Invariant Hermitian forms on vertex algebras

INVARIANT HERMITIAN FORMS ON VERTEX ALGEBRAS

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Abstract. We study invariant Hermitian forms on a conformal vertex algebra and on their (twisted) modules. We establish existence of a non-zero invariant Hermitian form on an arbitrary W–algebra. We show that for a minimal simple W–algebra $W_k(\mathfrak{g}, \theta/2)$ this form can be unitary only when its $\frac{1}{2}\mathbb{Z}$–grading is compatible with parity, unless $W_k(\mathfrak{g}, \theta/2)$ "collapses" to its affine subalgebra.

1. Introduction

In the present paper we study invariant Hermitian forms on a conformal vertex algebra V and its (possibly twisted) positive energy modules. By a conformal vertex algebra we mean a vector superspace V over $\mathbb{C}$, endowed with a structure of a vertex algebra (with state–field correspondence $a \mapsto Y(a, z)$), and a Virasoro vector L such that the eigenvalues of L_0 lie in $\frac{1}{2}\mathbb{Z}_+$, all eigenspaces are finite–dimensional, and the 0–th eigenspace consists of multiples of the vacuum vector (cf. Definition 1.1 in Section 2 and [11]).

Let ϕ be a conjugate linear involution of V. A Hermitian form $(\cdot\,,\cdot)$ on V is called *ϕ–invariant* if, for all $a \in V$, one has

$$(v, Y(a, z)u) = (Y(A(z)a, z^{-1})v, u), \quad u, v \in V. \tag{1.1}$$

Here $A(z) : V \to V((z))$ is defined by

$$A(z) = e^{zL_1} z^{-2L_0} g, \tag{1.2}$$

where

$$g(a) = e^{-\pi\sqrt{-1}(\frac{1}{2}p(a)+\Delta_a)}\phi(a), \quad a \in V, \tag{1.3}$$

and $p(a) = 0$ or 1 stands for the parity of a and Δ_a for its L_0–eigenvalue. The definition of a ϕ–invariant Hermitian form on a V–module M is similar (cf. Definition 6.6).

The operator $A(z)$ with $g = (-1)^{L_0}$ appeared first in [4] in the construction of the coadjoint module in the case when V is purely even and the eigenvalues of L_0 are integers. Under the same assumptions on V this operator was used in [9] for the construction of the dual to the V–modules.

Formula (1.2) with $g = (-1)^{L_0}\phi$ was used in [7] to define unitary structures on vertex operator algebras and this notion was generalized in [3] to vertex algebras with $\frac{1}{2}\mathbb{Z}_+$–grading compatible with parity, in which case formula (1.3) simplifies to (see (4.2))

$$g = (-1)^{L_0+2L_0^2}\phi.$$

As one can infer from the above remarks, the motivation for this definition stems from the observation that, given a V–module M, one has (as in the Lie algebra case), a bijective correspondence between ϕ–invariant Hermitian forms $(\cdot, \cdot)$ on V and V–module conjugate linear homomorphisms $\Theta : M \to M^\dagger$, where $M^\dagger$ is the conjugate linear dual to M, with V–module structure defined by

$$\langle Y_{M^\dagger}(a.z)m', m\rangle = \langle m', Y_M(A(z)a, z^{-1})m\rangle, \quad m \in M, m' \in M^\dagger. \tag{1.4}$$

Our first result, which generalizes [9, Theorem 5.2.1, Proposition 5.3.1] (with a similar proof), is Proposition 3.6: formula (1.4) indeed defines a structure of a V–module on the restricted dual superspace $M^\dagger$ of M. Our second result, which generalizes, with the same proof, that of [15] in the symmetric case, is Proposition 4.3, which describes ϕ–invariant Hermitian forms on V. Its Corollary 4.7 claims that a conformal vertex algebra V with a conjugate linear involution ϕ admits a (unique, up to a constant factor) ϕ–invariant Hermitian form if and only if any eigenvector of L_0 with eigenvalue 1 is annihilated by L_1 (see also Remark 4.4). As usual, such a Hermitian form can be expressed in terms of the expectation value on the vacuum (see formula (4.9)).

In Section 5 we construct invariant Hermitian forms of fermionic, bosonic, affine and lattice vertex algebras. In Section 6 we extend the results on invariant Hermitian forms on V to arbitrary positive–energy (twisted) modules M. Proposition 5.3 claims that the space of ϕ–invariant Hermitian forms on M is isomorphic to the set of ω–invariant Hermitian forms on the module M_0 over the Zhu algebra. Here M_0 is the lowest energy subspace of M and ω is the conjugate linear anti–involution of the Zhu algebra, induced by the endomorphism of the superspace V defined by

$$\omega(v) = A(1)v, \quad v \in V.$$

In Remark 6.8 we note that actually Proposition 4.3 is a special case of Proposition 6.7.

In Section 7 we construct an invariant Hermitian form on the W–algebras $W^k(\mathfrak{g}, x, f)$ [12], [13]. This construction is based on Proposition 7.4 (b), which says that the condition of Corollary 4.7, that all eigenvectors of L_0 with eigenvalue 1 of the vertex algebra are annihilated by L_1, holds, provided that the elements $h := 2x$ and f can be included in a $sl(2)$-triple $\{e, f, h\}$.

In conclusion of this section we briefly discuss unitarity (i.e., positive semi-definiteness) of this Hermitian form for minimal W–algebras $W^k(\mathfrak{g}, \theta/2)$. We show that the only interesting cases might occur when the $\frac{1}{2}\mathbb{Z}$–grading on the W–algebra is compatible with parity. In all the other cases we show that the W–algebra can be unitary only at collapsing levels [1], i.e. when the simple W-algebra $W_k(\mathfrak{g}, \theta/2)$ "collapses" to its affine subalgebra: see Propositions 7.9, 7.11. These are just the first steps towards classification of unitary minimal W–algebras.

Throughout the paper the base field is $\mathbb{C}$. We also denote by $\mathbb{Z}_+$ the set of nonnegative integers and by $\mathbb{N}$ the set of positive integers.

2. Setup

2.1. **Basic definitions.** Recall that a vector superspace is a $\mathbb{Z}/2\mathbb{Z}$–graded vector space $V = V_{\bar 0} \oplus V_{\bar 1}$. The elements in $V_{\bar 0}$ (resp. $V_{\bar 1}$) are called even (resp. odd). Set

$$p(v) = \begin{cases} 0 \in \mathbb{Z} & \text{if } v \in V_{\bar 0}, \\ 1 \in \mathbb{Z} & \text{if } v \in V_{\bar 1}, \end{cases}$$

i.e. we will regard $p(v)$ as an integer, not as a residue class. We will often use the notation

$$\sigma(u) = (-1)^{p(u)}u, \qquad p(u, v) = (-1)^{p(u)p(v)}. \tag{2.1}$$

Let V be a vertex algebra. We let

$$Y : V \to (\operatorname{End} V)[[z, z^{-1}]], \qquad v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_{(n)} z^{-n-1} \quad (v_{(n)} \in \operatorname{End} V), \tag{2.2}$$

be the state–field correspondence. We denote by $\mathbf{1}$ the vacuum vector in V and by T the translation operator (see e.g. [11] for details).

Definition 2.1. In the present paper we will call a vertex algebra V *conformal* if there exists a distinguished vector $L \in V_2$, called a Virasoro vector, satisfying the following conditions:

$$Y(L, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},\ [L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0} c\, I, \tag{2.3}$$

$$L_{-1} = T, \tag{2.4}$$

(2.5) L_0 is diagonalizable and its eigenspace decomposition has the form

$$V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_+} V_n, \tag{2.6}$$

where

$$\dim V_n < \infty \text{ for all } n \text{ and } V_0 = \mathbb{C}\mathbf{1}. \tag{2.7}$$

The number c is called the *central charge*.

Remark 2.2. Important examples of conformal vertex algebras are vertex operator superalgebras, namely the conformal vertex algebras for which decomposition (2.6) is compatible with parity, i.e. $\sigma(u) = (-1)^{2L_0} u$.

In the definition of [11] of conformal vertex algebras properties (2.6) and (2.7) are not required.

By an automorphism of a conformal vertex algebra V we mean a vertex algebra automorphism ϕ of V (i. e. $\phi(u_{(n)}v) = \phi(u)_{(n)}\phi(v)$ for all $n \in \mathbb{Z}$) with the property that $\phi(L) = L$. Consequently, $\phi(V_n) = V_n$.

The eigenvalues of L_0 on V are called *conformal weights*; the conformal weight of $v \in V$ is denoted by Δ_v, so that $v \in V_{\Delta_v}$. The eigenvector v of L_0 is called *quasiprimary* if $L_1 v = 0$ and *primary* if $L_n v = 0$ for $n \geq 1$. One has for v of conformal weight Δ_v:

$$[L_\lambda v] = (L_{-1} + \Delta_v \lambda)v + \sum_{n \geq 2} \tfrac{\lambda^n}{n!} L_{n-1} v. \tag{2.8}$$

Here and throughout the paper we use the formalism of λ-brackets, which are defined by

$$[u_\lambda v] = Res_z e^{z\lambda} Y(u, z)v, \quad u, v \in V.$$

Let Γ be an additive subgroup of $\mathbb{R}$ containing $\mathbb{Z}$. If $\gamma \in \mathbb{R}$, denote by $[\gamma]$ its coset $\gamma + \mathbb{Z}$.

Definition 2.3. Let V be a conformal vertex algebra. A $\Gamma/\mathbb{Z}$*–grading* on V is a map $\Upsilon : [\gamma] \mapsto V^{[\gamma]} \subseteq V$ such that V decomposes as

$$V \ = \bigoplus_{[\gamma] \in \Gamma/\mathbb{Z}} V^{[\gamma]} \tag{2.9}$$

and (2.9) is a vertex algebra grading, compatible with L_0, i.e.

$$V^{[\alpha]}{}_{(n)} V^{[\beta]} \subseteq V^{[\alpha+\beta]}\ , \quad L_0(V^{[\gamma]}) \subseteq V^{[\gamma]}\ .$$

If $a \in V^{[\gamma]}$ then $[\gamma]$ is called the *degree* of a. Given a vector $a \in V$ of conformal weight Δ_a and degree $[\gamma]$, denote by ϵ_a the maximal non–positive real number in the coset $[\gamma - \Delta_a]$. This number has the following properties [5]:

$$\epsilon_{\mathbf{1}} = 0\,, \quad \epsilon_{Ta} = \epsilon_a\,, \quad \epsilon_{a_{(n)}b} = \epsilon_a + \epsilon_b + \chi(a,b)\,, \tag{2.10}$$

where $\chi(a,b) = 1$ or 0, depending on whether $\epsilon_a + \epsilon_b \le -1$ or not.

Let $\gamma_a = \Delta_a + \epsilon_a$. Then

$$\gamma_{\mathbf{1}} = 0\,, \quad \gamma_{Ta} = \gamma_a + 1\,, \quad \gamma_{a_{(n)}b} = \gamma_a + \gamma_b + \chi(a,b) - n - 1\,. \tag{2.11}$$

2.2. **Twisted modules.**

Definition 2.4. Let Γ be an additive subgroup of $\mathbb{R}$ containing $\mathbb{Z}$, and let Υ be a $\Gamma/\mathbb{Z}$–grading on a conformal vertex algebra V. A Υ–*twisted module* for V is a vector superspace M and a parity preserving linear map from V to the space of End$M$–valued Υ–*twisted quantum fields* $a \mapsto Y^M(a,z) = \sum_{m\in[\gamma_a]} a^M_{(m)} z^{-m-1}$ (i.e. $a^M_{(m)} \in \mathrm{End}M$ and $a^M_{(m)}v = 0$ for each $v \in M$ and $m \gg 0$), such that the following properties hold:

$$\mathbf{1}^M_{(n)} = \delta_{n,-1} I_M\,, \tag{2.12}$$

$$\sum_{j\in\mathbb{Z}_+} \binom{m}{j} (a_{(n+j)}b)^M_{(m+k-j)} v \tag{2.13}$$

$$= \sum_{j\in\mathbb{Z}_+} (-1)^j \binom{n}{j} (a^M_{(m+n-j)} b^M_{(k+j)} - p(a,b)(-1)^n b^M_{(k+n-j)} a^M_{(m+j)}) v\,,$$

where $a \in V^{[\gamma_a]}$ ($\gamma_a \in \Gamma$), $m \in [\gamma_a]$, $n \in \mathbb{Z}$, $k \in [\gamma_b]$.

The following Lemma is known; we prove it for completeness.

Lemma 2.5. *The Borcherds identity* (2.13) *is equivalent to*

(2.14)

$$Res_u(i_{w,u} Y_M(Y(a,u)b,w)(w+u)^m u^n w^l) =$$
$$Res_z(i_{z,w} Y_M(a,z)Y_M(b,w) z^m (z-w)^n w^l - p(a,b) i_{w,z} Y_M(b,w) Y_M(a,z) z^m (z-w)^n w^l)$$

for all $n \in \mathbb{Z}$, $m \in [\gamma_a]$, $l \in [\gamma_b]$. As usual, $i_{x,y}$ means expanding in the domain $|x| > |y|$.

Proof. Computing the residues we find that (2.14) is equivalent to

$$\sum_{t\in\mathbb{Z}, j\in\mathbb{Z}_+} \binom{m}{j} (a_{(j+n)}b)^M_{(t-j+m+l)} w^{-t-1}$$
$$= \sum_{t\in\mathbb{Z}, j\in\mathbb{Z}_+} (-1)^j \binom{n}{j} \left(a^M_{(m+n-j)} b^M_{(t+j+l)} - p(a,b)(-1)^n b^M_{(t+n-j+l)} a^M_{(m+j)} \right) w^{-t-1}.$$

□

Since $V^{[\gamma]}$ is L_0–invariant, we have its eigenspace decomposition $V^{[\gamma]} = \oplus_\Delta V^{[\gamma]}_\Delta$, and we will write for $v \in V^{[\gamma]}_{\Delta_v}$,

$$Y_M(v,z) = \sum_{n\in[\gamma-\Delta_v]} v^M_n z^{-n-\Delta_v}.$$

Definition 2.6. A Υ–twisted V–module M is called a *positive energy* V–module if M has an $\mathbb{R}$–grading $M = \oplus_{j \geq 0} M_j$ such that

$$a_n^M M_j \subseteq M_{j-n}, \ a \in V_{\Delta_a}. \tag{2.15}$$

The subspace M_0 is called the *minimal energy subspace*. Then,

$$a_n^M M_0 = 0 \text{ for } n > 0 \text{ and } a_0^M M_0 \subseteq M_0\,. \tag{2.16}$$

2.3. **Zhu algebras.** Set

$$V_\Upsilon = span(a \in V \mid \epsilon_a = 0). \tag{2.17}$$

Define a subspace J_Υ of V as the span of elements

$$\sum_{j \in \mathbb{Z}_+} \binom{\gamma_a}{j} a_{(-2+\chi(a,b)+j)} b = Res_z z^{-2+\chi(a,b)} Y((1+z)^{\gamma_a} a, z) b, \tag{2.18}$$

with $\epsilon_a + \epsilon_b \in \mathbb{Z}$.

Let

$$a * b = \sum_{j \in \mathbb{Z}_+} \binom{\gamma_a}{j} a_{(-1+j)} b,$$

Then J_Υ is a two sided ideal in V_Υ with respect to the product $*$. The quotient $Zhu_\Upsilon(V) = V_\Upsilon / J_\Upsilon$ is an associative superalgebra with respect to the product $*$ (see [5] for a proof), which is called the *Zhu algebra* associated to the grading (2.9).

