

QUANTUM ELECTRODYNAMICS IN A DAMPED CAVITY

by

Subir Sachdev

Submitted to the Department of
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of the requirements for the
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ABSTRACT

We have made a theoretical study of the interaction of a two-level atom with a single mode of an electromagnetic cavity. The cavity is also coupled to a reservoir of atoms in thermal equilibrium through its walls. The damping and fluctuations this coupling produces in the atom-cavity system is examined through the Fokker-Planck and Langevin equations. Particular attention is paid to the exchange of energy between the atom and cavity and to the approach of the system to thermal equilibrium.

Thesis Supervisor: Daniel Kleppner

Professor of Physics

To my parents

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I. INTRODUCTION

It has recently become possible to study the interactions of atoms with isolated modes of the electromagnetic field in a cavity (Kleppner 1981). A Rydberg atom in a high m -state is an example of an atom through which it might be possible to study this interaction. Consider a sodium atom in the $n = 25$, $l = 24$, $m = 24$ state. In a single photon transition the selection rules imply that the atom can only decay to the $n = 24$, $l = 23$, $m = 23$, state. If such an atom is placed in a cavity whose fundamental frequency is close to the frequency of this transition, the atom will be coupled strongly to that mode of the cavity. It is the purpose of this thesis to study the interaction of such an atom with a single cavity mode. We will examine the exchange of energy between the atom and the cavity.

For reasons that will be explored further in chapter VIII, the resistive damping in the walls of the cavity is of crucial importance in determining the characteristics of the atom-cavity system. In other words, the cavity-atom system is influenced by its coupling through the atoms in the wall to a thermal reservoir. Eventually the energy in the excited atom is lost to the thermal reservoir and the whole system comes

into thermal equilibrium. This resistive damping is present even at zero temperature.

At first sight it may appear that we can represent this loss of energy by the atom-cavity system by introducing a phenomenological damping constant into the equations as is done in classical mechanics. However this introduces an inconsistency in the quantum mechanics of the system. The Heisenberg commutation relations are not maintained. For if,

$$p(t) = p(0)e^{-\gamma t} \quad (\text{I.A.1})$$

and

$$q(t) = q(0)e^{-\gamma t} \quad (\text{I.A.2})$$

Then it follows immediately that

$$[p(t), q(t)] = [p(0), q(0)]e^{-2\gamma t} \quad (\text{I.A.3})$$

$$\rightarrow 0 \text{ as } t \rightarrow \infty$$

which is a fundamental inconsistency. The reason for this is that we have ignored the fluctuations that always accompany the damping that is introduced by the reservoir. If a system is coupled to a thermal reservoir there are two major effects - (1) the system loses energy to the reservoir because the reservoir has a much larger number of degrees of freedom; and (2) thermal fluctuations that are introduced by the reser-

voir into the system. These effects are closely related to each other. Indeed, neither can occur without the other and the quantitative representation of this relationship is the Fluctuation - Dissipation Theorem (Callen and Welton 1951) We will attempt to study the quantum theory of such fluctuations and the effect they have upon the system in its approach to thermal equilibrium. We shall study the general theory of a quantum reservoir through the density matrix formalism.

To understand the intimate relation between fluctuation and dissipation, in chapter II we review the theory of Brownian motion. We introduce the two standard methods of treating fluctuation phenomena - the Langevin and Fokker-Planck equations and demonstrate the intimate relation between the two approaches. The importance of the correlation function of the fluctuating force and its relation to the mean thermal energy is also pointed out.

In chapter III, the density matrix is introduced and its usefulness in treating two coupled systems, only one of which is direct interest, is demonstrated. The equation of motion of the density matrix, after the fluctuations of the thermal reservoir have been averaged over, is derived. The correlation function of the operators of the reservoir is seen to determine the mean motion of the density matrix.

Chapter IV is concerned with the application of the techniques developed in chapter III to a system of fundamental importance - the harmonic oscillator. The density matrix equation is evaluated for the

case of the harmonic oscillator. For the purpose of solving this equation the coherent state representation of the harmonic oscillator is introduced. The associated anti-normal function of the density operator is used to derive the Fokker-Planck equation for the harmonic oscillator, which is then solved.

In chapter V, the ordering and associated complex function techniques, which were used for the harmonic oscillator, are extended to be applicable to an arbitrary system. These techniques are used to convert the density matrix equation into the Fokker-Planck equation for the system. The Langevin equations are then derived from the Fokker-Planck equation and the relation between the two approaches is noted.

In chapter VI we make a diversion to study the interaction between an atom and a cavity in the absence of any damping. Useful operator techniques are introduced for studying this interaction. The equations for the interaction between a 2-level system and a cavity mode are then solved exactly, in spite of the fact that they are non-linear.

Chapter VII is concerned with a system which is related to the atom-cavity system : we study two coupled harmonic oscillators, with one of them coupled to a thermal reservoir. The equations of motion for such a system can be solved exactly. The interaction of the system with the reservoir is characterized by the time taken for an energy interchange between the two oscillators. The behaviour of this time as a function of the damping constant and the detuning is examined. Chapter VIII returns

to the system under consideration - the atom-damped cavity system. The equations of this system cannot be solved exactly, but its features are examined by analogy to the system of chapter VII.

II. BROWNIAN MOTION

As an example of a system interacting with a thermal reservoir, we recapitulate here the theory of Brownian motion. Consider a particle immersed in a liquid, interacting with the molecules of the liquid. Typically the particle will undergo about 10^{13} collisions every second. To obtain a complete description of the motion of the particle is obviously a hopeless task. We will attempt to obtain some of the statistical properties of the motion of the particle in this chapter. The techniques we will use for this purpose will later on be extended to be applicable to quantum systems. In section II.A we present the most intuitive way of looking at the motion of the particle - the Langevin equation. The Langevin force $F(t)$ is introduced and its correlation function is evaluated. The correlation function is shown to be closely related to the mean kinetic energy of the particle. In section II.B we adopt a different approach. The probability distribution function of the particle in velocity space is introduced, and a differential equation for this probability distribution function is derived. This is the Fokker-Planck equation. This equation can be solved exactly. In section II.C the rela-

tion between the two approaches is stated explicitly.

II.A The Langevin equation

The equation of motion of the particle can be written as

$$m \frac{dv}{dt} = F(t) \quad (\text{II.A.1})$$

where $F(t)$ is a rapidly fluctuating force of the following form:

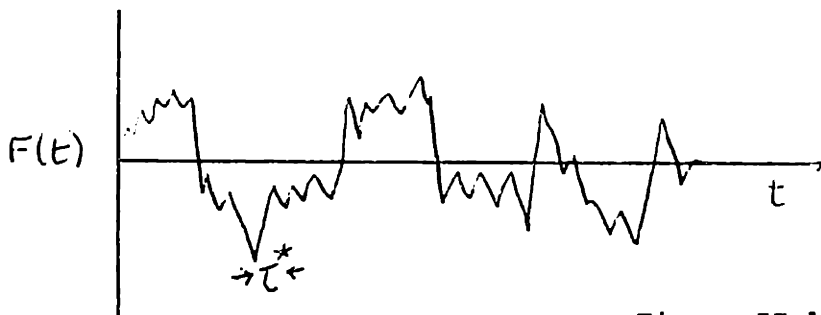


Figure II.1

τ^* , which is of the order of the time between two successive peaks of $F(t)$, is called the coherence time. In a typical situation τ^* is on the order of 10^{-13} seconds. Since the time scale of the phenomena we are interested in is much greater than 10^{-13} seconds, we will obtain all our results in the limit of zero coherence time. We may therefore integrate equation (II.A.1) and obtain

$$m \frac{v(t + \Delta t) - v(t)}{\Delta t} = \frac{1}{\Delta t} \int_t^{t + \Delta t} F(t') dt' \quad (\text{II.A.2})$$

where $\Delta t > \tau^*$.

The ensemble average over this equation is

$$m \left\langle \frac{\Delta v}{\Delta t} \right\rangle = \frac{1}{\Delta t} \int_t^{t + \Delta t} \langle F(t') \rangle dt' \quad (\text{II.A.3})$$

The ensemble in this case is a collection of a large number of hypothetical systems of a particle immersed in a liquid. The simplest approximation for $\langle F(t') \rangle$ from Fig.(I.1) would be $\langle F(t') \rangle = 0$. This however does not lead to any interesting results. We know from phenomenological observations that the particle feels a net resistive force equal to $-\gamma v(t)$. γ for a hard sphere is given by

$$\gamma = 6\pi\eta a \quad (\text{Stoke's Law}) \quad (\text{II.A.4})$$

where η is the coefficient of viscosity and a is the radius of the sphere. This indicates that there is a portion of $F(t)$ which depends upon $v(t)$. The resistive force then arises from the modification of $F(t)$ due to the reaction of the particle upon the molecules of the liquid.

We now make these considerations more quantitative. Let the probability that the particle has a velocity $v(t)$ at a time t be $W(t)$. Since there are a far greater number of degrees of freedom in the liquid than for the particle, this probability will be proportional to the number of accessible states in the liquid.

$$W(t) \propto \Omega(E) \quad (\text{II.A.5})$$

where E is the energy of the liquid.

Let the instantaneous force on the particle at a specified time t

be $F(t)$. Assume that the system is in an equilibrium situation at time t , with $\langle F(t) \rangle = 0$. After a time Δt the velocity will be $v(t + \Delta t)$ and the energy of the liquid will be $E + \Delta E$. (Again $\Delta t > \tau^*$)

$$W(t + \Delta t) \approx \Omega(E + \Delta E)$$

So

$$\frac{W(t + \Delta t)}{W(t)} = \frac{\Omega(E + \Delta E)}{\Omega(E)} = e^{\beta \Delta E} \quad (\text{II.A.6})$$

where $\beta = \frac{1}{kT} = \frac{\partial \ln \Omega}{\partial E}$ So

$$W(t + \Delta t) = W(t)e^{\beta \Delta E} \approx W(t)(1 + \beta \Delta E)$$

Therefore

$$\begin{aligned} \langle F \rangle &= \sum W(t + \Delta t) F \\ &= \sum W(t)(1 + \beta \Delta E) F = \beta \langle F \Delta E \rangle \end{aligned} \quad (\text{II.A.7})$$

where the \sum implies a summation over all the possible states of the liquid. Also $\Delta t = t' - t > \tau^*$

Clearly

$$\Delta E = - \int_t^{t'} v(t'') F(t'') dt'' \approx - v(t) \int_t^{t'} F(t'') dt'' \quad (\text{II.A.8})$$

where $v(t)$ is taken out of the integral sign because it is assumed to be slowly varying over a time less than τ^* . So from equation (II.A.7) we

obtain

$$\begin{aligned}
 \langle F(t') \rangle &= -\beta \langle F(t') v(t) \int_t^{t'} F(t'') dt'' \rangle \\
 &= -\beta \langle v(t) \rangle \int_t^{t'} dt'' \langle F(t') F(t'') \rangle
 \end{aligned}
 \tag{II.A.9}$$

We define the correlation function of $F(t)$, $K(s)$, by

$$K(s) \equiv \langle F(t) F(t + s) \rangle \tag{II.A.10}$$

K is assumed to be independent of t because the behaviour of the system cannot depend upon the origin of time. So equation (II.A.9) gives us

$$\langle F(t') \rangle = -\beta \langle v(t) \rangle \int_t^{t'} dt'' K(t'' - t') \tag{II.A.11}$$

Before we proceed we need to derive some properties of $K(s)$. Clearly

$$\begin{aligned}
 K(s) &= \langle F(t) F(t + s) \rangle = \langle F(t + s) F(t) \rangle \\
 &= \langle F(t) F(t - s) \rangle = K(-s)
 \end{aligned}
 \tag{II.A.12}$$

$$\rightarrow K(s) = K(-s)$$

Also since $F(t + \tau)$ bears no particular relation to $F(t)$ if $\tau > \tau^*$

(the coherence time), we must have

$$K(s) = 0 \text{ for } s > \tau^*$$

(II.A.13)

Also

$$\langle (F(t) \pm F(t + s))^2 \rangle \geq 0$$

$$\rightarrow \langle F^2(t) \rangle + \langle F^2(t + s) \rangle \pm 2\langle F(t)F(t + s) \rangle \geq 0$$

$$\rightarrow 2K(0) \pm 2K(s) \geq 0$$

(II.A.14)

$$\rightarrow |K(s)| \leq K(0)$$

(II.A.12), (II.A.13), (II.A.14) together imply that $|K(s)|$ is of the form

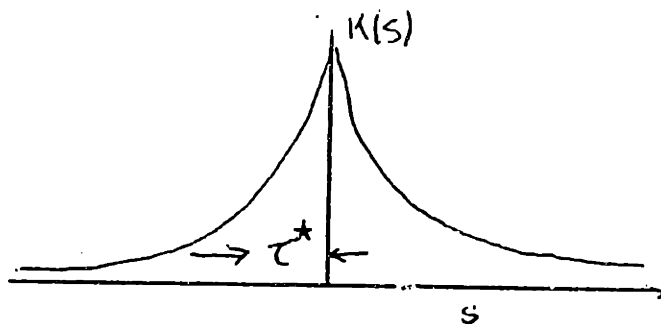


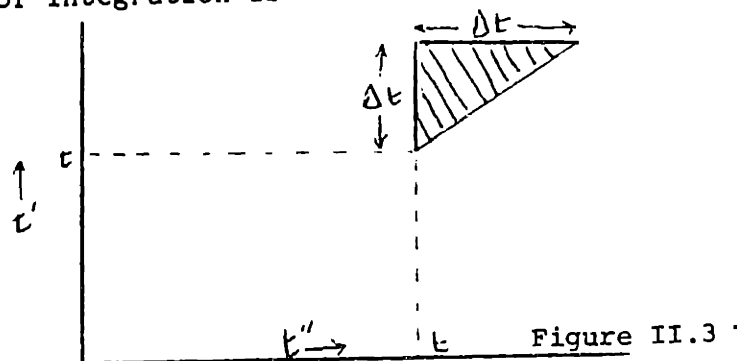
Figure II.2

In the limit of zero coherence time, the correlation function is a delta function.

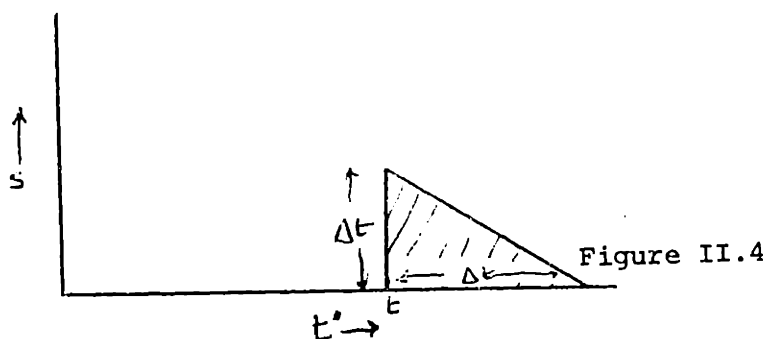
We now go back to equation (II.A.11). Substituting (II.A.11) into (II.A.3) we have

$$m \left\langle \frac{\Delta v}{\Delta t} \right\rangle = - \frac{1}{\Delta t} \beta \langle v(t) \rangle \int_t^{t+\Delta t} dt' \int_t^{t'} dt'' K(t'' - t') \quad (\text{II.A.15})$$

The region of integration is



If we change to the variables t'' , $t' - t'' = s$, it becomes



Therefore equation (II.A.15) now becomes

$$\begin{aligned} m \left\langle \frac{\Delta v}{\Delta t} \right\rangle &= - \frac{\beta}{\Delta t} \langle v(t) \rangle \int_0^{\Delta t} ds \int_t^{t+\Delta t-s} dt'' K(s) \\ &= - \frac{\beta}{\Delta t} \langle v(t) \rangle \int_0^{\Delta t} (\Delta t - s) K(s) ds \end{aligned} \quad (\text{II.A.16})$$

$$m \left\langle \frac{\Delta v}{\Delta t} \right\rangle = - \frac{\beta}{2} \langle v(t) \rangle \int_{-\Delta t}^{\Delta t} K(s) ds \quad (\text{II.A.17})$$

The last step follows from equation (II.A.12) and the fact that $s \ll \Delta t$ over the range $K(s)$ is appreciable.

Since $K(s) = 0$ for $t > \Delta t$, (II.A.17) is equivalent to

$$m \langle \frac{\Delta v}{\Delta t} \rangle = - \langle v(t) \rangle \frac{1}{2kT} \int_{-\infty}^{\infty} K(s) ds \quad (\text{II.A.18})$$

This gives us

$$\gamma = \frac{1}{2kT} \int_{-\infty}^{\infty} K(s) ds \quad (\text{II.A.19})$$

This is a special case of the fluctuation-dissipation theorem (Callen and Welton 1951).

We can rewrite (II.A.18) as

$$m \frac{dv}{dt} = - \gamma v + f(t)$$

$$\text{where } \langle f(t) \rangle = 0$$

(II.A.20)

In the limit of zero coherence time we have from (II.A.19)

$$K(s) = 2kT\gamma\delta(s)$$

$$\rightarrow \langle f(t)f(t') \rangle = 2kT\gamma\delta(t - t')$$

Therefore we have

$$m \frac{dv}{dt} = -\gamma v + f(t)$$

$$\langle f(t) \rangle = 0$$

(II.A.21)

$$\langle f(t)f(t') \rangle = 2kT\gamma\delta(t - t')$$

The relations (II.A.21) are called the Langevin equations of motion and represent all the information about the system that we shall need.

As a check, let us evaluate $\langle \frac{1}{2}mv^2(t) \rangle$. We have from (II.A.20)

$$v(t) = v(0)\exp(-\gamma t) + \frac{1}{m} \int_0^t dt_1 \exp\{-\gamma(t - t_1)\} f(t_1) \quad (\text{II.A.22})$$

Therefore

$$\begin{aligned} \langle v^2(t) \rangle = & \left\langle \left[v(0)\exp(-\gamma t) + \frac{1}{m} \int_0^t dt_1 \exp\{-\gamma(t - t_1)\} f(t_1) \right]^2 \right. \\ & \left. \left[v(0)\exp(-\gamma t) + \frac{1}{m} \int_0^t dt_2 \exp\{-\gamma(t - t_2)\} f(t_2) \right] \right\rangle \quad (\text{II.A.23}) \end{aligned}$$

The expectation value of the cross-term vanishes, and we get

$$\begin{aligned} \langle v^2(t) \rangle = & \langle v^2(0) \rangle \exp(-2\gamma t) \\ & + \frac{1}{m^2} \int_0^t dt_1 \int_0^t dt_2 \exp\{-\gamma(2t - t_1 - t_2)\} \langle f(t_1)f(t_2) \rangle \quad (\text{II.A.24}) \end{aligned}$$

Using (II.A.21) this reduces to

$$\langle v^2(t) \rangle = \langle v^2(0) \rangle \exp(-2\gamma t) + \frac{kT}{m} (1 - \exp(-2\gamma t)) \quad (\text{II.A.25})$$

Therefore the steady state value of

$$\langle \frac{1}{2} m v^2 \rangle = \frac{1}{2} kT$$

which is what would have been expected from thermodynamical considerations.

II.B The Fokker-Planck Equation

The Fokker-Planck equation offers an alternative, but equivalent, way of looking at the problem of Brownian motion. We attempt to develop a differential equation for $P(v,t)$, the probability that the particle has a velocity v at a time t . We find it useful to define $\Psi(v, \Delta v)$, the transition probability that a particle will undergo a change in velocity Δv from a velocity v . Then it is easy to see that the following relation must hold between P and Ψ

$$P(v, t + \Delta t) = \int_{-\infty}^{\infty} P(v - \Delta v, t) \Psi(v - \Delta v, \Delta v) d(\Delta v) \quad (\text{II.B.1})$$

Here $\Delta t > \tau^*$, the coherence time and the change in velocity, Δv , occurs over a time Δt . Since Δv and Δt are small, we can expand both sides in a Taylor series.

$$P(v, t) + \frac{\partial P}{\partial t} \Delta t = \int_{-\infty}^{\infty} \left[P(v, t) - \frac{\partial P}{\partial v}(\Delta v) + \frac{1}{2} \frac{\partial^2 P}{\partial v^2}(\Delta v)^2 + O(\Delta v)^3 \right]^* \left[\Psi(v, \Delta v) - \frac{\partial \Psi}{\partial v}(\Delta v) + \frac{1}{2} \frac{\partial^2 \Psi}{\partial v^2}(\Delta v)^2 + O(\Delta v)^3 \right] d(\Delta v) \quad (\text{II.B.2})$$

which simplifies to

$$\begin{aligned}
P(v,t) + \frac{\partial P}{\partial t} \Delta t &= \int_{-\infty}^{\infty} P(v,t) \Psi(v, \Delta v) d(\Delta v) \\
&\quad - \frac{\partial}{\partial v} \left[\int_{-\infty}^{\infty} P(v,t) \Psi(v, \Delta v) (\Delta v) d(\Delta v) \right] \\
&\quad + \frac{1}{2} \frac{\partial^2}{\partial v^2} \left[\int_{-\infty}^{\infty} P(v,t) \Psi(v, \Delta v) (\Delta v)^2 d(\Delta v) \right] \\
&\quad + O(\Delta v)^3
\end{aligned} \tag{II.B.3}$$

It is clear from the definition of $\Psi(v, \Delta v)$ that

$$\int \Psi(v, \Delta v) d(\Delta v) = 1$$

$$\int \Psi(v, \Delta v) (\Delta v) d(\Delta v) = \langle (\Delta v) \rangle$$

(II.B.4)

$$\int \Psi(v, \Delta v) (\Delta v)^2 d(\Delta v) = \langle (\Delta v)^2 \rangle$$

and so on. Therefore the equation (II.B.3) reduces to

$$\frac{\partial P}{\partial t} \Delta t = - \frac{\partial}{\partial v} \{ P \langle \Delta v \rangle \} + \frac{1}{2} \frac{\partial^2}{\partial v^2} \{ P \langle (\Delta v)^2 \rangle \} + \dots \tag{II.B.5}$$

This is the Fokker-Planck equation in its most general form. We have from equation (II.A.21)

$$m \frac{dv}{dt} = -\gamma v + f(t)$$

$$\Delta v = -\frac{\gamma}{m} v \Delta t + \frac{1}{m} \int_t^{t+\Delta t} f(t') dt' \quad (\text{II.B.6})$$

because we wish to average out the motion over times less than the coherence time. Equation (II.B.6) gives immediately

$$\langle \Delta v \rangle = -\left\langle \frac{\gamma}{m} v \Delta t \right\rangle \quad (\text{II.B.7})$$

and

$$\begin{aligned} \langle (\Delta v)^2 \rangle &= \left\langle \left[-\frac{\gamma}{m} v \Delta t + \frac{1}{m} \int_t^{t+\Delta t} f(t') dt' \right]^* \right. \\ &\quad \left. \left[-\frac{\gamma}{m} v \Delta t + \frac{1}{m} \int_t^{t+\Delta t} f(t'') dt'' \right] \right\rangle \\ \langle (\Delta v)^2 \rangle &= \frac{1}{m^2} \int_t^{t+\Delta t} dt' \int_t^{t+\Delta t} dt'' \langle f(t') f(t'') \rangle + O(\Delta t)^2 \\ &= \frac{2kT\gamma}{m} \Delta t + O(\Delta t)^2 \end{aligned} \quad (\text{II.B.8})$$

If we make the additional assumption that

$$\langle (\Delta v)^k \rangle = O(\Delta t)^2 \quad \text{for } k > 2 \quad (\text{II.B.9})$$

then equation (II.B.5) simplifies to the following differential equation for P

$$\frac{\partial P}{\partial t} = \frac{\gamma}{m} \frac{\partial}{\partial v}(Pv) + \frac{kT\gamma}{m^2} \frac{\partial^2}{\partial v^2}(P) \quad (\text{II.B.10})$$

We now attempt to find a solution to this equation, subject to the initial condition $v(t=0) = v_0$. Then

$$P(v,0) = \delta(v - v_0)$$

$$= \lim_{\sigma \rightarrow 0} \frac{1}{(2\pi\sigma)^{1/2}} \exp\left\{-\frac{(v - v_0)^2}{2\sigma}\right\} \quad (\text{II.B.11})$$

We make the ansatz

$$P(v,t) = \exp\left\{-\frac{G(v,t)}{2}\right\} \quad (\text{II.B.12})$$

Then (II.B.10) reduces to

$$-\frac{1}{2} \frac{\partial G}{\partial t} = \frac{\gamma}{m} - \frac{\gamma v}{2m} \frac{\partial G}{\partial v} - \frac{kT\gamma}{2m^2} \frac{\partial^2 G}{\partial v^2} + \frac{kT\gamma}{4m^2} \left(\frac{\partial G}{\partial v}\right)^2 \quad (\text{II.B.13})$$

At $t = 0$ we have from (II.B.11)

$$G(v,0) = \lim_{\sigma \rightarrow 0} \left\{ \frac{(v - v_0)^2}{\sigma} + \ln(2\pi\sigma) \right\} \quad (\text{II.B.14})$$

If we assume for $G(v,t)$ the form

$$G(v,t) = \frac{1}{a(t)}(v - b(t))^2 + \ln c(t) \quad (\text{II.B.15})$$

then we must have

$$a(0) = \lim_{\sigma \rightarrow 0} \sigma$$

$$b(0) = v_0$$

(II.B.16)

$$c(0) = \lim_{\sigma \rightarrow 0} 2\pi\sigma$$

Substituting (II.B.15) into (II.B.13) and equating the coefficients of equal powers of v , we easily get

$$\frac{da}{dt} + \frac{2\gamma}{m} a = \frac{2kT\gamma}{m^2}$$

$$\frac{db}{dt} + \frac{\gamma}{m} b = 0$$

(II.B.17)

$$\frac{1}{a} \frac{da}{dt} = \frac{1}{c} \frac{dc}{dt}$$

The equations (II.B.17) can be easily solved with the initial conditions (II.B.16) to give

$$a(t) = \frac{kT}{m} \left[1 - \exp \left\{ -\frac{2\gamma}{m} t \right\} \right]$$

$$b(t) = v_0 \exp \left\{ -\frac{\gamma}{m} t \right\}$$

(II.B.18)

$$c(t) = \frac{2\pi kT}{m} \left[1 - \exp \left\{ -\frac{2\gamma}{m} t \right\} \right]$$

Therefore we have

$$P(v,t) = \frac{1}{(c(t))^{1/2}} \exp\left[-\frac{(v - b(t))^2}{2a(t)}\right] \quad (\text{II.B.19})$$

which is the solution to (II.B.10) with the initial condition (II.B.11)

Let us now evaluate $\langle \frac{1}{2}mv^2 \rangle$

$$\langle \frac{1}{2}mv^2 \rangle = \int \frac{1}{2}mv^2 P(v,t) dv \quad (\text{II.B.20})$$

The above integral can be easily evaluated to give

$$\langle \frac{1}{2}mv^2 \rangle = \frac{1}{2}mv_0^2 \exp\left[-\frac{2\gamma t}{m}\right] + \frac{kT}{2} \left[1 - \exp\left[-\frac{2\gamma t}{m}\right]\right] \quad (\text{II.B.21})$$

which agrees with equation (II.A.25).

II.C Overview of the Two Approaches

In section B we derived the Fokker-Planck equation using the Langevin equation. It is clear that the reasoning can be reversed i.e. given a Fokker-Planck equation of the form

$$\frac{\partial P}{\partial t} = - \frac{\partial}{\partial v} (A(v)P) + \frac{\partial^2}{\partial v^2} (B(v)P) \quad (\text{II.C.1})$$

we can deduce that the Langevin equation obeyed by v is

$$\frac{dv}{dt} = A(v) + f(t)$$

where

$$\langle f(t)f(t') \rangle = 2\langle B(v) \rangle \delta(t - t') \quad (\text{II.C.2})$$

Equations (II.C.1) and (II.C.2) summarize the content of this chapter and are very similar to the quantum equations that we shall derive later.

Bibliography for Chapter II

The study of Brownian motion was initiated by Einstein (1905) and Smolouchowski (1906). Equation (II.A.19) is an alternative formulation of the well-known Einstein relation. The Langevin equation was first derived by Langevin (1908). The differential equation was first derived

by Fokker (1914) and Planck (1914). A complete theory of Brownian motion, with expressions for the expectation value of arbitrary functions of the velocity and position was first done by Uhlenbeck and Ornstein (1930) and Wang and Uhlenbeck (1945). A special case of the Fluctuation-Dissipation Theorem was derived by Nyquist (1928). The general formalism of this theorem was developed by Callen and Welton (1951), Bernard and Callen (1959), Callen and Greene (1952a,b). The theory of fluctuations was further developed by Lax (1960). Review articles may be found in Chandrasekhar (1943) and Rief (1965).

III. QUANTUM TREATMENT OF A THERMAL RESERVOIR

We will now try to describe the quantum theory of a thermal reservoir. We assume that the system under consideration is made up of two parts. One is the part of the system whose properties we are interested in determining. We will refer to this part as the system S. It has only a small number of degrees of freedom. The other part, which we will call the reservoir R, is assumed to be in thermal equilibrium at all times, and possessing a large number of degrees of freedom. It is weakly interacting with the system S. An example of a system and reservoir interacting in this manner is an electromagnetic cavity and a two level atom (the system S) interacting with the atoms in the walls of the cavity (the reservoir R). The behaviour of the reservoir will not be described in detail because it is assumed to be stochastic in nature. It shall be characterized chiefly by its correlation functions. In section III.A we will define the density operator of the system and demonstrate its usefulness in treating a reservoir which is stochastic. We will derive an equation of motion for the reduced density matrix of the system S. Section III.B will be similar to section III.A except that in it we will derive the equation of motion of any system operator.

