Genera via Deformation Theory and Supersymmetric Mechanics

by

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Abstract

We study naturally occurring genera (i.e. cobordism invariants) from the deformation theory inspired by supersymmetric quantum mechanics. First, we construct a canonical deformation quantization for symplectic supermanifolds. This gives a novel proof of the super-analogue of Fedosov quantization. Our proof uses the formalism of Gelfand-Kazhdan descent, whose foundations we establish in the super-symplectic setting.

In the second part of this thesis, we prove a super-version of Nest-Tsygan's algebraic index theorem, generalizing work of Engeli. This work is inspired by the appearance of the same genera in three related stories: index theory, trace methods in deformation theory, and partition functions in quantum field theory. Using the trace methodology, we compute the genus appearing in the story for supersymmetric quantum mechanics. This involves investigating supertraces on Weyl-Clifford algebras and deformations of symplectic supermanifolds.

Thesis Supervisor: Michael J. Hopkins
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Introduction

This thesis explores the interplay between quantum field theories, algebra, and topology. The algebra that will appear is deformation theory. Topologically, we will see cobordism invariants of manifolds. The quantum field theory we will be focused on is *supersymmetric quantum mechanics*.

Chapter 1 discusses the deformation problem for supersymmetric quantum mechanics and Chapter 2 produces the related genera. Here, we give a brief overview of the history of ideas surrounding this thesis. Each chapter has its own, more detailed, introduction.

The main theorem of Chapter 1 is Theorem 1.5.12. The main theorems of Chapter 2 are Theorem 2.5.6 and Theorem 2.8.38.

0.1 From Physics to Algebra

We begin by discussing the physical situation we would like to model. Before getting to quantum field theory, we address classical field theories.

Consider a particle moving around in a confined space X. For $I \subset \mathbb{R}$ a time interval, the space of maps $\mathsf{Map}(I, X)$ consists of all paths the particle could, theoretically, take. In a real world physical system, there are constrains on how particles can move. For example, particles may only travel on paths of least energy. That is, if X is given a metric, we may be only concerned with the subspace of geodesic paths.

Remark 0.1.1. Some field theories are indiscriminate of distances. Such theories are called *topological.*

The physical constrains on what paths a particle can take are called the *equations of motion* or the *Euler-Lagrange equations*. They are often presented as PDEs, and determine a map

$$S\colon \mathsf{Map}(I, X) \to \mathbb{R}$$

called the *action functional*.

We can make the following preliminary definition.

Definition 0.1.2. A *classical field theory* is the data of

- the space of *fields* $\mathsf{Map}(I, X)$, and
- the action functional S.

Depending on the ambient situation, the mapping space of field may be equipped with additional geometric structure (such as a symplectic form).

We let $\mathrm{EL} \subset \mathsf{Map}(I, X)$ denote the critical locus of S,

$$\mathrm{EL} = \{f\colon I \to X : (ds)(f) = 0\}.$$

Remark 0.1.3. So far, we have considered paths over an interval $I \subset \mathbb{R}$. More generally, one would like to allow for field theories where I is replaced by a more complicated *spacetime* manifold. For example, I may be replaced by S^1, or $I \times Y$ for another manifold Y. The *dimension* of the field theory is the dimension of the spacetime.

Example 0.1.4 (Classical Mechanics - massless free theory)**.** Let I be the interval $[a, b]$. Classical mechanics is the classical field theory with fields $\mathsf{Map}(I, \mathbb{R}^n)$ and action functional S defined by

$$S(f) = \int_a^b \langle f(t), \frac{d^2}{dt^2} f(t)\rangle dt.$$

The space EL consists of straight line paths. See [31, §1.1] for more details.

The field theory we will study in this thesis (supersymmetric mechanics) will be an analogue of the above example with $\mathbb{R}^n$ replaced by a symplectic supermanifold. It is a 1d topological field theory.

Example 0.1.5 (Gauge Theory)**.** Let G be a Lie group and Y be a manifold. From this data, one can form a gauge theory with fields given by the space of principal G-bundle on Y with connection, and Euler-Lagrange equations cutting out those bundles with flat connection. In an appropriate setting, one can form a stack $B_\nabla G$ so that the gauge theory fields are a mapping space

$$\mathsf{Map}(Y, B_\nabla G).$$

See [32, §3.3] for more details.

There are two styles of mathematical formulations of *quantum* field theories. One is the Schrödenger approach that focuses on the states of the theory. In this style, after Atiyah [3] and Segal [79], one defines a quantum field theory to be a type of symmetric monoidal functor out of a bordism category. The second approach is using the Heisenberg perspective and focuses instead on the observables of the theory. This is the formulation we will use below, following Costello-Gwilliam [31].

Given a classical field theory, the observables are the measurements one can make on the system. For example, one might ask what position the particle is in, or what its momentum is. In theory, one can perform all of these measurements simultaneously on a classical field theory. This changes in quantum field theory.

Principle (Heisenberg's Uncertainty Principle)**.** *In quantum field theory, one cannot precisely know both the position and the momentum of a particle at the same time.*

To formulate this uncertainty mathematically, let us first investigate further the properties of observables of a classical field theory. Following [31, Ch. 1 §1.2], we make the following preliminary definition.

Definition 0.1.6. Given a classical field theory

$$\mathrm{EL} \subset \mathsf{Map}(I, X),$$

the ring of *classical observables* is

$$\mathrm{Obs}^{\mathrm{cl}} = C^\infty(\mathrm{EL}, \mathbb{R}).$$

An observable $\mathrm{EL} \to \mathbb{R}$ corresponds to sending an allowable path to the value of the chosen measurement on that path.

Note that the classical observables form a commutative algebra. Physically this multiplication corresponds to performing two measurements at the same time. By Heisenberg's Uncertainty Principle, we can no longer do this in quantum mechanics.

Question 1. What replaces the commutative algebra structure on observables in quantum field theory?

Although we cannot take measurements simultaneously, we can take them at disjoint times. Say $I = [a, b]$ and $f, g \in \mathrm{Obs}^{\mathrm{cl}}$ are measurements. Given disjoint subintervals (t_1, t_2) and (t_3, t_4) with

$$a < t_1 < t_2 < t_3 < t_4 < b,$$

then even in quantum field theory, we should be able to the measurement f on (t_1, t_2) and g on (t_3, t_4).

More generally, on a spacetime $M = I \times Y$, we could perform different measurements on disjoint disks in M. This leads to the following notion, see [31, Ch. 6 §1].

Definition 0.1.7. A *factorization algebra* $\mathcal{A}$ on a manifold M is the following data:

- for every inclusion of disjoint disks
$$\coprod_{i=1}^{k} D_i \hookrightarrow M,$$
a vector space $A\left(\coprod_{i=1}^{k} D_i\right)$ and an isomorphism
$$\mathcal{A}\left(\coprod_{i=1}^{k} D_i\right) \simeq \bigotimes_{i=1}^{k} \mathcal{A}(D_i),$$
- and for every inclusion $\coprod_{i=1}^{k} D_i \hookrightarrow D \hookrightarrow M$, a partial multiplication map
$$\bigotimes_{i=1}^{k} \mathcal{A}(D_i) \simeq \mathcal{A}\left(\coprod_{i=1}^{k} D_i\right) \to \mathcal{A}(D)$$
satisfying compatibility conditions and a sheaf condition.

The following is contained in [31, 32].

Theorem 0.1.8 (Costello-Gwilliam). *The local observables of a quantum field theory form a factorization algebra* $\mathrm{Obs}^{\mathrm{q}}$.

These are called local observables as they consisted of measurements we could perform on some subspace of spacetime. The global observables -measurements we can perform on all of M- can be obtained by taking the factorization homology of M with coefficients in Obs^q, see [31].

Since the field theory we will be interested in is topological, we would like a condition on the observables that detects the metric invariance.

Definition 0.1.9. A factorization algebra $\mathcal{A}$ on M is *locally constant* if given an inclusion of disks $D_1 \subset D_2 \subset M$, the resulting map

$$\mathcal{A}(D_1) \to \mathcal{A}(D_2)$$

is an isomorphism.

In particular a locally constant factorization algebra does not care about the diameter of disks. It turns out that the observables of a topological field theory form a locally constant factorization algebra.

Theorem 0.1.10 (Lurie). *Locally constant factorization algebras on* $\mathbb{R}^n$ *are equivalent to* $\mathcal{E}_n$-*algebras.*

This is [63, Thm. 5.4.5.9].

For example, since supersymmetric quantum mechanics is a 1d topological field theory, its local quantum observables form an associative algebra.

Remark 0.1.11. Inspired by [6], in [31, Ch. 5], they show that over a $\mathbb{C}$, certain factorization algebras are related to vertex algebras. See [44] for an overview of vertex algebras.

Remark 0.1.12. The factorization algebra of an associative algebra A over S^1 is its Hochschild complex,

$$\int_{S^1} A = \mathrm{Hoch}_\bullet(A).$$

See [4, Thm. 3.19]. We will use this observation when forming manifold invariants in Chapter 2.

A *quantization* of a 1d classical field theory into a quantum field theory therefore corresponds to a *deformation* of the commutative algebra $\mathrm{Obs}^{\mathrm{cl}}$ into an associative algebra Obs^q.

Here, by a deformation we mean the following.

Definition 0.1.13. Let T be a commutative algebra. A *deformation* of T is an associative algebra structure on $T[[\hbar]]$ and an isomorphism of algebras

$$T[[\hbar]]/\hbar \simeq T.$$

Remark 0.1.14. If the field theory comes equipped with extra structure, such as a symplectic form or a group of symmetries, we will need to consider deformations that respect that structure.

The first part of this thesis, Chapter 1, is a study of the deformation problem corresponding to supersymmetric mechanics, using modern techniques.

0.2 Getting Manifold Invariants

Using the Costello-Gwilliam approach of studying quantum field theories by their algebras of observables, one gets a dictionary between physical questions and algebraic questions.

One interesting quantity one can sometimes extract from a quantum field theory is its *partition function*.

Example 0.2.1. In the Atiyah-Segal approach to quantum field theories, given an 2-dimensional QFT

$$Z\colon \mathsf{Bord}_2 \to \mathsf{Vect}_{\mathbb{C}},$$

the partition function of Z is the value of Z on a torus T. Since T is a closed 2-manifold, $Z(T) \in \mathbb{C}$ is a number. Removing a circle from T, we obtain a cylinder C. The value $Z(T)$ is the same as the trace of the linear map

$$Z(C)\colon Z(S^1) \to Z(S^1).$$

Usually this is considered for manifolds with a metric. The cylinder C will then have a length, say t. In practice, the map $Z(C)$ looks something like $\exp(-tH)$ for some operator H.

The interpretation of a partition function in the Costello-Gwilliam formalism is more complicated. If one sees a QFT as coming from formal elliptic moduli problem $B\mathcal{L}$, the partition function should be the volume of $B\mathcal{L}$ with respect to some volume form, see [32, §9.3]. In practice, computing the partition function in this formalism involves Feynman diagram manipulations.

Often times, one has a family of field theories, one for each manifold X.

Example 0.2.2. Given a Riemannian manifold X, one can consider a type of classical mechanics where the particle moves in X. That is, the classical field theory with fields $\mathsf{Map}(I, X)$ and action functional cutting out the space of geodesics.

In several of these examples, the partition function of the theory assigned to X is some genus evaluated at X.

The following example is considered in [55].

Example 0.2.3 (1d Chern-Simons Theory)**.** Let X be a manifold. There is a field theory, called 1d Chern-Simons theory, with fields $\mathsf{Maps}(S^1, T^*X)$. The partition function of this 1d Chern-Simons theory is $\hat{A}(X)$.

In [9, §4], Berwick-Evans shows that the $\hat{A}$-genus appears in the study of $1|1$-dimensional supersymmetric quantum mechanics. This is done using a Atiyah-Segal formulation of quantum field theory.

Example 0.2.4. Let Σ be a Riemann surface and Y a complex manifold. The curved $\beta\gamma$ system is a 2d field theory with fields $\mathsf{Maps}(\Sigma, Y)$ with allowable paths given by holomorphic maps. If Σ is an elliptic curve and Y is Kähler, then this field theory has a quantization whose partition function is the Witten genus of Y with respect to the elliptic curve Σ. This was shown by Costello in [30] and reproven in [51].

In the Atiyah-Segal style approach, Berwick-Evans has shown that the Witten genus appears in the study of a $2|1$-dimensional super Euclidean field theory, [9].

Example 0.2.5. In [11], Berwick-Evans shows that the L-genus appears in a $1|2$-dimensional supersymmetric quantum mechanics. Ignoring the supersymmetry, this field theory has dimension 1.

Remark 0.2.6. One might note that the in the above examples, the genera coming form a d-dimensional field theory have natural homes in cohomology theories of chromatic height d. Part of the Stolz-Teichner program is the philosophy that this phenomenon should always be true, [82, 83].

One might ask if there is a way of extracting these manifold invariants algebraically. That is, is there an analogous operation to "taking the partition function" that one can perform on the associated algebras of observables.

Expectation 1. *The expectation is that partition functions should correspond to a type of trace invariant on* Obs^q.

Remark 0.2.7. For an ordinary associative $\mathbb{R}$-algebra A, a trace on A is a map

$$\mathrm{HH}_0(A) \to \mathbb{R}.$$

Since $\mathrm{HH}_\bullet(A)$ is the factorization homology of A over S^1, we might expect that for a factorization algebra $\mathcal{A}$ on a manifold M, a trace on $\mathcal{A}$ is something like a map

$$\int_M \mathcal{A} \to \mathbb{R}.$$

The algebra of classical observables of the 1d Chern-Simons theory from Example 0.2.3 is $\mathcal{O}_{T^*X}$. The algebra of local quantum observables is the Rees algebra of differential operators on X. Note that this matches our observation that quantization corresponds to deformation, as $\mathrm{Rees}(\mathrm{Diff}_X)$ is the canonical deformation of $\mathcal{O}_{T^*X}$.

The Atiyah-Singer index theorem [2] says that one can recover $\hat{A}(X)$ by looking at the index of elliptic differential operators on X,

$$\mathrm{Index}(D) = \int_X \hat{A}(X)\mathrm{ch}(D).$$

There is therefore hope that the algebra $\mathrm{Rees}(\mathrm{Diff}_X)$ would know about $\hat{A}$.

More generally, functions on a symplectic manifold (M, ω) has a canonical deformation, called the Fedosov quantization [41].

Equipping the Fedosov deformation quantization with a trace, Bressler, Nest, and Tsygan have proven an algebraic version of the Atiyah-Singer index theorem, [18, 42, 70].

Theorem 0.2.8 (Algebraic Index Theorem)**.** *Let* (M, ω) *be a compact symplectic manifold. Let* Ω *be the Fedosov connection for* M*. There exists a unique normalized trace* t_M *on the Fedosov* *deformation quantization of $\mathcal{O}_M$ so that*

$$t_M(1) = \frac{1}{(2\pi i)^n} \int_M \hat{\mathrm{A}}(TM) \exp(-\Omega/\hbar).$$

In this Example 0.2.3, one can justify Expectation 1. In [53, 54], they show that the partition function computations of [55] match with the trace computations in the algebraic index theorem.

There is also evidence of Expectation 1 in dimension 2.

Example 0.2.9. The field theory from Example 0.2.4 has dimension 2, and therefore its algebra of local quantum observables Obs^q should be a factorization algebra over the manifold Σ. In [51], they show that Obs^q is given by the vertex algebra of chiral differential operators $\mathrm{CDO}(Y)$ on Y, see [64]. In [51], it is also shown that the Witten genus appears in the graded character of $\mathrm{CDO}(Y)$.

Chapter 2 of this thesis is proving a version of the algebraic index theorem for the algebra of local quantum observables for supersymmetric quantum mechanics.

0.3 Supersymmetric Quantum Mechanics

We highlight what changes in the super version of the quantum mechanics story.

In quantum mechanics, one considers deformations of functions on a symplectic manifold. Locally, symplectic manifolds are modeled by $T^*\mathbb{R}^n$. The canonical deformation is the Weyl algebra Weyl_{2n}.

In supersymmetric quantum mechanics, one instead deforms a symplectic supermanifold. By Batchelor's theorem [5], supermanifolds are noncanonically isomorphic to ones of the form $E[1]$ where $E \to X$ is a vector bundle on X. Similarly, symplectic supermanifolds come from quadratic vector bundles on symplectic manifolds, [76]. Locally, these are modeled by a quadratic vector space (V, Q) and $T^*\mathbb{R}^n$. The canonical deformation is the tensor product of the Weyl algebra and the Clifford algebra,

$$\mathrm{Weyl}_{2n} \otimes \mathrm{Cliff}(V, Q).$$

The quadratic vector bundle provides an extra direction of freedom that appears throughout this thesis.

Motivated by Example 0.2.5 and a special case computation by Engeli [38], we expect the super version of the algebraic index theorem to reveal the L-genus.

The L-genus on an oriented $4k$-dimensional manifold N is given by the *signature* $\mathrm{sign}(N)$. L-theory and the signature are built from the study of quadratic forms on vector spaces.

We thus see an interesting dynamic between the quadratic vector space (V, Q) appearing in the structure of the symplectic supermanifold, and the L-genus expected to appear in the partition function. This relationship incarnates in unexpected ways. For example, the L-genus only appears when the quadratic form Q has signature (a, a). The main theorem (Theorem 2.8.38) of Chapter 2 is to compute what replaces the L-genus in general signature.

Chapter 1

Deformation Quantization for Supermanifolds via Gelfand-Kazhdan Descent

1.1 Introduction

Given a symplectic manifold (M, ω), it is a classic question to ask whether there exists a deformation of the algebra of functions $\mathcal{O}_M$ compatible with the symplectic form ω. The space of such deformations was first described independently by De Wilde-Lecomte [35] and Fedosov [41]. This result was extended by Kontsevich to apply to all Poisson manifolds, [60].

Here, we give a new proof of the super-analogue of Fedosov's quantization result, showing that for symplectic supermanifolds there exists a deformation quantization. A symplectic supermanifold is a supermanifold $\mathbb{M}$ together with an even, closed, nondegenerate 2-form ω on $\mathbb{M}$, (Definition 1.2.6). In particular, we work with *even* symplectic supermanifolds.

Fedosov's quantization of a (non-super) symplectic manifold (M, ω) requires the data of a symplectic connection on M. In our formulation, for a symplectic supermanifold $(\mathbb{M}, \omega)$, the connection data is replaced with an *$\hbar$-formal exponential* (Definitions 1.3.27 and 1.5.7).

Theorem 1.1.1. *Let $(\mathbb{M}, \omega, \sigma)$ be a symplectic supermanifold with an $\hbar$-formal exponential σ. Then there exists a canonical deformation quantization $\mathcal{A}_\sigma(\mathbb{M})$ of the Poisson superalgebra $\mathcal{O}_{\mathbb{M}}$.*

This is Theorem 1.5.12 below. For (M, ω) a non-super symplectic manifold, a symplectic connection on M determines an $\hbar$-formal exponential. In this case, our theorem recovers Fedosov's canonical deformation. We discuss the relationship between the deformation quantization in Theorem 1.1.1 and the space of all deformation quantizations in Remark 1.1.2.

A *deformation quantization* of a Poisson $\Bbbk$-algebra $(A, \{-,-\})$ (such as $\Bbbk$-valued functions on a symplectic manifold) is an associative $\Bbbk[[\hbar]]$-algebra $(A_\hbar, \star)$ with a $\Bbbk[[\hbar]]$-module isomorphism $A_\hbar \simeq A[[\hbar]]$ such that

- for all $f, g \in A_\hbar$, we have

  $$f \star g = fg + \hbar B_1(f,g) + \hbar^2 B_2(f,g) + \cdots$$

  for bidifferential operators $B_i(-,-)$, and

- if $f, g \in A$, then $\frac{1}{\hbar}[f,g] = \{f,g\} \mod \hbar$.

The simplest example of a symplectic manifold is the cotangent bundle $T^*\mathbb{R}^n$. The cotangent bundle has a canonical quantization given by the Weyl algebra (Definition 1.5.17). By Darboux's Lemma, symplectic manifolds all locally look like $T^*\mathbb{R}^n$ for some n. Production of a deformation quantization on a general symplectic manifold usually proceeds by trying to globalize from the Weyl algebra on a Darboux chart. For example, given a manifold X, the canonical deformation quantization of T^*X given by the Rees algebra of differential operators on X can be produced locally using the Weyl algebra.

There is a similar story for symplectic supermanifolds, but the local structure in the odd direction has additional freedom. Locally, a symplectic supermanifold is specified by what we call it's *type*: a triple of numbers $(2n|a,b)$ where there are $2n$ even dimensions, $a+b$ odd dimensions, and the symplectic structure in the odd direction comes from a quadratic form Q of signature (a,b). The canonical local quantization is then a tensor product of Weyl and Clifford algebras,

$$\mathcal{A}(\mathbb{R}^{2n|r}) = \mathrm{Weyl}(T^*\mathbb{R}^n) \otimes \mathrm{Cliff}(\mathbb{R}^r, Q).$$

1.1.0.1 Comparison to other Deformation Quantizations

There have been many approaches to globalizing the canonical choice of local deformation quantization [12, 23, 36, 41, 42, 88]. In the super-case Bordemann [15, 16] constructed a deformation quantization for symplectic supermanifolds using Fedosov's approach. A similar result also appears in [38] using the methods of [42]. Our method of proof is similar, and inspired by, the methods used by Bezrukavnikov and Kaledin in the non-super case [12]. The formalism in [12] that we mimic here also works in the algebraic setting, and has even been extended to positive characteristic [13].

Here, we globalize the local quantization and prove Theorem 1.1.1, using techniques in formal geometry, first described by Gelfand-Kazhdan in [45], called *Gelfand-Kazhdan descent*. This is a special case of Harish-Chandra descent. Roughly speaking, this is a fancy version of the Borel construction that takes into account the connection data on the formal coordinate bundle ([12, §3.1], [57, §4.2], or §1.3.2.1 in the super-symplectic case below). We develop Gelfand-Kazhdan descent for symplectic supermanifolds in §1.3.3. Gelfand-Kazhdan descent is also used in a more modern computation of the Witten genus coming from the factorization algebra of chiral differential operators [51]. One benefit of using Gelfand-Kazhdan descent here is to make connections to Feynmann diagram computations in the BV formalism (as in [51]) more accessible, see §1.1.1.3.

Remark 1.1.2. Essentially, we construct deformation quantizations locally on a formal disk and use gluing data Glue to descend to a deformation quantization on the whole manifold. In [12], they classify the set $Q(M,\omega)$ of deformation quantizations of a (non-super) symplectic $2n$-manifold (M,ω) up to isomorphism using these techniques, [12, Lem. 3.4]. This is done by describing the set of all possible gluing datum, which involves considering the unwieldy pro-group $\mathrm{Aut}(\mathrm{Weyl}_{2n})$ of automorphisms of the Weyl algebra,

$$\mathsf{Glue}_{\mathrm{Aut}(\mathrm{Weyl}_{2n})} \xrightarrow{\sim} Q(M,\omega)$$

Here, we instead restrict the gluing datum to be linear. In the purely even case, this corresponds to requiring the data to come from the symplectic group $\mathrm{Sp}(2n)$. We get a factoring of the equivalence from [12],

$$\begin{array}{ccc} \mathsf{Glue}_{\mathrm{Aut}(\mathrm{Weyl}_{2n})} & \xrightarrow{\simeq} & Q(M,\omega) \\ \uparrow & \nearrow & \\ \mathsf{Glue}_{\mathrm{Sp}(2n)} & & \end{array}$$

The space $\mathsf{Glue}_{\mathrm{Sp}(2n)}$ is equivalent to the space of $\hbar$-formal exponentials (Remark 1.3.34), which is contractible, Lemmas 1.3.28 and 1.5.8. In summary, the added rigidity produces a contractible space of gluing data, and hence an essentially unique deformation quantization. See also Remark 1.5.15.

1.1.1 Motivation and Broader Perspective

Our present work is motivated by a larger program to relate genera to partition functions of field theories. In §1.1.1.1, we give a zoomed-out look at how this paper relates to manifold invariants of interest. The relationship between Fedosov quantizations and algebraic index theorems is discussed in §1.1.1.2. Lastly, in §1.1.1.3, we discuss the physical interpretation of this broader picture. Studying the questions raised here is ongoing joint work with Owen Gwilliam and Brian Williams.

1.1.1.1 Motivation: Manifold Invariants

Let $\mathsf{sMfld}^{\mathrm{Sp}}$ denote the category of symplectic supermanifolds, see Definition 1.2.6 below. There is a category $\mathsf{sGK}^{=}$ (Variation 1.3.31), fibered over $\mathsf{sMfld}^{\mathrm{Sp}}$, of pairs $(\mathbb{M},\sigma)$ of a symplectic supermanifold and an $\hbar$-formal exponential; that is, the necessary input data for Theorem 1.1.1. Roughly speaking, Theorem 1.1.1 provides a lift to the functor of $\mathbb{R}$-valued smooth functions,

$$\begin{array}{ccc} (\mathsf{sGK}^{=})^{\mathrm{op}} & \overset{\mathcal{A}_{(-)}}{\dashrightarrow} & \mathsf{Mod}_{\Omega^\bullet}(\mathsf{sAlg}_{\mathbb{R}[[\hbar]]}) \\ \downarrow & & \downarrow{\scriptstyle \hbar=0} \\ \left(\mathsf{sMfld}^{\mathrm{Sp}}\right)^{\mathrm{op}} & \xrightarrow[\mathcal{O}_{(-)}]{} & \mathsf{Mod}_{\Omega^\bullet}(\mathsf{sAlg}_{\mathbb{R}}) \end{array}$$

As mentioned above, given a manifold X and a quadratic vector bundle $E \to X$ with compatible connection, one can produce an even symplectic structure on the supermanifold $(\pi^*E)[1]$, where $\pi\colon T^*X \to X$ is the projection map. We obtain a functor

$$L_X \colon \mathsf{VB}_{/X}^{\mathrm{quad},\nabla} \to \left(\mathsf{sMfld}^{\mathrm{Sp}}\right)^{\mathrm{op}},$$

where $\mathsf{VB}_{/X}^{\mathrm{quad},\nabla}$ (Example 1.2.14) is the category with

- objects: triples (E, g, ∇) of vector bundles over X, equipped with a quadratic form and a compatible connection, and
- morphisms: a morphism $(E, g, \nabla) \to (E', g', \nabla')$ is a map of vector bundles $f \colon E \to E'$ that is a fiberwise isomorphism, intertwines the quadratic forms, and so that $f^*\nabla' = \nabla$.

Just as the cotangent bundle of an ordinary manifold has a canonical quantization, we will construct a deformation quantization for symplectic supermanifolds coming from $\mathsf{VB}_{/X}^{\mathrm{quad},\nabla}$, see Remark 1.3.37 and Lemma 1.5.14. This is done by constructing a lift of L_X to $\mathsf{sGK}^{=}$,

$$\begin{array}{ccc} & & \left(\mathsf{sGK}^{=}\right)^{\mathrm{op}} \\ & \overset{\tilde{L}_X}{\nearrow} & \downarrow \\ \mathsf{VB}_{/X}^{\mathrm{quad},\nabla} & \xrightarrow[L_X]{} & \left(\mathsf{sMfld}^{\mathrm{Sp}}\right)^{\mathrm{op}} \end{array}$$

Composing the lift $\tilde{L}_X$ with the deformation quantization functor $\mathcal{A}$ over T^*X, we obtain a functor (Remark 1.5.13)

$$\widetilde{A}_X \colon \mathsf{VB}_{/X}^{\mathrm{quad},\nabla} \to \mathsf{Mod}_{\Omega^\bullet_{T^*X}}(\mathsf{sAlg}_{\mathbb{R}[[\hbar]]}).$$

Further post-composing with the Hochschild cohomology functor $\mathrm{HH}^\bullet_{\mathbb{R}[[\hbar]]}(-; -^\vee)$, we obtain a functor

$$I_X \colon \mathsf{VB}_{/X}^{\mathrm{quad},\nabla} \to \mathsf{Mod}_{\Omega^\bullet_{T^*X}}(\mathsf{Ch}_{\mathbb{R}[[\hbar]]}).$$

Question 2. What invariant of quadratic vector bundles on X does I_X produce?

One well-studied invariant of quadratic vector bundles is the *Witt group*, [90]. It is natural to ask how I_X and $\mathrm{Witt}(X)$ are related. In particular, the Witt group is obtained by quotienting by the hyperbolic quadratic forms. Since Hochschild (co)homology is invariant under Morita equivalence, one might expect that $\widetilde{A}_X$ sends vector bundles with hyperbolic quadratic forms to Morita trivial algebras.

Question 3. How does I_X behave under stabilization by vector bundles with hyperbolic quadratic forms? In particular, does $\widetilde{A}_X$ send hyperbolic vector bundles to Morita trivial superalgebras?

Example 1.1.3. The answer to this question is "yes" when X is a point. In this case, we are considering the functor from vector spaces equipped with a quadratic form to superalgebras. The functor $\widetilde{A}_{\text{pt}}$ sends a quadratic vector space (V, Q) to the Clifford algebra $\text{Cliff}(V, Q)$. When Q is hyperbolic, the Clifford algebra is equivalent to a matrix algebra via the spinor representation, and hence is Morita trivial.

The Witt group $\text{Witt}(X)$ is closely related to the (quadratic) L-groups, $\mathbb{L}(X)_\bullet$, [74]. The L-groups are the natural home for the signature of X. As noted below (§1.1.1.2), we expect a super-version of the algebraic index theorem ([38]) applied to certain oriented vector bundles over X to recover the L-genus. There are also indications in the literature [11] that the 1d AKSZ theory relevant to $\widetilde{A}_X$ has partition function related to the L-genus (§1.1.1.3). The invariant I_X constructed here should therefore lead to interesting connections between super deformation quantization and the L-genus.

1.1.1.2 Motivation: Index Theory

An essential invariant of a differential operator is its *index*. One can ask how much the deformation $\text{Rees}(\text{Diff}_X)$ of T^*X knows about the topology of X. Famously, Atiyah and Singer [2] proved that the (analytic) index of an elliptic differential operator $\mathcal{D}$ on X is equivalent to its topological index. Bressler, Nest, and Tsygan have proven an algebraic index theorem [42,70] using deformation theory. The algebraic index theorem, equips the Fedosov quantization of (M, ω, ∇) (a symplectic manifold with symplectic connection ∇) with an interesting trace map Tr_M, and then gives a description of the trace evaluated at 1 involving known topological invariants,

$$\text{Tr}_M(1) = \int_M \hat{A}(TM)\exp(-\text{char}(\nabla)/\hbar).^1$$

[1] There is a scalar term here, which depends on a normalization condition. Here $\text{char}(\nabla)$ is the *characteristic class* of the deformation. See [42, §4].

In [38], Engeli proves a generalization of the algebraic index theorem of Bressler-Nest-Tsygan [42, 70] for certain symplectic supermanifolds of type $(2n|n, n)$. In Engeli's result [38, Thm. 2.26], one sees an invariant closely related to the multiplicative sequence for the L-genus replacing the $\hat{A}$-genus in the non-super version. Our techniques of super-Gelfand-Kazhdan descent could be used to reproduce and generalize Engeli's super algebraic index result. See §1.1.1.1 for more discussion along these lines.

We prove generalization of Engeli's result in Chapter 2.

1.1.1.3 Motivation: Quantum Field Theory

The deformation quantization of $T^*\mathbb{R}^n$ is the Weyl algebra. In quantum mechanics, this is the algebra of observables of a free bosonic system. The super-version, Theorem 1.1.1, corresponds to adding fermions. The resulting Weyl-Clifford algebra is the algebra of local observables of suspersymmetric quantum mechanics.

One can think of globalizing as going from the AKSZ theory for the formal super-disk to the theory for the symplectic supermanifold $\mathbb{M}$. On BV fields this is a process

$$\mathrm{Maps}(S^1, \hat{\mathbb{D}}^{2n|r}) \rightsquigarrow \mathrm{Maps}(S^1, \mathbb{M}).$$

In [55], Gwilliam and Grady construct 1d Chern-Simons theory in the BV formalism following Costello-Gwilliam [31, 33]. This 1d theory has quantum observables that agree with the Fedosov quantization of T^*X. We expect a super-analogue to [55] to show that the super-Fedosov quantization from Theorem 1.1.1 appears as the observables of supersymmetric quantum mechanics. Gelfand-Kazhdan descent for factorization algebras of observables has been developed in [51]. Assuming one uses these descent techniques to describe supersymmetric quantum mechanics in the BV formalism, our proofs of Theorem 1.1.1 below make one well-positioned to compare the algebraic and physical constructions. Such a comparison for 1d Chern-Simons theory is made in [53, 54].

1.1.2 Linear Overview

We give a brief overview of the structure of this paper.

In §1.2, we review the basics of symplectic supermanifold, including Rothstein's analogue [76] of Batchelor's structure theorem for supermanifolds [5]. We define super-Harish-Chandra pairs and construct the particular example of such that we will use for our descent in §1.3.1. Our descent functor is defined in §1.3.3, where we also give a few first examples of how descent works. In §1.3.3, we also prove several monoidal properties of our super-Gelfand-Kazhdan descent functor. The formalism developed in §1.3 is used in §1.4 and §1.5.0.1 to show that Gelfand-Kazhdan descent takes deformation quantizations to deformation quantizations. In §1.5, we prove Theorem 1.1.1, giving a deformation quantization of a symplectic supermanifold. We then describe the deformation quantization in terms of Weyl and Clifford algebras in §1.5.1.

1.1.3 Conventions

We set the following conventions for the paper.

Algebra Conventions.

- Let $\Bbbk$ be either $\mathbb{R}$ or $\mathbb{C}$
- K will be a Lie supergroup
- $\mathfrak{g}$ is a Lie superalgebra

Manifold Conventions.

All manifolds are real (i.e., not complex), smooth and without boundary. A *manifold with boundary* is a manifold with, possibly empty, boundary. We use the phrase "ordinary manifold" to distinguish from a supermanifold.

- X will denote an ordinary manifold
- $\mathbb{X}$ will denote a supermanifold
- (M, ω) will denote a ordinary symplectic manifold
- $(\mathbb{M}, \omega)$ will denote a symplectic supermanifold
- $\mathcal{O}_Y$ denotes smooth $\Bbbk$-valued functions on Y

- $\widehat{\mathbb{D}}^{2n|r}$ is the formal super-disk of dimension $2n|r$ whose ring of functions is

$$\mathcal{O}_{\widehat{\mathbb{D}}^{2n|r}} = \Bbbk[[p_1, \dots, p_n, q_1, \dots, q_n, \theta_1, \dots, \theta_r]].$$

- Given a $\Bbbk$-vector space V, the trivial vector bundle on $\mathbb{X}$ with fiber V is denoted $\underline{V}_{\mathbb{X}}$.

Further conventions are explained later, see Convention 1.3.38.

1.2 Review of Symplectic Supermanifolds

We review the basics of symplectic supermanifolds that we will use below. For more comprehensive discussions of supermanifolds, see [5, 62, 75].

Definition 1.2.1. A *supermanifold* is a $\mathbb{Z}/2$-graded ringed space $\mathbb{X}$ whose underlying space X is an n-manifold and such that the *ring of smooth functions* $\mathcal{O}_{\mathbb{X}}$ is locally isomorphic to

$$\mathcal{O}_{\mathbb{R}^n} \otimes \Lambda[\theta_1, \dots, \theta_r]$$

for some r. A *morphism of supermanifolds* $\mathbb{X} \to \mathbb{Y}$ is a graded map $\mathcal{O}_{\mathbb{Y}} \to \mathcal{O}_{\mathbb{X}}$ living over a smooth map $X \to Y$.

We will let $\mathsf{sMfld}_{n|r}$ denote the category of supermanifolds with n even dimensions and r odd dimensions.

Example 1.2.2 (Ordinary manifolds as supermanifolds)**.** Let X be an ordinary (i.e., not super) n-manifold. We can regard X as a supermanifold with 0 odd directions.

Example 1.2.3. We let $\mathbb{R}^{n|r}$ denote the supermanifold with underlying manifold $\mathbb{R}^n$ and functions

$$\mathcal{O}_{\mathbb{R}\backslash|\nabla} = \mathbb{R}[\S_\infty, \dots, \S_\backslash] \otimes *[\theta_\infty, \dots, \theta_\nabla].$$

Example 1.2.4 (Batchelor's theorem)**.** Let X be an ordinary n-manifold and $E \to X$ a rank k vector bundle on X. One can form a supermanifold $E[1]$ with underlying ordinary manifold X and functions

$$\mathcal{O}_{\mathcal{E}[\infty]} = -(\mathcal{X}, *^\bullet \mathcal{E}^\vee).$$

For the tangent bundle $TX \to X$, we use the notation $T[1]X$, which has $\mathcal{O}_{T[\infty]\mathcal{X}} = \otimes^{\#}_{\mathcal{X}}$, the underlying $\mathbb{Z}/2$-graded vector space of the de Rham complex. The notation ΠE is sometimes used for $E[1]$. By Batchelor's theorem [5, §3], every supermanifold is noncanonically isomorphic to one of the form $E[1]$.

Definition 1.2.5. Given a supermanifold $\mathbb{X}$, *vector fields* on $\mathbb{X}$ is the Lie superalgebra

$$\mathrm{Vect}(\mathbb{X})\colon = \mathrm{Der}(\mathcal{O}_{\mathbb{X}})$$

of graded derivations.

Forms of degree k on $\mathbb{X}$ is the space

$$\Omega^k_{\mathbb{X}}\colon = \Lambda^k(\mathrm{Vect}(\mathbb{X})^{\vee}).$$

The *de Rham complex* of $\mathbb{X}$ is $\Omega^{\bullet}_{\mathbb{X}}$ with differential defined to be the derivation of bidegree $(1, \mathrm{even})$ which locally on generators $x_i, \theta_j, dx_i, d\theta_j$ is given by $d(x_i) = dx_i$, $d(\theta_i) = d\theta_i$, $d(dx_i) = 0$, and $d(d\theta_i) = 0$.

Note that $\Omega^{\bullet}_{\mathbb{X}}$ inherits a $\mathbb{Z}/2$-grading, so we can speak of even and odd forms on $\mathbb{X}$.

Definition 1.2.6. A *symplectic supermanifold* is a pair $(\mathbb{M}, \omega)$ where $\mathbb{M}$ is a supermanifold and ω is an even, closed, non-degenerate 2-form on $\mathbb{M}$.

A *symplectomorphism* $\psi\colon (\mathbb{M}, \omega) \to (\mathbb{M}', \omega')$ is a morphism of supermanifolds $\mathbb{M} \to \mathbb{M}'$ that is a diffeomorphism on underlying manifolds and so that $\psi^*(\omega') = \omega$.

We let $\mathsf{sMfld}^{\mathrm{Sp}}_{n|r}$ denote the category of symplectic supermanifolds with n even dimensions and r odd dimensions.

Example 1.2.7 (Ordinary symplectic manifolds as symplectic supermanifolds)**.** Let (M, ω) be an ordinary symplectic manifold. Viewing M as a supermanifold with 0 odd directions, ω becomes an even 2-form. Thus (M, ω) can be seen as a symplectic supermanifold.

Example 1.2.8. By Darboux's theorem, [19, Thm. 8.1], every (ordinary) symplectic manifold

(M,ω) is locally isomorphic to $(\mathbb{R}^{2n},\omega_0)$ where, in coordinates $p_1,\dots,p_n,q_1,\dots,q_n$, the form ω_0 is

$$\omega_0 = \sum_{i=1}^{n} p_i \wedge q_i.$$

We can similarly give $\mathbb{R}^{2n|r}$ a symplectic structure but we need to make a choice of $Q=(\epsilon_1,\dots,\epsilon_r)$ where $\epsilon_i \in \{1,-1\}$. Given such a Q, we can define a symplectic form on $\mathbb{R}^{2n|r}$ by

$$\omega_Q = \sum_{i=1}^{n} dp_i \wedge dq_i + \sum_{i=1}^{r} \frac{\epsilon_i}{2} d\theta_i^2$$

where $\theta_1,\dots,\theta_r$ are the odd coordinates. Note that ω_Q is equivalent to the date of its *signature*, that is the number a of positive ϵ_i and the number b of negative ϵ_i. We have $a+b=r$.

Notation 1.2.9. We denote the symplectic supermanifold described in Example 1.2.8 by $(\mathbb{R}^{2n|r},\omega_Q)$.

Definition 1.2.10. A *symplectic super vector space* is a super vector space V together with a nondegenerate bilinear form $b\colon V\times V \to \Bbbk$ that is skew-symmetric in the even directions and symmetric in the odd directions,

$$b(x,y) = (-1)^{|x||y|} b(y,x).$$

Let (V,b) be a symplectic super vector space of dimension $2n|r$. Let Q be the quadratic form associated to the nondegenerate bilinear form b. Analogously to the purely even case [19, Thm. 1.1], there is an isomorphism between (V,b) and $(\mathbb{R}^{2n|r},\omega_Q)$ from Example 1.2.8.

Definition 1.2.11. The $(2n|a,b)$-symplectic group, denoted $\mathrm{Sp}(2n|a,b)$, is the group of linear symplectomorphisms of $(\mathbb{R}^{2n|r},\omega_Q)$ where Q has signature (a,b).

When $(a,b)=(r,0)$, this is sometimes called the symplectic-orthogonal group.

Remark 1.2.12. Define the super-transpose of a block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

to be

$$M^{sT} = \begin{bmatrix} A^T & -C^T \\ B^T & D^T \end{bmatrix}.$$

Let G denote the diagonal matrix $G = \mathrm{diag}(\epsilon_1, \dots, \epsilon_r)$ where $Q = (\epsilon_1, \dots, \epsilon_r)$. If $\mathrm{GL}(2n|r)$ denotes the general linear supergroup, then $\mathrm{Sp}(2n|a, b) \subset \mathrm{GL}(2n|r)$ consists of those matrices M so that

$$M^{sT} H M = H$$

where

$$H = \begin{bmatrix} \Omega & 0 \\ 0 & G \end{bmatrix}$$

and

$$\Omega = \begin{bmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{bmatrix}.$$

Note that when there are no odd dimensions we have $\mathrm{Sp}(2n|0, 0) \cong \mathrm{Sp}(2n)$.

Remark 1.2.13. If we replace G with a conjugate matrix, we obtain an isomorphic Lie group.

Example 1.2.14. Let (M, ω) be an ordinary symplectic manifold, $E \to M$ a vector bundle, and (g, ∇) a metric and compatible connection on E. The data (ω, g, ∇) defines a super-symplectic form on the supermanifold $E[1]$, see [76, Def. 1]. Let $\mathsf{VB}^{\mathrm{quad},\nabla}_{/\mathsf{Mfld}^{\mathrm{Sp}}}$ be the category of quadruples (M, E, g, ∇) and morphisms respecting this data. Explicitly, a morphism $(M, E, g, \nabla) \to (M', E', g', \nabla')$ is a map of vector bundles $f\colon E \to E'$ that is a fiberwise isomorphism, lives over a local symplectomorphism $M \to M'$, intertwines the quadratic forms, and so that $f^*\nabla' = \nabla$.

There is a symplectic analogue of Batchelor's theorem [5], due to Rothstein, [76].

Theorem 1.2.15 (Rothstein)**.** *Every symplectic supermanifold is non-canonically isomorphic to one of the form in Example 1.2.14.*

Corollary 1.2.16. *Let* $(\mathbb{M}, \omega)$ *be a symplectic supermanifold. Then the underlying manifold* M *inherits the structure of an ordinary symplectic manifold. In particular,* $\mathbb{M}$ *must have an even number of even directions.*

We also have a super-analogue of the Darboux theorem [61, Thm. 5.3]

Theorem 1.2.17 (Kostant). *Let $(\mathbb{M}, \omega) \in \mathsf{sMfld}^{\mathrm{Sp}}_{2n|r}$. Then there exists $Q = (\epsilon_1, \ldots, \epsilon_r)$ so that for every $x \in \mathbb{M}$, there exists an open neighborhood U of x that is symplectomorphic to $(\mathbb{R}^{2n|r}, \omega_Q)$.*

Here ω_Q is as in Example 1.2.8. See also [86, §3] and the references therein.

Notation 1.2.18. Let $(\mathbb{M}, \omega)$ be a symplectic supermanifold so that locally ω is of the form ω_Q with signature (a, b). We refer to $(\mathbb{M}, \omega)$ as having *type* $(2n|a, b)$.

