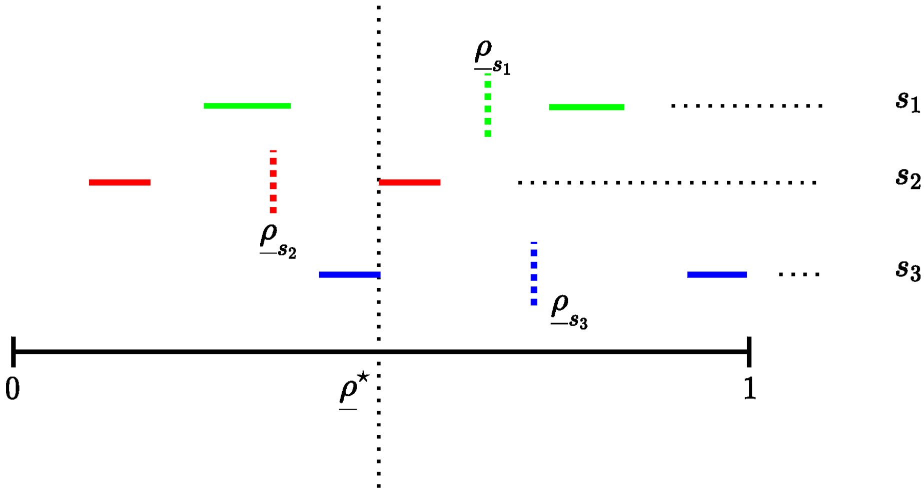


Figure 1-6: Efficiency

and s_3. The colored regions represent the posterior beliefs under these three signals, and cutoffs $\underline{\rho}_{s_1}$, $\underline{\rho}_{s_2}$ and $\underline{\rho}_{s_3}$ represent lenders' equilibrium lending cutoffs in these three equilibria. Consider signal s_1 with $\underline{\rho}_{s_1} > \underline{\rho}^{\star}$. Since in the subgame s_1, $\hat{G}$ borrowers will not be financed if $\rho \leq \underline{\rho}_{s_1}$, Lemma 1.5.5 implies that

$$(\pi(\rho|s_1)) \cap \left(\underline{\rho}^{\star}, \underline{\rho}_{s_1}\right] = \emptyset.$$

Similar results can be obtained for all other signal realizations.

Then from the unconditional (ex ante) perspective, $\hat{G}$ borrowers will be financed if and only if

$$\rho > \underline{\rho}^{\star}.$$

This condition confirms the efficiency of optimal policies, in which lenders utilize the feature $\hat{\theta}$ if and only if the relevance ρ is high enough. Besides, note that for almost all subgame s, Lemma 1.5.5 implies that, in equilibrium we have

$$\sup\left\{(\pi(\rho|s)) \cap [0, \underline{\rho}^{\star}]\right\} \leq \underline{\rho}_s \leq \inf\left\{(\pi(\rho|s)) \cap (\underline{\rho}^{\star}, 1]\right\}.$$

Then the support of the posterior belief is divided into two parts: the rejection region

$$(\pi(\rho|s)) \cap [0, \underline{\rho}^{\star}]$$

in which all loan applications are rejected, so the lending decision is independent of borrower data; and the approval region

$$(\pi(\rho|s)) \cap (\underline{\rho}^{\star}, 1]$$

in which $\hat{G}$ borrowers are financed, so the lending decision making is dependent on borrower data. The following lemma shows that, without loss of generality, we can focus on deterministic disclosure policies in which both regions are intervals for all posterior equilibria, and all subgames are ranked by the equilibrium data manipulation levels.

Lemma 1.5.6. *For any optimal disclosure policy $(\mathcal{S}, \tilde{\sigma})$, there must exist a deterministic optimal policy $(\mathcal{S}, \sigma)$ with the same signal space $\mathcal{S}$. Let $\left\{\tilde{f}(s), \tilde{\pi}(\rho|s)\right\}_{s\in\mathcal{S}}$ and $\{f(s), \pi(\rho|s)\}_{s\in\mathcal{S}}$ be the distribution of posteriors for the policy $(\mathcal{S}, \tilde{\sigma})$ and $(\mathcal{S}, \sigma)$ respectively, and let $\left(\tilde{k}_s, \tilde{\underline{\rho}}_s, \tilde{\bar{c}}_s\right)$ and $(k_s, \underline{\rho}_s, \bar{c}_s)$ be equilibrium outcomes for the policy $(\mathcal{S}, \tilde{\sigma})$ and $(\mathcal{S}, \sigma)$ respectively. Then the following properties hold:*

1. *$f(s) = \tilde{f}(s)$ and $\left(\tilde{k}_s, \tilde{\underline{\rho}}_s, \tilde{\bar{c}}_s\right) = (k_s, \underline{\rho}_s, \bar{c}_s)$ for almost all s, and the ex ante lending cutoffs defined in Lemma 1.5.5 are the same under these two policies, denoted as $\underline{\rho}^{\star}$;*

2. *for almost all $s \in \mathcal{S}$, both*

$$(\pi(\rho|s)) \cap [0, \underline{\rho}^{\star}]$$

and

$$(\pi(\rho|s)) \cap (\underline{\rho}^{\star}, 1]$$

are non-empty intervals;

3. *for almost all $s_1, s_2 \in S$ with $\bar{c}_{s_1} < \bar{c}_{s_2}$,*

$$\sup\{(\pi(\rho|s_1)) \cap [0, \underline{\rho}^{\star}]\} \leq \inf\{(\pi(\rho|s_2)) \cap [0, \underline{\rho}^{\star}]\} \tag{1.18}$$

and

$$\sup\{(\pi(\rho|s_1)) \cap (\underline{\rho}^{\star}, 1]\} \leq \inf\{(\pi(\rho|s_2)) \cap (\underline{\rho}^{\star}, 1]\}. \tag{1.19}$$

Lemma 1.5.6 simplifies the space of optimal disclosure policies. It shows that, for any optimal policy $(\mathcal{S}, \tilde{\sigma})$, we can find a payoff-equivalent deterministic policy $(\mathcal{S}, \sigma)$ which induces the same posterior lending market equilibria. And for the deterministic optimal disclosure policy, the posterior belief always consists of two intervals representing the rejection region and the approval region. For almost all signal realizations, the posterior equilibria are ranked by the equilibrium data manipulation levels (measured by $\bar{c}_s$). Based on the above observations, we characterize the structure of an optimal disclosure policy in the following theorem.

Theorem 1.5.1. *There exists a deterministic optimal policy* $(\mathcal{S}, \sigma)$ *which consists of*

1. *a signal space* $\mathcal{S} \subset [\bar{c}_{\min}, \bar{c}_{\max}]$*;*
2. *a message function* σ *and cutoff* $\underline{\rho}^\star \in (0,1)$ *such that both*

$$\sigma|_{[0,\underline{\rho}^\star]} : [0, \underline{\rho}^\star] \to \mathcal{S}$$

and

$$\sigma|_{(\underline{\rho}^\star,1]} : (\underline{\rho}^\star, 1] \to \mathcal{S}$$

are weakly increasing functions with the same range.

Under this optimal policy, for any subgame s*, the equilibrium cutoff of data manipulation cost is* $\bar{c}_s = s$*, and* $\hat{G}$ *borrowers will be financed if and only if* $\rho > \underline{\rho}^\star$ *for all* s*.*

Here we select a specific signal space such that the message sent to borrowers is actually the recommended data manipulation decision. Upon observing signal realization s, bad type borrowers are recommended to manipulate their features if and only if their manipulation cost satisfies $c \le s$. Note that in Theorem 1.5.1 we only characterize the general structure of the optimal message function σ but not provide the exact functional form of it.

Figure 1-7 is a graphical illustration of Theorem 1.5.1. For each signal realization s (for example, the signal $s = \bar{c}_{\min}$ or $\bar{c}_{24}$), the posterior belief is a union of two disjoint intervals[20] which can always be separated by the cutoff $\underline{\rho}^\star$. These two disjoint intervals represent the rejection region and approval region in lending decisions. For example, the red intervals represent the posterior belief of signal $s = \bar{c}_{\min}$, and when the true state of ρ is in the red region, the recommended cutoff of data manipulation cost $\bar{c}_{\min}$ is sent to the borrowers. Upon observing the signal $s = \bar{c}_{\min}$, bad type borrowers update their belief and choose to manipulate their features if their manipulation cost satisfies $c \le \bar{c}_{\min}$.

[20] Note that a single point is also a closed interval.

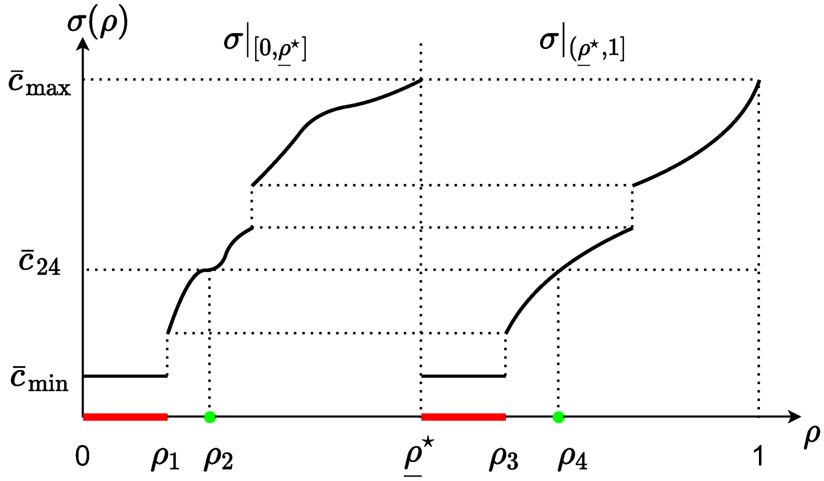


Figure 1-7: Graphical Illustration of Theorem 1.5.1

The optimal message function consists of two parts, denoted as

$$\sigma_L = \sigma|_{[0,\underline{\rho}^{\star}]}$$

and

$$\sigma_R = \sigma|_{(\underline{\rho}^{\star},1]}\,.$$

Then $\sigma_L(\rho)$ and $\sigma_R(\rho)$ can be viewed as the message functions for the rejection region and approval region respectively. For any $s \in \mathrm{Ran}(\sigma_L) = \mathrm{Ran}(\sigma_R)$[21], $\sigma_L^{-1}(s)$ (or $\sigma_R^{-1}(s)$) can either be an interval with positive length or a single point. In the first case, the signal is discrete and the equilibrium approval probability for $\hat{G}$ borrowers is

$$\frac{\mathrm{Prob}\left(\sigma_R^{-1}(s)\right)}{\mathrm{Prob}\left(\sigma_L^{-1}(s)\right) + \mathrm{Prob}\left(\sigma_R^{-1}(s)\right)}.$$

In the second case, the signal is continuous, and the equilibrium approval probability for $\hat{G}$

[21] Ran(f) means the range of a function f.

borrowers is[22]

$$\frac{\frac{1}{\sigma_R'\left(\sigma_R^{-1}(s)\right)}}{\frac{1}{\sigma_L'\left(\sigma_L^{-1}(s)\right)}+\frac{1}{\sigma_R'\left(\sigma_R^{-1}(s)\right)}},$$

where $\frac{1}{\sigma_R'\left(\sigma_R^{-1}(s)\right)}$ $\left(\frac{1}{\sigma_L'\left(\sigma_L^{-1}(s)\right)}\right)$ is an analog of $\text{Prob}\left(\sigma_R^{-1}(s)\right)$ $\left(\text{Prob}\left(\sigma_L^{-1}(s)\right)\right)$ in the previous case.

1.5.2 Properties of Optimal Policies

In this subsection, we discuss some properties of the posterior equilibria. First, the prior belief of ρ is $\rho \sim U[0,1]$, then unconditionally, the probability that $\hat{G}$ borrowers are financed is

$$\text{Prob}\left(\rho > \underline{\rho}^{\star}\right) = 1 - \underline{\rho}^{\star}.$$

Similarly, in the no disclosure equilibrium, the probability that $\hat{G}$ borrowers are financed is

$$\text{Prob}\left(\rho > \underline{\rho}_N\right) = 1 - \underline{\rho}_N.$$

The following Proposition shows that $\hat{G}$ borrowers are financed less frequently under the optimal disclosure policy compared to the no disclosure case.

Proposition 1.5.1. *Suppose $\underline{\rho}^{\star}$ is the cutoff described in Lemma 1.5.5, then $\underline{\rho}^{\star} > \underline{\rho}_N$.*

Proposition 1.5.1 implies that borrower data are used less frequently under the optimal disclosure policy compared to the no disclosure case, which confirms our intuition why no disclosure equilibrium is suboptimal, and why it can be improved. In the no disclosure equilibrium, the feature $\hat{\theta}$ is used too intensively, resulting in too much manipulation. To mitigate this problem, the regulator prefers the feature $\hat{\theta}$ to be used less frequently, and this is achieved by the optimal policy.

The second property is about the data manipulation levels in posterior equilibria.

Proposition 1.5.2. $\bar{c}_{\max} > \bar{c}_N > \bar{c}_{\min}$.

[22] In this case, the probability of observing a specific signal is always zero. Then the distribution of signal is represented by a density function $f(s)$:

$$f(s) = \frac{1}{\tilde{\sigma}_L'\left(\tilde{\sigma}_L^{-1}(s)\right)} + \frac{1}{\tilde{\sigma}_R'\left(\tilde{\sigma}_R^{-1}(s)\right)},$$

where $\frac{1}{\tilde{\sigma}_L'\left(\tilde{\sigma}_L^{-1}(s)\right)}$ and $\frac{1}{\tilde{\sigma}_R'\left(\tilde{\sigma}_R^{-1}(s)\right)}$ represent the weights of the rejection region and approval region, respectively.

$\bar{c}_{\max}$ and $\bar{c}_{\min}$ represents the highest and lowest equilibrium cutoffs of manipulation cost among all posterior equilibria. Proposition 1.5.2 explains the idea of differentiation of posterior equilibria. In the equilibrium with highest data manipulation level ($\bar{c}_s = \bar{c}_{\max}$), a higher ρ is required for the feature $\hat{\theta}$ to be used in lending decisions, and this deters the use of borrower data in this subgame equilibrium. The cost is the higher data manipulation level, and thus lenders have to finance more bad type borrowers, while the benefit is the less use of borrower data which discourages data manipulation unconditionally. As we discussed in Section 1.4, the positive effect dominates and surplus improves.

The last property is about the surplus in the posterior equilibria. Note in Section 1.4, we show that the inefficiency comes from states when ρ is close to ρ_N (see condition (1.17)). In these states the regulator's payoff is small, so the benefit of financing $\hat{G}$ borrowers cannot justify the negative externality it imposes on other states. The following Proposition shows that, to mitigate the negative externality, the surplus from lending activities must be large enough for any posterior equilibrium that occurs with positive probability.

Proposition 1.5.3. *(Positive surplus) Under any optimal policy characterized in Theorem 1.5.1, for any $\epsilon > 0$, there exists $\delta > 0$, such that for any posterior equilibrium with signal realization s satisfying*

$$Prob\,(s) > \epsilon,$$

the surplus from lending must be greater than δ for any $\rho > \underline{\rho}^{\star}$.

1.5.3 A Closed-Form Characterization

I characterize the general structure of optimal policies in Theorem 1.5.1, while leaving the functional form of message function $\sigma(\cdot)$ unsolved. In this subsection, I provide a closed-form characterization of the optimal policy by imposing a distributional assumption on the manipulation cost.

Assumption 1.5.1. *$xF_c(x)$ has at most one inflection point[23] on $[0, \bar{c}]$.*

Many commonly used distribution functions satisfy Assumption 1.5.1, including truncated normal distribution, uniform distribution, truncated exponential distribution, Beta distribution, Gamma distribution, Weibull distribution, etc. Since $xF_c(x)$ is locally convex

[23] Inflection points are points where the function changes concavity.

around $x = 0$, Assumption 1.5.1 means that $xF_c(x)$ is either a weakly convex function on $[0, \bar{c}]$, or there exists $\tilde{c} \in (0, \bar{c})$ such that $xF_c(x)$ is weakly convex on $[0, \tilde{c}]$ and weakly concave on $[\tilde{c}, \bar{c}]$. With this assumption, the optimal policy has a simpler structure. In Theorem 1.5.1, the message is the recommended data manipulation decision, while in the following theorem, without loss of generality, I choose a different signal space to make the results simpler.

Theorem 1.5.2. *When Assumption 1.5.1 is satisfied, there exists a deterministic optimal policy $(\mathcal{S}, \sigma)$ characterized by*

1. *three cutoffs $(\varrho_a, \varrho^\star, \varrho_b)$ satisfying $0 < \varrho_a < \varrho^\star < \varrho_b < 1$;*
2. *a signal space $\mathcal{S} = [\varrho_a, \varrho^\star]$;*
3. *a continuous, strictly increasing function $\gamma : [\varrho_b, 1] \to [\varrho_a, \varrho^\star]$ satisfying $\gamma(\varrho_b) = \varrho_a$ and $\gamma(1) = \varrho^\star$, such that the message functions $\sigma(\rho)$ is*

$$\sigma|_{[0,\varrho^\star]} = \begin{cases} \varrho_a & \text{if } \rho \in [0, \varrho_a] \\ \rho & \text{if } \rho \in (\varrho_a, \varrho^\star] \end{cases}$$

and

$$\sigma|_{(\varrho^\star,1]} = \begin{cases} \varrho_a & \text{if } \rho \in (\varrho^\star, \varrho_b] \\ \gamma(\rho) & \text{if } \rho \in (\varrho_b, 1] \end{cases}.$$

For any $s \in \mathcal{S}$, the equilibrium data manipulation decision $\bar{c}_s$ satisfies

$$\mu m(s) = (\mu + (1-\mu) F_c(\bar{c}_s)) I.$$

The optimal policy is a simplified version of our general result in Theorem 1.5.1. Both $\sigma|_{[0,\varrho^\star]}$ and $\sigma|_{(\varrho^\star,1]}$ are continuous and consist of a flat region and a strictly increasing region. In the signal space $\mathcal{S} = [\varrho_a, \varrho^\star]$, $s = \varrho_a$ is a discrete signal and the posterior belief is a uniform distribution conditional on $[0, \varrho_a] \cup (\varrho^\star, \varrho_b]$. For any $s \in (\varrho_a, \varrho^\star]$, the signal is continuous. Let $x = \gamma^{-1}(s)$, then the posterior distribution of relevance ρ is a lottery[24] with binary

[24] A lottery $\langle (x_1, x_2, ... x_N), (p_1, p_2, ... p_N) \rangle$ is a discrete random variable with probability function $\text{Prob}(x = x_i) = p_i$.

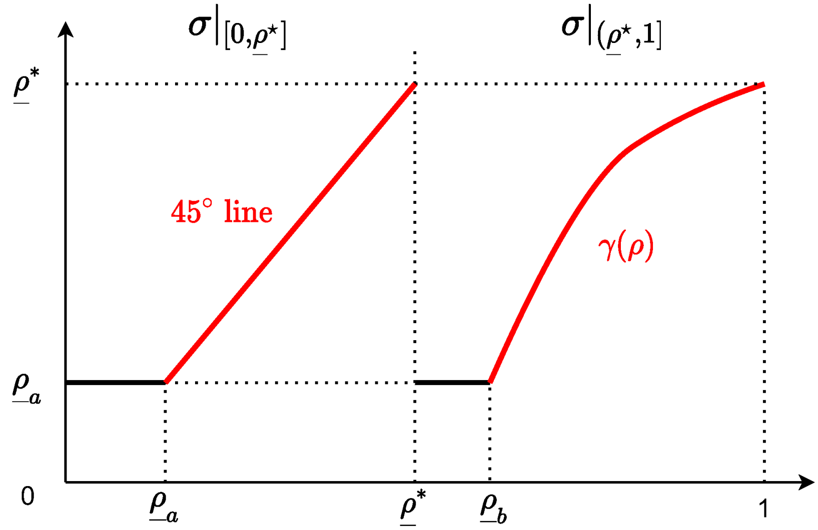


Figure 1-8: A closed-form characterization

outcomes:

$$\left\langle \left(\gamma\left(x\right),x\right), \frac{\left(\gamma'\left(x\right),1\right)}{\gamma'\left(x\right)+1} \right\rangle. \tag{1.20}$$

For any subgame s, the equilibrium lending cutoff $\underline{\rho}_s$ satisfies the following condition:

Lemma 1.5.7. *Under the deterministic optimal disclosure policy* $(\mathcal{S},\sigma)$ *characterized in Theorem 1.5.2, for any* $s \in \mathcal{S}$, *we must have* $\underline{\rho}_s = \sup\left\{\left(\pi\left(\rho|s\right)\right) \cap [0,\underline{\rho}^{\star}]\right\}$.

This means that the lending cutoff $\underline{\rho}_s$ is chosen such that it equals the highest value in the rejection region. Lemma 1.5.7 and all equilibrium conditions jointly imply that for all $\rho \in (\underline{\rho}_b, 1]$, the function $\gamma\left(\rho\right)$ satisfies the following ODE:

$$m\left(\gamma\left(\rho\right)\right) = 1 + \frac{1-\mu}{\mu} F_c\left(b \cdot \frac{1}{1+\gamma'\left(\rho\right)}\right), \tag{1.21}$$

with boundary conditions

$$\gamma\left(\underline{\rho}_b\right) = \underline{\rho}_a \text{ and } \gamma\left(1\right) = \underline{\rho}^{\star}.$$

The equilibrium condition under the discrete signal $s = \underline{\rho}_a$ implies

$$\gamma'\left(\underline{\rho}_b\right) = \frac{\underline{\rho}_a}{\underline{\rho}_b - \underline{\rho}^{\star}}.$$

With the above characterization, all of $\gamma\left(\rho\right)$, $\underline{\rho}_a$ and $\underline{\rho}_b$ can be solved as a function of the ex ante lending cutoff $\underline{\rho}^{\star}$, so the equilibrium is uniquely determined by a single variable

$\varrho^{\star}$. Then we can reduce the original infinite-dimensional optimization problem to a one-dimensional problem. The regulator's problem becomes

$$\begin{aligned}
&\underset{\varrho^{\star}}{\text{maximize}} \int_{\varrho^{\star}}^{1} m(x)\,dx - \int_{\varrho_b}^{1} m(\gamma(x))\,dx - (\varrho_b - \varrho^{\star})\, m(\varrho_a) \\
&\text{subject to } m(\gamma(x)) = 1 + \frac{1-\mu}{\mu} F_c\left(b \cdot \frac{1}{1+\gamma'(x)}\right), \\
&\qquad \gamma(\varrho_b) = \varrho_a, \gamma(1) = \varrho^{\star}, \\
&\qquad \gamma'(\varrho_b) = \frac{\varrho_a}{\varrho_b - \varrho^{\star}}.
\end{aligned} \tag{1.22}$$

Lemma 1.5.7 also implies that approval probability in posterior equilibrium is an increasing function of s. Note that for any $s \in (\varrho_a, \varrho^{\star})$, the posterior belief is a lottery represented by (1.20), then the approval probability is

$$\frac{1}{1+\gamma'(\gamma^{-1}(s))}.$$

These observations jointly implies that $\gamma'(\rho)$ is a strictly decreasing function of ρ. The following lemma states the formal result.

Lemma 1.5.8. *In the optimal disclosure policy characterized in Theorem 1.5.2, $\gamma(\rho)$ is a strictly concave function on $\rho \in (\varrho_b, 1)$.*

1.6 Extension: Costly Fraud Detection

In the main model, all bad type borrowers' manipulation decisions are unobservable to lenders. In practice, lenders can also costly identify fraudulent activities using various methods, which is another way of mitigating adverse selection. In this extension, I consider how the disclosure policy interacts with fraud detection in the regulator's problem.

Assume all lenders have the identical linear cost function of fraud detection, i.e., each lender can verify and reveal any borrower's true type by paying cost $t > 0$. Once the type of a borrower is verified, it becomes public information. To consider the optimal disclosure policy with this fraud detection technology, note that for any equilibrium with posterior belief $\pi(\rho|s)$ and total financing cost k_s, the net value of verifying a $\hat{G}$ borrower's true type is

$$W_V = \max\left\{\frac{\mu I}{k_s}(m(\rho) - I), 0\right\} - t.$$

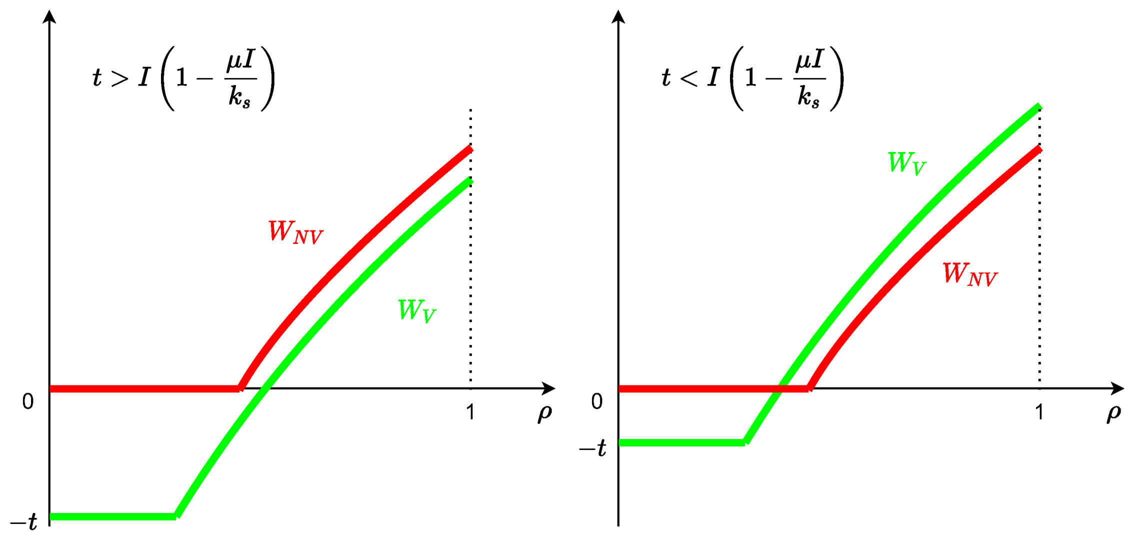


Figure 1-9: W_N vs W_{NV}

And the net value of not verifying the borrower's true type is

$$W_{NV} = \max\left\{\frac{\mu I}{k_s} m(\rho) - I, 0\right\}.$$

Figure 1-9 compares W_V and W_{NV}. When $t > I\left(1 - \frac{\mu I}{k_s}\right)$,

$$W_{NV} > W_V$$

for all ρ, then in this case, lenders will never verify any borrower's true type.

When $t < I\left(1 - \frac{\mu I}{k_s}\right)$,

$$W_{NV} < W_V \iff \rho > \rho_e,$$

where ρ_e solves

$$\frac{\mu I}{k_s}(m(\rho_e) - I) - t = 0.$$

Then lenders will verify $\hat{G}$ borrowers with probability 1 when $\rho > \rho_e$, and lend to $\hat{G}$ borrowers only when they pass the verification. However, this cannot be an equilibrium because in this case no bad type borrower has the incentive to manipulate (since approval is possible only when they pass the verification). As a best response, lenders have no incentive to verify, which is a contradiction.

When $t = I\left(1 - \frac{\mu I}{k_s}\right)$, ρ_e solves

$$\mu m\left(\rho_e\right) - k_s = 0,$$

so $\rho_e = \rho_s$. Moreover,

$$W_{NV} = (>) W_V \Longleftrightarrow \rho \geq (<) \rho_e.$$

In this case, lenders are indifferent between verifying or not when $\rho \geq \rho_s$, and will lend to $\hat{G}$ borrowers only when $\rho \geq \rho_s$.

In summary, if in subgame s, lenders verify any borrower's true type with positive probability, we must have

$$t = I\left(1 - \frac{\mu I}{k_s}\right) \Longleftrightarrow k_s = k^v = \frac{\mu I^2}{I - t}.$$

And the data manipulation level $\bar{c}^v$ is

$$k^v = \left(\mu + (1 - \mu) F_c\left(\bar{c}^v\right)\right) I \Longleftrightarrow \bar{c}^v = F_c^{-1}\left(\frac{\mu t}{(1 - \mu)(I - t)}\right).$$

In this subgame, when $\rho \geq m^{-1}\left(\frac{k^v}{\mu}\right)$, lenders verify $\hat{G}$ borrowers' true types with positive probability and lend to those who are not verified or verified to be good type borrowers. The verification probability p^v[25] satisfies the condition that bad type borrowers with cost $\bar{c}^v$ break even.

The following theorem presents the optimal disclosure policy with verification and confirms the robustness of our baseline result.

Theorem 1.6.1. *With costly verification, there exists t^v such that*

1. *when $t \geq t^v$, lenders will never use verification, and the optimal disclosure policy will not change;*

2. *when $t < t^v$, there exists $\rho^v \in (0, 1)$, such that the optimal disclosure is characterized as two steps:*

    (a) The regulator first reveals if the true state ρ is above ρ^v or not.

[25] Here I assume p^v to be constant for simplicity. The choice of verification probability p^v can depend on the true state ρ, and thus is not unique.

(b) If the true state $\rho > \rho^v$, then the lenders will verify all $\hat{G}$ borrowers with probability $p^v = 1 - \frac{\bar{c}^v}{B}$, and lend to $\hat{G}$ borrowers who are not verified or verified to be good type borrowers.

(c) If the true state $\rho \in [0, \rho^v]$, then information about ρ is disclosed according to a policy $(\mathcal{S}^v, \sigma^v)$, where $(\mathcal{S}^v, \sigma^v)$ is an optimal disclosure policy characterized in Theorem 1.5.1 with prior belief $\rho \sim U[0, \rho^v]$.

Theorem 1.6.1 shows that the disclosure policy and verification technology interact in a simple way: when the relevance ρ is sufficiently high ($\rho > \rho^v$), only verification is used to disincentivize manipulation behavior, and disclosure becomes irrelevant; while when the relevance ρ is not high enough ($\rho \leq \rho^v$), only disclosure policy is used to disincentivize the manipulation behavior and verification technology is never used.

1.7 Conclusion

I study the optimal algorithmic disclosure in a lending market. FinTech lenders use privately observed predictive algorithms to help make lending decisions. The input of the predictive algorithm is the data collected from borrowers, which is subject to a strategic manipulation problem. In the optimal public disclosure, the information about the predictive algorithm should be partially disclosed to the borrowers, which differentiates the posterior lending market equilibira by data manipulation levels. Under the optimal disclosure policy, lenders use borrower data less intensively in their lending decisions which decreases the average data manipulation level and improves efficiency.

There are some potential directions for future research. First, in my model, I abstract away the screening channel using contracts in the lending market. The joint design of information and contract will be a natural question for future research. Second, the feature in my model is a binary variable, and it will be interesting to consider a model with a general space for input. In a general model, all types of borrowers may signal their types by costly manipulation, and the interaction between the signaling and information design is also interesting. Thirdly, this paper mainly focuses on efficiency but not the distribution of surplus. Since fairness is also a crucial part of the regulator's objective, it will be interesting to consider the optimal algorithmic disclosure that achieves a particular surplus distribution. Finally, all lenders use the same predictive algorithm in my model, but it is natural to consider the setting where lenders use different but correlated algorithms, and in this case, algorithmic disclosure may change the lending market structure.

Chapter 2

Learning from Manipulable Signals

2.1 Introduction

Asymmetric information is pervasive in long-term relationships; meanwhile, learning often takes place during the interactions between different parties. For instance, venture capital (VC) firms face asymmetric information in their investments: startups often have better information about the odds of success of their projects than the investors (Leland and Pyle, 1977; Chan, 1983; Gompers and Lerner, 2004). Moreover, due to the private benefits from receiving continuous funding,[1] startups are willing to pursue projects that are less viable than what VCs are willing to invest in. VCs, upon agreeing to finance a startup, receive periodical performance reports (subscription growth, number of patents, media and user reviews, etc.) from the startup. These reports may provide information about the viability of the startup. However, the startup may undertake hidden actions to inflate the performance report, tampering with its informativeness. Examples include rideshare platforms who periodically announce their numbers of users and could inflate such statistics by specialized promotions, and Luckin Coffee and Theranos who have been under investigation for fabricating key performance data.

We analyze learning problems with asymmetric information and hidden actions, and investigate the equilibrium learning dynamics. In our model, a principal (VC) and an agent (startup) are engaged in a relationship that takes place in continuous time. Performance reports are modeled as public signals evolving according to a Brownian motion whose drift

[1] An extreme example is the former CEO of WeWork, Adam Neumann, who allegedly purchased a corporate jet with the company's money for personal use.

depends on the agent's privately-known type and action. If the agent is an *investible* type, then the drift is $\mu > 0$; if the agent is a *noninvestible* type, then the drift is 0 by default, but the agent can take a costly action to boost the drift up to μ. The signals serve only an informational role, and do not affect the principal's payoffs. The principal receives opportunities to terminate the relationship according to a Poisson process, and chooses whether to terminate the relationship whenever such an opportunity arises;[2] she prefers to continue the relationship with the investible type and to terminate the relationship against the noninvestible type.

We study Markov equilibria of this game where the state variable is the public belief that the agent is a noninvestible type. We call the complementary probability, i.e., the probability that the agent is an investible type, the agent's reputation. Our first result establishes the existence and uniqueness of Markov equilibrium.

In the unique equilibrium, the principal's termination strategy has a cutoff structure — the principal terminates the relationship if and only if the agent's reputation is sufficiently bad. The agent's equilibrium strategy depends on the magnitude of his discount rate. If his discount rate is greater than a cutoff (i.e., if he does not care much about the future), then he never engages in costly performance boosting. If his discount rate is less than the cutoff (i.e., if he is patient enough), then the agent does not engage in performance boosting when his reputation is very good or very bad, but will do so with intermediate reputation. In particular, the intensity of performance boosting is hump-shaped in the agent's reputation and *peaks* at the principal's termination cutoff.

Our first qualitative finding concerns the relationship between the agent's reputation and the expected performance, measured by the expected drift of the signal from an outsider's perspective. If the agent is so impatient that he never engages in performance boosting, then the expected performance is increasing in the agent's reputation (decreasing in the state variable). However, when the agent is more patient and engages in some performance boosting, the expected performance is non-monotone in the agent's reputation. Starting from an initial good reputation, as the agent's reputation deteriorates, the expected performance first declines, reaching a local minimum, and then it rises, reaching a local maximum precisely at the principal's termination cutoff, and decreases again thereafter (see Figure 2-3). This

[2]The Poisson arrival of stopping opportunities captures the frictions in the principal's decision making and implementation, and will technically help us avoid off-path histories in our equilibrium analysis.

finding may help explain why some startups deliver impressive performance reports, such as large sales growth (e.g., Luckin Coffee), extraordinary revenue flow (e.g., Theranos) or rapid expansion (e.g., WeWork), not long before investors pull their funds. It is also consistent with the observation that growing market suspicion and strong (expected) performance can coexist for a period of time.

Our second qualitative result concerns the relationship between the amount of information transmission and the transparency of the performance measure. Due to random events such as demand shocks and measurement errors, performance reports are imperfect signals of the agent's type and action, and we use the signal-to-noise ratio of the process to capture its transparency. In reality, transparency may be determined by the volatility of the product market and may also be affected by how much detail a startup is required to disclose. We show that, due to the agent's endogenous signal manipulation, the principal may be worse off as transparency improves. This result suggests that VCs can sometimes fare better when the startup initially operates in a more volatile market, and that policies that require disclosing too precise information may end up hurting the investors. We also find that exogenous delays in the principal's decision making can sometimes help her, as they facilitate information transmission by reducing the agent's incentive to manipulate the signal.

Specifically, if the opportunity to terminate the relationship arrives at a rate less than a cutoff (the high-friction case), then in equilibrium, the agent never engages in performance boosting too aggressively because termination is always unlikely. In this case, as the signal-to-noise ratio grows, the information flow in the principal's optimal stopping problem approaches immediate revelation of the agent's type, which benefits the principal.

On the other hand, if the termination opportunity arrives at a rate greater than the aforementioned cutoff (the low-friction case), then the agent has stronger incentives to engage in performance boosting. We find, perhaps surprisingly, that the principal's payoff is nonmonotone in the signal-to-noise ratio of the performance report, implying that the principal can be worse off when the performance measure becomes more transparent (less noisy). We obtain this result by looking at two extreme cases. At one extreme, if the performance report is independent of the agent's type and action (i.e., uninformative signals), then the principal can never learn about the agent's type and will receive her "no-information" value. At the other extreme, as the signal-to-noise ratio grows without bound, we show that the principal cannot utilize any information about the agent's type either. Intuitively, in this case the agent will engage in performance boosting aggressively, for otherwise his type would be revealed rapidly. Such aggressive performance boosting is anticipated by the principal, and thus largely reduces the informativeness of the signal. As a result, the principal's equilibrium payoff converges to her "no-information" value. In contrast to the extreme cases, for intermediate values of the signal-to-noise ratio, the principal will learn some information about the agent's type and get a payoff strictly above her "no-information" value.

Finally, we investigate the equilibrium outcomes as players get arbitrarily patient. We find a strong manifestation of the ratchet effect in the patient limit of our model. Since the principal cannot commit to refraining from using future information against the agent, a patient agent will engage in performance boosting with almost full intensity in order to maintain his reputation. In the limit, no useful information is revealed, and the principal's lack of commitment hurts her in the most extreme way.[3]

While the leading application of our model is the VC-startup relationships, we believe that the economic forces identified by our analysis are relevant in other scenarios, such as voter-politician, manager-worker and purchaser-supplier relationships, where learning with asymmetric information is a critical aspect. On the technical side, our choice of modeling this game in continuous time enables us to obtain semi-closed-form expressions that describe the key equilibrium properties.[4] However, our most substantive technical contributions lie in the asymptotic analysis, namely Theorems 2.6.1 and 2.7.1, where the lack of a fully closed-form solution presents additional challenges. To deal with them, we establish a new Learning Lemma (see Claim 17 and Lemma OA.5) that allows us to measure how frequently those beliefs under which the agent's mimicking intensity is low are visited. This result, while interesting on its own and useful in other settings, enables us to tackle discontinuities which arise in the limit of equilibria of our model. At the end of Section 2.6, we explain in more detail how our proofs combine the Learning Lemma with the semi-closed-form solution to obtain these limit results.

Related Literature. Our paper is most closely related to the reputation literature and the literature on dynamic games with stopping decisions.

[3]This result holds if both players get arbitrarily patient at the same rate, or if the agent gets patient at a faster rate than does the principal.

[4]Besides papers reviewed below, recent works that exploit the tractability of continuous-time methods include DeMarzo et al. (2012), Bonatti et al. (2017), Ortner (2017), Cisternas (2018) and Varas et al. (2020), among others.

Most of the reputation literature — starting with Kreps and Wilson (1982) and Milgrom and Roberts (1982), and later generalized by Fudenberg and Levine (1989, 1992) and most recently by Pei (2020) — investigates whether and how much a long-lived informed player can benefit from its private information in repeated games played against myopic opponents. The focus is typically on the case where the informed party is arbitrarily patient, and on bounding the informed player's equilibrium payoffs.[5] In contrast, our analysis fully characterizes (Markov) equilibrium behavior for all discount rates, and we uncover new qualitative features that the equilibrium dynamics exhibit.[6]

Faingold and Sannikov (2011) study reputation effects in games played in continuous time with one long-lived informed player against myopic opponents, and they characterize the set of sequential equilibria. Unlike Faingold and Sannikov (2011), the uninformed player in our game is forward-looking and can terminate the game. More importantly, the termination payoffs depend on the informed player's type, creating *interdependence* of payoffs between the players (similar to Pei (2020)) and thus making their characterization not applicable to our model.

There is a growing interest in dynamic games with stopping decisions. Daley and Green (2012), Kolb (2015, 2019), Dilmé (2019), Ekmekci and Maestri (2019) and Sun (2018) all study stopping games with two long-lived players, where the uninformed party receives information over time and obtains type-dependent payoffs. In Daley and Green (2012), Kolb (2015) and Dilmé (2019), the informed player makes the stopping decision while in our paper such decision is made by the uninformed player. This makes the incentives and equilibrium structure in our model quite different from theirs. In Kolb (2019), the agent can only influence the information process by irreversibly changing his type, while in our model the agent can directly manipulate the signal, with his type being persistent. Besides, the qualitative results on the equilibrium dynamics in our paper do not have a counterpart in these papers. Ekmekci and Maestri (2019) study a similar setting in discrete time, and focus solely on the limiting case with arbitrarily patient players. Our Theorem 2.7.1 is the continuous-time version of their main finding, while we obtain much richer equilibrium

[5] There are also papers that bound equilibrium payoff of the informed player with long-lived uninformed players, e.g., Schmidt (1993); Cripps and Thomas (1997); Celetani et al. (1996); Atakan and Ekmekci (2012, 2015)

[6] Studies on reputation dynamics include Mailath and Samuelson (2001); Phelan (2006); Liu (2011); Ekmekci (2011); Lee and Liu (2013); Liu and Skrzypacz (2014). However, these papers do not share similar equilibrium dynamics or qualitative results that we obtain partly because they look at repeated moral hazard games and/or the uninformed parties are myopic.

dynamics for any fixed discount rate. Finally, Sun (2018) studies dynamic censorship with Poisson news, wherein the agent can decide whether to show or hide the bad news after privately observing its realization.[7]

Aghion and Jackson (2016) and Kuvalekar and Lipnowski (2020b) also study dynamic games (in discrete and continuous time, respectively) between two long-run players with stopping decisions. However, the nature of uncertainty and agent's actions in their models are quite different from ours. Specifically, both papers look at a career-concern type of model with symmetric information between the two players, while the agent's actions affecting the signal process are *costless* to the agent and *observable* to the principal. By contrast, in our model the agent has private information about his type, and his action is *costly* and *hidden*. This necessarily makes the principal's inference problem more delicate, as she has to form a conjecture about the agent's action which need coincide with the agent's actual strategy in equilibrium. Moreover, in our model the agent's trade-off is between improving his reputation and saving the mimicking cost, while in their models the agent is optimizing over the speed of learning (i.e., variance, rather than drift, of the belief process). Orlov et al. (2020) also consider a dynamic setting with stopping decisions and symmetrically informed players, and they study the agent's optimal information disclosure policy in a persuasion game.

2.2 Model

2.2.1 Players, types, actions, and information flow

A principal (she) and an agent (he), both risk-neutral, interact in continuous time $t \in [0, \infty)$.

At any time t, an exogenous stopping opportunity arrives according to a Poisson process $\{J_t\}_{t\geq 0}$ with rate $\lambda > 0$. When the said opportunity arrives, the principal chooses whether to *continue* or irreversibly *stop* the game. The Poisson arrival of stopping opportunities captures the frictions in the principal's decision making and implementation.[8]

The agent can be one of two types, denoted by θ: an *investible* type ($\theta = I$), or a *non-*

[7]In Sun (2018), the equilibrium censoring intensity is monotone in the agent's reputation while in our model, the intensity of performance boosting is non-monotone. Besides, his analysis focuses on the welfare implications of the censoring activity, while we examine the welfare effects of better transparency and the ratchet effect at the patient limit.

[8]Technically, this assumption ensures that there is no off-equilibrium history/belief. In Remark 1 we discuss what happens as the frictions vanish, i.e. as $\lambda \to \infty$. None of our main results requires the frictions to be significant: they hold either for all λ, or when λ is sufficiently large.

investible type ($\theta = NI$). The agent's type is his private information. From the principal's viewpoint, the initial probability that the agent is a noninvestible type is $p_0 \in (0,1)$.

There is a public signal $\{X_t\}_{t\geq 0}$ that evolves over time. If the agent is an investible type, then the public signal evolves according to the process:

$$dX_t = \mu dt + \sigma dB_t,$$

where $\{B_t\}_{t\geq 0}$ is a standard Brownian motion. Without loss, we assume that $\mu > 0$ and $\sigma > 0$, and we define the *signal-to-noise ratio* ψ of the process as $\psi \equiv \mu/\sigma$. If the agent is a noninvestible type, he chooses an $\alpha_t \in [0,1]$ at any time t when the game has not stopped yet. In this case, his choice controls the drift of the public signal process:

$$dX_t = \mu\alpha_t dt + \sigma dB_t.$$

The model assumes that the investible type does not have any action choice, and the evolution of the public signal is exogenous conditional on this type (always having a drift of μ). Meanwhile, the noninvestible type chooses a *mimicking intensity*, which can be interpreted as the probability with which the noninvestible type acts the same as the investible type. In our leading application of VC investments, we can interpret the public signal as performance reports from the startup and the mimicking action taken by the noninvestible type as *performance boosting*.

2.2.2 Strategies

The investible type of the agent does not have an action choice. A strategy for the noninvestible type is a stochastic process $\{\alpha_t\}_{t\geq 0}$, which takes values in $[0,1]$ and is progressively measurable with respect to the filtration generated by $\{B_t\}_{t\geq 0}$. Let $\mathcal{A}$ be the set of strategies for the agent.

A strategy for the principal is a stochastic process $\beta \equiv \{\beta_t\}_{t\geq 0}$, progressively measurable with respect to the filtration generated by $\{X_t, J_t\}_{t\geq 0}$, which represents the probability with which the principal takes the stopping action conditional on the arrival of a stopping opportunity. Let $\mathcal{B}$ be the set of strategies for the principal.[9]

[9] We note that the principal only observes the public signal, while the agent knows his own past actions, and thus can recover $\{B_t\}_{t\geq 0}$ by removing the drift term.

Given a strategy profile (α, β) and a prior p_0, the principal updates her belief about the agent's type using Bayes' rule, and we let $\{p_t\}_{t\geq 0}$ denote the belief process defined by

$$p_t := P\{\theta = NI \mid \{X_s\}_{s\leq t}\}. \tag{2.1}$$

Note that the belief process p_t, conditional on a continuing relationship, is determined by the strategy of the agent and not affected by the strategy of the principal or the arrival of stopping opportunities.

2.2.3 Payoffs

If the game is stopped, the agent receives his outside option which we normalize to 0. If the game is not yet stopped, the noninvestible agent receives a flow payoff that depends on his action, $u + (1 - \alpha_t)c$, where $u > 0$ and $c > 0$.[10,11] That is, if the noninvestible agent does (not) mimic the investible type then his flow payoff in the relationship is u (resp., $u + c$); thus, c is the flow cost of mimicking. For a given strategy profile, (α, β), the expected discounted payoff of the noninvestible agent at time t is given by

$$U_1(t, \alpha, \beta) := E\left\{\int_t^T e^{-r_1(\tau - t)} r_1 \left[u + (1 - \alpha_\tau)c\right] d\tau \;\middle|\; \theta = NI, \{B_s\}_{s\leq t}\right\},$$

where T is the random time at which the game stops and the expectation is taken over T. This expression can be simplified to

$$U_1(t, \alpha, \beta) := E\left\{\int_t^\infty e^{-\Lambda_1(t,\tau,\beta)} r_1 \left[u + (1 - \alpha_\tau)c\right] d\tau \;\middle|\; \theta = NI, \{B_s\}_{s\leq t}\right\},$$

where we define the discounting exponent (taking into account the agent's discount rate r_1 and the termination probability)

$$\Lambda_1(t, \tau, \beta) := \int_t^\tau (r_1 + \lambda\beta_s) ds.$$

The principal's flow payoff does not depend on the agent's action or the public signal,[12]

[10]Interpreting the noninvestible type as choosing between mimicking ($A_t = 1$) or not ($A_t = 0$) and α_t as the probability of taking the mimicking action, we can think of the noninvestible type's flow payoff as defined by $u + 1_{\{A_t=0\}}c$.

[11]The flow payoff of the investible type in the relationship is always some positive constant, say, $u + c$.

[12]This assumption seems reasonable in our leading example of venture capital investments, wherein an investor's payoff is mainly driven by the viability (type) of the startup rather than its performance in the

and we normalize her flow payoff to zero. However, the principal receives a lump-sum payoff of $w_{NI} > 0$ if the game stops against a noninvestible type, and $w_I < 0$ if the game stops against an investible type. That is, relative to continuing the relationship, the principal prefers stopping against a noninvestible type but dislikes terminating an investible type. Thus, given a strategy profile (α, β), the expected discounted payoff of the principal at time t is given by

$$U_2(t,\alpha,\beta) := E\left\{\int_t^\infty e^{-\Lambda_2(t,\tau,\beta)}\lambda\beta_\tau\left(1_{\{\theta=NI\}}w_{NI} + 1_{\{\theta=I\}}w_I\right)d\tau \;\middle|\; \{X_s\}_{s\le t}\right\},$$

where we define the discounting exponent (taking into account the principal's discount rate r_2 and the termination probability)

$$\Lambda_2(t,\tau,\beta) := \int_t^\tau (r_2 + \lambda\beta_s)ds.$$

Note that $U_2(t,\alpha,\beta)$ is calculated conditional on the stopping opportunity *not* arriving (or having been forgone) at time t.

2.3 Discussion of Model Assumptions

The essential ingredients of our model are the following:

1. The agent has private information about his type, and wants to stay in the relationship for as long as possible.
2. The principal faces a learning problem about the agent's type; she prefers to terminate against the nonivestible type and to continue with the investible type.
3. The noninvestible type can manipulate the drift of a noisy signal at a cost to mimic the investible type's performance.[13]
4. The signal only serves an informational role and is payoff-irrelevant to the principal.

In addition to these assumptions, we adopt a normalization of flow payoffs and outside options to simplify the exposition. Below, we present an alternative but equivalent formu-

initial financing period, while the initial performance is still informative to the investor about the startup's type.

[13] Our equilibrium characterization in Theorem 2.4.1 is robust to introducing some strategic behavior to the investible type. See Remark 2 for a discussion.

lation, which fits better with our VC examples and provides a foundation to the principal's simplified payoff structure.

Suppose that the principal's outside option is independent of the agent's type and is equal to 0. By continuing the relationship the principal incurs a flow cost equal to $b > 0$. There is a *revealing event* that arrives according to a Poisson process with rate δ, independent of the agent's type and the signal process. When the event arrives, the game ends delivering a lump-sum payoff to the principal. This payoff is equal to $\pi_I > 0$ if the agent is investible and $\pi_{NI} = 0$ otherwise. The flow cost represents the continuous financial inputs that the VC contributes to the startup. The revealing event corresponds to the VC's realization of the startup's profitability (type), and the ensuing type-dependent lump-sum payoffs correspond to the value of the startup to the VC upon learning its type. As in the original formulation, the principal can terminate the relationship whenever a stopping opportunity arrives. The arrival follows a Poisson process at rate $\hat{\lambda}$, and is independent of the revealing event. These intermittent stopping opportunities capture the frictions that are inevitable in a VC's decision making and implementation. For example, the withdrawal of funding may be decided only through board meetings which are called upon once in a while; moreover, a VC that wants to liquidate its shares in a startup may have to wait some time until a buyer shows up.

The (noninvestible) agent's flow payoff is $\hat{u}$ if he engages in performance boosting and $\hat{u} + \hat{c}$ otherwise. The agent receives a payoff of 0 when the relationship ends, either because the principal terminates it or the revealing event occurs.[14] The discount rates of the agent and the principal are $\hat{r}_1$ and $\hat{r}_2$, respectively.

This formulation is strategically equivalent to the benchmark model with a type-dependent outside option for the principal. The equivalence is achieved through the following transformation of parameters, which can be verified by standard calculations. The agent's flow payoffs are identical across the two formulations, i.e., $u = \hat{u}$, $c = \hat{c}$, and so is the arrival rate of the principal's stopping opportunity, $\lambda = \hat{\lambda}$. The implied discount rates are augmented by the arrival rate of the revealing event, i.e., $r_1 = \hat{r}_1 + \delta$, $r_2 = \hat{r}_2 + \delta$. And finally, the

[14] We could also assume that the investible type gets a positive lump-sum reward when the revealing event occurs, but this will not change the game in any way because the revealing event is out of everyone's control.

principal's type-dependent outside options are given by

$$w_I = \frac{\hat{r}_2 b - \delta\pi_I}{\hat{r}_2 + \delta}, \quad w_{NI} = \frac{\hat{r}_2 b}{\hat{r}_2 + \delta}.$$

As long as $\pi_I > \frac{\hat{r}_2}{\delta} b$ (i.e., if the reward to the principal upon learning that the agent is investible is large enough), we have $w_I < 0 < w_{NI}$, as in our baseline model.

Finally, our model focuses solely on the adverse-selection aspect of the problem, by assuming that performance reports do not directly affect the principal's payoff. This seems reasonable in contexts such as VC-startup relationships, wherein an investor's payoff is mostly driven by the viability (type) of the startup rather than its performance in the initial financing period, though the initial performance is informative about the startup's type.[15]

2.4 Equilibrium Characterization

2.4.1 Equilibrium Concept

An **equilibrium** is a strategy profile (α, β) such that

$$\begin{aligned} U_1(t, \alpha, \beta) &\geq U_1(t, \tilde{\alpha}, \beta), \\ U_2(t, \alpha, \beta) &\geq U_2(t, \alpha, \tilde{\beta}), \end{aligned}$$

for all alternative strategies $\tilde{\alpha} \in \mathcal{A}$ and $\tilde{\beta} \in \mathcal{B}$, almost surely for all $t \geq 0$.

Let[16]

$$\mathcal{P} := \{f : (0,1) \to [0,1], f \text{ is right-continuous and piecewise Lipschitz}\}.$$

Recall that the belief process defined in (2.1) is determined by the agent's strategy. We say that a strategy α of the agent is *Markovian* if there exists *policy function* $a \in \mathcal{P}$ such that $\alpha_t = a(p_t)$ for all $t \geq 0$. An equilibrium (α, β) is **Markovian** if there exist policy functions $a, b \in \mathcal{P}$ such that $\alpha_t = a(p_t)$ and $\beta_t = b(p_t)$ for all $t \geq 0$. In this case, we say that the

[15] Moreover, we have verified that our equilibrium characterization still holds even after allowing for some dependence of the principal's flow payoff on the agent's action. This analysis is not included in the manuscript and is readily available upon request.

