

Multiplicative structures on Moore spectra

by

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Abstract

In this article we show that $\mathbb{S}/8$ is an \mathbb{E}_1 -algebra, $\mathbb{S}/32$ is an \mathbb{E}_2 -algebra, \mathbb{S}/p^{n+1} is an \mathbb{E}_n -algebra at odd primes and, more generally, for every h and n there exist generalized Moore spectra of type h which admit an \mathbb{E}_n -algebra structure.

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1 Introduction

A fundamental difference between ordinary commutative algebra and higher algebra is the difficulty of constructing quotient algebras. As an example, the quotient of \mathbb{Z} by 2 is the commutative algebra \mathbb{F}_2 , while the quotient of the sphere spectrum by 2 does not even admit a unital multiplication.¹ At odd primes \mathbb{S}/p admits a unital multiplication, but Toda showed in 1968 that there is no homotopy associative multiplication on $\mathbb{S}/3$ [22]. The issues continue at larger primes too; work of Kraines [14] and Kochman [13] implies² that \mathbb{S}/p admits an \mathbb{A}_{p-1} -algebra structure, but no \mathbb{A}_p -algebra structure. In particular, \mathbb{S}/p is never an \mathbb{E}_1 -algebra.

The situation improves somewhat if we look at the quotient by a power of p instead. In 1984, Oka constructed homotopy associative multiplications on $\mathbb{S}/4$ and $\mathbb{S}/9$ [18] and much more recently, Bhattacharya and Kitchloo constructed an \mathbb{A}_{p^q-1} -algebra structure on \mathbb{S}/p^q for p odd (on \mathbb{S}/p^{q+1} for $p = 2$) [4, 5]. Ultimately, the paucity of positive results has led to a rather negative outlook on the issue of quotient algebras—an outlook which led Mark Mahowald to conjecture that for $\alpha \in \pi_*\mathbb{S}$ the quotient \mathbb{S}/α admits an \mathbb{E}_1 -algebra structure if and only if $\alpha = 0$. In this article we show that this conjecture is completely false.

Theorem 1.1. *The Moore spectrum $\mathbb{S}/8$ admits an \mathbb{E}_1 -algebra structure.*

At odd primes \mathbb{S}/p^2 admits an \mathbb{E}_1 -algebra structure.

With an \mathbb{E}_1 -algebra structure on $\mathbb{S}/8$ in hand we are led to ask for more. Could there be Moore spectra which are \mathbb{E}_2 -algebras? The answer, rather shockingly, is yes.

Theorem 1.2. *Moore spectra admit the following multiplicative structures:*

- $\mathbb{S}/2^q$ admits an \mathbb{E}_n -algebra structure for $q \geq \frac{3}{2}(n+1)$ and
- \mathbb{S}/p^q admits an \mathbb{E}_n -algebra structure for $q \geq n+1$ and p odd.

The Moore spectra we have considered up to now are only the first examples of the generalized Moore spectra obtained by inductively taking quotients by higher chromatic self-maps and multiplicative structures on these objects have received a certain amount of attention (see [17, 19, 9, 8]). Here too we are able to exceed all expectations.

Theorem 1.3. *For each h and n there exist generalized Moore spectra of type h which admit an \mathbb{E}_n -algebra structure.*

Each of the theorems we have stated up to this point is proved as a corollary³ of the next theorem which allows us to construct quotient algebras in great generality.

Theorem 1.4. *Suppose we are given an \mathbb{E}_{m+1} -algebra $A \in \text{Sp}$ with $m \geq 2$ and a class $v \in \pi_{2*}(A)$ such that*

¹A unital multiplication on $\mathbb{S}/2$ can be ruled out by considering the Cartan formula for Steenrod squares.

²See [2, Example 3.3] for an explanation of this implication.

³Except for the 2-primary parts of Theorems 1.1 and 1.2 which are proved separately in Section 3.

- $Q_1(v) \equiv 0 \pmod{v}$ ⁴ or equivalently
- the cofiber A/v admits a unital multiplication.

Then, A/v^q admits an \mathbb{E}_n - A -algebra structure as long as $n \leq m$ and $q > n$.

This theorem too follows from an even more general result on quotients in higher algebra where we allow the category in which we work to vary.

Theorem 1.5. *Suppose we are given a stably \mathbb{E}_m -monoidal⁵ category \mathcal{C} with $m \geq 2$ an object $\mathcal{I} \in \mathcal{C}$ and a map $v : \mathcal{I} \rightarrow \mathbb{1}_{\mathcal{C}}$ such that the cofiber $\mathbb{1}_{\mathcal{C}}/v$ admits a right unital multiplication. Then for each $n \leq m$ there exists a tower of \mathbb{E}_n -algebras and \mathbb{E}_n -algebra maps*

$$\dots \longrightarrow \mathbb{1}_{\mathcal{C}}/v^{n+3} \longrightarrow \mathbb{1}_{\mathcal{C}}/v^{n+2} \longrightarrow \mathbb{1}_{\mathcal{C}}/v^{n+1}.$$

Moreover, each $\mathbb{1}_{\mathcal{C}}/v^q$ has a unique v -compatible (in the sense of Definition 5.1 below) \mathbb{E}_n -algebra structure.

Remark 1.6. If we let \mathcal{C}' denote the full subcategory of \mathcal{C} generated by $\mathbb{1}_{\mathcal{C}}$ and \mathcal{I} under tensor products and finite (co)limits, then there is an \mathbb{E}_m -monoidal left adjoint $\text{Ind}(\mathcal{C}') \rightarrow \mathcal{C}$ which is fully faithful on \mathcal{C}' .

This allows us to reduce the proof of Theorem 1.5 to the case where \mathcal{C} is stable, presentably \mathbb{E}_m -monoidal and $\mathbb{1}_{\mathcal{C}}$ and \mathcal{I} are compact \otimes -generators. \triangleleft

The proof of Theorem 1.5 has two main inputs: An obstruction theory for constructing \mathbb{E}_n -algebra structures on quotients which we develop in Section 2 and a categorification of the Adams spectral sequence constructed by Patchkoria and Pstrągowski in [20] which we review in Section 4. The proof of Theorem 1.5 is then quite direct: We construct a deformation of \mathcal{C} which categorifies the $\mathbb{1}_{\mathcal{C}}/v$ -Adams spectral sequence and an object $\nu\mathbb{1}_{\mathcal{C}}/\tilde{v}^q$ lying over $\mathbb{1}_{\mathcal{C}}/v^q$. Then we compute that the groups in which the obstructions to constructing an \mathbb{E}_n -algebra structure on $\nu\mathbb{1}_{\mathcal{C}}/\tilde{v}^q$ live vanish!

The proofs of the 2-primary parts of Theorems 1.1 and 1.2 are simpler than (and somewhat different from) the proof of Theorem 1.5 and are given in Section 3. In Appendix A, which may be of independent interest, we discuss bar-cobar duality for graded \mathbb{E}_n -algebras and use this duality to make the key construction used in Section 2.

The reader might naturally wonder whether our results are sharp. The following example shows that Theorem 1.4 at least cannot be too far from sharp.

Example 1.7. Let R be the free commutative \mathbb{F}_2 -algebra on a class x in degree zero. From Theorem 1.4 we know that $R/x^{2(2^k-1)}$ admits an \mathbb{E}_{2^k-2} -algebra structure. On

⁴Here Q_1 is the power operation carried by the second cell of $D_2^{\mathbb{E}_m}$ on an even sphere. Note that since $2Q_1(v) = 0$ this condition is automatic if 2 is invertible in R . Similarly, Q_1 satisfies a Cartan formula and therefore this condition is satisfied by squares of even degree classes (see Remark 5.5).

⁵Meaning that we ask that the binary tensor product $- \otimes -$ on \mathcal{C} commutes with finite (co)limits separately in each variable.

the other hand

$$Q_{2^k}(x^{2^{k+1}-2}) \equiv x^{2(2^k-2)}Q_1(x)^{2^k} \not\equiv 0 \pmod{x^{2^{k+1}-2}},$$

which implies that $R/x^{2(2^k-1)}$ is not an \mathbb{E}_{2^k+1} - R -algebra. \triangleleft

The issue of whether Theorem 1.2 is sharp is more delicate. We suspect it is sharp at odd primes, but that at $p = 2$ the function $\frac{3}{2}(n+1)$ can be replaced by $n + O(1)$.

Conventions

1. We fix three nested universes, referring to objects as small, large or huge depending on their size.
2. We write $\mathbb{1}_{\mathcal{E}}$ for the unit of a monoidal category \mathcal{E} .
3. We write $\mathbb{1}_{\mathcal{E}}\{X\}$ for the free \mathbb{E}_n -algebra on an object $X \in \mathcal{E}$.
4. Throughout the body of the paper \mathcal{C} will be a fixed stable, presentably \mathbb{E}_m -monoidal category with compact generators $\{\mathcal{I}^{\otimes k}\}_{k \geq 0}$ and $m \geq 2$.
5. We write $\mathcal{C}^{\mathbf{Gr}}$ for the \mathbb{E}_m -monoidal category of graded objects in \mathcal{C} . In $\mathcal{C}^{\mathbf{Gr}}$ we write $X(1)$ for the shift of X by 1 and X_k for the degree k component of X .
6. We write $\mathcal{C}^{\mathbf{Fil}}$ for the \mathbb{E}_m -monoidal category of filtered objects in \mathcal{C} and we take the convention that all filtrations are increasing. We write τ for the shift map $\tau : X(1) \rightarrow X$ on a filtered object.
7. We view graded objects as modules over the cofiber of the shift map τ in filtered objects and identify the associated graded functor with taking the cofiber by τ .
8. We say an \mathbb{E}_n -algebra or coalgebra in $\mathcal{C}^{\mathbf{Gr}}$ (or $\mathcal{C}^{\mathbf{Fil}}$) is positively graded if it vanishes in negative gradings and is given by the unit in degree 0.
9. Part of the data of a \mathbb{E}_n -monoidal structure on \mathcal{C} includes a k -fold tensor product functor $\mathbb{E}_n(k) \times_{\Sigma_k} \mathcal{C}^{\times k} \rightarrow \mathcal{C}$. Precomposing with the diagonal gives a functor $\mathcal{C} \rightarrow \text{Fun}(\mathbb{E}_n(k)_{h\Sigma_k}, \mathcal{C})$ and we write $D_k^{\mathbb{E}_n}(-)$ for the functor $\mathcal{C} \rightarrow \mathcal{C}$ obtained by taking the limit over $\mathbb{E}_n(k)_{h\Sigma_k}$ (the space of unordered configurations of k points in \mathbb{R}^n).
10. When working with \mathbb{F}_2 -synthetic spectra we use the convention that the k index of $\Sigma^{k,s}$ is the topological degree and the s -index is Adams filtration. This has the pleasant feature that (k, s) corresponds to (x, y) -coordinates in Adams spectral sequence charts.