Definition 1.2.19. Let $\mathsf{sMfld}_{2n|a,b}$ denote the category of symplectic supermanifolds of type $(2n|a, b)$ and local symplectomorphisms.

Next, we would like to discuss symplectic vector fields on a symplectic supermanifold. For motivation and to review, we first recall the notions on ordinary symplectic manifolds.

1.2.0.1 Symplectic Vector Fields: Ordinary Manifolds

Let (M, ω) be a symplectic manifold. The nondegenerate 2-form ω determines an isomorphism $TM \cong T^*M$, and thus an equivalence

$$\phi_\omega \colon \mathrm{Vect}(M) \cong \Omega^1(M).$$

Definition 1.2.20. Let (M, ω) be a symplectic manifold. The Lie algebra of *symplectic vector fields* is the sub-Lie algebra of $\mathrm{Vect}(M)$ consisting of those vector fields v such that $\phi_\omega(v)$ is closed. Denote by $\mathrm{Vect}^{\mathrm{symp}}(M)$ the Lie algebra of symplectic vector fields.

Say v is a *Hamiltonian vector field* if $\phi_\omega(v)$ is exact. In this case, we refer to a function h such that $dh = \phi_\omega(v)$ as a *Hamiltonian* of v.

We will describe a characterization of symplectic vector fields in terms of the ring of functions $\mathcal{O}_M$. To do this, we need to understand the structure the symplectic form ω induces on $\mathcal{O}_M$.

Definition 1.2.21. A *Poisson algebra* is an commutative algebra P equipped with a Lie bracket $\{-, -\}$ satisfying the Leibnitz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

for any $f, g, h \in P$.

A *Poisson derivation* of P is a linear map $d\colon P \to P$ so that for all $x, y \in P$ we have

- $d(xy) = d(x)y + xd(y)$, and
- $d(\{x, y\}) = \{d(x), y\} + \{x, d(y)\}$.

The following is [65, Lem. 1.1.18].

Lemma 1.2.22. *Let (M, ω) be a symplectic manifold. Then $\mathcal{O}_M$ is a Poisson algebra with bracket*

$$\{f, g\}_\omega = \phi_\omega^{-1}(df)(g).$$

Here, $\phi_\omega^{-1}(df)$ is the Hamiltonian vector field with Hamiltonian f.

Notation 1.2.23. Let (M, ω) be a symplectic manifold. We let $\mathrm{Der}_\omega(\mathcal{O}_M)$ denote the Lie algebra of Poisson derivations of the Poisson algebra $(\mathcal{O}_M, \{-,-\}_\omega)$.

The following is [19, Def. 18.2].

Lemma 1.2.24. *Let (M, ω) be a symplectic manifold. There is an equivalence of Lie algebras*

$$\mathrm{Vect}^{\mathrm{symp}}(M, \omega) \simeq \mathrm{Der}_\omega(\mathcal{O}_M).$$

1.2.0.2 Symplectic Vector Fields: Supermanifolds

For the super case, we mimic the description of symplectic vector fields as derivations of a Poisson algebra.

Definition 1.2.25. A *Poisson superalgebra* is a supercommutative superalgebra R equipped with a Lie superbracket $\{-,-\}$ such that

$$\{f, gh\} = \{f, g\}h + (-1)^{|f||g|}g\{f, h\}$$

for all $f, g, h \in R$.

The following is in [86, Pg. 244].

Lemma 1.2.26. *Let $(\mathbb{M}, \omega)$ be a symplectic supermanifold. Then ω induces an equivalence*

$$\phi_\omega \colon \mathrm{Vect}(\mathbb{M}) \cong \Omega^1(\mathbb{M}),$$

and $\mathcal{O}_{\mathbb{M}}$ is a Poisson superalgebra under the superbracket

$$\{f, g\}_\omega = \phi_\omega^{-1}(df)(g).$$

Example 1.2.27. Since we will be using formal geometry, we will often be interested in the formal super-disk $\widehat{\mathbb{D}}^{2n|r}$. If we give $\mathbb{R}^{2n|r}$ a symplectic form of type $(2n|a, b)$, then functions on the formal disk inherits a Poisson algebra structure from the completion of functions on $\mathbb{R}^{2n|r}$ at the point 0. In coordinates, the Poisson bracket on

$$\mathcal{O}_{\widehat{\mathbb{D}}^{2n|r}} = \Bbbk[[p_1, \ldots, p_n, q_1, \ldots, q_n, \theta_1, \ldots, \theta_a, \theta'_1, \ldots, \theta'_b]]$$

is given by

$$\{p_i, q_i\} = 1$$

$$\{\theta_i, \theta_j\} = 1$$

$$\{\theta'_i, \theta'_j\} = -1$$

and the rest zero. We denote this Poisson algebra by $\widehat{\mathcal{O}}_{2n|a,b}$.

Notation 1.2.28. Let $\mathrm{Der}_\omega(\mathcal{O}_{\mathbb{M}})$ denote the Lie superalgebra of Poisson derivations of the Poisson superalgebra $(\mathcal{O}_{\mathbb{M}}, \{-, -\}_\omega)$.

Definition 1.2.29. Let $(\mathbb{M}, \omega)$ be a symplectic supermanifold. The Lie superalgebra of *symplectic vector fields* on $\mathbb{M}$ is the Lie superalgebra of derivations

$$\mathrm{Vect}^{\mathrm{symp}}(\mathbb{M}, \omega) = \mathrm{Der}_\omega(\mathcal{O}_{\mathbb{M}}).$$

1.2.0.3 Bundles

For an overview of vector bundles and principal bundles on supermanifolds, see [77] or [20]. For a construction of the frame bundle of a supermanifold, see [86, §2]. Just as the structure group of the frame bundle of an ordinary symplectic manifold can be reduced to the symplectic group, we have the following,

Lemma 1.2.30. *Let* $(\mathbb{M}, \omega)$ *be a symplectic supermanifold of type* $2n|a, b$. *Then the structure group of the frame bundle* $\mathrm{Fr}_{\mathbb{M}} \to \mathbb{M}$ *can be reduced to* $\mathrm{Sp}(2n|a, b)$.

The fiber over $x \in \mathbb{M}$ will be

$$\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}|_x = \mathrm{Symp}\left((T_x\mathbb{M}, \omega|_x), (\mathbb{R}^{2n|r}, \omega_Q)\right),$$

the group of linear symplectomorphisms.

1.3 Gelfand-Kazhdan Descent for Symplectic Supermanifolds

We would like to construct a descent functor that allows us to study symplectic supermanifolds locally. The notion of descent we will consider is a variant of the Borel construction. For K a Lie group, the Borel construction takes a principal K-bundle $P \to X$ and a K-module V to the vector bundle $P \times_K V \to X$. Harish-Chandra descent is a generalization of this construction from K to a Harish-Chandra pair $(\mathfrak{g}, K)$. We will need a slightly more complicated version of Harish-Chandra descent, known as Gelfand-Kazhdan descent, [51, Def. 2.17].

Remark 1.3.1. The Gelfand-Kazhdan descent considered here generalizes that in [51, §2.4] in two ways. First, we work with supermanifolds. Second, our descent allows for general symplectic manifolds rather than just cotangent bundles. The symplectic case is also studied in [12].

1.3.1 Harish-Chandra Pair

We define the Harish-Chandra pair we will use for our super-Gelfand-Kazhdan descent.

Definition 1.3.2. A *super-Harish-Chandra* pair (sHC pair) is a pair $(\mathfrak{g}, K)$ where $\mathfrak{g}$ is a Lie superalgebra and K is a Lie supergroup together with

- an injective Lie superalgebra map $i \colon \mathrm{Lie}(K) \to \mathfrak{g}$
- an action of K on $\mathfrak{g}$, $\rho_K \colon K \to \mathrm{Aut}(\mathfrak{g})$

such that the action of $\mathrm{Lie}(K)$ on $\mathfrak{g}$ induced by ρ_K,

$$\mathrm{Lie}(\rho_K) \colon \mathrm{Lie}(K) \to \mathrm{Der}(\mathfrak{g}),$$

is the adjoint action induced from the embedding i.

In the purely even case, when $\mathfrak{g}$ is an ordinary Lie algebra and K is an ordinary Lie group, this recovers the usual (non-super) definition of an HC pair.

Definition 1.3.3. A *morphism of super-Harish-Chandra pairs* $(\mathfrak{f}, f) \colon (\mathfrak{g}, K) \to (\mathfrak{g}', K')$ is

- a map of Lie superalgebras $\mathfrak{f} \colon \mathfrak{g} \to \mathfrak{g}'$ and
- a map of Lie supergroups $f \colon K \to K'$

such that the diagram of Lie superalgebras

$$\begin{array}{ccc} \mathrm{Lie}(K) & \xrightarrow{\mathrm{Lie}(f)} & \mathrm{Lie}(K)' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ \mathfrak{g} & \xrightarrow[\mathfrak{f}]{} & \mathfrak{g}' \end{array}$$

commutes.

Example 1.3.4. The category of sHC pairs has an initial object $(0, e)$, where 0 is the 0-dimensional Lie superalgebra and e is the 0-dimensional Lie supergroup consisting of the identity point.

Example 1.3.5. Let K be a Lie supergroup. Then $(\mathrm{Lie}(K), K)$ is sHC pair, see [61, Thm. 3.5].

Example 1.3.6. Let G and K be Lie supergroups. Let $\iota \colon K' \subset K$ be a closed sub-supergroup. There is a unique sHC pair structure on $(\mathrm{Lie}(G), K')$ so that

$$(\mathrm{Id}_{\mathrm{Lie}(G)}, \iota) \colon (\mathrm{Lie}(G), K') \to (\mathrm{Lie}(G), K)$$

is a morphism of super-Harish-Chandra pairs. This is [51, Ex. 1.2].

More generally, if $(\mathfrak{g}, K)$ is an sHC pair and $K' \subset K$ is a closed sub-supergroup, then there is a unique sHC pairs structure on $(\mathfrak{g}, K')$ so that $(\mathrm{Id}_{\mathfrak{g}}, \iota)$ is a morphism of sHC pairs.

Example 1.3.7. Let $(\mathfrak{g}, K)$ be an HC pair. Given a central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ that is split over $\mathrm{Lie}(K)$, the pair $(\hat{\mathfrak{g}}, K)$ is an HC pair. This is in [7, §2.1.1].

The following lemmas allows us to produce more examples of sHC pairs.

Lemma 1.3.8. *Let $(\mathfrak{g}, K)$ be an sHC pair. Let $j\colon \mathfrak{g}' \subset \mathfrak{g}$ be a sub-Lie superalgebra. If the injective map $\mathrm{Lie}(K) \to \mathfrak{g}$ factors through a map $j'\colon \mathrm{Lie}(K) \to \mathfrak{g}'$, then there is a unique sHC pair structure on $(\mathfrak{g}', K)$ so that (j', Id_K) is a morphism of sHC pairs.*

Proof. Since $\mathrm{Lie}(K) \to \mathfrak{g}$ is injective, so is the factored map $j'\colon \mathrm{Lie}(K) \to \mathfrak{g}'$. To produce an action $\rho'_K\colon K \to \mathrm{Aut}(\mathfrak{g}')$ of K on $\mathfrak{g}$', note that the adjoint action of $\mathrm{Lie}(K)$ on $\mathfrak{g}$ (via the embedding $i\colon \mathrm{Lie}(K) \to \mathfrak{g}$) may be restricted to the adjoint action of $\mathrm{Lie}(K)$ on $\mathfrak{g}'$ (via the embedding j). Since $(\mathfrak{g}, K)$ is an sHC pair, the adjoint action of $\mathrm{Lie}(K)$ on $\mathfrak{g}$ is given by $\mathrm{Lie}(\rho_K)$. Thus, for $x \in \mathrm{Lie}(K)$ and $g \in \mathfrak{g}'$, the adjoint action of x on g is given by the formula

$$x(g) = \frac{d}{dt}|_{t=0} \exp(tx) \cdot g.$$

This is exactly the formula for the derivative of an action $\rho'_K\colon K \to \mathrm{Aut}(\mathfrak{g}')$. Thus $(\mathfrak{g}', K)$ is an sHC pair. The pair (j', Id_K) is a morphism of sHC pairs by construction. □

The following geometric incarnation of Example 1.3.5 will be the motivation from which we will construct our sHC pair of interest.

Example 1.3.9. Let $\mathbb{X}$ be a supermanifold. Then $(\mathrm{Vect}(\mathbb{X}), \mathrm{Diff}(\mathbb{X}))$ is almost an sHC pair. As $\mathrm{Diff}(\mathbb{X})$ is infinite-dimensional, this is not technically an example of an sHC pair. However, there is an injective map

$$\mathrm{Diff}(\mathbb{X}) \to \mathrm{Aut}(\mathcal{O}_{\mathbb{X}})$$

and one can think of the Lie algebra of $\mathrm{Aut}(\mathcal{O}_{\mathbb{X}})$ as being vector fields on $\mathbb{X}$,

$$\mathrm{Vect}(\mathbb{X}) = \mathrm{Der}(\mathcal{O}_{\mathbb{X}})\text{“} \simeq \text{”}\mathrm{Lie}(\mathrm{Aut}(\mathcal{O}_{\mathbb{X}})).$$

If $\mathbb{X}$ is an affine space, we can make a related precise statement. The linear diffeomorphisms $\mathrm{GL}(\mathbb{X})$ of $\mathbb{X}$ form a sub-supergroup of $\mathrm{Aut}(\mathcal{O}_{\mathbb{X}})$. Now, $\mathrm{GL}(\mathbb{X})$ is a Lie supergroup, and $(\mathrm{Vect}(\mathbb{X}), \mathrm{GL}(\mathbb{X}))$ is an sHC pair.

Example 1.3.10. Let $\mathbb{M} = \mathbb{R}^{2n|r}$ be the symplectic supermanifold with symplectic form ω_Q, as in Example 1.2.8. Then $\mathrm{Sp}(2n|a,b)$ is a closed sub-supergroup of $\mathrm{GL}(\mathbb{M})$, where (a,b) is the signature of Q, see Remark 1.2.12. By Example 1.3.6, we get an sHC pair

$$(\mathrm{Vect}(\mathbb{R}^{2n|r}), \mathrm{Sp}(2n|a,b)).$$

Recall the Lie sub-superalgebra of symplectic vector fields $j\colon \mathrm{Vect}^{\mathrm{symp}}(\mathbb{M},\omega) \subset \mathrm{Vect}(\mathbb{M})$ from Definition 1.2.20.

Corollary 1.3.11. *There is a unique sHC pair structure on* $(\mathrm{Vect}^{\mathrm{symp}}(\mathbb{M},\omega), \mathrm{Sp}(2n|a,b))$ *so that* (j, Id) *is a morphism of sHC pairs.*

Proof. This follows from Lemma 1.3.8. □

We would like to mimic the above story for $\mathbb{R}^{2n|r}$ replaced with the formal super-disk.

Definition 1.3.12. Let $\mathbb{X}$ be a supermanifold and $x \in \mathbb{X}$. Let $\mathcal{O}_{\mathbb{X},x}$ denote the superalgebra of germs of functions at x. Let $\mathfrak{m}_x$ be the ideal of functions vanishing at x. The ring of functions of the *formal neighborhood* of $\mathbb{X}$ at x is the limit

$$\widehat{\mathcal{O}}_{\mathbb{X},x} = \lim_i \mathcal{O}_{\mathbb{X},x}/\mathfrak{m}_x^i.$$

The Lie superalgebra of *formal vector fields* of $\mathbb{X}$ at x is the Lie superalgebra of derivations

$$\widehat{\mathrm{Vect}}_x(\mathbb{X}) = \mathrm{Der}(\widehat{\mathcal{O}}_{\mathbb{X},x}).$$

Remark 1.3.13. One can consider a subgroup $\mathrm{Aut}^{\mathrm{filt}}(\widehat{\mathcal{O}}_{\mathbb{X},x})$ of $\mathrm{Aut}(\widehat{\mathcal{O}}_{\mathbb{X},x})$ of filtration preserving automorphisms. This subgroup can be considered as a pro-Lie supergroup. Its Lie superalgebra consists of those vector fields that vanish at x. We have a non-canonical equivalence

$$\mathrm{Lie}(\mathrm{Aut}^{\mathrm{filt}}(\widehat{\mathcal{O}}_{\mathbb{X},x})) \oplus \mathbb{R}^{2n|r} \simeq \widehat{\mathrm{Vect}}_x(\mathbb{X}), \tag{1.1}$$

where $\mathbb{X}$ has dimension $2n|r$. This equivalence is given informally by taking a pair (v_0, y) of a vector field that vanishes at x and a vector $y \in \mathbb{R}^{2n|r}$ to the vector field that looks like v_0 translated by y. See [51, §2.1.1] for the non-super analogue.

Lemma 1.3.14. *The pair* $(\widehat{\mathrm{Vect}}_x(\mathbb{X}), \mathrm{Aut}^{\mathrm{filt}}(\widehat{\mathcal{O}}_{\mathbb{X},x}))$ *has the structure of a pro-sHC pair.*

Proof. This follows from the pro-version of Example 1.3.6. Indeed, $\widehat{\mathrm{Vect}}_x(\mathbb{X}) = \mathrm{Der}(\widehat{\mathcal{O}}_{\mathbb{X},x})$ is the Lie algebra of $\mathrm{Aut}(\widehat{\mathcal{O}}_{\mathbb{X},x})$, and filtration preserving automorphisms are a sub-pro-Lie group. □

As in the non-formal case, we would like to restrict to symplectic vector fields and linear symplectomorphisms.

Lemma 1.3.15. *Let* $(\mathbb{M}, \omega)$ *be a symplectic manifold and* $x \in \mathbb{M}$ *a point. The Poisson superalgebra structure on* $\mathcal{O}_{\mathbb{M}}$ *induces a Poisson superalgebra structure on* $\widehat{\mathcal{O}}_{\mathbb{M},x}$.

Proof. This follows from [39, §1.2]. □

In analogy with Lemma 1.2.24, we make the following definition.

Definition 1.3.16. Let $(\mathbb{M}, \omega)$ be a symplectic supermanifold and $x \in \mathbb{M}$ a point. The Lie superalgebra of *symplectic formal vector fields* on $\mathbb{M}$ at x is the Lie superalgebra of Poisson derivations of the Poisson superalgebra $(\widehat{\mathcal{O}}_{\mathbb{M},x}, \{-,-\}_\omega)$,

$$\widehat{\mathrm{Vect}}_x^{\mathrm{symp}}(\mathbb{M}, \omega) :\, = \mathrm{Der}_\omega(\widehat{\mathcal{O}}_{\mathbb{M},x}).$$

Consider the case when $\mathbb{M}$ is affine. That is, take a symplectic supermanifold of the form $(\mathbb{R}^{2n|r}, \omega_Q)$ from Example 1.2.8. We have a formal version of Corollary 1.3.11.

Lemma 1.3.17. *There is a unique sHC pair structure on*

$$(\widehat{\mathrm{Vect}}_0^{\mathrm{symp}}(\mathbb{R}^{2n|r}, \omega_Q), \mathrm{Sp}(2n|a, b))$$

so that the inclusion of symplectic vector fields and the inclusion of linear symplectic automorphisms induce morphisms of sHC pairs

$$\begin{aligned}\left(\widehat{\mathrm{Vect}}_0^{\mathrm{symp}}(\mathbb{R}^{2n|r},\omega_Q),\mathrm{Sp}(2n|a,b)\right) &\to \left(\widehat{\mathrm{Vect}}_0(\mathbb{R}^{2n|r},\omega_Q),\mathrm{Sp}(2n|a,b)\right)\\ &\to \left(\widehat{\mathrm{Vect}}_0(\mathbb{R}^{2n|r},\omega_Q),\mathrm{Aut}^{\mathrm{filt}}(\widehat{\mathcal{O}}_{\mathbb{R}^{2n|r},0})\right).\end{aligned}$$

Proof. This follows from the pro-version of Lemma 1.3.8 and Example 1.3.6. □

Convention 1.3.18. For the symplectic supermanifold $(\mathbb{R}^{2n|r},\omega_Q)$ with Q having signature (a,b), we set the following notation:

- $\mathrm{Aut}_{2n|a,b}$ is the pro-supergroup of filtration preserving automorphisms of Poisson superalgebras $\mathrm{Aut}^{\mathrm{filt}}_{\mathrm{Pois}}(\widehat{\mathcal{O}}_{\mathbb{R}^{2n|r},0},\omega_Q)$,
- $\widehat{\mathcal{O}}_{2n|a,b}$ is the Poisson superalgebra $(\widehat{\mathcal{O}}_{\mathbb{R}^{2n|r},0},\{-,-\}_{\omega_Q})$, and
- $\mathfrak{g}_{2n|a,b}$ is the Lie superalgebra $\widehat{\mathrm{Vect}}_0^{\mathrm{symp}}(\mathbb{R}^{2n|r},\omega_Q)$.

The pair $(\mathfrak{g}_{2n|a,b},\mathrm{Sp}(2n|a,b))$ will be the main sHC pair of interest to us.

1.3.2 Category of Manifolds

In the Borel construction, one considers the category of principal bundles. Analogously, we will make use of a category of principal $(\mathfrak{g},K)$-bundles.

Following [51, Def. 1.5], we define principal bundles for Harish-Chandra pairs as follows.

Definition 1.3.19. Let $\mathbb{X}$ be a supermanifold. Let $(\mathfrak{g},K)$ be a super-Harish-Chandra pair. A $(\mathfrak{g},K)$-*principal bundle* over $\mathbb{X}$ is a principal K-bundle $P\to\mathbb{X}$ together with a K-invariant $\mathfrak{g}$-valued 1-form $\nu\in\Omega^1(P;\mathfrak{g})$ such that

- for all $a\in\mathrm{Lie}(K)$, we have $\nu(\zeta_a)=a$ where ζ_a denotes the induced vector field on P, and
- ν satisfies the Maurer-Cartan equation

$$d_{\mathrm{dR}}\omega+\frac{1}{2}[\nu,\nu]=0.$$

We let $\mathsf{Bun}^{\text{flat}}_{(\mathfrak{g},K)}$ denote the category of $(\mathfrak{g},K)$-principal bundles and morphisms $(P,\omega)\to(P',\omega')$ bundle maps $F\colon P\to P'$ so that $F^*\omega'=\omega$.

Example 1.3.20. For the sHC pair $(\mathrm{Lie}(K),K)$, the notion of a principal $(\mathrm{Lie}(K),K)$-bundle recovers the notion of a principal K-bundle with connection.

Example 1.3.21. If $\mathfrak{g}$ is the Lie algebra of a Lie group G, and $K\subset G$ is a closed subgroup, then a principal $(\mathfrak{g},K)$-bundle is the same as a principal G-bundle with connection, together with a reduction of structure group from G to K. See [51, Pg. 8].

As noted in Convention 1.3.18, the sHC pair of interest to us is $(\mathfrak{g}_{2n|a,b},\mathrm{Sp}(2n|a,b))$. Let $(\mathbb{M},\omega)$ be a symplectic supermanifold of type $(2n|a,b)$. Recall from Lemma 1.2.30, that there is a principal $\mathrm{Sp}(2n|a,b)$-bundle $\mathrm{Fr}^{\mathrm{Sp}(2n|a,b)}_{\mathbb{M}}$ on $\mathbb{M}$. We will construct a $(\mathfrak{g}_{2n|a,b},\mathrm{Sp}(2n|a,b))$-principal bundle structure on $\mathrm{Fr}^{\mathrm{Sp}(2n|a,b)}_{\mathbb{M}}$. We will do this in two steps:

Step 1. first we construct a principal $\mathrm{Aut}_{2n|a,b}$-bundle $\mathbb{M}^{\text{coor}}$ on $\mathbb{M}$, and give it a $\mathfrak{g}_{2n|a,b}$-valued connection;

Step 2. second we move this structure from $\mathbb{M}^{\text{coor}}$ to $\mathrm{Fr}^{\mathrm{Sp}(2n|a,b)}_{\mathbb{M}}$ using a "formal exponential."

The results of this section are summarized in the following theorem.

Theorem 1.3.22. *There is a category with objects symplectic supermanifolds $(\mathbb{M},\omega)$ of type $(2n|a,b)$ equipped with a $(\mathfrak{g}_{2n|a,b},\mathrm{Sp}(2n|a,b))$-bundle structure on $\mathrm{Fr}^{\mathrm{Sp}(2n|a,b)}_{\mathbb{M}}$, which we denote $\mathsf{sGK}^{=}_{2n|a,b}$. A choice of formal exponential defines a lift of an object in $\mathsf{sMfld}_{2n|2,b}$ to an object of $\mathsf{sGK}^{=}_{2n|a,b}$.*

Moreover, there is a functor

$$\mathsf{sGK}^{=}_{2n|a,b}\to\mathsf{Bun}^{\text{flat}}_{(\mathfrak{g}_{2n|a,b},\mathrm{Sp}(2n|a,b))}$$

living above $\mathsf{sMfld}_{2n|a,b}$.

1.3.2.1 Formal Symplectic Coordinate Bundles

Given a symplectic supermanifold $(\mathbb{M},\omega)$ of type $(2n|a,b)$, we will construct a $\mathrm{Aut}_{2n|a,b}$-bundle $\mathbb{M}^{\text{coor}}$ on $\mathbb{M}$ with a $\mathfrak{g}_{2n|a,b}$-valued connection. Since $\mathrm{Aut}_{2n|a,b}$ is a pro-supergroup, $\mathbb{M}^{\text{coor}}$ will be a pro-supermanifold.

For $x \in \mathbb{M}$, by Lemma 1.3.15, $\widehat{\mathcal{O}}_{\mathbb{M},x}$ has the structure of a Poisson superalgebra. In analogy with [51, §2.2.1], we define the *formal symplectic coordinate bundle* of $(\mathbb{M}, \omega)$ to be the bundle with fiber over $x \in \mathbb{M}$ given by the group of isomorphisms of Poisson superalgebras,

$$\mathbb{M}_x^{\mathrm{coor}} = \mathrm{Isom}_{\mathrm{Pois}}\left(\widehat{\mathcal{O}}_{\mathbb{M},x}, \widehat{\mathcal{O}}_{2n|a,b}\right).$$

Since $\mathbb{M}$ is of type $(2n|a,b)$, we have an isomorphism of Poisson algebras $\widehat{\mathcal{O}}_{\mathbb{M},x} \to \widehat{\mathcal{O}}_{2n|a,b}$, so that $\mathbb{M}_x^{\mathrm{coor}}$ is nonempty. Moreover, this implies that $\mathbb{M}_x^{\mathrm{coor}}$ is non-canonically isomorphic to

$$\mathrm{Isom}_{\mathrm{Pois}}\left(\widehat{\mathcal{O}}_{2n|a,b}, \widehat{\mathcal{O}}_{2n|a,b}\right) = \mathrm{Aut}_{2n|a,b}.$$

See Convention 1.3.18.

As in [12, §3.1], one can construct $\mathbb{M}^{\mathrm{coor}}$ by the functor $\mathcal{F}_{\mathbb{M}^{\mathrm{coor}}}$ it represents. For T another supermanifold with a map $\eta\colon T \to \mathbb{M}$, let $T_\eta \subset T \times \mathbb{M}$ be the submanifold of pairs $(t, \eta(t))$. Let $\widehat{\mathcal{O}}_{\mathbb{M},\eta}$ be the ring of formal germs of functions on $T \times \mathbb{M}$ near T_η. The functor $\mathbb{M}^{\mathrm{coor}}$ represents sends an affine space T to the set of pairs

$$\mathcal{F}_{\mathbb{M}^{\mathrm{coor}}}(T) = \{(\eta, \phi : \eta\colon T \to \mathbb{M}, \phi\colon \widehat{\mathcal{O}}_{\mathbb{M},\eta} \cong \mathcal{O}_T \hat{\otimes} \widehat{\mathcal{A}}_{2n|a,b}\}.$$

Define an action of $\mathrm{Aut}_{2n|a,b}$ on $\mathbb{M}_x^{\mathrm{coor}}$ by post-composition. Under this action, $\mathbb{M}^{\mathrm{coor}} \to \mathbb{M}$ becomes a principal $\mathrm{Aut}_{2n|a,b}$-bundle.

Analogous to how a smooth map induces a map on frame bundles, we have the following.

Lemma 1.3.23. *Let $f\colon \mathbb{M}_1 \to \mathbb{M}_2$ be a morphism in* $\mathsf{sMfld}_{2n|a,b}$. *There is a morphism*

$$f^{\mathrm{coor}}\colon \mathbb{M}_1^{\mathrm{coor}} \to \mathbb{M}_2^{\mathrm{coor}}$$

of $\mathrm{Aut}_{2n|a,b}$*-bundles.*

Proof. We produce a map $f^{\mathrm{coor}}\colon \mathbb{M}_1^{\mathrm{coor}} \to \mathbb{M}_2^{\mathrm{coor}}$ from a natural transformation between the functors $\mathcal{F}_{\mathbb{M}_1^{\mathrm{coor}}}$ and $\mathcal{F}_{\mathbb{M}_2^{\mathrm{coor}}}$ that these spaces represent. Fix a supermanifold T. Let $(\eta, \phi) \in \mathcal{F}_{\mathbb{M}_1^{\mathrm{coor}}}(T)$. Set η' to be the composite

$$T \xrightarrow{\eta} \mathbb{M}_1 \xrightarrow{f} \mathbb{M}_2.$$

Then $f^*\colon \mathcal{O}_{\mathbb{M}_2} \to \mathcal{O}_{\mathbb{M}_1}$ descends to an equivalence on completions

$$\bar{f}^*\colon \widehat{\mathcal{O}}_{\mathbb{M}_2,\eta'} \to \widehat{\mathcal{O}}_{\mathbb{M}_1,\eta}$$

since $(\mathrm{Id}_T \times f)(T_\eta) = T_{\eta'}$. Thus, $(\eta', \phi \circ \bar{f}^*)$ defines an element of $\mathcal{F}_{\mathbb{M}_2^{\mathrm{coor}}}(T)$. This assignment on objects extends to a natural transformation, resulting in a morphism on representing spaces $f^{\mathrm{coor}}\colon \mathbb{M}_1^{\mathrm{coor}} \to \mathbb{M}_2^{\mathrm{coor}}$. □

As in [12, Lem. 3.2] and [51, Def. 2.2], we have the following.

Lemma 1.3.24. *Let $(\mathbb{M}, \omega)$ be a symplectic supermanifold of type $(2n|a,b)$. There is a connection 1-form $\nu^{\mathrm{coor}} \in \Omega^1(\mathbb{M}^{\mathrm{coor}}; \mathfrak{g}_{2n|a,b})$.*

Proof. The principal $\mathrm{Aut}_{2n|a,b}$-bundle $\pi\colon \mathbb{M}^{\mathrm{coor}} \to \mathbb{M}$ determines a short exact sequence of pro-vector bundles on the pro-supermanifold $\mathbb{M}^{\mathrm{coor}}$,

$$0 \to \ker(d\pi) \to T\mathbb{M}^{\mathrm{coor}} \to T\mathbb{M} \to 0.$$

The kernel $\ker(d\pi)$ is isomorphic to the trivial $\mathrm{Lie}(\mathrm{Aut}_{2n|a,b})$-bundle on $\mathbb{M}^{\mathrm{coor}}$. Hence, at each point $(x, \varphi) \in \mathbb{M}^{\mathrm{coor}}$, we get a short exact sequence

$$0 \to \mathrm{Lie}(\mathrm{Aut}_{2n|a,b}) \to T_{(x,\varphi)}\mathbb{M}^{\mathrm{coor}} \to T_x\mathbb{M} \to 0.$$

The isomorphism φ determines an equivalence $T_x\mathbb{M} \cong \mathbb{R}^{2n|r}$. This gives us an equivalence

$$\mathrm{Lie}(\mathrm{Aut}_{2n|a,b}) \oplus \mathbb{R}^{2n|r} \simeq T_{(x,\varphi)}\mathbb{M}^{\mathrm{coor}}.$$

Using the symplectic analogue of (1.1), we get an equivalence

$$\nu^{\mathrm{coor}}_{x,\varphi}\colon T_{(x,\varphi)}\mathbb{M}^{\mathrm{coor}} \xrightarrow{\sim} \mathfrak{g}_{2n|a,b}.$$

Sending a point $(x, \varphi) \in \mathbb{M}^{\mathrm{coor}}$ to the map $\nu^{\mathrm{coor}}_{x,\varphi}$ defines a one-form

$$\nu^{\mathrm{coor}} \in \Omega^1(\mathbb{M}^{\mathrm{coor}}, \mathfrak{g}_{2n|a,b}). \qquad \square$$

Corollary 1.3.25. *The connection ν^{coor} is flat.*

Proof. By construction, ν^{coor} is the inverse of a Lie superalgebra map

$$\mathfrak{g}_{2n|a,b} \to \mathrm{Vect}(\mathbb{M}^{\mathrm{coor}}),$$

and hence satisfies the Maurer-Cartan equation. $\square$

This completes Step 1 of our proof of Theorem 1.3.22.

Corollary 1.3.26. *The formal symplectic coordinate bundle $\mathbb{M}^{\mathrm{coor}} \to \mathbb{M}$ is a $(\mathfrak{g}_{2n|a,b}, \mathrm{Aut}_{2n|a,b})$-principal bundle.*

One can compare this to [12, Lem. 3.2] for the non-super version.

1.3.2.2 Formal Exponentials

We define the notion of a "formal exponential" which will allow us to move the structure defined in Corollary 1.3.26 from the formal symplectic coordinate bundle, to the symplectic frame bundle.

As in [51, Def. 2.4], we define a formal exponential as follows.

Definition 1.3.27. Let $(\mathbb{M}, \omega)$ be a symplectic supermanifold of type $(2n|a, b)$. A *formal exponential* on $\mathbb{M}$ is a section of the $\mathrm{Aut}_{2n|a,b}/\mathrm{Sp}(2n|a, b)$-bundle

$$\mathrm{Exp}(\mathbb{M}) := \mathbb{M}^{\mathrm{coor}}/\mathrm{Sp}(2n|a, b) \to \mathbb{M}.$$

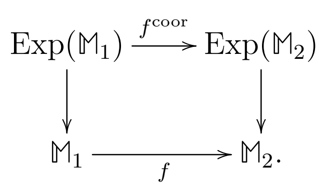
Lemma 1.3.28. *The space $\mathrm{Aut}_{2n|a,b}/\mathrm{Sp}(2n|a, b)$ is contractible, and thus formal exponentials always exist.*

Proof. Note that $\widehat{\mathcal{O}}_{2n|a,b}/\mathfrak{m}^2$ consists of linear functions on $\mathbb{R}^{2n|a+b}$. The image of $\mathrm{Aut}_{2n|a,b}$ in $\mathrm{Aut}_{\mathrm{Pois}}(\widehat{\mathcal{O}}_{2n|a,b}/\mathfrak{m}^2)$ is thus $\mathrm{Sp}(2n|a, b)$. Let $\mathrm{Aut}^{+}_{2n|a,b}$ be the kernel of the projection

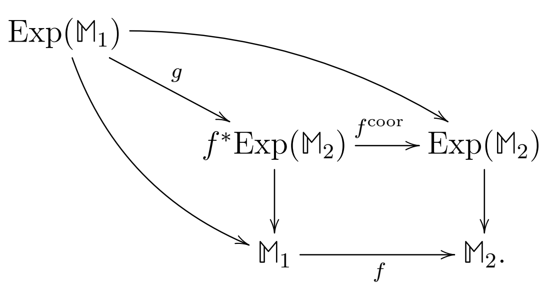
$$1 \to \mathrm{Aut}^+_{2n|a,b} \to \mathrm{Aut}_{2n|a,b} \to \mathrm{Sp}(2n|a,b) \to 1.$$

Since $\mathrm{Aut}^+_{2n|a,b}$ is pro-nilpotent, it is contractible. In fact, $\mathrm{Aut}^+_{2n|a,b}$ is a pro-vector space. The result follows. □

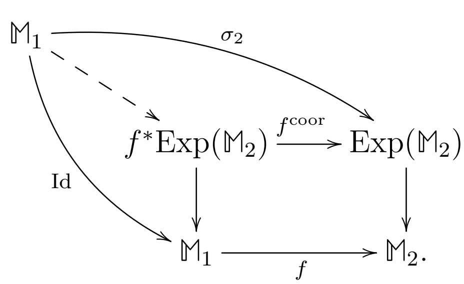
We will use the following to define a morphism between formal exponentials. Let $f\colon \mathbb{M}_1 \to \mathbb{M}_2$ be a morphism in $\mathsf{sMfld}_{2n|a,b}$. The map f^{coor} from Lemma 1.3.23 respects the action of $\mathrm{Aut}_{2n|a,b}$ on both sides by post-composition, and therefore descends to a map $\mathrm{Exp}(\mathbb{M}_1) \to \mathrm{Exp}(\mathbb{M}_2)$ of pro-supermanifolds, and a commuting diagram



We obtain the following diagram involving the pullback bundle $f^*\mathrm{Exp}(\mathbb{M}_2)$,



Given a formal exponential σ_1 on $\mathbb{M}_1$, we obtain a section of $f^{\mathrm{Exp}}(\mathbb{M}_2)$ by $g \circ \sigma_1$. On the other hand, given a formal exponential σ_2 on $\mathbb{M}_2$, we get a section of $f^*\mathrm{Exp}(\mathbb{M}_2)$ from the diagram



Definition 1.3.29. Let $\mathsf{sGK}_{2n|a,b}$ denote the category with

- objects are pairs $((\mathbb{M},\omega),\sigma)$ where $(\mathbb{M},\omega)$ is a symplectic supermanifold of type $(2n|a,b)$ and σ is a formal exponential on $\mathbb{M}$, and

- a morphism $(\mathbb{M}_1,\sigma_1) \to (\mathbb{M}_2,\sigma_2)$ is a morphism $f\colon \mathbb{M}_1 \to \mathbb{M}_2$ in $\mathsf{sMfld}_{2n|a,b}$, and a homotopy class of paths in the space $\Gamma(\mathbb{M}_1, f^*\mathrm{Exp}(\mathbb{M}_2))$ between the sections defined by σ_1 and σ_2.

One should compare the following with the discussion around [51, Def. 2.11].

Lemma 1.3.30. *The forgetful functor* $\mathsf{sGK}_{2n|a,b} \to \mathsf{sMfld}_{2n|a,b}$ *is an equivalence of categories.*

Proof. By our definition of morphism spaces in $\mathsf{sGK}_{2n|a,b}$, it suffices to show that for $f\colon \mathbb{M}_1 \to \mathbb{M}_2$ a morphism in $\mathsf{sMfld}_{2n|a,b}$, the space $\Gamma(\mathbb{M}_1, f^*\mathrm{Exp}(\mathbb{M}_2))$ is contractible. The bundle $\mathrm{Exp}(\mathbb{M}_2)$, and hence $f^*\mathrm{Exp}(\mathbb{M}_2)$, have fiber $\mathrm{Aut}_{2n|a,b}/\mathrm{Sp}(2n|a,b)$. By Lemma 1.3.28, this fiber is contractible. The space of sections of a bundle with contractible fiber is contractible. This completes the proof. □

We introduce a stricter variation on $\mathsf{sGK}_{2n|a,b}$.

Variation 1.3.31. Let $\mathsf{sGK}^{=}_{2n|a,b}$ be the category with

- objects: pairs $((\mathbb{M},\omega),\sigma)$ where $(\mathbb{M},\omega)$ is a symplectic supermanifold of type $(2n|a,b)$ and σ is a formal exponential on $\mathbb{M}$, and

- morphisms: a map $(\mathbb{M}_1,\sigma_1) \to (\mathbb{M}_2,\sigma_2)$ is a morphism $f\colon \mathbb{M}_1 \to \mathbb{M}_2$ in $\mathsf{sMfld}_{2n|a,b}$ such that the diagram

$$\begin{array}{ccc} \mathrm{Exp}(\mathbb{M}_1) & \xrightarrow{f^{\mathrm{coor}}} & \mathrm{Exp}(\mathbb{M}_2) \\ {\scriptstyle\sigma_1}\uparrow & & \uparrow{\scriptstyle\sigma_2} \\ \mathbb{M}_1 & \xrightarrow[f]{} & \mathbb{M}_2 \end{array}$$

  commutes.

Note that the condition on morphisms is equivalent to asking that $f\colon \mathbb{M}_1 \to \mathbb{M}_2$ be such that the sections in $\Gamma(\mathbb{M}_1, f^*\mathrm{Exp}(\mathbb{M}_2))$ defined by σ_1 and σ_2 are equal, as opposed to having a path between them.

Remark 1.3.32. There is an evident functor $\mathsf{sGK}^{=}_{2n|a,b} \to \mathsf{sGK}_{2n|a,b}$ that is the identity on objects and the inclusion of constant paths on morphisms, but this functor is not fully faithful.

The main use of a formal exponential is to put the structure of a $(\mathfrak{g}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$-bundle structure on the symplectic frame bundle. The following is analogous to [51, Prop. 2.6].

Proposition 1.3.33. *Let $\sigma \in \Gamma(\mathbb{M}, \mathrm{Exp}(\mathbb{M}))$ be a formal exponential. Then*

- *σ lifts to a $\mathrm{Sp}(2n|a,b)$-equivariant map $\tilde{\sigma}\colon \mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \to \mathbb{M}^{\mathrm{coor}}$,*

$$\begin{array}{ccc} \mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} & \xrightarrow{\tilde{\sigma}} & \mathbb{M}^{\mathrm{coor}} \\ \downarrow & & \downarrow \\ \mathbb{M} & \xrightarrow{\sigma} & \mathrm{Exp}(\mathbb{M}), \end{array}$$

- *and $\tilde{\sigma}^*(\nu^{\mathrm{coor}})$ is a flat $\mathfrak{g}_{2n|a,b}$-valued connection on $\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}$. With this connection, $\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}$ is a $(\mathfrak{g}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$-bundle.*

- *Any two choices of a formal exponential determine $(\mathfrak{g}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$-bundle structures on $\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}$ that are gauge-equivalent.*

Proof. The first claim follows from the the definition of $\mathrm{Exp}(\mathbb{M})$ as $\mathbb{M}^{\mathrm{coor}}/\mathrm{Sp}(2n|a,b)$ and the equivalence $\mathbb{M} \simeq \mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}/\mathrm{Sp}(2n|a,b)$. The second claim is just the statement that flat connections pullback, but in the context of pro-supermanifolds. The gauge-equivalence in the third claim can be produced using the contractibility of the space of formal exponentials. □

Remark 1.3.34. In the language of [12, §2.4], Proposition 1.3.33 shows that one can think of a formal exponential on $\mathbb{M}$ as determining a "lift" (similar to a reduction of structure group) of the $(\mathfrak{g}_{2n|a,b}, \mathrm{Aut}_{2n|a,b})$-bundle $(\mathbb{M}^{\mathrm{coor}}, \nu^{\mathrm{coor}})$ from Corollary 1.3.26 to the sHC pair $(\mathfrak{g}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$. Suppose $\mathbb{M}$ is purely even so that $M = \mathbb{M}$ is an ordinary symplectic manifold. In [12, Lem. 3.4], Bezrukavnikov-Kaledin describe an HC pair they call $(\mathrm{Der}(D), \mathrm{Aut}(D))$ (where D is the Weyl algebra) and show that the set of lifts of M^{coor} to a $(\mathrm{Der}(D), \mathrm{Aut}(D))$-bundle is in bijective correspondence with isomorphism classes of deformation quantizations $Q(M, \omega)$. See Remark 1.5.15 below and Remark 1.1.2 above for further discussion in this direction.