[16] A function $f : (0,1) \to [0,1]$ is piecewise Lipschitz if there exist $n \in N$ and $0 = x_1 < x_2 < ... < x_n = 1$, such that for each $i \in \{1, ..., n-1\}$, there exists a Lipschitz function f_i on $[x_i, x_{i+1}]$ such that $f_i(p) = f(p)$ for all $p \in (x_i, x_{i+1})$.

policy profile $(a, b) \in \mathcal{P}^2$ is *induced* by (α, β).

Given a Markovian equilibrium (α, β), let $\mathrm{SP}(\alpha)$ be the set of posteriors reached on the equilibrium path.[17]

Lemma 2.4.1. *Any Markovian equilibrium (α, β) with an induced policy profile (a, b) satisfies i) $\sup_{p\in(0,1)} a(p) < 1$, and ii) $SP(\alpha) = (0, 1)$.*

Lemma 2.4.1 says that in any Markovian equilibrium the agent's action is always bounded away from full mimicking and that every posterior belief is reached with positive probability. Intuitively, if the noninvestible type is expected to choose $a = 1$ at some belief $\hat{p}$, then the signal from that time on becomes uninformative, making $\hat{p}$ an absorbing state. But if the belief is not moving, the noninvestible type's best reply at state $\hat{p}$ is to choose $a = 0$, which violates the (implicit) requirement that the agent's equilibrium action must coincide with the principal's conjecture about his action. In fact, one can show that an equilibrium policy function $a(\cdot)$ must be bounded away from 1, and thus the variance of the belief process is always bounded away from 0.[18] Together with the Poisson arrival of stopping opportunities, this makes all interior beliefs reachable on the equilibrium path.

Given a Markovian equilibrium (α, β), the continuation payoff at time t depends only on the public belief p_t. Hence, we define the value function of the (noninvestible) agent as

$$V(p) := E\left\{U_1(t, \alpha, \beta) \mid p_t = p, \theta = NI\right\}$$

and the value function of the principal as

$$W(p) := E\left\{U_2(t, \alpha, \beta) \mid p_t = p\right\}$$

for every $p \in (0, 1)$.

We say that a value function is **regular** if it is continuously differentiable everywhere,

[17] Consider a Markovian equilibrium, (α, β), and the underlying probability space (Ω, F, P). For each $p \in (0,1)$, we define $\Phi(p) := \{\omega \in \Omega : \exists t \le T \text{ such that } p_t(\omega) = p\}$, where T is the equilibrium stopping time. The belief span, $\mathrm{SP}(\alpha)$, is the set of all p such that $P(\Phi(p)) > 0$. Because p_t in a continuing relationship depends only on the agent's strategy α and the principal's stopping opportunity may not arrive at any t, the belief span is also solely determined by α. Consequently, this notion of belief span can be defined for any (Markovian) strategy of the agent.

[18] This result holds for any fixed value of the parameters. The lower bound on the volatility of the belief process, $1 - \sup_{p\in(0,1)} a(p)$, depends on players' discount rates, the arrival rate of the Poisson process, and the signal to noise ratio. Therefore, the informativeness of the public signal can get arbitrarily close to zero in some cases (e.g., when the signal to noise ratio is very high or the agent is very patient, as we will see in Sections 2.6 and 2.7).

and twice continuously differentiable everywhere except perhaps at a finite number of points. We say that a Markovian equilibrium (α, β) is **smooth** if the associated value functions are regular and the agent's policy function $a(\cdot)$ is Lipschitz. We refer to smooth Markovian equilibria simply as **Markov equilibria**.[19] Moreover, when there is no confusion, we denote a Markov equilibrium by the policy profile (a, b) that it induces.

2.4.2 Characterization

We first introduce some terminology to define properties of policy functions for the principal and the agent. Recall that the state variable p is the principal's belief that the agent is noninvestible.

Definition 2.4.1. *The policy function $b \in \mathcal{P}$ for the principal has a* ***cutoff structure*** *if there exists $\tilde{p} \in [0, 1]$ such that $b(p) = 0$ for $p < \tilde{p}$, and $b(p) = 1$ for $p > \tilde{p}$. We refer to $\tilde{p}$ as the cutoff belief of b.*

Definition 2.4.2. *The policy function $a \in \mathcal{P}$ for the (noninvestible) agent is* ***fully separating*** *if $a(p) = 0$ for all $p \in (0, 1)$.*

Definition 2.4.3. *The policy function $a \in \mathcal{P}$ for the (noninvestible) agent is* ***hump-shaped*** *if a is continuous and there are cutoffs $0 < p_L < p^* < p_R < 1$ such that $a(p) = 0$ for $p \le p_L$, strictly increasing on (p_L, p^*), strictly decreasing on (p^*, p_R), and $a(p) = 0$ for $p \ge p_R$.*

Theorem 2.4.1. *There always exists a unique Markov equilibrium (a, b). In this equilibrium, b has a cutoff structure with some cutoff belief $p^* \in (0, 1)$. Moreover, there exists $r^* > 0$ such that*

1. *If $r_1 \ge r^*$, then a is fully separating.*

2. *If $r_1 < r^*$, then a is hump-shaped and is maximized at p^*.*

Theorem 2.4.1 characterizes the structure of the unique Markov equilibrium. First, the principal uses a cutoff strategy. This follows from the type-dependent stopping payoff of the principal, and the absence of flow payoffs.

[19] We emphasize that "smoothness" is built into our definition of the term "Markov equilibrium." We do not look for non-smooth Markovian equilibria in this paper, and any claim about equilibrium uniqueness does not rule out the possibility of non-smooth Markovian equilibria.

Second, the noninvestible agent's behavior depends on his discount rate. If he is impatient (i.e., with a high discount rate), then he never mimics the investible type, because he always finds the saving of the mimicking cost to outweigh the benefit of having a better reputation. A richer dynamics opens up if the agent is patient (i.e., with a low discount rate). In this case, his behavior can be described by three reputation phases: good, medium and bad, as depicted in Figure 2-1. Both in the good and the bad reputation phases, the noninvestible type does not mimic the investible type at all, but for different reasons: when his reputation is good ($p < p_L$), the relationship is highly stable, so the noninvestible agent gains little from further improving his reputation by mimicking; when his reputation is bad ($p > p_R$), termination is so imminent that the noninvestible agent gives up building reputation. In the intermediate phase, however, the noninvestible type first starts to mimic more often as his reputation worsens in order to slow down the principal's learning. We interpret this as a *"scramble-to-rescue" effect*: the agent increases his mimicking intensity (before p^*) as the relationship gets less stable. After certain point, he gradually gives up as the relationship becomes doomed. His mimicking intensity is highest at belief p^* when the principal's action switches from continuing the relationship to termination.[20]

That the agent's mimicking intensity reaches its peak around p^* can be understood from an equilibrium perspective. In general, the agent's incentive to manipulate depends on: i) how responsive the belief is to signal realizations; ii) how sensitive the principal's decision is to belief changes. If the agent manipulates more *in equilibrium*, then the principal believes that the signal is less informative and her belief is less responsive to the signal realizations, which reduces the agent's incentive to manipulate through (i). When the principal's decision is very sensitive to the agent's reputation (namely, at reputations around the cutoff p^*), the agent's manipulation has to be high in order to partially neutralize the effect of such sensitivity, and ensure that the marginal cost of manipulation equals the marginal benefit.

Finally, the cutoff discount rate, r^*, can be characterized in closed form. Specifically, r^* is the unique solution to the following equation:

$$r^*(\sqrt{1+8r^*/\psi^2}+\sqrt{1+8(r^*+\lambda)/\psi^2})+\lambda(\sqrt{1+8r^*/\psi^2}+1)=4\lambda\left(\frac{u}{c}+1\right). \tag{2.2}$$

[20] Even though the principal's optimal action switches at p^*, the agent's mimicking intensity does not immediately drop to 0 right after the belief passes p^*. This is because the relationship can only be terminated when a stopping opportunity arrives, leaving some hope for the agent to rebuild his reputation and avoid termination.

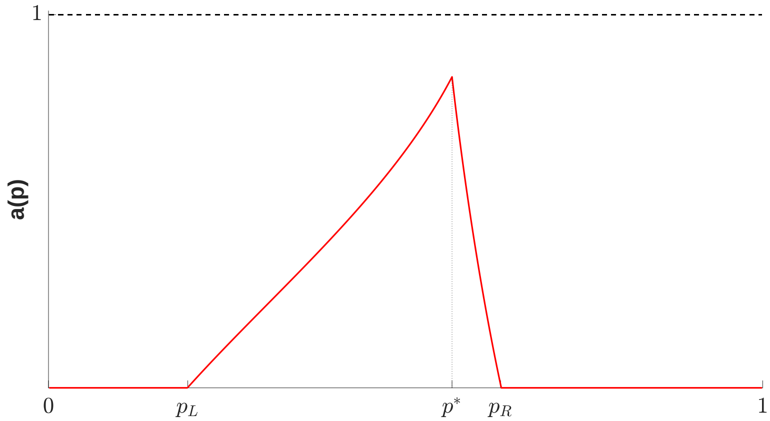


Figure 2-1: Agent's Equilibrium Policy Function When $r_1 < r^*$. This figure is plotted under the following parameter values: $r_1 = 0.5$, $r_2 = 0.5$, $\lambda = 2$, $\psi = 1.5$, $u = 1$, $c = 1$, $w_{NI} = 1$, $w_I = -1$. In equilibrium, $p_L \approx 0.195$, $p^* \approx 0.565$, $p_R \approx 0.633$.

Some comparative statics results are readily obtained from (2.2). In particular, r^* increases with λ, ψ and $\frac{u}{c}$. This is intuitive, because the noninvestible type will have a higher incentive to mimic if: *i)* the stopping opportunity arrives more frequently and thus the relationship is less stable; *ii)* the signal-to-noise ratio is higher and thus a manipulation of signal is more profitable; *iii)* mimicking is relatively less costly. We also note that r^* does not depend on the principal's payoff parameters (r_2, w_{NI} and w_I).

Remark 1. *As the stopping frictions vanish (i.e., as $\lambda \to \infty$), the cutoff discount rate r^* converges from below to a finite number $\bar{r}$. For any fixed $r < \bar{r}$, the agent's equilibrium policy function converges to one that resembles Figure 2-1 for $p < p^*$ and is equal to 0 for all $p > p^*$, that is, p_R^* converges to p^*, creating a discontinuity at p^*. This is intuitive because in the limit the agent's incentive when $p < p^*$ is similar to what we explained before, but once the belief is above p^* the agent expects the relationship to be terminated in the next instant regardless of what he does, and thus he should choose $a = 0$ to save the mimicking cost.*[21]

Remark 2. *The assumption that the investible type does not have an action choice makes him a "commitment/behavioral" type in the sense of the reputation literature. However, many*

[21] Technically, if λ is set to ∞, i.e., if we literally allow the principal to terminate the relationship whenever she wants, for that game additional refinement is needed to preserve equilibrium uniqueness, because once $p > p^*$ the agent expects the relationship to be terminated right away in which case his action choice in that instant has no payoff consequence. See Kuvalekar and Lipnowski (2020a) for a detailed discussion and a refinement that will select the limiting policy function we described in the limiting game. In our model, the Poisson arrival of stopping opportunities helps us avoid such complications and obtain equilibrium uniqueness for every finite λ without additional refinements.

of the insights of this paper are robust to certain forms of strategic behavior. For instance, if we allow the investible type to costlessly choose a drift, then the unique equilibrium we characterize in Theorem 2.4.1 remains an equilibrium in this modified game.[22]

Below we describe our approach to proving Theorem 2.4.1. Finding a Markov equilibrium amounts to finding a policy profile (a, b), and a conjecture that the principal holds about the agent's strategy such that: *i)* the principal's conjecture determines her interpretation of public signal histories into her beliefs about the agent's type; *ii)* the principal's policy b is optimal given her conjecture about the agent's strategy; *iii)* the agent's policy a is optimal given b and the principal's conjecture; *iv)* the principal's conjecture coincides with a.

Specifically, any Markov equilibrium (a, b) satisfies the optimality conditions stated below.[23]

Principal's Optimality:

$$b \in_{\tilde{b} \in \mathcal{P}} \hat{W}(p, a, \tilde{b}), \tag{2.3}$$

where

$$\hat{W}(p, a, \tilde{b}) := E\{e^{-r_2 \nu} \left(1_{\{\theta = NI\}} w_{NI} + 1_{\{\theta = I\}} w_I\right)\},$$

where $p_0 = p$, ν is the time when the game stops, controlled by both $\tilde{b}$ and $\{J_t\}_{t \geq 0}$, and the evolution of $\{p_t\}_{t \geq 0}$ is given by the SDE

$$dp_t = -\psi(1 - a_t)\gamma(p_t) d\tilde{B}_t, \tag{2.4}$$

In (2.4), ψ is the signal-to-noise ratio parameter, a_t is a function of p_t, $\gamma : [0, 1] \to R_+$ is defined by $\gamma(p) := p(1 - p)$, and $(\tilde{B}_t)_{t \geq 0}$ is the innovation process associated with the filtering of the principal, i.e.,

$$d\tilde{B}_t := \frac{dX_t - \mu(p_t a_t + 1 - p_t)}{\sigma} = \frac{dX_t}{\sigma} - \psi(p_t a_t + 1 - p_t) dt. \tag{2.5}$$

The optimality condition (2.3) requires that b maximizes the principal's payoff when the

[22] The equilibrium is no longer unique though. For example, both types choosing zero drift and the principal ignoring the signal is always an equilibrium in that case.

[23] The optimality conditions (2.3) and (2.6) restrict the players to maximize their payoffs over Markov controls in $\mathcal{P}$. This is for expository purposes and is without loss. In the proof of Theorem 2.4.1 we verify that the equilibrium strategies are mutual best replies among all strategies in $\mathcal{B}$ and $\mathcal{A}$.

agent is using policy a.

Agent's Optimality:

$$a \in_{\tilde{a}\in\mathcal{P}} \hat{V}(p, \tilde{a}, b; a), \tag{2.6}$$

and

$$\hat{V}(p, \tilde{a}, b; a) := E\left\{\int_0^{\nu} e^{-r_1\tau}\left\{r_1\left[(1-\tilde{a}(p_\tau))c + u\right]\right\} d\tau\right\}$$

where $p_0 = p$, ν is the time when the game stops, and the evolution of $\{p_t\}_{t\geq 0}$ is given by substituting $dX_t = \mu\tilde{a}_t dt + \sigma dB_t$ into equations (2.4) and (2.5). Specifically, from the noninvestible type's perspective, the belief process satisfies:

$$dp_t = \psi^2(1-a_t)[1-\tilde{a}_t - p_t(1-a_t)]\gamma(p_t)dt - \psi(1-a_t)\gamma(p_t)dB_t. \tag{2.7}$$

In a Markov equilibrium (a, b), the principal has a conjecture about the agent's behavior, which determines how she interprets any history of signal realizations into her belief about the agent's type. This conjecture has to coincide with the agent's policy a in equilibrium. If the agent contemplates a deviation from the equilibrium, this would not affect the processes in equations (2.4) and (2.5) (which jointly describe the dependence of beliefs on the public history), but would affect the process that governs the evolution of X_t (public histories). The necessary condition (2.6) requires that the agent does not have a profitable deviation from his equilibrium policy function a, when the principal conjectures that the agent is using this policy function.

We now build on the implications of the necessary conditions outlined above. We first show that in any Markov equilibrium, the principal's policy function has a cutoff structure: she terminates the relationship if and only if the agent's reputation is bad enough. We then show that the agent's equilibrium policy function must be either fully separating (i.e., never mimicking) or hump-shaped. Finally, the existence and uniqueness of Markov equilibrium follow from a fixed-point argument.

Let $R(p) := pw_{NI} + (1-p)w_I$ be the principal's expected payoff if the relationship is terminated at belief p. Define $p^{**} := R^{-1}(0) > 0$ and $p_H := R^{-1}\left(\frac{\lambda}{r_2+\lambda}w_{NI}\right) < 1$.

Lemma 2.4.2. *If (a, b) is a Markov equilibrium, then b has a cutoff structure with a cutoff belief $p^* \in [p^{**}, p_H]$.*

To prove this result, we utilize the optimality condition in (2.3). Observe that the equilibrium value function, $W(p)$, is such that $W(p) := \hat{W}(p, a, b)$. Then, the principal's value and policy functions must satisfy the following HJB equation:

$$r_2 W(p) = \max_{\tilde{b}\in[0,1]} \left\{ \frac{1}{2}\psi^2[1-a(p)]^2\gamma(p)^2 W''(p) + \lambda\tilde{b}\left[R(p) - W(p)\right] \right\}. \tag{2.8}$$

It is clear that $b(p) = 0$ whenever $R(p) < W(p)$, and $b(p) = 1$ whenever $R(p) > W(p)$. In the proof, we show that these functions have a unique intersection point. Moreover, because terminating the relationship when $p < p^{**}$ gives the principal a negative payoff, and because the stopping opportunity arrives only once in a while which bounds her payoff from waiting by $\frac{\lambda}{r_2+\lambda} w_{NI}$, the principal's optimal stopping threshold must be between p^{**} and p_H. See Figure 2-2 for an illustration.

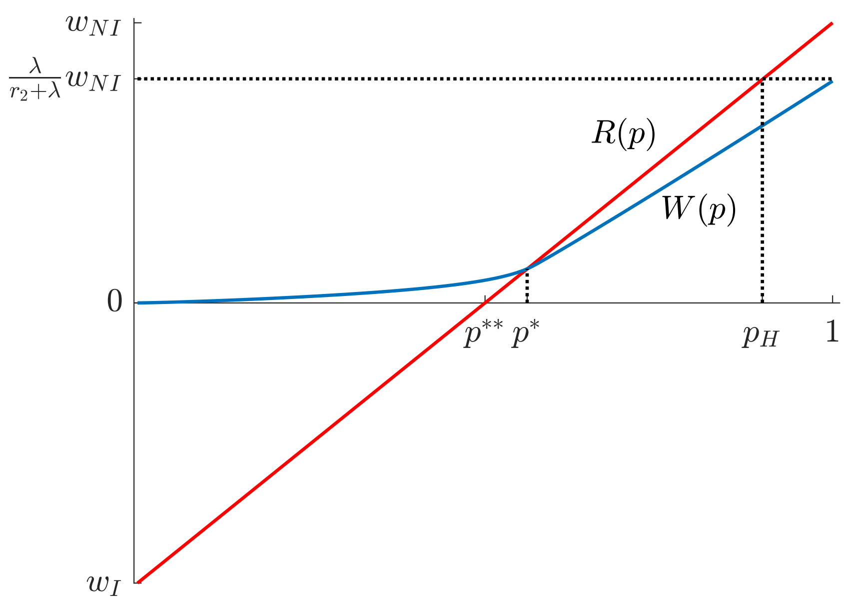


Figure 2-2: Principal's Equilibrium Cutoff. This figure is plotted under the following parameter values: $r_1 = 0.5$, $r_2 = 0.5$, $\lambda = 2$, $\psi = 1.5$, $u = 1$, $c = 1$, $w_{NI} = 1$, $w_I = -1$. In equilibrium, $p^{**} = 0.5$, $p^* \approx 0.565$, $p_H = 0.9$.

We now turn to the agent's behavior.

Lemma 2.4.3. *Suppose $b \in \mathcal{P}$ is a cutoff policy function for the principal with cutoff belief p^*. Then, there is a unique policy function $a \in \mathcal{P}$ for the agent such that i) $\hat{V}(p, a(p), b(p); a(p))$ is a regular function of p, ii) a is Lipschitz and $\sup_{p\in(0,1)} a(p) < 1$, and iii) a satisfies (2.6).*

Moreover, this unique policy function is fully separating if $r_1 \geq r^*$, and is hump-shaped if $r_1 < r^*$.

The proof of Lemma 2.4.3 is more involved. This is because finding a solution to program (2.6) is akin to finding a fixed point: the policy a for the agent is optimal when the principal holds the conjecture a. Nonetheless, we are able to characterize its unique solution in closed form.

Intuitively, if the agent is impatient, the short-run incentives determine his behavior, and the noninvestible type will never pay the cost to mimic the investible type, leading to a fully separating policy function. If the agent is patient, full separation can no longer be part of an equilibrium. This is because the fully separating policy function, if conjectured by the principal, generates opportunities to build a reputation rather fast; and when the agent cares enough about the future, it will give strict incentives to the noninvestible type to mimic.

What is the dynamics of the agent's mimicking intensity when he is patient? When the public belief p is very small, it takes a long time for the belief to increase all the way up to the termination cutoff. Hence, the limited benefit of further improving reputation cannot justify the mimicking cost. As a result, $a(p) = 0$ for low p. When p is very large, it takes so long for the agent to regain his reputation that he simply gives up. Consequently, $a(p) = 0$ for high p. For intermediate p, the agent's short-run temptation and long-run benefits are more balanced, so that $a(p) \in (0, 1)$. Specifically, after the good reputation phase, for $p \in (p_L, p^*)$, the noninvestible type starts to mimic more often as his reputation worsens (i.e., the scramble-to-rescue effect). Such an incentive peaks at p^* where the principal's action is most sensitive to a change in belief. After that, for $p \in (p^*, p_R)$, the noninvestible type gradually gives up restoring his reputation as termination becomes more imminent.

2.5 Non-Monotonicity of Expected Performance

We saw in Theorem 2.4.1 that the noninvestible type will engage in performance boosting whenever he is sufficiently patient, in which case $a(\cdot)$ peaks at the termination cutoff p^*. This has been referred to as the "scramble-to-rescue" effect: the agent increases his mimicking intensity (before p^*) as the relationship gets less stable.

What are the implications of this effect on the observables? An outsider (the principal or a modeler) does not see the agent's type or action, but can observe his performance, such as subscription growth or progress reports. In our model, the expected performance at time t is given by

$$EP_t := \frac{E[dX_t]}{dt} = \mu \left(\underbrace{1 - p_t}_{\text{investible}} + \underbrace{p_t a(p_t)}_{\text{noninvestible}} \right).$$

Holding constant $a < 1$, the expected performance decreases with p. We call this the *belief effect*: if the agent is more likely to be noninvestible, then the expected performance is lower. This is the entire story if the equilibrium is fully separating, in which case $a(p) = 0$ everywhere and $EP(p) = \mu(1 - p)$.

However, if the equilibrium is not fully separating, the noninvestible type's mimicking intensity a is no longer constant: it is increasing below p^* due to the scramble-to-rescue effect. Hence, whether the expected performance increases or decreases with the public belief depends on which of the two effects is stronger. The following theorem characterizes the evolution of the expected performance when the stopping opportunity arrives sufficiently fast.

Theorem 2.5.1. *Fixing all parameters of the model other than r_1 and λ, there exists $\bar{\lambda}$ such that for all $\lambda > \bar{\lambda}$, $EP(p)$ is non-monotone whenever the equilibrium is not fully separating (i.e., whenever $r_1 < r^*$). In particular, $EP(p)$ is*

- *strictly decreasing for $p \in [0, \underline{p})$, where $\underline{p}$ is in $[p_L, p^*)$;*
- *strictly increasing for $p \in (\underline{p}, p^*)$;*
- *strictly decreasing for $p \in (p^*, 1]$.*

Theorem 2.5.1 shows that if the arrival rate of the stopping opportunity is large enough, the scramble-to-rescue effect will dominate the belief effect when the public belief is less than but close to the termination cutoff p^*. As a result, the expected performance reaches a local maximum at p^* whenever the agent's equilibrium policy function is hump-shaped (see Figure 2-3).[24,25]

[24]The lower bound on λ is not crucial for this qualitative predication. In fact, even if λ is small, we can show that the expected performance is either decreasing or has the shape described in Theorem 2.5.1 (see Lemma B.1.1). Moreover, for any λ, we can find a $\bar{r}$ (less than r^*) such that the expected performance is non-monotone whenever $r_1 < \bar{r}$. The only difference for small λ is that we cannot say definitively what will happen when $r_1 \in (\bar{r}, r^*)$.

[25]The sudden drop of expected performance after p^* carries over to the time domain. Specifically, because $EP(p)$ is locally maximized (thus concave) at p^*, one can show that the stochastic process EP_t is a local supermartingale when $p_t = p^*$.

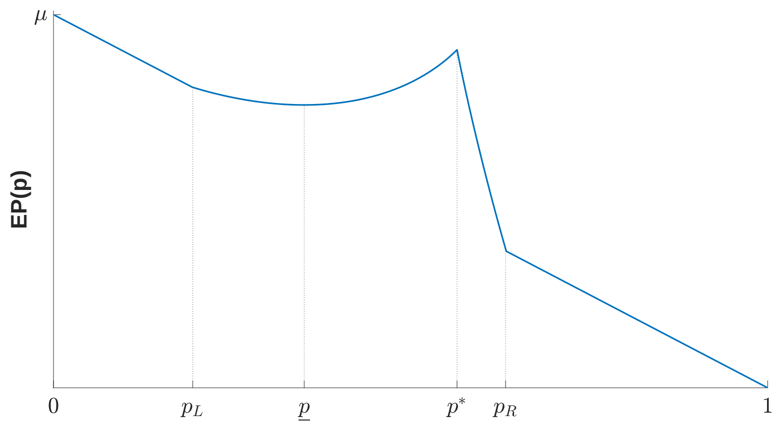


Figure 2-3: Agent's Expected Performance When $\lambda > \bar{\lambda}$ and $r_1 < r^*$. This figure is plotted under the following parameter values: $r_1 = 0.5$, $r_2 = 0.5$, $\lambda = 2$, $\psi = 1.5$, $u = 1$, $c = 1$, $w_{NI} = 1$, $w_I = -1$.

In the context of our applications, Theorem 2.5.1 offers an empirical prediction of our model: when performance boosting is expected to happen, terminations are preceded by a spike in expected performance. This seems consistent with a number of famous cases of corporate failure, such as Theranos, Luckin Coffee and WeWork: there were periods of time during which market suspicions about their business models grew, meanwhile the companies kept performing strongly and/or expanding aggressively prior to the crashes of their market values.[26]

2.6 Environments with Low Volatility / High Transparency

The noise in the performance measure may come from various random events such as temporary demand shocks, measurement errors, etc., making X_t only an imperfect signal of the agent's type. We say that a performance measure is more transparent if it is less affected by the noise component, and we can use the signal-to-noise ratio of the process to capture its *transparency*. In reality, transparency may be determined by the intrinsic volatility of the product market; it may also be affected by how much detail about the market or the project

[26]For example, in the third quarter of 2019, Luckin Coffee reported a 470.1% increase in the total items sold from 7.8 million in the same quarter of 2018. Its stock price was slashed by 75% in April 2020, following suspicion and then admission of fabricating sales data. Likewise, before scandals started to unravel, Theranos falsely claimed in 2014 that the company had annual revenues of $100 million, a thousand times more than the actual figure of $100,000. In the case of WeWork, the company once had expanded to over 86 cities in 32 countries, despite growing suspicion about its profitability. However, in September 2019, the companied delayed its IPO, followed by a 90% slash in valuation and enormous layoffs.

that a startup is required to disclose in a performance report. Indeed, one may expect that, other things equal, disclosing more details about the objective market conditions can help an investor better understand the numbers in the report.

In this section, we investigate the question: Do improvements in transparency always benefit the principal? We show that, under some parameters, improving transparency can inhibit learning and hurt the principal, due to the agent's endogenous response through signal manipulation. We also find that frictions in the principal's decision making (which cause decision delays) can sometimes help her, as they serve as a commitment of not terminating the relationship too quickly and thus reduce the agent's incentive to manipulate the signal.

Recall that the signal-to-noise ratio parameter is defined by $\psi = \mu/\sigma$. In what follows, we will fix all parameters other than ψ, and analyze the principal's payoff as ψ increases. Hence, we make explicit the dependence of any variable or function on ψ.[27]

As a benchmark, suppose that the principal never receives any information about the agent's type. Recall that $R(p) = pw_{NI} + (1-p)w_I$ and $p^{**} = R^{-1}(0)$. In this case, the principal would continue the relationship if $p < p^{**}$, and she would terminate the relationship at the first stopping opportunity if $p > p^{**}$. This leads to the following **"no-information value function"** for the principal:

$$\underline{W}(p) := \frac{\lambda}{r_2 + \lambda} \max\{0, pw_{NI} + (1-p)w_I\} = \frac{\lambda}{r_2 + \lambda} \max\{0, R(p)\}.$$

Note that when ψ is near 0, the signal process is close to pure noise regardless of the agent's action. So we have the following observation.

Observation 2.6.1. $\lim_{\psi \to 0} W(p_0; \psi) = \underline{W}(p_0)$ *and* $\lim_{\psi \to 0} p^*(\psi) = p^{**}$.

Now suppose that the agent's type is exogenously and immediately revealed to the principal. In this case, the principal will obtain her highest possible payoff for each belief, summarized by her **"full-information value function"**:

$$\overline{W}(p) := \frac{\lambda}{r_2 + \lambda} \left[p \ \max\{0, w_{NI}\} + (1-p) \max\{0, w_I\}\right] = \frac{\lambda}{r_2 + \lambda} pw_{NI}.$$

Note that the principal's equilibrium payoff is always strictly between $\underline{W}(p_0)$ and $\overline{W}(p_0)$.

[27] As is clear from the belief processes (2.4) and (2.7) and HJBs (2.8) and (B.2), in equilibrium μ and σ always affect the players' incentives and payoffs through ψ. So writing equilibrium objects as functions of ψ (while dropping μ and σ) is without loss.

This is because some learning will take place in equilibrium (as the noninvestible type never fully mimics), but the agent's type is not immediately revealed (as $\psi < \infty$).

Observation 2.6.2. *For all $\psi \in (0, \infty)$ and $p_0 \in (0, 1)$, $W(p_0; \psi) \in (\underline{W}(p_0), \overline{W}(p_0))$.*

The next theorem characterizes the limiting behavior of the principal's payoff as ψ goes to infinity.

Theorem 2.6.1. *Letting $\tilde{\lambda} := r_1\left(\frac{c}{u}\right)$,*

1. *If $\lambda < \tilde{\lambda}$, then $\lim_{\psi\to\infty} ||W(\cdot;\psi) - \overline{W}(\cdot)||_\infty = 0$ and $\lim_{\psi\to\infty} p^*(\psi) = 1$.*

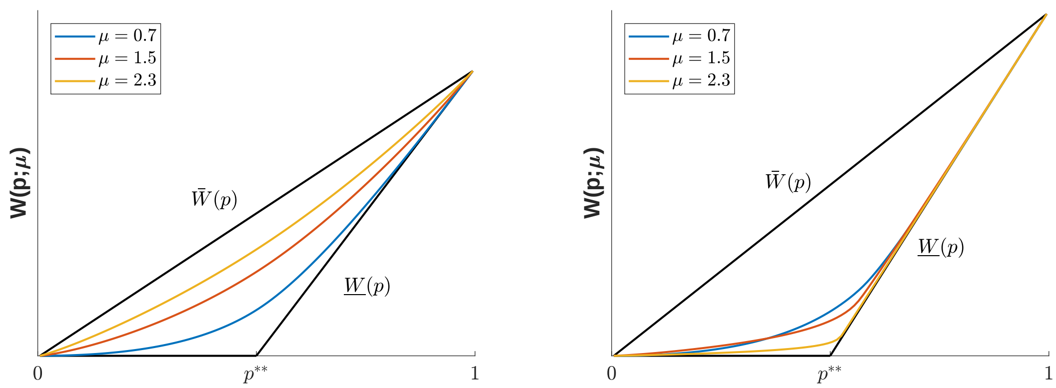
2. *If $\lambda > \tilde{\lambda}$, then $\lim_{\psi\to\infty} ||W(\cdot;\psi) - \underline{W}(\cdot)||_\infty = 0$ and $\lim_{\psi\to\infty} p^*(\psi) = p^{**}$.*

This result is heavily driven by the agent's equilibrium behavior. When the stopping opportunity arrives slowly ($\lambda < \tilde{\lambda}$), the noninvestible type's incentives to mimic are not strong. Intuitively, the relationship is relatively stable from the agent's viewpoint because, due to the lack of stopping opportunity, it will take a long time for the relationship to end even if the principal has decided to terminate it. As a result, the agent's equilibrium action is bounded away from "full mimicking" for all ψ.[28] As ψ grows without bound, the public signal becomes increasingly informative about the agent's type, and in the end, the agent's type is almost immediately revealed. Thus, the principal can afford to wait until being very certain that the agent is noninvestible, and her equilibrium value function converges to $\overline{W}$ (see Figure 2-4, left panel).

On the other hand, when the stopping opportunity arrives fast ($\lambda > \tilde{\lambda}$), the noninvestible type has stronger incentives to mimic the investible type. In particular, as ψ increases without bound, the equilibrium mimicking intensity at the termination cutoff converges to 1. The speed of this convergence is so fast that the variance of the belief process vanishes there (i.e., p^* becomes an almost absorbing state). Meanwhile, the equilibrium policy function $a(\cdot)$ converges to 1 also for $p < p^*$, and it converges to a function that is strictly less than 1 for $p > p^*$. In both of these regions, the principal will learn some information about the agent's type from the public signal. However, this information is not useful (payoff-relevant) for the principal since it does not lead to an action change. Hence, the principal's termination cutoff

[28] For sufficiently small λ, the noninvestible type does not mimic at all (i.e., $a(p) = 0$ for all p), and the principal's problem becomes identical to a standard two-armed bandit problem. For relatively large λ (still less than $\tilde{\lambda}$), some mimicking appears in equilibrium, but $a(\cdot)$ is still uniformly bounded away from 1 for all ψ.

converges to p^{**} and her equilibrium value function converges to $\underline{W}$—as if no information would ever arrive (see Figure 2-4, right panel).



(a) $\lambda < \tilde{\lambda}$

(b) $\lambda > \tilde{\lambda}$

Figure 2-4: Convergence of the Principal's Equilibrium Value Function. This figure is plotted under the following parameter values: $r_1 = 0.5$, $r_2 = 0.5$, $u = 1$, $c = 1$, $w_{NI} = 1$, $w_I = -1$ (so that $p^{**} = 0.5$, $\tilde{\lambda} = 0.25$); $\lambda = 0.1$ for (a), $\lambda = 2$ for (b).

To better understand the limiting equilibrium dynamics when $\lambda > \tilde{\lambda}$, note from equation (2.4) that the diffusion coefficient of the principal's belief process at time t is proportional to $\psi(1 - a_t)$. For any fixed p, we can write it as

$$\underbrace{\psi}_{\text{direct effect}} \quad [1 - \underbrace{a(p;\psi)}_{\text{equilibrium effect}}].$$

As ψ increases, the *direct effect* through the multiplier accelerates information revelation while the *equilibrium effect* through the agent's strategy slows down learning. Its limit depends on the value of p; in particular, one can show that

$$\lim_{\psi \to \infty} \psi[1 - a(p;\psi)] \in (0, \infty), \text{ for all } p \in (0, p^{**}),$$

$$\lim_{\psi \to \infty} \psi[1 - a(p;\psi)] = \infty, \text{ for all } p \in (p^{**}, 1),$$

$$\lim_{\psi \to \infty} \psi[1 - a(p^*(\psi);\psi)] = 0.$$

This suggests the following limiting equilibrium dynamics. If the prior belief is above p^{**}, there is an immediate split of belief to either 1 or very close to p^{**} and then it (almost) stops moving. If the prior belief is below p^{**}, then learning takes place gradually but becomes slower and slower as the posterior approaches p^{**}. In both cases the principal's learning does not stop despite the agent's extreme manipulation, but because her posterior never moves across p^{**}, the principal's action never changes with the information she learns, and so payoff-wise it is as if no information ever arrives.

The next two corollaries follow immediately from Theorem 2.6.1.

Corollary 2.6.1. *For any $\lambda > \tilde{\lambda}$ and $p_0 \in (0, 1)$, there exist ψ_1, ψ_2 such that $\psi_1 > \psi_2$ and $W(p_0; \psi_1) < W(p_0; \psi_2)$.*

Corollary 2.6.1 illustrates that a more transparent performance measure (higher ψ) can sometimes reduce the principal's payoff. This result indicates that VCs can sometimes fare better when the startup operates in a more volatile environment. Moreover, policies that require the startup to disclose too precise information may end up hurting the investors. Intuitively, when the signal-to-noise ratio is very high, the mimicking incentives of the non-investible type can be so strong that the public signal provides little useful information to the principal about the agent's type. In that case, introducing more noise to the public signal can lessen such perverse incentives of the noninvestible type, which leads to a more informative equilibrium signal process, benefiting the principal.[29] This result also suggests that there is usually a strictly positive but finite level of transparency that maximizes the principal's value.

Corollary 2.6.2. *For some large ψ and any $p_0 \in (0, 1)$, there exists λ_1, λ_2 such that $\lambda_1 > \lambda_2$ and $W(p_0; \psi, \lambda_1) < W(p_0; \psi, \lambda_2)$.*

Corollary 2.6.2 demonstrates that more frictions in the principal's decision making (i.e., less frequent arrivals of the stopping opportunity) may sometimes improve the principal's equilibrium payoff. Intuitively, such frictions instill some commitment to not terminating the relationship too soon in the principal's behavior, which, similar to before, weakens the noninvestible type's incentive to manipulate the signals. In turn, this indirect effect through the agent allows the signal process to provide more information to the principal and possibly increases her equilibrium payoff. Because of this, the principal does not always have an incentive to reduce the frictions that prevent prompt decisions.

[29] The possibility that better monitoring/more transparency may hurt a principal or relationship has appeared in other settings, such as career-concern models (**??**), contracting in insurance markets (**??**), contracting with moral hazard (**?**), and dynamic team production (**??**). In our model, this effect shows up for a different reason: better monitoring may give stronger incentives to the agent to engage in performance boosting, which depresses the informativeness of the public signals.

2.7 Information at the Patient Limit

We now investigate the equilibrium outcomes as players get arbitrarily patient. The purpose is to see more clearly the role of patience in the principal's incentives to wait for more information and in the agent's incentives to engage in performance boosting.

First, consider an extreme case where r_1 is constant and r_2 goes to 0 (i.e., the principal gets arbitrarily more patient than the agent). In this case, the agent's mimicking intensity (which is independent of r_2) stays bounded away from 1 everywhere, implying that the public signal always reveals some information about the agent's type. As the principal gets more patient, her marginal cost of waiting for new information becomes lower. Consequently, a patient principal will terminate the relationship only when p is very high. Indeed, the termination cutoff converges to 1 and the principal's payoff converges to $\overline{W}$.

Next, consider the other extreme case where r_2 is constant and r_1 goes to 0 (i.e., the agent gets arbitrarily more patient than the principal). As the agent gets more patient, he cares more about staying in the relationship for long and less about the instantaneous mimicking cost. Thus, the noninvestible type has stronger incentives to mimic the investible type, and the equilibirum mimicking intensity approaches one at and below the termination cutoff. In the limit, the outcome is similar to the case for large ψ and λ characterized in the previous section. That is, it is as if no information ever arrives, with the termination cutoff converging to p^{**} and the principal's value function converging to $\underline{W}$.

What happens in between the two extreme cases? Take a sequence $\{r_{1,n}, r_{2,n}\}_n$ of discount rates such that $r_{i,n} \to 0$ for both $i = 1, 2$ and $\lim_n \frac{r_{2,n}}{r_{1,n}} = \chi \in (0, \infty)$. Consider a sequence of games along which all other parameters are fixed, and let $\{W_n, V_n\}_n$ be the corresponding sequence of value functions for the principal and the agent, respectively. The following theorem displays their limits.

Theorem 2.7.1. *$W_n(\cdot)$ converges uniformly to $\max\{0, R(\cdot)\}$, and $V_n(\cdot)$ converges pointwise to $V^*(\cdot)$ which satisfies*

$$V^*(p) := \begin{cases} u, & \text{if } p < p^{**} \\ 0, & \text{if } p > p^{**} \end{cases}.$$

Theorem 2.7.1 shows that if both players get arbitrarily patient at comparable rates, then it is as if the principal does not receive any information.[30] Along the sequence, both

[30] Recall that the "no-information" value function $\underline{W}(\cdot) = \frac{\lambda}{r_2+\lambda} \max\{0, R(\cdot)\}$, so $\max\{0, R(\cdot)\}$ is the limiting

the agent's incentives to mimic and the principal's resolve to wait for more information get stronger, but it turns out that the former effect dominates the latter.

We view this result as a strong manifestation of the ratchet effect in the patient limit of our model. Since the principal cannot commit to not using future information against the agent, the noninvestible type will engage in performance boosting with almost full intensity in order to maintain his reputation. In the end, no useful information is ever revealed, and the principal's lack of commitment hurts her in the most extreme way. In our applications, this result suggests that the use of other instruments, such as some form of commitment (e.g., setting a deadline and/or grace period), additional screening devices (e.g., performance-based investment levels and/or salaries), or huge fines that increase the expected cost of performance boosting, may be necessary to help the principal get more information.

2.8 Concluding Remarks

In this paper, we study a stopping game with asymmetric information where the performance measures that reflect the fundamental can be manipulated by an agent at a cost. Despite the model being stylized, we obtain rich equilibrium dynamics. Our model illustrates that inflated performance can coexist with growing suspicion about a project's viability. Our analysis also implies that too much transparency may hinder the principal's ability to learn, by encouraging excessive performance boosting. This result suggests that some noise in the monitoring technology may be beneficial for the principal. Furthermore, we find an extreme form of ratchet effect in the patient limit, precluding any useful learning. This happens because the principal lacks the commitment to refrain from using the information obtained during the relationship against the agent, giving a highly patient noninvestible type strong incentives to boost performance and maintain his reputation.

Several ways to extend our analysis are worth mentioning. While our main focus is the adverse selection problem, in some settings moral hazard is a prominent issue. Thus, it would be interesting to allow the agent's action to directly influence the principal's payoff. Relatedly, in that setting the principal might want to use history-dependent flow payoffs to reward/punish the agent. Another possibility is to expand the choice set of the principal by allowing her to elevate the "status" of the relationship, such as promoting the agent or

"no-information" value function as r_2 tends to 0.

upgrading the terms of financing. Finally, optimal contracting in this setting remains an open problem. We leave all these aspects as interesting directions for future research.

Chapter 3

A Dynamic Delegated Investment Model of SPACs

3.1 Introduction

The past year (2020) has witnessed a remarkable rise of the special purpose acquisition company (SPAC). A SPAC is a company with no operations that offers securities for cash and places substantially all the offering proceeds into a trust or escrow account for future use in the acquisition of one or more private operating companies[1]. According to the calculation of Gahng et al. (2021), in 2020, "a total of 248 SPAC IPOs raised $75.3 billion" while 165 operating company IPOs raised $61.9 billion. As SPAC appears to be a major way that private companies raise money and go public, there emerges a heated debate among practitioners and the academia over the consequences and the future of SPAC. Proponents praise SPACs for their agility and flexibility to accommodate financing needs better than traditional ways. Opponents, citing the poor returns in the long history of blank-check companies, denounce SPACs as "bubbles" and "scams"[2]. Meanwhile, it is worth noting that SPAC is still a rapidly evolving industry. Practitioners are consistently experimenting with different practice, while the regulator is also pondering over how to ensure healthy growth of the industry. Therefore, understanding the economic mechanism of the current SPAC practice not only facilitates proper use of SPAC but also guides potential improvement of

[1] SEC website.

[2] "I have never found any blank-check investment vehicle attractive. No matter what the reputation or what the sponsor might be. . . . They are the ultimate in terms of lack of transparency."—Arthur Levitt, former SEC Chairman.

it.

Although there has been considerable empirical literature evaluating the performance of SPAC, we see little theoretical analysis on its underlying economic mechanisms. This paper intends to narrow the gap. We regard SPAC as a kind of delegated investment vehicles and focus on the strategic interaction between the SPAC sponsor and the SPAC investor. In theory, SPACs merit a special analysis because it differs from other common delegated investment vehicles such as private equity, hedge funds, and mutual funds in several aspects. First, the sponsor's payoff is not strongly linked to the actual performance of the investment, so the sponsor may prefer to do a deal unfavorable to the investor. Such systematic misalignment between the sponsor and the investor is minor in other cases. Second, SPACs feature a relatively short horizon. Typically, a SPAC will be liquidated absent a successful merger within 24 months while it is 10 years for private equity funds. Third, a SPAC leaves the final decision over investment to the investor, so the investor is heavily involved in the SPAC's operation.

Based on these unique features, we build a finite-horizon continuous-time model of the dynamic SPAC game with one sponsor and one representative investor. In the SPAC game, the sponsor receives projects stochastically over time and decides whether to propose one to the investor in the form of tender offer. When a project is proposed, the investor can choose to either invest in it or withdraw her money from the SPAC. In either case, the game ends, so the opportunity to propose is unique. If no project is proposed by a deadline, the game also ends, and the investor gets her money back. The tension between the two players rests on two points. First, the sponsor has informational advantage over the investor. He always observes the type of a project, which is either good or bad, but the investor only with a probability. Second, their interests are only partially aligned. The investor, who bears the cost of investment, prefers a good project to no project and further to a bad project. The sponsor, who only enjoys the payoff of investment, prefers a good project to a bad project and further to no project.

We derive a unique sequential equilibrium of the SPAC game. Generically, the equilibrium consists of two stages: in the first stage, the sponsor proposes only good projects, and the investor always invests in the proposed project; in the second stage, the sponsor proposes all the good projects and a fraction of the bad ones he receives, and the investor invests contingent on the information she observes. Since the sponsor has only one chance to propose projects, the opportunity cost of proposing a project is his continuation value, which is the expected payoff of proposing the projects he receives in the future. Note that the sponsor can obtain a higher expected payoff from proposing a good project than proposing a bad one because the investment in a good project brings more to the sponsor and is also more likely to be approved by the investor. As a result, the sponsor with a good project must propose because at best he can receive another good project in the future. As for the sponsor with a bad project, waiting is double-edged: he may be better off if a good project arrives and may be worse off if no project arrives. As the SPAC approaches its deadline, the downside becomes more and more dominant, and thus the sponsor's continuation value decreases. At a point, the sponsor starts to find proposing a bad project desirable. Concerned about the poor quality of the proposed project on average, the investor spontaneously chooses to invest more conservatively based on her information over time. Such conservatism effectively reduces the sponsor's expected payoff of proposing a bad project and in turn helps discipline the sponsor. By and large, the equilibrium is consistent with the conventional wisdom that the incentive misalignment gives rise to a moral hazard problem of the sponsor and it intensifies as the SPAC gets closer to the deadline.

Based on the equilibrium, we then analyze the nature of the sponsor's moral hazard problem—the central friction in the game. The sponsor's moral hazard is curbed by two forces. The first is the investor's screening based on her noisy information, and the second is the sponsor's continuation value. More importantly, the two forces intertwined with each other. On one hand, due to their substitution relationship in equilibrium, the investor's screening is decreasing in the sponsor's continuation value. On the other hand, the investor' screening reduces the possibility of investment and thus stifles the accumulation of the sponsor's continuation value. As a result, the sponsor's continuation value follows a kind of self-reinforcing dynamics: its accumulation rate is positively correlated with its current level. An important lesson is that both sides of the partial alignment in players' incentive are crucial to the equilibrium dynamics: while the misalignment side induces moral hazard, the alignment side helps mitigate it.

Next, we explore the welfare implications of current SPAC practice. A popular opinion is that the investor benefits from her control right over investment because it not only allows her to avoid investment in some bad projects but also discourages the sponsor from proposing bad projects in the first place. However, we find that when the investor's information is sufficiently noisy, this control arrangement actually exacerbates the sponsor's moral hazard problem and reduces the investor's welfare. What the popular opinion misses is that the investor's screening makes waiting less attractive for the sponsor, which incentivizes him to propose more bad projects. When the sponsor has the control right, he proposes any project he receives near the deadline. Though hurting the investor, such undisciplined behavior results in rapid accumulation of the sponsor's continuation value. Hence, in a long period following the beginning of the game, the continuation value is high enough to prevent the sponsor from proposing any bad project, and the investor can fully enjoy the payoff of the good project that arrives. When the investor has the control right, the sponsor's continuation value accumulates more slowly in a self-reinforcing manner as pointed out above. In the case that the investor's information is very noisy, a low level of the sponsor's continuation value translates into a low accumulation rate in the absolute sense. Since the continuation value accumulates from 0, it will be trapped by the self-reinforcing dynamics at a low level for a long period. In one extreme, during the whole game, the investor will exert stringent screening, miss most good projects that arrive, and earn little profit in expectation. This analysis uncovers that the investor's equilibrium screening, which is optimal ex post after the sponsor proposes a project, is too stringent ex ante because it has a negative externality on the sponsor's continuation value. Hence, regarding the design of SPAC, a natural question is whether there is a way to rein in the screening to strike a balance between the two disciplining forces. We find that it is sometimes helpful to make the control right contingent on certain public assessment, e.g. credit rating or auditing trusted by both players. Specifically, the investor should own the control right only if the result of public assessment is below a threshold.

Another issue regarding current SPAC practice we explore is one proposal vs. multiple proposals. Motivated by the recent trend of SPAC, we model the investor's decision-making process as a tender offer, which restricts the sponsor to proposing at most one project in the game. An alternative is to allow the sponsor to continue searching and proposing after a proposal is rejected by the investor until the deadline. Notably, multiple proposals can be naturally implemented if the investor's decision making is structured as voting. On one hand, the coercive termination feature of one proposal eliminates potential investment opportunities and hurts both parties. On the other hand, it enables the investor's screening to curb the sponsor's moral hazard and result in less stringent screening in equilibrium, so both parties benefit from it. We find that the sponsor's welfare is always higher under one proposal but the investor's is ambiguous. This intuition justifies the recent transition from voting to tender offers from an equilibrium perspective.

Finally, we discuss several extensions of the model. First, we explicitly incorporate entrepreneurs into the model and consider their strategic behavior. Entrepreneurs can raise funds through either the SPAC or a standard IPO. The opportunity cost of tapping the SPAC is that the deal may not be approved by the investor and the IPO process is also delayed. Hence, the investor's screening effectively discourages entrepreneurs from tapping the SPAC and diminishes the flow of projects received by the sponsor. Second, we consider the sponsor's endogenous effort to search for projects. We find that as the SPAC approaches its deadline, the sponsor's equilibrium effort first increases due to declining continuation value and then decreases due to intensifying screening of the investor. The two extensions further stoke our concern that the investor's control right exacerbates the moral hazard problem and may backfire. Third, we consider the case of long-lived projects where the sponsor can possibly keep a project for future proposals. It turns out that such possibility does not alter the equilibrium dynamics in the baseline setup. Fourth, we extend the model to multiple investors. Now, the investment in a project requires the approval of sufficient investors. An investor can infer other investors' information through the threshold in equilibrium. The equilibrium is similar to that with only one investor but has richer dynamics.

The paper proceeds as follows. The remainder of this section reviews the related literature. Section 2 describes the baseline setup. Section 3 characterizes the equilibrium. Section 4 analyzes current SPAC practice and discuss the design of SPAC. Section 5 extends the baseline setup along several dimensions. Section 6 concludes the paper. All proofs are given in Appendix.

Related Literature. This paper mainly contributes to two strands of the literature. First, there is a growing empirical literature examining the development, trend and performance of SPACs. Gahng et al. (2021) examine SPAC performance and show that SPAC IPO investors earn positive 9.3% per year, while post-merger returns are significantly negative. They also document that there is no cost advantage of SPACs compared with traditional IPO. Dimitrova (2017) shows that SPAC performance is worse for deals announced near the two-year deadline, which is consistent with our theoretical prediction. Examining the factors that influence approval probability, Cumming et al. (2014) find that the presence of active investors in a SPAC is negatively correlated with approval probability . Klausner et al. (2020) show that the post-merger performance is negatively correlated with dilution and cash shortfall. Blomkvist and Vulanovic (2020) show that the SPAC volume and SPAC share of total IPOs are negatively correlated with VIX and time-varying risk aversion, implying market condition is a key factor in SPAC development.

There is little work on the theoretical side despite SPAC has become more and more important in recent years. Bai et al. (2021) provide a model with endogenous segmented markets, and argues that SPAC is welfare improving as it works as certification intermediaries for risky firms who were unserved by the traditional IPO. Chatterjee et al. (2016) consider a security design problem and argue warrants in SPACs can help to mitigate the moral hazard problem in project selection. Our focus is how the partial alignment in incentive between SPAC sponsors and investors shapes their interaction in a dynamic setup and its welfare implications in SPAC lifecycle. To the best of our knowledge, this is the first theoretical paper examining the SPAC lifecycle in a dynamic environment. We also contribute to the literature by providing discussions on counterfactuals. SPACs is relatively new and has received less attention compared to traditional IPO. With SPAC developing in a fast-changing environment, it's crucial to understand the current practice of SPACs as well as counterfactuals. Our discussion on control rights and one proposal vs multiple proposals sheds light on the design of SPACs.