Example 2.7. If Γ is the subgroup of $\mathbb{R}$ spanned by the conformal weights Δ_a then one has a $\Gamma/\mathbb{Z}$–grading (2.9), for which

$$V^{[\gamma]} = \oplus_{\Delta \in [\gamma]} V_\Delta.$$

The corresponding Zhu algebra is called the L_0–twisted (or Ramond twisted) Zhu algebra and denoted by $Zhu_{L_0} V$. If $\Gamma = \mathbb{Z}$ then one has the trivial grading (2.9) by setting $V^{\mathbb{Z}} = V$. The corresponding Zhu algebra is denoted by $Zhu_{\mathbb{Z}} V$ and is called the non–twisted Zhu algebra ([5], Examples 2.14 and 2.15).

3. The conjugate contragredient module

In this section we adapt to our setting the proofs of Section 5 of [9], where the action of a vertex operator algebra on the linear dual of a module is defined. If $a \in V_{\Delta_a}$, set

$$(-1)^{L_0} a = e^{\pi \sqrt{-1} \Delta_a} a, \quad \sigma^{1/2}(a) = e^{\frac{\pi}{2} \sqrt{-1} p(a)} a. \tag{3.1}$$

Lemma 3.1. *Let g be a diagonalizable parity preserving conjugate linear operator on V with modulus 1 eigenvalues, such that $g(L) = L$. Then one has the relation*

$$g Y(a, z) g^{-1} b = p(a, b) Y(g(a), -z) b \tag{3.2}$$

if and only if the operator

$$\phi = (-1)^{L_0} \sigma^{1/2} g \tag{3.3}$$

is a conjugate linear automorphism of V. Moreover

$$g^2 = I \iff \phi^2 = I. \tag{3.4}$$

Proof. Assume that g satisfies (3.2). Then

$$\begin{aligned}\phi(a)_{(n)}\phi(b) &= ((-1)^{L_0}\sigma^{1/2}g)(a)_{(n)}((-1)^{L_0}\sigma^{1/2}g)(b) \\ &= e^{\pi\sqrt{-1}(\Delta_a+\Delta_b)}e^{\pi/2\sqrt{-1}(p(a)+p(b))}g(a)_{(n)}g(b).\end{aligned} \tag{3.5}$$

By (3.2), $g(a_{(n)}b) = (-1)^{n+1}p(a,b)g(a)_{(n)}g(b)$. Substituting in (3.5), and noting that $p(a) + p(b) + 2p(a)p(b) = p(a_{(n)}b) \mod 4\mathbb{Z}$, we obtain

$$\begin{aligned}\phi(a)_{(n)}\phi(b) &= e^{\pi\sqrt{-1}(\Delta_a+\Delta_b)}e^{\pi/2\sqrt{-1}(p(a)+p(b))}(-1)^{n+1}p(a,b)g(a_{(n)}b) \\ &= e^{\pi\sqrt{-1}\Delta_{a_{(n)}b}}e^{\pi/2\sqrt{-1}(p(a)+p(b)+2p(a)p(b))}g(a_{(n)}b) \\ &= e^{\pi\sqrt{-1}\Delta_{a_{(n)}b}}e^{\pi/2\sqrt{-1}p(a_{(n)}b)}g(a_{(n)}b) = \phi(a_{(n)}b).\end{aligned}$$

Reversing the argument we obtain the converse statement.

To prove (3.4) remark that $g(L) = L$, hence $L_0g(a) = g(L)_0g(a) = g(L_0a)$, so, since $\Delta_a \in \mathbb{R}$, $\Delta_{g(a)} = \Delta_a$. Moreover g is parity preserving and conjugate linear, hence

$$\begin{aligned}\phi^2(a) &= (-1)^{L_0}\sigma^{1/2}g(-1)^{L_0}\sigma^{1/2}g(a) = e^{\pi\sqrt{-1}(\Delta_a+\frac{1}{2}p(a))}ge^{\pi\sqrt{-1}(\Delta_a+\frac{1}{2}p(a))}g(a) \\ &= e^{\pi\sqrt{-1}(\Delta_a+\frac{1}{2}p(a))}e^{-\pi\sqrt{-1}(\Delta_a+\frac{1}{2}p(a))}g^2(a) = g^2(a).\end{aligned}$$

□

Definition 3.2. Let g be a diagonalizable parity preserving conjugate linear operator on V, satisfying (3.2) and such that $g^2 = I$. Define $A(z) : V \to V((z))$ by

$$A(z)v = e^{zL_1}z^{-2L_0}gv, \quad v \in V. \tag{3.6}$$

Lemma 3.3. *We have*

$$p(a,b)A(w)Y(a,z)A(w)^{-1}b = i_{w,z}Y\left(A(z+w)a, \frac{-z}{(z+w)w}\right)b \tag{3.7}$$

and

$$A(z^{-1}) = A(z)^{-1}. \tag{3.8}$$

Proof. It is clear that

$$w^{-2L_0}Y(a,z)w^{2L_0}b = Y(w^{-2L_0}a, z/w^2)b. \tag{3.9}$$

By (3.2)

$$p(a,b)gw^{-2L_0}Y(a,z)w^{2L_0}g^{-1}b = Y(gw^{-2L_0}a, -z/w^2)b. \tag{3.10}$$

Finally we use that, if $|wz| < 1$, then

$$e^{wL_1}Y(a,z)e^{-wL_1} = Y(e^{w(1-wz)L_1}(1-wz)^{-2L_0}a, \frac{z}{1-wz}) \tag{3.11}$$

(see (5.2.38) of [9] and (4.9.17) of [11]) to get, for $|z| < |w|$,

$$p(a,b)e^{wL_1}Y(gw^{-2L_0}a, -z/w^2)e^{-wL_1}b = Y(e^{(w+z)L_1}g(w+z)^{-2L_0}a, \frac{-z}{w(w+z)})b, \tag{3.12}$$

which is (3.7).

Since $g^2 = I$, (3.8) is equivalent to

$$A(z)a = g^{-1}z^{-2L_0}e^{-z^{-1}L_1}a = gz^{-2L_0}e^{-z^{-1}L_1}a. \tag{3.13}$$

Next observe that

$$\begin{aligned} gz^{-2L_0}e^{-z^{-1}L_1}a &= \sum_r z^{-2L_0}(-1)^r \tfrac{1}{r!} g(L_1^r a) z^{-r} \\ &= \sum_r z^{-2\Delta_a} z^{2r}(-1)^r \tfrac{1}{r!} g(L_1^r a) z^{-r} \\ &= \sum_r (-1)^r \tfrac{1}{r!} g(L_1^r a) z^{r-2\Delta_a}. \end{aligned}$$

Since $g(L_1 v) = -g(L)_1 g(v) = -L_1 g(v)$ we obtain

$$gz^{-2L_0}e^{-z^{-1}L_1}a = \sum_r \tfrac{1}{r!} L_1^r g(a) z^{r-2\Delta_a} = e^{zL_1} z^{-2L_0} g(a) = A(z)a.$$

□

Remark 3.4. Note that, by (3.13), if v is quasiprimary, we have

$$A(z)v = z^{-2\Delta_v} g(v). \tag{3.14}$$

If Υ is a $\Gamma/\mathbb{Z}$-grading on V, we let the *opposite* grading $-\Upsilon$ be the grading defined by setting

$$-\Upsilon([\gamma]) = \Upsilon(-[\gamma]).$$

We say that a $\Gamma/\mathbb{Z}$-grading is compatible with a map ϕ if $\phi(V^{[\gamma]}) \subseteq V^{[\gamma]}$.

Let M be a positive energy Υ–twisted module and let $M^\dagger$ denote the restricted conjugate dual of M, that is

$$M^\dagger = \bigoplus_{n\geq 0} M_n^\dagger \tag{3.15}$$

where $M_n^\dagger$ is the space of conjugate linear maps from M_n to $\mathbb{C}$.

Lemma 3.5. *If $M + K \in \mathbb{Z}$, then*

$$Res_z z^M w^N i_{z,w}(z+w)^K = (-1)^{K+M-1} Res_z z^{-2-K-M} w^{2+2K+M+N} i_{w,z}(z+w)^M. \tag{3.16}$$

Proof. If $M + K < -1$, both sides of (3.16) are zero. If $M + K \geq -1$, then

$$\begin{aligned} Res_z z^M w^N i_{z,w}(z+w)^K &= Res_z \sum_{j\in\mathbb{Z}_+} \binom{K}{j} z^{M+K-j} w^{N+j} \\ &= \binom{K}{M+K+1} w^{N+M+K+1}. \end{aligned}$$

On the other hand

$$\begin{aligned} Res_z z^m w^n i_{w,z}(z+w)^k &= Res_z \sum_{j\in\mathbb{Z}_+} \binom{k}{j} z^{m+j} w^{n+k-j} \\ &= \binom{k}{-m-1} w^{n+m+k+1} \\ &= (-1)^{-m-1} \binom{-m-1-k-1}{-m-1} w^{n+m+k+1}. \end{aligned}$$

Equality holds for $m = -2-K-M, n = 2+2K+M+N, k = M$. □

Theorem 3.6. *Let ϕ be a conjugate linear involution of a conformal vertex algebra V. Choose g as in Definition 3.2 and define $A(z)$ by (3.6). Let Υ be a $\Gamma/\mathbb{Z}$–grading on V compatible with ϕ. Let M be a Υ–twisted positive energy module. Then*

(a) *The map $Y_{M^\dagger}$ given by*

$$\langle Y_{M^\dagger}(v,z)m',m\rangle = \langle m', Y_M(A(z)v,z^{-1})m\rangle,\ \ m\in M, m'\in M^\dagger, \tag{3.17}$$

defines on $M^\dagger$ the structure of a $(-\Upsilon)$–twisted V–module.

(b) *If $\dim M_n<\infty$ for all n then $(M^\dagger)^\dagger$ is naturally isomorphic to M.*

Proof. Let $V=\oplus_{\gamma\in\Gamma/\mathbb{Z}}V^\gamma$ be the grading Υ. Write explicitly for $v\in V^\gamma_{\Delta_v}$,

$$Y_{M^\dagger}(v,z)=\sum_{n\in-\gamma-\Delta_v} v_n^{M^\dagger}z^{-n-\Delta_v}.$$

Then we have

$$\sum_n\langle v_n^{M^\dagger}m',m\rangle z^{-n-\Delta_v}=\sum_n\langle m',\sum_t \tfrac{1}{t!}(L_1^tg(v))_n^M m\rangle z^{n-\Delta_v}.$$

In other words, if $n\in-\gamma-\Delta_v$, then

$$\langle v_n^{M^\dagger}m',m\rangle=\langle m',\sum_t\tfrac{1}{t!}(L_1^tg(v))_{-n}^M m\rangle. \tag{3.18}$$

In particular, $v_n^{M^\dagger}M_j^\dagger\subseteq M^\dagger_{j-n}$. This proves that, by (3.15), $Y^{M^\dagger}$ is indeed a $(-\Upsilon)$–twisted quantum field.

Next observe that

$$\langle \mathbf{1}_{(n)}^{M^\dagger}m',m\rangle=\langle \mathbf{1}_{n+1}^{M^\dagger}m',m\rangle=\langle m',\mathbf{1}^M_{-n-1}m\rangle=\delta_{-n-1,0}\langle m',m\rangle, \tag{3.19}$$

hence (2.12) for $M^\dagger$ follows.

We now prove the Borcherds identity (2.14) for $M^\dagger$, that is

$$\begin{aligned}&Res_u\langle Y_{M^\dagger}(Y(a,u)b,w)i_{w,u}(w+u)^ku^nw^lm',m\rangle\\ &=Res_z(\langle Y_{M^\dagger}(a,z)Y_{M^\dagger}(b,w)i_{z,w}z^k(z-w)^nw^lm',m\rangle)\\ &-p(a,b)Res_z(\langle Y_{M^\dagger}(b,w)Y_{M^\dagger}(a,z)i_{w,z}z^k(z-w)^nw^lm',m\rangle)\end{aligned} \tag{3.20}$$

for all $n\in\mathbb{Z}$, $k\in[-\gamma_a]$, $l\in[-\gamma_b]$. Since

$$\begin{aligned}\langle Y_{M^\dagger}(a,z)Y_{M^\dagger}(b,w)m',m\rangle&=\langle m',Y_M(A(w)b,w^{-1})Y_M(A(z)a,z^{-1})m\rangle,\\ \langle Y_{M^\dagger}(b,w)Y_{M^\dagger}(a,z)m',m\rangle&=\langle m',Y_M(A(z)a,z^{-1})Y_M(A(w)b,w^{-1})m\rangle,\\ \langle Y_{M^\dagger}(Y(a,u)b,w)m',m\rangle&=\langle m',Y_M(A(w)Y(a,u)b,w^{-1})m\rangle,\end{aligned}$$

we have to prove that

$$\begin{aligned}&Res_u(\langle m',Y_M(A(w)Y(a,u)b,w^{-1})m\rangle i_{w,u}(w+u)^ku^nw^l)\\ &=Res_z(\langle m',Y_M(A(w)b,w^{-1})Y_M(A(z)a,z^{-1})m\rangle i_{z,w}z^k(z-w)^nw^l)\\ &-p(a,b)Res_z(\langle m',Y_M(A(z)a,z^{-1})Y_M(A(w)b,w^{-1})m\rangle i_{w,z}z^k(z-w)^nw^l).\end{aligned}$$

Hence we need to check that

$$\begin{aligned}&Res_u(Y_M(A(w)Y(a,u)b,w^{-1})i_{w,u}(w+u)^ku^nw^l)\\ &=Res_z(Y_M(A(w)b,w^{-1})Y_M(A(z)a,z^{-1})i_{z,w}z^k(z-w)^nw^l)\\ &-p(a,b)Res_z(Y_M(A(z)a,z^{-1})Y_M(A(w)b,w^{-1})i_{w,z}z^k(z-w)^nw^l).\end{aligned} \tag{3.21}$$

Changing variables in the Borcherds identity (2.14) for Y_M we obtain, for all $n\in\mathbb{Z}$, $m\in[\gamma_a]$, $l\in[\gamma_b]$,

$$\begin{aligned}&Res_tY_M(Y(a,t^{-1})b,w^{-1})i_{w^{-1},t^{-1}}(w^{-1}+t^{-1})^mt^{-n-2}w^{-l}\\&=Res_t(Y_M(a,t^{-1})Y_M(b,w^{-1})i_{t^{-1},w^{-1}}t^{-m-2}(t^{-1}-w^{-1})^nw^{-l})\\&-p(a,b)Res_t(Y_M(b,w^{-1})Y_M(a,t^{-1})i_{w^{-1},t^{-1}}t^{-m-2}(t^{-1}-w^{-1})^nw^{-l}),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&Res_t(Y_M(Y(a,t^{-1})b,w^{-1})i_{t,w}(w+t)^mt^{-n-2-m}w^{-l-m}\\&=Res_t(Y_M(a,t^{-1})Y_M(b,w^{-1})i_{w,t}t^{-m-n-2}(w-t)^nw^{-l-n})\\&-p(a,b)Res_t(Y_M(b,w^{-1})Y_M(a,t^{-1})i_{t,w}t^{-m-2-n}(w-t)^nw^{-l-n}).\end{aligned}\tag{3.22}$$