2 An obstruction theory for quotients

In order to motivate the developments of this section let us consider an example.

Example 2.1. Suppose we are given an object \mathcal{I} in \mathcal{C} and a map $v : \mathcal{I} \rightarrow \mathbb{1}_{\mathcal{C}}$ and we want to give the cofiber $\mathbb{1}_{\mathcal{C}}/v$ an \mathbb{E}_1 -algebra structure.

A natural first step in constructing such an \mathbb{E}_1 -algebra is to consider the \mathbb{E}_1 -cofiber of v which we write $\mathbb{1}_{\mathcal{C}}//v$.⁶ What we would like to do is continue attaching further \mathbb{E}_1 -cells to $\mathbb{1}_{\mathcal{C}}//v$ in order to eliminate the difference between $\mathbb{1}_{\mathcal{C}}/v$ and $\mathbb{1}_{\mathcal{C}}//v$. In order to organize this procedure we pass to the filtered setting where we consider the \mathbb{E}_1 -cofiber $\mathbb{1}_{\mathcal{C}\text{Fib}}//\tau v$ which we depict below

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{1}_{\mathcal{C}} \longrightarrow \mathbb{1}_{\mathcal{C}}/v \longrightarrow (?) \longrightarrow (?) \longrightarrow \cdots .$$

Through degree 1 this is what we would expect to see in an \mathbb{E}_1 -algebra structure on $\mathbb{1}_{\mathcal{C}\text{Fib}}/\tau v$, but $\mathbb{1}_{\mathcal{C}\text{Fib}}//\tau v$ begins to deviate from $\mathbb{1}_{\mathcal{C}\text{Fib}}/\tau v$ in degree 2. To analyze this deviation we note that since the associated graded functor is \mathbb{E}_m -monoidal $\text{gr}_{\bullet}(\mathbb{1}_{\mathcal{C}\text{Fib}}//\tau v)$ is a free graded \mathbb{E}_1 -algebra on a copy of $\Sigma\mathcal{I}$ in degree 1. This means that in degree 2 our algebra $\mathbb{1}_{\mathcal{C}\text{Fib}}//\tau v$ has an extra copy of $\Sigma^2\mathcal{I}^{\otimes 2}$ which we would like to eliminate. To do this we need to produce a lift

$$\begin{array}{ccc} & & \mathbb{1}_{\mathcal{C}\text{Fib}}//\tau v \\ & \nearrow \text{dashed} & \downarrow \\ \Sigma^2\mathcal{I}^{\otimes 2}(2) & \hookrightarrow & (\mathbb{1}_{\mathcal{C}\text{Fib}}//\tau v)/\tau \end{array}$$

which we can use to take a further \mathbb{E}_1 -cofiber. In this manner we encounter our first obstruction:

$$\overline{Q}_1(v) \in [\Sigma^1\mathcal{I}^{\otimes 2}, \mathbb{1}_{\mathcal{C}}/v].⁷$$

If $\overline{Q}_1(v)$ vanishes, then upon taking the \mathbb{E}_1 -cofiber of the associated lift we obtain a filtered \mathbb{E}_1 -algebra R which agrees with $\mathbb{1}_{\mathcal{C}}/\tau v$ in degrees 0, 1 and 2, but begins to deviate in degree 3. ◁

Our goal in this section is to extend the manipulations of Example 2.1 into an obstruction theory for constructing an \mathbb{E}_n -algebra structure on $\mathbb{1}_{\mathcal{C}}/v$. Before that we make a short digression on \mathbb{A}_2 -structures.

Lemma 2.2 (cf. [21, p.27-28], [11] and [17]). *In the situation described above, if $\overline{Q}_1(v)$ vanishes, then $\mathbb{1}_{\mathcal{C}}/v$ admits a unital multiplication.*

Proof. The desired unital multiplication appears as the degree ≤ 2 component of the multiplication on the filtered \mathbb{E}_1 -algebra R constructed in Example 2.1. ◻

⁶By this we mean the pushout of the span $\mathbb{1}_{\mathcal{C}} \leftarrow \mathbb{1}_{\mathcal{C}}\{\mathcal{I}\} \rightarrow \mathbb{1}_{\mathcal{C}}$ where \mathcal{I} maps to the unit by v on the left and by zero on the right.

⁷The reader can take this as the definition of $\overline{Q}_1(v)$.

Lemma 2.3. *Given an $X \in \mathcal{C}$ there exists a sequence of \mathbb{E}_n -algebras in $\mathcal{C}^{\mathbf{Gr}}$ converging to the trivial square zero extension of $\mathbb{1}_{\mathcal{C}^{\mathbf{Gr}}}$ by a copy of ΣX placed in degree 1*

$$\mathbb{1}_{\mathcal{C}^{\mathbf{Gr}}} = R^0 \xrightarrow{r_1} R^1 \xrightarrow{r_2} R^2 \rightarrow \cdots \rightarrow \mathbb{1}_{\mathcal{C}^{\mathbf{Gr}}} \oplus \Sigma X(1)$$

such that

1. R^k is equivalent to $\mathbb{1}_{\mathcal{C}^{\mathbf{Gr}}} \oplus \Sigma X(1)$ through degree k .
2. The map r_k fits into a pushout square of \mathbb{E}_n -algebras

$$\begin{array}{ccc} \mathbb{1}_{\mathcal{C}^{\mathbf{Gr}}} \{ \Sigma^{-1-n} D_k^{\mathbb{E}_n}(\Sigma^{n+1} X(1)) \} & \xrightarrow{\text{aug}} & \mathbb{1}_{\mathcal{C}^{\mathbf{Gr}}} \\ \downarrow s_k & & \downarrow \\ R^{k-1} & \xrightarrow{r_k} & R^k \end{array}$$

Proof. In proving this lemma we make heavy use of the material from Appendix A. The resolution is from Construction A.20. Lemma A.21 identifies the objects X^k from this construction with $\Sigma^{-1-n} \text{Bar}^{(n)}(\mathbb{1} \oplus \Sigma X(1))_k$. Then, the combination of Theorem A.6 and Lemma A.19 give us equivalences

$$\text{Bar}^{(n)}(\mathbb{1} \oplus \Sigma X(1))_k \simeq \text{coFree}(\Sigma^{n+1} X(1))_k \simeq D_k^{\mathbb{E}_n}(\Sigma^{n+1} X).$$

□

Proposition 2.4. *Given a map $r : X \rightarrow \mathbb{1}_{\mathcal{C}}$ in \mathcal{C} there exists a sequence of inductively defined obstructions*

$$\theta_k \in [\Sigma^{-2-n} D_k^{\mathbb{E}_n}(\Sigma^{n+1} X), \mathbb{1}_{\mathcal{C}}/r] \quad \text{for } k \geq 2$$

whose vanishing allows us to inductively construct a sequence of \mathbb{E}_n -algebras

$$\mathbb{1}_{\mathcal{C}} = \bar{R}^0 \xrightarrow{\bar{r}_1} \bar{R}^1 \xrightarrow{\bar{r}_2} \bar{R}^2 \rightarrow \cdots \rightarrow \mathbb{1}_{\mathcal{C}}/r$$

converging to an \mathbb{E}_n -algebra structure on $\mathbb{1}_{\mathcal{C}}/r$, where each map \bar{r}_k sits in a pushout square

$$\begin{array}{ccc} \mathbb{1}_{\mathcal{C}} \{ \Sigma^{-1-n} D_k^{\mathbb{E}_n}(\Sigma^{n+1} X(1)) \} & \xrightarrow{\text{aug}} & \mathbb{1}_{\mathcal{C}} \\ \downarrow \bar{s}_k & & \downarrow \\ \bar{R}^{k-1} & \xrightarrow{\bar{r}_k} & \bar{R}^k \end{array}$$

Proof. In order to construct an \mathbb{E}_n -algebra structure on $\mathbb{1}_{\mathcal{C}}/r$ we will construct an \mathbb{E}_n -algebra structure on the filtered object $\text{cof}(X(1) \xrightarrow{\tau} \mathbb{1}_{\mathcal{C}^{\mathbf{FII}}})$ whose associated graded is equipped with an equivalence with the square zero \mathbb{E}_n -algebra $\mathbb{1}_{\mathcal{C}^{\mathbf{Gr}}} \oplus \Sigma X(1)$. The desired \mathbb{E}_n -algebra is then obtained by inverting the filtration parameter τ .

Let R^k denote the graded \mathbb{E}_n -algebra from Lemma 2.3. We will inductively produce filtered \mathbb{E}_n -algebras \tilde{R}^k whose associated graded is R^k and whose underlying \mathbb{E}_n -algebra is the desired \bar{R}^k . For our base case we let \tilde{R}^1 be the \mathbb{E}_n -cofiber of the map $\tau r : X(1) \rightarrow \mathbb{1}_{\mathcal{C}^{\text{FH}}}$. On associated graded this is a free algebra on $\Sigma X(1)$ and we pick an identification of it with R^1 . Meanwhile, in degree 1 we have $(\tilde{R}^1)_1 \simeq \mathbb{1}_{\mathcal{C}}/r$ as desired.

