Large genus bounds for the distribution of triangulated surfaces in moduli space

by

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A.B., Harvard University (2017)

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 2022

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April 14, 2022

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Abstract

Triangulated surfaces are compact Riemann surfaces equipped with a conformal triangulation by equilateral triangles. In 2004, Brooks and Makover asked how triangulated surfaces are distributed in the moduli space of Riemann surfaces as the genus tends to infinity. Mirzakhani raised this question in her 2010 ICM address. We show that in the large genus case, triangulated surfaces are well distributed in moduli space in a fairly strong sense. We do this by proving upper and lower bounds for the number of triangulated surfaces lying in a Teichmüller ball in moduli space. In particular, we show that the number of triangulated surfaces lying in a Teichmüller unit ball is at most exponential in the number of triangles, independent of the genus.

Thesis Supervisor: Larry Guth
Title: Claude Shannon Professor of Mathematics

Acknowledgments

I thank all the professors who have taught me. First, I thank my advisor Larry Guth. Among other things, he has listened to my research ideas even when they were not so close to his own research, and taught me to formulate better questions. I try to reflect on his teaching so that I can become such a teacher. I also thank Chris Bishop, Yasha Eliashberg, Curt McMullen, Tom Mrowka, and Scott Sheffield.

During the past several years I have learned a lot from conversations with other students and postdocs at MIT and Harvard. Here, I mention a few related to the material in this thesis: participants in the extremely informal dynamics and geometry reading group at Harvard, including Yusheng Luo, Peter Park, Tina Torkaman, Jane Wang, Karl Winsor, David Yang and Yongquan Zhang; Morris Ang, for many conversations on random triangulations in probability theory; Robert Burklund, for comments on the writing; Pakawut Jiradilok, for conversations related to combinatorics; and Yilin Wang, for conversations related to probability theory as well as comments on the writing.

I have been supported by the MIT Praecis Presidential Fellowship, NSF Graduate Research Fellowship Program (under Grant No. 1745302), and Simons Foundation (under Larry Guth's Simons Investigator award).

Contents

Chapter 1

Introduction

In this thesis, we consider compact Riemann surfaces obtained by gluing together unit equilateral triangles. We call such surfaces triangulated surfaces. We study how triangulated surfaces are distributed among all compact Riemann surfaces.

Topologically, compact Riemann surfaces are classified by their genus. Moduli space $\mathcal{M}_g$ is the parameter space of all Riemann surfaces of genus g. It admits several natural metrics coming from the geometry of Riemann surfaces. One such metric is the Teichmüller metric. In this thesis, we answer the the question of how many triangulated surfaces lie in a Teichmüller metric ball in moduli space, as a function of the genus g and the number of triangles T.

The motivation to study the distrubution of triangulated surfaces in moduli space comes the random geometry of triangulated surfaces. Brooks and Makover studied this in [5] and asked:

Question 1. *What are the geometric properties of a random large genus triangulated surface?*

Brooks and Makover studied T-triangle genus g triangulated surfaces in the range $T \sim 4g$, as $g \to \infty$. Randomness is with respect to the counting measure in this range.

Assuming that a random triangulated surface has genus at least two, it admits a unique conformal hyperbolic metric. Brooks and Makover studied various properties of this metric, including the systole, diameter and Cheeger constant. The systole is the length of the shortest loop that is homotopically nontrivial. The diameter measures the maximum distance between two points on the surface. The Cheeger constant measures how difficult it is to cut the surface into two components of roughly equal area.

Brooks and Makover proved that the systole of a random triangulated surface is asymptotically almost surely bounded below by a constant. The Cheeger constant is also asymptotically almost surely bounded below by a constant. The diameter is asymptotically almost surely bounded above by around the logarithm of the genus. They proved these results by showing that the geometry of a typical triangulated surface in this range can be described by the combinatorics of its dual graph, a trivalent graph.

Subsequently, Guth, Parlier and Young [21] studied Question 1, but instead of the hyperbolic metric, they used the metric on triangulated surfaces coming from the Euclidean metric on each triangle. This is a flat metric with singularities. They defined the notion of combinatorial pair of pants, which is a combinatorial analogue of topological pair of pants. They studied the geometric quantity of total pants length. The total pants length measures how much total length one needs to decompose the triangulated surface into combinatorial pairs of pants. They proved that a random triangulated surface asymptotically almost surely has large total pants length.

Budzinski and Louf [10] also studied the canonical flat metric on triangulated surfaces. They studied triangulated surfaces in the range $T \sim \theta g$ (for any constant $\theta > 4$) and proved that a random point on a random triangulated surface asymptotically almost surely does not lie close to a short loop of nontrivial homotopy class.

If we wish to study the random geometry of surfaces we need not restrict to triangulated surfaces. We can study all compact Riemann surfaces. If the genus is at least two, then the Riemann surface admits a unique conformal hyperbolic metric, so it is equivalent to study hyperbolic surfaces. This motivates the following question:

Question 2. *What are the geometric properties of a random large genus hyperbolic surface?*

Mirzakhani studied Question 2 in [38] (following [35] and [36]). Similar to Brooks and Makover, she studied geometric quantities associated to the hyperbolic metric. However, to formulate the notion of a random hyperbolic surface we need to fix a measure on $\mathcal{M}_g$.

To do this, we first construct a measure on Teichmüller space $\mathcal{T}_g$, which is the universal cover of $\mathcal{M}_g$. Fix S_g, a smooth genus g surface. Geometrically, $\mathcal{T}_g$ consists of equivalence classes of pairs of a genus g Riemann surface along with a marking, that is, a diffeomorphism to S_g. Two marked surfaces are considered to be equivalent if they are related by a conformal map of Riemann surfaces that preserves the marking up to isotopy.

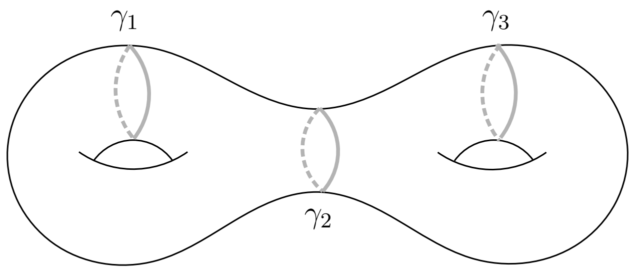

Figure 1-1: A topological pair of pants decomposition.

Teichmüller space admits Fenchel-Nielsen coordinates coming from the hyperbolic geometry of Riemann surfaces: Decompose S_g into $2g-2$ topological pairs of pants using $3g-3$ simple closed curves $\gamma_1, ..., \gamma_{3g-3}$. By an elementary result in hyperbolic geometry, a hyperbolic pair of pants with geodesic boundary components is completely determined by the lengths of its boundaries. To specify a hyperbolic structure on S_g, it suffices to specify the lengths of each curve in the pair of pants decomposition of S_g, as well as twist parameters for each curve that determine how to glue two pairs of pants together. These length $(\ell_1, ..., \ell_{3g-3})$ and twist $(\tau_1, ..., \tau_{3g-3})$ parameters give Fenchel-Nielsen coordinates on $\mathcal{T}_g$. The differential form $d\ell_1 \wedge ... \wedge d\ell_{3g-3} \wedge d\tau_1 \wedge ... \wedge d\tau_{3g-3}$ on $\mathcal{T}_g$ is actually independent of the choice of pair of pants decomposition, by a theorem of Wolpert [48]. In particular, it descends to a measure on $\mathcal{M}_g$. This is the Weil-Petersson measure.

Mirzakhani proved that with respect to the Weil-Petersson measure on $\mathcal{M}_g$, the systole of a random surface is bounded below by a constant with positive probability in the large genus limit. The Cheeger constant is asymptotically almost surely bounded below by a constant. The diameter is asymptotically almost surely bounded above by around the logarithm of the genus. Mirzakhani also proved that a random point on a random hyperbolic surface almost surely does not lie close to a short loop of nontrivial homotopy class.

Guth, Parlier and Young [21] also studied Question 2 with respect to the geometric quantity of total pants length. In this context, the total pants length is the total amount of curve length necessary to decompose a genus g hyperbolic surface into $2g-2$ hyperbolic pairs of pants. They proved that a random hyperbolic surface asymptotically almost surely has large total pants length.

A central obstruction to proving results about random hyperbolic surfaces is the following: we do not have a nice way to tell if two marked hyperbolic surfaces (two points on $\mathcal{T}_g$) correspond to the same underlying unmarked hyperbolic surface (point on $\mathcal{M}_g$). So while we can integrate the Weil-Petersson measure on $\mathcal{T}_g$, it is more difficult to integrate the Weil-Petersson measure on $\mathcal{M}_g$.

Mirzakhani circumvented this obstruction in the following way. Consider the following intermediate space $\mathcal{M}_g^\gamma$, which consists of pairs of an unmarked hyperbolic surface and a non-separating simple closed geodesic γ on it. There is a covering map $\mathcal{T}_g \to \mathcal{M}_g^\gamma$, which can be described as follows. Let γ_0 be a non-separating curve in our topological pair of pants decomposition of S_g. We simply send a marked surface to the underlying unmarked surface, along with the geodesic freely homotopic to γ_0. Now, suppose we have two points in $\mathcal{T}_g$ and wish to check if their images under the covering map are the same point in $\mathcal{M}_g^\gamma$. Cutting along the curve γ_0, we see that this problem is analogous to the original problem of checking if two points in Teichmüller space correspond to the same point in moduli space, but on a lower dimensional moduli space (of Riemann surfaces with boundary).

A key insight (due to McShane [33] and generalized by Mirzakhani [35]) is that studying the dynamics of all simple geodesics on a hyperbolic surface leads to unexpected identities on lengths of simple closed geodesics. These identities allowed Mirzakhani to reduce integration on $\mathcal{M}_g$ to integration on $\mathcal{M}_g^\gamma$. Since the latter can be viewed as a problem on a lower dimensional moduli space, this enables a recursive integration method and gets around the obstruction.

Further results on the geometry of random surfaces have been obtained in [8], [9], [11], [29], [40], [41], [42], [43], [50] and [51]. Similarities between answers to Question 1 and Question 2 motivate the following question.

Question 3. *How are triangulated surfaces distributed in the moduli space of Riemann surfaces, quantitatively?*

Question 3 was first asked by Brooks and Makover in [5]. Subsequently it has been raised in Mirzakhani's 2010 ICM address [37] as well as in [8], [13], [21] and [38].

We answer Question 3 by proving well distribution results for triangulated surfaces. We count how many triangulated surfaces lie in a Teichmüller metric ball in moduli space. Roughly speaking, the Teichmüller distance between two Riemann surfaces measures how far apart the conformal hyperbolic metrics are, in terms of bi-Lipschitz distance. One consequence of our main results (stated in Section 1.1) is the following simplified answer to Question 3.

Theorem 1.0.1. *In a Teichmüller* 1*-ball on* $\mathcal{M}_g$*, there are at most* C^T *number of* T*-triangle genus* g *triangulated surfaces, where* C *is a constant independent of* g *and* T*.*

Henceforth, all generic universal constants will be denoted by C. Theorem 1.0.1 implies that T-triangle genus g surfaces are well distributed in the thick part of $\mathcal{M}_g$. In [17], Fletcher, Kahn and Markovic proved that the number of Teichmüller 1-balls needed to cover the thick part of $\mathcal{M}_g$ is around g^{2g}. For $g = O(T)$, the number of T-triangle genus g surfaces is around $g^{2g}C^T$, which was computed by Budzinski and Louf in [10]. So if triangulated surfaces are well distributed in moduli space then we would expect there to be around C^T surfaces in each 1-ball. We also prove such a lower bound result (with different constants), which is stated in Section 1.1.

Note that the sphere admits a unique conformal structure, and the number of distinct T-triangle triangulations of the sphere is also around C^T. In this sense, Theorem 1.0.1 is genus independent. When we fix the conformal class, the higher genus case behaves just like the genus 0 case.

Theorem 1.0.1 is most interesting in the range $T \sim \theta g$ where $\theta \geq 4$ is a constant. In this range, we obtain that the number of triangulated surfaces in a Teichmüller 1-ball grows roughly exponentially in g as $g \to \infty$. This is similar to how integer points are distributed in high dimensional Euclidean space. In $\mathbb{R}^n$, a radius $\sim n^{1/2}$ ball has volume ~ 1. Such a ball contains around C^n integer points. It is not possible to give a better bound for the number of integer points as a small translation of the ball can change this number by an exponential multiplicative factor. So, the bound in Theorem 1.0.1 is likely optimal.

The difficulty in proving Theorem 1.0.1 is the following issue: given two particular triangulated surfaces, we do not have a nice way to tell if they are close in moduli space. To circumvent this obstruction, we attempt to encode the triangulation using holomorphic 1-forms. A holomorphic 1-form on a Riemann surface is a differential 1-form that locally has the form $f(z)dz$, for a local conformal coordinate z and a holomorphic function f. Holomorphic 1-forms have a geometric description: they determine a conformal flat metric with singularities at the zeros of the form. A triangulated surface also has a canonical conformal flat metric with singularities at some of the vertices. A triangulated surface is encoded by a holomorphic 1-form on the underlying Riemann surface if the corresponding flat metrics are the same. In this situation we can rebuild the triangulation from the holomorphic 1-form.

Unfortunately, not all triangulated surfaces can be encoded by a holomorphic 1-form. Let us disregard this fact for a moment and consider only those that can. Holomorphic 1-forms have an algebraic description: they are closed forms and therefore determine a coholomology class on the surface. The Hodge decomposition theorem implies that holomorphic 1-forms on a Riemann surface are determined by their cohomology class. This means that two triangulations on the same Riemann surface corresponding to the same cohomology class must be the same triangulation. We prove a quantitative version of this statement: two triangulations on Riemann surfaces close to each other in moduli space, with cohomology classes close to each other, must actually be combinatorially close in some sense. This fact allows us to get around the obstruction to prove Theorem 1.0.1.

In the rest of this chapter, we state the general form of our main results in Section 1.1, sketch the proofs in Section 1.2 and Section 1.3, and mention some additional comments in Section 1.4.

1.1 Statement of main results

We prove two results which describe the distrbution of triangulated surfaces in $\mathcal{M}_g$. The following lower bound describes when a surface in moduli space can be approximated by a triangulated surface.

Theorem 1.1.1. *Let $g \geq 2$. Let $X \in \mathcal{M}_g$. Denote by $\gamma_1,...,\gamma_n$ the simple closed geodesics* *on X having length less than $2\,\mathrm{arcsinh}(1)$ in the hyperbolic metric. Let*

$$R = \sum_{i=1}^{n} \mathrm{length}(\gamma_i)^{-1}.$$

Then for $r \in (0,1]$, there exists a $T(r)$-triangle triangulated surface inside a Teichmüller r-ball around X, where $T(r) \leq C(R+g)r^{-2}$. Here, C is a universal constant (independent of g, T, r and R).

Remark 1. In particular, if X lies in the thick part of $\mathcal{M}_g$, meaning $\mathrm{sys}\, X \geq 2\,\mathrm{arcsinh}(1)$, then there exists a Cgr^{-2}-triangle triangulated surface within r Teichmüller distance of X.

Remark 2. The condition that there should not be too short geodesics on X is necessary, as we shall see in Lemma 2.6.1.

We also have the following upper bound regarding the distribution of triangulated surfaces in $\mathcal{M}_g$.

Theorem 1.1.2. *There exists at most C^{T+rg} number of T-triangle triangulated surfaces in a Teichmüller r-ball in $\mathcal{M}_g$. Here, C is a universal constant (independent of g, T and r).*

Remark 3. By Euler characteristic conditions, $g = O(T)$. So substituting $r = 1$ in Theorem 1.1.2 we obtain Theorem 1.0.1.

Remark 4. In a Teichmüller 1-ball in the thick part of $\mathcal{M}_g$, the lower and upper bounds for the number of triangulated surfaces given by Theorem 1.1.1 and Theorem 1.1.2 differ by a multiple of $\exp(O(T))$ for T/g sufficiently large.

Remark 5. It is useful to first ask the combinatorial question: what is the number of T-triangle genus g triangulated surfaces, as a function of T and g? The best bounds for this question are proved in [10, Theorem 3], where there is a multiplicative error term of $\exp(o(T))$, in contrast to our slightly worse error of $\exp(O(T))$ as stated in Remark 4. (The constants in both error terms are independent of the genus.)

Remark 6. There are several possible choices of metrics on moduli space. In Teichmüller theory, the Teichmüller metric is a natural choice. Since we are also interested in the hyperbolic geometry of individual surfaces in moduli space, we may also consider the bi-Lipschitz metric on moduli spaces, in which the distance between two surfaces measures how far apart their hyperbolic metrics are. It turns out that the bi-Lipschitz metric is comparable to the Teichmüller metric, with genus independent constants (see Proposition 4.3.1). So Theorem 1.1.1 and Theorem 1.1.2 hold with respect to the bi-Lipschitz metric also. Another natural choice of metric on moduli space is the Weil-Petersson metric. Note that the Weil-Petersson volume of $\mathcal{M}_g$ is around g^{2g}, as computed by Penner in [44] and Grushevsky in [20], and subsequently improved by Mirzakhani and Zograf in [39]. Since up to an exponential factor of T this number is also the approximate number of T-triangle genus g triangulated surfaces, we may ask if an analogue of our main results hold for the Weil-Petersson metric as well. However, we do not know the answer to this question because we do not yet understand the large genus local geometry of the Weil-Petersson metric in a sufficiently detailed manner.

1.2 Key ideas in the proof of Theorem 1.1.1

Our approach is based on a characterization of the Teichmüller metric in [25] using extremal length and Jenkins-Strebel differentials, which we explain in Section 4. To show Theorem 1.1.1, given a surface in moduli space we construct a certain nicely behaved triangulation of it, take the associated triangulated surface, and use the characterization of the Teichmüller metric above to show that the triangulated surface is close to the original surface in moduli space. We do this in Section 5.

1.3 Key ideas in the proof of Theorem 1.1.2

There are roughly four parts to our proof:

1. Riemann surfaces equipped with a holomorphic 1-form are called translation surfaces. We first consider the subset of triangulated surfaces that are actually translation surfaces where the associated holomorphic 1-form is compatible with the triangulation. We call such surfaces combinatorial translation surfaces, a term we define precisely in Section 2. In this situation, the 1-form gives us a cohomology class, which we then

deal with using Hodge theory. If two combinatorial translation surfaces are close together in moduli space and so are their cohomology classes, we show that constraints coming from Hodge norms imply geometric constraints on how close or far apart vertices, edges, and faces of the two surfaces must be to each other. A combinatorial argument shows that these geometric constraints imply that the two triangulations are close except on a part of the surface with much smaller genus. So we reduce the counting problem for combinatorial translation surfaces to the counting problem for triangulated surfaces in a lower dimensional moduli space. As a result, we get bounds on combinatorial translation surfaces in terms of bounds on triangulated surfaces. We do this in Chapter 7.

2. Given any triangulated surface, there exists a degree six branched cover which is a combinatorial translation surface. We enumerate the number of possibilities for such covers and study the possibilities for where the branch points lie, to get bounds on triangulated surfaces given bounds on combinatorial translation surfaces. Combining with our previous bounds described in (1), we obtain recursive upper bounds for the number of triangulated surfaces lying in a ball in moduli space. We solve these recursive bounds to show Theorem 1.1.2. We do this in Chapter 8.

3. For the technicalities in (1) and (2) to work, we require the use of bounded degree triangulations instead of arbitrary triangulations, so in Chapter 6, we show that any triangulated surface may be approximated by a bounded degree triangulation in a way that increases the number of triangles by at most a constant factor.

4. Another requirement is to understand the large genus local geometry of the Teichmüller metric on Teichmüller space, the universal cover of moduli space. To this end, in Chapter 4 we provide a proof of the fact that the Teichmüller metric on Teichmüller space is comparable to the bi-Lipschitz metric on Teichmüller space with constants independent of the genus. We believe this is known to experts, but could not find the details written in the literature. Also in Chapter 4 we prove results on the geometry of Teichmüller balls in Teichmüller space that are variations of the bounds in [17, Theorem 1.5].

1.4 Comments and references

Non-quantitative versions of Question 3 have been studied in number theory. Belyi's theorem states that Riemann surfaces defined over the algebraic numbers $\overline{\mathbb{Q}}$ are exactly the Riemann surfaces which admit a branched cover (Belyi map) to $\mathbb{P}^1$ branched only at 0, 1 and ∞. Belyi maps give triangulations on the Belyi surface, and conversely, triangulations give Belyi maps. Note that Riemann surfaces defined over $\overline{\mathbb{Q}}$ are dense in $\mathcal{M}_g$, which is a non-quantitative answer to Question 3. Given this context, Theorem 1.1.1 may be interpreted as a quantitative version of this statement; it describes how well one can approximate an arbitrary surface in $\mathcal{M}_g$ by a Belyi surface with respect to the Teichmüller metric, in terms of the degree of the Belyi map. Another approach to approximating aribtrary Riemann surfaces by Belyi surfaces is described by Bishop in [3] and [4, Section 15].

The study of random triangulations is a central topic of research in probability theory. In the large genus setting, progress has been made in the study of local limits of triangulations. In the range $T \sim 4g$ (which is the probabilistically expected range, as shown by Gamburd in [18]), local limits do not exist since the expected surface has very few vertices and very high degrees of vertices. In the range $T \sim \theta g$ where $\theta > 4$, convergence does occur. Planar stochastic hyperbolic triangulations introduced by Curien in [14] are a family of random triangulations in the plane and were conjectured by Benjamini and Curien to be local limits of high genus random triangulations. This was proved by Budzinski and Louf in [10]. However, global questions, such as scaling limits, are difficult to understand in the large genus case. See [27], [30] and [34] for some results on scaling limits of random maps in the planar setting.

In [7], Buser and Sarnak related the homological systole of a hyperbolic Riemann surface to the systole of its Jacobian. In [12], Balacheff, Parlier and Sabourau gave a way to find a minimal homology basis on a hyperbolic Riemann surface, and used that to deduce more general results about the geometry of its Jacobian. The original method to prove Theorem 1.1.2 involved bounding the number of ways to express a hyperbolic Riemann surface as a triangulated surface by studying the geometry of its Jacobian, using [12]. Then, it turned out that these ideas were not necessary to prove Theorem 1.1.2, so they do not appear in the proofs henceforth.

Chapter 2

Preliminaries

2.1 Triangulated surfaces

Let S_g be a genus g surface, and $S_{g,b}$ a genus g surface with b boundary components. Let S be a metrized simplicial 2-complex wherein each 2-simplex is an oriented unit equilateral triangle and gluings of faces preserve orientations. We call S a triangulated surface if it is homeomorphic to S_g for some $g \geq 0$ and a triangulated surface with boundary if it is homeomorphic to $S_{g,b}$ for some $g, b \geq 0$. If S is a triangulated surface with or without boundary, we may consider each equilateral triangle as embedded in $\mathbb{C}$ with vertices at 0, 1 and $\frac{1}{2} + \frac{\sqrt{3}}{2}i$. The complex structure on each equilateral triangle of S is preserved when edges are glued. Extending the complex structure over the interior vertices of S, we obtain a canonical complex structure on S. In this way, we consider S as a Riemann surface.

Remark 7. The construction of triangulated surfaces (without boundary) in [5] is done slightly differently. Instead of gluing together equilateral triangles, hyperbolic ideal triangles are used, and the resulting surface with cusps is compactified. One may check that these two constructions result in the same Riemann surface.

Given a triangulated surface S (with or without boundary), we denote by $V(S)$ the vertices of S, $E(S)$ the edges of S, and $F(S)$ the triangular faces of S. Given $v \in V(S)$ the degree of v is the number of edges emanating from v. We also denote by $V_{>6}(S)$, $V_{\neq 6}(S)$ and $V_{<6}(S)$ the set of vertices of S of degree strictly greater than 6, not equal to 6 and strictly less than 6, respectively.

2.2 Marked surfaces

Let S_g be a genus g surface and $S_{g,b}$ a genus g surface with b boundary components. A marked surface X (without boundary) is a genus g surface, possibly with extra structure, that comes equipped with a homeomorphism $f_X : S_g \to X$. Let X and Y be marked surfaces equipped with markings $f_X : S_g \to X$ and $f_Y : S_g \to Y$, respectively. We call a map $f : X \to Y$ a map of marked surfaces if it is an embedding and the following diagram commutes up to isotopy.

$$\begin{array}{ccc} & S_g & \\ {}^{f_X}\swarrow & & \searrow^{f_Y} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

Replacing S_g with $S_{g,b}$, we may analogously define marked surfaces and maps between them for surfaces with boundary. In future sections, we may omit the word marked if it is clear from context.

2.3 Space of triangulated surfaces

We denote by $\mathrm{Comb}^T(\mathcal{M}_g)$ the set of all triangulated surfaces of genus g with T triangles up to orientation preserving simplicial isomorphism. We denote by $\mathrm{Comb}^T(\mathcal{T}_g)$ the set of all marked triangulated surfaces of genus g with T triangles up to simplicial isomorphism of marked surfaces. Since any genus g triangulated surface is a Riemann surface of genus g, we have a map

$$\Phi : \mathrm{Comb}^T(\mathcal{M}_g) \to \mathcal{M}_g$$

along with canonical biholomorphisms $\Phi_S : S \to \Phi(S)$ for all $S \in \mathrm{Comb}^T(\mathcal{M}_g)$. Similarly, any genus g marked triangulated surface is a marked Riemann surface of genus g, so we have a map

$$\Phi : \mathrm{Comb}^T(\mathcal{T}_g) \to \mathcal{T}_g$$

along with canonical biholomorphisms of marked surfaces $\Phi_S : S \to \Phi(S)$ for all $S \in \mathrm{Comb}^T(\mathcal{T}_g)$.

Remark 8. Here, $\mathcal{T}_g$ denotes the Teichmüller space of marked Riemann surfaces of genus g. See Section 3 for rigorous definitions of Teichmüller space and moduli space.

Note that $T \geq 2$ for $\mathrm{Comb}^T(\mathcal{T}_g)$ to be nonempty; moreover, Euler characteristic conditions imply that $T \geq 4g - 4$. Together these imply $g/T \leq 1/2$. In the future we will implicitly assume this condition.

Finally, we denote by $\mathcal{M}_{\leq g}$ (resp. $\mathcal{T}_{\leq g}$) the union of $\mathcal{M}_{g'}$ (resp. $\mathcal{T}_{g'}$) over all $g' \leq g$. Similarly, we denote by $\mathrm{Comb}^{\leq T}(\mathcal{T}_{\leq g})$ (resp. $\mathrm{Comb}^{\leq T}(\mathcal{M}_{\leq g})$) the union of $\mathrm{Comb}^{T'}(\mathcal{T}_{g'})$ (resp. $\mathrm{Comb}^{\leq T'}(\mathcal{M}_{\leq g'})$) over all $g' \leq g$, $T' \leq T$. We will use this type of notation with respect to other spaces we will define later as well.

2.4 Translation surfaces and combinatorial translation surfaces

A translation surface is a pair (X, ω) where X is a Riemann surface and ω is a holomorphic 1-form on X. The metric $|\omega|$ is a flat metric on X with singularities at zeros of ω. See [49] for an introduction to translation surfaces. The following result gives an alternative definition of translation surfaces.

Proposition 2.4.1 ([49], Proposition 1.6 and Proposition 1.8)**.** *Any translation surface (X, ω) can be expressed in the following manner: X is the union of a collection of polygons $P_1, ..., P_n$ in $\mathbb{C}$ together with a choice of identification of parallel boundary edges of equal length on opposite sides, and ω is the 1-form dz on each polygon. Similarly, any collection of polygons $P_1, ..., P_n \subset \mathbb{C}$ with edge identifications as above defines a translation surface.*

We shall see that some triangulated surfaces are canonically translation surfaces as well. To this end, we define a combinatorial translation structure on a triangulated surface. Let S be a triangulated surface. A combinatorial translation structure on S is an assignment, to each edge $e \in E(S)$ and vertex $v \in V(S)$ such that e emanates from v, a 6th root of unity $\zeta(e, v)$, called a directional weight, such that the following two properties are satisfied:

1. if e contains the vertices v and w, then $\zeta(e,v) = -\zeta(e,w)$ and

2. if e_1 and e_2 are two edges emanating from a vertex v that lie on a triangle of S such that e_1 lies counterclockwise from e_2 according to the orientation on S, then $\zeta(e_1,v) = e^{\pi i/3}\zeta(e_2,v)$.

Conditions 1 and 2 imply that for each triangle, there are only two possibilities for directional weights, which we label as Type A and Type B as seen in Fig. 2-1 and Fig. 2-2, respectively.

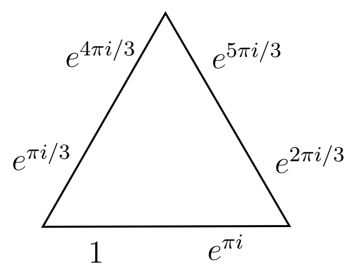

Figure 2-1: Type A triangle

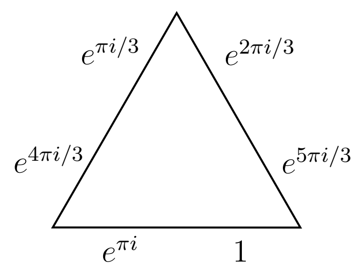

Figure 2-2: Type B triangle

Then, we have the following proposition.

Proposition 2.4.2. *Each triangulated surface S that admits a combinatorial translation structure is a translation surface wherein the associated flat metric agrees with the flat metric coming from the triangulation. Moreover, the associated holomorphic 1-form is canonical in the sense that it only depends on S and the combinatorial translation structure.*

Proof. Rotating as necessary, we identify Type A triangles with the equilateral triangle in $\mathbb{C}$ having vertices at 0, 1 and $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and Type B triangles with the equilateral triangle in $\mathbb{C}$ having vertices at 0, 1 and $\frac{1}{2} - \frac{\sqrt{3}}{2}i$. Conditions 1 and 2 in the definition of combinatorial translation surface imply that all edge identifications must be of a Type A triangle with a Type B triangle on opposite sides along parallel edges. By Proposition 2.4.1, S is a translation surface. Under the identification of Type A triangles and Type B triangles with triangles in $\mathbb{C}$ described above, the 1-forms dz on each triangle glue to give a 1-form ϕ on S, which only depends on S and the combinatorial translation structure. Finally $|dz|$ is simply the Euclidean metric on each triangle. $\square$

Note that given a combinatorial translation structure on S, then for $0 \leq i \leq 6$ we have another combinatorial translation structure on S obtained by multiplying each directional weight by $e^{\pi i/3}$. Moreover, these six structures are the only valid combinatorial translation structures on S, since once we assign directional weights to one triangle on S, there is only one choice for all other directional weights.

We define a combinatorial translation surface to be a triangulated surface equipped with a combinatorial translation structure. We define $\mathrm{Comb}^T(\mathcal{H}_g)$ to be the set of marked combinatorial translation surfaces of genus g with T triangles. By Proposition 2.4.2, any $S \in \mathrm{Comb}^T(\mathcal{H}_g)$ determines a canonical holomorphic 1-form on S that we denote by ϕ_S. We call the flat metric on S the S-metric. Its length element is $ds_S = |\phi_S|$, and area element is $|\phi_S|^2$. Distances in this metric shall be denoted by $d_S(\cdot,\cdot)$. As in the case of triangulated surfaces, we have denote by $\Phi : \mathrm{Comb}^T(\mathcal{H}_g) \to \mathcal{T}_g$ the map which sends a marked combinatorial translation surface to the underlying marked Riemann surface.

2.5 Extremal length on annuli

Let A be a Riemann surface that is topologically an annulus. By the uniformization theorem A is biholomorphic to $A(r) = \{z \in \mathbb{C} | 1 < |z| < r\}$ for some $r > 1$. The modulus of A, denoted $\mathrm{mod}(A)$, is the quantity $(1/2\pi)\log r$.

Now, denote by γ a generator of $H_1(A, \mathbb{Z})$. Given any Riemannian metric ρ on A, we define $\mathrm{length}_\rho(\gamma)$ to be the infimum of lengths in the ρ-metric over all curves representing γ. The extremal length of γ on A is defined to be

$$\mathrm{Ext}_A(\gamma) = \sup_\rho \frac{\mathrm{length}_\rho(\gamma)^2}{\mathrm{area}_\rho(A)}$$

where the supremum is taken over all conformal metrics ρ on A. The following result is proved in [1] (1.D).

Proposition 2.5.1. *We have,* $\mathrm{Ext}_A(\gamma) = \mathrm{mod}(A)^{-1}$.

2.6 Hyperbolic metric on a triangulated surface

In this section we prove two lemmas about the hyperbolic metric on a triangulated surface. The first lemma is about short geodesics.

Lemma 2.6.1. *Let S be a T-triangle triangulated surface of genus g, and $X = \Phi(S)$. Denote by ρ_X the hyperbolic metric on X. Suppose γ_1,...,γ_n are simple closed geodesics on X with* $\mathrm{length}_{\rho_X}(\gamma_i) \leq 2\,\mathrm{arcsinh}(1)$. *Then*

$$\sum_{i=1}^{n} \mathrm{length}_{\rho_X}(\gamma_i)^{-1} \leq CT,$$

where C is a universal constant.

Proof. By the collar theorem ([6], Theorem 4.1.1), each γ_i has an associated hyperbolic annular collar A_i of width $C\log(\mathrm{length}_{\rho_X}(\gamma_i)^{-1})$. The collars A_i are all mutually disjoint. The modulus of A_i is $C\,\mathrm{length}_{\rho_X}(\gamma_i)^{-1}$. By Proposition 2.5.1,

$$\frac{\mathrm{sys}_S(A_i)^2}{\mathrm{area}_S(A_i)} \leq C\,\mathrm{length}_{\rho_X}(\gamma_i).$$

Since $\mathrm{sys}_S(A_i) \geq 1$, we have

$$\mathrm{area}_S(A_i) \geq C\,\mathrm{length}_{\rho_X}(\gamma_i)^{-1}.$$

This is true for all $i \in \{1, ..., n\}$. Since the A_i are all disjoint,

$$\begin{aligned}\sum_{i=1}^{n} \mathrm{length}_{\rho_X}(\gamma_i)^{-1} &\leq C \sum_{i=1}^{n} \mathrm{area}_S(A_i) \\ &\leq CT\end{aligned}$$

as desired. □

Next, we show that the hyperbolic metric on a triangulated surface admits a nicely behaved covering by hyperbolic balls.

Lemma 2.6.2. *Let X be a hyperbolic surface of genus g and suppose X satisfies the property that if $\gamma_1, ..., \gamma_n$ are simple closed geodesics on (X, ρ_X) satisfying* $\text{length}_{\rho_X}(\gamma_j) \leq 2 \operatorname{arcsinh}(1)$, *then*

$$\sum_{j=1}^{n} \text{length}_{\rho_X}(\gamma_j)^{-1} \leq R.$$

There exist hyperbolic disks $U_1, ..., U_N$, $W_1, ..., W_N$ and $V_1, ..., V_N$ on X such that the following conditions are satisfied:

1. *The $\{V_i\}$ together cover X.*

2. $\text{center}(U_i) = \text{center}(W_i) = \text{center}(V_i)$ *and* $\text{radius}(U_i) = 2\,\text{radius}(W_i) = 4\,\text{radius}(V_i) \leq \operatorname{arcsinh}(1)/2$

3. *If U_i nontrivially intersects U_j, then* $\text{radius}(U_i) \leq C\,\text{radius}(U_j)$.