Corollary 1.3.35. *There is a functor*

$$\mathrm{Fr}\colon \mathsf{sGK}^{=}_{2n|a,b} \to \mathsf{Bun}^{\mathrm{flat}}_{(\mathfrak{g}_{2n|a,b}, \mathrm{Sp}(2n|a,b))}$$

sending a pair $((\mathbb{M}, \omega), \sigma)$ *to the symplectic frame bundle* $\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}$ *with* $(\mathfrak{g}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$*-bundle structure induced from* σ*.*

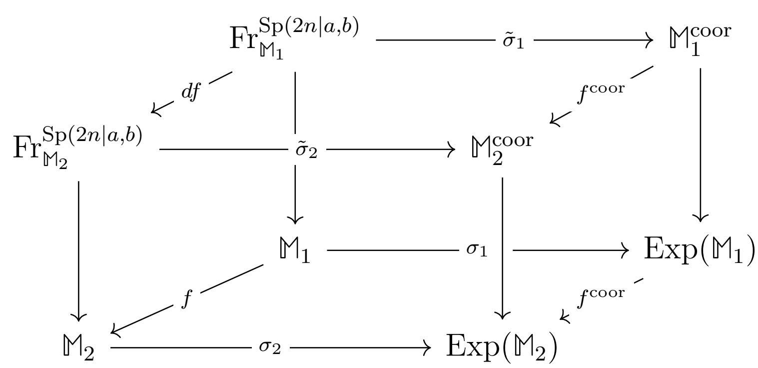
Proof. This functor is defined on objects by Proposition 1.3.33. We need to define Fr on morphisms. Let $(\mathbb{M}_1, \sigma_1)$ and $(\mathbb{M}_2, \sigma_2)$ be two objects in $\mathsf{sGK}^{=}_{2n|a,b}$, and $f\colon \mathbb{M}_1 \to \mathbb{M}_2$ a morphism between them. The map f induces a map of $\mathrm{Sp}(2n|a,b)$-bundles

$$df\colon \mathrm{Fr}_{\mathbb{M}_1}^{\mathrm{Sp}(2n|a,b)} \to \mathrm{Fr}_{\mathbb{M}_2}^{\mathrm{Sp}(2n|a,b)}.$$

For df to be a map of $(\mathfrak{g}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$-bundles, we need the $\mathfrak{g}_{2n|a,b}$-valued connection on $\mathrm{Fr}_{\mathbb{M}_2}^{\mathrm{Sp}(2n|a,b)}$ to pullback to the one on $\mathrm{Fr}_{\mathbb{M}_1}^{\mathrm{Sp}(2n|a,b)}$. By definition of the functor Fr on objects, the connection on $\mathrm{Fr}_{\mathbb{M}_i}^{\mathrm{Sp}(2n|a,b)}$ is $\tilde{\sigma}_i^*(\nu_{\mathbb{M}_i}^{\mathrm{coor}})$, for $i = 1, 2$. Thus it suffices to show that there is an equality

$$(df)^* \tilde{\sigma}_2^*(\nu_{\mathbb{M}_2}^{\mathrm{coor}}) = \tilde{\sigma}_1^*(\nu_{\mathbb{M}_1}^{\mathrm{coor}}).$$

This follows from the commutativity of the cube



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□

1.3.2.3 Cotangent Bundle Example

We discuss formal exponentials on symplectic supermanifolds of the form in Example 1.2.14. Recall from Theorem 1.2.15 that symplectic supermanifolds non-canonically look like $(E[1], \tilde{\omega})$ where $E \to M$ is a vector bundle on an ordinary symplectic manifold (M, ω), and $\tilde{\omega}$ is defined using a metric g on E and a compatible connection ∇.

Recall that a *symplectic connection* on an ordinary symplectic manifold (M, ω) is a torsion-free connection so that ω is constant with respect to the covariant derivative, see for example [14, Def. 2.1] or [46, Def. 2.1].

Lemma 1.3.36. *If $\mathbb{M} = E[1]$ is the symplectic supermanifold defined in Example 1.2.14 from the data $(M, \omega, E, g, \nabla)$, then a symplectic connection on M determines a formal exponential on $\mathbb{M}$.*

See [38, Pg. 3-4] for a description of the resulting differential on $\mathbf{desc}_{(\mathbb{M},\sigma)}(-)$.

Proof. Willwacher in [88, §2.5] has shown that a torsion-free connection on an ordinary manifold X gives a section of X^{coor}. So a torsion-free connection produces a compatible choice of, for each $x \in X$, an isomorphism

$$\Bbbk[[x_1, \ldots, x_n]] = \widehat{\mathcal{O}}_n \simeq \widehat{\mathcal{O}}_{X,x}.$$

Similarly, a symplectic connection on an ordinary symplectic manifold M gives a section of M^{coor}. So a symplectic connection produces a compatible choice of, for each $x \in M$, an isomorphism of Poisson algebras $\widehat{\mathcal{O}}_{2n} \simeq \widehat{\mathcal{O}}_{M,x}$.

In the purely odd case, a connection on a vector bundle $E \to X$ produces a compatible choice of, for each $x \in X$, an isomorphism of algebras

$$\Lambda^{\bullet}[[\theta_1, \ldots, \theta_r]] \simeq \Gamma(X, \Lambda^{\bullet} E)^{\wedge}_x.$$

Combining these, a symplectic connection on M and a metric connection on a quadratic vector bundle $E \to M$ produces a compatible choice of, for each $x \in X$, an isomorphism of Poisson super-algebras

$$\widehat{\mathcal{O}}_{2n|a,b} \simeq (\widehat{\mathcal{O}}_{E[1]})_x \simeq \Gamma(M, \Lambda^{\bullet} E)^{\wedge}_x.$$

This data is a formal exponential on $E[1]$. □

Remark 1.3.37. Given an ordinary manifold X, the symplectic manifold T^*X has a canonical symplectic connection.

Let $\pi \colon T^*X \to X$ be the projection. Consider the functor

$$T^* \colon \mathsf{VB}^{\mathrm{quad},\nabla}_{/X} \to \mathsf{VB}^{\mathrm{quad},\nabla}_{/T^*X}$$

sending a $E \to X$ to $\pi^* E \to T^* X$ and the metric and connection on E to the pullback metric and connection, respectively. Using the canonical symplectic connection on the cotangent bundle [17], Lemma 1.3.36 allows one to define a lift

$$\begin{array}{ccccc} & & & & \mathsf{sGK} \\ & & & \overset{\tilde{L}}{\nearrow} & \downarrow \\ \mathsf{VB}^{\text{quad},\nabla}_{/X} & \xrightarrow[T^*]{} & \mathsf{VB}^{\text{quad},\nabla}_{/T^*X} & \longrightarrow & \mathsf{sMfld}^{\text{Sp}} \end{array}$$

where the categories of manifolds here are not restricted to a particular type $(2n|a,b)$.

Just as the cotangent bundle T^*X has a canonical deformation quantization by differential operators on X, the lift $\tilde{L}$ will allow us to construct deformation quantizations for symplectic supermanifolds built from vector bundles over the cotangent bundle.

1.3.3 Descent Functor

We will discuss Harish-Chandra descent for the sHC pair $(\mathfrak{g}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$. After studying some monoidal properties of this descent functor, we will construct the super-Gelfand-Kazhdan descent functor that will be used in later sections.

Convention 1.3.38. Throughout this section, let

- $(\mathfrak{g}, K)$ be an sHC pair,
- $\mathsf{Mod}_{(\mathfrak{g},K)}$ denote the category of $(\mathfrak{g}, K)$-modules,
- $\mathsf{Mod}^{\text{fin}}_{(\mathfrak{g},K)}$ denote the category of finite-dimensional $(\mathfrak{g}, K)$-modules,
- $\mathsf{VB}^{\text{flat}}_{2n|a,b}$ denote the category, fibered over $\mathsf{sMfld}_{2n|a,b}$, of flat finite-dimensional vector bundles,
- $\mathsf{VB}^{\text{flat}}_{/\mathbb{M}}$ denote the category of flat finite-dimensional vector bundles over a symplectic supermanifold $\mathbb{M}$,
- $\mathsf{Pro}(\mathsf{VB}_{/\mathbb{M}})^{\text{flat}}$ denote the category of pro-objects in $\mathsf{VB}_{/\mathbb{M}}$ together with a flat connection, and
- $\mathsf{Mod}_{\Omega^\bullet}$ denote the category, fibered over $\mathsf{sMfld}_{2n|a,b}$, of symplectic supermanifolds $(\mathbb{M}, \omega)$ together with a module over the superalgebra $\Omega^\bullet_{\mathbb{M}}$.

Given a flat $(\mathfrak{g}, K)$-bundle $P \to \mathbb{M}$ with connection 1-form $\nu \in \Omega^1(P;\mathfrak{g})$ and a finite-dimensional $(\mathfrak{g}, K)$-module V, we obtain a vector bundle on $\mathbb{M}$ using the Borel construction, $P \times_K V$. We can equip $P \times_K V$ with a flat connection using ω and the action $\rho^V_{\mathfrak{g}}$ of $\mathfrak{g}$ on V as follows. The action of $\mathfrak{g}$ on V induces a map

$$\rho^V_{\mathfrak{g}}(\nu)\colon \Omega^\bullet(P;\underline{V}) \to \Omega^{\bullet+1}(P;\underline{V}) \tag{1.2}$$

defined by $\rho^V_{\mathfrak{g}}(\nu)(-) = \rho^V_{\mathfrak{g}}(\nu \wedge -)$. Now, $\nabla^{P,V} = d_{\mathrm{dR,P}} + \rho^V_{\mathfrak{g}}(\nu)$ defines a differential on the subalgebra of basic forms, and hence a flat connection on $P \times_K V$. See [51, Lem. 1.12] for the non-super case.

As in [51, Def. 1.14], given an sHC-pair $(\mathfrak{g}, K)$, *Harish-Chandra* descent is the resulting functor

$$\mathrm{desc}\colon \mathsf{Bun}^{\mathrm{flat}}_{(\mathfrak{g},K)} \times \mathsf{Mod}^{\mathrm{fin}}_{(\mathfrak{g},K)} \to \mathsf{VB}^{\mathrm{flat}}_{2n|a,b}$$

sending $(P \to \mathbb{M}, V)$ to $(P \times_K V \to \mathbb{M}, \nabla^{P,V})$.

Taking the de Rham complex of the flat vector bundle produces a functor

$$\mathbf{desc}\colon (\mathsf{Bun}^{\mathrm{flat}}_{(\mathfrak{g},K)})^{\mathrm{op}} \times \mathsf{Mod}^{\mathrm{fin}}_{(\mathfrak{g},K)} \to \mathsf{Mod}_{\Omega^\bullet}.$$

Example 1.3.39. Take $(\mathfrak{g}, K)$ to be the sHC pair $(\mathfrak{g}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$. Restricting along the functor Fr of Corollary 1.3.35, we obtain a descent functor

$$\mathsf{sGK}^{=}_{2n|a,b} \times \mathsf{Mod}^{\mathrm{fin}}_{(\mathfrak{g}_{2n|a,b},\mathrm{Sp}(2n|a,b))} \to \mathsf{VB}^{\mathrm{flat}}_{2n|a,b}.$$

1.3.3.1 Monoidal Properties of Descent

Let $(\mathfrak{g}, K)$ be an sHC pair. Restricting to a fixed $(\mathfrak{g}, K)$-bundle $(P \to \mathbb{M}, \nu)$, we have a functor

$$\mathrm{desc}_{P,\nu}\colon \mathsf{Mod}^{\mathrm{fin}}_{(\mathfrak{g},K)} \to \mathsf{VB}^{\mathrm{flat}}_{/\mathbb{M}}.$$

The category $\mathsf{Mod}^{\mathrm{fin}}_{(\mathfrak{g},K)}$ has a symmetric monoidal structure given by $\otimes_{\Bbbk}$. The category $\mathsf{VB}^{\mathrm{flat}}_{/\mathbb{M}}$ has a symmetric monoidal structure by taking tensor product of vector bundles and flat connections.

The following foundational observation allows us to deduce several nice properties of Harish-Chandra, and in particular super-Gelfand-Kazhdan, descent.

Proposition 1.3.40. *The functor* $\mathrm{desc}_{P,\nu}$ *is symmetric monoidal.*

Proof. Let $V, W \in \mathsf{Mod}^{\mathrm{fin}}_{(\mathfrak{g},K)}$. The Borel construction is symmetric monoidal,

$$P \times_K (V \otimes W) \simeq (P \times_K V) \otimes (P \times_K W),$$

as one can check on fibers. It therefore suffices to show that the connection on $\mathrm{desc}_{P,\nu}(V \otimes W)$ is the tensor product of the connection on $\mathrm{desc}_{(P,\nu)}(V)$ and on $\mathrm{desc}_{(P,\nu)}(W)$. From the construction of the connection, Equation (1.2) or [51, §1.3.2], we have $\nabla^{P,V\otimes W} = d_{\mathrm{dR},P} + \rho_{\mathfrak{g}}^{V\otimes W}$, where $\rho_{\mathfrak{g}}^{V\otimes W}(\nu)$ is defined using the action of $\mathfrak{g}$ on $V \otimes W$. Since the tensor product $V \otimes W$ is taken in $\mathsf{Mod}_{(\mathfrak{g},K)}$, we have $\rho_{\mathfrak{g}}^{V\otimes W}(\nu) = \rho_{\mathfrak{g}}^{V}(\nu) \otimes \rho_{\mathfrak{g}}^{W}(\nu)$. □

From this proposition, we will be able to deduce several corollaries of how the descent functors interact with algebraic structures.

Corollary 1.3.41. *The de Rham complex functor* $\mathbf{desc}_{(P,\nu)} \colon \mathsf{Mod}^{\mathrm{fin}}_{(\mathfrak{g},K)} \to \mathsf{Mod}_{\Omega^\bullet_{\mathbb{M}}}$ *is symmetric monoidal.*

Proof. The functor $\mathsf{VB}^{\mathrm{flat}}_{/\mathbb{M}} \to \mathsf{Mod}_{\Omega^\bullet_{\mathbb{M}}}$ is symmetric monoidal. □

Corollary 1.3.42. *Let* $\Bbbk \in \mathsf{Mod}_{(\mathfrak{g},K)}$ *be the unit module. Then* $\mathrm{desc}_{(P,\nu)}(\Bbbk)$ *is the trivial line bundle on* $\mathbb{M}$ *with connection given by the de Rham differential and* $\mathbf{desc}_{(P,\nu)}(\Bbbk)$ *is* $\Omega^\bullet_{\mathbb{M}}$*.*

Proof. Symmetric monoidal functors take units to units. The units of $\mathsf{VB}^{\mathrm{flat}}_{/\mathbb{M}}$ and $\mathsf{Mod}_{\Omega^\bullet_{\mathbb{M}}}$ are as described. □

Example 1.3.43. In particular, the space of horizontal sections of $\mathrm{desc}_{(P,\nu)}(\Bbbk)$ is $\mathcal{O}_{\mathbb{M}}$.

For a symmetric monoidal $\Bbbk$-linear category $\mathcal{V}$, let $\mathsf{Alg}(\mathcal{V})$ denote the category of algebra objects in $\mathcal{V}$.

Corollary 1.3.44. *The descent functors lifts to symmetric monoidal functors on the level of algebra objects,*

$$\mathsf{Alg}(\mathsf{Mod}^{\text{fin}}_{(\mathfrak{g},K)}) \to \mathsf{Alg}(\mathsf{VB}^{\text{flat}}_{/\mathbb{M}})$$

and

$$\mathsf{Alg}(\mathsf{Mod}^{\text{fin}}_{(\mathfrak{g},K)}) \to \mathsf{Alg}(\mathsf{Mod}_{\Omega^\bullet_{\mathbb{M}}}).$$

Note that an algebra object in $\Omega^\bullet_{\mathbb{M}}$-modules is just a $\Omega^\bullet_{\mathbb{M}}$-algebra.

Proof. Symmetric monoidal functors induce symmetric monoidal functors on categories of algebra objects. □

Example 1.3.45. Take $(\mathfrak{g}, K)$ to be the sHC pair $(\mathfrak{g}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$. Then $\widehat{\mathcal{O}}_{2n|a,b}$ is an object in

$$\mathsf{Alg}(\mathsf{Mod}_{(\mathfrak{g}_{2n|a,b},\mathrm{Sp}(2n|a,b))}),$$

but the underlying $(\mathfrak{g}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$-module of $\widehat{\mathcal{O}}_{2n|a,b}$ is not finite-dimensional. However, $\widehat{\mathcal{O}}_{2n|a,b}$ is a limit of finite-dimensional modules, Definition 1.3.12.

To include the above example, we extend the descent functors to pro-objects. For $\mathcal{C}$ a category, let $\mathsf{Pro}(\mathcal{C})$ denote the category of pro-objects in $\mathcal{C}$. Note that if $\mathcal{C}$ is a symmetric monoidal category, then so is Pro(C), with tensor product given levelwise, see [34, §4.2]. Since Pro$(-)$ is a functor between categories of categories, we obtain functors

$$\mathsf{Pro}\left(\mathsf{Alg}(\mathsf{Mod}^{\text{fin}}_{(\mathfrak{g},K)})\right) \to \mathsf{Pro}\left(\mathsf{Alg}(\mathsf{VB}^{\text{flat}}_{/\mathbb{M}})\right)$$

and

$$\mathsf{Pro}\left(\mathsf{Alg}(\mathsf{Mod}^{\text{fin}}_{(\mathfrak{g},K)})\right) \to \mathsf{Pro}\left(\mathsf{Alg}_{\Omega^\bullet_{\mathbb{M}}}\right).$$

By definition, $\widehat{\mathcal{O}}_{2n|a,b} = \lim_i \mathcal{O}_{\mathbb{R}^{2n|r},0}/\mathfrak{m}_0$ is a pro-object in algebras in $\mathsf{Mod}^{\text{fin}}_{(\mathfrak{g}_{2n|a,b},\mathrm{Sp}(2n|a,b))}$.

Example 1.3.46 (Jet Bundles)**.** Given a vector bundle $E \to \mathbb{M}$, the *infinite jet bundle* $J^\infty(E)$ is a pro-object in $\mathsf{VB}_{/\mathbb{M}}$, see [52, §A.2]. Moreover, given a flat connection on E, $J^\infty(E)$ has a canonical flat connection [52, Prop. A.8] so that $J^\infty(E) \in \mathsf{Pro}\left(\mathsf{VB}_{/\mathbb{M}}\right)^{\text{flat}}$. See also [24, §2].

Lemma 1.3.47. *Let* $((\mathbb{M},\omega),\sigma) \in \mathsf{sGK}^{=}_{2n|a,b}$. *Then descending* $\widehat{\mathcal{O}}_{2n|a,b}$ *along* $\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}$ *produces the jet bundle of the trivial line bundle* $\underline{k}_{\mathbb{M}}$ *with its canonical flat connection,*

$$\mathrm{desc}_{(\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)},\sigma)}(\widehat{\mathcal{O}}_{2n|a,b}) = J^\infty(\underline{\mathbb{k}}_{\mathbb{M}})$$

and thus

$$\mathbf{desc}_{(\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)},\sigma)}(\widehat{\mathcal{O}}_{2n|a,b}) = \Omega^\bullet_{\mathbb{M}}.$$

In particular, using [52, Prop. A.8], taking zero sections we see that $\widehat{\mathcal{O}}_{2n|a,b}$ descends to $\mathcal{O}_{\mathbb{M}}$.

One should compare this Lemma to [51, Prop. 2.20] or [12, Pg. 20].

Proof. The second claim follows from the first, so it suffices to produce an isomorphism of flat pro-bundles.

The bundle obtained by descending $\widehat{\mathcal{O}}_{2n|a,b}$ is

$$\mathrm{desc}_{(\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)},\sigma)}(\widehat{\mathcal{O}}_{2n|a,b}) = \mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} \widehat{\mathcal{O}}_{2n|a,b}.$$

A point in the right-hand side is an equivalence class of a point (x,ϕ) in the frame bundle and a function $\hat{f}$ on the formal disk. The frame ϕ determines an isomorphism between a neighborhood U_x of x in $\mathbb{M}$ and the space $\mathbb{R}^{2n|a+b}$. Composing $\hat{f}$ and ϕ, we obtain a germ of a function on U_x at x; that is, an element $\hat{f}_\phi$ of the completion $(\widehat{\mathcal{O}}_{U_x})_x$. Up to reparameterizations of U_x by elements of the group $\mathrm{Sp}(2n|a,b)$, the completed ring $(\widehat{\mathcal{O}}_{U_x})_x$ is the stalk of the infinite jet bundle $J^\infty(\underline{\mathbb{k}}_{\mathbb{M}})$.

The assignment $\left((x,\phi),\hat{f}\right) \mapsto \hat{f}_\phi$ therefore determines a map of bundles

$$\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} \widehat{\mathcal{O}}_{2n|a,b} \to J^\infty(\underline{\mathbb{k}}_{\mathbb{M}}).$$

One constructs an inverse to this map by sending a germ of a function $\hat{g}$ at $x \in \mathbb{M}$ to a neighborhood V_x on which $\hat{g}$ is defined.

□

Definition 1.3.48. Let $\mathcal{V}$ be a symmetric monoidal category. Let A be a pro-object in $\mathsf{Alg}(\mathcal{V})$. An *$A$-module* is an object $N \in \mathsf{Pro}(\mathcal{V})$ together with a map $A \otimes N \to N$ of pro-objects. A *morphism of*

A-modules is a morphism of pro-objects respecting the action map. We let $\mathsf{Mod}_A(\mathsf{Pro}(\mathcal{V}))$ denote the category of A-modules in $\mathcal{V}$.

One can define free, and finitely-generated modules over a pro-object in $\mathsf{Alg}(\mathcal{V})$ as in the ordinary case. Note that the underlying object of A is in $\mathsf{Pro}(\mathcal{V})$.

Corollary 1.3.49. *Let $A \in \mathsf{Pro}\left(\mathsf{Alg}(\mathsf{Mod}^{\mathrm{fin}}_{(\mathfrak{g},K)})\right)$. The descent functors induce symmetric monoidal functors*

$$\mathsf{Mod}_A(\mathsf{Pro}(\mathsf{Mod}^{\mathrm{fin}}_{(\mathfrak{g},K)})) \to \mathsf{Mod}_{\mathrm{desc}_{(P,\nu)}(A)}(\mathsf{Pro}(\mathsf{VB})^{\mathrm{flat}})$$

and

$$\mathsf{Mod}_A(\mathsf{Pro}(\mathsf{Mod}^{\mathrm{fin}}_{(\mathfrak{g},K)})) \to \mathsf{Mod}_{\mathbf{desc}_{(P,\nu)}(A)}(\mathsf{Pro}(\mathsf{Mod}_{\Omega^\bullet_{\mathbb{M}}})).$$

We can forget down

$$\mathsf{Mod}_{\mathrm{desc}_{(P,\nu)}(A)}(\mathsf{Pro}(\mathsf{Mod}_{\Omega^\bullet_{\mathbb{M}}})) \to \mathsf{Pro}(\mathsf{Mod}_{\Omega^\bullet_{\mathbb{M}}}).$$

However, the resulting functor

$$\mathsf{Mod}_A(\mathsf{Pro}(\mathsf{Mod}^{\mathrm{fin}}_{(\mathfrak{g},K)})) \to \mathsf{Pro}(\mathsf{Mod}_{\Omega^\bullet_{\mathbb{M}}})$$

is only lax-symmetric monoidal. The reader should compare this with [51, Lem. 2.18 and 2.19].

Example 1.3.50. Take $(\mathfrak{g}, K)$ to be the sHC pair $(\mathfrak{g}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$ and A to be $\widehat{\mathcal{O}}_{2n|a,b}$. Then we have lax-monoidal functors

$$\mathsf{Mod}_{\widehat{\mathcal{O}}_{2n|a,b}}(\mathsf{Mod}_{(\mathfrak{g}_{2n|a,b},\mathrm{Sp}(2n|a,b))}) \to \mathsf{Pro}(\mathsf{VB}_{/\mathbb{M}})^{\mathrm{flat}}$$

and

$$\mathsf{Mod}_{\widehat{\mathcal{O}}_{2n|a,b}}(\mathsf{Mod}_{(\mathfrak{g}_{2n|a,b},\mathrm{Sp}(2n|a,b))}) \to \mathsf{Mod}_{\Omega^\bullet_{\mathbb{M}}}.$$

Definition 1.3.51. The *super-Gelfand-Kazhdan descent functors* are the functors obtained from Example 1.3.50 by varying (P, ν) over $\mathsf{sGK}^{=}_{2n|a,b}$,

$$\mathrm{desc}^{\mathrm{sGK}} \colon \mathsf{sGK}^{=}_{2n|a,b} \times \mathsf{Mod}_{\widehat{\mathcal{O}}_{2n|a,b}}(\mathsf{Mod}_{(\mathfrak{g}_{2n|a,b},\mathrm{Sp}(2n|a,b))}) \to \mathsf{Pro}(\mathsf{VB})^{\mathrm{flat}}$$

and

$$\mathbf{desc}^{\mathrm{sGK}} \colon (\mathsf{sGK}^{=}_{2n|a,b})^{\mathrm{op}} \times \mathsf{Mod}_{\widehat{\mathcal{O}}_{2n|a,b}}(\mathsf{Mod}_{(\mathfrak{g}_{2n|a,b},\mathrm{Sp}(2n|a,b))}) \to \mathsf{Mod}_{\Omega^{\bullet}_{\mathbb{M}}}.$$

For $((\mathbb{M}, \omega), \sigma) \in \mathsf{sGK}_{2n|a,b}$, let $\mathbf{desc}_{\mathbb{M},\sigma}$ denote the resulting functor between module categories.

1.4 Deformation Quantization Descends

We would like to produce a deformation quantization for symplectic supermanifolds using super-Gelfand-Kazhdan descent. In this section, we explain what we mean by deformation quantization, and then show how the functor $\mathbf{desc}^{\mathrm{sGK}}$ of Definition 2.2.1 interacts with this process.

Definition 1.4.1. Let A be a supercommutative $\Bbbk$-superalgebra. A *deformation* of A is an associative $\Bbbk[[\hbar]]$-superalgebra $A_\hbar$ together with an isomorphism $A_\hbar/\hbar \simeq A$.

The commutative algebra we would like to deform is $\mathcal{O}_{\mathbb{M}}$ for $\mathbb{M}$ a symplectic supermanifold. By Lemma 1.2.26, $\mathcal{O}_{\mathbb{M}}$ has a Poisson superalgebra structure. We would like to consider deformations of $\mathcal{O}_{\mathbb{M}}$ that take into account this structure; that is, deformations of $\mathcal{O}_{\mathbb{M}}$ as a Poisson superalgebra. Historically this is done by asking for a deformation $A_\hbar$ of $\mathcal{O}_M$ whose associative product looks like

$$f \star g = fg + \hbar B_1(f, g) + \hbar^2 B_2(f, g) + \cdots$$

where the $B_i(-, -)$ are bilinear differential operators. Since the descent functor $\mathbf{desc}^{\mathrm{sGK}}$ lands in modules over a dg algebra, we would like a way to consider Poisson superalgebra in the differential graded setting. To do this, and to study deformations quantizations of Poisson dg superalgebras, we will use the language of operads.

Remark 1.4.2. We describe a rather general version of deformation of Poisson dg superalgebras below. We will only use the special case of $k = 1$ to prove our main result Theorem 1.1.1. The shifted cases when $k \neq 1$ are of interest for field theories over manifolds of dimension $k \neq 1$. The interaction between super-Gelfand-Kazhdan descent and deformation quantization holds in this larger generality, see Lemma 1.4.9.

The following is the super-version of [31, Def. 2.2.1] which can also be found in [25, Def. 1.1].

Definition 1.4.3. A $\mathcal{P}_k$*-algebra in* $\mathsf{Ch}_{\Bbbk}$ is a cochain complex A of super vector spaces with

- a supercommutative product $A \otimes A \to A$ of degree 0 and
- a Lie bracket
$$\{-,-\}\colon A[k-1] \otimes A[k-1] \to A[k-1]$$
so that, for every $a \in A$, the map $\{a,-\}$ is a graded superderivation.

See [80, Def. 2.9] for a construction of the operad $\mathcal{P}_k$ in terms of trees.

Example 1.4.4. When $k = 1$, a $\mathcal{P}_k$-algebra in $\mathsf{Ch}_{\Bbbk}$ is what one might call a Poisson dg algebra. In particular, there is no shift in the bracket.

Remark 1.4.5. For $k \geq 2$, there is an equivalence of operads $\mathcal{P}_k \simeq H_\bullet(\mathcal{E}_k)$, between the k-shifted Poisson operad and the homology of the little k-disks operad. By formality of the operad $\mathcal{E}_k$ [29], we have that $\mathcal{P}_k$-algebras in chain complexes over a field of characteristic zero are equivalent to algebras over the little k-disks operad $\mathcal{E}_k$. See for example [80, Thm. 4.9].

Next we describe the type of structure a deformation quantization of a $\mathcal{P}_k$-algebra should have. The following is [67, Def. 5.3].

Definition 1.4.6. A $\mathcal{BD}_1$*-algebra in* $\mathsf{Ch}_{\Bbbk[[\hbar]]}$ is a cochain complex R with

- an associative multiplication on R, and
- a Lie bracket on R,
$$\{-,-\}\colon R \otimes R \to R$$
so that, for every $a \in A$, the map $\{a,-\}$ is a graded superderivation, and

$$\hbar\{x,y\} = [x,y] \tag{1.3}$$

where $[x, y]$ is the graded supercommutator.

The structure of a $\mathcal{BD}_1$-algebra on a cochain complex R induces a $\mathcal{P}_1$-algebra structure on $R/\hbar$. This follows from Equation (1.3). One can use this to define an equivalence of operads $\mathcal{BD}_1/\hbar \simeq \mathcal{P}_1$.

Remark 1.4.7. More generally, one can define an operad $\mathcal{BD}_k$ for $k \geq 2$ to be the graded operad obtained from the Rees construction with respect to the Postnikov filtration on $\mathcal{E}_k$, see [67, §5.1]. One then has an equivalence of operads

$$\mathcal{BD}_k/\hbar \simeq \mathcal{P}_k$$

This follows, for example, from [67, Thm. 5.5].

Definition 1.4.8. Let A be a $\mathcal{P}_k$-algebra in $\mathsf{Ch}_{\Bbbk}$. A $\mathcal{BD}_1$-*deformation* of A is a $\mathcal{BD}_k$-algebra $A_\hbar$, together with an equivalence of $\mathcal{P}_k$-algebras $A_\hbar/\hbar \simeq A$.

Lemma 1.4.9. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a lax symmetric monoidal functor between symmetric monoidal categories tensored over $\Bbbk[[\hbar]]$. Let $\mathcal{C}_{\hbar=0}$ and $\mathcal{D}_{\hbar=0}$ denote the corresponding categories tensored over $\Bbbk$. Then F induces functors on algebra categories commuting with the quotient map $\Bbbk[[\hbar]] \to \Bbbk$,*

$$\begin{array}{ccc} \mathsf{Alg}_{\mathcal{BD}_k}(\mathcal{C}) & \xrightarrow{F} & \mathsf{Alg}_{\mathcal{BD}_k}(\mathcal{D}) \\ \downarrow{\scriptstyle \hbar=0} & & \downarrow{\scriptstyle \hbar=0} \\ \mathsf{Alg}_{\mathcal{P}_k}(\mathcal{C}_{\hbar=0}) & \xrightarrow[F]{} & \mathsf{Alg}_{\mathcal{P}_k}(\mathcal{D}_{\hbar=0}). \end{array}$$

Proof. Lax symmetric monoidal functors induce maps on algebra objects, given, for example, by

$$F(R) \otimes F(R) \to F(R \otimes R) \xrightarrow{F(m)} F(R)$$

where the first arrow is the lax monoidal structure, and $m\colon R \otimes R \to R$ is a multiplication.

More generally, given an operation $R^{\otimes j} \to R$, the lax structure gives a corresponding operation $F(R)^{\otimes j} \to F(R)$. Thus, lax symmetric monoidal functors induce functors between categories of algebras over operads. □

1.4.0.1 Star Products

When $k = 1$ and our Poisson algebra comes to us as functions on a symplectic supermanifold $(\mathcal{O}_{\mathbb{M}}, \{-,-\}_\omega)$, we would like our deformations to have an additional property involving the smooth structure on $\mathbb{M}$.

Definition 1.4.10. Let $(\mathbb{M}, \omega)$ be a symplectic supermanifold. A *deformation quantization* of $\mathbb{M}$ is a $\mathcal{BD}_1$-deformation $A_\hbar$ of $(\mathcal{O}_{\mathbb{M}}, \{-,-\}_\omega)$ with a $\Bbbk[[\hbar]]$-module isomorphism $A_\hbar \simeq A[[\hbar]]$ so that the associative product $\star$ on $f, g \in A_\hbar$ is of the form

$$f \star g = fg + \hbar B_1(f,g) + \hbar^2 B_2(f,g) + \cdots$$

where the $B_i(-,-)$ are bilinear differential operators on $\mathbb{M}$.

Such a product on $A_\hbar \simeq \mathcal{O}_{\mathbb{M}}[[\hbar]]$ is called a *star product*.

Since the super-Gelfand-Kazhdan descent functor starts from information over the *formal* disk, which is not a manifold, it does not make sense to ask if star products descend. The $\mathcal{BD}_1$-deformation we construct locally will have an obvious form that descends to differential operators globally, see the proof of Theorem 1.5.12.

1.5 Super-Fedosov Quantization

We would like to prove a super-analogue of Fedosov quantization. Recall that Fedosov quantization is the production of a canonical deformation quantization $\mathcal{A}_{\mathcal{D}}(\mathcal{M})$ of $\mathcal{O}_{\mathcal{M}}$ given a symplectic manifold M together with a sympelctic connection D. In this section, we will show that given a formal exponential $\sigma \in \Gamma(\mathbb{M}, \mathrm{Exp}(\mathbb{M}))$, one can construct a canonical deformation $\mathcal{A}_\sigma(\mathbb{M})$ of $\mathcal{O}_{\mathbb{M}}$ using super-Gelfand-Kazhdan descent. See Lemma 1.3.36 for the relation between a formal exponential on $\mathbb{M}$ and the data of a super-symplectic connection.

In other words, for σ a formal exponential on $\mathbb{M}$, we have an associative algebra $\mathcal{A}_\sigma(\mathbb{M})$ with an isomorphism of Poisson algebras

$$\mathcal{A}_\sigma(\mathbb{M})/\hbar \cong \mathcal{O}_{\mathbb{M}}.$$

We will construct $\mathcal{A}_\sigma(\mathbb{M})$ locally over the formal disk, and then use the descent construction from

Definition 2.2.1. By Lemma 1.4.9, the descent of a $\mathcal{BD}_1$-deformation is a $\mathcal{BD}_1$-deformation of the descended algebra.

Remark 1.5.1. By Lemma 1.3.28, the space $\Gamma(\mathbb{M}, \mathrm{Exp}(\mathbb{M}))$ is contractible. We therefore obtain an essentially unique deformation quantization of $(\mathbb{M}, \omega)$.

For motivation, we remind the reader of how this works in the non-super case.

Construction 1.5.2 (Local Fedosov Quantization)**.** We would like to deform

$$\widehat{\mathcal{O}}_{2n} = \mathbb{R}[[p_1, \ldots, p_n, q_1, \ldots, q_n]]$$

using the local symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$ where ω_0 is

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

In matrix form,

$$\omega_0(\zeta, \zeta') = -\langle \Omega\zeta, \zeta' \rangle = \zeta^T \Omega \zeta'$$

where

$$\Omega = \begin{bmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{bmatrix}.$$

The Poisson bracket on $\widehat{\mathcal{O}}_{2n}$ is

$$\{f, g\} = -(\nabla f)^T \Omega (\nabla g) = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i}.$$

From the Poisson bracket, we can abstract a bivector

$$\alpha = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial q_i} - \frac{\partial}{\partial q_i} \otimes \frac{\partial}{\partial p_i}.$$

The deformation of $\widehat{\mathcal{O}}_{2n}$ has underlying vector space

$$\widehat{\mathcal{A}}_{2n} = \mathbb{R}[[p_1, \ldots, p_n, q_1, \ldots, q_n, \hbar]]$$

with product given by

$$f \star g = m\left(\exp\left(\frac{\hbar}{2}\alpha\right)(f \otimes g)\right).$$

Here, m is multiplication of power series $\widehat{\mathcal{A}}_{2n} \otimes \widehat{\mathcal{A}}_{2n} \to \widehat{\mathcal{A}}_{2n}$. On generators, the product is given by

$$p_i \star q_j = \frac{\hbar}{2}\delta_{ij}$$

and

$$q_i \star p_i = -\frac{\hbar}{2}\delta_{ij}$$

with the rest of the products being zero. The algebra $\widehat{\mathcal{A}}_{2n}$ is sometimes called the *Weyl algebra.*

Construction 1.5.3 (Local Super-Fedosov Quantization)**.** We would like to replicate the above construction in the super context. Thus, we want to deform the Poisson superalgebra $\widehat{\mathcal{O}}_{2n|a,b}$, whose underlying superalgebra is

$$\widehat{\mathrm{Sym}}(p_1, \dots, p_n, q_1, \dots, q_n, \theta_1, \dots, \theta_r)$$

where $|p_i| = |q_i| = 0$ and $|\theta_i| = 1$. Our deformation will be constructed using the local picture of the symplectic supermanifold $(\mathbb{R}^{2n|r}, \omega_Q)$ from Example 1.2.8 where Q has signature (a, b). Here, Q is a symmetric, nondegenerate bilinear form, with corresponding matrix (g^{ij}), and ω_Q is

$$\omega_Q = \sum_{i=1}^{n} dp_i \wedge dq_i + \sum_{i,j=1}^{r} g^{ij} d\theta_i \otimes d\theta_j.$$

In matrix form,

$$\omega_Q(\xi, \xi') = \langle H_Q \xi, \xi' \rangle = -\xi^{sT} H_Q \xi'$$

where

$$H_Q = \begin{bmatrix} \Omega & 0 \\ 0 & G \end{bmatrix}$$

and $G = (g^{ij})$. The Poisson bracket on $\widehat{\mathcal{O}}_{2n|a,b}$ is given by

$$\{f, g\} = -(\nabla f)^{sT} H_Q (\nabla g),$$

we get an associated bivector

$$\tilde{\alpha} = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial q_i} - \frac{\partial}{\partial q_i} \otimes \frac{\partial}{\partial p_i} - \sum_{i=1}^{r} g^{ij} \left(\frac{\partial}{\partial \theta_i} \otimes \frac{\partial}{\partial \theta_j} \right).$$

Using the same idea as in the ordinary case, we make the following definition:

Definition 1.5.4. Let $\widehat{\mathcal{A}}_{2n|a,b}$ be the superalgebra with underlying super vector space

$$\widehat{\mathcal{A}}_{2n|a,b} = \widehat{\mathrm{Sym}}(p_1, \ldots, p_n, q_1, \ldots, q_n, \theta_1, \ldots, \theta_r, \hbar)$$

where $p_i, q_i, \hbar$ are even and θ_i are odd, and with product

$$f \star g = m \left(\exp \left(\frac{\hbar}{2} \tilde{\alpha} \right) (f \otimes g) \right).$$

On generators, the product is given by

$$\begin{aligned} p_i \star q_j &= \frac{\hbar}{2} \delta_{ij} \\ q_i \star p_j &= -\frac{\hbar}{2} \delta_{ij} \\ \theta_i \star \theta_j &= \theta_i \theta_j - \frac{\hbar}{2} g^{ij}. \end{aligned}$$

Proposition 1.5.5. *The superalgebra $\widehat{\mathcal{A}}_{2n|a,b}$ is a $\mathcal{BD}_1$-deformation of the Poisson superalgebra $\widehat{\mathcal{O}}_{2n|a,b}$.*

Proof. By construction, there is an equivalence of $\Bbbk[[\hbar]]$-modules

$$\widehat{\mathcal{A}}_{2n|a,b} = \widehat{\mathcal{O}}_{2n|a,b}[[\hbar]].$$

Quotienting by $\hbar$, the product on $\widehat{\mathcal{A}}_{2n|a,b}$ becomes the multiplication $m(f \otimes g)$ on $\widehat{\mathcal{O}}_{2n|a,b}$. Lastly, the super-commutator bracket on generators is given by

$$\begin{aligned} [p_i, q_i] &= \frac{\hbar}{2} \delta_{ij} + \frac{\hbar}{2} \delta_{ij} = \hbar \delta_{ij} = \hbar \{p_i, q_i\} \\ [\theta_i, \theta_j] &= \left(\theta_i \theta_j - \frac{\hbar}{2} g^{ij} \right) + \left(\theta_j \theta_i - \frac{\hbar}{2} g^{ij} \right) = \theta_i \theta_j - \theta_i \theta_j - \hbar g^{ij} = -\hbar g^{ij} = \hbar \{\theta_i, \theta_j\}. \end{aligned}$$

This is the Poisson bracket on $\widehat{\mathcal{O}}_{2n|a,b}$ form Example 1.2.27.

$\square$

1.5.0.1 Deformation Quantization and Super-Gelfand-Kazhdan Descent

For the super-Gelfand-Kazhdan descent functor of Definition 2.2.1, we would like to apply Lemma 1.4.9 in the case

$$\mathcal{C} = \mathsf{Mod}_{\widehat{\mathcal{O}}_{2n|a,b}}(\mathsf{Mod}_{(\mathfrak{g}_{2n|a,b},\mathrm{Sp}(2n|a,b))})(\mathrm{Vect}_{\Bbbk[[\hbar]]}),$$

$$\mathcal{D} = \mathsf{Mod}_{\Omega^\bullet_{\mathbb{M}}[[\hbar]]},$$

and F is descent $\mathbf{desc}^{\mathrm{sGK}}_{(\mathbb{M},\sigma)}$ on the level of $\Bbbk[[\hbar]]$-modules. In this case, given an algebra $A \in \mathcal{C}$, we would like to consider deformation quantizations $A_\hbar$ of A that live in $\mathcal{C}$. We will only use the deformations $A_\hbar$ that are isomorphic to $A_\hbar[[\hbar]]$ as $\Bbbk[[\hbar]]$-modules.

By Lemma 1.3.15, there is a Poisson superalgebra structure on formal functions $\widehat{\mathcal{O}}_{2n|a,b}$, making it an object

$$\widehat{\mathcal{O}}_{2n|a,b} \in \mathsf{Alg}_{\mathcal{P}_1}(\mathsf{Mod}_{\widehat{\mathcal{O}}_{2n|a,b}}(\mathsf{Mod}_{(\mathfrak{g}_{2n|a,b},\mathrm{Sp}(2n|a,b))})).$$

Let $(\mathbb{M}, \sigma) \in \mathsf{sGK}_{2n|a,b}$. By Lemma 1.3.47, we have an equivalence

$$\Gamma\left(\mathbb{M}, \mathbf{desc}^{\mathrm{sGK}}_{\mathbb{M},\sigma}\left(\widehat{\mathcal{O}}_{2n|a,b}\right)\right) = \mathcal{O}_{\mathbb{M}}.$$

By Construction 1.5.3, we have a deformation quantization $\mathcal{A}_{2n|a,b}$ of $\widehat{\mathcal{O}}_{2n|a,b}$. We would like to apply Lemma 1.4.9 to say that

$$\Gamma\left(\mathbb{M}, \mathbf{desc}^{\mathrm{sGK}}_{\mathbb{M},\sigma}\left(\mathcal{A}_{2n|a,b}\right)\right)$$

is a $\mathcal{BD}_1$-deformation of $\mathcal{O}_{\mathbb{M}}$. The one hiccup here is that $\widehat{\mathcal{A}}_{2n|a,b}$ is not a $\mathfrak{g}_{2n|a,b}$-module. We can fix this by replacing $\mathfrak{g}_{2n|a,b}$ with a Lie superalgebra involving $\hbar$.

Notation 1.5.6. Let $\mathfrak{g}^\hbar_{2n|a,b}$ be the Lie superalgebra of derivations $\mathrm{Der}(\widehat{\mathcal{A}}_{2n|a,b})$ of $\widehat{\mathcal{A}}_{2n|a,b}$ as a graded module over the graded algebra $\mathbb{K}$.

Now $\mathfrak{g}^{\hbar}_{2n|a,b}$ acts on $\widehat{\mathcal{A}}_{2n|a,b}$. Moreover, since $\widehat{\mathcal{A}}_{2n|a,b}/\hbar$ is $\widehat{\mathcal{O}}_{2n|a,b}$, we get an action by derivations of $\mathfrak{g}^{\hbar}_{2n|a,b}$ on $\widehat{\mathcal{O}}_{2n|a,b}$,

$$\mathrm{Aut}^{\mathrm{grad}}_{\mathbb{K}}(\widehat{\mathcal{A}}_{2n|a,b}) \xrightarrow{(-)\otimes \Bbbk} \mathrm{Aut}^{\mathrm{grad}}_{\Bbbk}(\widehat{\mathcal{O}}_{2n|a,b}) \xrightarrow{\text{forget}} \mathrm{Aut}_{\Bbbk}(\widehat{\mathcal{O}}_{2n|a,b}). \tag{1.4}$$

This action factors through the action by Poisson derivations. In other words, since the Lie superalgebra $\mathfrak{g}_{2n|a,b}$ is given by derivations of $\widehat{\mathcal{O}}_{2n|a,b}$ that respect the Poisson structure coming from the symplectic form ω_Q, there is a Lie superalgebra map

$$\mathfrak{g}^{\hbar}_{2n|a,b} \to \mathfrak{g}_{2n|a,b}.$$

See [12, §3.2] for similar statements in the purely even case. Note that, as in [12, §3.2], $\mathrm{Aut}^{\mathrm{grad}}_{\mathbb{K}}(\widehat{\mathcal{A}}_{2n|a,b})$ is a pro-algebraic super group.