III.A Equation of Motion for the Density Operator

Assume that the system S has a Hamiltonian H_0 and the reservoir R has a Hamiltonian R , and the interaction Hamiltonian is V . Then the total Hamiltonian H is given by

$$H = H_0 + R + V \quad (\text{III.A.1})$$

We will attempt to eliminate the coordinates of the reservoir. Let the state of S and R be represented by the ket $|\Psi(t)\rangle$. At a time t let the probability that S and R are in this state be given by p_Ψ . Then the density operator of the system, $\rho(t)$ is defined by

$$\rho(t) = \sum_{\Psi} |\Psi(t)\rangle p_{\Psi} \langle \Psi(t)| \quad (\text{III.A.2})$$

Let θ be any arbitrary operator. Then

$$\begin{aligned} \langle \theta(t) \rangle &= \sum_{\Psi} p_{\Psi} \langle \Psi(t) | \theta | \Psi(t) \rangle \\ &= \text{Trace} \{ \rho(t) \theta \} \end{aligned} \quad (\text{III.A.3})$$

where the trace is taken over both S and R. Thus from $\rho(t)$ we can derive the expectation value of any operator.

We have

$$\begin{aligned}
 \rho(t) &= \sum_{\Psi} |\Psi(t)\rangle p_{\Psi} \langle \Psi(t)| \\
 &= \sum_{\Psi} \exp\left\{-\frac{iHt}{\hbar}\right\} |\Psi(0)\rangle p_{\Psi} \langle \Psi(0)| \exp\left\{\frac{iHt}{\hbar}\right\} \quad (\text{III.A.4})
 \end{aligned}$$

from which we can easily derive

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho] \quad (\text{III.A.5})$$

which is the equation of motion of the density operator. We also find it convenient to define $\chi(t)$, the density operator in the interaction representation by

$$\rho(t) = \exp\left\{-\frac{i\bar{H}t}{\hbar}\right\} \chi(t) \exp\left\{\frac{i\bar{H}t}{\hbar}\right\} \quad (\text{III.A.6})$$

$$\text{where } \bar{H} = H_0 + R$$

Using (III.A.5) we can easily derive the equation of motion for $\chi(t)$

$$i\hbar \frac{\partial \chi}{\partial t} = [V(t), \chi]$$

$$\text{where } V(t) = \exp\left\{\frac{i\bar{H}t}{\hbar}\right\} V \exp\left\{-\frac{i\bar{H}t}{\hbar}\right\} \quad (\text{III.A.7})$$

Consider an operator A , which is only a function of the variables of the system S . Then

$\langle A \rangle = \text{Trace over } S \text{ and } R (\rho A)$

$$= \text{Trace}_S \left\{ \left[\text{Trace}_R \rho \right] A \right\}$$

We define a reduced density operator

$$S(t) = \text{Trace}_R \rho(t) \quad (\text{III.A.8})$$

Then

$$\langle A \rangle = \text{Trace}_S \{ S A \} \quad (\text{III.A.9})$$

Therefore we see that the reduced density operator offers a convenient formalism for eliminating the coordinates of the reservoir. We can also define a reduced density operator in the interaction representation by

$$S(t) = \exp \left[-\frac{iH_0 t}{\hbar} \right] \phi(t) \exp \left[\frac{iH_0 t}{\hbar} \right] \quad (\text{III.A.10})$$

It is easily seen that

$$\phi(t) = \text{Trace}_R \chi(t) \quad (\text{III.A.11})$$

We shall develop an equation of motion for $S(t)$ and $\phi(t)$, eliminating the coordinates of the reservoir. As in the case of Brownian motion, we shall only take into account second order correlations in the reservoir forces and ignore all higher order effects.

Since the reservoir is in thermal equilibrium, at $t=0$, the density

matrix $\rho(0)$ can be written as

$$\rho(0) = S(0) f(0) \quad (\text{III.A.12})$$

where

$$f(0) = f(t) = \frac{\exp(-\beta R)}{\text{Trace}(\exp(-\beta R))}$$

$$\text{and } \beta = \frac{1}{kT} \quad (\text{III.A.13})$$

We make the assumption that

$$V = \hbar \sum_i Q_i F_i \quad (\text{III.A.14})$$

where Q_i is an operator on the system S and F_i is an operator on the reservoir R. We also define

$$V(t) = \hbar \sum_i Q_i(t) F_i(t)$$

$$\text{where } Q_i(t) = \exp\left\{\frac{iH_0 t}{\hbar}\right\} Q_i \exp\left\{-\frac{iH_0 t}{\hbar}\right\} \quad (\text{III.A.15})$$

$$\text{and } F_i(t) = \exp\left\{\frac{iRt}{\hbar}\right\} F_i \exp\left\{-\frac{iRt}{\hbar}\right\}$$

Now $F_i(t)$ being a reservoir operator, is stochastic in nature. It is the analog of the force on a particle undergoing Brownian motion by the molecules of the liquid, differing only in that it is an operator upon an Hilbert space and not an ordinary function of time. As in the

previous chapter we define

$$\begin{aligned}
 K_{ij}(\tau) &= \text{Trace}_R \left\{ f(0) F_i F_j(\tau) \right\} \\
 &= \langle F_i F_j(\tau) \rangle
 \end{aligned}
 \tag{III.A.16}$$

As before the correlation function is assumed to be time translation invariant:

$$\langle F_i F_j(\tau) \rangle = \langle F_i(t) F_j(t + \tau) \rangle
 \tag{III.A.17}$$

We have from (III.A.16)

$$\begin{aligned}
 K_{ij}(-\tau) &= \langle F_i F_j(-\tau) \rangle \\
 &= \langle F_i(\tau) F_j \rangle \\
 &\neq K_{ij}(\tau)
 \end{aligned}
 \tag{III.A.18}$$

which is unlike the classical case. Also

$$K_{ij}(\tau) \neq K_{ji}(\tau)$$

because the operators F_i and F_j need not commute. We shall assume that $K_{ij}(\tau)$ is a sharply peaked function of time, with a width of the order of τ^* (the coherence time). The qualitative reasons for assuming this

are essentially the same as in the case of Brownian motion.

We recall the equation of motion of the density operator in the interaction representation.

$$i\hbar \frac{\partial \chi}{\partial t} = [V(t), \chi] \quad (\text{III.A.7})$$

which gives

$$\frac{\partial \chi}{\partial t} = -i \left[\sum_i Q_i(t) F_i(t), \chi(t) \right] \quad (\text{III.A.19})$$

We wish to "smooth out" all effects over times less than τ^*

and also ignore correlations in $F_i(t)$ beyond the second order. As a first approximation we have

$$\frac{\Delta \chi}{\Delta t} = -\frac{i}{\Delta t} \int_0^{\Delta t} dt' \left[\sum_i Q_i(t') F_i(t'), \chi(0) \right] \quad (\text{III.A.20})$$

where $\Delta t > \tau^*$

This approximation is too crude however, because it does not take into account the second order correlation in $F_i(t)$. To the next order we have therefore

$$\begin{aligned}
\frac{\Delta x}{\Delta t} &= -\frac{1}{\Delta t} \int_0^{\Delta t} dt' \left\{ \sum_i Q_i(t') F_i(t'), x(0) \right\} \\
&+ \frac{(-1)^2}{\Delta t} \int_0^{\Delta t} dt' \int_0^{t'} dt'' \\
&\left[\sum_i Q_i(t') F_i(t'), \left[\sum_j Q_j(t'') F_j(t''), x(0) \right] \right]
\end{aligned} \tag{III.A.21}$$

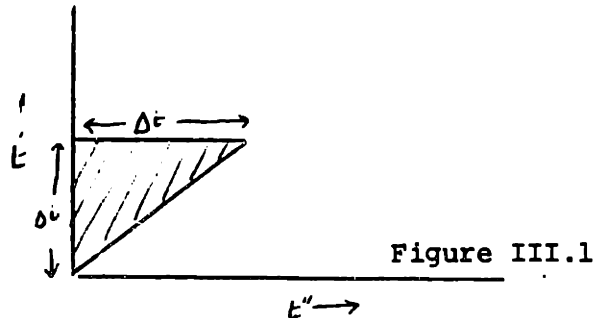
If we now trace over both sides of (III.A.21) over the reservoir R, and recalling equations (III.A.11) and (III.A.16), we have

$$\begin{aligned}
\frac{\Delta \phi}{\Delta t} &= -\frac{1}{\Delta t} \sum_i \int_0^{\Delta t} dt' \left\{ Q_i(t'), \phi(0) \right\} \langle F_i(t') \rangle dt' \\
&+ \frac{(-1)^2}{\Delta t} \sum_{i,j} \int_0^{\Delta t} dt' \int_0^{t'} dt'' \\
&\left[Q_i(t') Q_j(t'') \phi(0) - Q_j(t'') \phi(0) Q_i(t') \right] * K_{ij}(t'' - t') \\
&- \left[Q_i(t') \phi(0) Q_j(t'') - \phi(0) Q_j(t'') Q_i(t') \right] * K_{ji}(t' - t'')
\end{aligned}$$

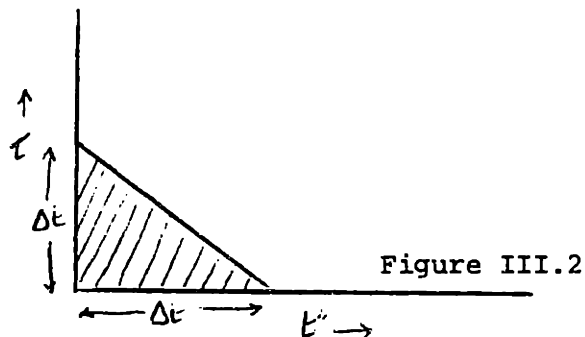
Now $\langle F_i(t') \rangle = 0$. Therefore the equation above simplifies to

$$\begin{aligned}
\frac{\partial \phi}{\partial t} &= \lim_{\Delta t \rightarrow 0} \frac{(-1)^2}{\Delta t} \sum_{i,j} \int_0^{\Delta t} dt' \int_0^{t'} dt'' \\
&\left[Q_i(t') Q_j(t'') \phi - Q_j(t'') \phi Q_i(t') \right] * K_{ij}(t'' - t') \\
&- \left[Q_i(t') \phi Q_j(t'') - \phi Q_j(t'') Q_i(t') \right] * K_{ji}(t' - t'')
\end{aligned}$$

The region of integration is shown in Figure III.1



If we define $\tau = t' - t''$, the region of integration is transformed to Figure III.2



We have therefore

$$\frac{\partial \phi}{\partial t} = \lim_{\Delta t \rightarrow 0} \frac{(-i)^2}{\Delta t} \sum_{i,j} \int_0^{\Delta t} dt'' \int_0^{t''} d\tau$$

$$\left\{ Q_1(\tau + t'') Q_j(t'') \phi - Q_j(t'') \phi Q_1(\tau + t'') \right\} * K_{ji}(-\tau)$$

$$- \left\{ Q_1(\tau + t'') \phi Q_j(t'') - \phi Q_j(t'') Q_1(\tau + t'') \right\} * K_{ji}(\tau)$$

(III.A.23)

For all the systems that we shall consider, the following property will hold for $Q_i(t)$

$$\begin{aligned} Q_i(t) &= \exp\left\{\frac{iH_0 t}{\hbar}\right\} Q_i \exp\left\{-\frac{iH_0 t}{\hbar}\right\} \\ &= \exp\{i\omega_i t\} Q_i \end{aligned} \quad (\text{III.A.24})$$

Using this property, (III.A.23) simplifies to

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \lim_{\Delta t \rightarrow 0} \sum_{i,j} -\frac{1}{\Delta t} \\ &\left\{ Q_i Q_j \phi - Q_j \phi Q_i \right\} * \int_0^{t''} \exp\{i\omega_i \tau\} K_{ij}(-\tau) d\tau \\ &- \left\{ Q_i \phi Q_j - \phi Q_j Q_i \right\} * \int_0^{t''} \exp\{i\omega_i \tau\} K_{ji}(\tau) d\tau \\ &* \left[\int_0^{\Delta t} dt'' \exp\{i(\omega_i + \omega_j)t''\} \right] \end{aligned} \quad (\text{III.A.25})$$

Here $\Delta t > \tau^*$ and $\Delta t > \frac{1}{\omega_i}$.

Since $K_{ij}(\tau)$ is sharply peaked with a width τ^* we make the approximation

$$\int_0^{t''} \exp\{i\omega_i \tau\} K_{ij}(-\tau) d\tau = \int_0^{\infty} \exp\{i\omega_i \tau\} K_{ij}(-\tau) d\tau$$

We define w_{ij}^+ and w_{ji}^- by

$$w_{ij}^+ = \int_0^{\infty} \exp\{i\omega_1\tau\} K_{ij}(-\tau) d\tau$$

$$w_{ji}^- = \int_0^{\infty} \exp\{i\omega_1\tau\} K_{ji}(\tau) d\tau \quad (\text{III.A.26})$$

Since $\Delta t > \frac{1}{\omega_1}$ the integral outside the brackets in (III.A.25) is

$$\int_0^{\Delta t} dt'' \exp\{i(\omega_1 + \omega_j)t''\} = \begin{cases} 0 & \text{if } \omega_1 + \omega_j \neq 0 \\ \Delta t & \text{if } \omega_1 + \omega_j = 0 \end{cases}$$

Therefore (III.A.25) now becomes

$$\frac{\partial \phi}{\partial t} = - \sum_{i,j} \delta_{i,j} \left\{ \left[Q_i Q_j \phi - Q_j \phi Q_i \right] w_{ij}^+ \right. \\ \left. - \left[Q_i \phi Q_j - \phi Q_j Q_i \right] w_{ji}^- \right\} \quad (\text{III.A.27})$$

We have defined $\delta_{i,j}$ by

$$\delta_{i,j} \equiv \begin{cases} 1 & \text{if } \omega_i + \omega_j = 0 \\ 0 & \text{otherwise} \end{cases}$$

(III.A.27) is the required equation of motion for the reduced density operator with the coordinates of the reservoir eliminated. The equation for the density operator $S(t)$ can now be easily seen to be

$$\frac{\partial S}{\partial t} = \frac{1}{i\hbar} [H_0, S] - \\ \sum_{i,j} \delta_{i,j} \left\{ \left[Q_i Q_j S - Q_j S Q_i \right] w_{ij}^+ \right. \\ \left. - \left[Q_i S Q_j - S Q_j Q_i \right] w_{ji}^- \right\}$$

This is the Master Equation for the density matrix of a system in contact with a thermal reservoir. We shall base our study of such systems upon this equation. The characteristics of the thermal reservoir shall be represented simply by the correlation functions w_{ij}^+ and w_{ji}^-

III.B Equation of Motion for a System Operator

The analysis in the previous section has been carried out in the Schroedinger picture, i.e. the density operator and the state vectors are time dependent, while the operators are stationary. The equation for the density matrix that we derived will be shown later to lead to the quantum mechanical Fokker-Planck equation. We shall ultimately be interested in deriving the the Langevin equations of motion for the system operators. To understand the relationship between the Langevin and Fokker-Planck approaches it is now necessary to derive the equations of motion for the system operators in the Heisenberg picture.

Let θ be any system operator. Then we know that

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{1}{i\hbar} [\theta, H] \\ &= \frac{1}{i\hbar} [\theta, H_0 + R + V] \\ &= \frac{1}{i\hbar} [\theta, H_0 + V] \end{aligned} \tag{III.B.1}$$

If we now transform to the Heisenberg-interaction representation through

$$\theta(t) = \exp\left[\frac{i\bar{H}t}{\hbar}\right] q(t) \exp\left[-\frac{i\bar{H}t}{\hbar}\right] \tag{III.B.2}$$

then it follows that the equation of motion for $q(t)$ is

$$\frac{dq}{dt} = \frac{1}{i\hbar} [q(t), V(-t)]$$

$$\text{where } V(t) = \exp\left\{\frac{i\bar{H}t}{\hbar}\right\} V \exp\left\{-\frac{i\bar{H}t}{\hbar}\right\} \quad (\text{III.B.3})$$

Equation (III.B.3) is very similar to equation (III.A.7), the equation of motion of the density operator. Using similar reasoning, we approximate (III.B.3) by

$$\begin{aligned} \frac{\Delta q}{\Delta t} = & -\frac{i}{\Delta t} \int_0^{\Delta t} dt' \left[q(0), \sum_i Q_i(-t') F_i(-t') \right] \\ & + \frac{(-i)^2}{\Delta t} \int_0^{\Delta t} dt' \int_0^{t'} dt'' \\ & \left[\left[q(0), \sum_i Q_i(-t') F_i(-t') \right], \sum_j Q_j(-t'') F_j(-t'') \right] \end{aligned} \quad (\text{III.B.4})$$

The simplification now proceeds in much the same manner as after equation (III.A.21). We multiply by $f(0)$, the reservoir density matrix, and take the trace over the reservoir states. Then since

$$\langle F_i(-t') \rangle_R = 0 \quad \text{and} \quad q(0) = \theta$$

$$\begin{aligned} \left\langle \frac{\Delta q}{\Delta t} \right\rangle_R = & -\frac{1}{\Delta t} \sum_{i,j} \int_0^{\Delta t} dt' \int_0^{t'} dt'' \\ & \left\langle \left[\left[\theta, Q_i(-t') \right] Q_j(-t'') K_{ij}(t' - t'') \right. \right. \\ & \left. \left. - Q_j(-t'') \left[\theta, Q_i(-t') \right] K_{ji}(t' - t'') \right] \right\rangle_R \end{aligned}$$

(III.B.5)

Making the change of variables as in equation (III.A.23) and carrying out similar simplifications we finally get

$$\left\langle \frac{\Delta q}{\Delta t} \right\rangle_{\mathcal{R}} = - \sum_{i,j} \delta_{i,j} \left\langle [\theta, Q_i] Q_j w_{ij}^+ - Q_j [\theta, Q_i] w_{ji}^- \right\rangle_{\mathcal{R}} \quad (\text{III.B.6})$$

Using equation (III.B.2) we get

$$\begin{aligned} \frac{d}{dt} \langle \theta \rangle_{\mathcal{R}} &= \left\langle \frac{1}{i\hbar} [\theta, H_0] \right\rangle_{\mathcal{R}} \\ &- \sum_{i,j} \delta_{i,j} \left\langle [\theta, Q_i] Q_j w_{ij}^+ - Q_j [\theta, Q_i] w_{ji}^- \right\rangle_{\mathcal{R}} \end{aligned} \quad (\text{III.B.7})$$

This is the Master Equation of motion for a system operator in the Heisenberg picture, with the degrees of freedom in the reservoir averaged over. Notice the similarity of this equation to equation (III.A.28). Equations (III.A.28) and (III.B.7) will form the basis of our study of quantum mechanical systems approaching thermal equilibrium.

Bibliography for Chapter III

Discussions on the utility of the density matrix in treating statistical systems may be found in Fano (1957) and ter Haar (1961). The master equations discussed in this chapter are sometimes presented in an altered form known as the transport equation. Original derivations of

the transport equation were made by Landau (1927), Pauli (1928) and Bloch (1928). These derivations all made use of the random phase approximation. An improved derivation, in which the importance of many degrees of freedom in the dissipation mechanism was explicitly recognized, was made by van Hove (1951). The master equation as presented in this chapter was first derived by Fain (1962), Lax (1963), Lax (1966), and Louisell and Walker (1965). Treatments may also be found in the books by Fain and Khanin (1969) and Louisell (1973).

IV. THE DAMPED HARMONIC OSCILLATOR

We shall now apply the techniques that have been developed in the previous chapter to a study of the damped harmonic oscillator. The harmonic oscillator offers a particularly simple illustration of these techniques and can also be solved exactly. Also we shall show later on that that harmonic oscillator can be used to represent the electromagnetic field in a cavity. Thus the study of the harmonic oscillator is of direct relevance. In section IV.A we will introduce all the operators needed and state their important properties. In section IV.B we will derive the equation of motion of the density matrix, (III.A.28) for the case of the harmonic oscillator. Section IV.C is a diversion from the other sections. Here we introduce the coherent state representation of the harmonic oscillator, and derive some of its useful properties. In section IV.D we shall use the coherent state representation of the harmonic oscillator to derive the Fokker-Planck equation, which will then be solved.

IV.A Definition of the Operators

We define the harmonic oscillator by the Hamiltonian and commutation relation

$$H_0 = \hbar\omega a^\dagger a \quad (\text{IV.A.1})$$

$$[a, a^\dagger] = 1 \quad (\text{IV.A.2})$$

In the Heisenberg picture, we then immediately have the solution

$$a^\dagger(t) = a^\dagger(0)\exp(i\omega t)$$

$$a(t) = a(0)\exp(-i\omega t) \quad (\text{IV.A.3})$$

It is now necessary to couple the oscillator to a thermal reservoir. We postulate an interaction of the form

$$R = \sum_j \hbar\omega_j R_{3j} \quad (\text{IV.A.4})$$

$$V = \frac{i\hbar\alpha}{2} \sum_j \left(R_{+j} a - a^\dagger R_{-j} \right) \quad (\text{IV.A.5})$$

So



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were not included in the copy of the
thesis deposited in the Institute Archives
by the author:

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$$f = \frac{\exp(-\beta R)}{\text{Trace}_R \exp(-\beta R)} \quad (\text{IV.A.14})$$

Since

$$R = \sum_j \hbar \omega_j R_{3j} \quad (\text{IV.A.4})$$

we have

$$\begin{aligned} \text{Trace}_R \exp(-\beta R) &= \text{Trace}_R \exp\left(-\sum_j \beta \hbar \omega_j R_{3j}\right) \\ &= \prod_j \text{Trace}_R \exp(-\beta \hbar \omega_j R_{3j}) \end{aligned} \quad (\text{IV.A.15})$$

Because the eigenvalues of R_{3j} are $+\frac{1}{2}$ and $-\frac{1}{2}$.

$$\begin{aligned} \text{Trace}_R \exp(-\beta R) &= \prod_j \left\{ \exp(-\beta \hbar \omega_j / 2) \right. \\ &\quad \left. + \exp(\beta \hbar \omega_j / 2) \right\} \end{aligned} \quad (\text{IV.A.16})$$

So

$$f = \prod_j \frac{\exp(-\beta \hbar \omega_j R_{3j})}{\exp(-\beta \hbar \omega_j / 2) + \exp(\beta \hbar \omega_j / 2)} \quad (\text{IV.A.17})$$

IV.B The Density Matrix for the Harmonic Oscillator

We are now ready to use the master equation for the density matrix that we derived in the last chapter. Recall that

$$\frac{\partial S}{\partial t} = \frac{1}{i\hbar} [H_0, S] - \sum_{i,j} \delta_{i,j} \left\{ \left[Q_i Q_j S - Q_j S Q_i \right] w_{ij}^+ - \left[Q_i S Q_j - S Q_j Q_i \right] w_{ji}^- \right\} \quad (\text{III.A.28})$$

From equations (IV.A.3) and (III.A.24), we have that the sum over i and j extends over the values $i=1,2$; $j=1,2$; and

$$\begin{aligned} \omega_1 &= -\omega \\ \omega_2 &= \omega \end{aligned} \quad (\text{IV.B.1})$$

By equation (III.A.14) we have

$$\begin{aligned} F_1 &= \frac{i\alpha}{2} \sum_j R_{+j} \\ F_2 &= -\frac{i\alpha}{2} \sum_j R_{-j} \end{aligned} \quad (\text{IV.B.2})$$

Using equation (III.A.15) we get therefore

$$\begin{aligned}
F_1(\tau) &= \exp \left\{ \sum_j i\omega_j R_{3j} \tau \right\} \left\{ \frac{i\alpha}{2} \sum_j R_{+j} \right\} \exp \left\{ - \sum_j i\omega_j R_{3j} \tau \right\} \\
&= \frac{i\alpha}{2} \sum_j \exp \left\{ i\omega_j R_{3j} \tau \right\} R_{+j} \exp \left\{ -i\omega_j R_{3j} \tau \right\}
\end{aligned} \tag{IV.B.3}$$

$$F_2(\tau) = - \frac{i\alpha}{2} \sum_j \exp \left\{ i\omega_j R_{3j} \tau \right\} R_{-j} \exp \left\{ -i\omega_j R_{3j} \tau \right\}$$

We now use the well known operator identity

$$\begin{aligned}
\exp(\xi A) B \exp(-\xi A) &= B + \xi [A, B] + \frac{\xi^2}{2!} [A, [A, B]] \\
&+ \frac{\xi^3}{3!} [A, [A, [A, B]]] + \dots
\end{aligned} \tag{IV.B.4}$$

and the commutation relations (IV.A.11) and (IV.B.12) to give us

$$\begin{aligned}
F_1(\tau) &= \frac{i\alpha}{2} \sum_j \left\{ R_{+j} + i\omega_j \tau [R_{3j}, R_{+j}] \right. \\
&\quad \left. + \frac{(i\omega_j \tau)^2}{2!} [R_{3j}, [R_{3j}, R_{+j}]] + \dots \right\} \\
&= \frac{i\alpha}{2} \sum_j R_{+j} \left\{ 1 + i\omega_j \tau + \frac{(i\omega_j \tau)^2}{2!} + \dots \right\}
\end{aligned} \tag{IV.B.5}$$

$$F_2(\tau) = - \frac{i\alpha}{2} \sum_j R_{-j} \exp(-i\omega_j \tau) \tag{IV.B.6}$$

Using definition (III.A.26) we get

$$w_{ij}^+ = \int_0^{\infty} \exp(i\omega_1 \tau) \langle F_1(\tau) F_j \rangle d\tau$$

$$w_{ji}^- = \int_0^{\infty} \exp(i\omega_1 \tau) \langle F_j F_1(\tau) \rangle d\tau \quad (\text{IV.B.7})$$

From equations (IV.B.5) and (IV.B.6) we have

$$\langle F_1(\tau) F_2 \rangle = \frac{\alpha^2}{4} \sum_j \frac{1}{A} \text{Trace}_R \left\{ R_{+j} R_{-j} \exp(-\beta \hbar \omega_j R_{3j}) \right\} \exp(i\omega_j \tau) \quad (\text{IV.B.8})$$

$$\text{where } A = \exp(-\beta \hbar \omega_j / 2) + \exp(\beta \hbar \omega_j / 2)$$

It is easiest to evaluate the trace in a basis in which R_{3j} is diagonal. We know from (IV.A.8) that the eigenvalues of R_{3j} are $+\frac{1}{2}$ and $-\frac{1}{2}$. The expression within the trace is thus

$$\begin{aligned} & \langle 1/2 | R_{+j} R_{-j} \exp(-\beta \hbar \omega_j / 2) | 1/2 \rangle \\ & + \langle -1/2 | R_{+j} R_{-j} \exp(\beta \hbar \omega_j / 2) | -1/2 \rangle \end{aligned} \quad (\text{IV.B.9})$$

Now by (IV.A.8), R_{+j} and R_{-j} are the raising and lowering operators :

$$R_{+j} |1/2\rangle = 0 \quad ; \quad R_{-j} |1/2\rangle = |-1/2\rangle$$

$$R_{+j} |-1/2\rangle = |1/2\rangle \quad ; \quad R_{-j} |-1/2\rangle = 0 \quad (\text{IV.B.10})$$

Therefore (IV.B.9) simplifies to

$$\exp(-\beta\hbar\omega_j/2) \quad (\text{IV.B.11})$$

Therefore

$$\langle F_1(\tau)F_2 \rangle = \sum_j \frac{\alpha^2}{4} \frac{\exp(i\omega_j\tau)}{\exp(\beta\hbar\omega_j) + 1} \quad (\text{IV.B.12})$$

In a similar fashion we can derive

$$\langle F_1F_2(\tau) \rangle = \sum_j \frac{\alpha^2}{4} \frac{\exp(-i\omega_j\tau)}{\exp(\beta\hbar\omega_j) + 1} \quad (\text{IV.B.13})$$

$$\langle F_2(\tau)F_1 \rangle = \sum_j \frac{\alpha^2}{4} \frac{\exp(-i\omega_j\tau)}{\exp(-\beta\hbar\omega_j) + 1} \quad (\text{IV.B.14})$$

$$\langle F_2F_1(\tau) \rangle = \sum_j \frac{\alpha^2}{4} \frac{\exp(i\omega_j\tau)}{\exp(-\beta\hbar\omega_j) + 1} \quad (\text{IV.B.15})$$

Therefore from (IV.B.7) we get

$$\begin{aligned} w_{12}^+ &= \int_0^{\infty} \exp(-i\omega\tau) \sum_j \frac{\alpha^2}{4} \frac{\exp(i\omega_j\tau)}{\exp(\beta\hbar\omega_j) + 1} \\ &= \sum_j \frac{\alpha^2}{4} \frac{1}{\exp(\beta\hbar\omega_j) + 1} \int_0^{\infty} \exp\{i(\omega_j - \omega)\tau\} d\tau \\ &= \sum_j \frac{\alpha^2}{4} \frac{1}{\exp(\beta\hbar\omega_j) + 1} * \\ &\quad * \left[\pi\delta(\omega_j - \omega) + iP \frac{1}{\omega_j - \omega} \right] \end{aligned}$$

We now convert the sum over j to an integral

$$= \int_0^{\infty} d\omega' \frac{\alpha^2 \rho(\omega')}{4 \exp(\beta \hbar \omega_j) + 1} * \left[\pi \delta(\omega' - \omega) + i P \frac{1}{\omega' - \omega} \right] \quad (\text{IV.B.16})$$

Since

$$\frac{\rho(\omega')}{\exp(\beta \hbar \omega') + 1}$$

is a slowly varying function of ω' , it can be taken out of the integral by replacing ω' by ω .