Write explicitly $A(w)a=\sum_{r\in\mathbb{Z}_+}C_r(a)w^{r-2\Delta_a}$, where $C_r(a)\in V$. Then

$$Y_M(A(t)a,t^{-1})=\sum_{r\in\mathbb{Z}_+,h\in[\gamma_a]}C_r(a)_{(h)}t^{h+1}t^{r-2\Delta_a}=\sum_{r\in\mathbb{Z}_+}Y_M(C_r(a),t^{-1})t^{r-2\Delta_a},$$

so, by (3.22),

$$\begin{aligned}&Res_t(Y_M(A(t)a,t^{-1})Y_M(A(w)b,w^{-1})i_{w,t}t^{-m-n-2}(w-t)^nw^{-l-n})\\&-p(a,b)Res_t(Y_M(A(w)b,w^{-1})Y_M(A(t)a,t^{-1})i_{t,w}t^{-m-2-n}(w-t)^nw^{-l-n})\\&=\sum_r Res_t(Y_M(C_r(a),t^{-1})Y_M(A(w)b,w^{-1})i_{w,t}t^{-m-n-2+r-2\Delta_a}(w-t)^nw^{-l-n})\\&-p(a,b)\sum_r Res_t(Y_M(A(w)b,w^{-1})Y_M(C_r(a),t^{-1})i_{t,w}t^{-m-2-n+r-2\Delta_a}(w-t)^nw^{-l-n})\\&=\sum_r Res_t(Y_M(Y(C_r(a),t^{-1})A(w)b,w^{-1})i_{t,w}(w+t)^{m-r+2\Delta_a}t^{-n-2-m+r-2\Delta_a}w^{-l-m+r-2\Delta_a}\\&=Res_t(Y_M(i_{t,w}Y(A(\frac{wt}{w+t})a,t^{-1})A(w)b,w^{-1})(w+t)^mt^{-n-2-m}w^{-l-m}.\end{aligned}$$

Therefore we have

$$\begin{aligned}&Res_z(Y_M(A(z)a,z^{-1})Y_M(A(w)b,w^{-1})i_{w,z}z^k(w-z)^nw^l)\\&-p(a,b)Res_z(Y_M(A(w)b,w^{-1})Y_M(A(z)a,z^{-1})i_{z,w}z^k(w-z)^nw^l)\\&=Res_t(Y_M(i_{t,w}Y(A(\frac{wt}{w+t})a,t^{-1})A(w)b,w^{-1})(w+t)^{-k-n-2}t^kw^{l+k+2n+2}.\end{aligned}$$

Hence (3.21) turns into

$$\begin{aligned}&Res_t(i_{t,w}Y_M(Y(A(\frac{wt}{w+t})a,t^{-1})A(w)b,w^{-1})(w+t)^{-k-n-2}t^kw^{l+k+2n+2}\\&=-p(a,b)(-1)^nRes_t(Y_M(A(w)Y(a,t)b,w^{-1})i_{w,t}(w+t)^kt^nw^l).\end{aligned}$$

Expand the L.H.S. above as

$$\begin{aligned}&Res_t(Y_M(i_{t,w}Y(A(\frac{wt}{w+t})a,t^{-1})A(w)b,w^{-1})(w+t)^{-k-n-2}t^kw^{l+k+2n+2})=\\&\sum_{p,q,r,s}Res_t((C_r(a)_{(p)}C_s(b))_{(q)}i_{t,w}(w+t)^{-r+2\Delta_a-k-n-2}t^{k+r-2\Delta_a+p+1}w^{l+k+2n+2+q+s-2\Delta_b+r-2\Delta_a}),\end{aligned}$$

and apply Lemma 3.5 to obtain

$$\begin{aligned}
&Res_t(Y_M(i_{t,w}Y(A(\frac{wt}{w+t})a,t^{-1})A(w)b,w^{-1})(w+t)^{-k-n-2}t^k w^{l+k+2n+2}\\
&= Res_t \sum_{p,q,r,s}(-1)^{p-n-1}(C_r(a)_{(p)}C_s(b))_{(q)}i_{w,t}(w+t)^{k+r-2\Delta_a+p+1}t^{n-p-1}w^{l+q+s+p+1-2\Delta_b}\\
&= (-1)^{n+1}Res_t\sum_p(-1)^p Y_M((A(w+t)a_{(p)}A(w)b),w^{-1})i_{w,t}(w+t)^{k+p+1}t^{n-p-1}w^{l+p}\\
&= (-1)^{n+1}Res_t(i_{w,t}Y_M(Y((A(w+t)a,\frac{-t}{w(w+t)})A(w)b),w^{-1})(w+t)^k t^n w^l).
\end{aligned}$$

Thus we are reduced to prove that

$$\begin{aligned}
&Res_t(i_{w,t}Y_M(Y(A(t+w)a,\frac{-t}{w(t+w)})A(w)b,w^{-1})w^l(t+w)^k t^n\\
&= p(a,b)Res_t(Y_M(A(w)Y(a,t)b,w^{-1})i_{w,t}(w+t)^k t^n w^l),
\end{aligned}$$

or equivalently

$$p(a,b)A(w)Y(a,t)b = i_{w,t}Y\left(A(t+w)a,\frac{-t}{(t+w)w}\right)A(w)b, \tag{3.23}$$

which is equation (3.7) with $A(w)b$ in place of b. Claim (a) follows.

Let us now check (b). We need only to check that the map $m\mapsto f_m\in(M^\dagger)^\dagger$ where $\langle f_m,m'\rangle=\overline{\langle m',m\rangle}$ is a V–module isomorphism. The map is clearly bijective since we are assuming $\dim M_n<\infty$. Now

$$\begin{aligned}
\langle (Y_{(M^\dagger)^\dagger}(v,z)f_m,m'\rangle &= \langle f_m, Y_{M^\dagger}(A(z)v,z^{-1})m'\rangle = \overline{\langle Y_{M^\dagger}(A(z)v,z^{-1})m',m\rangle}\\
&= \overline{\langle m', Y_M(A(z)A(z^{-1})v),z)m\rangle}.
\end{aligned}$$

Now use (3.8) to get

$$\langle (Y_{(M^\dagger)^\dagger}(v,z)f_m,m'\rangle = \overline{\langle m',Y_M(v,z)m\rangle} = \langle f_{Y_M(v.z)m},m'\rangle.$$

□

4. Invariant Hermitian forms on conformal vertex algebras

Let V be a conformal vertex algebra. By a Hermitian form on V we mean a map $(\cdot\,,\cdot):V\times V\to\mathbb{C}$ conjugate linear in the first argument and linear in the second, such that $(v_1,v_2)=\overline{(v_2,v_1)}$ for all $v_1,v_2\in V$.

Let ϕ be a conjugate linear parity preserving involution of V. Consider the conjugate linear operator (cf (3.3))

$$g=((-1)^{L_0}\sigma^{1/2})^{-1}\phi. \tag{4.1}$$

By (3.4), we have that $g^2=I$. Obviously g satisfies the hypothesis of Definition 3.2. Two instances of such a situation are the following.

(1) Recall from Remark 2.2 that, if V is a vertex operator superalgebra, then $(-1)^{2\Delta_a}=(-1)^{p(a)}$ for all $a\in V$. Set $s(a)=\Delta_a+\frac{1}{2}p(a)$ and note that in this case $s(a)$ is an

integer. Then

$$\begin{aligned}\Delta_a + 2\Delta_a^2 &= s(a) - \frac{1}{2}p(a) + 2(s(a) - \frac{1}{2}p(a))^2 \\ &= s(a) - \frac{1}{2}p(a) + 2s(a)^2 - 2s(a)p(a) + \frac{1}{2}p(a)^2.\end{aligned}$$

As $p(a) - p(a)^2 = 0$ and $p(a)$, $s(a)$ are integers, we see that

$$\Delta_a + 2\Delta_a^2 \equiv s(a) \mod 2$$

so that

$$g(a) = e^{-\pi\sqrt{-1}(\Delta_a + \frac{1}{2}p(a))}\phi(a) = (-1)^{s(a)}\phi(a) = (-1)^{\Delta_a + 2\Delta_a^2}\phi(a)$$

hence, if V is a vertex operator superalgebra,

$$g = (-1)^{L_0 + 2L_0^2}\phi. \tag{4.2}$$

(2) The vertex algebra of symplectic bosons provides an example of a conformal vertex algebra that is not a vertex operator superalgebra, where our definition applies. Let $R_{\mathbb{R}}$ be a real finite dimensional even vector space equipped with a bilinear non-degenerate symplectic form $\langle \cdot, \cdot \rangle$. Let $R = \mathbb{C} \otimes R_{\mathbb{R}}$. Equip R with the structure of a nonlinear conformal algebra with λ-bracket given by

$$[a_\lambda b] = \langle a, b \rangle.$$

Let V be the corresponding universal enveloping vertex algebra. The Virasoro vector is

$$L = \frac{1}{2}\sum :T(a^i)a_i:$$

with $\{a_i\}$, $\{a^i\}$ dual bases of R. The elements in R are primary of conformal weight $\frac{1}{2}$. Let $\phi(r) = \bar{r}$, where $\bar{r}$ is complex conjugation with respect to $R_{\mathbb{R}}$. Then, clearly,

$$[\phi(a)_\lambda \phi(b)] = \overline{\langle a, b \rangle},$$

hence ϕ extends to a conjugate linear involution of V. In this case

$$g(r) = -\sqrt{-1}\bar{r}, \ r \in R.$$

The following definition first appeared in [3] for vertex operator superalgebras, generalizing the definition, given in [7], for vertex operator algebras.

Definition 4.1. Let ϕ be a conjugate linear involution of a conformal vertex algebra V. Choose g as in Definition 3.2 and define $A(z)$ by (3.6). A Hermitian form $(\cdot\,,\cdot)$ on V is said to be *ϕ–invariant* if, for all $a \in V$,

$$(v, Y(a,z)u) = (Y(A(z)a, z^{-1})v, u), \quad u, v \in V. \tag{4.3}$$

Remark 4.2. If $v \in V$ is quasi-primary, then, due to (3.14), (4.3) becomes

$$(v, a_n u) = (g(a)_{-n}(v), u), \quad u, v \in V. \tag{4.4}$$

The statement of the main result of [15] can be extended to our setting as follows.

Theorem 4.3. *In the setting of Definition 4.1, the space of ϕ–invariant Hermitian forms on V is linearly isomorphic to the set of conjugate linear functionals $F \in V_0^\dagger$ such that $\langle F, L_1 V_1 \rangle = 0$ and $\langle F, g(v) \rangle = \overline{\langle F, v \rangle}$ for all $v \in V_0$.*

The proof is the same as in [15, Theorem 3.1]. In the following we simply check that the argument also works in our modified setting. Recall that an element m in a V–module M is called vacuum–like if $a_{(n)}m = 0$ for all $n \geq 0$ and all $a \in V$. By Proposition 2.3 of [15], a vector $m \in M$ is vacuum-like if and only if $L_{-1}m = 0$, i.e. the space of vacuum–like vectors is the space $M^{L_{-1}}$ of L_{-1}–invariants; moreover, if m is a vacuum–like vector in M, then $Y(u, z)m = e^{zL_{-1}}u_{(-1)}m$.

Consider the map

$$\Psi : Hom_V(V, M) \to M, \quad \Psi(\psi) = \psi(\mathbf{1}).$$

By Proposition 3.4 of [15], for any V–module M, Ψ is an isomorphism between $\mathrm{Hom}_V(V, M)$ and the space $M^{L_{-1}}$.

Proof of Theorem 4.3. Assume that $(\cdot\,,\cdot)$ is a ϕ–invariant Hermitian form on V. Note that, since $g(L) = L$, (4.4) implies that $(L_0 v, w) = (v, L_0 w)$. In particular the eigenspaces of L_0 are orthogonal. Define $F \in V_0^\dagger$ by $\langle F, v\rangle = (v, \mathbf{1})$. Then $(\cdot\,,\cdot)$ is uniquely determined by F, since, letting $u = \mathbf{1}$ and taking $Res_z z^{-1}$ of both sides of (4.3), we obtain

$$(v, a) = Res_z z^{-1}(Y(A(z)a, z^{-1})v, \mathbf{1}).$$

By Remark 4.2,

$$(L_1 v, \mathbf{1}) = (v, L_{-1}\mathbf{1}) = 0,\ (\mathbf{1}, L_1 v) = (L_{-1}\mathbf{1}, v) = 0, \tag{4.5}$$

hence, since $L_{-1}\mathbf{1} = 0$, we see that $\langle F, L_1 V_1\rangle = 0$.

Next we prove that, if $a \in V$, then

$$(g(a), \mathbf{1}) = (\mathbf{1}, a), \tag{4.6}$$

Since the form is Hermitian, we have $(\mathbf{1}, a) = \overline{(a, \mathbf{1})}$, so that (4.6) implies $\langle F, g(a)\rangle = \overline{\langle F, a\rangle}$. To prove (4.6) we observe that, since $g(L) = L$, g preserves the L_0–eigenspace decomposition. Since the eigenspaces of L_0 are orthogonal, we have that (4.6) is satisfied if $\Delta_a \neq 0$. We can therefore assume that $\Delta_a = 0$, so that

$$\begin{aligned}(\mathbf{1}, a) = Res_z z^{-1}(\mathbf{1}, Y(a, z)\mathbf{1}) &= Res_z z^{-1}(Y(A(z)a, z^{-1})\mathbf{1}, \mathbf{1})\\ &= \sum_r (\tfrac{1}{r!}(L_1^r g(a))_{\Delta_a}\mathbf{1}, \mathbf{1})\\ &= \sum_r \tfrac{1}{r!}((L_1^r g(a))_0\mathbf{1}, \mathbf{1}).\end{aligned}$$

By (4.5), in order to prove (4.6), we need only to prove that

$$(L_1^r g(a))_0\mathbf{1} \in L_1 V_1,\ r \geq 1, a \in V_0. \tag{4.7}$$

We prove by induction on r that

$$(L_1^r b)_0\mathbf{1} \in L_1 V_1,\ r \geq 1, b \in V_0.$$

If $r = 1$, then

$$[L_1, b_{-1}] = \sum_{j\in\mathbb{Z}_+} \binom{2}{j} (L_{(j)}b)_0 = (L_{-1}a)_0 + 2\Delta_b b_0 + (L_1 b)_0.$$

Since $\Delta_b = 0$, $(L_{-1}b)_0 = 0$, so

$$(L_1 b)_0\mathbf{1} = L_1(a_{-1}\mathbf{1}) - b_{-1}(L_1\mathbf{1}) = L_1(b_{-1}\mathbf{1}) \in L_1 V_1.$$

If $r>1$, then

$$\begin{aligned}[L_1,(L_1^{r-1}b)_{-1}] &= \sum_{j\in\mathbb{Z}_+}\binom{2}{j}(L_{(j)}(L_1^{r-1}b))_0\\ &= L_{-1}(L_1^{r-1}b)_0 - 2(r-1)(L_1^{r-1}b)_0 + (L_1^r b)_0\\ &= -(r-1)(L_1^{r-1}b)_0 + (L_1^r b)_0\,,\end{aligned}$$

so

$$\begin{aligned}(L_1^r b)_0\mathbf{1} &= L_1((L_1^{r-1}b)_{-1}\mathbf{1}) - (L_1^{r-1}b)_{-1}L_1\mathbf{1} + (r-1)(L_1^{r-1}b)_0\mathbf{1}\\ &= L_1((L_1^{r-1}b)_{-1}\mathbf{1}) + (r-1)(L_1^{r-1}b)_0\mathbf{1}.\end{aligned}$$

The claim now follows by the induction hypothesis.