4. *Any point $x \in X$ is contained in at most C of the U_i.*

5. *Any U_i nontrivially intersects at most C of the U_j.*

6. *$N \leq C(R + g)$.*

Here, C is a universal constant.

Proof. We divide X into the thick part (where the injectivity radius is at least $\operatorname{arcsinh}(1)$) and its complement the thin part, which is comprised of disjoint annuli we label $A_1, ..., A_n$.

On the thick part (which has area at most Cg), we take a maximal $\operatorname{arcsinh}(1)/16$-separated set (which has around Cg points). We let the V_i be radius $\operatorname{arcsinh}(1)/8$ disks around the points in the separated set, the W_i radius $\operatorname{arcsinh}(1)/4$ disks around these points and the U_i radius $\operatorname{arcsinh}(1)/2$ disks around these points.

On the thin part, by the collar theorem, each annulus A_j is the collar region around a geodesic of length $\text{length}_{\rho_X}(\gamma_j)$ and width approximately $C \log(\text{length}_{\rho_X}(\gamma_j)^{-1})$. We have coordinates (r, θ) on A_j where $r = 0$ on γ_j and $r \in [w(A_j)/2, w(A_j)/2]$ such that the hyperbolic metric on A_j is $d\rho_X^2 = dr^2 + \cosh^2 r d\theta^2$. The injectivity radius of A_j at any point (r, θ) (denoted $\text{injrad}_{(r,\theta)}(A_j)$) is around $C\,\text{length}_{\rho_X}(\gamma_j) \cosh r$.

Now, for $s \in \mathbb{N} \cap [-w(A_j)/2, w(A_j)/2]$, let $A_j^s = A_j \cap \{r \in [s-1, s+1]\}$. Denote by $\operatorname{injrad}(A_j^s)$ the injectivity radius of A_j^s. Choose a $\operatorname{injrad}(A_j^s)/16$-separated set on A_j^s to be the centers of V_i, so that the V_i are radius $\operatorname{injrad}(A_j^s)/8$ disks around these centers, the W_i are radius $\operatorname{injrad}(A_j^s)/4$ disks, and the U_i are radius $\operatorname{injrad}(A_j^s)/2$ disks. In each A_j^s there are at most $(C \operatorname{length}_{\rho_X}(\gamma_j) \cosh s)^{-1}$ centers. In total, there are at most

$$\begin{aligned} C \sum_{s=1}^{\lceil w(A_j) \rceil} (\operatorname{length}_{\rho_X}(\gamma_j) \cosh s)^{-1} &\leq C \operatorname{length}_{\rho_X}(\gamma_j)^{-1} \int_{s=0}^{\infty} (\cosh s)^{-1} \\ &\leq C \operatorname{length}_{\rho_X}(\gamma_j)^{-1} \end{aligned}$$

centers in A_j.

We simply take all the U_i, W_i and V_i constructed on the thick part and thin part. By construction, conditions 1, 2, 3, 4 and 5 are satisfied. The total number of U_i (or W_i or V_i) is at most

$$C \left(g + \sum_{j=1}^{n} \operatorname{length}_{\rho_X}(\gamma_j)^{-1} \right) \leq C(R+g)$$

which shows condition 6. □

2.7 Conformal doubles and triangulated surfaces

Denote by $S_{g,b}$ a surface of genus g with b boundary components. Let X be a Riemann surface of genus g with b boundary components, meaning that X is homeomorphic to $S_{g,b}$. The conformal mirror X^{-1} of X is the Riemann surface whose local coordinates are obtained by composing each local coordinate z on X with the anti-holomorphic map $z \to \overline{z}$. The conformal double X^d is obtained by gluing X and X^{-1} along their boundaries ∂X and ∂X^{-1} which are canonically identified. The complex structure on X^d in a neighborhood U of a point $p \in \partial X$ is given as follows: if z is a local coordinate on X under which $U \cap X$ is conformally a half-disk around p, then the local coordinate on U which is z on $U \cap X$ and $\overline{z}$ on $U \cap X^{-1}$ identifies U with a disk, thus giving a complex structure on U which can be checked to be independent of the choice of local coordinate z. In this way, we obtain a complex structure on X^d. Then X^d is a compact genus $2g+b-1$ surface which has an anti-holomorphic involution fixing $\partial X = \partial X^{-1} \subset X^d$. We have the following statement about conformal doubles of triangulated surfaces.

Proposition 2.7.1. *Suppose S is a triangulated surface with boundary. Then S^d is canonically a triangulated surface without boundary.*

Proof. Denote by S^{-1} the triangulated surface wherein each triangle of S is equipped with the opposite orientation. Since ∂S and ∂S^{-1} are naturally identified, gluing S and S^{-1} under this identification produces a triangulated surface S^d which can be checked to be the conformal double of S. □

2.8 Triangulated surfaces with boundedness properties

In this section we define several subsets of $\mathrm{Comb}^T(\mathcal{T}_g)$ that shall be useful in the proof of Theorem 1.1.2.

Given an equilateral triangle in $\mathbb{C}$ of side length ℓ, a k-subdivision of the equilateral triangle is the unique triangulation by k^2 equilateral triangles of side length ℓ/k. Given a triangulation of a surface by flat equilateral triangles of side length ℓ, a k-subdivision of the triangulation is the new triangulation of the surface formed by the k-subdivisions of each triangle in the original triangulation.

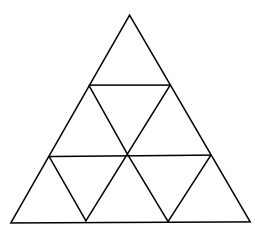

Figure 2-3: A 3-subdivision of an equilateral triangle.

Let $\mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{T}_g)$ (resp. $\mathrm{Comb}^{\leq T}_{\mathrm{lb}}(\mathcal{T}_{\leq g})$) to be the subset of $S \in \mathrm{Comb}^T(\mathcal{T}_g)$ (resp. $S \in \mathrm{Comb}^{\leq T}(\mathcal{T}_{\leq g})$) which satisfies the following two properties:

1. The maximum degree of any vertex of S is 7 and

2. There exists a triangulation S_{lb} of the surface S into equilateral triangles of side length 3 such that the triangulation S is a 3-subdivision S_{lb}.

We may call these surfaces locally bounded triangulated surfaces since the degrees of vertices are bounded. Similarly, we also define locally bounded combinatorial translation surfaces. First, we have a preliminary lemma that motivates the definition.

Lemma 2.8.1. *Let S be a combinatorial translation surface. Let $V_{>6}(S)$ be the set of vertices of S with degree greater than 6. If γ is an arc representing an element of $H_1(S, V_{>6}(S), \mathbb{Z})$, then $\int_\gamma \phi_S \in \mathbb{Z} + e^{\pi i/3}\mathbb{Z}$.*

Proof. Note that any arc γ on S with $\partial\gamma \subset V_{>6}(S)$ is homotopic to a piecewise smooth arc wherein each piece is an edge of S. Since dz integrates to an element of $\mathbb{Z} + e^{\pi i/3}\mathbb{Z}$ along sides of the equilateral triangle with vertices $0, 1, \frac{1}{2} + \frac{\sqrt{3}}{2}i$ in $\mathbb{C}$, $\int_\gamma \phi_S \in \mathbb{Z} + e^{\pi i/3}\mathbb{Z}$. □

Now, define $\mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g)$ (resp. $\mathrm{Comb}^{\leq T}_{\mathrm{lb}}(\mathcal{H}_{\leq g})$) to be the set of $S \in \mathrm{Comb}^T(\mathcal{H}_g)$ (resp. $S \in \mathrm{Comb}^{\leq T}(\mathcal{H}_{\leq g})$) which satisfy the following two properties:

1. The maximum degree of any vertex of S is 42 and
2. Given any two vertices $x, y \in V_{>6}(S)$ (not necessarily distinct) and γ any arc from x to y,
$$\int_\gamma \phi_S \in 3\mathbb{Z} + 3e^{\pi i/3}\mathbb{Z}.$$

A combinatorial translation surface which satisfies these two properties is called a locally bounded combinatorial translation surface.

Remark 9. Given Lemma 2.8.1, one would expect condition 2 in the definition of locally bounded combinatorial translation surfaces to be the translation surface version of condition 2 in the definition of locally bounded triangulated surfaces. More rigorously, we shall see in Section 8.4 that triangulated surfaces have a canonical branched 6-cover which is a combinatorial translation surface. Locally bounded combinatorial translation surfaces are defined so that the canonical branched 6-covers of locally bounded triangulated surfaces are locally bounded combinatorial translation surfaces. See Proposition 8.2.1.

It is also useful to consider triangulated surfaces for which the number of vertices of degree other than 6 is bounded. (Note that combinatorial translation surfaces of genus g

automatically satisfy the property that the number of vertices of degree other than 6 is bounded by Cg.)

Define $\mathrm{Comb}^{\leq T, \leq m}(\mathcal{T}_{\leq g})$ to be the set of $S \in \mathrm{Comb}^{\leq T}(\mathcal{T}_{\leq g})$ which satisfy the property that $|V_{\neq 6}(S)| \leq m$. Also, define $\mathrm{Comb}_{\mathrm{lb}}^{\leq T, \leq m}(\mathcal{T}_{\leq g})$ to be the set of $S \in \mathrm{Comb}_{\mathrm{lb}}^{\leq T}(\mathcal{T}_{\leq g})$ which satisfy the property that $|V_{\neq 6}(S)| \leq m$. Euler characteristic conditions imply that $|V(S)| \leq T/2$ when $g \geq 1$, so in the future, we will implicitly assume the condition $m/T \leq 1/2$ when $g \geq 1$.

2.9 Counting functions for the image of Φ and roadmap to prove Theorem 1.1.2

Define

$$N^{\mathcal{M}}(T, g, r) = \sup\{\#(\{S \in \mathrm{Comb}^{\leq T}(\mathcal{M}_{\leq g}) | \Phi(S) \in B_{d_T}(X, r)\}) | X \in \mathcal{M}_{\leq g}\},$$

$$N^{\mathcal{T}}(T, g, r) = \sup\{\#(\{S \in \mathrm{Comb}^{\leq T}(\mathcal{T}_{\leq g}) | \Phi(S) \in B_{d_T}(X, r)\}) | X \in \mathcal{T}_{\leq g}\},$$

$$N_{\mathrm{lb}}^{\mathcal{T}}(T, g, r) = \sup\{\#(\{S \in \mathrm{Comb}_{\mathrm{lb}}^{\leq T}(\mathcal{T}_{\leq g}) | \Phi(S) \in B_{d_T}(X, r)\}) | X \in \mathcal{T}_{\leq g}\},$$

$$N^{\mathcal{T}}(T, g, m, r) = \sup\{\#(\{S \in \mathrm{Comb}^{\leq T, \leq m}(\mathcal{T}_{\leq g}) | \Phi(S) \in B_{d_T}(X, r)\}) | X \in \mathcal{T}_{\leq g}\},$$

$$N_{\mathrm{lb}}^{\mathcal{T}}(T, g, m, r) = \sup\{\#(\{S \in \mathrm{Comb}_{\mathrm{lb}}^{\leq T, \leq m}(\mathcal{T}_{\leq g}) | \Phi(S) \in B_{d_T}(X, r)\}) | X \in \mathcal{T}_{\leq g}\}$$

and

$$N_{\mathrm{lb}}^{\mathcal{H}}(T, g, r) = \sup\{\#(\{S \in \mathrm{Comb}_{\mathrm{lb}}^{\leq T}(\mathcal{H}_{\leq g}) | \Phi(S) \in B_{d_T}(X, r)\}) | X \in \mathcal{T}_{\leq g}\}.$$

Remark 10. Here, d_T denotes the Teichmüller metric on moduli space $\mathcal{M}_g$ and Teichmüller space $\mathcal{T}_g$, explained in Section 4 in more detail. The Teichmüller metric is only defined for $g \geq 2$. For the purpose of notation, in our counting functions when we count genus g triangulated surfaces for $g = 0, 1$ we simply count all the surfaces and omit the radius variable.

To show Theorem 1.1.2, it is necessary to find an upper bound for $N^{\mathcal{M}}(T, g, r)$. We do by proving several bounds related to the other quantities defined above. In Chapter 4, we bound $N^{\mathcal{M}}$ in terms of $N^{\mathcal{T}}$. In Chapter 6, we bound $N^{\mathcal{T}}$ in terms of $N_{\mathrm{lb}}^{\mathcal{T}}$. In Chapter 7, we bound $N_{\mathrm{lb}}^{\mathcal{H}}$ in terms of $N^{\mathcal{T}}$. In Chapter 8, we bound $N_{\mathrm{lb}}^{\mathcal{T}}$ in terms of $N_{\mathrm{lb}}^{\mathcal{H}}$. Meanwhile, in Chapter 4 we prove bounds about the size of Teichmüller balls in $\mathcal{T}_g$ which shall be useful for the bounds in Chapter 7 and Chapter 8 as well. Finally, in Chapter 8, we combine all these bounds and use a recursive argument to prove Theorem 1.1.2.

Chapter 3

Properties of quasiconformal maps

3.1 Quasiconformal maps

Let U and V be Riemann surfaces and $K \geq 1$. An orientation preserving diffeomorphism $f : U \to V$ is K-quasiconformal if it satisfies

$$\left|\frac{\partial f}{\partial \overline{z}}\right| \leq k \left|\frac{\partial f}{\partial z}\right|$$

where $k = (K-1)/(K+1)$ and z is a holomorphic coordinate on U. Given conformal metrics on U and V, this means locally there exist oriented orthonormal bases on U and V with respect to which df has singular values λ_1 and λ_2 satisfying $K^{-1} \leq \lambda_1/\lambda_2 \leq K$. The smallest such quantity K is called the quasiconformal dilatation.

Remark 11. Quasiconformal maps are generally required to be homeomorphisms only. See [23, Sections 4.1 and 4.5] for definitions in this setting. In future sections, we explicitly use only the definition we have given above, but we implicitly use the definition in the homeomorphism setting in Section 3.4, Section 3.5 and Section 4.3.

3.2 Hodge norm on $H^1(X, \mathbb{C})$

Let X be a compact Riemann surface of genus $g \geq 2$ and ρ_X the hyperbolic metric on X. In this section we record certain constructions from Hodge theory on Riemann surfaces. See

[15] for a detailed exposition. Let $\mathcal{E}^1(X,\mathbb{C})$ denote the space of complex valued differential 1-forms on X; also denote closed and exact 1-forms by $Z^1(X,\mathbb{C})$ and $B^1(X,\mathbb{C})$, respectively. The Hodge star

$$*:\mathcal{E}^1(X,\mathbb{C})\to\mathcal{E}^1(X,\mathbb{C})$$

is an $\mathbb{C}$-linear map which fixes the space of harmonic 1-forms $\mathcal{H}^1(X,\mathbb{C})\subset\mathcal{E}^1(X,\mathbb{C})$. The Hodge inner product (a Hermitian inner product) on $\mathcal{E}^1(X,\mathbb{C})$ is defined as

$$\langle\omega_1,\omega_2\rangle_X=\int_X\omega_1\wedge*\overline{\omega_2}=\int_X\langle\omega_1,\omega_2\rangle_{\rho_X}d\rho_X^2$$

and Hodge norm is defined as

$$\|\omega\|_X^2=\int_X\omega\wedge*\overline{\omega}$$

for $\omega,\omega_1,\omega_2\in\mathcal{E}^1(X,\mathbb{C})$. By the Hodge decomposition theorem, the space of closed complex valued differential 1-forms on X splits as

$$\mathcal{Z}^1(X,\mathbb{C})=\mathcal{H}^1(X,\mathbb{C})\oplus\mathcal{B}^1(X,\mathbb{C}),$$

and the splitting is orthogonal with respect to the Hodge inner product. For a cohomology class

$$u\in\mathcal{Z}^1(X,\mathbb{C})/\mathcal{B}^1(X,\mathbb{C})\simeq H^1(X,\mathbb{C}),$$

we define the Hodge norm to be

$$\|u\|_X=\inf_{\omega\in u}\|\omega\|_X,$$

and the infimum is attained by the unique harmonic representative of u.

3.3 Quasiconformal maps and Hodge norm

In this section we study how the Hodge norm behaves under a quasiconformal map. All quasiconformal maps in this section are assumed to be diffeomorphisms. Let X and Y be compact Riemann surfaces of genus $g\geq 2$, with hyperbolic metrics ρ_X and ρ_Y, respectively.

Lemma 3.3.1. *If $f:X\to Y$ is a K-quasiconformal map and ω is a complex valued* *differential* 1*-form on* Y*, then*

$$(1/K)^{1/2}\|\omega\|_Y \leq \|f^*\omega\|_X \leq K^{1/2}\|\omega\|_Y.$$

Proof. We have,

$$\begin{aligned}\|f^*\omega\|_X^2 &= \int_X \langle f^*\omega, f^*\omega\rangle_{\rho_X}\rho_X^2 \\ &\leq \int_X \|df\|^2 f^*(\langle\omega,\omega\rangle_{\rho_Y})|\det(df)|^{-1} f^*\rho_Y^2 \\ &\leq K\int_Y \langle\omega,\omega\rangle_{\rho_Y}\rho_Y^2 \\ &= K\|\omega\|_Y^2.\end{aligned}$$

Applying an analogous argument to f^{-1}, which is also K-quasiconformal, we obtain $\|\omega\|_Y \leq K^{1/2}\|f^*\omega\|_X$. □

As a corollary, we have:

Corollary 3.3.2. *Let* $f : X \to Y$ *be a* K*-quasiconformal map. Let* $u \in H^1(Y, \mathbb{C})$*. Then*

$$(1/K)^{1/2}\|u\|_Y \leq \|f^*u\|_X \leq K^{1/2}\|u\|_Y.$$

Proof. Let ω be the harmonic form on Y representing u. Then by Lemma 3.3.1,

$$\begin{aligned}\|f^*u\|_X &\leq \|f^*\omega\|_X \\ &\leq K^{1/2}\|\omega\|_Y \\ &= K^{1/2}\|u\|_Y.\end{aligned}$$

The analogous argument applied to f^{-1}, gives $\|u\|_Y \leq K^{1/2}\|f^*u\|_X$. □

Finally, we show that the pullback of a harmonic form under a quasiconformal map is close to its harmonic representative.

Lemma 3.3.3. *Let $f : X \to Y$ be a K-quasiconformal map. Let ω be a harmonic 1-form on Y. Denote by $(f^*\omega)^h$ the unique harmonic 1-form on X cohomologous to $f^*\omega$. Then*

$$\|f^*\omega - (f^*\omega)^h\|_X \leq ((K^2-1)/K)^{1/2}\|\omega\|_Y.$$

Proof. We have,

$$\begin{aligned}\|f^*\omega - (f^*\omega)^h\|_X^2 &= \langle f^*\omega - (f^*\omega)^h, f^*\omega - (f^*\omega)^h\rangle_X \\ &= \langle f^*\omega, f^*\omega\rangle_X - \langle (f^*\omega)^h, (f^*\omega)^h\rangle_X - 2\langle f^*\omega - (f^*\omega)^h, (f^*\omega)^h\rangle_X.\end{aligned}$$

By Lemma 3.3.1 and Corollary 3.3.2,

$$\langle f^*\omega, f^*\omega\rangle_X - \langle (f^*\omega)^h, (f^*\omega)^h\rangle_X \leq ((K^2-1)/K)\|\omega\|_Y^2.$$

Exact forms are orthogonal to harmonic forms, so since $f^*\omega - (f^*\omega)^h$ is exact and $(f^*\omega)^h$ is harmonic,

$$2\langle f^*\omega - (f^*\omega)^h, (f^*\omega)^h\rangle_X = 0.$$

The lemma follows. □

3.4 Extending quasiconformal maps

In this section, we record two quasiconformal extension theorems.

Let (U, d_U) and (V, d_V) be metric spaces bi-Lipschitz to domains in $\mathbb{C}$. An embedding $f : U \to V$ is called K-weakly-quasisymmetric if for all $x, y, z \in U$,

$$d_V(f(x), f(y)) \leq K d_V(f(x), f(z))$$

if $d_U(x, y) \leq d_U(x, z)$. When U and V are oriented topological manifolds, we will assume f is orientation preserving. When U and V are simply domains in $\mathbb{C}$, the metrics on U and V are taken to be the restrictions of the Euclidean metric to U and V, respectively. Then weakly-quasisymmetric maps of the unit circle may be extended to quasiconformal maps of the unit disk:

Proposition 3.4.1 ([1], Theorem IV.B.2)**.** *Suppose that $f' : S^1 \to S^1$ is a K-weakly-quasisymmetric map. Then f' extends to a $C(K)$-quasiconformal map $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ where $C(K)$ only depends on K.*

The following statement describes when it is possible to extend a quasiconformal map on a subset of the plane to the entire plane.

Proposition 3.4.2 ([28], Theorem 2.8.1)**.** *Let $U, V \subset \mathbb{C}$ be open domains. Let $f' : U \to V$ be a K-quasiconformal mapping and $W \subset U$ a compact subset. Then there exists a $C(U, W, K)$-quasiconformal mapping from $\mathbb{C}$ to $\mathbb{C}$ which coincides with f on W. Here, $C(U, W, K)$ is a constant that depends only on K, the domain U and the set W.*

3.5 Quasicircles, quasiarcs and bounded-turning curves

In the complex plane, a K-quasicircle (resp. K-quasiarc) is the image of the unit circle (resp. unit line segment) under a K-quasiconformal homeomorphism of the plane.

Let (U, d_U) be a metric space bi-Lipschitz and homeomorphic to a domain in $\mathbb{C}$. When U is simply a domain in $\mathbb{C}$, d_U is taken to be the restriction of the Euclidean metric to U. A simple arc γ in U is a K-bounded-turning curve if it satisfies the condition that for all $x, y, z \in \gamma$ lying in order,

$$d_U(x, y) + d_U(y, z) \leq K d_U(x, z).$$

A simple closed curve γ in U is a K-bounded-turning curve if every point of γ lies in the interior of a subarc of γ that is a K-bounded-turning curve.

In the rest of this section, we note some results about quasicircles, quasiarcs, bounded-turning curves and quasisymmetric maps that shall be useful in later sections.

The following statement relates quasicircles and quasiarcs to the bounded-turning property. A proof can be found in [28, Sections 2.8.7, 2.8.8 and 2.8.9].

Proposition 3.5.1. *Suppose a simple closed curve (resp. simple arc) γ in $\mathbb{C}$ is a K-bounded-turning curve. Then γ is a $C(K)$-quasicircle (resp. $C(K)$-quasiarc) where $C(K)$ is a constant only depending on K. Similarly, suppose a simple closed curve (resp. simple arc) γ is* *a K-quasicircle (resp. K-quasiarc). Then it is a $C(K)$-bounded-turning curve.*

The following result asserts that bounded-turning curves are well-behaved under quasiconformal maps.

Proposition 3.5.2. *Let (U, d_U) and (V, d_V) be Riemann surfaces with boundary, equipped with metrics d_U and d_V such that there exists homeomorphisms of U and V to the closed unit disk that are bi-Lipschitz and quasiconformal. Suppose $f : U \to V$ is a K_1-quasiconformal embedding and $\gamma \subset \text{interior}\,(U)$ is a K_2-bounded-turning curve in U. Then $f(\gamma)$ is a $C(U, \gamma, V, K_1, K_2)$-bounded-turning curve in V, where $C(U, \gamma, V, K_1, K_2)$ is a constant that only depends on U, γ, V, K_1 and K_2.*

Proof. By assumption, there exist homeomorphism $f_U : U \to \overline{\mathbb{D}}$ and $f_V : V \to \overline{\mathbb{D}}$ such that f_U is $C(U)$-bi-Lipschitz between the d_U-metric on U and the Euclidean metric on $\mathbb{C}$, while f_V is $C(V)$-bi-Lipschitz between the d_V-metric on V and the Euclidean metric on $\mathbb{C}$. Since γ is K_2-bounded-turning in U, $f_U(\gamma)$ is $C(U, K_2)$-bounded-turning in $\mathbb{D}$. By Proposition 3.4.2 applied to $f_V \circ f \circ f_U^{-1} : \text{interior}\,(f_U(U)) \to \text{interior}\,(f_V(V))$ and the compact set $f_U(\gamma) \subset f_U(U)$, there exists a $C(U, \gamma, V, K_1, K_2)$-quasiconformal map $\mathbb{C} \to \mathbb{C}$ which agrees with $f_V \circ f \circ f_U^{-1}$ on $f_U(\gamma)$. This means $(f_V \circ f)(\gamma)$ is $C(U, \gamma, V, K_1, K_2)$-bounded-turning in $\mathbb{C}$. Hence $f(\gamma)$ is $C(U, \gamma, V, K_1, K_2)$-bounded-turning in V. □

We have the following extension result about quasisymmetric maps.

Lemma 3.5.3. *Let $U_1 \subset \mathbb{C}$ and $U_2 \subset \mathbb{C}$ be simply connected open domains such that ∂U_1 is a K_1-quasicircle and ∂U_2 is a K_2-quasicircle. Suppose $f' : \partial U_1 \to \partial U_2$ is a K_3-weakly-quasisymmetric homeomorphism. Then f' extends to a $C(K_1, K_2, K_3)$-quasiconformal map $f : \overline{U_1} \to \overline{U_2}$. Here, $C(K_1, K_2, K_3)$ is a constant that depends only on K_1, K_2 and K_3.*

Proof. Simply take $f_1 : \mathbb{C} \to \mathbb{C}$ to be a K_1-quasiconformal mapping taking $\overline{U_1}$ to $\overline{\mathbb{D}}$. Then f_1 is $C(K_1)$-weakly-quasisymmetric for some $C(K_1)$ only depending on K_1 (see [47], Theorem 18.1). Similarly, take $f_2 : \mathbb{C} \to \mathbb{C}$ to be a K_2-quasiconformal mapping taking $\overline{U_2}$ to $\overline{\mathbb{D}}$. Then f_2 is $C(K_2)$-weakly-quasisymmetric. Then identifying $\partial\mathbb{D} \simeq S^1$, the map $f_2 \circ f' \circ f_1^{-1} : S^1 \to S^1$ is $C(K_1, K_2, K_3)$-weakly-quasisymmetric. (Compositions and inverses of weakly-quasisymmetric maps on subsets of the plane are weakly-quasisymmetric, quantitatively, by

Proposition 10.6 and Corollary 10.22 in [22].) By Proposition 3.4.1, $f_2 \circ f' \circ f_1^{-1}$ extends to a $C(K_1, K_2, K_3)$-quasiconformal map $f'' : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. Taking $f = f_2^{-1} \circ f'' \circ f_1$ gives a $C(K_1, K_2, K_3)$-quasiconformal map from $\overline{U_1}$ to $\overline{U_2}$ that is an extension of f'. □

We have another extension result about quasisymmetric maps.

Lemma 3.5.4. *Let γ_1 be a K_1-quasicircle in the plane and let γ_2 be a K_2-quasicircle in the plane. Let $\gamma_1' \subset \gamma_1$ and $\gamma_2' \subset \gamma_2$ be connected subarcs which are closed subsets of γ_1 and γ_2, respectively. Let $f' : \gamma_1' \to \gamma_2'$ be a K_3-weakly-quasisymmetric homeomorphism. Then f' extends to a $C(K_1, K_2, K_3)$-weakly-quasisymmetric homeomorphism $f : \gamma_1 \to \gamma_2$. Here, $C(K_1, K_2, K_3)$ is a constant that only depends on K_1, K_2 and K_3.*

Proof. We first reduce to the case wherein γ_1 and γ_2 are S^1, the unit circle in $\mathbb{C}$. To do this, take $f_1 : \mathbb{C} \to \mathbb{C}$ to be a K_1-quasiconformal mapping taking γ_1 to S^1. Then f_1 is $C(K_1)$-weakly-quasisymmetric for some $C(K_1)$ only depending on K_1 (see [47], Theorem 18.1). Similarly, take $f_2 : \mathbb{C} \to \mathbb{C}$ to be a K_2-quasiconformal mapping taking γ_2 to S^1. Then f_2 is $C(K_2)$-weakly-quasisymmetric. So $f_2 \circ f' \circ f_1^{-1}$ is $C(K_1, K_2, K_3)$-weakly-quasisymmetric. Applying the lemma statement to $f_2 \circ f' \circ f_1^{-1} : f_1(\gamma_1') \subset S^1 \to f_2(\gamma_2') \subset S^1$ then composing the extension with f_2^{-1} and f_1 gives the statement in the case of general γ_1 and γ_2.

Henceforth, we assume $\gamma_1 = S^1$ and $\gamma_2 = S^1$. Denote $S^{1,+} = \{z \in S^1 | \operatorname{Im}(z) \geq 0\}$ and $S^{1,-} = \{z \in S^1 | \operatorname{Im}(z) \leq 0\}$. We may reduce to the case wherein $\gamma_1', \gamma_2' = S^{1,+}$. This can be done similar to the previous reduction, but instead by composing with two appropriate fractional linear transformations in $\operatorname{Isom}_+(\mathbb{D})$, one which takes γ_1' to $S^{1,+}$ and the other which takes $\gamma_2' \to S^{1,+}$. The arcs γ_1' and γ_2' can be arbitrarily small or with arbitrarily small complements in γ_1 and γ_2, so we may need to choose a fractional linear transformation that sends 0 arbitrarily close to S^1. All fractional linear transformations in $\operatorname{Isom}_+(\mathbb{D})$ are C-weakly-quasisymmetric on S^1.

So, we must now show the following statement. If $f' : S^{1,+} \to S^{1,+}$, then it extends to a $C(K_3)$-weakly-quasisymmetric map $f : S^1 \to S^1$. To do this, for $z \in S^{1,-}$, define $f(z) = \overline{f'(\overline{z})}$. To show that f is $C(K_3)$-weakly-quasisymmetric, we must show that for all $x, y, z \in S^1$, if $|x - y| \leq |x - z|$ then $|f(x) - f(y)| \leq C(K_3)|f(x) - f(z)|$. When $x, y, z \in S^{1,+}$, the weak-quasisymmetricity condition follows since f' is K_3-weakly-quasisymmetric, and similarly when $x, y, z \in S^{1,-}$. There are three nontrivial cases to check.

Case 1: $x, y \in S^{1,+}$ **and** $z \in S^{1,-}$**.** Suppose $|x-y| \le |x-z|$. Assume without loss of generality that $\operatorname{Re} f(x) \ge 0$ (the other case is analogous). Then since

$$\begin{aligned}|x-z| &\le |x-\overline{x}| + |\overline{x}-z| \\ &\le C|x-1| + |x-\overline{z}|,\end{aligned}$$

either

1. $|x-y| \le C|x-1|$ or
2. $|x-y| \le C|x-\overline{z}|$.

In the first scenario, Corollary 10.22 in [22] implies

$$\begin{aligned}|f(x)-f(y)| &\le C(K_3)|f(x)-1| \\ &\le C(K_3)|f(x)-f(z)|,\end{aligned}$$

where the second inequality follows from the fact that since $f(x) \in S^{1,+}$, $f(z) \in S^{1,-}$ and $\operatorname{Re} f(x) \ge 0$, the angle from $f(x)$ to 1 to $f(z)$ at the point 1 is bounded below by a constant. In the second scenario also, Corollary 10.22 in [22] implies

$$\begin{aligned}|f(x)-f(y)| &\le C(K_3)|f(x)-\overline{f(z)}| \\ &\le C(K_3)|f(x)-f(z)| + C(K_3)|f(z)-\overline{f(z)}| \\ &\le C(K_3)|f(x)-f(z)| + C(K_3)|f(z)-1| \\ &\le C(K_3)|f(x)-f(z)|.\end{aligned}$$

Case 2: $x, z \in S^{1,+}$ **and** $y \in S^{1,-}$**.** Suppose $|x-y| \le |x-z|$. Assume without loss of generality that $\operatorname{Re} x \ge 0$ (the other case is analogous). Then

$$\begin{aligned}|x-\overline{y}| &\le |x-y| \\ &\le |x-z|,\end{aligned}$$

where the first inequality follows from the fact that $x \in S^{1,+}$ and $y \in S^{1,-}$. Moreover, since $\operatorname{Re} x \geq 0$,

$$\begin{aligned} |x-1| &\leq C|x-y| \\ &\leq C|x-z|. \end{aligned}$$

By Corollary 10.22 in [22],

$$|f(x) - \overline{f(y)}| \leq C(K_3)|f(x) - f(z)|$$

and

$$|f(x) - 1| \leq C(K_3)|f(x) - f(z)|.$$

Thus,

$$\begin{aligned} |f(x) - f(y)| &\leq |\overline{f(x)} - f(y)| + |f(x) - \overline{f(x)}| \\ &\leq |f(x) - \overline{f(y)}| + C|f(x) - 1| \\ &\leq C(K_3)|f(x) - f(z)|. \end{aligned}$$

Case 3: $y, z \in S^{1,+}$ **and** $x \in S^{1,-}$**.** Suppose $|x-y| \leq |x-z|$. Assume without loss of generality that $\operatorname{Re} f(x) \geq 0$ (the other case is analogous). Then

$$\begin{aligned} |\overline{x} - y| &\leq |x-y| + |\overline{x} - x| \\ &\leq |x-z| + |\overline{x} - x| \\ &\leq |\overline{x} - z| + C|\overline{x} - x| \\ &\leq |\overline{x} - z| + C|\overline{x} - 1|, \end{aligned}$$

so either

1. $|\overline{x} - y| \leq C|\overline{x} - z|$ or

2. $|\overline{x} - y| \leq C|\overline{x} - 1|$.