Examining the pushout square from Lemma 2.3(2) we see that given a choice of \tilde{R}^{k-1} and a lift

$$\begin{array}{ccc} & & \tilde{R}^{k-1} \\ & \nearrow \tilde{s}_k & \downarrow \\ \mathbb{1}_{\mathcal{C}^{\text{FH}}} \{ \Sigma^{-1-n} D_k^{\mathbb{E}_n}(\Sigma^{n+1} X(1)) \} & \xrightarrow{s_k} & R^{k-1} \end{array}$$

we can construct an \tilde{R}^k as the pushout of \tilde{s}_k with the augmentation. Since we are mapping out of a free algebra and $R^{k-1} \cong \tilde{R}^{k-1}/\tau$ the obstruction to constructing the lift \tilde{s}_k is the composite of s_k with the τ -Bockstein

$$\theta_k : \Sigma^{-1-n} D_k^{\mathbb{E}_n}(\Sigma^{n+1} X(1)) \xrightarrow{s_k} R^{k-1} \xrightarrow{\delta_\tau} \Sigma \tilde{R}^{k-1}(1).$$

Now we note that since R^{k-1} vanishes in degrees $2, \dots, k-1$ by Lemma 2.3(1) we can identify $(\tilde{R}^{k-1})_{k-1}$ with $\mathbb{1}_{\mathcal{C}}/r$. Consequently, the obstruction θ_k lives in

$$[\Sigma^{-2-n} D_k^{\mathbb{E}_n}(\Sigma^{n+1} X), \mathbb{1}_{\mathcal{C}}/r].$$

□

Remark 2.5. The obstruction θ_k depends upon a choice of nullhomotopy of each of the previous obstructions. In particular, if $[\Sigma^{-1-n} D_k^{\mathbb{E}_n}(\Sigma^{n+1} X), \mathbb{1}_{\mathcal{C}}/r] = 0$, then there is at most one such nullhomotopy and therefore Proposition 2.4 produces at most one \mathbb{E}_n -algebra structure on the quotient. ◁

As computing maps out of $D_k^{\mathbb{E}_n}(\Sigma^{n+1} X)$ can be unwieldy in general, we end the section by noting that there is a simple resolution which lets us subdivide the obstruction θ_k into a collection of obstructions $\theta_{k,\alpha}$ whose sources are suspensions of $X^{\otimes k}$.

Lemma 2.6. $D_k^{\mathbb{E}_n}(Y)$ has a resolution by finitely many copies of $\Sigma^{-c} Y^{\otimes k}$ where $0 \leq c \leq (n-1)(k-1)$.

Proof. We can describe $D_k^{\mathbb{E}_n}(Y)$ as the limit over the diagram

$$F : \mathbb{E}_n(k)_{h\Sigma_k} \rightarrow \mathcal{C}$$

describing all ways of taking the k^{th} power of Y . The space $\mathbb{E}_n(k)_{h\Sigma_k}$ is equivalent to the space of unordered configurations of k points in \mathbb{R}^n and we can put a finite

cellular filtration on this object whose cells lie in the range $0, \dots, (n-1)(k-1)$.⁸ On the limit of the diagram F of shape $\mathbb{E}_n(k)_{h\Sigma_k}$ this induces a finite filtration whose associated graded is given by copies of $\Sigma^{-c}Y^{\otimes k}$ where $0 \leq c \leq (n-1)(k-1)$. \square

Corollary 2.7. *In the situation of Proposition 2.4 we can use the filtration on $D_k^{c\mathbb{E}_n}(-)$ from Lemma 2.6 to refine the obstructions θ_k to obstructions*

$$\theta_{k,\alpha} \in [\Sigma^{-2-n-c_\alpha}(\Sigma^{n+1}X)^{\otimes k}, \mathbb{1}_C/r]$$

where $k \geq 2$ and $0 \leq c_\alpha \leq (n-1)(k-1)$.

Remark 2.8. In the case $n = 1$ the space $\mathbb{E}_1(k)_{h\Sigma_k}$ is a single point and therefore we have obstructions

$$\theta_k \in [\Sigma^{-3}(\Sigma^2X)^{\otimes k}, \mathbb{1}_C/r].$$

◁

3 \mathbb{E}_n -algebra Moore spectra

The key idea in applying Proposition 2.4 to construct multiplicative structures on Moore spectra is that while the obstruction groups do not vanish in Sp , they do vanish in certain deformations of Sp . In this section we apply this idea using the category of \mathbb{F}_2 -synthetic spectra as our deformation.

Theorem 3.1. *$\mathbb{S}/8$ admits the structure of an \mathbb{E}_1 -algebra.*

Proof. In order to prove that $\mathbb{S}/8$ admits an \mathbb{E}_1 -algebra structure we will show that in the \mathbb{F}_2 -synthetic category $\nu\mathbb{S}/\widetilde{2}^3$ admits an \mathbb{E}_1 -algebra structure and then invert τ .

Applying the obstruction theory from Proposition 2.4 with the simplification from Remark 2.8 to the map $\widetilde{2}^3 : \mathbb{S}^{0,3} \rightarrow \nu\mathbb{S}$ we obtain a sequence of inductively defined obstructions

$$\theta_k \in [\Sigma^{-3,3}(\mathbb{S}^{2,1})^{\otimes k}, \nu\mathbb{S}/\widetilde{2}^3] \quad \text{for } k \geq 2$$

whose vanishing implies $\nu\mathbb{S}/\widetilde{2}^3$ admits an \mathbb{E}_1 -algebra structure. On the other hand, the vanishing line from [7, Prop. 15.8]^{9 10} says that

$$[\mathbb{S}^{w,s}, \nu\mathbb{S}/\widetilde{2}^3] = 0$$

when $s > \frac{1}{2}w + 3$. In particular, this implies that θ_k is zero because it lies in a zero group! \square

⁸For an explicit presentation of this space as a finite simplicial set of dimension $(n-1)(k-1)$ one can take the quotient of the (free) Σ_k action on the nerve of the poset of Fox–Neuwirth cells (see [3]).

⁹For the interested reader we note that the part of this proposition we use is essentially equivalent to Adams' vanishing line in the cohomology of the Steenrod algebra from [1].

¹⁰We warn the reader that our convention for the indexing of bigraded spheres (see convention (6)) differs from that used in loc. cit.

The \mathbb{F}_2 -synthetic homotopy groups of $\nu\mathbb{S}/\tilde{2}^3$.

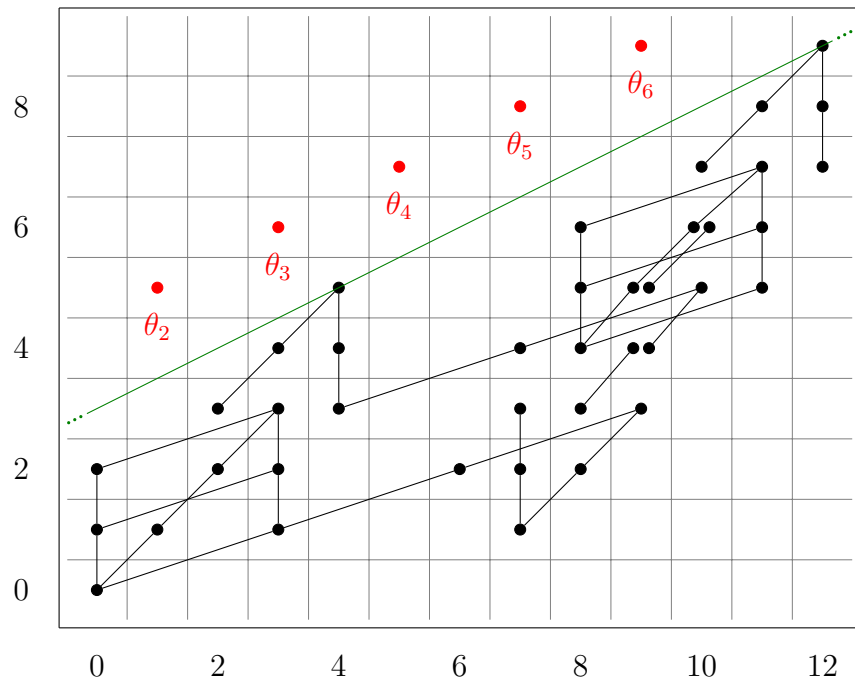


Figure 0-1: A picture of the \mathbb{F}_2 -synthetic homotopy groups of $\nu\mathbb{S}/\tilde{2}^3$. Black dots indicate non- τ -torsion classes and we suppress all τ -multiples. The green line is the vanishing line above which the bigraded homotopy groups are zero. The red classes are the obstructions θ_k .

Building on the proof of Theorem 3.1 we prove the more complicated Theorem 3.2 in an almost identical way.

Theorem 3.2. $\mathbb{S}/2^q$ admits the structure of an \mathbb{E}_n -algebra for every $q \geq \frac{3}{2}(n+1)$.

Proof. This time we will show that in the \mathbb{F}_2 -synthetic category $\mathbb{S}/\tilde{2}^q$ admits an \mathbb{E}_n -algebra structure. Applying the obstruction theory from Proposition 2.4 to the map $\tilde{2}^q$ we obtain a sequence of inductively defined obstructions

$$\theta_{k,\alpha} \in [\Sigma^{-2-n-c_\alpha, 2+n+c_\alpha}(\mathbb{S}^{1+n, q-1-n})^{\otimes k}, \mathbb{S}/\tilde{2}^q]$$

where $k \geq 2$ and $0 \leq c_\alpha \leq (n-1)(k-1)$ whose vanishing implies $\mathbb{S}/\tilde{2}^q$ admits an \mathbb{E}_n -algebra structure. Comparing the bidegree of $\theta_{k,\alpha}$ to the vanishing region from Lemma 3.3 below we again find that $\theta_{k,\alpha}$ lies in a zero group since

$$((q-1-n)k + 2 + n + c_\alpha) > \frac{1}{2}((n+1)k - 2 - n - c_\alpha) + q.$$

□

Lemma 3.3. If $s > \frac{1}{2}w + q$, then $\pi_{w,s}(\nu\mathbb{S}/\tilde{2}^q) = 0$.

Proof. In this lemma we invoke the machinery of vanishing lines from [7, §11] and our proof is patterned on the proof of [7, Prop. 15.8].

We proceed by induction with $q = 1$ as our base case (which is already covered by loc. cit.). For the inductive step we apply [7, Lem. 11.11] to the cofiber sequence

$$\Sigma^{0,1}\nu\mathbb{S}/\tilde{2}^{q-1} \rightarrow \nu\mathbb{S}/\tilde{2}^q \rightarrow \nu\mathbb{S}/\tilde{2}.$$

□

4 Deforming \mathcal{C}

In this section we use the machinery of [20] to construct a deformation of \mathcal{C} in which we can run the arguments of the previous section in order to prove Theorem 1.5.