We now prove the converse statement. Consider V as a $\Gamma/\mathbb{Z}$–graded vertex algebra with $\Gamma=\mathbb{Z}$ and the trivial grading $\Upsilon(\mathbb{Z})=V$. Then the state–field correspondence defines on V the structure of a Υ–twisted positive energy module. Since Υ is clearly compatible with ϕ, by Thoerem 3.6, we have a Υ–twisted module structure on $V^\dagger$. Fix $F\in V_0^\dagger$ which vanishes on L_1V_1. Then F is a vacuum–like vector in $V^\dagger$. In particular the map $\Phi_F:V\to V^\dagger$ defined by $\Phi_F(v)=v^\dagger_{(-1)}F$ is a V–module homomorphism. Here and in what follows we write for simplicity $a^\dagger_n$ instead of $a^{V^\dagger}_n$. Define

$$(u,v)=\langle v^\dagger_{(-1)}F,u\rangle=\langle\Phi_F(v),u\rangle.$$

Let us check that this form is ϕ–invariant:

$$\begin{aligned}(v,Y(a,z)u) &= \langle\Phi_F(Y(a,z)u),v\rangle\\ &= \langle Y_{V^\dagger}(a,z)\Phi_F(u),v\rangle\\ &= \langle\Phi_F(u),Y(A(z)a,z^{-1})v\rangle\\ &= (Y(A(z)a,z^{-1})v,u).\end{aligned}$$

It remains to show that, if $\langle F,a\rangle=\overline{\langle F,g(a)\rangle}$, then the form is Hermitian. Since the form is ϕ–invariant, by (4.6),

$$\overline{(a,\mathbf{1})}=\overline{\langle F,a\rangle}=\langle F,g(a)\rangle=(g(a),\mathbf{1})=(\mathbf{1},a).$$

We can now check that the form is Hermitian:

$$\begin{aligned}\overline{(u,v)} &= Res_z z^{-1}\overline{(u,Y(v,z)\mathbf{1})}\\ &= \mathrm{Res}_z z^{-1}\overline{(Y(A(z)v,z^{-1})u,\mathbf{1})}\\ &= Res_z z^{-1}(\mathbf{1},Y(A(z)v,z^{-1})u)\\ &= Res_z z^{-1}(Y(A(z)A(z^{-1})v,z)\mathbf{1},u)\\ &= Res_z z^{-1}(Y(v,z)\mathbf{1},u)=(v,u),\end{aligned}$$

where, in the last step, we used (3.8). □

Remark 4.4. Note that we didn't use in the proof the assumptions that $V_0=\mathbb{C}\mathbf{1}$ and that $\dim V_n=0$ for $n<0$. However, if $V_0=\mathbb{C}\mathbf{1}$, then Theorem 4.3 implies that there exists a non–zero ϕ–invariant Hermitian form on V if and only if V_1 consists of quasiprimary elements, and for this form $(\mathbf{1},\mathbf{1})\neq 0$. The last statement follows observing that the eigenspaces of L_0 are orthogonal to each other and the kernel of a ϕ–invariant Hermitian form is an ideal.

So if $(\mathbf{1},\mathbf{1})=0$, then $\mathbf{1}$ lies in the kernel, hence the kernel of the form is V. Also, such a Hermitian form, satisfying $(\mathbf{1},\mathbf{1})=1$, is unique.

Lemma 4.5. *Let M be a module over $sl(2):=span\{e,h,f\}$, such that h is diagonalisable with finite–dimensional eigenspaces and negative eigenvalues. Then M is a direct sum of Verma modules.*

Proof. Since the sum of h–eigenspaces with eigenvalues congruent mod 2 is a submodule, we may assume that all eigenvalues of h are congruent mod 2. Since the h–eigenspaces are finite–dimensional, U decomposes as the direct sum of the generalized eigenspaces for the Casimir operator Ω of $sl(2)$. We can therefore assume that Ω has only one eigenvalue. Any irreducible subquotient of M has negative highest weight, say n, and the eigenvalue of Ω is $\frac{1}{2}n^2+n$ on it. Hence if two irreducible subquotients with non–equal highest weights have the same Ω–eigenvalue, the sum of these highest weights is -2. Since all eigenvalues of h are negative and congruent mod 2, we deduce that all irreducible subquotients have the same highest weight n. So, on the space M^e of e–invariants (which is non-zero since the set of eigenvalues of h is bounded above), h has one eigenvalue n, and the same is true for any quotient of M. But on $N=M/U(sl(2))M^e$, h has eigenvalues strictly smaller that n, hence $N^e=0$ and $M=U(sl_2)M^e$. Since $n<0$, any vector from M^e generates an irreducible Verma module, so M is a direct sum of Verma modules with highest weight n. □

Proposition 4.6. *Let V be a conformal vertex algebra such that $L_1V_1=0$, i.e. V_1 consists of quasiprimary vectors. Let $\{v_1,v_2,\ldots\}$ be a minimal system of strong generators, which includes L, and consists of eigenvectors for L. Then, summing to the v_i elements from $L_{-1}V$, we can make these generators quasiprimary.*

Proof. By Lemma 4.5, applied to $U=\bigoplus_{n>0}V_n$ and $f=L_{-1},h=-2L_0,e=-\frac{1}{2}L_1$, we get

$$V=\mathbb{C}\mathbf{1}\oplus\sum_i M_i, \tag{4.8}$$

where the M_i are Verma modules for $sl(2)$ with highest weight vectors quasiprimary elements. We proceed by induction on the conformal weight of a generator. If the conformal weight is $\frac{1}{2}$ or 1 there is nothing to prove. Take now a generator v_i whose conformal weight is strictly greater than 1. By (4.8), we can write $v_i=v_i'+L_{-1}b$, where v_i' is quasiprimary and non-zero, due to minimality. By inductive assumption b lies in the subalgebra generated by quasiprimary generators. Hence we can replace v_i by v_i'. □

Recall (cf. [11]) that, since $V_0=\mathbb{C}\mathbf{1}$, one can define the expectation value $\langle v\rangle$ of v by the equation $P_{V_0}(v)=\langle v\rangle\mathbf{1}$ where P_{V_0} is the projection onto V_0 with respect to the decomposition $V=V_0\oplus(\sum_{n\neq 0}V_n)$.

Corollary 4.7. *Suppose that V is a conformal vertex algebra such that V_1 consists of quasiprimary vectors. Let ϕ be a conjugate linear involution of V. Then there exists a unique ϕ–invariant Hermitian form $(\cdot\,,\,\cdot)$ on V such that $(\mathbf{1},\mathbf{1})=1$. Moreover for any collection $\{U^i\mid i\in I\}$ of quasiprimary elements that strongly generate V (it exists by Proposition 4.6) we have*

$$\begin{aligned}&\Big((U^{i_1}_{j_1})^{m_1}\cdots(U^{i_t}_{j_t})^{m_t}\mathbf{1},(U^{i_1'}_{j_1'})^{m_1'}\cdots(U^{i_r'}_{j_r'})^{m_r'}\mathbf{1}\Big)\\&=\Big\langle((g(U^{i_t})_{-j_t})^{m_t}\cdots(g(U^{i_1})_{-j_1})^{m_1}(U^{i_1'}_{j_1'})^{m_1'}\cdots(U^{i_r'}_{j_r'})^{m_r'}\mathbf{1}\Big\rangle.\end{aligned} \tag{4.9}$$

Proof. Since $L_1V_1 = \{0\}$, the first statement follows from Theorem 4.3.

To prove the second statement, note that, by (4.4), for a quasiprimary element U, we have $(g(U)_n v, w) = (v, U_{-n}w)$ and

$$(U_n v, w) = \overline{(w, U_n v)} = \overline{(g(U)_{-n}w, v)} = (v, g(U)_{-n}w).$$

Since $V_0 = \mathbb{C}\mathbf{1}$, we have $(\mathbf{1}, a) = \langle a\rangle$, hence formula (4.9) follows. □

Definition 4.8. If the Hermitian form (4.9) is positive definite, the vertex algebra V is called *unitary.*

Lemma 4.9. *Let V be a conformal vertex algebra and let ϕ be a conjugate linear involution on V. If there is a ϕ–invariant positive definite Hermitian form on V and $a \in V$ is a non-zero quasiprimary element such that $\phi(a) = a$ then*

$$\langle a_{\Delta_a} a_{-\Delta_a}\mathbf{1}\rangle \in \mathbb{R}\backslash\{0\} \qquad \text{if } (-1)^{2L_0}\sigma(a) = a,$$
$$\langle a_{\Delta_a} a_{-\Delta_a}\mathbf{1}\rangle \in \sqrt{-1}\mathbb{R}\backslash\{0\} \qquad \text{if } (-1)^{2L_0}\sigma(a) = -a.$$

Proof. Since

$$(a, a) = e^{-\frac{\pi}{2}\sqrt{-1}(2\Delta_a + p(a))}\langle a_{\Delta_a} a_{-\Delta_a}\mathbf{1}\rangle,$$

and $(a, a) > 0$, we see that $\langle a_{\Delta_a} a_{-\Delta_a}\mathbf{1}\rangle$ is real and non-zero if $(-1)^{2\Delta_a}\sigma(a) = a$, while it is purely imaginary and non-zero otherwise. □

In conclusion of this section we discuss invariant Hermitian forms on tensor products of vertex algebras. Recall from [11] that if V, W are vertex algebras, their tensor product is the vertex algebra having $V \otimes W$ as space of states, $\mathbf{1}\otimes\mathbf{1}$ as vacuum vector and $T\otimes I + I\otimes T$ as translation operator. The state–field correspondence is given by

$$Y(u \otimes v, z) = Y(u, z) \otimes Y(v, z).$$

If V, W are conformal vertex algebras, also $V \otimes W$ is conformal: its Virasoro vector is $L = L_V \otimes \mathbf{1} + \mathbf{1} \otimes L_W$.

Let ϕ_V, ϕ_W be conjugate linear involutions of V, W and set

$$g_V = ((-1)^{(L_V)_0}\sigma_V^{1/2})^{-1}\phi_V, \quad g_W = ((-1)^{(L_W)_0}\sigma_W^{1/2})^{-1}\phi_W,$$
$$g = g_V \otimes g_W, \quad \phi = \phi_V \otimes \phi_W.$$

Observe that

$$\phi = (-1)^{L_0}\sigma^{1/2}g.$$

Moreover

$$A(z) = e^{zL_1}z^{-2L_0}g = (e^{z(L_V)_1} \otimes e^{z(L_W)_1})(z^{-2(L_V)_0} \otimes z^{-2(L_W)_0})(g_V \otimes g_W) = A_V(z) \otimes A_W(z).$$

If $(\,.\,,\,.\,)_V$, $(\,.\,,\,.\,)_W$ are invariant Hermitian forms on V, W, respectively, we can induce an invariant Hermitian form $(\,.\,,\,.\,)_{V\otimes W}$ on $V \otimes W$ by setting

$$(v_1 \otimes w_1, v_2 \otimes w_2)_{V\otimes W} = (v_1, v_2)_V (w_1, w_2)_W.$$

Indeed,

$$\begin{aligned}
&(v_1 \otimes v_2, Y(a \otimes b, z)(w_1 \otimes w_2)) = (v_1, Y(a, z)w_1)_V (v_2, Y(b, z)(w_2))_W \\
&= (Y(A_V(z)a, z^{-1})v_1, w_1)_V (Y(A_W(z)b, z^{-1})v_2, w_2)_W \\
&= (Y(A_V(z) \otimes A_W(z)(a \otimes b), z^{-1})(v_1 \otimes v_2), w_1 \otimes w_2) \\
&= (Y(A(z)(v_1 \otimes v_2), w_1 \otimes w_2).
\end{aligned}$$

5. Examples of invariant Hermitian forms

In this Section we apply Corollary 4.7 to fermionic, bosonic, affine, and lattice vertex algebras.

5.1. **Superfermions.** Consider a superspace $A = A_{\bar{0}} \oplus A_{\bar{1}}$ endowed with a non-degenerate even skew-supersymmetric bilinear form $(\,.\,|\,.\,)$. Let $V(A)$ be the universal vertex algebra of the Lie conformal superalgebra $A \oplus \mathbb{C}K$ with λ-bracket

$$[a_\lambda b] = (a|b)K,$$

K being an even central element. Let F be the fermionic vertex algebra:

$$F = V(A)/(K - \mathbf{1}).$$

Let ϕ be a conjugate linear involution of A such that

$$(\phi(a)|\phi(b)) = \overline{(a|b)}.$$

By setting $\phi(K) = K$ we can extend ϕ to a conjugate linear involution of $A \oplus \mathbb{C}K$. Indeed

$$[\phi(a)_\lambda \phi(b)] = (\phi(a)|\phi(b)) = \overline{(a|b)}K = \phi((a|b)K).$$

This implies that ϕ extends to a conjugate linear involution of $V(A)$, hence, since $\phi(K-\mathbf{1}) = K - \mathbf{1}$, to an involution of F.

Fix a basis $\{a^i\}$ of A and let $\{b^i\}$ be its dual basis w.r.t. $(\,.\,|\,.\,)$ (i.e. $(a^i|b^j) = \delta_{i,j}$). The Virasoro vector is [11]

$$L = \frac{1}{2}\sum_{i=1}^{n} :(Tb^i)a^i: \ . \tag{5.1}$$

It is easy to see that $\phi(L) = L$. We embed A in F by identifying v with $:v\mathbf{1}:$. It is easily checked that $v \in A$ is a primary element of F of conformal weight $1/2$. Set

$$g_A = ((-1)^{L_0}\sigma^{1/2})^{-1}\phi. \tag{5.2}$$

By (3.4), we have that $g_A^2 = I$. Note that

$$g_A(a) = -\sqrt{-1}\phi(a),\ a \in A_{\bar{0}}, \quad g_A(a) = -\phi(a),\ a \in A_{\bar{1}}. \tag{5.3}$$

The set $\{a^i\}$ strongly and freely generates F. This means that, if we order $(-\frac{1}{2} - \mathbb{Z}_+) \times \{1, \dots, m+n\}$ lexicographically, then the set

$$B = \bigcup_r \{(a^{i_1}_{j_1})^{h_1} \cdots (a^{i_r}_{j_r})^{h_r}\mathbf{1} \mid (j_1, i_1) < \cdots < (j_r, i_r), h_s = 1 \text{ if } p(a^{i_s}) = 1\}$$

is a basis of F. With this choice one easily checks that

$$F_0 = \mathbb{C}\mathbf{1}, \quad F_1 = span_{\mathbb{C}}(\{:a^i a^j:\}).$$

Since, by Wick formula [11],

$$\begin{aligned}
[L_\lambda :a^i a^j:] &=: T(a^i)a^j: + :a^i T(a^j): + \lambda :a^i a^j: + \int_0^\lambda ([T(a^i)_\mu a^j] + \frac{1}{2}\lambda[a^i_\mu a^j])d\mu \\
&= T(:a^i a^j:) + \lambda :a^i a^j: - \frac{1}{2}\lambda^2(a^i|a^j) + \frac{1}{2}\lambda^2(a^i|a^j) \\
&= T(:a^i a^j:) + \lambda :a^i a^j:,
\end{aligned}$$

we see that $L_1(F_1) = \{0\}$, hence Corollary 4.7 applies. Let $(\cdot\,,\cdot)$ be the unique invariant Hermitian form on F such that $(\mathbf{1},\mathbf{1}) = 1$. By (5.3) and (4.4), the invariance amounts to

$$(v_j a, b) = -\sqrt{-1}(a, \phi(v)_{-j} b),\ v \in A_{\bar 0},\quad (v_j a, b) = -(a, \phi(v)_{-j} b),\ v \in A_{\bar 1}$$

for all $a, b \in F$, $j \in \frac{1}{2} + \mathbb{Z}$.