Second, our paper contributes to the literature on delegation and authority in organizations (Crawford and Sobeli 1982; Agh 1997; Dessein 2002; Grenadier et al. 2016; Guo 2016). In this literature, the principal (the SPAC investor in our model) cannot commit to a decision rule and the allocation of control matters. There are several trade-offs identified in this literature, including the trade-off between informativeness vs bias (Dessein 2002) and information acquisition of different players (Agh 1997). In our discussion on control rights, we extend the model to the case when the sponsor has the control right and compare it with our baseline case when the investor makes the final decision. The allocation of control endogenously changes the shape of sponsor's bias in the SPAC lifecycle. When delegating the investment decision to the sponsor, the investment decision is efficient for a longer period but it deteriorates when the SPAC approaches the deadline. This new trade-off is a direct result of the dynamic nature and the hard deadline of decision making in our model, which is novel in the literature. As for dynamic setups, Grenadier et al. (2016) considers a model

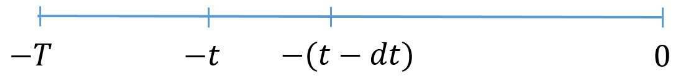


Figure 3-1: Time Flow

in which the principal exercises an option and relies on an informed but biased agent. Guo (2016) considers a dynamic delegation model with experimentation in which the principal and agent have different preferences on project riskiness. Guo (2016) is not a stopping-time game and thus fundamentally different from ours.

3.2 A Dynamic Model of SPAC

3.2.1 Model Setup

Consider a SPAC with one penniless sponsor (he) and one investor (she). They are both risk neutral and have common discount rate $r = 0$[3]. Motivated by the practice in reality, we model the SPAC as a finite-horizon continuous-time dynamic game unfolding over the period $[-T, 0]$. Figure 3-1 is a representation of the time flow. Both t and T are non-negative, and physical time moves forward as t decreases from T to 0. As we will show later, it's easier to consider our model backward, which corresponds to t increasing from 0 to T.

Projects Since the paper is primarily focused on the strategic interaction between the sponsor and the investor, we abstract away entrepreneurs' strategic behavior and assume an exogenous process of projects[4]. Per unit of time, the sponsor receives projects at the rate λ. The type of a project ω can be either good (G) or bad (B), and the probability (odds) of receiving a good project is $\text{Prob}\,(G) = p_0$ $(\theta_0 = p_0/(1-p_0))$. The arrivals and the types of projects are independent over time. Both the good and bad projects require the same investment $I = 1$ and generate gross return R_G and R_B respectively. We made the following assumptions on R_G and R_B:

Assumption 3.2.1. *(1) (positive gross returns)* $R_G > 1 > R_B > 0$; *(2) (negative NPV)* $p_0 R_G + (1-p_0) R_B < 1$.

[3] We assume no discounting merely to simplify the exposition. The main results hold for a positive discount rate.

[4] In Section 3.5.1, we explicitly model entrepreneurs' strategic behavior and examine its impact on the equilibrium dynamics.

The first assumption states that the investment in a good project generates a higher return than that in a bad project and both returns are positive. The second one resonates with the concern that potential SPAC targets are of poor quality on average.

When a project arrives, the sponsor decides whether to propose it to the investor. For the baseline setup, we assume that projects are short-lived. That is, if the sponsor does not propose the project he receives, the project will disappear or become unavailable immediately. With this assumption, the state of the sponsor with respect to whether he has a project and what type of project he has is completely independent over time[5].

The investor's decision making A salient feature of SPACs is that it is the investors who finally decide whether to make an investment. Traditionally, after the sponsor proposes a project, the investors vote on acquisition approval. The acquisition is approved if and only if a sufficient fraction of investors vote for it. However, in the recent wave of SPACs, tender offer becomes the most popular way to structure the investors' decision making. Shachmurove and Vulanovic (2017) claims that "these post financial crisis SPACs are almost exclusively structured as tender offers". Motivated by the trend, we model the sponsor's proposal as a tender offer. The investor can choose to either invest $I = 1$ and receive a prespecified fraction of shares of the project, or withdraw from the SPAC. Because of the nature of a tender offer, the game ends immediately after a proposal. Essentially, the sponsor has only one opportunity to propose a project to the investor in the lifecycle of a SPAC.

Information Both the arrivals and the types of projects are observable to the sponsor[6] but not to the investor. When the sponsor proposes a project, the investor observes the true type of the project with the probability q and nothing otherwise. We denote the investor's observations as $\{H, M, L\}$, whose probabilistic structure follows Table 3.1. Hence, q stands for the quality of the investor's information, and $q < 1$ captures information asymmetry between the sponsor and the investor. Notice that with Assumption 3.2.1, if the sponsor proposes any project he receives, the investor will have negative expected profit of investing upon observing M.

[5] In Section 3.5.3, we study the case that projects are long-lived and thus the state of the sponsor is positively correlated over time.

[6] The assumption that arrivals are privately observed by the sponsor is not important. The equilibrium will be the same even if the arrivals are publicly observable.

	H	M	L
Prob $(\cdot; G)$	q	$1-q$	0
Prob $(\cdot; B)$	0	$1-q$	q

Table 3.1: The investor's observations

Investment	G	B	withdrawal
The sponsor's payoff	v_G	v_B	0
The investor's payoff	u_G	u_B	1

Table 3.2: The payoff structure

The payoff structure Depending on the investment result, the sponsor's and the investor's payoffs follow Table 3.2. If the investor chooses to invest and the type of the project is ω, she receives u_ω and the sponsor receives $v_\omega = R_\omega - u_\omega$ from the project. If the investor chooses to withdraw from the SPAC, she keeps her money $I = 1$ and the sponsor receives 0. If the sponsor does not make a proposal by the time 0, the investor automatically withdraws. We make the following assumptions regarding the payoff structure:

Assumption 3.2.2. *(partial alignment)* $v_G > v_B > 0$, $u_G > 1 > u_B$.

This assumption stems from the famous contractual arrangement of SPAC: the shares granted to the sponsor is not contingent on the value of the project. Typically, the sponsor can obtain 20% of the shares of the target firm owned by the SPAC, and the investors the rest 80%. As a result, when comparing investing in a bad project with investing in no project, the sponsor prefers the former to the latter, while the investor opposite. As recognized by both the academia and practitioners, this preference misalignment underlies the fundamental moral hazard problem in SPACs[7]. On the other hand, it should not be ignored that the contractual arrangement also has an alignment side: both the sponsor and the investor prefer investing in a good project to investing in a bad project or no project. As shown later, both sides of the partial alignment play important roles in equilibrium dynamics.

Timeline Although the game is in continuous time, heuristically, conditional on the game continues at time $-t$, each instantaneous "period" $[-t, -(t-dt))$ consists of events occurring

[7]Aware that potential agency problems may discourage investors, some SPAC sponsors try to tie the shares they get more closely to the ex post value of the firm through deferred grant or clawback. Also, it becomes more popular to let the sponsor have some skin-in-the game. However, these remedies are still far from eliminating the misalignment.

in the following order:

1. With the probability λdt, the sponsor receives a project and observes its type;
2. Receiving a project, the sponsor can propose it or not;
3. If the sponsor proposes a project, the investor receives a signal and chooses to invest in the project or withdraw; then the game ends, and both players receive their payoffs;
4. If the sponsor does not propose a project, the game continues to $-(t-dt)$.

3.2.2 Equilibrium Concept

We focus on the sequential equilibria of the game. First, we characterize the players' strategies and beliefs. Since the game has a finite horizon, time is naturally a state variable that their strategies are based on. The sponsor has only one action in the game: whether to propose the project he receives. Hence, his strategy can be characterized by $(\alpha_\omega(-t))_{\omega\in\{G,B\}}$, where $\alpha_\omega(-t)$ represents the probability that the sponsor proposes the project of the type ω at the time $-t$. The investor also has only one action in the game: whether to invest in the project proposed by the sponsor based on her signal. Therefore, her strategy can be characterized by $(\eta_s(-t))_{s\in\{H,L,M\}}$, where $\eta_s(-t)$ represents the probability that the investor invests at the time $-t$ when observing the signal s.

The players' beliefs can characterized accordingly. Let $(\tilde{\eta}_s(-t))_{s\in\{H,L,M\}}$ be the sponsor's belief about the investor's strategy. Then by proposing a project of the type ω to the investor at $-t$, the sponsor's expected payoff is

$$F_\omega(-t) \equiv \begin{cases} [q\tilde{\eta}_H(-t) + (1-q)\,\tilde{\eta}_M(-t)]\,v_G, & \text{if } \omega = G \\ [(1-q)\,\tilde{\eta}_M(-t) + q\tilde{\eta}_L(-t)]\,v_B, & \text{if } \omega = B \end{cases}.$$

Let $\tilde{\theta}(-t)$ be the investors' prior belief of the odds of a good project before observing the signal. As required by sequential equilibria, these beliefs should be consistent with the strategies on the equilibrium path according to Bayes' rule. However, in this model, sequential equilibria have no effective restriction on the beliefs off the equilibrium paths. Specifically, if $\alpha_G(-t) = \alpha_B(-t) = 0$ at a time $-t$, $\tilde{\theta}(-t)$ can take any nonnegative value. This gives rise to multiplicity of equilibria[8]. To obtain sharper prediction of the equilibrium, we impose D1 refinement: the investor believes that the project must be good if it is proposed

[8] Besides the equilibrium we derive later, another obvious equilibrium is that $\alpha_G(-t) = \alpha_B(-t) = 0$ and $\tilde{\theta}(-t) = 0$ for all time points.

by the sponsor at a time when no project should be proposed in equilibrium.

Below is the equilibrium concept used throughout the paper.

Definition 3.2.1. *An (sequential) equilibrium consists of the sponsor's proposal strategy* $(\alpha_\omega(-t))_{\omega\in\{G,B\}}$*, the investor's investment strategy* $(\eta_s(-t))_{s\in\{H,L,M\}}$*, the sponsor's belief* $(\tilde{\eta}_s(-t))_{s\in\{H,L,M\}}$*, and investor's belief* $\tilde{\theta}(-t)$ *such that at any time* $-t \in [-T, 0]$ *and conditional on no proposal before* $-t$*, the following conditions hold:*

1. $(\alpha_\omega(-\tau))_{\omega\in\{G,B\}}$ *after* $-t$ *maximizes the sponsor's continuation value at* $-t$*:*

$$V(-t) = \max_{(\alpha_\omega(-\tau))_\omega} \int_0^t P(-\tau;-t)\cdot\lambda\left[p_0\alpha_G(-\tau)\cdot F_G(-\tau) + (1-p_0)\,\alpha_B(-\tau)\cdot F_B(-\tau)\right]d\tau, \tag{3.1}$$

where $P(-\tau;-t) \equiv e^{-\int_\tau^t \lambda[p_0\alpha_G(-\xi)+(1-p_0)\alpha_B(-\xi)]d\xi}$ *is the probability that the game still continues at time* $-\tau > -t$ *conditional on that the game continues at time* $-t$*.*

2. *For any* $s \in \{H, M, L\}$*, the investor's investment strategy* $\eta_s(-t)$ *maximizes her expected profit based on the prior belief* $\tilde{\theta}(-t)$ *and the signal* s*:*

$$\eta_s(-t)\left\{\frac{\tilde{\theta}(-t)\,\frac{Prob(s;G)}{Prob(s;B)}}{1+\tilde{\theta}(-t)\,\frac{Prob(s;G)}{Prob(s;B)}}(u_G - u_B) + u_B - 1\right\}.$$

3. *Rational beliefs and D1 refinement:*

   (a) $\tilde{\eta}_s(-t) = \eta_s(-t)$ *for all* $-t$ *and* $s \in \{H, L, M\}$*;*

   (b) $\tilde{\theta}(-t) = \frac{p_0}{1-p_0}\frac{\alpha_G(-t)}{\alpha_B(-t)}$ *for all* $-t$ *satisfying* $\alpha_G(-t) + \alpha_B(-t) > 0$*;*

   (c) $\tilde{\theta}(-t) = +\infty$ *if* $\alpha_G(-t) = \alpha_B(-t) = 0$*.*

3.3 Model Solution

3.3.1 Equilibrium Characterization

We first analyze the investor's problem. When the investor observes the signal H (L), her posterior probability of the project proposed being good becomes 1 (0), and her net payoff from investing in the project is $u_G - 1 > 0$ $(u_B - 1 < 0)$. Thus her equilibrium strategy must be $\eta_H(-t) = 1$ and $\eta_L(-t) = 0$ for all $-t$. To characterize the investor's equilibrium strategy, we can focus on that when she observes the signal M, i.e., $\eta_M(-t)$. For simplicity, we get rid of the subscript of η_M, and let $\eta(-t) \equiv \eta_M(-t)$. It is easy to see that the investor's problem can be reduced to

$$\max_{\eta(-t)} \eta(-t) \left\{ \tilde{\theta}(-t) - \frac{1-u_B}{u_G-1} \right\},$$

where $\tilde{\theta}(-t)$ is the investor's posterior belief of the odds of a good project. Then we obtain the following lemma.

Lemma 3.3.1. *In equilibrium, at any time $-t$,*

1. $\eta_H(-t) = 1$*, and* $\eta_L(-t) = 0$ *;*
2. *When* $\tilde{\theta}(-t) > (<) \frac{1-u_B}{u_G-1}$, $\eta(-t) = 1\,(0)$*; when* $\tilde{\theta}(-t) = \frac{1-u_B}{u_G-1}$, $\eta(-t) \in [0,1]$.

Next, we turn to the sponsor's problem. According to Lemma 3.3.1 and rational beliefs in equilibrium, if the sponsor proposes a project of type ω at time $-t$, his expected payoff is

$$F_\omega(-t) = \begin{cases} [q + (1-q)\,\eta(-t)]\, v_G, & \text{if } \omega = G \\ (1-q)\,\eta(-t)\, v_B, & \text{if } \omega = B \end{cases}.$$

At any time $-t$, the sponsor's continuation value $V(-t)$ satisfies the HJB equation

$$\frac{dV(-t)}{dt} = \max_{\alpha_G(-t),\alpha_B(-t)} \lambda p_0 \cdot \alpha_G(-t) \cdot [F_G(-t) - V(-t)] + \lambda(1-p_0) \cdot \alpha_B(-t) \cdot [F_B(-t) - V(-t)]. \quad (3.2)$$

In addition, at the last instant of the game, it is almost sure that the sponsor will not receive a project, so the continuation value at $-t = 0$ must be 0, i.e., $V(0) = 0$. eq. (3.2) reflects an important feature of the game: the sponsor has only one opportunity to propose a project. When proposing a project of the type ω at $-t$, the sponsor can get expected payoff $F_\omega(-t)$. However, he also loses the opportunity to receive and propose new projects in the future, whose value amounts to $V(-t)$ in expectation. Therefore, the sponsor's equilibrium strategy $\alpha_\omega(-t)$ must satisfy

$$\alpha_\omega(-t) \begin{cases} = 1 & \text{if } F_\omega(-t) - V(-t) > 0 \\ \in [0,1] & \text{if } F_\omega(-t) - V(-t) = 0 \\ = 0 & \text{if } F_\omega(-t) - V(-t) < 0 \end{cases}$$

for $\omega \in \{G, B\}$.

A critical observation of the game is that the sponsor always has more incentive to propose a good project than a bad one. On one hand, $F_G(-t) > F_B(-t)$ always holds because a good project not only gives the sponsor a higher payoff than a bad one but also is more likely to be approved by the investor. On the other hand, the opportunity cost of proposing a project at time $-t$ is $V(-t)$, which is independent of the type of the project that the sponsor receives. An implication of the observation is that in equilibrium, it is always strictly better for the sponsor to propose a good project than not. The sponsor's continuation value is decreasing over time because as the time passes, he is less likely to receive and propose a good project.

Lemma 3.3.2. *In equilibrium, for any $-t$, $F_G(-t) > V(-t)$. Further, $\alpha_G(-t) = 1$, and $V(-t)$ strictly decreases to 0 as $-t$ increases to 0.*

Since the sponsor always propose the good project he receives, the perceived quality of the proposed project depends on his incentive of proposing bad projects. Lemma 3.3.3 implies that whenever the sponsor receives a bad project, he must choose not to propose it with positive probability. This relies on the key assumption that the potential projects of SPACs have negative NPV on average, i.e., $p_0 R_G + (1-p_0)R_B < 1$. If the sponsor surely proposes the bad project he receives at a time point, the investor must withdraw surely when observing M because she has negative expected profit of investing. Then the sponsor should have no incentive to propose a bad project. Hence, this situation cannot take place in equilibrium.

Lemma 3.3.3. *When $V(-t) < (1-q)\,v_B$ and $-t < 0$, $\alpha_B(-t) \in (0,1)$. When $V(-t) > (1-q)\,v_B$, $\alpha_B(-t) = 0$.*

Combining Lemma 3.3.1, Lemma 3.3.2, and Lemma 3.3.3, we obtain a unique equilibrium of the game.

Proposition 3.3.1. *The unique equilibrium of the SPAC game has potentially two stages, the transition time between which is $-t^*$.*

- *The second stage spans the period $(-t^*, 0]$, in which*
  - *the investor's equilibrium strategy $\eta(-t)$ makes the sponsor indifferent to whether to propose a bad project or not, i.e.,*

$$V(-t) = F_B(-t) = (1-q)\eta(-t)v_B;$$

  - *the sponsor's equilibrium strategy $(\alpha_\omega(-t))_{\omega\in\{G,B\}}$ satisfies $\alpha_G(-t) = 1$ and* *makes the investor indifferent to whether to invest or withdraw when observing M, i.e.,*

$$\frac{p_0}{1-p_0}\frac{\alpha_G(-t)}{\alpha_B(-t)} = \frac{1-u_B}{u_G-1};$$

- *the sponsor's continuation value satisfies*

$$\frac{dV(-t)}{dt} = \lambda p_0 \cdot \left[F_G\left(-t\right) - V(-t)\right]$$

with the boundary condition $V(0) = 0$.

- *The first stage spans the period $[-T, -t^*)$, in which*
  - *the investor always invest when observing M, i.e. $\eta(-t) = 1$;*
  - *the sponsor proposes only good projects, i.e., $\alpha_G(-t) = 1$ and $\alpha_B(-t) = 0$;*
  - *the sponsor's continuation value satisfies*

$$\frac{dV(-t)}{dt} = \lambda p_0 \cdot \left[v_G - V(-t)\right].$$

- *The transition time $-t^*$ satisfies $V(-t^*) = (1-q)v_B$, and $V(-t)$ is continuous at $-t^*$. If $-T \geq -t^*$, the first stage will be degenerate, and the equilibrium has only the second stage.*

The misalignment of the two players' payoffs is key to the equilibrium dynamics. Due to the finite horizon of SPAC, as time passes, the sponsor has less chance to receive to a project, and thus his continuation value decreases. Note that the continuation value is also the opportunity cost of proposing a project, which dampens the sponsor's desire to propose a bad project. In the early stage of the game, the continuation value is high enough to prevent the sponsor from proposing any bad project, even though the investor imposes the least stringent screening, $\eta(-t) = 1$. Later on, the sponsor starts to find proposing a bad project desirable. Because of the poor quality of potential projects on average, the investor is concerned about an undisciplined sponsor and spontaneously chooses to invest more conservatively. Such conservatism imposes more stringent screening, $\eta(-t) < 1$, which in turn helps discipline the sponsor.

However, the alignment of their payoffs also plays an important role here. Although the investor imposes more stringent screening in the future, the sponsor's continuation value, which accumulates from potential investment in the future, can always be no less than his expected payoff of proposing a bad project currently. That is because the sponsor may receive good projects in the future. According to the payoff structure of SPACs, a good project gives the sponsor a higher payoff than a bad one. Moreover, since the investor also prefers a good project to a bad one, her screening automatically makes investment in the former more likely than that in the latter. This further enhances the attractiveness of waiting for a good project relative to proposing a bad project currently.

3.3.2 The Welfare

Proposition 3.3.2 provides a characterization of the sponsor's welfare and the investor's welfare based on the properties of the equilibrium. Denote the investor's continuation value at $-t$ by $U(-t)$. Then the sponsor's welfare and the investor's welfare are $V(-T)$ and $U(-T)$ respectively.

Proposition 3.3.2.

- *Given the investor's equilibrium strategy $\eta(-t)$, $V(-T)$ is equal to the sponsor's expected payoff if he proposes only good projects to the investor, i.e.,*

$$V(-T) = v_G \int_{t=0}^{T} \lambda p_0 e^{-\lambda p_0 (T-t)} \left[q + (1-q)\eta(-t) \right] dt. \tag{3.3}$$

- *Given the sponsor's equilibrium strategy $\alpha_G(-t) = 1$ and $\alpha_B(-t)$, $U(-T)$ is linear in the unconditional probabilities that the sponsor proposes good projects in the two stages, i.e.,*

$$U(-T) = (u_G - 1) \cdot (P_1^* + q \cdot P_2^*) + 1, \tag{3.4}$$

where

$$P_1^* = \int_{\min\{T,t^*\}}^{T} \lambda p_0 e^{-\lambda p_0 (T-t)} dt,$$

$$P_2^* = \int_{t=0}^{\min\{T,t^*\}} \lambda p_0 e^{-\lambda p_0 (T-t)} \cdot e^{-\lambda (1-p_0) \int_t^{\min\{T,t\}} \alpha_B(-\tau) d\tau} dt.$$

Since the sponsor is always indifferent to whether to propose a bad project or not in the second stage, we can use the equilibrium path in which the sponsor never proposes a bad project to calculate his welfare. A useful property of the sponsor manifested by the representation (3.3) is that his welfare depends on only how likely a proposed good project is invested by the investor. The probability is $q + (1-q)\eta(-t)$ at $-t$, which depends on the quality of the investor's information as well as her screening in equilibrium.

The statement about the investor's welfare stems from the observation that the investor always breaks even in expectation except that she knows the proposed project is surely good. In the first stage, only good projects are proposed, and the probability that it happens is P_1^*. In the second stage, the investor's expected profit is equal to her outside option 1 when observing M or L, and she knows the project is surely good when observing H. With probability P_2^*, the sponsor proposes a good project in the second stage, and conditional on that, the investor observes H with the probability q. Note that we always have $V(-t^*) = (1-q)\,v_B$ at the start of second stage. The representation (3.4) implies that the key elements in investor's welfare are: the length of the second stage t^*, the probability that the sponsor proposes a bad project in the second stage $\alpha_B(\cdot)$, and the quality of her information q.

3.3.3 Moral Hazard in Equilibrium

In this subsection, we take a closer look at the sponsor's moral hazard problem. Notice that the sponsor acts in the investor's best interest in the first stage and his moral hazard problem is present in only the second stage. Hence, there are two dimensions regarding the degree of his moral hazard in equilibrium: the duration of moral hazard, which is represented by the length of the second stage t^*, and the intensity of moral hazard, which is represented by the probability of proposing a bad project in the second stage $\alpha_B(-t)$.

Proposition 3.3.3.

- *t^* satisfies*

$$\left[e^{\lambda p_0\left(\frac{v_G}{v_B}-1\right)t^*}-1\right]\frac{1}{\frac{v_G}{v_B}-1}\cdot q\cdot v_G=(1-q)\cdot v_B. \tag{3.5}$$

  t^* is decreasing in v_G/v_B and q.

- *For $t < t^*$,*

$$\alpha_B(-t)=\frac{p_0}{1-p_0}\frac{u_G-1}{1-u_B}. \tag{3.6}$$

The length of the second stage depends on two factors. The first is the sponsor's continuation value, and the second is the maximum expected payoff the sponsor can receive from proposing a bad project. He can receive at most $(1-q)v_B$ from proposing a bad project, since the investor must withdraw when observing the signal L. In the first stage, the former is greater than the latter but keeps decreasing. When the former meets the latter, the second stage starts, and lasts until time $-t = 0$. When we look at the model backward from time

$-t = 0$, the sponsor's continuation value is simply all the value $dV(-t)/dt$ accumulated from time 0 to time $-t$. As implied by Proposition 3.3.2, the sponsor's continuation value essentially relies on only the proposal of good projects and consists of two parts. First, upon observing M, the investor invests in the project with the probability $\eta(-t)$ at $-t$, which results in investment in good projects occurring at the rate of $\lambda p_0(1-q)\eta(-t)$. This part corresponds to

$$\left[e^{\lambda p_0\left(\frac{v_G}{v_B}-1\right)t^*}-1\right]\frac{1}{\frac{v_G}{v_B}-1}$$

in eq. (3.5). Second, upon observing H, the investor invests in the project with the probability 1, which results in investment in good projects occurring at the rate of $\lambda p_0 q$. This part corresponds to the q in the left-hand side of eq. (3.5).

Rather than an additive relationship implied by their origination, the parts are convoluted in an multiplicative manner. Note that in equilibrium, $\eta(-t)$ makes the sponsor indifferent to whether to propose a bad project or not, so it satisfies

$$(1-q)\eta(-t)\cdot v_B = V(-t).$$

That means, the first part accumulates by an amount proportional to the level of the sponsor's continuation value. Due to such self-reinforcing dynamics, the sponsor's continuation value becomes very sensitive to q.

The probability of proposing a bad project $\alpha_B(-t)$ is actually a constant in the second stage. The investor is indifferent between withdrawing and investing upon observing signal M, so her posterior belief upon observing signal M must be a constant. As a result, $\alpha_B(-t)$ depends on the quality of the project pool and the investor's payoff structure.

3.4 Welfare Implications and the Design of SPAC

3.4.1 The control arrangement

As shown above, a typical SPAC suffers from moral hazard problems since the sponsor and the investor are not fully aligned about what projects should be invested in. Such systematic misalignment is actually rare in other delegated investment vehicles usually considered to be comparable to SPAC. Regarding private equity, hedge fund, and mutual fund, the sponsor's objective is primarily to maximize the value of the whole fund and thus consistent with the investor's. This is also an important reason why we observe that only SPAC investors can directly decide whether to invest in a project. Such arrangement of investment control rights is meant to mitigate moral hazard problems and facilitate investors' participation in the game in the first place.

In this subsection, we present a welfare analysis of SPAC with respect to investment control rights. We characterize the equilibrium when the sponsor can directly decide whether to invest. Surprisingly, our analysis implies that in some cases, the SPAC investor can be better off if the sponsor has the control right.

Suppose that the sponsor can directly decides whether to invest. Since the sponsor's proposal guarantees investment, his payoff is v_ω if he proposes a project of the type ω. Let $V_s(-t)$ represent his continuation value at time $-t$. It is easy to see that as the time passes, the continuation value must be weakly decreasing and always smaller than v_G. At the last instant of the SPAC life cycle, the continuation value must be 0. Similar to the case that the investor has the control right, the game is divided into two stages in equilibrium. Denote the transition time as $-t_s^*$. In the first stage where $-t < -t_s^*$, $v_B < V_s(-t) < v_G$, so the sponsor proposes only the good project he receives. In the second stage where $-t > -t_s^*$, $V_s(-t) < v_B < v_G$, and the sponsor proposes any project he receives. Let $U_s(-t)$ represent the investor's continuation value at the time $-t$. We readily obtain the following properties about the equilibrium.

Lemma 3.4.1.

- *t_s^* is finite and satisfies*

$$\left(1 - e^{-\lambda t_s^*}\right)[p_0 v_G + (1 - p_0) v_B] = v_B$$

- *There exists $T_s^* > 0$ such that $U_s(-T) > 1$ if and only if $T > T_s^*$.*

The first point of Lemma 3.4.1 states that although the sponsor has full discretion over investment, he only acts at odds with the investor's interest in a later period of the game. Similar to that in the baseline setup, the equilibrium has two stages. In the second stage, the sponsor's continuation value accumulates at the rate of $\lambda [p_0 v_G + (1 - p_0) v_B]$ while his expected payoff from proposing a bad project is always v_B. The second stage starts when the former meets the latter, and lasts until $-t = 0$. The key force behind the equilibrium is that the sponsor also prefers a good project to a bad one as the investor does. With only one opportunity to invest, if he expects that the remaining time allows him to receive a good project with a sufficiently high probability, he would prefer to forgo the bad project at hand despite the risk that he may end up with no project.

The second point stems from the following fact: in the second stage, the sponsor invests in all the projects he receives, so the investor loses money from investment in expectation because of the poor quality of potential SPAC projects on average, i.e.,

$$p_0 u_G + (1 - p_0) u_B < 1.$$

In the first stage, the sponsor invests in only the good projects he receives, so the investor gets positive profit from investment in this stage. As a result, when T is large enough, going backward from the last instant $-t = 0$ to $-t = -T$, the investor's continuation value $U_s(-t)$ first decreases and then increases.

Next, we focus on how the ownership of the control right affects the two players' welfare.

Proposition 3.4.1.

- $V_s(-T) > V(-T)$ *for any* T.
- *Suppose* $T > T_s^*$. *There exists* q_s^* *such that* $U_s(-T) > U(-T)$ *if and only if* $q < q_s^*$.

The first point is straightforward. When the sponsor have full control over investment, his expected payoff must dominates his payoff in the baseline setup. The second point implies that if the investor's information is very noisy, the investor can be better off if the sponsor has the control right and T is large enough. The key to the result is that when q is small, the investor's control right exacerbates the sponsor's moral hazard problem and prolongs the second stage.

Next, we explain how the investor's control right affects the length of the second stage. When the investor has control right, at any time $-t$, a proposed project will be rejected with probability 1 if the signal L is observed and will be rejected with the probability $1 - \eta(-t)$ if the signal M is observed. Compared to the case when the sponsor has the control right, such potential rejection directly reduces the sponsor's expected payoff from proposing a bad project to $(1 - q)\eta(-t) \cdot v_B$ and thus reduces its maximum to $(1 - q) \cdot v_B$. Apparently, this direct effect shortens the second stage. It is consistent with the conventional wisdom that with the control right, the investor's profit-maximizing decision can naturally discipline the sponsor's behavior. However, the rejection also impedes the accumulation of the sponsor's continuation value. Recall that according to Proposition 3.3.2, the accumulation essentially comes from only the proposals of good projects. Hence, at any time $-t$ in the second stage, the sponsor's continuation value accumulates at the rate of $\lambda p_0 \left[q + (1-q)\eta(-t)\right] \cdot v_G$ in the case when the investor has the control right, as opposed to $\lambda \left[p_0 v_G + (1-p_0)v_B\right]$ in the case when sponsor has the control right. So when the investor has control right, the sponsor's continuation value accumulates more slowly, which can potentially make the second stage longer.

Then what is the net effect of the investor's control right when q is small? The reduction in the maximum expected payoff from proposing a bad project is proportional to q and thus small, but that in the accumulation of the sponsor's continuation value could be very large. As pointed out in Section 3.3.3, the sponsor's continuation value follows a self-reinforcing dynamics. With a small q, a low level of the sponsor's continuation value directly translates into a low accumulation rate. Since the accumulation starts at 0, the self-reinforcing dynamics essentially trap it at a low level for a long period. As a result, it takes long for the sponsor to accumulate sufficient continuation value to leave the second stage.

In this SPAC setup, there are potentially two forces that determine the degree of the sponsor's moral hazard problem in equilibrium. As argued previously, the investor and the sponsor are partially aligned: they both prefer good projects to bad projects. Such partial alignment, which works through the accumulation of the sponsor's continuation value, naturally motivates the sponsor to act in the investor's interest to some extent. When the investor has the control right, she exerts screening based on her information. The screening directly disciplines the sponsor's behavior, yet it also dampens the effect of partial alignment. Our analysis suggests that the investor's equilibrium screening, which is optimal ex post after the sponsor proposes a project, is too stringent ex ante because the investor does not consider the negative externality on the sponsor's continuation value. The precision of the investor's information determines which side of the excessive screening dominates.

3.4.2 Public assessment and contingent control right

Public assessment of assets or projects plays an important role in various financing activities. For example, credit rating in bond issuance and auditing in syndicated loans. Apart from providing trustworthy information for investors[9], another potential function of public

[9]This welfare impliaction of this function will be examined in the next subsection.

assessment is to provide public signals used for contracting. As implied by our previous analysis, giving the control right to either party may both incur severe welfare loss. In this subsection, we show that making the control right contingent on public signals can be more favorable in some situations.

Suppose there is a public assessment agency that always publicly, truthfully discloses what it observes about the project proposed by the sponsor. Its information structure is similar to the investor's: if the project is good (bad), it observes H (L) with the probability $\hat{q}$ and M otherwise. We focus on monotone contingent allocation of the control right[10]: the sponsor can decide whether to invest if and only if the public signal is more favorable than a threshold. Notice that the two players always prefer the same decision upon observing H. If the sponsor (investor) has the control right upon observing M and L, the case is equivalent to that he (she) has the full control right. Therefore, we only need to deal with the case that the sponsor has the control right when the public signal is M and the investor has the control right when it is L.

It is straightforward to see that the sponsor's proposal is certainly rejected when the public signal is L. Hence, his expected payoff is v_G if he proposes a good project and $(1-\hat{q})v_B$ if he proposes a bad one. Let $V_c(-t)$ represent his continuation value at $-t$. The game is divided into two stages in equilibrium. Denote the transition time as $-t_c^*$. In the first stage where $-t < -t_c^*$, $(1-\hat{q})v_B < V_c(-t) < v_G$, so the sponsor proposes only the good projects he receives. In the second stage where $-t > -t_c^*$, $V_c(-t) < (1-\hat{q})v_B < v_G$, so the sponsor proposes any project he receives. Let $U_c(-t)$ represent the investor's continuation value at the time $-t$. We obtain the following results.

Proposition 3.4.2.

- *The sponsor's welfare is increasing in the control right he has, i.e.,* $V_s(-T) > V_c(-T) > V(-T)$.
- *The investor's welfare is higher when the control right is contingent than when the sponsor has the control right, i.e.,* $U_c(-T) > U_s(-T)$.
- *The total welfare of the two players is higher when the control right is contingent than when the sponsor has the control right, i.e.,* $V_c(-T) + U_c(-T) > V_s(-T) + U_s(-T)$.

The first point is straightforward: more control right is always beneficial to the sponsor. The second point says the investor is always better off with the contingent control right

[10] It can be shown that other allocation is weakly dominated by this family of allocations .

than none. Compared to no control right, the contingent control right effectively reject the investment in bad projects with the probability $\hat{q}$ (when the public signal is L). First, it increases the investor's expected payoff at each instant in the second stage. Second, it also shortens the length of the second stage. The sponsor's expected payoff from proposing a bad project is reduced by a factor of $\hat{q}$ to $(1-\hat{q})v_B$, and his accumulation of continuation value is also reduced to $\lambda\left[p_0 \cdot v_G + (1-p_0)\cdot(1-\hat{q})v_B\right]$. The reduction in the latter is disproportionately low relative to the former because the former depends on only the investment in bad projects while the latter depends on that in both types of projects. This property also holds for more general signal structures: the beneficial disciplining effect of allowing the investor to reject the investment when observing sufficiently unfavorable signals probably outweighs its adverse effect on the accumulation of the sponsor's continuation value. For the same reason, the total welfare of the two players is higher with the contingent control right because it reduces investment in bad projects without affecting that in good projects.

The comparison between the contingent control right and the investor having the control right is ambiguous. Besides the forces related to control rights discussed in Section 3.4.1, it also depends on the quality of the public information compared to the investor's ($\hat{q}$ vs. q).

3.4.3 One proposal vs. multiple proposals

Motivated by the recent trend of SPAC, we model the investor's decision-making process as tender offers for the baseline setup. Once the sponsor proposes a project, the game enters into the investor's decision making stage and ends immediately after that. Hence, the sponsor has only one opportunity to propose projects before the deadline. Another possible, also natural way to structure a SPAC is to let the sponsor continue to search for projects until the deadline if the investor is not willing to invest in the current one. Then the sponsor will essentially have multiple opportunities to propose. For convenience, we regard the two regimes as one proposal and multiple proposals respectively. In this subsection, we examine the impact of allowing multiple proposals on the equilibrium and the players' welfare.

Notably, multiple proposals can be naturally implemented if the investor's decision making is structured as voting, which was a very popular practice until recently. Voting in SPACs proceeds as follows. After the sponsor proposes a project, the investors vote on acquisition approval. If a sufficient fraction of investors vote for it, the deal is approved. Then the investors who vote against the deal are offered the right to redeem their shares[11]. Investors who are not offered or do not exercise the right will invest. If the deal is not approved, the SPAC will continue, and the sponsor searches for new projects. In our single-investor setup, the investor will either approve the deal and invest, or disapprove it and let the SPAC continue, consistent with the regime of multiple proposals. Therefore, the comparison between the two regimes also shed light on that between tender offers and voting.

We consider a derivation of our baseline setup and assume that the sponsor can continue to search if the proposed project is rejected. All other assumptions are unchanged. Let $V_v(-t)$ represent the sponsor's continuation value at $-t$ in this new game. Likewise, $V_v(-t)$ is weakly decreasing, always smaller than v_G, and equal 0 at the deadline of the SPAC. By proposing a project of the type ω at $-t$, the sponsor enjoys v_ω if the investor approves the deal and $V_v(-t)$ otherwise. Hence, given the investor's strategy, his marginal benefit of proposing a project of the type ω is proportional to $v_\omega - V_v(-t)$. To obtain sharp equilibrium prediction, we assume that the sponsor does not use weakly dominated strategies. That is, he proposes a project of the type ω at $-t$ with the probability 1 if $v_\omega - V_v(-t) > 0$ and 0 if $v_\omega - V_v(-t) < 0$. It is easy to see the game is still divided into two stages in equilibrium. Denote the transition time as $-t_v^*$. In the first stage where $-t < -t_v^*$, $v_B < V_v(-t) < v_G$, so the sponsor proposes only the good project he receives. In the second stage where $-t > -t_v^*$, $V_v(-t) < v_B < v_G$, so the sponsor proposes any project he receives. Let $U_v(-t)$ be the investor's continuation value at $-t$. We obtain the following results.

Proposition 3.4.3. *The sponsor always has a lower welfare under multiple proposals than under one proposal, i.e.,* $V_v(-T) < V(-T)$. *But the comparison about the investor's welfare between the two regimes is ambiguous.*

Surprisingly, the sponsor is worse off under multiple proposals. Since the first stage proceeds in the same way under both regimes, to understand the intuition, we can focus on the second. On one hand, under multiple proposals, his continuation value accumulates at a lower rate in the second stage, which is

$$\begin{aligned}\frac{dV_v(-t)}{dt} &= \lambda p_0 \cdot q\left[v_G - V_v(-t)\right] \\ &= \lambda p_0 \cdot \left[q v_G + (1-q)\, V_v(-t) - V_v(-t)\right],\end{aligned}$$

[11] This is required by stock exchange listing rules. In many cases, SPACs offer all investors the redemption rights.

as opposed to

$$\frac{dV(-t)}{dt} = \lambda p_0 \cdot [qv_G + (1-q)\,\eta\,(-t)\,v_G - V(-t)]$$

under one proposal. The difference between the two accumulation rates is that when M is observed, the sponsor receives $V(-t)$ in expectation under multiple proposals while he receives v_G with the probability $\eta(-t)$ under one proposal. Recall that $(1-q)\eta(-t)\cdot v_B = V(-t)$, so

$$\eta\,(-t)\,v_G = \frac{v_G}{(1-q)\cdot v_B}V(-t) > V(-t).$$

This simple observation relies on two points. First, since the signal L has helped screen out the fraction q of bad projects, the investor can exert lesser screening when observing M. Second, the sponsor strictly prefers a good project to a bad one. On the other hand, the sponsor needs a higher continuation value to leave the second stage under multiple proposals, $V_v(-t) = v_B$, as opposed to $V(-t) = (1-q)v_B$ under one proposal. Hence, allowing multiple proposals also prolongs the second stage.

The underlying economic intuition is that the coercive termination feature of one proposal enables the investor's screening to have ex ante disciplining effect on the sponsor. Under multiple proposals, the investor's disapproval when observing M or L cannot suppress the sponsor's incentive to propose bad projects at all because disapproval is not worse than not proposing. To protect herself from the sponsor's undisciplined behavior, the investor has to reject any investment unless she observes the clear-cut good signal H. As pointed out in Section 3.4.1, this rational response further restricts the accumulation of the sponsor's continuation value and prolongs the second stage. This intuition also helps justify the recent transition from voting to tender offers from an equilibrium perspective.

However, the comparison about the investor's welfare is ambiguous. Allowing multiple proposals affects the welfare in two opposite ways. On one hand, it prolongs the less efficient second stage. On the other hand, it increases the rate at which the investor's continuation value accumulates in the second stage. In both regimes, the investor can earn the positive profit $u_G - 1$ in the second stage only when she observes H. Under multiple proposals, the game ends only in this event by the deadline, while under one proposal, the game may end when she observes M or L. Hence, this event occurs more likely under multiple proposals.

3.5 Extensions

3.5.1 Strategic entrepreneurs

In the baseline setup, we deliberately abstract away from entrepreneurs' strategic behavior to better focus on the strategic interaction between the sponsor and the investor. Usually, the entrepreneurs can choose to bring the project public through either SPAC or standard IPO, and in more general settings, entrepreneurs can choose other financing strategies. In our baseline setup, the equilibrium features a (weakly) decreasing probability of approval for both good and bad projects. Then in a more general setting when entrepreneurs are strategic, the choice between SPAC and standard IPO will be endogenous, and thus the supply of projects from entrepreneurs may not be constant in SPAC lifecycle. In this subsection, we introduce strategic entrepreneurs into the model and examine its impact on the equilibrium dynamics.

There are many entrepreneurs, each of whom is endowed with one project. Projects can be either good (G) or bad (B), and the fraction of good projects is p_0. Each entrepreneur observes the type of her own project. There is only one SPAC in the market operating in the way modeled in the baseline setup. At each instantaneous "period" $[-t, -(t-dt))$, a liquidity shock arrives with probability λdt, and a randomly chosen entrepreneur needs to raise $I = 1$ to continue her project. If the project is not funded instantly, it may fail or shrink over time. The entrepreneur hit by a liquidity shock can choose to bring her project public through either SPAC or standard IPO. There are three possible scenarios:

1. she chooses IPO directly;
2. she taps SPAC, and the project is funded by the SPAC investor;
3. she taps SPAC, but the project is not funded by the SPAC investor and she turns to IPO.

Denote an entrepreneur's payoff as π_{IPO} if the project is funded through IPO directly and as π_{SPAC} if it is funded through SPAC. In the case that she chooses IPO after unsuccessful SPAC financing, her payoff is $\rho \cdot \pi_{IPO}$. We assume $\rho < 1$ because preparing for SPAC delays the project's IPO process and the project may fail or shrink due to lack of funding during that period. We do not impose a particular probabilistic structure on ρ, π_{SPAC}, and π_{IP0}: they can vary with projects in any reasonable way. The relationship between π_{IPO} and π_{SPAC} is ambiguous and depends heavily on project specifics. π_{SPAC} could be greater than

π_{IPO} for several reasons. For example, projects can be funded through SPAC more quickly; projects that cannot access standard IPO may go public through SPAC.

Due to the stringent screening process and regulatory requirement of IPO, bad projects can hardly be funded through it, so we assume $\pi_{IPO} = 0$ for bad projects. Then it is easy to see that entrepreneurs with bad projects will always tap SPAC. Now consider those with good projects. Suppose an entrepreneur expects that if she taps SPAC, her project can be funded through SPAC with the probability x. Then she will tap SPAC if and only if

$$\begin{aligned} x \cdot \pi_{SPAC} + (1-x) \cdot \rho\pi_{IPO} &> \pi_{IPO} \\ \Leftrightarrow \frac{(1-\rho)\pi_{IPO}}{\pi_{SPAC} - \rho\pi_{IPO}} &< x. \end{aligned} \tag{3.7}$$

Certainly, since unsuccessful SPAC financing causes costly delay to IPO, if a project is more likely to be funded through SPAC, the entrepreneur is more willing to choose SPAC over IPO. Let $\Phi(\cdot)$ represent the CDF of the random variable $\frac{(1-\rho)\pi_{IPO}}{\pi_{SPAC}-\rho\pi_{IPO}}$, and assume that $\Phi(\cdot)$ is strictly increasing. Then at each instant, the sponsor receives good projects at the rate $\lambda p_0 \Phi(x)$.

Now we characterize the equilibrium of the SPAC game with strategic entrepreneurs. At the time $-t$, the sponsor receives bad projects at the rate $\lambda(1-p_0)$ and good projects at the rate $\lambda p_0 \Phi(x(-t))$, where

$$x(-t) \equiv \alpha_G(-t) \cdot [q + (1-q)\,\eta\,(-t)]$$

is the probability that the project can be approved if the entrepreneur chooses SPAC. Lemma 3.3.2 and Lemma 3.3.3 still hold because the projects the sponsor receives at each instant have negative NPV on average, i.e.,

$$\frac{p_0\Phi(x(-t))R_G + (1-p_0)R_B}{p_0\Phi(x(-t)) + 1 - p_0} \leq p_0 R_G + (1-p_0)R_B < 1.$$

The equilibrium is very similar to that in Proposition 3.3.1 except that the sponsor receives good projects at the rate $\lambda p_0 \Phi(q + (1-q)\,\eta\,(-t))$.

Proposition 3.5.1. *As the SPAC approaches its deadline, a decreasing fraction of the entrepreneurs with good projects choose to tap the SPAC.*

As the SPAC approaches its deadline, the investor becomes more concerned about the sponsor's moral hazard problem and exert more stringent screening. The screening effectively discourages the entrepreneurs with good projects, who can also access standard IPO, from tapping the SPAC. That means, the investor's screening abates not only the sponsor's expected payoff from proposing good projects but also the probability that he receives good projects. As analyzed in Section 3.4.1, this additional effect further dampens the accumulation of the sponsor's continuation value and exacerbates the sponsor's moral hazard problem. Entrepreneurs' potential strategic behavior actually stokes our concern that giving less informed investors the control right to reassure them may backfire.

3.5.2 Endogenous Effort to Search for Projects

In reality, the search for projects also depends on the sponsor's effort. To prepare investment proposals to the investor, the sponsor needs to spend time, energy, and money in finding projects and negotiating deals. Such effort can hardly be observed or enforced, so it is mainly determined by the sponsor's utility maximization. Since the marginal benefit of proposing a project is not constant over the lifecycle of the SPAC, he may optimally exert different amount of effort. In this subsection, we incorporate the sponsor's endogenous effort into the model.

At each instant $-t$, the sponsor can choose to exert a flow effort $\kappa(-t)$ to search for projects. It increases the arrival rate of projects from λ to $\lambda + \kappa(-t)$ without changing the probability of a good one. Meanwhile, it incurs a private flow cost $C(\kappa(-t))$ to the sponsor. $C(\cdot)$ is an increasing, convex function, and $C(0) = 0$. The equilibrium is very similar to that in Proposition 3.3.1 except that the sponsor receives projects at the rate $\lambda + \kappa^*(-t)$. $\kappa^*(-t)$ is chosen by the sponsor to maximize his continuation value, i.e.,

$$\kappa^*(-t) = \arg\max_{\kappa} \ (\lambda + \kappa)\, p_0 \cdot [F_G(-t) - V(-t)] - C(\kappa),$$

so in equilibrium it satisfies

$$C'(\kappa^*(-t)) = p_0 \left[F_G(-t) - V(-t)\right].$$

Plugging $F_G(-t) \equiv [q + (1-q)\,\eta(-t)]\, v_G$ into the equation, we obtain that in the second stage,

$$C'(\kappa^*(-t)) = p_0 \left[v_G q + \left(\frac{v_G}{v_B} - 1 \right) V(-t) \right]$$

and in the first stage,

$$C'(\kappa^*(-t)) = p_0 \left[v_G - V(-t) \right].$$

Proposition 3.5.2. *As the SPAC approaches its deadline, the sponsor exerts more effort in the first stage but less in the second stage.*

At every instant, the sponsor's endogenous effort is motivated by the difference between the expected payoff of proposing a good project and his continuation value. In the first stage, the benefit is always v_G, but his continuation value keeps decreasing. It implies that failing to find a good project, his situation deteriorates. Hence, he has more incentive to exert effort to search for projects. In the second stage, his situation still deteriorates, but the benefit also shrinks over time because of the investor's intensifying screening. Because a good project is more valuable than a bad one to the sponsor, $v_G > v_B$, the decrease in the sponsor's expected payoff of proposing a good project is more dramatic than that in his continuation value in an absolute basis. As a result, his incentive to exert effort is greater over time in the second stage. Similar to that on strategic entrepreneurs, this analysis also uncovers a channel through which the investor's control right may further exacerbate the sponsor's moral hazard problem.

3.5.3 Long-lived projects

For the baseline setup, we assume that projects are short-lived: if the sponsor doesn't propose the project he receives, the project will disappear or become unavailable immediately. With this assumption, the state of the sponsor with respect to whether he has a project and what type he has is completely independent over time. In this subsection, we explore the case of long-lived projects where the sponsor can possibly keep a project for future proposals. It turns out that such possibility does not alter the equilibrium dynamics in our setup.

The new setup is the same as the baseline one except that the projects the sponsor has received but not yet proposed still exists. Such projects are called old project. At each instant, the sponsor can choose to revisit one of the old projects. The revisit makes the project ready for proposal again at a rate of γ. It is easy to see that the sponsor must choose to revisit the best project he has received so far, so his continuation value depends on its type. Denote the sponsor's continuation value at $-t$ as $V^{\sigma}(-t)$ if the best project he has received is of the type $\sigma \in \{G, B\}$. Then in equilibrium, $V^{G}(-t) > V^{B}(-t)$. Heuristically, conditional on the game continues at time $-(t+dt)$, each instantaneous "period" $(-(t+dt), -t]$ consists of events occurring in the following order:

1. The initial state is $\sigma(-t-dt) \in \{G, B\}$;
2. With probability λdt, the sponsor receives a new project and observes its type ω', and the new project is ready for proposal;
3. With probability γdt, the best old project becomes ready;
4. If there is at least one project ready, denote the type of the best of them as $\omega(-t) \in \{G, B\}$, the sponsor proposes the best one with probability $\alpha_{\omega}(-t)$;
5. If the sponsor proposes a project, the game enters into the decision making stage and ends after that; if no project is proposed, the state is updated to $\sigma(-t)$, and the game moves on to the next period.

Since the opportunity to propose is unique, a sponsor with a project of the type ω ready for proposal faces a trade-off between $V^{\sigma}(-t)$ and $F_{\omega}(-t)$, the expected payoff of proposing it right away.

Recall that in the baseline setup, a critical observation is that the sponsor always has more incentive to propose a good project than a bad one. It follows that his expected payoff of proposing a good project is higher than that of proposing a bad one but his opportunity cost is the same for both. Although the second half does not hold in the new setup (since the sponsor's continuation value depends on the type of the projects he has received so far), we can show that this critical observation still holds.

Lemma 3.5.1. *In equilibrium, for any $-t$, $F_G(-t) > V^{G}(-t)$, so $\alpha_G(-t) = 1$.*

Notice that if the sponsor has a good project ready for proposal, his continuation value must be $V^{G}(-t)$. $F_G(-t) \leq V^{G}(-t)$ implies that the investor must have less stringent screening at some points in the future, which can compensate for the possibility that the sponsor may not have a good project ready for proposal again. However, less stringent screening increases the probability of investment in a bad project disproportionately more than that in a good one. Hence, $F_B(-t) < V^{B}(-t) < V^{G}(-t)$ must holds. Then the rest follows the proof of Lemma 3.3.2.

Proposition 3.5.3. *The new setup has a unique equilibrium, and it is the same as the one characterized by Proposition 3.3.1.*

In the baseline setup, the sponsor receives too high a fraction of bad projects so that if he proposes any bad project he receives, the investor has a negative expected profit of investing when observing M; hence, in equilibrium, the investor exerts screening that induces the sponsor to propose bad projects at only the rate

$$\lambda(1-p_0)\alpha_B(-t) = \lambda p_0 \frac{u_G - 1}{1 - u_B}.$$

In the new setup, as implied by Lemma 3.5.1, the sponsor still proposes any good project he receives right away, so revisit does not change the rate that good projects are proposed. But revisit increases the amount of bad projects ready for proposal, which makes the investor even more concerned about the average quality of proposed projects. So, the investor will exert the same screening, and the sponsor will propose bad projects at the same rate. Notably, the sponsor does not benefit from revisit. As pointed out by Proposition 3.3.2, his expected payoff depends on only the proposals of good projects in equilibrium.

3.5.4 Multiple agents

In the baseline setup, we assume only one investor to simplify the characterization of the equilibrium dynamics. Here we extend the model to multiple investors. If one investor's decision does not affect other investors' payoff, then the game is essentially the same as the one-investor baseline game. However, there may be externalities between investors in practice. For a deal to be approved, it requires a sufficient fraction of investors willing to invest. In some cases, the SPAC prospectus specifies a threshold beforehand while in others, the threshold is set later to meet the minimum investment required by the project. Such a threshold allows an investor to infer information from approved investment and thus add a layer of strategic interaction between investors to the game. In this subsection, we examine the impact of this strategic interaction on the equilibrium dynamics. As shown later, the new equilibrium consists of three stages, and the new stage (the third stage) is a direct result of information aggregation through the threshold.

There are N investors in this new game. After the sponsor proposes a project, each investor observes a signal, which has the same ternary signal structure modeled in Table 3.1.

Conditional on the type of project, the signal realizations are independent across all investors. After observing their own signals, each investor chooses to withdraw or not. There is an exogenous threshold K such that the proposal is approved if at least K investors choose not to withdraw. If the project is approved, only the investors who choose not to withdraw invest. If the project is not approved, all investors withdraw. The investors all have the same payoff structure as in the baseline setup; namely, an investor will get u_ω if she invests in a project of the type ω and get 1 if she withdraws. The sponsor's payoff is proportional to the size of the investment: if x investors invest, the sponsor receives $x \cdot v_\omega$[12].

As in the baseline setup, an investor still chooses to withdraw if he observes the signal L and not if H, so an investor's strategy is captured by the probability not to withdraw when observing M at $-t$, $\eta(-t)$. We focus on symmetric mixed-strategy equilibria where all investors have the same $\eta(-t)$ at each instant. Let $\tilde{\theta}(-t)$ be the investors' prior belief of the odds of a good project before observing any signal,. Consider an investor who observes M. If he chooses not to withdraw, then he invests if and only if at least $K-1$ of the other $N-1$ investors make the same decision. Conditional on this information, the odds of a good project is

$$\frac{\Gamma\left(q+(1-q)\eta(-t)\right)}{\Gamma\left((1-q)\eta(-t)\right)}\tilde{\theta}(-t), \tag{3.8}$$

where

$$\Gamma(y) \equiv \sum_{x=K-1}^{N-1} \binom{N-1}{x} y^x (1-y)^{N-1-x}.$$

Comparing this odds to that in the baseline setup, we can see it has an additional term due to the information inferred from the threshold in equilibrium. Similar to Lemma 3.3.1, when this odds is greater (smaller) than $\frac{1-u_B}{u_G-1}$, $\eta(-t)$ is equal to 1 (0); when it is equal to $\frac{1-u_B}{u_G-1}$, $\eta(-t)$ is between 0 and 1.