Recollection 4.1. Given an epimorphism class Q on a stable category \mathcal{E} such that \mathcal{E} has enough Q -injectives Patchkoria and Pstrągowski define a prestable category $\mathcal{D}_{\geq 0}^\omega(\mathcal{E}; Q)$ with associated stable category $\mathcal{D}^\omega(\mathcal{E}; Q)$ which fits into a diagram

$$\begin{array}{ccc} & \mathcal{D}_{\geq 0}^\omega(\mathcal{E}; Q) & \\ \nu \nearrow & & \searrow (-)^\tau=1 \\ \mathcal{E} & \xrightarrow{\text{Id}} & \mathcal{E} \end{array}$$

such that

1. $\mathcal{D}_{\geq 0}^\omega(\mathcal{E}; Q)$ has finite limits and colimits.

2. ν is fully faithful.
3. $\mathcal{D}_{\geq 0}^{\omega}(\mathcal{E}; Q)$ is generated under finite colimits by the image of ν .
4. ν preserves those cofiber sequences $a \rightarrow b \rightarrow c$ for which $b \rightarrow c$ is Q -epi.
5. $\mathcal{D}_{\geq 0}^{\omega}(\mathcal{E}; Q)$ is equipped with an automorphism [1] and an equivalence

$$\nu(-)[1] \simeq \nu(\Sigma-).$$

6. $(-)^{\tau=1}$ is the localization of $\mathcal{D}^{\omega}(\mathcal{E}; Q)$ at (integer suspensions of) the assembly maps

$$\tau_X : \Sigma\nu X[-1] \rightarrow \nu X.$$

7. If \mathcal{I} is Q -injective and $X \in \mathcal{E}$, then $[\Sigma^{-s}\nu X, \nu\mathcal{I}] = 0$ for $s > 0$.

For these claims see [20, §5], specifically 5.32, 5.34, 5.37, 5.47 and 5.60. ◁

For the proof of our main theorem we will also need an \mathbb{E}_m -monoidal structure on our deformation of \mathcal{C} . Although [20] only considers monoidal structures in the case of Sp , as it turns out the general case is no more difficult and our treatment follows [20, §5.5] closely.

Definition 4.2. We say that an epimorphism class Q on a stably \mathbb{E}_m -monoidal category \mathcal{E} (with $m \geq 2$) is \otimes -compatible if for every Q -epi map $X \rightarrow Y$ and $Z \in \mathcal{E}$ the map $X \otimes Z \rightarrow Y \otimes Z$ is Q -epi as well.¹¹ ◁

Proposition 4.3. *If Q is a \otimes -compatible epimorphism class on a stably \mathbb{E}_m -monoidal category \mathcal{E} with $m \geq 2$, then $\mathcal{D}^{\omega}(\mathcal{E}; Q)$ admits an exact \mathbb{E}_m -monoidal structure, compatible with the prestable structure, such that ν and $(-)^{\tau=1}$ are \mathbb{E}_m -monoidal.*

Proof. The construction of the \mathbb{E}_m -monoidal structure, compatibility with the prestable structure and the \mathbb{E}_m -monoidality of ν are proved in the same way as in [20, Prop. 5.69]. Using the fully-faithfulness of ν we can identify τ_X with $\tau_{\mathbb{1}} \otimes \nu X$ (and consequently we drop the subscript from $\tau_{\mathbb{1}}$ going forward). The \mathbb{E}_m -monoidality of $(-)^{\tau=1}$ now follows from describing this localization as inverting the map τ in a monoidal way. □

We are now ready to introduce the specific deformation of interest to us.

Definition 4.4. Let \mathcal{Q} denote the epimorphism class of maps $X \rightarrow Y \in \mathcal{C}^{\omega}$ which are split epi upon tensoring with $\mathbb{1}_{\mathcal{C}}/v$.¹² ◁

Lemma 4.5. *The epimorphism class \mathcal{Q} enjoys the following properties:*

¹¹Note that because $m \geq 2$ it doesn't matter whether Z is on the left or right in this definition. The potential discrepancy between left \otimes -compatible and right \otimes -compatible in the \mathbb{E}_1 -monoidal case is the root of our restriction to $m \geq 2$.

¹²That these maps form an epimorphism class follows from [20, Examples 3.4 and 3.6].

1. Every object of the form $\mathbb{1}_{\mathcal{C}}/v \otimes X$ is \mathcal{Q} -injective.
2. The map $X \rightarrow \mathbb{1}_{\mathcal{C}}/v \otimes X$ is \mathcal{Q} -mono.
3. \mathcal{C}^ω has enough \mathcal{Q} -injectives.
4. \mathcal{Q} is \otimes -compatible.

Proof. Using the right unital multiplication on $\mathbb{1}_{\mathcal{C}}/v$ conclusions (1) and (2) follow from the same argument as in the proof of [20, Lem. 5.67]. (3) follows from (1) and (2). Again using the right unital multiplication on $\mathbb{1}_{\mathcal{C}}/v$, conclusion (4) follows from the same argument as in [20, Lem. 5.68]. \square

There is one more modification we need to make in order to link up with the material from Section 2: We want to have a *presentable* category deforming \mathcal{C} .

Construction 4.6. Recall that we have already arranged in Remark 1.6 that $\mathbb{1}$ and \mathcal{I} are compact and \mathcal{C} is generated under tensor products and finite (co)limits by these generators. We let $\text{Def}(\mathcal{C}; \mathcal{Q})$ denote the ind-completion of $\mathcal{D}^\omega(\mathcal{C}^\omega; \mathcal{Q})$. This renormalization fits into a diagram of presentably \mathbb{E}_m -monoidal categories and \mathbb{E}_m -monoidal, filtered colimit preserving functors

$$\begin{array}{ccc}
 & \text{Def}(\mathcal{C}; \mathcal{Q}) & \\
 \nu \nearrow & & \searrow (-)^\tau=1 \\
 \mathcal{C} & \xrightarrow{\text{Id}} & \mathcal{C}
 \end{array}$$

which agrees with the one from 4.1 on compact objects. \triangleleft

We end the section by proving a vanishing lemma which serves as the analog of Lemma 3.3 in $\text{Def}(\mathcal{C}; \mathcal{Q})$.

Construction 4.7. Since $\mathbb{1}_{\mathcal{C}} \rightarrow \mathbb{1}_{\mathcal{C}}/v$ is $(\mathbb{1}_{\mathcal{C}}/v)$ -split mono we have a cofiber sequence

$$\nu \mathbb{1}_{\mathcal{C}} \rightarrow \nu(\mathbb{1}_{\mathcal{C}}/v) \rightarrow \nu(\Sigma \mathcal{I}).$$

We write \tilde{v} for the associated boundary map

$$\tilde{v} : \Sigma^{-1} \nu \mathcal{I}[1] \rightarrow \nu \mathbb{1}_{\mathcal{C}}.$$

\triangleleft

Lemma 4.8. *The cofiber of the map $\tilde{v}^q : (\Sigma^{-1} \nu \mathcal{I}[1])^{\otimes q} \rightarrow \nu \mathbb{1}_{\mathcal{C}}$ in $\text{Def}(\mathcal{C}; \mathcal{Q})$ has*

$$[\Sigma^{-s} \nu X, \nu \mathbb{1}_{\mathcal{C}}/\tilde{v}^q] = 0$$

for every $X \in \mathcal{C}^\omega$ and $s \geq q$.

Proof. The restriction to $X \in \mathcal{C}^\omega$ lets us move back to $\mathcal{D}^\omega(\mathcal{C}^\omega; \mathcal{Q})$. We proceed by induction on q using the cofiber sequence

$$(\Sigma^{-1}\nu\mathcal{I}[1])^{\otimes q-1} \otimes \nu\mathbb{1}_{\mathcal{C}}/\tilde{v} \rightarrow \nu\mathbb{1}_{\mathcal{C}}/\tilde{v}^q \rightarrow \nu\mathbb{1}_{\mathcal{C}}/\tilde{v}^{q-1}.$$

Using the equivalence between $\nu(\mathbb{1}_{\mathcal{C}}/v)$ and $\nu\mathbb{1}_{\mathcal{C}}/\tilde{v}$ from Construction 4.7 and the fact that ν is monoidal we have an equivalence

$$(\nu\mathcal{I}[1])^{\otimes q-1} \otimes \nu\mathbb{1}_{\mathcal{C}}/\tilde{v} \simeq \nu(\Sigma^{q-1}\mathcal{I}^{\otimes q-1} \otimes \mathbb{1}_{\mathcal{C}}/v).$$

Since $\Sigma^{q-1}\mathcal{I}^{\otimes q-1} \otimes \mathbb{1}_{\mathcal{C}}/v$ is \mathcal{Q} -injective (see Lemma 4.5) we then have that

$$[\Sigma^{-s}\nu X, (\Sigma^{-1}\nu\mathcal{I}[1])^{\otimes q-1} \otimes \nu\mathbb{1}_{\mathcal{C}}/\tilde{v}] = 0$$

(see 4.1(7)) for every $X \in \mathcal{C}^\omega$ and $s \geq q$. \square

5 \mathbb{E}_n -algebra quotients

With all the preparation finished we are now ready to prove the main theorem. Our proof follows the pattern established in Section 3 with the key difference being that we use the stable, presentably \mathbb{E}_n -monoidal deformation $\text{Def}(\mathcal{C}; \mathcal{Q})$ of \mathcal{C} from the previous section in place of $\text{Syn}_{\mathbb{F}_2}$.