We now discuss the unitarity of F. Assume that F is unitary and $A_{\bar 0} \neq 0$. Choose $a \neq 0$ in $A_{\bar 0}$; we can assume $\phi(a) = a$. Then, by Lemma 4.9, $\langle a_{1/2} a_{-1/2}\mathbf{1}\rangle \neq 0$ but $\langle a_{1/2} a_{-1/2}\mathbf{1}\rangle = (a|a) = 0$. It follows that, if F is unitary, then $A = A_{\bar 1}$. Set $A_{\mathbb{R}} = \{a \in A \mid \phi(a) = -a\}$. Then, if $a \in A_{\mathbb{R}}$,

$$0 < (a, a) = \langle a_{1/2} a_{-1/2}\mathbf{1}\rangle = (a|a),$$

so $(\,.\,|\,.\,)_{A_{\mathbb{R}} \times A_{\mathbb{R}}}$ must be positive definite. In such a case, choose $\{a^i\}$ to be an orthonormal basis of $A_{\mathbb{R}}$. It can be checked (say by induction on r) that

$$\left\langle a^{i_t}_{-j_t} \cdots a^{i_1}_{-j_1} a^{i'_1}_{j'_1} \cdots a^{i'_r}_{j'_r}\mathbf{1}\right\rangle = \delta_{r,t} \prod_{s=1}^{r} \delta_{i_s, i'_s} \prod_{s=1}^{r} \delta_{j_s, j'_s}$$

so the invariant Hermitian form is the form defined by declaring the basis B to be orthonormal. Hence F is a unitary conformal vertex algebra if and only if A is purely odd.

5.2. **Superbosons.** Let $\mathfrak{h}$ be a vector superspace equipped with a supersymmetric even bilinear form $(\,.\,|\,.\,)$. Let $V(\mathfrak{h})$ be the universal vertex algebra of the Lie conformal superalgebra $\mathfrak{h} \oplus \mathbb{C}K$ with λ-bracket

$$[v_\lambda w] = \lambda(v|w)K,$$

K being an even central element. Let $M(\mathfrak{h})$ be the vertex algebra:

$$M(\mathfrak{h}) = V(\mathfrak{h})/(K - \mathbf{1}).$$

Let ϕ be a conjugate linear involution of $\mathfrak{h}$. As in the previous example, if

$$(\phi(a)|\phi(b)) = \overline{(a|b)}.$$

we can extend ϕ to a conjugate linear involution of $M(\mathfrak{h})$.

Fix a basis $\{a^i\}$ of $\mathfrak{h}$ and let $\{b^i\}$ be its dual basis w.r.t. $(\,.\,|\,.\,)$ (i.e. $(a^i|b^j) = \delta_{i,j}$). The Virasoro vector is

$$L = \tfrac{1}{2}\sum_{i=1}^{n} :b^i a^i: \ . \tag{5.4}$$

It is easy to see that $\phi(L) = L$.

We embed $\mathfrak{h}$ in $M(\mathfrak{h})$ by identifying h with $:h\mathbf{1}:$. It is easily checked that $h \in \mathfrak{h}$ is a primary element of $M(\mathfrak{h})$ of conformal weight 1.

Set

$$g_{\mathfrak{h}} = ((-1)^{L_0}\sigma^{1/2})^{-1}\phi. \tag{5.5}$$

By (3.4), we have that $g_{\mathfrak{h}}^2 = I$. Note that

$$g_{\mathfrak{h}}(a) = -\phi(a),\ a \in A_{\bar 0},\quad g_{\mathfrak{h}}(a) = \sqrt{-1}\phi(a),\ a \in A_{\bar 1}. \tag{5.6}$$

As in the previous example we can apply Corollary 4.7, thus there is a unique ϕ–invariant Hermitian form $(\cdot\,,\cdot)$ on $M(\mathfrak{h})$ such that $(\mathbf{1},\,\mathbf{1}) = 1$.

We now discuss the unitarity of $M(\mathfrak{h})$. Assume that $M(\mathfrak{h})$ is unitary and $\mathfrak{h}_{\bar 1} \neq 0$. Choose $h \neq 0$ in $\mathfrak{h}_{\bar 1}$; we can assume $\phi(h) = h$. Then, by Lemma 4.9, $\langle h_1 h_{-1}\mathbf{1}\rangle \neq 0$ but $\langle h_1 h_{-1}\mathbf{1}\rangle = (h|h) = 0$. It follows that, if $M(\mathfrak{h})$ is unitary, then $\mathfrak{h} = \mathfrak{h}_{\bar 0}$. If this is the case, set $\mathfrak{h}_{\mathbb{R}} = \{h \in \mathfrak{h} \mid \phi(h) = -h\}$. Then, as in Subsection 5.1, we must have that $(\,.\,|\,.\,)_{\mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}}$ is positive definite.

We choose an orthonormal basis $\{a^i\}$ of $\mathfrak{h}_{\mathbb{R}}$, ; the ϕ–invariant Hermitian form is therefore given by

$$\Big((a^{i_1}_{j_1})^{m_1}\cdots(a^{i_t}_{j_t})^{m_t}\mathbf{1},(a^{i'_1}_{j'_1})^{m'_1}\cdots(a^{i'_r}_{j'_r})^{m'_r}\mathbf{1}\Big)=\Big\langle(a^{i_t}_{-j_t})^{m_t}\cdots(a^{i_1}_{-j_1})^{m_1}(a^{i'_1}_{j'_1})^{m'_1}\cdots(a^{i'_r}_{j'_r})^{m'_r}\mathbf{1}\Big\rangle.$$

If we order $(-\mathbb{N})\times\{1,\dots,\dim\mathfrak{h}\}$ lexicographically, then the set

$$B=\bigcup_r\{a^{i_1}_{j_1}\cdots a^{i_r}_{j_r}\mathbf{1}\mid(j_1,i_1)<\cdots<(j_r,i_r)\}$$

is a basis of $M(\mathfrak{h})$. As in Example 5.1, one can check that the basis B is orthogonal; moreover the norm of each element is positive, so $M(\mathfrak{h})$ is a unitary vertex operator superalgebra, if and only if $\mathfrak{h}$ is purely even.

5.3. **Affine vertex algebras.** Let $\mathfrak{g}$ be a simple Lie algebra or a basic classical simple finite–dimensional Lie superalgebra and let $(\,.\,|\,.\,)$ be a supersymmetric non-degenerate even invariant bilinear form on $\mathfrak{g}$.

We normalize the form $(\,.\,|\,.\,)$ on $\mathfrak{g}$ by choosing an even highest root θ of $\mathfrak{g}$ as in [13] or [1], and requiring $(\theta|\theta)=2$. If $\mathfrak{g}=D(2,1,a)$, we assume $a\in\mathbb{R}$.

Let ϕ be a conjugate linear involution of $\mathfrak{g}$. We assume that

$$(\phi(x)|\phi(y))=\overline{(x|y)},$$

noting that, if $\mathfrak{g}$ is a Lie algebra, then the above assumption always holds.

Let $\mathrm{Cur}\,\mathfrak{g}=\mathfrak{g}\oplus\mathbb{C}K$ be the current Lie conformal algebra associated to $\mathfrak{g}$ [11]. We extend ϕ to $\mathrm{Cur}\,\mathfrak{g}$ by setting $\phi(K)=K$. Since

$$[\phi(x)_\lambda\phi(y)]=[\phi(x),\phi(y)]+\lambda(\phi(x)|\phi(y)K=\phi([x,y])+\lambda\overline{(x|y)}K=\phi([x_\lambda y]),$$

ϕ is a conjugate linear involution of $\mathrm{Cur}\,\mathfrak{g}$, hence we can extend ϕ to a conjugate linear involution of the universal enveloping vertex algebra $V(\mathfrak{g})$ of $\mathrm{Cur}\,\mathfrak{g}$.

Choosing $k\in\mathbb{R}$, we note that $\phi(K-k\mathbf{1})=K-k\mathbf{1}$, so ϕ pushes down to a conjugate linear involution of the the universal affine vertex algebra of level k.

We identify $a\in\mathfrak{g}$ with $:a\mathbf{1}:\in V^k(\mathfrak{g})$. Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$, i.e. the eigenvalue of the Casimir operator $\sum_i b^ia^i$ on $\mathfrak{g}$ divided by 2, where $\{a^i\}$ and $\{b^i\}$ are dual bases of $\mathfrak{g}$, i. e. $(a^i|b^j)=\delta_{ij}$.

A Virasoro vector is provided by the Sugawara construction (defined for $k\neq-h^\vee$), see e.g. [11]:

$$L^{\mathfrak{g}}=\frac{1}{2(k+h^\vee)}\sum_{i=1}^{\dim\mathfrak{g}}:b^ia^i:\ . \tag{5.7}$$

It is easy to see that $\phi(L^{\mathfrak{g}})=L^{\mathfrak{g}}$ provided that $k\in\mathbb{R}$.

Set

$$g_{\mathfrak{g}}=((-1)^{L_0}\sigma^{1/2})^{-1}\phi.$$

Explicitly

$$g_{\mathfrak{g}}(a)=-\phi(a),\ a\in\mathfrak{g}_{\bar 0},\quad g_{\mathfrak{g}}(a)=\sqrt{-1}\phi(a),\ a\in\mathfrak{g}_{\bar 1}.$$

It is well known that $a\in\mathfrak{g}$ is a primary element of $V^k(\mathfrak{g})$ of conformal weight 1 (see e.g. [11]). Moreover, the set $\{a^i\}$ strongly and freely generates $V^k(\mathfrak{g})$. It follows that $V^k(\mathfrak{g})_0=\mathbb{C}\mathbf{1}$ and

$L_1V^k(\mathfrak{g})_1 = 0$. By Corollary 4.7, there exists a unique ϕ–invariant Hermitian form on $V^k(\mathfrak{g})$, given by

$$\Big((a^{i_1}_{j_1})^{m_1}\cdots(a^{i_t}_{j_t})^{m_t}\mathbf{1},(a^{i'_1}_{j'_1})^{m'_1}\cdots(a^{i'_r}_{j'_r})^{m'_r}\mathbf{1}\Big)$$
$$=\Big\langle(g_{\mathfrak{g}}(a^{i_t})_{-j_t})^{m_t}\cdots(g_{\mathfrak{g}}(a^{i_1})_{-j_1})^{m_1}(a^{i'_1}_{j'_1})^{m'_1}\cdots(a^{i'_r}_{j'_r})^{m'_r}\mathbf{1}\Big\rangle.$$

If $k\neq -h^\vee$, the vertex algebra $V^k(\mathfrak{g})$ has a unique simple quotient that we denote by $V_k(\mathfrak{g})$. We now discuss the unitarity of $V_k(\mathfrak{g})$. Assume that there is a conjugate linear involution ϕ such that the corresponding ϕ–invariant form on $V_k(\mathfrak{g})$ is positive definite. If $\mathfrak{g}$ is not a Lie algebra then there is $a\in\mathfrak{g}_{\bar 1}$, $a\neq 0$. Since ϕ is parity preserving we can assume $\phi(a)=a$. Then

$$(a,a)=(a_{-1}\mathbf{1},a_{-1}\mathbf{1})=\sqrt{-1}\langle a_1a_{-1}\mathbf{1}\rangle=\sqrt{-1}k(a|a)=0.$$

If $V_k(\mathfrak{g})$ is unitary, then a is in the maximal ideal of $V^k(\mathfrak{g})$, hence $k=0$ and $V_k(\mathfrak{g})=\mathbb{C}$.

Assume now that $\mathfrak{g}$ is a Lie algebra. Since ϕ is a conjugate linear involution of $V^k(\mathfrak{g})$ then $\phi_{|\mathfrak{g}}$ is a conjugate linear involution of $\mathfrak{g}$. Let $\mathfrak{g}_\mathbb{R}$ be the corresponding real form. As shown above, if $a\in\mathfrak{g}_\mathbb{R}$, then

$$0<(a,a)=(a_{-1}\mathbf{1},a_{-1}\mathbf{1})=-\langle a_1a_{-1}\mathbf{1}\rangle=-k(a|a),$$

hence $(\,.\,|\,.\,)_{|\mathfrak{g}_\mathbb{R}\times\mathfrak{g}_\mathbb{R}}$ is either positive or negative definite. Let $\overset{\circ}{\omega}_0$ be a compact conjugate linear involution of $\mathfrak{g}$ such that $\phi\overset{\circ}{\omega}_0=\overset{\circ}{\omega}_0\phi$. Let $\mathfrak{k}_\mathbb{R}$ be the corresponding compact real form. Then

$$\mathfrak{g}_\mathbb{R}=\mathfrak{g}_\mathbb{R}\cap\mathfrak{k}_\mathbb{R}\oplus\mathfrak{g}_\mathbb{R}\cap(\sqrt{-1}\mathfrak{k}_\mathbb{R}).$$

Since $(\,.\,|\,.\,)_{|\mathfrak{k}_\mathbb{R}\times\mathfrak{k}_\mathbb{R}}$ is negative definite and $\mathfrak{k}_\mathbb{R}\cap\mathfrak{g}_\mathbb{R}\neq\{0\}$, we see that $(\,.\,|\,.\,)_{|\mathfrak{g}_\mathbb{R}\times\mathfrak{g}_\mathbb{R}}$ is negative definite so $\phi=\overset{\circ}{\omega}_0$. Let ω_0 be the conjugate linear involution of the affinization $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ which extends $\overset{\circ}{\omega}_0$ as in §7.6 of [10]. Then the $\overset{\circ}{\omega}_0$–invariant Hermitian form on $V^k(\mathfrak{g})$ is defined by the property that

$$(a_jx,y)=-(x,\overset{\circ}{\omega}_0(a)_{-j}y),\ a\in\mathfrak{g}.$$

It follows from Theorem 11.7 of [10] combined with the formula for ω_0 given at page 103 of loc. cit., that the $\overset{\circ}{\omega}_0$–invariant Hermitian form on $V^k(\mathfrak{g})$ is positive semi–definite if and only if $k\in\mathbb{Z}_+$.