Let $\tilde{\eta}(-t)$ be the sponsor's belief about the investors' strategy. By proposing a project of the type ω to the investors at $-t$, the sponsor's expected payoff is

$$F_\omega(-t) \equiv \begin{cases} v_G \cdot \Lambda\left(q+(1-q)\tilde{\eta}(-t)\right), & \text{if } \omega = G \\ v_B \cdot \Lambda\left((1-q)\tilde{\eta}(-t)\right), & \text{if } \omega = B \end{cases},$$

[12]The equilibrium structure stay unchanged as long as the sponsor's payoff is weakly increasing in x.

where

$$\Lambda(y) \equiv \sum_{x=K}^{N} \begin{pmatrix} N \\ x \end{pmatrix} y^{x} (1-y)^{N-1-x} x.$$

Last, we close the model by imposing rational beliefs and D1 refinement as in Definition 3.2.1. The following proposition characterizes the unique symmetric equilibrium of the game.

Proposition 3.5.4. *The unique equilibrium of the SPAC game has three stages, and the transition time points between two consecutive stages are $-t_1^* < -t_2^*$ respectively.*

- *The third stage spans the period $(-t_2^*, 0]$, in which*
  - *$\eta(-t)$ solves*
    $$\frac{\Gamma(q+(1-q)\eta(-t))}{\Gamma((1-q)\eta(-t))} \frac{p_0}{1-p_0} = \frac{1-u_B}{u_G-1};$$
  - *$\alpha_G(-t) = \alpha_B(-t) = 1$;*
  - *the sponsor's continuation value satisfies*
    $$\frac{dV(-t)}{dt} = \lambda p_0 \cdot [F_G(-t) - V(-t)] + \lambda(1-p_0) \cdot [F_B(-t) - V(-t)]$$
    and the boundary condition $V(0) = 0$.
- *The second stage spans the period $(-t_1^*, -t_2^*]$, in which*
  - *$\eta(-t)$ satisfies*
    $$V(-t) = F_B(-t);$$
  - *$\alpha_G(-t) = 1$ and*
    $$\frac{\Gamma(q+(1-q)\eta(-t))}{\Gamma((1-q)\eta(-t))} \frac{p_0}{1-p_0} \frac{\alpha_G(-t)}{\alpha_B(-t)} = \frac{1-u_B}{u_G-1};$$
  - *the sponsor's continuation value satisfies*
    $$\frac{dV(-t)}{dt} = \lambda p_0 \cdot [F_G(-t) - V(-t)].$$
- *The first stage spans the period $[-T, -t_1^*)$, in which*
  - *$\eta(-t) = 1$;*
  - *$\alpha_G(-t) = 1$ and $\alpha_B(-t) = 0$;*

– *the sponsor's continuation value satisfies*

$$\frac{dV(-t)}{dt} = \lambda p_0 \cdot [v_G - V(-t)] \, .$$

- *The transition time* t_2^* *satisfies* $V(-t_2^*) = F_B(-t_2^*)$*, the transition time* t_1^* *satisfies* $V(-t_1^*) = v_B \cdot \Lambda((1-q))$*, and* $V(-t)$ *is continuous at* $-t_1^*$ *and* $-t_2^*$*.*

Proposition 3.5.4 has the underlying forces similar to Proposition 3.3.1: the partial alignment of the two players' payoffs. A salient difference is the existence of the third stage in which the sponsor proposes any project he receives instantly. This cannot occur in the baseline setup because the investor will respond with the most stringent screening, $\eta(-t) = 0$, which will in turn eliminate the sponsor's desire to propose any bad project. However, with multiple investors present, an investor's confidence in the project can be bolstered by others' screening through the threshold. Note that the additional term in eq. (3.8), $\frac{\Gamma(q+(1-q)\eta(-t))}{\Gamma((1-q)\eta(-t))}$, goes to infinity as $\eta(-t)$ goes to 0. Even if the sponsor proposes any project he receives, the investors will choose a positive $\eta(-t)$ and thus approve a bad project with a positive probability. Although this probability could be very low, it can still induce the sponsor to propose bad projects near the deadline.

3.6 Concluding Remark

Studying SPAC from a perspective of delegated investment, this paper focuses on the the strategic interaction between the sponsor and the investor. Consistent with the conventional wisdom, the incentive misalignment of the two parties gives rise to a moral hazard problem of the sponsor. However, this is not the whole story. The alignment side of the two parties' incentives helps mitigate the problem. A key takeaway is that giving less informed investors much control right may exacerbate the moral hazard problem and make everyone worse off.

SPAC in reality is a complicated business that involves many parties and interactions. To better illustrate our main idea, we abstract away from several elements of SPAC. Here are some that we think are important and merit more research.

1. The secondary market of SPAC shares. SPAC shares are publicly traded, which aggregates investors' information and affect their decisions.
2. Private Investment in Public Equity (PIPE). An SPAC sponsor frequently invites a

PIPE investment as a part of the business combination, which further complicates their incentives.

3. Investors' demand for liquidity. Besides profitability, liquidity is another critical reason why some investors favor SPACs. Potentially, concern or demand for liquidity may affect investors' decisions as well.

Appendix A

Appendix: Algorithmic Transparency

A.1 When All Borrowers Can Manipulate

In this section, we consider the case that all borrowers can manipulate their features $\hat{\theta}$, while keeping the space of feature to be binary. In this section, we show that in any lending market equilibrium, the good type borrowers never manipulate their features.

Suppose any borrower can manipulate his characteristic by paying cost c_i, i.e., a good (bad) type borrower can change his characteristic to $\hat{\theta} = \hat{B}$ ($\hat{\theta} = \hat{G}$) by paying cost c_i, which follows a continuous distribution $D^{\theta}(c)$, for $\theta \in \{G, B\}$. So the distribution of manipulation cost is type-dependent. Similar to our baseline model, c_i is observable to borrowers but not to lenders. Under posterior belief $\pi(\rho|s)$, for borrowers with type $\theta \in \{G, B\}$, denote lender j's lending decision as $\left\{I_j^{s,\hat{\theta}}(\rho), D_j^{s,\hat{\theta}}(\rho)\right\}$, where $I_j^{s,\hat{\theta}}(\rho) \in [0,1]$ represents the probability that lender j approves the loan applications from borrowers with feature $\hat{\theta}$ conditional on the true state being ρ and the signal disclosed to borrowers being s; and $D_j^{s,\hat{\theta}}(\rho)$ represents the face value of debt that lender j offers to borrowers with feature $\hat{\theta}$, conditional on the true state being ρ and the signal disclosed to borrowers being s. We only focus on symmetric equilibria.

Similar to the baseline model, in this extension, the borrowers' manipulation strategy is summarized by a cutoff $\bar{c}^{\theta}$, such that borrowers with type θ will choose to manipulate if and only if their manipulation cost type c_i is no greater than $\bar{c}^{\theta}$. The following lemma shows that the good type borrowers will never manipulate in any subgame equilibrium, so our assumption in the baseline model that good type borrowers are not able to manipulate is without loss of generality.

Lemma A.1.1. *Given $\pi(\rho|s)$, in any equilibrium, $I_j^{s,\hat{B}}(\rho)=0$ for all j and $\rho\in supp(\pi(\rho|s))$, and no good type borrower chooses to manipulate, i.e., $\bar{c}^G=0$.*

Proof. Suppose the posterior belief is $\pi(\rho|s)$, and in equilibrium, all lenders choose contract $\left\{I_j^{s,\hat{\theta}}(\rho), D_j^{s,\hat{\theta}}(\rho)\right\}$. Then the fraction of borrowers with different types and features are summarized in the following table:

	$\hat{\theta}=\hat{G}$	$\hat{\theta}=\hat{B}$
$\theta=G$	$\mu\left(1-F_c\left(\bar{c}^G\right)\right)$	$\mu F_c\left(\bar{c}^G\right)$
$\theta=B$	$(1-\mu)F_c\left(\bar{c}^B\right)$	$(1-\mu)\left(1-F_c\left(\bar{c}^B\right)\right)$

.

Table A.1: Fraction of borrowers.

In equilibrium, lenders will lend to borrowers with feature $\hat{\theta}=\hat{G}$ if and only if $\rho\geq\varrho^G$, where ϱ^G is solved by

$$\mu\left(1-F_c\left(\bar{c}^G\right)\right)m\left(\varrho^G\right)-\left[\mu\left(1-F_c\left(\bar{c}^G\right)\right)+(1-\mu)F_c\left(\bar{c}^B\right)\right]I=0. \tag{A.1}$$

And for all $\rho\geq\varrho^G$, the equilibrium debt contract $D^{s,\hat{G}}(\rho)$ is solved by

$$E\left(\min\left\{\tilde{V},D^{s,\hat{G}}(\rho)\right\}|s,\rho,\theta=G\right)-\frac{\mu\left(1-F_c\left(\bar{c}^G\right)\right)+(1-\mu)F_c\left(\bar{c}^B\right)}{\mu\left(1-F_c\left(\bar{c}^G\right)\right)}I=0. \tag{A.2}$$

Similarly, lenders will lend to borrowers with feature $\hat{\theta}=\hat{B}$ if and only if $\rho\geq\varrho^B$, where ϱ^B is solved by

$$m\left(\varrho^B\right)-\frac{\mu F_c\left(\bar{c}^G\right)+(1-\mu)\left(1-F_c\left(\bar{c}^B\right)\right)}{\mu F_c\left(\bar{c}^G\right)}I=0. \tag{A.3}$$

And for all $\rho\geq\varrho^B$, the equilibrium debt contract $D^{s,\hat{B}}(\rho)$ is solved by

$$E\left(\min\left\{\tilde{V},D^{s,\hat{B}}(\rho)\right\}|s,\rho,\theta=G\right)-\frac{F_c\left(\bar{c}^G\right)+(1-\mu)\left(1-F_c\left(\bar{c}^B\right)\right)}{\mu F_c\left(\bar{c}^G\right)}I=0. \tag{A.4}$$

If $\varrho^G\geq\varrho^B$, from (A.1) and (A.3), we have

$$\frac{\mu\left(1-F_c\left(\bar{c}^G\right)\right)+(1-\mu)F_c\left(\bar{c}^B\right)}{\mu\left(1-F_c\left(\bar{c}^G\right)\right)}\geq\frac{\mu F_c\left(\bar{c}^G\right)+(1-\mu)\left(1-F_c\left(\bar{c}^G\right)\right)}{\mu F_c\left(\bar{c}^G\right)}.$$

However, in this case, for all the bad type borrowers, it's strictly profitable not to manipulate, which means in equilibrium we must have $\bar{c}^B = 0$, and thus

$$\frac{\mu\left(1 - F_c\left(\bar{c}^G\right)\right) + (1-\mu)\, F_c\left(\bar{c}^B\right)}{\mu\left(1 - F_c\left(\bar{c}^G\right)\right)} = 1 < \frac{\mu F_c\left(\bar{c}^B\right) + (1-\mu)}{\mu F_c\left(\bar{c}^B\right)},$$

a contradiction!

If $\underline{\rho}^G < \underline{\rho}^B$, by (A.1) and (A.3), we must have

$$\frac{\mu\left(1 - F_c\left(\bar{c}^G\right)\right) + (1-\mu)\, F_c\left(\bar{c}^B\right)}{\mu\left(1 - F_c\left(\bar{c}^G\right)\right)} < \frac{\mu F_c\left(\bar{c}^G\right) + (1-\mu)\left(1 - F_c\left(\bar{c}^G\right)\right)}{\mu F_c\left(\bar{c}^G\right)},$$

substitute this condition into (A.2) and (A.4), we can show that, for any $\rho \geq \underline{\rho}^B$ we must have

$$D^{s,\hat{B}}(\rho) > D^{s,\hat{G}}(\rho)\,.$$

Then for all good type borrowers, manipulating is strictly dominated by not manipulating, and thus in equilibrium we must have $\bar{c}^G = 0$.

So in this equilibrium we must have $\bar{c}^G = 0$, which is the same as our baseline model. Then allowing all borrowers to manipulate their features doesn't change the equilibrium for any posterior belief $\pi(\rho, s)$, and thus it doesn't change our results. □

A.2 Proofs

A.2.1 Proofs in Section 1.2

No Disclosure On The Relevance ρ

Based on the distributional assumptions on the manipulation cost c and the relevance, the two equilibrium conditions are

$$B\left(1-\varrho_N\right)=\bar{c}_N$$

and

$$\mu\varrho_N V=\left(\mu+(1-\mu)\bar{c}_N\right)I.$$

The unique solution is

$$\left(\varrho_N=\frac{I}{\mu V+(1-\mu)I},\bar{c}_N=B\cdot\frac{\mu(V-I)}{\mu V+(1-\mu)I}\right).$$

Full Transparency

In this case, when $\rho\geq\frac{I}{V}$, the surplus from lending to $\hat{G}$ borrowers must be zero. To see this, if lenders lend to $\hat{G}$ borrowers for sure, since $B\geq 1$, all of the bad type borrowers must choose to manipulate their features. In this case, it's not profitable to lend to $\hat{G}$ borrowers for any $\rho<1$ because $\mu V\leq I$, a contradiction!

A.2.2 Proof of Proposition 1.4.1

The no disclosure policy is implemented by a signal $(\mathcal{S},\tilde{\sigma})$ with only one element in the signal space $\mathcal{S}=\{s_N\}$, and the mapping $\tilde{\sigma}\left(s|\rho\right)$ is trivial. The lending market equilibrium is characterized by $(k_N,\varrho_N,\bar{c}_N)$ which satisfy conditions in Definition 1.3.3 under the prior belief of ρ. Let the regulator's payoff be W_N in the no disclosure case. Now let's consider the following deterministic disclosure policy $(\mathcal{S}',\sigma')$, where $\mathcal{S}'=\{s_1',s_2'\}$, and

$$\sigma'\left(\rho\right)=\begin{cases}s_1' & \rho\in[0,\rho_1']\cup[\varrho_N,\varrho_N+\epsilon_1]\cup[1-\epsilon_2,1]\\ s_2' & \rho\in(\rho_1',\varrho_N)\cup(\varrho_N+\epsilon_1,1-\epsilon_2)\end{cases}$$

where $\rho_1' < \varrho_N$ satisfies

$$\frac{\text{Prob}\left(\rho \in [\varrho_N, \varrho_N + \epsilon_1] \cup [1-\epsilon_2, 1]\right)}{\text{Prob}\left(\rho \in [0, \rho_1'] \cup [\varrho_N, \varrho_N + \epsilon_1] \cup [1-\epsilon_2, 1]\right)} = \bar{c}_N.$$

Denote the equilibria under signals s_1' and s_2' as $\left(k_{s_1'}, \varrho_{s_1'}, \bar{c}_{s_1'}\right)$ and $\left(k_{s_2'}, \varrho_{s_2'}, \bar{c}_{s_2'}\right)$, respectively, it's easy to verify

$$\left(k_{s_1'}, \varrho_{s_1'}, \bar{c}_{s_1'}\right) = \left(k_{s_2'}, \varrho_{s_2'}, \bar{c}_{s_2'}\right) = \left(k_N, \varrho_N, \bar{c}_N\right).$$

Then introducing the policy $(\mathcal{S}', \sigma')$ doesn't change the regulator's payoff, i.e., $W_N = W'$, where the regulator's payoff under disclosure policy $(\mathcal{S}', \sigma')$ can be written as

$$\begin{aligned} W' =& \text{Prob}\left(\rho \in \left[0, \rho_1'\right] \cup [\varrho_N, \varrho_N + \epsilon_1] \cup [1-\epsilon_2, 1]\right) \cdot E^{s_1'}\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_{s_1'}\right)\right) I\right) \cdot 1_{\left\{\rho \geq \rho_{s_1'}\right\}}\right] \\ &+ \text{Prob}\left(\rho \in \left(\rho_1', \varrho_N\right) \cup (\varrho_N + \epsilon_1, 1-\epsilon_2)\right) \cdot E^{s_2'}\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_{s_2'}\right)\right) I\right) \cdot 1_{\left\{\rho \geq \rho_{s_2'}\right\}}\right] \end{aligned} \tag{A.5}$$

Then, let's construct a new disclosure policy based on $(\mathcal{S}', \sigma')$, and show that the new disclosure policy increases regulator's payoff. Let's consider the deterministic disclosure policy $(\mathcal{S}'', \sigma'')$, with $S'' = \{s_1'', s_2'', s_3''\}$, and

$$\sigma'' = \begin{cases} s_1'' & \rho \in [\varrho_N, \varrho_N + \epsilon_1] \\ s_2'' & \rho \in [0, \rho_1'] \cup [1-\epsilon_2, 1] \\ s_3'' & \rho \in (\rho_1', \varrho_N) \cup (\varrho_N + \epsilon_1, 1-\epsilon_2) \end{cases}.$$

The signal realization s_3'' is "equivalent" to the signal realization s_2' in disclosure policy $(\mathcal{S}', \sigma')$, both induce the same posterior belief in $(\rho_1', \varrho_N) \cup (\varrho_N + \epsilon_1, 1-\epsilon_2)$. The difference is that policy $(\mathcal{S}'', \sigma'')$ further reveals if the true state is in $[\varrho_N, \varrho_N + \epsilon_1]$ or not. Note that the regulator's payoff in state $[\varrho_N, \varrho_N + \epsilon_1]$ is close to zero in the no disclosure case, as ϱ_N is the equilibrium cutoff in lending decisions. So revealing this information only changes the regulator's payoff marginally in states $\rho \in [\varrho_N, \varrho_N + \epsilon_1]$. However, the increase in regulator's payoff is non-trivial. Note that the approval probability is lower under s_2'' compared to the no disclosure case, so the equilibrium data manipulation level is lower under s_2''. As what we will show later, this is the dominating effect thus the regulator's payoff increases under the disclosure policy $(\mathcal{S}'', \sigma'')$. To see this, note that the regulator's payoff under $(\mathcal{S}'', \sigma'')$ is

$$\begin{aligned} W'' =& \text{Prob}\left(\rho \in [\varrho_N, \varrho_N + \epsilon_1]\right) \cdot E^{s_1''}\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_{s_1''}\right)\right) I\right) \cdot 1_{\left\{\rho \geq \rho_{s_1''}\right\}}\right] \\ &+ \text{Prob}\left(\rho \in \left[0, \rho_1'\right] \cup [1-\epsilon_2, 1]\right) \cdot E^{s_2''}\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_{s_2''}\right)\right) I\right) \cdot 1_{\left\{\rho \geq \rho_{s_2''}\right\}}\right] \\ &+ \text{Prob}\left(\rho \in \left(\rho_1', \varrho_N\right) \cup (\varrho_N + \epsilon_1, 1-\epsilon_2)\right) \cdot E^{s_3''}\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_{s_3''}\right)\right) I\right) \cdot 1_{\left\{\rho \geq \rho_{s_3''}\right\}}\right]. \end{aligned} \tag{A.6}$$

It's obvious that the last term in (A.6) is equal to the last term in (A.5), because equilibria under signal realizations s_3'' and s_2' are the same. Then

$$\begin{aligned} & W'' - W' \\ =& \text{Prob}\left(\rho \in [\varrho_N, \varrho_N + \epsilon_1]\right) \cdot E^{s_1''}\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_{s_1''}\right)\right) I\right) \cdot 1_{\left\{\rho \geq \rho_{s_1''}\right\}}\right] \\ &+ \text{Prob}\left(\rho \in \left[0, \rho_1'\right] \cup [1-\epsilon_2, 1]\right) \cdot E^{s_2''}\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_{s_2''}\right)\right) I\right) \cdot 1_{\left\{\rho \geq \rho_{s_2''}\right\}}\right] \\ &- \text{Prob}\left(\rho \in \left[0, \rho_1'\right] \cup [\varrho_N, \varrho_N + \epsilon_1] \cup [1-\epsilon_2, 1]\right) \cdot E^{s_1'}\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_{s_1'}\right)\right) I\right) \cdot 1_{\left\{\rho \geq \rho_{s_1'}\right\}}\right] \\ \geq& \text{Prob}\left(\rho \in \left[0, \rho_1'\right] \cup [1-\epsilon_2, 1]\right) \cdot E^{s_2''}\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_{s_2''}\right)\right) I\right) \cdot 1_{\left\{\rho \geq \rho_{s_2''}\right\}}\right] \\ &- \text{Prob}\left(\rho \in \left[0, \rho_1'\right] \cup [\varrho_N, \varrho_N + \epsilon_1] \cup [1-\epsilon_2, 1]\right) \cdot E^{s_1'}\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_{s_1'}\right)\right) I\right) \cdot 1_{\left\{\rho \geq \rho_{s_1'}\right\}}\right]. \end{aligned}$$

Note that $\rho_{s_1'} = \varrho_N$, we know

$$\begin{aligned} & \text{Prob}\left(\rho \in \left[0, \rho_1'\right] \cup [\varrho_N, \varrho_N + \epsilon_1] \cup [1-\epsilon_2, 1]\right) \cdot E^{s_1'}\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_{s_1'}\right)\right) I\right) \cdot 1_{\left\{\rho \geq \rho_{s_1'}\right\}}\right] \\ =& \begin{array}{c} \text{Prob}\left(\rho \in [\varrho_N, \varrho_N + \epsilon_1]\right) \cdot E\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_N\right)\right) I\right) | \rho \in [\varrho_N, \varrho_N + \epsilon_1]\right] \\ +\text{Prob}\left(\rho \in [1-\epsilon_2, 1]\right) \cdot E\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_N\right)\right) I\right) | \rho \in [1-\epsilon_2, 1]\right]. \end{array} \end{aligned}$$

Then

$$\begin{aligned} W'' - W' =& \text{Prob}\left(\rho \in [1-\epsilon_2, 1]\right) \cdot \left[(1-\mu) I\left(F_c\left(\bar{c}_N\right) - F_c\left(\bar{c}_{s_2''}\right)\right)\right] \\ &- \text{Prob}\left(\rho \in [\varrho_N, \varrho_N + \epsilon_1]\right) \cdot E\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_N\right)\right) I\right) | \rho \in [\varrho_N, \varrho_N + \epsilon_1]\right]. \end{aligned}$$

In the equilibrium of subgame s_1'',

$$\begin{aligned} \bar{c}_{s_2''} &= \frac{\text{Prob}\left(\rho \in [1-\epsilon_2, 1]\right)}{\text{Prob}\left(\rho \in \left[0, \rho_1'\right] \cup [1-\epsilon_2, 1]\right)} \\ &= \frac{\bar{c}_N - x}{1-x} \end{aligned}$$

where

$$x = \frac{\text{Prob}\left(\rho \in [\varrho_N, \varrho_N + \epsilon_1]\right) \cdot (1 - \bar{c}_N)}{\text{Prob}\left(\rho \in [0, \rho_1']\right)}.$$

Consider the case when fixing ϵ_2, and let $\epsilon_1 \to 0$, then $x \to 0$ and $\bar{c}_{s_2''} = \bar{c}_N - x\,(1 - \bar{c}_N) + o(x)$.

$$\begin{aligned}
& W'' - W' \\
=& \text{Prob}\left(\rho \in [\varrho_N, \varrho_N + \epsilon_1]\right) \cdot \left[\begin{array}{c} \text{Prob}\left(\rho \in [1-\epsilon_2, 1]\right) \cdot \frac{(1-\mu) I \left(F_c(\bar{c}_N) - F_c\left(\bar{c}_{s_2''}\right)\right)}{\text{Prob}(\rho \in [\varrho_N, \varrho_N + \epsilon_1])} - \\ E\left[\left(\mu \tilde{V} - (\mu + (1-\mu) F_c(\bar{c}_N)) I\right) | \rho \in [\varrho_N, \varrho_N + \epsilon_1]\right] \end{array} \right] \\
\approx& \text{Prob}\left(\rho \in [\varrho_N, \varrho_N + \epsilon_1]\right) \cdot \left[\begin{array}{c} \frac{\text{Prob}(\rho \in [1-\epsilon_2, 1])(1-\bar{c}_N)}{\text{Prob}\left(\rho \in \left[0, \rho_1'\right]\right)} \cdot \frac{(1-\mu) I (F_c(\bar{c}_N) - F_c(\bar{c}_N - x(1-\bar{c}_N)))}{x} - \\ E\left[\left(\mu \tilde{V} - (\mu + (1-\mu) F_c(\bar{c}_N)) I\right) | \rho \in [\varrho_N, \varrho_N + \epsilon_1]\right] \end{array} \right] \\
\approx& \text{Prob}\left(\rho \in [\varrho_N, \varrho_N + \epsilon_1]\right) \cdot \left[\begin{array}{c} \frac{\text{Prob}(\rho \in [1-\epsilon_2, 1])(1-\bar{c}_N)^2 (1-\mu) I}{\text{Prob}\left(\rho \in \left[0, \rho_1'\right]\right)} \cdot F_c'(\bar{c}_N) - \\ E\left[\left(\mu \tilde{V} - (\mu + (1-\mu) F_c(\bar{c}_N)) I\right) | \rho \in [\varrho_N, \varrho_N + \epsilon_1]\right] \end{array} \right].
\end{aligned}$$

Since $\epsilon_1 \to 0$, we must have

$$E\left[\left(\mu \tilde{V} - (\mu + (1-\mu) F_c(\bar{c}_N)) I\right) | \rho \in [\varrho_N, \varrho_N + \epsilon_1]\right] \to 0,$$

because the equilibrium condition in the no disclosure case is

$$E\left[\left(\mu \tilde{V} - (\mu + (1-\mu) F_c(\bar{c}_N)) I\right) | \rho = \varrho_N\right] = 0.$$

With $F_c'(\bar{c}_N) > 0$, then we must have

$$W'' - W' > 0$$

which means that the no disclosure policy is dominated by our new disclosure policy $(\mathcal{S}'', \sigma'')$.

A.2.3 Proof of Lemma 1.5.1

The full disclosure policy $(\mathcal{S}, \sigma)$ can be implemented by space $\mathcal{S} = [0, 1]$ and a deterministic message function $\sigma(\rho) = \rho$. In this case, the true state ρ is perfectly revealed to the public. Denote ϱ as the solution of

$$m(\varrho) = I,$$

For any $s = \rho > m^{-1}(I)$, the lending market equilibrium of subgame s, $(k_s, \varrho_s, \bar{c}_s)$, must satisfy $\varrho_s = \rho$, and thus

$$\mu m(\rho) - k_s = 0. \tag{A.7}$$

To see this, suppose $\mu(\rho) > k_s$, in equilibrium all $\hat{G}$ borrowers must be approved, and all bad type borrowers must choose to manipulate because of Assumption 1.3.2. Then the regulator's payoff of financing all $\hat{G}$ borrowers is

$$\mu m(\rho) - I \leq \mu m(1) - I \leq 0,$$

and the equality holds only when $\rho = 1$. As as result, lenders will not lend to $\hat{G}$ borrowers for all $\rho < 1$, a contradiction. So in equilibrium condition (A.7) must hold. And this condition implies that the regulator's payoff is zero.

Next, it's obvious that when $\rho \leq \varrho$, lender will never lend to any borrowers. In summary, regulator's payoff is zero for any $s \in \mathcal{S}$ thus the regulator's total payoff is $W_F = 0$ under full disclosure policy.

A.2.4 Proof of Lemma 1.5.2

The results are directly derived from the definition of lending market equilibrium. For any two equlibria $\left(k_{s_1}, \varrho_{s_1}, \bar{c}_{s_1}\right)$ and $\left(k_{s_2}, \varrho_{s_2}, \bar{c}_{s_2}\right)$, the first condition in Definition 1.3.3

$$\mu m(\varrho_s) = k_s$$

implies

$$k_{s_1} \geq k_{s_2} \Longleftrightarrow \varrho_{s_1} \geq \varrho_{s_2}, \tag{A.8}$$

because $m(\cdot)$ is an increasing function. The third condition

$$k_s = [\mu + (1-\mu) F_c(\bar{c}_s)] I$$

implies that

$$k_{s_1} \geq k_{s_2} \Longleftrightarrow \bar{c}_{s_1} \geq \bar{c}_{s_2}. \tag{A.9}$$

Then (A.8) and (A.9) complete the proof.

A.2.5 Proof of Lemma 1.5.3

We just need to verify that the regulator's payoff is unchanged under the new disclosure policy $(\mathcal{S}', \tilde{\sigma}')$. Notice that

$$\tilde{\sigma}'(s|\rho) = \tilde{\sigma}(s|\rho)$$

for any $\rho \in [0,1]$ and $s \in \mathcal{S} \setminus \{s_1, s_2\} = \mathcal{S}' \setminus \{s_0'\}$. Then for any $s \in \mathcal{S} \setminus \{s_1, s_2\} = \mathcal{S}' \setminus \{s_0'\}$, the posterior beliefs are the same under the two policies, i.e., for any $s \in \mathcal{S} \setminus \{s_1, s_2\} = \mathcal{S}' \setminus \{s_0'\}$, we have

$$\pi(\rho|s) = \pi'(\rho|s).$$

So the lending market equilibira are the same for any $s \in \mathcal{S} \setminus \{s_1, s_2\} = \mathcal{S}' \setminus \{s_0'\}$ in these two policies. Besides, the regulator's payoff from signal realization s_0' in the new disclosure policy is the sum of that under signal realizations s_1 and s_2 in policy $(\mathcal{S}, \sigma)$, this is because the lending market equilibrium under s_0, s_1 and s_2 are all the same, and the probability of observing s_0' in the new policy is the sum of the probabilities of observing s_1 and s_2 in policy $(\mathcal{S}, \tilde{\sigma})$. Since policy $(\mathcal{S}, \tilde{\sigma})$ is optimal, the new policy $(S', \tilde{\sigma}')$ must also be optimal.

A.2.6 Proof of Lemma 1.5.4

Given any policy $(\mathcal{S}, \tilde{\sigma})$ with distribution of posteriors $\{f(s), \pi(\rho|s)\}_{s \in \mathcal{S}}$, for any subgame s,

$$\bar{c}_s = 0 \Longleftrightarrow \underline{\rho}_s = m^{-1}(I) \Longleftrightarrow k_s = I.$$

In this equilibrium, there is no manipulation, and lenders always reject all loan applications. The posterior belief must satisfy

$$\sup\{(\pi(\rho|s))\} \leq m^{-1}(I).$$

Since $m(1) > I$, there must exist at least one signal realization s_1, such that

$$\bar{c}_{s_1} > 0.$$

Suppose there also exists another signal realization s_2, such that

$$\bar{c}_{s_2} = 0.$$

Here assume both the probabilities of s_1 and s_2 are positive[1], then let's consider a new policy $(\mathcal{S}', \sigma')$ with distribution of posteriors $\{f'(s), \pi'(\rho|s)\}_{s\in\mathcal{S}'}$, signal space $\mathcal{S}' = \{s_0'\} \cup \mathcal{S}\backslash\{s_1, s_2\}$, and

$$\tilde{\sigma}'(s|\rho) = \tilde{\sigma}(s|\rho)\, 1_{\mathcal{S}\backslash\{s_1,s_2\}}(s) + (\tilde{\sigma}(s_1|\rho) + \tilde{\sigma}(s_2|\rho))\, 1_{\{s_0'\}}(s).$$

Obviously, any signal realization $s \in \mathcal{S}\backslash\{s_1, s_2\}$ must exist in the signal spaces of both disclosure polices, and induce the same lending market equilibrium. Besides, for signal realization s_0' in $(\mathcal{S}', \sigma')$ and $\{s_1.s_2\}$ in $(\mathcal{S}, \sigma)$, we have

$$f'(s_0) = f(s_1) + f(s_2),$$

and

$$\pi'\left(\rho|s_0'\right) = \frac{1}{f(s_1) + f(s_2)}\left(f(s_1)\,\pi(\rho|s_1) + f(s_2)\,\pi(\rho|s_2)\right).$$

The equilibrium conditions in Definition 1.3.3 implies that $\bar{c}_s$ satisfies

$$\text{Prob}\left(\rho \geq m^{-1}\left(\frac{[\mu + (1-\mu)\,F_c(\bar{c}_s)]\,I}{\mu}\right)|s\right) \geq \frac{\bar{c}_s}{B} \geq \text{Prob}\left(\rho > m^{-1}\left(\frac{[\mu + (1-\mu)\,F_c(\bar{c}_s)]\,I}{\mu}\right)|s\right).$$

Note that

$$\Pi\left(\rho|s_0'\right) > \Pi(\rho|s_1)$$

for any $\rho > m^{-1}(I)$, and both $m^{-1}(\cdot)$ and $F_c(\cdot)$ are increasing functions, we conclude that

$$\bar{c}_{s_0'} < \bar{c}_{s_1}.$$

Since the regulator's payoff is always zero for any $\rho \leq m^{-1}(I)$, the difference of regula-

[1]The proof for the case when the signal is continuous is similar, in that case, we just need to deal with density functions.

tor's payoffs under $(\mathcal{S}', \sigma')$ and $(\mathcal{S}, \sigma)$ is

$$
\begin{aligned}
W' - W = \; & \begin{array}{c} f\left(s_0'\right) E^{s_0'}\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_{s_0'}\right)\right) I\right) 1_{\left(\varrho_{s_0'},1\right]\cap\left(\pi\left(\rho|s_0'\right)\right)}(s)\right] - \\ f\left(s_1\right) E^{s_1}\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_{s_1}\right)\right) I\right) 1_{\left(\varrho_{s_1},1\right]\cap\left(\pi\left(\rho|s_1\right)\right)}(s)\right] \end{array} \\
= \; & \begin{array}{c} f\left(s_1\right) E^{s_1}\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_{s_0'}\right)\right) I\right) 1_{\left(\varrho_{s_0'},1\right]\cap\left(\pi\left(\rho|s_0'\right)\right)}(s)\right] - \\ f\left(s_1\right) E^{s_1}\left[\left(\mu\tilde{V} - \left(\mu + (1-\mu) F_c\left(\bar{c}_{s_1}\right)\right) I\right) 1_{\left(\varrho_{s_1},1\right]\cap\left(\pi\left(\rho|s_1\right)\right)}(s)\right] \end{array} .
\end{aligned}
$$

Since $\bar{c}_{s_0} < \bar{c}_{s_1}$ and $m^{-1}(I) < \varrho_{s_0'} < \varrho_{s_1}$, we have

$$\left(\varrho_{s_0'}, 1\right] \cap \left(\pi\left(\rho|s_0'\right)\right) = \left(\varrho_{s_1}, 1\right] \cap \left(\pi\left(\rho|s_1\right)\right),$$

and thus

$$W' - W > 0.$$

A.2.7 Proof of Lemma 1.5.5

Suppose there exists an optimal disclosure policy $(\mathcal{S}, \tilde{\sigma})$ and it induces the distribution of posteriors $\{f(s), \pi(\rho|s)\}_{s\in\mathcal{S}}$. If $\mathcal{S}$ is a singleton, then the policy is simply the no information policy. In this case, let $\varrho^{\star} = \varrho_N$ and this lemma is obviously true. Otherwise, if the statement is not true, there must exist $s_1, s_2 \in \mathcal{S}$, with lending market equilibria $\left(k_{s_1}, \varrho_{s_1}, \bar{c}_{s_1}\right)$ and $\left(k_{s_2}, \varrho_{s_2}, \bar{c}_{s_2}\right)$, such that

$$\rho_1 > \rho_2,$$

$$\rho_1 \in \left[0, \varrho_{s_1}\right] \cap \left(\pi\left(\rho|s_1\right)\right)$$

and

$$\rho_2 \in \left(\varrho_{s_2}, 1\right] \cap \left(\pi\left(\rho|s_2\right)\right).$$

Then there must exist intervals B_1, B_2[2], such that

$$\rho_1 \in B_1 \subset \left[0, \varrho_{s_1}\right] \cap \left(\pi\left(\rho|s_1\right)\right),$$

$$\rho_2 \in B_2 \subset \left(\varrho_{s_2}, 1\right] \cap \left(\pi\left(\rho|s_2\right)\right),$$

[2]Note that a single point is also a closed interval.

$$\inf B_1 > \sup B_2,$$

$$\text{Prob}\left(B_1|s_1\right) = K_1 > 0,$$

and

$$\text{Prob}\left(B_2|s_2\right) = K_2 > 0.$$

For this proof, let's assume both s_1 and s_2 occur with positive probability, then $f(s_1)$ and $f(s_s)$ represent the associated probabilities. The other case is when either s_1 or s_2 occur with zero probability and $f(s_1)$ or $f(s_2)$ represent the density functions. The proof strategy is basically the same.

In this case, if $f(s_1) K_1 \geq f(s_2) K_2$, then let's consider the following distribution of posteriors: $\left\{\hat{f}(s), \hat{\pi}(\rho|s)\right\}_{s\in\tilde{\mathcal{S}}}$, where $\hat{\mathcal{S}} = \mathcal{S}$, $\hat{f}(s) = f(s)$, and

$$\hat{\pi}(\rho|s) = \begin{cases} \pi\{\rho|s_1\} + \frac{f(s_2)}{f(s_1)}\pi\{\rho|s_2\} 1_{B_2}(\rho) - \frac{f(s_2)K_2}{f(s_1)K_1}\pi\{\rho|s_1\} 1_{B_1} & \text{if } s = s_1 \\ \pi\{\rho|s_2\} - \pi\{\rho|s_2\} 1_{B_2} + \frac{K_2}{K_1}\pi\{\rho|s_1\} 1_{B_1} & \text{if } s = s_2 \\ \pi(\rho|s) & o.w. \end{cases}$$

We can check that $\left\{\hat{f}(s), \hat{\pi}(\rho|s)\right\}_{s\in\tilde{\mathcal{S}}}$ is Bayes-plausible, and there exists a disclosure policy $\left(\hat{S}, \hat{\sigma}\right)$ that induces this distribution of posteriors. But now in the new policy $\left(\hat{S}, \hat{\sigma}\right)$, $\rho_2 \notin (\pi(\rho|s_2))$. And the regulator's payoff is weakly increasing under the new policy $\left(\hat{\mathcal{S}}, \hat{\sigma}\right)$ because

1. $\hat{f}(s) = f(s)$for all $s \in \hat{\mathcal{S}} = \mathcal{S}$;

2. the lending market equilibria $(k_s, \varrho_s, \bar{c}_s)$ are the same under the two policies for any $s \in \mathcal{S} = \hat{\mathcal{S}}$;

3. the regulator's payoff under any signal realizations except for s_2 is unchanged;

4. the regulator's payoff under signal realization s_2 increases.

The last point holds because with the new disclosure policy $\left(\hat{S}, \hat{\sigma}\right)$, under the signal realization s_2, the equilibrium variables $\left(k_{s_2}, \varrho_{s_2}, \bar{c}_{s_2}\right)$ is the same compared to that with policy

$(\mathcal{S}, \sigma)$, so the total financing cost is unchanged, which is k_s. But the total payoff generated from projects increases by

$$f(s_2) \cdot K_2 \cdot [E[\mu m(\rho)|s_1, B_1] - E[\mu m(\rho)|s_2, B_2]]$$

which is positive because $\inf B_1 > \sup B_2$.

A.2.8 Proof of Lemma 1.5.6

For the optimal disclosure policy $(\mathcal{S}, \tilde{\sigma})$, if $\mathcal{S}$ is a singleton, this lemma is obviously true. Otherwise, there exist two different signals s_1 and s_2 with probabilities(densities) $\tilde{f}(s_1)$ and $\tilde{f}(s_2)$, respectively. For simplicity, let's assume that both $\tilde{f}(s_1)$ and $\tilde{f}(s_2)$ are positive, the proof for other cases are basically the same. Denote the lending market equilibrium variables as $\left(\tilde{k}_{s_1}, \tilde{\varrho}_{s_1}, \tilde{\bar{c}}_{s_1}\right)$ and $\left(\tilde{k}_{s_2}, \tilde{\varrho}_{s_2}, \tilde{\bar{c}}_{s_2}\right)$ under these two signals. Without loss of generality, let's assume $\tilde{\varrho}_{s_1} < \tilde{\varrho}_{s_2}$. Denote the ex ante lending cutoff as $\varrho^\star$ in this case. Suppose for s_1, s_2, the condition

$$\sup\{(\tilde{\pi}(\rho|s_1)) \cap (\varrho^\star, 1]\} \leq \inf\{(\tilde{\pi}(\rho|s_2)) \cap (\varrho^\star, 1]\} \tag{A.10}$$

is not satisfied, let

$$B = [\inf\{(\tilde{\pi}(\rho|s_2)) \cap (\varrho^\star, 1]\}, \sup\{(\tilde{\pi}(\rho|s_1)) \cap (\varrho^\star, 1]\}].$$

Then there must exist two non-negative functions v_1, v_2, such that

$$\tilde{f}(s_1) v_1(\rho) + \tilde{f}(s_2) v_2(\rho) = \tilde{f}(s_1) \tilde{\pi}\{\rho|s_1\} \cdot 1_B(\rho) + \tilde{f}(s_2) \tilde{\pi}\{\rho|s_2\} \cdot 1_B(\rho), \tag{A.11}$$

$$\sup\{(v_1(\rho)) \cap (\varrho^\star, 1]\} \leq \inf\{(v_2(\rho)) \cap (\varrho^\star, 1]\}$$

and

$$\int v_1(\rho) d\rho = \int \pi\{\rho|s_1\} \cdot 1_B(\rho) d\rho. \tag{A.12}$$

Now let's consider the following distribution of posterior beliefs with signal space $\mathcal{S}$:

$\left\{\hat{f}(s), \hat{\pi}(\rho|s)\right\}_{s\in\mathcal{S}}$, where $\hat{f}(s) = f(s)$ and

$$\hat{\pi}(\rho|\tilde{s}) = \begin{cases} \tilde{\pi}\{\rho|s_1\} - \tilde{\pi}\{\rho|s_1\} \cdot 1_B(\rho) + v_1(\rho) & \text{if } \tilde{s} = s_1 \\ \tilde{\pi}\{\rho|s_2\} - \tilde{\pi}\{\rho|s_2\} 1_B(\rho) + v_2(\rho) & \text{if } \tilde{s} = s_2 \\ \tilde{\pi}(\rho|s) & o.w. \end{cases}.$$

We can check that the new distribution of posteriors $\left\{\hat{f}(s), \hat{\pi}(\rho|s)\right\}_{s\in\mathcal{S}}$ is still Bayes-plausible, because

$$\int v_1(\rho)\, d\rho = \int \tilde{\pi}\{\rho|s_1\} \cdot 1_B(\rho)\, d\rho$$

and

$$\int v_2(\rho)\, d\rho = \int \tilde{\pi}\{\rho|s_2\} \cdot 1_B(\rho)\, d\rho.$$

The second condition is a direct result of (A.11) and (A.12). And we can check that $\left\{\hat{f}(s), \hat{\pi}(\rho|s)\right\}_{s\in\mathcal{S}}$ can be induced by a disclosure policy $(\mathcal{S}, \hat{\sigma})$. Now the condition (A.10) is not violated anymore in the new policy. Then we just need to show that the regulator's payoff is unchanged under the new policy, and thus it is still optimal. To see this, with policy $(\mathcal{S}, \tilde{\sigma})$, we know under posterior belief $\tilde{\pi}(\rho|s_1)$

$$\text{Prob}^{(\mathcal{S},\tilde{\sigma})}\left(\rho > \underline{\rho}_{s_1}|s_1\right) = \frac{\bar{c}_{s_1}}{B}.$$

Note that since $\underline{\rho}_{s_1} < \underline{\rho}_{s_2}$, we know

$$\inf\{(\pi(\rho|s_2)) \cap (\underline{\rho}^\star, 1]\} \geq \underline{\rho}_{s_2} > \underline{\rho}_{s_1}.$$

Then for any $\rho \in B$, we must have $\rho > \underline{\rho}_{s_1}$. Then under posterior belief $\hat{\pi}(\rho|s_1)$, we know

$$\begin{aligned} \text{Prob}^{(\mathcal{S},\hat{\sigma})}\left(\rho > \underline{\rho}_{s_1}|s_1\right) &= \text{Prob}^{(\mathcal{S},\tilde{\sigma})}\left(\rho > \underline{\rho}_{s_1}|s_1\right) - \int \tilde{\pi}\{\rho|s_1\} \cdot 1_B(\rho)\, d\rho + \int v_1(\rho)\, d\rho \\ &= \text{Prob}^{(\mathcal{S},\tilde{\sigma})}\left(\rho > \underline{\rho}_{s_1}|s_1\right) \\ &= \frac{\bar{c}_{s_1}}{B}. \end{aligned}$$

The second equality comes from condition (A.12). Based on this, we can check all other equilibrium conditions are also satisfied, and this implies $\left(\hat{\phi}_{s_1}, \hat{\rho}_{s_1}, \hat{\Delta}_{s_1}\right) = (\phi_{s_1}, \rho_{s_1}, \Delta_{s_1})$.

Similarly, we can check $\left(\tilde{k}_s, \tilde{\varrho}_s, \tilde{\bar{c}}_s\right) = (k_s, \varrho_s, \bar{c}_s)$. For all other $s \in \mathcal{S} \backslash \{s_1, s_2\}$, it's obvious that the lending market equilibria are all the same under these two disclosure policies. Then we can easily show that the regulator's payoff is the same under those two policies.

The proof strategy still works if condition (A.12) is not satisfied in the optimal policy $(\mathcal{S}, \sigma)$. Besides, note that the third property in Lemma 1.5.6 implies the second property in Lemma 1.5.6, and these two jointly imply that the disclosure policy must be deterministic. Since all the posterior lending market equilibira are the same, the ex ante lending cutoff $\varrho^\star$ must be unchanged.

A.2.9 Proof of Theorem 1.5.1

Lemma 1.5.6 shows that for any optimal policy, there exists a deterministic optimal policy $(\mathcal{S}, \sigma)$ that induces almost equivalent lending market equilibria. Our Criterion 1.5.1 implies that for any two distinct signal realizations $s_1, s_2 \in \mathcal{S}$, we must have

$$\bar{c}_{s_1} \neq \bar{c}_{s_2}.$$

Then we consider a new signal space $\mathcal{S}' = [\bar{c}_{\min}, \bar{c}_{\max}]$, where

$$\bar{c}_{\min} = \inf_{s \in \mathcal{S}} \{\bar{c}_s\}$$

and

$$\bar{c}_{\max} = \sup_{s \in \mathcal{S}} \{\bar{c}_s\},$$

and a message function

$$\sigma'(\rho) = \bar{c}_{\sigma(\rho)}.$$

Then obviously $(\mathcal{S}', \sigma')$ is also a deterministic optimal policy, with the same lending market equilibira as $(\mathcal{S}, \sigma)$. And the cutoff $\varrho^\star$ will be the same under these two optimal policies. For any two signals $s_1', s_2' \in \mathcal{S}'$, and $s_1' < s_2'$ where both $\sigma'^{-1}(s_1')$ and $\sigma'^{-1}(s_2')$ are nonempty. Based on the construction of the new policy, we must have

$$\sup\left\{\sigma^{-1}\left(s_1'\right) \cap [0, \varrho^\star]\right\} \leq \inf\left\{\sigma^{-1}\left(s_2'\right) \cap [0, \varrho^\star]\right\}$$

and

$$\sup\left\{\tilde{\sigma}^{-1}\left(s_1'\right)\cap\left(\varrho^{\star},1\right]\right\}\leq\inf\left\{\tilde{\sigma}^{-1}\left(s_2'\right)\cap\left(\varrho^{\star},1\right]\right\}.$$

This means that for any $\rho\in[0,\varrho^{\star}]$ or $\rho\in(\varrho^{\star},1]$, $\sigma'(\rho)$ is a weakly increasing function, with $\inf\sigma'(\rho)=\bar{c}_{\min}$ and $\sup\sigma'(\rho)=\bar{c}_{\max}$.

A.2.10 Proof of Proposition 1.5.1

For any optimal policy, Lemma 1.5.6 shows that there exists another optimal policy that has the same ex ante lending cutoff $\varrho^{\star}$, the same lending market equilibria, and satisfies conditions (1.18) and (1.19). So without loss of generality, we just need to focus on optimal policies that satisfy conditions (1.18) and (1.19). Let's introduce the following lemmas to establish our results.

Lemma A.2.1. *For any two posterior beliefs $\pi(\rho|s_1)$ and $\pi(\rho|s_2)$, with positive probabilities (densities) $f(s_1)$ and $f(s_2)$, and lending market equilibira $\left(k_{s_1},\varrho_{s_1},\bar{c}_{s_1}\right)$ and $\left(k_{s_2},\varrho_{s_2},\bar{c}_{s_2}\right)$ satisfying $\varrho_{s_1}<\varrho_{s_2}$. Let $\hat{s}$ be the "combined"signal with posterior belief*

$$\pi(\rho|\hat{s})=\frac{f(s_1)\,\pi(\rho|s_1)+f(s_2)\,\pi(\rho|s_2)}{f(s_1)+f(s_2)},$$

Then the lending market equilibrium $(k_{\hat{s}},\varrho_{\hat{s}},\bar{c}_{\hat{s}})$ satisfies

$$k_{s_1}<k_{\hat{s}}<k_{s_2},$$

$$\varrho_{s_1}<\varrho_{\hat{s}}<\varrho_{s_2}$$

and

$$\bar{c}_{s_1}<\bar{c}_{\hat{s}}<\bar{c}_{s_2}.$$

Proof. First, it's impossible to have $\varrho_{\hat{s}}\leq\varrho_{s_1}$. Note that for the equilibria under s_1 and s_2, the equilibrium conditions are[3]

$$\mu m\left(\varrho_{s_1}\right)=\left[\mu+(1-\mu)\,F_c\left(\bar{c}_{s_1}\right)\right]I$$

[3] This is solved by the equilibrium conditions in Definition 1.3.3.

and

$$\mu m\left(\varrho_{s_2}\right)=\left[\mu+(1-\mu)\,F_c\left(\bar{c}_{s_2}\right)\right]I.$$

For $\hat{s}$, we have

$$\mu m\left(\varrho_{\hat{s}}\right)=\left[\mu+(1-\mu)\,F_c\left(\bar{c}_{\hat{s}}\right)\right]I.$$

If $\varrho_{\hat{s}}\leq\varrho_{s_1}$, Lemma 1.5.2 implies

$$\bar{c}_{\hat{s}}\leq\bar{c}_{s_1}<\bar{c}_{s_2}.$$

Following the equilibrium conditions in Definition 1.3.3, we have

$$\begin{aligned}\frac{\bar{c}_{\hat{s}}}{B}&=E\left(I^{\hat{s}}|\hat{s}\right)\\&=\frac{f\left(s_1\right)}{f\left(s_1\right)+f\left(s_2\right)}E\left(I^{\hat{s}}|s_1\right)+\frac{f\left(s_2\right)}{f\left(s_1\right)+f\left(s_2\right)}E\left(I^{\hat{s}}|s_2\right).\end{aligned}\tag{A.13}$$

Since

$$\varrho_{\hat{s}}\leq\varrho_{s_1}<\varrho_{s_2},$$

we must have

$$I^{\hat{s}}\left(\rho\right)\geq I^{s_1}\left(\rho\right)\geq I^{s_2}\left(\rho\right)$$

for all $\rho\in[0,1]$. Then

$$E\left(I^{\hat{s}}|s_1\right)\geq E\left(I^{s_1}|s_1\right)=\frac{\bar{c}_{s_1}}{B}$$

and

$$E\left(I^{\hat{s}}|s_1\right)\geq E\left(I^{s_2}|s_1\right)=\frac{\bar{c}_{s_2}}{B}.$$

Then condition (A.13) implies

$$\frac{\bar{c}_{\hat{s}}}{B}\geq\frac{f\left(s_1\right)}{f\left(s_1\right)+f\left(s_2\right)}\frac{\bar{c}_{s_1}}{B}+\frac{f\left(s_2\right)}{f\left(s_1\right)+f\left(s_2\right)}\frac{\bar{c}_{s_2}}{B}>\frac{\bar{c}_{s_1}}{B}\Rightarrow\bar{c}_{\hat{s}}>\bar{c}_{s_1},$$

contradiction!

The same proof strategy works for the case $\varrho_{\hat{s}}\geq\varrho_{s_2}$. So the equilibrium must satisfy

$$\varrho_{s_1}<\varrho_{\hat{s}}<\varrho_{s_2}.$$

□

The following lemma provides an intermediate results about the structure of the deterministic optimal policy characterized in Lemma 1.5.6.