Also defining

$$\gamma' = \frac{\pi \rho(\omega) \alpha^2}{4}$$

$$\Delta \omega' = P \int_0^{\infty} d\omega' \frac{\rho(\omega') \alpha^2}{4(\omega' - \omega)} \quad (\text{IV.B.17})$$

we get

$$w_{12}^+ = \frac{1}{\exp(\beta \hbar \omega) + 1} (\gamma' + i \Delta \omega') \quad (\text{IV.B.18})$$

$$w_{21}^+ = \frac{1}{\exp(-\beta \hbar \omega) + 1} (\gamma' - i \Delta \omega') \quad (\text{IV.B.19})$$

$$w_{12}^- = \frac{1}{\exp(\beta \hbar \omega) + 1} (\gamma' - i \Delta \omega') \quad (\text{IV.B.20})$$

$$w_{21}^- = \frac{1}{\exp(-\beta \hbar \omega) + 1} (\gamma' + i \Delta \omega')$$

(IV.B.21)

We are now ready to use equation (III.B.7) for the motion of an operator in the Heisenberg picture. Let us derive the equation of motion for $\langle a(t) \rangle$. Plugging in

$$\theta = a, Q_1 = a, Q_2 = a^\dagger, \text{ and } H_0 = \hbar\omega a^\dagger a,$$

and equations (IV.B.18) to (IV.B.21) into equation (III.B.7) and after algebraic simplification we get,

$$\begin{aligned} \frac{d\langle a \rangle}{dt} = \langle a \rangle & \left[-i\omega + i\Delta\omega' \frac{\exp(\beta\hbar\omega) - 1}{\exp(\beta\hbar\omega) + 1} \right. \\ & \left. - \gamma' \frac{\exp(\beta\hbar\omega) - 1}{\exp(\beta\hbar\omega) + 1} \right] \end{aligned} \quad (\text{IV.B.22})$$

Therefore

$$\langle a(t) \rangle = \langle a(0) \rangle \exp\{-\gamma t - i(\omega - \Delta\omega)t\} \quad (\text{IV.B.23})$$

where by comparison with (IV.A.3) we have defined a damping constant

$$\begin{aligned} \gamma &= \gamma' \frac{\exp(\beta\hbar\omega) - 1}{\exp(\beta\hbar\omega) + 1} \\ &= \frac{\pi\rho(\omega)\alpha^2}{4} \frac{\exp(\beta\hbar\omega) - 1}{\exp(\beta\hbar\omega) + 1} \end{aligned} \quad (\text{IV.B.24})$$

and damping frequency shift

$$\begin{aligned} \Delta\omega &= \Delta\omega' \frac{\exp(\beta\hbar\omega) - 1}{\exp(\beta\hbar\omega) + 1} \\ &= P \int_0^{\infty} d\omega' \frac{\rho(\omega') \alpha^2}{4(\omega' - \omega)} \frac{\exp(\beta\hbar\omega) - 1}{\exp(\beta\hbar\omega) + 1} \end{aligned} \quad (\text{IV.B.25})$$

Similarly we have the conjugate equation

$$\langle a^\dagger(t) \rangle = \langle a^\dagger(0) \rangle \exp\{-\gamma t + i(\omega - \Delta\omega)t\} \quad (\text{IV.B.26})$$

We can also derive the time development of the energy H_0 by equation (III.B.7). Carrying out this procedure, we obtain after minor algebraic simplification

$$\frac{d\langle a^\dagger a \rangle}{dt} = -2\gamma \left\{ \langle a^\dagger a \rangle - \frac{1}{\exp(\beta\hbar\omega) - 1} \right\} \quad (\text{IV.B.27})$$

The solution to this equation is

$$\begin{aligned} \langle a^\dagger a(t) \rangle &= \langle a^\dagger a(0) \rangle \exp(-2\gamma t) \\ &+ \frac{1}{\exp(\beta\hbar\omega) - 1} (1 - \exp(-2\gamma t)) \end{aligned} \quad (\text{IV.B.28})$$

which gives us

$$\lim_{t \rightarrow \infty} \langle a^\dagger a(t) \rangle = \frac{1}{\exp(\beta\hbar\omega) - 1} \quad (\text{IV.B.29})$$

which is what we would have expected from ordinary statistical considerations at equilibrium. Thus the thermal reservoir, in addition to introducing a damping in the harmonic oscillator, introduces fluctua-

tions which ensure that the oscillator has the right thermal equilibrium energy. We define

$$\bar{n} = \frac{1}{\exp(\beta\hbar\omega) - 1} \quad (\text{IV.B.30})$$

We can also derive a corresponding equation for the density matrix from equation (III.A.28). We get after minor algebraic simplification

$$\begin{aligned} \frac{\partial S}{\partial t} = & -i(\omega - \Delta\omega) [a^\dagger a, S] + \gamma [aS, a^\dagger] \\ & + \gamma [a, Sa^\dagger] + 2\gamma\bar{n} [a^\dagger, [S, a]] \end{aligned} \quad (\text{IV.B.31})$$

which is the Master Equation of motion of the reduced density matrix of the harmonic oscillator.

IV.C The Coherent State Representation

It is the purpose of this chapter to solve equation (IV.B.31) for the density operator. To do this we need to develop the appropriate operator algebra necessary. We therefore make a short diversion in this section to introduce some useful techniques in solving such equations. We note

$$[a, (a^\dagger)^l] = l(a^\dagger)^{l-1} = \frac{\partial (a^\dagger)^l}{\partial a^\dagger} \quad (\text{IV.C.1})$$

and

$$[a^\dagger, (a)^l] = -l(a)^{l-1} = -\frac{\partial (a)^l}{\partial a} \quad (\text{IV.C.2})$$

we therefore postulate

$$[a, f(a, a^\dagger)] = \frac{\partial f}{\partial a^\dagger} \quad (\text{IV.C.3})$$

$$[a^\dagger, f(a, a^\dagger)] = -\frac{\partial f}{\partial a} \quad (\text{IV.C.4})$$

where $f(a, a^\dagger)$ is any arbitrary function of a and a^\dagger .

Let us now investigate the eigenstates of the operator a . Firstly a is not hermitian, we therefore don't expect the eigenstates to form a complete orthonormal set. Let $|\alpha\rangle$ be an eigenstate of a , with eigenvalue α :

$$a |\alpha\rangle = \alpha |\alpha\rangle \quad (\text{IV.C.5})$$

Then

$$a \exp(\xi a^\dagger) |\alpha\rangle = \exp(\xi a^\dagger) \exp(-\xi a^\dagger) a \exp(\xi a^\dagger) |\alpha\rangle$$

where ξ is any complex number. But using (IV.B.4) we have

$$\exp(-\xi a^\dagger) a \exp(\xi a^\dagger) = a + \xi \quad (\text{IV.C.6})$$

$$\begin{aligned} a \exp(\xi a^\dagger) |\alpha\rangle &= \exp(\xi a^\dagger) (a + \xi) |\alpha\rangle \\ &= \exp(\xi a^\dagger) (\alpha + \xi) |\alpha\rangle \\ &= (\alpha + \xi) \exp(\xi a^\dagger) |\alpha\rangle \end{aligned} \quad (\text{IV.C.7})$$

So if $|\alpha\rangle$ is an eigenstate of a with eigenvalues α , then $\exp(\xi a^\dagger) |\alpha\rangle$ is an eigenstate with eigenvalue $\alpha + \xi$. Since ξ is any complex number, it follows that the eigenvalues of a cover the entire complex plane.

Let us expand $|\alpha\rangle$ in the energy basis. Suppose

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n(\alpha) |n\rangle \quad (\text{IV.C.8})$$

Plugging this into (IV.C.5) and equating coefficients of $|n\rangle$ we obtain the recursion relation

$$C_{n+1}(\alpha) \sqrt{n+1} = \alpha C_n(\alpha) \quad (\text{IV.C.9})$$

Using (4.57) and the fact that $\langle \alpha | \alpha \rangle = 1$, we easily obtain

$$C_n(\alpha) = \exp(-1/2|\alpha|^2) \frac{\alpha^n}{\sqrt{n!}} \quad (\text{IV.C.10})$$

$$|\alpha\rangle = \exp(-1/2|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (\text{IV.C.11})$$

Using (IV.C.11) we can calculate in a straightforward fashion

$$|\langle \beta | \alpha \rangle|^2 = \exp\{-|\alpha - \beta|^2\} \quad (\text{IV.C.12})$$

i.e. the $|\alpha\rangle$ states are not orthogonal. Also

$$\int |\alpha\rangle \langle \alpha| \frac{d^2\alpha}{\pi} = 1 \quad (\text{IV.C.13})$$

$$\text{Here } d^2\alpha \equiv d(\text{Re } \alpha) d(\text{Im } \alpha) \quad (\text{IV.C.14})$$

i.e. the $|\alpha\rangle$ states are complete, with (IV.C.13) the completeness relation. Such a collection of states which are complete but not orthogonal is called an overcomplete basis. The $|\alpha\rangle$ basis is also referred to as the coherent basis and the $|\alpha\rangle$ state as the coherent state.

Any arbitrary function $f(a, a^\dagger)$ of the creation and destruction operators is said to be in normal order $f^N(a, a^\dagger)$ if it is written in a form in which all the creation operators lie to the left of the destruction operators. Thus

$$f(a, a^\dagger) = f^n(a, a^\dagger) = \sum_{p, q} f_{pq}^n(a^\dagger)^p a^q \quad (\text{IV.C.15})$$

We define an associated normal function $\bar{f}^n(\alpha, \alpha^*)$ for every $f(a, a^\dagger)$ such that

$$\bar{f}^n(\alpha, \alpha^*) = \sum_{p, q} f_{pq}^n(\alpha^*)^p (\alpha)^q \quad (\text{IV.C.16})$$

To find the associated normal function of any operator we then have the following procedure. Express the function in normal order and then replace every a^\dagger by α^* and a by α . We may also define a normal ordering operator N by

$$N \left\{ \bar{f}^n(\alpha, \alpha^*) \right\} = f(a, a^\dagger) \quad (\text{IV.C.17})$$

It is clear that there is a one-to-one correspondence between associated normal functions and a function of the creation and destruction operators, i.e. N is a one-to-one operator.

In an analogous manner we may define an anti-normal order in which a is placed to the left of every a^\dagger . We may also define the associated anti-normal function and the anti-normal ordering operator A by the equations

$$f(a, a^\dagger) = f^a(a, a^\dagger) = \sum_{r, s} f_{rs}^a(a)^r (a^\dagger)^s \quad (\text{IV.C.18})$$

$$\bar{f}^a(\alpha, \alpha^*) = \sum_{r, s} f_{rs}^a(\alpha)^r (\alpha^*)^s$$

(IV.C.19)

$$A \left\{ \bar{f}^a(\alpha, \alpha^*) \right\} = f(a, a^\dagger) \quad (\text{IV.C.20})$$

We will attempt to transform equation (IV.B.31) to a differential equation for the associated anti-normal function $S^a(\alpha, \alpha^*)$ of the density operator. The reason for choosing the anti-normal ordering will become apparent later. For the moment we note the following extremely useful properties of anti-normal ordering.

[1]

$$\begin{aligned} f(a, a^\dagger) &= \sum_{r,s} f_{rs}^a(a)^r (a^\dagger)^s \\ &= \sum_{r,s} f_{rs}^a \int (a)^r |\alpha\rangle \frac{d^2\alpha}{\pi} \langle\alpha| (a^\dagger)^s \\ &= \sum_{r,s} f_{rs}^a \int (\alpha)^r |\alpha\rangle \frac{d^2\alpha}{\pi} \langle\alpha| (a^*)^s \\ &= \int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle\alpha| \bar{f}^a(\alpha, \alpha^*) \\ f(a, a^\dagger) &= \int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle\alpha| \bar{f}^a(\alpha, \alpha^*) \quad (\text{IV.C.21}) \end{aligned}$$

[2]

$$\text{Trace } |a\rangle \langle b| = \langle a|b\rangle$$

(IV.C.22)

$$\text{Trace } f(a, a^\dagger) = \int \frac{d^2\alpha}{\pi} \bar{f}^a(\alpha, \alpha^*) \quad (\text{IV.C.23})$$

from (IV.C.21)

[3]

$$\begin{aligned} & \text{Trace} \left\{ \rho(a, a^\dagger) f(a, a^\dagger) \right\} \\ &= \text{Trace} \left\{ \sum_{r,s} \rho_{rs}^a (a)^r (a^\dagger)^s \sum_{p,q} f_{pq}^n (a^\dagger)^p (a)^q \right\} \\ &= \sum_{r,s} \sum_{p,q} \rho_{rs}^a f_{pq}^n \text{Trace} \left\{ (a)^r (a^\dagger)^{s+p} (a)^q \right\} \\ &= \sum_{r,s} \sum_{p,q} \rho_{rs}^a f_{pq}^n \text{Trace} \left\{ (a)^{r+q} (a^\dagger)^{s+p} \right\} \\ &= \sum_{r,s} \sum_{p,q} \rho_{rs}^a f_{pq}^n \int \frac{d^2\alpha}{\pi} (a)^{r+q} (\alpha^*)^{s+p} \end{aligned}$$

$$\text{Trace} \left\{ \rho(a, a^\dagger) f(a, a^\dagger) \right\}$$

$$= \int \frac{d^2\alpha}{\pi} \bar{\rho}^a(\alpha, \alpha^*) \bar{f}^n(\alpha, \alpha^*) \quad (\text{IV.C.24})$$

[4]

$$\begin{aligned}
& \text{Trace} \left\{ f(a, a^\dagger) \delta(\alpha^* - a^\dagger) \delta(\alpha - a) \right\} \\
&= \int \frac{d^2\beta}{\pi} \bar{f}^a(\beta, \beta^*) \delta(\alpha^* - \beta^*) \delta(\alpha - \beta) \\
&= \frac{1}{\pi} \bar{f}^a(\alpha, \alpha^*)
\end{aligned}$$

$$\bar{f}^a(\alpha, \alpha^*) = \pi \text{Trace} \left\{ f(a, a^\dagger) \delta(\alpha^* - a^\dagger) \delta(\alpha - a) \right\} \quad (\text{IV.C.25})$$

IV.D Solution of the Density Matrix Equation

We now attempt to solve the equation of motion for the density operator

$$\begin{aligned} \frac{\partial S}{\partial t} = & -i(\omega - \Delta\omega) [a^\dagger a, S] + \gamma [aS, a^\dagger] \\ & + \gamma [a, Sa^\dagger] + 2\gamma\bar{n} [a^\dagger, [S, a]] \end{aligned} \quad (\text{IV.B.31})$$

We shall attempt to put both sides of this equation into anti-normal order. Let us consider the first term $[a^\dagger a, S]$

$$\begin{aligned} [a^\dagger a, S] &= a^\dagger a S - S a^\dagger a \\ &= a a^\dagger S - S - S a a^\dagger + S \\ &= a \left[S a^\dagger - \frac{\partial S}{\partial a} \right] - \left[a S - \frac{\partial S}{\partial a^\dagger} \right] a^\dagger \\ &= \frac{\partial S}{\partial a^\dagger} a^\dagger - a \frac{\partial S}{\partial a} \end{aligned} \quad (\text{IV.D.1})$$

where we have used equations (IV.C.3) and (IV.C.4). The final expression in (IV.D.1) is in anti-normal order. In a similar fashion we can convert all the other terms into anti-normal order by using the relations (IV.C.3) and (IV.C.4). If we carry this out, we obtain after algebraic simplification

$$\begin{aligned} \frac{\partial S}{\partial t} = & -i(\omega - \Delta\omega) \left[\frac{\partial S}{\partial a^\dagger} a^\dagger - a \frac{\partial S}{\partial a} \right] \\ & + \gamma \left[\frac{\partial S}{\partial a^\dagger} a^\dagger + a \frac{\partial S}{\partial a} + 2S \right] + 2\gamma\bar{n} \frac{\partial^2 S}{\partial a \partial a^\dagger} \end{aligned} \quad (\text{IV.D.2})$$

Since both sides of (IV.D.2) are in anti-normal order, we can apply the anti-normal ordering operator to both sides and obtain

$$\begin{aligned} \frac{\partial P}{\partial t} = & (\gamma + i(\omega - \Delta\omega)) \frac{\partial}{\partial \alpha} (\alpha P) \\ & + (\gamma - i(\omega - \Delta\omega)) \frac{\partial}{\partial \alpha^*} (\alpha^* P) + 2\gamma\bar{n} \frac{\partial P}{\partial \alpha \partial \alpha^*} \end{aligned} \quad (\text{IV.D.3})$$

$$\text{where } P = \bar{S}^a(\alpha, \alpha^*, t)$$

This is the Fokker-Planck equation for the associated anti-normal function of the density matrix. The techniques used to derive this equation are rather specialized to the harmonic oscillator. In the next chapter we shall develop more general techniques which can be used for any arbitrary set of operators.

Equation (IV.D.3) can be solved exactly in much the same manner we used to solve equation (II.B.10). We solve it with the initial condition

$$P(\alpha, \alpha^*, 0) = \delta(\alpha - \beta) \delta(\alpha^* - \beta^*)$$

i.e. we will find the Green's function for the equation. We can then find the solution to any arbitrary initial condition by integration.

$$\begin{aligned}
 P(\alpha, \alpha^*, 0) &= \delta(\alpha - \beta) \delta(\alpha^* - \beta^*) \\
 &= \lim_{\epsilon \rightarrow \infty} \frac{\epsilon}{\pi} \exp \left\{ -\epsilon(\alpha - \beta)(\alpha^* - \beta^*) \right\}
 \end{aligned}
 \tag{IV.D.4}$$

As before we make the ansatz

$$P(\alpha, \alpha^*, 0) = \exp \{ G(t) \} \tag{IV.D.5}$$

where $G(t)$ is of the form

$$G(t) = \frac{1}{A(t)} (\alpha - B(t)) \left\{ \alpha^* - B^*(t) \right\} + \ln C(t) \tag{IV.D.6}$$

Then

$$A(0) = \lim_{\epsilon \rightarrow \infty} \frac{1}{\epsilon}$$

$$B(0) = \beta$$

$$B^*(0) = \beta^*$$

(IV.D.7)

$$C(0) = \lim_{\epsilon \rightarrow \infty} \frac{\epsilon}{\pi}$$

Substituting (IV.D.5) and (IV.D.6) into (IV.D.3) and equating the coefficients of α , α^* , and $\alpha^* \alpha$ we have the equations

$$\frac{dA}{dt} = -2\gamma A + 2\gamma \bar{n}$$

(IV.D.8)

$$\frac{dB}{dt} = (-\gamma - i\bar{\omega}) B \quad (\text{IV.D.9})$$

$$\frac{1}{C} \frac{dC}{dt} = -\frac{1}{A} \frac{dA}{dt} \quad (\text{IV.D.10})$$

$$\text{where } \bar{\omega} = \omega - \Delta\omega \quad (\text{IV.D.11})$$

which can easily be solved for the initial conditions to give

$$A(t) = \bar{n} (1 - \exp(-2\gamma t)) \quad (\text{IV.D.12})$$

$$B(t) = \beta \exp(-\gamma t - i\bar{\omega}t) \quad (\text{IV.D.13})$$

$$C(t) = \frac{1}{\pi \bar{n} (1 - \exp(-2\gamma t))} \quad (\text{IV.D.14})$$

$$P(\alpha, \alpha^*, t) = C(t) \exp \left[-\frac{|\alpha - B(t)|^2}{A(t)} \right] \quad (\text{IV.D.15})$$

As $t \rightarrow \infty$ we have

$$P(\alpha, \alpha^*, t) = \frac{1}{\pi \bar{n}} \exp \left[-\frac{|\alpha|^2}{\bar{n}} \right] \quad (\text{IV.D.16})$$

Therefore the density matrix is independent of the initial conditions and represents Gaussian noise.

Suppose at $t=0$, the system is in its ground state.

$$S(t=0) = |0\rangle\langle 0| \quad (\text{IV.D.17})$$

We have to find the anti-normal form of this state. We have from equation (IV.C.25)

$$\begin{aligned} A^{-1} (|0\rangle\langle 0|) &= \pi \text{Trace} \left\{ |0\rangle\langle 0| \delta(\alpha^{\#} - a^{\dagger}) \delta(\alpha - a) \right\} \\ &= \pi \langle 0| \delta(\alpha^{\#} - a^{\dagger}) \delta(\alpha - a) |0\rangle \end{aligned} \quad (\text{IV.D.18})$$

We note the useful property of the delta function

$$\delta(a - b) = \exp \left\{ -b \frac{\partial}{\partial a} \right\} \delta(a) \quad (\text{IV.D.19})$$

Using (IV.D.19) we have for (IV.D.18)

$$\begin{aligned} A^{-1} (|0\rangle\langle 0|) &= \\ &= \pi \langle 0| \exp \left[a^{\dagger} \frac{\partial}{\partial \alpha^{\#}} \right] \exp \left[a \frac{\partial}{\partial \alpha} \right] |0\rangle \delta(\alpha) \delta(\alpha^{\#}) \\ &= \pi \langle 0| 1 - a^{\dagger} \frac{\partial}{\partial \alpha^{\#}} - a \frac{\partial}{\partial \alpha} + a^{\dagger} a \frac{\partial^2}{\partial \alpha \partial \alpha^{\#}} + \dots |0\rangle \delta(\alpha) \delta(\alpha^{\#}) \end{aligned}$$

All the terms, but the first, have zero expectation value and we get

$$A^{-1} (|0\rangle\langle 0|) = \pi \delta(\alpha) \delta(\alpha^{\#}) \quad (\text{IV.D.20})$$

Therefore the solution to (4.76) with the initial condition

$$P(\alpha, \alpha^*, 0) = \pi \delta(\alpha) \delta(\alpha^*)$$

is given by

$$\begin{aligned} P(\alpha, \alpha^*, t) &= \int d^2\beta \pi \delta(\beta) \delta(\beta^*) C(t) \exp \left[- \frac{|\alpha - B(t)|^2}{A(t)} \right] \\ &= \frac{1}{A(t)} \exp \left[- \frac{|\alpha|^2}{A(t)} \right] \end{aligned} \quad (\text{IV.D.21})$$

Let us now determine the expectation value of $a^\dagger a$

$$\langle a^\dagger a \rangle = \text{Trace}(S a^\dagger a)$$

by IV.C.24

$$\begin{aligned} &= \int \frac{d^2\alpha}{\pi} \bar{S}^a(\alpha, \alpha^*) \alpha^* \alpha \\ &= \frac{1}{\pi A(t)} \int d^2\alpha |\alpha|^2 \exp \left[- \frac{|\alpha|^2}{A(t)} \right] \\ &= \bar{n} (1 - \exp(-2\gamma t)) \end{aligned} \quad (\text{IV.D.22})$$

which agrees with equation (IV.B.28)

Bibliography for Chapter IV

The 'R' operators in section IV.A were introduced by Dicke (1954).
The form of the reservoir interaction has been used by Senitzky (1960)

and (1961). The coherent state representation was introduced by Glauber (1963) and has since been used extensively in the literature. Louisell (1973) also has a treatment of this subject. Louisell and Walker (1965) and Louisell and Marburger (1967) first obtained the solution of the damped oscillator density matrix equation.

V. THE QUANTUM FOKKER-PLANCK AND LANGEVIN EQUATIONS

In the previous chapter we derived the Fokker-Planck equation for a damped harmonic oscillator from the equation for the density operator. In this chapter we will derive this equation for an arbitrary system. In section V.A we shall develop the necessary operator algebra to do this. In section V.B the Fokker-Planck equation shall be derived for an arbitrary system. Section V.C will illustrate the general techniques developed for the case of the harmonic oscillator. In section V.D we will use the equation of motion for a system operator that was derived in chapter III and transform it with the help of the operator algebra developed in section V.A. Finally, in section V.E we will derive the Langevin equations of motion and point out their connection with the Fokker-Planck equation.

V.A Operator Algebra

Let the system under consideration be described by the complete set of non-commuting operators

$$a_1, a_2, \dots, a_n$$

By the description complete we mean that these operators, with their commutation relations form a Lie algebra. In the case of the harmonic oscillator we had the operators a^\dagger and a forming such a set. By analogy with normal and anti-normal ordering we define an arbitrary chosen order of the operators. Let this chosen order be

$$a_1 a_2 \dots a_n$$

To simplify the notation we introduce here a method of representing a vector of n elements. A bar over any symbol will represent a vector with n elements. Thus we have the following representations

$$a_1, a_2, \dots, a_n \rightarrow \bar{a}$$

$$a_1^{r_1} a_2^{r_2} \dots a_n^{r_n} \rightarrow \bar{a}^{\bar{r}}$$

(V.A.1)

$$\delta(\alpha_1 - a_1) \delta(\alpha_2 - a_2) \dots \delta(\alpha_n - a_n) \rightarrow \bar{\delta}(\alpha - a)$$

With each operator a_i we associate a real number α_i . If the operator is non-hermitian, the associated number is chosen to be complex, and with the hermitian conjugate of the operator, we associate the complex conjugate of the number. Thus if

$$\alpha_i \rightarrow a_i$$

then

$$\alpha_i^* \rightarrow a_i^\dagger \tag{V.A.2}$$

Let Q be any operator which can be expressed as a function of this complete set of operators. Q is said to be in chosen order if every term in the power series of Q is in chosen order. Thus

$$\begin{aligned} Q(\bar{a}) &= Q^c(\bar{a}) \\ &= \sum_{\bar{r}} q_{\bar{r}}(\bar{a}) \bar{r} \end{aligned} \tag{V.A.3}$$

The superscript c on Q indicates that it is in chosen order. The chosen order in the case of the harmonic oscillator was the normal or anti-normal ordering.

We also define the associated chosen-order function $\bar{Q}^c(\bar{a})$, and the chosen-ordering operator by the equations

$$\bar{Q}^c(\bar{a}) = \sum_{\bar{r}} q_{\bar{r}}(\bar{a}) \bar{r} \tag{V.A.4}$$

where the $q_{\bar{r}}$ are the coefficients in equation (V.A.3), and

$$c^{-1}(Q(\bar{a})) = \bar{Q}^c(\bar{a})$$

(V.A.5)

The action of the chosen ordering operator can be formally represented by

$$\begin{aligned} Q(\bar{a}) &= c \left[\bar{Q}^c(\bar{a}) \right] \\ &= \int \dots \int \bar{Q}^c(\bar{a}) \delta(\alpha - a) d\bar{a} \end{aligned} \quad (\text{V.A.6})$$

where the delta functions are written in chosen order. Also the expectation value of the operator Q is given by

$$\langle Q(\bar{a}) \rangle = \text{Trace} \{ \rho(t) Q(\bar{a}) \} \quad (\text{V.A.7})$$

where $\rho(t)$ is the density operator for the system. Using (V.A.6) we can write (V.A.7) as

$$\begin{aligned} \langle Q(\bar{a}) \rangle &= \text{Trace} \left\{ \rho(t) \int \dots \int \bar{Q}^c(\bar{a}) \delta(\alpha - a) d\bar{a} \right\} \\ &= \int \dots \int \bar{Q}^c(\bar{a}) \text{Trace} \{ \rho(t) \delta(\alpha - a) \} d\bar{a} \end{aligned} \quad (\text{V.A.8})$$

If we now define the distribution function by

$$P(\bar{a}, t) = \text{Trace} \{ \rho(t) \delta(\alpha - a) \} \quad (\text{V.A.9})$$

equation (V.A.8) becomes

$$\langle Q(\bar{\alpha}) \rangle = \int \dots \int \bar{Q}^C(\bar{\alpha}) P(\bar{\alpha}, t) d\bar{\alpha} \quad (\text{V.A.10})$$

Thus if we know the distribution function as a function of time we can find the expectation value of any operator. We already have an equation of motion for the density operator (III.A.28). Using the relation (V.A.9) we will convert this equation into a differential equation for $P(\bar{\alpha}, t)$. This will be the quantum analog of the Fokker-Planck equation.