5.4. **Lattice vertex algebras.** Let Q be a positive definite integral lattice and V_Q be its associated lattice vertex superalgebra (see e.g. [11, §5.4]). Set $\mathfrak{h}=\mathbb{C}\otimes_\mathbb{Z}Q$. Recall that the free bosons vertex operator algebra $M(\mathfrak{h})$ embeds in $V_Q=\oplus_{\alpha\in Q}(M(\mathfrak{h})\otimes\mathbb{C}e^\alpha)$ with parity $p(M(\mathfrak{h})\otimes e^\alpha)=(\alpha|\alpha)\mod 2$. Let $\{a^1,\dots,a^l\}$ be an orthogonal basis of $\mathbb{R}\otimes_\mathbb{Z}Q$ and let $\{b^1,\dots,b^l\}$ be the dual basis of $\mathfrak{h}$ with respect to the form $(\,.\,|\,.\,)$ linearly extended from the form on Q. The Virasoro vector of V_Q is

$$L=\frac{1}{2}\sum_{i=1}^{l}:a^ib^i:.$$

There are primary elements $e^\alpha\in V_Q$, $\alpha\in Q$ of conformal weight $\frac12(\alpha|\alpha)$, such that a basis of V_Q is

$$B=\bigcup_{r,\alpha}\{a^{i_1}_{j_1}\cdots a^{i_r}_{j_r}e^\alpha\mid(j_1,i_1)<\cdots<(j_r,i_r)\},$$

where, as in Example 5.2, $(-\mathbb{N})\times\{1,\dots,\dim\mathfrak{h}\}$ is ordered lexicographically.

Following [7], we define a conjugate linear involution ϕ of V_Q by setting

$$\phi(a^{i_1}_{-j_1}\cdots a^{i_r}_{-j_r}e^{\alpha})=(-1)^r a^{i_1}_{-j_1}\cdots a^{i_r}_{-j_r}e^{-\alpha}. \tag{5.8}$$

It is immediate to see that $\phi(L)=L$. Since the conformal weight of e^α is $\frac{1}{2}(\alpha|\alpha)$, we have that $(-1)^{2L_0}\sigma=I$ so, if $g=((-1)^{L_0}\sigma^{1/2})^{-1}\phi$, then

$$g=(-1)^{L_0+2L_0^2}\phi.$$

We have

$$(V_Q)_0=\mathbb{C}\mathbf{1},\quad (V_Q)_1=span_{\mathbb{C}}(\{a^i\}\cup\{e^\alpha\mid(\alpha|\alpha)=2\}).$$

Since the a^i, as well as the e^α, are primary, we see that Corollary 4.7 applies. In particular the explicit expression for the ϕ–invariant Hermitian form is

$$\left((a^{i_1}_{j_1})^{m_1}\cdots(a^{i_t}_{j_t})^{m_t}e^\alpha,(a^{i'_1}_{j'_1})^{m'_1}\cdots(a^{i'_r}_{j'_r})^{m'_r}e^\beta\right)=\delta_{\alpha,-\beta}\left\langle((a^{i_t}_{-j_t})^{m_t}\cdots(a^{i_1}_{-j_1})^{m_1}(a^{i'_1}_{j'_1})^{m'_1}\cdots(a^{i'_r}_{j'_r})^{m'_r}\mathbf{1}\right\rangle.$$

As in Example 5.2 one can check that the basis B is orthogonal and consists of elements of positive norm, so V_Q is unitary.

6. Invariant Hermitian forms on modules

Let V be a conformal vertex algebra. Recall from (3.6) the definition of $A(z)$. We let

$$\omega(v)=A(1)v,\ v\in V. \tag{6.1}$$

Assume that V is $\Gamma/\mathbb{Z}$–graded and let Υ be a $\Gamma/\mathbb{Z}$–grading compatible with ϕ.

Proposition 6.1. *$\omega(J_\Upsilon)\subseteq J_{-\Upsilon}$ so ω induces a conjugate linear anti–isomorphism of associative algebras $\omega: Zhu_\Upsilon(V)\to Zhu_{-\Upsilon}(V)$. Moreover $\omega^2=Id$.*

Proof. By (2.18), we have

$$\omega\left(\sum_{j\in\mathbb{Z}_+}\binom{\gamma_a}{j}a_{(-2+\chi(a,b)+j)}b\right)=Res_w(w^{-2+\chi(a,b)}A(z)(Y((1+w)^{\gamma_a}a,w)b)_{|z=1}). \tag{6.2}$$

By (3.2)

$$\begin{aligned}&p(a,b)A(z)Y((1+w)^{\gamma_a}a,w)b=\\&=p(a,b)e^{zL_1}z^{-2L_0}gY((1+w)^{\gamma_a}a,w)b\\&=e^{zL_1}z^{-2L_0}Y((1+w)^{\gamma_a}g(a),-w)g(b).\end{aligned}$$

By (3.9)

$$e^{zL_1}z^{-2L_0}Y((1+w)^{\gamma_a}g(a),w)g(b)=e^{zL_1}Y((1+w)^{\gamma_a}z^{-2L_0}g(a),-w/z^2)z^{-2L_0}g(b).$$

By (3.11)

$$\begin{aligned}&e^{zL_1}Y((1+w)^{\gamma_a}z^{-2L_0}g(a),-w/z^2)z^{-2L_0}g(b)\\&=Y(e^{(z+w)L_1}(z+w)^{-2L_0}(1+w)^{\gamma_a}g(a),\tfrac{-w}{z(z+w)})e^{zL_1}z^{-2L_0}g(b),\end{aligned}$$

which means that

$$\begin{aligned}&p(a,b)A(z)Y((1+w)^{\gamma_a}a,w)b=\\&=Y(e^{(z+w)L_1}(z+w)^{-2L_0}(1+w)^{\gamma_a}g(a),\tfrac{-w}{z(z+w)})A(z)b,\end{aligned}$$

so that, since the grading is compatible with ϕ and $g(L)=L$,

$$(p(a,b)A(z)Y((1+w)^{\gamma_a}a,w)b)_{|z=1}=Y(e^{(1+w)L_1}(1+w)^{-L_0+\epsilon_a}g(a),\tfrac{-w}{(1+w)})\omega(b)$$
$$=Y(e^{(1+w)L_1}(1+w)^{-L_0+\epsilon}g(a),\tfrac{-w}{(1+w)})\omega(b)$$

Note that

$$e^{(1+w)L_1}(1+w)^{-L_0+\epsilon}=(1+w)^{-L_0+\epsilon}e^{L_1}.$$

Indeed, if $a\in V$,

$$e^{(1+w)L_1}(1+w)^{-L_0+\epsilon}a=e^{(1+w)L_1}(1+w)^{-\Delta_a+\epsilon_a}a=(1+w)^{-\Delta_a+\epsilon_a}\sum_{r\geq 0}(1+w)^r\tfrac{1}{r!}L_1^r a$$
$$=\sum_{r\geq 0}(1+w)^{-\Delta_a+r+\epsilon_a}\tfrac{1}{r!}L_1^r a=(1+w)^{-L_0+\epsilon}\sum_{r\geq 0}\tfrac{1}{r!}L_1^r a$$
$$=(1+w)^{-L_0+\epsilon}e^{L_1}a.$$

Hence,

$$(p(a,b)A(z)Y((1+w)^{L_0+\epsilon}a,w)b)_{|z=1}=Y((1+w)^{-L_0+\epsilon}e^{L_1}g(a),\tfrac{-w}{(1+w)})\omega(b)$$
$$=Y((1+w)^{-L_0+\epsilon}\omega(a),\tfrac{-w}{(1+w)})\omega(b)=(1+w)^{-L_0}Y((1+w)^{\epsilon}\omega(a),-w)(1+w)^{L_0}\omega(b). \tag{6.3}$$

Set

$$\varpi_a=\begin{cases}-\epsilon_a-1 & \text{if } \epsilon_a\neq 0,\\ 0 & \text{if } \epsilon_a=0.\end{cases}$$

Note that ϖ is the function ϵ defined in Section 2 corresponding to the grading $-\Upsilon$.

Since $\epsilon_a+\epsilon_b\in\mathbb{Z}$, we have that $\varpi_a=-\chi(a,b)-\epsilon_a$ and $\chi(a,b)=1$ if and only if $\varpi_a+\varpi_b\leq -1$. It follows that

$$Res_w(w^{-2+\chi(a,b)}(1+w)^{-L_0}Y((1+w)^{\epsilon}a,-w)(1+w)^{L_0}b)$$
$$=Res_w(w^{-2+\chi(a,b)}\sum_{n,j}(-1)^n\binom{-\Delta_a+\epsilon_a+n+1}{j}(a_{(n)}b)w^{-n-1+j})$$
$$=\sum_j(-1)^j\binom{-\Delta_a+\epsilon_a+j+\chi(a,b)-1}{j}(a_{(-2+\chi(a,b)+j)}b)$$
$$=\sum_j\binom{\Delta_a-\epsilon_a-\chi(a,b)}{j}(a_{(-2+\chi(a,b)+j)}b)$$
$$=\sum_j\binom{\Delta_a+\varpi_a}{j}(a_{(-2+\chi(a,b)+j)}b)=Res_w(w^{-2+\chi(a,b)}Y((1+w)^{L_0+\varpi}a,w)b.$$

Since Υ is compatible with ϕ, we have that $\epsilon_{\omega(a)}=\epsilon_a$ (hence $\chi(a,b)=\chi(\omega(a),\omega(b))$. We find that

$$Res_w(w^{-2+\chi(a,b)}A(z)(Y((1+w)^{L_0}a,w)b)_{|z=1})$$
$$=p(a,b)Res_w(w^{-2+\chi(\omega(a),\omega(b))}Y((1+w)^{L_0+\varpi}\omega(a),w)\omega(b)),$$

hence, by (6.2), $\omega(J_\Upsilon)\subset J_{-\Upsilon}$.

Next we prove that ω is an anti-automorphism. If $a \in V_\Upsilon$ (cf. (2.17)) then $\epsilon_a = \epsilon_{\omega(a)} = 0$, thus, if $a, b \in V_\Upsilon$, by (6.3),

$$p(a,b)\omega(a*b) = p(a,b)Res_w(w^{-1}A(z)(Y((1+w)^{L_0}a,w)b)_{|z=1})$$
$$= Res_w w^{-1}(1+w)^{-L_0}Y(\omega(a),-w)(1+w)^{L_0}\omega(b).$$

Now use skew–symmetry $Y(a,z)b = p(a,b)e^{zL_{-1}}Y(b,-z)a$ (see e.g. [11]) to get

$$\begin{aligned}
&\omega(a*b)\\
&= Res_w(w^{-1}(1+w)^{-L_0}e^{-wL_{-1}}Y((1+w)^{L_0}\omega(b),w)\omega(a))\\
&= Res_w \sum_{n,j,r}(-1)^r\binom{-\Delta_{\omega(a)}+n+1-r}{j}\tfrac{1}{r!}L_{-1}^r(\omega(b)_{(n)}\omega(a))w^{-n-2+j+r})\\
&= \sum_{j,r}(-1)^r\binom{-\Delta_{\omega(a)}+j}{j}\tfrac{1}{r!}L_{-1}^r(\omega(b)_{(-1+j+r)}\omega(a))\\
&= \sum_{r,j}(-1)^r\binom{-\Delta_{\omega(a)}+j}{j}\binom{-\Delta_{\omega(b)}-\Delta_{\omega(a)}+j+r}{r}(\omega(b)_{(-1+j+r)}\omega(a)\\
&= \sum_{r,j}(-1)^{r+j}\binom{\Delta_{\omega(a)}-1}{j}\binom{-\Delta_{\omega(b)}-\Delta_{\omega(a)}+j+r}{r}(\omega(b)_{(-1+j+r)}\omega(a))\\
&= \sum_{r,j}(-1)^{r+j}\binom{\Delta_{\omega(a)}-1}{j}\binom{-\Delta_{\omega(b)}-\Delta_{\omega(a)}+j+r}{r}(\omega(b)_{(-1+j+r)}\omega(a))\\
&= \sum_{n\geq r}(-1)^{n}\binom{\Delta_{\omega(a)}-1}{n-r}\binom{-\Delta_{\omega(b)}-\Delta_{\omega(a)}+n}{r}(\omega(b)_{(-1+n)}\omega(a))\\
&= \sum_{n}(-1)^{n}\binom{-\Delta_{\omega(b)}+n-1}{n}(\omega(b)_{(-1+n)}\omega(a))\\
&= \sum_{n}\binom{\Delta_{\omega(b)}}{n}(\omega(b)_{(-1+n)}\omega(a)) = \omega(b)*\omega(a).
\end{aligned}$$

We used the fact that in $Zhu_\Upsilon V$ we have (cf. [5, (2.35)])

$$\frac{1}{r!}L_{-1}^r a = \binom{-\Delta_a}{r}a.$$

and the Vandermonde identity on binomial coefficients.

Finally, by (3.8),

$$\omega^2(a) = A(1)^2 a = a.$$

hence $\omega^2 = I$. □

Remark 6.2. We now make explicit the map ω in the examples dealt with in Section 4. In general, if a is quasi-primary, we have, by (6.1)

$$\omega(a) = g(a). \tag{6.4}$$

(1) Let $V = F$ be the fermionic vertex algebra associated to a superspace A as in Example 5.1. According to [5, Theorem 3.25], $Zhu_{L_0}(V)$ is the Clifford algebra of A, i.e. the unital associative algebra generated by A with relations

$$[a,b] = (a|b), \quad a,b \in A.$$

Then, according to (6.4) and (5.3),

$$\omega(a) = -\sqrt{-1}\phi(a),\ a \in A_{\bar 0}, \quad \omega(a) = -\phi(a),\ a \in A_{\bar 1}. \tag{6.5}$$

(2) Let $V = M(\mathfrak{h})$ be the vertex algebra of superbosons associated to a superspace $\mathfrak{h}$ as in Example 5.2. According to [5, Theorem 3.25], $Zhu_{L_0}(V)$ is the (super)symmetric algebra of A. Then, according to (6.4) and (5.6),

$$\omega(a) = -\phi(a),\ a \in A_{\bar 0}, \quad \omega(a) = \sqrt{-1}\phi(a),\ a \in A_{\bar 1}.$$

(3) If $V = V^k(\mathfrak{g})$ (cf. Example 5.3), then $Zhu_{L_0}(V) = U(\mathfrak{g})$ (see e.g. [5]). Then, according to (6.4),

$$\omega(a) = -\phi(a),\ a \in \mathfrak{g}_{\bar 0}, \quad \omega(a) = \sqrt{-1}\phi(a),\ a \in \mathfrak{g}_{\bar 1}.$$

(4) If $V = V_Q$ is a lattice vertex algebra (cf. Example 5.4), formulas (5.8) and (6.4) give

$$\omega(e^\alpha) = (-1)^{\frac{(\alpha|\alpha)((\alpha|\alpha)+1)}{2}} e^{-\alpha}, \quad \omega(h) = -\bar h,\ h \in \mathfrak{h}.$$

Here $\bar h$ is the conjugatie of $h \in \mathfrak{h}$ with respect to $\mathbb{R} \otimes_{\mathbb{Z}} Q$. If Q is even, $Zhu_{L_0}(V_Q)$ has been proved in [6] to be isomorphic to a generalized Smith algebra, denoted there by $\overline{A(Q)}$. The algebra $\overline{A(Q)}$ is generated by elements E_α, $\alpha \in Q, h \in \mathfrak{h}$, and the explicit formula for the isomorphism $Zhu_{L_0} V_Q \cong \overline{A(Q)}$ given in [6, Theorem 3.4] implies that

$$\omega(E_\alpha) = (-1)^{\frac{(\alpha|\alpha)}{2}} E_{-\alpha}, \quad \omega(h) = -\bar h,\ \ h \in \mathfrak{h},$$

is a conjugate linear anti–automorphism of $\overline{A(Q)}$.