Lemma A.2.2. *Suppose* $(\mathcal{S}, \sigma)$ *is a deterministic optimal policy, then for almost any* $s \in \mathcal{S}$ *, if there exists a constant* $\epsilon_s > 0$*, such that*

$$(\pi(\rho|s)) \cap (\varrho_s - \epsilon_s, \varrho_s) = \emptyset \tag{A.14}$$

and

$$\mathit{Prob}\left(\rho \le \varrho_s - \epsilon_s | s\right) > 0, \tag{A.15}$$

then there must exist a constant $\delta_s > 0$*, such that*

$$(\pi(\rho|s)) \cap [\varrho_s, \varrho_s + \delta_s] = \emptyset.$$

Proof. Suppose for the sake of contradiction that there exists $s_0 \in \mathcal{S}$ satisfying conditions (A.14) and (A.15), and at least one of the following two scenarios is true:

1. There exists a constant δ, such that for any $0 < x < \delta$,

$$\left(\varrho_{s_0}, \varrho_{s_0} + x\right) \subset (\pi(\rho|s_0)) \cap (\varrho^\star, 1]$$

   and

$$\text{Prob}\left(\rho \in \left(\varrho_{s_0}, \varrho_{s_0} + x\right) | s_0\right) > 0.$$

2. The first condition doesn't hold and $\text{Prob}\left(\rho = \varrho_{s_0} | s_0\right) > 0$.

If the first scenario is true, then let's consider another deterministic disclosure policy $(\mathcal{S}', \sigma')$ with signal space $\mathcal{S}' = \mathcal{S} \backslash \{s_0\} \cup \{s_0', s'\}$, and

$$\sigma'(\rho) = \begin{cases} \sigma(\rho) & \text{if } \rho \notin [0,1] \setminus (\pi(\rho|s_0)) \\ s_0' & \text{if } \rho \in (\pi(\rho|s_0)) \setminus \left(\varrho_{s_0}, \varrho_{s_0} + x\right) \\ s' & \text{if } \rho \in \left(\varrho_{s_0}, \varrho_{s_0} + x\right) \end{cases},$$

where $x < \delta$.

The increase in regulator's payoff under this new policy is

$$
\begin{aligned}
&\Delta W \\
=&\text{Prob}\left(\rho \in \left(\varrho_{s_0}, \varrho_{s_0}+x\right)\right) \cdot E^{s_1', G}\left[\left(\mu \tilde{V}-\left(\mu+(1-\mu) F_c\left(\bar{c}_{s_1'}\right)\right)\right) \cdot 1_{\left\{\rho \geq \varrho_{s_1'}\right\}}\right] \\
&+\text{Prob}\left(\rho \in (\pi(\rho|s_0)) \setminus \left(\varrho_{s_0}, \varrho_{s_0}+x\right)\right) \cdot E^{s_0', G}\left[\left(\mu \tilde{V}-\left(\mu+(1-\mu) F_c\left(\bar{c}_{s_0'}\right)\right)\right) \cdot 1_{\left\{\rho \geq \varrho_{s_0'}\right\}}\right] \\
&-\text{Prob}\left(\rho \in (\pi(\rho|s_0))\right) \cdot E^{s_0, G}\left[\left(\mu \tilde{V}-\left(\mu+(1-\mu) F_c\left(\bar{c}_{s_0}\right)\right)\right) \cdot 1_{\left\{\rho \geq \varrho_{s_0}\right\}}\right] \\
\geq&\text{Prob}\left(\rho \in (\pi(\rho|s_0)) \setminus \left(\varrho_{s_0}, \varrho_{s_0}+x\right)\right) \cdot E^{s_0', G}\left[\left(\mu \tilde{V}-\left(\mu+(1-\mu) F_c\left(\bar{c}_{s_0'}\right)\right)\right) \cdot 1_{\left\{\rho \geq \varrho_{s_0'}\right\}}\right] \\
&-\text{Prob}\left(\rho \in (\pi(\rho|s_0))\right) \cdot E^{s_0, G}\left[\left(\mu \tilde{V}-\left(\mu+(1-\mu) F_c\left(\bar{c}_{s_0}\right)\right)\right) \cdot 1_{\left\{\rho \geq \varrho_{s_0}\right\}}\right] \\
=&\text{Prob}\left(\rho \in (\varrho^{\star}, 1] \cap (\pi(\rho|s_0)) \setminus \left(\varrho_{s_0}, \varrho_{s_0}+x\right)\right) \cdot (1-\mu)\left[F_c\left(\bar{c}_{s_0}\right)-F_c\left(\bar{c}_{s_0'}\right)\right] \\
&-\text{Prob}\left(\rho \in \left(\varrho_{s_0}, \varrho_{s_0}+x\right)\right) \cdot E^{G}\left[\left(\mu \tilde{V}-\left(\mu+(1-\mu) F_c\left(\bar{c}_{s_0}\right)\right)\right) | \rho \in \left(\varrho_{s_0}, \varrho_{s_0}+x\right)\right].
\end{aligned}
$$

Similar to the proof of Proposition 1.4.1, we know

$$
\text{Prob}\left(\rho \in (\varrho^{\star}, 1] \cap (\pi(\rho|s_0)) \setminus \left(\varrho_{s_0}, \varrho_{s_0}+x\right)\right) \cdot (1-\mu)\left[F_c\left(\bar{c}_{s_0}\right)-F_c\left(\bar{c}_{s_0'}\right)\right] = O(x)
$$

and

$$
\text{Prob}\left(\rho \in \left(\varrho_{s_0}, \varrho_{s_0}+x\right)\right) \cdot E^{G}\left[\left(\mu \tilde{V}-\left(\mu+(1-\mu) F_c\left(\bar{c}_{s_0}\right)\right)\right) | \rho \in \left(\varrho_{s_0}, \varrho_{s_0}+x\right)\right] = O(x^2).
$$

So we must have

$$
\Delta W > 0
$$

when x is sufficiently small, and this means the new disclosure policy generates higher regulator's payoff, a contradiction!

If the second scenario is true, the same proof strategy applies and we can also find a disclosure policy (not deterministic) that generates higher regulator's payoff, a contradiction.

□

The next lemma in this proof presents a property about the "worse" subgame (the subgame with highest data manipulation level in equilibrium).

Lemma A.2.3. *For any deterministic optimal policy $(\mathcal{S}, \sigma)$ that satisfies properties in*

Lemma 1.5.6, we must have

$$\sup_{s\in\mathcal{S}} \varrho_s = \varrho^\star. \tag{A.16}$$

Proof. First, we must have $\sup_{s\in\mathcal{S}} \varrho_s \geq \varrho^\star$, otherwise, there exists $\delta > 0$, such that for any s, $\varrho_s < \varrho^\star - \delta$. If this is true, consider any state $\rho \in (\varrho^\star - \delta, \varrho^\star)$. Since $\varrho_s < \varrho^\star - \delta$ for any s, lenders will finance $\hat{G}$ borrowers when the true state $\rho \in (\varrho^\star - \delta, \varrho^\star)$. However, by our definition of $\varrho^\star$, lenders will reject all loan applications if the state $\rho < \varrho^\star$, a contradiction. So we must have $\sup_{s\in\mathcal{S}} \varrho_s \geq \varrho^\star$.

Now we want to show it's impossible to have $\sup_{s\in\mathcal{S}} \varrho_s > \varrho^\star$. Suppose for the sake of contradiction that there exists $\delta > 0$, such that

$$\sup_{s\in\mathcal{S}} \varrho_s > \varrho^\star + \delta.$$

Then there must exist a signal realization, denoted as s_m, such that $\varrho_{s_m} > \varrho^\star + \delta$. From Lemma A.2.2, we know that there exists an interval $\left(\varrho_{s_m}, \varrho_{s_m} + \epsilon_m\right)$ such that

$$\left(\varrho_{s_m}, \varrho_{s_m} + \epsilon_m\right) \cap \left(\pi\left(\rho|s_m\right)\right) = \emptyset,$$

and

$$\text{Prob}\left(\rho \in \left(\varrho_{s_m}, \varrho_{s_m} + \epsilon_m\right)\right) < \text{Prob}\left(\left[\varrho_{s_m} + \epsilon_m, 1\right] \cap \left(\pi\left(\rho|s_m\right)\right)\right).$$

Then we can find an interval B, and a one to one mapping

$$z : \left(\varrho_{s_m}, \varrho_{s_m} + \epsilon_m\right) \to B,$$

with $z'(x) \equiv 1$, such that

$$B \subseteq \left[\varrho_{s_m} + \epsilon_m, 1\right] \cap \left(\pi\left(\rho|s_m\right)\right).$$

Now let's consider the following deterministic disclosure policy $(\mathcal{S}', \sigma')$ with $\mathcal{S}' = \mathcal{S}$, and

$$\sigma'(\rho) = \begin{cases} \sigma(\rho) & \text{if } \rho \notin B \cup \left(\varrho_{s_m}, \varrho_{s_m} + \epsilon_m\right) \\ \sigma(z(\rho)) & \text{if } \rho \in \left(\varrho_{s_m}, \varrho_{s_m} + \epsilon_m\right) \\ \sigma\left(z^{-1}(\rho)\right) & \text{if } \rho \in B \end{cases}.$$

It's easy to check that all lending market equilibria are unchanged. Then the regulator's payoff is unchanged. However, under the new disclosure policy, for the signal realization s_m, we have

$$\rho_{s_m} = \inf\left\{\left(\pi\left(\rho|s_m\right)\right) \cap \left(\varrho^{\star}, 1\right]\right\}.$$

But this violates Lemma A.2.2, a contradiction. So it's impossible to have

$$\sup_{s \in \mathcal{S}} \varrho_s > \varrho^{\star},$$

and thus we must have

$$\sup_{s \in \mathcal{S}} \varrho_s = \varrho^{\star}.$$

□

Then Lemma 1.5.1 is a direct result of Lemma A.2.1 and A.2.3. Suppose for the sake of contradiction that $\varrho^{\star} \leq \varrho_N$, then Lemma A.2.3 implies that

$$\varrho_s \leq \varrho_N$$

for all $s \in \mathcal{S}$. Note that the signal in no information case is a "combined" signal of all signals in the optimal policy $(\mathcal{S}, \sigma)$, Lemma A.2.1 implies that

$$\varrho_N < \sup_{s \in \mathcal{S}} \varrho_s,$$

a contradiction! So we must have

$$\varrho^{\star} > \varrho_N.$$

A.2.11 Proof of Proposition 1.5.2

Suppose the deterministic optimal policy is $(\mathcal{S}, \sigma)$. Since there are at least two signals in the optimal policy, we must have

$$\bar{c}_{\max} > \bar{c}_{\min}.$$

Note that the no disclosure is the "combined" signal of optimal policy $(\mathcal{S}, \sigma)$. Then Lemma A.2.1 implies that

$$\bar{c}_{\max} > \bar{c}_N > \bar{c}_{\min}.$$

A.2.12 Proof of Proposition 1.5.3

Consider an optimal disclosure policy $(\mathcal{S}, \tilde{\sigma})$ with distribution of posteriors

$$\{f(s), \pi(\rho|s)\}_{s\in\mathcal{S}}.$$

Since the prior belief of ρ is a continuous distribution, for any s_0 satisfying

$$\text{Prob}(s_0) > \epsilon,$$

the equilibrium conditions imply that there must exist $\epsilon_{s_0} > 0$ and $\delta_{s_0} > 0$, such that

$$\text{Prob}(\rho \leq \varrho^\star - \epsilon_{s_0}|s_0) = \delta_{s_0} > 0.$$

Let $M = \frac{\bar{c}_{s_0}}{B}$. Denote T as the solution of

$$\frac{T \cdot \text{Prob}(\rho \geq \varrho^\star|s_0)}{\delta_{s_0}} = \frac{M}{1-M}.$$

Then let's consider a new signal space

$$\mathcal{S}_a = \mathcal{S} \backslash \{s_0\} \cup \{s_{a1}, s_{a2}\}$$

and a distribution of posteriors $\{f_a(s), \pi_a(\rho|s)\}_{s\in\mathcal{S}_a}$ where

$$f_a(s) = \begin{cases} f(s) & \text{if } s \in \mathcal{S}\backslash\{s_0\} \\ \frac{\delta_{s_0}}{1-M} f(s_0) & \text{if } s = s_{a1} \\ \left(1 - \frac{\delta_{s_0}}{1-M}\right) f(s_0) & \text{if } s = s_{a2} \end{cases},$$

and

$$\pi_a(\rho|s) = \begin{cases} \pi(\rho|s) & \text{if } s \in \mathcal{S}\backslash\{s_0\} \\ \frac{1-M}{\delta_{s_0}} \left[\pi(\rho|s_0) 1_{[0,\varrho^\star-\epsilon_{s_0}]}(\rho) + T \cdot \pi(\rho|s_0) 1_{[\varrho^\star,1]}(\rho)\right] & \text{if } s = s_{a1} \\ \frac{1}{1-\frac{\delta_{s_0}}{1-M}} \left[\pi(\rho|s_0) 1_{(\varrho^\star-\epsilon_{s_0},\varrho^\star)}(\rho) + (1-T) \cdot \pi(\rho|s_0) 1_{[\varrho^\star,1]}(\rho)\right] & \text{if } s = s_{a2} \end{cases}.$$

We can check the distribution of posteriors $\{f_a(s), \pi_a(\rho|s)\}_{s\in\mathcal{S}_a}$ is still Bayes-plausible, and there exists a disclosure policy that can induce this distribution of posteriors. Besides, we can check that the equilibrium variables $\{k_s, \varrho_s, \bar{c}_s\}$ are all the same for equilibria under signal s_0, s_{a1} and s_{a2}. Then by Lemma A.2.2[4], there must exists $t_s > 0$, such that

$$(\pi(\rho|s_{a2})) \cap \left[\varrho_{s_0}, \varrho_{s_0} + t_s\right] = \emptyset,$$

which implies

$$(\pi(\rho|s_0)) \cap \left[\varrho_{s_0}, \varrho_{s_0} + t_s\right] = \emptyset$$

because of our construction of π_a. Then the surplus from lending must be greater than

$$\mu\left(m\left(\varrho_{s_0} + \frac{t_s}{2}\right) - m\left(\varrho_{s_0}\right)\right) > 0$$

for any $\rho > \varrho^\star$ in this posterior equilibrium s_0.

A.2.13 Proof of Theorem 1.5.2

The proof of Theorem 1.5.2 is established by three lemmas.

Lemma A.2.4. *(Pooling at the bottom) When Assumption 1.5.1 is satisfied, in any deterministic optimal policy $(\mathcal{S}, \sigma)$ characterized in Theorem 1.5.1, there must exist $\epsilon > 0$, such that for any $\rho_1, \rho_2 \in (0, \epsilon) \cup (\varrho^\star, \varrho^\star + \epsilon)$, we have $\sigma(\rho_1) = \sigma(\rho_2)$.*

Proof. Suppose $(\mathcal{S}, \sigma)$ is a deterministic optimal policy characterized in Theorem 1.5.1. Note Lemma A.2.3 implies $\varrho_s \le \varrho^\star$ for all s, then let

$$\mathcal{S}_1 = \left\{ s \middle| \sup\left\{(\pi(\rho|s)) \cap (0, \varrho^\star)\right\} < m^{-1}(I) \,\&\, \varrho_s < \frac{1}{2}\left(m^{-1}(I) + \varrho^\star\right) \right\},$$

$$B_1 = \cup_{s \in \mathcal{S}_1} (\pi(\rho|s)),$$

and

$$\bar{c}_1 = \sup\{\bar{c}_s | s \in \mathcal{S}_1\}.$$

Suppose for the sake of contradiction that it doesn't satisfy the *Pooling at the bottom* property. Then there are infinite elements in $\mathcal{S}_1$. Regulator's ex ante surplus from all $s \in \mathcal{S}_1$

[4] Although Lemma A.2.2 only considers deterministic optimal policies, it can be shown that it also holds for general optimal policies.

is

$$\begin{aligned}
\tilde{W}_1 &= \int_{s\in\mathcal{S}_1} f(s)\cdot E^s\left[\left(\mu m(\rho) - (\mu + (1-\mu) F_c(\bar{c}_s)) I\right)^+\right] ds \\
&= \begin{array}{c} \int_{s\in\mathcal{S}_1} f(s)\cdot \text{Prob}\left(\rho > \varrho^\star | s\right)\cdot \mu E^s\left(m(\rho) - I | \rho > \varrho^\star\right) ds - \\ \int_{s\in\mathcal{S}_1} f(s)\cdot \text{Prob}\left(\rho > \varrho^\star | s\right)\cdot (1-\mu) F_c(\bar{c}_s) I ds \end{array} \\
&= \int_{s\in\mathcal{S}_1} f(s)\cdot \frac{\bar{c}_s}{B}\cdot \mu E^s\left(m(\rho) - I | \rho > \varrho^\star\right) ds - \int_{s\in\mathcal{S}_1} f(s)\cdot \frac{\bar{c}_s}{B}\cdot (1-\mu) F_c(\bar{c}_s) I ds, \\
&= \begin{array}{c} \text{Prob}\left(\rho \in (\varrho^\star, 1] \cap B_1\right) \mu E\left(m(\rho) - I | \rho \in (\varrho^\star, 1] \cap B_1\right) \\ - \int_{s\in\mathcal{S}_1} f(s)\cdot \frac{\bar{c}_s}{B}\cdot (1-\mu) F_c(\bar{c}_s) I ds \end{array}
\end{aligned}$$

Here we use the equilibrium condition $\text{Prob}\left(\rho > \varrho^\star | s\right) = \frac{\bar{c}_s}{B}$ in the last equality. Then we show that the regulator's payoff increases under another disclosure policy that satisfies the *Pooling at the bottom* property. To see this, in the above equilibrium,

$$\begin{aligned}
\int_{s\in\mathcal{S}_1} f(s)\cdot \frac{\bar{c}_s}{B}\cdot ds &= \int_{s\in\mathcal{S}_1} f(s)\cdot \text{Prob}\left(\rho > \varrho^\star | s\right)\cdot ds \\
&= \text{Prob}\left(\rho \in (\varrho^\star, 1] \cap B_1\right).
\end{aligned}$$

Let $\bar{c}_0$ be the solution of

$$\left(\int_{s\in\mathcal{S}_1} f(s)\cdot ds\right)\frac{\bar{c}_0}{B} = \text{Prob}\left(\rho \in (\varrho^\star, 1] \cap B_1\right) = \int_{s\in\mathcal{S}_1} f(s)\cdot \frac{\bar{c}_s}{B}\cdot ds,$$

obviously $\bar{c}_0 < \sup_{s\in\mathcal{S}_1} \bar{c}_s$. Based on Assumption 1.5.1, and using the concavification method (Kamenica and Gentzkow (2011)), we know there exist $\bar{c}_1 \leq \bar{c}_2 \leq \sup_{s\in\mathcal{S}_1} \bar{c}_s$, and two positive numbers p_1, p_2 satisfying $p_1 + p_2 = 1$, such that

$$p_1 + p_2 = 1,$$

$$p_1 \frac{\bar{c}_1}{B} + p_2 \frac{\bar{c}_2}{B} = \frac{\bar{c}_0}{B},$$

and

$$\left(\int_{s\in\mathcal{S}_1} f(s)\cdot ds\right)(1-\mu)\left[p_1 \frac{\bar{c}_1}{B} F_c(\bar{c}_1) + p_2 \frac{\bar{c}_2}{B} F_c(\bar{c}_2)\right] < \int_{s\in\mathcal{S}_1} f(s)\cdot \frac{\bar{c}_s}{B}\cdot (1-\mu) F_c(\bar{c}_s) ds. \tag{A.17}$$

Here $\bar{c}_1$ and $\bar{c}_2$ represent the equilibrium data manipulation cutoffs for two signals $\hat{s}_1$ and $\hat{s}_2$. From the ex ante perspective. The regulator's payoff from financing good projects are unchanged for all states $\rho \in B_1$, while the ex ante surplus loss from financing bad projects decreases with the binary signals $\hat{s}_1$ and $\hat{s}_2$ because of condition (A.17). Then the new disclosure policy with signals $\hat{s}_1$ and $\hat{s}_2$ improves regulator's payoff, and this policy satisfies the *Pooling at the bottom* property. □

Then we want to show that there exists at most one discrete signal s satisfying $\text{Prob}\,(s) > 0$. To get this result, let first provide an intermediate result:

Lemma A.2.5. *Suppose $(\mathcal{S}, \sigma)$ is a deterministic optimal policy, then for any $s \in \mathcal{S}$ such that $Prob\,(s) > 0$, function $xF_c\,(x)$ can not be strictly concave at $x = \bar{c}_s$.*

Proof. Suppose $(\mathcal{S}, \sigma)$ is a deterministic optimal policy, and there exists $s_0 \in \mathcal{S}$ such that $\text{Prob}\,(s_0) > 0$. Suppose for the sake of contradiction that $xF_c\,(x)$ is strictly concave at $x = \bar{c}_{s_0}$. In this equilibrium, since $\text{Prob}\,(s_0) > 0$, there must exists $\epsilon_0 > 0$ and $\delta_0 > 0$ such that

$$\text{Prob}\left(\rho \in \left(0, \varrho_{s_0} - \epsilon_0\right) \cap \left(\pi\left(\rho|s_0\right)\right)\right) > \delta_0$$

and

$$\text{Prob}\left(\rho \in \left(\varrho_{s_0} + \epsilon_0, 1\right) \cap \left(\pi\left(\rho|s_0\right)\right)\right) > \delta_0.$$

Then there exists two sets L_1 and R_1, such that

$$L_1 \subset \left(0, \varrho_{s_0} - \epsilon_0\right) \cap \left(\pi\left(\rho|s_0\right)\right),$$

$$R_1 \subset \left(\varrho_{s_0} + \epsilon_0, 1\right) \cap \left(\pi\left(\rho|s_0\right)\right),$$

$$\text{Prob}\,(L_1) > 0,$$

$$\text{Prob}\,(R_1) > 0,$$

and

$$\frac{\text{Prob}\,(L_1)}{\text{Prob}\,(R_1)} = \frac{\text{Prob}\left(\left(0, \varrho_{s_0}\right) \cap \left(\pi\left(\rho|s_0\right)\right)\right)}{\text{Prob}\left(\left(\varrho^{\star}_{s_0}, 1\right) \cap \left(\pi\left(\rho|s_0\right)\right)\right)}.$$

Then consider the following deterministic policy with signal space $\mathcal{S}' = \mathcal{S} \setminus \{s_0\} \cup \{s'_1, s'_2\}$

and message function

$$\sigma'(\rho) = \begin{cases} \sigma(\rho) & \text{if } \rho \notin (\pi(\rho|s_0)) \\ s_1' & \text{if } \rho \in L_1 \cup R_1 \\ s_2' & \text{if } \rho \in (\pi(\rho|s_0)) \setminus (L_1 \cup R_1) \end{cases}.$$

The lending market equilibria under s_1' and s_2' are the same as the equilibrium under s_0 with equilibrium variables $\left(k_{s_0}, \varrho^{\star}_{s_0}, \bar{c}_{s_0}\right)$. However we can improve the regulator's payoff from states $\rho \in L_1 \cup R_1$ by disclosing additional information based on s_1'. Since $xF_c(x)$ is strictly concave at $x = \bar{c}_{s_0}$, there exists $\bar{\epsilon}_2 > 0$, such that for all $\epsilon < \bar{\epsilon}_2$, there exist two numbers $\bar{c}_1$ and $\bar{c}_2$ satisfying $\bar{c}_1, \bar{c}_2 \in (\bar{c}_{s_0} - \epsilon, \bar{c}_{s_0} + \epsilon)$ and two positive numbers p_1, p_2 satisfying $p_1 + p_2 = 1$ such that

$$p_1\bar{c}_1 + p_2\bar{c}_2 = \bar{c}_{s_0},$$

and

$$p_1\bar{c}_1 F_c(\bar{c}_1) + p_2\bar{c}_2 F_c(\bar{c}_2) < \bar{c}_{s_0} F_c(\bar{c}_{s_0}). \tag{A.18}$$

Let ϱ_1 and ϱ_2 be

$$\varrho_1 = m^{-1}\left(\frac{\mu + (1-\mu) F_c(\bar{c}_1)}{\mu} I\right)$$

and

$$\varrho_2 = m^{-1}\left(\frac{\mu + (1-\mu) F_c(\bar{c}_2)}{\mu} I\right),$$

then we can choose ϵ small enough, such that

$$\varrho_1, \varrho_2 \in \left(\varrho_{s_0} - \epsilon_0, \varrho_{s_0} + \epsilon_0\right).$$

Then there must exist L_{11}, L_{12}, R_{11}, R_{12} such that

$$L_{11} \cup L_{12} = L_1$$

$$R_{11} \cup R_{12} = R_1,$$

$$\frac{\text{Prob}(\rho \in R_{11})}{\text{Prob}(\rho \in R_{11}) + \text{Prob}(\rho \in L_{11})} = \frac{\bar{c}_1}{B}$$

and

$$\frac{\text{Prob}\left(\rho \in R_{12}\right)}{\text{Prob}\left(\rho \in R_{12}\right)+\text{Prob}\left(\rho \in L_{12}\right)}=\frac{\bar{c}_2}{B}.$$

Then let's consider the following deterministic policy with signal space $\mathcal{S}_1' = \mathcal{S}\setminus\{s_0\}\cup\{s_2'\}\cup\{s_{11}', s_{12}'\}$ and message function

$$\sigma_1'\left(\rho\right)=\begin{cases}\sigma\left(\rho\right) & \text{if } \rho \notin \left(\pi\left(\rho|s_0\right)\right)\\ s_{11}' & \text{if } \rho \in L_{11}\cup R_{11}\\ s_{12}' & \text{if } \rho \in L_{12}\cup R_{12}\\ s_2' & \text{if } \rho \in \left(\pi\left(\rho|s_0\right)\right)\setminus\left(L_1\cup R_1\right)\end{cases}.$$

It can be verified that, compared to the disclosure policy $(\mathcal{S},\sigma)$, the regulator's payoff is unchanged under the new policy $(\mathcal{S}_1',\sigma_1')$ for all states $\rho \in [0,1]\setminus(L_1\cup R_1)$. And for states $\rho \in L_1\cup R_1$, the regulator's payoff under $(\mathcal{S},\sigma)$ is

$$\begin{aligned}W_0 &= \text{Prob}\left(\rho\in R_1\right)\mu E\left(m\left(\rho\right)-I|\rho\in R_1\right)-\text{Prob}\left(\rho\in R_1\right)\left(1-\mu\right)F_c\left(\bar{c}_{s_0}\right)I\\ &= \text{Prob}\left(\rho\in R_1\right)\mu E\left(m\left(\rho\right)-I|\rho\in R_1\right)-\text{Prob}\left(\rho\in L_1\cup R_1\right)\frac{\bar{c}_{s_0}}{B}\left(1-\mu\right)F_c\left(\bar{c}_{s_0}\right)I\end{aligned}$$

while the regulator's payoff under $(\mathcal{S}_1',\sigma_1')$ is

$$W_{12}=\begin{array}{c}\text{Prob}\left(\rho\in R_1\right)\mu E\left(m\left(\rho\right)-I|\rho\in R_1\right)-\text{Prob}\left(\rho\in L_{11}\cup R_{11}\right)\frac{\bar{c}_1}{B}\left(1-\mu\right)F_c\left(\bar{c}_1\right)I\\ -\text{Prob}\left(\rho\in L_{12}\cup R_{12}\right)\frac{\bar{c}_2}{B}\left(1-\mu\right)F_c\left(\bar{c}_2\right)I\end{array}.$$

Since

$$\text{Prob}\left(\rho\in L_{11}\cup R_{11}\right)+\text{Prob}\left(\rho\in L_{12}\cup R_{12}\right)=\text{Prob}\left(\rho\in L_1\cup R_1\right),$$

condition (A.18) implies that

$$W_{12} > W_0,$$

which implies that the regulator's payoff under $(\mathcal{S}_1',\sigma_1')$ is greater than her payoff under $(\mathcal{S},\sigma)$, a contradiction! ☐

Lemma A.2.6. *For any deterministic optimal policy characterized in Theorem 1.5.1, there exists a payoff-equivalent deterministic optimal policy $(\mathcal{S},\sigma)$, such that there exists only one*

$s \in \mathcal{S}$ *that satisfies Prob* $(s) > 0$.

Proof. Suppose $(\mathcal{S}, \sigma)$is a deterministic optimal policy, and suppose for the sake of contradiction that there exists two signals s_1, $s_2 \in \mathcal{S}$, such that

$$\text{Prob}\,(s_1) > 0,$$

and

$$\text{Prob}\,(s_2) > 0.$$

Denote the equilibrium variables under these two signals are $\left(k_{s_1}, \varrho_{s_1}, \bar{c}_{s_1}\right)$ and $\left(k_{s_2}, \varrho_{s_2}, \bar{c}_{s_2}\right)$, respectively. Without loss of generality assume $\bar{c}_{s_1} < \bar{c}_{s_2}$. Using the proof techniques in Lemma A.2.5, we can create two signals s_1' and s_2' based on s_1 and s_2, such that equilibrium under s_1' (s_2') is the same as the equilibrium under s_1 (s_2), and there exists two constant $\epsilon > 0$, such that

$$\left(\pi\left(\rho|s_1'\right)\right) \cap \left(\varrho_{s_1} - \epsilon, \varrho_{s_1} + \epsilon\right) = \emptyset$$

and

$$\left(\pi\left(\rho|s_2'\right)\right) \cap \left(\varrho_{s_2} - \epsilon, \varrho_{s_2} + \epsilon\right) = \emptyset.$$

Lemma A.2.5 implies that function $xF_c(x)$ is weakly convex at both $\bar{c}_{s_1}$ and $\bar{c}_{s_2}$.

If the function $xF_c(x)$ is convex on $[\bar{c}_{s_1}, \bar{c}_{s_2}]$. Then for any $\delta > 0$ that is small enough, there exists positive numbers p_1, p_2, $\bar{c}_1 \in (\bar{c}_{s_1}, \bar{c}_{s_1} + \delta)$, $\bar{c}_2 \in (\bar{c}_{s_2} - \delta, \bar{c}_{s_2})$, such that

$$p_1 + p_2 = \text{Prob}\left(s_1'\right) + \text{Prob}\left(s_2'\right)$$

and

$$p_1\bar{c}_1 + p_2\bar{c}_2 = \text{Prob}\left(s_1'\right)\bar{c}_{s_1} + \text{Prob}\left(s_2'\right)\bar{c}_{s_2}.$$

Let ϱ_1 and ϱ_2 be

$$\varrho_1 = m^{-1}\left(\frac{\mu + (1-\mu)\,F_c\left(\bar{c}_1\right)}{\mu} I\right)$$

and

$$\varrho_2 = m^{-1}\left(\frac{\mu + (1-\mu)\,F_c\left(\bar{c}_2\right)}{\mu} I\right).$$

Then we can choose δ small enough, such that

$$\varrho_1 \in \left(\varrho_{s_1} - \epsilon, \varrho_{s_1} + \epsilon\right)$$

and

$$\varrho_2 \in \left(\varrho_{s_2} - \epsilon, \varrho_{s_2} + \epsilon\right).$$

Then following the proof strategy in Lemma A.2.5we can create another deterministic disclosure policy that generates higher regulator's payoff by creating two signals with equilibrium cutoffs ϱ_1 and ϱ_2.

If the function $xF_c(x)$ is not always convex on $[\bar{c}_{s_1}, \bar{c}_{s_2}]$, based on Assumption (1.5.1), we must have $(xF_c(x))''\big|_{x=\bar{c}_{s_2}} = 0$. Let

$$L_1 = (\pi(\rho|s_2)) \cap (\varrho^\star, 1],$$

and

$$C_1 = \left[F_c^{-1}\left(\frac{\mu}{1-\mu}\left(\frac{m(\inf L_1)}{I} - 1\right)\right), F_c^{-1}\left(\frac{\mu}{1-\mu}\left(\frac{m(\sup L_1)}{I} - 1\right)\right)\right].$$

if $xF_c(x)$ is linear on C_1, then we can show that there exists a disclosure policy such that the message function is strictly increasing on L_1. If $xF_c(x)$ is not linear on C_1, then there must exist $c_2 \in C_1$, such that $xF_c(x)$ is strictly concave at c_2. The using the proof strategy in Lemma (A.2.5), we can show this disclosure policy must be suboptimal, a contradiction. □

Besides,the based on the general characterization in Theorem 1.5.1, there must exists cutoff ϱ_a, ϱ_b and $\varrho^\star$ and a signal space $[\bar{c}_{\min}, \bar{c}_{\max}]$ such that the message function is weakly increasing on $[0, \varrho^\star]$ and $(\varrho^\star, 1]$. Since message function is $\sigma(\rho) = \bar{c}_{\sigma(\rho)}$, we can consider a different signal space $\mathcal{S}' = [\varrho_a, \varrho^\star]$, such that new message function is

$$\sigma'\big|_{[0,\varrho^\star]} = \begin{cases} \varrho_a & \text{if } \rho \in [0, \varrho_a] \\ \rho & \text{if } \rho \in (\varrho_a, \varrho^\star] \end{cases},$$

and

$$\sigma'\big|_{(\varrho^\star,1]} = \begin{cases} \varrho_a & \text{if } \rho \in (\varrho^\star, \varrho_b] \\ \gamma(\rho) & \text{if } \rho \in (\varrho_b, 1] \end{cases},$$

where $\gamma(x) = (\pi(\rho|s = \sigma(x))) \cap [0, \varrho^\star]$. Then there must exist a deterministic policy that has the structure characterized in Theorem 1.5.2.

A.2.14 Proof of Lemma 1.5.7

Suppose $(\mathcal{S}, \sigma)$ is the deterministic optimal signal characterized in Theorem 1.5.2. For any $\rho \in (\varrho_a, \varrho^\star)$, the signal is $s = \rho$, and

$$(\pi(\rho|s)) \cap [0, \varrho^\star] = s.$$

If

$$\varrho_s = \sup\{(\pi(\rho|s)) \cap [0, \varrho^\star]\}$$

doesn't hold for $\rho_0 \in (\varrho_a, \varrho^\star)$, there must exist an interval $B_0 \in (\varrho_a, \varrho^\star)$ and a constant $\epsilon_0 > 0$, such that

$$\varrho_x > \sup\{(\pi(\rho|x)) \cap [0, \varrho^\star]\} + 2\epsilon_0$$

for all $x \in B$. Besides, Lemma A.2.3 implies that there exists $B \in B_0$ and a constant $\epsilon < \epsilon_0$, such that

$$\varrho_x < \inf\{(\pi(\rho|x)) \cap (\varrho^\star, 1]\} - 2\epsilon.$$

Then for all $x \in B$, we have

$$\varrho_x \in (\sup\{(\pi(\rho|x)) \cap [0, \varrho^\star]\} + 2\epsilon, \inf\{(\pi(\rho|x)) \cap (\varrho^\star, 1]\} - 2\epsilon).$$

Theorem 1.5.2 implies that there exists s_0 with

$$\text{Prob}(s_0) > 0,$$

and $\varrho_{s_0} < \varrho_x$ for any $x \in B$. Without loss of generality, based on Assumption 1.5.1, we can focus on the cases when $xF_c(x)$ is concave on $x \in B$ or it's convex on $x \in B$. This is because there is only one inflection point for function $xF_c(x)$, so if this condition doesn't hold, we can always "truncate" it such that the concavity of function $xF_c(x)$ is unchanged on $x \in B$.

If $xF_c(x)$ is convex on $x \in B$, since $B \cap (\pi(\rho|s_0) \cap [0, \varrho^\star]) = \emptyset$, we can find $\bar{c}_1 > \bar{c}_{s_0}$, and two functions $f_n(\rho)$ and $\bar{c}_n(\rho)$ on $\rho \in B$, such that

$$\int_{\rho \in B} f_n(\rho)\, d\rho = \int_{\rho \in B} f(\rho)\, d\rho$$

$$\text{Prob}(s_0) \cdot \bar{c}_1 + \int_{x \in B} f_n(x)\, \bar{c}_n(x)\, d\rho = \text{Prob}(s_0) \cdot \bar{c}_{s_0} + \int_{x \in B} f(x)\, \bar{c}_x dx$$

and

$$\begin{aligned} &\text{Prob}(s_0) \cdot \bar{c}_1 F_c(\bar{c}_1) + \int_{x \in B} f_n(x)\, \bar{c}_n(x)\, F_c(\bar{c}_n(x))\, d\rho \\ < &\text{Prob}(s_0) \cdot \bar{c}_{s_0} F_c(\bar{c}_{s_0}) + \int_{x \in B} f(x)\, \bar{c}_x F_c(\bar{c}_x)\, dx, \end{aligned} \tag{A.19}$$

where the last condition is from the convexity of function $xF_c(x)$. We can always find $(\bar{c}_1, f_n(\rho), \bar{c}_n(\rho))$ such that

$$\bar{c}_1 < \inf_{x \in B} \bar{c}_n(x),$$

and

$$\inf_{x \in B} m^{-1} \left(\left(1 + \frac{1-\mu}{\mu} F_c(\bar{c}_n(x)) \right) I \right) > \sup_{x \in B} \{ (\pi(\rho|x)) \cap [0, \varrho^\star] \}.$$

This proof strategy replicates the idea in the proof of Lemma A.2.6, basically we want to design a new disclosure policy that generates higher regulator's payoff. And the conditions we impose here guarantee that under the new disclosure policy, the regulator's payoff from financing good projects is unchanged, while the cost from financing bad projects decreases because of the condition (A.19). The complete proof is omitted here because the rest is the same as the proof of Lemma A.2.6.

If $xF_c(x)$ is concave on $x \in B$, then we can follow the idea in proof of Lemma A.2.5 and show this is suboptimal. To see this, note we can find two functions $f_m(\rho)$ and $\bar{c}_m(\rho)$ on $\rho \in B$, such that

$$\int_{\rho \in B} f_m(\rho)\, d\rho = \int_{\rho \in B} f(\rho)\, d\rho$$

$$\int_{x \in B} f_m(x)\, \bar{c}_m(x)\, d\rho = \int_{x \in B} f(x)\, \bar{c}_x dx$$

and

$$\int_{x\in B} f_m(x)\,\bar{c}_m(x)\,F_c(\bar{c}_m(x))\,d\rho < \int_{x\in B} f(x)\,\bar{c}_x F_c(\bar{c}_x)\,dx \tag{A.20}$$

where the last condition is from the concavity of function $xF_c(x)$. We can always find $(f_m(\rho), \bar{c}_m(\rho))$ such that

$$\inf_{x\in B} m^{-1}\left(\left(1+\frac{1-\mu}{\mu}F_c(\bar{c}_m(x))\right)I\right) > \sup_{x\in B}\{(\pi(\rho|x))\cap[0,\varrho^{\star}]\}.$$

This proof strategy replicates the idea in the proof of Lemma A.2.5, basically we want to design a new disclosure policy that generates higher regulator's payoff. And the conditions we impose here guarantee that under the new disclosure policy, the regulator's payoff from financing good projects is unchanged, while the cost from financing bad projects decreases because of the condition (A.20). The complete proof is omitted here because the rest is the same as the proof of Lemma A.2.5.

A.2.15 Proof of Theorem 1.6.1

First, it's obvious that when verification cost t is sufficiently high, the verification technology will never be used. In our analysis, we already show that in any equilibrium s that the verification is used, we must have

$$k_s = k^v = \frac{\mu I^2}{I-t},$$

and the data manipulation level is

$$\bar{c}^v = F_c^{-1}\left(\frac{\mu t}{(1-\mu)(I-t)}\right).$$

Then the lending market equilibrium variables $(k_s, \varrho_s, \bar{c}_s)$ are uniquely determined whenever there is verification used in equilibrium. Suppose the disclosure policy is $(\mathcal{S}, \tilde{\sigma})$, then there is at most one signal s under which verification is used. Suppose under $s_0 \in \mathcal{S}$ there is verification used in equilibrium, and $\mathrm{Prob}(s_0) > 0$, then we must have

$$(\pi(\rho|s_0)) \cap \left(0, m^{-1}\left(\frac{k^v}{\mu}\right)\right) = \emptyset.$$

To see this, suppose for the sake of contradiction that

$$(\pi(\rho|s_0)) \cap \left(0, m^{-1}\left(\frac{k^v}{\mu}\right)\right) = B,$$

and $\text{Prob}(B|s_0) > 0$. It's clear that lenders will never lend to any borrowers if $\rho \in B$ in equilibrium s_0. Then let's consider a new disclosure policy which keeps everything unchanged except disclosing whether the true state $\rho \in B$ or not if the signal realization is s_0 in the previous policy. It's clear that if the true state $\rho \in B$, the regulator's payoff from these states is zero under the old policy, and is non-negative under the new policy, so it weakly improves. The regulator's payoff from other states are unchanged, because lenders are always indifferent between verifying types or not under this equilibrium, and thus the regulator's payoff will be unchanged from these states. Then the regulator's payoff weakly increases under the new policy. Besides, we know that for all $s \in \mathcal{S}\backslash\{s_0\}$, we have $\underline{\rho}_s < m^{-1}\left(\frac{k^v}{\mu}\right)$. Then without loss of generality, we can consider the policy such that the signal s_0 reveals if the true state is above a threshold or not. Formally speaking,

Lemma A.2.7. *There exists an optimal disclosure policy* $(\mathcal{S}, \tilde{\sigma})$ *and a cutoff* ρ^v *such that*

$$(\pi(\rho|s_0)) = (\rho^v, 1],$$

and

$$(\pi(\rho|s)) \subset [0, \rho^v]$$

for any $s \in \mathcal{S}\backslash\{s_0\}$, *where* s_0 *is the signal under which verification is used with positive probability.*

Then all signals $s \in \mathcal{S}\backslash\{s_0\}$ can only reveal information about states below ρ^v. The following lemma shows that the disclosure policy conditional on $\mathcal{S}\backslash\{s_0\}$ is the optimal disclosure policy when the prior belief is $\rho \sim U[0, \rho^v]$.

Lemma A.2.8. *Suppose* $(\mathcal{S}, \tilde{\sigma})$ *is an optimal disclosure policy characterized in Lemma A.2.7, then the disclosure policy* $(\mathcal{S}_1, \tilde{\sigma}_1)$ *where*

$$\mathcal{S}_1 = \mathcal{S}\backslash\{s_0\}$$

and

$$\tilde{\sigma}_1(s|\rho) = \tilde{\sigma}(s|\rho)|_{\rho\in[0,\rho^v]}$$

is an optimal disclosure policy when the prior $\rho \sim U[0,\rho^v]$*.*

The proof of Lemma A.2.8 is intuitive. Suppose $(\mathcal{S}_2, \tilde{\sigma}_2)$ is an optimal disclosure policy under prior belief $\rho \sim U[0,\rho^v]$. If

$$\sup_{s\in\mathcal{S}} \rho_s \le \rho^v,$$

then this optimal disclosure policy is consistent with the constraint of no verification: $\rho \le \rho^v$, and thus this is optimal. If

$$\sup_{s\in\mathcal{S}} \rho_s > \rho^v,$$

then including verification can actually increase the regulator's payoff from states $\rho \in [0,\rho^v]$, which means that $(\mathcal{S}, \tilde{\sigma})$ is not optimal, a contradiction!

The last part of the proof is to show that for any cost t_x, if when $t = t_x$, verification is used with positive probability under the optimal disclosure policy, then verification will always be used under optimal disclosure policy for any $t < t_x$. This result is straightforward. Suppose W_{NV} is the regulator's payoff when there is no verification technology available, and $W_V(t)$ is regulator's payoff when verification cost is available and the cost parameter is t. It's easy to show that $W_V(t)$ is decreasing in t, so if

$$W_V(t_x) > W_{NV},$$

we must have

$$W_V(t) > W_{NV}$$

for any $t < t_x$. This means that when t is below a threshold (denoted as t^v), verification will always be used under optimal disclosure. The above observation, together with Lemma A.2.7 and Lemma A.2.8, complete the proof.

Appendix B

Appendix: Learning From Manipulable Signals

B.1 Appendix: Proofs

Remark. Because only the noninvestible type of the agent has an active choice to make, whenever there is no confusion we simply refer to the "noninvestible-type agent" as the agent.

B.1.1 Equilibrium Characterization: Toward a Proof of Theorem 2.4.1

To establish Theorem 2.4.1, we use the results that the equilibrium belief process must have full support (Lemma 2.4.1), and that the principal's equilibrium strategy must have a cutoff structure (Lemma 2.4.2). These two lemmas are proved in the Online Appendix. The main proof characterizes the agent's (pseudo-)best reply to any cutoff termination rule (Lemma 2.4.3). Finally, we prove equilibrium existence and uniqueness using a fixed point argument.

Proof of Lemma 2.4.3

In light of Lemma 2.4.2, let us fix a cutoff termination rule of the principal. We define a new state variable $Z_t := \log \frac{p_t}{1-p_t}$, which is a strictly increasing transformation of p_t. Note that Z_t is defined on $(-\infty, \infty)$.

Given the principal's conjecture $a(\cdot)$ about the agent's policy function and the agent's actual policy function $\tilde{a}(\cdot)$, the law of motion of p_t is given by (2.7). By Itô's lemma, the law of motion of Z_t is

$$dZ_t =^2 (1-a_t)\left[1-\tilde{a}_t - \tfrac{1}{2}(1-a_t)\right]dt - (1-a_t)dB_t. \tag{B.1}$$

Now, suppose that the principal uses a particular cutoff policy function b with cutoff belief $p^* \in (0,1)$. Suppose also that the noninvestible type's policy function a satisfies the conditions in Lemma 2.4.3: $a(\cdot)$ is Lipschitz, $\sup_{p\in(0,1)} a(p) < 1$, and it satisfies (2.6), i.e., $a \in_{\tilde{a}\in\mathcal{P}} \hat{V}(p,\tilde{a},b;a)$; moreover, the resulting $V(p) = \hat{V}(p,a(p),b(p);a(p))$ is regular. **For brevity, we call any $a(\cdot)$ that satisfies these conditions a *pseudo-best reply* to $b(\cdot)$.**[1] Lipschitz continuity of $a(\cdot)$ implies that, for any control in $\mathcal{P}$ or $\mathcal{A}$, the controlled process p_t or Z_t in the agent's problem always admits a unique strong solution.

Let Z_t be the new state variable and define $v(z) := V\left(\frac{e^z}{1+e^z}\right)$ and $z^* := \log\frac{p^*}{1-p^*}$. Because we work with Z_t most of the time in this appendix, *we will write $a(z)$ to mean $a\left(\frac{e^z}{1+e^z}\right)$ whenever there is no confusion.* The HJB for the agent is[2]

$$[r_1 + b(z)\lambda]\,v(z) = \max_{\tilde{a}\in[0,1]} r_1[u+(1-\tilde{a})c] +^2 [1-a(z)]\left[1-\tilde{a}-\tfrac{1}{2}(1-a(z))\right]v'(z) + \tfrac{1}{2}{}^2[1-a(z)]^2 v''(z). \tag{B.2}$$

The following sequence of claims establishes some necessary properties of any pseudo-best reply $a(\cdot)$.

Claim 1. $a(z) = 1 - \frac{r_1 c}{\max\{r_1 c, -^2 v'(z)\}}$, *for all* $z \in (-\infty,\infty)$.

Proof. Since the RHS of (B.2) is affine in the choice variable, optimality requires that, for almost every $z \in R$,[3]

$$a(z)\begin{cases} = 0, & \text{if } r_1 c +^2 [1-a(z)]v' > 0 \\ \in [0,1] & \text{if } r_1 c +^2 [1-a(z)]v' = 0 \\ = 1, & \text{if } r_1 c +^2 [1-a(z)]v' < 0 \end{cases}\cdot$$

[1] We call such $a(\cdot)$ pseudo-best reply because $b(\cdot)$ by itself does not lead to a well-defined strategy of the principal; the principal's interpretation of the observed signal into her posterior belief depends on (her conjecture of) the agent's strategy. The equilibrium condition that the principal's conjecture coincides with the agent's actual strategy is imposed as part of the definition of a pseudo-best reply.

[2] Since v is regular, v is C^2 except at possibly finite points. This HJB holds on any interval over which v is C^2.

[3] Since $\sup_{z\in R} a(z) < 1$ by definition of a pseudo-best reply (and by Lemma 2.4.1), the law of motion (B.1) implies that the distribution of Z_t has full support for any $t > 0$, i.e., $\text{supp}(Z_t) = R, \forall t > 0$.

This implies that, for almost every $z \in R$, we have

$$a(z) = 1 - \frac{r_1 c}{\max\{r_1 c, -^2 v'(z)\}}. \tag{B.3}$$

Since both sides of (B.3) are continuous in z (recall that $a(\cdot)$ is Lipschitz by assumption, and $v(\cdot)$ is C^1 by assumption), we conclude that (B.3) must hold for every $z \in R$.[4] □

Claim 2. *Fix any $z_1 < z_2 \leq z^*$ and suppose that $a(z) = 0$ for all $z \in (z_1, z_2)$. Then,*

$$v(z) = u + c + A_1 e^{\xi_L z} + A_2 e^{\xi'_L z}, \forall z \in (z_1, z_2) \tag{B.4}$$

for some $A_1, A_2 \in R$, where $\xi_L > 0 > \xi'_L$ are the two roots of the characteristic equation $\xi^2 + \xi = 2r_1/^2$.

Proof. Since $b(z) = 0$ and $a(z) = 0$ for all $z \in (z_1, z_2)$, equation (B.2) becomes

$$r_1 v(z) = r_1(u + c) + \frac{1^2}{2}[v'(z) + v''(z)].$$

It is easy to verify that its general solution is given by (B.4). □

Claim 3. *Fix any $z^* \leq z_1 < z_2$ and suppose that $a(z) = 0$ for all $z \in (z_1, z_2)$. Then,*

$$v(z) = \frac{r_1}{r_1 + \lambda}(u + c) + B_1 e^{\xi_R z} + B_2 e^{\xi'_R z}, \forall z \in (z_1, z_2) \tag{B.5}$$

for some $B_1, B_2 \in R$, where $\xi_R < 0 < \xi'_R$ are the two roots of the characteristic equation $\xi^2 + \xi = 2(r_1 + \lambda)/^2$.

Proof. Since $b(z) = 1$ and $a(z) = 0$ for all $z \in (z_1, z_2)$, equation (B.2) becomes

$$(r_1 + \lambda)v(z) = r_1(u + c) + \frac{1^2}{2}[v'(z) + v''(z)].$$

It is easy to verify that its general solution is given by (B.5). □

Now, let us denote by Φ and ϕ the CDF and PDF of the standard normal distribution, respectively.

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[4] This is because any continuous function that is 0 almost everywhere is equal to 0 everywhere.

Claim 4. *Fix any $z_1 < z_2 \leq z^*$ and suppose that $a(z) \in (0,1)$ for all $z \in (z_1, z_2)$. Then,*

$$v(z) = u + \sqrt{\kappa_L}\Phi^{-1}(C_1 e^z + C_2), \forall z \in (z_1, z_2) \tag{B.6}$$

and

$$a(z) = 1 + \frac{\sqrt{2r_1}}{} \frac{\phi\left(\Phi^{-1}(C_1 e^z + C_2)\right)}{C_1 e^z} \tag{B.7}$$

for some $C_1 < 0$ and $C_2 \in R$, where $\kappa_L := \frac{r_1 c^2}{2^2}$.

Moreover, $a(z)$ is strictly increasing, or strictly decreasing, or first strictly decreasing and then strictly increasing on (z_1, z_2).

Proof. Fix any $z_1 < z_2 \leq z^*$ such that $a(z) \in (0,1)$ for all $z \in (z_1, z_2)$. Claim 1 implies that

$$a(z) = 1 + \frac{r_1 c}{{}_2 v'(z)}, \forall z \in (z_1, z_2). \tag{B.8}$$

Substituting (B.8) into (B.2) and setting $b(z) = 0$, we have

$$v(z) = u + \kappa_L \frac{v''(z) - v'(z)}{v'(z)^2}. \tag{B.9}$$

It is easy to verify that its general solution is given by (B.6), and that the resulting $a(\cdot)$ implied by (B.8) is given by (B.7). Moreover, since $a(z) \in (0,1)$, we must have $v'(z) < 0$, i.e., $C_1 < 0$.

To analyze the monotonicity of $a(\cdot)$ on (z_1, z_2), we first establish the following equality which links $a'(z)$ to $a(z)$:

$$1 - a(z) - a'(z) = 2\left(\frac{v(z) - u}{c}\right). \tag{B.10}$$

By (B.8),

$$1 - a(z) = -\frac{r_1 c}{{}_2 v'(z)}.$$

Differentiating this expression, we obtain

$$-a'(z) = \frac{r_1 c}{2} \frac{v''(z)}{v'(z)^2} = -\frac{v''(z)}{v'(z)}[1 - a(z)].$$

Recall, from the agent's HJB (B.9) in this case, that

$$\frac{v''(z)}{v'(z)} = 1 + \frac{[v(z) - u]\, v'(z)}{\kappa_L} = 1 - 2\left(\frac{v(z) - u}{c}\right)\frac{1}{1 - a(z)},$$

where the second equality follows from (B.8) and $\kappa_L = \frac{r_1 c^2}{2^2}$. Equation (B.10) then follows immediately.

Equation (B.10) implies that

(i) If $a'(\tilde{z}) < 0$ for some $\tilde{z} \in (z_1, z_2)$, then $a'(z) < 0$ for all $z \in (z_1, \tilde{z})$.

(ii) There does not exist an interval $I \subseteq (z_1, z_2)$ s.t. $a'(z) = 0$ for all $z \in I$.

(iii) If $a(\tilde{z}) \geq 0$ for some $\tilde{z} \in (z_1, z_2)$, then $a(\cdot)$ is strictly increasing on $(\tilde{z}, z_2)$.

To see (i), suppose that $a'(\tilde{z}) < 0$ for some $\tilde{z} \in (z_1, z_2)$. Since $a(\cdot)$ given by (B.7) is a smooth function on (z_1, z_2), we can define $\underline{z} = \inf\left\{z \in [z_1, \tilde{z}) : a'(\cdot)|_{(z,\tilde{z}]} < 0\right\}$. Result (i) is proved if $\underline{z} = z_1$. Suppose (for a contradiction) that $\underline{z} > z_1$. Continuity of a' implies that $a'(\underline{z}) = 0$. Moreover, since $a'(z) < 0$ for all $z \in (\underline{z}, \tilde{z}]$, we have $a(\underline{z}) > a(\tilde{z})$. Then,

$$2\left(\frac{v(\underline{z}) - u}{c}\right) = 1 - a(\underline{z}) - a'(\underline{z}) < 1 - a(\tilde{z}) - a'(\tilde{z}) = 2\left(\frac{v(\tilde{z}) - u}{c}\right)$$

where the equalities follow from (B.10) and the strict inequality follows from $a(\underline{z}) > a(\tilde{z})$ and $a'(\underline{z}) = 0 > a'(\tilde{z})$. But this is a contradiction to $v(\underline{z}) > v(\tilde{z})$ because $C_1 < 0$ and v is strictly decreasing on (z_1, z_2).