For the case of the harmonic oscillator we have from equation (IV.C.25) that

$$\bar{\rho}^a(\alpha, \alpha^*) = \pi \text{Trace} \left\{ \rho(a, a^\dagger) \delta(\alpha^* - a^\dagger) \delta(\alpha - a) \right\}$$

Using equation (V.A.9) this becomes

$$P(\alpha, \alpha^*, t) = \frac{1}{\pi} \bar{\rho}^a(\alpha, \alpha^*) \quad (\text{V.A.11})$$

Thus the distribution function for the harmonic oscillator is directly related to the anti-normal form of the density matrix. This was the reason for choosing the anti-normal ordering in chapter IV. Therefore from equation (IV.D.3) we can immediately derive the Fokker-Planck equation for the harmonic oscillator.

$$\frac{\partial P}{\partial t} = (\gamma + i(\omega - \Delta\omega)) \frac{\partial}{\partial \alpha} (\alpha P)$$

$$+ (\gamma - i(\omega - \Delta\omega)) \frac{\partial}{\partial \alpha^*} (\alpha^* P) + 2\gamma \bar{n} \frac{\partial P}{\partial \alpha \partial \alpha^*}$$

$$\text{where } P = \bar{S}^a(\alpha, \alpha^*, t)$$

V.B The Fokker-Planck Equation

We recall equation (III.A.28) for the density operator

$$\frac{\partial S}{\partial t} = \frac{1}{i\hbar} [H_0, S] - \sum_{i,j} \delta_{i,j} \left[\left(Q_i Q_j S - Q_j S Q_i \right) w_{ij}^+ - \left(Q_i S Q_j - S Q_j Q_i \right) w_{ji}^- \right] \quad (\text{III.A.28})$$

We multiply both sides of this equation by

$$\delta(\alpha_1 - a_1) \dots \delta(\alpha_n - a_n) \equiv \bar{\delta}(\alpha - a)$$

and trace over the system. Then the first term on the left hand side becomes

$$\text{Trace} \left[\frac{\partial S}{\partial t} \bar{\delta}(\alpha - a) \right] = \frac{\partial P}{\partial t} \quad (\text{V.B.1})$$

by equation (V.A.9). The first term on the right hand side is

$$\begin{aligned}
\text{Trace} \left\{ \frac{1}{i\hbar} [H_0, S] \bar{\psi}(\alpha - a) \right\} &= \\
&= \text{Trace} \left\{ \frac{1}{i\hbar} (H_0 S \bar{\psi}(\alpha - a) - S H_0 \bar{\psi}(\alpha - a)) \right\} \\
&= \text{Trace} \left\{ \frac{1}{i\hbar} (S \bar{\psi}(\alpha - a) H_0 - S H_0 \bar{\psi}(\alpha - a)) \right\}
\end{aligned}$$

using the cyclic properties of the traces

(V.B.2)

$$= \text{Trace} \left\{ \frac{1}{i\hbar} S [\bar{\psi}(\alpha - a), H_0] \right\}$$

The second term is

$$\text{Trace} \left\{ \left(Q_i Q_j S - Q_j S Q_i \right) w_{ij}^+ \bar{\psi}(\alpha - a) \right\} =$$

(V.B.3)

$$= \text{Trace} \left\{ S [\bar{\psi}(\alpha - a), Q_i] Q_j \right\} w_{ij}^+$$

where we again used the cyclic property of the traces. We similarly operate on the third term on the right hand side and equation (III.A.28) finally simplifies to

$$\begin{aligned}
\frac{\partial P}{\partial t} &= \text{Trace} S \left\{ \frac{1}{i\hbar} [\bar{\psi}(\alpha - a), H_0] \right. \\
&- \sum_{i,j} \delta_{ij} \left\{ [\bar{\psi}(\alpha - a), Q_i] Q_j w_{ij}^+ \right. \\
&\quad \left. \left. - Q_j [\bar{\psi}(\alpha - a), Q_i] w_{ji}^- \right\} \right\}
\end{aligned}$$

We have to express the right hand side as an operator on P. Before proceeding we define for convenience the following shorthand :

$$\Pi_i \exp \left(-a_i \frac{\partial}{\partial \alpha_i} \right) \equiv \overline{\exp} \left(-a \frac{\partial}{\partial \alpha} \right)$$

Then using equation (IV.D.19) on (V.B.21) we get

$$\begin{aligned} \frac{\partial P}{\partial t} &= \text{Trace S} \left[\frac{1}{i\hbar} \left[\overline{\exp} \left(-a \frac{\partial}{\partial \alpha} \right), H_0 \right] \right. \\ &\quad - \sum_{i,j} \delta_{ij} \left[\overline{\exp} \left(-a \frac{\partial}{\partial \alpha} \right), Q_i \right] Q_j w_{ij}^+ \\ &\quad \left. - Q_j \left[\overline{\exp} \left(-a \frac{\partial}{\partial \alpha} \right), Q_i \right] w_{ji}^- \right] \bar{\sigma}(\alpha) \end{aligned} \quad (\text{V.B.5})$$

We now define

$$\begin{aligned} X \left(\bar{a}, \frac{\partial}{\partial \alpha} \right) &= \frac{1}{i\hbar} \left[\overline{\exp} \left(-a \frac{\partial}{\partial \alpha} \right), H_0 \right] \\ &\quad - \sum_{i,j} \delta_{ij} \left[\overline{\exp} \left(-a \frac{\partial}{\partial \alpha} \right), Q_i \right] Q_j w_{ij}^+ \\ &\quad - Q_j \left[\overline{\exp} \left(-a \frac{\partial}{\partial \alpha} \right), Q_i \right] w_{ji}^- \end{aligned} \quad (\text{V.B.6})$$

Then equation (V.B.5) can be written simply as

$$\frac{\partial P}{\partial t} = \text{Trace} \left[S(t) X \left[\bar{a}, \frac{\partial}{\partial \bar{a}} \right] \right] \bar{\delta}(\alpha) \quad (\text{V.B.7})$$

$$= \text{Trace} \left[S(t) \int \dots \int \bar{X}^c \left[\bar{B}, \frac{\partial}{\partial \bar{a}} \right] \bar{\delta}(\beta - \alpha) d\bar{B} \bar{\delta}(\alpha) \right] \quad (\text{V.B.8})$$

where we have used equation (V.A.6) and \bar{L}^c is the associated chosen order function.

$$\begin{aligned} &= \int \dots \int \text{Trace} \left[S(t) \bar{\delta}(\beta - \alpha) \right] \bar{X}^c \left[\bar{B}, \frac{\partial}{\partial \bar{a}} \right] \bar{\delta}(\alpha) d\bar{B} \\ &= \int \dots \int P(\bar{B}, t) \bar{X}^c \left[\bar{B}, \frac{\partial}{\partial \bar{a}} \right] \bar{\delta}(\alpha) d\bar{B} \end{aligned} \quad (\text{V.B.9})$$

We now make the assumption that $\bar{X}^c \left[\bar{B}, \frac{\partial}{\partial \bar{a}} \right]$ can be written in the form

$$\bar{X}^c \left[\bar{B}, \frac{\partial}{\partial \bar{a}} \right] = \Xi \left[\bar{B}, \frac{\partial}{\partial \bar{a}} \right] \exp \left[-\beta \frac{\partial}{\partial \alpha} \right] \quad (\text{V.B.10})$$

This assumption will be to be valid for all the cases that we shall consider. Then equation (V.B.9) gives us

$$\frac{\partial P}{\partial t} = \int \dots \int P(\bar{B}, t) \Xi \left[\bar{B}, \frac{\partial}{\partial \bar{a}} \right] \exp \left[-\beta \frac{\partial}{\partial \alpha} \right] \bar{\delta}(\alpha) d\bar{B} \quad (\text{V.B.11})$$

$$\frac{\partial P}{\partial t} = \int \dots \int P(\bar{B}, t) \Xi \left[\bar{B}, \frac{\partial}{\partial \bar{a}} \right] \bar{\delta}(\alpha - \beta) d\bar{B}$$

$$\frac{\partial P}{\partial t} = \Xi \left[\bar{\alpha}, \frac{\partial}{\partial \bar{a}} \right] P(\bar{\alpha}, t) \quad (\text{V.B.12})$$

which is the equation we wished to derive. In all the cases that we shall consider, $\Xi \left[\bar{\alpha}, \frac{\partial}{\partial \bar{a}} \right]$ will only have second order derivatives in α in

which case (V.B.12) is the Fokker-Planck equation for the system.

V.C The Harmonic Oscillator Revisited

To illustrate the techniques developed in this chapter we will derive the function $\Xi \left(\bar{\alpha}, \frac{\partial}{\partial \bar{\alpha}} \right)$ for a harmonic oscillator. We have already derived the Fokker-Planck equation using other techniques, but this section will be a confirmation of equation (V.A.12). Note that

$$a_1 \equiv a^\dagger ; a_2 = a$$

$$Q_2 \equiv a^\dagger ; Q_1 = a$$

For notational convenience we make the following definitions

$$\underline{e}^{-a^\dagger} = \exp \left[-a^\dagger \frac{\partial}{\partial \alpha^*} \right]$$

$$\underline{e}^{-a} = \exp \left[-a \frac{\partial}{\partial \alpha} \right]$$

From equation (V.B.6) we have for $X \left(\bar{\alpha}, \frac{\partial}{\partial \bar{\alpha}} \right)$

$$\begin{aligned}
X \left(\bar{a}, \frac{\partial}{\partial \bar{a}} \right) &= \frac{\omega}{I} [\underline{e^{-a^\dagger}} \underline{e^{-a}}, a^\dagger a] \\
&- [\underline{e^{-a^\dagger}} \underline{e^{-a}}, a] a^\dagger \frac{(\gamma + i\Delta\omega)}{\exp(\beta\hbar\omega) - 1} \\
&- a^\dagger [\underline{e^{-a^\dagger}} \underline{e^{-a}}, a] \frac{(\gamma + i\Delta\omega)}{\exp(\beta\hbar\omega) - 1} \\
&- [\underline{e^{-a^\dagger}} \underline{e^{-a}}, a] a \frac{(\gamma - i\Delta\omega)}{\exp(\beta\hbar\omega) - 1} \\
&- a [\underline{e^{-a^\dagger}} \underline{e^{-a}}, a] \frac{(\gamma + i\Delta\omega)}{\exp(\beta\hbar\omega) - 1}
\end{aligned} \tag{V.C.1}$$

To derive $\Xi \left(\bar{a}, \frac{\partial}{\partial \bar{a}} \right)$ we need to find the normal form of the above expression. For this we need the normal form of the following expressions

$$\begin{aligned}
&\underline{e^{-a^\dagger}} \underline{e^{-a}} a^\dagger a \\
&= \underline{e^{-a^\dagger}} \underline{e^{-a}} a^\dagger \underline{e^a} \underline{e^{-a}} a
\end{aligned}$$

Using equation (IV.B.4) we have

$$\begin{aligned}
&= \underline{e^{-a^\dagger}} \left[a^\dagger - \frac{\partial}{\partial \alpha} [a, a^\dagger] \right. \\
&\quad \left. + \frac{1}{2} \frac{\partial^2}{\partial \alpha^2} [a, [a, a^\dagger]] \dots \right] \underline{e^{-a}} a \\
&= \underline{e^{-a^\dagger}} \left[a^\dagger - \frac{\partial}{\partial \alpha} \right] \underline{e^{-a}} a
\end{aligned}$$

(V.C.2)

Similarly

$$\begin{aligned}
 a^\dagger a \underline{e^{-a^\dagger}} \underline{e^{-a}} \\
 &= a^\dagger \underline{e^{-a^\dagger}} \underline{e^{a^\dagger}} a \underline{e^{-a^\dagger}} \underline{e^{-a}} \\
 &= a^\dagger \underline{e^{-a^\dagger}} \left(a - \frac{\partial}{\partial \alpha^\dagger} \right) \underline{e^{-a}}
 \end{aligned}$$

(V.C.3)

and

$$\begin{aligned}
 a \underline{e^{-a^\dagger}} \underline{e^{-a}} a^\dagger \\
 &= \underline{e^{-a^\dagger}} \left(a - \frac{\partial}{\partial \alpha^\dagger} \right) \left(a^\dagger - \frac{\partial}{\partial \alpha} \right) \underline{e^{-a}}
 \end{aligned}$$

(V.C.4)

We insert (V.C.2), (V.C.3) and (V.C.4) into (V.C.1) and obtain after algebraic simplification.

$$\begin{aligned}
& X \left(\bar{a}, \frac{\partial}{\partial \bar{a}} \right) \\
&= -i(\omega - \Delta\omega) \underline{e}^{-a^\dagger} \left(\frac{\partial}{\partial \alpha^*} a^\dagger - \frac{\partial}{\partial \alpha} a \right) \underline{e}^{-a} \\
&+ \gamma \underline{e}^{-a^\dagger} \left(\frac{\partial}{\partial \alpha^*} a^\dagger + \frac{\partial}{\partial \alpha} a \right) \underline{e}^{-a} \\
&+ 2\gamma \bar{n} \frac{\partial^2}{\partial \alpha \partial \alpha^*} \underline{e}^{-a^\dagger} \underline{e}^{-a}
\end{aligned} \tag{V.C.5}$$

This expression is in normal order. Therefore

$$\begin{aligned}
& \bar{X}^c \left(\bar{B}, \frac{\partial}{\partial \bar{B}} \right) \\
&= \left(-i(\omega - \Delta\omega) \left(\frac{\partial}{\partial \alpha^*} \beta^* - \frac{\partial}{\partial \alpha} \beta \right) \right. \\
&+ \gamma \left(\frac{\partial}{\partial \alpha^*} \beta^* + \frac{\partial}{\partial \alpha} \beta \right) \\
&\left. + 2\gamma \bar{n} \frac{\partial^2}{\partial \alpha \partial \alpha^*} \right)^* \\
&* \exp \left(-\beta^* \frac{\partial}{\partial \alpha^*} \right) \exp \left(-\beta \frac{\partial}{\partial \alpha} \right)
\end{aligned} \tag{V.C.6}$$

which gives for $\Xi \left(\bar{\alpha}, \frac{\partial}{\partial \bar{\alpha}} \right)$

$$\begin{aligned}
&= \left(\alpha, \frac{\partial}{\partial \alpha} \right) \\
&= -i(\omega - \Delta\omega) \left(\frac{\partial}{\partial \alpha^*} \alpha^* - \frac{\partial}{\partial \alpha} \alpha \right) \\
&+ \gamma \left(\frac{\partial}{\partial \alpha^*} \alpha^* + \frac{\partial}{\partial \alpha} \alpha \right) \\
&+ 2\gamma\bar{n} \frac{\partial^2}{\partial \alpha \partial \alpha^*}
\end{aligned} \tag{V.C.7}$$

Therefore from equation (V.B.12) we get

$$\begin{aligned}
\frac{\partial P}{\partial t} &= -i(\omega - \Delta\omega) \left(\frac{\partial}{\partial \alpha^*} \alpha^* - \frac{\partial}{\partial \alpha} \alpha \right) P \\
&+ \gamma \left(\frac{\partial}{\partial \alpha^*} \alpha^* + \frac{\partial}{\partial \alpha} \alpha \right) P + 2\gamma\bar{n} \frac{\partial^2}{\partial \alpha \partial \alpha^*} P
\end{aligned} \tag{V.C.8}$$

which is identical to equation (V.A.12)

V.D Equation of Motion for a System Operator

It is our purpose in this chapter to derive the Langevin equations and to study their relation to the Fokker-Planck equation. To do this we begin with the equation for the reservoir expectation value for a system operator (III.B.7). We will derive the differential equation for the associated chosen order function of the system operator. We restate equation (III.B.7)

$$\frac{d}{dt} \langle \theta(\bar{a}) \rangle_R = \left\langle \frac{1}{i\hbar} [\theta, H_0] \right\rangle_R - \sum_{i,j} \delta_{i,j} \left\langle [\theta, Q_i] Q_j w_{ij}^+ - Q_j [\theta, Q_i] w_{ji}^- \right\rangle_R \quad (\text{III.B.7})$$

We have from equation (V.A.6)

$$\theta(\bar{a}) = \int \dots \int \bar{\theta}^c(\bar{\alpha}, t) \bar{\delta}(\alpha - a) d\bar{\alpha} \quad (\text{V.D.1})$$

Inserting this into (III.B.7)

$$\frac{d}{dt} \langle \theta(\bar{a}, t) \rangle_R = \int \dots \int \bar{\theta}^c(\bar{\alpha}) \frac{1}{i\hbar} \left\langle [\bar{\delta}(\alpha - a), H_0] \right\rangle_R - \sum_{i,j} \delta_{i,j} \left\langle [\bar{\delta}(\alpha - a), Q_i] Q_j w_{ij}^+ - Q_j [\bar{\delta}(\alpha - a), Q_i] w_{ji}^- \right\rangle_R d\bar{\alpha} \quad (\text{V.D.2})$$

Using equation (IV.D.19) this gives us

$$\begin{aligned}
\frac{d}{dt} \langle \theta(\bar{a}, t) \rangle_{\mathcal{R}} &= \int \dots \int \bar{\theta}^c(\bar{\alpha}, t) \frac{1}{i\hbar} \left\langle \left[\exp \left[-a \frac{\partial}{\partial \alpha} \right], H_0 \right] \right\rangle_{\mathcal{R}} \\
&- \sum_{i,j} \delta_{i,j} \left\langle \left[\exp \left[-a \frac{\partial}{\partial \alpha} \right], Q_i \right] Q_j w_{ij}^+ \right. \\
&\quad \left. - Q_j \left[\exp \left[-a \frac{\partial}{\partial \alpha} \right], Q_i \right] w_{ji}^- \right\rangle_{\mathcal{R}} \bar{\delta}(\alpha) d\bar{\alpha}
\end{aligned} \tag{V.D.3}$$

From definition (V.B.6) we have

$$\frac{d}{dt} \langle \theta(\bar{a}, t) \rangle_{\mathcal{R}} = \int \bar{\theta}^c(\bar{\alpha}, t) \langle X \left[\bar{a}, \frac{\partial}{\partial \bar{\alpha}} \right] \rangle_{\mathcal{R}} \bar{\delta}(\alpha) d\bar{\alpha} \tag{V.D.4}$$

Thus the similarity between equations (III.A.28) and (III.B.7) for the density matrix and a system operator has been used here. Now we convert $X \left[\bar{a}, \frac{\partial}{\partial \bar{\alpha}} \right]$ into chosen order and use equation (V.A.6) again to get

$$X \left[\bar{a}, \frac{\partial}{\partial \bar{\alpha}} \right] = \int \dots \int \bar{X}^c \left[\bar{\beta}, \frac{\partial}{\partial \bar{\alpha}} \right] \bar{\delta}(\beta - a) d\bar{\beta} \tag{V.D.5}$$

Inserting this into (V.D.4) we have

$$\begin{aligned}
\frac{d}{dt} \langle \theta(\bar{a}, t) \rangle &= \int \bar{\theta}^c(\bar{\alpha}, t) \bar{X}^c \left[\bar{\beta}, \frac{\partial}{\partial \bar{\alpha}} \right] \\
&\quad * \langle \bar{\delta}(\beta - a) \rangle_{\mathcal{R}} \langle \bar{\delta}(\alpha - a) \rangle_{\mathcal{R}} d\bar{\beta} d\bar{\alpha}
\end{aligned} \tag{V.D.6}$$

If we use the ansatz (V.B.10) this gives us

$$\frac{d}{dt} \langle \theta(\bar{a}, t) \rangle_{\mathcal{R}} = \int \bar{\theta}^C(\bar{\alpha}, t) \equiv \left[\beta, \frac{\partial}{\partial \bar{\alpha}} \right] \langle \bar{\theta}(\beta - a) \rangle_{\mathcal{R}} \\ * \exp \left[-\beta \frac{\partial}{\partial \alpha} \right] \langle \bar{\theta}(\alpha - a) \rangle_{\mathcal{R}} d\beta d\alpha \quad (\text{V.D.7})$$

If we now use the property (IV.D.19) of the delta functions we have

$$\frac{d}{dt} \langle \theta(\bar{a}, t) \rangle_{\mathcal{R}} = \int \bar{\theta}^C(\bar{\alpha}, t) \equiv \left[\beta, \frac{\partial}{\partial \bar{\alpha}} \right] \langle \bar{\theta}(\beta - a) \rangle_{\mathcal{R}} \\ * \bar{\theta}(\alpha - \beta) d\beta d\alpha \quad (\text{V.D.8})$$

We can easily carry out the β integrals and we get

$$\frac{d}{dt} \langle \theta(\bar{a}, t) \rangle_{\mathcal{R}} = \int \bar{\theta}^C(\bar{\alpha}, t) \equiv \left[\bar{\alpha}, \frac{\partial}{\partial \bar{\alpha}} \right] \langle \bar{\theta}(\alpha - a) \rangle_{\mathcal{R}} d\bar{\alpha} \quad (\text{V.D.9})$$

This then is the equation of motion for the system operator. In order to convert it into a more convenient form we need to assume a specific form for $\equiv \left[\bar{\alpha}, \frac{\partial}{\partial \bar{\alpha}} \right]$. Recall equation (V.B.12), the Fokker-Planck equation for the density matrix. By analogy with (II.C.1) we assume that

$$\equiv = - \frac{\partial}{\partial \bar{\alpha}} A(\bar{\alpha}) + \frac{\partial^2}{\partial \bar{\alpha}_1 \partial \bar{\alpha}_2} B(\bar{\alpha}) \quad (\text{V.D.10})$$

where we made the definitions

$$\sum_i \frac{\partial}{\partial \alpha_i} A_i(\bar{\alpha}) \equiv \frac{\partial}{\partial \bar{\alpha}} A(\bar{\alpha}) \\ \sum_{i,j} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} B_{ij}(\bar{\alpha}) \equiv \frac{\partial^2}{\partial \bar{\alpha}_1 \partial \bar{\alpha}_2} B(\bar{\alpha})$$

(V.D.11)

We insert this into (V.D.9) and integrate by parts

$$\frac{d}{dt} \langle \theta(\bar{a}, t) \rangle_{\mathcal{R}} = \int \bar{\theta}^C(\bar{x}, t) \left[-\frac{\partial}{\partial \bar{x}} A(\bar{x}) + \frac{\partial^2}{\partial \bar{x}_1 \partial \bar{x}_2} B(\bar{x}) \right] \langle \bar{\tau}(\alpha - a) \rangle_{\mathcal{R}} d\bar{x} \quad (\text{V.D.12})$$

$$= \int \dots \int A(\bar{x}) \langle \bar{\tau}(\alpha - a) \rangle_{\mathcal{R}} \frac{\partial}{\partial \bar{x}} \bar{\theta}^C(\bar{x}, t) d\bar{x} - \int \dots \int \frac{\partial}{\partial \bar{x}_1} \left[B(\bar{x}) \langle \bar{\tau}(\alpha - a) \rangle_{\mathcal{R}} \right] \frac{\partial}{\partial \bar{x}_2} \left[\bar{\theta}^C(\bar{x}, t) \right] d\bar{x} \quad (\text{V.D.13})$$

We used the fact that the delta functions vanish at the limits of integration. We integrate the second part of the right hand side by parts again and obtain

$$\frac{d}{dt} \langle \theta(\bar{a}, t) \rangle_{\mathcal{R}} = \int \dots \int \left[A(\bar{x}) \frac{\partial}{\partial \bar{x}} \left[\bar{\theta}^C(\bar{x}, t) \right] + B(\bar{x}) \frac{\partial^2}{\partial \bar{x}_1 \partial \bar{x}_2} \left[\bar{\theta}^C(\bar{x}, t) \right] \right] \langle \bar{\tau}(\alpha - a) \rangle_{\mathcal{R}} d\bar{x} \quad (\text{V.D.14})$$

If we now apply equation (V.A.6) to the left hand side of this equation

$$\begin{aligned}
& \int \dots \int \left\langle \frac{d}{dt} \theta(\alpha, t) \right\rangle_{\mathcal{R}} d\alpha \\
&= \int \dots \int \left\langle \left(A(\alpha) \frac{\partial}{\partial \alpha} \left[\bar{\theta}^C(\alpha, t) \right] \right. \right. \\
&+ \left. \left. B(\alpha) \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} \left[\bar{\theta}^C(\alpha, t) \right] \right) \right\rangle_{\mathcal{R}} \bar{\theta}(\alpha - a) d\alpha
\end{aligned} \tag{V.D.15}$$

We can equate the arguments of the integral and obtain

$$\begin{aligned}
\frac{d}{dt} \langle \theta(\alpha, t) \rangle_{\mathcal{R}} &= \left\langle \sum_i A_i(\alpha) \frac{\partial}{\partial \alpha_i} \left[\bar{\theta}^C(\alpha, t) \right] \right. \\
&+ \left. \sum_{ij} B_{ij}(\alpha) \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} \left[\bar{\theta}^C(\alpha, t) \right] \right\rangle_{\mathcal{R}}
\end{aligned} \tag{V.D.16}$$

This is the equation of motion for the system operator which we wished to derive. Note its close relation to the Fokker-Planck equation

$$\begin{aligned}
\frac{\partial}{\partial t} P(\alpha, t) &= - \sum_i \frac{\partial}{\partial \alpha_i} \left\{ A_i(\alpha) P(\alpha, t) \right\} \\
&+ \sum_{ij} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} \left\{ B_{ij}(\alpha) P(\alpha, t) \right\}
\end{aligned} \tag{V.D.17}$$

V.E The Langevin Equations

In this section using equation (V.D.16) , we will develop the Langevin equations for the system operators in a manner similar to that of chapter II. Let

$$\frac{d}{dt} \alpha_{\mu} = A_{\mu}(\alpha) \quad \text{where } \alpha_{\mu} \in \{ \alpha \} \quad (\text{V.E.1})$$

Then by equation (V.D.16) we have

$$\frac{d}{dt} \langle \alpha_{\mu} \rangle_{\mathcal{R}} = \langle A_{\mu}(\alpha) \rangle_{\mathcal{R}} \quad (\text{V.E.2})$$

We wish to remove the reservoir averages. We can therefore add a fluctuating random Langevin force on the right hand side of equation (V.E.2)

$$\frac{d}{dt} \alpha_{\mu} = A_{\mu}(\alpha) + G_{\mu}(t) \quad (\text{V.E.3})$$

with property

$$\langle G_{\mu}(t) \rangle_{\mathcal{R}} = 0 \quad (\text{V.E.4})$$

The equations (V.E.3) and (V.E.4) are then entirely equivalent to equation (V.E.2). We may apply the chosen ordering operator C to both sides of (V.E.3) and obtain

$$\frac{d}{dt} a_{\mu} = A_{\mu}(\bar{a}) + \hat{G}_{\mu}(t) \quad (\text{V.E.5})$$

where $A_{\mu}(\bar{a})$ is in chosen order. As in chapter II, the important property

of the Langevin force that we need to obtain is its correlation function. From equation (V.D.16) we obtain the following equation of motion for $\alpha_\mu \alpha_\nu$

$$\frac{d}{dt} \langle \alpha_\mu \alpha_\nu \rangle_R = \langle A_\mu(\bar{\alpha}) \alpha_\nu \rangle_R + \langle \alpha_\mu A_\nu(\bar{\alpha}) \rangle_R + \langle B_{\mu\nu}(\bar{\alpha}) \rangle_R \quad (\text{V.E.6})$$

We have to find the relationship between this equation and equation (V.E.3). If we "smooth out" fluctuations in (V.E.3) which are smaller than the coherence time, we get

$$\begin{aligned} \frac{\Delta \alpha_\mu}{\Delta t} &= A_\mu + \frac{1}{\Delta t} \int_t^{t+\Delta t} G_\mu(s) ds \\ \frac{\Delta \alpha_\nu}{\Delta t} &= A_\nu + \frac{1}{\Delta t} \int_t^{t+\Delta t} G_\nu(s') ds' \end{aligned} \quad (\text{V.E.7})$$

where $\Delta t > \tau^*$. But we have

$$\begin{aligned} \frac{\Delta \alpha_\mu \alpha_\nu}{\Delta t} &= \frac{(\alpha_\mu + \Delta \alpha_\mu)(\alpha_\nu + \Delta \alpha_\nu) - \alpha_\mu \alpha_\nu}{\Delta t} \\ &= \frac{\Delta \alpha_\mu}{\Delta t} \alpha_\nu + \alpha_\mu \frac{\Delta \alpha_\nu}{\Delta t} + \frac{\Delta \alpha_\mu \Delta \alpha_\nu}{\Delta t} \end{aligned} \quad (\text{V.E.8})$$

Using equation (V.E.7) and taking the limit as $\Delta t \rightarrow 0$ we obtain

$$\begin{aligned} \frac{d}{dt} \langle \alpha_\mu \alpha_\nu \rangle_R &= \langle A_\mu(\bar{\alpha}) \alpha_\nu \rangle_R + \langle \alpha_\mu A_\nu(\bar{\alpha}) \rangle_R \\ &+ \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} ds \int_t^{t+\Delta t} ds' \langle G_\mu(s) G_\nu(s') \rangle \end{aligned} \quad (\text{V.E.9})$$

Using equation (V.E.6) this gives us immediately

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} ds \int_t^{t+\Delta t} ds' \langle G_\mu(s) G_\nu(s') \rangle_R = \langle B_{\mu\nu}(\bar{\alpha}) \rangle_R \quad (\text{V.E.10})$$

In the limit of zero coherence time this is equivalent to

$$\langle G_\mu(s) G_\nu(s') \rangle_R = \langle B_{\mu\nu}(\bar{\alpha}) \rangle_R \delta(s - s') \quad (\text{V.E.11})$$

If $\mu < \nu$, we may apply the chosen ordering operator and obtain

$$\langle \hat{G}_\mu(s) \hat{G}_\nu(s') \rangle_R = \langle B_{\mu\nu}(\bar{a}) \rangle_R \delta(s - s') \quad (\text{V.E.12})$$

This is the required correlation function. We recapitulate the important equations derived in this chapter to stress the intimate connection between the Langevin and Fokker-Planck approaches

Fokker-Planck equation

$$\begin{aligned} \frac{\partial}{\partial t} P(\bar{\alpha}, t) &= - \sum_i \frac{\partial}{\partial \alpha_i} \left\{ A_i(\bar{\alpha}) P(\bar{\alpha}, t) \right\} \\ &+ \sum_{ij} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} \left\{ B_{ij}(\bar{\alpha}) P(\bar{\alpha}, t) \right\} \end{aligned} \quad (\text{V.E.13})$$

The Langevin equations

$$\frac{d}{dt} a_\mu = A_\mu(\bar{a}) + \hat{G}_\mu(t) \quad (\text{V.E.14})$$

$$\text{where } \langle \hat{G}_\mu(t) \rangle_R = 0 \quad (\text{V.E.15})$$

$$\langle \hat{G}_\mu(s) \hat{G}_\nu(s') \rangle_R = \langle B_{\mu\nu}(\bar{a}) \rangle_R \delta(s-s') \quad (\text{V.E.16})$$

Notice the similarity of these equations to equations (II.C.1) and (II.C.2). We will derive the Fokker-Planck equation for the system under consideration and then use the above equations to derive the Langevin equations.