In the first scenario, by Corollary 10.22 in [22], we have

$$|\overline{f(x)} - f(y)| \leq C(K_3)|\overline{f(x)} - f(z)|$$

so

$$\begin{aligned}
|f(x) - f(y)| &\leq |\overline{f(x)} - f(y)| + |\overline{f(x)} - f(x)| \\
&\leq |\overline{f(x)} - f(y)| + C|\overline{f(x)} - 1| \\
&\leq C(K_3)|\overline{f(x)} - f(z)| + C|f(x) - 1| \\
&\leq C(K_3)|f(x) - f(z)|
\end{aligned}$$

where the last inequality follows from the assumption that $\operatorname{Re} f(x) > 0$; since $f(x)$ lies on S^1, the angle between $f(x)$, 1 and $f(z)$ at the point 1 must be bounded below. In the second scenario, By Corollary 10.22 in [22], we have

$$|\overline{f(x)} - f(y)| \leq C(K_3)|\overline{f(x)} - 1|$$

so

$$\begin{aligned}
|f(x) - f(y)| &\leq |\overline{f(x)} - f(y)| + |\overline{f(x)} - 1| \\
&\leq C(K_3)|\overline{f(x)} - 1| \\
&\leq C(K_3)|f(x) - 1| \\
&\leq C(K_3)|f(x) - f(z)|.
\end{aligned}$$

The other three cases $x, y \in S^{1,-}$ and $z \in S^{1,+}$; $x, z \in S^{1,-}$ and $y \in S^{1,+}$; and $y, z \in S^{1,-}$ and $x \in S^{1,+}$ follow analogously. Thus $f : S^1 \to S^1$ is $C(K_3)$-weakly-quasisymmetric. □

As a corollary, we have:

Corollary 3.5.5. *Let (U, d_U) and (V, d_V) be Riemann surfaces with boundary, equipped with metrics d_U and d_V such that there exist homeomorphisms of U and V to the closed unit disk that are bi-Lipschitz and quasiconformal. Let $W \subset U$ and $Z \subset V$ be simply connected open* *domains such that ∂W is a K_1-bounded-turning curve in U and ∂Z is a K_2-bounded-turning curve in V. Let $\gamma_W \subset \partial W$ and $\gamma_Z \subset \partial Z$ be connected arcs that are closed subsets of ∂W and ∂Z, respectively. Suppose $f' : \gamma_W \to \gamma_Z$ is a K_3-weakly-quasisymmetric homeomorphism. Then f' extends to a $C(U, V, K_1, K_2, K_3)$-quasiconformal homeomorphism $f : W \to Z$. Here, $C(U, V, K_1, K_2, K_3)$ is a constant that depends only on U, V, K_1, K_2 and K_3.*

Proof. By assumption, there exists $f_U : U \to \overline{\mathbb{D}}$, a $C(U)$-quasiconformal and $C(U)$-bi-Lipschitz homeomorphism between the d_U-metric on U and the Euclidean metric on $\mathbb{C}$. There also exists $f_V : V \to \overline{\mathbb{D}}$, a $C(V)$-quasiconformal and $C(V)$-bi-Lipschitz homeomorphism between the d_V-metric on V and the Euclidean metric on $\mathbb{C}$. Now, $f_U(\partial W)$ is $C(U, K_1)$-bounded-turning in $\mathbb{D}$, therefore a $C(U, K_1)$-quasicircle. Similarly, $f_V(\partial Z)$ is $C(V, K_2)$-bounded-turning in $\mathbb{D}$, therefore a $C(V, K_2)$-quasicircle. We now claim that $f_V \circ f' \circ f_U^{-1} : f_U(\gamma_W) \to f_V(\gamma_Z)$ is $C(U, V, K_3)$-weakly-quasisymmetric; this follows from the fact that f_U and f_V are bi-Lipschitz along with Corollary 10.22 in [22]. By Lemma 3.5.4, we see that $f_V \circ f' \circ f_U^{-1}$ extends to a $C(U, V, K_1, K_2, K_3)$-quasisymmetric map from $f_U(\partial W)$ to $f_V(\partial Z)$; by Lemma 3.5.3, we have an extension to a $C(U, V, K_1, K_2, K_3)$-quasiconformal map from $f_U(W)$ to $f_V(Z)$. Precomposing with f_V^{-1} and postcomposing with f_U we obtain the desired extension of f' in the lemma statement. □

Chapter 4

Large genus geometry of Teichmüller space

In this chapter we give a brief review of definitions in Teichmüller theory and describe the large genus geometry of Teichmüller space. Detailed exposition can be found in [1], [19] and [23].

4.1 Teichmüller space and Teichmüller metric

Let $g \geq 2$ and S_g a smooth oriented genus g surface. Let X and Y be marked Riemann surfaces of genus g. Two marked surfaces X and Y are considered equivalent if there is a biholomorphism of marked surfaces $f : X \to Y$. Then $\mathcal{T}_g$ is the set of equivalence classes of marked Riemann surfaces. The Teichmüller distance, denoted d_T, is given by

$$d_T(X, Y) = \inf \left\{ \frac{1}{2} \log K | f : X \to Y \text{ is } K\text{-quasiconformal} \right\}$$

where $f : X \to Y$ is a diffeomorphism of marked surfaces. Teichmüller's theorem asserts that the infimum is attained and is unique. The unique map that attains the infimum is called a Teichmüller map.

The space $\mathcal{T}_g$ admits a natural complex structure under which it is a complex manifold of dimension $3g - 3$. The cotangent space at a point X may be naturally identified with

$Q(X)$, the space of holomorphic quadratic differentials on X. Under this identification, the Teichmüller metric is the L^1-norm on $Q(X)$, given by

$$\|\phi\| = \int_X |\phi|$$

for a holomorphic quadratic differential ϕ on X.

Moduli space $\mathcal{M}_g$ can be obtained from $\mathcal{T}_g$ by quotienting by the action of the mapping class group Mod_g, under which the Teichmüller metric also descends to $\mathcal{M}_g$.

4.2 Extremal length and Teichmüller metric

The Teichmüller metric has a description in terms of extremal length, as we shall explain now. Let X be a Riemann surface of genus g. Let γ be simple closed curve (resp. the free homotopy class of a simple closed curve) on X. Given any Riemannian metric ρ on X, we define $\text{length}_\rho(\gamma)$ to be the infimum of lengths in the ρ-metric over all curves freely homotopic to (resp. representing) γ.

The extremal length of γ on X is defined to be

$$\text{Ext}_X(\gamma) = \sup_\rho \frac{\text{length}_\rho(\gamma)^2}{\text{area}_\rho(X)}$$

where the supremum is taken over all conformal metrics ρ on X. The next theorem describes when the supremum is achieved.

Theorem 4.2.1 (Jenkins [24], Strebel [46], see also Theorem 3.1 in [25])**.** *The supremum of*

$$\sup_\rho \frac{\text{length}_\rho(\gamma)^2}{\text{area}_\rho(X)}$$

is achieved when ρ is the flat metric $|\phi|$ corresponding to particular holomorphic quadratic differentials $\phi \in Q(X)$ called Jenkins-Strebel differentials.

Remark 12. A holomorphic quadratic differential has associated to it a horizontal foliation and vertical foliation [19, Section 2.1]. Jenkins-Strebel differentials are quadratic differentials for which almost every leaf of the horizontal foliation is closed. Here, we only need the fact that the extremal length is achieved by a flat metric associated to a quadratic differential, not the extra Jenkins-Strebel property.

The Teichmüller distance has the following description due to Kerckhoff ([25], Theorem 4).

Theorem 4.2.2. *For $X, Y \in \mathcal{T}_g$,*

$$d_T(X,Y) = \frac{1}{2}\log\sup_{\gamma}\frac{\mathrm{Ext}_Y(\gamma)}{\mathrm{Ext}_X(\gamma)},$$

where the supremum is taken over all free homotopy classes of a simple closed curve on X.

Remark 13. Since X and Y are marked surfaces, γ (which was initially defined as a free homotopy class on X) is also automatically a free homotopy class on Y.

4.3 bi-Lipschitz metric

Let $g \geq 2$. The bi-Lipschitz metric d_L on $\mathcal{T}_g$ is defined by

$$d_L(X,Y) = \inf\left\{\log K | f : X \to Y \text{ is a K-bi-Lipschitz map}\right\}$$

where the bi-Lipschitz constant is measured with respect to the unique hyperbolic metrics on X and Y, and the infimum is over all diffeomorphisms $f : X \to Y$ of marked surfaces. The metric d_L descends to $\mathcal{M}_g$ as well.

The following result establishes a comparison between d_T and d_L.

Proposition 4.3.1. *There exists a universal constant C such that for all g,*

$$d_T \leq d_L \leq C d_T$$

on $\mathcal{T}_g$.

Proof. The inequality $d_T \leq d_L$ follows from the fact that any K-bi-Lipschitz map is automatically K^2-quasiconformal. We must show that $d_L \leq C d_T$. This follows from the visual extension described in [31, Appendices A and B] as we shall explain now.

Let $V(S^1)$ be the space of continuous vector fields on $S^1 \simeq \partial\mathbb{D}$ and $V(\mathbb{D})$ the space of smooth vector fields on $\mathbb{D}$. Equipping $\mathbb{D}$ with the Poincare metric, a smooth strain field is a smooth section of $\mathrm{Sym}^2(T\mathbb{D})$. Let $S(\mathbb{D})$ be the space of smooth strain fields on $\mathbb{D}$. The space $V(S^1)$ admits an L^∞-norm coming from the standard Euclidean metric on S^1 considered as a unit circle in $\mathbb{R}^2$. The space $V(\mathbb{D})$ admits an L^∞-norm coming from the Poincare metric on $\mathbb{D}$. A fiber of a strain in $S(\mathbb{D})$ at a particular point admits an operator norm coming from the Poincare metric on $\mathbb{D}$. Taking the essential supremum over all points in $\mathbb{D}$ gives an L^∞-norm on $S(\mathbb{D})$. We denote all three norms described above by $\|\cdot\|_\infty$.

Now, there exists a nontrivial map $S : V(\mathbb{D}) \to S(\mathbb{D})$, unique up to scaling, that measures to what extend a vector field generates a quasiconformal flow. This map is constructed in [31] (A.2) and satisfies the following properties.

1. The map S is linear and continuous.

2. If $T \in \mathrm{Isom}_+(\mathbb{D})$ is a fractional linear transformation, then $T \circ S = S \circ T$.

3. The infinitesimal homeomorphism generated by $v \in V(\mathbb{D})$ is quasiconformal if and only if $\|Sv\|_\infty \leq \infty$, in which case the dilatation of the flow of v is given by $\|Sv\|_\infty$.

We say that v is a quasiconformal vector field if v satisfies one of the equivalent conditions in Property 3. If $Sv = 0$, then v is called a conformal vector field.

We have another nontrivial map $\mathrm{av} : V(S^1) \to V(\mathbb{D})$, unique up to scaling, called the visual extension which satisfies the following properties. This map is constructed in [31] (A).

4. The map av is linear and continuous.

5. If $T \in \mathrm{Isom}_+(\mathbb{D})$ is a fractional linear transformation, then $T \circ \mathrm{av} = \mathrm{av} \circ T$.

6. The infinitesimal flow generated by $\mathrm{av}\, v$ is area preserving with respect to the Poincare metric on $\mathbb{D}$.

(Property 6 is true by the proof of Theorem B.10 in [31]. While the theorem is stated only for higher dimensional spheres, the proof that $\mathrm{av}(v)$ is area-preserving also works in the case of S^1.)

On an infinitesimal level, Teichmüller maps from a point $X \in \mathcal{T}_g$ can be described by a continuous vector field v on the surface X generating a quasiconformal flow. Lift v to $\mathbb{D}$ (identified with $\mathbb{H}$) under a uniformization map $\mathbb{H} \to X$. By Theorem A.10 in [31], there exists a conformal vector field w on $\mathbb{D}$ such that $\|v - w\|_\infty \leq \|Sv\|_\infty$. (The theorem is stated for vector fields on $\mathbb{P}^1$ among other cases, but the proof also works in the case of $\mathbb{D}$.) Then w is an infinitesimal fractional linear transformation. Both v and w continuously extend to $\partial\mathbb{D} \simeq S^1$, giving continuous vector fields v' and w' (respectively) on S^1 satisfying $\|v' - w'\|_\infty \leq \|Sv\|_\infty$. Then $\mathrm{av}(v')$ gives another extension of v' to $\mathbb{D}$ that is smooth on the interior. Moreover, by Corollary B.6 in [31], $\mathrm{av}(w')$ is a conformal vector field on $\mathbb{D}$ hence $S\,\mathrm{av}(w') = 0$. Finally, $\mathrm{av}(v')$ is quasiconformal:

$$\begin{aligned}\|S\,\mathrm{av}(v')\|_\infty &= \|S\,\mathrm{av}(v' - w')\|_\infty \\ &\leq C\|v' - w'\|_\infty \\ &\leq C\|Sv\|_\infty\end{aligned}$$

for a universal constant C, by continuity of the map $S\,\mathrm{av}$. Combining with Property 6, we see that $\mathrm{av}(v')$ generates a $C\|Sv\|_\infty$-Lipschitz flow, and Property 5 implies that $\mathrm{av}(v')$ descends to a vector field on X. This vector field corresponds to a tangent direction of $\mathcal{T}_g$ at X that agrees with the tangent direction given by v, since $\mathrm{av}(v')$ and v agree on S^1. In this way, one may take an infinitesimal quasiconformal map on X and generate an infinitesimal bi-Lipschitz map on X with bi-Lipschitz constant bounded by the quasiconformal constant of the original map times a universal constant. This proves that $d_L \leq C d_T$ on $\mathcal{T}_g$. □

4.4 Kobayashi metric

Let M be a complex manifold of arbitrary dimension. Roughly speaking, the Kobayashi pseudometric is the largest pseudometric on M such that all holomorphic maps into M are distance decreasing. See [26, Section 4.1] for a rigorous construction. In general, the Kobayashi pseudometric may not be a metric (i.e. may not separate points). However, we shall see that in all cases of M relevant to us, the Kobayashi metric exists. In particular, the Kobayashi metric exists when M is a bounded domain in $\mathbb{C}^n$ [26, Corollary 4.4.6]. The Kobayashi metric satisfies the following important property.

Proposition 4.4.1 ([26], Proposition 4.1.1)**.** *Let M and N be complex manifolds. Suppose they admit Kobayashi metrics d_M and d_N, respectively. Let $f : M \to N$ be a holomorphic map. Then for all $x, y \in M$,*

$$d_M(x, y) \geq d_N(f(x), f(y)).$$

If f is a biholomorphism, then equality holds.

We also have the following theorem due to Royden ([45], Theorem 3).

Theorem 4.4.2. *On $\mathcal{T}_g$, the Kobayashi metric exists and is the Teichmüller metric d_T.*

4.5 Bers embedding of $\mathcal{T}_g$

Fix X, a compact hyperbolic Riemann surface of genus g. Recall that X^{-1} denotes the conformal mirror of X. Let $Q^\infty(X^{-1})$ denote the space of quadratic differentials on X^{-1}, equipped with the norm

$$\|\phi\|_\infty = \sup_{x \in X^{-1}} \frac{|\phi(x)|}{|\rho_{X^{-1}}|^2}.$$

(Here, $\rho_{X^{-1}}$ is the hyperbolic metric on X^{-1}.) The Bers embedding (see [19], 5.4) is a holomorphic embedding of $\mathcal{T}_g$ into $Q^\infty(X^{-1})$ sending $X \in \mathcal{T}_g$ to the origin in $Q^\infty(X^{-1})$.

Theorem 4.5.1 ([19], Theorem 5.4.1)**.** *The Bers embedding $\beta_X : \mathcal{T}_g \to Q^\infty(X^{-1})$ satisfies*

$$B_\infty(0, 1/2) \subset \beta_X(\mathcal{T}_g) \subset B_\infty(0, 3/2)$$

where $B_\infty(0, r)$ denotes the norm ball of radius r in the space $Q^\infty(X^*)$.

Remark 14. The constants in [19, Theorem 4.5.1] are different because the normalization of the Teichmüller metric is different. See [32, Theorem 2.2] for normalization and constants that agree with ours.

4.6 Asymptotic geometry of Teichmüller space

In this section, we use the Bers embedding to obtain bounds on the geometry of d_T on $\mathcal{T}_g$. This technique has been used to prove [32, Theorem 8.2] and [17, Theorem 1.5]. Our bounds are variations of the bounds in the latter.

Fix $X \in \mathcal{T}_g$. Denote by $Q = B_\infty(0,1)$ the open unit norm ball in $Q^\infty(X^{-1})$. Since $Q^\infty(X^{-1})$ is $3g-3$-dimensional, Q is an open $3g-3$ complex manifold. Let d_Q denote the Kobayashi metric on Q. Putting together Proposition 4.4.1, Theorem 4.4.2 and Theorem 4.5.1, we have the following two metric comparison results.

Lemma 4.6.1. *Let $Y, Z \in \mathcal{T}_g$. Then $\|\beta_X(Y)\|_\infty, \|\beta_X(Z)\|_\infty \leq 3/2$. Moreover,*

$$d_Q((2/3)\beta_X(Y), (2/3)\beta_X(Z)) \leq d_T(Y, X).$$

Lemma 4.6.2. *There exists a holomorphic map $\beta_X^{-1} : B_\infty(0,1/2) \to \mathcal{T}_g$ such that $\beta_X \circ \beta_X^{-1} =$ id on $B_\infty(0,1/2)$. Moreover, for $\phi_1, \phi_2 \in B_\infty(0,1/2)$,*

$$d_T(\beta_X^{-1}(\phi_1), \beta_X^{-1}(\phi_2)) \leq d_Q(2\phi_1, 2\phi_2).$$

The following lemma gives us an approximation for d_Q.

Lemma 4.6.3. *For $\phi_1, \phi_2 \in B_\infty(0,1/8)$,*

$$C_1\|\phi_1 - \phi_2\|_\infty \leq d_Q(\phi_1, \phi_2) \leq C_2\|\phi_1 - \phi_2\|_\infty$$

for universal constants C_1 and C_2.

Proof. First, we show that $d_Q(\phi_1, \phi_2) \geq C_1\|\phi_1 - \phi_2\|_\infty$. To see this, let $x \in X^{-1}$ (the conformal mirror of X), and let v be a unit tangent vector at x with respect to the hyperbolic metric $\rho_{X^{-1}}$ on X^{-1}. We define a map $I_x : B_\infty(0,1) \to \mathbb{D}$ that sends ϕ to the evaluation of ϕ at $v \otimes v$. Note that for all $\phi \in B_\infty(0,1)$,

$$|I_x(\phi)| = \frac{|\phi|}{|\rho_{X^{-1}}^2|}(x)$$

$$\leq 1$$

and for all $\phi \in B_\infty(0,1/2)$, $|I_x(\phi)| \leq 1/2$. Furthermore, I_x is complex linear, thus holomorphic. Therefore, by Proposition 4.4.1, I_x is distance decreasing with respect to d_Q on $B_\infty(0,1)$ and the Kobayashi metric on $\mathbb{D}$ which is the Poincare metric. So for $\phi_1, \phi_2 \in B_\infty(0,1/2)$,

$$\begin{aligned} d_Q(\phi_1,\phi_2) &\geq \rho_{\mathbb{D}}(I_x(\phi_1), I_x(\phi_2)) \\ &\geq C_1|I_x(\phi_1) - I_x(\phi_2)| \\ &= C_1\frac{|\phi_1 - \phi_2|}{|\rho_X|^2}(x). \end{aligned}$$

Since this is true for all $x \in X^{-1}$, $d_Q(\phi_1,\phi_2) \geq C_1\|\phi - \phi_2\|_\infty$ as desired.

Next, we show that for $\phi_1, \phi_2 \in B_\infty(0,1/8)$, $d_Q(\phi_1,\phi_2) \leq C_2\|\phi_1 - \phi_2\|_\infty$. To do this, consider the map $I : \mathbb{D} \to B_\infty(0,1)$ sending $\zeta \in \mathbb{D}$ to

$$\phi_1 + \zeta\frac{\phi_2 - \phi_1}{2\|\phi_2 - \phi_1\|_\infty}.$$

The image of I is contained in $B_\infty(0,1)$ because

$$\begin{aligned} \left\|\phi_1 + \zeta\frac{\phi_2 - \phi_1}{2\|\phi_2 - \phi_1\|_\infty}\right\|_\infty &\leq \|\phi_1\|_\infty + \left|\frac{\zeta}{2}\right| \\ &\leq 1. \end{aligned}$$

Note that I is complex linear and therefore holomorphic. Moreover, $I(0) = \phi_1$ and $I(2\|\phi_1 - \phi_2\|_\infty) = \phi_2$. By Proposition 4.4.1, I must be distance decreasing with respect to the Poincare metric on $\mathbb{D}$. Since $\phi_1, \phi_2 \in B_\infty(0,1/8)$, $2\|\phi_1 - \phi_2\|_\infty \leq 1/2$, therefore

$$\begin{aligned} d_Q(\phi_1,\phi_2) &\leq d_{\rho_{\mathbb{D}}}(0, 2|\phi_1 - \phi_2\|_\infty) \\ &\leq C_2\|\phi_1 - \phi_2\|_\infty, \end{aligned}$$

as desired $\square$

Lemma 4.6.1, Lemma 4.6.2 and Lemma 4.6.3 together imply:

Lemma 4.6.4. *Let $\phi_1, \phi_2 \in B_\infty(0, 1/16)$. Then*

$$C_3\|\phi_1 - \phi_2\|_\infty \leq d_T(\beta_X^{-1}(\phi_1), \beta_X^{-1}(\phi_2)) \leq C_4\|\phi_1 - \phi_2\|_\infty.$$

Proof. By Lemma 4.6.2,

$$d_T(\beta_X^{-1}(\phi_1), \beta_X^{-1}(\phi_2)) \leq d_Q(2\phi_1, 2\phi_2).$$

Since $2\phi_1, 2\phi_2 \in B_\infty(0, 1/8)$, by Lemma 4.6.3,

$$d_Q(2\phi_1, 2\phi_2) \leq C_4\|2\phi_1 - 2\phi_2\|_\infty \leq C_4\|\phi_1 - \phi_2\|_\infty.$$

This gives one direction of the comparison in the lemma statement. To obtain the other direction, by Lemma 4.6.1,

$$d_T(\beta_X^{-1}(\phi_1), \beta_X^{-1}(\phi_2)) \geq d_Q((2/3)\phi_1, (2/3)\phi_2).$$

Since $(2/3)\phi_1, (2/3)\phi_2 \in B_\infty(0, 1/8)$, Lemma 4.6.3 implies

$$d_Q((2/3)\phi_1, (2/3)\phi_2) \geq C_3\|(2/3)\phi_1 - (2/3)\phi_2\|_\infty \geq C_3\|\phi_1 - \phi_2\|_\infty,$$

as desired. □

Finally, we have:

Lemma 4.6.5. *There exist universal constants C_3 and C_4 so that for $Y, Z \in B_{d_T}(X, C_3/20)$,*

$$C_3\|\beta_X(Y) - \beta_X(Z)\|_\infty \leq d_T(Y, Z) \leq C_4\|\beta_X(Y) - \beta_X(Z)\|_\infty.$$

Proof. Consider the map $\beta_X^{-1} : B_\infty(0, 1/20) \to \mathcal{T}_g$. This is a biholomorphism onto its image. Since $C_3\|\phi_1 - \phi_2\|_\infty \leq d_T(\beta_X^{-1}(\phi_1), \beta_X^{-1}(\phi_2))$ for $\phi_1, \phi_2 \in B_\infty(0, C_3/16)$ (by Lemma 4.6.4), in particular, $C_3\|\phi\|_\infty \leq d_T(0, \beta_X^{-1}(\phi))$ for $\phi \in B_\infty(0, C_3/16)$. Thus $\partial\beta_X^{-1}(B_\infty(0, 1/20))$ bounds the point $X \in \mathcal{T}_g$ and intersects trivially with $B_{d_T}(X, C_3/20)$ which means $\beta_X^{-1}(B_\infty(0, 1/20))$

contains $B_{d_T}(X, C_3/20)$. Now Lemma 4.6.4 gives the desired inequalities. ☐

Lemma 4.6.6. *Let C_3 be as in Lemma 4.6.5 and suppose $r_1, r_2 \in \mathbb{R}_+$ such that $r_1 \leq C_3/20$ and $r_1 \leq r_2$. Let $X_1, ..., X_N \in B_{d_T}(X, r_2) \subset \mathcal{T}_g$ such that $d_T(X_i, X_j) \geq r_1$ for all $i, j \in \{1, ..., N\}$ distinct. If $r_2 \leq C_3/20$, then $N \leq (C/r_1)^{6g-6}$. For all $r_2 \geq C_3/20$, $N \leq (C/r_1)^{Cr_2(6g-6)}$. Here, C is a universal constant.*

Proof. First we treat the case $r_2 \leq C_3/20$. For all $i, j \in \{1, ..., N\}$ distinct, $r_1/C_4 \leq \|\beta_X(X_i) - \beta_X(X_j)\|_\infty$ by Lemma 4.6.5, therefore norm balls of radius $r_1/(2C_4)$ around the X_i are disjoint. Moreover, $\|\beta_X(X_i)\|_\infty \leq r_2/C_3$ for all $i \in \{1, ..., N\}$. Denote by $H(r)$ the $6g-6$-dimension Hausdorff measure of a norm ball of radius r in the $Q^\infty(X^{-1})$. Then

$$H(r_1/(2C_4)) \geq ((4C_4r_2)/(C_3r_1))^{-(6g-6)}H(2r_2/C_3)$$

since a ball of radius $2r_2/C_3$ is simply a scaled copy of a ball of radius $r_1/(2C_4)$. Since $B_\infty(\beta_X(X_i), r_1/(2C_4)) \subset B_\infty(0, 2r_2/C_3)$ (because $X_i \in \overline{B_\infty(0, r_2/C_3)}$ and $r_1 \leq r_2$, $C_3 \leq C_4$) for all $i \in \{1, ..., N\}$ and the former are also mutually disjoint, $H(2r_2/C_3) \geq NH(r_1/(2C_4))$. Therefore $N \leq (C/r_1)^{6g-6}$. This proves the first part of the lemma.

Now, to prove the second part, let us assume that $r_2 \geq C_3/20$. We have that at most $(C/r_1)^{6g-6}$ of the X_i are contained in $B_{d_T}(X, C_3/20)$. Also, at most $(C/r_1)^{6g-6}$ of the X_j are contained in $B_{d_T}(X_i, C_3/20)$ for all X_i. By induction we obtain the desired result. ☐

4.7 Upper bounds for marked triangulated surfaces by unmarked triangulated surfaces

In this section, we note the following result.

Lemma 4.7.1. *We have,*

$$N^{\mathcal{M}}(T, g, r) \leq N^{\mathcal{T}}(T, g, r)$$

for $g \geq 2$.

Proof. Let $X \in \mathcal{M}_g$. Choosing a marking $S_g \to X$, we obtain a lift $\widetilde{X} \in \mathcal{T}_g$. If $S \in \mathrm{Comb}^{\leq T}(\mathcal{M}_g)$ and $\Phi(S) \in B_{d_T}(X, r)$, then there exists a e^{2r}-quasiconformal map $X \to$

$\Phi(S)$ which, when composed with the marking on X, gives a marking on $\Phi(S)$ and thus also on S. We denote by $\widetilde{S}$ the corresponding marked triangulated surface. Under this correspondence, distinct triangulated surfaces are mapped to distinct marked triangulated surfaces. By construction, $\Phi(\widetilde{S}) \in B_{d_T}(\widetilde{X}, r)$ in $\mathcal{T}_g$. Hence

$$\sup\{\#(\{S \in \mathrm{Comb}^{\leq T}(\mathcal{M}_g) | \Phi(S) \in B_{d_T}(X, r)\}) | X \in \mathcal{M}_g\}$$
$$\leq \sup\{\#(\{S \in \mathrm{Comb}^{\leq T}(\mathcal{T}_g) | \Phi(S) \in B_{d_T}(X, r)\}) | X \in \mathcal{T}_g\}.$$

Summing over all $g' \leq g$ gives the lemma statement. □

Remark 15. The difference between $N^{\mathcal{M}}$ and $N^{\mathcal{T}}$ is apparent in the following situation. Suppose S is a genus g triangulated surface and $f : S \to S$ is a e^{2r}-quasiconformal map which is not isotopic to the identity. Then S will be counted only once in $N^{\mathcal{M}}(\cdot, \cdot, r)$, but at least twice in $N^{\mathcal{T}}(\cdot, \cdot, r)$.

Chapter 5

Lower bounds

In this chapter, we prove Theorem 1.1.1. Let $X \in \mathcal{M}_g$ satisfying the following property: if $\gamma_1, ..., \gamma_n$ are simple closed geodesics on (X, ρ_X) satisfying $\text{length}(\gamma_j) \leq 2\,\text{arcsinh}(1)$, then

$$\sum_{j=1}^{n} \text{length}_{\rho_X}(\gamma_j)^{-1} \leq R.$$

Choose $U_1, ..., U_N$, $W_1, ..., W_N$ and $V_1, ..., V_N$ as in Lemma 2.6.2, where $N \leq C(R+g)$. Let $\mathbb{D}_{\tanh(\text{radius}(V_i)/2)} = \{z \in \mathbb{D} | z < \tanh(\text{radius}(V_i)/2)\}$, and let $f_i : V_i \to \mathbb{D}_{\tanh(\text{radius}(V_i)/2)}$ be a conformal map (unique up to rotation) that sends $\text{center}(V_i)$ to 0 that is C-bi-Lipschitz between V_i with ρ_X and $\mathbb{D}_{\tanh(\text{radius}(V_i)/2)}$ with the Euclidean metric. The circles $L_i = \partial V_i$ divide X into M connected components that we label $P_1, ..., P_M$. Each ∂P_j is a union of hyperbolic circular arcs, and there are at most C arcs because P_j is contained in U_i which may intersect at most C of the $U_1, ..., U_N$ (by condition 5 of Lemma 2.6.2). For the same reason, each V_i is a union of at most C of the $P_1, ..., P_M$.

Now, we construct triangulations T_r of X by triangulating each P_j as follows.

1. For each P_j, choose an arbitrary $V_{s(j)}$ such that $P_j \subset V_{s(j)}$.

2. Divide $f_{s(j)}(P_j)$ into $P_j^B = B_{\text{Euc}}(\partial(f_{s(j)}(P_j)), r\,\text{radius}(V_{s(j)}))$ and $P_j^I = f_{s(j)}(P_j) \setminus P_j^B$. We triangulate each P_j separately.

3. Triangulate a subset of $f_{s(j)}(P_j)$ by the standard hexagonal triangulation with side length $r\,\text{radius}(V_{s(j)})$ such that P_j^I is covered by the triangulation.

4. Pullback by $f_{s(j)}$ to obtain a partial triangulation of a subset $X_E \subset X$.

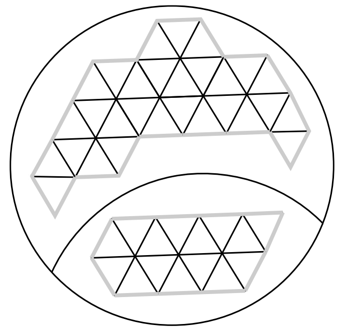

Figure 5-1: Two of the P_j contained in the same V_i, with the partial triangulation constructed in Step 3. The gray edges are replaced in Step 5.

5. In the triangulation of X_E, replace all border edges between two vertices by the shortest hyperbolic geodesic connecting these two vertices.

6. Choose a set of vertices on $X \setminus X_E$ that is a $C_1 r\,\mathrm{radius}(V_i)$-separated set and a $C_2 r\,\mathrm{radius}(V_i)$-net on each V_i. Using these chosen vertices, complete the triangulation of $X \setminus X_E$ by hyperbolic triangles such that each triangle is contained in some W_i, and within W_i is contained in a $C_3 r\,\mathrm{radius}(V_i)$-radius hyperbolic ball.

We have the following lemma regarding Step 5 of the triangulation process described.

Lemma 5.0.1. *There exists a universal constant r_0 such that for all $r < r_0$, the following statements hold. Let $x, y \in V_i$ such that $f_i^* d_{\mathrm{Euc}}(x, y) \leq r\,\mathrm{radius}(V_i)$. Let γ_{Euc} be the geodesic in V_i connecting x and y in the metric $f_i^* d_{\mathrm{Euc}}$, and γ_X be the geodesic in V_i connecting x and y in the metric ρ_X. The angles between $f_i(\gamma_X)$ and $f_i(\gamma_{\mathrm{Euc}})$ are at most $\pi/7$. Moreover, the curvatures of $f_i(\gamma_X)$ and $f_i(\gamma_{\mathrm{Euc}})$ are bounded above by a universal constant.*

Proof. The map f_i maps V_i conformally to the disk $\mathbb{D}_{\tanh(\mathrm{radius}(V_i)/2)}$ (sending center(V_i) to 0) such that the restriction of the Poincare metric $\rho_{\mathbb{D}}$ on $\mathbb{D}$ is ρ_X on V_i. Geodesics with respect to d_{Euc} correspond to straight lines in $\mathbb{D}_{\tanh(\mathrm{radius}(V_i)/2)}$, whereas geodesics with respect to $\rho_{\mathbb{D}}$ are circular arcs which meet $\partial\mathbb{D}$ at a right angle. Since $\mathrm{radius}(V_i) \leq \mathrm{arcsinh}(1)/8$ (see condition 2 of Lemma 2.6.2), $\tanh(\mathrm{radius}(V_i)/2)$ is bounded away from 1 so a geodesic of $\rho_{\mathbb{D}}$ passing through $\mathbb{D}_{\tanh(\mathrm{radius}(V_i)/2)}$ must have curvature bounded above by a universal constant C. If two points in $\mathbb{D}$ are sufficiently close, then the line between them and constant curvature arc (of curvature at most C) between them meet at an angle of at most $\pi/7$. The lemma follows. □

Thus we obtain, for each $0 < r < r_0$, a triangulation of X that we label T_r. There are two types of edges in T_r: those constructed in Step 3 and left unchanged (which we call flat edges), and those constructed in Step 5 or 6 (which we call hyperbolic edges). There are three types of triangles: those constructed in Step 3 and left unchanged (which we call equilateral triangles), those constructed in Step 5 (which we call semi-equilateral triangles), and those constructed in Step 6 (which we call hyperbolic triangles). We denote by T_E the union of the equilateral triangles of T_r, T_{SE} the union of the semi-equilateral triangles of T_r, and T_H the union of the hyperbolic triangles of T_r.

Denote by T_r^1 the 1-skeleton of the triangulation T_r. We define a metric $d_{T_r^1}$ on T_r^1 as follows. On a hyperbolic edge, we define $d_{T_r^1}$ to the restriction of d_{ρ_X} (the hyperbolic distance on X) to the hyperbolic edge. On a flat edge that was constructed by triangulating $f_{s(j)}(P_j)$ in Step 3, we define $d_{T_r^1}$ to be the restriction of the pullback under $f_{s(j)}$ of the Euclidean distance d_{Euc} on $f_{s(j)}(P_j) \subset \mathbb{D}$ to the flat edge.

We let S_r be the unique triangulated surface with the same underlying triangulation as T_r. Note that T_r has at most $Cr^{-2}(R+g)$ triangles, since the number of equilateral and semi-equilateral triangles is bounded by CNr^{-2} and the number of hyperbolic triangles is bounded by $Cr^{-1}\sum_i \text{length}_{\rho_X}(L_i)\,\text{radius}(V_i)^{-1} \leq Nr^{-1}$. Hence $S_r \in \text{Comb}^{\leq Cr^{-2}(R+g)}(\mathcal{T}_g)$. We define $X_r = \Phi(S_r) \in \mathcal{M}_g$.

We will now show:

Theorem 5.0.2. *For $0 < r < r_0$, $d_T(X, X_r) < Cr$. Here C is a universal constant.*

This directly implies Theorem 1.1.1. First, we have a preliminary lemma.