To see (ii), suppose (for a contradiction) that there exists an interval $I \subseteq (z_1, z_2)$ s.t. $a'(z) = 0$ for all $z \in I$. Then, the LHS of (B.10) is constant on I while the RHS is strictly decreasing, a contradiction.

To see (iii), suppose that $a'(\tilde{z}) \geq 0$ for some $\tilde{z} \in (z_1, z_2)$. Then, we must have $a'(z) \geq 0$ for all $z \in (\tilde{z}, z_2)$, for otherwise we would reach a contradiction to (i). Further, take any z_3, z_4 s.t. $\tilde{z} \leq z_3 < z_4 \leq z_2$. Since $a'(z) \geq 0$, we know that $a(z_3) \leq a(z_4)$. But this inequality must be strict, for otherwise $a(z_3) = a(z) = a(z_4)$ for all $z \in (z_3, z_4)$ contradicting Result (ii). So, $a(\cdot)$ is strictly increasing on $(\tilde{z}, z_2)$.

Finally, to establish the monotonicity of $a(\cdot)$, suppose first that $a'(z) \geq 0$ for all $z \in (z_1, z_2)$. The same argument for (iii) above proves that a must be strictly increasing on (z_1, z_2). Suppose now that $a'(\tilde{z}) < 0$ for some $\tilde{z} \in (z_1, z_2)$. Let $Z \subseteq (z_1, z_2)$ be the largest interval containing $\tilde{z}$ s.t. $a'(z) < 0$ for all $z \in Z$. Result (i) immediately implies that $\inf Z = z_1$. If $\sup Z = z_2$, then $a(\cdot)$ is strictly decreasing on (z_1, z_2). If $\sup Z < z_2$, continuity of a' implies that $a'(\sup Z) = 0$. Then, Result (iii) implies that $a(\cdot)$ is strictly increasing on $(\sup Z, z_2)$. In summary, $a(\cdot)$ is either strictly increasing, or strictly decreasing, or first strictly decreasing and then strictly increasing on (z_1, z_2). □

Claim 5. *Fix any $z^* \leq z_1 < z_2$ and suppose that $a(z) \in (0,1)$ for all $z \in (z_1, z_2)$. Then,*

$$v(z) = \frac{r_1}{r_1 + \lambda} u + \sqrt{\kappa_R} \Phi^{-1}(D_1 e^z + D_2), \forall z \in (z_1, z_2) \tag{B.11}$$

and

$$a(z) = 1 + \frac{\sqrt{2(r_1 + \lambda)}}{} \frac{\phi\left(\Phi^{-1}(D_1 e^z + D_2)\right)}{D_1 e^z} \tag{B.12}$$

for some $D_1 < 0$ and $D_2 \in R$, where $\kappa_R := \frac{r_1^2 c^2}{2(r_1+\lambda)^2}$.

Moreover, $a(z)$ is strictly increasing, or strictly decreasing, or first strictly decreasing and then strictly increasing on (z_1, z_2).

Proof. The idea of this proof is completely analogous to that of Claim 4. Fix any $z^* \leq z_1 < z_2$ such that $a(z) \in (0,1)$ for all $z \in (z_1, z_2)$. In this case, equation (B.8) still holds. Substituting it into (B.2) and setting $b(z) = 1$, we have

$$v(z) = \frac{r_1}{r_1 + \lambda} u + \kappa_R \frac{v''(z) - v'(z)}{v'(z)^2}.$$

It is easy to verify that its general solution is given by (B.11), and that the resulting a implied by (B.8) is given by (B.12). Moreover, since $a(z) \in (0,1)$, we must have $v'(z) < 0$, i.e., $D_1 < 0$.

Analogous to (B.10), the following equation links $a(z)$ to $a'(z)$ in this case:

$$1 - a(z) - a'(z) = 2 \left(\frac{v(z) - \frac{r_1}{r_1+\lambda} u}{c} \right) \left(\frac{r_1 + \lambda}{r_1} \right). \tag{B.13}$$

The proof of equation (B.13) and the monotonicity of $a(\cdot)$ is along the same lines as in the proof of Claim 4, and is thus omitted. □

Claim 6. *If $a(\tilde{z}) > 0$, then there are z_L, z_R such that $z_L < \tilde{z} < z_R$ and $a(z_L) = a(z_R) = 0$.*

Proof. Let $a(\tilde{z}) > 0$ and suppose, seeking a contradiction, that $a(z) > 0$ for all $z > \tilde{z}$. Then, for all such z, we would have (by Claim 1) $v'(z) = -\frac{r_1 c}{2[1-a(z)]} \leq -\frac{r_1 c}{2}$. Taking the limit for arbitrary large z, we obtain $\lim_{z \to +\infty} v(z) = -\infty$, a contradiction as v is always nonnegative. Similarly, if $a(z) > 0$ for all $z < \tilde{z}$, then $\lim_{z \to -\infty} v(z) = +\infty$, which contradicts that v is bounded above by $u + c$. □

Claim 7. *For any z_1, z_2 such that either $z_1 < z_2 \leq z^*$ or $z^* \leq z_1 < z_2$, $a(z_1) = a(z_2) = 0$ implies $a(z) = 0$ for all $z \in [z_1, z_2]$.*

Proof. Fix any $z_1 < z_2 \leq z^*$ such that $a(z_1) = a(z_2) = 0$. Suppose (for a contradiction) that there exists $\tilde{z} \in (z_1, z_2)$ s.t. $a(\tilde{z}) \in (0, 1)$. Let Z be the largest interval containing $\tilde{z}$ such that $a(z) \in (0, 1)$ for all $z \in Z$. Obviously, $z_1 \leq \inf Z < \sup Z \leq z_2$, and $a(\inf Z) = a(\sup Z) = 0$ because $a(\cdot)$ is continuous. By Claim 4, $a(\cdot)$ is strictly increasing, or strictly decreasing, or first strictly decreasing and then strictly increasing on Z. Since $a(\inf Z) = 0$, $a(\cdot)$ can only be strictly increasing on Z, but this contradicts the continuity of $a(\cdot)$ at $\sup Z$. An analogous argument which invokes Claim 5 establishes same result for any $z^* \leq z_1 < z_2$ such that $a(z_1) = a(z_2) = 0$. □

Corollary B.1.1. *One of the following must hold for any pseudo-best reply $a(\cdot)$:*

- *$a(z) = 0$ for all $z \in R$;*
- *$a(z)$ is hump-shaped (and maximized at z^*).*

Proof. Suppose that $a(\cdot)$ is not always equal to 0. Then there exists $\tilde{z}$ s.t. $a(\tilde{z}) > 0$. Let Z be the largest interval containing $\tilde{z}$ such that $a(z) > 0$ for all $z \in Z$. Let $z_L = \inf Z$ and $z_R = \sup Z$. By Claim 6, $-\infty < z_L < z_R < \infty$. By continuity of a, $a(z_L) = a(z_R) = 0$. Then we must have $z^* \in (z_L, z_R)$, for otherwise we would reach a contradiction to Claim 7. Moreover, for any $z \in (z_L, z_R)^c$, we must have $a(z) = 0$, for otherwise we can construct another Z' which also contains z^* such that $Z' - Z \neq \emptyset$, contradicting the maximality of Z. Since $a(z_L) = 0$, Claim 4 implies that $a(\cdot)$ must be strictly increasing on (z_L, z^*). Similarly, since $a(z_R) = 0$, Claim 5 implies that a must be strictly decreasing on (z^*, z_R).

In summary, if $a(\cdot)$ is not always equal to 0, then there exist $z_L < z_R$ s.t. $a(\cdot) = 0$ on $(-\infty, z_L) \cup (z_R, \infty)$, $a(\cdot)$ is strictly increasing on (z_L, z^*), and strictly decreasing on (z^*, z^R); that is, $a(\cdot)$ is hump-shaped and maximized at z^*. □

Claim 8. *$a(\cdot)$ is hump-shaped if and only if $r_1 < r^*$, where r^* is the unique solution to (2.2).*

Proof. **("Only if" part)** Suppose first that $a(\cdot)$ is hump-shaped. We will show that this implies $r_1 < r^*$. By Definition 2.4.3, there exist z_L, z_R s.t. $-\infty < z_L < z^* < z_R < \infty$ such that $a(z) = 0$ on $(-\infty, z_L] \cup [z_R, \infty)$ and $a(z) \in (0,1)$ on (z_L, z_R). Let

$$v^* := v(z^*), \quad v_L := v(z_L), \quad v_R := v(z_R).$$

We now calculate the undetermined coefficients in Claims 2 to 5 in various parts of the agent's policy and value, as functions of v^* and model parameters. First, consider $z < z^*$. As $z \to -\infty$, the agent's value function is given in Claim 2 by (B.4). Because $v(\cdot)$ is bounded, we must have

$$A_2 = 0. \tag{B.14}$$

Note that Claim 1 implies that $r_1 c = -^2 v'(z_L)$, that is, $v'(z_L) = -\frac{r_1 c}{2}$. By Claims 2 and 4, the value function $v(\cdot)$ must satisfy the following value-matching and smooth-pasting conditions at z_L and z^*:

$$\begin{aligned}
u + c + A_1 e^{\xi_L z_L} &= v_L, && \text{(value-matching at } z_L) \\
u + \sqrt{\kappa_L}\Phi^{-1}\left(C_1 e^{z_L} + C_2\right) &= v_L, && \text{(value-matching at } z_L) \\
u + \sqrt{\kappa_L}\Phi^{-1}\left(C_1 e^{z^*} + C_2\right) &= v^*, && \text{(value-matching at } z^*) \\
A_1 \xi_L e^{\xi_L z_L} &= -\tfrac{r_1 c}{2}, && \text{(smooth-pasting at } z_L) \\
\frac{\sqrt{\kappa_L} C_1 e^{z_L}}{\phi\left(\frac{v_L - u}{\sqrt{\kappa_L}}\right)} &= -\tfrac{r_1 c}{2}. && \text{(smooth-pasting at } z_L)
\end{aligned}$$

These five conditions can uniquely pin down the undetermined vector $(v_L, z_L, A_1, C_1, C_2)$

as:

$$v_L = u + c - \frac{r_1 c}{\xi_L^2}, \tag{B.15}$$

$$e^{z_L} = e^{z^*}\left[\frac{\sqrt{2r_1}\,\phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right)}{\sqrt{2r_1}\,\phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right)+\Phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right)-\Phi\left(\frac{v^*-u}{\sqrt{\kappa_L}}\right)}\right], \tag{B.16}$$

$$A_1 = -\frac{r_1 c}{\xi_L^2} e^{-\xi_L z_L} \tag{B.17}$$

$$C_1 = e^{-z^*}\left[\Phi\left(\frac{v^*-u}{\sqrt{\kappa_L}}\right) - \Phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right) - \sqrt{2r_1}\,\phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right)\right], \tag{B.18}$$

$$C_2 = \Phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right) + \sqrt{2r_1}\,\phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right). \tag{B.19}$$

Now, consider $z > z^*$. As $z \to \infty$, the agent's value function is given in Claim 3 by (B.5). Because $v(\cdot)$ is bounded, we must have

$$B_2 = 0. \tag{B.20}$$

Note that Claim 1 implies that $r_1 c = -^2 v'(z_R)$, that is, $v'(z_R) = -\frac{r_1 c}{2}$. By Claims 3 and 5, the value function $v(\cdot)$ must satisfy the following value-matching and smooth-pasting conditions at z_R and z^*:

$$(u+c)\frac{r_1}{r_1+\lambda} + B_1 e^{\xi_R z_R} = v_R, \qquad \text{(value-matching at } z_R\text{)}$$

$$\frac{r_1}{r_1+\lambda}u + \sqrt{\kappa_R}\Phi^{-1}\left(D_1 e^{z_R} + D_2\right) = v_R, \qquad \text{(value-matching at } z_R\text{)}$$

$$\frac{r_1}{r_1+\lambda}u + \sqrt{\kappa_R}\Phi^{-1}\left(D_1 e^{z^*} + D_2\right) = v^*, \qquad \text{(value-matching at } z^*\text{)}$$

$$B_1 \xi_R e^{\xi_R z_R} = -\frac{r_1 c}{2}, \qquad \text{(smooth-pasting at } z_R\text{)}$$

$$\frac{\sqrt{\kappa_R} D_1 e^{z_R}}{\phi\left(\frac{v(z_R)-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right)} = -\frac{r_1 c}{2}. \qquad \text{(smooth-pasting at } z_R\text{)}$$

These five conditions can uniquely pin down the undetermined vector $(v_R, z_R, B_1, D_1, D_2)$

as:

$$v_R = \frac{r_1}{r_1+\lambda}(u+c) - \frac{r_1 c}{\xi_R^2}, \tag{B.21}$$

$$e^{z_R} = e^{z^*}\left[\frac{\sqrt{2(r_1+\lambda)}\,\phi\left(\frac{v_R-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right)}{\sqrt{2(r_1+\lambda)}\,\phi\left(\frac{v_R-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right)+\Phi\left(\frac{v_R-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right)-\Phi\left(\frac{v^*-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right)}\right], \tag{B.22}$$

$$B_1 = -\frac{r_1 c}{\xi_R^2}e^{-\xi_R z_R}, \tag{B.23}$$

$$D_1 = e^{-z^*}\left[\Phi\left(\frac{v^*-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right)-\Phi\left(\frac{v_R-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right)-\sqrt{2\left(r_1+\lambda\right)}\,\phi\left(\frac{v_R-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right)\right], \tag{B.24}$$

$$D_2 = \Phi\left(\frac{v_R-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right)+\sqrt{2\left(r_1+\lambda\right)}\,\phi\left(\frac{v_R-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right). \tag{B.25}$$

Given v^* and model parameters, equations (B.4) through (B.7), (B.11) and (B.12) with coefficients given by (B.14) through (B.25) fully determine the agent's policy function $a(\cdot)$ and his value function $v(\cdot)$ on R. Since $a(z) \in (0,1)$ on (z_L, z_R), Claim 1 implies that $v'(z) < 0$ on (z_L, z_R), which in turn implies that $v_L > v_R$. Note that v_L and v_R are given by (B.15) and (B.21), both of which are independent of v^*. In particular, $v_L > v_R$ amounts to $u + c - \frac{r_1 c}{\xi_L^2} > \frac{r_1}{r_1+\lambda}(u+c) - \frac{r_1 c}{\xi_R^2}$. Straightforward calculation shows that this is equivalent to

$$r_1(\sqrt{1+8r_1/^2}+\sqrt{1+8(r_1+\lambda)/^2})+\lambda(\sqrt{1+8r_1/^2}+1) < 4\lambda\left(\tfrac{u}{c}+1\right),$$

that is, $r_1 < r^*$ (see condition (2.2)). Therefore, we have shown that "$a(\cdot)$ is hump-shaped $\Rightarrow r_1 < r^*$", so the "only if" part of the claim is proved.

("If" part) Suppose now that $a(\cdot)$ is not hump-shaped. We will show that this implies $r_1 \geq r^*$. By Corollary B.1.1, we know that $a(z) = 0$ for all $z \in R$. Then, by Claims 2 and 3, $v(\cdot)$ is given by (B.4) for $z < z^*$ and by (B.5) for $z > z^*$. We now pin down the undetermined coefficients (A_1, A_2, B_1, B_2) as functions of model parameters. Since $v(\cdot)$ is bounded as $z \to -\infty$ or $z \to +\infty$, we must have

$$A_2 = 0, \tag{B.26}$$

$$B_2 = 0. \tag{B.27}$$

Also, the value function $v(\cdot)$ must satisfy the following value-matching and smooth-pasting condition at z^*:

$$u + c + A_1 e^{\xi_L z^*} = \tfrac{r_1}{r_1+\lambda}(u+c) + B_1 e^{\xi_R z^*}, \qquad \text{(value-matching at } z^*\text{)}$$
$$A_1 \xi_L e^{\xi_L z^*} = B_1 \xi_R e^{\xi_R z^*}. \qquad \text{(smooth-pasting at } z^*\text{)}$$

These two conditions uniquely pin down (A_1, B_1) as:

$$A_1 = \tfrac{\xi_R}{\xi_L - \xi_R} \tfrac{\lambda}{r_1+\lambda}(u+c) e^{-\xi_L z^*}, \tag{B.28}$$
$$B_1 = \tfrac{\xi_L}{\xi_L - \xi_R} \tfrac{\lambda}{r_1+\lambda}(u+c) e^{-\xi_R z^*}. \tag{B.29}$$

Since $a(z) = 0$ for all $z \in R$, by Claim 1 we must have $r_1 c \geq -^2 v'(z)$ for all $z \in R$. In particular, this should hold at z^*: $r_1 c \geq -\frac{\xi_R \xi_L}{\xi_L - \xi_R} \frac{\lambda}{r_1+\lambda}(u+c)^2$. Straightforward calculation shows that this is equivalent to

$$r_1(\sqrt{1+8r_1/^2} + \sqrt{1+8(r_1+\lambda)/^2}) + \lambda(\sqrt{1+8r_1/^2}+1) \geq 4\lambda\left(\tfrac{u}{c}+1\right),$$

that is, $r_1 \geq r^*$ (see condition (2.2)). Therefore, we have shown that

"$a(\cdot)$ is not hump-shaped $\Rightarrow r_1 \geq r^*$",

so the "if" part of the claim is also proved. □

In Claim 8, we find that if $r_1 < r^*$, then $a(\cdot)$ is hump-shaped, in which case we can express all coefficients and cutoffs in the agent's policy and value functions in closed form with respect to $v^* := v(z^*)$ and model parameters. However, v^* itself is an endogenous object that needs to be determined. Our next claim, proved in the Online Appendix, paves the final way for establishing Lemma 2.4.3.

Claim 9. *Suppose that $r_1 < r^*$, and let v_L and v_R be given by (B.15) and (B.21), respec-*

tively. Define $a^*_-, a^*_+ : [v_R, v_L] \to R$ *by*

$$a^*_-(x) := 1 - \frac{\frac{\sqrt{2r_1}}{}\phi\left(\frac{x-u}{\sqrt{\kappa_L}}\right)}{\frac{\sqrt{2r_1}}{}\phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right) + \Phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right) - \Phi\left(\frac{x-u}{\sqrt{\kappa_L}}\right)}, \tag{B.30}$$

$$a^*_+(x) := 1 - \frac{\frac{\sqrt{2(r_1+\lambda)}}{}\phi\left(\frac{x-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right)}{\frac{\sqrt{2(r_1+\lambda)}}{}\phi\left(\frac{v_R-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) + \Phi\left(\frac{v_R-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) - \Phi\left(\frac{x-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right)}. \tag{B.31}$$

Then, $a^*_-(\cdot)$ *is strictly decreasing on* $[v_R, v_L]$ *with* $a^*_-(v_R) \in (0,1)$ *and* $a^*_-(v_L) = 0$*;* $a^*_+(\cdot)$ *is strictly increasing on* $[v_R, v_L]$ *with* $a^*_+(v_R) = 0$ *and* $a^*_+(v_L) \in (0,1)$*.*

Proof. See Online Appendix. □

Proof of Lemma 2.4.3. Suppose first that $r_1 \geq r^*$. By Corollary B.1.1 and Claim 8, any pseudo-best reply $a(\cdot)$ must be such that $a(z) = 0$ for all $z \in R$. Obviously, such function is unique. To verify that $a(z) = 0$ for all $z \in R$ is indeed a solution to (2.6), note that we have shown in the proof of Claim 8 that, together with the $v(\cdot)$ given by (B.4) and (B.5) and the coefficients given by (B.26) through (B.29), it satisfies the agent's HJB equation (B.2).[5] Since $v(\cdot)$ is bounded, we have $\lim_{t\to\infty} e^{-r_1 t} E\left[v(z_t)\, 1_{\{\tau \geq t\}}\right] = 0$, where τ is the stopping time when the relationship is terminated. Then by Ross (2008, Theorem 3.3.5), $a(z) = 1$ for all $z \in R$ is indeed a solution to (2.6). In addition, $v(\cdot)$ is regular because the functions given by (B.4) and (B.5) are smooth, and value-matching and smooth-pasting conditions are imposed at z^*.

Suppose now that $r_1 < r^*$. By Claim 8, any psuedo-best reply $a(\cdot)$ is hump-shaped. We first show that such function, if it exists, must be unique. By Claims 2 to 5 and the proof of the "only if" part of Claim 8, such a policy function and the associated value function must satisfy (B.4) through (B.7), (B.11) and (B.12) with coefficients given by (B.14) through (B.25), given the value $v^* := v(z^*)$.

Recall that the functions a^*_- and a^*_+ are defined in (B.30) and (B.31), respectively. By Claim 4 and equations (B.18) and (B.19), it is easy to verify that $\lim_{z\uparrow z^*} a(z; v^*) = a^*_-(v^*)$. Similarly, by Claim 5 and equations (B.24) and (B.25), it is easy to verify that

[5] Everywhere except at z^* where v'' does not exist.

$\lim_{z\downarrow z^*} a(z;v^*) = a_+^*(v^*)$. Since $a(\cdot)$ is continuous at z^*, v^* must satisfy

$$a_-^*(v^*) = a_+^*(v^*). \tag{B.32}$$

By Claim 9, there is a unique $v^* \in (v_R, v_L)$ satisfying (B.32), rendering the unique (candidate) policy function. Finally, take such unique v^*, and let $a(\cdot)$ and $v(\cdot)$ be defined by (B.4) through (B.7), (B.11) and (B.12) with coefficients given by (B.14) through (B.25). Exactly the same verification argument as in the case of $r_1 \geq r^*$ confirms that $a(\cdot)$ indeed solves (2.6). In addition, $v(\cdot)$ is regular because the functions given by (B.4), (B.5), (B.6) and (B.11) are smooth, and value-matching and smooth-pasting conditions are imposed at z_L, z^* and z_R. □

Proof of Theorem 2.4.1

Lemmas 2.4.2 and 2.4.3 establish the unique *structure* of Markov equilibria. To prove Theorem 2.4.1, we still need an argument for equilibrium existence and uniqueness.

Proof of Theorem 2.4.1. Suppose first that $r_1 \geq r^*$. By Lemma 2.4.3, the agent's pseudo-best reply to any cutoff termination rule satisfies that $a(p) = 0$ for all $p \in (0,1)$. In fact, the verification theorem we invoke in proving Lemma 2.4.3 tells us that such $a(\cdot)$ satisfies the agent's optimality condition (2.6) in a stronger sense, even if we allow him to maximize over all strategies in $\mathcal{A}$ instead of over Markov controls in $\mathcal{P}$. On the other hand, given this Markovian strategy of the agent under which the belief span is $(0,1)$, the proof of Lemma 2.4.2 (in the Online Appendix) can be used verbatim to show that the principal has a unique best reply whose policy function b admits a cutoff $p^* \in (0,1)$. Hence, such (a,b) is the unique Markov equilibrium in this case.

Suppose now that $r_1 < r^*$. Lemmas 2.4.2 and 2.4.3 imply that any Markov equilibrium (a,b) must be such that a is hump-shaped and b has a cutoff structure. We now show that there exists a unique Markov equilibrium.

To show that a Markov equilibrium exists, note that the pseudo-best reply of the noninvestible agent can be described by a function φ_1 that maps a conjecture $\tilde{p}^*$ about the cutoff p^* used by the principal into a policy function $a(\cdot)$ defined on (0,1), the probability-domain version of the policy function in $z-$space constructed in Claim 8's proof. The function φ_1

is continuous.[6] Moreover, the best reply of the principal can be described by a function φ_2 that maps any Markov strategy α of the noninvestible agent into a unique cutoff p^*, and Lemma 2.4.2 also tells us that $p^* \in [p^{**}, p_H] \subset (0,1)$.[7]

Define the composition $\varphi := \varphi_2 \cdot \varphi_1$ that maps each conjecture $\tilde{p}^*$ into the associated optimal cutoff p^*.[8] The mapping φ satisfies

$$\varphi(\tilde{p}^*) =_{p' \in [p^{**}, p_H]} \hat{M}(p, p', \tilde{p}^*),$$

where

$$\hat{M}(p, p', \tilde{p}^*) := r_2 \int_0^{+\infty} \int_{p'}^{1} e^{-r_2 t} R(q) d\Gamma\left(t, q | \tilde{p}^*, p\right)$$

and Γ is the joint probability measure of getting the first Poisson shock in the stopping region $[p', 1)$ at time t and state $\tilde{p}$, when the prior belief at time zero is p. Notice that $\varphi(p^*)$ is the unique solution to this maximization problem and is independent of the prior due to its Markovian nature. Moreover, $\hat{M}(p, p', \tilde{p}^*)$ is jointly continuous in $(p', \tilde{p}^*)$. Since the choice space is compact and the objective is continuous in both the choice variable p' and the "parameter" $\tilde{p}^*$, the Maximum Theorem implies that φ is continuous.

Since φ is a continuous function mapping from $(0,1)$ to $[p^{**}, p_H]$ such that

$$\liminf_{p \to 0} [\varphi(p) - p] \geq p^{**} > 0$$

and

$$\limsup_{p \to 1} [\varphi(p) - p] \leq p_H - 1 < 0,$$

the intermediate value theorem implies that there must be $p^* \in (0,1)$ such that $\varphi(p^*) = p^*$. Taking any such p^*, it is easy to verify that $\left(a^* := \varphi_1(p^*), b^* := 1_{\{p_t \geq p^*\}}\right)$ represents a Markov equilibrium. This proves equilibrium existence.

To show that the Markov equilibrium is unique, let (a, b) be a Markov equilibrium in which the principal uses a threshold p^*, associated with a likelihood z^*. Consider the payoff

[6] This follows from Lemma OA.1 in the Online Appendix, i.e., the translation invariance of the agent's problem.

[7] Lemma 2.4.2 is stated for an equilibrium. However, its proof is applicable to the principal's best reply to any Markovian strategy of the agent.

[8] Note that $SP[\varphi_1(\tilde{p}^*)] = (0,1)$ for all $\tilde{p}^* \in (0,1)$, because the pseudo-best reply delivered by Lemma 2.4.3 (derived in Claim 8's proof) is always bounded away from 1 by a positive number and thus the diffusion coefficient of the belief process is bounded away from 0.

of the principal when deviating to a different threshold z' when the state is z.

$$M(z,z',z^*) := p(z)E\left\{e^{-r_2 T_{NI}(z,z',z^*)}\right\}w_{NI} + (1-p(z))E\left\{e^{-r_2 T_I(z,z',z^*)}\right\}w_I,$$

where $T_\theta(z,z',z^*)$ is the random time of occurrence of the first Poisson shock that arrives while the state lies in the stopping interval $[z',+\infty)$, provided the initial state is z and the dynamics is conditioned on type $\theta \in \{I, NI\}$. By the conditional translation invariance property proved in Lemma OA.2 of the Online Appendix,

$$E\left\{e^{-r_2 T_\theta(z,z',z^*)}\right\} = E\left\{e^{-r_2 T_\theta(0,z'-z,z^*-z)}\right\}$$

for all $z, z', z^* \in R$. The FOC of the principal is

$$\frac{\partial M(z,z',z^*)}{\partial z'} = p(z)D_{NI}(z,z',z^*)w_{NI} + (1-p(z))D_I(z,z',z^*)w_I = 0,$$

where we define $D_\theta(z,z',z^*) := \frac{\partial E\left\{e^{-r_2 T_\theta(z,z',z^*)}\right\}}{\partial z'}$ for each $\theta \in \{I, NI\}$, $z, z', z^* \in R$. Note that this condition should hold for every $z \in R$. In equilibrium, the principal's choice of z' must coincide with z^*, so the FOC becomes: $p(z)D_{NI}(z,z^*,z^*)w_{NI} + (1 - p(z))D_I(z,z^*,z^*)w_I = 0$, which can be rewritten as:

$$p(z) = \frac{D_C(z,z^*,z^*)w_C}{D_C(z,z^*,z^*)w_C - D_S(z,z^*,z^*)w_S}.$$

Evaluating the limit from below as $z \uparrow z^*$, we have

$$p^* = p(z^*) = \frac{D_C(z^*,z^*,z^*)w_C}{D_C(z^*,z^*,z^*)w_C - D_S(z^*,z^*,z^*)w_S} = \frac{D_C(0,0,0)w_C}{D_C(0,0,0)w_C - D_S(0,0,0)w_S},$$

where the last equality follows from Lemma OA.2 in the Online Appeneix, i.e., the conditional translation invariance of the principal's payoff function. Since the RHS is independent of p^*, we conclude that there can be at most one value p^* consistent with a Markov equilibrium, establishing the uniqueness claim. □

Corollary B.1.2. *The following hold:*

1. *$W(\cdot)$ is (weakly) increasing, nonnegative and convex on $(0,1)$, and it satisfies*

$$\lim_{p\to 0} W(p) = 0$$

and

$$\lim_{p\to 1} W(p) = \frac{\lambda}{r_2+\lambda} w_{NI}.$$

2. *$v(\cdot)$ is strictly decreasing on R, and it satisfies*

$$\lim_{z\to -\infty} v(z) = u + c$$

and

$$\lim_{z\to \infty} v(z) = \frac{r_1}{r_1+\lambda}(u+c).$$

Moreover, $v(\cdot)$ is concave on $(-\infty, z^*)$ and convex on (z^*, ∞).

Proof. See Online Appendix. □

B.1.2 Expected Performance: Toward a Proof of Theorem 2.5.1

In this section, we prove Theorem 2.5.1 which is about the non-monotonicity of the expected performance.

Given a Markov equilibrium (a, b) (where the equilibrium policy functions are defined on the $z-$space), let v be the agent's value function, z^* be the principal's termination cutoff, and recall that the agent's expected performance is given by

$$EP(z) = [1 - (1 - a(z))p(z)], \tag{B.33}$$

where $p(z) = \frac{e^z}{1+e^z}$. Our analysis in this section fixes all model parameters, except r_1 and/or λ.[9]

Lemma B.1.1. *If $r_1 \geq r^*$, then $EP(\cdot)$ is strictly decreasing on R. If $r_1 < r^*$, then either $EP(\cdot)$ is strictly decreasing on R, or $EP(\cdot)$ is*

- *strictly decreasing for $z < \underline{z}$, where $\underline{z}$ is some number in $[z_L, z^*)$;*

[9] In Section 2.5, we defined $EP_t = \mu[1 - (1 - a_t)p_t]$. To ease notation, here we divide the original expression by σ, which is completely equivalent because for this exercise μ and σ are fixed numbers.

- *strictly increasing for* $z \in (\underline{z}, z^*)$*;*

- *strictly decreasing for* $z > z^*$*.*

Proof. If $r_1 \geq r^*$, Theorem 2.4.1 tells us that $a(z) = 0$ for all $z \in R$. Thus, $EP(z) = [1 - p(z)]$, which is strictly decreasing on R.

If $r_1 < r^*$, by Theorem 2.4.1 the agent's equilibrium policy function a is hump-shaped, with cutoffs denoted by z_L and z_R such that $a(z) > 0$ if and only if $z \in (z_L, z_R)$. Obviously, $EP(\cdot)$ is strictly decreasing on $(-\infty, z_L)$ and on (z^*, ∞), because on each of these intervals $a(\cdot)$ is weakly decreasing and $p(\cdot)$ is strictly increasing in z. We now focus on the monotonicity of $EP(\cdot)$ on (z_L, z^*).

First, analogous to (B.10), we establish the following equality which links $EP(z)$ to $EP'(z)$:

$$-EP(z) - \frac{EP'(z)}{p(z)} = 2\left(\frac{v(z) - u}{c}\right). \tag{B.34}$$

To see this, note first that since $a(z) > 0$ on (z_L, z^*), by equation (B.33) and Claim 1 we have

$$-EP(z) = [1 - a(z)]p(z) = -\frac{r_1 c}{}\frac{p(z)}{v'(z)}. \tag{B.35}$$

Differentiating this expression, we obtain

$$-EP'(z) = -\frac{r_1 c}{}\left(-\frac{v''(z)p(z)}{v'(z)^2} + \frac{p(z)[1-p(z)]}{v'(z)}\right) = [1-a(z)]p(z)\left(-\frac{v''(z)}{v'(z)} + 1 - p(z)\right).$$

where the first equality follows from $p'(z) = p(z)[1 - p(z)]$ and the second equality follows from (B.35). Recall, from the agent's HJB (B.9) in this case, that

$$\frac{v''(z)}{v'(z)} = 1 + \frac{[v(z) - u]\,v'(z)}{\kappa_L} = 1 - 2\left(\frac{v(z) - u}{c}\right)\frac{1}{1 - a(z)},$$

where the second equality follows from Claim 1 and $\kappa_L = \frac{r_1 c^2}{2^2}$. Thus,

$$\begin{aligned}
-\frac{EP'(z)}{p(z)} &= [1 - a(z)]\left[2\left(\frac{v(z) - u}{c}\right)\frac{1}{1 - a(z)} - p(z)\right] \\
&= 2\left(\frac{v(z) - u}{c}\right) - [1 - a(z)]p(z) \\
&= 2\left(\frac{v(z) - u}{c}\right) - [-EP(z)],
\end{aligned}$$

where the last equality follows from (B.35). Equation (B.34) then follows immediately.

From equation (B.34), we can apply the same argument as that after equation (B.10) to show that $EP(\cdot)$ must be either strictly increasing, or strictly decreasing, or first strictly decreasing and then strictly increasing on (z_L, z^*), a property that echoes what we have shown for the policy function $a(\cdot)$ in Claim 4. Since we know that $EP(\cdot)$ is strictly decreasing on $(-\infty, z_L)$ and on (z^*, ∞), the result in the lemma follows. □

Corollary B.1.3. *$EP(\cdot)$ is non-monotone if and only if $r_1 < r^*$ and $EP'_-(z^*) > 0$.*

Lemma B.1.2. *$EP(\cdot)$ is non-monotone if and only if $r_1 < r^*$ and $[1 - a(z^*)]p(z^*) > 2\left(\frac{v^*-u}{c}\right)$, where $v^* := v(z^*)$.*

Proof. Since $a(z) > 0$ on $(z_L, z^*]$, by Claim 1 and equation (B.6) we have

$$1 - a(z) = -\frac{r_1 c}{2 v'(z)} \propto \frac{\phi\left(\frac{v(z)-u}{\sqrt{\kappa_L}}\right)}{e^z}, \forall z \in (z_L, z^*]. \tag{B.36}$$

Then,

$$\begin{aligned}
& EP'_-(z^*) > 0 \\
\iff\quad & \frac{d}{dz}\left[(1-a(z))p(z)\right]\Big|_{z=z^*_-} < 0 && \text{(by (B.33))} \\
\iff\quad & -a'(z)p(z) + [1-a(z)]p(z)[1-p(z)]\Big|_{z=z^*_-} < 0 && \text{(because } p'(z) = p(z)[1-p(z)]\text{)} \\
\iff\quad & [1-a(z)]p(z)\left[-\tfrac{a'(z)}{1-a(z)} + 1 - p(z)\right]\Big|_{z=z^*_-} < 0 \\
\iff\quad & [1-a(z)]p(z)\left[\tfrac{d\ln[1-a(z)]}{dz} + 1 - p(z)\right]\Big|_{z=z^*_-} < 0 \\
\iff\quad & \tfrac{d\ln[1-a(z)]}{dz} + 1 - p(z)\Big|_{z=z^*_-} < 0 && \text{(because } a(z) < 1 \text{ for all } z\text{)} \\
\iff\quad & \tfrac{d}{dz}\left[\ln\phi\left(\tfrac{v(z)-u}{\sqrt{\kappa_L}}\right) - z\right] + 1 - p(z)\Big|_{z=z^*_-} < 0 && \text{(by (B.36))} \\
\iff\quad & \left(-\tfrac{v^*-u}{\sqrt{\kappa_L}}\right)\left(\tfrac{v'(z^*)}{\sqrt{\kappa_L}}\right) - p(z^*) < 0 && \text{(because } d\ln\phi(x)/dx = -x\text{)} \\
\iff\quad & \tfrac{v^*-u}{\kappa_L} < -\tfrac{p(z^*)}{v'(z^*)} && \text{(because } -v'(z^*) > 0\text{)} \\
\iff\quad & 2\left(\tfrac{v^*-u}{c}\right) < [1-a(z^*)]p(z^*) && \text{(by } \kappa_L = \tfrac{r_1 c^2}{2^2} \text{ and (B.36))}
\end{aligned}$$

By Corollary B.1.3, the result follows. □

Corollary B.1.4. *$EP(\cdot)$ is non-monotone if $r_1 < r^*$ and $v^* < u$.*

For a given λ, recall that $r^*(\lambda)$ is the unique solution to (2.2). It is easy to see that there exists a unique $\lambda_1 > 0$ such that

$$r^*(\lambda_1) =^2 . \tag{B.37}$$

Consequently, $r^*(\lambda) >^2$ if and only if $\lambda > \lambda_1$.

The following lemma deals with the agent's discount rates that are close to $r^*(\lambda)$.

Lemma B.1.3. *If $\lambda > \lambda_1$ and $^2 < r_1 < r^*(\lambda)$, then $EP(\cdot)$ is non-monotone.*

Proof. Recall that, in the proof of Claim 8, we have shown that if $a(\cdot)$ is hump-shape, then $v_L := v(z_L)$ and $v_R := v(z_R)$ are calculated in (B.15) and (B.21), respectively. Moreover, $v_L > v_R$ if (and only if) $r_1 < r^*(\lambda)$, and since $v'(\cdot) < 0$, we have $v^* \in (v_R, v_L)$.

From (B.15), it is easy to verify that $v_L < u$ if and only if $r_1 >^2$. Therefore, if $\lambda > \lambda_1$ and $^2 < r_1 < r^*(\lambda)$, we have $v_R < v^* < v_L < u$. By Corollary B.1.4, $EP(\cdot)$ is non-monotone. □

What about $r_1 \in (0, \ ^2]$? For $\lambda > \lambda_1$ and $r_1 \leq^2$, recall that the functions $a^*_-(\cdot; r_1), a^*_+(\cdot; r_1, \lambda) : [v_R, v_L] \to R$ are defined by (B.30) and (B.31), respectively. Recall also, from the proof of Lemma 2.4.3, that $v^* \in (v_R, v_L)$ is the unique solution to $a^*_-(x; r_1) = a^*_+(x; r_1, \lambda)$.

Claim 10. *If $\lambda > \lambda_1$ and $r_1 \leq^2$, then $u \in [v_R, v_L]$. Moreover, $v^* < u$ if and only if $a^*_-(u; r_1) < a^*_+(u; r_1, \lambda)$.*

Proof. Suppose that $\lambda > \lambda_1$ and $r_1 \leq^2$. First, by definition of λ_1 in (B.37), we have $r^*(\lambda) >^2 \geq r_1$, so the equilibrium $a(\cdot)$ is hump-shaped and $v_L > v_R$. Substituting the expressions of ξ_L and ξ_R into (B.15) and (B.21), we can rewrite v_L and v_R as

$$v_L(r_1) = u + c\left(1 - \frac{\sqrt{1 + 8r_1/^2} + 1}{4}\right),$$

$$v_R(r_1, \lambda) = \frac{r_1}{r_1 + \lambda}\left[u + c\left(1 + \frac{\sqrt{1 + 8(r_1 + \lambda)/^2} - 1}{4}\right)\right].$$

It is easy to verify that v_L is strictly decreasing in r_1 and v_R is strictly increasing in r_1. Thus, for $\lambda > \lambda_1$ and $r_1 \leq^2$,

$$v_R(r_1, \lambda) < v_R(r^*(\lambda), \lambda) = v_L(r^*(\lambda)) < v_L(^2) \leq v_L(r_1)$$

where the inequalities follow from the monotonicity of v_L and v_R in r_1, and the equality follows from the definition of r^*. Note that $v_L(^2) = u$, so we have $u \in [v_R, v_L]$.

The fact that $u \in [v_R, v_L]$ implies that $a^*_-(u;r_1)$ and $a^*_+(u;r_1,\lambda)$ are well-defined. By Claim 9, the function $g : [v_R, v_L] \to R$ defined by $g(x;r_1,\lambda) := a^*_-(x;r_1) - a^*_+(x;r_1,\lambda)$ is strictly decreasing in x, and v^* is the unique zero point of $g(\cdot;r_1,\lambda)$ on $[v_R, v_L]$. Therefore,

$$v^* < u \iff g(u;r_1,\lambda) < 0 \iff a^*_-(u;r_1) < a^*_+(u;r_1,\lambda),$$

as desired. □

The next lemma deals with the case where the agent's discount rate r_1 is small. There exist $\lambda_2 \geq \lambda_1$ and $\underline{r} > 0$, such that if $\lambda > \lambda_2$ and $0 < r_1 < \underline{r}$, then $EP(\cdot)$ is non-monotone.

Proof. See Online Appendix. □

We note that the bound $\underline{r}$ obtained in Lemma B.1.2 is a fixed number independent of λ. We now turn to the last case where $r_1 \in [\underline{r},^2]$.

Claim 11. *There exist $\lambda_3' \geq \lambda_1$ and $A > 1$ such that if $\lambda > \lambda_3'$, then*

$$a^*_+(u;r_1,\lambda) > 1 - A\exp\left(-\frac{2}{c^2 r_1^2}\frac{\lambda^2}{r_1+\lambda}u^2\right), \forall r_1 \in [\underline{r},^2]. \quad \text{(B.38)}$$

Proof. See Online Appendix. □

Lemma B.1.4. *There exists $\lambda_3 \geq \lambda_1$ such that if $\lambda > \lambda_3$ and $\underline{r} \leq r_1 \leq^2$, then $EP(\cdot)$ is non-monotone.*

Proof. Let $\lambda_3' \geq \lambda_1$ and $A > 1$ be delivered by Claim 11. For each $r_1 \in [\underline{r},^2]$, let $\lambda(r_1)$ be the unique solution on R_+ to $A\exp\left(-\frac{2}{c^2 r_1^2}\frac{\lambda^2}{r_1+\lambda}u^2\right) = 1 - a^*_-(u;r_1)$. Note that $\lambda(r_1)$ is well-defined for all $r_1 \in [\underline{r},^2]$ because the LHS is strictly decreasing in λ while the RHS is independent of λ.[10] Hence, we have

$$A\exp\left(-\frac{2}{c^2 r_1^2}\frac{\lambda^2}{r_1+\lambda}u^2\right) < 1 - a^*_-(u;r_1), \forall \lambda > \lambda(r_1). \quad \text{(B.39)}$$

Also, $\lambda(r_1)$ is continuous in r_1 by the implicit function theorem. Let $\lambda_3'' := \max_{r\in[\underline{r},^2]} \lambda(r)$

[10] Moreover, when $\lambda = 0$, the LHS is equal to $A > 1 - a^*_-(u;r_1)$; when $\lambda \to \infty$, the LHS converges to $0 < 1 - a^*_-(u;r_1)$.

and $\lambda_3 := \max\{\lambda_3', \lambda_3''\}$. Combining (B.38) and (B.39), we have

$$a_+^*(u; r_1, \lambda) > 1 - A \exp\left(-\frac{2}{c^2 r_1^2} \frac{\lambda^2}{r_1 + \lambda} u^2\right) > a_-(u; r_1), \text{ for all } \lambda > \lambda_3 \text{ and } r_1 \in [\underline{r}, {}^2].$$

Then, by Claim 10 and Corollary B.1.4, we conclude that $EP(\cdot)$ is non-monotone if $\lambda > \lambda_3$ and $r_1 \in [\underline{r}, {}^2]$. □

Proof of Theorem 2.5.1. Let λ_1 be defined in (B.37), and let λ_2 and λ_3 be delivered by Lemmas B.1.2 and B.1.4, respectively. Define $\bar{\lambda} := \max\{\lambda_1, \lambda_2, \lambda_3\}$. Lemmas B.1.3 through B.1.4 imply that if $\lambda > \bar{\lambda}$ and $r_1 < r^*(\lambda)$, then $EP(\cdot)$ is non-monotone. Lemma B.1.1 then leads to the conclusion of the theorem. □

B.1.3 Effects of Better Transparency: Toward a Proof of Theorem 2.6.1

In this section, we prove Theorem 2.6.1 which is about the convergence of the principal's equilibrium value function when the signal-to-noise ratio grows without bound.

Take any sequence $\{_n\}_n$ such that $\lim_n {}_n = +\infty$. For each $n \in N$, take the unique Markov equilibrium (a_n, b_n) associated with the signal-to-noise ratio $_n$. Let $V_n(\cdot)$ be the agent's value function in the equilibrium (a_n, b_n) and $W_n(\cdot)$ be the principal's value function. We will often use $z \equiv \log(p/1-p)$ as state variable when analyzing the agent's behavior. When doing so, we denote by $v_n(z) := V_n(p(z))$ the agent's value function in the $z-$space. Write z_n^* for the principal's equilibrium cutoff. Write $z_{L,n}$ for the infimum belief z at which the agent plays $a_n(z) > 0$ and write $z_{R,n}$ for the supremum. Write T for the equilibrium stopping time that stops the play of the game. *Without labeling explicitly, we note that the distribution of T depends on n and the current state z.* For $i = 1, 2$, let $E_n^\theta\left\{e^{-r_i T}\right\}$ be the expected discount factor when the stopping action is taken in the equilibrium (a_n, b_n) discounted at rate r_i and given the equilibrium strategy of type $\theta \in \{NI, I\}$. When the game starts at state z, let

$$E_n\left\{e^{-r_i T}\right\} := p(z) E_n^{NI}\left\{e^{-r_i T}\right\} + (1 - p(z)) E_n^I\left\{e^{-r_i T}\right\}.$$

Case 1: $\lambda < r_1\left(\frac{c}{u}\right)$, i.e., $u < \frac{r_1}{r_1+\lambda}(u+c)$

Proof of Part 1 of Theorem 2.6.1. We first show that there is $\varepsilon > 0$ such that $a_n(z_n^*) < 1-\varepsilon$ for all n. Without loss, assume that $a_n(z_n^*) > 0$ for all n. Then condition (B.10) implies that

$$a_n(z_n^*) = 1 - 2\left(\frac{v_n(z_n^*) - u}{c}\right) - a_n'\left(z_{n-}^*\right) < 1 - 2\left(\frac{\frac{r_1}{r_1+\lambda}(u+c) - u}{c}\right),$$

where the inequality follows from $v_n(\cdot) \geq \frac{r_1}{r_1+\lambda}(u+c)$ and $a_n'(z_{n-}^*) > 0$. By the assumption of Case 1, we can set $\varepsilon = 2\left(\frac{\frac{r_1}{r_1+\lambda}(u+c)-u}{c}\right) > 0$.

Since $a_n(\cdot)$ is maximized at z_n^*, the above implies that $a_n(\cdot)$ is uniformly bounded away from 1. Consequently, as $n \to \infty$, the principal learns the agent's type almost immediately, and thus the principal's equilibrium value function $W_n(\cdot)$ converges uniformly to her full-information value function $\overline{W}(\cdot)$. □

Case 2: $\lambda > r_1\left(\frac{c}{u}\right)$**, i.e.,** $u > \frac{r_1}{r_1+\lambda}(u+c)$

Claim 12. *There exists $N \in N$ such that whenever $n \geq N$, $a_n(\cdot)$ is hump-shaped.*

Proof. From condition (2.2), it is easily verified that $\lim_n r_n^* = \lambda\left(\frac{2u}{c} + 1\right)$. The assumption of Case 2 implies that $r_1 < \lambda\left(\frac{u}{c}\right) < \lambda\left(\frac{2u}{c} + 1\right)$. By Theorem 2.4.1, the result follows. □

We assume $n \geq N$ for the rest of the proofs in Case 2.

Claim 13. *Take any compact set $[p_1, p_2] \subset (0,1)$. We have*

$$\limsup_{n\to\infty}\left[\max_{z\in[z(p_1),z(p_2)]} v_n(z)\right] \leq u.$$

Proof. Because $p^* \in [p^{**}, p_H]$ (by Lemma 2.4.2), we can without loss assume that $p_n^* \in [p_1, p_2]$. Since $v_n(\cdot)$ is decreasing, it suffices to show that $\limsup v_n\left(z\left(\frac{p_1}{2}\right)\right) \leq u$.

By Claim 12, $a_n(\cdot)$ is hump-shaped for all n. First assume that $a_n(z_n^*) \to 1$. Take any $\varepsilon > 0$, and let z_n^ε be the smallest z such that $a_n(z) = 1 - \varepsilon$, which is well-defined for every large n such that $a_n(z_n^*) > 1 - \varepsilon$. Consider the stochastic process Z_t in the equilibrium (a_n, b_n) under the noninvestible-type strategy and the initial condition $Z_0 = z\left(\frac{p_1}{2}\right)$. Let $T_n^\dagger$ be the stopping time that stops the game at the first time that $Z_t \geq z_n^\varepsilon$. From the law of motion (B.1), as $n \to \infty$, we have $E_n^{NI}\left[e^{-r_1 T_n^\dagger}\right] \to 1$ and hence $v_n\left(z\left(\frac{p_1}{2}\right)\right) \to v_n(z_n^\varepsilon)$. Moreover, since $v_n(\cdot)$ is decreasing and concave to the left of z_ε^n (by Corollary B.1.2), we have

$$\begin{aligned} r_1 v_n(z_\varepsilon^n) &= r_1\left[u + (1 - a_n(z_\varepsilon^n))c\right] + \frac{1}{2}^2\left[1 - a_n(z_\varepsilon^n)\right]^2\left[v'\left(a_n\left(z_{\varepsilon-}^n\right)\right) + v''\left(a_n\left(z_{\varepsilon-}^n\right)\right)\right] \\ &\leq r_1\left[u + (1 - a_n(z_\varepsilon^n))c\right], \end{aligned}$$

which implies $v_n(z_\varepsilon^n) \le u + \varepsilon c$, delivering the result as ε is arbitrary.

Next assume that $\liminf a_n(z_n^*) < 1$. Take an $\epsilon > 0$ such that $u > \frac{r_1}{r_1+\lambda}(u+c) + \epsilon$; such an ϵ exists because of the assumption of Case 2. Notice that we can find a $z^\dagger$ sufficiently large such that $v_n(z^\dagger) < \frac{r_1}{r_1+\lambda}(u+c) + \epsilon, \forall n$. [11] Let $T_n^\dagger$ be the stopping time that stops the game at the first time that $Z_t = z^\dagger$. As $_n \to \infty$, we have $v_n\left(z\left(\frac{p_1}{2}\right)\right) \to v_n(z^\dagger) < \frac{r_1}{r_1+\lambda}(u+c) + \epsilon < u$. ☐

Claim 14. $\lim_{n\to\infty} z_{L,n} = -\infty$.

Proof. Recall that $v_n(z_{L,n}) = v_{L,n}$ whose expression is given by (B.15), i.e., $v_{L,n} = u + c\left(1 - \frac{\sqrt{1+8r_1/_n^2}+1}{4}\right)$. It is easy to see that $\lim_n v_n(z_{L,n}) = u + \frac{c}{2}$. Assume toward a contradiction, taking a subsequence if necessary, that $\lim_n z_{L,n} = \underline{z} > -\infty$. Then for any $\varepsilon > 0$, we have $z_{L,n} \in [\underline{z} - \varepsilon, \underline{z} + \varepsilon]$ and $v_n(z_{L,n}) \ge u + \frac{c}{4}$ when n is sufficiently large. Take any $\varepsilon < (0, c/4)$. By Claim 13 and the monotonicity of $v_n(\cdot)$, there exists n^* such that for every $n > n^*$ and for every $z \in [\underline{z} - \varepsilon, \underline{z} + \varepsilon]$, we have $v_n(z) < u + \varepsilon < u + \frac{c}{4}$, a contradiction to $v_{L,n} \to u + \frac{c}{2}$ and $z_{L,n} \to \underline{z}$. ☐

Claim 15. *For any* $\kappa > 0$, *we have* $\lim_{n\to\infty} a_n(z_n^* - \kappa) = 1$.

Proof. Assume toward a contradiction that, taking a subsequence if necessary, $\lim a_n(z_n^* - \kappa) = 1 - 2\varepsilon$ for some $\varepsilon > 0$. Take any large $M > 0$ and notice that Claim 14 tells us that $z_{L,n} < z_n^* - \kappa - M$ for large n. Since $a_n(\cdot)$ is increasing on $[z_n^* - \kappa - M, z_n^* - \kappa]$ and $\lim a_n(z_n^* - \kappa) = 1 - 2\varepsilon$, we know that $a_n(z) < 1 - \varepsilon$ (infinitely often) for all $z \in [z_n^* - \kappa - M, z_n^* - \kappa]$. Recall from condition (B.10) that for $z \in [z_n^* - \kappa - M, z_n^* - \kappa]$,

$$a_n'(z) = 1 - a_n(z) - 2\left(\frac{v_n(z) - u}{c}\right),$$

implying, in light of Claim 13, that for n sufficiently large, we have $a_n'(z) > \frac{\varepsilon}{2}$ for all $z \in [z_n^* - \kappa - M, z_n^* - \kappa]$. But then, we can take M large enough such that $a_n(z_n^* - \kappa - M) < 0$, a contradiction. ☐

Claim 16. *For any* $\kappa > 0$, *we have* $\lim_{n\to\infty} v_n(z_n^* - \kappa) = u$.

[11] To see this, note first that p_n^* is bounded above by $p_H < 1$ (Lemma 2.4.2). Next observe that the posterior is a submartingale according to the strategy of the noninvestible type. This implies that for every $\eta > 0$ we can find $p_\eta < 1$ such that, conditional on the noninvestible-type strategy, $p \in (p_\eta, 1)$ implies that the posterior goes below p_H with probability less than η. This immediately easily implies the existence of said $z^\dagger$.