We would like to apply a variant of super-Gelfand-Kazhdan descent for the sHC pair

$$(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$$

instead of $(\mathfrak{g}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$. To do so, we need an analogue of the functor

$$\mathrm{Fr}\colon \mathsf{sGK}^{=}_{2n|a,b} \to \mathsf{Bun}^{\mathrm{flat}}_{(\mathfrak{g}_{2n|a,b},\mathrm{Sp}(2n|a,b))}$$

from Corollary 1.3.35. That is, we need a way of equipping the symplectic frame bundle $\mathrm{Fr}^{\mathrm{Sp}(2n|a,b)}_{\mathbb{M}}$ with the structure of a $(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$-bundle. This is done by replacing the principal $\mathrm{Aut}_{2n|a,b}$-bundle $\mathbb{M}^{\mathrm{coor}} \to \mathbb{M}$ with the principal $\mathrm{Aut}^{\mathrm{filt}}(\widehat{\mathcal{A}}_{2n|a,b})$-bundle $\mathbb{M}^{\mathrm{coor}}_{\hbar}$ whose fiber over a point $x \in \mathbb{M}$ is

$$\mathrm{Isom}^{\mathrm{grad}}_{\mathbb{K}}(\widehat{\mathcal{O}}_{\mathbb{M},x}[[\hbar]], \widehat{\mathcal{A}}_{2n|a,b}).$$

See also [42, Pg. 18] in the purely even case.

We obtain a map $\mathbb{M}^{\mathrm{coor}}_{\hbar} \to \mathbb{M}^{\mathrm{coor}}$ over $\mathbb{M}$ given by the map (1.4) fiberwise.

Just as in Lemma 1.3.24, $\mathbb{M}_\hbar^{\mathrm{coor}}$ has a flat connection $\nu_\hbar^{\mathrm{coor}}$, which now takes values in

$$\mathrm{Lie}(\mathrm{Aut}_{\mathbb{K}}^{\mathrm{grad}}(\widehat{\mathcal{A}}_{2n|a,b})) = \mathfrak{g}_{2n|a,b}^{\hbar}.$$

As before, we use a type of formal exponential to pullback this connection to a connection on $\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}$.

Definition 1.5.7. Let $(\mathbb{M}, \omega)$ be a symplectic supermanifold of type $(2n|a, b)$. An *$\hbar$-formal exponential* on $\mathbb{M}$ is a section of the bundle

$$\mathrm{Exp}^{\hbar}(\mathbb{M}) = \mathbb{M}_\hbar^{\mathrm{coor}}/\mathrm{Sp}(2n|a, b).$$

See also [42, Pg. 18].

In the super case, we have the following analogue of Lemma 1.3.28.

Lemma 1.5.8. *The space* $\mathrm{Aut}_{\mathbb{K}}^{\mathrm{grad}}(\widehat{\mathcal{A}}_{2n|a,b})/\mathrm{Sp}(2n|a, b)$ *is contractible, and thus $\hbar$-formal exponentials always exist.*

Proof. As in [12, Pg. 24], we have a short exact sequence

$$1 \to \ker(P) \to \mathrm{Aut}_{\mathbb{K}}^{\mathrm{grad}}(\widehat{\mathcal{A}}_{2n|a,b}) \xrightarrow{P} \mathrm{Sp}(2n|a, b) \to 1$$

and $\ker(P)$ is pro-unipotent, hence pro-nilpotent and contractible. As in Lemma 1.3.28, $\ker(P)$ is a pro-vector space. □

As in Proposition 1.3.33, given an $\hbar$-formal exponential $\sigma_\hbar$ on $\mathbb{M}$, we get a $(\mathfrak{g}_{2n|a,b}^{\hbar}, \mathrm{Sp}(2n|a, b))$-bundle structure on $\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}$.

Lemma 1.5.9. *An $\hbar$-formal exponential $\sigma_\hbar$ on $\mathbb{M}$ induces a formal exponential σ on $\mathbb{M}$. Moreover, the induced connection 1-form $\tilde{\sigma}_\hbar^*(\nu_\hbar^{\mathrm{coor}}))$ on $\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}$ hits the induced connection 1-form $\tilde{\sigma}^*(\nu^{\mathrm{coor}}))$ under the map*

$$\Omega^1(\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}; \mathfrak{g}_{2n|a,b}^{\hbar}) \to \Omega^1(\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}; \mathfrak{g}_{2n|a,b}).$$

Proof. The map $\mathbb{M}_\hbar^{\mathrm{coor}} \to \mathbb{M}^{\mathrm{coor}}$ induces a map $\mathrm{Exp}^{\hbar}(\mathbb{M}) \to \mathrm{Exp}(\mathbb{M})$. Composing with this map takes an $\hbar$-formal exponential to a formal exponential. The second claim follows from the fact that these bundle maps and the map $\mathfrak{g}_{2n|a,b}^{\hbar} \to \mathfrak{g}_{2n|a,b}$ are both defined by the map (1.4). □

We can now consider super-Gelfand-Kazhdan descent for $(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$-modules,

$$\mathbf{desc}^{\mathrm{sGK}}_{(\mathbb{M},\sigma_\hbar)} \colon \mathsf{Mod}_{(\mathfrak{g}^{\hbar}_{2n|a,b},\mathrm{Sp}(2n|a,b))} \to \mathsf{Mod}_{\Omega^\bullet_{\mathbb{M}}[[\hbar]]}.$$

Using the same notation for this variant is somewhat justified by the following lemma.

Lemma 1.5.10. *Let $\sigma_\hbar$ be an $\hbar$-formal exponential on $\mathbb{M}$ with induced formal exponential σ. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathsf{Mod}_{(\mathfrak{g}^{\hbar}_{2n|a,b},\mathrm{Sp}(2n|a,b))} & \xrightarrow{\mathbf{desc}^{\mathrm{sGK}}_{(\mathbb{M},\sigma_\hbar)}} & \mathsf{Mod}_{\Omega^\bullet_{\mathbb{M}}[[\hbar]]} \\ r\uparrow & & s\uparrow \\ \mathsf{Mod}_{\widehat{\mathcal{O}}_{2n|a,b}}(\mathsf{Mod}_{(\mathfrak{g}_{2n|a,b},\mathrm{Sp}(2n|a,b))}) & \xrightarrow[\mathbf{desc}^{\mathrm{sGK}}_{(\mathbb{M},\sigma)}]{} & \mathsf{Mod}_{\Omega^\bullet_{\mathbb{M}}} \end{array}$$

where the left vertical arrow is given by restricting the module structure along $\mathfrak{g}^{\hbar}_{2n|a,b} \to \mathfrak{g}_{2n|a,b}$ and the right vertical arrow is given by restriction along the map setting $\hbar = 0$.

Proof. Let V be in $\mathsf{Mod}_{\widehat{\mathcal{O}}_{2n|a,b}}(\mathsf{Mod}_{(\mathfrak{g}_{2n|a,b},\mathrm{Sp}(2n|a,b))})$. Then both $\mathbf{desc}^{\mathrm{sGK}}_{(\mathbb{M},\sigma)}(V)$ and $\mathbf{desc}^{\mathrm{sGK}}_{(\mathbb{M},\sigma_\hbar)}(r(V))$ are given by taking horizontal forms of the vector bundle $\mathrm{Fr}^{\mathrm{Sp}(2n|a,b)}_{\mathbb{M}} \times_{\mathrm{Sp}(2n|a,b)} V$, but with respect to possibly different flat connections. By the discussion around Equation (1.2), in the first case, the differential induced by the flat connection is

$$\rho^V_{\mathfrak{g}}(\tilde{\sigma}^*(\nu^{\mathrm{coor}})) + d_{\mathrm{dR}}$$

and in the latter case by

$$\rho^{r(V)}_{\mathfrak{g}^{\hbar}_{2n|a,b}}(\tilde{\sigma}^*_\hbar(\nu^{\mathrm{coor}}_\hbar)) + d_{\mathrm{dR}}.$$

By Lemma 1.5.9, the action of $\tilde{\sigma}^*_\hbar(\nu^{\mathrm{coor}}_\hbar))$ on $r(V)$ is the same as the action of $\tilde{\sigma}^*(\nu^{\mathrm{coor}}))$ on V. Thus, the flat connections are the same and therefore the diagram commutes. □

One consequence of this result is that Lemma 1.3.47 still holds when viewing $\widehat{\mathcal{O}}_{2n|a,b}$ as a $(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$-module. Now $\widehat{\mathcal{A}}_{2n|a,b}$ is a deformation of $\widehat{\mathcal{O}}_{2n|a,b}$ in $(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$-modules, and by Lemma 1.4.9 should descend to a deformation of $\mathcal{O}_{\mathbb{M}}$.

In summary, we have the following,

Corollary 1.5.11. *Given a $\hbar$-formal exponential $\sigma_\hbar$ on $\mathbb{M}$ super-Gelfand-Kazhdan descent for $(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$-modules takes a $\mathcal{BD}_1$-deformation $\widehat{\mathcal{A}}_{2n|a,b}$ of $\widehat{\mathcal{O}}_{2n|a,b}$ to a $\mathcal{BD}_1$-deformation of $\mathcal{O}_{\mathbb{M}}$.*

1.5.0.2 Main Theorem

Theorem 1.5.12. *Let $(\mathbb{M}, \omega)$ be a symplectic supermanifold. The assignment*

$$\sigma \mapsto \Gamma(\mathbb{M}, \mathbf{desc}^{\mathrm{sGK}}_{(\mathbb{M},\sigma)}(\widehat{\mathcal{A}}_{2n|a,b}))$$

defines a map

$$\mathcal{A}_{(-)}(\mathbb{M})\colon \Gamma(\mathbb{M}, \mathrm{Exp}^{\hbar}(\mathbb{M})) \to Q(\mathbb{M}, \omega)$$

from the set of $\hbar$-formal exponentials of $\mathbb{M}$ to the set of equivalence classes of deformation quantizations of $(\mathbb{M}, \omega)$.

Proof. Let $(\mathbb{M}, \sigma)$ be a symplectic supermanifold and $\hbar$-formal exponential. By Lemma 1.3.47, the degree zero piece of $\mathbf{desc}^{\mathrm{sGK}}_{(\mathbb{M},\sigma)}(\widehat{\mathcal{O}}_{2n|a,b})$ is $\mathcal{O}_{\mathbb{M}}$. By Lemma 1.4.9, $\mathcal{O}_{\mathbb{M}}$ is a Poisson superalgebra. Let $\mathcal{A}_\sigma(\mathbb{M})$ be the degree zero piece of $\mathbf{desc}^{\mathrm{sGK}}_{(\mathbb{M},\sigma)}(\widehat{\mathcal{A}}_{2n|a,b})$. By Lemma 1.4.9, $\mathcal{A}_\sigma(\mathbb{M})$ is a $\mathcal{BD}_1$-deformation of the Poisson superalgebra $\mathcal{O}_{\mathbb{M}}$.

It remains to check that the product on $\mathcal{A}_\sigma(\mathbb{M})$ is a star product. One can check this locally, where $\mathbb{M}$ looks like $\mathbb{R}^{2n|a+b}$. Here, the $\hbar$ terms in the product on $\mathcal{A}_\sigma(\mathbb{M})$ are given in terms of the partial derivatives $\frac{\partial}{\partial p_i}, \frac{\partial}{\partial q_i}, \frac{\partial}{\partial \theta_i}$, which are differential operators. $\square$

Example 1.5.13. By Remark 1.3.37, we have a functor

$$\tilde{L}\colon \mathsf{VB}^{\mathrm{quad},\nabla}_{/X} \to \mathsf{sGK},$$

so that every symplectic supermanifold of the form $(\pi^*E)[1]$ from Example 1.2.14 has a natural choice of formal exponential σ_E. We can upgrade this choice to an $\hbar$-formal exponential.

Lemma 1.5.14. *If $\mathbb{M} = E[1]$ is the symplectic supermanifold defined in Example 1.2.14 from the data $(M, \omega, E, g, \nabla)$, then a symplectic connection on M determines an $\hbar$-formal exponential on $\mathbb{M}$.*

Proof. The argument is the same as in the proof of Lemma 1.3.36 after noting that a symplectic connection on M defines not just a compatible choice of isomorphisms of Poisson algebras

$$\widehat{\mathcal{O}}_{2n} \simeq \widehat{\mathcal{O}}_{M,x},$$

but isomorphisms of $\mathbb{K}$-modules

$$\mathrm{Weyl}_{2n} \simeq \widehat{\mathcal{O}}_{M,x}[[\hbar]].$$

Similarly, the metric connection ∇ on E induces a compatible family of isomorphisms

$$\widehat{\mathcal{A}}_{2n|a,b} \simeq \widehat{\mathcal{O}}_{\mathbb{M},x}.$$

of an $\hbar$-formal exponential. □

The assignment $E \mapsto \mathcal{A}_{\sigma_E}((\pi^* E)[1])$ defines a functor

$$\tilde{A}_X \colon \mathsf{VB}^{\mathrm{quad},\nabla}_{/X} \to \mathsf{sAlg}(\mathsf{Ch}_{\Bbbk[[\hbar]]}),$$

as discussed in §1.1.1.1.

Remark 1.5.15. If, following [12, Lem. 3.4], one wishes to construct all deformation quantizations, one would apply the same process but using descent for a much larger sHC pair. The descent functor we use (from Definition 2.2.1) uses the sHC pair $(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$. As explained in Remarks 1.1.2 and 1.3.34, this corresponds to only allowing our gluing data to come from *linear* maps $\mathrm{Sp}(2n|a,b)$. To surject onto $Q(\mathbb{M},\omega)$, one would like to consider lifts of $\mathbb{M}^{\mathrm{coor}}$ from a $(\mathfrak{g}_{2n|a,b}, \mathrm{Aut}_{2n|a,b})$-bundle to a $(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Aut}(\widehat{\mathcal{A}}_{2n|a,b}))$-bundle instead of choosing $\mathbb{M}^{\mathrm{coor}}_{\hbar}$ and pulling back along an $\hbar$-formal exponential. Here, $(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Aut}(\widehat{\mathcal{A}}_{2n|a,b})$ is a super-version of the HC pair $(\mathrm{Der}(D), \mathrm{Aut}(D))$ in [12, Lem. 3.4]; that is, derivations and automorphism of the algebra $\widehat{\mathcal{A}}_{2n|a,b}$.

1.5.1 Description in terms of Weyl and Clifford Algebras

We review the basic definitions of Weyl and Clifford algebras as a means of establishing notation. Then, we give a description of the super-Fedosov quantization in terms of these algebras.

Definition 1.5.16. Fix n. The *Heisenberg Lie superalgebra* over $\Bbbk[[\hbar]]$, denoted $\mathfrak{h}_{2n}$, is the Lie superalgebra with even generators $p_1, \ldots, p_n, q_1, \ldots, q_n$ and Lie brackets given by $[p_i, q_i] = \hbar$, and zero otherwise.

Definition 1.5.17. The *Weyl algebra* of a symplectic vector space (V, ω) is the quotient

$$\mathrm{Weyl}(V, \omega) \colon = T(V[\hbar, \hbar^{-1}])/I_\omega,$$

where $T(V)$, the tensor algebra, is taken over $\Bbbk[\hbar, \hbar^{-1}]$, and I_ω is the ideal generated by the set

$$\{u \otimes v - v \otimes u - \omega(u, v)\hbar : u, v \in V\}.$$

Example 1.5.18. With notation as in Example 1.2.8, the Weyl algebra of the symplectic vector space $(T_0^*\mathbb{R}^n, \omega_0)$ is the enveloping algebra of the Heisenberg Lie algebra,

$$\mathrm{Weyl}(T_0^*\mathbb{R}^n, \omega_0) = U(\mathfrak{h}_{2n}).$$

Definition 1.5.19. For fixed a, b, the *Clifford Lie superalgebra* over $\Bbbk[[\hbar]]$, denoted $\mathfrak{cl}_{a,b}$, is the Lie algebra with odd generators $\gamma_1, \ldots, \gamma_a, \psi_1, \ldots, \psi_b$ and brackets zero except

$$[\gamma_i, \gamma_i] = \hbar$$

and

$$[\psi_i, \psi_i] = -\hbar.$$

Definition 1.5.20. The *Clifford algebra* of a super vector space equipped with a quadratic function (V, Q) is the quotient

$$\mathrm{Cliff}(V, Q) \colon = T(V[\hbar, \hbar^{-1}])/I_Q$$

where I_Q is the ideal generated by the set

$$\{v \otimes v - Q(v)\hbar : v \in V\}.$$

Example 1.5.21. With notation as in Example 1.2.8, let $a + b = r$ and Q be a quadratic function on $\mathbb{R}^r$ with signature (a, b). Then the Clifford algebra of the symplectic super vector space $(\mathbb{R}^{0|r}, \omega_Q)$ is the enveloping algebra of the Clifford Lie algebra,

$$\mathrm{Cliff}(\mathbb{R}^{0|a,b}, Q) = U(\mathfrak{cl}_{a,b}).$$

The deformation of $\widehat{\mathcal{O}}_{2n|a,b}$ that we are interested in is a mixture of Weyl and Clifford algebras. Since we are working with formal functions, we are interested in power series rings. Replacing the tensor algebra with the completed tensor algebra in the definitions of the Weyl, Clifford, and enveloping algebras, we obtain notations of a completed Weyl algebra $\widehat{\mathrm{Weyl}}$, a completed Weyl algebra $\widehat{\mathrm{Cliff}}$, and a completed enveloping algebra $\widehat{U}$. Since the generators of the Clifford algebra, which are odd, square to zero, completing does not change the algebra.

Lemma 1.5.22. *Let Q be a quadratic function on $\mathbb{R}^r$ with signature (a, b). There is an equivalence of superalgebras*

$$\widehat{\mathrm{Weyl}}(\mathbb{R}^{2n|0}, \omega_0) \otimes \mathrm{Cliff}(\mathbb{R}^{0|r}, \omega_Q) \xrightarrow{\sim} \widehat{\mathcal{A}}_{2n|a,b}.$$

One can make further (notational) identifications,

$$\widehat{\mathcal{A}}_{2n|a,b} \cong \widehat{U}(\mathfrak{h}_{2n}) \otimes_{\mathbb{k}[[\hbar]]} U(\mathfrak{cl}_{a,b}).$$

Proof. The underlying super vector spaces are both

$$\widehat{\mathrm{Sym}}(p_1, \ldots, p_n, q_1, \ldots, q_n, \theta_1, \ldots, \theta_r, \hbar).$$

By the proof of Proposition 1.5.5, the super-commutator bracket in $\widehat{\mathcal{A}}_{2n|a,b}$ agrees with the Lie bracket of $\mathfrak{h}_{2n}$ and $\mathfrak{cl}_{a,b}$. By the universal property of enveloping algebras, we obtain a map of algebras

$$\widehat{U}(\mathfrak{h}_{2n}) \otimes_{\mathbb{k}[[\hbar]]} U(\mathfrak{cl}_{a,b}) \to \widehat{\mathcal{A}}_{2n|a,b}$$

which is an isomorphism on underlying super vector spaces, and hence an isomorphism of algebras. □

One should compare this to [38, §1.4].

Chapter 2

Genera from an algebraic index theorem for Supermanifolds

2.1 Introduction

The Atiyah-Singer index theorem [2] states that the analytic index of an elliptic differential operator agrees with its topological index. Since its announcement in the early 1960s, various proofs and generalizations of the index theorem have been given, [1, 47, 48, 68, 72]. See [43] for a nice overview. An algebraic analogue of the index theorem [40, 42, 70] was given in the early 1990s, replacing the analytic index with a trace on the algebra of differential operators. In 1996, Nest and Tsygan [71] showed that the algebraic index theorem implies Atiyah-Singer's result.

Here, we prove a super-version of the algebraic index theorem. A superalgebraic index theorem was proven for a specific class of symplectic supermanifolds (those of type $(2n|a, a)$) by Engeli in [38]. In the super-case, deformation theory of symplectic manifolds is replaced with deformation theory of symplectic supermanifolds, which was first studied by Bordemann [15, 16]. We studied this theory in Chapter 1 using Gelfand-Kazhdan descent.

Let X be a manifold. The Rees algebra of differential operators on X is a deformation of the symplectic manifold T^*X. More generally, De Wilde-Lecomte [35] and Fedosov [41]. described the space of deformations of any symplectic manifold. This was generalized to all Poisson manifolds by Kontsevich, [60]. The Fedosov deformation is a global version of the Weyl algebra. In his original paper, Fedosov also defines an interesting (derived) trace on the Weyl algebra [41, Def. 5.1]. This trace is essentially the trace appearing in the algebraic index theorem.

Theorem 2.1.1 (Algebraic Index Theorem). *Let (M, ω) be a compact symplectic manifold. Let Ω be the Fedosov connection for M. There exists a unique normalized trace t_M on the Fedosov deformation quantization of $\mathcal{O}_M$ so that*

$$t_M(1) = \frac{1}{(2\pi i)^n} \int_M \hat{\mathrm{A}}(TM) \exp(-\Omega/\hbar).$$

Remark 2.1.2. Other proofs of the algebraic index theorem can be found in [26, 73]. In [73], they also prove a version of the algebraic index theorem for orbifolds. In [87], an algebraic index theorem for Cherednik algebras is shown.

Comparing the algebraic index theorem to the Atiyah-Singer index theorem, we have a bridge between deformation theory and index theory.

Deformation theory is also fundamentally related to the problem of quantizing field theories. For example, the quantum observables of a quantum field theory form a deformation of the classical observables of the associated classical field theory, [31]. The relationship between Fedosov and Kontsevich's deformation theory and perturbative methods in field theory go back further, see for example [22].

Using this translation between field theory and deformation theory, Grady-Li-Li and Gui-Li-Xu [53, 54] reproved the algebraic index theorem by computing the partition function of 1d Chern Simons theory using the BV formalism. See [55] for an analysis of 1d Chern Simons theory in the BV formalism. This provides an exciting new interplay between trace methods in deformation theory and partition function results in quantum field theory.

We thus see three interconnected stories: the index theorem, the trace methods in deformation theory, and the partition functions of quantum field theories. The goal of this paper is to study the super-analogue of the algebraic index theorem part of these stories. Just as [53, 54] showed that the algebraic index theorem was related to quantum mechanics, the super-version should be related to *supersymmetric quantum mechanics*.

Remark 2.1.3. An important class of theories one can study in the BV formalism is given by those equipped with a $\mathbb{Z}/2$-grading: supersymmetric theories. The possible applications to supersymmetry is in large part our motivation to study a super analogue of the algebraic index theorem.

Additionally, there are interesting relationships between supersymmetric field theories and chromatic homotopy theory, following Stolz-Teichner [10, 27, 82, 83]. Here and in Chapter 1, symplectic supermanifolds have *even* symplectic form. Odd symplectic supermanifolds also have interesting connections to the BV formalism, see for example [49].

2.1.0.1 Manifold Invariants

In the Atiyah-Singer index theorem, the algebraic index theorem, and the partition function of 1d Chern-Simons theory, we see the same cobordism invariant of manifolds appearing: the $\hat{A}$-genus.

For example, the topological index of an elliptic differential operator D on a compact manifold X is

$$\int_X \mathrm{Td}(X)\mathrm{ch}(D),$$

the integral of the Todd class and the Chern character. Viewed as an invariant of real, rather than complex bundles, the $\widehat{A}$-genus of a spin manifold Y is $e^{-c_1(Y)/2}\mathrm{Td}(Y)$, see [56, Pg . 165].

From the relationships explained above, the $\widehat{A}$-genus can also be extracted from the trace on the Fedosov deformation and from the partition function of 1d Chern-Simons theory.

Remark 2.1.4. One dimension higher, the 2d $\beta\gamma$ holomorphic sigma model (also called holomorphic Chern-Simons theory) has partition function related to the Witten genus, see [30, 51]. In this 2d story, the algebra of differential operators is replaced with the vertex algebra of chiral differential operators [64]. See [28] for a discussion of how this story relates to supermanifolds.

One of the main goals of this paper is to discover what genus replaces $\widehat{A}$ in the super-version of the algebraic index theorem. In [11], Berwick-Evans shows that the partition function of supersymmetric quantum mechanics is related to Hirzebruch's L-genus. The L-genus assigns to a $4m$-dimensional manifold its signature, [56, §4.1]. It has associated characteristic series

$$\frac{\sqrt{z}}{\tanh(\sqrt{z})}.$$

A very similar formal power series also appears in Engeli's special case of the superalgebraic index theorem [38, Lem. 2.25]. See also [89, Appendix E]. We therefore expect the L-genus to replace the $\widehat{A}$-genus in some cases.

Remark 2.1.5. Note that the L-genus and the $\widehat{A}$-genus, which appeared in partition functions of field theories of dimension 1, naturally land in cohomology theories of chromatic height 1. Analogously, the Witten genus, coming from a 2d theory, naturally lands in a cohomology theory of height 2, [37]. This paradigm of chromatic height relating to field theory dimension is expected by the Stolz-Teichner program, [83].

The L-genus is closely related to the theory of quadratic forms and their signatures, [74]. Given a quadratic vector space, one has an associated Clifford algebra. These Clifford algebras appear in the odd part of the deformations of symplectic supermanifolds. One also has a parameterized analogue for quadratic vector bundles, see §1.1.1.1. The case considered in [38, Lem. 2.25] is when the Clifford algebra comes from a quadratic vector space of signature (a, a). The superalgebraic index theorem of Engeli thus relates quadratic forms of a *fixed* signature to the L-genus. A natural question to ask is what happens for quadratic forms of other signatures, which is exactly the generalization of Engeli's result that we prove here.

Remark 2.1.6. While our motivation comes from a variety of areas, we hope this paper is relatively accessible. In particular, no knowledge of chromatic homotopy theory or physics is necessary to understand the content of this paper.

2.1.0.2 Overview of Results

Let $(\mathbb{M}, \omega)$ be a symplectic supermanifold of type $(2n|a, b)$, see 1.2.18. In Chapter 1 we construct a version of Gelfand-Kazhdan descent for $\mathbb{M}$ that depends on an *$\hbar$-formal exponential* σ, 1.5.7. This descent functor sends a deformation $\widehat{\mathcal{A}}_{2n|a,b}$ of functions on the formal disk to a deformation $\mathcal{A}_\sigma(\mathbb{M})$ of $\mathbb{M}$. The superalgebra $\widehat{\mathcal{A}}_{2n|a,b}$ looks like a Weyl algebra tensored with a Clifford algebra. The goal of this paper is to study supertraces on the deformed superalgebra $\mathcal{A}_\sigma(\mathbb{M})$.

Corollary 2.3.10 shows that a trace on the Weyl-Clifford algebra induces a trace on the deformation $\mathcal{A}_\sigma(\mathbb{M})$. By a straightforward Hochschild cohomology computation, a trace on $\widehat{\mathcal{A}}_{2n|a,b}$ is unique up to a scalar. We discuss this computation and a unique normalization condition of traces on $\mathcal{A}_\sigma(\mathbb{M})$ in §2.4.

The algebraic index theorem computes the value of the normalized trace on the unit. Analogously, we define an invariant Ev of normalized traces given by evaluating on a volume form. There is a local version of this invariant Ev_{loc} for supertraces on $\widehat{\mathcal{A}}_{2n|a,b}$.

The underpinnings of algebraic index type theorems is the ability to compute Ev of a descended supertrace in terms of Chern-Weil style characteristic classes using $\mathsf{Ev}_{\mathrm{loc}}$ of the trace. In the purely even case, this style of result is proven in [42, Thm. 4.3], by appealing to uniqueness results.

Below, in Theorem 2.5.6, we give an alternative proof of a more general result.

Theorem 2.1.7. *Let $t_{2n|a,b}$ be a $2n$-derived relative supertrace on $\widehat{\mathcal{A}}_{2n|a,b}$. For $(\mathbb{M},\sigma)$ a symplectic supermanifold of type $(2n|a,b)$ and an $\hbar$-formal exponential, let $t_{\mathbb{M}}$ denote the supertrace on $\mathcal{A}_\sigma(\mathbb{M})$ induced from $t_{2n|a,b}$ using Corollary 2.4.2. Then*

$$\mathsf{Ev}_{\mathbb{M}}(t_{\mathbb{M}}) = \int_{\mathbb{M}} \mathrm{char}_{(\mathbb{M},\sigma)}(\mathbb{K})(\mathsf{Ev}_{\mathrm{loc}}(t_{2n|a,b})).$$

Here, $\mathrm{char}_{(\mathbb{M},\sigma)}$ is the characteristic map from [51, Def. 1.18] which provides characteristic classes for flat principal bundles over the relevant Harish-Chandra pair. This map is related to the classical Chern-Weil map as well as the Chern-Weil map studied in [42, §5.1], a new observation that we record in Lemma 2.8.36 below.

The computational heart of this paper is an explicit description of the invariant $\mathsf{Ev}_{\mathrm{loc}}$ on our chosen supertrace on $\widehat{\mathcal{A}}_{2n|a,b}$. This local superalgebraic index theorem appears as Theorem 2.8.13 below. Our proof of this more general result follows ideas in [42] and [38]. In the special case considered in [38], we obtain a slightly different computation, see Remark 2.8.35. We also include as many details as possible in our proof so the reader may follow along with the computation.

Relying on the local computations and Theorem 2.5.6, we are able to compute Ev on the unique normalized supertrace on the deformed superalgebra $\mathcal{A}_\sigma(\mathbb{M})$.

Let $t = \lfloor \frac{b-a}{2} \rfloor$. Given a connection on the symplectic frame bundle $\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}$, write its curvature as a sum $R_1+S_2+S_3$ so that R_1 is a $\mathfrak{sp}_{2n}$-valued form, S_1 is a $\mathfrak{so}_a$-valued form, and S_2 is a $\mathfrak{so}_t$-valued form.

Theorem 2.1.8 (Superalgebraic Index Theorem)**.** *The evaluation of the unique normalized supertrace $\mathrm{Tr}_{\mathbb{M}}$ on the volume form $1 \otimes \Theta_{\mathbb{M}}$ is*

$$\mathsf{Ev}_{\mathbb{M}}(\mathrm{Tr}_{\mathbb{M}}) = (-1)^{n+a+t}\hbar^n e \int_{\mathbb{M}} \widehat{A}(R_1)\widehat{B}(S_1)\widehat{C}(S_2)$$

where $\widehat{A}, \widehat{B}$, and $\widehat{C}$ have characteristic power series $\frac{z}{\sinh z}$, $\cosh(z)\frac{e^z-1}{z}$, and $\frac{z^2}{\sinh(z)}\cot(z)\cos(z)$, *respectively.*

This is Theorem 2.8.38 below.

2.1.1 Linear Overview

This paper is broken into two parts. The first part establishes the conceptual framework we use, proving the general results we will use in the second part. The second part contains all of the computations and constructions.

We begin in §2.2 by reviewing the main results of Chapter 1. In §2.2.1, we recall the super version of Gelfand-Kazhdan descent (sGK descent) that will be our main tool for globalizing local results. Subsequently, in Theorem 2.2.2 we introduce the deformation of functions on a symplectic supermanifold and then record some basic results on integration over supermanifolds in §2.2.1.3.

In §2.3, we prove that sGK descent takes supertraces to supertraces. We then introduce normalization conditions in §2.4 that uniquely determine the traces we will construct in Part 2. The supertrace invariants we will be interested in are defined in §2.5. Therein, we also prove that the local version of this invariant determines the global invariant of the descended trace, see Theorem 2.5.6.

Part 2 begins in §2.6 with a review of quadratic forms, including useful results about their corresponding Clifford algebras. In §2.7, we give a formula for the supertrace on the Weyl-Clifford algebra following [38]. We justify how this rather complicated formula comes from a more obviously canonical one in §2.7.1. The bulk of computational content of this paper is contained in §2.8. Therein, we compute the invariants defined in §2.5. The superalgebraic index theorem is proven in §2.8.3.

2.1.2 Conventions

Notation 2.1.9. We set the following notation

- We let $\mathrm{HH}_\bullet(A)$ denote Hochschild homology with coefficients in A, and $\mathrm{Hoch}_\bullet(A)$ the Hochschild chains.

- We let $\mathrm{HH}^\bullet(A)$ denote Hochschild cohomology with coefficients in A^*, and $\mathrm{Hoch}^\bullet(A)$ the Hochschild cochains.
- Let $\Bbbk$ be either $\mathbb{K}$ or $\mathbb{C}$.
- $\hbar$ is a free variable we use as our deformation parameter.
- Let $\mathbb{K} = \Bbbk[[\hbar]]$
- Given an object R with a $\mathbb{Z}/2$-grading, we let ΠR denote the parity shift of R with opposite grading. In particular, $\Pi\mathbb{K} = \Bbbk^{0|1}[[\hbar]]$.
- We let $(\mathbb{M}, \omega)$ denote a symplectic supermanifold of type $(2n|a, b)$.
- The coordinates of $\mathbb{R}^{2n|a,b}$ are given by $p_1, \ldots, p_n, q_1, \ldots, q_n, \theta_1, \ldots, \theta_{a+b}$ where p_i, q_i are even and θ_i are odd.

Part I

Formal Theory

2.2 Background

We recall the results of Chapter 1 that we will use below. For a review of symplectic supermanifolds, see 1.2. In §2.2.1, we review the super version of Gelfand-Kazhdan descent, which allows us to globalize results from the formal disk to an entire supermanifold. We then review the algebras whose supertraces we will be interested in, §2.2.1.1. We end this section by discussing how to integrate over a symplectic supermanifold and defining an odd volume form.

2.2.1 Super-Gelfand-Kazhdan Descent

All symplectic supermanifolds $(\mathbb{M}, \omega)$ will be assumed to be of type $(2n|a,b)$, see 1.2.18. That is, $\mathbb{M}$ has $2n$ even dimensions, $a+b$ odd dimensions, and the symplectic structure in the odd direction comes from a quadratic form of signature (a,b).

Recall the super-Harish-Chandra (sHC) pair $(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$ from Convention 1.3.18 and Notation 1.5.6 where $\mathrm{Sp}(2n|a,b)$ is the Lie supergroup of linear automorphisms of a symplectic super vector space, and $\mathfrak{g}^{\hbar}_{2n|a,b}$ is the Lie superalgebra of derivations of the Weyl-Clifford algebra.

Super-Gelfand-Kazhdan descent is a fancy version of the Borel construction for modules over the sHC pair $(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$. By Theorem 1.3.22 and §1.5.0.1 there is a category $\mathsf{sGK}^{=,\hbar}_{2n|a,b}$ of symplectic supermanifolds equipped with an $\hbar$-formal exponential and a functor from $\mathsf{sGK}^{=,\hbar}_{2n|a,b}$ to principal $(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$-bundles,

$$\mathsf{sGK}^{=,\hbar}_{2n|a,b} \to \mathsf{Bun}^{\mathrm{flat}}_{(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Sp}(2n|a,b))}. \tag{2.1}$$

Given a pair $((\mathbb{M}, \omega), \sigma) \in \mathsf{sGK}^{=,\hbar}_{2n|a,b}$ of a symplectic supermanifold $(\mathbb{M}, \omega)$ and $\hbar$-formal exponential σ, one assigns the symplectic frame bundle $\mathrm{Fr}^{\mathrm{Sp}(2n|a,b)}_{\mathbb{M}}$ and a flat connection

$$A \in \Omega^1(\mathrm{Fr}^{\mathrm{Sp}(2n|a,b)}_{\mathbb{M}}; \mathfrak{g}^{\hbar}_{2n|a,b}) \tag{2.2}$$

determined by σ.

Composing the functor (2.1) with a version of the Borel construction, we obtain our desired functor.

Definition 2.2.1. The *super-Gelfand-Kazhdan descent functors* are the functors obtained from Example 1.3.50 and §1.5.0.1 by varying (P,ν) over $\mathsf{sGK}^{=,\hbar}_{2n|a,b}$,

$$\mathrm{desc}^{\mathrm{sGK}}\colon \mathsf{sGK}^{=,\hbar}_{2n|a,b} \times \mathsf{Mod}_{\widehat{\mathcal{O}}_{2n|a,b}}(\mathsf{Mod}_{(\mathfrak{g}^{\hbar}_{2n|a,b},\mathrm{Sp}(2n|a,b))}) \to \mathsf{Pro}(\mathsf{VB})^{\mathrm{flat}}$$

and

$$\mathbf{desc}^{\mathrm{sGK}}\colon (\mathsf{sGK}^{=,\hbar}_{2n|a,b})^{\mathrm{op}} \times \mathsf{Mod}_{\widehat{\mathcal{O}}_{2n|a,b}}(\mathsf{Mod}_{(\mathfrak{g}^{\hbar}_{2n|a,b},\mathrm{Sp}(2n|a,b))}) \to \mathsf{Mod}_{\Omega^{\bullet}_{\mathbb{M}}}.$$

For $((\mathbb{M},\omega),\sigma) \in \mathsf{sGK}_{2n|a,b}$, let $\mathbf{desc}_{(\mathbb{M},\sigma)}$ denote the resulting functor between module categories.

This is the version of Definition 2.2.1 considered in §1.5.0.1.

More concretely, $\mathbf{desc}_{(\mathbb{M},\sigma)}$ sends a $\widehat{\mathcal{O}}_{2n|a,b}$-module V in $(\mathfrak{g}^{\hbar}_{2n|a,b},\mathrm{Sp}(2n|a,b))$-modules to the de Rham forms with differential induced from the connection ∇,

$$\mathbf{desc}_{(\mathbb{M},\sigma)}(V) = (\Omega^{\bullet}(\mathbb{M},\mathrm{Fr}^{\mathrm{Sp}(2n|a,b)}_{\mathbb{M}} \times_{\mathrm{Sp}(2n|a,b)} V), d_{\nabla}), \tag{2.3}$$

where ∇ is a connection on the Borel construction induced via A from the $\hbar$-formal exponential σ.

2.2.1.1 Weyl and Clifford Algebras

The main application in Chapter 1 of super-Gelfand-Kazhdan descent was to produce a deformation of $\mathcal{O}_{\mathbb{M}}$ using a deformation of functions on the formal disk. Here, we will produce supertraces on these formally local and global deformations. Fix an object $(\mathbb{M},\sigma)$ in $\mathsf{sGK}^{=,\hbar}_{2n|a,b}$.

The deformation of $\widehat{\mathcal{O}}_{2n|a,b}$ that is studied in Chapter 1 is

$$\widehat{\mathcal{A}}_{2n|a,b} \cong \widehat{\mathrm{Weyl}}(T^*\mathbb{R}^n,\omega_0) \otimes \mathrm{Cliff}(\mathbb{R}^{0|a+b},Q).$$

Here, $\widehat{\mathrm{Weyl}}$ is the completed Weyl algebra and Cliff is the Clifford algebra, see Definition 1.5.4 and §subsec-WeylClifford.

The following is Theorem 1.5.12.

Theorem 2.2.2. *The algebra*

$$\mathcal{A}_\sigma(\mathbb{M}) := \Gamma_\nabla\left(\mathbb{M}, \mathbf{desc}_{(\mathbb{M},\sigma)}(\widehat{\mathcal{A}}_{2n|a,b})\right)$$

is a deformation of the super Poisson algebra $\mathcal{O}_\mathbb{M}$.

Note that the algebra $\mathcal{A}_\sigma(\mathbb{M})$ is the zeroth cohomology of the $\Omega^\bullet_\mathbb{M}[[\hbar]]$-algebra $\mathbf{desc}_{(\mathbb{M},\sigma)}(\widehat{\mathcal{A}}_{2n|a,b})$.

2.2.1.2 Connections

The Lie superalgebra $\mathfrak{g}^\hbar_{2n|a,b}$ is defined to be derivations of $\widehat{\mathcal{A}}_{2n|a,b}$,

$$\mathfrak{g}^\hbar_{2n|a,b} = \mathrm{Der}(\widehat{\mathcal{A}}_{2n|a,b}).$$

As in general, we have a central extension of Lie superalgebras

$$0 \to Z \to \widehat{\mathcal{A}}_{2n|a,b} \to \mathrm{Der}(\widehat{\mathcal{A}}_{2n|a,b}) \to 0$$

where Z is the center of $\widehat{\mathcal{A}}_{2n|a,b}$, viewed as an abelian Lie algebra. The connection 1-form A from (2.2) can be lifted to a connection 1-form $\tilde{A}$ with values in $\widehat{\mathcal{A}}_{2n|a,b}$. The curvature $F_{\tilde{A}}$ of $\tilde{A}$ is then a closed 2-form with values in the center Z of $\widehat{\mathcal{A}}_{2n|a,b}$. Moreover, $F_{\tilde{A}}$ is a $\mathrm{Sp}(2n|a,b)$-basic form. We can therefore view $F_{\tilde{A}}$ as an element

$$F_{\tilde{A}} \in H^2(\mathbb{M}; \mathrm{Fr}_\mathbb{M}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} Z).$$

This class is independent of the choice of lift $\tilde{A}$ of A. Since $Z \simeq \mathbb{K}$, then $F_{\tilde{A}}$ becomes a class in $H^2(\mathbb{M}; \mathbb{K})$. This $\mathbb{K}$-valued de Rham form is what is called the *characteristic class* of the deformation $\widehat{\mathcal{A}}_{2n|a,b}$.

One should compare the above to [42, Pg. 18] and [12, §4].

2.2.1.3 Integration and Orientations

Ordinary symplectic manifolds (M, ω) are orientable. If M has dimension $2n$, then ω^n is a common choice of orientation. The underlying manifold of a symplectic supermanifold is symplectic, and hence orientable; let $[M]$ denote the fundamental class. The main use of an orientation for us will be to integrate over the manifold. See [78] for an introduction to integrating over supermanifolds. One can integrate along symplectic supermanifolds using a combination of integration along the underlying manifold and a Berezin integral. We define this locally in coordinates.

Locally, $\mathbb{M}$ is modeled on the symplectic supermanifold $\mathbb{R}^{2n|a+b}$. Let $\theta_1, \ldots, \theta_{a+b}$ be local odd coordinates. Given a compactly supported function f on $\mathbb{R}^{2n|a,b}$, we take

$$\int_{\mathbb{R}^{2n|a,b}} f d\mathbb{M} = \int_{\mathbb{R}^{2n}} d[M] \int_{\Lambda^{a+b}} f d\Theta$$

where $\int_{\Lambda^{a+b}} d\Theta$ is the Berezin integral on $a+b$ Grassmann variables such that

$$\int_{\Lambda^{a+b}} \theta_1 \cdots \theta_n d\Theta = 1,$$

See [8, Part 1. Ch. 2. §2].

To check that this globalizes to define an integration on $\mathbb{M}$, we need to check that it is invariant under coordinate changes coming from local super symplectomorphisms ϕ. The change of variables formula for the Berezin integral can be found in [8, Thm. 2.1, Part 2. Ch. 2. §2]. The Berezin transforms under ϕ by the Berezinian ([8, 2.2.11]) of the Jacobian of ϕ. It therefore suffices to check that the Berezinian of the Jacobian matrix of ϕ is 1, and that $\theta_1 \cdots \theta_{a+b}$ globalizes to a function on $\mathbb{M}$.

The Jacobian of a super symplectomorphism ϕ is a super symplectic matrix M_ϕ, analogously to the even case. Just as the determinant of a symplectic matrix is 1, the Berezinian of a super symplectic matrix is 1. Indeed, since M_ϕ is a super symplectic matrix, we have

$$M_\phi^{\mathrm{sT}} H_Q M_\phi = H_Q$$

where H_Q is as in §2.6. See also Remark 1.2.12. Since the Berezinian of a product is the product of the Berezinians, we may reduce to computing $\mathrm{Ber}(H_Q)$. Then Berezinian of the block matrix H_Q maybe computed by the formula [8, §2]

$$\mathrm{Ber}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \frac{\det(A - BD^{-1}C)}{\det(D)}.$$

We defer the following computation to §2.6.