Lemma 5.0.3. *Let Q be a hyperbolic triangle contained in a hyperbolic ball of radius $r < 1/4$. Then there exists a $r^2\,\text{area}(Q)^{-1}$-quasiconformal map from Q (equipped with the hyperbolic metric) to the unit equilateral triangle T (equipped with the Euclidean metric) which sends vertices of Q to vertices of T and is a scaling map on each of the three boundary edges.*

Proof. The hyperbolic triangle Q admits a C-quasiconformal map to some flat triangle Q_1, as follows. Consider Q as contained in $\mathbb{D}$ (equipped with the Poincare metric $\rho_{\mathbb{D}}$) with one of the vertices the origin. Constructing a unit hemisphere S^2 over $\mathbb{D}$, we may take the inverse stereographic projection of Q onto S^2, and then the orthographic projection of the image of Q back onto $\mathbb{D}$, to obtain a map from Q to a Euclidean triangle Q_1. Away from the boundary of the hemisphere, the stereographic projection is C-bi-Lipschitz between the spherical metric on S^2 and the hyperbolic metric on $\mathbb{D}$, and the orthographic projection is C-bi-Lipschitz between the spherical metric on S and the Euclidean metric on $\mathbb{D}$. Since Q is contained in a hyperbolic ball of radius $r < 1/4$, we have a C-bi-Lipschitz map F_1 from Q (with the hyperbolic metric) to Q_1 (with the Euclidean metric). Let Q_2 be Q_1 scaled by r^{-2}; then we have a map $F_2 : Q_1 \to Q_2$ that is conformal. The map F_2 satisfies the property that F_2 is r^{-2}-Lipschitz and F_2^{-1} is r^2-Lipschitz. Finally, note that Q_2 is contained in a C-radius ball in the Euclidean metric. There exists an $\text{area}(Q_2)^{-1}$-quasiconformal affine map F_3 from Q_2 to T, which sends vertices of Q_2 to vertices of vertices of T and is scaling on each boundary edge of ∂Q_2. Now, since $\text{area}(Q_2)$ is around $Cr^{-2}\,\text{area}(Q)$, $F_3 \circ F_2 \circ F_1 : Q \to T$ is $Cr^2\,\text{area}(Q)^{-1}$-quasiconformal. Let $\text{sc} : \partial Q \to \partial T$ be a map that sends vertices to vertices and is scaling on each boundary edge, with respect to the hyperbolic metric on ∂Q and the Euclidean metric on ∂T. Then by construction, $\text{sc} \circ (F_3 \circ F_2 \circ F_1)^{-1} : \partial T \to \partial T$ is C-bi-Lipschitz with respect to the Euclidean metric, therefore C-weakly-quasisymmetric, and thus by Lemma 3.5.3 extends to a C-quasiconformal map from T to T. Preomposing this map with $F_3 \circ F_2 \circ F_1$ gives the map desired in the lemma statement. □

Proof of Theorem 5.0.2. We construct a map $F_r : X \to X_r$ triangle-by-triangle as follows. On vertices, F_r is naturally defined. On edges, we define F_r to be scaling (with respect to the metrics $d_{T_r^1}$ on T_r^1 and d_{S_r}, the canonical flat metric given by S_r, on X_r). Then F_r conformally extends to all equilateral triangles (triangles in T_E) of T_r. To define F_r on T_{SE}, note that by construction each semi-equilateral triangle Q is contained in V_i for some i. By Lemma 5.0.1, Step 6 of the triangulation construction and the C-bi-Lipschitz property of f_i, the triangular region $f_i(Q)$ has angles bounded below by $\pi/21$ and has edges with lengths bounded above and below by $Cr\,\text{radius}(V_i)$ and curvature bounded above by C. Therefore $f_i(\partial Q)$ is a C-quasicircle. Let $\text{sc} : \partial Q \to \partial T$ denote a scaling map, with respect to the $d_{T_r^1}$ on ∂Q and Euclidean metric on ∂T, that sends vertices of ∂Q to vertices of ∂T. Since $\text{sc} \circ f_i^{-1} : f_i(\partial Q) \to \partial T$ is $Cr^{-1}\,\text{radius}(V_i)^{-1}$-Lipschitz with a $Cr\,\text{radius}(V_i)$-Lipschitz inverse (with respect to the Euclidean metrics), it is C-weakly-quasisymmetric. Hence by Lemma 3.5.3, $\mathrm{sc} \circ f_i^{-1}$ extends to a C-quasiconformal map from $f_{s(i)}(Q)$ to T which means F_r extends to a C-quasiconformal map on Q. Finally, by Lemma 5.0.3, F_r extends to a $Cr^2\,\mathrm{area}(Q)^{-1}$-quasiconformal map on each hyperbolic triangle Q in T_H. In this way, we construct a map $F_r : X \to X_r$.

We use the characterization of the Teichmüller metric in terms of extremal length from Theorem 4.2.2. We have on $\mathcal{M}_g$,

$$d_T(X, X_r) = \frac{1}{2} \log \inf_f \sup_\gamma \frac{\mathrm{Ext}_X(\gamma)}{\mathrm{Ext}_{X_r}(f(\gamma))}$$

where the infimum runs through all quasiconformal maps $f : X \to X_r$ and the supremum runs through all free homotopy classes of simple closed curves γ on X. To show this quantity is bounded by Cr, it suffices to show that for all free homotopy classes γ of a simple closed curve on X,

$$\mathrm{Ext}_{X_r}(F_r(\gamma)) \geq (1 - Cr)\,\mathrm{Ext}_X(\gamma).$$

Recall that

$$\mathrm{Ext}_{X_r}(F_r(\gamma)) = \sup_{\rho_r} \frac{\mathrm{length}_{\rho_r}(F_r(\gamma))^2}{\mathrm{area}_{\rho_r}(X_r)}$$

where the supremum ranges over all conformal metrics ρ_r on X_r. Similarly,

$$\mathrm{Ext}_X(\gamma) = \sup_\rho \frac{\mathrm{length}_\rho(\gamma)^2}{\mathrm{area}_\rho(X)}$$

where the supremum ranges over all conformal metrics ρ on X. By Theorem 4.2.1, the ρ which achieves this supremum is given by $|\phi|^{1/2}$ for a holomorphic quadratic differential ϕ on X. To show

$$\mathrm{Ext}_{X_r}(F_r(\gamma)) \geq (1 - Cr)\,\mathrm{Ext}_X(\gamma),$$

it suffices to exhibit a conformal metric ρ_r on X_r such that

$$\frac{\mathrm{length}_{\rho_r}(F_r(\gamma))^2}{\mathrm{area}_{\rho_r}(X_r)} \geq (1 - Cr)\frac{\mathrm{length}_\phi(\gamma)^2}{\mathrm{area}_\phi(X)}.$$

To do this, we consider the metric $(F_r)_*|\phi|^{1/2}$ (which is not a conformal metric on X_r), and let ρ_r be the smallest conformal metric on X_r such that $\rho_r \geq (F_r)_*|\phi|^{1/2}$. Then

$$\operatorname{length}_{\rho_r}(F_r(\gamma))^2 \geq \operatorname{length}_\phi(\gamma)^2$$

by construction. It remains to see that

$$\operatorname{area}_{\rho_r}(X_r) \leq (1 + Cr)\operatorname{area}_\phi(X).$$

To see this, for each $1 \leq i \leq N$ let

$$m_i = \sup_{x \in W_i} \frac{|\phi|}{\rho_X^2}(x).$$

By the mean value property,

$$\int_{U_i} |\phi| \geq C\operatorname{radius}(U_i)^2 m_i.$$

Therefore, by condition 4 of Lemma 2.6.2,

$$\operatorname{area}_\phi(X) \geq C\sum_{i=1}^{N}\operatorname{radius}(U_i)^2 m_i. \tag{5.1}$$

Now, for any equilateral triangle Q that is a face of T_r, $\operatorname{area}_{\rho_r}(F_r(Q)) = \operatorname{area}_\phi(Q)$ because F_r is conformal on Q. Therefore,

$$\operatorname{area}_{\rho_r}(F_r(T_E)) \leq \operatorname{area}_\phi(X) \tag{5.2}$$

Any triangle Q in T_{SE} is contained in W_i for some $1 \leq i \leq N$. By C-quasiconformality of F_r on Q,

$$\begin{aligned} \operatorname{area}_{\rho_r}(F_r(Q)) &\leq C\operatorname{area}_\phi(Q) \\ &\leq Cm_i\operatorname{area}_{\rho_X}(Q). \end{aligned} \tag{5.3}$$

Now, by Step 5 in the triangulation construction semi-equilateral triangles contained in W_i

are actually contained in a $Cr\,\mathrm{radius}(U_i)$ hyperbolic neighborhood of $\left(\cup_{i=1}^N L_i\right) \cap W_i$, and the latter has length at most $C\,\mathrm{radius}(U_i)$ in the hyperbolic metric by condition 3 and condition 5 of Lemma 2.6.2. Summing Eq. (5.3) over all semi-equilateral triangles Q we obtain

$$\begin{aligned}\mathrm{area}_{\rho_r}(F_r(T_{SE})) &\leq Cr \sum_{i=1}^{N} \mathrm{radius}(U_i)^2 m_i \\ &\leq Cr\,\mathrm{area}_\phi(X)\end{aligned} \tag{5.4}$$

by Eq. (5.1).

Finally, any triangle Q in T_H is contained in W_i for some $1 \leq i \leq N$. By $Cr^2\,\mathrm{area}_{\rho_X}(Q)^{-1}$-quasiconformality of F_r on Q, we have

$$\begin{aligned}\mathrm{area}_{\rho_r}(F_r(Q)) &\leq Cr^2\,\mathrm{area}_{\rho_X}(Q)^{-1}\,\mathrm{area}_\phi(Q) \\ &\leq Cr^2 m_i.\end{aligned} \tag{5.5}$$

Since by Step 6 of the triangulation construction at most $C\,\mathrm{radius}(U_i)^2 r^{-1}$ hyperbolic triangles Q are contained in any W_i, summing Eq. (5.5) over all hyperbolic triangles Q we obtain

$$\begin{aligned}\mathrm{area}_{\rho_r}(F_r(T_H)) &\leq Cr \sum_{i=1}^{N} \mathrm{radius}(U_i)^2 m_i \\ &\leq Cr\,\mathrm{area}_\phi(X)\end{aligned} \tag{5.6}$$

by Eq. (5.1).

Eq. (5.2), Eq. (5.4) and Eq. (5.6) together give

$$\begin{aligned}\mathrm{area}_{\rho_r}(X_r) &= \mathrm{area}_{\rho_r}(F_r(T_E)) + \mathrm{area}_{\rho_r}(F_r(T_{SE})) + \mathrm{area}_{\rho_r}(F_r(T_H)) \\ &\leq (1 + Cr)\,\mathrm{area}_\phi(X)\end{aligned}$$

which completes the proof. □

Chapter 6

Approximating triangulated surfaces with bounded degree triangulations

6.1 Approximation theorem

The goal of this section is to show the following theorem:

Theorem 6.1.1. *There exists a map* $B : \mathrm{Comb}^T(\mathcal{T}_g) \to \mathrm{Comb}^{\leq \sigma T}(\mathcal{T}_g)$ *that satisfies:*

1. *There is a* C*-quasiconformal map* $f_S : S \to B(S)$*,*
2. *The maximum degree of any vertex of* $B(S)$ *is* 7*,*
3. *Given any two vertices* $x, y \in V_{\neq 6}(B(S))$ *(not necessarily distinct), and* γ *any arc from* x *to* y *that is homotopically nontrivial in* $B(S) \setminus \{x, y\}$*, we have*
$$\int_\gamma |\psi_{B(S)}|^{1/6} \geq 3,$$
4. $|V_{\neq 6}(B(S))| \leq \mu(|V_{\neq 6}(S)| + g)$*, and*
5. *The fiber of* B *over* $B(S)$ *has cardinality at most* $C^{|V_{\neq 6}(B(S))|}$*.*

Here, C*,* σ *and* μ *are universal constants.*

To show Theorem 6.1.1, we first construct a local replacement for high degree vertices as follows.

For $d \in \mathbb{N}$, define the triangulated disk TD_d to be the following triangulation of a topological disk by unit equilateral triangles: d unit equilateral triangles are glued together to form a topological disk with one interior vertex of degree d. The boundary of TD_d, denoted ∂TD_d, consists of d edges. We thus obtain a triangulated surface with boundary that we also call TD_d, which comes with a flat metric that may have a singularity at the interior vertex.

Lemma 6.1.2. *There is a conformal map $f : TD_d \to \overline{\mathbb{D}}$ such that $f : \partial TD_d \to S^1$ is a scaling map with respect to the restriction of the flat metric on TD_d to ∂TD_d, and the Euclidean metric on S^1.*

Remark 16. Scaling map here means all distances get scaled by a constant factor which is

$$\frac{\text{length}(\partial TD_d)}{\text{length}(S^1)}.$$

Proof. Denote by $W_{\pi/3}$ the closed sector of the unit circle with angle $\pi/3$, and denote by T the unit equilateral triangle. By Lemma 3.5.3, there exists a C-quasiconformal map $W_{\pi/3} \to T$ that sends boundary vertices of $W_{\pi/3}$ to boundary vertices of T and is scaling on each boundary edge. Gluing, we obtain a conformal map from TD_d to the unit cone of angle $d\pi/3$, which is scaling on the boundary. The unit cone of angle $d\pi/3$ admits a conformal map to $\overline{\mathbb{D}}$ which is scaling on the boundary. Composing these two maps, we obtain the statement in the lemma. □

For $d \geq 8$, we now construct the triangulated hyperbolic disk TH_d to be another topological disk formed by gluing unit equilateral triangles satisfying $\partial TH_d \simeq \partial TD_d$ (that is, TH_d also has d boundary edges). However, TH_d will have more interior points, and therefore more triangles. The number of triangles in TH_d will be bounded above by Cd. We construct TH_d inductively by constructing annular layers starting from its boundary.

Label the boundary points of $\partial TH_d \simeq \partial TD_d$ by $x_{0,0}, ..., x_{0,d-1}$. If $d \geq 8$, construct TA_1, a triangulated annulus with outer and inner boundaries ∂TA_1^+ and ∂TA_1^-, respectively, such that

1. $\partial TA_1^+ = \partial TH_d$,

2. $\deg_{TA_1}(x_{0,j}) = 3$ if j is even and 4 if j is odd,

3. TA_1 has no interior vertices.

(Recall that the degree of a vertex of a triangulated surface with boundary is the number of edges emanating from the vertex.) These conditions determine TA_1 uniquely. That is, ∂TA_1^- consists of d_1 edges and vertices where $d_1 = \lfloor d/2 \rfloor$. The vertices of ∂TA_1^- may be labelled $x_{1,0}, ..., x_{1,d_1-1}$ (in cyclic order), such that the following holds.

When d is even, in TA_1, $x_{0,j}$ is connected by an edge to $x_{1,j/2}$ if j is even, and $x_{0,j}$ is connected by an edge to $x_{1,(j-1)/2}$ and $x_{1,(j+1)/2}$ if $j < d-1$ is odd. Finally, $x_{0,d-1}$ is connected by an edge to $x_{1,(d-2)/2}$ and $x_{1,0}$.

When d is odd, $x_{0,j}$ is connected by an edge to $x_{1,j/2}$ if $j < d-1$ is even, and $x_{0,j}$ is connected by an edge to $x_{1,(j-1)/2}$ and $x_{1,(j+1)/2}$ if $j < d-2$ is odd. Moreover, $x_{0,d-1}$ is connected by an edge to $x_{1,0}$, while $x_{0,d-2}$ is connected by an edge to $x_{1,(d-3)/2}$ and $x_{1,0}$.

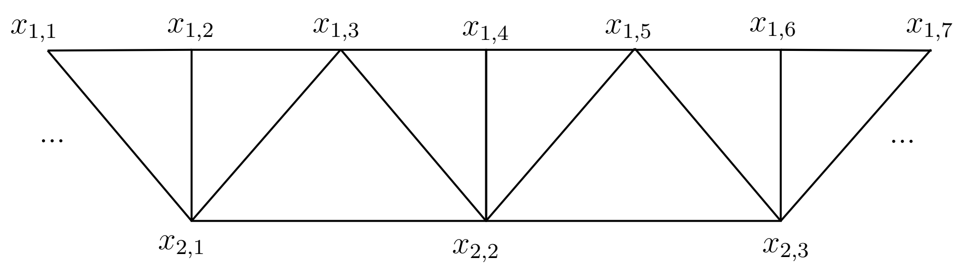

Figure 6-1: A piece of TA_1, with the triangles not to scale.

We now describe the second inductive step, constructing another annulus TA_2 with outer and inner boundaries ∂TA_2^+ and ∂TA_2^-, respectively, such that

1. $\partial TA_2^+ = \partial TA_1^-$,

2. $\deg_{TA_2}(x_{1,j}) = 3$ if j is even and 4 if j is odd,

3. TA_2 has no interior vertices.

We continue this inductive process. At the ith inductive step, wherein the vertices of TA_{i-1}^- are denoted $x_{i-1,0}, ..., x_{i-1,d_{i-1}-1}$ in cyclic order (where $d_{i-1} \sim d/2^{i-1}$), we construct annulus TA_i with outer and inner boundaries ∂TA_i^+ and ∂TA_i^- such that

1. $\partial TA_i^+ = \partial TA_{i-1}^-$,
2. $\deg_{TA_i}(x_{i,j}) = 3$ if j is even and 4 if j is odd,
3. A_i has no interior vertices.

Here, ∂TA_i^- consists of d_i edges and vertices where $d_i = \lfloor d_{i-1}/2 \rfloor$. The vertices of ∂TA_i^- may by labelled $x_{i,0}, ..., x_{i,d_i-1}$ (in cyclic order) such that the following holds.

When d_{i-1} is even, in TA_i, $x_{i-1,j}$ is connected by an edge to $x_{i,j/2}$ if j is even, and $x_{i-1,j}$ is connected by an edge to $x_{i,(j-1)/2}$ and $x_{i,(j+1)/2}$ if $j < d_{i-1} - 1$ is odd. Finally, $x_{i-1,d_{i-1}-1}$ is connected by an edge to $x_{i,(d_{i-1}-2)/2}$ and $x_{i,0}$.

When d_{i-1} is odd, $x_{i-1,j}$ is connected by an edge to $x_{i,j/2}$ if $j < d_{i-1} - 1$ is even, and $x_{i-1,j}$ is connected by an edge to $x_{i,(j-1)/2}$ and $x_{i,(j+1)/2}$ if $j < d_{i-1} - 2$ is odd. Moreover, $x_{i,d_{i-1}-1}$ is connected by an edge to $x_{i,0}$, while $x_{i-1,d_{i-1}-2}$ is connected by an edge to $x_{i,(d_{i-1}-3)/2}$ and $x_{i,0}$.

We continue until we reach the kth step wherein $d_{k-1} \leq 7$. We let $TA_k = TD_{d_{k-1}}$ and glue TA_k to TA_{k-1} along the boundary $T\partial A_{k-1}^-$. Our construction of the $TA_1, ..., TA_k$ naturally identifies ∂TA_{i-1}^- with ∂TA_i^+. We define TH_d to be the union of the $TA_1, ..., TA_k$. We call the unique interior vertex of TA_k the center of TH_d. From the construction described above, the following lemma is evident.

Lemma 6.1.3. *Let v be the center of TH_d. For $2 \leq i \leq k$, the closed star of the union of $TA_i, ..., TA_k$ is the union of $TA_{i-1}, ..., TA_k$, and TA_k is the closed star of v. In particular, the kth successive closed star of v is TH_d.*

We also have the following lemma.

Lemma 6.1.4. *There are at most Cd vertices of TH_d. All interior vertices of TH_d have degree at most 7 and all boundary vertices have degree at most 4. Moreover, each TA_i for $2 \leq i \leq k$ contains at least one outer boundary vertex that has degree 7 in TH_d.*

Proof. The total number of vertices of TA_i is at most $Cd/2^i$, so the number of vertices of TH_d is at most Cd. By construction, outer boundary vertices of TA_i have degree at most 4 (in particular, $x_{i-1,0}$ has degree 3) and inner boundary vertices of TA_i have degree at most 5, except for vertex $x_{i,0}$ which has degree 6. From this we obtain the second claim in the lemma. Finally, for $2 \leq i \leq k$, $x_{i-1,0}$ has degree 7 in TH_d. □

Our next goal is to show the following lemma.

Lemma 6.1.5. *There exists a C-quasiconformal map $f : TH_d \to TD_d$ which on the boundaries agrees with an identification $\partial TH_d \simeq \partial TD_d$.*

To do this, we have the following preliminary lemma.

Lemma 6.1.6. *Each triangulated annulus TA_i admits a C-quasiconformal map to a closed annulus*

$$A(r, 1) = \{x \in \mathbb{C} | r \leq |x| \leq 1\}$$

whose restriction $\partial TA_i \to \partial A(r, 1)$ is a scaling map. Here, C is a universal constant independent of TA_i.

Proof. We will construct a quasiconformal map to $A(\alpha, \beta) = \{x \in \mathbb{C} | \alpha \leq |x| \leq \beta\}$. Then, composing with a scaling map gives the lemma statement. We will choose α and β later. We first partition $A(\alpha, \beta)$ into triangular regions. Each triangular region consists of three boundary components, two of which are straight lines and one which is a circular arc. Let (r, θ) be polar coordinates on $\mathbb{C}$ centered at 0. Denote by $C_- = \{(r, \theta) | r = \alpha\}$ and $C_+ = \{(r, \theta) | r = \beta\}$ the two boundary circles of $A(\alpha, \beta)$ of radius α and β respectively. Recall that $x_{i-1,0}, ..., x_{i,d_{i-1}-1}$ are the vertices of ∂TA_i^+ and $x_{i,0}, ..., x_{i,d_i-1}$ are the vertices of ∂TA_i^-, where $d_i = \lfloor d_{i-1}/2 \rfloor$. Now, let $y_{i-1,j}$ denote the vertex $(r, \theta) = (\beta, 2\pi j/d_{i-1})$ (for $0 \leq j \leq d_{i-1} - 1$) and $y_{i,j}$ denote the vertex $(r, \theta) = (\alpha, 2\pi j/d_i)$ (for $0 \leq j \leq d_i - 1$). We construct a triangulation of $A(\alpha, \beta)$ as follows. We let the vertices of the triangulation be the $y_{i-1,j}$ and $y_{i,j}$. Two vertices y_{i-1,j_1} and y_{i,j_2} are connected by an edge if x_{i-1,j_1} and x_{i,j_2} in TA_i are connected by an edge. In this case, the edge between y_{i-1,j_1} and y_{i,j_2} is a straight line.

Now, for d_i sufficiently large, we take $\alpha = d_{i-1} - 1$ and $\beta = d_{i-1}$, and this decomposition of $A(\alpha, \beta)$ into vertices and edges gives a triangulation of $A(\alpha, \beta)$. This is true since by

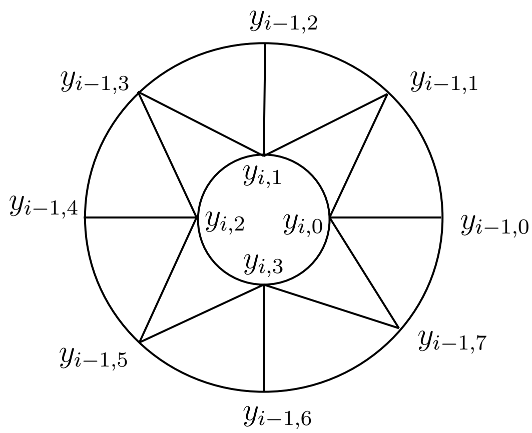

Figure 6-2: Triangulation of $A(\alpha, \beta)$ when $d_{i-1} = 8$, which is combinatorially equivalent to TA_i when $d_{i-1} = 8$.

the construction of TA_i, two vertices y_{i-1,j_1} and y_{i,j_2} are connected by an edge only if the distance between them is at most C. For small d_i but still greater than 7, we take $\alpha = 1/10$ and $\beta = 1$ to obtain a triangulation where the angles of each triangular region are nonzero. We claim that when d_i is large, the angles of each triangular region are bounded below by a universal constant. We also claim that when d_i is large, the lengths of the triangular sides are bounded below and above by universal constants. To see this, note that each triangular region Q consists of two straight lines along with a circular arc, and associated to Q is a genuine flat triangle Q' which may be obtained by replacing the circular arc with a straight line, which we call the base of Q'. By construction, the circular arc has length bounded below and above by universal constants, and since d_i is assumed to be sufficiently large, the straight line approximates the circular arc. So the length of the base of the triangle is bounded below and above by universal constants. Since the two circles C_α and C_β are distance 1 apart, the height of each triangle is also bounded below and above by universal constants. Hence, ∂Q has three sides, two of which are straight lines and the third a circular arc, each with lengths bounded below and above by universal constants. Moreover the angles of ∂Q are bounded below by a universal constant because the angles of $\partial Q'$ are bounded below by a universal constant. Therefore for all d_i, each triangular region Q is a C-quasicircle, where C is independent of d_i. The map from ∂Q to ∂T (the boundary of an equilateral triangle) which sends vertices of to vertices and is a scaling map on each of the three sides is C-bi-Lipschitz with respect to the Euclidean metric, therefore also C-weakly-quasisymmetric. By

Lemma 3.5.3 on each triangular region to obtain a map to the unit equilateral triangle that is scaling on each boundary component, and gluing the inverses of these maps together, we obtain a map from TA_i to $A(\alpha, \beta)$ which is a scaling map on the boundary. □

Proof of Lemma 6.1.5. We already have shown that there exists a C-quasiconformal map $TD_d \to \overline{\mathbb{D}}$ which is a scaling map on the boundary. To show Lemma 6.1.5, it suffices to construct a C-quasiconformal map $TH_d \to \overline{\mathbb{D}}$ that is a scaling map on the boundary. We do this inductively as follows. First, by Lemma 6.1.6, TA_1 admits a C-quasiconformal map to the annulus $A(r_1, 1)$ which is a scaling map on the boundary. Then TA_2 admits a C-quasiconformal map to the annulus $A(r_1 r_2, r_1)$ which is a scaling map on the boundary. In general, for $i \in \{1, ..., k-1\}$, TA_i admits a C-quasiconformal map to the annulus $A(r_1...r_i, r_1...r_{i-1})$ which is scaling on the boundary. Finally, by Lemma 6.1.2 TA_k (which is a triangulated disk) admits a C-quasiconformal map to the Euclidean disk of radius $r_1...r_{k-1}$ which is a scaling map on the boundary. These maps (after possible rotations) glue together to give the desired map $TH_d \to \overline{\mathbb{D}}$. □

Proof of Theorem 6.1.1. First, we replace S with a surface S_1 in $\mathrm{Comb}^{16T}(\mathcal{T}_g)$ which is the 4-subdivision of S rescaled. Then we may replace the closed star around every vertex in S_1 of degree d greater than 7 (which is a copy of TD_d) by TH_d to obtain a triangulated surface S_2. (These closed stars are all disjoint.) Then, take the rescaled 3-subdivision of S_2 to obtain $B(S)$, a genus g triangulated surface with at most σT triangles, for a constant σ independent of T and g. The marking on $B(S)$ is obtained from the marking on S. By construction, conditions 2 and 3 in the statement of Theorem 6.1.1 are satisfied. Applying Lemma 6.1.5 for each triangulated disk replacement and gluing, we have a C-quasiconformal map $f : S \to S_2$, and S_2 is naturally conformally equivalent to $B(S)$. This shows condition 1 in the statement of Theorem 6.1.1.

Now, $|V_{\neq 6}(S)| = |V_{\neq 6}(S_1)|$ and $|V_{\neq 6}(B(S))| = |V_{\neq 6}(S_2)|$. Since S_2 is formed by replacing disjoint copies of TD_d in S_1 with TH_d, by Lemma 6.1.4, we have

$$|V_{\neq 6}(S_2)| \leq |V_{<6}(S_1)| + C \sum_{x \in V_{>6}(S_1)} \deg x.$$

By Euler characteristic considerations,

$$\sum_{x \in V_{>6}(S_1)} \deg x \leq C|V_{<6}(S_1)| + Cg.$$

Hence

$$\begin{aligned} |V_{\neq 6}(S_2)| &\leq C|V_{<6}(S_1)| + Cg \\ &\leq C|V_{\neq 6}(S_1)| + Cg, \end{aligned}$$

which means $|V_{\neq 6}(B(S))| \leq C(|V_{\neq 6}(S)| + g)$, showing condition 4 in the statement of Theorem 6.1.1.

It remains to bound the cardinality of the fibers of B. Given $B(S)$, we may recover S_2 by choosing a vertex of $B(S)$ with degree strictly greater than 6 (which must exist by Euler characteristic considerations) which must also be a vertex of S_2, and inductively reconstructing S_2. Recall that S_1 may be constructed from S_2 by replacing certain (disjoint) copies of TH_d by copies of TD_d. Each copy TD_d contains its center, which is a vertex of degree $d > 6$, and these centers must be distinct. There are therefore at most $C^{|V_{\neq 6}(B(S))|}$ total choices for the set of centers, and the set of centers has cardinality at most $|V_{\neq 6}(B(S))|$.

Lemma 6.1.7. *Suppose $v \in V(S_2)$ is the center of a copy of TH_d. Given S_2 and v, there are at most two possible choices for d.*

Proof. Suppose that TH_d, $TH_{d'}$ and $TH_{d''}$, with $d < d' < d''$ are all contained in S_2 and centered at v. By construction of S_2, all the boundary vertices of the closed star of TH_d have degree 6 in S_2. (This is true because S_1 is the rescaled 4-subdivision of S, and to construct S_2 from S_1 we only replace stars of vertices of degree $d \geq 8$ by copies of TH_d.) Since $TH_{d'}$ and $TH_{d''}$ are also subsets of S_2 centered at v and $d < d' < d''$, by Lemma 6.1.3 we must have that the closed star of TH_d in S_2 is contained in the interior of $TH_{d''}$ (meaning boundary vertices of the closed start of TH_d are interior vertices of $TH_{d''}$), but this is a contradiction to Lemma 6.1.4. □

Given a set of centers, by Lemma 6.1.7 there are at most $C^{|V_{\neq 6}(B(S))|}$ possibilities for the choice of triangulated hyperbolic disks TH_d centered at these centers. Once these disks are chosen, there is a unique choice of replacement (TD_d) for each triangulated hyperbolic disk, hence S_1 may be reconstructed. Finally, given S_1, S can be recovered by choosing a vertex of S_1 with degree strictly greater than 6 (which must exist by Euler characteristic considerations), which must also be a vertex of S, and inductively reconstructing S starting from the chosen vertex. Thus, given $B(S)$, there are at most $C^{|V_{\neq 6}(B(S))|}$ possibilities for S. This shows condition 5 in the statement of Theorem 6.1.1. $\square$

6.2 Upper bounds for triangulated surfaces in terms of locally bounded surfaces

We have the following corollary of Theorem 6.1.1.

Corollary 6.2.1. *There exists a universal constant C such that*

$$N^{\mathcal{T}}(T, g, m, r) \leq C^{m+g} N^{\mathcal{T}}_{\text{lb}}(\sigma T, g, \mu(m+g), r+C)$$

for $g \geq 2$. Here, σ and μ are the constants defined in the statement of Theorem 6.1.1.

Proof. Let $X \in \mathcal{T}_g$. Suppose $S \in \text{Comb}^{\leq T}(\mathcal{T}_g)$ such that $\Phi(S) \in B_{d_T}(X, r)$. By Theorem 6.1.1, there exists $B(S) \in \text{Comb}^{\leq \sigma T}(\mathcal{T}_g)$ such that $d_T(\Phi(B(S)), \Phi(S)) \leq C$ and $B(S)$ is in $\text{Comb}^{\leq \sigma T, \leq \mu(m+g)}_{\text{lb}}(\mathcal{T}_g)$. Here, B is the map defined in statement of Theorem 6.1.1. Since $|V_{\neq 6}(B(S))| \leq \mu(|V_{\neq 6}(S)| + g) \leq \mu(m+g)$, the fibers of B have cardinality at most C^{m+g}. Summing over all $g' \leq g$, we have $N^{\mathcal{T}}(T, g, m, r) \leq C^{m+g} N^{\mathcal{T}}_{\text{lb}}(\sigma T, g, \mu(m+g), r+C)$. $\square$

Chapter 7

Upper bounds for combinatorial translation surfaces via triangulated surfaces

In this chapter, we shall prove the following result.

Lemma 7.0.1. *There exists a universal constant C such that*

$$N_{\text{lb}}^{\mathcal{H}}(T,g,r) \le (T/g)^{C(1+r)g} \sum_{\substack{n\le(1/100)g\\ T_1+\ldots+T_n\le T\\ g_1+\ldots+g_n\le\mu^{-1}(1/100)g\\ g_1,\ldots,g_n\ge 2\\ m_1+\ldots+m_n\le\mu^{-1}(1/100)g}} \prod_{i=1}^{n} N^{\mathcal{T}}(2T_i,g_i,m_i,r+C)$$

for $g \ge 2$. Here, μ is the constant defined in the statement of Theorem 6.1.1.

7.1 Hodge norms and roadmap to prove Lemma 7.0.1

Let $X \in \mathcal{T}_g$. We will first compute $|\{S_Y \in \text{Comb}_{\text{lb}}^T(\mathcal{H}_g)|\Phi(S_Y) \in B_{d_T}(X,r)\}|$. To do this, we will compute $|\{S_Y \in \text{Comb}_{\text{lb}}^T(\mathcal{H}_g)|\Phi(S_Y) \in B_{d_T}(X,(g/T)^{\kappa_0})\}|$ for an appropriate integer $\kappa_0 \in \mathbb{N}$ to be chosen later, and then use Lemma 4.6.6 to compute $|\{S_Y \in \text{Comb}_{\text{lb}}^T(\mathcal{H}_g)|\Phi(S_Y) \in B_{d_T}(X,r)\}|$. For any $Y \in B_{d_T}(X,(g/T)^{\kappa_0})$, there exists a diffeomorphism of marked surfaces $f : X \to Y$ such that f is $1+\alpha_0(g/T)^{\kappa_0}$-quasiconformal. Here, α_0 is a universal constant.

We define a map

$$H_X : \{S_Y \in \mathrm{Comb}_{\mathrm{lb}}^T(\mathcal{H}_g) | \Phi(S_Y) \in B_{d_T}(X, (g/T)^{\kappa_0})\} \to H^1(X, \mathbb{C})$$

that sends S_Y to the cohomology class represented by $f^*\phi_{S_Y}$. We count the quantity $|\{S_Y \in \mathrm{Comb}_{\mathrm{lb}}^T(\mathcal{H}_g) | \Phi(S_Y) \in B_{d_T}(X, (g/T)^{\kappa_0})\}|$ in two steps. First, we compute the number of $S_Y \in \mathrm{Comb}_{\mathrm{lb}}^T(\mathcal{H}_g)$ such that $\Phi(S_Y) \in B_{d_T}(X, (g/T)^{\kappa_0})$ and $H_X(S_Y)$ is close in the Hodge metric to a fixed cohomology class in $H^1(X, \mathbb{C})$. Then we bound the number of cohomology classes, quantitatively.