Proof. From Claim 13 and Lemma 2.4.2 , we know $\limsup v_n(z_n^* - \kappa) \leq u$. Assume toward a contradiction, taking a subsequence if necessary, that $\lim_{n\to\infty} v_n(z_n^* - \kappa) = u - 2\varepsilon$ for some $\varepsilon > 0$. This implies that $v_n(z_n^* - \kappa) < u - \varepsilon$ for n sufficiently large. Note also that Claim 14 tells us that $z_n^* - \kappa > z_{L,n}$ for n sufficiently large. From condition (B.10), $a_n'(z) = 1 - a_n(z) - 2\left(\frac{v_n(z)-u}{c}\right)$ for all $z \in [z_n^* - \kappa, z_n^* - \frac{\kappa}{2}]$. Then by Claim 15 and the monotonicity of $a_n(\cdot)$, we know that for n sufficiently large, $a_n'(z) > \frac{\varepsilon}{c}$ for all $z \in [z_n^* - \kappa, z_n^* - \frac{\kappa}{2}]$. But then, we have $a_n(z_n^* - \kappa) < 1 - \left(\frac{\kappa}{2}\right)\left(\frac{\varepsilon}{c}\right)$, a contradiction to Claim 15. ☐

Claim 17. *Fix a prior $p_0 \in (0,1)$ and some $\bar{p} \in (p_0, 1)$. For each > 0, consider an adapted Markov function $\alpha(\cdot)$ and a belief process defined by substituting $\alpha_(\cdot)$ into (2.7). Take $\varepsilon > 0$ and let $\bar{T}$ be the random time that stops the play in the first time that $p \geq \bar{p}$. Then we have:*

$$\limsup_{\uparrow\infty} E^{NI}\left\{r_1 \int_0^{\bar{T}} e^{-r_1 t} I_{\{\alpha(p_t)\leq 1-\varepsilon\}} dt\right\} = 0.$$

Proof. See Online Appendix (Section B.2.3, and Lemma OA.5 in Section B.2.4). In words, this lemma says that if the noninvestible type does not mimic too often, then as the noise in the signal vanishes, the principal can learn the agent's type almost immediately. ☐

Lemma B.1.5. *Fix any $\kappa > 0$ and, for each $n \in N$, assume that the game starts at the prior $z_n^* - \kappa$. Let T_n^* be the stopping time that stops the play in the first time that a posterior reaches $[z_n^*, \infty)$. We have:*

$$\limsup_{n\to\infty} E_n^{NI}\left[e^{-r_1 T_n^*}\right] = 0.$$

Proof. Let $\hat{z}_n := z_n^* - \kappa$ and $\underline{z}_n := z_n^* - 2\kappa$. Let $T_n(\underline{z}_n)$ be the stopping time that stops the play in the first time that the posterior reaches $\underline{z}_n$ and $T_n(z_n^*)$ be the stopping time that stops the play in the first time that the posterior reaches z_n^*. Observe that for any n, $P_n^{NI}[T_n(\underline{z}_n) < \infty] + P_n^{NI}[T_n(z_n^*) < \infty] = 1$.

By Claim 15, we know that for any $\varepsilon > 0$, there exists $n_1 \in N$ such that $n > n_1$ implies that $v_n(\hat{z}_n^*)$ is bounded above by:

$$\begin{aligned} &P_n^{NI}[T_n(\underline{z}_n) < \infty] E_n^{NI}\left[\int_0^{T_n(\underline{z}_n)} ue^{-r_1 t}dt + e^{-r_1 T_n(\underline{z}_n)} v_n(\underline{z}_n) \mid T_n(\underline{z}_n) < \infty\right] \\ &\quad + P_n^{NI}[T_n(z_n^*) < \infty] E_n^{NI}\left[\int_0^{T_n(z_n^*)} ue^{-r_1 t}dt + e^{-r_1 T_n(z_n^*)} v_n(z_n^*) \mid T_n(z_n^*) < \infty\right] + \varepsilon. \end{aligned} \tag{B.40}$$

Next we obtain an upper bound for $v_n(z_n^*)$. For that, we let T_λ be the random time of the next Poisson shock. Note that

$$v_n(z_n^*) = P_n^{NI}\left[z_{T_\lambda} > z_n^*\right] E_n^{\theta_S}\left[\int_0^{T_\lambda} e^{-r_1 t}\left[u + (1 - a_n(z_t))\, c\right] dt \mid z_{T_\lambda} > z_n^*\right]$$
$$+ P_n^{NI}\left[z_{T_\lambda} \leq z_n^*\right] E_n^{\theta_S}\left[\int_0^{T_\lambda} e^{-r_1 t}\left[u + (1 - a_n(z_t))\, c\right] dt + e^{-r_1 T_\lambda} v_n\left(z_{T_\lambda}\right) \mid z_{T_\lambda} \leq z_n^*\right].$$

Now we use the following facts to bound the expected value above:

i) Because $a_n(z_t) \geq 0$,

$$E_n^{NI}\left[\int_0^{T_\lambda} e^{-r_1 t}\left[u + (1 - a_n(z_t))\, c\right] dt \mid z_{T_\lambda} > z_n^*\right] \leq E_n^{NI}\left[\int_0^{T_\lambda} e^{-r_1 t}(u + c) dt \mid z_{T_\lambda} > z_n^*\right].$$

ii) From Claim 13,

$$\limsup E_n^{NI}\left[I_{\{z_{T_\lambda} \leq z_n^*\}} v_n\left(z_{T_\lambda}\right)\right] \leq \limsup E_n^{NI}\left[I_{\{z_{T_\lambda} \leq z_n^*\}} u\right].$$

iii) For every $\varepsilon > 0$, from Claim 17,

$$\limsup P_n^{NI}\left[\{z_{T_\lambda} \leq z_n^*\} \cap \left\{\int_0^{T_\lambda} e^{-r_1 t}(1 - a(z_t)) dt > \varepsilon\right\}\right] = 0.$$

Conditions (i), (ii) and (iii) above imply that for every $\epsilon > 0$, we can find $n_2 \in N$ with $n_2 > n_1$ such that $n > n_2$ implies

$$v_n(z_n^*) \leq P_n^{NI}\left[z_{T_\lambda} > z_n^*\right] E_n^{NI}\left[\int_0^{T_\lambda} e^{-r_1 t}(u + c) dt \mid z_{T_\lambda} > z_n^*\right] + P_n^{\theta_S}\left[z_{T_\lambda} \leq z_n^*\right] u + \epsilon.$$

Since Poisson shocks are independent of the Brownian motion,

$$E_n^{NI}\left[\int_0^{T_\lambda} e^{-r_1 t}(u + c) dt \mid z_{T_\lambda} > z_n^*\right] = \left(\frac{r}{r + \lambda}\right)(u + c) < u.$$

Moreover, notice that the z_t (and p_t) are submartingales conditional on $\theta = NI$, so

$$P_n^{NI}\left[z_{T_\lambda} > z_n^*\right] \geq \frac{1}{2}.$$

Therefore, the last two observations imply

$$v_n(z_n^*) \leq \frac{1}{2}\left(\frac{r_1}{r_1+\lambda}\right)(u+c) + \frac{1}{2}u + \epsilon.$$

Since $\left(\frac{r_1}{r_1+\lambda}\right)(u+c) < u$ and ϵ is arbitrary, we conclude that $\limsup v_n(z_n^*) < u$.

Going back to the upper bound (B.40) for $v_n(\hat{z}_n^*)$, we have shown that $\limsup v_n(z_n^*) < u$, and from Claim 16, we know that $\lim v_n(\hat{z}_n^*) = \lim v_n(\underline{z}_n^*) = u$. So for (B.40) to be an valid upper bound of $v_n(\hat{z}_n^*)$ for arbitrary ε, we must have $\limsup_{n\to\infty} E_n^{NI}\left[e^{-r_1 T_n(z_n^*)}\right] = 0$, i.e., $\limsup_{n\to\infty} E_n^{NI}\left[e^{-r_1 T_n^*}\right] = 0$, as desired. □

Lemma B.1.6. $\lim_{n\to\infty} z_n^* = z^{**}$, *where* z^{**} *is the myopic cutoff satisfying* $R(p(z^{**})) = 0$

Proof. Recall that we always have $z_n^* \geq z^{**}$. Hence assume (toward a contradiction) that we can find some $\varepsilon > 0$ such that, taking a subsequence if necessary, $z_n^* > z^{**} + \varepsilon$ for every n. For every n, consider the game starts at $z_n^* - \frac{\varepsilon}{2} > z^{**} + \frac{\varepsilon}{2}$. Let T_n^* be the stopping time that stops the play in the first time that a posterior reaches $[z_n^*, \infty)$. By Lemma B.1.5, we have $E_n^{NI}\left[e^{-r_1 T_n^*}\right] \to 0$, implying that $\limsup W_n\left(p\left(z_n^* - \frac{\varepsilon}{2}\right)\right) \leq 0$. But since $z_n^* - \frac{\varepsilon}{2} > z^{**} + \frac{\varepsilon}{2}$, the principal can get a strictly positive payoff by terminating the relationship when the next stopping opportunity arrives. So the principal has a profitable deviation at $z_n^* - \frac{\varepsilon}{2}$ when n is sufficiently large, a contradiction. □

Proof of Part 2 of Theorem 2.6.1. In light of Corollary B.1.2, we extend each W_n continuously from $(0,1)$ to $[0,1]$ by setting $W_n(0) = 0$ and $W_n(1) = \frac{\lambda}{r_2+\lambda} w_{NI}$.

We first show that $W_n(\cdot)$ converges to $\underline{W}(p^{**}) = 0$. By Lemmas B.1.5 and B.1.6, it is easy to see that $\lim_{n\to\infty} W_n(p) = 0$ for all $p < p^{**}$.[12] Suppose toward a contradiction, taking a subsequence if necessary, that $\lim_{n\to\infty} W_n(p^{**}) = \delta > 0$. Let $\epsilon = \frac{\delta(1-p^{**})}{2w_{NI}}$. For n large enough, we have $W_n(p^{**} - \epsilon) < \frac{\delta}{2}$. Since $W_n(\cdot)$ is convex (by Corollary B.1.2), we have $\frac{W_n(1) - W_n(p^{**})}{1-p^{**}} \geq \frac{W_n(p^{**}) - W_n(p^{**}-\epsilon)}{\epsilon} \geq \frac{w_{NI}}{1-p^{**}}$, which implies $W_n(1) > w_{NI}$, a contradiction.

Next, for each n, because $W_n(0) = \underline{W}(0)$, $W_n(1) = \underline{W}(1)$, and $W_n(\cdot)$ is increasing and convex, we have $0 \leq W_n'(\cdot) \leq \frac{\lambda}{r_1+\lambda}$. But then, we always have $p^{**} =_{p\in[0,1]} |W_n(p) - \underline{W}(p)|$. Hence, uniform convergence of W_n follows immediately from its pointwise convergence at p^{**}. □

[12]Note that Lemma B.1.5 enables us to conclude that $\limsup_n E_n\left[e^{-r_2 T}\right] = 0$. This is because the principal only derives positive payoff from terminating against the noninvestible type. As a result, if $\limsup_n E_n^{NI}\left[e^{-r_2 T}\right] = 0$, then we must have $\limsup_n E_n^{I}\left[e^{-r_2 T}\right] = 0$, for otherwise the principal's equilibrium payoff would be negative.

B.2 Online Appendix

B.2.1 Omitted Proofs for Theorem 2.4.1

Proofs of Lemmas 2.4.1 and 2.4.2

Consider a Markovian equilibrium, (α, β), and the underlying probability space (Ω, F, P). For each $p \in (0,1)$, we define $\Phi(p) := \{\omega \in \Omega : \exists t \leq T \text{ such that } p_t(\omega) = p\}$, where T is the random stopping time of the relationship induced by (α, β). The belief span, $\mathrm{SP}(\alpha)$, is the set of all p such that $P(\Phi(p)) > 0$. Clearly, $\mathrm{SP}(\alpha)$ is a connected set because the sample path of X_t is almost surely continuous. Let $\underline{p} := \inf \mathrm{SP}(\alpha)$, and $\bar{p} := \sup \mathrm{SP}(\alpha)$. Define the principal's value function W as in the main text on the domain of $\mathrm{SP}(\alpha)$. The function W is continuous because the agent's equilibrium policy function $a \in \mathcal{P}$.

Claim OA.1. *$SP(\alpha)$ is an open interval. That is, $SP(\alpha) = (\underline{p}, \bar{p})$.*

Proof. Since $SP(\alpha)$ is a connected set, we only need to show that $\bar{p}, \underline{p} \notin SP(\alpha)$. Suppose, toward a contradiction, that $\bar{p} \in SP(\alpha)$. Then, consider a history that leads to the belief $\bar{p}$ and the continuation play starting from this history. Since the belief process is a martingale, we must have $p_t = \bar{p}$ for all $t \leq T$ and almost all sample paths. Agent's optimality then implies $a(\bar{p}) = 0$,and thus the diffusion coefficient of the belief process at $\bar{p}$ is strictly positive. This contradicts $p_t = \bar{p}$ for all $t \leq T$ and almost all sample paths. The same argument proves that $\underline{p} \notin SP(\alpha)$. □

Claim OA.2. *The principal's equilibrium policy function b has a cutoff structure on $SP(\alpha)$. That is, there exists a unique $p^* \in [\underline{p}, \bar{p}]$ such that $p \in (\underline{p}, p^*)$ implies $b(p) = 0$ and $p \in (p^*, \bar{p})$ implies $b(p) = 1$.*

Proof. Recall that $R(p) := pw_{NI} + (1-p)w_I$ is the principal's expected payoff if the relationship is terminated when her belief is p. For any $p \in SP(\alpha)$, define $F(p) := W(p) - R(p)$. At any time t such that the stopping opportunity arrives, given her belief $p_t = p \in SP(\alpha)$, if the principal terminates the relationship, her expected payoff is $R(p)$; if the principal continues the relationship, her continuation value is $W(p)$. Thus, principal's optimality requires that $b(p) = 1$ if $F(p) < 0$ and that $b(p) = 0$ if $F(p) > 0$.

We first establish two useful properties of F.

Property 1: If $F(\tilde{p}) > 0$ at some $\tilde{p} \in SP(\alpha)$, then $F(p) > 0$ for all $p < \tilde{p}$.

To see this, suppose that $F(\tilde{p}) > 0$ at some $\tilde{p} \in SP(\alpha)$. Let (p_a, p_b) be the largest interval containing $\tilde{p}$ such that $F(p) > 0$ for all $p \in (p_a, p_b)$. We want to show that $p_a = \underline{p}$. Suppose, toward a contradiction, that $p_a > \underline{p}$. Since W is continuous, we have $F(p_a) = 0$, i.e., $W(p_a) = R(p_a)$. Moreover, principal's optimality requires that $b(p) = 0$ for all $p \in (p_a, p_b)$. We consider two cases, and will reach a contradiction in each of these cases.

Case 1: $p_b < \bar{p}$.

In this case, continuity of W also implies that $F(p_b) = 0$, i.e., $W(p_b) = R(p_b)$. Consider now a history that leads to the belief $\tilde{p}$ and the continuation play starting from this history. Let $T^\dagger$ be the first time that the posterior belief reaches $(p_a, p_b)^c$ (setting $T^\dagger = \infty$ if this event is not reached in finite time). Let φ represent the probability measure (from the principal's perspective) induced by the distribution of $p_{T^\dagger}$. Then,

$$\begin{aligned} R(\tilde{p}) < W(\tilde{p}) &= \int_{p_a}^{p_b} W(p) E\left[e^{-r_2 T^\dagger} \mid p_{T^\dagger} = p\right] \varphi\,(dp) \\ &= \int_{p_a}^{p_b} R(p) E\left[e^{-r_2 T^\dagger} \mid p_{T^\dagger} = p\right] \varphi\,(dp) < \int_{p_a}^{p_b} R(p) \varphi\,(dp) = R(\tilde{p}). \end{aligned}$$

The second equality trivially holds if $E\left[e^{-r_2 T^\dagger} \mid p_{T^\dagger} = p\right] = 0$, and otherwise if $T^\dagger < \infty$, then it holds because $W(p_{T^\dagger}) = R(p_{T^\dagger})$. The last inequality uses the facts that $0 \leq W(p_a) = R(p_a)$ implies $R(p) > 0$ for all $p > p_a$, and that $T^\dagger > 0$ almost surely. The final equality holds because p_t is a bounded martingale and $R(\cdot)$ is an affine function. But then, we have an obvious contradiction.

Case 2: $p_b = \bar{p}$.

In this case, we have $b(p) = 0$ for all $p \in (p_a, \bar{p})$. Consider again a history that leads to the belief $\tilde{p}$ and the continuation play starting from this history. Let $T^\dagger$ be the first time that the posterior reaches p_a (setting $T^\dagger = \infty$ if this event is not reached in finite time). Let φ represent the probability measure (from the principal's perspective) induced by the distribution of $p_{T^\dagger}$. Then,

$$R(\tilde{p}) < W(\tilde{p}) = W(p_a) E\left(e^{-rT^\dagger}\right) < R(p_a),$$

which contradicts R being increasing.

Property 2: Let $p^* := \sup\{p \in SP(\alpha) : F(p) > 0\}$.[13] Then, $F(p) < 0$ for all $p > p^*$.

[13] By convention, if $\{p \in SP(\alpha) : F(p) > 0\} = \emptyset$, we set $\sup\{p \in SP(\alpha) : F(p) > 0\} = \underline{p}$.

By definition of p^*, we know that $F(p) \leq 0$ for all $p \geq p^*$, i.e., $W(p) \leq R(p)$ for all $p \geq p^*$, so it is weakly optimal for the principal to terminate the relationship whenever $p \in (p^*, \bar{p})$. Suppose, toward a contradiction, that $F(\tilde{p}) = 0$ for some $\tilde{p} > p^*$. Consider a history that leads to the belief $\tilde{p}$ and the continuation play starting from this history. Let $T^\dagger$ be the first time that the stopping opportunity arrives or the posterior reaches p^* (setting $T^\dagger = \infty$ if this event is not reached in finite time). Let φ represent the probability measure (from the principal's perspective) induced by the distribution of $p_{T^\dagger}$. Then,

$$
\begin{aligned}
R(\tilde{p}) = W(\tilde{p}) &= \int_{p^*}^{\bar{p}} W(p) E\left[e^{-r_2 T^\dagger} \mid p_{T^\dagger} = p\right] \varphi(dp) \\
&\leq \int_{p^*}^{\bar{p}} R(p) E\left[e^{-r_2 T^\dagger} \mid p_{T^\dagger} = p\right] \varphi(dp) < \int_{p^*}^{\bar{p}} R(p) \varphi(dp) = R(\tilde{p}).
\end{aligned}
$$

The first equality follows from the contradiction assumption that $F(\tilde{p}) = 0$, the first inequality follows from the definition of p^* such that $F(p) \leq 0$ for all $p \geq p^*$, the last inequality holds because $0 \leq W(p^*) \leq R(p^*)$ implies that $R(p) > 0$ for all $p > p^*$, and the final equality holds because p_t is a bounded martingale and $R(\cdot)$ is an affine function. But then, we have an obvious contradiction, establishing Property 2.

These two properties of F immediately deliver our result. Specifically, let $p^* := \sup\{p \in SP(\alpha) : F(p) > 0\}$. Then, Property 1 implies that $F(p) > 0$ (and thus $b(p) = 0$) for all $p \in (\underline{p}, p^*)$, and Property 2 implies that $F(p) < 0$ (and thus $b(p) = 1$) for all $p \in (p^*, \bar{p})$. □

We continue with a technical result that will be later used.

Claim OA.3. *Fix a positive integer T. For any $\varepsilon > 0$ there exists $\eta > 0$ satisfying the following property: Take any pair of adapted processes $dY_t^1 = \mu_{1,t} dt + \sigma dB_t$ and $dY_t^2 = \mu_{2,t} dt + \sigma dB_t$ such that $\mu_{j,t} \in [0,1]$ for $j = 1, 2$ and for every t. Let P_1 and P_2 be the probability distributions over $(C([0,T]), B(C([0,T])))$*[14] *generated by such stochastic processes. If $A \in B(C([0,T]))$ is such that $E_{P_1}[I_A] < \eta$ then $E_{P_2}[I_A] < \varepsilon$.*

Proof. Dividing both processes by σ and subtracting the same drift from both processes if necessary we may assume that $dY_t^1 = dB_t$ and $dY_t^2 = \mu_{2,t} dt + dB_t$ with $\mu_{2,t} \in \left[-\sigma^{-1}, \sigma^{-1}\right]$. Since the drift is bounded we can invoke Girsanov's theorem to obtain

$$
E_{P_2}[I_A] = E_{P_1}[I_A M_T],
$$

[14] B stands for the Borel sigma-field, and $C([0,T])$ is the set of continuous functions over $[0,T]$.

where $M_T = \exp\left(\int_0^T \mu_{2,t} dB_t - \frac{1}{2}\int_0^T \mu_{2,t}^2 dt\right)$. Notice that $M_T \leq F_{\mu_2} := \exp\left(\int_0^T \mu_{2,t} dB_t\right)$. Since this class of processes is uniformly integrable we can take $n^* \in N$ such that

$$E_{P_1}\left[F_{\mu_2} I_{\{F^{\mu_2}>n^*\}}\right] < \frac{\varepsilon}{2}$$

holding for every process in this class and consequently

$$E_{P_1}\left[M_T I_{\{M_T>n^*\}}\right] \leq E_{P_1}\left[F_{\mu_2} I_{\left\{F_{\mu_2}>n^*\right\}}\right] < \frac{\varepsilon}{2}.$$

Therefore, taking $\eta = \frac{\varepsilon}{2n^*}$, we obtain that

$$E_{P_2}[I_A] = E_{P_1}\left[I_A M_T I_{\{M_T\leq n^*\}}\right] + E_{P_1}\left[I_A M_T I_{\{M_T>n^*\}}\right] \leq E_{P_1}\left[I_A M_T I_{\{M_T\leq n^*\}}\right] + \frac{\varepsilon}{2}$$
$$< n^* E_{P_1}[I_A] + \frac{\varepsilon}{2} < \varepsilon. \qquad \square$$

Claim OA.4. $\bar{p} = 1$

Proof. Assume towards a contradiction that $\bar{p} < 1$.

Case 1: $\bar{p} > p^*$.

The belief process p_t is a martingale, so for every $\varepsilon > 0$ there exists an $\epsilon > 0$ such that if $p_t > \bar{p} - \epsilon$, then $P\left(\inf_{s>t} p_s > p^* + \varepsilon \mid \theta = NI\right) > 1 - \varepsilon$. This implies that

$$P\left(T = T_\lambda \mid \theta = NI\right) > 1 - \varepsilon$$

where T_λ is the arrival of the next Poisson-shock. Notice that for every $\eta > 0$ we can take $\varepsilon_\eta > 0$ such that the agent's payoff at p_t is no more than $(u+c)\left(\frac{r_1}{r_1+\lambda}\right) + \eta$. This implies that for every $\nu > 0$ we can take η small enough (taking ε_η to satisfy the condition above) so that $E\left(\int_t^{\min\{t+1,T\}} (a(p_t))dt \mid \theta = NI\right) < \nu$. Hence, there exists $\varpi > 0$ such that $E\left(\int_t^{\min\{t+1,T\}}(1-a(t))dt \mid \theta = NI\right) > \varpi$.

Consider the law of motion (2.7) when $p_t \in [p^*, \bar{p}]$. Observe that the instantaneous variance of the belief process when the (noninvestible type) agent plays $a(\cdot)$ is bounded below by a positive constant times $(1-a_t)^2 \min\left\{p^*(1-p^*), \bar{p}(1-\bar{p})\right\}^2 > 0$. Because $\bar{p} < 1$ and because $p_t - p_0 = \int_0^t dp_t$, we obtain that $E\left[\left|p_{\min\{t+1,T\}} - p_t\right|^2\right] \geq \delta$ for some positive constant δ, hence $E\left[\left|p_{\min\{t+1,T\}} - p_t\right|\right] \geq \delta$. Because $\left(p_{\min\{t+1,T\}} - p_t\right)$ has mean zero, we obtain that $E\left[\left(p_{\min\{t+1,T\}} - p_t\right)^+\right] \geq \frac{\delta}{2}$. Taking $\varepsilon < \frac{\delta}{4}$ we conclude that $P\left(p_{\min\{t+1,T\}} > \bar{p} + \frac{\varepsilon}{2}\right) > 0$, which is a contradiction.

Case 2: $\bar{p} \leq p^*$.

Assume $\bar{p} \leq p^*$. Then, $b(p) = 0$ for all $p \in SP(\alpha, \beta)$. Claim OA.3 implies that, for every $T > 0$, if the noninvestible agent plays $a_t = 0$ for every $t \in [0, T]$, then the relationship terminates before T with probability zero, which implies that the agent's best response must satisfy $a_t = 0$ for every $t > 0$. This contradicts the assumption that $\bar{p}$ is never reached. □

Claim OA.5. $\underline{p} = 0$.

Proof. Assume towards a contradiction that $\underline{p} > 0$.

Case 1: $\underline{p} < p^*$.

Step 1: For every $\eta \in \left(0, \frac{p^* - \underline{p}}{2}\right)$ there exists $\epsilon > 0$ such that if $p_t < \underline{p} + \epsilon$, then

$$P\left(\sup_{s>t} p_s \geq p^* \mid \theta = NI\right) < \eta,$$

and consequently $P\left(\sup_{s>t} p_s \geq p^*\right) < \eta$.

This follows from the martingale property of the belief process.

Step 2: For every $\varepsilon > 0$ and $T \in N$ there exists $\epsilon > 0$ such that if $p_t < \underline{p} + \epsilon$ and if the agent plays a strategy $\tilde{\sigma}$ that plays $a_t = 0$ for all $t > 0$, then $P^{\tilde{\sigma}}(\{\mathcal{T} < T\}) < \varepsilon$.

This follows from Step 1 and Claim OA.3.

Step 3: There exists $T^* \in N$ and $\varepsilon > 0$ such that if $P^{\tilde{\sigma}}(\{\mathcal{T} < T^*\}) < \varepsilon$, then

$$E\left(\int_t^{\min\{t+T^*, \mathcal{T}\}} (1 - a_t) dt \mid \theta = NI\right) \geq 1.$$

To see this, take an arbitrary $T \in N$. If $E\left(\int_t^{\min\{t+T, \mathcal{T}\}} (1 - a_t)\, dt \mid \theta = NI\right) < 1$, then agent gets no more than

$$(u+c)\left(\int_0^1 r_1 e^{-r_1 t} ds\right) + u\left(\int_1^T r_1 e^{-r_1 t} ds\right) + (u+c)\left(\int_T^\infty r_1 e^{-r_1 t} ds\right) < (u+c)\left(\int_0^T r e^{-r_1 t} ds\right)$$

for T large enough. We can thus let T^* be the smallest positive integer satisfying the inequality above and then pick ε so that

$$(u+c)\left(\int_0^1 r_1 e^{-r_1 t} ds\right) + u\left(\int_1^{T^*} r_1 e^{-r_1 t} ds\right) + (u+c)\left(\int_{T^*}^\infty r_1 e^{-r_1 t} ds\right) < (1-\varepsilon)(u+c)\left(\int_0^{T^*} r_1 e^{-r_1 t} ds\right),$$

in which case $E\left(\int_t^{\min\{t+T^*, \mathcal{T}\}} (1 - a_t)\, dt \mid \theta = NI\right) < 1$ and $P^{\tilde{\sigma}}(\{\mathcal{T} < T^*\}) < \varepsilon$ would imply that $\tilde{\sigma}$ is a profitable deviation.

Step 4: There exists $\epsilon^* > 0$ and $\varepsilon^* > 0$ such that if:

1. $p_t < \underline{p} + \epsilon^*$,
2. $P\left(\sup_{s>t} p_s \geq p^*\right) < \varepsilon^*$,
3. $E\left(\int_t^{\min\{t+T^*,T\}}(1-a(t))dt \mid \theta = NI\right) \geq 1$,

then $P\left(\inf_{s>t} p_s < \underline{p}\right) > 0$.

The argument is analogous to that used in the Case 1 of Claim OA.4's proof, and is thus omitted.

Step 5: Step 3 guarantees that we can find $\varepsilon^* > 0$ and T^* such that if $P^{\tilde{\sigma}}\left(\{T < T^*\}\right) < \varepsilon^*$ then $E\left(\int_t^{\min\{t+T^*,T\}}(1-a_t)\,dt \mid \theta = NI\right) \geq 1$. Steps 1 and 2 guarantee that we can find $\epsilon^* > 0$ such that if $p_t < \underline{p}+\epsilon^*$ then $P^{\tilde{\sigma}}\left(\{T < T^*\}\right) < \varepsilon^*$. Therefore, using Step 4, we conclude that $P\left(\inf_{s>t} p_s < \underline{p}\right) > 0$, which contradicts the definition of $\underline{p}$.

Case 2: $p^* \leq \underline{p}$. This case is analogous to the Case 2 of Claim OA.4's proof, and is thus omitted. □

Claim OA.6. *In any Markovian equilibrium policy profile* (a, b)*,* $\lim_{p\to 1} a(p) < 1$.

Proof. Assume towards a contradiction that $\lim_{p\to 1} a(p) = 1$. Fix an $\varepsilon > 0$. Take $T_\varepsilon \in N$ such that $e^{-r_1 T_\varepsilon} < \varepsilon$ and $P\left(T_\lambda > T_\varepsilon\right) < \varepsilon$ where T_λ is the random time that the next stopping opportunity arrives. Let $Z(p) := \ln(p/(1-p))$, for $p \in (0,1)$. Under the contradiction assumption, take $\hat{Z} > Z(p^*)$ such that $z > \hat{Z}$ implies $a(z) > \frac{1}{2}$.[15] Recall from the law of motion (B.1) that Z_t has bounded drift. So there exists $Z_\varepsilon > \hat{Z}$ such that $Z_0 \geq Z_\varepsilon$ implies $P\left(\inf_{t<T_\varepsilon} Z_t < \hat{Z} \mid \theta = NI\right) < \varepsilon$.

Under the contradiction assumption, if $Z_0 \geq Z_\varepsilon$, the payoff of the agent is no greater

[15] Here, p^* is the principal's equilibrium termination threshold delivered by Claim OA.2. It satisfies $p^* < 1$, for otherwise the principal would never stop the game and the agent would choose $a_t = 0$, to which the principal's best reply would be to stop when p is close to 1.

than

$$\begin{aligned} & P\left(T_\lambda \leq T_\varepsilon\right) \left[\begin{array}{c} P\left(\inf_{t<T_\varepsilon} Z_t < \hat{Z} \mid \theta = NI\right)(u+c) \\ +P\left(\inf_{t<T_\varepsilon} Z_t > \hat{Z} \mid \theta = NI\right) \int_0^{T_\varepsilon} \lambda e^{-\lambda t}\left(1-e^{-r_1 t}\right)\left[\frac{1}{2}u + \frac{1}{2}(u+c)\right] dt \end{array} \right] \\ & +P\left(T_\lambda > T_\varepsilon\right)(u+c) \\ < \; & (1-\varepsilon)^2 \left(\frac{r_1}{r_1+\lambda}\right)\left(\frac{1}{2}u + \frac{1}{2}(u+c)\right) + \left[1-(1-\varepsilon)^2\right](u+c) \\ < \; & \left(\frac{r_1}{r_1+\lambda}\right)(u+c) \end{aligned}$$

for ε small, where the expression in the first line uses the fact that $a(z) > \frac{1}{2}$ for $z > \hat{Z}$. This leads to a contradiction as the agent can guarantee him a payoff of $\left(\frac{r_1}{r_1+\lambda}\right)(u+c)$ by never mimicking. □

Claim OA.7. *In any Markovian equilibrium policy profile* (a, b)*,* $\lim_{p\to 0} a(p) < 1$.

Proof. Assume towards a contradiction that $\lim_{p\to 0} a(p) = 1$. Fix an $\varepsilon > 0$. Under the contradiction assumption, take $\hat{Z} < Z(p^*)$ such that $Z < \hat{Z}$ implies $a(Z) > \frac{1}{2}$.[16] By Claim OA.3 and because Z_t has bounded drift, there exists $Z_\varepsilon < \hat{Z}$ such that $Z_0 < Z_\varepsilon$ implies $P\left(\sup_{t<T_\varepsilon} Z_t \geq \hat{Z} \mid \theta = NI\right) < \varepsilon$ and $P^{\tilde{\sigma}}\left(\sup_{t<T_\varepsilon} Z_t \geq \hat{Z} \mid \theta = NI\right) < \varepsilon$ where $\tilde{\sigma}$ is the strategy that prescribes $a_t = 0$ for every $t > 0$.

Under the contradiction assumption, if $Z_0 < Z_\varepsilon$, the payoff of the agent is no greater than

$$\begin{aligned} & P\left(\sup_{t<T_\varepsilon} Z_t < \hat{Z} \mid \theta = NI\right)\left(\varepsilon\left[\left(\frac{1}{2}\right)(u+c) + \left(\frac{1}{2}\right)u\right] + (1-\varepsilon)(u+c)\right) \\ & +P\left(\sup_{t<T_\varepsilon} Z_t \geq \hat{Z} \mid \theta = NI\right)(u+c) \\ < \; & (1-\varepsilon)\left(\varepsilon\left[\left(\frac{1}{2}\right)(u+c) + \left(\frac{1}{2}\right)u\right] + (1-\varepsilon)(u+c)\right) + \varepsilon(u+c). \end{aligned}$$

On the other hand, the strategy $\tilde{\sigma}$ yields a payoff at least as large as $(1-\varepsilon)^2(u+c)$. So taking ε such that

$$(1-\varepsilon)\left(\varepsilon\left[\left(\frac{1}{2}\right)(u+c) + \left(\frac{1}{2}\right)u\right] + (1-\varepsilon)(u+c)\right) + \varepsilon(u+c) < (1-\varepsilon)^2(u+c),$$

we conclude that the agent can profitably deviate by playing $\tilde{\sigma}$. □

[16] Note that $p^* \geq p^{**} > 0$.

Proof of Lemma 2.4.1. That $SP(\alpha) = (0, 1)$ follows directly from Claims OA.4 and OA.5. Consequently, $a(p) < 1$ for all $p \in (0, 1)$, for otherwise there would be an absorbing state, contradicting $SP(\alpha) = (0, 1)$. By definition of an Markovian equilibrium, $a(\cdot)$ is piecewise Lipschitz, so $a(\cdot)$ only has finite discontinuities and its one-sided limit always exists. Hence, $\sup_{p\in(0,1)} a(p) < 1$ if and only if $\lim_{p\to 1} a(p) < 1$ and $\lim_{p\to 0} a(p) < 1$, so by Claims OA.6 and OA.7 we are done. □

Proof of Lemma 2.4.2. Take any Markov equilibrium (a, b). By Claim OA.2 and Lemma 2.4.1, b has a cutoff structure on $(0, 1)$. So we only need to argue that the cutoff belief p^* satisfies $0 < p^* < 1$. First, since $R(p) < 0$ for all $p \in (0, p^{**})$, principal's optimality requires that $b(p) = 0$ for all such p, and so $p^* \geq p^{**} > 0$. Moreover, since W is bounded above by $\frac{\lambda}{r_2+\lambda} w_{NI}$,[17] $R(p) > W(p)$ for all $p \in (p_H, 1)$, so principal's optimality requires that $b(p) = 1$ for all $p \in (p_H, 1)$, and thus $p^* \leq p_H < 1$. □

Properties of $a_+^*(x)$ and $a_-^*(x)$

Proof of Claim 9. Recall the definition of $a_-^*(\cdot)$ and $a_+^*(\cdot)$ in (B.30) and (B.31). In this proof, we focus on the properties of $a_+^*(\cdot)$, which is the more difficult case. The properties of $a_-^*(\cdot)$ can be established analogously.

For ease of notation, define $q := \frac{x - \frac{r_1}{r_1+\lambda} u}{\sqrt{\kappa_R}}$ and $q_R := \frac{v_R - \frac{r_1}{r_1+\lambda} u}{\sqrt{\kappa_R}}$. Consequently, we can rewrite $a_+^*(\cdot)$ as

$$a_+^*(x(q)) = 1 - \frac{\frac{}{\sqrt{2(r_1+\lambda)}}\phi(q)}{\frac{}{\sqrt{2(r_1+\lambda)}}\phi(q_R) + \Phi(q_R) - \Phi(q)}.$$

First, notice that

$$q_R = \sqrt{\frac{2}{r_1+\lambda}}\left(1 - \frac{r_1+\lambda}{\xi_R^2}\right) = \sqrt{\frac{2}{r_1+\lambda}}\left(1 - \frac{1+\xi_R}{2}\right) > \frac{}{\sqrt{2(r_1+\lambda)}}, \tag{B.41}$$

where the first equality uses the definition of v_R in (B.21) and the fact that $\kappa_R = \frac{r_1^2 c^2}{2(r_1+\lambda)^2}$, and the rest follows from the fact that ξ_R is the positve root of $\xi^2 + \xi = \frac{2(r_1+\lambda)}{2}$ so that $\frac{r_1+\lambda}{\xi_R^2} = \frac{1+\xi_R}{2} < \frac{1}{2}$.

[17] Recall that $W(p)$ is the principal's value at p conditional on the stopping opportunity *not* arriving.

Next, notice that given (B.41), we have

$$\left[\frac{\sqrt{2\left(r_1+\lambda\right)}}{}\phi\left(q\right)+\Phi\left(q\right)\right]'=\phi\left(q\right)\left[1-\frac{\sqrt{2\left(r_1+\lambda\right)}}{}q\right]<0,\ \forall q\in[q_R,\infty). \tag{B.42}$$

Now, let us take derivative of $1-a_+^*(x(q))$ with respect to q. Using the fact that $\phi'(q)=-q\phi(q)$, we have

$$\begin{aligned}\left[\frac{\frac{\sqrt{2(r_1+\lambda)}}{}\phi\left(q\right)}{\frac{\sqrt{2(r_1+\lambda)}}{}\phi\left(q_R\right)+\Phi\left(q_R\right)-\Phi\left(q\right)}\right]'&=\frac{-\frac{\sqrt{2(r_1+\lambda)}}{}q\phi\left(q\right)\left[\frac{\sqrt{2(r_1+\lambda)}}{}\phi\left(q_R\right)+\Phi\left(q_R\right)-\Phi\left(q\right)\right]+\frac{\sqrt{2(r_1+\lambda)}}{}\phi^2\left(q\right)}{\left[\frac{\sqrt{2(r_1+\lambda)}}{}\phi\left(q_R\right)+\Phi\left(q_R\right)-\Phi\left(q\right)\right]^2}\\&=\frac{\sqrt{2\left(r_1+\lambda\right)}}{}\phi\left(q\right)\frac{-q\left[\frac{\sqrt{2(r_1+\lambda)}}{}\phi\left(q_R\right)+\Phi\left(q_R\right)-\Phi\left(q\right)\right]+\phi\left(q\right)}{\left[\frac{\sqrt{2(r_1+\lambda)}}{}\phi\left(q_R\right)+\Phi\left(q_R\right)-\Phi\left(q\right)\right]^2}.\end{aligned}$$

So the sign of $a_+^{*\prime}(x(q))$ is the same as that of

$$S(q):=q\left[\frac{\sqrt{2\left(r_1+\lambda\right)}}{}\phi\left(q_R\right)+\Phi\left(q_R\right)-\Phi\left(q\right)\right]-\phi\left(q\right).$$

Note that, given condition (B.41), we have $S\left(q_R\right)=\left[\frac{\sqrt{2(r_1+\lambda)}}{}q_R-1\right]\Phi\left(q_R\right)>0$. Moreover, given condition (B.42), we have $S'(q)=\left[\frac{\sqrt{2(r_1+\lambda)}}{}\phi\left(q_R\right)+\Phi\left(q_R\right)-\Phi\left(q\right)\right]>0$. Therefore, $S(q)>0$ for all $q\geq q_R$, implying $a_+^*(\cdot)$ is strictly increasing on $[v_R,v_L]$.

Finally, inspecting (B.31) we have $a_+^*(v_R)=0$; applying condition (B.42) we have $0<a_+^*(v_R)<1$. □

Translation Invariance

The analysis in Claim 8 implies that value function and pseudo-best reply of the agent is *translation-invariant*: Should the principal displace her threshold from z^* to say $z^*+\epsilon$, then the agent's previous value and mixing behavior at z would coincide with the new value and mixing behavior at $z+\epsilon$. More formally, this an implication of the (more general) translation invariance of agent's payoff function in the z-space.

Lemma B.2.1 (Translation Invariance)**.** *Fix an arbitrary strategy profile (α,β) and some $\epsilon\in R$. Consider a new profile (α',β'), defined by $\alpha_t':=\alpha_t\Big|\{Z_s-\epsilon\}_{s\leq t}$ and $\beta_t':=\beta_t\Big|\{Z_s-\epsilon\}_{s\leq t}$.*

Then, the payoff of the noninvestible agent satisfies

$$E\left\{U_1(t,\alpha,\beta)|Z_0 = z\right\} = E\left\{U_1(t,\alpha',\beta')\big|Z_0 = z+\epsilon\right\},$$

almost surely, for all $t \geq 0$ *and* $z \in R$.

Proof. The law of motion of the process $\{Z_t\}_{t\geq 0}$ from the perspective of the noninvestible agent is:

$$dZ_t = \frac{1}{2}^2(1-\alpha_t)^2 dt + (1-\alpha_t)dB_t.$$

This means that, if the principal perturbs her strategy using a constant displacement, the agent can maintain his distribution of payoffs intact by imitating the perturbation. □

The payoff of the principal is not translation-invariant in the z-space. Note, however, that we can write her payoff conditioning on the type of the agent:

$$U_2(t,\alpha,\beta) = p_t E\left\{U_2(t,\alpha,\beta)|\theta = NI\right\} + (1-p_t)E\left\{U_2(t,\alpha,\beta)|\theta = I\right\}.$$

The conditional payoff inside the outer expectation satisfy a *conditional* translation invariance property.

Lemma B.2.2 (Conditional Translation Invariance)**.** *Fix an arbitrary strategy profile* (α,β) *and some* $\epsilon \in R$. *Consider a new profile* (α',β')*, defined by* $\alpha'_t := \alpha_t\Big|\{Z_s-\epsilon\}_{s\leq t}$ *and* $\beta'_t := \beta_t\Big|\{Z_s-\epsilon\}_{s\leq t}$ *for all* $t \geq 0$. *Then, the conditional payoffs of the principal satisfy*

$$E\left\{U_2(t,\alpha,\beta)|\theta = NI, Z_t = z\right\} = E\left\{U_2(t,\alpha',\beta')\big|\theta = NI, Z_t = z+\epsilon\right\},$$

$$E\left\{U_2(t,\alpha,\beta)|\theta = I, Z_t = z\right\} = E\left\{U_2(t,\alpha',\beta')\big|\theta = I, Z_t = z+\epsilon\right\},$$

almost surely, for all $t \geq 0$ *and* $z \in R$.

Proof. In the case of conditioning on $\theta = NI$, the law of motion of $\{Z_t\}_{t\geq 0}$ is as in the proof of Lemma B.2.1. In the case of conditioning on $\theta = I$, the dynamics of $\{Z_t\}_{t\geq 0}$ satisfies

$$dZ_t = -\frac{1}{2}^2(1-\alpha_t)^2 dt + (1-\alpha_t)dB_t.$$

In both cases the dynamics are linear given α, so if the agent perturbs his strategy using a constant displacement in $z-$space, the principal can maintain her payoff distribution intact by imitating the perturbation. $\square$

Monotonicity and Curvature of Value Functions

Proof of Corollary B.1.2. We first establish the claimed properties of $W(\cdot)$. From the law of motion of p_t given by (2.4), we know that the diffusion coefficient converges to 0 as $p \to 0$ or $p \to 1$. So it is easy to verify that $\lim_{p\to 0} W(p) = 0$ and $\lim_{p\to 1} W(p) = \frac{\lambda}{r_2+\lambda} w_{NI}$. Also, since the principal can always ignore any information, $W(p)$ is bouneded below by $\underline{W}(p) \equiv \frac{\lambda}{r_2+\lambda} \max\{0, R(p)\}$. Then, from the principal's HJB given by (2.8), we always have $W''(p) \geq 0$ no matter whether $W(p) > R(p)$ or $W(p) < R(p)$; that is, $W(\cdot)$ is convex on $(0,1)$. Since $\lim_{p\to 0} W(p) = 0$ and $W(\cdot) \geq 0$, $W(\cdot)$ must be (weakly) increasing at 0, and because it is convex, $W(\cdot)$ is increasing on $(0,1)$.

Now we turn to $v(\cdot)$. Suppose first that $r_1 \geq r^*$, so that in equilibrium $a(\cdot) \equiv 0$. From Claim 2 and conditions (B.26) and (B.28), $v(\cdot)$ is strictly decreasing and concave on $(-\infty, z^*)$, with $\lim_{z\to -\infty} v(z) = u + c$. From Claim 3 and conditions (B.27) and (B.29), $v(\cdot)$ is strictly decreasing and convex on (z^*, ∞), with $\lim_{z\to\infty} v(z) = \frac{r_1}{r_1+\lambda}(u+c)$. Suppose now that $r_1 < r^*$, so that in equilibrium $a(\cdot)$ is hump-shaped. In light of Claims 2 and 3 and conditions (B.14), (B.17), (B.20) and (B.23), it suffices to show that $v(\cdot)$ is strictly decreasing and concave on (z_L, z^*), and strictly decreasing and convex on (z^*, z_R). But these properties follow immediately from Claim 1 and the fact that $a(\cdot)$ is hump-shaped with $0 < a(z) < 1$ for $z \in (z_L, z_R)$. $\square$

B.2.2 Omitted Proofs for Theorem 2.5.1

Proof of Lemma B.1.2. Recall from the proof of Claim 8 that $r_1 < r^*$ implies $v_R < v_L$. Then by Claim 10 and Corollary B.1.4, we are done if we can find an $\lambda_2 \geq \lambda_1$ and an $\underline{r}$ such that $\lambda > \lambda_2$ and $r_1 < \underline{r}$ imply that $a^*_-(u; r_1) < a^*_+(u; r_1, \lambda)$.

Using (B.30) and (B.31), we have

$$\begin{aligned} & a^*_-\left(u; r_1\right) < a^*_+\left(u; r_1, \lambda\right) \\ \iff & \frac{\frac{\sqrt{2}}{}\phi\left(0\right)}{\frac{\sqrt{2r_1}}{}\phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right) + \Phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right) - \Phi\left(0\right)} > \frac{\frac{1}{\sqrt{r_1}}\phi\left(\sqrt{\frac{2}{r_1+\lambda}}\frac{\lambda}{r_1 c}u\right)}{\phi\left(\frac{v_R-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) + \frac{}{\sqrt{2(r_1+\lambda)}}\left[\Phi\left(\frac{v_R-\frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) - \Phi\left(\sqrt{\frac{2}{r_1+\lambda}}\frac{\lambda}{r_1 c}u\right)\right]} \end{aligned} \tag{B.43}$$

Since $\phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right) \leq \phi(0)$ and $\Phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right) \leq 1$, we can find a lower bound for the LHS of (B.43) whenever $r_1 < 1$:

$$\frac{\frac{\sqrt{2}}{}\phi(0)}{\frac{\sqrt{2r_1}}{}\phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right) + \Phi\left(\frac{v_L-u}{\sqrt{\kappa_L}}\right) - \Phi(0)} \geq \frac{\frac{\sqrt{2}}{}\phi(0)}{\frac{\sqrt{2}}{}\phi(0) + 1 - \Phi(0)}.$$

Now let us find an upper bound for the RHS of (B.43). First, when $\lambda \geq 1$, we know

$$\frac{1}{\sqrt{r_1}}\phi\left(\sqrt{\frac{2}{r_1+\lambda}}\frac{\lambda}{r_1 c}u\right) \leq \frac{1}{\sqrt{r_1}}\phi\left(\sqrt{\frac{2}{r_1+1}}\frac{}{r_1 c}u\right). \tag{B.44}$$

Second, by direct calculation we have

$$\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}} = \frac{\sqrt{2}}{4}\left(\frac{3}{\sqrt{\lambda+r_1}} + \sqrt{\frac{1}{r_1+\lambda} + \frac{8}{{}^2}}\right),$$

and when $\lambda \geq 1$, we have

$$\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}} \leq \frac{\sqrt{2}}{4}\left(\frac{3}{\sqrt{1+0}} + \sqrt{\frac{1}{1+0} + \frac{8}{{}^2}}\right) = \frac{\sqrt{2}}{4}\left(3 + \sqrt{{}^2+8}\right).$$

Then

$$\phi\left(\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) \geq \phi\left(\frac{\sqrt{2}}{4}\left(3 + \sqrt{{}^2+8}\right)\right). \tag{B.45}$$

Third,

$$\frac{}{\sqrt{2(r_1+\lambda)}}\left[\Phi\left(\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) - \Phi\left(\sqrt{\frac{2}{r_1+\lambda}}\frac{\lambda}{r_1 c}u\right)\right] \geq -\frac{}{\sqrt{2\lambda}}(1 - \Phi(0)).$$

Apparently, there exists $\underline{\lambda} \geq \lambda_1$, such that for all $\lambda > \underline{\lambda}$, we have

$$\phi\left(\frac{\sqrt{2}}{4}\left(3 + \sqrt{{}^2+8}\right)\right) - \frac{}{\sqrt{2\lambda}}(1 - \Phi(0)) \geq \frac{1}{2}\phi\left(\frac{\sqrt{2}}{4}\left(3 + \sqrt{{}^2+8}\right)\right). \tag{B.46}$$

Defining $\lambda_2 = \max\{1, \underline{\lambda}\}$ and applying conditions (B.44), (B.45) and (B.46), we know that whenever $\lambda > \lambda_2$, we have the following upper bound for the RHS of (B.43):

$$\frac{\frac{1}{\sqrt{r_1}}\phi\left(\sqrt{\frac{2}{r_1+\lambda}}\frac{\lambda}{r_1 c}u\right)}{\phi\left(\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) + \frac{}{\sqrt{2(r_1+\lambda)}}\left[\Phi\left(\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) - \Phi\left(\sqrt{\frac{2}{r_1+\lambda}}\frac{\lambda}{r_1 c}u\right)\right]} \leq \frac{\frac{1}{\sqrt{r_1}}\phi\left(\sqrt{\frac{2}{r_1+1}}\frac{}{r_1 c}u\right)}{\frac{1}{2}\phi\left(\frac{\sqrt{2}}{4}\left(3 + \sqrt{{}^2+8}\right)\right)}.$$

Now we compare the lower bound for the LHS of (B.43) with the upper bound for the RHS of (B.43). Taking limit $r_1 \to 0$, we have

$$\lim_{r_1 \to 0} \frac{\frac{1}{\sqrt{r_1}}\phi\left(\sqrt{\frac{2}{r_1+1}}\frac{}{r_1 c}u\right)}{\frac{1}{2}\phi\left(\frac{\sqrt{2}}{4}\left(3+\sqrt{^2+8}\right)\right)} = 0 < \frac{\frac{\sqrt{2}}{}\phi(0)}{\frac{\sqrt{2}}{}\phi(0)+1-\Phi(0)},$$

recalling that ϕ is the pdf of the standand normal distribution. So there exists $\underline{r}' > 0$ such that for all $r < \underline{r}'$,

$$\frac{\frac{1}{\sqrt{r_1}}\phi\left(\sqrt{\frac{2}{r_1+1}}\frac{}{r_1 c}u\right)}{\frac{1}{2}\phi\left(\frac{\sqrt{2}}{4}\left(3+\sqrt{^2+8}\right)\right)} < \frac{\frac{\sqrt{2}}{}\phi(0)}{\frac{\sqrt{2}}{}\phi(0)+1-\Phi(0)}.$$

Letting $\underline{r} = \min\{1, \underline{r}', r^*\}$, we have $a_-^\star(u; r_1) < a_+^\star(u; r_1, \lambda)$ whenever $\lambda > \lambda_2$ and $r_1 < \underline{r}$, as desired. ☐

Proof of Claim 11. Note that

$$a_+^*(u; r_1, \lambda) = 1 - \frac{1}{\phi\left(\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) + \frac{}{\sqrt{2(r_1+\lambda)}}\left[\Phi\left(\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) - \Phi\left(\sqrt{\frac{2}{r_1+\lambda}}\frac{\lambda}{r_1 c}u\right)\right]}\frac{1}{\sqrt{2\pi}}e^{-\frac{\lambda^2}{\lambda+r_1}\frac{2}{r_1^2 c^2}u^2}. \tag{B.47}$$

Note also that, for all $\lambda > \lambda_1$,

$$\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}} = \frac{\sqrt{2}}{4}\left(\frac{3}{\sqrt{\lambda+r_1}} + \sqrt{\frac{1}{\lambda+r_1}+\frac{8}{2}}\right) \leq \frac{\sqrt{2}}{4}\left(\frac{3}{\sqrt{\lambda_1}} + \sqrt{\frac{1}{\lambda_1}+\frac{8}{2}}\right)$$

Thus for all $\lambda > \lambda_1$,

$$\begin{aligned} & \phi\left(\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) + \frac{}{\sqrt{2(r_1+\lambda)}}\left[\Phi\left(\frac{v_R - \frac{r_1}{r_1+\lambda}u}{\sqrt{\kappa_R}}\right) - \Phi\left(\sqrt{\frac{2}{r_1+\lambda}}\frac{\lambda}{r_1 c}u\right)\right] \\ \geq \quad & \phi\left(\frac{\sqrt{2}}{4}\left(\frac{3}{\sqrt{\lambda_1}} + \sqrt{\frac{1}{\lambda_1}+\frac{8}{2}}\right)\right) - \frac{}{\sqrt{2\lambda}}. \end{aligned} \tag{B.48}$$

Let A' to be such that

$$\frac{1}{\sqrt{2\pi}A'} = \frac{1}{2}\phi\left(\frac{\sqrt{2}}{4}\left(\frac{3}{\sqrt{\lambda_1}} + \sqrt{\frac{1}{\lambda_1}+\frac{8}{2}}\right)\right).$$

Since $\lim_{\lambda \to \infty} \frac{}{\sqrt{2\lambda}} = 0$, there must exist $\lambda_3' \geq \lambda_1$, such that for all $\lambda > \lambda_3'$,

$$\phi\left(\frac{\sqrt{2}}{4}\left(\frac{3}{\sqrt{\lambda_1}} + \sqrt{\frac{1}{\lambda_1} + \frac{8}{2}}\right)\right) - \frac{}{\sqrt{2\lambda}} > \frac{1}{\sqrt{2\pi}A'}. \tag{B.49}$$

Finally, let $A = \max\{A', 2\}$. Conditions (B.47), (B.48) and (B.49) then tell us that whenever $\lambda > \lambda_3'$, we have

$$a_+^*\left(u; r_1, \lambda\right) > 1 - A' e^{-\frac{\lambda^2}{\lambda + r_1}\frac{2}{r_1^2 c^2}u^2} \geq 1 - A e^{-\frac{\lambda^2}{\lambda + r_1}\frac{2}{r_1^2 c^2}u^2},$$

as desired. □

B.2.3 Omitted Proofs for Theorem 2.6.1

Proof of Claim 17. The proof is almost identical to Lemma B.2.4's proof in the next section, and is thus omitted. □

B.2.4 Patient Limit: Toward a Proof of Theorem 2.7.1

In this section, we prove Theorem 2.7.1 which is about the convergence of equilibrium value functions when players get arbitrarily patient at comparable rates.