In the case of the harmonic oscillator we obtain from equations (V.C.8), (V.E.14), (V.E.15) and (V.E.16)

$$\begin{aligned} \frac{da}{dt} &= -i(\omega - \Delta\omega)a - \gamma a + G_a \\ \frac{da^\dagger}{dt} &= -i(\omega - \Delta\omega)a^\dagger - \gamma a^\dagger + G_{a^\dagger} \end{aligned} \quad (\text{V.E.17})$$

where

$$\langle G_{a^\dagger}(s) G_a(s') \rangle = 2\gamma \bar{n} \delta(s-s') \quad (\text{V.E.18})$$

Let $\omega - \Delta\omega \equiv \bar{\omega}$. Then

$$\begin{aligned} a(t) &= a(0) \exp\{-i\bar{\omega}t - \gamma t\} \\ &\quad + \int_0^t G_a(\tau) \exp\{-(\gamma + i\bar{\omega})(t-\tau)\} d\tau \\ a^\dagger(t) &= a^\dagger(0) \exp\{i\bar{\omega}t - \gamma t\} \\ &\quad + \int_0^t G_{a^\dagger}(\tau) \exp\{-(\gamma - i\bar{\omega})(t-\tau)\} d\tau \end{aligned}$$

(V.E.19)

So

$$\frac{d}{dt} \langle a^\dagger a(t) \rangle = a^\dagger(t) \frac{da}{dt} + \frac{da^\dagger}{dt} \quad (\text{V.E.20})$$

Using equation (V.E.17)

$$\begin{aligned} \frac{d}{dt} \langle a^\dagger a(t) \rangle &= -2\gamma \langle a^\dagger a(t) \rangle \\ &+ \langle a^\dagger(t) G_a(t) + G_{a^\dagger}(t) a(t) \rangle \end{aligned} \quad (\text{V.E.21})$$

Using (V.E.19) we have

$$\begin{aligned} \langle a^\dagger(t) G_a(t) \rangle &= \int_0^t \langle G_{a^\dagger}(\tau) G_a(t) \rangle \exp \left\{ -(\gamma - i\bar{\omega})(t - \tau) \right\} d\tau \\ &= 2\gamma \bar{n} \int_0^t \delta(t - \tau) \exp \left\{ -(\gamma - i\bar{\omega})(t - \tau) \right\} d\tau \\ &= \gamma \bar{n} \end{aligned} \quad (\text{V.E.22})$$

because only "half" of the delta function is considered (the correlation function is an even function of time).

$$\frac{d}{dt} \langle a^\dagger a(t) \rangle = -2\gamma \left[\langle a^\dagger a(t) \rangle - \bar{n} \right] \quad (\text{V.E.23})$$

which agrees with equation (IV.B.27)

Bibliography for Chapter V

The distribution function was introduced by Lax (1968). The conversion of the distribution function into the Fokker-Planck equation was achieved by Lax (1968) and Marburger and Louisell (1969). The techniques of conversion are illustrated in Louisell (1973). Another example may be found in Lax and Yeun (1968).

VI. THE ELECTROMAGNETIC INTERACTION

In this chapter we make a diversion and turn our attention to the quantization of the electromagnetic field. In section VI.A we will present the standard method of quantizing the free-space electromagnetic field. In section VI.B the interaction of a two-level atom with the electromagnetic field will be studied. The interaction will be represented in terms of the creation and destruction operators and the "R" operators. The Hamiltonian of this interaction shall be solved in section VI.C

VI.A Quantization of the Electromagnetic Field

We study the electromagnetic field in free space in the absence of any space charges or currents. Maxwell's homogeneous equations can be satisfied by a vector potential \vec{A} by

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

(VI.A.1)

The inhomogeneous equations then give

$$\vec{\nabla} \cdot \vec{A} = 0$$

$$\vec{\nabla}^2 \vec{A} = \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} \quad (\text{VI.A.2})$$

We can express \vec{A} as a sum of standing waves

$$\vec{A} = \sqrt{4\pi c^2} \sum_{\lambda} q_{\lambda}(t) \vec{u}_{\lambda}(\vec{r}) \quad (\text{VI.A.3})$$

So

$$\int_V \vec{u}_{\lambda}(\vec{r}) \cdot \vec{u}_{\mu}(\vec{r}) dV = \delta_{\lambda\mu} \quad (\text{VI.A.4})$$

where V is the volume of the electromagnetic cavity under consideration. Here the $\vec{u}_{\lambda}(\vec{r})$ form a complete orthonormal set, and hence any function $\vec{A}(\vec{r}, t)$ can be expanded in a series of the $\vec{u}_{\lambda}(\vec{r})$. We impose periodic boundary conditions on $\vec{u}_{\lambda}(\vec{r})$ and require that the tangential component of $\vec{u}_{\lambda}(\vec{r})$ shall vanish on the walls of the cavity

$$\vec{u}_{\lambda}(\vec{r}) |_{\text{tan}} = 0 \quad (\text{VI.A.5})$$

from which it follows that

$$\vec{\nabla} \times \vec{u}_{\lambda}(\vec{r}) |_{\text{norm}} = 0 \quad (\text{VI.A.6})$$

After inserting (VI.A.3) into (VI.A.2) we get

$$\vec{\nabla} \cdot \vec{u}_{\lambda}(\vec{r}) = 0$$

(VI.A.7)

$$\frac{d^2 q_\lambda(t)}{dt^2} + \omega_\lambda^2 q_\lambda(t) = 0 \quad (\text{VI.A.8})$$

and

$$\nabla^2 \vec{u}_\lambda(\vec{r}) + \frac{\omega_\lambda^2}{c^2} \vec{u}_\lambda(\vec{r}) = 0 \quad (\text{VI.A.9})$$

where ω_λ is a separation parameter which is used to separate the independent functions $\vec{u}_\lambda(\vec{r})$ and $q_\lambda(t)$. From equation (VI.A.8) we identify ω_λ to be the frequency of the standing wave mode.

The classical Hamiltonian for the electromagnetic field is given by

$$\begin{aligned} H &= \frac{1}{8\pi} \int dV (\vec{E}^2 + \vec{B}^2) \\ &= \frac{1}{8\pi} \int dV \left[\frac{1}{c^2} \left(\frac{\partial \vec{A}}{\partial t} \right)^2 + (\nabla \times \vec{A})^2 \right] \end{aligned} \quad (\text{VI.A.10})$$

Using the standing wave expansion this becomes

$$\begin{aligned} &= \frac{1}{2} \int dV \sum_{\lambda, \mu} \dot{q}_\lambda(t) \dot{q}_\mu(t) \vec{u}_\lambda(\vec{r}) \vec{u}_\mu(\vec{r}) \\ &\quad + \frac{c^2}{2} \int dV \sum_{\lambda, \mu} q_\lambda(t) q_\mu(t) (\nabla \times \vec{u}_\lambda(\vec{r})) \cdot (\nabla \times \vec{u}_\mu(\vec{r})) \end{aligned} \quad (\text{VI.A.11})$$

The first term, using (VI.A.4) collapses to

$$\frac{1}{2} \sum_{\lambda} \dot{q}_\lambda^2$$

while for the second term we use the relation

$$\begin{aligned} & \left[\vec{\nabla} \times \vec{u}_\lambda(\vec{r}) \right] \cdot \left[\vec{\nabla} \times \vec{u}_\mu(\vec{r}) \right] \\ &= \vec{u}_\mu(\vec{r}) \cdot \left[\vec{\nabla} \times \left[\vec{\nabla} \times \vec{u}_\lambda(\vec{r}) \right] \right] + \vec{\nabla} \cdot \left[\vec{u}_\mu(\vec{r}) \times \left[\vec{\nabla} \times \vec{u}_\lambda(\vec{r}) \right] \right] \end{aligned} \quad (\text{VI.A.12})$$

The second term on the right hand side of (VI.A.12) can be converted to a surface integral by Gauss' Theorem and vanishes by (VI.A.5) and (VI.A.6). The first term gives

$$\begin{aligned} & \vec{u}_\mu(\vec{r}) \cdot \left[\vec{\nabla} \left[\vec{\nabla} \cdot \vec{u}_\lambda(\vec{r}) \right] - \vec{\nabla}^2 \vec{u}_\lambda(\vec{r}) \right] \\ &= \frac{\omega_\lambda^2}{c^2} \vec{u}_\mu(\vec{r}) \cdot \vec{u}_\lambda(\vec{r}) \end{aligned} \quad (\text{VI.A.13})$$

where we have used equations (VI.A.9) and (VI.A.7). Using the orthonormality relations, (VI.A.11) now simplifies to

$$H = \frac{1}{2} \sum_\lambda \left\{ \dot{q}_\lambda^2(t) + \omega_\lambda^2 q_\lambda^2(t) \right\} \quad (\text{VI.A.14})$$

If we define

$$p_\lambda(t) = \dot{q}_\lambda(t)$$

then

$$H(p_\lambda(t), q_\lambda(t)) = \frac{1}{2} \sum_\lambda \left[p_\lambda^2 + \omega_\lambda^2 q_\lambda^2 \right] \quad (\text{VI.A.15})$$

and the Hamilton equations

$$\frac{\partial H}{\partial p_\lambda(t)_\lambda} = \dot{q}_\lambda(t) \quad ; \quad \frac{\partial H}{\partial q_\lambda(t)_\lambda} = -\dot{p}_\lambda$$

merely give us (VI.A.8).

To quantize this Hamiltonian we postulate the commutation relations

$$[p_\lambda(t), q_\mu(t)] = i\hbar \delta_{\lambda\mu} \quad (\text{VI.A.16})$$

If we define

$$a_\lambda(t) = \frac{1}{\sqrt{2\hbar\omega_\lambda}} \left[\omega_\lambda q_\lambda(t) + ip_\lambda(t) \right] \quad (\text{VI.A.17})$$

$$a_\lambda^\dagger(t) = \frac{1}{\sqrt{2\hbar\omega_\lambda}} \left[\omega_\lambda q_\lambda(t) - ip_\lambda(t) \right] \quad (\text{VI.A.18})$$

we get immediately

$$H = \sum_\lambda \hbar\omega_\lambda \left[a_\lambda^\dagger(t) a_\lambda(t) + \frac{1}{2} \right] \quad (\text{VI.A.19})$$

In terms of the creation and destruction operators, the electric and magnetic field operators are given by

$$\begin{aligned}
 \vec{A} &= \sum_{\lambda} \left[\frac{2\pi\hbar c^2}{\omega_{\lambda}} \right]^{\frac{1}{2}} \left\{ a_{\lambda}(t) + a_{\lambda}^{\dagger}(t) \right\} \vec{u}_{\lambda}(\vec{r}) \\
 \vec{E} &= \sum_{\lambda} i \left[2\pi\hbar\omega_{\lambda} \right]^{\frac{1}{2}} \left\{ a_{\lambda}(t) - a_{\lambda}^{\dagger}(t) \right\} \vec{u}_{\lambda}(\vec{r}) \\
 \vec{B} &= \sum_{\lambda} \left[\frac{2\pi\hbar c^2}{\omega_{\lambda}} \right]^{\frac{1}{2}} \left\{ a_{\lambda}(t) + a_{\lambda}^{\dagger}(t) \right\} \left[\vec{\nabla} \times \vec{u}_{\lambda}(\vec{r}) \right]
 \end{aligned}
 \tag{VI.A.20}$$

The eigenstates of the field are the eigenstates of the Hamiltonian (6.19) and can therefore be represented by a set of occupation numbers $\{n_{\lambda}\}$.

$$|\Psi_E\rangle = |n_0, n_1, n_2, \dots\rangle$$

$$E = \sum_{\lambda} \hbar\omega_{\lambda} n_{\lambda}$$

(VI.A.21)

VI.B Interaction of an Atom with the Electromagnetic Field

The Rydberg electron in an alkali atom can be described by the Hamiltonian

$$H = \frac{\vec{p}^2}{2m} + V(\vec{r}) \quad (\text{VI.B.1})$$

where $V(\vec{r})$ is the potential due to the core and the nucleus, and is approximately coulombic. Any state of the electron can be described by a sum from the complete set of energy eigenstates $|E_i\rangle$. If the electron is in the presence of an electromagnetic field in a cavity, the total Hamiltonian for the field-electron system is

$$H = \frac{1}{2m} \left\{ \vec{p} - \frac{e\vec{A}}{c} \right\}^2 + V(\vec{r}) + \frac{1}{8\pi} \int dV \left[\frac{1}{c^2} \left\{ \frac{\partial \vec{A}}{\partial t} \right\}^2 + \left\{ \vec{\nabla} \times \vec{A} \right\}^2 \right] \quad (\text{VI.B.2})$$

Now the eigenstates of the field, given by (VI.A.21), and the energy eigenstates $|E_i\rangle$ of the electron form a complete set of states in terms of which we can describe any state of the atom-cavity system. Under the conditions described in chapter I : with the cavity near cut-off and two energy eigenstates of the electron separated by a frequency close to the cavity frequency, only the two levels of the atom and the fundamental mode of the cavity are important. We denote these two levels of the atom by $|+\rangle$ and $|-\rangle$, referring to the higher and lower states respectively.

We also drop the subscript from the cavity operators, since we are referring to only one mode of the cavity. Then the hamiltonian H can be written as

$$H = |+\rangle\langle+| M |+\rangle\langle+| + |+\rangle\langle+| M |-\rangle\langle-| \\ + |-\rangle\langle-| M |+\rangle\langle+| + |-\rangle\langle-| M |-\rangle\langle-| + \hbar\omega \left\{ a^\dagger a + \frac{1}{2} \right\} \quad (\text{VI.B.3})$$

$$\text{where } M = \frac{1}{2m} \left(\vec{p} - \frac{e}{c} \vec{A} \right)^2 + V(\vec{r}) \quad (\text{VI.B.4})$$

But in the radiation gauge ($\vec{\nabla} \cdot \vec{A} = 0$)

$$M = \frac{\vec{p}^2}{2m} + V(\vec{r}) - \frac{e}{mc} \vec{p} \cdot \vec{A} + \frac{e^2}{2mc^2} \vec{A}^2 \quad (\text{VI.B.5})$$

Also because the $|+\rangle$ and $|-\rangle$ states are energy eigenstates of the isolated atom

$$\langle+| \frac{\vec{p}^2}{2m} + V(\vec{r}) |+\rangle = E_+ \\ \langle-| \frac{\vec{p}^2}{2m} + V(\vec{r}) |-\rangle = E_- \quad (\text{VI.B.6})$$

Also

$$\left[\frac{\vec{p}^2}{2m} + V(\vec{r}), \vec{r} \right] = -i\hbar \frac{\vec{p}}{m} \quad (\text{VI.B.7})$$

So

$$\begin{aligned}
& \langle + | -\frac{e}{mc} \vec{p} \cdot \vec{A} | + \rangle \\
&= -\frac{ie}{\hbar c} \left[\langle + | \left[\frac{\vec{p}^2}{2m} + V(\vec{r}) \right] \vec{r} - \vec{r} \left[\frac{\vec{p}^2}{2m} + V(\vec{r}) \right] | + \rangle \right] \cdot \vec{A} \\
&= -\frac{ie}{\hbar c} \left[E_+ \langle + | \vec{r} | + \rangle - E_+ \langle + | \vec{r} | + \rangle \right] \cdot \vec{A} \\
&= 0
\end{aligned} \tag{VI.B.8}$$

$$\text{Similarly } \langle - | -\frac{e}{mc} \vec{p} \cdot \vec{A} | - \rangle = 0$$

and

$$\begin{aligned}
\langle + | -\frac{e}{mc} \vec{p} \cdot \vec{A} | - \rangle &= \langle - | -\frac{e}{mc} \vec{p} \cdot \vec{A} | + \rangle^* \\
&= -\frac{ie}{\hbar c} \left[\langle + | \left[\frac{\vec{p}^2}{2m} + V(\vec{r}) \right] \vec{r} - \vec{r} \left[\frac{\vec{p}^2}{2m} + V(\vec{r}) \right] | - \rangle \right] \cdot \vec{A} \\
&= -\frac{ie}{\hbar c} \left[E_+ - E_- \right] \langle + | \vec{r} | - \rangle \cdot \vec{A}
\end{aligned} \tag{VI.B.9}$$

We set

$$\vec{d} = \langle + | \vec{r} | - \rangle$$

the dipole moment of the atom. The $\frac{e^2}{2mc^2} \vec{A}^2$ term clearly has no off-diagonal matrix elements. It has small diagonal terms which introduce an energy shift and can be absorbed into the values of E_+ and E_- . Equation

(VI.B.3) can now be written

$$\begin{aligned}
 H = & |+\rangle E_+ \langle +| - |-\rangle E_- \langle -| + \hbar\omega \left\{ a^\dagger a + \frac{1}{2} \right\} \\
 & + \frac{ie}{\hbar c} (E_+ - E_-) \left\{ \vec{d}^* \cdot \vec{A} |-\rangle \langle +| - \vec{d} \cdot \vec{A} |+\rangle \langle -| \right\} \quad (\text{VI.B.10})
 \end{aligned}$$

If we define $E_+ - E_- = \hbar\omega_a$, (VI.B.10) becomes

$$\begin{aligned}
 H = & \frac{E_+ + E_-}{2} (|+\rangle \langle +| + |-\rangle \langle -|) + \frac{\hbar\omega_a}{2} (|+\rangle \langle +| - |-\rangle \langle -|) \\
 & - \frac{ie\omega_a}{c} (|+\rangle \langle -| - |-\rangle \langle +|) \vec{d} \cdot \vec{A} + \hbar\omega \left\{ a^\dagger a + \frac{1}{2} \right\} \quad (\text{VI.B.11})
 \end{aligned}$$

We have assumed above that \vec{d} is real. This does not alter any physical features but simplifies the algebra. We note

$$|+\rangle \langle +| + |-\rangle \langle -| = 1$$

and drop all the constant terms in H (by changing the zero of energy) to obtain

$$\begin{aligned}
 H = & \frac{\hbar\omega}{2} (|+\rangle \langle +| - |-\rangle \langle -|) \\
 & - \frac{ie\omega_a}{c} (|+\rangle \langle -| - |-\rangle \langle +|) \vec{d} \cdot \vec{A} + \hbar\omega a^\dagger a \quad (\text{VI.B.12})
 \end{aligned}$$

If we now calculate the matrix representation of the operators $(|+\rangle \langle +| - |-\rangle \langle -|)$, $(|+\rangle \langle +|)$, $(|-\rangle \langle -|)$ in the basis with the elements $|+\rangle$ and $|-\rangle$ we see from definition (IV.A.7) that

$$\frac{1}{2} (|+\rangle\langle +| - |-\rangle\langle -|) = R_3 \quad (\text{VI.B.13})$$

$$|+\rangle\langle -| = R_+ \quad (\text{VI.B.14})$$

$$|-\rangle\langle +| = R_- \quad (\text{VI.B.15})$$

Therefore

$$H = h\omega_a R_3 + h\omega_a^\dagger a - \frac{ie\omega_a}{c} \{ R_+ - R_- \} \vec{d} \cdot \vec{A} \quad (\text{VI.B.16})$$

$$H = h\omega_a R_3 + h\omega_a^\dagger a - ie\omega_a \sqrt{\frac{2\pi\hbar}{\omega}} \vec{d} \cdot \vec{u} \left\{ R_+ a + R_+ a^\dagger - a R_- - a^\dagger R_- \right\} \quad (\text{VI.B.17})$$

where we have used equation (VI.A.20). Define

$$\lambda = -\frac{2e\omega_a}{\hbar} \sqrt{\frac{2\pi\hbar}{\omega}} \vec{d} \cdot \vec{u} \quad (\text{VI.B.18})$$

So

$$H = h\omega_a R_3 + h\omega_a^\dagger a + \frac{i\hbar\lambda}{2} \left\{ R_+ a + R_+ a^\dagger - a R_- - a^\dagger R_- \right\} \quad (\text{VI.B.19})$$

If there was no interaction between the field and atom, we would have

$$R_+(t) = R_+(0) \exp \left\{ i\omega_a t \right\}$$

$$R_-(t) = R_-(0) \exp \left\{ -i\omega_a t \right\}$$
(VI.B.20)

and

$$a^\dagger(t) = a^\dagger(0) \exp(i\omega t)$$

$$a(t) = a(0) \exp(-i\omega t)$$
(VI.B.21)

Therefore to zeroth order, the $(R_+ a - a^\dagger R_-)$ term will oscillates at frequency $\omega - \omega_a$, while the $(R_+ a^\dagger + a R_-)$ term will oscillates at a frequency $\omega + \omega_a$. Since the detuning is assumed to be small

$$\omega + \omega_a \gg \omega - \omega_a$$

Thus we can neglect the $(R_+ a^\dagger + a R_-)$ term because it oscillates rapidly and averages to zero. This is the Rotating Wave Approximation.

Finally, we have

$$H = h\omega_a R_3 + h\omega a^\dagger a + \frac{i\hbar\lambda}{2} \left\{ R_+ a - a^\dagger R_- \right\}$$
(VI.B.22)

This is the Hamiltonian that we shall use to represent the atom-field interaction. Notice the similarity of this Hamiltonian to (IV.A.6). This is the justification for (IV.A.6) representing the interaction of one mode of the radiation field with a large number of two-level atoms.

VI.C Solution of the Heisenberg Equations

The Hamiltonian (VI.B.22) can be solved exactly. This is despite the fact that the equations of motion are non-linear. We recall

$$\frac{d\theta}{dt} = \frac{1}{i\hbar} [\theta, H] \quad (\text{VI.C.1})$$

which gives us

$$i \frac{dR_3}{dt} = \frac{i\lambda}{2} \left(R_+ a + a^\dagger R_- \right) \quad (\text{VI.C.2})$$

$$i \frac{dR_+}{dt} = -\omega_a R_+ - i\lambda a^\dagger R_3 \quad (\text{VI.C.3})$$

$$i \frac{dR_-}{dt} = \omega_a R_- - i\lambda R_3 a \quad (\text{VI.C.4})$$

$$i \frac{da}{dt} = \omega_a - \frac{i\lambda}{2} R_- \quad (\text{VI.C.5})$$

$$i \frac{da^\dagger}{dt} = -\omega_a^\dagger - \frac{i\lambda}{2} R_+ \quad (\text{VI.C.6})$$

We define



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$$\langle R_3(t) \rangle = \frac{1}{2} - \frac{\lambda^2(n+1)}{\Gamma^2} \sin^2 \frac{\Gamma}{2} t \quad (\text{VI.C.10})$$

$$\langle a^\dagger a(t) \rangle = n + \frac{\lambda^2(n+1)}{\Gamma^2} \cos^2 \frac{\Gamma}{2} t \quad (\text{VI.C.11})$$

where we have defined

$$\Gamma \equiv \sqrt{(\omega_a - \omega)^2 + \lambda^2(n+1)} \quad (\text{VI.C.12})$$

Therefore the energy oscillates between the atoms and the field with the Rabi frequency Γ .

The rate of loss of energy from the atom is proportional to $\frac{dR_3}{dt}$.

For small t , we obtain from (VI.C.10)

$$\frac{dR_3}{dt} = -\frac{\lambda^2}{2}(n+1)t \quad (\text{VI.C.13})$$

We can separate this into two parts - the part proportional to n , which is the rate of stimulated emission, and the the part independent of n , which is the spontaneous emission rate.

Bibliography for Chapter VI

The treatment in sections VI.A and VI.B may be found in most quantum mechanics text books, e.g. Heitler (1960). The operator method of treating the interaction was introduced by Dicke (1954). The interaction Hamiltonian was first solved by Jaynes and Cummings (1963). The importance of using the Heisenberg picture in Q.E.D. was pointed out by Dirac

(1965). The phenomena of spontaneous emission and Lamb shift can be treated simply in this picture as has been done by Ackerhalt, Knight and Eberly (1972), and Ackerhalt and Eberly (1974). A discussion may be found in Allen and Eberly (1975).

VII. Two Coupled Damped Harmonic Oscillators

In the previous chapters we have recapitulated all the formal theory required for the purpose of studying the physical situation we are interested in. We want to solve the system described by the Hamiltonian (VI.B.22) with a damping mechanism coupled onto the field mode. However as we have noted, the equations of motion are non-linear. When damping is added onto the system the equations cannot be solved in a simple form. To examine the general features of the problem, we examine in this chapter a simplified problem. Instead of coupling the harmonic oscillator to a two-level atom and then coupling the harmonic oscillator to a thermal reservoir, we will consider two coupled harmonic oscillators and then couple one of them to some damping. Therefore we replace the Hamiltonian (VI.B.22), by the Hamiltonian

$$H = \hbar\omega_a a^\dagger a + \hbar\omega_b b^\dagger b + \hbar\kappa \left(a^\dagger b + b^\dagger a \right) \quad (\text{VII.A.1})$$

Thus we are considering two harmonic oscillators with frequencies ω_a and ω_b and an interaction $\hbar\kappa \left(a^\dagger b + b^\dagger a \right)$.

In section VII.A we will solve this Hamiltonian without any damping. The

Fokker-Planck equation for the system, after a damping has been added, will be derived in section VII.B. We will then use the Langevin equations to derive coupled equations for the energy variables, after averaging out the Langevin forces in section VII.C. Sections VII.D and VII.E contain solutions of the equations for small and large detuning respectively. We present some computer generated solutions in section VII.F

VII.A Solution of the Undamped Hamiltonian

Using the relation

$$\frac{d\theta}{dt} = \frac{1}{i\hbar} [\theta, H]$$

we can derive following equations of motion for the operators of the system

$$\frac{da^\dagger}{dt} = i\omega_a a^\dagger + i\kappa b^\dagger \quad (\text{VII.A.2})$$

$$\frac{da}{dt} = -i\omega_a a - i\kappa b \quad (\text{VII.A.3})$$

$$\frac{db^\dagger}{dt} = i\omega_b b^\dagger + i\kappa a^\dagger \quad (\text{VII.A.4})$$

$$\frac{db}{dt} = -i\omega_b b - i\kappa a \quad (\text{VII.A.5})$$

Unlike equations (VI.C.2-VI.C.6) these equations are linear, and can

therefore be easily solved. As before we have the constants of motion

$$N = a^\dagger a + b^\dagger b \quad (\text{VII.A.6})$$

and

$$\Omega = \omega_a a^\dagger a + \omega_b b^\dagger b + \kappa \left(a^\dagger b + b^\dagger a \right) \quad (\text{VII.A.7})$$

N is the number of photons in the system. Its conservation indicates that one oscillator can be excited to a higher state only if the other oscillator decays to a lower state. It is necessary for the Hamiltonian to only have terms containing an equal number of creation and destruction operators for N to be conserved. Ω is the total energy of the system and must obviously be a constant of the motion. Since we are interested in the energy transfer between the two systems, we have

$$\begin{aligned} \frac{d \left(a^\dagger a \right)}{dt} &= a^\dagger \frac{da}{dt} + \frac{da^\dagger}{dt} a \\ &= -i\kappa \left(a^\dagger b - b^\dagger a \right) \\ &= - \frac{d \left(b^\dagger b \right)}{dt} \end{aligned} \quad (\text{VII.A.8})$$

Differentiating (VII.A.8) again

$$\begin{aligned} \frac{d^2 \langle a^\dagger a \rangle}{dt^2} + \left[(\omega_a - \omega_b)^2 + 4\kappa^2 \right] \langle a^\dagger a \rangle \\ = \left[\omega_a - \omega_b \right] \langle \Omega \rangle - \left[\omega_b (\omega_a - \omega_b) - 2\kappa^2 \right] \langle N \rangle \end{aligned} \quad (\text{VII.A.9})$$

At $t=0$, let us assume that there are n_a photons in the ω_a oscillator, and n_b photons in the ω_b oscillator.

$$\langle n_a, n_b | a^\dagger a | n_a, n_b \rangle = n_a \quad (\text{VII.A.10})$$

$$\begin{aligned} \langle n_a, n_b | \frac{d \langle a^\dagger a \rangle}{dt} | n_a, n_b \rangle &= \langle n_a, n_b | i\kappa (a^\dagger b - b^\dagger a) | n_a, n_b \rangle \\ &= 0 \end{aligned} \quad (\text{VII.A.11})$$

$$\langle a^\dagger a(0) \rangle = n_a$$

$$\left\langle \frac{d \langle a^\dagger a \rangle}{dt} (0) \right\rangle = 0$$

With these initial conditions we can find the solution to equation (VII.A.9). We note that

$$\begin{aligned} \langle \Omega \rangle &= n_a \omega_a + n_b \omega_b \\ \langle N \rangle &= n_a + n_b \end{aligned} \quad (\text{VII.A.12})$$

So

$$\langle a^\dagger a(t) \rangle = n_a - \frac{4\kappa^2 \{ n_a - n_b \}}{\Gamma^2} \sin^2 \frac{\Gamma}{2} t \quad (\text{VII.A.13})$$

where we have defined Γ by

$$\Gamma \equiv \sqrt{\{ \omega_a - \omega_b \}^2 + 4\kappa^2} \quad (\text{VII.A.14})$$

Therefore the energy oscillates between the two oscillators with the Rabi frequency Γ . For small t we have

$$\frac{d \langle a^\dagger a(t) \rangle}{dt} = -2\kappa^2 \{ n_a - n_b \} t \quad (\text{VII.A.15})$$

i.e. the rate of loss of energy is proportional to the square of the coupling constant and the difference in the number of photons.