Lemma 7.1.1. *Let $X \in \mathcal{T}_g$ and suppose $S_X \in \mathrm{Comb}_{\mathrm{lb}}^T(\mathcal{H}_g)$ such that $\Phi(S_X) = X$. Then there are at most*

$$(T/g)^{Cg} \sum_{\substack{n \leq (1/100)g \\ T_1 + \ldots + T_n \leq T \\ g_1 + \ldots + g_n \leq \mu^{-1}(1/100)g \\ g_1, \ldots, g_n \geq 2 \\ m_1 + \ldots + m_n \leq \mu^{-1}(1/100)g}} \prod_{i=1}^{n} N^{\mathcal{T}}(2T_i, g_i, m_i, r + C)$$

number of $S_Y \in \mathrm{Comb}_{\mathrm{lb}}^T(\mathcal{H}_g)$ that satisfy the following properties:

1. *$Y = \Phi(S_Y) \in B_{d_T}(X, (g/T)^{\kappa_0})$ and*
2. *$\|\phi_{S_X} - f^*\phi_{S_Y}\|_X \leq \alpha_1 (g/T)^{\kappa_1} g^{1/2}$*

Here, α_1 is a sufficiently small universal constant and $\kappa_1 < \kappa_0$ is a sufficiently large universal constant. We choose these constants in Section 7.9.

As a corollary we have:

Corollary 7.1.2. *We have,*

$$|\{S_Y \in \mathrm{Comb}_{\mathrm{lb}}^T(\mathcal{H}_g) | \Phi(S_Y) \in B_{d_T}(X, (g/T)^{\kappa_2}))\}|$$
$$\leq (T/g)^{Cg} \sum_{\substack{n \leq (1/100)g \\ T_1 + \ldots + T_n \leq T \\ g_1 + \ldots + g_n \leq \mu^{-1}(1/100)g \\ g_1, \ldots, g_n \geq 2 \\ m_1 + \ldots + m_n \leq \mu^{-1}(1/100)g}} \prod_{i=1}^{n} N^{\mathcal{T}}(2T_i, g_i, m_i, r + C)$$

where κ_2 is a sufficiently large universal constant chosen in Section 7.9.

First, a preliminary lemma.

Lemma 7.1.3. *The Hodge norm squared of ϕ_S is*

$$\int_X \phi_S \wedge *\overline{\phi_S} = (\sqrt{3}/4)T.$$

Proof. On the equilateral triangle with vertices $0, 1, \frac{1}{2} + \frac{\sqrt{3}}{2}i$ in $\mathbb{C}$, the Hodge norm squared of dz is $\frac{\sqrt{3}}{4}$. Since there are T triangles in S, the lemma follows. □

Proof of Corollary 7.1.2. The key idea is to reduce Corollary 7.1.2 to Lemma 7.1.1 by using Lemma 3.3.3 which allows us to deduce from a condition about cohomology classes being close condition 2 of Lemma 7.1.1, a much stronger condition about the individual forms being close averaged over the surface.

For any $S_Y \in \mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g)$ satisfying $\Phi(S_Y) \in B_{d_T}(X, (g/T)^{\kappa_2})$, by Lemma 7.1.3 and Corollary 3.3.2, $\|H_X(S_Y)\| \leq CT^{1/2}$. Now, cover the $CT^{1/2}$ radius Hodge norm ball in $H^1(X, \mathbb{C})$ centered at 0 by $(T/g)^{Cg}$ number of $(g/T)^{\kappa_3} g^{1/2}$ radius balls $B_1, ..., B_{(T/g)^{Cg}}$, where κ_3 is a constant to be chosen. Given $S_Y, S_Z \in \mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g)$ satisfying $\Phi(S_Y), \Phi(S_Z) \in B_{d_T}(X, (g/T)^{\kappa_2})$ and $H_X(S_Y), H_X(S_Z) \in B_k$, let $f : X \to Y$ and $g : Y \to Z$ be $1+8(g/T)^{\kappa_2}$-quasiconformal maps. Let $(f^*\phi_{S_Y})^h$ be the harmonic form on X representing the cohomology class of $f^*\phi_{S_Y}$ which is $H_X(S_Y)$. Let $((g \circ f)^*\phi_{S_Z})^h$ be the harmonic form on X representing the cohomology class of $(g \circ f)^*\phi_{S_Z}$, which is $H_X(S_Z)$.

By Lemma 3.3.3 and Lemma 7.1.3,

$$\|f^*\phi_{S_Y} - (f^*\phi_{S_Y})^h\|_X \leq 4(\sqrt{3}/4)^{1/2}(g/T)^{\kappa_2/2}T^{1/2}$$

and

$$\|(g \circ f)^*\phi_{S_Z} - ((g \circ f)^*\phi_{S_Z})^h\|_X \leq 8(\sqrt{3}/4)^{1/2}(g/T)^{\kappa_2/2}T^{1/2}.$$

Since $H_X(S_Y), H_X(S_Z) \in B_k$,

$$\|(f^*\phi_{S_Y})^h - (f^*\phi_{S_Z})^h\| \leq 2(g/T)^{\kappa_3}g^{1/2}.$$

Summing, we obtain

$$\begin{aligned}\|f^*\phi_{S_Y} - (g \circ f)^*\phi_{S_Z}\| &\leq (20(g/T)^{(\kappa_2-1)/2} + 2(g/T)^{\kappa_3})g^{1/2} \\ &\leq 40(g/T)^{(\kappa_2-1)/2}g^{1/2}\end{aligned}$$

assuming $\kappa_3 \geq (\kappa_2 - 1)/2$. Pulling back to Y under f^{-1}, by Lemma 3.3.1 we obtain

$$\|\phi_{S_Y} - g^*\phi_{S_Z}\| \leq 100(g/T)^{(\kappa_2-1)/2}g^{1/2}.$$

Since $g/T \leq 1/2$, assuming $100\alpha_1^{-1}(1/2)^{(\kappa_2-1)/2-\kappa_1} \leq 1$, condition 2 of Lemma 7.1.1 is satisfied. Assuming $\kappa_2 \geq \kappa_0$, condition 1 of Lemma 7.1.1 is satisfied. Applying Lemma 7.1.1, we obtain

$$|(H_X)^{-1}(B_k)| \leq (T/g)^{Cg} \sum_{\substack{n \leq (1/100)g \\ T_1+\ldots+T_n \leq T \\ g_1+\ldots+g_n \leq \mu^{-1}(1/100)g \\ g_1,\ldots,g_n \geq 2 \\ m_1+\ldots+m_n \leq \mu^{-1}(1/100)g}} \prod_{i=1}^{n} N^{\mathcal{T}}(2T_i, g_i, m_i, r + C)$$

for all $k \in \{1, ..., (T/g)^{Cg}\}$. The lemma statement follows. □

Corollary 7.1.2 now implies Lemma 7.0.1:

Proof of Lemma 7.0.1. By Lemma 4.6.6, we may cover $B_{d_T}(X, r)$ with $(T/g)^{C\kappa_2(1+r)(3g-3)}$ many $B_{d_T}(\cdot, (g/T)^{\kappa_2})$ balls. Applying Corollary 7.1.2 to each ball, then summing over all $T' \leq T$ and $g' \leq g$ gives the desired bound on $N^{\mathcal{H}}(T, g, r)$. □

We now turn to the proof of Lemma 7.1.1, which will take the rest of Chapter 7. Condition 2 in the statement of Lemma 7.1.1 can be written as

$$\int_X |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X}^2 |\phi_{S_X}|^2 \leq \alpha_2(g/T)^{\kappa_4} g, \tag{7.1}$$

where $\alpha_2 = \alpha_1^2$, $\kappa_4 = 2\kappa_1$, and $|\cdot|_{S_X}$ here denotes the operator norm of a 1-form at a particular point of X with respect to the S_X-metric.

Remark 17. Roughly speaking, Eq. (7.1) gives a measure of how Lipschitz the map f is with respect to the metrics d_{S_X} and d_{S_Y}, averaged over the entire surface X. However, Eq. (7.1) also implies that on most of X, the integral of ϕ_X on a curve is close to the integral of the 1-form ϕ_Y on the image of the curve.

The first step to prove Lemma 7.1.1 is to show that Eq. (7.1) implies most vertices of S_X and S_Y of degree strictly greater than 6 must be close to each other under f, many edges of S_X and S_Y must be close to each other under f, and many faces of S_X and S_Y must be close to each other under f. We do this precisely in Section 7.2, Section 7.3 and Section 7.4 below. The second step to prove Lemma 7.1.1 is to show that the geometric conditions about many vertices, edges and faces being close together under f imply that f is close to a simplicial isomorphism on all of S_X except for a part that has much lower genus. To do this, we decompose S_X into around g parallelograms of length and width at most around T/g and use the geometric conditions to say that f is close to a simplicial isomorphism on most of the parallelograms. The remaining parallelograms form a surface of much smaller genus, which allows us to reduce the problem of counting combinatorial translation surfaces in Teichmüller space to the problem of counting triangulated surfaces in a lower dimensional Teichmüller space. We do this in Section 7.5, Section 7.6 and Section 7.7. Finally in Section 7.8 we prove Lemma 7.1.1.

7.2 Vertices

In this section, we show that most vertices of S_X of degree strictly greater than 6 are close under f to vertices of S_Y of degree strictly greater than 6.

Let $0 < \varepsilon_0 < 1/2$, to be chosen later. Let V_X be the set of vertices $x \in V(S_X)$ such that there exists $y \in V(S_Y)$ satisfying $d_{S_Y}(y, f(x)) < \varepsilon_0$. Note that there is at most one choice of such y, since vertices of S_Y are at least distance 1 apart in the d_{S_Y}-metric.

Lemma 7.2.1. *We have, $|V_{>6}(S_X) \setminus V_X| \leq \alpha_3 g$ where $\alpha_3 = 10^{10}\alpha_2(g/T)^{\kappa_4}\varepsilon_0^{-10}$.*

Note that α_3 is not a universal constant, since it depends on T/g. To show Lemma 7.2.1, we first show the following lemma.

Lemma 7.2.2. *Suppose $x \in V_{>6}(S_X)$ and*

$$\int_{B_{S_X}(x,1/2)} |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} |\phi_{S_X}|^2 \leq \beta.$$

Then there exists $y \in V_{>6}(S_Y)$ satisfying $d_{S_Y}(y, f(x)) < \eta(\beta)$, assuming $\eta(\beta) = 10\beta^{1/10} < 1/2$.

Proof. In the following argument we write $\eta = \eta(\beta)$. Suppose the contrary, that $f(x)$ is not within a η-neighborhood of any vertex in $V_{>6}(S_Y)$. Let $a = \deg x/6$. Since S_X is a locally bounded combinatorial translation surface, $a \leq 7$. Let (r_X, θ_X) (where $0 \leq \theta_X < 2\pi a$) be polar coordinates on $B_{S_X}(x, \eta)$ around x such that $\phi_{S_X} = e^{i\theta_X} dr_X + i r_X e^{i\theta_X} d\theta_X$. Let (r_Y, θ_Y) (where $0 \leq \theta < 2\pi$) be polar coordinates on $B_{S_Y}(f(x), \eta)$ around y such that $\phi_{S_Y} = e^{i\theta_Y} dr_Y + i r_Y e^{i\theta_Y} d\theta_Y$. Let $C_r = \{(r_X, \theta_X) | r_X = r\}$. By assumption,

$$\int_{r=\eta/200}^{\eta/100} \int_{C_r} |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} r_X dr_X d\theta_X \leq \beta.$$

Therefore for some $u \in [\eta/200, \eta/100]$, we have

$$\int_{C_u} |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} d\theta_X \leq 40000\beta\eta^{-2}.$$

By Cauchy-Schwarz,

$$\begin{aligned}\int_{C_u} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X} d\theta_X &\leq \left(\int_{C_u} |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} d\theta_X\right)^{1/2} (2\pi a\eta)^{1/2} \\ &\leq 400\eta^{-1/2}(\pi a\beta)^{1/2}.\end{aligned} \tag{7.2}$$

Therefore,

$$\left|\int_{C_u} \left|f^*\phi_{S_Y}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right)\right| d\theta_X - \int_{C_u} \left|\phi_{S_X}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right)\right| d\theta_X\right|$$

$$\leq \int_{C_u} \left| f^*\phi_{S_Y}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right) - \phi_{S_X}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right)\right| d\theta_X$$
$$\leq \int_{C_u} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X} d\theta_X$$
$$\leq 400\eta^{-1/2}(\pi a\beta)^{1/2}.$$

Now,

$$\text{length}_{S_X}(C_u) = u\int_{C_u} \left|\phi_{S_X}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right)\right| d\theta_X$$

and

$$\text{length}_{S_Y}(f(C_u)) = u\int_{C_u} \left|f^*\phi_{S_Y}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right)\right| d\theta_X.$$

Thus

$$2\pi au - 400\eta^{-1/2}(\pi a\beta)^{1/2} \leq \text{length}_{S_Y}(f(C_u)) \leq 2\pi au + 400\eta^{-1/2}(\pi a\beta)^{1/2}.$$

Since $u \in [\eta/200, \eta/100]$, $a \leq 7$ and $\beta \leq \eta^{10}/10^{10}$, $f(C_u)$ is contained in $B_{S_Y}(f(x), \eta)$.

For $v \in [0, 2a\pi)$, let x_v be a point on C_u wherein $r_X = u$ and $\theta_X = v$. Letting $C_u(x_0, x_v)$ be an arc of C_u from x_0 to x_v, we have

$$\left|(ue^{iv} - u) - \int_{C_u(x_0,x_v)} f^*\phi_{S_Y}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right)\right|$$
$$= \left|\int_{C_u(x_0,x_v)} f^*\phi_{S_Y}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right) - \int_{C_u(x_0,x_v)} \phi_{S_X}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right)\right|$$
$$\leq \int_{C_u(x_0,x_v)} \left|f^*\phi_{S_Y}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right) - \phi_{S_X}\left(\frac{1}{r_X}\frac{\partial}{\partial\theta_X}\right)\right|$$
$$\leq \int_{C_u} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X} d\theta_X$$
$$\leq 400\eta^{-1/2}(\pi a\beta)^{1/2}.$$

by Eq. (7.2). Hence, $|f(x_v) - f(x_0) - (ue^{iv} - u)|_{S_Y} \leq 400\eta^{-1/2}(\pi a\beta)^{1/2}$. This means $f(C_u)$ lies in a $400\eta^{-1/2}(\pi a\beta)^{1/2}$-neighborhood of a radius u-ball passing through $f(x_0)$ (in the S_Y-metric). Therefore,

$$\text{area}_{S_Y}(f(B_{S_X}(x, u))) \leq \pi(u + 400\eta^{-1/2}(\pi a\beta)^{1/2})^2. \tag{7.3}$$

Finally, by $1+\alpha_0(g/T)^{\kappa_0}$-quasiconformality of f,

$$\begin{aligned}
\mathrm{area}_{S_X}(B_{S_X}(x,u))^{1/2} - (1+\alpha_0(g/T)^{\kappa_0})^{1/2}\,\mathrm{area}_{S_Y}(f(B_{S_X}(x,u)))^{1/2} \\
&\le \left(\int_{B_{S_X}(x,u)} |\phi_{S_X}|^2\right)^{1/2} - \left(\int_{B_{S_X}(x,u)} |f^*\phi_{S_Y}|^2_{S_X}|\phi_{S_X}|^2\right)^{1/2} \\
&\le \left(\int_{B_{S_X}(x,u)} |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X}|\phi_{S_X}|^2\right)^{1/2} \\
&\le \beta^{1/2}.
\end{aligned}$$

Therefore $\mathrm{area}_{S_Y}(f(B_{S_X}(x,u)))^{1/2} \ge (1-\alpha_0(g/T)^{\kappa_0})^{1/2}((a\pi)^{1/2}u - \beta^{1/2})$. Since $a \ge 2$, $g/T \le 1/2$, $\beta \le \eta^{10}/10^{10}$ and $u \in [\eta/200, \eta/100]$, assuming $\alpha_0(1/2)^{\kappa_0} \le (1/10)^{10}$ we have a contradiction with Eq. (7.3). □

Proof of Lemma 7.2.1. Combining Eq. (7.1) and Lemma 7.2.2, we obtain the desired result. □

7.3 Edges

In this section, we show that many edges of S_X and S_Y are close to each other under f.

Lemma 7.3.1. *Let $0 < \varepsilon \le 1/1000$. Suppose $x_0 \in V(S_X)$ and $f(x_0) \in B_{S_Y}(y_0, \varepsilon)$ for a vertex $y_0 \in V(S_Y)$. Suppose $e(x_0, x_1)$ is an edge in S_X from x_0 to x_1. Suppose further that*

$$\int_{B_{S_X}(e(x_0,x_1),1/2)} |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X}|\phi_{S_X}|^2 \le \beta, \tag{7.4}$$

where β satisfies $\tau(\varepsilon,\beta) = 100\beta^{1/4}\varepsilon^{-1} + \varepsilon \le 1/1000$. Then $f(x_1) \in B_{S_Y}(y_1, \tau(\varepsilon,\beta))$ for some vertex y_1 in S_Y. Also, y_1 is connected to y_0 by an edge $e(y_0, y_1)$ such that $f(e(x_0,x_1)) \subset B_{S_Y}(e(y_0,y_1), 10\tau(\varepsilon,\beta))$.

Proof. Let T be a triangle in S_X containing edge $e(x_0,x_1)$, and let T' be the other triangle containing edge $e(x_0,x_1)$. As shown in Fig. 7-1, identify T with the triangle in $\mathbb{C}$ with vertices at 0, 1 and $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, with x_0 and x_1 identified with 0 and 1. Then T' is naturally identified with the triangle in $\mathbb{C}$ with vertices at 0, 1 and $\frac{1}{2} - \frac{\sqrt{3}}{2}i$. Let (r_0, θ_0) be polar coordinates on $T \cup T'$ centered at 0, and (r_1, θ_1) be polar coordinates on $T \cup T'$ centered at 1. Note that the form ϕ_{S_X} may be written as $\zeta e^{i\theta_0} dr_0 + i\zeta r_0 e^{i\theta_0} d\theta_0$ in the (r_0, θ_0) coordinates and $-\zeta e^{-i\theta_0} dr_0 + i\zeta r_0 e^{-i\theta_0} d\theta_0$ in the (r_1, θ_1)-coordinates, for some 6th root of unity ζ. Let $\upsilon < 1/4$ be sufficiently small, to be chosen later in this proof. For $\theta \in [-\pi/3, \pi/3]$, define

$$L_\theta = \{(r_0, \theta_0) \in T | \theta_0 = \theta, \upsilon \leq r_0 \leq (1/2)(\cos\theta)^{-1}\},$$

and

$$R_\theta = \{(r_1, \theta_1) \in T | \theta_1 = \pi - \theta, \upsilon \leq r_1 \leq (1/2)(\cos\theta)^{-1}\}.$$

We assume that L_θ (resp. R_θ) is oriented so that r_0 (resp. r_1) is increasing.

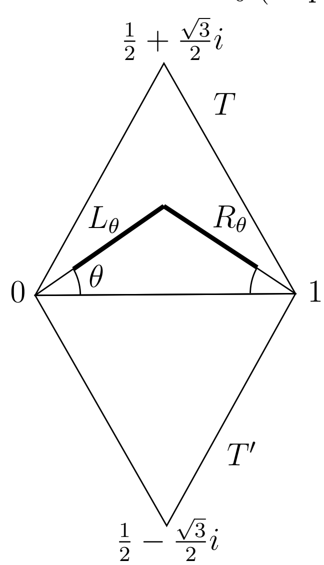

Figure 7-1: A diagram of T and T' with L_θ and R_θ in bold.

By elementary trigonometry, we see that for $\theta \in [-\varepsilon, \varepsilon]$, $L_\theta \cup R_\theta \subset B_{S_X}(e(x_0, x_1), \varepsilon)$. By the co-area formula along with Eq. (7.4),

$$\int_{\theta=0}^{\varepsilon} \int_{L_\theta} r_0 |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} dr_0 d\theta_0 + \int_{\theta=0}^{\varepsilon} \int_{R_\theta} r_1 |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} dr_1 d\theta_1 \leq \beta.$$

This means for some $u \in [0, \varepsilon]$,

$$\int_{L_u} r_0 |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} dr_0 + \int_{R_u} r_1 |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} dr_1 \leq \beta\varepsilon^{-1}.$$

By Cauchy-Schwarz,

$$\int_{L_u} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X} dr_0 + \int_{R_u} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X} dr_1$$
$$\leq \left(\int_{L_u} r_0 |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} dr_0\right)^{1/2} \left(\int_{r_0=\upsilon}^{(1/2)(\cos\theta)^{-1}} r_0^{-1}\right)^{1/2}$$
$$+ \left(\int_{R_u} r_1 |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} dr_1\right)^{1/2} \left(\int_{r_1=\upsilon}^{(1/2)(\cos\theta)^{-1}} r_1^{-1}\right)^{1/2} \quad (7.5)$$

thus

$$\int_{L_u} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X} dr_0 + \int_{R_u} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X} dr_1 \leq 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2}. \quad (7.6)$$

Since

$$\left| f_*\phi_{S_Y}\left(\frac{\partial}{\partial r_0}\right) - \phi_{S_X}\left(\frac{\partial}{\partial r_0}\right)\right| \leq |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X}$$

on L_u and

$$\left| f_*\phi_{S_Y}\left(\frac{\partial}{\partial r_1}\right) - \phi_{S_X}\left(\frac{\partial}{\partial r_1}\right)\right| \leq |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X}$$

on R_u, combining with Eq. (7.6) we obtain

$$\int_{L_u} \left| f_*\phi_{S_Y}\left(\frac{\partial}{\partial r_0}\right) - \phi_{S_X}\left(\frac{\partial}{\partial r_0}\right)\right| dr_0 + \int_{R_u} \left| f_*\phi_{S_Y}\left(\frac{\partial}{\partial r_1}\right) - \phi_{S_X}\left(\frac{\partial}{\partial r_1}\right)\right| dr_1$$
$$\leq 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2}. \quad (7.7)$$

For an appropriate 6th root of unity ζ,

$$\phi_{S_Y}\left(\frac{\partial}{\partial r_0}\right) = e^{iu}\zeta$$

on L_u and

$$\phi_{S_Y}\left(\frac{\partial}{\partial r_1}\right) = -e^{-iu}\zeta$$

on R_u. Plugging into Eq. (7.7) we have

$$\int_{L_u} \left| f^*\phi_{S_Y}\left(\frac{\partial}{\partial r_0}\right) - e^{iu}\zeta \right| dr_0 + \int_{R_u} \left| f^*\phi_{S_Y}\left(\frac{\partial}{\partial r_1}\right) + e^{-iu}\zeta \right| dr_1 \leq 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2}.$$

This means

$$\left| \int_{L_u \cup R_u} f^*\phi_{S_Y} - \int_{L_u} e^{iu}\zeta dr_0 - \int_{R_u} e^{-iu}\zeta dr_1 \right| \leq 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2}. \tag{7.8}$$

Note that when integrating over $L_u \cup R_u$, the orientation of R_u switches, hence there is a sign change. By construction of L_u and R_u,

$$\int_{L_u} e^{iu}\zeta dr_0 = ((1/2)(\cos u)^{-1} - \upsilon)\zeta e^{iu}$$

and

$$\int_{R_u} e^{-iu}\zeta dr_1 = ((1/2)(\cos u)^{-1} - \upsilon)\zeta e^{-iu},$$

hence combining with Eq. (7.8) we obtain

$$\left| \zeta - \int_{L_u \cup R_u} f^*\phi_{S_Y} \right| \leq 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 2\upsilon. \tag{7.9}$$

Pushing forward to Y we have

$$\left| \zeta - \int_{f(L_u \cup R_u)} \phi_{S_Y} \right| \leq 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 2\upsilon. \tag{7.10}$$

From Eq. (7.9) we also have

$$\int_{L_u} \left| f^*\phi_{S_Y}\left(\frac{\partial}{\partial r_0}\right) \right| dr_0 + \int_{R_u} \left| f^*\phi_{S_Y}\left(\frac{\partial}{\partial r_1}\right) \right| dr_1 \leq 1 + 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 2\upsilon.$$

The left-hand-side is the length of $f(L_u \cup R_u)$ in the S_Y-metric. Hence,

$$\text{length}_{S_Y}(f(L_u \cup R_u)) \leq 1 + 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 2\upsilon. \tag{7.11}$$

Now, let $C_{0,r}$ be the circle of radius r around x_0, with respect to the S_X-metric. Similarly, let $C_{1,r}$ be the circle of radius r around x_1, with respect to the S_X-metric. By assumption, for $\upsilon < 1/4$ we have

$$\int_{r=\upsilon}^{2\upsilon} \int_{C_{0,r}} |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} r_0 dr_0 d\theta_0 \leq \beta.$$

Therefore for some $w \in [\upsilon, 2\upsilon]$, we have

$$\int_{C_{0,w}} |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} d\theta_0 \leq \beta\upsilon^{-2}.$$

By Cauchy-Schwarz,

$$\begin{aligned}\int_{C_{0,w}} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X} d\theta_0 &\leq \left(\int_{C_{0,w}} |f^*\phi_{S_Y} - \phi_{S_X}|^2_{S_X} d\theta_0\right)^{1/2} (2\pi\upsilon)^{1/2} \\ &\leq (2\pi)^{1/2}\upsilon^{-1/2}\beta^{1/2}.\end{aligned}$$

Moreover,

$$\begin{aligned}&\left|\int_{C_{0,w}} \left|f^*\phi_{S_Y}\left(\frac{1}{r_0}\frac{\partial}{\partial\theta_0}\right)\right| d\theta_0 - \int_{C_{0,w}} \left|\phi_{S_X}\left(\frac{1}{r_0}\frac{\partial}{\partial\theta_0}\right)\right| d\theta_0\right| \\ &\qquad\leq \int_{C_{0,w}} \left|f^*\phi_{S_Y}\left(\frac{1}{r_0}\frac{\partial}{\partial\theta_0}\right) - \phi_{S_X}\left(\frac{1}{r_0}\frac{\partial}{\partial\theta_0}\right)\right| d\theta_0 \\ &\qquad\leq \int_{C_{0,w}} |f^*\phi_{S_Y} - \phi_{S_X}|_{S_X} d\theta_0 \\ &\qquad\leq (2\pi)^{1/2}\upsilon^{-1/2}\beta^{1/2}\end{aligned}$$

Now,

$$\mathrm{length}_{S_X}(C_{0,w}) = w\int_{C_{0,w}} \left|\phi_{S_X}\left(\frac{1}{r_0}\frac{\partial}{\partial\theta_0}\right)\right| d\theta_0$$

and

$$\mathrm{length}_{S_Y}(f(C_{0,w})) = w\int_{C_{0,w}} \left|f^*\phi_{S_Y}\left(\frac{1}{r_0}\frac{\partial}{\partial\theta_0}\right)\right| d\theta_0.$$

Thus

$$2\pi w - (2\pi)^{1/2}\upsilon^{-1/2}\beta^{1/2} \leq \mathrm{length}_{S_Y}(f(C_{0,w})) \leq 2\pi w + (2\pi)^{1/2}\upsilon^{-1/2}\beta^{1/2}.$$

Since $w \in [\upsilon, 2\upsilon]$ and assuming $\beta \leq \upsilon^4$, $f(C_{0,w})$ is contained in $B_{S_Y}(f(x_1), 20\upsilon)$. Since $f(C_{0,\upsilon})$ is enclosed by $f(C_{0,w})$, $f(C_{0,\upsilon})$ is also contained in $B_{S_Y}(f(x_0), 20\upsilon)$. Analogously, we may show that $f(C_{1,\upsilon})$ is contained in $B_{S_Y}(f(x_1), 20\upsilon)$.

The arc $L_u \cup R_u$ has two endpoints, one wherein $r_0 = \upsilon$ and $\theta_0 = u$, which we label $x_{0,u}$, and the other wherein $r_1 = \upsilon$ and $\theta_1 = \pi - u$, which we label $x_{1,u}$. Since $f(x_{0,u})$ lies on $f(C_{0,\upsilon})$ and $f(x_{1,u})$ lies on $f(C_{1,\upsilon})$, $f(x_{0,u}) \in B_{S_Y}(f(x_0, 20\upsilon))$ and $f(x_{1,u}) \in B_{S_Y}(f(x_1, 20\upsilon))$. By Eq. (7.10) we have,

$$d_{S_Y}(f(x_0), f(x_1)) \leq 1 + 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 50\upsilon,$$

so

$$\begin{aligned} d_{S_Y}(y_0, f(x_1)) &\leq d_{S_Y}(y_0, f(x_0)) + d_{S_Y}(f(x_0), f(x_1)) \\ &\leq 1 + 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 50\upsilon + \varepsilon. \end{aligned}$$

Let y_1 be a vertex of S_Y nearest to $f(x_1)$. Since $f(x_1)$ lies on a triangle in S_Y,

$$d_{S_Y}(y_1, f(x_1)) \leq \sqrt{3}^{-1}. \tag{7.12}$$

Therefore

$$\begin{aligned} d_{S_Y}(y_0, y_1) &\leq d_{S_Y}(y_0, f(x_1)) + d_{S_Y}(y_0, f(x_1)) \\ &\leq 1 + 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 50\upsilon + \varepsilon + \sqrt{3}^{-1}. \end{aligned}$$

Choosing υ later such that $4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 50\upsilon + \varepsilon < 1/1000$, we must have that either $d_{S_Y}(y_0, y_1) = 0$ or $d_{S_Y}(y_0, y_1) = 1$. The former possibility is ruled out by Eq. (7.10), therefore the latter equation is true. This necessarily means that y_0 and y_1 are adjacent vertices on S_Y.

We must show that $d(x_1, f(y_1)) \leq \tau(\varepsilon, \beta)$ and also find an edge of S_Y connecting y_0 and y_1. To do this, letting q be a shortest path from $f(x_1)$ to y_1, we have

$$\begin{aligned} d_{S_Y}(f(x_1), y_1) &= \left| \int_q \phi_Y \right| \\ &\leq \sqrt{3}^{-1} \end{aligned} \tag{7.13}$$

by Eq. (7.12). Let p be a shortest path from y_0 to $f(x_0)$ in S_Y. Then

$$\left|\int_p \phi_{S_Y}\right| \leq \varepsilon. \tag{7.14}$$

Let s_0 be a shortest path from $f(x_{0,u})$ to $f(x_0)$ and s_1 be a shortest path from $f(x_{1,u})$ to $f(x_1)$. Finally, let t be a shortest path from y_0 to y_1 that is homotopic to the path $p \cup s_0 \cup f(L_u \cup R_u) \cup s_1 \cup q$ (which also has the endpoints y_0 and y_1). By construction q is homotopic to $t^{-1} \cup p \cup s_0 \cup f(L_u \cup R_u) \cup s_1$, where here t^{-1} denotes the path t with the opposite orientation. So to compute

$$\int_q \phi_Y,$$

it suffices to compute

$$\int_{t^{-1}\cup p\cup s_0\cup f(L_u\cup R_u)\cup s_1} \phi_Y = \int_p \phi_Y + \int_{s_0} \phi_Y + \int_{f(L_u\cup R_u)} \phi_Y + \int_t \phi_Y + \int_{s_1} \phi_Y.$$

Since $s_0 \subset B(f(x_1), 20\upsilon)$,

$$\left|\int_{s_0} \phi_Y\right| \leq 20\upsilon. \tag{7.15}$$

Similarly,

$$\left|\int_{s_1} \phi_Y\right| \leq 20\upsilon. \tag{7.16}$$

From Eq. (7.10), Eq. (7.13), Eq. (7.14), Eq. (7.15) and Eq. (7.16), we have

$$\left|\zeta - \int_t \phi_Y\right| \leq 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 50\upsilon + \varepsilon + \sqrt{3}^{-1}.$$

Since $\int_t \phi_Y \subset \mathbb{Z} + e^{\pi i/3}\mathbb{Z}$, we must necessarily have $\int_t \phi_Y = \zeta$. Combining with Eq. (7.10), Eq. (7.14), Eq. (7.15) and Eq. (7.16) we obtain

$$\begin{aligned} d(y_1, f(x_1)) &= \left|\int_q \phi_Y\right| \\ &\leq \left|\int_{t^{-1}\cup p\cup s_0\cup f(L_u\cup R_u)\cup s_1} \phi_Y\right| \\ &\leq 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 50\upsilon + \varepsilon, \end{aligned}$$

as desired. Finally, since t is a shortest path from y_0 to y_1 and $\int_t \phi_Y = \zeta$, t must be an edge which we denote $e(y_0, y_1)$.

It remains to show that $f(e(x_0, x_1)) \subset B_{S_Y}(e(y_0, y_1), 10\tau(\varepsilon, \beta))$. To do this, we have shown by Eq. (7.11) the existence of $u \in [0, \varepsilon]$ such that

$$\text{length}_{S_Y}(f(L_u \cup R_u)) \leq 1 + 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 2\upsilon.$$

Analogously, there exists $u' \in [-\varepsilon, 0]$ such that

$$\text{length}_{S_Y}(f(L_{u'} \cup R_{u'})) \leq 1 + 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 2\upsilon.$$

Since the endpoints of $f(L_u \cup R_u)$ and $f(L_{u'} \cup R_{u'})$ are contained in a 20υ-ball around $f(x_0)$ or $f(x_1)$ in the S_Y-metric, and $f(x_0) \in B_{S_Y}(y_0, \varepsilon)$ while

$$f(x_1) \in B_{S_Y}(y_1, 4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 50\upsilon + \varepsilon),$$

both $f(L_u \cup R_u)$ and $f(L_{u'} \cup R_{u'})$ must be contained in

$$B_{S_Y}(e(y_0, y_1), 40\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 500\upsilon + 10\varepsilon).$$

Since $L_u \cup R_u$ and $L_{u'} \cup R_{u'}$ along with arcs of $C_{0,\upsilon}$ and $C_{1,\upsilon}$ bound a closed simply connected region R in S_X such that $e(x_0, x_1) \subset B_{S_X}(x_0, \upsilon) \cup R \cup B_{S_X}(x_1, \upsilon)$,

$$\begin{aligned} f(e(x_0, x_1)) &\subset f(B_{S_X}(x_0, \upsilon)) \cup f(R) \cup f(B_{S_X}(x_1, \upsilon)) \\ &\subset B_{S_Y}(f(x_0), 20\upsilon) \cup f(R) \cup B_{S_Y}(f(x_1), 20\upsilon)). \end{aligned}$$

Therefore

$$f(e(x_0, x_1)) \subset B_{S_Y}(e(y_0, y_1), 40\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 500\upsilon + 10\varepsilon)$$

as well. Finally, choosing $\upsilon = \beta^{1/4}$ satisfies $\beta \leq \upsilon^4$ and $4\beta^{1/2}\varepsilon^{-1/2}(\log(1/\upsilon))^{1/2} + 50\upsilon + \varepsilon \leq 100\beta^{1/4}\varepsilon^{-1} + \varepsilon \leq 1/1000$, which we assumed in the proof. $\square$

7.4 Faces

In this section, we note that if vertices and edges of S_X are close to vertices and edges of S_Y under f, then so are the faces they bound.

Lemma 7.4.1. *Let* $0 < \varepsilon \leq 1/4$. *Suppose* $x_1, x_2, x_3 \in V(S_X)$ *and* $y_1, y_2, y_3 \in V(S_Y)$ *such that* $f(x_1) \in B_{S_Y}(y_1, \varepsilon)$, $f(x_2) \in B_{S_Y}(y_2, \varepsilon)$, *and* $f(x_3) \in B_{S_Y}(y_3, \varepsilon)$. *Suppose* x_1, x_2, x_3 *bound a triangle in* S_X *with edge* $e(x_i, x_j)$ *connecting* x_i *and* x_j. *Suppose additionally that for each* $i, j \in \{1, 2, 3\}$ *satisfying* $i \neq j$, *there is an edge* $e(y_i, y_j)$ *in* S_Y *such that* $f(e(x_i, x_j)) \subset B_{S_Y}(e(y_i, y_j), \varepsilon)$. *Then* y_1, y_2, y_3 *along with the* $e(y_i, y_j)$ *bound a triangle in* S_Y.