Definition 5.1. We say that an \mathbb{E}_n -algebra structure on $\mathbb{1}_{\mathcal{C}}/v^q$ is *v-compatible* if it is equivalent to $(\nu\mathbb{1}_{\mathcal{C}}/\tilde{v}^q)^{\tau=1}$ for some \mathbb{E}_n -algebra structure on the quotient $\nu\mathbb{1}_{\mathcal{C}}/\tilde{v}^q$ in $\text{Def}(\mathcal{C}; \mathcal{Q})$. \triangleleft

Theorem 5.2 (Theorem 1.5). *If $\mathbb{1}_{\mathcal{C}}/v$ admits a right unital multiplication, then there exists a unique up to equivalence v-compatible \mathbb{E}_n -algebra $\mathbb{1}/v^q$ for each $q > n$ and these \mathbb{E}_n -algebras fit into a tower*

$$\dots \longrightarrow \mathbb{1}/v^{n+3} \longrightarrow \mathbb{1}/v^{n+2} \longrightarrow \mathbb{1}/v^{n+1}.$$

Proof. We begin by producing an \mathbb{E}_n -algebra structure on $\nu\mathbb{1}_{\mathcal{C}}/\tilde{v}^q$ for $q > n$. Applying the obstruction theory from Proposition 2.4 and Corollary 2.7 to the map \tilde{v}^q we obtain a sequence of inductively defined obstructions

$$\theta_{k,\alpha} \in [\Sigma^{-2-n-c_\alpha}(\Sigma^{n+1}(\Sigma^{-1}\nu\mathcal{I}[1])^{\otimes q})^{\otimes k}, \nu\mathbb{1}_{\mathcal{C}}/\tilde{v}^q]$$

where $k \geq 2$ and $0 \leq c_\alpha \leq (n-1)(k-1)$ whose vanishing implies $\nu\mathbb{1}_{\mathcal{C}}/\tilde{v}^q$ admits an \mathbb{E}_n -algebra structure. Since ν is \mathbb{E}_n -monoidal we can rewrite the source of $\theta_{k,\alpha}$ as

$$\Sigma^{(n+1-q)k-n-2-c_\alpha}\nu((\Sigma\mathcal{I})^{\otimes qk}).$$

Since $q \geq n+1$ and $n+2+c_\alpha \geq n+1$, Lemma 4.8 now tells us that the group in which $\theta_{k,\alpha}$ lives is trivial.

In order to prove the uniqueness statement and produce the desired tower we now examine the space of maps from the \mathbb{E}_n -algebra $\nu\mathbb{1}_C/\tilde{v}^q$ constructed above to any other \mathbb{E}_n -algebra R with the underlying object $\nu\mathbb{1}_C/\tilde{v}^w$. The pushout squares from Proposition 2.4 can be interpreted as providing an obstruction theory for producing \mathbb{E}_n -algebra maps $\nu\mathbb{1}_C/\tilde{v}^q \rightarrow R$. Using Corollary 2.7 we can sub-divide these obstructions into obstructions

$$\gamma_{k,\alpha} \in [\Sigma^{(n+1-q)k-n-1-c_\alpha} \nu((\Sigma\mathcal{I})^{\otimes qk}), \nu\mathbb{1}_C/\tilde{v}^w]$$

where $k \geq 1$ and $0 \leq c_\alpha \leq (n-1)(k-1)$. When $q \geq w$ Lemma 4.8 tells us that these groups are trivial and therefore we obtain the desired map. In the case $q = w$ the obstruction γ_1 is the composite

$$\Sigma^{-q} \nu((\Sigma\mathcal{I})^{\otimes q}) \xrightarrow{\tilde{v}^q} \nu\mathbb{1}_C \rightarrow \nu\mathbb{1}_C/\tilde{v}^q$$

and if we pick the nullhomotopy of γ_1 to be one which makes this into a cofiber sequence, then we will obtain an equivalence of \mathbb{E}_n -algebras $\nu\mathbb{1}_C/\tilde{v}^q \rightarrow R$, proving the uniqueness assertion. \square

We conclude by deducing the remaining theorems from the introduction as corollaries of the main theorem.

Corollary 5.3 (Theorem 1.4). *Suppose we are given an \mathbb{E}_{m+1} -algebra $A \in \mathrm{Sp}$ with $m \geq 2$ and a class $v \in \pi_*(A)$ such that A/v admits a unital multiplication. Then A/v^q admits an \mathbb{E}_n - A -algebra structure for each $n \leq m$ and $q > n$.*

Proof. Since A is an \mathbb{E}_{m+1} -algebra it has an \mathbb{E}_m -monoidal category of left modules in which we can apply Theorem 1.5 to v . \square

Note that the statement proved here is slightly different from the one which appeared in the introduction. In order to bridge the gap we include the next lemma, which is likely well known to experts, but for which we could not find a reference in the literature.

Lemma 5.4. *Given an \mathbb{E}_{m+1} -algebra $A \in \mathrm{Sp}$ with $m \geq 2$ and a class $v \in \pi_{2w}(A)$ we can identify the obstruction $\overline{Q}_1(v)$ to A/v admitting a unital multiplication with the reduction mod v of $Q_1(v)$.*

Proof. In proving this lemma we pick up where Lemma 2.2 left off, passing to the filtered setting. We write $X[\tau]$ for the image of $X \in \mathrm{Sp}$ under the unit map $\mathrm{Sp} \rightarrow \mathrm{Sp}^{\mathrm{Fil} 13}$.

The \mathbb{E}_1 -cofiber $A[\tau]//\tau v$ admits a second description as the relative tensor product of $A[\tau]$ with $\mathbb{S}[\tau]$ over a free filtered \mathbb{E}_2 -algebra on a class x in degree $2w$ and filtration 1 which maps to τv in $A[\tau]$ and to zero in $\mathbb{S}[\tau]$. The arity 2 component of the free \mathbb{E}_2 -algebra on \mathbb{S}^{2w} is given by $\mathbb{S}^{4w} \oplus \mathbb{S}^{4w+1}$ where the bottom cell is x^2 and the top

¹³This notation is meant to evoke that the underlying graded object of $X[\tau]$ looks like a free module over $\mathbb{S}[\tau]$ on a copy of X placed in degree 0.

cell is $Q_1(x)$. Thus, through filtration 2, we can replace the free \mathbb{E}_2 -algebra with a free \mathbb{E}_1 -algebra on two classes x and $Q_1(x)$ such that x maps to τv and $Q_1(x)$ maps to $\tau^2 Q_1(v)$. This lets us identify the filtration 2 component of $A[\tau]//\tau v$ with

$$\text{cof} \left(\Sigma^{2w} A \oplus \Sigma^{4w+1} A \xrightarrow{(v, Q_1(v))} A \right).$$

Unrolling the definition we see that the attaching map of the top cell to the copy of A/v is the obstruction $\overline{Q_1}(v)$ of Lemma 2.2 and we can thereby identify $\overline{Q_1}(v)$ with the reduction mod v of $Q_1(v)$. \square

Remark 5.5. The Cartan formula from [6, Prop. V.1.10]¹⁴ tells us that for an \mathbb{E}_{m+1} -algebra A with $m \geq 2$ and $x, y \in \pi_{2*}(A)$ we have

$$Q_1(xy) \equiv Q_1(x)y^2 + x^2Q_1(y) + c\eta x^2y^2$$

for some integer c . In particular we find that

$$\overline{Q_1}(x^2) \equiv Q_1(x^2) \equiv 2x^2Q_1(x) + c\eta x^4 \equiv 0 \pmod{x^2}.$$

Thus, the square of an even dimensional class always satisfies the conditions of Theorem 1.4. \triangleleft

Corollary 5.6 (Theorem 1.2, odd primes). *Applying Theorem 1.4 with $\mathcal{C} = \text{Sp}$ and $v = p$ we obtain an \mathbb{E}_n -algebra structure on \mathbb{S}/p^{n+1} .*

Remark 5.7. Although $\mathbb{S}/2$ does not admit a unital multiplication, $\overline{Q_1}(4) = 0$ and therefore $\mathbb{S}/4$ admits a unital multiplication. If we apply Theorem 1.4 with $v = 4$, then we obtain an \mathbb{E}_n -algebra structure on $\mathbb{S}/2^{2(n+1)}$. Note that this is *less* structure than is provided by Theorem 3.2. This discrepancy suggests that there is still room for improvement at the prime 2. Specifically, we suspect that the optimal value of q for which $\mathbb{S}/2^q$ is an \mathbb{E}_n -algebra is not much larger than n in general. \triangleleft

Corollary 5.8 (Theorem 1.3). *For each h and n there exists a generalized Moore spectrum $\mathbb{S}/(p^{i_0}, \dots, v_{h-1}^{i_{h-1}})$ of type h which admits an \mathbb{E}_n -algebra structure.*

Proof. We proceed by induction on h . Suppose, by induction, that we have an \mathbb{E}_{n+1} -algebra structure on a type $h - 1$ generalized Moore spectrum M . The periodicity theorem of [12] guarantees we can find a v_{h-1} -self map $v \in \pi_*M$. To conclude we apply Theorem 1.4 to v^2 (see Remark 5.5). \square

A Bar-cobar duality for graded \mathbb{E}_n -algebras

In this appendix we show that Lurie's bar-cobar duality for \mathbb{E}_n -algebras in an \mathbb{E}_n -monoidal category can be upgraded to an equivalence in the positively graded setting. Our proof follows [10, §4] closely enough that it is worth pointing out why a simple citation is insufficient.

¹⁴Here we are using that we are in the stable range where \mathbb{E}_∞ and \mathbb{E}_{m+1} power operations agree.

- (a) In [10] the underlying category \mathcal{E} is *symmetric monoidal*.
- (b) In [10] Koszul duality has target divided power coalgebras over the Koszul dual operad.

By contrast, Lurie’s (iterated) bar-cobar duality has the advantage that it is defined for \mathbb{E}_n -monoidal categories, but the disadvantage that it is not immediate that this duality is the same one one would expect to obtain using operadic Koszul duality (together with the Koszul self-duality of the \mathbb{E}_n -operad).

Remark A.1. In the long-run the author would like to see this appendix supplanted by an extension of operadic Koszul duality to algebras in categories which are not symmetric monoidal. ◁

Convention A.2. Throughout this appendix \mathcal{E} will denote a stable, presentably \mathbb{E}_n -monoidal category. ◁

As in the body of the paper $\mathcal{E}^{\mathbf{Gr}}$ denotes the category of graded objects in \mathcal{E} and this category is also stable and presentably \mathbb{E}_n -monoidal. Crucially for this appendix, in $\mathcal{E}^{\mathbf{Gr}}$ limits and colimits are computed component-wise. In particular we have the following lemma:

Lemma A.3. *Given a collection of objects $\{X_\alpha\}_{\alpha \in A}$ in $\mathcal{E}^{\mathbf{Gr}}$ such that only finitely many of the X_α are non-zero in each degree, the natural map*

$$\bigoplus_A X_\alpha \rightarrow \prod_A X_\alpha$$

is an equivalence.

We will also need several variants of $\mathcal{E}^{\mathbf{Gr}}$

Definition A.4.