VII.B The Fokker-Planck Equation

We now couple the ω_a oscillator to a large number of atoms in thermal equilibrium. This introduces a damping in the oscillator similar to that studied in chapter IV. The total Hamiltonian is

$$\begin{aligned}
 H = & \hbar\omega_a a^\dagger a + \hbar\omega_b b^\dagger b + \hbar\kappa \left\{ a^\dagger b + b^\dagger a \right\} \\
 & + \frac{i\hbar\alpha}{2} \sum_j \left\{ R_{+j} a - a^\dagger R_{-j} \right\}
 \end{aligned}
 \tag{VII.B.1}$$

Since our purpose is to derive the Fokker-Planck equation of the system, we need to derive in detail the equation corresponding to the equation of motion for the density operator for this system. Since the coupling between the oscillator and the reservoir is the same as in chapter IV, the constants w_{ij}^\dagger and w_{ji}^- are still given by equations (IV.B.18) to (IV.B.21). We also need to define a chosen order on the system and chose the associated complex variables. We choose the order

$$a^\dagger, b^\dagger, b, a$$

and the associated variables

$$\alpha^*, \beta^*, \beta, \alpha$$

For notational convenience we make the following definitions

$$\underline{e^{-a^\dagger}} \equiv \exp \left[-a^\dagger \frac{\partial}{\partial \alpha^*} \right]$$

$$\underline{e^{-a}} \equiv \exp \left[-a \frac{\partial}{\partial \alpha} \right]$$

$$\underline{e^{-b^\dagger}} \equiv \exp \left[-b^\dagger \frac{\partial}{\partial \beta^*} \right]$$

$$\underline{e^{-b}} \equiv \exp \left[-b \frac{\partial}{\partial \beta} \right]$$

(VII.B.2)

We now have from equation (V.B.6)

$$\begin{aligned} X \left(\bar{a}, \frac{\partial}{\partial \bar{\alpha}} \right) &= \frac{1}{i\hbar} \left[\overline{\exp \left(-\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right)}, \hbar \omega_a a^\dagger a \right] \\ &+ \frac{1}{i\hbar} \left[\overline{\exp \left(-\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right)}, \hbar \omega_b b^\dagger b \right] \\ &- \frac{1}{i\hbar} \left[\overline{\exp \left(-\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right)}, \hbar \kappa \left(a^\dagger b + b^\dagger a \right) \right] \\ &- \sum_{i,j} \delta_{ij} \left[\overline{\exp \left(-\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right)}, Q_i \right] Q_j w_{ij}^\dagger \\ &- Q_j \left[\overline{\exp \left(-\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right)}, Q_i \right] w_{ji} \end{aligned} \quad \text{(VII.B.3)}$$

Using expression (V.C.5) for a damped harmonic oscillator, this expression becomes

$$\begin{aligned}
X \left(\bar{a}, \frac{\partial}{\partial \bar{\alpha}} \right) &= -i(\omega_a - \Delta\omega_a) \underline{e^{-a^\dagger}} \underline{e^{-b^\dagger}} \left(\frac{\partial}{\partial \alpha^*} a^\dagger - \frac{\partial}{\partial \alpha} a \right) \underline{e^{-b}} \underline{e^{-a}} \\
&+ \gamma \underline{e^{-a^\dagger}} \underline{e^{-b^\dagger}} \left(\frac{\partial}{\partial \alpha^*} \frac{\partial}{\partial \alpha} a^\dagger + \frac{\partial}{\partial \alpha} a \right) \underline{e^{-b}} \underline{e^{-a}} \\
&+ 2\gamma \bar{n} \frac{\partial^2}{\partial \alpha \partial \alpha^*} \underline{e^{-a^\dagger}} \underline{e^{-b^\dagger}} \underline{e^{-b}} \underline{e^{-a}} \\
&+ \frac{1}{i\hbar} \left[\overline{\exp \left(-\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right)}, \hbar \omega_b b^\dagger b \right] \\
&+ \frac{1}{i\hbar} \left[\overline{\exp \left(-\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right)}, \hbar \kappa \left(a^\dagger b + b^\dagger a \right) \right]
\end{aligned} \tag{VII.B.4}$$

By analogy with ω_a oscillator we see that

$$\begin{aligned}
&\frac{1}{i\hbar} \left[\overline{\exp \left(-\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right)}, \hbar \omega_b b^\dagger b \right] \\
&= -i\omega_b \underline{e^{-a^\dagger}} \underline{e^{-b^\dagger}} \left(\frac{\partial}{\partial \beta^*} b^\dagger - \frac{\partial}{\partial \beta} b \right) \underline{e^{-b}} \underline{e^{-a}} \tag{VII.B.5}
\end{aligned}$$

We now need to find the chosen order expression for the interaction term. We have

$$\begin{aligned}
&\underline{e^{-a^\dagger}} \underline{e^{-b^\dagger}} \underline{e^{-b}} \underline{e^{-a}} a^\dagger b \\
&= \underline{e^{-a^\dagger}} \underline{e^{-b^\dagger}} \underline{e^{-b}} \underline{e^{-a}} a^\dagger \underline{e^a} \underline{e^{-a}} b \\
&= \underline{e^{-a^\dagger}} \underline{e^{-b^\dagger}} \underline{e^{-b}} \left(a^\dagger - \frac{\partial}{\partial \alpha} \right) \underline{e^{-a}} b
\end{aligned}$$

(VII.B.6)

where we have used equation (IV.B.4). Similarly

$$\begin{aligned}
 a^\dagger b \underline{e^{-a^\dagger}} \underline{e^{-b^\dagger}} \underline{e^{-b}} \underline{e^{-a}} \\
 = a^\dagger \underline{e^{-a^\dagger}} \underline{e^{-b^\dagger}} \left(b - \frac{\partial}{\partial \beta^*} \right) \underline{e^{-b}} \underline{e^{-a}}
 \end{aligned}
 \tag{VII.B.7}$$

Therefore

$$\begin{aligned}
 [\overline{\exp \left\{ -\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right\}}, a^\dagger b] = \\
 \frac{\partial}{\partial \beta^*} a^\dagger \overline{\exp \left\{ -\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right\}} \\
 - \overline{\exp \left\{ -\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right\}} \frac{\partial}{\partial \alpha} b
 \end{aligned}
 \tag{VII.B.8}$$

Similarly

$$\begin{aligned}
 [\overline{\exp \left\{ -\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right\}}, b^\dagger a] = \\
 \frac{\partial}{\partial \alpha^*} b^\dagger \overline{\exp \left\{ -\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right\}} \\
 - \overline{\exp \left\{ -\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right\}} \frac{\partial}{\partial \beta} a
 \end{aligned}
 \tag{VII.B.9}$$

Using equations (V.B.10), (VII.B.4), (VII.B.5), (VII.B.8) and (VII.B.9)

we have finally

$$\begin{aligned}
 \Xi \left(\bar{\alpha}, \frac{\partial}{\partial \bar{\alpha}} \right) &= -i(\omega_a - \Delta\omega_a) \left(\frac{\partial}{\partial \alpha^*} \alpha^* - \frac{\partial}{\partial \alpha} \alpha \right) \\
 &\quad - i\omega_b \left(\frac{\partial}{\partial \beta^*} \beta^* - \frac{\partial}{\partial \beta} \beta \right) \\
 &\quad + i\gamma \left(\frac{\partial}{\partial \alpha^*} \alpha^* + \frac{\partial}{\partial \alpha} \alpha \right) \\
 &\quad - i\kappa \left(\frac{\partial}{\partial \alpha^*} \beta^* + \frac{\partial}{\partial \beta^*} \alpha^* - \frac{\partial}{\partial \alpha} \beta - \frac{\partial}{\partial \beta} \alpha \right) \\
 &\quad + 2\gamma\bar{n} \frac{\partial^2}{\partial \alpha \partial \alpha^*}
 \end{aligned} \tag{VII.B.10}$$

from which, using (V.B.12), we have the Fokker-Planck equation

$$\begin{aligned}
 \frac{\partial P}{\partial t} &= -i(\omega_a - \Delta\omega_a) \left(\frac{\partial}{\partial \alpha^*} \alpha^* - \frac{\partial}{\partial \alpha} \alpha \right) P \\
 &\quad - i\omega_b \left(\frac{\partial}{\partial \beta^*} \beta^* - \frac{\partial}{\partial \beta} \beta \right) P \\
 &\quad + i\gamma \left(\frac{\partial}{\partial \alpha^*} \alpha^* + \frac{\partial}{\partial \alpha} \alpha \right) P \\
 &\quad - i\kappa \left(\frac{\partial}{\partial \alpha^*} \beta^* + \frac{\partial}{\partial \beta^*} \alpha^* - \frac{\partial}{\partial \alpha} \beta - \frac{\partial}{\partial \beta} \alpha \right) P \\
 &\quad + 2\gamma\bar{n} \frac{\partial^2}{\partial \alpha \partial \alpha^*} P
 \end{aligned} \tag{VII.B.11}$$

Finally, if we use equations (V.E.13) and (V.E.14) we have the Langevin

equations for the system

$$\frac{da^\dagger}{dt} = \left(i\bar{\omega}_a - \gamma \right) a^\dagger + i\kappa b^\dagger + f_a^\dagger \quad (\text{VII.B.12})$$

$$\frac{da}{dt} = \left(-i\bar{\omega}_a - \gamma \right) a - i\kappa b + f_a \quad (\text{VII.B.13})$$

$$\frac{db^\dagger}{dt} = i\omega_b b^\dagger + i\kappa a^\dagger \quad (\text{VII.B.14})$$

$$\frac{db}{dt} = -i\omega_b b - i\kappa a \quad (\text{VII.B.15})$$

where

$$\langle f_a^\dagger(s) f_a(s') \rangle = 2\gamma \bar{n} \delta(s - s') \quad (\text{VII.B.16})$$

and

$$\bar{\omega}_a \equiv \omega_a - \Delta\omega_a \quad (\text{VII.B.17})$$

VII.C Equations for the Energy Variables

We are interested in the energy transfer between the ω_a and ω_b oscillators. Therefore we define the two new variables

$$n_a(t) = \langle a^\dagger a(t) \rangle \quad (\text{VII.C.1})$$

$$n_b(t) = \langle b^\dagger b(t) \rangle \quad (\text{VII.C.2})$$

Then using (VII.B.12) to (VII.B.15) we get

$$\frac{dn_a}{dt} = -2\gamma n_a + \kappa \chi + \langle f_a^\dagger(t)a(t) + a^\dagger(t)f_a(t) \rangle \quad (\text{VII.C.3})$$

$$\frac{dn_b}{dt} = -\kappa \chi \quad (\text{VII.C.4})$$

$$\text{where } \chi = \langle i \{ b^\dagger a - a^\dagger b \} \rangle \quad (\text{VII.C.5})$$

We now need to evaluate the expression in the brackets in equation (VII.C.3). The only part of $a(t)$ that will give us a non-zero expectation value is the part that depends upon $f_a^\dagger(t)$

$$\begin{aligned} \langle f_a^\dagger(t)a(t) \rangle &= \int_0^t \langle f_a^\dagger(t)f_a(\tau) \rangle \exp\{(\gamma + i\omega)(t - \tau)\} d\tau \\ &= \gamma \bar{n} \end{aligned}$$

(VII.C.6)

as in equation (V.E.22).

$$\langle f_a^\dagger(t)a(t) + a^\dagger(t)f_a(t) \rangle = 2\gamma\bar{n} \quad (\text{VII.C.7})$$

To determine the time development we need to derive an equation of motion for χ . Using equations (VII.B.12) to (VII.E.15) we have

$$\begin{aligned} \frac{d\chi}{dt} &= i \frac{db^\dagger}{dt} a + b^\dagger \frac{da}{dt} - \frac{da^\dagger}{dt} b - a \frac{db}{dt} \\ &= -2\kappa n_a + 2\kappa n_b - \gamma\chi \end{aligned} \quad (\text{VII.C.8})$$

$$+ \left[\bar{\omega}_a - \omega_b \right] \langle b^\dagger a + a^\dagger b \rangle + i \langle b^\dagger f_a - f_a^\dagger \rangle$$

But

$$i \langle b^\dagger f_a - f_a^\dagger \rangle = 0 \quad (\text{VII.C.9})$$

Because no part of b^\dagger or b depends upon the Langevin forces.

We define the detuning

$$\Delta = \bar{\omega}_a - \omega_b$$

and



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ω_a oscillator is in thermal equilibrium and is brought in contact with the ω_b oscillator, which is in an excited state. Assume for definiteness that there is one photon in the ω_b oscillator. The anti-normal form of the density matrix of the ω_a oscillator is given by

$$\bar{S}^a(\alpha, \alpha^*) = \frac{1}{\pi \bar{n}} \exp \left[-\frac{|\alpha|^2}{\bar{n}} \right] \quad (\text{VII.C.14})$$

from equation (IV.D.16). The antinormal form of the density matrix in the ω_b oscillator is given by (using (IV.C.25))

$$\begin{aligned} \bar{S}^b(\beta, \beta^*) &= \pi \text{Trace} \left[|1\rangle \langle 1| \delta(\beta^* - b^\dagger) \delta(\beta - b) \right] \\ &= \pi \langle 1| \delta(\beta^* - b^\dagger) \delta(\beta - b) |1\rangle \\ &= \pi \langle 1| e^{-b^\dagger} e^{-b} |1\rangle \delta(\beta) \delta(\beta^*) \\ &= \pi \langle 1| \left[1 - b^\dagger \frac{\partial}{\partial \beta^*} - b \frac{\partial}{\partial \beta} + b^\dagger b \frac{\partial^2}{\partial \beta \partial \beta^*} + \dots \right] |1\rangle \delta(\beta) \delta(\beta^*) \\ &= \pi \delta(\beta) \delta(\beta^*) + \pi \delta'(\beta) \delta'(\beta^*) \end{aligned} \quad (\text{VII.C.15})$$

because the matrix elements of all the other terms vanish. So at $t=0$, the density matrix is given by

$$S = A \left[\frac{1}{\bar{n}} \exp \left[-\frac{|\alpha|^2}{\bar{n}} \right] \left\{ \delta(\beta) \delta(\beta^*) + \delta'(\beta) \delta'(\beta^*) \right\} \right] \quad (\text{VII.C.16})$$

Using (IV.C.24) we have

$$n_a(t=0) = \frac{1}{\pi^2} \int d^2\alpha d^2\beta \alpha^* \alpha$$

$$* \left[\frac{1}{\bar{n}} \exp \left[-\frac{|\alpha|^2}{\bar{n}} \right] \left(\delta(\beta) \delta(\beta^*) + \delta'(\beta) \delta'(\beta^*) \right) \right]$$

(VII.C.17)

$$= \bar{n}$$

Similarly

$$n_b(0) = 1 \tag{VII.C.18}$$

$$x(0) = 0 \tag{VII.C.19}$$

$$E(0) = 0 \tag{VII.C.20}$$

The eigenvalue equation for (VII.C.13) is in general a quartic. The complete solution to (VII.C.13) with the initial conditions (VII.C.17) to (VII.C.20) exists and can easily be found. However the form of the solution is too complicated to be enlightening. We examine the solutions to (VII.C.13) under two special conditions in the following sections.

VII.D Solutions for Small Detuning

We first examine the solutions to (VII.C.13) for $\Delta = 0$.

$$\frac{dn_a}{dt} = 2\gamma (n - n_a) + \kappa\chi$$

$$\frac{dn_b}{dt} = -\kappa\chi$$

$$\frac{d\chi}{dt} = \Delta E - 2\kappa n_a + 2\kappa n_b - \gamma\chi$$

(VII.D.1)

$$\frac{dE}{dt} = -\Delta\chi - \gamma E$$

For convenience we define the variable

$$T = \sqrt{4\kappa^2 - \gamma^2}$$

(VII.D.2)

For $\Delta = 0$, the eigenvalues of the above matrix are

$$-\gamma \text{ and } -\gamma \pm iT$$

and four possible corresponding eigenvectors are

$$\begin{pmatrix} \kappa \\ \kappa \\ \gamma \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \gamma - iT \\ \gamma + iT \\ 4\kappa \\ 0 \end{pmatrix}, \begin{pmatrix} \gamma + iT \\ \gamma - iT \\ 4\kappa \\ 0 \end{pmatrix}$$

(VII.D.3)

For the initial conditions (VII.C.17) to (VII.C.20) we have the solution

$$n_a(t) = \bar{n} + \exp(-\gamma t) \frac{4\kappa^2(1-\bar{n})}{\tau^2} \sin^2 \frac{\tau}{2} t \quad (\text{VII.D.4})$$

$$n_b(t) = \bar{n} + \exp(-\gamma t) \frac{4\kappa^2(1-\bar{n})}{\tau^2} \cos^2 \frac{\tau}{2} t$$

$$- \exp(-\gamma t) \frac{(1-\bar{n})\gamma}{\tau^2} \{ \gamma \cos \tau t - \tau \sin \tau t \} \quad (\text{VII.D.5})$$

The above solution is for the case $\Delta = 0$. The time for the system to undergo one complete oscillation of energy is proportional to $\frac{2}{\tau}$. From equation (VII.A.13) we see that with no damping this time was proportional to $\frac{1}{\kappa}$ i.e. the presence of damping has increased the time of oscillation. We also note that as

$$t \rightarrow \infty, n_a(t) = \bar{n} \quad \text{and} \quad n_b(t) = \bar{n}$$

i.e the system reaches thermal equilibrium with a time constant $\frac{1}{\gamma}$.

From (VII.D.2) we obtain that if

$$\gamma < 2\kappa$$

τ is real. This implies that the system undergoes energy oscillation before reaching thermal equilibrium. If γ is greater than this value, the system is overdamped and decays monotonically to thermal equilibrium.

Let us now calculate the solution to (VII.D.1) for small Δ i.e.

$$\Delta \ll \kappa, \gamma$$

The eigenvectors which are correct to zeroth order in Δ , correspond to eigenvalues which are first order in Δ . To lowest order in Δ the eigenvalues of the matrix (VII.D.1) are easily found to be

$$\begin{aligned}
 & -\gamma + \frac{\gamma\Delta}{T} \quad , \quad -\gamma - \frac{\gamma\Delta}{T} \\
 & -\gamma + i \left[T + \frac{2\kappa^2\Delta^2}{T^3} \right] \quad , \quad -\gamma - i \left[T + \frac{2\kappa^2\Delta^2}{T^3} \right] .
 \end{aligned} \tag{VII.D.6}$$

The corresponding eigenvectors to lowest order in Δ are

$$\begin{aligned}
 & \begin{pmatrix} \kappa \\ \kappa \\ \gamma \\ -T \end{pmatrix} , \begin{pmatrix} \kappa \\ \kappa \\ \gamma \\ T \end{pmatrix} \\
 & \begin{pmatrix} \gamma - iT \\ \gamma + iT \\ 4\kappa \\ 0 \end{pmatrix} , \begin{pmatrix} \gamma + iT \\ \gamma - iT \\ 4\kappa \\ 0 \end{pmatrix}
 \end{aligned} \tag{VII.D.7}$$

Then with the initial conditions (VII.C.17) to (VII.C.20), the solutions to (VII.D.1) to second order in Δ are given by

$$\begin{aligned}
 n_a(t) &= \bar{n} + \exp(-\gamma t) \frac{2\kappa^2(1-\bar{n})}{T^2} \left\{ \cosh \frac{\gamma\Delta}{T} t - \cos \Omega t \right\} \\
 n_b(t) &= \bar{n} + \exp(-\gamma t) \frac{2\kappa^2(1-\bar{n})}{T^2} \left\{ \cosh \frac{\gamma\Delta}{T} t + \cos \Omega t \right\} \\
 &\quad - \exp(-\gamma t) \frac{(1-\bar{n})\gamma}{T^2} \left\{ \gamma \cos \Omega t - T \sin \Omega t \right\}
 \end{aligned} \tag{VII.D.8}$$

(VII.D.9)

where we have defined

$$\Omega = \left[T + \frac{2\kappa^2 \Delta^2}{T^3} \right] \quad (\text{VII.D.10})$$

For small Δ , therefore the time for the system to undergo one complete oscillation of energy is approximately proportional to

$$\begin{aligned} \frac{2}{\Omega} &= \frac{2}{\left[T + \frac{2\kappa^2 \Delta^2}{T^3} \right]} \\ &= \frac{8\kappa \left[1 - \frac{\gamma^2}{4\kappa^2} \right]^{1.5}}{8\kappa \left[1 - \frac{\gamma^2}{4\kappa^2} \right]^2 + \Delta^2} \quad (\text{VII.D.11}) \end{aligned}$$

VII.E Solutions for Large Detuning

In this section we examine solutions for

$$\Delta \gg \gamma, \kappa$$

We will perturb the original equation in orders of γ . The eigenvalues of the matrix to first order in γ are found to be

$$-\gamma + \frac{\Delta\gamma}{\Gamma}, \quad -\gamma - \frac{\Delta\gamma}{\Gamma}, \quad -\gamma + i\Gamma, \quad -\gamma - i\Gamma \quad (\text{VII.E.1})$$

where

$$\Gamma = \sqrt{\Delta^2 + 4\kappa^2} \quad (\text{VII.E.2})$$

The corresponding eigenvectors to zeroth order in γ are

$$\begin{pmatrix} \kappa\Gamma(\Gamma - \Delta) \\ \kappa\Gamma(\Gamma + \Delta) \\ 0 \\ -\Gamma(\Gamma^2 - \Delta^2) \end{pmatrix}, \begin{pmatrix} \kappa\Gamma(\Gamma + \Delta) \\ \kappa\Gamma(\Gamma - \Delta) \\ 0 \\ \Gamma(\Gamma^2 - \Delta^2) \end{pmatrix}$$

$$\begin{pmatrix} \kappa \\ -\kappa \\ i\Gamma \\ -\Delta \end{pmatrix}, \begin{pmatrix} \kappa \\ -\kappa \\ -i\Gamma \\ -\Delta \end{pmatrix} \quad (\text{VII.E.3})$$

The solution with initial conditions (VII.C.17) to (VII.C.20) is given by

$$n_a(t) = \bar{n} + \exp(-\gamma t) \frac{2\kappa}{\Gamma^2} (1 - \bar{n}) \left\{ \cosh \frac{\Delta\gamma}{\Gamma} t - \cos \Gamma t \right\}$$

$$n_b(t) = \bar{n} + \exp(-\gamma t) \frac{2\kappa}{\Gamma^2} (1 - \bar{n}) \left\{ \cosh \frac{\Delta\gamma}{\Gamma} t + \cos \Gamma t \right\}$$

(VII.E.4)

$$+ \exp(-\gamma t) \frac{(1 - \bar{n})\Delta}{\Gamma\kappa} \left\{ \frac{\Delta}{\Gamma} \cosh \frac{\Delta\gamma}{\Gamma} t + \sinh \frac{\Delta\gamma}{\Gamma} t \right\}$$

The time for an energy exchange is seen to be proportional to

$$\frac{1}{\kappa \sqrt{1 + \frac{\Delta^2}{4\kappa^2}}}$$

(VII.E.5)

VII.F Computer Generated Solutions

The equations (VII.D.1) have been solved on the computer for varying values of $\frac{\gamma}{\kappa}$ and $\frac{\Delta}{\kappa}$. Figures VII.1 to VII.3 represent solutions for three different pairs of values. We have chosen a value of \bar{n} of 0.9. Reasons for this estimate will be given later. Figure VII.1 is the underdamped case for $\Delta = 0$. We see that there is an energy exchange between the two oscillators, before both of them damp out completely and acquire \bar{n} photons each. Figure VII.2 is the overdamped case for zero detuning. It is overdamped because $\gamma > 2\kappa$. In figure VII.3 we have solutions for non-zero detuning. We see that there is an oscillation of energy but it is not complete. As before, ultimately both oscillators acquire \bar{n} photons each.

Figure VII.4 is a manifold of the oscillation time as a function of the damping constant and the detuning. The oscillation time is the time between two successive minima of $n_a(t)$, and is used as an estimate of the time required for a complete oscillation of energy. The stars represent the values of the detuning and the damping constant for which the system becomes overdamped. For zero detuning this obviously occurs at $\gamma = 2\kappa$.

In figure VII.5 we have plotted the oscillation time as a function of the detuning for a given value of the damping constant gamma (0.8). Note that this function is monotonically decreasing and is given