Proof. First, we claim that $f(e(x_i, x_j))$ and $e(y_i, y_j)$ are homotopic relative to their boundaries in the quotient space $S_Y / \cup_{i \in \{1,2,3\}} B_{S_Y}(y_i, \varepsilon)$. This is true because by supposition they are homotopic relative to their boundaries in $B_{S_Y}(e(y_i, y_j), \varepsilon)/(B_{S_Y}(y_i, \varepsilon) \cup B_{S_Y}(y_j, \varepsilon))$ which lies inside $S_Y / \cup_{i \in \{1,2,3\}} B_{S_Y}(y_i, \varepsilon)$. So $f(e(x_1, x_2)) \cup f(e(x_2, x_3)) \cup f(e(x_3, x_1))$ and $e(y_1, y_2) \cup e(y_2, y_3) \cup e(y_3, y_1)$ are freely homotopic to each other in $S_Y / \cup_{i \in \{1,2,3\}} B_{S_Y}(y_i, \varepsilon)$, which means they are also freely homotopic to each other in S_Y, since $S_Y / \cup_{i \in \{1,2,3\}} B_{S_Y}(y_i, \varepsilon)$ and S_Y are homotopy equivalent. The curve $f(e(x_1, x_2)) \cup f(e(x_2, x_3)) \cup f(e(x_3, x_1))$ is contractible since it is the image of $e(x_1, x_2) \cup e(x_2, x_3) \cup e(x_3, x_1)$ (which bounds a triangle in S_X) under f which is a diffeomorphism. Therefore $e(y_1, y_2) \cup e(y_2, y_3) \cup e(y_3, y_1)$ is contractible in S_Y, which means that it bounds a triangle in S_Y with vertices y_1, y_2 and y_3. □

7.5 Parallelogram decomposition

In this section, given $S \in \mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g)$, we decompose S into parallelograms wherein each parallelogram contains a vertex of degree greater than 6. This decomposition applied to S_X will allow us to use Lemma 7.2.1 and Lemma 7.3.1 repeatedly to show in Section 7.6 that most edges of S_X are close to edges of S_Y under f.

To construct a parallelogram decomposition of $S \in \mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g)$, we construct a 2-polytope B of which S is a refinement. The faces of B will be the parallelograms. To construct B, we first build a certain 1-complex A as follows. We do this by constructing 1-complexes A_0, A_1 and A_2 inductively and taking their union. First, we construct A_0 to be the union of all edge trajectories in S of direction 1 starting from vertices of degree greater than 6. That is, we start by adding all vertices in $V_{>6}(S)$ to A_0. Then, for any $v \in V(S) \cap A_0$ and edge $e(v,w)$ connecting v and $w \in V(S)$ such that $\zeta(e(v,w),v) = 1$, we add $e(v,w)$ and w to A_0. We repeat this inductive step until it is redundant to obtain a 1-complex A_0. Next, we construct A_1 to be the union of all edge trajectories in S of direction $e^{\pi i/3}$ starting from vertices of degree greater than 6, but the trajectories stop once they hit A_0. That is, we build A_1 inductively by adding all vertices in $V_{>6}(S)$ to A_1. At each inductive step, for each $v \in ((V(S) \cap A_1) \setminus ((V(S) \cap A_0) \cap (V(S) \cap A_1))) \cup V_{>6}(S)$ and edge $e(v,w)$ connecting v and $w \in V(S)$ such that $\zeta(e(v,w),v) = e^{\pi i/3}$, we add $e(v,w)$ and w to A_1. We repeat this inductive step until it is redundant, to obtain a 1-complex A_1. Finally, we construct A_2 to be the union of all edge trajectories in S of direction $-e^{\pi i/3}$ starting from vertices of degree greater than 6, but the trajectories stop once they hit A_0. That is, we build A_2 inductively by adding all vertices in $V_{>6}(S)$ to A_2. At each inductive step, for each $v \in ((V(S) \cap A_2) \setminus ((V(S) \cap A_0) \cap (V(S) \cap A_2))) \cup V_{>6}(S)$ and edge $e(v,w)$ connecting v and $w \in V(S)$ such that $\zeta(e(v,w),v) = -e^{\pi i/3}$, we add $e(v,w)$ and w to A_2. Now, let $A = A_0 \cup A_1 \cup A_2$. By construction A is a simplicial 1-complex such that each vertex of A has degree at least 2. Then A is naturally the 1-skeleton of a 2-polytope B which is homeomorphic to S. The vertices $V(B)$ of B are the vertices $x \in V(S) \cap A$ for which

1. there exists an edge $e \in E(S) \cap A$ emanating from x such that $\zeta(e,x) = 1$ or -1 and

2. there exists an edge $e \in E(S) \cap A$ emanating from x such that $\zeta(e,x) = e^{\pi i/3}$ or $-e^{\pi i/3}$.

Two vertices $v, w \in V(B)$ have an edge in $E(B)$ between them if there exists a collection of vertices $v_1, ..., v_{k+1} \in V(S) \cap A$ and edges $e_1, ..., e_k \in E(S_X) \cap A$ such that

1. $v_1 = v$ and $v_{k+1} = w$,

2. e_i connects v_i to v_{i+1} for all $i \in \{1, ..., k\}$,

3. $\zeta(e_i, v_i) = \zeta(e_{i+1}, v_{i+1})$ for all $i \in \{1, ..., k-1\}$ and

4. $v_i \notin V(B)$ for $i \neq 1, k+1$.

Note that all edges of S contained in A are contained in some edge of B. The faces $F(B)$ are the regions of S bounded by the edges in $E(B)$ and vertices in $V(B)$.

Lemma 7.5.1. *The polytope B satisfies the following properties.*

1. *If $e \in E(S)$ is contained in an edge in $E(B)$ and $x \in V(S)$ is a vertex which is an endpoint of e, then $\zeta(e, x) = 1$, -1, $e^{\pi i/3}$ or $-e^{\pi i/3}$.*

2. *If $x \in V_{>6}(S)$ and $e \in E(S)$ is an edge emanating from x satisfying $\zeta(e, x) = 1$, -1, $e^{\pi i/3}$ or $-e^{\pi i/3}$, then e is contained in an edge in $E(B)$.*

3. *If $x \in V(B)$ and $e \in E(S)$ is an edge emanating from x satisfying $\zeta(e, x) = 1$ or -1, then e is contained in an edge in $E(B)$.*

4. *Given two vertices $x, y \in V(B)$ and a path γ from x to y on S, $\int_\gamma \phi_S \in 3\mathbb{Z} + 3e^{\pi i/3}\mathbb{Z}$.*

Proof. Property 1 is true because all edges of S contained in A_0 have directional weights 1 or -1, and all edges of S contained in A_1 and A_2 have directional weights $e^{\pi i/3}$ or $-e^{\pi i/3}$. All edges of S contained in an edge of B are contained in $A = A_0 \cup A_1 \cup A_2$.

To show property 2, let $x \in V_{>6}(S)$. If e is an edge of S emanating from x satisfying $\zeta(e, x) = 1$ (resp. $\zeta(e, x) = e^{\pi i/3}$ or $\zeta(e, x) = -e^{\pi i/3}$), then e is added to A_0 (resp. A_1 or A_2) in the first step of the inductive construction. So e is contained in A, therefore contained in an edge in $E(B)$. Now, we show property 2 in the case wherein $\zeta(e, x) = -1$. Since $x \in V_{>6}(S)$, there exists a sequence of vertices and edges $x_0, ..., x_k = x$ and $e_1, ..., e_k$ such that $x_0 \in V_{>6}(S)$, e_i has endpoints x_{i-1} and x_i, $e_k = e$ and $\zeta(e_i, x_{i-1}) = 1$. (It is possible that $x_0 = x$.) Following the inductive construction of A_0 starting from x_0, we see that e is contained in A_0 as well. So e is contained in A, therefore also contained in an edge in $E(B)$.

To show property 3, let $x \in V(B)$ and suppose e is an edge emanating from x satisfying $\zeta(e, x) = 1$. Then x satisfies property 1 in the definition of $V(B)$, so x lies in A_0. Therefore, at some point in the inductive construction of A_0, e is added to A_0 as well, implying that e is contained in an edge of B. Now, suppose $\zeta(e, x) = -1$. Again, x lies in A_0. Assuming $x \in V_{=6}(S)$ (the other case is covered by property 2), e is the only edge emanating from x satisfying $\zeta(e, x) = -1$, hence during the inductive construction of A_0, x must have been added to A_0 along with e. Therefore e lies in A_0 which means e is contained in an edge of B.

Finally, we show property 4. Since S is a locally bounded combinatorial translation surface, property 4 is automatically true for $x, y \in V_{>6}(S)$. Now, suppose $x \in V(B)$ and $y \in V_{>6}(S)$. By properties 1 and 2 in the definition of $V(B)$, x is contained in both A_0 and $A_1 \cup A_2$. By the inductive construction of A_0, there exists a sequence of vertices and edges $x_0^0, ..., x_k^0 = x$ and $e_1^0, ..., e_k^0$ such that $x_0^0 \in V_{>6}(S)$, e_i^0 has endpoints x_{i-1}^0 and x_i^0, and $\zeta(e_i^0, x_{i-1}^0) = 1$. Now, assume $x \in A_1$. (The case where $x \in A_2$ is analogous). By the inductive construction of A_1, there exists a sequence of vertices and edges $x_0^1, ..., x_j^1 = x$ and $e_1^1, ..., e_j^1$ such that $x_0^1 \in V_{>6}(S)$, e_i^1 has endpoints x_{i-1}^1 and x_i^1, and $\zeta(e_i^1, x_{i-1}^1) = e^{\pi i/3}$. Let γ be the path from x_0^0 to x_0^1 first traversing along e_1^0,...,e_k^0, then traversing along e_j^1,...,e_1^1. Then $\int_\gamma \phi_S = k - je^{\pi i/3} \in 3\mathbb{Z} + 3e^{\pi i/3}\mathbb{Z}$, hence k and j are both divisible by 3. In particular, letting γ_0 be the path from x_0^0 to x traversing along e_1^0,...,e_k^0, we have $\int_{\gamma_0} \phi_S \in 3\mathbb{Z} + 3e^{\pi i/3}\mathbb{Z}$. We already know that the integral of ϕ_S over any path from x_0^0 to y is in $3\mathbb{Z} + 3e^{\pi i/3}\mathbb{Z}$. Hence, this is true for x and y as well. Finally, in the case where $x, y \in V(B)$, we simply take an arbitrary $z \in V_{>6}(S)$, and since the integral of ϕ_S over any path from x to z and y to z is in $3\mathbb{Z} + 3e^{\pi i/3}\mathbb{Z}$, so is the integral of ϕ_S over any path from x to y. □

As a result, we have:

Lemma 7.5.2. *If $R \in F(B)$ is a face of B, then R is a planar parallelogram with respect to the flat metric on S.*

The edges of R as a parallelogram need not correspond to edges of B; they may be unions of edges of B in the same direction. Similarly, R may have more than four vertices of B in its boundary. We say that $v \in V(B)$ is a corner vertex vertices of R if two boundary edges of R meet at v in non-parallel directions.

Proof of Lemma 7.5.2. Suppose $v \in V(B_X)$ is a corner vertex of R. Let b_1 and b_2 be the two corresponding edges of $V(B)$ that lie on the boundary of R and meet at v. Note that b_1 and b_2 are both unions of edges of S. Let $e_1 \subset b_1$ be the edge of S that emanates from v and $e_2 \subset b_2$ be the edge of S that emanates from v. We assume b_1 and b_2 (or e_1 and e_2) are labelled so that the face R exists in the clockwise direction from b_1 (or e_1) with respect to the orientation on S.

We call an ordered pair of 6th roots of unity (ζ_1, ζ_2) an allowed pair if it is possible that $\zeta(e_1, v) = \zeta_1$ and $\zeta(e_2, v) = \zeta_2$. First, we enumerate all the allowed pairs. Allowed pairs must have roots that are all 1, -1, $e^{\pi i/3}$, $-e^{\pi i/3}$ by property 1 of Lemma 7.5.1. By property 2 of Lemma 7.5.1, $(1,1)$, $(-1,-1)$, $(e^{\pi i/3}, e^{\pi i/3})$, $(-e^{\pi i/3}, -e^{\pi i/3})$, $(1, e^{\pi i/3})$, $(-1, -e^{\pi i/3})$, $(e^{\pi i/3}, -e^{\pi i/3})$, $(e^{\pi i/3}, -1)$, $(-e^{\pi i/3}, e^{\pi i/3})$ and $(-e^{\pi i/3}, 1)$ are not allowed pairs. Since v is a corner vertex of R, $(-1,1)$ and $(1,-1)$ are not allowed pairs. Therefore, the only allowed pairs are $(1, -e^{\pi i/3})$, $(-e^{\pi i/3}, -1)$, $(-1, e^{\pi i/3})$ and $(e^{\pi i/3}, 1)$. So if $\zeta(e_1, v) = 1$, then $\zeta(e_2, v) = -e^{\pi i/3}$; if $\zeta(e_1, v) = -1$, then $\zeta(e_2, v) = e^{\pi i/3}$; if $\zeta(e_1, v) = -e^{\pi i/3}$, then $\zeta(e_2, v) = -1$; and if $\zeta(e_1, v) = e^{\pi i/3}$, then $\zeta(e_2, v) = 1$. By property 1 of Lemma 7.5.1, the angle between e_1 and e_2 is $2\pi/3$ in the first two cases and $\pi/3$ in the last two cases.

So the direction of the boundary of R changes $4n$ times for some n, alternating between a change in angle by $\pi/3$ and $2\pi/3$. Applying Gauss-Bonnet to the flat metric: the total sum of turning angles around R is $2\pi n$. Since no vertices $v \in V_{>6}(S)$ lie in the interior of R (by property 3 of Lemma 7.5.1), R is flat. Since the Euler characteristic of the interior of R can be 2π at most, we must have $n = 1$. Therefore, R is a planar parallelogram, as desired. □

Given a parallelogram $R \in F(B)$, we define its length $\ell(R)$ to be the length between two adjacent corner vertices along a component of the boundary of R in the direction 1 or -1. Similarly, we define its width $w(R)$ to be the length between two adjacent corner vertices along a component of the boundary of R in the direction $e^{\pi i/3}$ or $-e^{\pi i/3}$.

Lemma 7.5.3. *For all $R \in F(B)$, $\ell(R), w(R) \geq 3$.*

Proof. This follows directly from property 4 of Lemma 7.5.1. □

We also have:

Lemma 7.5.4. *For all $R \in F(B)$, R contains a corner vertex in $V_{>6}(S)$.*

Proof. Let $x_0, ..., x_k \in V(S)$ and $e_1, ..., e_k \in E(S)$ such that e_i is an edge between x_{i-1} and x_i, and $\zeta(e_i, x_{i-1}) = e^{\pi i/3}$. Suppose $x_0, ..., x_k$ and $e_1, ..., e_k$ correspond to a component of the boundary of R so that x_0 and x_k are corner vertices of R. (In particular, $x_0, x_k \in V(B)$, so

$x_0, x_k \in A_0$ as well.) We claim that either $x_0 \in V_{>6}(S)$ or $x_k \in V_{>6}(S)$. To see this, note first that for $j \in \{1, ..., k-1\}$, $x_j \notin V(B)$ by property 3 of Lemma 7.5.1. Since $x_0, ..., x_k$ and $e_1, ..., e_k$ are in A, there are two possibilities: either they are in A_1 or they are in A_2. In the first case we must have had $x_0 \in ((V(S) \cap A_1) \setminus ((V(S) \cap A_0) \cap (V(S) \cap A_1))) \cup V_{>6}(S)$ at some point in the inductive construction of A_1; since $x_0 \in A_0$, this means $x_0 \in V_{>6}(S)$. In the second case, we must have $x_k \in ((V(S) \cap A_2) \setminus ((V(S) \cap A_0) \cap (V(S) \cap A_2))) \cup V_{>6}(S)$ at some point in the inductive construction of A_2; since $x_k \in A_0$, this means $x_k \in V_{>6}(S)$. □

Lemma 7.5.5. *We have,* $|F(B)| \leq 12(g-1)$.

In other words, there are at most $12(g-1)$ parallelograms in the decomposition of S we have constructed.

Proof of Lemma 7.5.5. Since S is a combinatorial translation surface, all vertices of S have degree a multiple of 6, so

$$\begin{aligned} \sum_{x \in V_{>6}(S)} \deg x &\leq -12|V(S)| + 4|E(S)| \\ &= -12\chi(S) \\ &\leq 24(g-1). \end{aligned} \tag{7.17}$$

By Lemma 7.5.4, the lemma follows. □

7.6 Lower genus triangulated surface from parallelogram decomposition

In Section 7.5, we constructed a parallelogram decomposition for combinatorial translation surfaces. The goal of this section is to apply this decomposition to S_X and show that for most of these parallelograms, vertices, edges and triangular faces in the parallelogram are mapped under f to a small neighborhood of vertices, edges and triangular faces in an isomorphic parallelogram in S_Y. As a consequence, we will see that the part of S_X on which f is not close to a simplicial isomorphism is a lower genus surface.

Decompose S_X into parallelograms as in Section 7.5 and let B_X denote the polytope constructed in Section 7.5 applied to the combinatorial translation surface S_X. Enumerate the parallelograms in $F(B_X)$ by $R_1, ..., R_N$. Let E_0 be the set of R_i which contain a vertex in V_X.

Lemma 7.6.1. *We have,* $N - |E_0| \leq 42\alpha_3 g$.

Proof. By Lemma 7.2.1 we have that $|V_{>6}(S_X) \setminus V_X| \leq \alpha_3 g$. Now, if $R_i \notin E_0$, then by Lemma 7.5.4 R_i contains a vertex in $V_{>6}(S_X) \setminus V_X$, which means R_i also contains a triangle with a vertex belonging to $V_{>6}(S_X) \setminus V_X$. Since S_X is locally bounded, the number of such triangles is at most $42\alpha_3 g$. Since the interiors of the R_i are all disjoint, at most $42\alpha_3 g$ number of R_i can contain such a triangle. The lemma follows. □

Let $\delta \in \mathbb{N}$, to be chosen later. Let $E_1 \subset E_0$ be the subset of R_i in E_0 which satisfy the additional property that $\ell(R_i), w(R_i) \leq \delta(T/g)$.

Lemma 7.6.2. *We have,* $|E_0| - |E_1| \leq \delta^{-1} g/2$.

Proof. If either $\ell(R_i)$ or $w(R_i)$ is greater than $\delta(T/g)$, then $\text{area}_{S_X}(R_i) \geq (\sqrt{3}/2)\delta(T/g)$. Since $\text{area}_{S_X}(S_X) = (\sqrt{3}/4)T$, there can be at most $\delta^{-1} g/2$ such R_i. □

Let $0 < \varepsilon_1 < 1/1000$ be sufficiently small, to be chosen later. Let $E_2 \subset E_1$ be the subset of R_i in E_1 which satisfies

$$\int_{B_{S_X}(R_i, 2+\varepsilon_1)} |f^*\phi_{S_Y} - \phi_{S_X}|^2 |\phi_{S_X}|^2 \leq (\varepsilon_1 \delta^{-1}/80)^8 (g/T)^{36}.$$

Lemma 7.6.3. *If $R_i \in E_2$, then R_i satisfies the following two properties:*

1. *If $x \in V(S_X) \cap \overline{B_{S_X}(R_i, 2)}$, then there exists a unique vertex $y \in V(S_Y)$ (henceforth denoted $\mathfrak{f}(x)$) such that $f(x) \in B_{S_Y}(y, \varepsilon_1)$.*

2. *If $x_1, x_2 \in V(S_X) \cap \overline{B_{S_X}(R_i, 2)}$ and $e(x_1, x_2) \in E(S_X) \cap R_i$ is an edge connecting x_1 and x_2, there exists a unique edge $e(\mathfrak{f}(x_1), \mathfrak{f}(x_2)) \in E(S_Y)$ (henceforth denoted $\mathfrak{f}(e(x_1, x_2))$) connecting $\mathfrak{f}(x_1)$ and $\mathfrak{f}(x_2)$ such that $f(e(x_1, x_2)) \subset B_{S_Y}(\mathfrak{f}(e(x_1, x_2)), \varepsilon_1)$.*

Remark 18. The uniqueness condition in property 1 automatically follows from the existence condition in property 1 since any two distinct vertices in S_Y are at least distance 1 apart in the S_Y-metric. The uniqueness condition in property 2 automatically follows from property 1 and the existence condition in property 2 since two distinct vertices in S_Y can have at most one edge between them.

Proof. Suppose $R_i \in E_2$ and does not satisfy one of the properties in the lemma statement. Let $\xi = \varepsilon_1 \delta^{-1}/80$. Note that since $\delta > 1$ and $g/T \leq 1/2$ by construction,

$$(g/T)(8 + 2\delta(T/g))\xi \leq \varepsilon_1/10. \tag{7.18}$$

Define $\xi_k = (g/T)(1+k)\xi$ and $\beta = \xi^8 (g/T)^{36}$. It is easy to check that

$$\tau(\xi_k, \beta) \leq \xi_{k+1}, \tag{7.19}$$

where τ is as defined in Lemma 7.3.1. Now, since $R_i \in E_1$, there exists $x \in V_X \cap \overline{R_i}$. For any $x' \in V(S_X) \cap \overline{B_{S_X}(R_i, 2)}$, denote by $d_G(x', x)$ the graph distance between x and x' with respect to the 1-skeleton of S_X intersected with $\overline{B_{S_X}(R_i, 2)}$. Note that for $x' \in V(S_X) \cap \overline{B_{S_X}(R_i, 2)}$,

$$d_G(x, x') \leq 2\delta(T/g) + 4 \tag{7.20}$$

as $R_i \in E_0$. By supposition, Eq. (7.18) and Eq. (7.20), there exist adjacent $x_1, x_2 \in V(S_X) \cap \overline{B_{S_X}(R_i, 2)}$, with edge $e(x_1, x_2)$ between them in R_i, with a unique vertex $y_1 \in V(S_Y)$ satisfying $f(x_1) \in B_{S_Y}(y_1, \xi_{d_G(x_1,x)})$, but either

1. there does not exist a vertex $y_2 \in V(S_Y)$ such that

   $$f(x_2) \in B_{S_Y}(y_2, \xi_{d_G(x_1,x)+1})$$

   or

2. condition 1 holds but there does not exist an edge $e(y_1, y_2) \in E(S_Y)$ connecting y_1 and y_2 such that $f(e(x_1, x_2)) \subset B_{S_Y}(e(y_1, y_2), 10\xi_{d_G(x_1,x)+1})$.

Then by Lemma 7.3.1

$$\int_{B_{S_X}(e(x_1,x_2),\xi_{d_G(x_1,x)})} |f^*\phi_{S_Y} - \phi_{S_X}|^2|\phi_{S_X}|^2 > \beta. \tag{7.21}$$

(Note that Lemma 7.3.1 requires β to be sufficiently small compared to $\xi_{d_G(x_1,x)}$ which holds in this case because of Eq. (7.18), Eq. (7.19) and Eq. (7.20) combined.) Eq. (7.21) implies that

$$\begin{aligned}\int_{B_{S_X}(R_i,2+\varepsilon_1)} |f^*\phi_{S_Y} - \phi_{S_X}|^2|\phi_{S_X}|^2 > \beta& \\ &= \xi^8(g/T)^{36} \\ &= (\varepsilon_1\delta^{-1}/80)^8(g/T)^{36}.\end{aligned}$$

Therefore $R_i \notin E_2$, which is a contradiction. □

Lemma 7.6.4. *We have,* $|E_1| - |E_2| \leq 10^5(\varepsilon_1\delta^{-1}/80)^{-8}\alpha_2(g/T)^{\kappa_4-36}g$.

Proof. By Eq. (7.1),

$$\int_X |f^*\phi_{S_Y} - \phi_{S_X}|^2|\phi_{S_X}|^2 \leq \alpha_2(g/T)^{\kappa_4}g.$$

By definition, if $R_i \in E_1 \setminus E_2$,

$$\int_{B_{S_X}(R_i,2+\varepsilon_1)} |f^*\phi_{S_Y} - \phi_{S_X}|^2|\phi_{S_X}|^2 \geq (\varepsilon_1\delta^{-1}/80)^8(g/T)^{36}.$$

Since $S_X \in \mathrm{Comb}_{\mathrm{lb}}^T(\mathcal{H}_g)$, degrees of vertices in S_X are bounded above by 42 and distances between distinct vertices of degree $\neq 6$ are bounded below by 3. Therefore, any radius $2+\varepsilon_1$-ball in the flat metric on S_X centered at a vertex of S_X nontrivially intersects at most 10^5 vertices, edges and triangles. Combined with the fact that the R_is are disjoint, this means any point in S_X is contained in at most 10^5 of the $B_{S_X}(R_i, 2+\varepsilon_1)$. The lemma follows. □

Denote by G_X the 2-subcomplex of S_X given by $\cup_{R_i\in E_2} ST(ST(R_i))$.

Lemma 7.6.5. *There exists a 2-subcomplex G_Y of S_Y along with a simplicial map $\mathfrak{f} : G_X \to G_Y$ such that*

1. $\mathfrak{f}$ is an isomorphism,

2. for every vertex $x \in V(G_X)$, $f(x) \in B_{S_Y}(\mathfrak{f}(x), \varepsilon_1)$, and

3. for every edge $e \in E(G_X)$, $f(e) \subset B_{S_Y}(\mathfrak{f}(e), \varepsilon_1)$ and

4. for every triangular face $t \in F(G, X)$, $f(t) \subset B_{S_Y}(\mathfrak{f}(t), \varepsilon_1)$.

Proof. In Lemma 7.6.3 we constructed $\mathfrak{f}$ on vertices and edges of G_X. By Lemma 7.4.1, $\mathfrak{f}$ extends to faces of G_X as well, is a simplicial map, and satisfies conditions 2, 3 and 4 in the statement of the lemma. We show now that $\mathfrak{f}$ is injective onto its image. Suppose the contrary. Then there are distinct triangular faces $t_{X,1}, t_{X,2} \in F(G_X)$ such that $\mathfrak{f}(t_{X,1}) = \mathfrak{f}(t_{X,2}) = t_Y \in F(S_Y)$. Conditions 2 and 3 imply that $f(\partial t_{X,1}), f(\partial t_{X_2}) \subset B_{S_Y}(\partial t_Y, \varepsilon_1)$ while condition 4 implies that $f(t_{X,1}), f(t_{X,2}) \subset B_{S_Y}(t_Y, \varepsilon_1)$. As a result the intersection $f(\text{interior}\,(t_{X,1})) \cap f(\text{interior}\,(t_{X,2}))$ must be nonempty. However, this implies f is not injective, which is a contradiction. Therefore $\mathfrak{f}$ is injective onto its image. Define $G_Y = \mathfrak{f}(G_X)$. The lemma follows. □

Denote by $V_{\text{cor}}(F(B_X) \setminus E_2)$ the collection of corner vertices of R_i over all $R_i \in F(B_X) \setminus E_2$. From $\cup_{R_i \in E_2} R_i$, remove stars of all vertices in $V_{\text{cor}}(F(B_X) \setminus E_2) \cap \partial\,(\cup_{R_i \in E_2} R_i)$ (i.e. boundary vertices that are also in $V_{\text{cor}}(F(B_X) \setminus E_2)$), to obtain a simplicial 2-complex G'_X. (By Lemma 7.5.3, these stars are all disjoint.) Let $J_X = \overline{S_X \setminus G'_X}$. Also, define $G'_Y = \mathfrak{f}(G'_X)$, and $J_Y = \overline{S_Y \setminus G'_Y}$. By construction and Lemma 7.5.3, we have the following lemma.

Lemma 7.6.6. *Each connected component of the simplicial complexes G'_X, G'_Y, J_X and J_Y is a triangulated surface with boundary. Moreover, suppose x is a vertex of G'_X lying on $\partial G'_X$. Then the $ST(ST(x))$ (in S_X) is contained in G_X and $ST(ST(\mathfrak{f}(x)))$ (in S_Y) is contained in G_Y. Finally, $\overline{S_X \setminus (\cup_{R_i \in E_2} R_i)}$ is homotopy equivalent to J_X.*

Proof. First, we show that each connected component of G'_X is a triangulated surface with boundary. To do this it suffices to show that each vertex in $V(B_X)$ lying on $\partial G'_X$ has a neighborhood in G'_X homeomorphic to a half-disk. (Neighborhoods of the other boundary points are automatically homeomorphic to half-disks by construction.) Suppose the contrary. Then $\partial G'_X$ contains a vertex $v \in V(B_X)$ such that at least three edges e_1, e_2, e_3 emanating from v are contained in $\partial G'_X$. By property 4 of Lemma 7.5.1, v is not contained in the closed star of any other vertex in $V(B_X)$, so e_1, e_2 and e_3 are contained in edges in $E(B_X)$. Suppose $v \notin V_{>6}(S_X)$. Then by properties 1 and 2 of Lemma 7.5.1, v is a corner vertex of all parallelograms that contain it, which means that $v \in V_{\text{cor}}(F(B_X) \setminus E_2)$. This is a contradiction. On the other hand, suppose $v \in V_{=6}(S_X)$. Then since the three edges e_1, e_2, e_3 in $\partial G'_X$ emanate from v, by property 1 of Lemma 7.5.1 at least two of the edges point in non-parallel directions. So v is the corner vertex for some parallelogram in $F(B_X) \setminus E_2$, which is again a contradiction. Thus, connected components of G'_X are triangulated surfaces with boundary. Since $J_X = S_X \setminus G'_X$, connected components of J_X are also triangulated surfaces with boundary. Since $G'_Y \simeq G'_X$ and $J_Y = G'_Y \setminus J_Y$, connected components of G'_Y and J_Y are triangulated surfaces with boundary as well.

Next, since $G_X = \cup_{R_i \in E_2} ST(ST(R_i))$ and $G'_X \subset \cup_{R_i \in E_2} R_i$, $ST(ST(G'_X)) \subset G_X$. Since $\mathfrak{f} : G_X \to G_Y$ is an isomorphism, $ST(ST(\mathfrak{f}(G'_X))) \subset G_Y$.

Finally, we show the last statement in the lemma. The subcomplex J_X is obtained by adding the closed star of some boundary vertices of $\overline{S_X \setminus (\cup_{R_i \in E_2} R_i)}$ to it. These additions are all disjoint and deformation retract onto the boundary of $\overline{S_X \setminus (\cup_{R_i \in E_2} R_i)}$, hence J_X deformation retracts onto $\overline{S_X \setminus (\cup_{R_i \in E_2} R_i)}$. □

Now, suppose J_X has n connected components, with the ith boundary component J_X^i homeomorphic to S_{h_i,b_i}. Fix a marking $S_{h_i,b_i} \to J_X^i$. By Lemma 7.6.5, since $\partial J_X^i \subset G_X$, f maps J_X^i into an ε_1-neighborhood (in the S_Y-metric) of a connected component of J_Y that we label J_Y^i. The ε_1-neighborhood of J_Y^i is isotopic to J_Y^i in S_Y, giving a natural homeomorphism from J_X^i to J_Y^i which endows J_Y^i with a marking as well. Since f is a homeomorphism from X to Y which takes G'_X into an ε_1-neighborhood of G'_Y (in the S_Y-metric) which is isotopic to G'_Y in S_Y, J_X and J_Y are homeomorphic as well so all connected components of J_Y arise in the manner described.

Lemma 7.6.7. *The Euler characteristic of J_X satisfies $\chi(J_X) \geq -\alpha_4 g$ where*

$$\alpha_4 = 96(42\alpha_3 + \delta^{-1}/2 + 10^5(\varepsilon_1\delta^{-1}/80)^{-8}\alpha_2(g/T)^{\kappa_4 - 36}).$$

Similarly, $\chi(J_Y) \geq -\alpha_4 g$. Moreover, J_X and J_Y each have at most $\alpha_4 g$ connected compo-

nents.

Proof. By Lemma 7.6.1, Lemma 7.6.2 and Lemma 7.6.4,

$$\begin{aligned} |F(B_X) \setminus E_2| &= N - |E_2| \\ &\leq (42\alpha_3 + \delta^{-1}/2 + 10^5(\varepsilon_1\delta^{-1}/80)^{-8}\alpha_2(g/T)^{\kappa_4 - 36})g. \end{aligned} \tag{7.22}$$

Now, $\overline{S_X \setminus (\cup_{R_i \in E_2} R_i)}$ is naturally a subpolytope of B_X. We denote this subpolytope by B'_X. The vertices of B'_X (denoted $V(B'_X)$) are the vertices of B_X in $\overline{S_X \setminus (\cup_{R_i \in E_2} R_i)}$, the edges of B'_X (denoted $E(B'_X)$) are the edges of B_X in $\overline{S_X \setminus (\cup_{R_i \in E_2} R_i)}$, and the faces of B'_X are the $R_i \in F(B_X) \setminus E_2$.

Given a vertex $v \in V(B'_X)$, let the degree of v in B'_X denote the number of edges of B'_X emanating from it (with loops contributing 2 to the degree). Denote by $V_{=2}(B'_X)$ the set of vertices of B'_X that have degree 2 in B'_X, and $V_{>2}(B'_X)$ the set of vertices of B'_X that have degree strictly greater than 2 in B'_X.

We claim that $V_{>2}(B'_X) \subset V_{\text{cor}}(F(B_X) \setminus E_2)$. To see this, note that if $v \in V_{>2}(B'_X)$ and $v \in V_{>6}(S_X)$, then by properties 1 and 2 of Lemma 7.5.1, v is a corner vertex of one of the R_i in $F(B_X) \setminus E_2$. If $v \in V_{>2}(B'_X)$ and $v \in V_{=6}(S_X)$, then v has unique edges e_1, e_2, e_3 and e_4 in $E(S_X)$ emanating from v satisfying $\zeta(e_1, v) = 1$, $\zeta(e_2, v) = -1$, $\zeta(e_3, v) = e^{\pi i/3}$ and $\zeta(e_4, v) = -e^{\pi i/3}$. By property 3 of Lemma 7.5.1, e_1 and e_2 are contained in edges of $E(B'_X)$. Since $v \in V_{>2}(B'_X)$, either e_3 or e_4 is contained in an edge of $E(B'_X)$; in either case, v is a corner vertex of some R_i in $F(B_X) \setminus E_2$.