- $\mathcal{E}_{\geq 0}^{\mathbf{Gr}}$ is the full subcategory of $\mathcal{E}^{\mathbf{Gr}}$ on objects which vanish in negative degrees.
- $\mathcal{E}_+^{\mathbf{Gr}}$ is the full subcategory of $\mathcal{E}_{\mathbb{1}/-/\mathbb{1}}^{\mathbf{Gr}}$ on the objects which vanish in negative degrees and are equivalent to the unit in degree 0. We say that the objects of $\mathcal{E}_+^{\mathbf{Gr}}$ are *positively graded*.
- Given an object $X \in \mathcal{E}_{\mathbb{1}/-/\mathbb{1}}^{\mathbf{Gr}}$ we write \overline{X} for the object obtained by splitting off the copy of the unit.¹⁵
- We say an object in $\mathcal{E}^{\mathbf{Gr}}$ is *thin* if it vanishes in all but finitely many degrees. We write $(\mathcal{E}^{\mathbf{Gr}})^{\text{thin}}$, $(\mathcal{E}_{\geq 0}^{\mathbf{Gr}})^{\text{thin}}$ and $(\mathcal{E}_+^{\mathbf{Gr}})^{\text{thin}}$ for the respective subcategories of thin objects.

◁

¹⁵The functor which sends $X \in \mathcal{E}_+^{\mathbf{Gr}}$ to \overline{X} gives an equivalence between $\mathcal{E}_+^{\mathbf{Gr}}$ and the category of graded objects concentrated in positive degrees. Note however, that this equivalence is not monoidal.

Recollection A.5. In [16, §5.2.3] Lurie constructs a bar-cobar adjunction

$$\mathrm{Bar}^{(n)} : \mathrm{Alg}_{\mathbb{E}_n}^{\mathrm{aug}}(\mathcal{E}^{\mathrm{Gr}}) \rightleftarrows \mathrm{coAlg}_{\mathbb{E}_n}^{\mathrm{aug}}(\mathcal{E}^{\mathrm{Gr}}) : \mathrm{Cobar}^{(n)}$$

(see [16, 5.2.3.6 and 5.2.3.9] specifically). Since $\mathcal{E}_+^{\mathrm{Gr}}$ is a full subcategory of $\mathcal{E}_{\mathbb{1}/-/\mathbb{1}}^{\mathrm{Gr}}$ closed under tensor products, limits and colimits, the bar-cobar adjunction restriction to an adjunction between these subcategories

$$\mathrm{Bar}^{(n)} : \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{E}_+^{\mathrm{Gr}}) \rightleftarrows \mathrm{coAlg}_{\mathbb{E}_n}(\mathcal{E}_+^{\mathrm{Gr}}) : \mathrm{Cobar}^{(n)}$$

(see [16, 5.2.3.11]). In the case $n = 1$, Bar (resp. Cobar) is computed by a bar (cobar) construction [16, 5.2.2.17]. \triangleleft

The main theorem of this appendix is that in the positively graded setting we can upgrade Lurie's bar-cobar adjunction to an equivalence.

Theorem A.6. *The bar-cobar adjunction*

$$\mathrm{Bar}^{(n)} : \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{E}_+^{\mathrm{Gr}}) \rightleftarrows \mathrm{coAlg}_{\mathbb{E}_n}(\mathcal{E}_+^{\mathrm{Gr}}) : \mathrm{Cobar}^{(n)}$$

is an equivalence.

Remark A.7. Similar ideas were considered in [15] where, following [10], Krause proves the $n = 1$, $\mathcal{E} = \mathrm{Mod}(\mathbb{Z})$ case of Theorem A.6. \triangleleft

A.1 The proof of Theorem A.6

The proof of Theorem A.6 will proceed by induction on n and the bulk of the work lies in handling the base-case $n = 1$. The preparation for this proof will occupy us for the next couple pages.

Construction A.8. Given a positively graded \mathbb{E}_1 -algebra A in \mathcal{E} we can consider the bar filtration on the underlying object of $\mathrm{Bar}(A)$,

$$\begin{array}{ccccccc} \mathrm{colim}_{\Delta_{\leq 0}^{\mathrm{op}}} A^{\otimes \bullet} & \longrightarrow & \mathrm{colim}_{\Delta_{\leq 1}^{\mathrm{op}}} A^{\otimes \bullet} & \longrightarrow & \mathrm{colim}_{\Delta_{\leq 2}^{\mathrm{op}}} A^{\otimes \bullet} & \longrightarrow & \cdots \longrightarrow \mathrm{Bar}(A) \\ \downarrow \simeq & & \downarrow & & \downarrow & & \\ \mathbb{1} & & \Sigma \bar{A} & & (\Sigma \bar{A})^{\otimes 2} & & \end{array}$$

which is an ω -indexed filtration of the underlying object of $\mathrm{Bar}(A)$ with associated graded given by $(\Sigma \bar{A})^{\otimes k}$.

Dually, given a positively graded \mathbb{E}_1 -coalgebra C in \mathcal{E} we can consider the cobar filtration of $\mathrm{Cobar}(C)$, which is an ω -indexed tower with limit the underlying object of $\mathrm{Cobar}(C)$ and associated graded given by $(\Sigma^{-1} \bar{C})^{\otimes k}$. \triangleleft

Remark A.9. The key observation in this appendix is that $(\Sigma^{-1}\overline{C})^{\otimes k}$ is concentrated in degrees $\geq k$, therefore in any fixed degree $\text{Cobar}(C)$ is computed by a *finite* limit. \triangleleft

Lemma A.10. *The functor $\text{Cobar} : \text{coAlg}(\mathcal{E}) \rightarrow \text{Alg}(\mathcal{E})$ commutes with sifted colimits.*

Proof. Since the underlying object functor $\text{Alg}(\mathcal{E}) \rightarrow \mathcal{E}$ is conservative and commutes with sifted colimits it will suffice to show that the composite of Cobar with this functor commutes with sifted colimits.

Suppose we have a sifted diagram $F : D \rightarrow \text{coAlg}(\mathcal{E})$. Using that fact that Cobar is computed by a cobar construction we have equivalences

$$\begin{aligned} \left(\text{colim}_{d \in D} \text{Cobar}(F(d)) \right)_k &\simeq \text{colim}_{d \in D} \varprojlim_s \left(\lim_{\Delta_{\leq s}} F(d)^{\otimes \bullet} \right)_k \simeq \varprojlim_s \text{colim}_{d \in D} \left(\lim_{\Delta_{\leq s}} F(d)^{\otimes \bullet} \right)_k \\ &\simeq \varprojlim_s \lim_{\Delta_{\leq s}} \left(\text{colim}_{d \in D} F(d)^{\otimes \bullet} \right)_k \end{aligned}$$

where the key step is using the fact that the cobar filtration stabilizes in finitely many steps (see Remark A.9) to commute the colimit and infinite limit. Using the assumption that D is sifted and the fact that the tensor product on \mathcal{E}^{Gr} commutes with colimits separately in each variable we have

$$\text{colim}_{d \in D} F(d)^{\otimes s} \simeq \text{colim}_{(d_1, \dots, d_s) \in D^{\times s}} F(d_1) \otimes \dots \otimes F(d_s) \simeq \left(\text{colim}_{d \in D} F(d) \right)^{\otimes s}.$$

Feeding this into the previous equivalence we obtain the desired equivalence

$$\left(\text{colim}_{d \in D} \text{Cobar}(F(d)) \right)_k \simeq \varprojlim_s \lim_{\Delta_{\leq s}} \left(\left(\text{colim}_{d \in D} F(d) \right)^{\otimes \bullet} \right)_k \simeq \text{Cobar} \left(\text{colim}_{d \in D} F(d) \right)_k.$$

□

Lemma A.11. *A map of postively graded \mathbb{E}_1 -algebras $A \rightarrow B$ in \mathcal{E} is an equivalence through degree k iff the map of postively graded \mathbb{E}_n -coalgebras $\text{Bar}(A) \rightarrow \text{Bar}(B)$ is an equivalence through degree k .*

Dually, a map of postively graded \mathbb{E}_1 -coalgebras $C \rightarrow D$ in \mathcal{E} is an equivalence through degree k iff the map of postively graded \mathbb{E}_n -algebras $\text{Cobar}(C) \rightarrow \text{Cobar}(D)$ is an equivalence through degree k .

In particular this implies that both Bar and Cobar are conservative.

Proof. Suppose the map $A \rightarrow B$ is an equivalence through degree k , but is not an equivalence in degree $k + 1$. Examining the bar filtration we see that the cofiber of the map $A \rightarrow B$ has a filtration with associated graded given by

$$X_{s,j} := \text{cof} \left((\Sigma^s \overline{A}^{\otimes s})_j \rightarrow (\Sigma^s \overline{B}^{\otimes s})_j \right)$$

in degree j . Using the fact that \bar{A} and \bar{B} are concentrated in degrees ≥ 1 and the map between them is an equivalence in degrees $\leq k$ we can read off that $X_{s,j} = 0$ for $j \leq s + k - 1$ and $X_{1,k+1} \simeq \text{cof}(\Sigma \bar{A}_{k+1} \rightarrow \Sigma \bar{B}_{k+1})$. In particular, this implies that

- (a) the map $\text{Bar}(A) \rightarrow \text{Bar}(B)$ is an equivalence through degree k and
- (b) in degree $k + 1$ we have

$$\text{cof}(\text{Bar}(A) \rightarrow \text{Bar}(B))_{k+1} \simeq \text{cof}(\Sigma \bar{A}_{k+1} \rightarrow \Sigma \bar{B}_{k+1}) \not\simeq 0.$$

The argument for Cobar is dual to the argument for Bar. \square

Construction A.12. After passing to categories of large pro-objects the natural inclusion $(\mathcal{E}^{\text{Gr}})^{\text{thin}} \rightarrow \mathcal{E}^{\text{Gr}}$ picks up a left adjoint and we write e for the composite

$$e : \mathcal{E}^{\text{Gr}} \rightarrow \text{Pro}(\mathcal{E}^{\text{Gr}}) \rightarrow \text{Pro}((\mathcal{E}^{\text{Gr}})^{\text{thin}})$$

of this left adjoint with the Yoneda embedding. Similarly, we have functors $e_{\geq 0}$ and e_+ in the positively graded setting. \triangleleft

Lemma A.13. *There is a natural \mathbb{E}_n -monoidal equivalence*

$$\text{Pro}((\mathcal{E}^{\text{Gr}})^{\text{thin}}) \simeq \text{Pro}(\mathcal{E})^{\text{Gr}}$$

which restricts to a similar equivalence in the positively graded setting. In particular this means every object in $\text{Pro}((\mathcal{E}^{\text{Gr}})^{\text{thin}})$ is both the coproduct and the product of its components in each degree.