Lemma 2.2.3. *One can choose an ordering of local odd coordinates $\theta_1, \dots, \theta_{a+b}$ on the symplectic supermanifold $\mathbb{R}^{2n|a+b}$ so that the function*

$$\Theta\colon \mathbb{R}^{2n|a,b} \to \mathbb{R}$$

given by $\Theta = \theta_1 \cdots \theta_{a+b}$ is invariant under local symplectomorphisms of $\mathbb{M}$ and therefore defines a global function on $\mathbb{M}$,

$$\Theta_{\mathbb{M}} \in \mathcal{O}_{\mathbb{M}}.$$

The term *volume form* is sometimes used to mean the data needed to integrate against. As $\mathcal{A}_\sigma(\mathbb{M})$ is a deformation of $\mathcal{O}_{\mathbb{M}}$, we have an equivalence of modules

$$\mathcal{A}_\sigma(\mathbb{M}) \simeq \mathcal{O}_{\mathbb{M}}[[\hbar]].$$

We may therefore view $\Theta_{\mathbb{M}}$ as an element of $\mathcal{A}_\sigma(\mathbb{M})$.

Definition 2.2.4. The *volume form* on $\mathbb{M}$ is $\Theta_{\mathbb{M}} \in \mathcal{A}_\sigma(\mathbb{M})$.

We can see the volume form in terms of the Weyl-Clifford algebra as well.

Remark 2.2.5. Recall that $\mathcal{A}_\sigma(\mathbb{M})$ is the result of applying super-Gelfand-Kazhdan descent to the tensor product

$$\widehat{\mathcal{A}}_{2n|a,b} \cong \widehat{\mathrm{Weyl}}(T^*\mathbb{R}^n, \omega_0) \otimes \mathrm{Cliff}(\mathbb{R}^{0|a+b}, Q).$$

After possibly scaling, the element $\Theta_{\mathbb{M}}$ locally looks like the unit in the completed Weyl algebra tensored with

$$\Theta = \theta_1 \cdots \theta_{a+b}$$

in the Clifford algebra.

Example 2.2.6. Identify $\mathbb{M}$ as $E[1]$ for $E \to M$ a quadratic vector bundle on a symplectic manifold as in Rothstein's theorem [76]. Then functions on $\mathbb{M}$ are given by sections of the exterior product bundle, $\mathcal{O}_{\mathbb{M}} = \Gamma(\mathbb{M}, \Lambda^\bullet E)$. If E is oriented, we can take $\Theta_{\mathbb{M}}$ to be given by the corresponding section of $\Lambda^{\mathrm{rank}(E)}E$, sending a point of M to the top dimensional form on E from the orientation. This is also explained in, for example, [38, Pg. 26].

Definition 2.2.7. Let $\mathbb{M}$ be a symplectic supermanifold with reduced manifold M. Let $d[M]$ denote the volume form on M induced from the symplectic structure. *Integration* over $\mathbb{M}$ is given by the map $\int_{\mathbb{M}} \colon \Omega^{2n}_{\mathrm{dR}}(\mathbb{M}) \to \mathbb{K}$ defined by

$$\int_{\mathbb{M}} = \int \left(\int (-) d\Theta_{\mathbb{M}} \right) d[M].$$

2.3 Descending Supertraces

The goal of this section is to show how the notion of supertraces interacts with the super-Gelfand-Kazhdan descent functor. In particular, we would like a way of procuring supertraces on the deformed superalgebra $\mathcal{A}_\sigma(\mathbb{M})$ of Theorem 2.2.2. By definition, the superalgebra $\mathcal{A}_\sigma(\mathbb{M})$ is obtained by applying super-Gelfand-Kazhdan descent to the superalgebra $\widehat{\mathcal{A}}_{2n|a,b}$. We will show that a derived supertrace on $\widehat{\mathcal{A}}_{2n|a,b}$ descends to a supertrace on $\mathcal{A}_\sigma(\mathbb{M})$.

We begin by defining the types of maps we want.

Note that for a superalgebra A over $\mathbb{K}$, the Hochschild homology groups $\mathrm{HH}^{\mathbb{K}}_i(A; A)$ are $\mathbb{Z}/2$-graded as well.

Definition 2.3.1. Let $(\mathfrak{g}, K)$ be an super-Harish-Chandra pair and let $A \in \mathrm{Alg}(\mathrm{Mod}_{(\mathfrak{g},K)})$. A ($i$-derived) *supertrace* on A is a morphism $t\colon A^{\otimes_{\mathbb{K}} i+1} \to \mathbb{K}$ in $\mathrm{Mod}_{(\mathfrak{g},K)}$ such that the composition of t with the differential ∂ of the Hochschild complex,

$$A^{\otimes i+2} \xrightarrow{\partial} A^{\otimes i+1} \xrightarrow{t} \mathbb{K},$$

is zero.

We allow t to be either an even or odd map.

Example 2.3.2. The superalgebra $\widehat{\mathcal{A}}_{2n|a,b}$ is an algebra object in $\mathrm{Mod}_{(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Sp}(2n|a,b))}$. We construct a $2n$-derived supertrace $(\widehat{\mathcal{A}}_{2n|a,b})^{\otimes 2n+1} \to \mathbb{K}$ in Theorem 2.7.9.

We will eventually restrict to supertraces satisfying an additional property.

Definition 2.3.3. Let t be an i-derived supertrace on $A \in \mathrm{Alg}(\mathrm{Mod}_{(\mathfrak{g},K)})$. Let $\rho\colon \mathfrak{g} \to \mathrm{End}(A)$ denote the action of $\mathfrak{g}$. Say t is a *relative* supertrace if for every $x \in \mathrm{Lie}(K)$ and $a_0, \dots, a_{i-1} \in A$ we have

$$\sum_{j=1}^{i} (-1)^j t(a_0 \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes a \otimes a_i \otimes \cdots \otimes a_{i-1}) = 0$$

where $a = \rho(x) \cdot 1$.

We will see later (Lemma 2.5.3) that relative supertraces correspond to relative Lie algebra cohomology classes.

Definition 2.3.4. Let R be a $\mathbb{K}$-superalgebra. Let $B \in \mathrm{Alg}(\mathrm{Mod}_R)$. An ($i$-derived) *supertrace* on B is a morphism $t\colon B^{\otimes_R i+1} \to R$ in Mod_R such that the composition of t with the differential ∂ of the Hochschild complex,

$$B^{\otimes i+2} \xrightarrow{\partial} B^{\otimes i+1} \xrightarrow{t} \mathbb{K},$$

is zero.

Example 2.3.5. Take $R = \Omega^\bullet_{\mathrm{dR}}(\mathbb{M}; \mathbb{K})$. Then

$$\mathbf{desc}_{(\mathbb{M},\sigma)}(\widehat{\mathcal{A}}_{2n|a,b}) = \Omega^\bullet(\mathbb{M}; \mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} \widehat{\mathcal{A}}_{2n|a,b})$$

is an algebra in Mod_R. Note that we have an isomorphism

$$\mathbf{desc}_{(\mathbb{M},\sigma)}(\widehat{\mathcal{A}}_{2n|a,b}) \otimes_R \mathbf{desc}_{(\mathbb{M},\sigma)}(\widehat{\mathcal{A}}_{2n|a,b}) \cong \Omega^\bullet \left(\mathbb{M}; \mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} (\widehat{\mathcal{A}}_{2n|a,b} \otimes_{\mathbb{K}} \widehat{\mathcal{A}}_{2n|a,b}) \right).$$

Example 2.3.6. Take $R = \mathbb{K}$. Then $\mathcal{A}_\sigma(\mathbb{M}) = \Gamma_\nabla(\mathbb{M}, \mathbf{desc}_{(\mathbb{M},\sigma)}(\widehat{\mathcal{A}}_{2n|a,b}))$ is an algebra in $\mathrm{Mod}_{\mathbb{K}}$.

Lemma 2.3.7. *The functor*

$$\mathbf{desc}_{(\mathbb{M},\sigma)}\colon \mathrm{Alg}(\mathrm{Mod}_{(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Sp}(2n|a,b))}) \to \mathrm{Alg}(\mathrm{Mod}_{\Omega^\bullet_{\mathbb{M}}})$$

from Equation (2.1) sends derived supertraces to derived supertraces.

Proof. Let $A \in \mathrm{Alg}(\mathrm{Mod}_{(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Sp}(2n|a,b))})$ and $t\colon A \to \mathbb{K}$ be a supertrace on A. Applying $\mathbf{desc}_{(\mathbb{M},\sigma)}$ to t, we obtain a map

$$\mathbf{desc}_{(\mathbb{M},\sigma)}(t)\colon \mathbf{desc}_{(\mathbb{M},\sigma)}(A) \to \mathbf{desc}_{(\mathbb{M},\sigma)}(\mathbb{K}).$$

By Corollary 1.3.47 and Lemma 1.5.10, $\mathbf{desc}_{(\mathbb{M},\sigma)}(\mathbb{K}) \cong \Omega^{\bullet}_{\mathrm{dR}}(\mathbb{M};\mathbb{K})$. Thus $\mathbf{desc}_{(\mathbb{M},\sigma)}(t)$ has the right domain and codomain to be a supertrace on $\mathbf{desc}_{(\mathbb{M},\sigma)}(A)$, as in Definition 2.3.4. Since t is a supertrace, we have a commutative diagram

$$\begin{array}{ccccc} A \otimes A & \xrightarrow{m} & A & & \\ & & & \searrow^{t} & \\ \Big\downarrow \mathrm{swap} & & & & \mathbb{K} \\ & & & \nearrow_{t} & \\ A \otimes A & \xrightarrow[m]{} & A & & \end{array}$$

where m is the multiplication map for A. Since the functor $\mathbf{desc}_{(\mathbb{M},\sigma)}$ is symmetric monoidal, it takes the swap map to the swap map, and m to the multiplication map m' for $\mathbf{desc}_{(\mathbb{M},\sigma)}(A)$. Therefore, applying $\mathbf{desc}_{(\mathbb{M},\sigma)}$ to this commutative diagram, we obtain a new commutative diagram

$$\begin{array}{ccccc} \mathbf{desc}_{(\mathbb{M},\sigma)}(A) \otimes \mathbf{desc}_{(\mathbb{M},\sigma)}(A) & \xrightarrow{m'} & \mathbf{desc}_{(\mathbb{M},\sigma)}(A) & & \\ & & & \searrow^{\mathbf{desc}_{(\mathbb{M},\sigma)}(t)} & \\ \Big\downarrow \mathrm{swap} & & & & \Omega^{\bullet}_{\mathbb{M}}[[\hbar]] \\ & & & \nearrow_{\mathbf{desc}_{(\mathbb{M},\sigma)}(t)} & \\ \mathbf{desc}_{(\mathbb{M},\sigma)}(A) \otimes \mathbf{desc}_{(\mathbb{M},\sigma)}(A) & \xrightarrow[m']{} & \mathbf{desc}_{(\mathbb{M},\sigma)}(A) & & \end{array}$$

which implies that $\mathbf{desc}_{(\mathbb{M},\sigma)}(t)$ is a supertrace.

For a derived supertrace $T\colon A^{\otimes i+1} rta \mathbb{K}$, the functoriality of $\mathbf{desc}_{(\mathbb{M},\sigma)}$ implies that the face and degeneracy maps in the Hochschild complex of A map to those in $\mathrm{Hoch}_{\bullet}(\mathbf{desc}_{(\mathbb{M},\sigma)}(A))$. Thus, the composition

$$(\mathbf{desc}_{(\mathbb{M},\sigma)}(A))^{\otimes i+1}) \xrightarrow{\partial} (\mathbf{desc}_{(\mathbb{M},\sigma)}(A))^{\otimes i+1} \xrightarrow{\mathbf{desc}_{(\mathbb{M},\sigma)}(T)} \mathbf{desc}_{(\mathbb{M},\sigma)}(\mathbb{K}).$$

Which shows that $\mathbf{desc}_{(\mathbb{M},\sigma)}(T)$ is a derived supertrace on $\mathbf{desc}_{(\mathbb{M},\sigma)}(A)$. □

Example 2.3.8. Let $t_{2n|a,b}$ be a $2n$-derived supertrace on $\widehat{\mathcal{A}}_{2n|a,b}$. Then $\mathbf{desc}_{(\mathbb{M},\sigma)}(t_{2n|a,b})$, as a map

$$\mathbf{desc}_{(\mathbb{M},\sigma)}(t_{2n|a,b})\colon \mathbf{desc}_{(\mathbb{M},\sigma)}(\widehat{\mathcal{A}}_{2n|a,b})^{\otimes 2n+1} \to \Omega^\bullet_{\mathrm{dR}}(\mathbb{M};\mathbb{K})$$

is a $2n$-derived trace on $\mathbf{desc}_{(\mathbb{M},\sigma)}(\widehat{\mathcal{A}}_{2n|a,b})$.

We would like $t_{2n|a,b}$ to determine an *underived* trace on $\mathcal{A}_\sigma(\mathbb{M})$. For this, we will need the internal and external product maps on Hochschild homology. First, note that

$$\mathbf{desc}_{(\mathbb{M},\sigma)}(\widehat{\mathcal{A}}_{2n|a,b}) = \Omega^\bullet(\mathbb{M};\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} \widehat{\mathcal{A}}_{2n|a,b})$$

is graded by degree of forms. Its Hochschild complex is therefore bigraded: one grading from degree of forms and one grading from the Hochschild complex.

Let d_{dR} denote the de Rham differential on $\Omega^\bullet_{\mathrm{dR}}(\mathbb{M};\mathbb{K})$, and d_∇ the differential on

$$\mathbf{desc}_{(\mathbb{M},\sigma)}(\widehat{\mathcal{A}}_{2n|a,b})^{\otimes 2n+1}$$

induced from the covariant derivative. By the proof of [38, Thms. 2.14 and 2.15], we have

$$d_{\mathrm{dR}} \circ \mathbf{desc}_{(\mathbb{M},\sigma)}(t_{2n|a,b}) = \mathbf{desc}_{(\mathbb{M},\sigma)}(t_{2n|a,b}) \circ d_\nabla.$$

Thus, $\mathbf{desc}_{(\mathbb{M},\sigma)}(t_{2n|a,b})$ descends to a map on cohomology

$$H^\bullet_\nabla\left(\mathbb{M};\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} (\widehat{\mathcal{A}}_{2n|a,b})^{\otimes 2n+1}\right) \to H^\bullet_{\mathrm{dR}}(\mathbb{M};\mathbb{K})$$

which does not preserve degree, as we have moved the $\otimes 2n+1$ inside.

We would like to create an *underived* trace on $\mathcal{A}_\sigma(\mathbb{M})$ from $\mathbf{desc}_{(\mathbb{M},\sigma)}(t_{2n|a,b})$. To do this, we will use the product structure on forms on $\mathbb{M}$ valued in $\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} (\widehat{\mathcal{A}}_{2n|a,b})$.

Let

$$\tilde{A} \in \Omega^1(\mathbb{M};\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} \widehat{\mathcal{A}}_{2n|a,b}) \simeq \Omega^1(\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)};\widehat{\mathcal{A}}_{2n|a,b})_{\mathrm{basic}}$$

denote the connection 1-form on the flat pro-bundle $\mathrm{desc}_{(\mathbb{M},\sigma)}(\widehat{\mathcal{A}}_{2n|a,b})$, see §2.2.1.2.

Wedging with the form $\tilde{A}^{\wedge 2n}$ determines a degree $2n$ map

$$\beta_{\tilde{A}}\colon \Omega^0\left(\mathbb{M}; \mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} \widehat{\mathcal{A}}_{2n|a,b}\right) \to \Omega^{2n}\left(\mathbb{M}; \mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} (\widehat{\mathcal{A}}_{2n|a,b})^{\otimes 2n+1}\right)$$

of superalgebras. Here, the algebra structure on both sides is induced from the algebra structure on $\widehat{\mathcal{A}}_{2n|a,b}$. One should compare $\beta_{\tilde{A}}$ to part of what is denoted χ_0 in [38, §§2.5.1-2.5.2]. Taking cohomology, the domain of $\beta_{\tilde{A}}$ becomes $\mathcal{A}_\sigma(\mathbb{M})$. We can then form the composite

$$\mathcal{A}_\sigma(\mathbb{M}) \xrightarrow{\beta_{\tilde{A}}} H^\bullet_\nabla\left(\mathbb{M}; \mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} (\widehat{\mathcal{A}}_{2n|a,b})^{\otimes 2n+1}\right) \xrightarrow{\mathbf{desc}_{(\mathbb{M},\sigma)}(t_{2n|a,b})} H^\bullet_{\mathrm{dR}}(\mathbb{M}; \mathbb{K}).$$

The next proposition follows from [38, Thm. 2.14].

Proposition 2.3.9. *Let $t_{2n|a,b}$ be a $2n$-derived supertrace on $\widehat{\mathcal{A}}_{2n|a,b}$. The map on cohomology*

$$\mathcal{A}_\sigma(\mathbb{M}) \to H^{2n}_{\mathrm{dR}}(\mathbb{M}; \mathbb{K}),$$

induced from $\mathbf{desc}_{(\mathbb{M},\sigma)}(t_{2n|a,b}) \circ \beta_{\tilde{A}}$, is linear and factors through $\mathrm{HH}_0(\mathcal{A}_\sigma(\mathbb{M}))$.

To obtain a $\mathbb{K}$-valued supertrace on $\mathcal{A}_\sigma(\mathbb{M})$, we need to get from $H^{2n}_{\mathrm{dR}}(\mathbb{M}; \mathbb{K})$ to $\mathbb{K}$. We will do this by integrating out the odd directions with a Berezin integral, and then using the orientation on the underlying manifold M_0 of $\mathbb{M}$. See Definition 2.2.7.

Corollary 2.3.10. *The composite*

$$t_{\mathbb{M}} = \int_{\mathbb{M}} \left(\mathbf{desc}_{(\mathbb{M},\sigma)}(t_{2n|a,b}) \circ \beta_{\tilde{A}}\right) : \mathcal{A}_\sigma(\mathbb{M}) \to \mathbb{K}$$

is a supertrace of $\mathbb{K}$-algebras.

Proof. The map $\int_{\mathbb{M}}$ is $\mathbb{K}$-linear. By Proposition 2.3.9, the composite $t_{\mathbb{M}}$ is linear and factors through $\mathrm{HH}_0(\mathcal{A}_\sigma(\mathbb{M})$. □

2.4 Uniqueness of supertraces

We discuss reasonable properties we would like our supertraces (locally on $\widehat{\mathcal{A}}_{2n|a,b}$ and globally on $\mathcal{A}_\sigma(\mathbb{M})$) to satisfy, and show that these properties uniquely determine such a supertrace.

2.4.1 Local Uniqueness

Here, we show that there is a unique derived supertrace on $\widehat{\mathcal{A}}_{2n|a,b}$ up to scalar multiple.

We can reinterpret a supertrace as an element of the Hochschild cohomology using the following lemma. For $A \in \mathrm{Alg}_{\mathbb{K}}(\mathrm{Mod}_{(\mathfrak{g},K)})$, let A^* denote the dual, $A^* = \mathrm{Hom}(A, \mathbb{K})$. The following can be found after Theorem 2.1 in [42].

Lemma 2.4.1. *Let $A \in \mathrm{Alg}_{\mathbb{K}}(\mathrm{Mod}_{(\mathfrak{g},K)})$. There is an equivalence*

$$\mathrm{Hom}\left(\mathrm{Hoch}_\bullet^{\mathbb{K}}(A; A), \mathbb{K}\right) \cong \mathrm{Hoch}_{\mathbb{K}}^\bullet(A; A^*).$$

To show that there is a unique derived supertrace on $\widehat{\mathcal{A}}_{2n|a,b}$ up to scalar multiple, it suffices to prove that $\mathrm{HH}_\bullet^{\mathbb{K}}(\widehat{\mathcal{A}}_{2n|a,b})$ is one-dimensional.

By §1.5.1, we have an equivalence

$$\widehat{\mathcal{A}}_{2n|a,b} \cong \widehat{\mathrm{Weyl}}(T^*\mathbb{R}^n, \omega_0) \otimes_{\mathbb{K}} \mathrm{Cliff}(\mathbb{R}^{0|a+b}, Q).$$

Hochschild homology satisfies a Künneth formula. That is, as the Hochschild complex of a tensor product is the tenors product of the Hochschild complexes, we have

$$\mathrm{HH}_\bullet^{\mathbb{K}}(\widehat{\mathcal{A}}_{2n|a,b}) \cong \mathrm{HH}_\bullet^{\mathbb{K}}\left(\widehat{\mathrm{Weyl}}(T^*\mathbb{R}^n, \omega_0)\right) \otimes_{\mathbb{K}} \mathrm{HH}_\bullet^{\mathbb{K}}\left(\mathrm{Cliff}(\mathbb{R}^{a+b}, Q)\right).$$

The Hochschild homology of the Weyl algebra is computed in [42, Thm. 2.1], where $\mathrm{Weyl}(T^*\mathbb{R}^n, \omega_0)$ is denoted $\mathcal{A}_{2n}^{\mathrm{pol}}$. We have an isomorphism

$$\mathrm{HH}_\bullet^{\mathbb{K}}(\mathrm{Weyl}(T^*\mathbb{R}^n, \omega_0)) \cong \mathbb{K}[2n].$$

A computation of the Hochschild homology of the Clifford algebra can be found in [58, §6 Proof of Prop. 1] where it is shown that

$$\mathrm{HH}_\bullet^{\mathbb{K}}(\mathrm{Cliff}(\mathbb{R}^{a+b}, Q) = \begin{cases} \mathbb{K}[0] & a+b \text{ is even} \\ \Pi\mathbb{K}[0] & a+b \text{ is odd} \end{cases}.$$

See also [38, Thm. 2.10].

Putting this together, we have the following computation.

Corollary 2.4.2. *Let $\widehat{\mathcal{A}}_{2n|a,b}$ be the super Fedosov quantization as in Theorem 2.2.2. There are isomorphisms*

$$\mathrm{HH}_\bullet^{\mathbb{K}}(\widehat{\mathcal{A}}_{2n|a,b}) \cong \begin{cases} \mathbb{K}[2n] & a+b \text{ is even} \\ \Pi\mathbb{K}[2n] & a+b \text{ is odd} \end{cases}.$$

See also [38, §2.3].

In particular, this means that for $a+b$ even, there is a unique (up to scalar) supertrace on $\widehat{\mathcal{A}}_{2n|a,b}$ that is an *even* map, and for $a+b$ odd, there is a unique (up to scalar) *odd* supertrace on $\widehat{\mathcal{A}}_{2n|a,b}$.

2.4.2 Normalization Condition

We would like to put conditions on the type of supertraces $\mathcal{A}_\sigma(\mathbb{M}) \to \mathbb{K}$ that will uniquely determine it. Note that $\mathcal{A}_\sigma(\mathbb{M})$ is a deformation of $\mathcal{O}_{\mathbb{M}}$ in sheaves of algebras on $\mathbb{M}$. We will specify what our supertrace should look like locally over $\mathbb{M}$.

Let $p_1, \ldots, p_n, q_1, \ldots, q_n, \theta_1, \ldots, \theta_{a+b}$ be local coordinates for $\mathbb{M}$ as in Lemma 2.2.3.

Definition 2.4.3. Let $\mathcal{A}$ be a deformation of $\mathcal{O}_{\mathbb{M}}$ in sheaves of algebras on $\mathbb{M}$. A supertrace $t_{\mathbb{M}} \colon \mathcal{A} \to \mathbb{K}$ on $\mathcal{A}$ is *normalized* if on sufficiently small neighborhoods $\mathbb{R}^{2n|a+b} \subset \mathbb{M}$, the map is given by

$$(t_{\mathbb{M}})|_U(f) = \left((-1)^{n+a+t}\hbar^n \int_{\mathbb{R}^{2n}} \left(\int f d\Theta\right) dq_1 \wedge dp_1 \wedge \cdots dq_n \wedge dp_n\right).$$

By [38, §2.6] and [42, Thm. 4.2], a normalized supertrace on $\mathcal{A}_\sigma(\mathbb{M})$ is unique. Below, in Theorem 2.7.9, we show the existence of such a normalized supertrace.

2.5 Evaluation on a Volume Form

In this section, we will define and study an invariant of symplectic supermanifolds using their normalized supertraces. In §2.8.1 we will compute this invariant.

Recall from §2.2.1.3 that we have chosen a volume form $\Theta_{\mathbb{M}} \in \mathcal{A}_\sigma(\mathbb{M})$.

Definition 2.5.1. Let $t_{\mathbb{M}}$ be a supertrace on $\mathcal{A}_\sigma(\mathbb{M})$. The *evaluation of $t_{\mathbb{M}}$ on the volume form $\Theta_{\mathbb{M}}$* is

$$\mathsf{Ev}_{\mathbb{M}}(t_{\mathbb{M}}) = t_{\mathbb{M}}(\Theta_{\mathbb{M}}) \in \mathbb{K}.$$

Note that the definition of $\mathsf{Ev}_{\mathbb{M}}$ depends on a choice of volume form, though we suppress this from the notation.

Although $\mathsf{Ev}_{\mathbb{M}}$ may be defined for any supertrace on $\mathcal{A}_\sigma(\mathbb{M})$, we will only show that it is an invariant for *normalized* supertraces.

Lemma 2.5.2. *Let $\varphi\colon (\mathbb{M}, \omega, \sigma) \to (\mathbb{M}', \omega', \sigma)$ be a morphism in $\mathsf{sGK}^{=,\hbar}_{2n|a,b}$. If T and T' are normalized supertraces on $\mathbb{M}$ and $\mathbb{M}'$, respectively, then $\mathsf{Ev}_{\mathbb{M}}(T) = \mathsf{Ev}_{\mathbb{M}'}(T')$.*

Proof. The local symplectomorphism φ induces a pullback map $\varphi^*\colon \mathcal{O}_{\mathbb{M}'} \to \mathcal{O}_{\mathbb{M}}$. By the functoriality of $\mathbf{desc}_{(\mathbb{M},\sigma)}$, the pullback extends to a map

$$\varphi^*\colon \mathcal{A}_{\sigma'}(\mathbb{M}') \to \mathcal{A}_\sigma(\mathbb{M}).$$

We can check that the diagram

$$\begin{array}{ccc} \mathcal{A}_\sigma(\mathbb{M}) & \xleftarrow{\varphi^*} & \mathcal{A}_{\sigma'}(\mathbb{M}') \\ \downarrow T & \swarrow T' & \\ \mathbb{K} & & \end{array}$$

commutes by examining it locally in $\mathbb{M}$ and $\mathbb{M}'$. Over sufficiently small open subsets, the normalization condition guarantees that supertraces T and T' agree.

Since φ is a local symplectomorphism, $\varphi^*(\Theta_{\mathbb{M}'}) = \Theta_{\mathbb{M}}$. Thus, we have

$$\mathsf{Ev}_{\mathbb{M}'}(T') = T'(\Theta_{\mathbb{M}'}) = T\varphi^*(\Theta_{\mathbb{M}'}) = T(\Theta_{\mathbb{M}}) = \mathsf{Ev}_{\mathbb{M}}(T).$$

□

We will show that, for normalized supertraces, $\mathsf{Ev}_{\mathbb{M}}(t_{\mathbb{M}})$ can be, in a way, computed locally. In §2.5.1, we define an analogue of $\mathsf{Ev}_{\mathbb{M}}$ for supertraces on $\widehat{\mathcal{A}}_{2n|a,b}$. One should think of this as making sense of $\mathsf{Ev}_{\widehat{\mathbb{D}}^{2n|a,b}}$. Then, in §2.5.2, we show how this local analogue descends to $\mathsf{Ev}_{\mathbb{M}}$.

2.5.1 Formally Local Invariant

Consider a derived supertrace $t_{2n|a,b}$ on $\widehat{\mathcal{A}}_{2n|a,b}$ as in Definition 2.3.1. If we think of $\widehat{\mathcal{A}}_{2n|a,b}$ as a deformation of the formal disk $\widehat{\mathbb{D}}^{2n|a,b}$, then a formally local analogue of Definition 2.5.1 would be to evaluate $t_{2n|a,b}$ on a volume form for $\widehat{\mathbb{D}}^{2n|a+b}$. If we write

$$\mathcal{O}_{\widehat{\mathbb{D}}^{2n|a+b}} = \mathcal{O}_{\widehat{\mathbb{D}}^{2n}} \otimes \Lambda[\theta_1, \ldots, \theta_{a+b}],$$

then our volume form will be $1 \otimes \theta_1 \cdots \theta_{a+b} = 1 \otimes \Theta$.

If t is an underived supertrace, then we may take $t_{2n|a,b}(1 \otimes \Theta)$ to obtain an element of $\mathbb{K}$. Taking in to consideration that $t_{2n|a,b}$ is morphism of $(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$-modules, we will see that, in general, evaluation of $t_{2n|a,b}$ on $1 \otimes \Theta$ naturally lives in $C^{\bullet}_{\mathrm{Lie}}\left(\mathfrak{g}^{\hbar}_{2n|a,b}; \mathfrak{sp}_{2n|a,b}; \mathbb{K}\right)$.

To interpolate between derived supertraces and Lie algebra cochains, we need the following observation. Given an algebra A, we have an anti-symmetrization map

$$(-)^{\mathrm{Lie}} \colon \mathrm{Hoch}^{\bullet}(A; A^*) \to C^{\bullet}_{\mathrm{Lie}}(A; A^*)$$

where on the right-hand side, we are viewing A as a Lie algebra under the commutator. See, for example, [42, Pg. 11]. The map $(-)^{\mathrm{Lie}}$ is given by

$$(b^{\mathrm{Lie}})(a_1 \otimes \cdots a_k)(a_0) = \sum_{s \in \Sigma_k} \mathrm{sign}(s) b(a_0 \otimes a_{s(1)} \otimes \cdots a_{s(k)}).$$

We would like to use this map to view a derived trace $t_{2n|a,b} \in \mathrm{HH}^{\bullet}(\widehat{\mathcal{A}}_{2n|a,b})$ as a Lie algebra cocycle. To do so, we need a Hochschild cocycle representative $\widetilde{t}_{2n|a,b}$ of $t_{2n|a,b}$. The Lie algebra cocycle $\widetilde{t}^{\mathrm{Lie}}_{2n|a,b}$ will, in fact, live in a *relative* Lie algebra group. For an introduction to relative Lie algebra (co)homology, see [81, §2.3] or [38, §2.8.1].

In particular, for a Lie subalgebra $\mathfrak{h} \to \mathfrak{g}$, we have a map

$$C^{\bullet}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{h}) \to C^{\bullet}_{\mathrm{Lie}}(\mathfrak{g}).$$

For a $\mathfrak{g}$-module M, the group $C^\bullet_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{h}; M)$ is

$$C^p_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{h}; M) = \mathrm{Hom}_{\mathfrak{h}}(\wedge^p(\mathfrak{g}/\mathfrak{h}), M)$$

consisting of those cochains $c \in C^p_{\mathrm{Lie}}(\mathfrak{g}; M)$ so that

- $c(x_1 \wedge \cdots \wedge x_p) = 0$ if $x_i \in \mathfrak{h}$ for any $i = 1, \ldots, p$, and
- c is an $\mathfrak{h}$ invariant map.

See [81, Def. 2.17] for the definition of relative Lie algebra cohomology with coefficients.

Recall the notion of relative supertraces from Definition 2.3.3.

Lemma 2.5.3. *Let Z denote the center of $\widehat{\mathcal{A}}_{2n|a,b}$. If $t_{2n|a,b}$ is a derived relative supertrace on $\widehat{\mathcal{A}}_{2n|a,b}$, then the element $t^{\mathrm{Lie}}_{2n|a,b} \in C^\bullet_{\mathrm{Lie}}(\widehat{\mathcal{A}}_{2n|a,b}; (\widehat{\mathcal{A}}_{2n|a,b})^*)$ is in the image of the map*

$$C^\bullet_{\mathrm{Lie}}\left(\widehat{\mathcal{A}}_{2n|a,b}, \mathfrak{sp}_{2n|a,b} \oplus Z; (\widehat{\mathcal{A}}_{2n|a,b})^*\right) \to C^\bullet_{\mathrm{Lie}}\left(\widehat{\mathcal{A}}_{2n|a,b}; (\widehat{\mathcal{A}}_{2n|a,b})^*\right).$$

For the purely even case, see [42, §4.2 (ii)-(iii)].

Proof. The invariance under $\mathfrak{sp}_{2n|a,b}$ follows from the fact that $t_{2n|a,b}$ is $(\mathfrak{g}^\hbar_{2n|a,b}, \mathrm{Sp}(2n|a,b))$-module map, see [38, Thm. 2.11(ii)]. The condition that $t_{2n|a,b}$ is a relative supertrace translates to $\widetilde{t}^{\mathrm{Lie}}_{2n|a,b}$ vanishing on $\mathfrak{sp}_{2n|a,b}$ after noting that one can reduce from the sum over all elements of the symmetric group to just transpositions.

Reduction to a cocycle relative to Z follows from the fact that $Z \subset A$ is an abelian Lie subalgebra. □

By Notation 1.5.6, $\mathfrak{g}^\hbar_{2n|a,b}$ is the Lie algebra of derivations of $\widehat{\mathcal{A}}_{2n|a,b}$. Using the more general fact that $\mathrm{Der}(A) = A/Z$, we see that the pair $(\mathrm{Der}(A), \mathfrak{h})$ is equivalent to $(A, \mathfrak{h} \oplus Z)$. We may therefore view $\widetilde{t}^{\mathrm{Lie}}_{2n|a,b}$ as an element in

$$C^\bullet_{\mathrm{Lie}}\left(\mathfrak{g}^\hbar_{2n|a,b}, \mathfrak{sp}_{2n|a,b}; \left(\widehat{\mathcal{A}}_{2n|a,b}\right)^*\right).$$

Given an element $u \in \widehat{\mathcal{A}}_{2n|a,b}$, we have an evaluation map

$$\mathrm{ev}_u \colon C^{\bullet}_{\mathrm{Lie}}\left(\mathfrak{g}^{\hbar}_{2n|a,b}; \mathfrak{sp}_{2n|a,b}; (\widehat{\mathcal{A}}_{2n|a,b})^*\right) \to C^{\bullet}_{\mathrm{Lie}}\left(\mathfrak{g}^{\hbar}_{2n|a,b}; \mathfrak{sp}_{2n|a,b}; \mathbb{K}\right).$$

We will be interested in $\mathrm{ev}_u(\widetilde{t}^{\mathrm{Lie}}_{2n|a,b})$ for $u = 1 \otimes \Theta$.

Definition 2.5.4. Given a k-derived relative supertrace $t_{2n|a,b}$ on $\widehat{\mathcal{A}}_{2n|a,b}$ with cocycle representative $\widetilde{t}_{2n|a,b}$, the *evaluation of $t_{2n|a,b}$ on the formal volume form* $1 \otimes \Theta$ is

$$\mathsf{Ev}_{\mathrm{loc}}(\widetilde{t}_{2n|a,b}) = \mathrm{ev}_{1\otimes\Theta}(t^{\mathrm{Lie}}_{2n|a,b}) \in C^{k}_{\mathrm{Lie}}\left(\mathfrak{g}^{\hbar}_{2n|a,b}; \mathfrak{sp}_{2n|a,b}; \mathbb{K}\right).$$

Example 2.5.5. If $t_{2n|a,b} \colon \widehat{\mathcal{A}}_{2n|a,b} \to \mathbb{K}$ is underived, then $\mathsf{Ev}_{\mathrm{loc}}(t_{2n|a,b}$ lives in the zeroth cocycle group

$$C^{0}_{\mathrm{Lie}}\left(\mathfrak{g}^{\hbar}_{2n|a,b}; \mathfrak{sp}_{2n|a,b}; \mathbb{K}\right) = \mathrm{Hom}(\mathbb{K}, \mathbb{K}) \simeq \mathbb{K},$$

see [81, Prop. 2.16(1)]. We can identify $\mathsf{Ev}_{\mathrm{loc}}(t_{2n|a,b})$ with $t_{2n|a,b}(1 \otimes \Theta)$. Indeed, the map $(-)^{\mathrm{Lie}}$ is the identity in degree zero.

2.5.2 Globalizing the Invariant

Let $t_{2n|a,b}$ be a derived supertrace on $\widehat{\mathcal{A}}_{2n|a,b}$. By Corollary 2.4.2, $t_{2n|a,b}$ must be in degree $2n$,

$$t_{2n|a,b} \in \mathrm{HH}^{2n}(\widehat{\mathcal{A}}_{2n|a,b}, (\widehat{\mathcal{A}}_{2n|a,b})^*).$$

By Corollary 2.3.10, $t_{2n|a,b}$ determines a supertrace $t_{\mathbb{M}}$ on $\mathcal{A}_{\sigma}(\mathbb{M})$. The goal of this section is to recover $\mathsf{Ev}_{\mathbb{M}}(t_{\mathbb{M}})$ from $\mathsf{Ev}_{\mathrm{loc}}(t_{2n|a,b})$. This will be done using a variation of the Chern-Weil map.

Let $(\mathfrak{g}, K)$ be an HC pair. Let P be a $(\mathfrak{g}, K)$-bundle. As in [51, Def. 1.18], we have a natural transformation

$$\mathrm{char}_P \colon C^{\bullet}_{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Lie}(K); -) \Rightarrow \mathbf{desc}_P(-)$$

between functors $\mathrm{Mod}_{(\mathfrak{g},K)} \to \mathrm{Ch}_{\mathbb{K}}$.

Consider now the case of the sHC pair $(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Sp}(2n|a,b))$. Let $\mathrm{char}_{(\mathbb{M},\sigma)}$ denote the natural transformation coming from the principal $(\mathfrak{g}^\hbar_{2n|a,b}, \mathrm{Sp}(2n|a,b))$-bundle structure on $\mathrm{Fr}^{\mathrm{Sp}(2n|a,b)}_{\mathbb{M}}$, see Corollary 1.3.35 and §1.5.0.1. Then $\mathrm{char}_{(\mathbb{M},\sigma)}$ is given as follows. Let $A \in \Omega^1(\mathrm{Fr}^{\mathrm{Sp}(2n|a,b)}_{\mathbb{M}}; \mathfrak{g}^\hbar_{2n|a,b})$ be the flat connection 1-form so that

$$A^{\wedge i} \in \Omega^i(\mathrm{Fr}^{\mathrm{Sp}(2n|a,b)}_{\mathbb{M}}; \Lambda^i \mathfrak{g}^\hbar_{2n|a,b}).$$

View an element $r \in C^i_{\mathrm{Lie}}(\mathfrak{g}^\hbar_{2n|a,b}, \mathrm{sp}_{2n|a,b}; V)$ as a map

$$r \colon \Lambda^i(\mathfrak{g}^\hbar_{2n|a,b}/\mathfrak{sp}_{2n|a,b}) \to V.$$

Then r induces a map on $\mathrm{Sp}(2n|a,b)$-basic forms

$$r_* \colon \Omega^\bullet(\mathrm{Fr}^{\mathrm{Sp}(2n|a,b)}_{\mathbb{M}}; \Lambda^i \mathfrak{g}^\hbar_{2n|a,b})_{\mathrm{basic}} \to \Omega^\bullet(\mathrm{Fr}^{\mathrm{Sp}(2n|a,b)}_{\mathbb{M}}; V)_{\mathrm{basic}} = \mathbf{desc}_{(\mathbb{M},\sigma)}(V).$$

By definition, $\mathrm{char}_{(\mathbb{M},\sigma)}(r) = r_*(A^{\wedge i})$.

Theorem 2.5.6. *Let $t_{2n|a,b}$ be a $2n$-derived relative supertrace on $\widehat{\mathcal{A}}_{2n|a,b}$. For $(\mathbb{M},\sigma)$ and object in $\mathsf{sGK}^{=,\hbar}_{2n|a,b}$, let $t_{\mathbb{M}}$ denote the supertrace on $\mathcal{A}_\sigma(\mathbb{M})$ induced from $t_{2n|a,b}$ using Corollary 2.4.2. Then*

$$\mathsf{Ev}_{\mathbb{M}}(t_{\mathbb{M}}) = \int_{\mathbb{M}} \mathrm{char}_{(\mathbb{M},\sigma)}(\mathbb{K})(\mathsf{Ev}_{\mathrm{loc}}(t_{2n|a,b})).$$

Proof. By naturality, we get a commutative diagram

$$\begin{array}{ccc} C^\bullet_{\mathrm{Lie}}(\mathfrak{g}^\hbar_{2n|a,b}, \mathfrak{sp}_{2n|a,b}; ((\widehat{\mathcal{A}}_{2n|a,b})^*) & \xrightarrow{\mathrm{char}_{(\mathbb{M},\sigma)}((\widehat{\mathcal{A}}_{2n|a,b})^*)} & \mathbf{desc}_{(\mathbb{M},\sigma)}((\widehat{\mathcal{A}}_{2n|a,b})^*) \\ \downarrow{\scriptstyle \mathrm{ev}_{1\otimes\Theta}} & & \downarrow{\scriptstyle \mathrm{char}_{(\mathbb{M},\sigma)}(\mathrm{ev}_{1\otimes\Theta})} \\ C^\bullet_{\mathrm{Lie}}(\mathfrak{g}^\hbar_{2n|a,b}, \mathfrak{sp}_{2n|a,b}; \mathbb{K}) & \xrightarrow[\mathrm{char}_{(\mathbb{M},\sigma)}(\mathbb{K})]{} & \mathbf{desc}_{(\mathbb{M},\sigma)}(\mathbb{K}) \end{array}.$$

We can construct a morphism of $\Omega^\bullet_{\mathrm{dR}}(\mathbb{M};\mathbb{K})$-modules

$$B \colon \mathbf{desc}_{(\mathbb{M},\sigma)}\left((\widehat{\mathcal{A}}_{2n|a,b})^*\right) \to \mathrm{Hom}_{\Omega^\bullet_{\mathrm{dR}}(\mathbb{M};\mathbb{K})}\left(\mathbf{desc}_{(\mathbb{M},\sigma)}(\widehat{\mathcal{A}}_{2n|a,b}), \mathbf{desc}_{(\mathbb{M},\sigma)}(\mathbb{K})\right)$$

as follows.

By Construction (2.3), the descent functor is given by

$$\mathbf{desc}_{(\mathbb{M},\sigma)}(-) = \Omega^\bullet(\mathbb{M}; \mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} (-)).$$

Since the bundle of homomorphisms is formed fiberwise, we have an identification

$$\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} (\widehat{\mathcal{A}}_{2n|a,b})^* \simeq \mathrm{Hom}_{\mathrm{Bun}}(\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} \widehat{\mathcal{A}}_{2n|a,b}, \mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} \mathbb{K}).$$

Letting E denote the bundle $\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} \widehat{\mathcal{A}}_{2n|a,b}$ and $E^\vee$ the bundle dual, we are therefore looking for a map

$$B\colon \Omega^\bullet(\mathbb{M}; E^\vee) \to \mathrm{Hom}_{\Omega^\bullet(\mathbb{M};\mathbb{K})}\left(\Omega^\bullet(\mathbb{M}; E), \Omega^\bullet(\mathbb{M}; \mathbb{K})\right).$$

By adjunction, this is the same as a map

$$\Omega^\bullet(\mathbb{M}; E^\vee) \otimes_{\Omega^\bullet(\mathbb{M};\mathbb{K})} \Omega^\bullet(\mathbb{M}; E) \to \Omega^\bullet(\mathbb{M}; \mathbb{K}).$$

Such a map is given by the product of forms together with the evaluation map $E^\vee \otimes E \to \underline{\mathbb{K}}_{\mathbb{M}}$. Note that this map sends a degree i form and a degree j form to a degree $i+j$ form.

We can therefore view $\left(B \circ \mathrm{char}_{(\mathbb{M},\sigma)}\left((\widehat{\mathcal{A}}_{2n|a,b})^*\right)\right)(t^{\mathrm{Lie}}_{2n|a,b})$ as a map

$$\Omega^\bullet(\mathbb{M}; \mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} \widehat{\mathcal{A}}_{2n|a,b}) \to \Omega^{\bullet+2n}_{\mathrm{dR}}(\mathbb{M}; \mathbb{K}),$$

where the degree shift is since $t^{\mathrm{Lie}}_{2n|a,b}$ is a coycle of degree $2n$.