equations (VII.D.11) and (VII.E.5) for large and small values of Δ

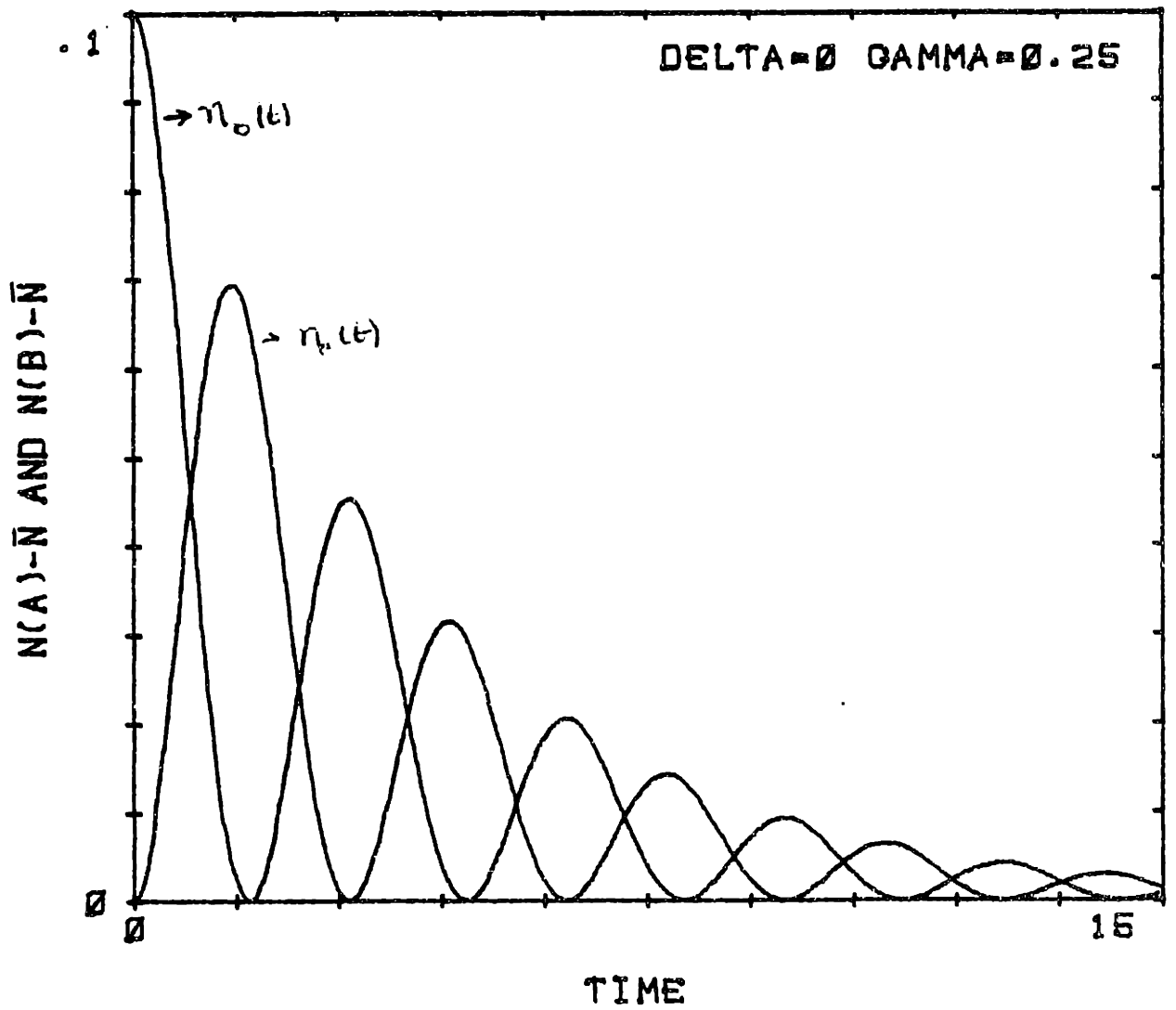


FIGURE VII.1

Time measured in units of ~~1/lambda~~ $1/\kappa$

Delta and gamma measured in units of ~~lambda~~ κ

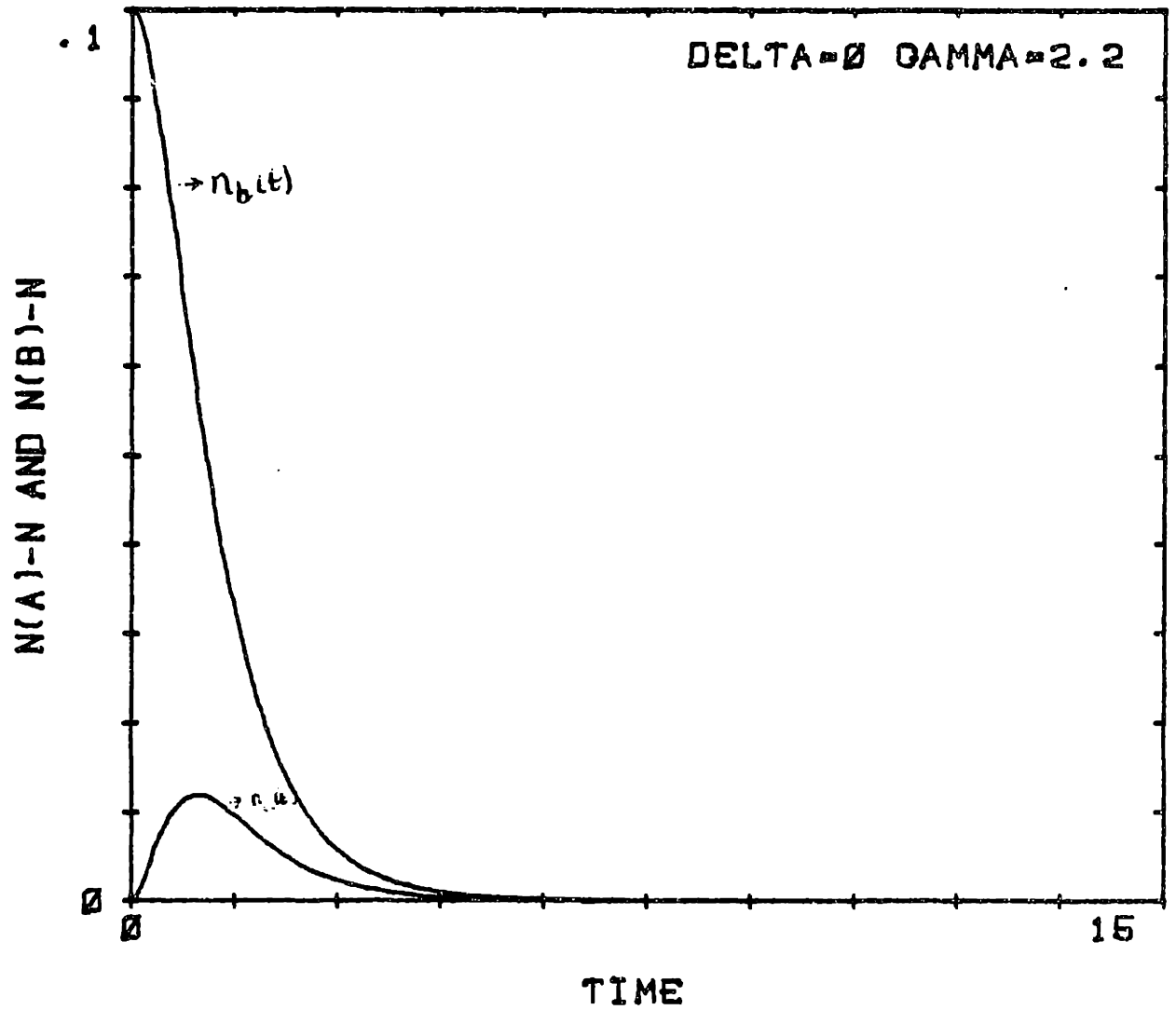


FIGURE VII.2

Time measured in units of $\lambda / (\lambda + \mu + \nu) / \kappa$

Delta and gamma measured in units of $\lambda / (\lambda + \mu + \nu) \kappa$

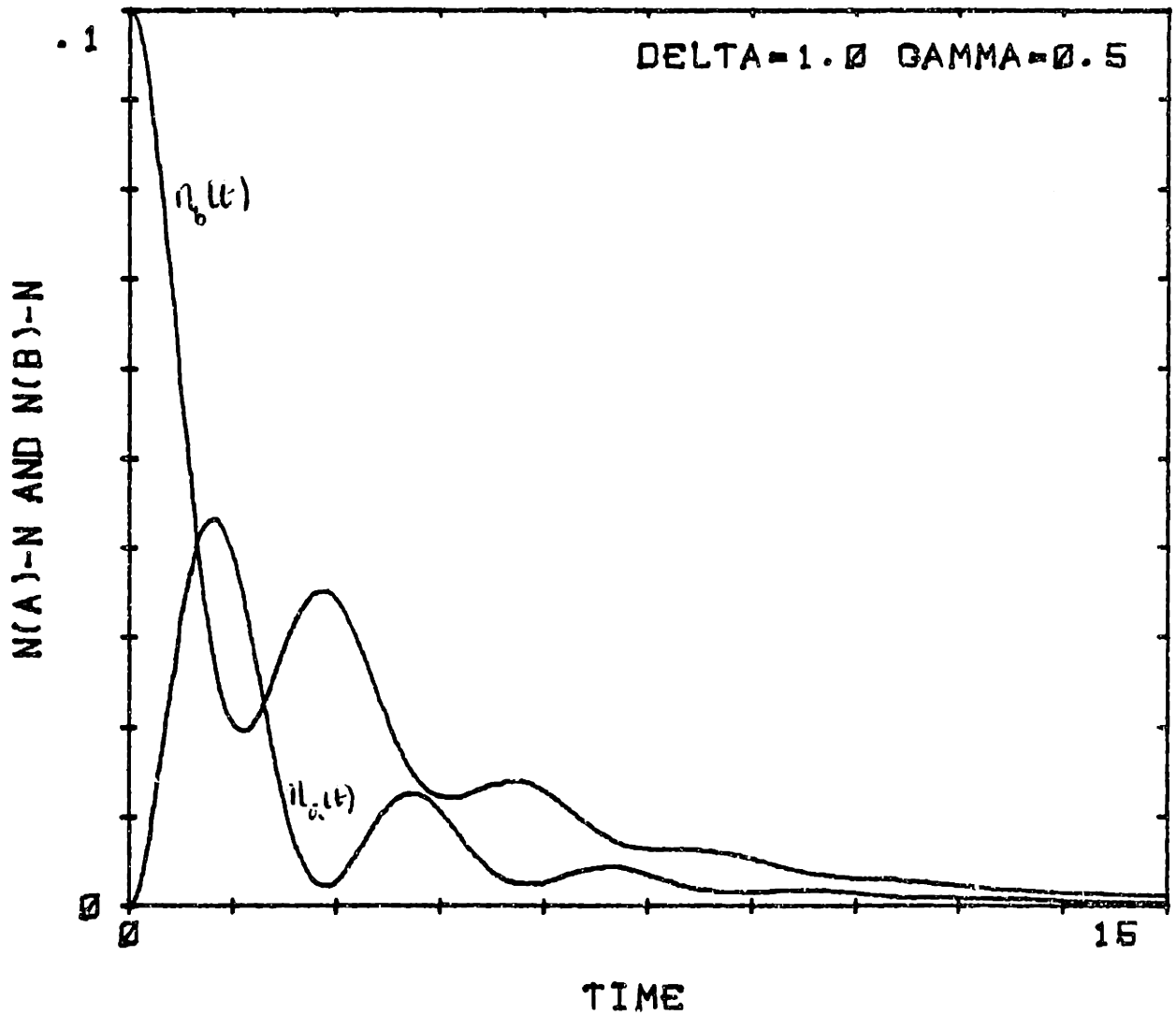


FIGURE VII.3

Time measured in units of ~~1/kappa~~ 1/kappa

Delta and gamma measured in units of ~~1/kappa~~ kappa

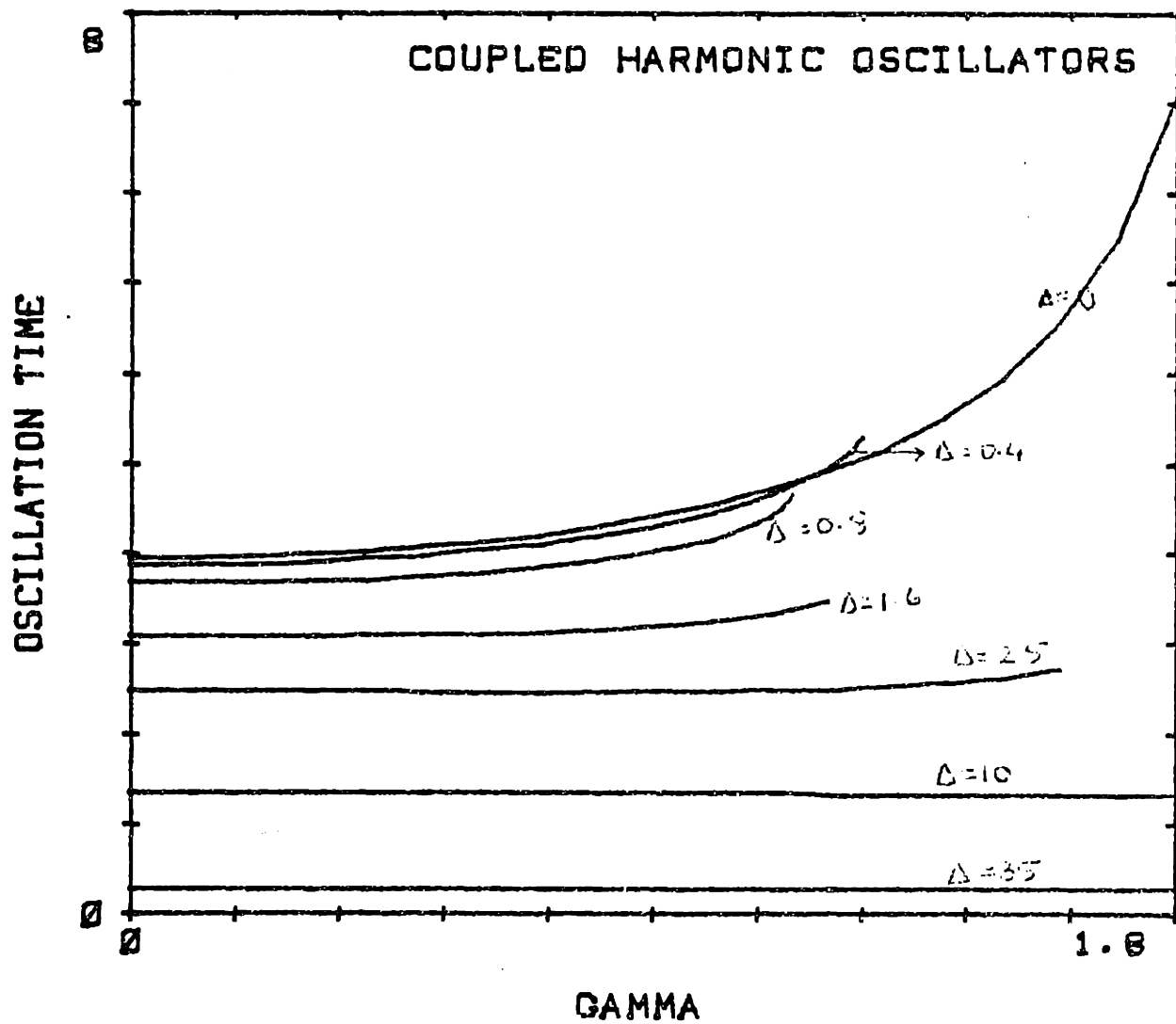


FIGURE VII.4

Time measured in units of ~~1/kappa~~ 1/kappa

Delta and gamma measured in units of ~~kappa~~ kappa

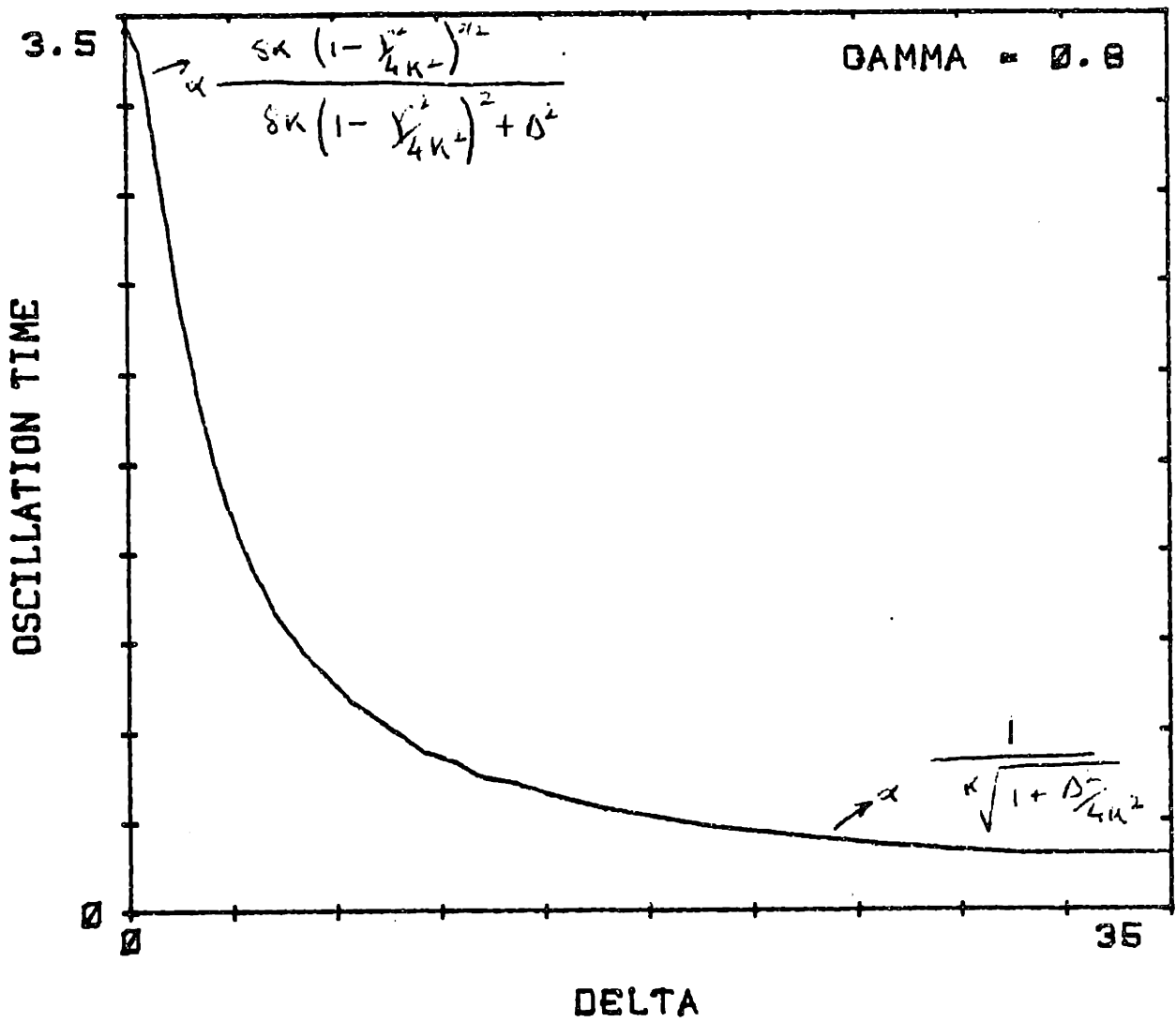


FIGURE VII.5

Time measured in units of ~~1/kappa~~ 1/kappa
 Delta and gamma measured in units of ~~kappa~~ kappa

VIII. DAMPED CAVITY AND TWO-LEVEL ATOM

We now return to our study of the two-level atom and cavity interaction as described by the Hamiltonian (VI.B.22). We are interested in coupling a thermal reservoir to the harmonic oscillator by the interaction described by (IV.A.5). In section VIII.A we derive the Fokker-Planck equation of the system using which we write down the Langevin equations. We convert these operator equations to equations for the energy variables and eliminate the Langevin forces. In section VIII.B we will make a numerical estimate of the parameters involved. The equations will be solved numerically using the computer in section VIII.C

VIII.A The Fokker-Planck Equation

As in section VII.B we will now derive the Fokker-Planck equation of the system. The total Hamiltonian is

$$\begin{aligned}
 H = & \hbar\omega_a R_3 + \hbar\omega a^\dagger a + \frac{\hbar\lambda}{2} \left\{ R_+ a - a^\dagger R_- \right\} \\
 & + \frac{\hbar\alpha}{2} \sum_j \left\{ R_{+j} a - a^\dagger R_{-j} \right\}
 \end{aligned}
 \tag{VIII.A.1}$$

For notational convenience we make the following definitions

$$\underline{e}^{-a} \equiv \exp\left[-a \frac{\partial}{\partial \alpha}\right]$$

$$\underline{e}^{-a^\dagger} \equiv \exp\left[-a^\dagger \frac{\partial}{\partial \alpha^*}\right]$$

$$\underline{e}^{-R_3} \equiv \exp\left[-R_3 \frac{\partial}{\partial \Lambda}\right]$$

$$\underline{e}^{-R_-} \equiv \exp\left[-R_- \frac{\partial}{\partial \Omega}\right]$$

$$\underline{e}^{-R_+} \equiv \exp\left[-R_+ \frac{\partial}{\partial \Omega^*}\right]$$

(VIII.A.2)

We define the following chosen order of the operators

$$a^\dagger, R_+, R_3, R_-, a$$

and choose the associated complex variables

$$\alpha^*, \Omega^*, \Lambda, \Omega, \alpha$$

We have from (V.B.6)

$$\begin{aligned}
x \left(\bar{a}, \frac{\partial}{\partial \bar{\alpha}} \right) &= \frac{1}{i\hbar} \left[\overline{\exp \left(-\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right)}, \hbar \omega_a R_3 \right] \\
&+ \frac{1}{i\hbar} \left[\overline{\exp \left(-\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right)}, \hbar \omega a^\dagger a \right] \\
&+ \frac{1}{i\hbar} \left[\overline{\exp \left(-\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right)}, \frac{i\hbar \lambda}{2} (R_+ a - a^\dagger R_-) \right] \\
&+ \sum_{i,j} \delta_{ij} \left[\left[\overline{\exp \left(-\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right)}, Q_i \right] Q_j w_{ij}^\dagger \right. \\
&\quad \left. - Q_j \left[\overline{\exp \left(-\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right)}, Q_i \right] w_{ji}^\dagger \right]
\end{aligned} \tag{VIII.A.3}$$

The value of the expression involving only the cavity has already been evaluated. It is from (V.C.5)

$$\begin{aligned}
&= -i(\omega - \Delta\omega) \underline{e}^{-a^\dagger} \underline{e}^{-R_+} \underline{e}^{-R_3} \underline{e}^{-R_-} \left(\frac{\partial}{\partial \alpha^*} a^\dagger - \frac{\partial}{\partial \alpha} a \right) \underline{e}^{-a} \\
&+ \gamma \underline{e}^{-a^\dagger} \underline{e}^{-R_+} \underline{e}^{-R_3} \underline{e}^{-R_-} \left(\frac{\partial}{\partial \alpha^*} a^\dagger + \frac{\partial}{\partial \alpha} a \right) \underline{e}^{-a} \\
&+ 2\gamma \bar{n} \underline{e}^{-a^\dagger} \underline{e}^{-R_+} \underline{e}^{-R_3} \underline{e}^{-R_-} \underline{e}^{-a} \frac{\partial^2}{\partial \alpha \partial \alpha^*}
\end{aligned} \tag{VIII.A.4}$$

Let us now evaluate $\frac{1}{i\hbar} \left[\overline{\exp \left(-\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right)}, \hbar \omega_a R_3 \right]$. We have

$$\begin{aligned}
\underline{e}^{-R_+} \underline{e}^{-R_3} \underline{e}^{-R_-} R_3 &= \underline{e}^{-R_+} \underline{e}^{-R_3} \underline{e}^{-R_-} R_3 \underline{e}^{R_-} \underline{e}^{-R_-} \\
&= \underline{e}^{-R_+} \underline{e}^{-R_3} \left\{ R_3 - \frac{\partial}{\partial \Omega} [R_-, R_3] + \dots \right\} \underline{e}^{-R_-} \\
&= \underline{e}^{-R_+} \underline{e}^{-R_3} \left\{ R_3 - \frac{\partial}{\partial \Omega} R_- \right\} \underline{e}^{-R_-}
\end{aligned}
\tag{VIII.A.5}$$

where we have used (IV.B.4). Similarly we can derive

$$\begin{aligned}
R_3 \underline{e}^{-R_+} \underline{e}^{-R_3} \underline{e}^{-R_-} &= \\
&= \underline{e}^{-R_+} \left\{ R_3 - \frac{\partial}{\partial \Omega} R_+ \right\} \underline{e}^{-R_3} \underline{e}^{-R_-}
\end{aligned}
\tag{VIII.A.6}$$

which gives us

$$\begin{aligned}
\frac{1}{i\hbar} \left[\exp \left\{ -\bar{a} \frac{\partial}{\partial \bar{\alpha}} \right\}, \hbar \omega_a R_3 \right] &= \\
&= -i\omega_a \underline{e}^{-a^\dagger} \underline{e}^{-R_+} \frac{\partial}{\partial \Omega} R_+ \underline{e}^{-R_3} \underline{e}^{-R_-} \underline{e}^{-a} \\
&+ i\omega_a \underline{e}^{-a^\dagger} \underline{e}^{-R_+} \underline{e}^{-R_3} \frac{\partial}{\partial \Omega} R_- \underline{e}^{-R_-} \underline{e}^{-a}
\end{aligned}
\tag{VIII.A.7}$$

Now let us place the interaction term in chosen order. We will need the following equations

$$\begin{aligned}
& \underline{e}^{-a^\dagger} \underline{e}^{-R_+} \underline{e}^{-R_3} \underline{e}^{-R_-} \underline{e}^{-a} R_+ a \\
&= \underline{e}^{-a^\dagger} \underline{e}^{-R_+} \left\{ R_+ - \frac{\partial}{\partial \Omega} [R_-, R_+] \right. \\
&\quad \left. + \frac{1}{2} \frac{\partial^2}{\partial \Omega^2} [R_-, [R_-, R_+]] + \dots \right\} \underline{e}^{-R_-} \underline{e}^{-a} a
\end{aligned} \tag{VIII.A.8}$$

where we have again used (IV.B.4)

$$\begin{aligned}
&= \underline{e}^{-a^\dagger} \underline{e}^{-R_+} \underline{e}^{-R_3} \left\{ R_+ + 2 \frac{\partial}{\partial \Omega} R_3 + \frac{\partial^2}{\partial \Omega^2} R_- \right\} \underline{e}^{-R_-} \underline{e}^{-a} a \\
&= \underline{e}^{-a^\dagger} \underline{e}^{-R_+} \left\{ R_+ - \frac{\partial}{\partial \Lambda} [R_3, R_+] \right. \\
&\quad \left. + \frac{1}{2} \frac{\partial^2}{\partial \Lambda^2} [R_3, [R_3, R_+]] + \dots \right\} \underline{e}^{-R_3} \underline{e}^{-R_-} \underline{e}^{-a} a \\
&\quad + 2 \underline{e}^{-a^\dagger} \underline{e}^{-R_+} \underline{e}^{-R_3} \frac{\partial}{\partial \Omega} R_3 \underline{e}^{-R_-} \underline{e}^{-a} a \\
&\quad + \underline{e}^{-a^\dagger} \underline{e}^{-R_+} \underline{e}^{-R_3} \frac{\partial^2}{\partial \Omega^2} R_- \underline{e}^{-R_-} \underline{e}^{-a} a
\end{aligned} \tag{VIII.A.9}$$

from which we obtain

$$\begin{aligned}
&= \underline{e}^{-a^\dagger} \underline{e}^{-R_+} R_+ \underline{e}^{-R_-} \underline{e}^{-a} a \\
&\quad - \underline{e}^{-a^\dagger} \underline{e}^{-R_+} \frac{\partial}{\partial \Lambda} R_+ \underline{e}^{-R_3} \underline{e}^{-R_-} \underline{e}^{-a} a \\
&\quad + \frac{1}{2} \underline{e}^{-a^\dagger} \underline{e}^{-R_+} \frac{\partial^2}{\partial \Lambda^2} R_+ \underline{e}^{-R_3} \underline{e}^{-R_-} \underline{e}^{-a} a \\
&\quad + 2 \underline{e}^{-a^\dagger} \underline{e}^{-R_+} \underline{e}^{-R_3} \frac{\partial}{\partial \Omega} R_3 \underline{e}^{-R_-} \underline{e}^{-a} a \\
&\quad + \underline{e}^{-a^\dagger} \underline{e}^{-R_+} \underline{e}^{-R_3} \frac{\partial^2}{\partial \Omega^2} R_- \underline{e}^{-R_-} \underline{e}^{-a} a
\end{aligned} \tag{VIII.A.10}$$

In a similar manner we can convert all the expressions in the interaction term into chosen order. After performing this conversion and using equations (VIII.A.4), (VIII.A.7), (VIII.A.10), and (V.B.10) we get

$$\begin{aligned}
i \left[\bar{\alpha}, \frac{\partial}{\partial \bar{\alpha}} \right] &= -i(\omega - \Delta\omega) \left[\frac{\partial}{\partial \alpha^*} \alpha^* - \frac{\partial}{\partial \alpha} \alpha \right] \\
&+ \gamma \left[\frac{\partial}{\partial \alpha^*} \alpha^* + \frac{\partial}{\partial \alpha} \alpha \right] \\
&- i\omega_a \left[\frac{\partial}{\partial \Omega^*} \Omega^* - \frac{\partial}{\partial \Omega} \Omega \right] \\
&+ \lambda \left[\frac{\partial}{\partial \Omega^*} \alpha^* \Lambda + \frac{\partial}{\partial \Omega} \alpha \Lambda \right] \\
&+ \frac{\lambda}{2} \left[\frac{\partial}{\partial \alpha^*} \Omega^* - \frac{\partial}{\partial \alpha} \Omega \right] \\
&+ \frac{\lambda}{2} \frac{\partial}{\partial \Lambda} \left[\alpha \Omega^* + \alpha^* \Omega \right] \\
&+ \frac{\lambda}{4} \frac{\partial^2}{\partial \Lambda^2} \left[\alpha \Omega^* + \alpha^* \Omega \right] \\
&- \frac{\lambda}{2} \left[\frac{\partial^2}{\partial \Omega^2} \Omega + \frac{\partial^2}{\partial \Omega^{*2}} \Omega^* \right] \\
&+ 2\gamma \bar{n} \frac{\partial^2}{\partial \alpha \partial \alpha^*}
\end{aligned}$$

(VIII.A.11)

which gives us the Fokker-Planck equation

$$\begin{aligned}
\frac{\partial P}{\partial t} = & \frac{\partial}{\partial \alpha} \left\{ (\gamma + i\bar{\omega})\alpha + \frac{\lambda}{2} \Omega \right\} \\
& + \frac{\partial}{\partial \alpha^*} \left\{ (\gamma - i\bar{\omega})\alpha + \frac{\lambda}{2} \Omega^* \right\} \\
& + \frac{\partial}{\partial \Omega} \left\{ i\omega_a \Omega + \lambda \alpha \Lambda \right\} \\
& + \frac{\partial}{\partial \Omega^*} \left\{ -i\omega_a \Omega^* + \lambda \alpha^* \Lambda \right\} \\
& + \frac{\partial}{\partial \Lambda} \frac{\lambda}{2} \left\{ -\alpha \Omega^* - \alpha^* \Omega \right\} \\
& + \frac{\partial^2}{\partial \Lambda^2} \frac{\lambda}{4} \left\{ \alpha \Omega^* + \alpha^* \Omega \right\} \\
& - \frac{\partial^2}{\partial \Omega^2} \frac{\lambda}{2} \Omega - \frac{\partial^2}{\partial \Omega^{*2}} \frac{\lambda}{2} \Omega^* + 2\gamma \bar{n} \frac{\partial^2}{\partial \alpha \partial \alpha^*} P
\end{aligned} \tag{VIII.A.12}$$

Using equations (V.E.13) and (V.E.14), we can now write down the Langevin equations of motion

$$\frac{da}{dt} = -(\gamma + i\bar{\omega})a - \frac{\lambda}{2} R_- + f_a \tag{VIII.A.13}$$

$$\frac{da^\dagger}{dt} = -(\gamma + i\bar{\omega})a^\dagger - \frac{\lambda}{2} R_+ + f_{a^\dagger} \tag{VIII.A.14}$$

$$\frac{dR_-}{dt} = -i\omega_a R_- - \lambda a R_3 + f_{R_-} \tag{VIII.A.15}$$

$$\frac{dR_+}{dt} = i\omega_a R_+ - \lambda a^\dagger R_3 + f_{R_+} \quad (\text{VIII.A.16})$$

$$\frac{dR_3}{dt} = \frac{\lambda}{2} \left[R_+ a + a^\dagger R_- \right] + f_{R_3} \quad (\text{VIII.A.17})$$

where the expectation values of all the Langevin forces vanish and

$$\langle f_{a^\dagger}(t) f_a(s) \rangle = 2\gamma\bar{n}\delta(t-s) \quad (\text{VIII.A.18})$$

$$\langle f_{R_3}(t) f_{R_3}(s) \rangle = \frac{\lambda}{4} \langle a^\dagger R_- - R_+ a \rangle \delta(t-s) \quad (\text{VIII.A.19})$$

$$\langle f_{R_-}(t) f_{R_-}(s) \rangle = -\frac{\lambda}{2} \langle R_- \rangle \delta(t-s) \quad (\text{VIII.A.20})$$

$$\langle f_{R_+}(t) f_{R_+}(s) \rangle = -\frac{\lambda}{2} \langle R_+ \rangle \delta(t-s) \quad (\text{VIII.A.21})$$

all other correlation functions being zero.