Proof. The key point in this lemma is that both of these categories are the opposite of a huge presentable category with a large collection of compact objects. To prove the lemma it therefore suffices to argue that the full subcategory of compact objects in $\text{Ind}(\mathcal{E}^{\text{op}})^{\text{Gr}}$ is equivalent to $((\mathcal{E}^{\text{Gr}})^{\text{thin}})^{\text{op}}$. For this we observe that since $\text{Ind}(\mathcal{E}^{\text{op}})^{\text{Gr}} \simeq \text{Sp}^{\text{Gr}} \otimes \text{Ind}(\mathcal{E}^{\text{op}})$ the compact objects are generated under finite colimits by objects of the form $X(k)$ with $X \in \mathcal{E}^{\text{op}}$.

The second conclusion follows from Lemma A.3 which applies since the category in question is now the opposite of a category of graded objects in a (huge) presentable category. \square

Lemma A.14. *The functor e_+ of Construction A.12 is fully faithful, \mathbb{E}_n -monoidal and colimit preserving.*

Proof. The Yoneda embedding $\mathcal{E}_+^{\text{Gr}} \rightarrow \text{Pro}(\mathcal{E}_+^{\text{Gr}})$ is \mathbb{E}_n -monoidal and preserves colimits. The functor

$$\text{Pro}(\mathcal{E}_+^{\text{Gr}}) \rightarrow \text{Pro}((\mathcal{E}_+^{\text{Gr}})^{\text{thin}})$$

is the left adjoint of an \mathbb{E}_n -monoidal functor and is therefore oplax monoidal.¹⁶ In order to show that e_+ is actually \mathbb{E}_n -monoidal we use the fact that e_+ is \mathbb{E}_n -monoidal

¹⁶It is easier to think in terms of opposite categories here, since they are presentable and then this a lax monoidal right adjoint.

after restricting to thin objects and Lemma A.3 which lets us convert between sums and product.

$$\begin{aligned}
e(X \otimes Y) &\simeq e \left(\left(\bigoplus_{j \geq 0} X_k(j) \right) \otimes \left(\bigoplus_{k \geq 0} Y_k(k) \right) \right) \simeq e \left(\bigoplus_{j,k \geq 0} X_k(j) \otimes Y_k(k) \right) \\
&\simeq \bigoplus_{j,k \geq 0} e(X_k(j) \otimes Y_k(k)) \simeq \bigoplus_{j,k \geq 0} e(X_k(j)) \otimes e(Y_k(k)) \simeq \prod_{j,k \geq 0} e(X_k(j)) \otimes e(Y_k(k)) \\
&\simeq \left(\prod_{j \geq 0} e(X_k(j)) \right) \otimes \left(\prod_{k \geq 0} e(Y_k(k)) \right) \simeq \left(\bigoplus_{j \geq 0} e(X_k(j)) \right) \otimes \left(\bigoplus_{k \geq 0} e(Y_k(k)) \right) \\
&\simeq e \left(\bigoplus_{j \geq 0} X_k(j) \right) \otimes e \left(\bigoplus_{k \geq 0} Y_k(k) \right)
\end{aligned}$$

The key step in these manipulations is the point where we converted the sum over $j, k \geq 0$ into a product over $j, k \geq 0$ and it is at this point that we used our restriction to the positively graded setting. Note also that in pro-objects it is products and not sums which distribute over the tensor product. The manipulations used to prove that e (and therefore e_+) is fully faithful are similar.

$$\begin{array}{ccc}
\mathrm{Map}_{\mathcal{E}\mathrm{Gr}} \left(\bigoplus_j X_j(j), \bigoplus_k Y_k(k) \right) & \longrightarrow & \mathrm{Map}_{\mathrm{Pro}(\mathcal{E})\mathrm{Gr}} \left(e \left(\bigoplus_j X_j(j) \right), e \left(\bigoplus_k Y_k(k) \right) \right) \\
\downarrow \simeq & & \downarrow \simeq \\
& & \mathrm{Map}_{\mathrm{Pro}(\mathcal{E})\mathrm{Gr}} \left(\bigoplus_j e(X_j(j)), \bigoplus_k e(Y_k(k)) \right) \\
& & \downarrow \simeq \\
\mathrm{Map}_{\mathcal{E}\mathrm{Gr}} \left(\bigoplus_j X_j(j), \prod_k Y_k(k) \right) & \longrightarrow & \mathrm{Map}_{\mathrm{Pro}(\mathcal{E})\mathrm{Gr}} \left(\bigoplus_j e(X_j(j)), \prod_k e(Y_k(k)) \right) \\
\downarrow \simeq & & \downarrow \simeq \\
\prod_{j,k} \mathrm{Map}_{\mathcal{E}\mathrm{Gr}} (X_j(j), Y_k(k)) & \longrightarrow & \prod_{j,k} \mathrm{Map}_{\mathrm{Pro}(\mathcal{E})\mathrm{Gr}} (e(X_j(j)), e(Y_k(k))) \\
\downarrow \simeq & & \downarrow \simeq \\
\prod_i \mathrm{Map}_{\mathcal{E}} (X_i, Y_i) & \xrightarrow{\simeq} & \prod_i \mathrm{Map}_{\mathcal{E}} (X_i, Y_i)
\end{array}$$

□

Lemma A.15. *Bar and Cobar each send objects in the image of e_+ to objects in the image of e_+ .*

Proof. For Bar this follows from Lemma A.14 since the bar construction is composed of tensor products and a colimit. For Cobar this follows from the fact that because we are in the positively graded setting the tot tower in Construction A.8 is finite in each degree and therefore $\mathrm{Cobar}(C)$ is a constant pro-object in each degree (i.e. it is in the image of e_+). □

Lemma A.16. *Suppose we are given a monoidal left adjoint*

$$f : ((\mathcal{E}_1)_+^{\mathrm{Gr}})^{\mathrm{thin}} \rightarrow ((\mathcal{E}_2)_+^{\mathrm{Gr}})^{\mathrm{thin}},$$

then there is a right adjointable square

$$\begin{array}{ccc} \mathrm{Alg}((\mathcal{E}_1)_+^{\mathbf{Gr}}) & \xrightarrow{\tilde{f}} & \mathrm{Alg}((\mathcal{E}_2)_+^{\mathbf{Gr}}) \\ \downarrow \mathrm{Bar} & & \downarrow \mathrm{Bar} \\ \mathrm{coAlg}((\mathcal{E}_1)_+^{\mathbf{Gr}}) & \xrightarrow{\tilde{f}} & \mathrm{coAlg}((\mathcal{E}_2)_+^{\mathbf{Gr}}). \end{array}$$

Proof. Kan extending f to all of $\mathcal{E}_+^{\mathbf{Gr}}$ we obtain a monoidal left adjoint,

$$\tilde{f} : (\mathcal{E}_1)_+^{\mathbf{Gr}} \rightarrow (\mathcal{E}_2)_+^{\mathbf{Gr}}.$$

From [16, 5.2.3.11] we now obtain the desired square. In order to show this square is right adjointable we use the embedding into pro-thin objects of Construction A.12.

The functor $\mathrm{Pro}(f) : \mathrm{Pro}(\mathcal{E}_1)_+^{\mathbf{Gr}} \rightarrow \mathrm{Pro}(\mathcal{E}_2)_+^{\mathbf{Gr}}$ is an \mathbb{E}_n -monoidal left adjoint which preserves all limits. Consequently we can apply [16, 5.2.3.11] to obtain a right adjointable square

$$\begin{array}{ccc} \mathrm{Alg}(\mathrm{Pro}(\mathcal{E}_1)_+^{\mathbf{Gr}}) & \xrightarrow{\mathrm{Pro}(f)} & \mathrm{Alg}(\mathrm{Pro}(\mathcal{E}_2)_+^{\mathbf{Gr}}) \\ \downarrow \mathrm{Bar} & & \downarrow \mathrm{Bar} \\ \mathrm{coAlg}(\mathrm{Pro}(\mathcal{E}_1)_+^{\mathbf{Gr}}) & \xrightarrow{\mathrm{Pro}(f)} & \mathrm{coAlg}(\mathrm{Pro}(\mathcal{E}_2)_+^{\mathbf{Gr}}). \end{array}$$

Using Lemma A.14 and [16, 5.2.3.11] we can extend this square to a cube via the colimit preserving, fully faithful embeddings into pro-thin objects. At this point right adjointability follows from Lemma A.15 which says that Cobar and Bar send objects in the image of e_+ to objects in the image of e_+ . \square

Lemma A.17. *In $\mathrm{Sp}^{\mathbf{Gr}}$ the unit map $\mathbb{1}\{\mathbb{S}^0(1)\} \rightarrow \mathrm{Cobar}(\mathrm{Bar}(\mathbb{1}\{\mathbb{S}^0(1)\}))$ is an equivalence.*

Proof. Applying Lemma A.16 to the map $\mathrm{Sp} \rightarrow \mathrm{Mod}(\mathbb{Z})$ and using the fact that the bar and cobar produce objects which are levelwise finite (see Construction A.8) it suffices to observe that the unit is an equivalence in the graded \mathbb{Z} -linear case where this is the usual Koszul duality between polynomial and exterior algebras. \square

Lemma A.18. *Given an $X \in \mathcal{E}^{\mathbf{Gr}}$ concentrated in positive degrees the unit map $\mathbb{1}\{X\} \rightarrow \mathrm{Cobar}(\mathrm{Bar}(\mathbb{1}\{X\}))$ is an equivalence.*

Proof. We begin by handling the case where X is thin. Using the fact that $\mathrm{Sp}_{\geq 0}^{\mathbf{Gr}}$ is the free \mathbb{E}_1 -monoidal category on an object $\mathbb{S}^0(1)$ we can construct a monoidal left adjoint

$$\tilde{f}^* : \mathrm{Sp}_{\geq 0}^{\mathbf{Gr}} \rightarrow \mathcal{E}_{\geq 0}$$

which sends $\mathbb{S}^0(1)$ to X whose right adjoint \tilde{f}_* sends $Y \in \mathcal{E}_{\geq 0}^{\mathbf{Gr}}$ to $\left(k \mapsto \mathrm{Map}_{\mathcal{E}^{\mathbf{Gr}}}^{\mathrm{Sp}}(X^{\otimes k}, Y)\right)$. Since X is thin and concentrated in positive degrees \tilde{f}^* and \tilde{f}_* restrict to an adjunction

$$f^* : (\mathrm{Sp}_+^{\mathbf{Gr}})^{\mathrm{thin}} \rightleftarrows (\mathcal{E}_+^{\mathbf{Gr}})^{\mathrm{thin}} : f_*.$$

Since f^* is a monoidal left adjoint we have $f^*(\mathbb{1}\{Y\}) \simeq \mathbb{1}\{f^*Y\}$. Lemma A.16 now allows us to reduce showing that the unit map $\mathbb{1}\{X\} \rightarrow \text{Cobar}(\text{Bar}(\mathbb{1}\{X\}))$ is an equivalence to Lemma A.17. The general case now follows by writing X as a filtered colimit of thin objects and appealing to Lemma A.10. \square

Proof (of Theorem A.6). We proceed by induction on n with $n = 1$ as our base case.