Since each R_i has at most 4 corner vertices, there are at most

$$4(42\alpha_3 + \delta^{-1}/2 + 10^5(\varepsilon_1\delta^{-1}/80)^{-8}\alpha_2(g/T)^{\kappa_4 - 36})g$$

vertices in $V_{>2}(B'_X)$. The total number of pairs (e, v) where $v \in V_{=2}(B'_X)$ and $e \in E(B'_X)$ is an edge emanating from v is at most $2|V_{=2}(B'_X)|$. The total number of pairs (e, v) where $v \in V_{>2}(B'_X)$ and $e \in E(B'_X)$ is an edge emanating from v is at most $42|V_{>2}(B'_X)|$ because all vertices of S_X have degree at most 42 in S_X. So the total number of edges in B'_X is at most $|V_{=2}(B'_X)| + 24|V_{>2}(B'_X)|$. Thus,

$$\begin{aligned}\chi(B'_X) &\geq |V(B'_X)| - |E(B'_X)| \\ &\geq -24|V_{>2}(B'_X)| \\ &\geq -96(42\alpha_3 + \delta^{-1}/2 + 10^5(\varepsilon_1\delta^{-1}/80)^{-8}\alpha_2(g/T)^{\kappa_4-36})g \\ &= -\alpha_4 g.\end{aligned}$$

By Lemma 7.6.6, B'_X and J_X are homotopy equivalent, so $\chi(J_X) \geq -\alpha_4 g$. Since G'_X is a triangulated surface with boundary, the Euler characteristic of $\chi(G'_X)$ is at most $\chi(S_X)+\alpha_4 g$. Since G'_X and G'_Y are homeomorphic as are S_X and S_Y, $\chi(G'_Y) = \chi(G'_X)$ and $\chi(J_Y) = \chi(J_X)$. The number of connected components of J_X is at most $|F(B_X) \setminus E_2|$ since each face in $F(B_X) \setminus E_2$ belongs to at most one connected component. Hence this number, along with the number of connected components of J_Y, is at most $\alpha_4 g$. □

Since the J_X^i and J_Y^i are marked triangulated surfaces of genus h_i with b_i boundary components, by Proposition 2.7.1, their conformal doubles each admit canonical triangulations. Denote these canonical triangulations by K_X^i and K_Y^i, respectively. The markings on J_X^i and J_Y^i give markings on their doubles as well, so K_X^i and K_Y^i are marked triangulated surfaces (without boundary) of genus $g_i = 2h_i + b_i - 1$.

Lemma 7.6.8. *We have,*

$$\sum_{i=1}^{n} g_i \leq \alpha_5 g$$

where g_i denotes the genus of K_X^i which is equal to the genus of K_Y^i. Here, $\alpha_5 = 2\alpha_4$.

Proof. By Lemma 7.6.7, $\chi(K_X) = 2\chi(J_X) \geq -2\alpha_4 g$. Now, $\chi(K_X^i) = 2 - 2g_i$, so

$$\begin{aligned}2n - 2\sum_{i=1}^{n} g_i &= \chi(K_X) \\ &\geq -2\alpha_4 g.\end{aligned}$$

Since $n \leq \alpha_4 g$ by Lemma 7.6.7, we have

$$\sum_{i=1}^{n} g_i \leq 2\alpha_4 g$$

as desired. $\square$

Next, we bound the number of vertices of K_X and K_Y that have degree not equal to 6.

Lemma 7.6.9. *We have, $|V_{\neq 6}(K_X)| \leq \alpha_6 g$ and $|V_{\neq 6}(K_Y)| \leq \alpha_6 g$ where $\alpha_6 = 500\alpha_4$.*

Proof. First, we compute $|V_{<6}(K_X)|$. Since K_X is the double of J_X and J_X is a subset of S_X (which, being in $\mathrm{Comb}^T(\mathcal{H}_g)$, does not have any vertices of degree strictly less than 6), all vertices of K_X of degree less than 6 must lie on $\partial J_X \subset K_X$. Such a vertex will have degree less than 4 considered as a vertex of J_X, so it suffices to compute the number ofvertices in ∂J_X that have degree strictly less than 4 in J_X. Denote by $V_{<4}(\partial J_X)$ the set of vertices in ∂J_X that have degree strictly less than 4 in J_X. (Recall that degree for a vertex in a triangulated surface with boundary is defined as the number of edges emanating from the vertex, which for a vertex on the boundary, is different from the number of faces containing the vertex.)

Since $J_X = \overline{S_X \setminus G'_X}$ and G'_X was created by removing stars of boundary vertices of $\cup_{R_i \in E_2} R_i$ that are also in $V_{\mathrm{cor}}(F(B_X) \setminus E_2)$, any vertex in $V_{<4}(\partial J_X)$ is either in $V_{\mathrm{cor}}(F(B_X) \setminus E_2)$ or adjacent to a vertex in $V_{\mathrm{cor}}(F(B_X) \setminus E_2)$. Since by Eq. (7.22),

$$\begin{aligned} |V_{\mathrm{cor}}(F(B_X) \setminus E_2)| &\leq 4|F(B_X) \setminus E_2| \\ &\leq \alpha_4 g \end{aligned}$$

and each vertex in J_X has degree at most 42, $|V_{<4}(\partial J_X)| \leq 43\alpha_4 g$. Therefore

$$|V_{<6}(K_X)| \leq 43\alpha_4 g \tag{7.23}$$

as well. Note that

$$\chi(K_X) = |V(K_X)| - |E(K_X)| + |F(K_X)|$$

$$= |V(K_X)| - (1/3)|E(K_X)|.$$

Since there are at least $3|V_{=6}(K_X)| + (7/2)|V_{>6}(K_X)|$ edges in K_X, $\chi(K_X) \leq |V_{<6}(K_X)| - (1/6)|V_{>6}(K_X)|$ which means $|V_{>6}(K_X)| \leq -6\chi(K_X) + 6|V_{<6}(K_X)|$. From Lemma 7.6.8 and Eq. (7.23), we obtain

$$\begin{aligned} |V_{\neq 6}(K_X)| &\leq |V_{<6}(K_X)| + |V_{>6}(K_X)| \\ &\leq 500\alpha_4 g. \end{aligned}$$

Since ∂J_X is identified with ∂J_Y, a similar argument shows that $|V_{<6}(K_Y)| \leq 43\alpha_4 g$. Since $\chi(K_X) = \chi(K_Y)$, we obtain $|V_{\neq 6}(K_Y)| \leq 500\alpha_4 g$ too. □

7.7 Quasiconformal extension on a triangulated surface

The goal of this section is to show:

Lemma 7.7.1. *There exists a universal constant C such that $d_T(K_X^i, K_Y^i) \leq C$ for all i satisfying $g_i \geq 2$.*

We will do this by using the quasiconformal extension results from Section 3.4.

To show Lemma 7.7.1, it suffices to show the existence of a C-quasiconformal map of marked surfaces between J_X and J_Y that takes ∂J_X to ∂J_Y, as gluing would then give a C-quasiconformal map K_X to K_Y as desired. Restricting this map to each connected component would then give the lemma statement.

Observe that $\partial J_X \subset G_X$. So by Lemma 7.6.5, $f(\partial J_X) \subset B_{S_Y}(\mathfrak{f}(\partial J_X), \varepsilon_1) = B_{S_Y}(\partial J_Y, \varepsilon_1)$. Recall that $\varepsilon_1 < 1/1000$.

We construct a C-quasiconformal map H from $f(J_X)$ to J_Y. Since f is C-quasiconformal, this implies the existence of the desired C-quasiconformal map from J_X to J_Y. On

$$f(J_X) \setminus (f(J_X) \cap B_{S_Y}(\partial J_Y, 1/4)),$$

which by Lemma 7.6.5 may be identified with

$$J_Y \setminus (J_Y \cap B_{S_Y}(\partial J_Y, 1/4)),$$

we let H be the identity. It remains to define H from

$$f(J_X) \cap B_{S_Y}(\partial J_Y, 1/4)$$

to

$$J_Y \cap B_{S_Y}(\partial J_Y, 1/4).$$

We consider the neighborhood around each boundary component $\partial^k J_Y$ of J_Y separately. Let $y_1, ..., y_n$ be the vertices of $\partial^k J_Y$ in cyclic order, and let $e(y_i, y_{i+1})$ be the edge between vertices y_i and y_{i+1} (with $e(y_n, y_0)$ the edge between y_n and y_0). Let m_i be the midpoint of $e(y_i, y_{i+1})$. The surface $J_Y \cap B_{S_Y}(\partial^k J_Y, 1/4)$ has two boundary components. One is $\partial^k J_Y$, and the other we denote $\partial^k_{1/4} J_Y$. Denote by p_i the perpendicular bisector of $e(y_i, y_{i+1})$, extended until it hits $\partial^k_{1/4} J_Y$ at a point m_i'. Doing this for all the $e(y_i, y_{i+1})$ on $\partial^k J_Y$, we divide

$$J_Y \cap B_{S_Y}(\partial^k J_Y, 1/4)$$

into quadrilaterals A_i each containing vertex y_i. Note that A_i is contained in the closed star of y_i that we call $ST(y_i)$. By Lemma 7.6.6, $ST(y_i) \subset G_Y$.

Since ∂J_Y is identified with ∂J_X, we label boundary components of ∂J_X such that $\partial^k J_X$ is identified with $\partial^k J_Y$. We also label the boundary components of $\partial f(J_X)$ so that $\partial^k f(J_X) = f(\partial^k J_X)$. Now,

$$f(J_X) \cap B_{S_Y}(\partial^k J_Y, 1/4),$$

has two boundary components. One of them is $\partial^k f(J_X)$, the other we label $\partial^k_{1/4} f(J_X)$. The point m_i' exists on $\partial^k_{1/4} f(J_X)$ since it is identified with $\partial^k_{1/4} J_Y$. We let q_i be the shortest straight line from m_i' to $\partial^k f(J_X)$. The q_i divide $f(J_X) \cap B_{S_Y}(\partial^k J_Y, 1/4)$ into quadrilaterals B_i each a subset of $ST(y_i)$. Since $\partial^k_{1/4} f(J_X) \subset B_{S_Y}(\partial^k J_Y, \varepsilon_1)$, $B_i \subset B_{S_Y}(y_i, 9/10)$.

The region $B_{S_Y}(ST(ST(y_i)), \varepsilon_1)$ is a Riemann surface with boundary homeomorphic to

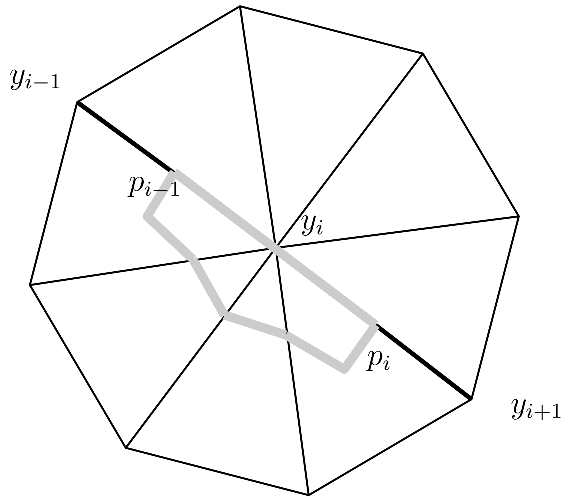

Figure 7-2: $ST(y_i)$ for a degree 8 vertex $y_i \in V(S_Y)$ with ∂J_Y in bold and ∂A_i in gray. Triangles are equilateral triangles, which is not depicted in the figure.

the closed unit disk, equipped with the S_Y-metric. It admits a map to the closed unit disk that is quasiconformal and bi-Lipschitz with respect to the S_Y metric on $B_{S_Y}(ST(ST(y_i)), \varepsilon_1)$ and the Euclidean metric on the closed unit disk. Since degrees of vertices of S_Y are at most 42, there are a bounded number of choices for $B_{S_Y}(ST(ST(y_i)), \varepsilon_1)$.

Lemma 7.7.2. *The curves ∂A_i and ∂B_i are C-bounded turning in $B_{S_Y}(ST(ST(y_i)), \varepsilon_1)$ for a universal constant C.*

Proof. It is evident that ∂A_i is C-bounded-turning, as ∂A_i consists of a bounded number of straight lines with lengths bounded above and below by constants and angles between them bounded below by a constant. We now show that ∂B_i is C-bounded-turning. To do this, it suffices to show that for $x, y, z \in \partial B_i$,

$$d_{S_Y}(x, y) + d_{S_Y}(y, z) \leq C d_{S_Y}(x, z). \tag{7.24}$$

We may divide ∂B_i into four components. Two components are q_{i-1} and q_i. Another component is contained in $\partial^k f(J_X)$; we label this component $\partial_0 B_i$. The final component is contained in $\partial^k_{1/4} J_Y$; we label this component $\partial_{1/4} B_i$. Note that q_{i-1} and q_i are straight lines, while $\partial_{1/4} B_i$ consist of a bounded number of straight lines lengths bounded below with angles between them bounded below by a constant. So if $x, y, z \in q_i \cup q_{i-1} \cup \partial_{1/4} B_i$, Eq. (7.24) is satisfied.

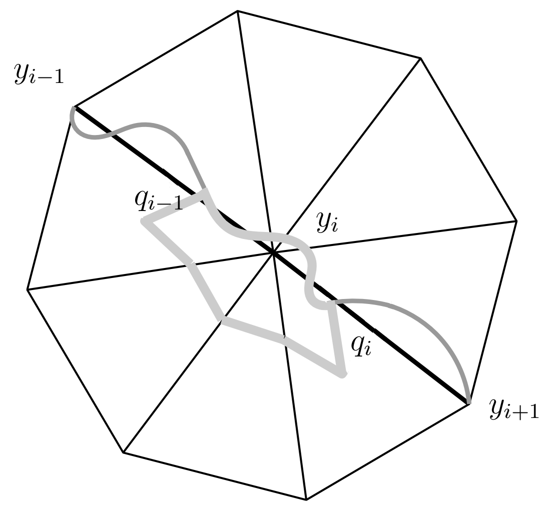

Figure 7-3: $ST(y_i)$ for a degree 8 vertex $y_i \in V(S_Y)$ with ∂J_Y in bold, $\partial f(J_X)$ in dark gray and and ∂B_i in gray. Triangles are equilateral triangles, which is not depicted in the figure.

Let us consider the case where $x, y, z \in \partial_0 B_i$. Let $x_i = \mathfrak{f}^{-1}(y_i)$, and $e(x_i, x_{i+1}) = \mathfrak{f}^{-1}(e(y_i, y_{i+1}))$. The region $ST(ST(x_i))$ is a Riemann surface with boundary, equipped with the S_X-metric. It admits a homeomorphism to the closed unit disk that is quasiconformal and bi-Lipschitz with respect to the S_X metric on $ST(ST(x_i))$ and the Euclidean metric on the closed unit disk. Since degrees of vertices of S_X are at most 42, there are a bounded number of choices for $ST(ST(x_i))$. By Lemma 7.6.5 and Lemma 7.6.6, $f(ST(ST(x_i))) \subset B_{S_Y}(ST(ST(y_i)), \varepsilon_1)$. The curve $e(x_{i-1}, x_i) \cup e(x_i, x_{i+1}) \subset ST(ST(x_i))$ is bounded turning in $ST(ST(x_i))$. By Proposition 3.5.2, $f(e(x_{i-1}, x_i) \cup e(x_i, x_{i+1}))$ is C-bounded-turning in $B_{S_Y}(ST(ST(y_i)), \varepsilon_1)$. Because $\partial_0 B_i \subset f(\partial^k J_X) \cap B_{S_Y}(y_i, 9/10)$, by Lemma 7.6.5, we have $\partial_0 B_i \subset f(e(x_{i-1}, x_i) \cup e(x_i, x_{i+1}))$. So $\partial_0 B_i$ is also C-bounded-turning in $B_{S_Y}(ST(ST(y_i)), \varepsilon_1)$. This means that for $x, y, z \in \partial_0 B_i$, Eq. (7.24) is satisfied.

We must now show Eq. (7.24) in the cases where some of the x, y, z lie on $\partial_0 B_i$ and the others lie on other boundary components. Since the diameter of B_i is bounded, Eq. (7.24) is only nontrivial to show when $d_{S_Y}(x, z)$ is small. So there are only two remaining nontrivial cases. One is when $x, y \in \partial_0 B_i$ and $z \in q_i$ (and the analogous $x, y \in \partial_0 B_i$ and $z \in q_{i-1}$). The other is when $x \in \partial_0 B_i$ and $y, z \in q_i$ (and the analogous $x \in \partial_0 B_i$ and $y, z \in q_{i-1}$).

To show these cases, first consider the case wherein $x \in \partial_0 B_i$, $z \in q_i$, and y is the meeting point between q_i and $\partial_0 B_i$. Since q_i was constructed to be the shortest path from m_i' to $\partial_0 B_i$,

$$d_{S_Y}(y,z) \le d_{S_Y}(x,z).$$

By the triangle inequality,

$$\begin{aligned} d_{S_Y}(x,y) &\le d_{S_Y}(x,z) + d_{S_Y}(y,z) \\ &\le 2d_{S_Y}(x,z). \end{aligned}$$

Summing, we obtain Eq. (7.24) in this case.

Now, suppose that $x, y \in \partial_0 B_i$ and $z \in q_i$. Let w be the meeting point between q_i and $\partial_0 B_i$. We already know that the triples x, w, z and x, y, w satisfy satisfy Eq. (7.24). This means $d_{S_Y}(x,w) \le Cd_{S_Y}(x,z)$ and $d_{S_Y}(x,y) \le Cd_{S_Y}(x,w)$. By the triangle inequality we have

$$\begin{aligned} d_{S_Y}(x,y) + d_{S_Y}(y,z) &\le d_{S_Y}(x,y) + d_{S_Y}(y,x) + d_{S_Y}(x,z) \\ &\le Cd_{S_Y}(x,z). \end{aligned}$$

Finally, suppose that $x \in \partial_0 B_i$ and $y, z \in q_i$. Let w be the meeting point between q_i and $\partial_0 B_i$. We already know that the triples x, w, z and w, y, z satisfy satisfy Eq. (7.24). This means $d_{S_Y}(w,z) \le Cd_{S_Y}(x,z)$ and $d_{S_Y}(y,z) \le Cd_{S_Y}(w,z)$. By the triangle inequality we have

$$\begin{aligned} d_{S_Y}(x,y) + d_{S_Y}(y,z) &\le d_{S_Y}(x,z) + d_{S_Y}(z,y) + d_{S_Y}(y,z) \\ &\le Cd_{S_Y}(x,z). \end{aligned}$$

We have shown all nontrivial cases of Eq. (7.24). □

Now, ∂A_i may be divided into four components; two are p_{i-1} and p_i, another component (which we label $\partial_0 A_i$) is contained in $\partial^k J_Y$, and the final component (which we label $\partial_{1/4} A_i$)

is contained in $\partial^k_{1/4} J_Y$. Define

$$H : p_i \cup p_{i+1} \cup \partial_{1/4} A_i \to q_i \cup q_{i+1} \cup \partial_{1/4} B_i$$

so that H that takes p_{i-1} to q_{i-1}, p_i to q_i, and $\partial_{1/4} A_i$ to $\partial_{1/4} B_i$ and is scaling on each component. (Note that H was already defined on $\partial_{1/4} A_i$ and agrees with the new definition on this component.) Since both $p_{i-1} \cup p_i \cup \partial_{1/4} A_i$ and $q_{i-1} \cup q_i \cup \partial_{1/4} B_i$ are unions of a bounded number of straight lines each with lengths bounded below and above by constants, with angles between them bounded below by a constant, H is C-weakly-quasisymmetric on $p_i \cup p_{i+1} \cup \partial_{1/4} A_i \subset B_{S_Y}(ST(ST(y_i)), \varepsilon_1)$.

Corollary 3.5.5 (which applies because of Lemma 7.7.2) implies that H extends to a C-quasiconformal map from A_i to B_i. Gluing these maps together over all the i and repeating this procedure for all connected components of ∂J_Y gives the desired extension of H. This completes the proof of Lemma 7.7.1.

7.8 Proof of Lemma 7.1.1

We must compute the number of possible S_Y. To do this, it suffices to compute the number of possible G'_Y, the number of possible J_Y and the number of possible gluings of J_Y with G'_Y. Since G'_Y admits a simplicial isomorphism to G'_X, which itself is a subcomplex of S_X that is uniquely determined by the subset $E_2 \subset F(B_X)$, the number of possibilities of G'_Y is the number of possibilities of G'_X which is bounded above by the number of possibilities for E_2. Since $|F(B_X)| \leq Cg$ by Lemma 7.5.5, this number of possibilities for E_2 is bounded above by C^g.

Next, we compute the number of possibilites for K_Y. We first treat the case of the connected components which have genus 0 or 1. Relabelling as necessary, we may assume that $g_1, ..., g_{n'}$ are 0 or 1 and $g_{n'+1}, ..., g_n$ at least 2. First, we show:

Lemma 7.8.1. *There are at most $(T/g)^{Cg}$ total choices for the K^1_Y,..., $K^{n'}_Y$.*

Proof. Let $J^{1,...,n'}_Y = \cup_{i=1}^{n'} J^i_Y$ and $K^{1,...,n'}_Y = \cup_{i=1}^{n'} K^i_Y$. We decompose $J^{1,...,n'}_Y$ into a combination of cylinders and planar domains as follows. To do this, we first construct a 1-complex A^J_Y

on $J_Y^{1,\ldots,n'}$ inductively. For each vertex $x \in J_Y^{1,\ldots,n'}$ which satisfies $x \in V_{\neq 6}(K_Y)$, we add edge trajectories of direction 1 and -1 starting from x to A_Y^J. That is, we first add $V_{\neq 6}(K_Y)$ to A_Y^J, then at every induction step, for every vertex $x \in V_{\neq 6}(K_Y) \cap J_Y^{1,\ldots,n'}$ and adjacent vertex $y \in J_Y^{1,\ldots,n'}$ such that $e(x,y) \in J_Y^{1,\ldots,n'}$ and $\zeta(e(x,y),x) = \pm 1$, we add $e(x,y)$ and y to A_Y^J. In this way we obtain a 1-complex A_Y^J on $J_Y^{1,\ldots,n'}$. Taking the conformal mirror, we have a 1-complex $(A_Y^J)^{-1}$ on $(J_Y^{1,\ldots,n'})^{-1}$. The union $A_Y^J \cup \partial J_Y \cup (A_Y^J)^{-1}$ is a 1-complex A_Y on $K_Y^{1,\ldots,n'}$, which is naturally the 1-skeleton of a 2-polytope B_Y on $K_Y^{1,\ldots,n'}$. On $K_Y^{1,\ldots,n'}$, directional weights of edges are defined up to sign. We define an edge of A_Y (or B_Y) to be a maximal trajectory of edges of S_Y in A_Y in the same direction up to sign.

We claim that each face of R of B_Y is a flat cylinder or embeds isometrically in the plane. To see this, first note that by construction of A_Y and B_Y, the angles of ∂R are at most π, and the interior of R is flat. Hence, by Gauss-Bonnet, $\chi(R) \geq 0$ which means $\chi(R) = 0$ or $\chi(R) = 1$. In the case $\chi(R) = 0$, ∂R must not contain any angles less than π, hence R is a flat cylinder. In the case $\chi(R) = 1$, R is a flat topological disk with piecewise geodesic boundary such that ∂R does not contain any angles less than π, so R is geodesically convex. Therefore the exponential map gives an isometry between a planar domain and R.

By Lemma 7.6.9, the total number of edges of A_Y is at most Cg. The total number of edges of S_Y contained in A_Y is at most CT. Since A_Y is a graph on $K_Y^{1,\ldots,n'}$ which is a disjoint union of genus 0 or 1 surfaces, the number of possibilities for A_Y as a graph embedded on $K_Y^{1,\ldots,n'}$ is at most C^g. (See [2], in which a bound for the number of rooted maps with prescribed number of edges on a surface is given. Any graph is contained in a rooted map.) The number of assignments of length and directional weights is at most $(T/g)^{Cg}$. Since each face of B_Y is a flat cylinder or embeds isometrically in the plane, the lengths and directional weights on A_Y determine the triangulation $K_Y^{1,\ldots,n'}$. The lemma follows. □

By Lemma 7.6.8, Lemma 7.7.1, Lemma 7.6.9 and Lemma 7.8.1 there are at most

$$(T/g)^{Cg} \sum_{\substack{n \leq \alpha_4 g \\ T_1+\ldots+T_n \leq T \\ g_1+\ldots+g_n \leq \alpha_5 g \\ g_1,\ldots,g_n \geq 2 \\ m_1+\ldots+m_n \leq \alpha_6 g}} \prod_{i=1}^{n} N^{\mathcal{T}}(2T_i, g_i, m_i, r+C)$$

choices of K_Y.

We must compute how many choices for J_Y there are given K_Y. Since K_Y is the conformal double of J_Y, it suffices to compute the number of choices for $\partial J_Y \subset K_Y$. Denote by $V_{\neq 4}(\partial J_Y)$ the set of vertices in ∂J_Y of degree not equal to 4 in J_Y. Since $V_{\neq 4}(\partial J_Y) \subset V_{\neq 6}(\partial K_Y)$, by Lemma 7.6.9, there are at most $C^{\alpha_6 g}$ possibilities for $V_{\neq 4}(\partial J_Y)$. Since J_Y is a triangulated surface with boundary, each vertex of ∂J_Y is contained in exactly two edges of ∂J_Y. Given $V_{\neq 4}(\partial J_Y)$ there are at most C possibilities for the two edges of ∂J_Y emanating out of each vertex, because degrees of vertices of K_Y are bounded above by 84 (as degrees of vertices of J_Y are bounded by 42 since $J_Y \subset S_Y$ which is in $\mathrm{Comb}^T_{\mathrm{lb}}(\mathcal{H}_g)$).

Locally, ∂J_Y near a vertex in $V_{\neq 4}(\partial J_Y)$ is a straight line in the S_Y-metric. So, the set $|V_{\neq 6}(\partial J_Y)|$, along with the edges of ∂J_Y containing these vertices, determine the entire ∂J_Y as a subset of $V(K_Y) \cup E(K_Y)$. Therefore given K_Y there are a total of C^g possibilities for ∂J_Y, and thus also for J_Y.

So there are a total of

$$(T/g)^{Cg} \sum_{\substack{n \leq \alpha_4 g \\ T_1+\ldots+T_n \leq T \\ g_1+\ldots+g_n \leq \alpha_5 g \\ g_1,\ldots,g_n \geq 2 \\ m_1+\ldots+m_n \leq \alpha_6 g}} \prod_{i=1}^{n} N^{\mathcal{T}}(2T_i, g_i, m_i, r+C)$$

choices for J_Y at most. Finally, since the boundary components of G'_X are identified with boundary components of G'_X and the markings on J_Y and G'_Y come from the markings on J_X and G'_X, boundary components of J_Y are also identified with boundary components of G'_Y. To complete the gluings, the number of ways to glue each boundary component of J_Y to the corresponding boundary component of G'_Y is the number of edges of the boundary component. Assume there are b boundary components of J_Y, with the number of edges $\ell_1, \ldots, \ell_b$. Note that $b \leq \alpha_4 g$ since $\chi(J_Y) \geq -\alpha_4 g$ by Lemma 7.6.7. Since the total number of edges of ∂J_Y is CT, $\sum \ell_i \leq CT$. The total number of ways to complete these gluings is

$$\begin{aligned} \ell_1 \cdots \ell_b &\leq (T/b)^{Cb} \\ &\leq (T/g)^{Cg} \end{aligned}$$

by the arithmetic mean-geometric mean inequality. Finally, as long as $\alpha_4 \leq 1/100$ and $\alpha_5, \alpha_6 \leq \mu^{-1}(1/100)$ (see Section 7.9), we have that the number of choices for S_Y is at most

$$(T/g)^{Cg} \sum_{\substack{n \leq (1/100)g \\ T_1+\ldots+T_n \leq T \\ g_1+\ldots+g_n \leq \mu^{-1}(1/100)g \\ g_1,\ldots,g_n \geq 2 \\ m_1+\ldots+m_n \leq \mu^{-1}(1/100)g}} \prod_{i=1}^{n} N^{\mathcal{T}}(2T_i, g_i, m_i, r+C).$$

This completes the proof of Lemma 7.1.1.

7.9 Constants

In this section, we show that the constants α_1, δ, ε_0, ε_1, κ_0, κ_1, κ_2 and κ_3 can be appropriately chosen. We list the relationships between these constants below:

1. $\mu \in \mathbb{N}$ is a universal constant
2. $\kappa_0 \in \mathbb{N}$, to be chosen
3. $\alpha_0 > 0$ is a universal constant
4. $\kappa_1 \in \mathbb{N}$, to be chosen
5. $\kappa_2 \in \mathbb{N}$, to be chosen
6. $\kappa_3 \in \mathbb{N}$, to be chosen
7. $\alpha_1 > 0$, to be chosen
8. $\alpha_2 = \alpha_1^2$
9. $\kappa_4 = 2\kappa_1$
10. $\varepsilon_0 > 0$, to be chosen
11. $\alpha_3 = 10^{10}\alpha_2(g/T)^{\kappa_4}\varepsilon_0^{-10}$
12. ε_1 sufficiently small satisfying $\varepsilon_0 < \varepsilon_1 < 1/1000$, to be chosen

13. $\delta \in \mathbb{N}$, to be chosen

14. $\alpha_4 = 96(42\alpha_3 + \delta^{-1}/2 + 10^5(\varepsilon_1\delta^{-1}/80)^{-8}\alpha_2(g/T)^{\kappa_4-36})$

15. $\alpha_5 = 2\alpha_4$

16. $\alpha_6 = 500\alpha_4$

17. $\alpha_4 \leq 1/100$

18. $\alpha_5 \leq (1/100)\mu^{-1}$

19. $\alpha_6 \leq (1/100)\mu^{-1}$

20. $100\alpha_1^{-1}(1/2)^{(\kappa_2-1)/2-\kappa_1} \leq 1$

21. $\kappa_3 \geq (\kappa_2 - 1)/2$

22. $\kappa_2 \geq \kappa_0$

23. $\alpha_0(1/2)^{\kappa_0} \leq (1/10)^{10}$

Noting that $g/T \leq 1/2$, it is clear that κ_0, κ_1, κ_2, κ_3, α_1, ε_0, ε_1 and δ can be chosen to satisfy conditions 17 through 23, in the following way. First, choose ε_0 and ε_1 sufficiently small. Then, choose δ sufficiently large. Finally, choose α_1 sufficiently small and κ_0 and κ_1 sufficiently large so that $\kappa_4 > 36$, α_3, α_4, α_5 and α_6 are sufficiently small and condition 23 is satisfied. Finally, choose κ_2 and κ_3 sufficiently large so that conditions 20 through 22 are satisfied.

Chapter 8

Upper bounds for triangulated surfaces via combinatorial translation surfaces

The first goal of this chapter is to show:

Lemma 8.0.1. *For $g \geq 2$, we have,*

$$N_{\text{lb}}^{\mathcal{T}}(T, g, m, r) \leq C^{(1+r)g}(T/m)^{Cm} N_{\text{lb}}^{\mathcal{H}}(6T_i, 6g + 5m, r + C)$$

for a universal constant C.

Note that if $m = 0$, then $g = 1$. See also Remark 10.

To show Lemma 8.0.1, in Section 8.1 and Section 8.2 we associate to each triangulated surface a branched 6-cover which is a combinatorial translation surface. In Section 8.3 we enumerate the number of combinatorial possibilities for the branched 6-cover given the branch points. In Section 8.4, Section 8.5 and Section 8.6, we study where the branch points lie. In Section 8.7, we prove Lemma 8.0.1 by showing that the connected components of the branched 6-cover of a marked triangulated surface lie in the union of a relatively small number of balls of radius around a constant in a higher dimensional Teichmüller space.

The second goal of this chapter is to complete the proof of Theorem 1.1.2, which follows from Lemma 4.7.1, Corollary 6.2.1, Lemma 7.0.1 and Lemma 8.0.1. We do this in Section 8.8.

8.1 Triangulated surfaces and 6-differentials

Given a Riemann surface X, a meromorphic 6-differential is a global section of the sixth tensor power of the sheaf of meromorphic differentials on X. Locally, in a neighborhood in X with holomorphic coordinate z, a 6-differential behaves like $f(z)dz^6$ where f is a holomorphic function. Although generally triangulated surfaces may not admit a combinatorial translation structure (and therefore do not admit a canonical holomorphic 1-form), they do admit a canonical meromorphic 6-differential. Given a triangulated surface S, we may associate a meromorphic 6-differential ψ_S as follows: we identify each triangle of S with the unit equilateral triangle in $\mathbb{C}$ with vertices at 0, 1 and $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and define $\psi_S = dz^6$ under this identification. Gluings of triangles must preserve the form dz^6, and ψ_S extends holomorphically over vertices of S, therefore ψ_S is a globally defined 6-differential on S. It has a zero/pole of order $(\deg x) - 6$ at a point $x \in V_{\neq 6}(S)$. The flat metric on S (equivalently denoted S-metric) coming from the Euclidean metric on each individual unit equilateral triangle has length element given by $ds_S = |\psi_S|^{1/6}$, and area element $|\psi_S|^{1/3}$. Distances in this metric shall be denoted by $d_S(\cdot,\cdot)$.

8.2 Canonical cover

A triangulated surface S has an associated canonical holomorphic differential on an appropriate 6-degree branched cover that we construct as follows. We cover $S \setminus V_{\neq 6}(S)$ (which is also $S \setminus \{\text{zeros and poles of } \psi_S\}$) by open balls $\{U_i\}$ with transition functions $f_{i,i'}$ on $U_i \cap U_{i'}$ whenever the latter is nonempty. To each U_i, we associate $U_{i,1}, ..., U_{i,6}$, each consisting of the pair $(U_i, \omega_{i,j})$ where $\omega_{i,j} = \zeta^j \psi_S^{1/6}$ for some branch of $\psi_S^{1/6}$ on U_i. We glue $U_{i,j}$ and $U_{i',j'}$ if $U_i \cap U_{i'} \neq \emptyset$ and $f_{i,i'}^* \omega_{i,j} = \omega_{i',j'}$ on $U_i \cap U_{i'}$. Compactifying, we obtain a (possibly disconnected) degree 6 branched cover $F : \widetilde{S} \to S$ such that $F^*(\psi_S) = \widetilde{\phi}_S^6$ for some holomorphic 1-form $\widetilde{\phi}_S$ on $\widetilde{S}$.

Proposition 8.2.1. *Let S be a locally bounded $\leq T$-triangle triangulated surface of genus $\leq g$ with $|V_{\neq 6}(S)| \leq m$. Then $\widetilde{S}$ has at most six connected components that we label $\widetilde{S}^1, ..., \widetilde{S}^6$ (where some components may be empty). For each $i \in \{1, ..., 6\}$, $\widetilde{S}^i$ is a locally bounded*

$\leq 6T$*-triangle combinatorial translation surface of genus* $\leq 6g+5m$.

Proof. Since $\widetilde{S}$ is a degree 6 branched cover of S, $\widetilde{S}$ has at most six connected components. If $\widetilde{S}^i$ is a connected component, then $F|_{\widetilde{S}^i}$ is a degree d branched cover for some $d \leq 6$. Denote by $c_1, ..., c_{n'}$ the critical points on $\widetilde{S}^i$ of this covering map, and $p_1, ..., p_n$ the branch points on S. Since branch points must be vertices of S of degree not equal to 6, $n \leq m$. Since each branch point has a preimage which is a critical point, $n' \geq n$. By the Riemann-Hurwitz formula,

$$2\,\mathrm{genus}(\widetilde{S}^i) - 2 + n' = d(2g-2) + dn.$$

Thus $\widetilde{S}^i$ has genus at most $6g+5m$. Since all branch points of F are vertices of S, the pullback of the triangulation S under F gives a triangulation of $\widetilde{S}^i$ with corresponding 1-form $\widetilde{\phi}_S$ by construction. Hence $\widetilde{S}^i$ is a combinatorial translation surface. Since $F : \widetilde{S} \to S$ is a degree 6 branched cover, $\widetilde{S}^i$ has at most $6T$ triangles.