We now wish to convert equations (VIII.A.13) to (VIII.A.17) to real variables and to average out all the Langevin forces. Since we are interested in the energy transfer between the two systems, we define the two new variables

$$E(t) = \langle R_3(t) \rangle \quad (\text{VIII.A.22})$$

$$N(t) = \langle a^\dagger a(t) \rangle \quad (\text{VIII.A.23})$$

We also define, for later convenience

$$\begin{aligned}
 M(t) &= \frac{1}{\lambda} \frac{dE}{dt} = \frac{1}{\lambda} \frac{d\langle R_3 \rangle}{dt} \\
 &= \frac{1}{2} \langle R_+ a + a^\dagger R_- \rangle
 \end{aligned}
 \tag{VIII.A.24}$$

and

$$Q(t) = \frac{1}{2} \langle R_+ a - a^\dagger R_- \rangle
 \tag{VIII.A.25}$$

Then using equations (VIII.A.13) through (VIII.A.17) we have

$$\begin{aligned}
 \frac{dM}{dt} &= \frac{\gamma}{2} \langle R_+ a + a^\dagger R_- \rangle + \left\{ \omega_a - \bar{\omega} \right\} \frac{i}{2} \langle R_+ a - a^\dagger R_- \rangle \\
 &\quad - \lambda \langle a^\dagger a R_3 \rangle - \frac{\lambda}{2} R_+ R_- \\
 &\quad + \frac{1}{2} \langle R_+ f_a + f_a^\dagger R_- \rangle + \frac{1}{2} \langle a^\dagger f_{R_-} + f_{R_+} a \rangle
 \end{aligned}
 \tag{VIII.A.26}$$

From the correlation functions (VIII.A.18) through (VIII.A.21) we see that

$$\frac{1}{2} \langle R_+ f_a + f_a^\dagger R_- \rangle = 0
 \tag{VIII.A.27}$$

$$\frac{1}{2} \langle a^\dagger f_{R_-} + f_{R_+} a \rangle = 0
 \tag{VIII.A.28}$$

Also from (VI.C.8) we have

$$R_+ R_- = R_3 + \frac{1}{2}$$

and we approximate

$$\begin{aligned} \langle a^\dagger a R_3 \rangle &\approx \langle a^\dagger a \rangle \langle R_3 \rangle \\ &= NE \end{aligned} \tag{VIII.A.29}$$

Using (VIII.A.27), (VIII.A.28), (VI.C.8) and (VIII.A.29), (VIII.A.26) becomes

$$\frac{dM}{dt} = -\gamma M + \left\{ \omega_a - \bar{\omega} \right\} Q - \frac{\lambda}{2} E + \frac{\lambda}{4} - \lambda NE \tag{VIII.A.30}$$

Similarly we can evaluate the derivative of Q

$$\begin{aligned} \frac{dQ}{dt} &= -\gamma Q - \left\{ \omega_a - \bar{\omega} \right\} M \\ &+ \frac{1}{2} \langle R_+ f_a - f_a^\dagger R_- \rangle - \frac{1}{2} \langle a^\dagger f_{R_-} - f_{R_+} a \rangle \end{aligned} \tag{VIII.A.31}$$

As before, the average over the Langevin forces is zero, and we have

$$\frac{dQ}{dt} = -\gamma Q - \left\{ \omega_a - \bar{\omega} \right\} M \tag{VIII.A.32}$$

We also find

$$\frac{dN}{dt} = -2\gamma N - \lambda M + \langle a^\dagger f_a + f_a^\dagger a \rangle$$

(VIII.A.33)

From equation (V.E.22) we have

$$\langle a^\dagger f_a + f_a^\dagger a \rangle = 2\gamma\bar{n} \quad (\text{VIII.A.34})$$

So

$$\frac{dN}{dt} = -2\gamma(N - \bar{n}) - \lambda M \quad (\text{VIII.A.35})$$

The system is therefore described by the following set of equations

$$\frac{dE}{dt} = \lambda M$$

$$\frac{dM}{dt} = -\gamma M + \Delta Q - \frac{\lambda}{2} E + \frac{\lambda}{4} - \lambda N E$$

$$\frac{dQ}{dt} = -\gamma Q - \Delta M$$

(VIII.A.36)

$$\frac{dN}{dt} = -2\gamma(N - \bar{n}) - \lambda M$$

$$\text{where } \Delta \equiv \omega_a - \bar{\omega}$$

Let us examine the steady state values. At steady state we have

$$\frac{dE}{dt} = \frac{dM}{dt} = \frac{dQ}{dt} = \frac{dN}{dt} = 0 \quad (\text{VIII.A.37})$$

which implies that

$$M = Q = 0$$

$$N = \bar{n}$$

(VIII.A.38)

$$E = \frac{1}{2 + 4\bar{n}} = \frac{1 \exp(\beta \hbar \omega) - 1}{2 \exp(\beta \hbar \omega) + 1}$$

which is what we would have expected from thermodynamic considerations for a two-level system.

VIII.B Numerical Estimation of Parameters

As discussed in chapter I, a possible transition for study is the $n=25, l=24, m=24$ to $n=24, l=23, m=23$ transition in sodium. For this transition

$$\omega \propto \frac{1}{n^3}$$

and

$$\omega = 2.81 \times 10^{12} \text{ Hz} \quad (\text{VIII.B.1})$$

This corresponds to a wavelength of 0.07 cms. Also we have from (IV.B.30), at a temperature of around 25 K

$$n = 0.9 \quad (\text{VIII.B.2})$$

The coupling constant between the atom and the cavity is given by

$$\lambda = -\frac{2e\omega}{\hbar} a \sqrt{\frac{2\pi\hbar}{\omega}} \vec{d} \cdot \vec{u} \quad (\text{VIII.B.3})$$

using equation (VI.B.18). But

$$\vec{d} = \langle 25, 24, 24 | \vec{r} | 24, 23, 23 \rangle$$

We can easily evaluate the matrix element. We have

$$\langle 25,24,24 | z | 24,23,23 \rangle = 0 \quad (\text{VIII.B.4})$$

$$\langle 25,24,24 | x + iy | 24,23,23 \rangle = 0 \quad (\text{VIII.B.5})$$

$$\langle 25,24,24 | x - iy | 24,23,23 \rangle = \sqrt{\frac{48}{49}} R_{25,24}^{24,23} \quad (\text{VIII.B.6})$$

We can easily calculate

$$R_{25,24}^{24,23} = 587.7 \text{ atomic units}$$

$$\langle x \rangle = \frac{1}{2} \sqrt{\frac{48}{49}} R_{25,24}^{24,23}$$

$$\langle y \rangle = \frac{1}{2} \sqrt{\frac{48}{49}} R_{25,24}^{24,23}$$

(VIII.B.7)

$$|\vec{d}| = \frac{1}{\sqrt{2}} \sqrt{\frac{48}{49}} R_{25,24}^{24,23} = 2.18 \times 10^{-6}$$

The \vec{u} in (VIII.B.3) is the solution to (VI.A.7) and (VI.A.9) and normalized by (VI.A.4). Let us assume for simplicity that the cavity is cubical in shape. Then for the lowest mode in the cavity, the solution to (VI.A.4), (VI.A.7) and (VI.A.9) is

$$\vec{u}(x,y,z) = \sqrt{\frac{4}{L^3}} \sin \frac{\pi x}{L} \sin \frac{\pi y}{L} \hat{k} \quad (\text{VIII.B.8})$$

where L is the side of the cavity. If we assume that the frequency of the two-level system is the same as the fundamental frequency of the

cavity, we have

$$L = \sqrt{2} \frac{\pi c}{\omega} = 0.048 \text{ cms} \quad (\text{VIII.B.9})$$

which gives us

$$u_{\max} = \sqrt{\frac{4}{L^3}} = 190.2 \text{ cm}^{-3/2}$$

Also

$$|\vec{d} \cdot \vec{u}|_{\max} = |\vec{d}| u_{\max}$$

Inserting into (VIII.B.3) we get

(VIII.B.10)

$$\lambda = -5.15 \times 10^7 \text{ Hz}$$

For a cavity with a Q of 10^5 , we have

$$\gamma = \frac{\omega}{Q} = 10^7 \text{ Hz} \quad (\text{VIII.B.11})$$

Thus the coupling constant and the damping constant are of approximately the same magnitude. We will look at solutions in the next section for values of γ which are of the same order of magnitude as λ . The detuning Δ , can effectively be adjusted to any arbitrary value by properly choosing the frequency of the cavity.

VIII.C Computer Generated Solutions

As in the last chapter, we will now solve the equations (VIII.A.36) on the computer. We find that the qualitative behaviour of the system is similar. We start with the initial conditions that the cavity is in thermal equilibrium and the atom is in an excited state. This gives us immediately

$$N(0) = \bar{n}$$

and

$$E(0) = 0.5$$

(VIII.C.1)

As in equation (VII.C.17), we use equations (VIII.A.24) and (VIII.A.25) and the facts that the cavity is in thermal equilibrium and the interaction between the atom and the cavity is turned off at $t=0$, to obtain

$$M(0) = 0$$

$$Q(0) = 0$$

(VIII.C.2)

We first examine the solutions for zero detuning. For small values of the damping constant γ , we find that there is an oscillation of energy between the atom and cavity. Eventually the energy is dissipated

in the thermal reservoir, and the system reaches thermal equilibrium. Figure VIII.1 represents such an energy oscillation. The final values of N and E are in agreement with equation (VIII.A.38). As we increase the damping constant, the period of the energy oscillation increases and eventually reaches infinity. Thus, if the damping constant is large enough, there is no oscillation of energy and the system decays monotonically to thermal equilibrium. Figure VIII.2 is an example of such a decay. The critical value of γ which represents the transition from the underdamped to the overdamped case is numerically found to be

$$\gamma = 2.38\lambda \quad (\text{VIII.C.3})$$

Figure VIII.3 is an example of energy oscillation with non-zero detuning. Again we find the energy oscillation is not complete. Figure VIII.4 is a manifold of the oscillation time as function of the damping constant for different values of the detuning. The stars represent the values of the parameters at which the system becomes overdamped. For $\Delta = 0$ this is given by (VIII.C.1). The general behaviour of the parameters at which the transition takes place is similar to the behaviour of the coupled harmonic oscillators. This may be seen by a comparison of figures (VII.4) and (VII.5)

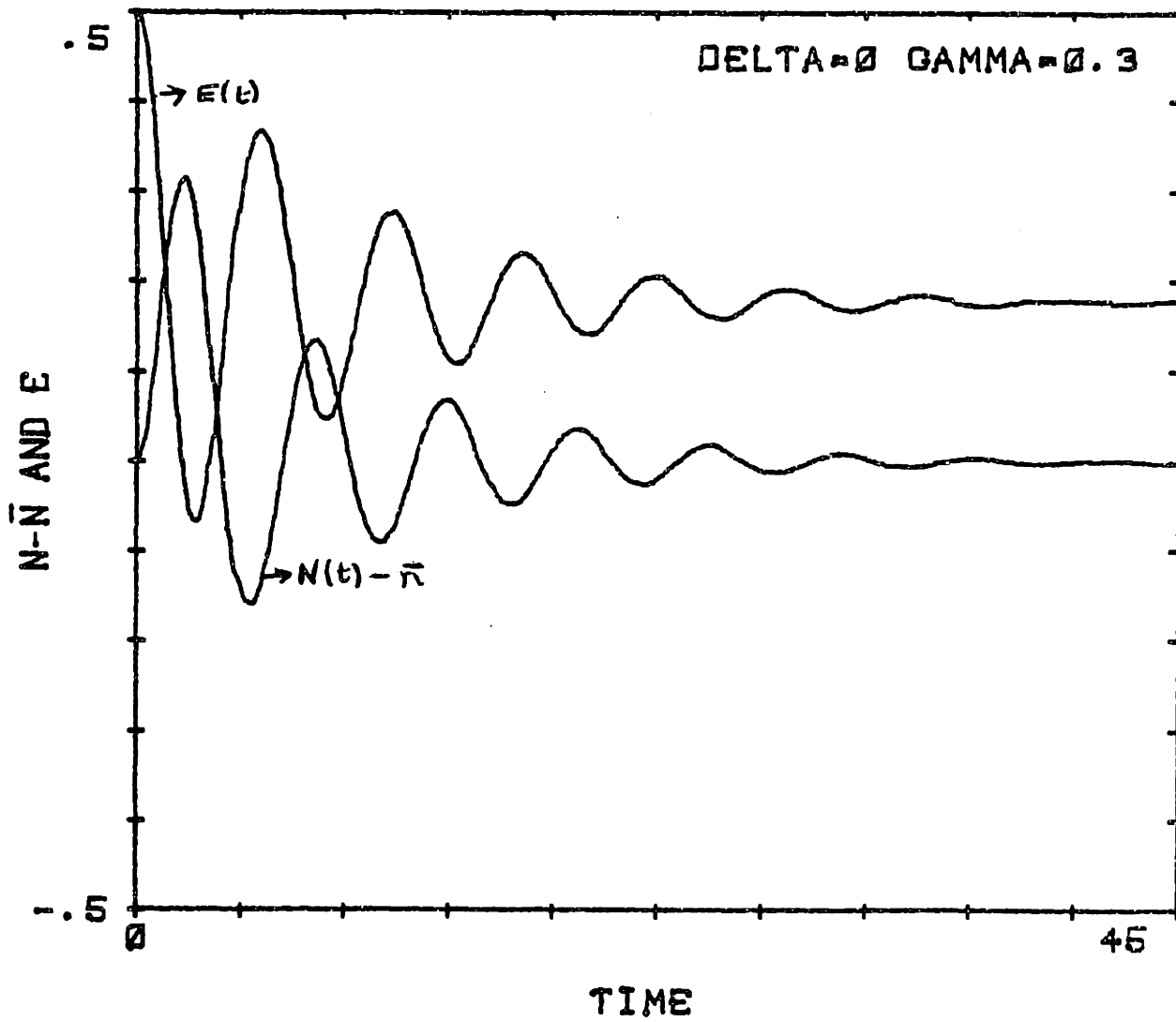


FIGURE VIII.1

Time measured in units of $1/\lambda$

Delta and gamma measured in units of λ

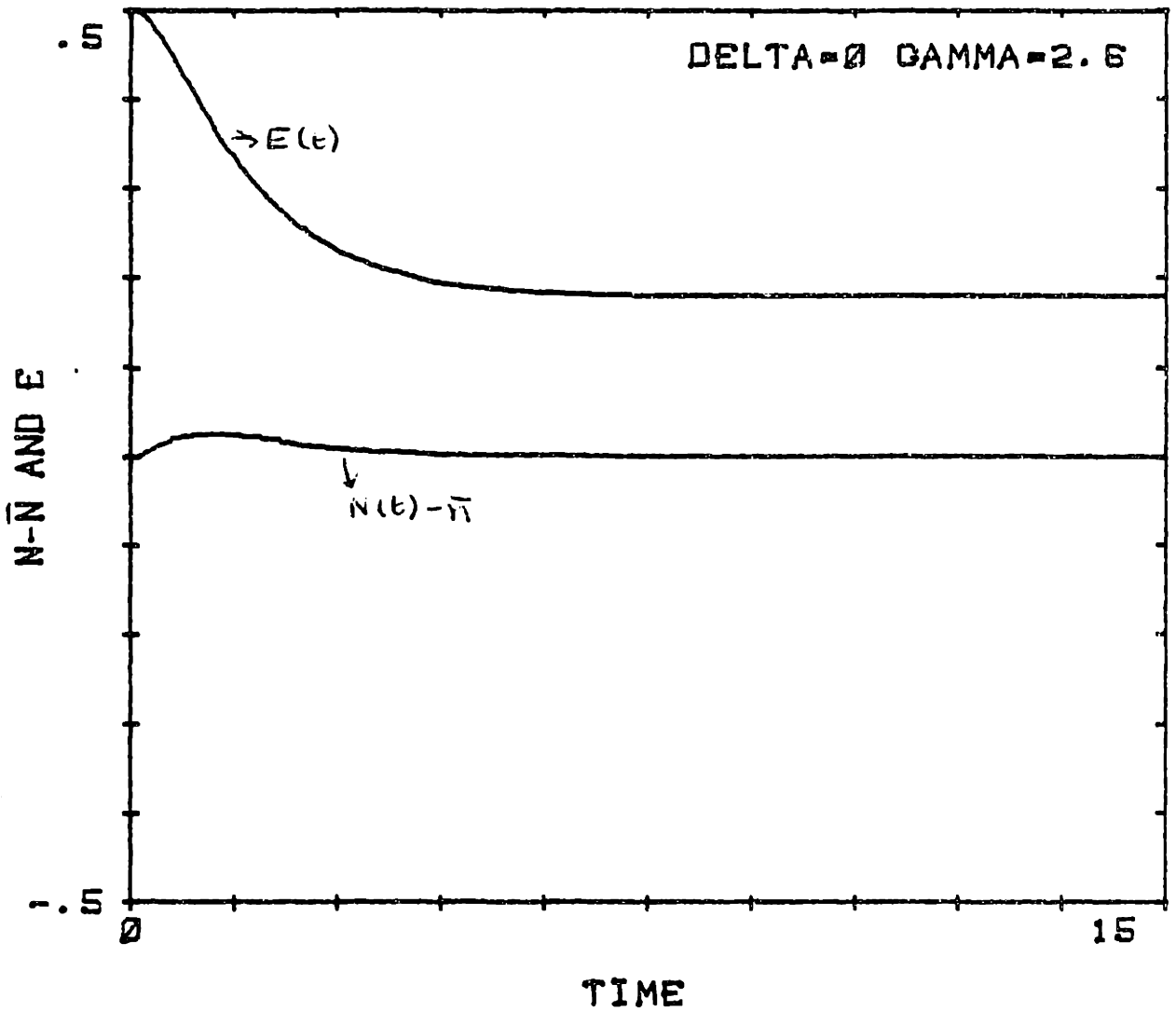


FIGURE VIII.2

Time measured in units of $1/\lambda$

Delta and Gamma measured in units of λ

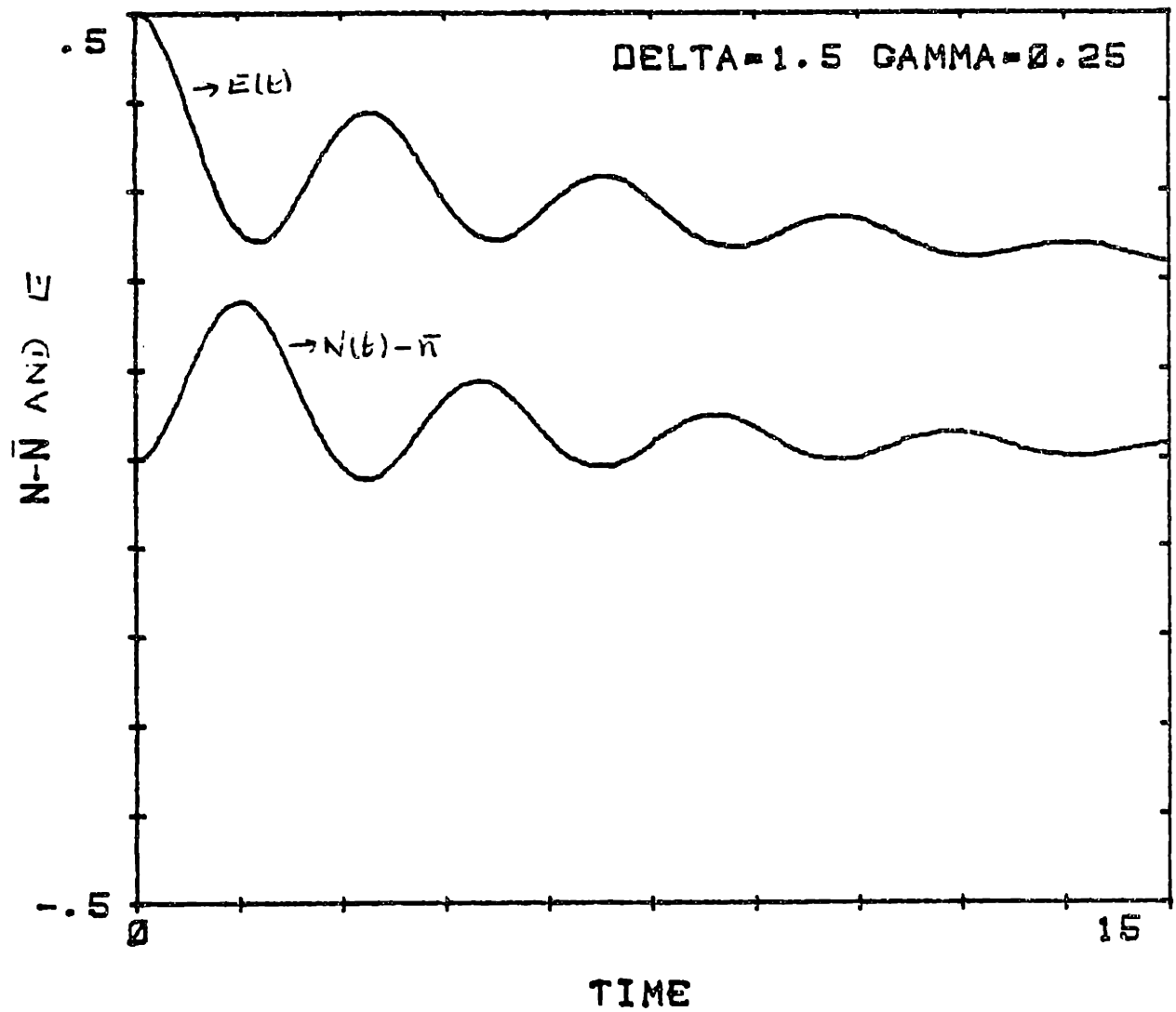


FIGURE VIII.3

Time measured in units of $1/\lambda$

Delta and gamma measured in units of λ

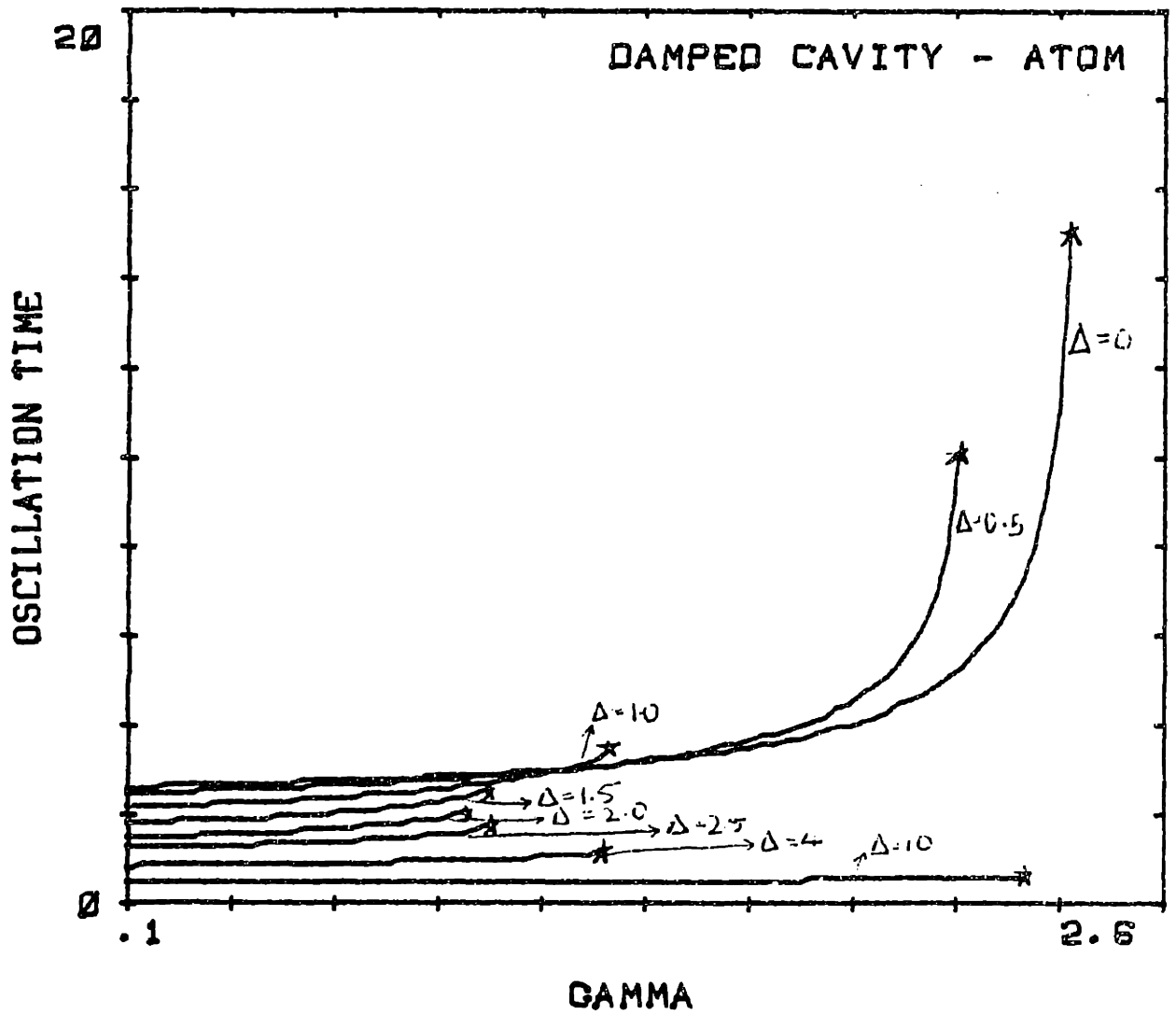


FIGURE VIII.4

Time measured in units of $1/\lambda$

Delta and gamma measured in units of λ

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REFERENCES

- (1) J.R. Ackerhalt and J.H. Eberly, Phys. Rev. D, 10, 3350, 1974.
- (2) J.R. Ackerhalt, P.L. Knight, and J.H. Eberly, Phys. Rev. Lett., 30, 1973.
- (3) L. Allen and J.H. Eberly, Optical Resonance and Two-Level Atoms, Wiley, N.Y., 1975.
- (4) W. Bernard and H.B. Callen, Rev. Mod. Phys., 31, 1017, 1951.
- (5) F. Bloch, Physik Z., 29, 58, 1928.
- (6) H.B. Callen and R.F. Greene, Phys. Rev., 86, 702, 1952a.
- (7) H.B. Callen and R.F. Greene, Phys. Rev., 88, 1387, 1952b.
- (8) H.B. Callen and T.A. Welton, Phys. Rev., 83, 34, 1951.
- (9) S. Chandrasekhar, Rev. Mod. Phys., 15, 1, 1943.
- (10) R.H. Dicke, Phys. Rev., 93, 99, 1954.
- (11) P.A.M. Dirac, Phys. Rev., 139, B684, 1965.
- (12) A. Einstein, Ann. d. Physik, 17, 549, 1905.
- (13) V.M. Fain, JETP, 15, 743, 1962.
- (14) V.M. Fain and Ya I. Khanin, Quantum Electronics vol 1, The M.I.T. Press, 1969.
- (15) U. Fano, Rev. Mod. Phys., 29, 74, 1957.
- (16) A.D. Fokker, Ann. d. Physik, 43, 812, 1914.
- (17) R.J. Glauber, Phys. Rev., 131, 2766, 1963.

- (18) D. ter Haar, Rept. Progr. Phys., 24, 304, 1961.
- (19) W. Heitler, The Quantum Theory of Radiation, Oxford U.P., Oxford, 1960.
- (20) L. van Hove, Physica, 21, 517, 1955.
- (21) E.T. Jaynes and F.W. Cummings, Proc. IEEE, 51, 89, 1963.
- (22) D. Kleppner, Phys. Rev. Lett., 47, 233, 1981.
- (23) L. Landau, Z. Physik, 45, 430, 1927.
- (24) P. Langevin, Comptes rendus, 146, 530, 1908.
- (25) M. Lax, J. Phys. Chem. Solids, 25, 487, 1964.
- (26) M. Lax, Phys. Rev., 145, 110, 1966.
- (27) M. Lax, Phys. Rev., 172, 350, 1968.
- (28) M. Lax and H. Yuen, Phys. Rev., 172, 362, 1968.
- (29) W.H. Louisell Quantum Statistical Properties of Radiation, J. Wiley and Sons, N.Y., 1973.
- (30) W.H. Louisell and J.H. Marburger, IEEE Journ. of Quan. Elec., QE-3, 348, 1967.
- (31) W.H. Louisell and L.R. Walker, Phys. Rev., 137, B204, 1965.
- (32) J.H. Marburger and W.H. Louisell, Phys. Rev., 186, 174, 1969.
- (33) W. Pauli, Festschrift rum 60. Geburtstage A. Sommerfelds, Hirzel, Leipzig, p30, 1928
- (34) M. Planck, Sitz. der preuss. Akad, 324, 1917.
- (35) F. Rief, Fundamentals of Statistical and Thermal Physics, Chap. 15, McGraw Hill N.Y., 1965.

- (36) H. Nyquist, Phys. Rev., 32, 110, 1928.
- (37) I.R. Senitzky, Phys. Rev., 119, 670, 1960.
- (38) I.R. Senitzky, Phys. Rev., 124, 642, 1961.
- (39) M. Smoluchowski, Ann. d. Physik, 21, 756, 1906.
- (40) G.E. Uhlenbeck and L.S. Ornstein, Phys. Rev., 36, 823, 1930.
- (41) M.C. Wang and G.E. Uhlenbeck, Rev. Mod. Phys., 17, 323, 1945.