In degree 0, this map is given by sending $a \in \Omega^0(\mathbb{M}; \mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)} \times_{\mathrm{Sp}(2n|a,b)} \widehat{\mathcal{A}}_{2n|a,b})$ to the $2n$ form $(t^{\mathrm{Lie}}_{2n|a,b})_*(A^{\wedge 2n})(a)$. Note that fiberwise we are evaluating the antisymmetrization $t^{\mathrm{Lie}}_{2n|a,b}$ on a class $A^{\wedge 2n}$ in the diagonal,

$$(t^{\mathrm{Lie}}_{2n|a,b})_*(A^{\wedge 2n})(a) = \sum_{s\in\Sigma_{2n}} \mathrm{sign}(s) t_{2n|a,b}(a \otimes A \otimes \cdots \otimes A) = t_{2n|a,b}(a \otimes A \otimes \cdots \otimes A).$$

Since $\tilde{A}$ is a lift of A (see §2.2.1.2), the map defined fiberwise by $t_{2n|a,b}(a \otimes A \otimes \cdots \otimes A)$ is exactly $\mathbf{desc}_{(\mathbb{M},\sigma)}(t_{2n|a,b}) \circ \beta_{\tilde{A}}$ where $\beta_{\tilde{A}}$ is as in Proposition 2.3.9. Thus, in degree zero we have an equivalence of maps

$$\left(B \circ \mathrm{char}_{(\mathbb{M},\sigma)}\left((\widehat{\mathcal{A}}_{2n|a,b})^*\right)\right)(t^{\mathrm{Lie}}_{2n|a,b}) = \mathbf{desc}_{(\mathbb{M},\sigma)}(t_{2n|a,b}) \circ \beta_{\tilde{A}}.$$

Lastly, we claim that the diagram

$$\begin{array}{ccc} \mathbf{desc}_{(\mathbb{M},\sigma)}\left((\widehat{\mathcal{A}}_{2n|a,b})^*\right) & \xrightarrow{B} & \mathrm{Hom}_{\Omega^\bullet_{\mathrm{dR}}(\mathbb{M};\mathbb{K})}\left(\mathbf{desc}_{(\mathbb{M},\sigma)}(\widehat{\mathcal{A}}_{2n|a,b}), \mathbf{desc}_{(\mathbb{M},\sigma)}(\mathbb{K})\right) \\ \Big\downarrow{\scriptstyle \mathrm{char}_{(\mathbb{M},\sigma)}(\mathrm{ev}_{1\otimes\Theta})} & \swarrow{\scriptstyle \mathrm{ev}_{\Theta_{\mathbb{M}}}} & \\ \Omega^\bullet_{\mathrm{dR}}(\mathbb{M};\mathbb{K}) & & \end{array}$$

commutes. Indeed, $\mathrm{char}_{(\mathbb{M},\sigma)}(\mathrm{ev}_{1\otimes\Theta})$ is given by fiberwise evaluating on $1 \otimes \Theta$. The composition $\mathrm{ev}_{\Theta_{\mathbb{M}}} \circ B$ is likewise given fiberwise over $x \in \mathbb{M}$ by evaluating on $\Theta_{\mathbb{M}}$ restricted to x. The volume form $\Theta_{\mathbb{M}}$ was defined to restrict to $1 \otimes \Theta$ over each point, see Definition 2.2.4.

By definition (Corollary 2.4.2), the supertrace $t_{\mathbb{M}}$ is given by the formula

$$t_{\mathbb{M}} = \int_{\mathbb{M}} \left(\mathbf{desc}_{(\mathbb{M},\sigma)}(t_{2n|a,b}) \circ \beta_{\tilde{A}}\right).$$

Putting this all together, we have

$$\begin{aligned} \mathsf{Ev}_{\mathbb{M}}(t_{\mathbb{M}}) &= \mathrm{ev}_{\Theta_{\mathbb{M}}}(t_{\mathbb{M}}) \\ &= \mathrm{ev}_{\Theta_{\mathbb{M}}}\left(\int_{\mathbb{M}} \left(\mathbf{desc}_{(\mathbb{M},\sigma)}(t_{2n|a,b}) \circ \beta_{\tilde{A}}\right)\right) \\ &= \int_{\mathbb{M}} \mathrm{ev}_{\Theta_{\mathbb{M}}}\left(\mathbf{desc}_{(\mathbb{M},\sigma)}(t_{2n|a,b}) \circ \beta_{\tilde{A}}\right) \\ &= \int_{\mathbb{M}} \left(\mathrm{ev}_{\Theta_{\mathbb{M}}} \circ B \circ \mathrm{char}_{(\mathbb{M},\sigma)}\left((\widehat{\mathcal{A}}_{2n|a,b})^*\right)\right)(t^{\mathrm{Lie}}_{2n|a,b}) \\ &= \int_{\mathbb{M}} \mathrm{char}_{(\mathbb{M},\sigma)}(\mathrm{ev}_{1\otimes\Theta})\left(\mathrm{char}_{(\mathbb{M},\sigma)}\left((\widehat{\mathcal{A}}_{2n|a,b})^*\right)(t^{\mathrm{Lie}}_{2n|a,b})\right) \\ &= \int_{\mathbb{M}} \mathrm{char}_{(\mathbb{M},\sigma)}(\mathbb{K})(\mathrm{ev}_{1\otimes\Theta}(t^{\mathrm{Lie}}_{2n|a,b})) \\ &= \int_{\mathbb{M}} \mathrm{char}_{(\mathbb{M},\sigma)}(\mathbb{K})(\mathsf{Ev}_{\mathrm{loc}}(t_{2n|a,b})). \end{aligned}$$

□

Below, in §2.8.3, we will see that the characteristic functor $\mathrm{char}_{(\mathbb{M},\sigma)}$ relates to the classical

Chern-Weil map [69, Appendix C].

Part II

Computations and Constructions

In this part, we will construct a normalized supertrace on $\mathcal{A}_\sigma(\mathbb{M})$ and compute its evaluation on the volume form. The purely even portion of these computations can be found in [42]. We begin with preliminaries on Clifford algebras that we will need in our computations.

Given a quadratic vector space (V, Q), we can form several different structures

- the special orthogonal group $\mathrm{SO}(V, Q)$ and its Lie algebra $\mathfrak{so}(V, Q)$,
- the Clifford algebra $\mathrm{Cliff}(V, Q)$, and
- a symplectic structure ω_Q on the supermanifold $\mathbb{R}^{0|\dim(V)}$.

These objects are related. For example, the Lie algebra $\mathfrak{so}(V, Q)$ embeds in the Clifford algebra. The following can be found, for example, in [59, Pg. 61].

Lemma 2.5.7. *Consider the Lie subalgebra*

$$[V, V]_- = \{vw - wv : v, w \in \mathrm{Cliff}(V, Q)\}$$

in $\mathrm{Cliff}(V, Q)$*. There is an isomorphism of Lie algebras*

$$\Phi\colon [V, V]_- \to \mathfrak{so}(V, Q)$$

given by sending $w \in [V, V]_-$ *to the endomorphism* $[-, w]_- : V \to V$*.*

Note that Φ allows us to view $\mathfrak{so}(V, Q)$ as an *even* subspace of $\mathrm{Cliff}(V, Q)$.

Just as the Weyl algebra is a deformation of a polynomial algebra, the Clifford algebra is a deformation of an exterior algebra. The local results in Chapter 1 (and [38]) show that the symplectic supermanifold $(\mathbb{R}^{0|\dim(V)}, \omega_Q)$ has a canonical deformation by $\mathrm{Cliff}(V, Q)$, as $\mathrm{SO}(V, Q)$-modules.

2.6 Quadratic Forms of Signature (a, b)

Over $\mathbb{R}$, a quadratic form Q on a vector space V is determined by its signature (a, b). In this section, we will analyze the various constructions (orthogonal groups, Clifford algebras, symplectic superspaces) for signature (a, b).

Notation 2.6.1. Unless otherwise noted, Q will denote a quadratic form on $\mathbb{R}^{a+b}$ of signature (a,b). A quadratic form Q on a vector space V has associated matrix H_Q with $Q(v) = v^T H_Q v$, and bilinear form B_Q with $Q(v) = B_Q(v,v)$.

We will use the following shorthands:

$$\begin{aligned}
\mathrm{Cliff}(V,Q) &= \mathrm{Ciff}_{a,b} \\
\mathrm{SO}(V,Q) &= \mathrm{SO}(a,b) \\
\mathfrak{so}(V,Q) &= \mathfrak{so}_{a,b}.
\end{aligned}$$

Without loss of generality, we may assume $a \leq b$. Indeed, there are equivalences of Lie groups

$$\mathrm{SO}(a,b) \simeq \mathrm{SO}(b,a)$$

and of superalgebras

$$\mathrm{Cliff}_{a,b} \simeq \mathrm{Cliff}_{b,a}.$$

An explicit isomorphism can be found right above [50, Def. 1.1.1].

Let $t = \lfloor \frac{b-a}{2} \rfloor$. Note that if $b-a$ is odd, then $2a+2t+1 = a+b$ is the dimension of V. We will fix a basis

$$\{\zeta_1, \ldots, \zeta_a, \eta_1, \ldots, \eta_a, \xi_1, \ldots, \xi_t, \mu_1, \ldots, \mu_t, \upsilon\}$$

of $\mathbb{R}^{0|a+b}$ (where υ is only included if $b-a$ is odd) with

$$\begin{aligned}
&B_Q(\zeta_i, \eta_i) = 1 \text{ for all } i = 1, \ldots, a \\
&B_Q(\xi_i, \xi_i) = -1 \text{ for all } i = 1, \ldots, t \\
&B_Q(\mu_i, \mu_i) = -1 \text{ for all } i = 1, \ldots, t \\
&B_Q(\upsilon, \upsilon) = -1 \text{ for all } i = 1, \ldots, t.
\end{aligned}$$

In our chosen basis, the matrix H_Q associated to Q is

$$H_Q = (h_Q^{ij}) = \left[\begin{array}{cc|c} 0 & \mathrm{Id}_a & \multirow{2}{*}{0} \\ \mathrm{Id}_a & 0 & \\ \hline \multicolumn{2}{c|}{C} & -\mathrm{Id}_{b-a} \end{array}\right].$$

As noted in Definition 2.4.3, the supertraces we will define will depend on an orientation.

Notation 2.6.2. We will give $\mathbb{R}^{0|a+b}$ the orientation

$$\Theta = \zeta_1\eta_1 \cdots \zeta_a\eta_a\xi_1\mu_1 \cdots \xi_t\mu_t\upsilon.$$

We will sometimes also use the notation $\theta_1, \ldots, \theta_{a+b}$ so that $\theta_1 = \zeta_1$, $\theta_2 = \eta_1$, and so on giving $\Theta = \theta_1 \cdots \theta_{a+b}$. Note that this is the same choice as was made in §2.5.1.

Proof of Lemma 2.2.3. Take $\theta_1, \ldots, \theta_{a+b}$ as in Notation 2.6.2. □

2.6.0.1 Cartan Subalgebra

We describe a Cartan subalgebra of $\mathfrak{so}_{a,b}$. We will use this later in the proof of Theorem 2.8.13.

Lemma 2.6.3. *A Cartan subalgebra of* $\mathfrak{so}_{a,b}$ *is given by the subset*

$$\{\mathfrak{h} = \begin{bmatrix} \mathrm{diag}(h_1, \ldots, h_a) & 0 & 0 & 0 \\ 0 & -\mathrm{diag}(h_1, \ldots, h_a) & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{diag}(u_1, \ldots, u_t) \\ 0 & 0 & \mathrm{diag}(u_1, \ldots, u_t) & 0 \end{bmatrix} : r_i, s_i \in \mathbb{K}\}$$

if $b - a$ is even. If $b - a$ is odd, there is an additional row and column of zeroes, corresponding to the υ basis element.

See [84, Pg. 402].

Let $E_{i,j}$ denote the matrix with a 1 in the ijth entry and zeroes elsewhere. A basis for our chosen Cartan subalgebra is given by

$$\mathcal{B} = \{E_{i,i} - E_{a+i,a+i}, E_{2a+j,2a+t+j} - E_{2a+t+j,2a+j}\}_{i=1,\dots,a}^{j=1,\dots,t}$$

We would like to view these basis elements in the Clifford algebra under the isomorphism Φ from Lemma 2.5.7. In other words, given a basis element $U \in \mathcal{B}$, we would like find an element $v \in \mathrm{Cliff}_{a,b}$ so that the morphisms $\Phi(v) = [-, v]_-$ and $U : V \to V$ agree.

By direct computation, one obtains the following.

Lemma 2.6.4. *For each $i = 1, \dots, a$ we have*

$$\Phi(\eta_i \zeta_i) = E_{i,i} - E_{a+i,a+i}$$

and for each $j = 1, \dots, t$ we have

$$\Phi(-\xi_j \mu_j) = E_{2a+j,2a+t+j} - E_{2a+t+j,2a+j}.$$

2.6.0.2 Symplectic Form

The symplectic form on $\mathbb{R}^{0|a+b}$ determined by Q is given by

$$\omega_Q = \sum_{i,j} \frac{1}{2} h_Q^{ij} \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial \theta_j}$$

for entries (h_Q^{ij}) of the matrix H_Q, see Example 1.2.8.

Using our naming convention (Notation 2.6.2), we can rewrite ω_Q as

$$\omega_Q = \frac{1}{2} \sum_{i=1}^{a} \left(\frac{\partial}{\partial \zeta_i} \otimes \frac{\partial}{\partial \eta_i} + \frac{\partial}{\partial \zeta_i} \otimes \frac{\partial}{\partial \eta_i} \right) - \frac{1}{2} \sum_{j=1}^{t} \left(\frac{\partial}{\partial \xi_j} \otimes \frac{\partial}{\partial \xi_j} - \frac{\partial}{\partial \mu_j} \otimes \frac{\partial}{\partial \mu_j} \right) - \left(\frac{\partial}{\partial \upsilon} \right)^{\otimes 2}.$$

2.7 Constructing the supertrace

The goal of this section is to construct a supertrace on $\widehat{\mathcal{A}}_{2n|a,b}$. This construction will depend on a choice of orientation of $\mathbb{R}^{2n|a+b}$, see §2.2.1.3. By Corollary 2.3.10, this supertrace will descend to define a supertrace on $(\mathbb{M}, \omega)$. Recall that $\widehat{\mathcal{A}}_{2n|a,b}$ is the tensor product of a Weyl and a Clifford algebra. By Lemma 2.4.1, we would like to construct an element of

$$\mathrm{HH}^n(\mathrm{Weyl}_{2n} \otimes \mathrm{Cliff}_{a,b}) = \mathrm{HH}^n(\mathrm{Weyl}_{2n}) \otimes \mathrm{HH}^0(\mathrm{Cliff}_{a,b})$$

In fact, we would like a cocycle representative of the cohomology class so we may apply Definition 2.5.4.

In the purely even case, an appropriate element of $\tau_{2n} \in \mathrm{Hoch}^n(\mathrm{Weyl}_{2n})$ was constructed in [42]. We review the definition of τ_{2n} below. The element is constructed using Kontsevich formality, see [42, Rmk. Pg. 7].

In the purely odd case, the Clifford algebra $\mathrm{Cliff}_{a,b}$ has a canonical underived supertrace.

Lemma 2.7.1. *Let (V, Q) be a quadratic real vector space. View V as an even space. Given an orientation $\det(V) \simeq \mathbb{K}$, the quotient map*

$$\mathrm{Cliff}(V, Q) \to \mathrm{Cliff}(V, Q)/\mathrm{Cliff}_{(n-1)}(V, Q) \simeq \det(V) \simeq \mathbb{K}$$

defines a supertrace on $\mathrm{Cliff}(V, Q)$.

For a proof, see [66, Prop. 2.10].

Remark 2.7.2. If we use $\Theta \in \det(V)$ to identify $\det(V)$ with $\mathbb{K}$, then the supertrace of Lemma 2.7.1 is the same map as the Berezin integral $\int(-)d\Theta$.

Remark 2.7.3. When $a = b$, and we choose the orientation $\zeta_1\eta_1 \cdots \zeta_a\eta_a$ (as in Notation 2.6.2) this supertrace can also be described using the spinor representation, [36]. We can identify the quadratic space (V, Q) with $(W \oplus W^*, \mathrm{ev})$. The spinor representation of the Clifford algebra $\mathrm{Cliff}(V, Q)$ is then a map

$$\rho_{\mathrm{spin}} \colon \mathrm{Cliff}(V, Q) \to \mathrm{End}(\mathrm{Sym}(W[1])).$$

This map (of algebras) is an isomorphism. As $W[1]$ is odd, we can identify $\mathrm{End}(\mathrm{Sym}(W[1]))$ with finite dimensional matrices. Taking the supertrace of matrices, we obtain a supertrace

$$t_2 \colon \mathrm{Cliff}(V, Q) \to \mathbb{K}.$$

Note that t_2 is the map inducing the Morita equivalence between the Clifford algebra and $\mathbb{K}$.

We claim that t_2 agrees with the supertrace in Lemma 2.7.1, up to a scalar. As

$$\mathrm{Cliff}_{a,a} = (\mathrm{Cliff}_{1,1})^{\otimes a},$$

it suffices to prove this when $a = 1$. When $a = 1$, the map

$$\rho_{\mathrm{spin}} \colon \mathrm{Cliff}_{1,1} \to M_{2\times 2}(\mathbb{R})$$

given by

$$\begin{aligned} \rho_{\mathrm{spin}}(\zeta_1) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \rho_{\mathrm{spin}}(\eta_1) &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ \rho_{\mathrm{spin}}(\zeta_1\eta_1) &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Thus, $t_2(\zeta_1\eta_1) = t_2(\Theta) = -2$, which is -2 times the Berezin integral $\int(\Theta)d\Theta = 1$.

This supertrace determines an element $\tau_{0|a,b} \in \mathrm{Hoch}^0(\mathrm{Cliff}_{a,b})$. Together, τ_{2n} and $\tau_{0|a,b}$ determine a cohomology class

$$[\tau_{2n}] \otimes [\tau_{0|a,b}] \in \mathrm{HH}^n(\mathrm{Weyl}_{2n} \otimes \mathrm{Cliff}_{a,b})$$

In §2.7.1, we produce a cocycle representative of this class. That is, a map

$$(\widehat{\mathcal{A}}_{2n|a,b})^{\otimes 2n+1} \to \mathbb{K}$$

that has a corresponding Hochschild cocycle $[\tau_{2n}] \otimes [\tau_{0|a,b}]$. As $\tau_{0|a,b}$ was defined from a map out of $\mathrm{Cliff}_{a,b}$, we need a way of lifting this map to a map out of the $(2n+1)$-fold tensor product of the Clifford algebra.

2.7.1 Lifting to Degree $2n$

We have a class in $\mathrm{HH}^{2n}(\mathrm{Weyl}_{2n} \otimes \mathrm{Cliff}_{a,b})$. We would like a class in $\mathrm{Hoch}^{2n}(\mathrm{Weyl}_{2n} \otimes \mathrm{Cliff}_{a,b})$.

2.7.1.1 General Argument

In general, one has an external product map in Hochschild cohomology

$$\vee\colon \mathrm{HH}^i(\Lambda, B) \otimes \mathrm{HH}^j(\Gamma, B') \to \mathrm{HH}^{i+j}(\Lambda \otimes \Gamma, B \otimes B')$$

which is constructed in [21, Ch. XI §6] and defined by what they call g on [21, Pg. 219 (3)].

We are interested in the case

$$\mathrm{HH}^{2n}(\mathrm{Weyl}_{2n}; (\mathrm{Weyl}_{2n})^*) \otimes \mathrm{HH}^0(\mathrm{Cliff}_{a,b}; (\mathrm{Cliff}_{a,b})^*) \to \mathrm{HH}^{2n}(\mathrm{Weyl}_{2n} \otimes \mathrm{Cliff}_{a,b}; (\mathrm{Weyl}_{2n} \otimes \mathrm{Cliff}_{a,b})^*).$$

In this case, the product is given by

$$(f \vee g)(w_0 \otimes c_0 | \cdots | w_{2n} \otimes c_{2n}) = f(w_0 | \cdots | w_{2n}) \otimes g(c_0 \cdots c_{2n})$$

where $w_i \in \mathrm{Weyl}_{2n}$, $c_i \in \mathrm{Cliff}_{a,b}$, and the bars denote the tensor product in the bar complex defining Hochschild cohomology.

Completing the Weyl algebra, we obtain a class $[\tau_{2n}] \vee [\tau_{0|a,b}]$ in $\mathrm{HH}^{2n}(\widehat{\mathcal{A}}_{2n|a,b}; (\widehat{\mathcal{A}}_{2n|a,b})^*)$ which involves evaluating $\tau_{0|a,b}$ on a product $c_0 \cdots c_{2n}$. The product in $\widehat{\mathcal{A}}_{2n|a,b}$, as defined in Definition 1.5.4 or [38, §1.4], is given by

$$x \star y = m\left(\left(\exp\left(\frac{\hbar}{2}(\alpha + g)\right)\right)(x \otimes y)\right)$$

where $\alpha + g$ is the bivector

$$\alpha + g = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial q_i} - \frac{\partial}{\partial q_i} \otimes \frac{\partial}{\partial p_i} - \sum_{i=1}^{a+b} h_Q^{ij} \left(\frac{\partial}{\partial \theta_i} \otimes \frac{\partial}{\partial \theta_j}\right). \tag{2.4}$$

Here, for $x = p_i, q_i, \theta_i$, the endomorphism $\frac{\partial}{\partial x}$ of $\widehat{\mathcal{A}}_{2n|a,b}$ is given by identifying the underlying module of $\widehat{\mathcal{A}}_{2n|a,b}$ with $\widehat{\mathcal{O}}_{2n|a,b}[[\hbar]]$ using the standard Poincaré-Birkhoff-Witt isomorphism in characteristic zero given by (super)-symmetrization, then taking partial derivatives of polynomials as usual.

Remark 2.7.4. Note that $\frac{\partial}{\partial\theta_i}$ is an odd map on $\widehat{\mathcal{A}}_{2n|a,b}$ since θ_i is an odd coordinate. This will be important for our calculations in §2.8.

Manipulating the exponential power series appearing in $c_0 \star \cdots \star c_{2n}$ will result in a complicated formula that appears in $\omega_{2n|a,b}$ below, §2.7.1.2.

Remark 2.7.5. One could use this same argument to reduce from type $(2n|a,b)$ to type $(2|1,1)$ and type $(2|0,2)$ since $\mathrm{Weyl}_{2n} = \mathrm{Weyl}_2^{\otimes n}$ and similarly for $\mathrm{Cliff}_{a,b}$. Applying this process to the Weyl algebra gives some explanation for the complicated formula for τ_{2n} given below and in [42].

2.7.1.2 Description in Our Case

We now describe our desired cocycle, which we will denote $\tau_{2n|a,b} \in \mathrm{Hoch}^{2n}(\widehat{\mathrm{Weyl}}_{2n} \otimes \mathrm{Cliff}_{a,b})$.

The supertrace cocycle is a map

$$\tau_{2n|a,b} \colon (\widehat{\mathcal{A}}_{2n|a,b})^{\otimes 2n+1} \to \mathbb{K}.$$

This map should look like τ_{2n} on the Weyl algebra pieces, and applying $\tau_{0|a,b}$ to the product of the Clifford algebra pieces. Recall from Remark 2.7.2 that $\tau_{0|a,b}$ is a Berezin integral $\int(-)d\Theta$.

We will construct $\tau_{2n|a,b}$ as the composite of three maps:

1. the counit of the Hopf algebra $\widehat{\mathrm{Weyl}}2n \otimes \mathrm{Cliff}_{a,b} = U(\mathfrak{h}_{2n}) \otimes_{\mathbb{K}} U(\mathrm{cl}_{a,b})$ which is a Berezin integral (which depends on a choice of orientation Θ)

$$\Upsilon_{2n|a,b} \colon (\widehat{\mathcal{A}}_{2n|a,b})^{\otimes 2n+1} \to \mathbb{K},$$

2. a complicated combination of the bidifferential operators coming from ω_Q, mixed with configuration space integrals (which appear from Kontsevich formality. See [42, Rmk. Pg. 7].)

$$\int_{\Delta_{2n}} \omega_{2n|a,b} \colon (\widehat{\mathcal{A}}_{2n|a,b})^{\otimes 2n+1} \to (\widehat{\mathcal{A}}_{2n|a,b})^{\otimes 2n+1},$$

3. and a map

$$\pi_{2n|a,b}\colon (\widehat{\mathcal{A}}_{2n|a,b})^{\otimes 2n+1} \to (\widehat{\mathcal{A}}_{2n|a,b})^{\otimes 2n+1}.$$

Spelled out, we will consider the composite

$$(\widehat{\mathcal{A}}_{2n|a,b})^{\otimes 2n+1} \xrightarrow{\pi_{2n|a,b}} (\widehat{\mathcal{A}}_{2n|a,b})^{\otimes 2n+1} \xrightarrow{\int_{\Delta_{2n}} \omega_{2n|a,b}} (\widehat{\mathcal{A}}_{2n|a,b})^{\otimes 2n+1} \xrightarrow{\Upsilon_{2n|a,b}} \mathbb{K}$$

which we denote $\tau_{2n|a,b}$.

We will construct and study each of these three maps individually and then show that $\tau_{2n|a,b}$ satisfies the necessary properties. In particular, we will show that there is an equivalence of cohomology classes

$$[\tau_{2n|a,b}] = [\tau_{2n}] \otimes [\tau_{0|a,b}].$$

Berezin Integral. Define the map $\Upsilon_{2n|a,b}\colon (\widehat{\mathcal{A}}_{2n|a,b})^{\otimes 2n+1} \to \mathbb{K}$ by

$$\Upsilon_{2n|a,b}(a_0 \otimes \cdots \otimes a_{2n}) = \int (a_0 \cdots a_{2n})(y,u)|_{y=0} d\theta_1 \cdots d\theta_{a+b}.$$

The notation $\mu_{2n|a,b}$ is used for $\Upsilon_{2n|a,b}$ in [38]. We avoid this notation to prevent conflicts with the elements μ_i in $\widehat{\mathcal{A}}_{2n|a,b}$.

The notation $a(y,u)|_{y=0}$ means the following. Assume $a \in \widehat{\mathcal{A}}_{2n|a,b}$ is the tensor product $w \otimes c$ of $w \in \widehat{\mathrm{Weyl}}_{2n}$ and $c \in \mathrm{Cliff}_{a,b}$. Then w can be viewed as a power series in $2n$ even variables y and c can be viewed as a polynomial in $a+b$ odd variables u. Then $a(y,u)|_{y=0}$ means $w(0) \otimes c$, where we evaluated the power series w at 0.

Then $\Upsilon_{2n|a+b}$ is the counit for the Hopf superalgebra $\widehat{\mathcal{A}}_{2n|a,b} = U(\mathfrak{h}_{2n}) \otimes_{\mathbb{K}} U(\mathfrak{cl}_{a,b})$.

Bidifferential Operators and Configuration Space Integrals. For fixed k and $1 \le i < j \le k$, let α_{ij} be the endomorphism of $\left(\widehat{\mathcal{A}}_{2n|a,b}\right)^{\otimes k+1}$ sending $a_0 \otimes \cdots \otimes a_k$ to

$$\frac{1}{2}\sum_{l=1}^{n} a_0 \otimes \cdots \otimes \frac{\partial}{\partial p_l} a_i \otimes \cdots \otimes \frac{\partial}{\partial q_l} a_j \otimes \cdots \otimes a_k - a_0 \otimes \cdots \otimes \frac{\partial}{\partial q_l} a_i \otimes \cdots \otimes \frac{\partial}{\partial p_l} a_j \otimes \cdots \otimes a_k.$$

For $i < j$, we let $\alpha_{ji} = -\alpha_{ij}$. Essentially, α_{ij} acts by applying the two-form α from (2.4) to the ijth factor $a_i \otimes a_j$. Similarly, we define $g_{ij} \in \mathrm{End}\left(\mathcal{A}_{2n|a,b}^{\otimes k+1}\right)$ by

$$g_{ij}(a_0 \otimes \cdots \otimes a_k) = \frac{1}{2} \sum_{m,l} h_Q^{ml} a_0 \otimes \cdots \frac{\partial}{\partial \theta_m} a_i \otimes \cdots \otimes \frac{\partial}{\partial \theta_l} a_j \otimes \cdots \otimes a_k.$$

For $i < j$, we let $g_{ji} = g_{ij}$.

Let Δ_{2n} be the space

$$\Delta_{2n} = \{(v_1, \dots, v_{2n}) \in [0,1]^{2n} \colon j < k \text{ implies } v_j < v_k\}.$$

Let $\omega_{2n|a,b}$ be the endomorphism of $(\widehat{\mathcal{A}}_{2n|a,b})^{2n+1}$ given by

$$\omega_{2n|a,b} = \exp\left(\sum_{1 \le i < j \le 2n} \hbar \psi(v_i - v_j)(\alpha_{ij} + g_{ij}) \right)$$

where ψ is the 1-periodic function so that $\psi(v) = 2v + 1$ for $-1 \le v < 0$. That is, $\psi(v) = 2B_1(v)$ for $B_1(v)$ the 1st Bernoulli polynomial.

Third Map. Let $\pi_{2n} \in \mathrm{End}((\widehat{\mathcal{A}}_{2n|a,b})^{2n+1})$ be the map given by

$$\pi_{2n} = \frac{1}{n!} \left(\sum_{1 \le j < k \le 2n} \alpha_{ij} dv_j \wedge dv_k \right)^n.$$

Definition 2.7.6. Let $\tau_{2n|a,b}$ denote the Hochschild cocycle corresponding to the map

$$\tau_{2n|a,b} = \Upsilon_{2n|a,b} \int_{\Delta_{2n}} \omega_{2n|a,b} \circ \pi_{2n|a,b}.$$

The map $\tau_{2n|a,b}$ is an even map if $a + b$ is even and an odd map if $a + b$ is odd, see Corollary 2.4.2.

We now need to check that the cocycle $\tau_{2n|a,b}$ is a representative of our chosen cohomology class.

Lemma 2.7.7. *The cocycle $\tau_{2n|a,b}$ has cohomology class $[\tau_{2n}] \otimes [\tau_{0|a,b}]$.*

Proof. Setting $a = b = 0$, we obtain the formula for τ_{2n} given in [42, §2.3 (2)]. Taking $n = 0$, the terms $\omega_{0|a,b}$ and π_0 become the identity map. We are left with the Berezin integral $\int(-)d\Theta = \tau_{0|a,b}$. □

Proposition 2.7.8. *The cocycle $\tau_{2n|a,b}$ corresponds to a derived relative supertrace on $\widehat{\mathcal{A}}_{2n|a,b}$ as an object of* $\mathrm{Alg}(\mathrm{Mod}_{(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathrm{Sp}(2n|a,b))})$.

Proof. This is [38, Thm. 2.11 (ii) and (iii)]. □

Theorem 2.7.9. *The cocycle* $\tau_{2n|a,b}$ *determines a* $2n$*-derived supertrace on* $\widehat{\mathcal{A}}_{2n|a,b}$*. Descending* $\tau_{2n|a,b}$ *to* $\mathbb{M}$ *gives a normalized supertrace* $\mathrm{Tr}_{\mathbb{M}}$ *on* $\mathcal{A}_\sigma(\mathbb{M})$*.*

Proof. By Corollary 2.3.10, $\tau_{2n|a,b}$ determines a supertrace on $\mathcal{A}_\sigma(\mathbb{M})$ by the map on cohomology

$$\mathbf{desc}_{(\mathbb{M},\sigma)}(\tau_{2n|a,b}) \circ \beta_{\tilde{A}}.$$

The normalization condition is shown in [38, Thm. 2.11(i)]. □

2.8 Computation of Evaluation on a Volume Form

The goal of this section is to compute $\mathrm{Tr}_{\mathbb{M}}(\Theta_{\mathbb{M}})$ for the supertrace $\mathrm{Tr}_{\mathbb{M}}$ as in Theorem 2.7.9. Motivated by Theorem 2.5.6, we will first compute $\mathrm{ev}_\Theta(\tau^{\mathrm{Lie}}_{2n|a,b})$. We view this as the local computation, which is stated below as Theorem 2.8.13. We end this section by proving the superalgebraic index theorem, generalizing Engeli's results [38, Thm. 2.26].

2.8.1 Local Superalgebraic Index Theorem: Set Up

By Proposition 2.7.8, the class

$$\tau_{2n|a,b} \in \mathrm{Hoch}^{2n}(\mathcal{A}_{2n|a,b})$$

defines a $\mathfrak{sp}_{2n|a,b}$ equivariant class. Using Definition 2.5.4, we get a class

$$\mathsf{Ev}_{\mathrm{loc}}(\tau_{2n|a,b}) \in C^\bullet_{\mathrm{Lie}}(\mathfrak{g}^\hbar_{2n|a,b}, \mathfrak{sp}_{2n|a,b}).$$

As our eventual goal is to compute the global invariant $\mathsf{Ev}_{\mathbb{M}}(\mathrm{Tr}_{\mathbb{M}})$, we only need to know the cohomology class of $\mathsf{Ev}_{\mathrm{loc}}(\tau_{2n|a,b}) = \mathrm{ev}_\Theta(\tau^{\mathrm{Lie}}_{2n|a,b})$. Indeed, by Theorem 2.5.6, the value of $\mathsf{Ev}_{\mathbb{M}}(\mathrm{Tr}_{\mathbb{M}})$ is given by integrating over a term determined by $\mathsf{Ev}_{\mathrm{loc}}(\tau_{2n|a,b})$. By Stoke's theorem, the integral only depends on the cohomology class.

We will show that the class $[\mathsf{Ev}_{\mathrm{loc}}(\tau_{2n|a,b})] \in H^{2n}_{\mathrm{Lie}}(\mathfrak{g}^\hbar_{2n|a,b}, \mathfrak{sp}_{2n|a,b})$ comes from an invariant polynomial on $\mathfrak{sp}_{2n|a,b}$.

The following can be found in [42, §5.1] or [38, §2.8.1 (2.3)].

Definition 2.8.1. Let $\mathfrak{h} \subset \mathfrak{g}$ an inclusion of Lie superalgebras. Let $\mathrm{pr}\colon \mathfrak{g} \to \mathfrak{h}$ be a projection map (on underlying super vector spaces). The *curvature* of pr is the map $C \in \mathrm{Hom}(\Lambda^2\mathfrak{g}, \mathfrak{h})$ given by

$$C(v \wedge w) = [\mathrm{pr}(v), \mathrm{pr}(w)] - \mathrm{pr}([v, w]).$$

Let $\chi\colon \mathrm{Sym}^m(\mathfrak{h}^*)^{\mathfrak{h}} \to H^{2m}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{h})$ denote the map sending an ad-invariant polynomial P to the cocycle defined by

$$\chi(P)(v_1 \wedge \cdots \wedge v_{2m}) = \frac{1}{m!} \sum_{s \in \Sigma_{2m}/(\Sigma_2)^{\times m}} \mathrm{sign}(s) P\left(C(v_{\sigma(1)}, v_{\sigma(2)}), \ldots, C(v_{\sigma(2m-1)}, v_{\sigma(2m)})\right).$$

The curvature C measures how far pr is from being a Lie superalgebra map. Just as the usual Chern-Weil map is independent of the choice of connection, the map χ is independent of the choice of projection pr, [38, §2.8.1].

Remark 2.8.2. We have encountered three Chern-Weil style maps: the map χ, the functor

$$\mathrm{char}_{(\mathbb{M},\sigma)}(\mathbb{K})$$

from §2.5.2, and the classical Chern-Weil map [69, Appendix C]. In Lemma 2.8.36 below, we will describe how these three maps are related.

Example 2.8.3. In our case, we take the projection $\mathrm{pr}\colon \mathfrak{g}^{\hbar}_{2n|a,b} \to \mathfrak{sp}_{2n|a,b}$ by projecting onto the homogeneous degree 2 piece. One should compare this with Lemma 2.5.7.

We will describe this cohomology class by giving an explicit description of a polynomial P_n so that

$$(-1)^n[\chi(P_n)] = [\mathsf{Ev}_{\mathrm{loc}}(\tau_{2n|a,b})].$$

Let Φ denote the map $\mathfrak{sp}_{2n|a,b} \to \widehat{\mathcal{A}}_{2n|a,b}$ extending the map from Lemma 2.5.7 from the Clifford algebra to the Weyl algebra tensor the Clifford algebra. For $u \in \widehat{\mathcal{A}}_{2n|a,b}$, consider the ad-invariant function P_n on $\mathfrak{sp}_{2n|a,b}$ of degree n defined by the formula

$$P_n(a_1, \ldots, a_n) = \Upsilon_{2n|a,b} \circ \int_{[0,1]^n} \omega_{2n|a,b}(\Theta \otimes \Phi(a_1) \otimes \cdots \otimes \Phi(a_n)) dv_1 \cdots dv_n. \tag{2.5}$$

for $a_1, \ldots, a_n \in \mathfrak{sp}_{2n|a,b}$, and $v_1, \ldots, v_n$ the coordinates for $[0,1]^n$. Note here that the bidifferential operators α_{ij} and g_{ij} in $\omega_{2n|a,b}$ are acting on $(\widehat{\mathcal{A}}_{2n|a,b})^{\otimes n+1}$.

By [38, Lem. 2.24], we have the following.

Lemma 2.8.4. *There is an equality in Lie cohomology*

$$(-1)^n[\chi(P_n)] = [\mathsf{Ev}_{\mathrm{loc}}(\tau_{2n|a,b})].$$

Note that we are not using any bijectivity of χ, just the computation of $\chi(P_n)$.

We would like a nice description of the ad-invariant function P_n. This will be done in terms of characteristic series of genera. For an overview of genera and their characteristic series, see [56].

2.8.1.1 Characteristic Series

We define the characteristic series of interest.

Convention 2.8.5. For the remainder of this section, we let $\mathfrak{k} = \mathfrak{sp}_{2n|a,b}$.

An ad-invariant function on the Lie algebra $\mathfrak{k}$ is determined by its value on a Cartan subalgebra. It therefore suffices to show that a Cartan subalgebra of $\mathfrak{k} = \mathfrak{sp}_{2n|a,b}$ is as described. Note that $\mathfrak{sp}_{2n|a,b} = \mathfrak{sp}_{2n} \times \mathfrak{so}_{a,b}$. A Cartan subalgebra of $\mathfrak{sp}_{2n}$ is given by the diagonal matrices. These matrices correspond to the elements $q_l p_l$ of $\widehat{\mathcal{A}}_{2n|a,b}$ for $l = 1, \ldots, n$. By Lemmas 2.6.3 and 2.6.4, there is a basis for a Cartan subalgebra of $\mathfrak{so}_{a,b}$ whose image under Φ is $\{\eta_i\zeta_i, -\xi_j\mu_j\}$ for $i = 1, \ldots, a$ and $j = 1, \ldots, t$

Notation 2.8.6. Let $t_i \in \mathfrak{k}$ denote the element corresponding to $q_i p_i$. Let $s_i \in \mathfrak{k}$ denote the element corresponding to $\eta_i\zeta_i$. Let $r_i \in \mathfrak{k}$ denote the element corresponding to $-\xi_i\mu_i$.

Example 2.8.7. Consider the ad-invariant function $\widehat{A}(-)$ on $\mathfrak{k}$ determined by the polynomial

$$\prod_{i=1}^{n} \frac{t_i/2}{\sinh(t_i/2)}.$$

This is the characteristic series for the $\widehat{A}$-genus.

Example 2.8.8. Let $\widehat{B}(-)$ denote the ad-invariant function on $\mathfrak{k}$ determined by the polynomial

$$\prod_{i=1}^{a} \cosh(s_i/2) \frac{e^{s_i} - 1}{s_i}.$$

Remark 2.8.9. The characteristic series of the Todd genus is $\frac{z}{e^z - 1}$. As the Todd class is multiplicative, $\mathrm{Td}(E \oplus E') = \mathrm{Td}(E)\mathrm{Td}(E')$, the power series $\frac{e^s - 1}{s}$ in Example 2.8.8 looks like the characteristic series of the Todd genus of $-E$. For example, the power series $\frac{e^s-1}{s}$ on the tangent bundle determines the Todd genus of the stable normal bundle.

Remark 2.8.10. The characteristic series for the L-genus is

$$\frac{z}{\tanh(z)} = \frac{z}{\sinh(z)} \cosh(z).$$

We see this power series in the product of the power series in $\widehat{A}$ and $\widehat{B}$.

Example 2.8.11. We get an ad-invariant function $\widehat{C}(-)$ on $\mathfrak{k}$ from the polynomial

$$\prod_{i=1}^{t} \frac{r_i/2}{\sinh(r_i/2)} (r_i/2) \cot(r_i/2) \cos(r_i).$$

Example 2.8.12. We get an ad-invariant function $\widehat{BC}(-)$ on $\mathfrak{k}$ from the polynomial

$$\prod_{i=1}^{a} \cosh(s_i/2) \frac{e^{s_i} - 1}{s_i} \prod_{i=1}^{t} \frac{r_i/2}{\sinh(r_i/2)} (r_i/2) \cot(r_i/2) \cos(r_i.$$

2.8.1.2 Theorem Statement

By Lemma 2.8.4, the evaluation of $\tau_{2n|a,b}$ on the volume form Θ is given by the formula

$$[\mathsf{Ev}_{\mathrm{loc}}(\tau_{2n|a,b})] = (-1)^n [\chi(P_n)].$$

To give an explicit description of $\mathsf{Ev}_{\mathrm{loc}}(\tau_{2n|a,b})$, we need to compute the power series P_n from (2.5). In the case of type $(2n|0,0)$, one should compare the following with [42, Pg. 28]. In the case of type $(2n|a,a)$, one should compare the following with [38, Lem. 2.25].

Theorem 2.8.13 (Local Superalgebraic Index Theorem). *The ad invariant power series P_n from (2.5) satisfies the equation*

$$P_n(x,\ldots,x) = (-1)^{a+t}e\left[\det\left(\widehat{A}(\hbar x_1)\widehat{B}(\hbar y_1)\widehat{C}(\hbar y_2)\right)^{1/2}\right]_n.$$

where $x = x_1 + y_1 + y_2$ with $x_1 \in \mathfrak{sp}_{2n}$, $y_1 \in \mathfrak{so}_a$ and $y_2 \in \mathfrak{so}_t$. The notation $[-]_n$ denotes the degree n piece.

This is proven in §2.8.2 below.

In the notation of §2.8.1.1, Theorem 2.8.13 says that $P_n(x,\ldots,x)$ is equal to

$$(-1)^{a+t}e\left[\det\left(\frac{\hbar x_1/2}{\sinh(\hbar x_1/2)}\cosh(\hbar y_1/2)\left(\frac{e^{\hbar y_1}-1}{\hbar y_1}\right)\frac{(\hbar y_2/2)^2}{\sinh(\hbar y_2/2)}\cot(\hbar y_2/2)\cos(\hbar y_2)\right)^{1/2}\right]_n$$

It suffices to compute P_n on a Cartan subalgebra of $\mathfrak{sp}_{2n|a,b}$. Let x be in this Cartan subalgebra and $X = \Phi(x)$. We will prove some preliminary lemmas that will be useful in the proof of Theorem 2.8.13. Before we do this, we need the general set-up for the proof.

2.8.1.3 Proof Set-Up

Note that the generators $q_l p_l, \eta_i\zeta_i, -\xi_j\mu_j$ of the Cartan subalgebra are of degree at most two in the variables $q_l, p_l, \eta_i.\zeta_i.\xi_j.\mu_j$. Thus, only derivatives of order at most two from the α_{ij} and g_{ij} appearing in the exponential $\omega_{2n|a,b}$ contribute, and cross terms $\alpha_{ij}g_{i'j'}$ vanish. The remaining piece of $\omega_{2n|a,b}$ that may not vanish is

$$\omega^{\leq 2:}_{2n|a,b} = \prod_{0\leq i\leq j\leq n}\left(1 + \hbar\psi(v_i - v_j)(\alpha_{ij} + g_{ij}) + \frac{1}{2}\hbar^2\psi(v_i - v_j)^2(\alpha_{ij}^2 + g_{ij}^2)\right).$$

We would like a workable description of the expanded product of $\omega^{\leq 2}_{2n|a,b}(\Theta\otimes X^{\otimes n})$. Following [42, Pg. 26] and [38, Lem. 2.25], we associate each summand in the expanded product to a labeled graph on $n+1$ vertices. For notational consistency, we refer to these vertices as the 0th through nth. The zeroth vertex will be labeled by Θ. The remaining n vertices are labeled by the n copies of X. A summand of the product expansion of $\omega^{\leq 2}_{2n|a,b}(\Theta \otimes X^{\otimes n})$ is obtained by, for each ij with

$0 \leq i < j \leq 2n$, choosing either 1, $\hbar\psi(v_i - v_j)(\alpha_{ij} + g_{ij})$ or the quadratic term $\frac{1}{2}\hbar^2\psi(v_i - v_j)^2(\alpha_{ij}^2 + g_{ij}^2)$. One then adds edges to the $n+1$ vertex graph according to the following rules:

For $i > 0$, if for the ij term,

- one chose the constant term 1, add no edges.
- one chose the linear term $\hbar\psi(v_i - v_j)(\alpha_{ij} + g_{ij})$, add an edge between the i and jth vertices (which are labeled by X).
- one chose the quadratic term $\frac{1}{2}\hbar^2\psi(v_i - v_j)^2(\alpha_{ij}^2 + g_{ij}^2)$, add two edges between the i and jth vertices (which are labeled by X).