Definition 6.3. Let R be an associative superalgebra over $\mathbb{C}$ with a conjugate linear anti–involution ω, and let M be an R–module. A Hermitian form $(\cdot\,,\,\cdot)$ on M is called ω–invariant if

$$(\omega(a)m_1, m_2) = (m_1, a\, m_2),\ a \in R,\ m_1, m_2 \in M.$$

Assume for the rest of this Section that $\Gamma = \mathbb{Z}$ or $\Gamma = \frac{1}{2}\mathbb{Z}$, so that $Zhu_\Upsilon = Zhu_{-\Upsilon}$. The following is the natural extension of Definition 4.1 to V–modules.

Definition 6.4. Let ϕ be a conjugate linear involution of the vertex algebra V. A Hermitian form $(\cdot\,,\,\cdot)$ on a Υ–twisted V–module M is called ϕ–invariant if, for all $v \in V$,

$$(m_1, Y_M(a,z)m_2) = (Y_M(A(z)a, z^{-1})m_1, m_2). \tag{6.6}$$

From now on we assume that the module M is a positive energy module (see Definition 2.6).

Remark 6.5. The space of ϕ–invariant Hermitian forms on M is linearly isomorphic to

$$\{\Theta \in Hom_V(M, M^\dagger) \mid \langle \Theta(m_1), m_2\rangle = \overline{\langle \Theta(m_2), m_1\rangle}\}$$

Indeed, given $\Theta : M \to M^\dagger$ a V–module homomorphism, then setting, for $m_1, m_2 \in M$

$$(m_1, m_2)_\Theta = \langle \Theta(m_2), m_1\rangle$$

defines a ϕ–invariant hermitian form on M. In fact

$$\begin{aligned}(m_1, Y_M(a,z)m_2)_\Theta &= \langle \Theta(Y_M(a,z)m_2), m_1\rangle = \langle Y_{M^\dagger}(a,z)\Theta(m_2), m_1\rangle \\ &= \langle \Theta(m_2), Y_M(A(z)v, z^{-1})m_1\rangle = (Y_M(A(z)v, z^{-1})m_1, m_2)_\Theta.\end{aligned}$$

Conversely, let $F : M\times M\to\mathbb{C}$ be a ϕ–invariant hermitian form; then $\Theta_F : M\to M^\dagger$ defined by $\langle\Theta_F(m_1),m_2\rangle = F(m_2,m_1)$ is a V–homomorphism from M to $M^\dagger$. Indeed

$$\begin{aligned}\langle\Theta_F(Y_M(a,z)m_1),m_2\rangle &= F(m_2,Y_M(a,z)m_1) = F(Y_M(A(z)a,z^{-1})m_2,m_1)\\ &= \langle\Theta_F(m_1),Y_M(A(z)a,z^{-1})m_2\rangle = \langle Y_{M^\dagger}(a,z)\Theta_F(m_1),m_2\rangle.\end{aligned}$$

Recall that a positive energy Υ–twisted V–module M is said quasi–irreducible if it is generated by M_0 and there are no non-zero submodules $N\subset M$ such that $N\cap M_0=\{0\}$.

By [5, Lemma 2.2], if M is a positive energy Υ-twisted V–module, then the map $a\mapsto (a_0^M)_{|M_0}$ descends to define a $Zhu_\Upsilon V$–module structure on M_0.

Lemma 6.6. *If M is quasi–irreducible then $M^\dagger$ is quasi–irreducible.*

Proof. Set $N=VM_0^\dagger$. Then $N^\perp$ is graded and $\langle F,v\rangle=0$ for all $v\in N_0^\perp$, $F\in M_0^\dagger$. This implies that $N_0^\perp=\{0\}$, so $N^\perp=\{0\}$, hence $N=M^\dagger$.

If N is a graded submodule of $M^\dagger$ with $N_0=\{0\}$ then $N^\perp$ is a graded submodule of M containing M_0. Since M_0 generates M, it follows that $N^\perp=M$ hence $N=\{0\}$. □

Proposition 6.7. *Let M be a Υ-twisted positive–energy V–module generated by M_0. Then the space of ϕ–invariant Hermitian forms on M is linearly isomorphic to the set of ω–invariant Hermitian forms on the $Zhu_\Upsilon V$–module M_0.*

Proof. If $(\cdot\,,\cdot)$ is a ϕ–invariant Hermitian form on M, then $(\cdot\,,\cdot)_0=(\cdot\,,\cdot)_{|M_0\times M_0}$ is a ω–invariant Hermitian form on M_0 by Proposition 6.1.

Let $(\cdot\,,\cdot)_0$ be a ω–invariant Hermitian form on the $Zhu_\Upsilon V$–module M_0. Let N be the sum of all graded submodules N' of M such that $N'\cap M_0=\{0\}$. Then M/N is quasi–irreducible and $(M/N)_0=M_0$. Define $\Phi_0: M_0\to M_0^\dagger$ by setting $\Phi_0(m_1)(m_2)=(m_2,m_1)_0$. Since the form $(\cdot\,,\cdot)_0$ is ω–invariant, we have

$$\begin{aligned}\Phi_0(v_0^M m_1)(m_2) &= (m_2,v_0^M m_1)_0 = (\omega(v)_0^M m_2,m_1)_0 = \Phi_0(m_1)(\omega(v)_0^M m_2)\\ &= (v_0^{M^\dagger}\Phi_0)(m_2)(m_1),\end{aligned}$$

so Φ_0 is a $Zhu_\Upsilon(V)$–module map between M_0 and $M_0^\dagger$. By Lemma 6.6 and [5, Theorem 2.30], there is a V–module map $\Phi: M/N\to(M/N)^\dagger$ such that $\Phi_{|M_0}=\Phi_0$. Define, for $m_1,m_2\in M$,

$$(m_1,m_2)=\Phi(m_2+N)(m_1+N).$$

It is clear that the form $(\cdot\,,\cdot)$ is ϕ–invariant and that $(\cdot\,,\cdot)_0=(\cdot\,,\cdot)_{|M_0\times M_0}$. It remains to check that the form is Hermitian.

Consider the form $(\cdot\,,\cdot)'$ defined by $(m_1,m_2)'=\overline{(m_2,m_1)}$. Note that $(\cdot\,,\cdot)'$ is ϕ–invariant:

$$\begin{aligned}(m_1,Y_M(a,z)m_2)' &= \overline{(Y_M(a,z)m_2,m_1)} = \overline{(Y_M(A(z)A(z^{-1})a,z)m_2,m_1)}\\ &= \overline{(m_2,Y_M(A(z)a,z^{-1})m_1)} = (Y_M(A(z)a,z^{-1})m_1,m_2)'.\end{aligned}$$

Since $(\cdot\,,\cdot)_0$ is Hermitian, then

$$(\cdot\,,\cdot)'_{|M_0\times M_0}=(\cdot\,,\cdot)_{|M_0\times M_0},$$

hence $(\cdot\,,\cdot)'=(\cdot\,,\cdot)$. □

Remark 6.8. Theorem 4.3 is a consequence of Proposition 6.7. Indeed, the space of ω–invariant Hermitian forms on V_0 is linearly isomorphic to $(V_0/L_1V_1)^\dagger$. The isomorphism is defined by mapping $(\cdot\,,\cdot)_0$ to $F_{(\cdot\,,\cdot)_0}$ where $F_{(\cdot\,,\cdot)_0}(v)=(v,\mathbf{1})_0$. To prove that this map is well defined, let us check that $F_{(\cdot\,,\cdot)_0}(L_1V_1)=0$. If $v\in V_1$, then

$$L_1v=(L_1v)_0\mathbf{1}=(v_0+(L_1v)_0)\mathbf{1})=\omega(g(v))_0\mathbf{1},$$

so

$$F_{(\cdot\,,\cdot)_0}(L_1v)=(L_1v,\mathbf{1})_0=((\omega(g(v))_0\mathbf{1},\mathbf{1})_0=-(\mathbf{1},g(v)_0\mathbf{1})_0=0.$$

The inverse is the map $F\mapsto(\cdot\,,\cdot)_F$, where $(v,w)_F=F(\omega(w)_0v)$. Let us check that $(\cdot\,,\cdot)_F$ is ω–invariant. If $u,v\in V_0$ and $w\in V_{\mathbb{Z}}$, then $(u,w_0v)_F=F(\omega(w_0v)_0u)$ and $(\omega(w)_0u,v)_F=F(\omega(v)_0\omega(w)_0u)$. Viewing F as an element of $V^\dagger$, we observe that

$$F(\omega(w_0v)_0u)=((w_0v)_0^{V^\dagger}F)(u),\ F(\omega(v)_0\omega(w)_0u)=(w_0^{V^\dagger}v_0^{V^\dagger}F)(u),$$

so it is enough to check that

$$(w_0v)_0^{V^\dagger}F=w_0^{V^\dagger}v_0^{V^\dagger}F. \tag{6.7}$$

Observe that, since $\langle F,L_1V_1\rangle=0$, $L_{-1}F=0$, F is a vacuum–like element of $V^\dagger$. It follows from Proposition 3.4 of [15] that the map $\Phi:V\to V^\dagger$ defined by $\Phi(a)=a_{(-1)}^{V^\dagger}F$ is a V–module map. In particular,

$$\Phi(a_{(n)}b)=a_{(n)}^{V^\dagger}\Phi(b)=a_{(n)}^{V^\dagger}(b_{(-1)}^{V^\dagger}F).$$

On the other hand

$$\Phi(a_{(n)}b)=(a_{(n)}b)_{(-1)}^{V^\dagger}F$$

so

$$a_{(n)}^{V^\dagger}(b_{(-1)}^{V^\dagger}F)=(a_{(n)}b)_{(-1)}^{V^\dagger}F.$$

Since $\Delta_v=\Delta_{w_0v}=0$, we find $v_{(-1)}^{V^\dagger}F=v_0^{V^\dagger}F$ and $(w_0v)_{(-1)}^{V^\dagger}F=(w_0v)_0^{V^\dagger}F$, so (6.7) follows.

7. Invariant Hermitian forms on W–algebras

We adopt the setting and notation of Section 1 of [13]. We let $W^k(\mathfrak{g},x,f)$ be the universal W–algebra of level $k\in\mathbb{R}$ associated to the datum $(\mathfrak{g},x,f)$, where $\mathfrak{g}$ is a simple finite–dimensional Lie superalgebra with a reductive even part and a non-zero even invariant supersymmetric bilinear form $(.|.)$, x is an ad–diagonalizable element of $\mathfrak{g}$ with eigenvalues in $\frac{1}{2}\mathbb{Z}$, f is an even element of $\mathfrak{g}$ such that $[x,f]=-f$ and the eigenvalues of $\operatorname{ad}x$ on the centralizer $\mathfrak{g}^f$ of f in $\mathfrak{g}$ are non-positive. Recall that we are assuming that $a\in\mathbb{R}$ for $\mathfrak{g}=D(2,1;a)$. We call the datum $(\mathfrak{g},x,f)$ a *Dynkin datum* if there is a $sl(2)$–triple $\{f,h,e\}$ containing f and $x=\frac{1}{2}h$.

Let

$$\mathfrak{g}=\bigoplus_{j\in\frac{1}{2}\mathbb{Z}}\mathfrak{g}_j \tag{7.1}$$

be the grading of $\mathfrak{g}$ by $ad(x)$–eigenspaces. We assume that $k\neq-h^\vee$ so that $W^k(\mathfrak{g},x,f)$ has a Virasoro vector. Then $W^k(\mathfrak{g},x,f)$ is a conformal vertex algebra in the sense of Definition 2.1.

Remark 7.1. It is easy to show that a datum $(\mathfrak{g},x,f)$ as above is independent, up to isomorphism, from the choice of f, hence we may use notation $W^k(\mathfrak{g},x)$.

Remark 7.2. An important special case is when f is a minimal nilpotent element of the even part of $\mathfrak{g}$, i.e. f is the root vector $e_{-\theta}$ corresponding to a maximal even root θ. In this case, the invariant bilinear form $(.|.)$ is normalized so that $(\theta|\theta)=2$. Choose the root vector $e_\theta \in \mathfrak{g}_\theta$ in such a way that $(e_\theta|e_{-\theta})=\frac{1}{2}$. Setting $x=[e_\theta,e_{-\theta}]$, it is clear that $(\mathfrak{g},x,e_{-\theta})$ is a Dynkin datum. Identifying the Cartan subalgebra $\mathfrak{h}$ with its dual using $(.|.)$, one has $x=\theta/2$. The algebra $W^k(\mathfrak{g},\theta/2)$ is called a *minimal* W-algebra.

Lemma 7.3. *Let ϕ be a conjugate linear involution of $\mathfrak{g}$ such that*

$$\phi(f)=f, \quad \phi(x)=x. \tag{7.2}$$

Assume also, as in Subsection 5.3, that

$$\overline{(\phi(X)|\phi(Y))}=(X|Y), \tag{7.3}$$

so that ϕ extends to a conjugate linear involution of $V^k(\mathfrak{g})$. Then ϕ descends to an involution of the vertex algebra $W^k(\mathfrak{g},x,f)$.

Proof. Let A be the superspace $\Pi(\sum_{j>0}\mathfrak{g}_j)$ where Π is the reverse parity functor. Let A^* be the linear dual of A and set $A_{ch}=A\oplus A^*$. Define the form $\langle\cdot,\cdot\rangle_{ch}$ on A_{ch} by setting, for $a,b\in A$, $a',b'\in A^*$,

$$\langle a,b\rangle_{ch}=\langle a',b'\rangle_{ch}=0, \quad \langle a,b'\rangle_{ch}=b'(a), \quad \langle b',a\rangle_{ch}=-p(a,b')a'(b).$$

Let A_{ne} be the superspace $\mathfrak{g}_{1/2}$ equipped with the form $\langle\cdot,\cdot\rangle_{ne}$ defined by

$$\langle a,b\rangle_{ne}=(f|[a,b]).$$

Since $\phi(f)=f$,

$$\langle\phi(a),\phi(b)\rangle_{ne}=(f|[\phi(a),\phi(b)])=(\phi(f)|\phi([a,b]))=\overline{(f|[a,b])}=\overline{\langle a,b\rangle}_{ne}.$$

It follows that ϕ extends to a conjugate linear involution of $F(A_{ne})$. Similarly, setting $\phi(b^*)(a)=\overline{b^*(\phi(a))}$ for $b^*\in A^*$ and $a\in A$, we have

$$\langle\phi(a),\phi(b^*)\rangle_{ch}=\phi(b^*)(\phi(a))=\overline{b^*(a)}=\overline{\langle a,b^*\rangle}_{ch},$$

so ϕ extends to a conjugate linear involution of $F(A_{ch})$. It follows that ϕ is a conjugate linear involution of the vertex algebra $\mathcal{C}(\mathfrak{g},f,x,k)=V^k(\mathfrak{g})\otimes F(A_{ch})\otimes F(A_{ne})$.