Finally, we claim that $\widetilde{S}^i$ is locally bounded. To see this, note that if x is a vertex of the triangulation of $\widetilde{S}^i$, then $F(x)$ is a vertex of S. If $F(x)$ is not a branch point, then $\deg x = \deg(F(x)) \leq 7$ as S is locally bounded. If $F(x)$ is a branch point, then the ramification index of x, denoted $e(x)$, is at most 6, so $\deg x = e(x)\deg(F(x)) \leq 42$. Thus $\widetilde{S}^i$ satisfies condition 1 in the definition of local boundedness for combinatorial translation surfaces. To show condition 2: since S is locally bounded, we also have a triangulation S_{lb} of S by equilateral triangles of side length 3 in the flat metric, such that S is a 3-subdivision of S_{lb}. The pullback of the triangulation S_{lb} under F gives a triangulation $\widetilde{S}^i_{\mathrm{lb}}$ of the surface $\widetilde{S}^i$ wherein each triangle has side length 3, such that the triangulation $\widetilde{S}^i$ is the 3-subdivision of the triangulation $\widetilde{S}^i_{\mathrm{lb}}$. Thus $\widetilde{S}^i$ is a locally bounded combinatorial translation surface. □

8.3 Combinatorics of branched 6-covers

In this section, we enumerate the number of degree 6 branched covers of a fixed surface with fixed branched points.

Lemma 8.3.1. *Let* S_g *by a topological surface of genus* g *and* $P \subset S$ *a nonempty, unordered set of* n *marked points on* S_g*. There are at most* C^{g+n} *choices of branched* 6*-covers* $\widetilde{S}_g \to S_g$

(up to isomorphism of branched covers), such that the branch points are contained in the set P.

Proof. Cut S along elements of $H_1(S_g, P, \mathbb{Z})$ until we obtain S'_g, a simply connected surface with boundary such that $P \subset \partial S'_g$. The boundary $\partial S'_g$ may be decomposed into $4g+2n-2$ curves which come in pairs $\gamma^{+,1}, \gamma^{-,1}, ..., \gamma^{+,2g+n-1}, \gamma^{-,2g+n-1}$ that are glued together to form S'_g. Here, each $\gamma^{\pm,i}$ represents one element of $H_1(S_g, P, \mathbb{Z})$. Any degree 6 branched cover $\widetilde{S}_g$ with branched points in the set P necessarily admits a decomposition into isomorphic copies of S'_g denoted by $S'^1_g, ..., S'^6_g$. Boundary curve $\gamma_j^{+,i}$ (for $j \in \{1, ...6\}$ must necessarily be glued to $\gamma_{j'}^{-,i}$ for some $j' \in \{1, ..., 6\}$) under this decomposition of $\widetilde{S}_g$. Each decomposition corresponds to an isomorphism class of branched 6-covers $\widetilde{S}_g \to S_g$. For a fixed i, there are C choices to glue all the $\gamma_j^{+,i}$ and $\gamma_{j'}^{-,i}$, and i ranges from 1 to $2g+n-1$. Hence, the total number of gluing choices (and total number of branched covers) is bounded by C^{g+n}. □

For $n > 1$, we denote by $\mathcal{F}_{g,n}$ the set of topological branched covers enumerated above.

8.4 Mean value property and 6-differentials

In this section, we show a rough mean value property for 6-differentials arising from triangulated surfaces, which will be useful in Section 8.5.

Lemma 8.4.1. *Let S be a triangulated surface and let $X = \Phi(S)$. Suppose $v, w \in V(S)$, $v \neq w$ and let $\mathbb{D} \subset X$ a conformal identification of $\mathbb{D}$ with a subset of X containing v and w such that 0 is identified with v. Suppose further that $|w| \leq r < 3/4$ in $\mathbb{D}$. Then*

$$\int_{\mathbb{D}} |\psi_S|^{1/3} \geq Cr^{-2}.$$

Proof. Denote by C_α the circle of radius α around 0 in $\mathbb{D}$. Note that for $r \leq \alpha \leq 1$ the length of C_α in the d_S metric must be at least 1. Thus, writing $\psi = g(z)dz^6$ for a meromorphic function g on $\mathbb{D}$, we have that $g(z_0)^{1/6} \geq Cr^{-1}$ for some $z_0 \in C_r$. Let $a = g(z_0)^{1/6}$, and define

$$\psi' = \frac{g(z)}{(z-z_0)^6} dz^6,$$

a meromorphic 6-differential on $\mathbb{D}$. On a small neighborhood around z_0, $\psi' = \phi'^6$ for a 1-form ϕ', and locally

$$\phi' = \frac{g(z)^{1/6}}{(z-z_0)}dz$$

(note that $g(z)^{1/6}$ makes sense on a small neighborhood of z_0). Therefore the residue of ϕ' at z_0 is a. Let $f : \widetilde{X} \to X$ be the canonical 6-cover associated to ψ (constructed in Section 8.1). This means that $f^*(\psi') = \widetilde{\phi}'^6$ for a meromorphic 1-form ϕ defined on $f^{-1}(\mathbb{D})$. The 1-form $\widetilde{\phi}'$ is holomorphic except for a pole at $f^{-1}(z_0)$. On a neighborhood of $f^{-1}(z_0)$, $f^{-1}(\phi') = \widetilde{\phi}'$ so the residue of $\widetilde{\phi}'$ at $f^{-1}(z_0)$ is a. Since $r < 3/4$, for $7/8 \leq \alpha \leq 1$ we have

$$\int_{f^{-1}(C_\alpha)} \widetilde{\phi}' = 2\pi i a$$

so

$$\int_{f^{-1}(C_\alpha)} |\widetilde{\phi}'| \geq Ca.$$

Pushing forward this integral to $\mathbb{D}$ we obtain

$$\begin{aligned}\int_{C_\alpha} |g(z)|^{1/6}|dz| &\geq C\int_{C_\alpha} \frac{|g(z)|^{1/6}}{|z-z_0|}|dz| \\ &\geq Ca,\end{aligned}$$

since for $z \in C_\alpha$, $|z - z_0| \geq 1/8$. By Jensen's inequality, we have

$$\begin{aligned}\int_{C_\alpha} |g(z)|^{1/3}|dz| &\geq C\left(\int_{C_\alpha} |g(z)|^{1/6}|dz|\right)^2 \\ &\geq Ca^2.\end{aligned}$$

Then,

$$\begin{aligned}\int_{\mathbb{D}} |g(z)|^{1/3}|dz|^2 &\geq \int_{\alpha=7/8}^{1}\int_{C_\alpha} |g(z)|^{1/3}|dz|d\alpha \\ &\geq C\int_{\alpha=7/8}^{1} a^2 d\alpha \\ &\geq Ca^2\end{aligned}$$

$$\geq Cr^{-2}$$

as desired. ∎

We also have a quantitative version:

Lemma 8.4.2. *Let S be a triangulated surface and let $X = \Phi(S)$. Denote by ρ_X the hyperbolic metric on X, and $\rho_{\mathbb{D}}$ the Poincare metric on $\mathbb{D}$. Suppose $v, w \in V(S)$, $v \neq w$ and let $U \subset X$ be a region containing v and w with $f : U \to B_{\rho_{\mathbb{D}}}(0, s)$ a K-bi-Lipschitz diffeomorphism (with respect to ρ_X on U and $\rho_{\mathbb{D}}$ on $B_{\rho_{\mathbb{D}}}(0, s)$) such that $f(v) = 0$. Suppose further that $d_{\rho_X}(v, w) \leq (1/100)rs/K$ where $r < 3/4$. Then*

$$\int_U |\psi_S|^{1/3} \geq Cr^{-2}$$

where C is a universal constant.

Proof. Since $f : U \to B_{\rho_{\mathbb{D}}}(0, s)$ is a K-bi-Lipschitz map, U contains $B_{\rho_X}(v, s/K)$, the ball of radius s/K around v. Note that $w \in B_{\rho_X}(v, s/K)$ too. This ball may be isometrically identified with

$$\mathbb{D}_{\tanh(s/2K)} = \{z \in \mathbb{D} | |z| \leq \tanh(s/2K)\}$$

(v identified with 0) equipped with the restriction of the Poincare metric $\rho_{\mathbb{D}}$. Now

$$\begin{aligned} d_{\rho_X}(v, w) &\leq (1/100)rs/K \\ &\leq 2\tanh^{-1}(r \tanh(s/(2K))) \end{aligned}$$

since $s \leq 1$, $r < 3/4$ and $K \geq 1$, so $|w| \leq r \tanh(s/(2K))$ in $\mathbb{D}_{\tanh(s/(2K))}$. By Lemma 8.4.1,

$$\int_{B_{\rho_X}(v,s/K)} |\psi|^{1/3} \geq Cr^{-2}.$$

Since $B_{\rho_X}(v, s/K) \subset U$, the lemma statement follows. ∎

8.5 Location of branch points

Fix an arbitrary constant $r_0 > 0$. Let $X \in \mathcal{T}_g$. In this section, to each locally bounded triangulated surface whose conformal class lies in $B_{d_T}(X, r_0)$, we associate combinatorial data that is a discrete measure of where the vertices of degree not equal to 6 are located. Suppose $S_1, S_2 \in \mathrm{Comb}_{\mathrm{lb}}^{T,m}(\mathcal{T}_g)$ such that $\Phi(S_1), \Phi(S_2) \in B_{d_T}(X, r_0)$. We shall say that $V_{\neq 6}(S_1)$ and $V_{\neq 6}(S_2)$ are close to each other if S_1 and S_2 are associated to the same combinatorial data. In Section 8.6, we shall show that if $V_{\neq 6}(S_1)$ and $V_{\neq 6}(S_2)$ are close to each other, and additionally their canonical covers have the same underlying topological branched cover, then connected components of their canonical covers are close in a higher dimensional Teichmüller space.

Let ρ_X be the hyperbolic metric on X. We take hyperbolic disks $U_1, ..., U_N$, $W_1, ..., W_N$ and $V_1, ..., V_N$ as in Lemma 2.6.2. By Lemma 2.6.1 and Lemma 2.6.2, we may assume $N \leq CT$. For each $i \in \{1, ..., N\}$ and $M \geq 1$, let $\{V_{i,j}^M\}|_{j \in \{1,...,M\}}$ be a collection of hyperbolic balls on X satisfying the following properties:

1. For all $j, j' \in \{1, ..., M\}$, $\mathrm{radius}(V_{i,j}^M) = \mathrm{radius}(V_{i,j'}^M) \leq CM^{-1/2}\,\mathrm{radius}(V_i)$ where C is a universal constant,

2. $V_{i,j}^M \subset W_i$ and $\mathrm{center}(V_{i,j}^M) \in V_i$,

3. the $\{B_{\rho_X}(\mathrm{center}(V_{i,j}^M), \mathrm{radius}(V_{i,j}^M)/2)\}_{j \in \{1,...,M\}}$ cover V_i and

4. each $x \in U_i$ is contained in at most C of the $V_{i,j}^M$ for a universal constant C.

Such a collection may be constructed by taking $\sim M^{-1/2}\,\mathrm{radius}(V_i)$ radius balls around a maximal $\sim (1/8)M^{-1/2}\,\mathrm{radius}(V_i)$-separated set on V_i.

Define $\mathcal{D}$ to be the set of all values for the following data of (I, L, V_L):

1. subset $I \subset \{1, ..., N\}$ with $|I| \leq \alpha m$

2. function $L : I \to \mathbb{N}$ such that $\sum_{i \in I} L(i) \leq \alpha T$ and

3. subset V_L of $\cup_i \{V_{i,j}^{\kappa L(i)}\}|_{j \in \{1,...,\kappa L(i)\}}$ with $|V_L| \leq \alpha m$.

Here, $\alpha, \kappa > 1$ are sufficiently large universal constants which will be chosen in the proof of Lemma 8.5.2 later in this section.

Lemma 8.5.1. *We have* $|\mathcal{D}| \leq (\alpha\kappa T/m)^{C\alpha\kappa m}$.

Proof. The number of subsets I is bounded above by $(T/m)^{C\alpha m}$. Given I, the number of functions L is bounded above by $(T/m)^{C\alpha m}$. Given I and L, the number of subsets of $\cup_i \{V_{i,j}^{L(i)}\}|_{j\in\{1,\ldots,\kappa L(i)\}}$ with cardinality at most αm is bounded above by $(\alpha\kappa T/m)^{C\alpha\kappa m}$ since $|\cup_{i\in I} \{V_{i,j}^{L(i)}\}|_{j\in\{1,\ldots,\kappa L(i)\}}| \leq \alpha\kappa T$. Therefore $|\mathcal{D}| \leq (\alpha\kappa T/m)^{C\alpha\kappa m}$. □

Choose a marking of the Riemann surface X so that it is a point in $\mathcal{T}_g$. For all $Y \in B_{d_T}(X, r_0)$, by Proposition 4.3.1, we choose $f : X \to Y$ a Ce^{2r_0}-quasiconformal map that is Ce^{2r_0}-bi-Lipschitz with respect to the hyperbolic metrics ρ_X on X and ρ_Y on Y.

Lemma 8.5.2. *Let* $S \in \mathrm{Comb}^{T,m}(\mathcal{T}_g)$ *and suppose* $Y = \Phi(S) \in d_T(X, r_0)$*. Then there exist an element* $(I, L, V_L) \in \mathcal{D}$ *associated to* S *satisfying*

1. $i \in I$ *if and only if* V_i *contains an vertex in* $f^{-1}(V_{\neq 6}(S))$*,*

2. *each ball* $B \in V_L$ *contains a unique vertex in* $f^{-1}(V_{\neq 6}(S))$*, and this vertex lies in* $B_{\rho_X}(\mathrm{center}(B), \mathrm{radius}(B)/2)$*,*

3. *each point* $x \in X$ *is contained in at most* C *of the* $V \in V_L$*.*

Proof of Lemma 8.5.2. We let I be the subset of $i \in \{1, ..., N\}$ for which $f(V_i)$ contains an element of $V_{\neq 6}(S)$. Condition 1 in the statement of Lemma 8.5.2 is automatically satisfied. We also let $L(i) = \lceil \int_{f(U_i)} |\psi_S|^{1/3} \rceil$. Note that $|I| \leq |V_{\neq 6}(S)| \leq m$. Also $\sum_{i\in I} L(i) \leq \alpha T$ for a sufficiently large constant α since

$$\int_Y |\psi_S|^{1/3} \leq CT$$

and each point of Y is contained in at most C of the $f(U_i)$ (by condition 4 of Lemma 2.6.2). Finally, we let V_L be the subset of B in $\cup_{i\in I}\{V_{i,j}^{L(i)}\}_{j\in 1,\ldots,\kappa(L(i))}$ which satisfy the property that $B_{\rho_X}(\mathrm{center}(B), \mathrm{radius}(B)/2)$ contains a vertex in $f^{-1}(V_{\neq 6}(S))$. We now show the uniqueness part of condition 2 in the statement of Lemma 8.5.2. Suppose the contrary; that there exists $B \in V_L$ containing $f^{-1}(x)$ and $f^{-1}(y)$ for vertices $x, y \in V_{\neq 6}(S)$. Then $f^{-1}(x), f^{-1}(y) \in W_i$

for some $i \in \{1, ..., N\}$ and for sufficiently large κ, radius$(B) \leq$ radius(V_i) implying $f^{-1}(y) \in B_{\rho_X}(f^{-1}(x), 2\,\text{radius}(V_i)) \subset U_i$. Now, Lemma 8.4.2 applied to

$$f^{-1} : f(B_{\rho_X}(f^{-1}(x), 2\,\text{radius}(V_i))) \to B_{\rho_X}(f^{-1}(x), 2\,\text{radius}(V_i))$$

implies

$$d_{\rho_Y}(x, y) \geq C(L(i))^{-1/2}\,\text{radius}(U_i)/e^{2r_0}$$

which means

$$d_{\rho_X}(f^{-1}(x), f^{-1}(y)) \geq C(L(i))^{-1/2}\,\text{radius}(U_i)/e^{4r_0},$$

which is greater than $CL(i)^{-1/2}\kappa^{-1/2}\,\text{radius}(U_i) \geq 2\,\text{radius}(B)$ for κ sufficiently large, hence we have a contradiction to the assumption that $f^{-1}(x), f^{-1}(y) \in B$. So condition 2 in the statement of Lemma 8.5.2 is satisfied. Finally, condition 3 in the statement of Lemma 8.5.2 follows from condition 4 of Lemma 2.6.2 along with condition 4 in the construction of the set $\{V_{i,j}^{\kappa L(i)}\}$. □

For $S \in \text{Comb}_{\text{lb}}^{T,m}(\mathcal{T}_g)$ such that $Y = \Phi(S) \in B_{d_T}(X, r_0)$, we choose $(I, L, V_L) \in \mathcal{D}$ satisfying the conditions in the statement of Lemma 8.5.2. We label this particular choice of element in $\mathcal{D}$ by (I_S, L_S, V_{L_S}). Given $S_1, S_2 \in \text{Comb}_{\text{lb}}^{T,m}(\mathcal{T}_g)$ such that $\Phi(S_1), \Phi(S_2) \in B_{d_T}(X, r_0)$, we say that $V_{\neq 6}(S_1)$ and $V_{\neq 6}(S_2)$ are close to each other if $(I_{S_1}, L_{S_1}, V_{L_{S_1}}) = (I_{S_2}, L_{S_2}, V_{L_{S_2}})$.

8.6 Branched covers and Teichmüller distance

In this section we show that if two locally bounded marked triangulated surfaces satisfy the property that their conformal classes are close in Teichmüller space, their sets of vertices of degree not equal to 6 are close to each other, and their respective canonical covering maps are topologically isomorphic, then connected components of their respective canonical branched 6-covers are close in a higher dimensional Teichmüller space.

Let $X \in \mathcal{T}_g$. Suppose $S_1, S_2 \in \text{Comb}^{T,m}(\mathcal{T}_g)$ and $\Phi(S_1), \Phi(S_2) \in B_T(X, r_0)$. Let $f_1 : X \to S_1$ and $f_2 : X \to S_2$ be the associated Ce^{2r_0}-quasiconformal, Ce^{2r_0}-bi-Lipschitz maps.

Suppose $(I_{S_1}, L_{S_1}, V_{L_{S_1}}) = (I_{S_2}, L_{S_2}, V_{L_{S_2}})$. Suppose $\mathfrak{f} \in \mathcal{F}_{g,\leq m}$ is a topological branched cover underlying both the covers $\widetilde{S}_1 \to S_1$ and $\widetilde{S}_2 \to S_2$ (i.e. the diagram

$$\begin{array}{ccc} \widetilde{S}_1 & \longrightarrow & \widetilde{S}_2 \\ {\scriptstyle \mathfrak{f}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{f}} \\ S_1 & \xrightarrow{f_2 \circ f_1^{-1}} & S_2 \end{array}$$

of unmarked topological surfaces commutes for some choice of top map labelled $\widetilde{f_2 \circ f_1^{-1}}$). The markings on S_1 and S_2 along with the topological branched cover $\mathfrak{f}$ give markings on the connected components of $\widetilde{S}_1$ and $\widetilde{S}_2$. We label components of $\widetilde{S}_1$ and $\widetilde{S}_2$ as $\widetilde{S}_1^k$ and $\widetilde{S}_2^k$, respectively, for $1 \leq k \leq 6$, so that the labeling is compatible with $\mathfrak{f}$, meaning that the diagram

$$\begin{array}{ccc} \widetilde{S}_1^k & \xrightarrow{\widetilde{f_2 \circ f_1^{-1}}} & \widetilde{S}_2^k \\ {\scriptstyle \mathfrak{f}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{f}} \\ S_1 & \xrightarrow{f_2 \circ f_1^{-1}} & S_2 \end{array}$$

of marked topological surfaces commutes for all $1 \leq k \leq 6$.

Lemma 8.6.1. *We have, $d_T(\widetilde{S}_1^k, \widetilde{S}_2^k) \leq C$ for all $1 \leq k \leq 6$ for which the surfaces $\widetilde{S}_1^k$ and $\widetilde{S}_2^k$ are nonempty.*

Before proving Lemma 8.6.1, we have an elementary lemma.

Lemma 8.6.2. *There exists a universal constant C such that for all $z_1, z_2 \in \mathbb{D}$ satisfying $|z_1|, |z_2| \leq 3/4$, there exists a diffeomorphism $f_{z_1,z_2} : \mathbb{D} \to \mathbb{D}$ satisfying the following conditions:*

1. *$f_{z_1,z_2}|_{\partial\mathbb{D}}$ is the identity map,*

2. *$Df_{z_1,z_2}|_{\partial\mathbb{D}}$ is the identity map,*

3. *$f_{z_1,z_2}(z_1) = z_2$ and*

4. *f_{z_1,z_2} is C-quasiconformal.*

Proof. We first treat the case where $z_1 = 0$. Denote by $f_{0,3/4}$ any chosen diffeomorphism $\mathbb{D} \to \mathbb{D}$ satisfying conditions (1), (2) and (3) above for $z = 3/4$, and let C be the quasiconformal dilatation of $f_{0,3/4}$. Note that C is finite since f is defined on a compact set. Now, suppose $z \in \mathbb{R}$ such that $z \leq 3/4$. The function $f_{0,z}(w) = (4/3)zf_{0,3/4}((3/4)z^{-1}w)$ for $|w| \leq (4/3)z$ and $f_{0,z}(w) = w$ satisfies the conditions in the lemma statement. Suppose next that $z \in \mathbb{D}$ with $|z| \leq (3/4)$. We have already constructed $f_{0,|z|}$, since $|z| \in \mathbb{R}$. We simply take

$$f_{0,z}(w) = \frac{z}{|z|} f_{0,|z|}\left(\left(\frac{z}{|z|}\right)^{-1} w\right)$$

and note that $f_{0,z}$ satisfies the desired properties. Now removing the assumption that $z_1 = 0$, we simply take $f_{z_1,z_2} = f_{0,z_2} \circ f_{0,z_1}^{-1}$. □

Proof of Lemma 8.6.1. It suffices to construct a quasi-conformal map between $S_1 \setminus V_{\neq 6}(S_1)$ and $S_2 \setminus V_{\neq 6}(S_2)$, since $V_{\neq 6}(S_1)$ and $V_{\neq 6}(S_2)$ are the branch points of the respective canonical 6-covers. Then, lifting via $\mathfrak{f}$ gives the desired quasiconformal map $\widetilde{S}_1^k \to \widetilde{S}_2^k$. To do this, it suffices to construct a C-quasiconformal map between $X \setminus P_1$ and $X \setminus P_2$ where $P_1 = f_1^{-1}(V_{\neq 6}(S_1))$ and $P_2 = f_2^{-1}(V_{\neq 6}(S_2))$. By assumption, $V_{L_{S_1}} = V_{L_{S_2}} = V$, and let $B_1, ..., B_N$ be the elements of V. Recall that the B_i are hyperbolic disks of maximal radius arcsinh(1), and by condition 2 in the statement of Lemma 8.5.2, each B_i contains exactly one element of P_1 (which we denote by $p_{1,i}$), and exactly one element of P_2 (which we denote by $p_{2,i}$). By condition 2 in the statement of Lemma 8.5.2 along with the upper bound on radius(B_i), under a conformal identification of B_i with $\mathbb{D}$ sending its center to 0, both $p_{1,i}$ and $p_{2,i}$ lie inside $\mathbb{D}_{3/4} = \{z \in \mathbb{D} | |z| < 3/4\}$. Thus there exists $f_{p_{1,i},p_{2,i}}$ satisfying the conditions in the statement of Lemma 8.6.2, and $f_{p_{1,i},p_{2,i}}$ extends to a C-quasiconformal map $f^i : X \to X$ that is the identity outside B_i. Composing all such maps f^i (for $i \in \{1, ..., N\}$) in an arbitrary order we obtain a map $F : X \to X$ such that $F(P_1) = P_2$. Now, by condition 3 of Lemma 8.5.2, the set $B_1, ..., B_N$ satisfies the property that any $x \in X$ is contained in at most C of the B_i. Therefore F is C-quasiconformal. □

8.7 Proof of Lemma 8.0.1

First, a preliminary lemma.

Lemma 8.7.1. *Let* $S_1, S_2 \in \mathrm{Comb}_{\mathrm{lb}}^{T,m}(\mathcal{T}_g)$. *Suppose the topological branched covers underlying* $\widetilde{S}_1 \to S_1$ *and* $\widetilde{S}_2 \to S_2$ *are isomorphic and represented by the element* $\mathfrak{f} \in \mathcal{F}_{g,m}$. *Label the components of* $\widetilde{S}_1$ *and* $\widetilde{S}_2$ *as* $\widetilde{S}_1^{\,k}$ *and* $\widetilde{S}_2^{\,k}$, $1 \leq k \leq 6$ *such that the labelling is compatible with* $\mathfrak{f}$. *If the there exists a simplicial isomorphism of marked surfaces that preserves the underlying triangulations of* $\widetilde{S}_1^{\,k}$ *and* $\widetilde{S}_2^{\,k}$, *then there exists a simplicial isomorphism of marked surfaces between* S_1 *and* S_2 *as well.*

Proof. Let $\widetilde{f} : \widetilde{S}_1^{\,k} \to \widetilde{S}_2^{\,k}$ be such a simplicial isomorphism. Homotoping $\widetilde{f}$ as necessary we may assume $\widetilde{f}$ is a biholomorphism and an isometry with respect to the flat metrics on $\widetilde{S}_1^{\,k}$ and $\widetilde{S}_2^{\,k}$. So there exists a map $f : S_1 \to S_2$ such that the following diagram of marked triangulated surfaces commutes.

$$\begin{array}{ccc} \widetilde{S}_1^k & \xrightarrow{\widetilde{f}} & \widetilde{S}_2^k \\ \mathfrak{f}\downarrow & & \downarrow\mathfrak{f} \\ S_1 & \xrightarrow{f} & S_2 \end{array}$$

Since $\widetilde{f}$ is a biholomorphism, f is a biholomorphism. Since the two branched covering maps are simplicial maps and $\widetilde{f}$ is a simplicial map, f is also a simplicial map. Thus f is a simplicial isomorphism. □

We now prove Lemma 8.0.1. First let $m > 0$ and $X \in \mathcal{T}_g$. We must count the number of locally bounded marked triangulated surfaces lying in $B_{d_T}(X, r_0)$ that have at most m vertices of degree not equal to 6. By Proposition 8.2.1, any such marked triangulated surface has associated to it a topological branched cover in $\mathcal{F}_{g,\leq m}$, and by Lemma 8.3.1, there are at most C^{g+m} such topological branched covers. Any such triangulated surface has associated to it a triple (I, L, V) satisfying the conditions in the statement of Lemma 8.5.2, and by Lemma 8.5.1, there are at most $(T/m)^{Cm}$ such triples. If two triangulated surface have the same associated topological branched cover and associated triple, then by Lemma 8.6.1, two connected components of their covers are close together in a higher genus Teichmüller space. By Lemma 8.7.1 it suffices to count combinatorial translation surfaces lying in a ball in the higher genus Teichmüller space. Thus we obtain

$$N^{\mathcal{T}}_{\text{lb}}(T, g, m, r_0) \le C^g (T/m)^{Cm} N^{\mathcal{H}}_{\text{lb}}(6T_i, 6g + 5m, C).$$

Combining with Lemma 4.6.6 gives Lemma 8.0.1.

8.8 Proof of Theorem 1.1.2

Recall that for $g \ge 2$,

$$N^{\mathcal{T}}(T, g, m, r) \le C^{m+g} N^{\mathcal{T}}_{\text{lb}}(\sigma T, g, \mu(m+g), r + C)$$

by Corollary 6.2.1,

$$N^{\mathcal{T}}_{\text{lb}}(T, g, m, r) \le C^{(1+r)g} (T/m)^{Cm} N^{\mathcal{H}}_{\text{lb}}(6T, 6g + 5m, r + C)$$

by Lemma 8.0.1 and

$$N^{\mathcal{H}}_{\text{lb}}(T, g, r) \le (T/g)^{C(1+r)g} \sum_{\substack{n \le (1/100)g \\ T_1 + \ldots + T_n \le T \\ g_1 + \ldots + g_n \le \mu^{-1}(1/100)g \\ g_1, \ldots, g_n \ge 2 \\ m_1 + \ldots + m_n \le \mu^{-1}(1/100)g}} \prod_{i=1}^{n} N^{\mathcal{T}}(2T_i, g_i, m_i, r + C)$$

by Lemma 7.0.1.

Choosing r_0 sufficiently small (independent of T and g), we may apply Lemma 4.6.6 to obtain

$$N^{\mathcal{T}}(T, g, m, r_0) \le C^{m+g} N^{\mathcal{T}}_{\text{lb}}(\sigma T, g, \mu(m+g), r_0), \tag{8.1}$$

$$N^{\mathcal{T}}_{\text{lb}}(T, g, m, r_0) \le C^g (T/m)^{Cm} N^{\mathcal{H}}_{\text{lb}}(6T, 6g + 5m, r_0) \tag{8.2}$$

and

$$N_{\mathrm{lb}}^{\mathcal{H}}(T,g,r) \le (T/g)^{Cg} \sum_{\substack{n\le(1/100)g\\ T_1+\ldots+T_n\le T\\ g_1+\ldots+g_n\le\mu^{-1}(1/100)g\\ g_1,\ldots,g_n\ge 2\\ m_1+\ldots+m_n\le\mu^{-1}(1/100)g}} \prod_{i=1}^{n} N^{\mathcal{T}}(2T_i,g_i,m_i,r_0).$$

Together, these three bounds give

$$\begin{aligned}
&N_{\mathrm{lb}}^{\mathcal{H}}(T,g,r_0)\\
&\le (T/g)^{Cg} \sum_{\substack{n\le(1/100)g\\ T_1+\ldots+T_n\le T\\ g_1+\ldots+g_n\le\mu^{-1}(1/100)g\\ g_1,\ldots,g_n\ge 2\\ m_1+\ldots+m_n\le\mu^{-1}(1/100)g}} \prod_{i=1}^{n} N^{\mathcal{T}}(2T_i,g_i,m_i,r_0)\\
&\le (T/g)^{Cg} \sum_{\substack{n\le(1/100)g\\ T_1+\ldots+T_n\le T\\ g_1+\ldots+g_n\le(1/100)g\\ g_1,\ldots,g_n\ge 2\\ m_1+\ldots+m_n\le(1/50)g}} \prod_{i=1}^{n} C^{m_i+g_i} N_{\mathrm{lb}}^{\mathcal{T}}(2\sigma T_i,g_i,m_i,r_0)\\
&\le (T/g)^{Cg} \sum_{\substack{n\le(1/100)g\\ T_1+\ldots+T_n\le T\\ g_1+\ldots+g_n\le(1/100)g\\ g_1,\ldots,g_n\ge 2,\\ m_1+\ldots+m_n\le(1/50)g\\ m_1,\ldots,m_n\ge 1}} \prod_{i=1}^{n} (T_i/m_i)^{Cm_i} N_{\mathrm{lb}}^{\mathcal{H}}(12\sigma T_i,6g_i+5m_i,r_0)\\
&\le (T/g)^{Cg} \sum_{\substack{n\le(1/100)g\\ T_1+\ldots+T_n\le 12\sigma T\\ g_1+\ldots+g_n\le g/4}} \prod_{i=1}^{n} N_{\mathrm{lb}}^{\mathcal{H}}(T_i,g_i,r_0).
\end{aligned} \tag{8.3}$$

where the last inequality follows from renaming variables along with the arithmetic mean-geometric mean inequality.

We claim $N_{\mathrm{lb}}^{\mathcal{H}}(T,g,r_0) \le \Theta^g (T/g)^{\Omega g}$ for universal constants Ω and Θ that we choose later. We prove this by induction. The quantity $N_{\mathrm{lb}}^{\mathcal{H}}(T,4,r_0)$ is bounded by a polynomial in T. (See [16] for more precise results.) To show the induction hypothesis, assume $N_{\mathrm{lb}}^{\mathcal{H}}(T,g',r_0) \le$

$\Theta^{g'}(T/g')^{\Omega g'}$ for all $g' \leq g$. We have,

$$\begin{aligned}
N^{\mathcal{H}}_{\text{lb}}(T,g,r_0) &\leq (T/g)^{Cg} \sum_{\substack{n\leq(1/100)g\\ T_1+\ldots+T_n\leq 12\sigma T\\ g_1+\ldots+g_n\leq g/4}} \prod_{i=1}^{n} N^{\mathcal{H}}_{\text{lb}}(T_i,g_i,r_0) && \text{Eq. (8.3)}\\
&\leq (T/g)^{Cg} \sum_{\substack{n\leq(1/100)g\\ T_1+\ldots+T_n\leq 12\sigma T\\ g_1+\ldots+g_n\leq g/4}} \prod_{i=1}^{n} \Theta^{g_i}(T_i/g_i)^{\Omega g_i}\\
&\leq (T/g)^{Cg} \sum_{\substack{n\leq(1/100)g\\ T_1+\ldots+T_n\leq 12\sigma T\\ g_1+\ldots+g_n\leq g/4}} \Theta^{g/4}(48\sigma T/g)^{\Omega g/4} && \text{arithmetic mean-geometric mean}\\
&\leq (T/g)^{Cg}\Theta^{g/4}(48\sigma T/g)^{\Omega g/4}
\end{aligned}$$

where the last inequality follows since the sum contains at most $(T/g)^{Cg}$ terms. Noting that $T/g \geq 2$, there exists a choice of constants Θ and Ω such that

$$(T/g)^{Cg}\Theta^{g/4}(48\sigma T/g)^{\Omega g/4} \leq \Theta^{g}(T/g)^{\Omega g}.$$

This completes the induction step. Therefore, we have

$$\begin{aligned}
N^{\mathcal{H}}_{\text{lb}}(T,g,r_0) &\leq \Theta^{g}(T/g)^{\Omega g}\\
&\leq C^{T}.
\end{aligned} \tag{8.4}$$

Finally,

$$\begin{aligned}
N^{\mathcal{M}}(T,g,r_0) &\leq N^{\mathcal{T}}(T,g,r_0) && \text{Lemma 4.7.1}\\
&\leq N^{\mathcal{T}}(T,g,3T,r_0)\\
&\leq C^{T} N^{\mathcal{T}}_{\text{lb}}(\sigma T,g,3\mu T,r_0) && \text{Eq. (8.1)}\\
&\leq C^{T} N^{\mathcal{H}}_{\text{lb}}(6\sigma T,6g+15\mu T,r_0) && \text{Eq. (8.2)}\\
&\leq C^{T} && \text{Eq. (8.4)}
\end{aligned}$$

where the second inequality follows from the fact that a T-triangle triangulated surface has at most $3T$ vertices. Combining with Lemma 4.6.6 gives

$$N^{\mathcal{M}}(T, g, r) \leq C^{T+rg},$$

which completes the proof of Theorem 1.1.2.

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