If for the $0j$ term,

- one chose the constant term 1, add no edges.
- one chose the linear term $\hbar\psi(v_0 - v_j)(\alpha_{0j} + g_{0j})$, add an edge between the 0 vertex (labeled by Θ) and the jth vertex (labeled by X).
- one chose the quadratic term $\frac{1}{2}\hbar^2\psi(v_0 - v_j)^2(\alpha_{0j}^2 + g_{0j}^2)$, add two edges between the 0 vertex (labeled by Θ) and the jth vertex (labeled by X).

Example 2.8.14. The graph that is a disjoint union of two cycles, one between the 0th and 1st vertices, and one between the 2nd, 3rd, and 4th vertices corresponds to the summand

$$\frac{1}{2}\hbar^2\psi(v_0 - v_1)^2(\alpha_{01} + g_{01})^2\hbar\psi(v_2 - v_3)(\alpha_{23} + g_{23})\hbar\psi(v_3 - v_4)(\alpha_{34} + g_{34})\hbar\psi(v_2 - v_4)(\alpha_{24} + g_{24}).$$

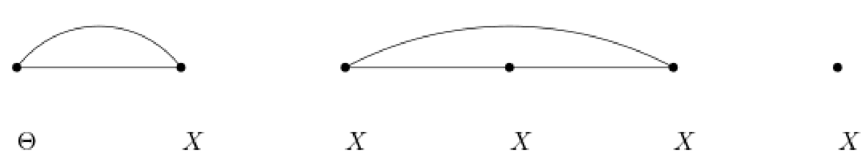

Figure 2-1: Example Graph

Note that a single graph consisting of a disjoint union of subgraphs corresponds to a product in $\omega_{2n|a,b}^{\leq 2}$.

2.8.1.4 Graphs with Vanishing Terms

We can rule out the following types of graphs.

Lemma 2.8.15. *Graphs containing a loop on a single vertex as a connected component correspond to a vanishing summand of* $\omega^{\leq 2}_{2n|a,b}(\Theta \otimes X^{\otimes n})$.

Proof. Say the loop is on the ith vertex. Then the loop subgraph corresponds to the term

$$\frac{1}{2}\hbar^2\psi(v_i - v_i)(\alpha_{ii} + g_{ii})^2.$$

As the disjoint union of subgraphs correspond to a product in $\omega^{\leq 2}_{2n|a,b}$, it suffices to show that

$$\left(\frac{1}{2}\hbar^2\psi(v_i - v_i)(\alpha_{ii} + g_{ii})^2\right)(\Theta \otimes X^{\otimes n}) = 0.$$

Since partial derivatives commute, the bidifferential operator α_{ii} applies

$$\sum_{l=1}^{n} \frac{\partial}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial}{\partial q_i}\frac{\partial}{\partial p_i} = 0$$

to the ith term in the tensor product, and hence vanishes.

Similarly, using the fact that $\frac{\partial}{\partial \theta_i}$ is an odd degree operator, we see that the bidifferential operator g_{ii} applies

$$\sum_{j=1}^{a} \frac{\partial}{\partial \zeta_i}\frac{\partial}{\partial \eta_i} + \frac{\partial}{\partial \eta_i}\frac{\partial}{\partial \zeta_i} = \sum_{j=1}^{a} \frac{\partial}{\partial \zeta_i}\frac{\partial}{\partial \eta_i} - \frac{\partial}{\partial \zeta_i}\frac{\partial}{\partial \eta_i} = 0$$

to the ith term in the tensor product, and hence vanishes. □

Lemma 2.8.16. *Graphs containing an X labeled vertex of valence strictly more than two correspond to a vanishing summand of* $\omega^{\leq 2}_{2n|a,b}(\Theta \otimes X^{\otimes n})$.

Proof. The vertex of valence more than two corresponds to applying more than two partial derivatives to X. These vanish as each summand of X has degree at most two in the basis elements. □

Recall that $\psi(t) = 2B_1(v)$. We will need the following identities for the Bernoulli polynomials $B_n(t)$ which can be found in [**?**].

$$B_n * B_m(v) = \int_0^1 B_n(u)B_m(v-u)du = -\frac{n!m!}{(n+m)!}B_{n+m}(v). \tag{2.6}$$

In particular, for $v = 0$ we have

$$\int_0^1 B_n(u)B_m(-u)du = -\frac{n!m!}{(n+m)!}B_{n+m},$$

where B_{n+m} is the $(n+m)$th Bernoulli *number*.

Moreover, we have

$$\int_x^{x+1} B_n(u)du = x^n. \tag{2.7}$$

Lemma 2.8.17. *Graphs containing a connected component that is a linear subgraph correspond to a vanishing summand of* $\omega^{\leq 2}_{2n|a,b}(\Theta \otimes X^{\otimes n})$.

Proof. Say the linear subgraph has length j. Either the Θ labeled vertex is in the linear subgraph or not.

In the first case, when all vertices of the linear subgraph are labeled by X, after possibly reordering vertices, we may assume we are dealing with the graph corresponding to the summand

$$\Upsilon_{2m|amb} \int_{[0,1]^j} \psi(v_1 - v_2) \cdots \psi(v_{j-1} - v_j) dv_1 \cdots dv_j \left(\hbar(\alpha_{12} - g_{12}) \cdots \hbar(\alpha_{j-1j} - g_{j-1j})\right)$$

evaluated on $(\Theta \otimes X \otimes \cdots \otimes X)$.

Following [38], we can use the convolution identity (2.6) and see that the integral

$$\int_{[0,1]^j} \psi(v_1 - v_2) \cdots \psi(v_{j-1} - v_j) dv_1 \cdots dv_j$$

is proportional to the zeroth Fourier coefficient of the 1-periodic Bernoulli polynomial $B_{j-1}(-)$, which vanishes.

Now assume that the Θ labeled vertex is a part of the linear subgraph. Say there are i X

labeled vertices to one side of Θ and $j-i$ to the other side. This subgraph then corresponds to the summand with integral

$$\int_{[0,1]^j} \psi(v_0-v_1)\cdots\psi(v_{i-1}-v_i)\psi(v_0-v_{i+1})\cdots\psi(v_{j-1}-v_j)dv_1\cdots dv_j$$
$$=\left(\int_{[0,1]^{j-i}} \psi(-v_1)\cdots\psi(v_{i-1}-v_i)dv_{i+1}\cdots dv_j\right)\left(\int_{[0,1]^i} \psi(-v_{i+1})\cdots\psi(v_{j-1}-v_j)dv_1\cdots dv_i\right).$$

Using the convolution identity (2.6), this integral is proportional to

$$\left(\int_0^1 B_{j-i}(v_j)dv_j\right)\left(\int_0^1 B_i(v_i)dv_i\right).$$

By the identity (2.7), both these integrals vanish.

Thus, contributions from all linear subgraphs, containing Θ or not, vanish. □

The remaining types of graphs are disjoint unions of cycles on the X labeled vertices or flowers whose center is the Θ labeled vertex and whose petals are cycles from the Θ vertex to X labeled vertices.

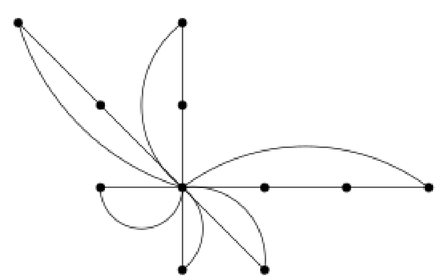

Figure 2-2: Θ-flower

Below, we will simply refer to this second type of graph as a Θ-flower.

Our next step is to compute the non-vanishing contributions of cycles and Θ-flowers on $\Theta\otimes X^{\otimes k}$. For this, it will be useful to break X into a sum of three types of terms.

Notation 2.8.18. Say $X = X_1 + Y_1 + Y_2$ with

$$X_1 = \sum_{i=1}^{n} \gamma_i q_i p_i$$
$$Y_1 = \sum_{i=1}^{a} \lambda_i \eta_i \zeta_i$$
$$Y_2 = \sum_{i=1}^{t} -\kappa_i \xi_i \mu_i$$

for some scalars $\gamma_i, \lambda_i, \kappa_i \in \mathbb{K}$. In sections §2.8.1.5, §2.8.1.6, §2.8.1.7 we compute the contributions of the X_1, Y_1, and Y_2 pieces respectively in terms corresponding to cycle and Θ-flower graphs.

2.8.1.5 Computations for X_1 Terms

The following appears in [38, Pg. 34]

Lemma 2.8.19. *In $\mathcal{A}_{2n|a,b}^{\otimes k+1}$, we have*

$$\alpha_{12} \cdots \alpha_{j-1j}\alpha_{j1}(\Theta \otimes X_1^{\otimes k}) = \frac{1}{2^{j-1}} \sum_{i=1}^{n} \gamma_i^j (\Theta \otimes 1^{\otimes j} \otimes X_1^{\otimes k-j})$$

if j is even. This term vanishes if j is odd.

Lemma 2.8.20. *In $\mathbb{K}$ we have*

$$\Upsilon_{2n|a,b}(\Theta \otimes 1^{\otimes j} \otimes X_1^{\otimes k-j}) = \delta_{jk}$$

Proof. We have

$$\Upsilon_{2n|a,b}(\Theta \otimes 1^{\otimes j} \otimes X_1^{\otimes k-j}) = \int \left(\Theta X_1^{k-j}\right)|_{y=0}$$

where y represents the even variables. If $k-j$ is nonzero, then $X_1|_{y=0} = 0$ and this term vanishes. When $k-j=0$, we have $\int \Theta = 1$. □

Corollary 2.8.21. *If k is even, the X_1 contribution of a cycle of length k is*

$$\Upsilon_{2n|a,b}(\alpha_{12} \cdots \alpha_{k-1k}\alpha_{k1}(\Theta \otimes X_1^{\otimes k})) = \frac{1}{2^{k-1}} \sum_{i=1}^{n} \gamma_i^k.$$

This term vanishes if k is odd.

For example, if $k = 6$ graph corresponding to such a cycle looks like the following.

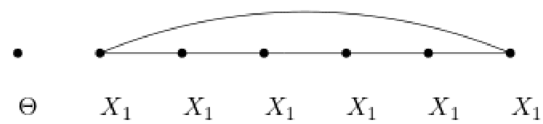

Figure 2-3: X_1 cycle

2.8.1.6 Computations for Y_1 Terms

Lemma 2.8.22. *In $\mathcal{A}_{2n|a,b}^{\otimes k+1}$, for $j = 2, \ldots, k$, we have*

$$g_{12} \cdots g_{j-1j} g_{j1}(\Theta \otimes Y_1^{\otimes k}) = \frac{-1}{2^{j-1}} \sum_{r=1}^{a} \lambda_r^j (\Theta \otimes 1^{\otimes j} \otimes Y_1^{\otimes k-j})$$

and

$$g_{11}(\Theta \otimes Y_1^{\otimes k}) = 0.$$

Proof. Since Y_1 does not involve any μ_i or ξ_i terms, $\frac{\partial}{\partial \mu_i} Y_1$ and $\frac{\partial}{\partial \xi_i} Y_1$ vanish. Thus, we can replace the bidifferential operators g_{ij} by the map sending $(a_0 \otimes \cdots \otimes a_k)$ to

$$\frac{1}{2} \sum_{r=1}^{a} a_0 \otimes \cdots \frac{\partial}{\partial \zeta_r} a_i \otimes \cdots \otimes \frac{\partial}{\partial \eta_r} a_j \otimes \cdots \otimes a_k + a_0 \otimes \cdots \frac{\partial}{\partial \eta_r} a_i \otimes \cdots \otimes \frac{\partial}{\partial \zeta_r} a_j \otimes \cdots \otimes a_k$$

for this computation.

We have

$$\frac{\partial}{\partial \zeta_r} Y_1 = \frac{\partial}{\partial \zeta_r} \left(\sum_{i=1}^{a} \lambda_i \eta_i \zeta_i \right) = -\lambda_r \eta_r$$
$$\frac{\partial}{\partial \eta_r} Y_1 = \frac{\partial}{\partial \eta_r} \left(\sum_{i=1}^{a} \lambda_i \eta_i \zeta_i \right) = \lambda_r \zeta_r.$$

For $j = 2$, using $g_{21} = g_{12}$, we have

$$\begin{aligned}
&g_{12}g_{21}(\Theta \otimes Y_1^{\otimes k}) \\
&= g_{12}\left(\frac{1}{2}\sum_{r=1}^{a} \Theta \otimes \frac{\partial}{\partial \zeta_r}Y_1 \otimes \frac{\partial}{\partial \eta_r}Y_1 \otimes Y_1^{\otimes k-2} + \Theta \otimes \frac{\partial}{\partial \eta_r}Y_1 \otimes \frac{\partial}{\partial \zeta_r}Y_1 \otimes Y_1^{\otimes k-2}\right) \\
&- g_{12}\left(\frac{1}{2}\sum_{r=1}^{a} \Theta \otimes (-\lambda_r\eta_r) \otimes \lambda_r\zeta_r \otimes Y_1^{\otimes k-2} + \Theta \otimes \lambda_r\zeta_r \otimes (-\lambda_r\eta_r) \otimes Y_1^{\otimes k-2}\right) \\
&= -\frac{1}{2}\sum_{r=1}^{a} \lambda_r^2 g_{12}\left(\Theta \otimes \eta_r \otimes \zeta_r \otimes Y_1^{\otimes k-2} + \Theta \otimes \zeta_r \otimes \eta_r \otimes Y_1^{\otimes k-2}\right) \\
&= -\frac{1}{4}\sum_{r=1}^{a} \lambda_r^2(\Theta \otimes \otimes 1 \otimes 1 \otimes Y_1^{\otimes k-2} + \Theta \otimes 1 \otimes 1 \otimes Y_1^{\otimes k-2}) \\
&= -\frac{1}{2}\sum_{r=1}^{a} \lambda_r^2(\Theta \otimes 1^{\otimes 2} \otimes Y_1^{\otimes k-2}).
\end{aligned}$$

For $j = 3$ we have

$$g_{12}g_{23}g_{31}(\Theta \otimes Y_1^{\otimes 3}) = \frac{1}{2^3}\sum_{r=1}^{a} \Theta \otimes (-\lambda_r) \otimes \lambda_r \otimes \lambda_r + \Theta \otimes \lambda_r \otimes (-\lambda_r) \otimes (-\lambda_r) = 0.$$

This parity continues in general, with the end term either canceling or doubling depending on a sign. For general j, only the first j copies of Y_1 in $\Theta \otimes Y_1^{\otimes k}$ are acted upon by the bidifferential operators $g_{12}\cdots g_{j-1j}g_{j1}$. Each such copy of Y_1 is acted on twice: the ith copy for $i = 2, \ldots, j-1$ is acted on by g_{i-1i} and g_{ii+1}, the first copy is acted on by g_{12} and g_{j1}, and the jth copy is acted on by g_{j-1j} and g_{j1}.

We therefore have

$$\begin{aligned}
g_{12}\cdots g_{j-1j}g_{j1}(\Theta \otimes Y_1^{\otimes k}) &= \frac{1}{2^j}\sum_{r=1}^{a} -\Theta \otimes \lambda_r^{\otimes j} \otimes Y_1^{\otimes k-j} + (-1)^{j-1}\Theta \otimes \lambda_r^{\otimes j} \otimes Y_1^{\otimes k-j} \\
&= \begin{cases} \frac{-1}{2^{j-1}}\sum_{i=1}^{a} \lambda_r^j(\Theta \otimes 1^{\otimes j} \otimes Y_1^{\otimes k-j}) & j \text{ even} \\ = 0 & j \text{ odd.} \end{cases}
\end{aligned}$$

Lastly, we compute $g_{11}(\Theta \otimes Y_1^{\otimes k})$:

$$\begin{aligned} g_{11}(\Theta \otimes Y_1^{\otimes k}) &= \frac{1}{2}\sum_{r=1}^{a}\left(\Theta \otimes \frac{\partial}{\partial \zeta_r}\frac{\partial}{\partial \eta_r} Y_1 \otimes Y_1^{\otimes k-1} + \Theta \otimes \frac{\partial}{\partial \eta_r}\frac{\partial}{\partial \zeta_r} Y_1 \otimes Y_1^{\otimes k-1}\right) \\ &= \sum_{r=1}^{a}\left(\Theta \otimes (-\lambda_r) \otimes Y_1^{\otimes k-1}\right) + \left(\Theta \otimes \lambda_r \otimes Y_1^{\otimes k-1}\right) \\ &= 0. \end{aligned}$$

□

Recall that $\Theta = \zeta_1\eta_1 \cdots \zeta_a\eta_a\xi_1\mu_1\cdots\xi_t\mu_t\upsilon$ (where the υ only appears if $b-a$ is odd).

Lemma 2.8.23. *In $\mathcal{A}_{2n|a,b}^{\otimes k+1}$, for $j = 1, \ldots, k$ odd, we have*

$$g_{01}\cdots g_{j-1j}g_{j0}(\Theta \otimes Y_1^{\otimes k}) = \frac{1}{2^j}\sum_{r=1}^{a}\lambda_r^j\left(\frac{\Theta}{\zeta_r\eta_r} \otimes 1^{\otimes j} \otimes Y_1^{\otimes k-j}\right).$$

This term vanishes if j is even. Moreover $g_{00}(\Theta \otimes Y_1^{\otimes k}) = 0$.

Proof. The proof is the similar to that of Lemma 2.8.22, after noting that $\frac{\partial}{\partial \eta_r}\Theta = -\frac{\Theta}{\eta_r}$ and $\frac{\partial}{\partial \zeta_r}\Theta = \frac{\Theta}{\zeta_r}$.

For example, we have

$$\begin{aligned} g_{01}g_{10}(\Theta \otimes Y_1) &= \frac{1}{2}g_{01}\left(\sum_{r=1}^{a}\frac{\Theta}{\zeta_r} \otimes \lambda_r\zeta_r + \left(-\frac{\Theta}{\eta_r}\right) \otimes (-\lambda_r\eta_r)\right) \\ &= \frac{1}{2^2}\sum_{r=1}^{a}\frac{\Theta}{\eta_r\zeta_r} \otimes \lambda_r + \frac{\Theta}{\zeta_r\eta_r} \otimes \lambda_r \\ &= \frac{1}{2}\sum_{r=1}^{a}\frac{\Theta}{\zeta_r\eta_r} \otimes \lambda_r. \end{aligned}$$

In general, we have

$$g_{01}\cdots g_{j-1j}g_{j0}(\Theta \otimes Y_1^{\otimes k}) = \frac{1}{2^{j+1}}\sum_{r=1}^{a}\frac{\partial}{\partial \eta_r}\frac{\partial}{\partial \zeta_r}\Theta \otimes \lambda_r^{\otimes j} \otimes Y_1^{\otimes k-j} + (-1)^j\frac{\partial}{\partial \zeta_r}\frac{\partial}{\partial \eta_r}\Theta \otimes \lambda_r^{\otimes j} \otimes Y_1^{\otimes k-j}.$$

As $\frac{\partial}{\partial\zeta_r}\frac{\partial}{\partial\zeta_r} = -\frac{\partial}{\partial\zeta_r}\frac{\partial}{\partial\eta_r}$, the right hand side vanishes if j is even and becomes

$$\frac{1}{2^j}\sum_{r=1}^{a}\lambda_r^j\left(\frac{\Theta}{\eta_r\zeta_r}\otimes 1^{\otimes j}\otimes Y_1^{\otimes k-j}\right)$$

if j is odd.

For g_{00}, the bidifferential operators are only acting on Θ. As Θ contains all basis elements $\mu_i, \eta_i, \xi_i, \zeta_i, \upsilon_i$, we would apriori need to consider the full form of g_{00}. However, the μ_i and ξ_i terms vanish here:

$$\begin{aligned}
g_{00}(\Theta\otimes Y_1^{\otimes k}) &= \frac{1}{2}\sum_{ml} h_Q^{ml}\frac{\partial}{\partial\theta_m}\frac{\partial}{\partial\theta_l}\Theta\otimes Y_1^{\otimes k}\\
&= \frac{1}{2}\sum_{r=1}^{a}\left(\frac{\partial}{\partial\zeta_r}\frac{\partial}{\partial\eta_r}\Theta\otimes Y_1^{\otimes k} + \frac{\partial}{\partial\eta_r}\frac{\partial}{\partial\zeta_r}\Theta\otimes Y_1^{\otimes k}\right)\\
&\quad - \frac{1}{2}\sum_{r=1}^{t}\left(\frac{\partial}{\partial\xi_r}\frac{\partial}{\partial\xi_r}\Theta\otimes Y_1^{\otimes k} + \frac{\partial}{\partial\mu_r}\frac{\partial}{\partial\mu_r}\Theta\otimes Y_1^{\otimes k}\right) - \frac{1}{2}\frac{\partial}{\partial\upsilon}\frac{\partial}{\partial\upsilon}\Theta\otimes Y_1^{\otimes k}.
\end{aligned}$$

Now $\frac{1}{2}\frac{\partial}{\partial\upsilon}\frac{\partial}{\partial\upsilon}\Theta$, $\frac{1}{2}\frac{\partial}{\partial\xi}\frac{\partial}{\partial\xi}\Theta$, and $\frac{1}{2}\frac{\partial}{\partial\mu}\frac{\partial}{\partial\mu}\Theta$ all vanish, and we are left with

$$g_{00}(\Theta\otimes Y_1^{\otimes k}) = \frac{1}{2}\sum_{r=1}^{a} -\frac{\Theta}{\zeta_r\eta_r}\otimes Y_1^{\otimes k} + \frac{\Theta}{\eta_r\zeta_r}\otimes Y_1^{\otimes k} = 0$$

as desired.

□

Note that the computation of $g_{00}(\Theta\otimes Y_1)$ holds for Y_1 replaced with any element of the Cartan subalgebra, in particular for X_1 or Y_2.

The term computed in Lemma 2.8.23 corresponds to a single petal of length j on a Θ-flower. Say we have a Θ-flower with l petals of length $j_1, j_2, \ldots, j_l$ on a total of j X-labeled vertices. This graph corresponds to the following computation.

Corollary 2.8.24. *Consider a partition $j_1 + \cdots j_l = j$. Then*

$$(g_{01}\cdots g_{j_1-1j_1}g_{j_10})(g_{0j_1+1}\cdots g_{j_1+j_2-1j_1+j_2}g_{j_1+j_20})\cdots(g_{0j-j_l}\cdots g_{j-1k}g_{j0})(\Theta\otimes Y_1^{\otimes k})$$

vanishes unless $j_1, \dots, j_l$ are odd in which case it is equal to

$$\frac{1}{2^{j_1} \cdots 2^{j_l}} \sum_{|R|=l} \lambda_{r_1}^{j_1} \cdots \lambda_{r_l}^{j_l} \left(\frac{\Theta}{\zeta_{r_1}\eta_{r_1} \cdots \zeta_{r_l}\eta_{r_l}} \otimes 1^{\otimes j} \otimes Y_1^{k-j} \right)$$

where $R = (r_1, \dots, r_l)$ ranges over ordered subsets of $\{1, \dots, a\}$ of size l.

In particular, we may have $R = (i, j)$ and $R = (j, i)$.

Lemma 2.8.25. *In $\mathbb{K}$ for $j = 0, \dots, k$, we have*

$$\Upsilon_{2n|a,b}(\Theta \otimes 1^{\otimes j} \otimes Y_1^{\otimes k-j}) = \delta_{jk}$$

vanishes unless $j = k$.

Proof. Since $\zeta_i^2 = 0 = \eta_i^2$ we have

$$\zeta_i\eta_i\eta_i\zeta_i = 0 \tag{2.8}$$

Now since $\Theta = \zeta_1\eta_1 \cdots \zeta_a\eta_a\xi_1\mu_1 \cdots \xi_t\mu_t\upsilon$, for any i we have

$$\Theta\eta_i\zeta_i = 0,$$

as we can commute the $\eta_i\zeta_i$ past the $\eta_j\zeta_j$ for $i \neq j$, and then use the above observation (2.8).

We have

$$\begin{aligned} \Upsilon_{2n|a,b}(\Theta \otimes 1^{\otimes j} \otimes Y_1^{\otimes k-j}) &= \int \Theta \left(\sum_{r=1}^{a} \lambda_r \eta_r \zeta_r \right)^{k-j} \\ &= \sum_{|I|=k-j} \lambda_I \int \Theta(\eta_{i_1}\zeta_{i_1} \cdots \eta_{i_{k-j}}\zeta_{i_{k-j}}) \\ &= 0. \end{aligned}$$

□

Lemma 2.8.26. *Consider a subset $R = (r_1, \dots, r_l)$ of $\{1, \dots, a\}$. In $\mathbb{K}$, if $j \neq k$, we have*

$$\Upsilon_{2n|a,b}\left(\frac{\Theta}{\zeta_{r_1}\eta_{r_1}\cdots\zeta_{r_l}\eta_{r_l}} \otimes 1^{\otimes j} \otimes Y_1^{k-j}\right) = \sum_{\sum s_i = k-j} \frac{(k-j)!}{s_1! \cdots s_l!}(-1)^l \hbar^{k-j-l} \lambda_{r_1}^{s_1} \cdots \lambda_{r_l}^{s_l}$$

where $S = (s_1, \dots, s_l)$ is a partition of $k-j$. This term vanishes if $j = k$.

In particular, for $l = 1$ we have

$$\Upsilon_{2n|a,b}\left(\frac{\Theta}{\zeta_r\eta_r} \otimes 1^{\otimes j} \otimes Y_1^{\otimes k-j}\right) = \begin{cases} -\hbar^{k-j-1}\lambda_r^{k-j} & j = 0, \dots, k-1 \\ 0 & j = k. \end{cases}$$

Proof. Let $\Theta_R = \frac{\Theta}{\zeta_{r_1}\eta_{r_1}\cdots\zeta_{r_l}\eta_{r_l}}$. We have

$$\Upsilon_{2n|a,b}(\Theta_R \otimes 1^{\otimes j} \otimes Y_1^{\otimes k-j}) = \int \Theta_R Y_1^{k-j} = \sum_{|I|=k-j} \lambda_I \int \Theta_R(\eta_{i_1}\zeta_{i_1} \cdots \eta_{i_{k-j}}\zeta_{i_{k-j}}).$$

If there is a term $r' \in I$ with $r' \notin R$, then we see a $\zeta_{r'}\eta_{r'}\eta_{r'}\zeta_{r'} = 0$ in the product

$$\Theta_R(\eta_{i_1}\zeta_{i_1} \cdots \eta_{i_{k-j}}\zeta_{i_{k-j}}).$$

These summands therefore vanish, and so we may assume I consists only of terms in R. Note that $[\zeta_r, \eta_r] = \hbar$ so that $\eta_r\zeta_r = \hbar - \zeta_r\eta_r$. We therefore have

$$\frac{\Theta}{\zeta_r\eta_r}\eta_r\zeta_r = \frac{\Theta}{\zeta_r\eta_r}(\hbar - \zeta_r\eta_r) = \hbar\frac{\Theta}{\zeta_r\eta_r} - \Theta$$

and

$$\left(\hbar\frac{\Theta}{\zeta_r\eta_r} - \Theta\right)(\eta_r\zeta_r) = \hbar^2\frac{\Theta}{\zeta_r\eta_r} - \hbar\Theta.$$

Thus inductively we see that,

$$\frac{\Theta}{\zeta_r\eta_r}(\eta_r\zeta_r)^l = \hbar^l\frac{\Theta}{\zeta_r\eta_r} - \hbar^{l-1}\Theta.$$

If $j = k$, or $R \not\subset I$, then there is some $r \in R$ that is not in I. We would then be taking the Berezin integral of $\frac{\Theta}{\zeta_r\eta_r}$, which is zero.

Thus, for $|R| = 1$, we must have $I = (r, r, \ldots, r)$. We get

$$\Upsilon_{2n|a,b}\left(\frac{\Theta}{\zeta_r\eta_r} \otimes 1^{\otimes j} \otimes Y_1^{\otimes k-j}\right) = \lambda_r^{k-j} \int \hbar^{k-j} \frac{\Theta}{\eta_r\zeta_r} - \hbar^{k-j-1}\Theta = -\hbar^{k-j-1}\lambda_r^{k-j}.$$

More generally, given a partition $S = (s_1, \ldots, s_l)$ with $s_1 + \cdots s_l = k - j$, we have

$$\Theta_R(\eta_{r_1}\zeta_{r_1})^{s_1} \cdots (\eta_{r_l}\zeta_{r_l})^{s_l} = \left(\hbar^{s_1}\Theta_R - \hbar^{s_1-1}\Theta_{R\setminus\{r_1\}}\right)(\eta_{r_2}\zeta_{r_2})^{s_2} \cdots (\eta_{r_l}\zeta_{r_l})^{s_l}.$$

The term Θ_R on the right-hand side will vanish under the Berezin integral as it contains no $\zeta_{r_1}\eta_{r_1}$. We can therefore ignore it. Replacing R with $R \setminus \{r_1\}$, we are reiterating the same computation, but with a negative sign. Continuing through $i = 2, \ldots, l$, we obtain

$$\Upsilon_{2n|a,b}\Theta_R(\eta_{r_1}\zeta_{r_1})^{s_1} \cdots (\eta_{r_l}\zeta_{r_l})^{s_l} = (-1)^l\Upsilon_{2n|a,b}\hbar^{s_1-1} \cdots \hbar^{s_l-1}\Theta = (-1)^l\hbar^{k-j-l}.$$

We therefore obtain

$$\Upsilon_{2n|a,b}\left(\frac{\Theta}{\zeta_{r_1}\eta_{r_1} \cdots \zeta_{r_l}\eta_{r_l}} \otimes 1^{\otimes j} \otimes Y_1^{k-j}\right) = \sum_{\sum s_i = k-j} \frac{(k-j)!}{s_1! \cdots s_l!}(-1)^l\hbar^{k-j-l}\lambda_{r_1}^{s_1} \cdots \lambda_{r_l}^{s_l},$$

as

$$\left(\sum_{i=1}^{l} \lambda_{r_i}\right)^{k-j} = \sum_{\sum s_i = k-j} \frac{(k-j)!}{s_1! \cdots s_l!}\lambda_{r_1}^{s_1} \cdots \lambda_{r_l}^{s_l}.$$

□

In summary, we have shown the following:

Corollary 2.8.27. *For $j = 2, \ldots, k$, the Y_1 contribution from a cycle of length j is*

$$\Upsilon_{2n|a,b}\left(g_{12} \cdots g_{j-1}g_{j1}(\Theta \otimes Y_1^{\otimes j})\right) = \frac{-1}{2^j}\sum_{r=1}^{a} \lambda_r^j$$

if j is even and vanishes if j is odd.

Moreover, for $j = 1, \ldots, k-1$, a Θ-flower with l petals of length $j_1, j_2, \ldots, j_l$ on a total of j X-labeled vertices has Y_1 contribution

$$\Upsilon_{2n|a,b}\left((g_{01}\cdots g_{j_1-1j_1}g_{j_10})(g_{0j_1+1}\cdots g_{j_1+j_2-1j_1+j_2}g_{j_1+j_20})\cdots(g_{0j-j_l}\cdots g_{j-1k}g_{j0})(\Theta\otimes Y_1^{\otimes k})\right)$$

which is equal to

$$\frac{(-1)^l}{2^{j_1}\cdots 2^{j_l}}\sum_{|R|=l}\sum_{\sum s_i=k-j}\frac{(k-j)!}{s_1!\cdots s_l!}\hbar^{k-j-l}\lambda_{r_1}^{j_1+s_1}\cdots\lambda_{r_l}^{j_l+s_l}.$$

Here $R=(r_1,\ldots,r_l)$ is a subset of $\{1,\ldots,a\}$ and $S=(s_1,\ldots,s_l)$ is a partition of $k-j$. This term vanishes if $j=k$.

For example, if $l=5$, $j_1=3$, $j_2=1$, $j_3=2$, $j_4=1$, $j_5=2$ and $k=14$, the Θ-flower considered here looks like

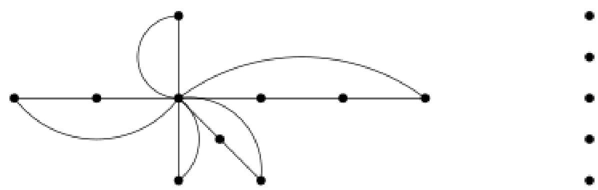

Figure 2-4: Y_1 Θ-flower

where all the vertices are labeled Y_1, except the center of the flower which is labeled Θ. The partition S of the $k-j=5$ spare vertices corresponds graphically to assigning spare vertices to the petals.

2.8.1.7 Computations for Y_2 Terms

Lemma 2.8.28. *In $\mathcal{A}_{2n|a,b}^{\otimes k+1}$, for $j=2,\ldots,k$ even we have*

$$g_{12}\cdots g_{j-1j}g_{j1}(\Theta\otimes Y_2^{\otimes k})=\frac{1}{2^{j-1}}\sum_{r=1}^{t}\kappa_r^j(\Theta\otimes 1^{\otimes j}\otimes Y_2^{\otimes k-j}).$$

This term vanishes if j is odd. Moreover, $g_{11}(\Theta\otimes Y_2^{\otimes k})=0$.

Proof. Since Y_2 does not involve any η_i, ζ_i, or υ terms, $\frac{\partial}{\partial\eta_i}Y_2$, $\frac{\partial}{\partial\zeta_i}Y_2$, and $\frac{\partial}{\partial\upsilon}Y_2$ vanish. Thus, we can replace the bidifferential operators g_{ij} by the map sending $(a_0\otimes\cdots\otimes a_k)$ to

$$\frac{1}{2}\sum_{r=1}^{t} a_0 \otimes \cdots \frac{\partial}{\partial \xi_r} a_i \otimes \cdots \otimes \frac{\partial}{\partial \xi_r} a_j \otimes \cdots \otimes a_k + a_0 \otimes \cdots \frac{\partial}{\partial \mu_r} a_i \otimes \cdots \otimes \frac{\partial}{\partial \mu_r} a_j \otimes \cdots \otimes a_k$$

for this computation.

In the expression for $g_{11}(\Theta \otimes Y_2^{\otimes k})$, the bidifferential operators $\frac{\partial}{\partial \xi_r}\frac{\partial}{\partial \xi_r}$ and $\frac{\partial}{\partial \mu_r}\frac{\partial}{\partial \mu_r}$ act on the first copy of Y_2. Since $Y_2 = \sum_{i=1}^{t} -\kappa_i \xi \mu$ only has one copy of each ξ_i and μ_i, these operators are zero. This proves the second claim that $g_{11}(\Theta \otimes Y_2^{\otimes k})$ vanishes.

We have $\frac{\partial}{\partial \xi_r} Y_2 = -\kappa_r \mu_r$ and $\frac{\partial}{\partial \mu_r} Y_2 = \kappa_r \xi_r$.

For $j = 2$, using $g_{21} = -g_{12}$, we have

$$\begin{aligned}
&g_{12}g_{21}(\Theta \otimes Y_2^{\otimes k}) \\
&= g_{12}\left(\frac{1}{2}\sum_{r=1}^{t} \Theta \otimes \frac{\partial}{\partial \xi_r} Y_2 \otimes \frac{\partial}{\partial \xi_r} Y_2 \otimes Y_2^{\otimes k-2} + \Theta \otimes \frac{\partial}{\partial \mu_r} Y_2 \otimes \frac{\partial}{\partial \mu_r} Y_2 \otimes Y_2^{\otimes k-2}\right) \\
&= g_{12}\left(\frac{1}{2}\sum_{r=1}^{t} \Theta \otimes (-\kappa_r \mu_r) \otimes (-\kappa_r \mu_r) \otimes Y_2^{\otimes k-2} + \Theta \otimes (\kappa_r \xi_r) \otimes (\kappa_r \xi_r) \otimes Y_2^{\otimes k-2}\right) \\
&= \frac{1}{2}\sum_{r=1}^{t} \kappa_r^2 g_{12}\left(\Theta \otimes \mu_r \otimes \mu_r \otimes Y_2^{\otimes k-2} + \Theta \otimes \xi_r \otimes \xi_r \otimes Y_2^{\otimes k-2}\right) \\
&= \frac{1}{4}\sum_{r=1}^{t} \kappa_r^2 (\Theta \otimes \otimes 1 \otimes 1 \otimes Y_2^{\otimes k-2} + \Theta \otimes 1 \otimes 1 \otimes Y_2^{\otimes k-2}) \\
&= \frac{1}{2}\sum_{r=1}^{t} \kappa_r^2 (\Theta \otimes 1^{\otimes 2} \otimes Y_1^{\otimes k-2}).
\end{aligned}$$

In general, we have

$$g_{12}\cdots g_{j-1j}g_{j1}(\Theta \otimes Y_2^{\otimes k}) = \frac{1}{2^j}\sum_{r=1}^{t} (-1)^j \Theta \otimes \kappa_r^{\otimes j} \otimes Y_2^{\otimes k-j} + \Theta \otimes \kappa_r^{\otimes j} \otimes Y_2^{\otimes k-j}$$

which vanishes if j is odd and gives the desired computation when j is even.

□

Lemma 2.8.29. *In $\mathcal{A}_{2n|a,b}^{\otimes k+1}$, we have, for $j = 1, \ldots, k$ odd*

$$g_{01} \cdots g_{j-1j} g_{j0}(\Theta \otimes Y_2^{\otimes k}) = \frac{(-1)^{(j+1)/2}}{2^j} \sum_{r=1}^{t} \kappa_r^j \left(\frac{\Theta}{\xi_r \mu_r} \otimes 1^{\otimes j} \otimes Y_2^{\otimes k-j} \right).$$

This term vanishes if j is even. Moreover $g_{00}(\Theta \otimes Y_2^{\otimes k}) = 0$.

The claim for g_{00} follows from the proof of Lemma 2.8.23.

Proof. Note that $\frac{\partial}{\partial \xi_r}\Theta = \frac{\Theta}{\xi_r}$ and $\frac{\partial}{\partial \mu_r}\Theta = -\frac{\Theta}{\mu_r}$. For $j = 1$ we have

$$\begin{aligned} g_{01} g_{10}(\Theta \otimes Y_2^{\otimes k}) &= \frac{1}{2} g_{01} \left(\sum_{r=1}^{t} \frac{\partial}{\partial \xi_r}\Theta \otimes \frac{\partial}{\partial \xi_r} Y_2 \otimes Y_2^{\otimes k-1} + \frac{\partial}{\partial \mu_r}\Theta \otimes \frac{\partial}{\partial \mu_r} Y_2 \otimes Y_2^{\otimes k-1} \right) \\ &= \frac{1}{2} g_{01} \left(\sum_{r=1}^{t} \frac{\Theta}{\xi_r} \otimes (-\kappa_r \mu_r) \otimes Y_2^{\otimes k-1} + \left(-\frac{\Theta}{\mu_r} \right) \otimes (\kappa_r \xi_r) \otimes Y_2^{\otimes k-1} \right) \\ &= -\frac{1}{4} \sum_{r=1}^{t} \frac{\Theta}{\xi_r \mu_r} \otimes \kappa_r \otimes Y_2^{\otimes k-1} + \frac{\Theta}{\xi_r \mu_r} \otimes \kappa_r \otimes Y_2^{\otimes k-1} \\ &= -\frac{1}{2} \sum_{r=1}^{t} \kappa_r (\frac{\Theta}{\xi_r \mu_r} \otimes 1 \otimes Y_2^{\otimes k-1}). \end{aligned}$$

When $j = 2$, we have

$$g_{01} g_{12} g_{20}(\Theta \otimes Y_2^{\otimes 2}) = \frac{1}{2^3} \sum_{r=1}^{t} \frac{\partial}{\partial \xi_r} \frac{\partial}{\partial \xi_r}\Theta \otimes \kappa_r \otimes (-\kappa_r) + \frac{\partial}{\partial \mu_r} \frac{\partial}{\partial \mu_r}\Theta \otimes (-\kappa_r) \otimes \kappa_r = 0.$$

This parity pattern continues. If j is even, then we have

$$g_{01} \cdots g_{j0}(\Theta \otimes Y_2^{\otimes j}) = \frac{1}{2^{j+1}} \sum_{r=1}^{t} (-1)^{j/2} \frac{\partial}{\partial \xi_r} \frac{\partial}{\partial \xi_r}\Theta \otimes \kappa_r^{\otimes j} + (-1)^{j/2} \frac{\partial}{\partial \mu_r} \frac{\partial}{\partial \mu_r}\Theta \otimes \kappa_r^{\otimes j} = 0.$$

If j is odd, then we have

$$
\begin{aligned}
g_{01}\cdots g_{j0}(\Theta\otimes Y_2^{\otimes j}) &= \frac{1}{2^{j+1}}\sum_{r=1}^{t}(-1)^{(j+1)/2}\frac{\partial}{\partial\mu_r}\frac{\partial}{\partial\xi_r}\Theta\otimes\kappa_r^{\otimes j}+(-1)^{(j-1)/2}\frac{\partial}{\partial\xi_r}\frac{\partial}{\partial\mu_r}\Theta\otimes\kappa_r^{\otimes j}\\
&= \frac{1}{2^{j+1}}\sum_{r=1}^{t}(-1)^{(j+1)/2}\frac{\Theta}{\xi_r\mu_r}\otimes\kappa_r^{\otimes j}-(-1)^{(j-1)/2}\frac{\Theta}{\xi_r\mu_r}\otimes\kappa_r^{\otimes j}\\
&= \frac{(-1)^{(j+1)/2}}{2^j}\sum_{r=1}^{t}\frac{\Theta}{\xi_r\mu_r}\otimes\kappa_r^{\otimes j}.
\end{aligned}
$$

□

Corollary 2.8.30. *Consider a partition $j_1+\cdots j_l=j$. If $j_1,\ldots,j_l$ are odd, then*

$$
(g_{01}\cdots g_{j_1-1j_1})(g_{j_10}g_{0j_1+1}\cdots g_{j_1+j_2-1j_1+j_2}g_{j_1+j_20})\cdots(g_{0j-j_l}\cdots g_{j-1k}g_{j0})(\Theta\otimes Y_2^{\otimes k})
$$

is equal to

$$
\frac{(-1)^{\frac{j_1+1}{2}}\cdots(-1)^{\frac{j_l+1}{2}}}{2^{j_1}\cdots 2^{j_l}}\sum_{|R|=l}\kappa_{r_1}^{j_1}\cdots\kappa_{r_l}^{j_l}\left(\frac{\Theta}{\xi_{r_1}\mu_{r_1}\cdots\xi_{r_l}\mu_{r_l}}\otimes 1^{\otimes j}\otimes Y_2^{k-j}\right)
$$

where $R=(r_1,\ldots,r_l)$ ranges over subsets of $\{1,\ldots,t\}$ of size l. This term vanishes if any j_i is even.

Lemma 2.8.31. *In $\mathbb{K}$ we have*

$$
\Upsilon_{2n|a,b}(\Theta\otimes 1^{\otimes j}\otimes Y_2^{\otimes k-j}) = \begin{cases}\sum\limits_{|J|=s}\frac{(k-j)!}{2s!}\left(\frac{-\hbar^2}{4}\right)^s\kappa_J^2 & k-j=2s \text{ is even}\\ 0 & k-j \text{ is odd.}\end{cases}
$$

Proof. We have

$$
Y_2^{k-j}=\left(\sum_{r=1}^{t}-\kappa_r\xi_r\mu_r\right)^{k-j}=\sum_{|I|=k-j}(-1)^{k-j}\kappa_I\xi_{i_1}\mu_{i_1}\cdots\xi_{i_{k-j}}\mu_{i_{k-j}}
$$

where I ranges over all ordered subsets $I=(i_1,\ldots,i_{k-j})$ of $\{1,\ldots,t\}$ and $\kappa_I=\kappa_{i_1}\cdots\kappa_{i_l}$.