In order to prove that $\text{Bar} : \text{Alg}(\mathcal{E}_+^{\text{Gr}}) \rightarrow \text{coAlg}(\mathcal{E}_+^{\text{Gr}})$ is an equivalence we will show that it is fully faithful and its right adjoint is conservative. Recall that we already proved Cobar is conservative in Lemma A.11. We prove Bar is fully faithful by showing that the unit map

$$A \rightarrow \text{Cobar}(\text{Bar}(A))$$

is an equivalence for every $A \in \text{Alg}(\mathcal{E}_+^{\text{Gr}})$. From Lemma A.18 we know the unit is an equivalence when A is a free algebra. We also know that both Bar and Cobar commute with geometric realizations (see Lemma A.10), therefore it suffices to argue that every A can be written as a geometric realization of a simplicial diagram of free algebras. This, in turn, follows from the fact that the free-underlying adjunction on $\text{Alg}(\mathcal{E}_+^{\text{Gr}})$ is monadic.

Now we handle the inductive step. Applying [16, 5.2.3.12 and 5.2.3.14] we are able to construct the following diagram where we have indicated the maps which are equivalences based on our inductive hypothesis.

$$\begin{array}{ccc} \text{Alg}_{/\mathbb{E}_{a+b}}(\mathcal{E}_+^{\text{Gr}}) & \xrightarrow{\text{Bar}^{(a+b)}} & \text{coAlg}_{/\mathbb{E}_{a+b}}(\mathcal{E}_+^{\text{Gr}}) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Alg}_{/\mathbb{E}_a} \left(\text{Alg}_{\mathbb{E}_b/\mathbb{E}_{a+b}}(\mathcal{E}_+^{\text{Gr}}) \right) \xrightarrow{\widetilde{\text{Bar}}^{(a)}} & \text{coAlg}_{/\mathbb{E}_a} \left(\text{Alg}_{\mathbb{E}_b/\mathbb{E}_{a+b}}(\mathcal{E}_+^{\text{Gr}}) \right) \xrightarrow{\text{Bar}^{(b)}} & \text{coAlg}_{/\mathbb{E}_a} \left(\text{coAlg}_{\mathbb{E}_b/\mathbb{E}_{a+b}}(\mathcal{E}_+^{\text{Gr}}) \right) \\ \downarrow & & \downarrow \\ \text{Alg}_{/\mathbb{E}_a}(\mathcal{E}_+^{\text{Gr}}) & \xrightarrow[\simeq]{\text{Bar}^{(a)}} & \text{coAlg}_{/\mathbb{E}_a}(\mathcal{E}_+^{\text{Gr}}) \end{array}$$

In order to complete the proof we must show that $\widetilde{\text{Bar}}^{(a)}$ is an equivalence. The underlying object functor $\text{Alg}_{\mathbb{E}_b/\mathbb{E}_{a+b}}(\mathcal{E}_+^{\text{Gr}}) \rightarrow \mathcal{E}_+^{\text{Gr}}$ preserves both limits and geometric realizations, therefore by [16, 5.2.3.11] the bottom square is right adjointable. The vertical arrows in the square are conservative, therefore the (co)unit map of the $\widetilde{\text{Bar}}^{(a)}$ - $\widetilde{\text{Cobar}}^{(a)}$ adjunction is an equivalence at an object X iff it is an equivalence on underlying. This follows from the inductive assumption that $\text{Bar}^{(a)}$ is an equivalence. \square

A.2 Using bar-cobar duality

We end the appendix with a couple lemmas focused on exposing the consequences of bar-cobar duality necessary for Section 2.

Lemma A.19. *The underlying object functor $\text{coAlg}_{\mathbb{E}_n}(\mathcal{E}) \rightarrow \mathcal{E}$ has a right adjoint*

$$\text{coFree} : \mathcal{E} \rightarrow \text{coAlg}_{\mathbb{E}_n}(\mathcal{E})$$

whose composite with $\text{Cobar}^{(n)}$ sends an object $X(k)$ to the square-zero algebra $\mathbb{1} \oplus \Sigma^{-n}X(k)$. The underlying object of $\text{coFree}(Y)$ is given by $\prod_k D_k^{\text{c}\mathbb{E}_n}(Y)$.

Proof. $\text{Pro}(\mathcal{E})_+^{\text{Gr}}$ is the opposite of a presentably \mathbb{E}_n -monoidal category and therefore by [16, 3.1.3.13] we have an underlying-cofree adjunction

$$\text{coAlg}_{\mathbb{E}_n}(\text{Pro}(\mathcal{E})_+^{\text{Gr}}) \rightleftarrows \text{Pro}(\mathcal{E})_+^{\text{Gr}} : \widehat{\text{coFree}}$$

where $\widehat{\text{coFree}}(X)$ is given by $\prod_k D_k^{\text{c}\mathbb{E}_m}(X)$.

The underlying object functors commutes with the \mathbb{E}_n -monoidal embedding e_+ of Construction A.12 giving us a square

$$\begin{array}{ccc} \text{coAlg}_{\mathbb{E}_n}(\mathcal{E}_+^{\text{Gr}}) & \longrightarrow & \mathcal{E}_+^{\text{Gr}} \\ \downarrow e_+ & & \downarrow e_+ \\ \text{coAlg}_{\mathbb{E}_n}(\text{Pro}(\mathcal{E})_+^{\text{Gr}}) & \longrightarrow & \text{Pro}(\mathcal{E})_+^{\text{Gr}}. \end{array}$$

As the two vertical maps are fully faithful, the underlying object functor will admit a right adjoint if the cofree \mathbb{E}_n -coalgebra functor on $\text{Pro}(\mathcal{E})_+^{\text{Gr}}$ sends objects in the image of e_+ to objects in the image of e_+ . In order to check this we observe that if X is concentrated in positive degrees, then the infinite product $\prod_k D_k^{\text{c}\mathbb{E}_n}(e(X))$ is a finite product in any fixed degree and the limit used to compute $D_k^{\text{c}\mathbb{E}_n}(e(X))$ is finite.

In order to compute $\text{Cobar}^{(n)}(\text{coFree}(X(k)))$ we again observe that if we compute what this object is in pro-thin objects and the result is in the image of e_+ , then this is the correct answer. In pro-thin objects we can use [16, 5.2.3.15] to conclude that the underlying object of $\text{Cobar}^{(n)}(\widehat{\text{coFree}}(e(X(k))))$ is $\mathbb{1} \oplus \Sigma^{-n}e(X(k))$. In particular, this is in the image of e_+ as desired.

In order to identify the algebra structure on $\text{Cobar}^{(n)}(\text{coFree}(X(k)))$ we observe that all augmented \mathbb{E}_n -algebras with underlying object $\mathbb{1} \oplus \Sigma^{-n}e(X(k))$ are equivalent, as they can all be obtained by truncating a free algebra to lie in degrees $\leq k$. \square

Construction A.20. Given a positively graded \mathbb{E}_n -algebra R in \mathcal{E} we can inductively produce a filtration

$$\mathbb{1}_{\mathcal{E}^{\text{Gr}}} = R^0 \xrightarrow{r_1} R^1 \xrightarrow{r_2} R^2 \rightarrow \dots \rightarrow R$$

converging to R such that

1. the map $R^k \rightarrow R$ is an equivalence through degree k and
2. the map r_k fits into a pushout square of positively graded \mathbb{E}_n -algebras

$$\begin{array}{ccc} \mathbb{1}_{\mathcal{E}^{\text{Gr}}} \{X^k(k)\} & \xrightarrow{\text{aug}} & \mathbb{1}_{\mathcal{E}^{\text{Gr}}} \\ \downarrow s_k & & \downarrow \\ R^{k-1} & \xrightarrow{r_k} & R^k. \end{array}$$

for some object $X^k \in \mathcal{E}$.

via the following procedure: Given R^{k-1} we let $X^k := \text{fib}((R^{k-1})_k \rightarrow (R^k)_k)$ this choice of X^k naturally comes equipped with a map $X^k \rightarrow (R^{k-1})_k$ and a nullhomotopy of the composite $X^k \rightarrow (R^{k-1})_k \rightarrow (R^k)_k$ which allows us to define the map s_k and the factorization of the map $R^{k-1} \rightarrow R$ through r_k . Since the free algebra we used started in grading k we have a cofiber sequence

$$X^k \rightarrow (R^{k-1})_k \rightarrow (R^k)_k$$

from which (1) follows. Convergence of the R^k to R follows from condition (1). \triangleleft

Lemma A.21. *Let R be a positively graded \mathbb{E}_n -algebra in \mathcal{E} . We can identify the object X^k from Construction A.20 with $\Sigma^{-1-n}\text{Bar}^{(n)}(R)_k$.*

Proof. Upon applying $\text{Bar}^{(n)}$ to the resolution of Construction A.20 we obtain a resolution of $\text{Bar}^{(n)}(R)$ which at its k^{th} term is a pushout under the square-zero \mathbb{E}_n -coalgebra $\mathbb{1} \oplus \Sigma^n X^k(k)$ (see Lemma A.19). Since pushouts of coalgebras are computed on underlying, this allows us to read off that

$$\text{Bar}^{(n)}(R)_k \simeq \Sigma^{n+1} X^k. \quad \square$$

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