For each $n \in N$, take the unique Markov equilibrium (a_n, b_n) associated with the discount factor $r_{i,n}$ for $i = 1, 2$. Assume that $\lim_{n\to\infty} r_{i,n} = 0$ and $\lim_{n\to\infty}\left(r_{2,n}/r_{1,n}\right) := \chi \in (0, \infty)$. Let $V_n\left(\cdot\right)$ be the agent's value function in the equilibrium (a_n, b_n) and $W_n\left(\cdot\right)$ be the principal's value function. We will often use $z \equiv \log(p/1-p)$ as state variable when analyzing the agent's behavior. When doing so, we denote by $v_n(z) := V_n(p(z))$ the agent's value function in the $z-$space. Write z_n^* for the principal's equilibrium cutoff. Write $z_{L,n}$ for the infimum belief z at which the agent plays $a_n\left(z\right) > 0$ and write $z_{R,n}$ for the supremum. Write T for the equilibrium stopping time that stops the play of the game. *Without labeling explicitly, we note that the distribution of T depends on n and the current state z.* For $i = 1, 2$, let $E_n^\theta\left\{e^{-r_{i,n}T}\right\}$ be the expected discount factor when the stopping action is taken in the equilibrium (a_n, b_n) discounted at rate $r_{i,n}$ and given the equilibrium strategy of type

$\theta \in \{NI, I\}$.[18] When the game starts at state z, let

$$E_n\left\{e^{-r_{i,n}T}\right\} := p(z)E_n^{NI}\left\{e^{-r_{i,n}T}\right\} + (1-p(z))E_n^{I}\left\{e^{-r_{i,n}T}\right\}.$$

Claim OA.8. *Take $(r_1, r_2) \in R^2_{++}$ and let τ be any stopping time. Assume that $E\{e^{-r_1\tau}\} = \xi \in (0,1)$.*

i) If $r_2 \leq r_1$ then

$$\xi \leq E\left\{e^{-r_2\tau}\right\} \leq \xi^{(r_2/r_1)}.$$

ii) If $r_2 > r_1$ then

$$\xi^{(r_2/r_1)} \leq E\left\{e^{-r_2\tau}\right\} \leq \xi.$$

Moreover, for each $\xi \in (0,1)$ and any inequality above, there exists a distribution over stopping times for which this inequality is tight.

Proof. Assume that $E\{e^{-r_1\tau}\} = \xi \in (0,1)$. Let F be the CDF of τ. Let $y = e^{-r_1\tau}$ and H be its CDF. We have $\tau = -(\log y)/r_1$ and hence $\int_0^\infty e^{-r_2 t}dF(t) = \int_0^1 y^{(r_2/r_1)}dH(y)$.

i) Suppose that $r_2 \leq r_1$. On the one hand, since $y^{(r_2/r_1)}$ is concave, we have

$$\int_0^1 y^{(r_2/r_1)}dH(y) \leq \left(\int_0^1 ydH(y)\right)^{(r_2/r_1)} = \xi^{(r_2/r_1)},$$

with equality if H has an atom of mass one.

On the other hand, $y^{(r_2/r_1)} \geq y$ for every $y \in [0,1]$ and hence $\int_0^1 y^{(r_2/r_1)}dH(y) \geq \int_0^1 ydH(y)$, with equality if the support of H is $\{0,1\}$.

ii) Suppose that $r_2 \geq r_1$. On the one hand, since $y^{(r_2/r_1)}$ is convex, we have

$$\int_0^1 y^{(r_2/r_1)}dH(y) \geq \left(\int_0^1 ydH(y)\right)^{(r_2/r_1)} = \xi^{(r_2/r_1)},$$

with equality if H has an atom of mass one.

On the other hand, $y^{(r_2/r_1)} \leq y$ for every $y \in [0,1]$ and hence $\int_0^1 y^{(r_2/r_1)}dH(y) \leq \int_0^1 ydH(y)$, with equality if the support of H is $\{0,1\}$. □

Claim OA.9. *For every $\varepsilon > 0$ there exists $z^\dagger \in R$ and $\tilde{n}_1 \in N$ such that, if $z \geq z^\dagger$ and $n \geq \tilde{n}_1$, then the continuation payoff of the agent at z is less than ε in the equilibrium*

[18] Here, we interpret the strategy of the investible type as always setting $a = 1$.

(a_n, b_n).

Proof. Otherwise we can find a sequence of equilibria (a_n, b_n) starting at $(z_n) \to +\infty$ in which the agent obtains a payoff weakly greater than ε. Since the agent's equilibrium payoff is bounded above by $(u+c)\left[1 - E_n^{NI}\left(e^{-r_{1,n}T}\right)\right]$, we have

$$(u+c)\left[1 - E_n^{NI}\left(e^{-r_{1,n}T}\right)\right] \geq \varepsilon \quad \Rightarrow \quad E_n^{NI}\left(e^{-r_{1,n}T}\right) \leq \left(1 - \tfrac{\varepsilon}{u+c}\right),$$

and hence the principal's payoff in equilibrium (a_n, b_n) at z_n is at most

$$\max\left\{\left(1 - \tfrac{\varepsilon}{u+c}\right), \left(1 - \tfrac{\varepsilon}{u+c}\right)^{(r_{2,n}/r_{1,n})}\right\} p\left(z_n\right) w_{NI},$$

which is always strictly less than w_{NI}. Meanwhile, since $r_{2,n} \to 0$ and $z_n \to \infty$ as $n \to \infty$, the principal's payoff at z_n by terminating the relationship in the first opportunity satisfies

$$\lim_{n\to\infty}\left(\tfrac{\lambda}{r_{2,n}+\lambda}\right)\left[(1-p(z_n))w_I + p(z_n)w_{NI}\right] = w_{NI}.$$

So the principal has a profitable deviation when n is sufficiently large. □

We assume that $n \geq \tilde{n}_1$ for the remainder of this proof.

Claim OA.10. *For every fixed z_0, we have $\limsup_{n\to\infty} v_n\left(z_0\right) \leq u$.*

Proof. Take any small $\varepsilon \in (0, u/2)$. For each $n \in N$, let $z_\varepsilon^n := \inf\left\{z | a_n\left(z\right) = 1 - \varepsilon\right\}$. There are two cases to consider. Let $z^\dagger$ be defined and delivered by Claim OA.9. Every sequence can be split into (at most) two subsequence, each one of them satisfying one of the cases below.

Case 1 $z_\varepsilon^n \leq z^\dagger$ for every $n \in N$.

In this case, take $m \in N$ such that $z^\dagger - m < z_0$ and let $z_0^n := z_\varepsilon^n - m$. Since $v_n(\cdot)$ is decreasing, it suffices to show that $\limsup_{n\to\infty} v_n\left(z_0^n\right) \leq u$.

Take any $\zeta > 0$. Suppose that the game starts at z_0^n and consider the stopping time $\hat{T}_n$ that stops the play of the game at the first time $Z_n(t) = z_\varepsilon^n$ (setting $\hat{T}_n = +\infty$ if this event does not happen in finite time). Note that $Z_n\left(t\right)$ is a submartingale under the strategy of the noninvestible type and that $a_n\left(z\right) \leq 1 - \varepsilon$ with probability one before $\hat{T}_n$. Using this observation and $Z_n(t)$'s law of motion (B.1), it is straightforward to show that $\hat{T}_n < +\infty$ with probability one under the strategy of the noninvestible type and that $E_n^{NI}\left[e^{-r_{1,n}\hat{T}_n}\right] \to 1$.

Take $n^{**} \in N$ for which $n > n^{**}$ implies $E_n^{NI}\left[e^{-r_{1,n}\hat{T}_n}\right] > 1 - \varepsilon$. Next notice that, at the state z_ε^n, v_n is decreasing and concave (by Corollary B.1.2), and hence

$$\begin{aligned} r_{1,n} v_n\left(z_\varepsilon^n\right) &= r_{1,n}\left[u + \left(1 - a_n\left(z_\varepsilon^n\right)\right) c\right] + \frac{1}{2}^2\left[1 - a_n\left(z_\varepsilon^n\right)\right]^2\left[v'\left(a_n\left(z_\varepsilon^n\right)\right) + v''\left(a_n\left(z_\varepsilon^n\right)\right)\right] \\ &\leq r_{1,n}\left[u + \left(1 - a_n\left(z_\varepsilon^n\right)\right) c\right], \end{aligned}$$

which implies $v_n\left(z_n^\varepsilon\right) \leq u + \varepsilon c$, because $a_n(z_n^\varepsilon) = 1 - \varepsilon$. It follows that the payoff of the noninvestible type converges to a number not greater than $(1-\varepsilon)\left(u + \varepsilon c\right) + \varepsilon(u + c)$, which proves the result as ε is arbitrary.

Case 2 $z_\varepsilon^n > z^\dagger$ for every $n \in N$.

We may assume that $z_0 < z^\dagger$ for every n as otherwise the claim follows from Claim OA.9. Suppose that the game starts at z_0 and consider the stopping time $\hat{T}_n$ that stops the play of the game at $z^\dagger$. As in Case 1, we have $\hat{T}_n < +\infty$ with probability one under the noninvestible-type's strategy and $E_n^{NI}\left\{e^{-r_{1,n}\hat{T}_n}\right\} \to 1$. Since $\limsup_{n\to\infty} v_n\left(z^\dagger\right) \leq \varepsilon < u/2$, the rest of the proof follows the same argument as in Case 1. □

Claim OA.11. $\lim_{n\to\infty} z_{L,n} = -\infty$.

Proof. The proof follows verbatim from Claim 14's proof. □

For every $z_0 < \liminf z_n^*$, we have $\lim_{n\to\infty} a_n\left(z_0\right) = 1$.

Proof. By Claim OA.11, $z_0 \in (z_{L,n}, z_n^*)$ for n sufficiently large. Then from condition (B.10), we know that $a_n(\cdot)$ eventually satisfies the following differential equation

$$a_n'\left(z\right) = 1 - a_n(z) - 2\left(\tfrac{v_n(z) - u}{c}\right). \tag{B.50}$$

Assume toward a contradiction that we can find a subsequence such that

$$\lim_{n\to\infty} a_n\left(z_0\right) = \bar{a} < 1.$$

Take $m \in N$ such that $\frac{1-\bar{a}}{4} m > 2$. Claim OA.11 implies that $[z_0 - m, z_0] \subset (z_{L,n}, z_n^*)$ for n sufficiently large. Claim OA.10 and the monotonicity of $v_n(\cdot)$ imply that we can find $n^\dagger \in N$ such that for every $n \geq n^\dagger$, for every $z \in [z_0 - m, z_0]$, we have $2\left(\frac{v_n(z) - u}{c}\right) < \frac{1-\bar{a}}{4}$. Given the contradiction assumption, we can find $n^\dagger \in N$ such that for every $n \geq n^\dagger$ we have

$a_n(z_0) < \frac{1+\bar{a}}{2}$. Since $a_n(\cdot)$ is strictly increasing on $[z_0 - m, z_0]$, this implies $a_n(z) < \frac{1+\bar{a}}{2}$ for all $z \in [z_0 - m, z_0]$. So (B.50) implies $a'_n(z) > \frac{1-\bar{a}}{4}$ for all $z \in [z_0 - m, z_0]$ and hence

$$a_n(z_0 - m) < a_n(z_0) - \tfrac{1-\bar{a}}{4} m < a_n(z_0) - 2 < 0,$$

which leads to a contradiction as a_n is bounded below by 0. □

Lemma B.2.3. *For every $z_0 < \liminf_{n\to\infty} z^*_n$, we have $\lim_{n\to\infty} v_n(z_0) = u$.*

Proof. By Claim OA.11, $z_0 \in (z_{L,n}, z^*_n)$ for n sufficiently large. Take $\vartheta > 0$ such that $z_0 + 2\vartheta < \liminf z^*_n$ and, taking a subsequence if necessary, assume that $z_0 + \vartheta < z^*_n$ for each one of its elements.

Assume toward a contradiction, taking a subsequence if necessary, that

$$\lim_{n\to\infty} v_n(z_0) < u - \varepsilon,$$

for some $\varepsilon > 0$. Because $v_n(\cdot)$ is strictly decreasing, we may take n^* such that $n \geq n^*$ implies $v_n(z) < u - \frac{\varepsilon}{2}$ for all $z \in \left[z_0, z_0 + \frac{\vartheta}{2}\right]$. In this case, we have $a'_n(z) = 1 - a_n(z) - 2\left(\frac{v_n(z)-u}{c}\right) \geq \frac{\varepsilon}{c}$ for every $z \in \left[z_0, z_0 + \frac{\vartheta}{2}\right]$. This implies that $\limsup_n a_n(z_0) \leq 1 - \left(\frac{\vartheta}{2}\right)\frac{\varepsilon}{c}$, contradicting Lemma B.2.4. □

Lemma B.2.4. *Fix a prior $p_0 \in (0,1)$ and some $\bar{p} \in (p_0, 1)$. For each $r > 0$, consider an adapted Markov function $\alpha_r(\cdot)$ and a belief process defined by substituting $\alpha_r(\cdot)$ into (2.7). Take $\varepsilon > 0$ and let $\bar{T}$ be the random time that stops the play in the first time that $p \geq \bar{p}$. Then we have:*

$$\limsup_{r\downarrow 0} E^{NI}\left\{ r \int_0^{\bar{T}} e^{-rt} I_{\{\alpha_r(p_t) \leq 1-\varepsilon\}} dt \right\} = 0.$$

Proof. Take a small $\epsilon > 0$. Next, take $\zeta > 0$ and let T^ζ be the stopping time that stops the play in the first time that the posterior reaches $(\zeta, \bar{p})^c$. Using the martingale property of beliefs whose law of motion is given by (2.7), it is straightforward to show that we can take ζ small enough so that

$$P^{NI}\left\{ T^\zeta < \infty, p(T^\zeta) = \zeta \right\} < \frac{\epsilon}{2}.$$

Therefore, we have:

$$\begin{aligned}
E^{NI}\left\{r\int_0^{\bar{T}} e^{-rt}I_{\{\alpha_r(p_t)\leq 1-\varepsilon\}}dt\right\} &= E^{NI}\left\{r\int_0^{\bar{T}} e^{-rt}I_{\left\{\inf_{t\leq\bar{T}} p_t\leq\zeta\right\}}I_{\{\alpha_r(p_t)\leq 1-\varepsilon\}}dt\right\} \\
&\quad + E^{NI}\left\{r\int_0^{\bar{T}} e^{-rt}I_{\left\{\inf_{t\leq\bar{T}} p_t>\zeta\right\}}I_{\{\alpha_r(p_t)\leq 1-\varepsilon\}}dt\right\} \\
&\leq \frac{\epsilon}{2} + E^{NI}\left\{r\int_0^{\bar{T}} e^{-rt}I_{\left\{\inf_{t\leq\bar{T}} p_t>\zeta\right\}}I_{\{\alpha_r(p_t)\leq 1-\varepsilon\}}dt\right\} \\
&\leq \frac{\epsilon}{2} + E^{NI}\left\{r\int_0^{T^\zeta} e^{-rt}I_{\{\alpha_r(p_t)\leq 1-\varepsilon\}}dt\right\} \\
&\leq \frac{\epsilon}{2} + E^{NI}\left\{r\int_0^{T^\zeta} I_{\{\alpha_r(p_t)\leq 1-\varepsilon\}}dt\right\}.
\end{aligned}$$

We must then show that $\limsup_{r\downarrow 0} E^{NI}\left\{r\int_0^{T^\zeta} I_{\{\alpha_r(p_t)\leq 1-\varepsilon\}}dt\right\} < \frac{\epsilon}{2}$. Let $\xi_r(t)$ be a function that is 1 whenever $a_r\left(p_t\right) \leq 1-\varepsilon$ and 0 otherwise. It suffices to show that

$$\limsup_{r\downarrow 0} rE^{NI}\left\{\int_0^{T^\zeta} \xi_r(t)dt\right\} < \frac{\epsilon}{2}.$$

For that we will consider a different stopping time T^* and a new process $\xi_r^*(t)$ which are built from T^ζ and $\xi_r(t)$ in the following way. Whenever $T^\zeta < \infty$ and $\int_0^{T^\zeta} \xi_r(t)dt \in (m-1,m)$ for some $m \in N$, we will set

$$\xi_r^*(t) := \begin{cases} \xi_r(t) & t \leq T^\zeta, \\ 1 & t > T^\zeta. \end{cases}$$

We will also set $T^* := T^\zeta + \tilde{t}$, where $\tilde{t}$ is defined by $\int_0^{T^\zeta} \xi_r(t)dt + \tilde{t} = m$. Whenever $T^\zeta < +\infty$ and $\int_0^{T^\zeta} \xi_r(t)dt = m-1$ for some $m \in N$, we set $\xi_r^*(t) := \xi_r(t)$ and $T^* := T^\zeta$. Clearly it suffices to show that

$$\limsup_{r\downarrow 0} E^{NI}\left\{r\int_0^{T^*} \xi_r^*(t)dt\right\} < \frac{\epsilon}{2}.$$

Next, we build a family of stochastic processes $\left\{\xi_{r,m}^*(t)\right\}_{m\in N}$ from $\xi_r^*(t)$ by setting

$$\xi_{r,m}^*(t) := \begin{cases} \xi_r^*(t) & \int_0^t \xi^*(t)dt \in (m-1,m], \\ 0 & \text{otherwise.} \end{cases}$$

This immediately implies that $rE^{NI}\left\{\int_0^{T^*}\xi_r^*(t)dt\right\} = r\sum_{m=1}^{\infty}E^{NI}\left\{\int_0^{T^*}\xi_{r,m}^*(t)dt\right\}$.

Next, observe that, conditional on $\theta = NI$, p_t is a bounded submartingale. Thus, for any adapted function $\tilde{\xi}(t) \in \{0,1\}$ and any stopping time $\tilde{T}$, we have

$$1 \geq p_T - p_0 = E^{NI}\left[\int_0^T dp_t\right] = E^{NI}\left[\int_0^T \tilde{\xi}(t)\,dp_t\right] + E^{NI}\left[\int_0^T \left(1-\tilde{\xi}(t)\right)dp_t\right]$$
$$\geq E^{NI}\left[\int_0^T \tilde{\xi}(t)\,dp_t\right]$$

because $\left(1-\tilde{\xi}(t)\right)$ being an adapted process and p_t being a submartingale jointly imply that $E^{NI}\left[\int_0^T\left(1-\tilde{\xi}(t)\right)dp_t\right] \geq 0$. As a result, we have

$$1 \geq E^{NI}\left[\int_0^T \xi_r^*(t)\,dp_t\right] = \sum_{m=1}^{\infty}E^{NI}\left[\int_0^T \xi_{r,m}^*(t)\,dp_t\right]. \quad \text{(B.51)}$$

Next, since $0 < \zeta < \bar{p} < 1$, from condition (2.7) it is straightforward to show that there exists a positive constant $\vartheta > 0$ such that, for any $m \in N$, we have

$$E^{NI}\left\{\int_0^{T^*}\xi_{r,m}^*(t)dp_t\right\} \geq \vartheta E^{NI}\left\{\int_0^{T^*}\xi_{r,m}^*(t)dt\right\}. \quad \text{(B.52)}$$

Therefore, combining (B.51) and (B.52) we have

$$\sum_{m=1}^{\infty}E^{NI}\left\{\int_0^{T^*}\xi_{r,m}^*(t)dt\right\} \leq \frac{1}{\vartheta}\sum_{m=1}^{\infty}E^{NI}\left\{\int_0^{T^*}\xi_{r,m}^*(t)dp_t\right\} \leq \frac{1}{\vartheta},$$

implying that $\sum_{m=1}^{\infty}rE^{NI}\left\{\int_0^{T^*}\xi_{r,m}^*(t)dt\right\} \leq \frac{r}{\vartheta}$, which is smaller than $\frac{\epsilon}{2}$ when r is sufficiently small. □

Lemma B.2.5. $\lim_{n\to\infty} z_n^* = z^{**}$.

Proof. Suppose toward a contradiction that we can find a subsequence for which $\lim_{n\to\infty} z_n^* := \bar{z} > z^{**}$. Let z^m be the midpoint between $\bar{z}$ and z^{**}. Take $\varepsilon > 0$. Consider the game starting at z^m. Notice that Lemma B.2.3 implies that $\lim_{n\to\infty} v_n(z^m) = u$, while Lemma B.2.4 implies that, for each $\nu > 0$, we have

$$\lim_{n\to\infty} E_n^{NI}\left\{r_{1,n}\int_0^T e^{-r_{1,n}t}I_{\{a_n(z_t)\leq 1-\nu\}}dt\right\} = 0.$$

These two observations imply that $\lim_{n\to\infty} E_n^{NI}\left(e^{-r_{1,n}T}\right) = 0$, which, by the same argument as Claim OA.8's proof, implies that $\lim_{n\to\infty} E_n^{NI}\left(e^{-r_{2,n}T}\right) = 0$; that is, conditional on $\theta = NI$, the principal derives zero discounted payoff from the game. It follows that the principal obtains a limit payoff bounded above by zero at z^m. But then, for n sufficiently large, if the stopping opportunity arrives at $z = z^m$, the principal can profitably deviate by stopping the game to obtain $p(z^m) w_{NI} + (1 - p(z^m)) w_I > 0$, a contradiction. □

Lemma B.2.6. *For every $z_0 > z^{**}$ and $i = 1, 2$, we have $\lim_{n\to\infty} E_n\left\{e^{-r_{i,n}T}\right\} = 1$.*

Proof. Fix $z_0 > z^{**}$. By Claim OA.8, it suffices to show that $\lim_{n\to\infty} E_n\left\{e^{-r_{2,n}T}\right\} = 1$. Taking a subsequence if necessary, assume toward a contradiction that

$$\lim_{n\to\infty} E_n\left\{e^{-r_{2,n}T}\right\} < 1.$$

Let $\tilde{\tau}$ be the stopping time that stops the play in the first time that either the state reaches $[0, p(z_n^*)]$ or when T happens. Let $x = e^{-r_{2,n}t}$. Let Q_n be the distribution of $p_{\tilde{\tau}}$ and $H_n(\cdot \mid p_{\tilde{\tau}})$ be the conditional distribution of x given $p_{\tilde{\tau}}$.

Step 1. We show that the contradiction assumption implies that, the discounted amount of time that the relationship continues with beliefs close to $p(z_n^*)$ is nonnegligible (i.e., condition (B.55) holds).

Note that

$$W_n(p(z_0)) = \int_{p(z_n^*)}^1 \int_0^1 x\left[I_{\{p_{\tilde{\tau}} > p(z_n^*)\}} R(p_{\tilde{\tau}}) + I_{\{p_{\tilde{\tau}} \le p(z_n^*)\}} W_n(p(z_n^*))\right] H_n(dx \mid p_{\tilde{\tau}})\, dQ_n(dp_{\tilde{\tau}}).$$

Because $\lim_{n\to\infty} W_n(p(z_n^*)) = \lim_{n\to\infty} R(p(z_n^*)) = 0$, we have

$$\limsup_{n\to\infty} W_n(p(z_0)) = \limsup_{n\to\infty} \int_{p(z_n^*)}^1 \int_0^1 x R(p_{\tilde{\tau}}) H_n(dx \mid p_{\tilde{\tau}})\, Q_n(dp_{\tilde{\tau}}). \tag{B.53}$$

Moreover, since $R(p^{**}) = 0$ and $p(z_n^*) \to p^{**}$ (by Lemma B.2.5), for every $\varepsilon > 0$ there exists $\zeta > 0$ such that when n is sufficiently large, $R(p) > \zeta$ for every $p > p(z_n^*) + \varepsilon$. Combining this observation with condition (B.53), it is easy to show that, for every $\varepsilon > 0$, if

$$\limsup_{n\to\infty} \int_{p(z_n^*)+\varepsilon}^1 \int_0^1 (1 - x) H_n(dx \mid p_{\tilde{\tau}})\, Q_n(dp_{\tilde{\tau}}) > 0,$$

then we would have $\limsup_{n\to\infty} W_n(p(z_0)) < R(p(z_0))$, which contradicts $b_n(z_0) = 1$

(principal's optimality) when n is sufficiently large. Hence, for every $\varepsilon > 0$, we have

$$\limsup_{n\to\infty} \int_{p(z_n^*)+\varepsilon}^{1} \int_0^1 (1-x)\, H_n\left(dx \mid p_{\tilde{\tau}}\right) Q_n(dp_{\tilde{\tau}}) = 0. \tag{B.54}$$

Therefore, the assumption that $\lim_{n\to\infty} E_n\left(e^{-r_{2,n}T}\right) < 1$ implies that, for every $\varepsilon > 0$, we have

$$\limsup_{n\to\infty} \int_{p(z_n^*)}^{p(z_n^*)+\varepsilon} \int_0^1 (1-x)\, H_n\left(dx \mid p_{\tilde{\tau}}\right) Q_n(dp_{\tilde{\tau}}) > 0. \tag{B.55}$$

For the remainder of this proof, we take $\varepsilon > 0$ such that $p\left(z_n^*\right) + \varepsilon < \left(\frac{z_0 + z^{**}}{2}\right)$.

Step 2. We show that condition (B.55) implies that, the noninvestible type has a profitable deviation by fully mimicking the investible type.

Lemma B.2.4 implies that if we let $\bar{T}_m$ be the random time that stops the play in the first time that the posterior leaves $\left(m^{-1}, 1-m^{-1}\right)$ or that T happens, then for each $\upsilon > 0$, we have

$$\lim_{n\to\infty} E_n^{NI}\left\{ r_{1,n} \int_0^{\bar{T}_m} e^{-r_{1,n}t} I_{\{a_n(p_t)\leq 1-\upsilon\}} dt \right\} = 0.$$

By the martingale property of beliefs we can take $m \in N$ large enough to make

$$\limsup_{n\to\infty} P^{NI}\left\{ \inf_{t\leq T} p_t \leq m^{-1} \right\}$$

as small as we want. Analogously, we can take m large enough to guarantee that whenever the posterior starts at $\left(1-m^{-1}, 1\right)$ then $\limsup_{n\to\infty} P^{NI}\left\{\inf_{t\leq T} p_t \leq p\left(z_n^*\right) + \varepsilon\right\}$ is as small as we want. These two observations then imply that

$$\limsup_{n\to\infty} E_n^{NI}\left\{ r_{1,n} \int_0^{T} e^{-r_{1,n}t} (1 - a_n\left(p_t\right)) dt \right\} = 0. \tag{B.56}$$

Next, let $y = e^{-r_{1,n}t}$. For $\theta \in \{NI, I\}$, let Q_n^θ stand for the distribution of p_T (not $p_{\tilde{\tau}}$ as above) given the strategy of type θ and let $H_n^\theta\left(\cdot \mid p_T\right)$ stand for the conditional distribution of y given p_T and the strategy of type θ. On the one hand, using (B.54) and (B.56), it is straightforward to see that, taking a subsequence if necessary, the limit payoff of the noninvestible type from following his equilibrium strategy is given by:

$$\lim_{n\to\infty} u \int_0^{p(z_n^*)+\varepsilon} \int_0^1 (1-y)\, H_n^{NI}\left(dy \mid p_T\right) Q_n^{NI}(dp_T) > 0, \tag{B.57}$$

where the positive sign follows from (B.55). On the other hand, the limit payoff of the noninvestible type from following the strategy of the investible type (i.e., always boosting performance with probability 1) is given by:

$$\lim_{n\to\infty} u \int_0^{p(z_n^*)+\varepsilon} \int_0^1 (1-y)\, H_n^I\,(dy \mid p_T)\, Q_n^I(dp_T) > 0. \tag{B.58}$$

Next, a straightforward application of Bayes rule implies that $H_n^{NI}\,(\cdot \mid p_T) = H_n^I\,(\cdot \mid p_T)$ for every $p_T \in (0,1)$. Moreover, using $p\,(z_n^*) + \varepsilon < \left(\frac{z_0+z^{**}}{2}\right)$ and Bayes rule, one can find $\xi > 1$ such that $Q_n^I(A) \geq \xi Q_n^{NI}(A)$ for every (Borel-measurable) $A \subset [0, p\,(z_n^*) + \varepsilon]$. Hence, subtracting (B.57) from (B.58) we obtain an expression at least as large as

$$\lim_{n\to\infty} (\xi - 1)\, u \int_0^{p(z_n^*)+\varepsilon} \int_0^1 (1-y)\, H_n^{NI}\,(dy \mid p_T)\, Q_n^{NI}(dp_T) > 0.$$

This implies that the noninvestible type can profitably deviate by fully mimicking, which leads to a contradiction and concludes the proof. □

Proof of Theorem 2.7.1. First, for the agent, Lemmas B.2.3 and B.2.5 tell us that

$$\lim_{n\to\infty} v_n(z) = u$$

for all $z < z^{**}$, and Lemma B.2.6 implies that $\lim_{n\to\infty} v_n(z) = 0$ for all $z > z^{**}$.

Next, for the principal, we first argue that $W_n(\cdot)$ converges pointwise to $\max\{0, R(\cdot)\}$. In light of Corollary B.1.2, we continuously extend $W_n(\cdot)$ from $(0,1)$ to $[0,1]$ by setting $W_n(0) = 0$ and $W_n(1) = \frac{\lambda}{r_{2,n}+\lambda} w_{NI}$. Lemma B.2.6 implies that $\lim_{n\to\infty} W_n(p(z)) = R(p(z))$ for all $z > z^{**}$. We now show that $\lim_{n\to\infty} W_n(p(z)) = 0$ for all $z \leq z^{**}$. To see this, fix any $z \leq z^{**}$ and take any $\varepsilon > 0$. Since $R(p(z^{**})) = 0$, there exists $\delta > 0$ such that $R(p(z^{**})+\delta) < \frac{\varepsilon}{2}$. But then, we can find n^* such that for every $n > n^*$, $W_n(p(z^{**})+\delta) < \varepsilon$. Since $W_n(\cdot)$ is increasing, it follows that $W_n(p(z)) < \varepsilon$ for every $n > n^*$. So we must have $\lim_{n\to\infty} W_n(p(z)) = 0$, because ε is arbitrary and $W_n(\cdot)$ is bounded below by 0.

To show uniform convergence, note that for any fixed n, $W_n(\cdot)$ is bounded below by $\frac{\lambda}{r_{2,n}+\lambda} \max\{0, R(\cdot)\}$ such that $W_n(1) = \frac{\lambda}{r_{2,n}+\lambda} R(1)$. Because $W_n(\cdot)$ is convex and increasing, $|W_n'(\cdot)|$ is bounded above by $(w_{NI} - w_I)$, and hence $\{W_n\}_n$ is uniformly equicontinuous. Since W_n converges pointwise to $\max\{0, R\}$, invoking Arzelà–Ascoli theorem we conclude that the convergence is uniform. □

Appendix C

Appendix: A Dynamic Delegated Investment Model of SPAC

C.1 Proofs

C.1.1 Proof of Lemma 3.3.2

Suppose $F_G(-t) \leq V(-t)$ at a time $-t$. Then $F_B(-t) < V(-t)$, so $\alpha_B(t) = 0$. If $\alpha_G(t) > 0$, by the investor's rational belief in equilibrium, $\tilde{\theta}(-t) = +\infty$. According to Lemma 3.3.1, $\eta(-t) = 1$, so $F_G(-t) = v_G$. Since looking forward in the future, the sponsor always expects a positive probability of no investment, $V(-t) < v_G = F_G(-t)$. Contradiction! If $\alpha_G(t) = 0$, by D1 refinement, $\tilde{\theta}(-t) = +\infty$. Following the same argument, we encounter contradiction. Finally, $F_G(-t) > V(-t)$ directly implies $\alpha_G(-t) = 1$ and strictly decreasing continuation value over time.

C.1.2 Proof of Lemma 3.3.3

Suppose $V(-t) < v_B(1-q)$ and $t > 0$. If $\alpha_B(-t) = 0$, following the proof of Lemma 3.3.2, we have $\eta(-t) = 1$, so $F_B(-t) = (1-q)\, v_B > V(-t)$. Contradiction! If $\alpha_B(-t) = 1$,

$$\tilde{\theta}(-t) \leq \frac{p_0}{1-p_0} < \frac{1-R_B}{R_G-1} < \frac{1-u_B}{u_G-1},$$

which implies $\eta(-t) = 0$ according to Lemma 3.3.1. Then $F_B(-t) = 0 < V(-t)$. Contradiction! Therefore, $\alpha_B(-t) \in (0,1)$. $V(-t) > (1-q)\, v_B$ implies $V(-t) > F_B(-t)$, so

$\alpha_B(-t) = 0$.

C.1.3 Proof of Proposition 3.3.3

For $t < t^*$,

$$\frac{dV(-t)}{dt} = \lambda p_0 \cdot [F_G(-t) - V(-t)].$$

Plugging $F_G(-t) \equiv q + (1-q)\,\eta(-t)$ and the equilibrium condition $V(-t) = (1-q)\eta(-t)v_B$ into the equation, we obtain

$$\frac{dV(-t)}{dt} = \lambda p_0 \cdot \left[\left(\frac{v_G}{v_B} - 1 \right) V(-t) + q v_G \right].$$

Combining with the boundary condition $V(0) = 0$, we obtain

$$V(-t) = \left[e^{\lambda p_0 \left(\frac{v_G}{v_B} - 1 \right) t} - 1 \right] \frac{1}{\frac{v_G}{v_B} - 1} \cdot q \cdot v_G.$$

So, t^* satisfies eq. (3.5).

C.1.4 Proof of Lemma 3.4.1

In the second stage where $-t > -t_s^*$,

- the sponsor's continuation value $V_s(-t)$ satisfies

$$\frac{dV_s(-t)}{dt} = \lambda \left[p_0 v_G + (1-p_0) v_B - V_s(-t) \right]$$

  and two boundary conditions $V_s(0) = 0$ and $V_s(-t_s^*) = v_B$;

- the investor's continuation value $U_s(-t)$ satisfies

$$\frac{dU_s(-t)}{dt} = \lambda \cdot \left[p_0 u_G + (1-p_0) u_B - U_s(-t) \right]$$

  and one boundary condition $U_s(0) = 1$.

In the first stage where $-t < -t_s^*$,

- $V_s(-t)$ satisfies

$$\frac{dV_s(-t)}{dt} = \lambda p_0 \cdot \left[v_G - V_s(-t) \right]$$

  and one boundary condition $V_s(-t_s^*) = v_B$;

- $U_s(t)$ satisfies
$$\frac{dU_s(-t)}{dt} = \lambda p_0 \cdot [u_G - U_s(-t)]$$
and one boundary condition that requires $U_s(-t)$ is continuous at $-t_s^*$.

According to the evolution of the sponsor's continuation value, we obtain that for $-t \geq -t_s^*$,

$$V_s(-t) = \left(1 - e^{-\lambda t}\right)[p_0 v_G + (1-p_0)v_B].$$

The first point is directly implied by the boundary condition $V_s(-t_s^*) = v_B$.

According to the evolution of the investor's continuation value, we obtain that for $-t \geq -t_s^*$,

$$U_s(t) = \left(1 - e^{-\lambda t}\right)[p_0 u_G + (1-p_0)u_B] + e^{-\lambda t}$$

and for $-t < -t_s^*$,

$$U_s(-t) = \left(1 - e^{-\lambda p_0 (t - t_s^*)}\right) u_G + e^{-\lambda p_0 (t-t_s^*)} U_s(-t_s^*).$$

Notice that $U_s(-t) < 1$ for $t \geq -t_s^*$ and $U_s(-t)$ increases to u_G as t increases from t_s^* to $+\infty$. T_s^* exists, and $T_s^* > t_s^*$.

C.1.5 Proof of Proposition 3.4.1

Here we prove the second point. Since $T > T_s^*$, $T > t_s^*$.

First, since $U_s(-t_s^*) < 1$,

$$\begin{aligned}
U_s(-T) &= \left(1 - e^{-\lambda p_0 (T - t_s^*)}\right) u_G + e^{-\lambda p_0 (T-t_s^*)} U_s(-t_s^*) \\
&< \left(1 - e^{-\lambda p_0 (T - t_s^*)}\right) u_G + e^{-\lambda p_0 (T-t_s^*)} \\
&< \left(1 - e^{-\lambda p_0 T}\right) u_G + e^{-\lambda p_0 T}.
\end{aligned}$$

Second, $U(-T)$ is strictly increasing in q. According to Proposition 3.3.3, t^* is strictly decreasing in q. If $t^* \geq T$,

$$U(-T) = \left(1 - e^{-\lambda p_0 \frac{u_G - u_B}{1-u_B} T}\right) \frac{1-u_B}{u_G - u_B} q \cdot (u_G - 1) + 1,$$

which is strictly increasing in q.

Suppose $t^* < T$. Consider $\tilde{q}$ marginally smaller than q such that its corresponding $\tilde{t}^*$ is smaller than T as well. To reflect different q, we write the investor's continuation value as $U(-t;q)$. Since $0 < t^* < \tilde{t}^*$,

$$\begin{aligned} U(-T;q) &= \left(1 - e^{-\lambda p_0 (T-t^*)}\right) u_G + e^{-\lambda p_0 (T-t^*)} U(-t^*;q) \\ &= \left(1 - e^{-\lambda p_0 (T-\tilde{t}^*)}\right) u_G + e^{-\lambda p_0 (T-\tilde{t}^*)} U(-\tilde{t}^*;q) \end{aligned}$$

We just need to show $U(-\tilde{t}^*;q) > U(-\tilde{t}^*;\tilde{q})$. Denote $e^{\lambda p_0 (\tilde{t}^* - t^*)}$ as a and $\frac{u_G - u_B}{1-u_B}$ as x.

$$\begin{aligned} U(-\tilde{t}^*;\tilde{q}) - 1 &= \left(1 - e^{-\lambda p_0 \frac{u_G-u_B}{1-u_B}\tilde{t}^*}\right) \frac{1}{\frac{u_G-u_B}{1-u_B}} \tilde{q}\cdot(u_G - 1) \\ &= \left(1 - e^{-\lambda p_0 \frac{u_G-u_B}{1-u_B}(\tilde{t}^*-t^*)}\right) \frac{1}{\frac{u_G-u_B}{1-u_B}} \tilde{q}\cdot(u_G - 1) + e^{-\lambda p_0 \frac{u_G-u_B}{1-u_B}(\tilde{t}^*-t^*)} \left(U(-t^*;\tilde{q}) - 1\right) \\ &= \left(1 - a^{-x}\right) \frac{\tilde{q}\cdot(u_G-1)}{x} + a^{-x}\left(U(-t^*;\tilde{q}) - 1\right). \end{aligned}$$

Since $a > 1$ and $x > 1$, $a^{-x} < a^{-1}$ and

$$\frac{1 - a^{-x}}{x} < 1 - a^{-1}.$$

So,

$$\begin{aligned} U(-\tilde{t}^*;\tilde{q}) - 1 &< \left(1 - a^{-1}\right)\tilde{q}\cdot(u_G - 1) + a^{-1}\left(U(-t^*;\tilde{q}) - 1\right) \\ &< \left(1 - a^{-1}\right)(u_G - 1) + a^{-1}\left(U(-t^*;q) - 1\right) \\ &= U(-\tilde{t}^*;q) - 1. \end{aligned}$$

As $q \to 0$, $t^* \to +\infty$, so

$$U(-T) \to \lim_{q\to 0}\left[\left(1 - e^{-\lambda p_0 \frac{u_G-u_B}{1-u_B}T}\right)\frac{1-u_B}{u_G-u_B} q\cdot(u_G - 1) + 1\right] = 1.$$

As $q \to 1$, $t^* \to 0$, so

$$U(-T;q) \to \lim_{q\to 1}\left[\left(1 - e^{\lambda p_0 (t^*-T)}\right) u_G + e^{\lambda p_0 (t^*-T)} U(-t^*;q)\right] = \left(1 - e^{-\lambda p_0 T}\right) u_G + e^{-\lambda p_0 T}.$$

We obtain the second point.

C.1.6 Proof of Proposition 3.4.2

In the second stage where $-t > -t_c^*$,

- $V_c(-t)$ in this stage satisfies

$$\frac{dV_c(-t)}{dt} = \lambda \left[p_0 \cdot v_G + (1-p_0) \cdot (1-\hat{q})v_B - V_c(-t)\right]$$

  and two boundary conditions $V_c(0) = 0$ and $V_c(-t_c^*) = (1-\hat{q})v_B$;

- $U_c(-t)$ satisfies

$$\frac{dU_c(-t)}{dt} = \lambda \cdot \left[p_0 u_G + (1-p_0)(1-\hat{q})u_B + (1-p_0)\hat{q} - U_c(-t)\right]$$

  and one boundary condition $U_c(0) = 1$.

In the first stage where $-t < -t_c^*$,

- $V_c(-t)$ in this stage satisfies

$$\frac{dV_c(-t)}{dt} = \lambda p_0 \cdot \left[v_G - V_c(-t)\right]$$

  and one boundary condition $V_c(-t_c^*) = (1-\hat{q})v_B$.

- $U_c(-t)$ satisfies

$$\frac{dU_c(-t)}{dt} = \lambda p_0 \cdot \left[u_G - U_c(-t)\right]$$

  and one boundary condition that requires $U_c(-t)$ is continuous at $-t = -t_c^*$.

First, $t_c^* < t_s^*$. They satisfy respectively

$$\left(1 - e^{-\lambda t_s^*}\right)\left[p_0 v_G + (1-p_0)v_B\right] = v_B$$
$$\left(1 - e^{-\lambda t_c^*}\right)\left[p_0 v_G + (1-p_0)(1-\hat{q})v_B\right] = (1-\hat{q})v_B.$$

So,

$$1 - e^{-\lambda t_s^*} > 1 - e^{-\lambda t_c^*} \Leftrightarrow t_s^* > t_c^*.$$

For $t \in (0, t_c^*]$,

$$\begin{aligned} U_c(-t) &= \left(1 - e^{-\lambda t}\right)\left[p_0 u_G + (1-p_0)(1-\hat{q})u_B + (1-p_0)\hat{q}\right] + e^{-\lambda t} \\ &> \left(1 - e^{-\lambda t}\right)\left[p_0 u_G + (1-p_0)u_B\right] + e^{-\lambda t} \\ &= U_s(-t). \end{aligned}$$

For $t \in (t_c^*, t_s^*]$,

$$\begin{aligned} U_c(-t) &= \left(1 - e^{-\lambda p_0 (t - t_c^*)}\right) u_G + e^{-\lambda p_0 (t-t_c^*)} U_c(-t_c^*) \\ &> \left(1 - e^{-\lambda p_0 (t - t_c^*)}\right)\left[p_0 u_G + (1-p_0)u_B\right] + e^{-\lambda p_0 (t-t_c^*)} U_s(-t_c^*) \\ &= U_s(-t). \end{aligned}$$

For $t \in (t_s^*, +\infty)$,

$$\begin{aligned} U_c(-t) &= \left(1 - e^{-\lambda p_0 (t - t_s^*)}\right) u_G + e^{-\lambda p_0 (t-t_s^*)} U_c(-t_s^*) \\ &> \left(1 - e^{-\lambda p_0 (t - t_s^*)}\right) u_G + e^{-\lambda p_0 (t-t_s^*)} U_s(-t_s^*) \\ &= U_s(-t). \end{aligned}$$

Therefore, $U_c(-T) > U_s(-T)$.

C.1.7 Proof of Proposition 3.4.3

In the second stage where $-t > -t_v^*$,

- $V_v(-t)$ satisfies
$$\frac{dV_v(-t)}{dt} = \lambda p_0 \cdot q\left[v_G - V_v(-t)\right]$$
and two boundary conditions $V_v(0) = 0$ and $V_v(-t_v^*) = v_B$.;
- $U_v(-t)$ satisfies
$$\frac{dU_v(-t)}{dt} = \lambda p_0 \cdot q\left[u_G - U_v(-t)\right]$$
and one boundary condition $U_v(0) = 1$.

In the first stage where $-t < -t_v^*$,

- $V_v(-t)$ satisfies
$$\frac{dV_v(-t)}{dt} = \lambda p_0 \cdot [v_G - V_v(-t)]$$
and one boundary condition $V_v(-t_v^*) = v_B$;
- $U_v(t)$ satisfies
$$\frac{dU_v(-t)}{dt} = \lambda p_0 \, [u_G - U_v(-t)]$$
and one boundary condition that requires $U_v(-t)$ is continuous at $-t_v^*$.

First, for $-t > \max\{-t^*, -t_v^*\}$, $V_v(-t) < V(-t)$.

$$V(-t) = \left[e^{\lambda p_0 \left(\frac{v_G}{v_B} - 1\right) t} - 1\right] \frac{1}{\frac{v_G}{v_B} - 1} \cdot q \cdot v_G$$
$$V_v(-t) = \left(1 - e^{-\lambda p_0 q t}\right) v_G.$$

Note that for $a > 1$, $\frac{a^x - 1}{x}$ is increasing in x and $\frac{1 - a^{-x}}{x}$ is decreasing in x. Since $\frac{v_G}{v_B} - 1 > 0$,

$$\begin{aligned} V(-t) &> \lim_{x \downarrow 0} \left[e^{\lambda p_0 t \cdot x} - 1\right] \frac{1}{x} \cdot q \cdot v_G \\ &= \lambda p_0 t \cdot q \cdot v_G. \end{aligned}$$

On the other hand,

$$\begin{aligned} V_v(-t) &= \frac{1 - e^{-\lambda p_0 q t}}{\lambda p_0 q t} \lambda p_0 q t v_G \\ &< \lim_{x \downarrow 0} \frac{1 - e^{-x}}{x} \lambda p_0 q t v_G \\ &= \lambda p_0 q t v_G. \end{aligned}$$

So, $V_v(-t) < V(-t)$.

Second, $t^* < t_v^*$. They satisfy respectively

$$V(-t^*) = (1 - q) v_B$$
$$V_v(-t_v^*) = v_B.$$

$V(-t^*) < V_v(-t_v^*)$ implies $t^* < t_v^*$.

For $t \in (0, t^*]$, obviously $V(-t) > V_v(-t)$.

For $t \in (t^*, t_v^*]$, since $v_G > V(-t^*) > V_v(-t^*)$,

$$\begin{aligned} V(-t) &= \left(1 - e^{-\lambda p_0 (t-t^*)}\right) v_G + e^{-\lambda p_0 (t-t^*)} V(-t^*) \\ &> \left(1 - e^{-\lambda p_0 q (t-t^*)}\right) v_G + e^{-\lambda p_0 q (t-t^*)} V(-t^*) \\ &> \left(1 - e^{-\lambda p_0 q (t-t^*)}\right) v_G + e^{-\lambda p_0 q (t-t^*)} V_v(-t_c^*) \\ &= V_v(-t). \end{aligned}$$

For $t \in (t_v^*, +\infty)$,

$$\begin{aligned} V(-t) &= \left(1 - e^{-\lambda p_0 (t-t_v^*)}\right) v_G + e^{-\lambda p_0 (t-t_v^*)} V(-t_v^*) \\ &> \left(1 - e^{-\lambda p_0 (t-t_v^*)}\right) v_G + e^{-\lambda p_0 (t-t_v^*)} V_v(-t_v^*) \\ &= V_v(-t). \end{aligned}$$

Therefore, $V(-T) > V_v(-T)$.

C.1.8 Proof of Lemma 3.5.1

Consider any $-t$ and suppose the type of the best project the sponsor has received until that is σ. The sponsor's proposal strategy $(\alpha_\omega(\cdot))_{\omega \in \{G,B\}}$[1] implies a pair of functions $(f_\omega(\cdot))_{\omega \in \{G,B\}}$: $f_\omega(-\tau)$ represents the unconditional probability density that the sponsor proposes a project of the type ω at $-\tau$. Accordingly, the sponsor's expected payoff by adopting this strategy is

$$\tilde{V}(-t, f_G, f_B) \equiv \int_0^t F_G(-\tau) f_G(-\tau) d\tau \cdot v_G + \int_0^t F_B(-\tau) f_B(-\tau) d\tau \cdot v_B.$$

Specifically, denote the densities resulting from the sponsor's optimal proposal strategy as f_G^σ and f_B^σ respectively. Then

$$V^\sigma(-t) = \tilde{V}\left(-t, f_G^\sigma, f_B^\sigma\right).$$

[1]Note that in the new setup, the sponsor's strategy is also based on the type of the best project he has received until then besides the time $-t$.

Next, suppose $F_G(-t) \leq V^G(-t)$. Then

$$[q+(1-q)\eta(-t)]\cdot v_G \leq \int_0^t [q+(1-q)\eta(-\tau)]\, f_G^G(-\tau)d\tau\cdot v_G + \int_0^t (1-q)\eta(-\tau)\, f_B^G(-\tau)d\tau\cdot v_B. \quad \text{(C.1)}$$

Since $v_G > v_B$ and

$$q+(1-q)\eta(-\tau) \geq \eta(-\tau) \geq (1-q)\eta(-\tau),$$

it further implies

$$q+(1-q)\eta(-t) \leq \int_0^t [q+(1-q)\eta(-\tau)]\left[f_G^G(-\tau)+f_B^G(-\tau)\right]d\tau.$$

Since it is always possible that the sponsor may not have any project ready for proposal in the future,

$$\int_0^t \left[f_G^G(-\tau)+f_B^G(-\tau)\right]d\tau < 1.$$

Hence,

$$\eta(-t) < \int_0^t \eta(-\tau)\left[f_G^G(-\tau)+f_B^G(-\tau)\right]d\tau.$$

We claim that $F_B(-t) < V^B(-t)$ must hold. Consider the sponsor with $\sigma = B$ at $-t$. Imagine that he mistakenly regards one of his old projects as good, always revisits it, and plays the optimal proposal strategy of the sponsor with $\sigma = G$ at $-t$. Let f_G and f_B represent the the true unconditional probability densities implied by this strategy. The sponsor thinks he will end up with the unconditional probability densities f_G^σ and f_B^σ, but some "good" projects he proposes are actually bad. Therefore, for any $-\tau \in (-t, 0]$,

$$f_G(-\tau)+f_B(-\tau) = f_G^G(-\tau)+f_B^G(-\tau),$$
$$f_G(-\tau) \leq f_G^G(-\tau).$$

Note that the sponsor's optimal strategy should be no worse than this mimicking strategy.

So,

$$
\begin{aligned}
V^B(-t) &\geq \int_0^t \left[q + (1-q)\eta\left(-\tau\right)\right] f_G(-\tau) d\tau \cdot v_G + \int_0^t (1-q)\eta\left(-\tau\right) f_B(-\tau) d\tau \cdot v_B \\
&\geq \int_0^t (1-q)\eta\left(-\tau\right) f_G(-\tau) d\tau \cdot v_B + \int_0^t (1-q)\eta\left(-\tau\right) f_B(-\tau) d\tau \cdot v_B \\
&= (1-q) \cdot v_B \cdot \int_0^t \eta\left(-\tau\right) \left[f_G^G(-\tau) + f_B^G(-\tau)\right] d\tau \\
&> (1-q)\eta(-t) \cdot v_B = F_B\left(-t\right).
\end{aligned}
$$

Following the proof of Lemma 3.3.2, we will encounter contradiction. So, $F_G\left(-t\right) > V^G(-t)$. and $\alpha_G(-t) = 1$.

C.1.9 Proof of Proposition 3.5.3

First, $V^B(-t)$ strictly decreases to 0 as $-t$ increases to 0 because

$$
\frac{dV^B(-t)}{dt} \geq \lambda p_0 \cdot \left[F_G\left(-t\right) - V^B(-t)\right] > 0.
$$

Second, following the logic similar to Lemma 3.3.3, we obtain that when $V^B(-t) < (1-q)\, v_B$ and $t > 0$, $\alpha_B(-t) \in (0,1)$; when $V^B(-t) > (1-q)\, v_B$, $\alpha_B(-t) = 0$. Combining the two, we obtain a unique equilibrium of the game.

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