Since $\xi_r^2=-\frac{1}{2}\hbar=\mu_r^2$ and $\xi_r\mu_r=-\mu_r\xi_r$, we have

$$
\xi_r\mu_r\xi_r\mu_r=-\frac{\hbar^2}{4}.
$$

Hence,

$$\int \Theta \xi_r \mu_r = -\frac{\hbar^2}{4} \int \frac{\Theta}{\xi_r \mu_r} = 0.$$

Thus, for each $r = 1, \ldots, t$, we must have an even number of copies of $\xi_r \mu_r$ in the term

$$\xi_{i_1} \mu_{i_1} \cdots \xi_{i_{k-j}} \mu_{i_{k-j}}$$

for

$$\int \Theta \xi_{i_1} \mu_{i_1} \cdots \xi_{i_{k-j}} \mu_{i_{k-j}}$$

to be nonzero. In particular, if $k - j$ is odd, we get

$$\Upsilon_{2n|a,b}(\Theta \otimes 1^{\otimes j} \otimes Y_2^{\otimes k-j}) = 0.$$

If $k - j$ is even and we set $s = \frac{1}{2}(k - j)$, we have

$$\begin{aligned}
\Upsilon_{2n|a,b}(\Theta \otimes 1^{\otimes j} \otimes Y_2^{\otimes k-j}) &= \int \Theta Y_2^{k-j} \\
&= \sum_{|I|=k-j} (-1)^{k-j} \kappa_I \int \Theta \xi_{i'_1} \mu_{i_1} \cdots \xi_{i'_{k-j}} \mu_{i_{k-j}} \\
&= \sum_{|J|=s} \frac{(k-j)!}{2s!} \kappa_J^2 \int \Theta (\xi_{i'_1} \mu_{i'_1} \cdots \xi_{i'_s} \mu_{i'_s})^2 \\
&= \sum_{|J|=s} \frac{(k-j)!}{2s!} \kappa_J^2 \int \Theta \left(\frac{-\hbar^2}{4} \right)^s \\
&= \sum_{|J|=s} \frac{(k-j)!}{2s!} \left(\frac{-\hbar^2}{4} \right)^s \kappa_J^2
\end{aligned}$$

where $J = (i'1, \ldots, i'_s)$ and $(k-j)!/2s!$ is the number of ways of ordering the $2s$ elements $J \sqcup J$ into a set I. □

Lemma 2.8.32. *Consider a subset $R = (r_1, \ldots, r_l)$ of $\{1, \ldots, t\}$. In $\mathbb{K}$, if $k - j - 1 = 2s'$ is even,* *we have*

$$\Upsilon_{2n|a,b}\left(\frac{\Theta}{\xi_{r_1}\mu_{r_1}\cdots\xi_{r_l}\mu_{r_l}}\otimes 1^{\otimes j}\otimes Y_2^{k-j}\right)=$$
$$(-1)^l\kappa_{r_1}\cdots\kappa_{r_l}\sum_{\sum 2u_i}\left(\frac{-1}{4}\right)^{\frac{k-j-l}{2}}\frac{(k-j-l)!}{(2u_1)!\cdots(2u_l)!}\hbar^{k-j-l}\kappa_1^{2u_1}\cdots\kappa_t^{2u_t}$$

where the sum is taken over all partitions $k-j-l=2u_1+\cdots+2u_t$. This term vanishes if $k-j-1$ is odd.

Proof. Let $I=(i_1,\ldots,i_{k-j})$ with $i_l\in\{1,\ldots,t\}$. If r appears in I, say $r=i_{k-j}$, then

$$\frac{\Theta}{\xi_r\mu_r}\xi_{i_1}\mu_{i_1}\cdots\xi_{i_{k-j}}\mu_{i_{k-j}}=\Theta\xi_{i_1}\mu_{i_1}\cdots\xi_{i_{k-j-1}}\mu_{i_{k-j-1}}$$

and we are in the situation of Lemma 2.8.31.

If r does not appear in I, then $\frac{\Theta}{\xi_r\mu_r}xi_{i_1}\mu_{i_1}\cdots\xi_{i_{k-j}}\mu_{i_{k-j}}$ contains no $\xi_r\mu_r$ term, and its Berezin integral therefore vanishes.

We therefore must have $R\subset I$ for the term to not vanish.

Thus if $k-j-1=2s'$ is even, then

$$\Upsilon_{2n|a,b}\left(\frac{\Theta}{\xi_r\mu_r}\otimes 1^{\otimes j}\otimes Y_2^{\otimes k-j}\right)=-\kappa_r\sum_{|J'|=s'}\left(\frac{-\hbar^2}{4}\right)^{s'}\kappa_{J'}^2$$

and if $k-j-1$ is odd, then

$$\Upsilon_{2n|a,b}\left(\frac{\Theta}{\xi_r\mu_r}\otimes 1^{\otimes j}\otimes Y_2^{\otimes k-j}\right)=0$$

and similarly for $\frac{\Theta}{\xi_{r_1}\mu_{r_1}\cdots\xi_{r_l}\mu_{r_l}}$. □

Lemma 2.8.33. *In $\mathbb{K}$, for $j=0,\ldots,k$, we have*

$$\Upsilon_{2n|a,b}\left(\frac{\Theta}{\zeta_r\eta_r}\otimes 1^{\otimes j}\otimes Y_2^{\otimes k-j}\right)=0.$$

Proof. We have

$$\Upsilon_{2n|a,b}\left(\frac{\Theta}{\zeta_r\eta_r}\otimes 1^{\otimes j}\otimes Y_2^{\otimes k-j}\right)=\int\frac{\Theta}{\zeta_r\eta_r}Y_2^{k-j}.$$

Since Y_2 contains no η_r or ζ_r terms, we will be taking the Berezin integral of something with no η_r or ζ_r, which vanishes. □

In summary, we have shown the following:

Corollary 2.8.34. *For $j = 2, \dots, k$, a cycle of length j has Y_2 contribution*

$$\Upsilon_{2n|a,b}\left(g_{12}\cdots g_{j-1j}g_{j1}(\Theta \otimes Y_2^{\otimes k})\right) = \frac{1}{2^{j-1}}\sum_{r=1}^{t}\kappa_r^j\left(\sum_{|J|=s}\frac{(k-j)!}{2s!}\left(\frac{-\hbar^2}{4}\right)^s\kappa_J^2\right)$$

if $k-j=2s$ is even, and vanishes if k or j is odd.

Moreover,

$$\Upsilon_{2n|a,b}(g_{11}(\Theta \otimes Y_2^{\otimes k})) = 0.$$

Consider a partition $j_1 + \cdots j_l = j$. Then a Θ-flower with l petals of length $j_1, \dots, j_l$ has Y_2 contribution

$$\Upsilon_{2n|a,b}\left((g_{01}\cdots g_{j_1-1j_1})(g_{j_10}g_{0j_1+1}\cdots g_{j_1+j_2-1j_1+j_2}g_{j_1+j_20})\cdots(g_{0j-j_l}\cdots g_{j-1k}g_{j0})(\Theta\otimes Y_2^{\otimes k})\right).$$

If $k-j-l$ is even, this is equal to

$$\sum_{|R|=l}\sum_{\sum 2u_i}\frac{(-1)^{k/2}(-1)^l}{2^{k-l}}\frac{(k-j-l)!}{(2u_1)!\cdots(2u_l)!}\hbar^{k-j-l}\kappa_{r_1}^{j_1+1}\cdots\kappa_{r_l}^{j_l+1}(\kappa_1^{2u_1}\cdots\kappa_t^{2u_t}), \tag{2.9}$$

where $R=(r_1,\dots,r_l)$ ranges over subsets of $\{1,\dots,t\}$ of size l and the second sum is over all partitions $k-j-l = 2u_1+\cdots+2u_t$. This vanishes if $k-j-l$ is odd.

Moreover,

$$\Upsilon_{2n|a,b}(g_{00}(\Theta \otimes Y_2^{\otimes k})) = 0.$$

2.8.2 Proof of Theorem 2.8.13

We now combine the preliminary lemmas summarized in Corollaries 2.8.21, 2.8.27, and 2.8.34 to prove Theorem 2.8.13.

Recall that we are trying to compute

$$P_n(x, \ldots, x) = \Upsilon_{2n|a,b} \int_{[0,1]^n} \omega^{\leq 2}_{2n|a,b}(\Theta \otimes X^{\otimes n}) dv_1 \cdots dv_n. \tag{2.10}$$

Proof of Theorem 2.8.13. We need to classify all possible graph types and piece together the corresponding contributions from X.

By §2.8.1.4, the only terms that do not vanish after applying $\omega^{\leq 2}_{2n|a,b}$ are disjoint unions of cycles on X labeled vertices and possibly a Θ-flower. We must also consider vertices with no edges. Each of the n vertices labeled X belongs to one of these three types: a cycle, a flower, or a vertex with no edges. Correspondingly, to each graph we have a partition of n into three numbers, $n = n_1 + n_2 + n_3$ where

n_1 is the number of X-labeled vertices in cycles,

n_2 is the number of X-labeled vertices in the Θ-flower, and

n_3 is the number of solo X-labeled vertices.

Note that if $n_2 = 0$ then the Θ-labeled vertex has no edges.

The cycles part of the graph is determined by a partition of the n_1 vertices

$$n_1 = \sum_{j=2}^{n_1} j l_j$$

where l_j denotes the number of cycles of length j. Note that this sum starts at $j = 2$ since a cycle of length 1 is a vertex with no edges. Let $\mathcal{P}(n_1)$ denote the set of such partitions of n_1.

The Θ-flower part of the graph is determined by a partition of the n_2 X-labeled vertices

$$n_2 = \sum_{i=1}^{n_2} i \tilde{l}_i$$

where $\tilde{l}_i$ denotes the number of petals with i X-labeled vertices. This sum starts at $i = 1$ since a petal with one X-labeled vertex is allowed. Let $\tilde{\mathcal{P}}(n_2)$ denote the set of such partitions of n_2.

The data of a partition of n into $n_1 + n_2 + n_3$ along with the further partitions of n_1 and n_2 determine the graph. We will therefore be taking the sum over all such choices.

By §2.8.1.3, each graph corresponds to a summand in the expanded product of $\omega^{\leq 2}_{2n|a,b}$. We would like to compute (2.10). Since $\Upsilon_{2n|a,b}$ and integration are linear, we can pull out the sum over all graphs. Given a graph G, let C_G denote the contribution from the corresponding summand of (2.10). Let G_{aut} be the automorphism group of G. We then have

$$P_n = \sum_{n=n_1+n_2+n_3} \sum_{\mathcal{P}(n_1)} \sum_{\mathcal{P}(n_2)} \frac{n!}{|G_{\text{aut}}|} C_G.$$

To compute $|G_{\text{aut}}|$ and C_G, let us fix some notation. Say G corresponds to the graph $n = n_1+n_2+n_3$ and the partitions

$$n_1 = \sum_{j=2}^{n_1} j l_j$$

and

$$n_2 = \sum_{i=1}^{n_2} i \tilde{l}_i.$$

Then we have

$$|G_{\text{aut}}| = (n_3)! 2^{\sum_{i=3}^{n_1} l_i} 2^{\sum_{i=3}^{n_2} \tilde{l}_i} \prod_{j=2}^{n_1} j^{l_j} \prod_{j=2}^{n_2} j^{\tilde{l}_j}.$$

The $(n_3)!$ comes from permuting the vertices with no edges. We get a factor of j from permuting the vertices within a cycle of length j. As there are l_j of these cycles, we get a factor of j^{l_j}. Similarly, the term $j^{\tilde{l}_j}$ comes from permuting the vertices within petals of length j. Lastly, the powers of 2 come from the reflection (or mirror) symmetry of cycles and petals of length ≥ 3. One can compare this computation with [42, Pg. 27] and [38, Pg. 35].

The term C_G is the product of the contributions of the connected subgraphs of G,

$$C_G = \left(\prod_{j=2}^{n_1} (\text{length } j \text{ cycle contribution})^{l_j} \right) (\Theta\text{-flower contribution}).$$

We first analyze the length j cycle contribution. Recall that $X = X_1 + Y_1 + Y_2$ and let $\mathcal{X}_1^j, \mathcal{Y}_1^j$, and $\mathcal{Y}_2^j$ denote the contribution to a length j cycle from the X_1, Y_1, and Y_2 pieces, respectively.

Expanding the product of the sum of these terms, we get the following

$$\prod_{j=2}^{n_1}(\text{length } j \text{ cycle contribution})^{l_j} = \sum_{l_j=l_j(X_1)+l_j(Y_1)+l_j(Y_2)} \prod_{j=2}^{n_1} C_j \left(\mathcal{X}_1^j\right)^{l_j(X_1)} \left(\mathcal{Y}_1^j\right)^{l_j(Y_1)} \left(\mathcal{Y}_2^j\right)^{l_j(Y_2)}$$

where

$$C_j = \frac{1}{(l_j(X_1))!\,(l_j(Y_1))!\,(l_j(Y_2))!}$$

accounts for the additional graph automorphisms.

In each of the terms $\mathcal{X}_1^j, \mathcal{Y}_1^j$, and $\mathcal{Y}_2^j$ we see the same integral

$$I_j = \int_{[0,1]^j} \psi(v_1 - v_2) \cdots \psi(v_{j-1} - v_j)\psi(v_j - v_1) dv_1 \cdots dv_j.$$

By [42, Lem. 5.4] or [38, Pg. 34], we have

$$I_j = \begin{cases} -\frac{(-2)^j}{j!} B_j & j \text{ even} \\ 0 & j \text{ odd} \end{cases} \tag{2.11}$$

where B_j is the jth Bernoulli number. Thus, for $j \geq 2$ odd, the term I_j vanishes. Assume j is even.

By Corollary 2.8.21, we have

$$\mathcal{X}_1^j = \hbar^j I_j \Upsilon_{2n|a,b}(\alpha_{12} \cdots \alpha_{j-1j}\alpha_{j1}(\Theta \otimes X_1^{\otimes j}) = \frac{\hbar^j}{2^{j-1}} I_j \sum_{i=1}^{n} \gamma_i^j.$$

By Corollary 2.8.27, we have

$$\mathcal{Y}_1^j = \hbar^j I_j \Upsilon_{2n|a,b}\left(g_{12} \cdots g_{j-1}g_{j1}(\Theta \otimes Y_1^{\otimes j})\right) = \frac{-\hbar^j}{2^{j-1}} I_j \sum_{r=1}^{a} \lambda_r^j.$$

By Corollary 2.8.34, we have

$$\mathcal{Y}_2^j = \hbar^j I_j \Upsilon_{2n|a,b}\left(g_{12} \cdots g_{j-1}g_{j1}(\Theta \otimes Y_2^{\otimes j})\right) = \frac{\hbar^j}{2^{j-1}} I_j \sum_{s=1}^{t} \kappa_s^j.$$

Since I_j vanishes for j odd, we may assume j is even and the $(-2)^{j-1}$ contribution in $\mathcal{Y}_1^j$ and $\mathcal{Y}_2^j$ may be replaced with a -2^{j-1}.

Ignoring the Θ-flowers for a moment, we can write the cycles piece of P_n as

$$P_n^{\text{cycle}} = \sum_{\mathcal{P}(n)} \frac{n!}{(n_3)!} \prod_{j\geq 2} C_j \left(\frac{I_j}{2^j j} \sum_{i=1}^{n} (\hbar\gamma_i)^j \right)^{l_j(X_1)} \left(\frac{-I_j}{2^j j} \sum_{r=1}^{a} (\hbar\lambda_r)^j \right)^{l_j(Y_1)} \left(\frac{I_j}{2^j j} \sum_{s=1}^{t} (\hbar\kappa_s)^j \right)^{l_j(Y_2)}$$

where $\mathcal{P}(n)$ ranges over partitions of n as

$$n = \sum_{j\geq 2} j(l_j(X_1) + l_j(Y_1) + l_j(Y_2)) + n_3$$

and we brought in the copies of $\frac{1}{j}$ and powers of 2 from $|G_{\text{aut}}|$.

Setting $P_0^{\text{cycle}} = 1$, let

$$P^{\text{cycle}} = \sum_{n=0}^{\infty} \frac{1}{n!} P_n^{\text{cycle}}.$$

Then, we have

$$P^{\text{cycle}} = \exp\left(1 + \sum_{j\geq 2} \frac{I_j}{2^j j} \sum_{i=1}^{n} (\hbar\gamma_i)^j - \frac{I_j}{2^j j} \sum_{r=1}^{a} (\hbar\lambda_r)^j + \frac{I_j}{2^j j} \sum_{s=1}^{t} (\hbar\kappa_s)^j \right).$$

Using the identity

$$\sum_{j\geq 2} \frac{I_j}{2^j j} x^j = \log\left(\frac{x/2}{\sinh(x/2)} \right)$$

found in [42, Pg. 27] and [38, Pg. 36], we have

$$P^{\text{cycle}} = e \prod_{i=1}^{n} \left(\frac{\hbar\gamma_i/2}{\sinh(\hbar\gamma_i/2)} \right) \prod_{r=1}^{a} \left(\frac{\sinh(\hbar\lambda_r/2)}{\hbar\lambda_r/2} \right) \prod_{s=1}^{t} \left(\frac{\hbar\kappa_s/2}{\sinh(\hbar\kappa_s/2)} \right).$$

For the Θ-flower contribution, only Y_1 and Y_2 parts contribute. Let $\widetilde{\mathcal{Y}}_1$ and $\widetilde{\mathcal{Y}}_2$ denote their contributions so that

$$(\Theta\text{-flower contribution}) = \left(\widetilde{\mathcal{Y}}_1\right)\left(\widetilde{\mathcal{Y}}_2\right).$$

To describe the Θ-flower contribution term, we need to reorganize the data of our partition of n_2.

Let

$$l = \sum_{j=1}^{n_2} l_j$$

denote the number of petals of the Θ-flower. Order the petals from smallest length to largest and let j_i denote the length of the ith petal. For example, we have $j_i = 1$ for $i = 1, \dots, l_1$ corresponding to the l_1 petals of length 1. Note that $j_1 + \cdots + j_l = n_2$, the total number of X-labeled vertices in the flower.

Rewriting the Y_1 contribution from Corollary 2.8.27 we have

$$\widetilde{\mathcal{Y}}_1 = n_3! \sum_{n_3 = \sum_{i=1}^{l} s_i} \sum_{|R|=l} \prod_{i=1}^{l} \frac{-\widetilde{I}_{j_i} j_i}{2^{j_i} s_i!} (\hbar\lambda_{r_i})^{j_i+s_i}.$$

The term $j_1 \cdots j_l$ appears because each petal in the Θ-flower could be connected to Θ at any one of its j_i X-labeled vertices, see the top of [38, Pg. 34].

Here, $\tilde{I}_j$ is the integral

$$\tilde{I}_j = \int_{[0,1]^j} \psi(v_0 - v_1)\psi(v_1 - v_2) \cdots \psi(v_{j-1} - v_j)\psi(v_j - v_0) dv_1 \cdots dv_j.$$

By [42, Lem. 5.4] or [38, Pg. 34], we have $\tilde{I}_j = -I_{j+1}$.

Ignoring the cycle and $\widetilde{Y}_2$ contributions for a moment, let

$$\begin{aligned} P_n^{\Theta_1} &= n! \sum_{\widetilde{P}(n)} \left(\frac{1}{l! 2^l \left(\sum_{i=1}^{l} s_i\right)! \prod_{i=1}^{l} j_i} \right) \left(\sum_{i=1}^{l} s_i \right)! \left(\sum_{|R|=l} \prod_{i=1}^{l} \frac{-\widetilde{I}_{j_i} j_j}{2^{j_i} s_i!} (\hbar\lambda_{r_i})^{j_i+s_i} \right) \\ &= n! \sum_{\widetilde{\mathcal{P}}(n)} \frac{1}{l! 2^l} \sum_{|R|=l} \prod_{i=1}^{l} \frac{I_{j_i+1}}{2^{j_i} s_i!} (\hbar\lambda_{r_i})^{j_i+s_i}. \end{aligned}$$

where $\widetilde{\mathcal{P}}(n)$ ranges over all partitions

$$n = \sum_{i=1}^{l} j_i + s_i$$

where j_i and s_i are all nonzero. We can view $\widetilde{P}(n)$ as the set of all decorated Θ-flower graphs. Indeed, given such a partition, we have a corresponding Θ-flower with l petals of length $j_1, \dots, j_l$ and $s_1 + \cdots + s_l$ disjoint vertices, with s_i spare vertices assigned to "decorate" the ith petal. The combinatorial term

$$l!2^l\left(\sum_{i=1}^{l} s_i\right)!\prod_{i=1}^{l} j_i$$

is the automorphism group of such a decorated Θ-flower.

Set $P_0^{\Theta_1}=1$ and let

$$P^{\Theta_1}=\sum_{n=0}^{\infty}\frac{1}{n!}P_n^{\Theta_1}.$$

Now,

$$\sum_{n=0}^{\infty}\sum_{\widetilde{\mathcal{P}}(n)}\frac{1}{l!}\sum_{|R|=l}\prod_{i=1}^{a}\lambda_i^{j_i+s_i}=\prod_{i=1}^{a}\left(\sum_{m=0}^{\infty}\sum_{m=j+s}\lambda_i^{j+s}\right).$$

Thus we have

$$\begin{aligned}
P^{\Theta_1}&=\sum_{n=0}^{\infty}\sum_{\widetilde{\mathcal{P}}(n)}\frac{1}{l!2^l}\sum_{|R|=l}\prod_{i=1}^{l}\frac{I_{j_i+1}}{2^{j_i}s_i!}(\hbar\lambda_{r_i})^{j_i+s_i}\\
&=\sum_{n=0}^{\infty}\sum_{\widetilde{\mathcal{P}}(n)}\frac{1}{l!}\sum_{|R|=l}\prod_{i=1}^{l}\frac{I_{j_i+1}}{2^{j_i+1}s_i!}(\hbar\lambda_{r_i})^{j_i+s_i}\\
&=\prod_{i=1}^{a}\left(\sum_{m=0}^{\infty}\sum_{m=j+s}\frac{I_{j+1}}{2^{j+1}s!}(\hbar\lambda_i)^{j+s}\right)\\
&=\prod_{i=1}^{a}\left(1+\sum_{j=0}^{\infty}\frac{I_{j+1}}{2^{j+1}}(\hbar\lambda_i)^{j+1}\right)\left(\sum_{s=1}^{\infty}\frac{(\hbar\lambda_i)^{s-1}}{s!}\right)\\
&=\prod_{i=1}^{a}\left(1+\sum_{j=0}^{\infty}\frac{I_{j+1}}{2^{j+1}}(\hbar\lambda_i)^{j+1}\right)\left(\frac{e^{\hbar\lambda_i}-1}{\hbar\lambda_i}\right)
\end{aligned}$$

Since the contribution vanishes for $j+1$ odd and we have set $P_0^{\Theta}=1$, after substituting Equation (2.11), this expression becomes

$$\prod_{i=1}^{a}\left(\sum_{m=0}^{\infty}-B_{2m}\frac{(\hbar\lambda_i)^{2m}}{(2m)!}\right)\left(\frac{e^{\hbar\lambda_i}-1}{\hbar\lambda_i}\right)=(-1)^a\prod_{i=1}^{a}(\hbar\lambda_i/2)\coth(\hbar\lambda_i/2)\left(\frac{e^{\hbar\lambda_i}-1}{\hbar\lambda_i}\right).$$

Lastly, we compute the $\mathcal{Y}_2$ contribution. Using the description (2.9) we have

$$\widetilde{Y}_2 = \sum_{\sum 2u_i} \sum_{|R|=l} \frac{(-1)^{(n_2+n_3)/2} n_3!}{2^{n_2+n_3-l}} \left(\prod_{i=1}^{l} -\widetilde{I}_{j_i} j_i (\hbar \kappa_{r_i})^{j_i+1} \right) \left(\prod_{s=1}^{t} \frac{1}{(2u_s)!} (\hbar \kappa_s)^{2u_s} \right).$$

Note that this term only appears when $n_2 + n_3$ is even.

Let

$$\begin{aligned} P_n^{\Theta_2} &= n! \sum_{\widetilde{P}_2(n)} \left(\frac{\left(\sum_{s=1}^{t} 2u_s\right)! \left(\frac{(-1)^{n/2}}{2^{n-l}}\right)}{l! 2^l \left(\sum_{i=1}^{t} 2u_s\right)! \prod_{i=1}^{l} j_i} \right) \left(\sum_{|R|=l} \left(\prod_{i=1}^{l} -\widetilde{I}_{j_i} j_i (\hbar \kappa_{r_i})^{j_i+1} \right) \left(\prod_{s=1}^{t} \frac{1}{(2u_s)!} (\hbar \kappa_s)^{2u_s} \right) \right) \\ &= n! \sum_{\widetilde{P}_2(n)} \frac{(-1)^{n/2}}{l! 2^n} \sum_{|R|=l} \left(\prod_{i=1}^{l} -\widetilde{I}_{j_i} (\hbar \kappa_{r_i})^{j_i+1} \right) \left(\prod_{s=1}^{t} \frac{1}{(2u_s)!} (\hbar \kappa_s)^{2u_s} \right) \\ &= n! \sum_{\widetilde{P}_2(n)} \frac{1}{l!} \sum_{|R|=l} \left(\prod_{i=1}^{l} -\widetilde{I}_{j_i} (-1)^{\frac{j_i+1}{2}} \left(\frac{\hbar \kappa_{r_i}}{2} \right)^{j_i+1} \right) \left(\prod_{s=1}^{t} \frac{(-1)^{u_s}}{(2u_s)!} \left(\frac{\hbar \kappa_s}{2} \right)^{2u_s} \right) \end{aligned}$$

where $\widetilde{\mathcal{P}}_2(n)$ ranges over all partitions

$$n = \sum_{i=1}^{l} (j_i + 1) + \sum_{s=1}^{t} 2u_s$$

where j_i and s_i are all nonzero.

Set $P_0^{\Theta_2} = 1$ and let

$$P^{\Theta_2} = \sum_{n=0}^{\infty} \frac{1}{n!} P_n^{\Theta_2}.$$

We have

$$
\begin{aligned}
P^{\Theta_2} &= \sum_{n=0}^{\infty} \sum_{\widetilde{P}_2(n)} \frac{1}{l!} \sum_{|R|=l} \left(\prod_{i=1}^{l} -\widetilde{I}_{j_i} (-1)^{\frac{j_i+1}{2}} \left(\frac{\hbar\kappa_{r_i}}{2}\right)^{j_i+1} \right) \left(\prod_{s=1}^{t} \frac{(-1)^{u_s}}{(2u_s)!} \left(\frac{\hbar\kappa_s}{2}\right)^{2u_s} \right) \\
&= \prod_{i=1}^{t} \left(\sum_{n=0}^{\infty} \sum_{n=j+2u} I_{j+1} (-1)^{\frac{j+1}{2}} \left(\frac{\hbar\kappa_i}{2}\right)^{j+1} \frac{(-1)^u}{(2u)!} \left(\frac{\hbar\kappa_i}{2}\right)^{2u} \right) \\
&= \prod_{i=1}^{t} \left(1 + \sum_{j=0}^{\infty} I_{j+1} (-1)^{\frac{j+1}{2}} \left(\frac{\hbar\kappa_i}{2}\right)^{j+1} \right) \left(\sum_{u=0}^{\infty} \frac{(-1)^u}{(2u)!} \left(\frac{\hbar\kappa_i}{2}\right)^{2u} \right) \\
&= \prod_{i=1}^{t} \left(\sum_{j=0}^{\infty} -B_{2m} (-1)^m \frac{(\hbar\kappa_i)^{2m}}{(2m)!} \right) \cos(\hbar\kappa_i) \\
&= (-1)^t \prod_{i=1}^{t} (\hbar\kappa_i/2) \cot(\hbar\kappa_i/2) \cos(\hbar\kappa_i)
\end{aligned}
$$

Putting the contributions from cycles and Θ-flowers together, let

$$
P = \sum_{n=0}^{\infty} n! P_n.
$$

Then P is equal to

$$
\begin{aligned}
&(-1)^{a+t} e \prod_{i=1}^{n} \frac{\hbar\gamma_i/2}{\sinh(\hbar\gamma_i/2)} \prod_{r=1}^{a} \frac{\sinh(\hbar\lambda_r/2)}{\hbar\lambda_r/2} \frac{1}{2} \coth(\hbar\lambda_r/2)(e^{\hbar\lambda_r} - 1) \prod_{s=1}^{t} \frac{(\hbar\kappa_s/2)^2}{\sinh(\hbar\kappa_s/2)} \cot(\hbar\kappa_s/2)\cos(\hbar\kappa_s) \\
&= (-1)^{a+t} e \prod_{i=1}^{n} \frac{\hbar\gamma_i/2}{\sinh(\hbar\gamma_i/2)} \prod_{r=1}^{a} \cosh(\hbar\lambda_r/2) \frac{e^{\hbar\lambda_r} - 1}{\hbar\lambda_r} \prod_{s=1}^{t} \frac{(\hbar\kappa_s/2)^2}{\sinh(\hbar\kappa_s/2)} \cot(\hbar\kappa_s/2)\cos(\hbar\kappa_s).
\end{aligned}
$$

□

Remark 2.8.35. In type $(2n|a,a)$, Theorem 2.8.13 differs from Engeli's computation [38, Lem. 2.25] by the sign $(-1)^a$ and the term

$$
\frac{e^{\hbar\lambda_r} - 1}{\hbar\lambda_r}
$$

in Theorem 2.8.13. This difference is traceable to Lemma 2.8.26. Engeli's result only takes into account the computation of Lemma 2.8.26 when $k = j+l$. Graphically, this corresponds to decorated Θ-flowers with exactly one spare vertex assigned to each petal. Here, we see contributions from decorated Θ-flowers with multiple spare vertices assigned to each petal.

2.8.3 Global Superalgebraic Index Theorem

Consider the supertrace $\mathrm{Tr}_{\mathbb{M}}$ on $\mathcal{A}_\sigma(\mathbb{M})$ from Theorem 2.7.9. We would like to compute $\mathsf{Ev}_{\mathbb{M}}(\mathrm{Tr}_{\mathbb{M}})$, the evaluation of $\mathrm{Tr}_{\mathbb{M}}$ on the volume form from Definition 2.5.1.

For this, we need a way of relating the characteristic class homomorphism $\mathrm{char}_{(\mathbb{M},\sigma)}$ from §2.5.2, the map χ from Definition 2.8.1, and the classical Chern-Weil map [69, Appendix C].

Let $\mathrm{pr}\colon \mathfrak{g}^{\hbar}_{2n|a,b} \to \mathfrak{sp}_{2n|a,b}$ be the map used to define χ and $A \in \Omega^1(\mathrm{Fr}^{\mathrm{Sp}(2n|a,b)}_{\mathbb{M}}; \mathfrak{g}^{\hbar}_{2n|a,b})$ the connection 1-form on $\mathrm{Fr}^{\mathrm{Sp}(2n|a,b)}_{\mathbb{M}}$ used to define $\mathrm{char}_{(\mathbb{M},\sigma)}$. Then A is a flat connection by [51, Def. 1.7]. Since A is $\mathrm{Sp}(2n|a,b)$ invariant and satisfies [51, Def. 1.7(1)], the 1-form $\mathrm{pr}(A)$ is a connection 1-form on $\mathrm{Fr}^{\mathrm{Sp}(2n|a,b)}_{\mathbb{M}}$ valued in $\mathfrak{sp}_{2n|a,b}$.

Since A is flat, its curvature F_A is zero. However, since pr is not a Lie algebra map, the connection pr_A may not be flat. Let $F_{\mathrm{pr}(A)} \in \Omega^2$ denote the curvature of this connection. We use the notation $\mathrm{CW}_{\mathbb{M}}$ for the map

$$\mathrm{CW}_{\mathbb{M}}\colon (\widehat{\mathrm{Sym}}^{\bullet}(\mathfrak{sp}^*_{2n|a,b}))^{\mathfrak{sp}_{2n|a,b}} \to H^{2\bullet}_{\mathrm{dR}}(\mathbb{M};\mathbb{K})$$

given by evaluating an invariant polynomial on $F_{\mathrm{pr}(A)}$. Note that, as we are viewing $\mathrm{CW}_{\mathbb{M}}$ as a map landing in cohomology, it is independent of the choice of connection whose curvature on which we evaluate polynomials.

Lemma 2.8.36. *The diagram*

$$\begin{array}{ccc} H^{2\bullet}_{\mathrm{Lie}}(\mathfrak{g}^{\hbar}_{2n|a,b}, \mathfrak{sp}_{2n|a,b}; \mathbb{K}) & \xrightarrow{\mathrm{char}_{(\mathbb{M},\sigma)}(\mathbb{K})} & H^{2\bullet}_{\mathrm{dR}}(\mathbb{M};\mathbb{K}) \\ \uparrow{\scriptstyle\chi} & \nearrow{\scriptstyle \mathrm{CW}_{\mathbb{M}}} & \\ (\widehat{\mathrm{Sym}}^{\bullet}(\mathfrak{sp}^*_{2n|a,b}))^{\mathfrak{sp}_{2n|a,b}} & & \end{array}$$

commutes.

Remark 2.8.37. Lemma 2.8.36 holds for any super Harish-Chandra pair $(\mathfrak{g}, K)$ and principal $(\mathfrak{g}, K)$-bundle P, with the same proof.

Proof. Let $P \in \mathrm{Sym}^m(\mathfrak{sp}^*_{2n|a,b})^{\mathfrak{sp}_{2n|a,b}}$. Let $C \in \mathrm{Hom}(\Lambda^2 \mathfrak{g}^\hbar_{2n|a,b}, \mathfrak{sp}_{2n|a,b})$ be the curvature of the projection pr used to define χ, see Definition 2.8.1. Then, $\mathrm{char}_{(\mathbb{M},\sigma)}(\mathbb{K})(\chi(P))$ is the cohomology class of the $2m$-form $\chi(P)_*(A^{\wedge 2m})$ where $\chi(P)_*$ is the map

$$\chi(P)_*\colon \Omega^{2m}(\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}; \Lambda^{2m}\mathfrak{g}^\hbar_{2n|a,b}/\mathfrak{sp}_{2n|a,b}) \to \Omega^{2m}(\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}; \mathbb{K}).$$

Using the definition of $\chi(P)$, we have

$$\chi(P)_*(A^{\wedge 2m}) = \frac{1}{m!} \sum_{s \in \Sigma_{2m}/(\Sigma_2)^{\times m}} \mathrm{sign}(s) P(C(A,A), \dots, C(A,A)).$$

The permutation has no effect on the term, so we may rewrite this as

$$\chi(P)_*(A^{\wedge 2m}) = \frac{1}{m!} P(C(A,A), \cdots, C(A,A)).$$

This is the definition of the Chern-Weil map $\mathrm{CW}_{\mathbb{M}}$, assuming that the 2-form $F_{\mathrm{pr}(A)}$ on $\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}$ is given by $C(A,A)$.

To see this, note that the curvature $F_{\mathrm{pr}(A)}$ measures the failure of $\mathrm{pr}(A)$ to satisfy the Maurer-Cartan equation. Now $\mathrm{pr}(A)$ is a Maurer-Cartan element exactly when the corresponding morphism

$$\mathrm{pr}(A)\colon C^\bullet_{\mathrm{Lie}}(\mathfrak{g}^\hbar_{2n|a,b}) \to \Omega^\bullet(\mathbb{M}; \mathbb{K})$$

is an algebra map. The map pr induces a map

$$\mathrm{pr}^*\colon C^\bullet_{\mathrm{Lie}}(\mathfrak{g}^\hbar_{2n|a,b}) \to C^\bullet_{\mathrm{Lie}}(\mathfrak{sp}_{2n|a,b})$$

which is an algebra map if and only if pr is a map of Lie algebras. The failure of pr to be a Lie algebra map is measured by $C(A,A)$. Since the diagram

$$\begin{array}{ccc} C^\bullet_{\mathrm{Lie}}(\mathfrak{g}^\hbar_{2n|a,b}) & \xrightarrow{A} & \Omega^\bullet(\mathbb{M}; \mathbb{K}) \\ {\scriptstyle \mathrm{pr}^*}\uparrow & \nearrow_{\mathrm{pr}(A)} & \\ C^\bullet_{\mathrm{Lie}}(\mathfrak{sp}_{2n|a,b}) & & \end{array}$$

commutes, $\mathrm{pr}(A)$ is an algebra map if and only if pr^* is; that is, $F_{\mathrm{pr}(A)}$ and $C(A, A)$ are the same measurement. □

Write $F_{\mathrm{pr}(A)} = R_1 + S_2 + S_3$ so that R_1 is a $\mathfrak{sp}_{2n}$-valued form, S_1 is a $\mathfrak{so}_a$-valued form, and S_2 is a $\mathfrak{so}_t$-valued form.

Theorem 2.8.38 (Superalgebraic Index Theorem)**.** *The evaluation of the unique normalized supertrace* $\mathrm{Tr}_{\mathbb{M}}$ *on the volume form* $1 \otimes \Theta_{\mathbb{M}}$ *is*

$$\mathsf{Ev}_{\mathbb{M}}(\mathrm{Tr}_{\mathbb{M}}) = (-1)^{n+a+t} \hbar^n e \int_{\mathbb{M}} \widehat{A}(R_1) \widehat{B}(S_1) \widehat{C}(S_2).$$

Note that the $\widehat{A}$ appearing in Theorem 2.8.38 is not the $\hat{A}$-genus of the supermanifold $\mathbb{M}$ as in [85, Def. 3.1(6)], but rather closer to the $\hat{A}$ genus of the reduced manifold M of $\mathbb{M}$, see Example 2.8.40.

Proof. Since $\mathrm{Tr}_{\mathbb{M}}$ is defined by descending the supertrace $\tau_{2n|a,b}$, by Theorem 2.5.6, we have

$$\mathsf{Ev}_{\mathbb{M}}(\mathrm{Tr}_{\mathbb{M}}) = \int_{\mathbb{M}} \mathrm{char}_{(\mathbb{M},\sigma)}(\mathbb{K})(\mathsf{Ev}_{\mathrm{loc}}(\tau_{2n|a,b})).$$

We saw in Lemma 2.8.4 that

$$\mathsf{Ev}_{\mathrm{loc}}(\tau_{2n|a,b}) = (-1)^n \chi(P_n).$$

By Lemma 2.8.36, we can relate $\mathrm{char}(\chi)$ to the Chern-Weil map and obtain the following

$$\begin{aligned} \int_{\mathbb{M}} \mathrm{char}_{(\mathbb{M},\sigma)}(\mathbb{K})(\mathsf{Ev}_{\mathrm{loc}}(\tau_{2n|a,b})) &= \int_{\mathbb{M}} \mathrm{char}_{(\mathbb{M},\sigma)}(\mathbb{K}) \left((-1)^n \chi(P_n)\right) \\ &= (-1)^n \int_{\mathbb{M}} \mathrm{CW}_{\mathbb{M}}(P_n). \end{aligned}$$

The Chern-Weil map $\mathrm{CW}_{\mathbb{M}}$ evaluates an ad invariant polynomial on the curvature $F_{\mathrm{pr}(A)}$. Using the description in Theorem 2.8.13, of the polynomial P_n, we have

$$\mathsf{Ev}_{\mathbb{M}}(\mathrm{Tr}_{\mathbb{M}}) = (-1)^{n+a+t} e \int_{\mathbb{M}} \left[\widehat{A}(\hbar R_1) \widehat{B}(\hbar S_1) \widehat{C}(\hbar S_2) \right]_n.$$

The degree n homogeneous part is

$$\left[\widehat{A}(\hbar R_1)\widehat{B}(\hbar S_1)\widehat{C}(\hbar S_2)\right]_n = \hbar^n \left[\widehat{A}(R_1)\widehat{B}(S_1)\widehat{C}(S_2)\right]_n .$$

□

In the purely even case, Theorem 2.8.38 recovers the algebraic index theorem of [42].

2.8.4 Examples

We can rephrase Theorem 2.8.38 in terms of the reduced (non-super) manifold M of $\mathbb{M}$. Note that the $(\mathfrak{g}^\hbar_{2n|a,b}, \mathrm{Sp}(2n|a,b))$-bundle $\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}$ determines a $(\mathfrak{g}^\hbar_{2n|a,b}, \mathrm{Sp}(2n|a,b))$-bundle F_M on M given (as a space) by the reduced manifold of $\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}$. The connection 1-form A on $\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}$ is sent to the connection 1-form A_{red} on F_M by the Berezin integral

$$\int (-) d\Theta \colon \Omega^1(\mathrm{Fr}_{\mathbb{M}}^{\mathrm{Sp}(2n|a,b)}; \mathfrak{g}^\hbar_{2n|a,b}) \to \Omega^1(F_M; \mathfrak{g}^\hbar_{2n|a,b}).$$

Since the characteristic map char_P from §2.5.2 is defined in terms of the connection 1-form on the principal bundle P, we have a commutative diagram

$$\begin{array}{ccc} C^\bullet_{\mathrm{Lie}}(\mathfrak{g}^\hbar_{2n|a,b}, \mathrm{sp}_{2n|a,b}) & \xrightarrow{\mathrm{char}_{(\mathbb{M},\sigma)}} & \Omega^\bullet(\mathbb{M}; \mathbb{K}) \\ & \searrow^{\mathrm{char}_{F_M}} & \downarrow{\scriptstyle \int(-)d\Theta} \\ & & \Omega^\bullet(M; \mathbb{K}). \end{array}$$

Corollary 2.8.39. *There is an equivalence of maps* $C^\bullet_{\mathrm{Lie}}(\mathfrak{g}^\hbar_{2n|a,b}, \mathrm{sp}_{2n|a,b}) \to \mathbb{K}$ *by*

$$\int_{\mathbb{M}} \mathrm{char}_{(\mathbb{M},\sigma)} = \int_M \mathrm{char}_{F_M}.$$

We can therefore interpret Theorem 2.8.38 in terms of characteristic classes for the bundle $F_M \to M$. A particularly nice expression is obtained when the symplectic supermanifold $\mathbb{M}$ is “split."

A Theorem of Rothstein [76] says that all symplectic supermanifolds are non-canonically isomorphic to one of the form $E[1]$ where $E \to M$ is a quadratic vector bundle on a symplectic manifold

M. Call $\mathbb{M}$ *split* if we have chosen an identification $\mathbb{M} = E[1]$.

Example 2.8.40. Let $\mathbb{M} = E[1]$ be a split symplectic supermanifold. Then Theorem 2.8.38 gives the following computation:

$$\mathsf{Ev}_{\mathbb{M}}(\mathrm{Tr}_{\mathbb{M}}) = (-1)^{n+a+t}\hbar^n e \int_M \widehat{A}(M)\widehat{BC}(E)$$

in terms of the characteristic series for th $\widehat{A}$-genus of M and the characteristic series $\widehat{BC}$ from Example 2.8.12 of the vector bundle E.

Example 2.8.41 (L-genus)**.** As a special case of the above example, consider the vector bundle $\pi\colon TM \to M$. Since M is a symplectic manifold, we get an identification $T^*M \cong TM$. Using this identification, we can consider the evaluation pairing

$$TM \otimes TM \cong T^*M \otimes TM \xrightarrow{\mathrm{ev}} \mathbb{R}.$$

With this pairing, TM becomes a quadratic vector bundle on M. We use the notation $T^*[1]M$ for the associated symplectic supermanifold. Note that $T^*[1]M$ has type $(2n|n, n)$. Moreover, the tangent bundle of TM restricted to M is

$$\pi^*(TTM) = TM \oplus TM.$$

Example 2.8.40 then becomes

$$\mathsf{Ev}_{T^*[1]M}(\mathrm{Tr}_{T^*[1]M}) = (-1)^n e\hbar^n \int_M \widehat{A}(M)\widehat{B}(M).$$

The characteristic series for $\widehat{A}\widehat{B}$ is

$$\frac{t_i}{\sinh(t_i)}\cosh(t_i) = \frac{t_i}{\tanh(t_i)},$$

which is the characteristic series for the L-genus times the Todd genus of the stable normal bundle, see Remarks 2.8.9 and 2.8.10. Thus,

$$\mathsf{Ev}_{T^*[1]M}(\mathrm{Tr}_{T^*[1]M}) = e\hbar^n \int_M L(M)\mathrm{Td}(-TM).$$

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