Recall that there is an element $d\in\mathcal{C}(\mathfrak{g},f,x,k)$ such that d_0 is an odd derivation and $d_0^2=0$, making $\mathcal{C}(\mathfrak{g},f,x,k)$ a complex. It is easy to see that $\phi(d)=d$, hence the involution ϕ descends to an involution of the vertex algebra $W^k(\mathfrak{g},x,f)=H^0(\mathcal{C}(\mathfrak{g},f,x,k),d)$ [12], [13]. □

Recall from [13] that the vertex algebra $W^k(\mathfrak{g},x,f)$ is strongly and freely generated by fields $J^{\{x_i\}}$ with $\{x_i\}$ a basis of $\mathfrak{g}^f$, the centralizer of f in $\mathfrak{g}$. We can clearly assume that the elements x_i are homogeneous with respect to the gradation $\mathfrak{g}^f=\oplus_j\mathfrak{g}^f_j$. Let $\mathfrak{g}_\mathbb{R}$ be the fixed point set of ϕ. By (7.3), we see that $(.|.)_{\mathfrak{g}_\mathbb{R}\times\mathfrak{g}_\mathbb{R}}$ is a real bilinear form. Since $\phi(x)=x$, we see that $\mathfrak{g}_j=(\mathfrak{g}_j\cap\mathfrak{g}_\mathbb{R})\oplus(\sqrt{-1}\mathfrak{g}_j\cap\mathfrak{g}_\mathbb{R})$. Moreover $\langle\cdot,\cdot\rangle_{ne}$ is real when restricted to $\mathfrak{g}_{1/2}\cap\mathfrak{g}_\mathbb{R}$. Likewise, we can identify the real dual of $\mathfrak{g}_+\cap\mathfrak{g}_\mathbb{R}$ with the set of $b^*\in A^*$ such that $\phi(b^*)=b^*$. It follows that we can identify the algebra $\mathcal{C}(\mathfrak{g}_\mathbb{R},f,x,k)$ as a real subalgebra of $\mathcal{C}(\mathfrak{g},f,x,k)$. We can therefore carry out the construction of the fields $J^{\{a\}}$ for $a\in\mathfrak{g}^f_\mathbb{R}$ inside $\mathcal{C}(\mathfrak{g}_\mathbb{R},f,x,k)$ and therefore obtain that $\phi(J^{\{a\}})=J^{\{a\}}$. As $a\in\mathfrak{g}^f$ can be written as $a=a_\mathbb{R}+ib_\mathbb{R}$ with $a_\mathbb{R},b_\mathbb{R}\in\mathfrak{g}^f_\mathbb{R}$, we see that we can construct the field $J^{\{a\}}$ so that $\phi(J^{\{a\}})=J^{\{\phi(a)\}}$.

Let $L^\mathfrak{g}$ the Virasoro vector for $V^k(\mathfrak{g})$ defined in (5.7). The vertex algebra $W^k(g,x,f)$ carries a Virasoro vector L, making it a conformal vertex algebra, which is the homology class of $L^\mathfrak{g}+T(x)+L^{ch}+L^{ne}$ (see [12]).

In particular, by the above discussion and the explicit expressions for $L^{\mathfrak{g}}$, L^{ch}, L^{ne}, we obtain that $\phi(L) = L$. Following (3.3) we set

$$g = ((-1)^{L_0}\sigma^{1/2})^{-1}\phi.$$

If $x_i \in \mathfrak{g}^f_j$, then the conformal weight of $J^{\{x_i\}}$ is $1 - j$. It follows that

$$W^k(\mathfrak{g}, x, f)_0 = \mathbb{C}\mathbf{1}, \quad W^k(\mathfrak{g}, x, f)_1 = span(\{J^{\{x_i\}} \mid x_i \in \mathfrak{g}^f_0\}).$$

Theorem 7.4. *(a) Let $v \in \mathfrak{g}^f_0$. If $J^{\{v\}} \in W^k(\mathfrak{g}, x, f)_1$ is quasiprimary for more than one $k \in \mathbb{C}$, then*

$$(x|v) = 0. \tag{7.4}$$

(b) If the datum $(\mathfrak{g}, x, f)$ is a Dynkin datum, then the elements $J^{\{v\}}$ are primary for all $v \in \mathfrak{g}^f_0$ and $k \in \mathbb{C}$ $(k \neq -h^\vee)$. In particular, by Corollary 4.7, there is a unique ϕ–invariant Hermitian form $(\cdot\,,\cdot)$ on $W^k(\mathfrak{g}, x, f)$ such that $(\mathbf{1}, \mathbf{1}) = 1$.

(c) Assume that $\mathfrak{g}$ is a Lie algebra. If (7.4) holds for a datum $(\mathfrak{g}, x, f)$ and all $v \in \mathfrak{g}^f_0$, then it is a Dynkin datum.

Proof. By [12, Theorem 2.4b], if $v \in \mathfrak{g}^f_0$, then

$$[L_\lambda J^{\{v\}}] = (T + \lambda)J^{\{v\}} + \lambda^2(\tfrac{1}{2}str_{\mathfrak{g}_+}(ad\, v) - (k + h^\vee)(v|x)),$$

hence claim (a) follows immediately.

If the datum $(\mathfrak{g}, x, f)$ is a Dynkin datum, then $2(x|v) = ([e, f]|v) = (e|[f, v]) = 0$ if $v \in \mathfrak{g}^f$. Hence for (b) it suffices to show $str_{\mathfrak{g}_j}(ad\, v) = 0$ for all $j \in \frac{1}{2}\mathbb{N}$ and $v \in \mathfrak{g}^f_0$.

Consider the following bilinear form on $\mathfrak{g}_j$:

$$< a, b >= ((ad\, f)^{2j}a|b).$$

By $sl(2)$–representation theory, $(ad\, f)^{2j} : \mathfrak{g}_j \to \mathfrak{g}_{-j}$ is injective for $j > 0$, hence $< \cdot, \cdot >$ is non-degenerate. The form is clearly $ad\, \mathfrak{g}^f_0$–invariant. The form is super (resp. skew–super) symmetric if $j \in \mathbb{Z}$ (resp. $j \in \frac{1}{2} + \mathbb{Z}$):

$$< a, b >= ((ad\, f)^{2j}a|b) = (-1)^{2j}(a|(ad\, f)^{2j}b) = (-1)^{2j}p(a, b) < b, a > .$$

Hence for $v \in \mathfrak{g}^f_0$, $ad\, v$ lies in $osp(\mathfrak{g}_j)$ (resp. $spo(\mathfrak{g}_j)$) if $j \in \mathbb{Z}$ (resp. $j \in \frac{1}{2} + \mathbb{Z}$). Hence in either case its supertrace is 0. This proves (b).

By Theorem 1.1 from [8], $x = \frac{1}{2}h + c$, where $\{e, h, f\}$ is an $sl(2)$–triple for some $e \in \mathfrak{g}_1$ and c is a semisimple central element from the centralizer of this triple. We may assume that c is defined over $\mathbb{R}$. But then $(x|c) = (\frac{1}{2}h + c|c) = (c|c)$. Since we are assuming that $\mathfrak{g}$ is a simple Lie algebra, (7.4) implies that $c = 0$, proving (c). □

Remark 7.5. Let $\mathfrak{g}$ be a simple Lie algebra. It follows from Theorem 7.4 that a datum $(\mathfrak{g}, x, f)$ is Dynkin if and only if $(x|\mathfrak{g}^f_0) = 0\,(\iff (x|\mathfrak{g}^f) = 0)$. In other words a $\frac{1}{2}\mathbb{Z}$–grading of $\mathfrak{g}$ is Dynkin iff $f \in \mathfrak{g}_{-1}$, all eigenvalues of adx on $\mathfrak{g}^f$ are non-positive and $(x|\mathfrak{g}^f) = 0$.

Example 7.6. Let $\mathfrak{g} = sl(3)$ with the data $(\mathfrak{g}, \frac{1}{2}(E_{11} - E_{33}), E_{31}, k)$ and $(\mathfrak{g}, -2E_{11} + E_{22} + E_{33}, E_{31}, k)$. The first one is a Dynkin datum corresponding to the minimal W–algebra $W^k(\mathfrak{g}, \theta/2)$. The second one is not Dynkin: indeed, if $v = E_{11} - 2E_{22} + E_{33}$, then $v \in \mathfrak{g}^f_0$ and $(x|v) \neq 0$.

Corollary 7.7. *Assume that $(\mathfrak{g}, x, f)$ is a Dynkin datum. Then there is a unique ϕ–invariant Hermitian form $(\,.\,,\,.\,)$ on $W^k(\mathfrak{g}, x, f)$ such that $(\mathbf{1}, \mathbf{1}) = 1$.*

Proof. By Theorem 7.4 (b), we can apply Corollary 4.7. □

We now describe the ϕ–invariant Hermitian form more explicitly using formula (4.9). Fix a basis $\{x^i\}$ of $\mathfrak{g}^f$. Set $\Delta_i = \Delta_{x^i}$ and $p_i = p(x^i)$. By Proposition 4.6 we may assume that the fields $J^{\{x_i\}}$ are quasiprimary for all i. We can clearly assume that $\phi(x^i) = x^i$ for all i. Since $\phi(L) = L$, the proof of Lemma 4.5, hence of Proposition 4.6, can be done over $\mathbb{R}$, so $\phi(J^{\{x^i\}}) = J^{\{x^i\}}$ and let $J^{\{x^i\}}(z) = \sum_{n\in -\Delta_i+\mathbb{Z}} J^{\{x^i\}}_n z^{-n-\Delta_i}$.

Order the set

$$\{(j,i) \in \tfrac{1}{2}\mathbb{Z}_+ \times \{0,\dots,\dim \mathfrak{g}^f - 1\} \mid j \in \Delta_i + \mathbb{Z}_+\}$$

lexicographically. Then the set

$$\{(J^{\{x^{i_1}\}}_{-j_1})^{m_1} \cdots (J^{\{x^{i_t}\}}_{-j_t})^{m_t}\mathbf{1} \mid m_i = 0 \text{ or } 1 \text{ if } x^i \text{ is odd}\} \tag{7.5}$$

is a basis of $W^k(\mathfrak{g}, x, f)$. Since

$$g(J^{\{x^i\}}) = (-\sqrt{-1})^{2\Delta_i + p_i} J^{\{x^i\}},$$

formula (4.9) gives that

$$\begin{aligned}(7.6)\qquad &\left((J^{\{x^{i_1}\}}_{j_1})^{m_1} \cdots (J^{\{x^{i_t}\}}_{j_t})^{m_t}\mathbf{1}, (J^{\{x^{i'_1}\}}_{j'_1})^{m'_1} \cdots (J^{\{x^{i'_t}\}}_{j'_r})^{m'_t}\mathbf{1}\right)\\ &= (-\sqrt{-1})^{\sum_r m_r(2\Delta_{i_r}+p_{i_r})} \left\langle (J^{\{x^{i_t}\}}_{-j_t})^{m_t} \cdots (J^{\{x^{i_1}\}}_{-j_1})^{m_1} (J^{\{x^{i'_1}\}}_{j'_1})^{m'_1} \cdots (J^{\{x^{i'_r}\}}_{j'_r})^{m'_r} \right\rangle.\end{aligned}$$

Remark 7.8. Set $R = span(T^k(J^{\{x^i\}}), k \in \mathbb{Z}^+)$. Let π_Z be the quotient map from $W^k(\mathfrak{g}, x, f)$ to $Zhu_{L_0}(W^k(\mathfrak{g}, x, f))$. Set $\mathfrak{w} = span(\pi_Z(J^{\{x^i\}}))$. By (7.5) the set

$$\{: (T^{k_1} J^{\{x^{i_1}\}})^{m_1} \cdots (T^{k_t} J^{\{x^{i_t}\}})^{m_t} :\mid m_i = 0 \text{ or } 1 \text{ if } x^{j_i} \text{ is odd}\}$$

is a basis of $W^k(\mathfrak{g}, x, f)$. It follows from Theorem 3.25 of [5] that

$$R/(L_{-1} + L_0)R \simeq \mathfrak{w}$$

has the structure of a nonlinear Lie superalgebra and that $Zhu_{L_0}(W^k(\mathfrak{g}, x, f))$ is its universal enveloping algebra. In particular the set

$$\{(\pi_Z J^{\{x^{i_1}\}})^{m_1} * \cdots * (\pi_Z J^{\{x^{i_t}\}})^{m_t} \mid m_i = 0 \text{ or } 1 \text{ if } x^{j_i} \text{ is odd}\}$$

is a basis of $Zhu_{L_0}(W^k(\mathfrak{g}, x, f))$. Since, by Proposition 4.6, $J^{\{x_i\}}$ can be chosen to be quasiprimary for all i, it is clear that the involution ω in this basis is given by

$$\omega((\pi_Z J^{\{x^{i_1}\}})^{m_1} * \cdots * (\pi_Z J^{\{x^{i_t}\}})^{m_t}) = (-\sqrt{-1})^{\sum_r m_r(2\Delta_{i_r}+p_{i_r})} (\pi_Z J^{\{x^{i_t}\}})^{m_t} * \cdots * (\pi_Z J^{\{x^{i_1}\}})^{m_1}.$$

We now restrict to the case of a minimal W–algebra $W^k(\mathfrak{g}, \theta/2)$ (see Remark 7.2) where one has a more explicit description of $Zhu_{L_0}(W^k(\mathfrak{g}, \theta/2))$ and its involution.

Set $\mathfrak{g}^\natural = \mathfrak{g}^f_0$. Then $\mathfrak{g}^f = \mathfrak{g}^\natural \oplus \mathfrak{g}_{-1/2} \oplus \mathbb{C}f$. The elements $J^{\{v\}}$ are uniquely determined for $v \in \mathfrak{g}^\natural \oplus \mathfrak{g}_{-1/2}$ and have been computed explicitly in [12]. One usually denotes $J^{\{v\}}$ by $G^{\{v\}}$ if $v \in \mathfrak{g}_{-1/2}$. We also write $\mathfrak{g}^\natural = \oplus_{i=0}^r \mathfrak{g}_i$ with $\mathfrak{g}_0$ the (possibly zero) center and $\mathfrak{g}_i$ a simple ideal for $i > 0$.

Set, for $u, v \in \mathfrak{g}_{-1/2}$,

$$\langle u, v\rangle = (e_\theta | [u, v])$$

and note that $\langle\cdot,\cdot\rangle$ is a $\mathfrak{g}^\natural$–invariant skew–supersymmetric bilinear form on $\mathfrak{g}_{-1/2}$. Fix a basis $\{a_i\}$ of $\mathfrak{g}^\natural$ and a basis $\{u_i\}$ of $\mathfrak{g}_{-1/2}$. Then $W^k(\mathfrak{g},\theta)$ has as set of free generators

$$\{J^{\{a_i\}}\}\cup\{G^{\{u_i\}}\}\cup\{L\}.$$

Moreover the λ–brackets between generators is known explicitly [12], [13], [1], [14], and Section 8: L is the Virasoro vector and its central charge is $\frac{k\,\mathrm{sdim}\mathfrak{g}}{k+h^\vee}-6k+h^\vee-4$, the $J^{\{u\}}$ are primary of conformal weight 1, the $G^{\{v\}}$ are primary of conformal weight $\frac{3}{2}$ and

(1) $[J^{\{a\}}{}_\lambda J^{\{b\}}]=J^{\{[a,b]\}}+\lambda\delta_{ij}(k+\frac{h^\vee-h^\vee_{0,i}}{2})(a|b)$ for $a\in\mathfrak{g}^\natural_i$, $b\in\mathfrak{g}^\natural_j$;
(2) $[J^{\{a\}}{}_\lambda G^{\{u\}}]=G^{\{[a,u]\}}$ for $u\in\mathfrak{g}_{-1/2}$, $a\in\mathfrak{g}^\natural$;
(3)

$$[G^{\{u\}}{}_\lambda G^{\{v\}}]=-2(k+h^\vee)\langle u,v\rangle L+\langle u,v\rangle\sum_{\alpha=1}^{\dim\mathfrak{g}^\natural}:J^{\{a^\alpha\}}J^{\{a_\alpha\}}:+$$
$$2\sum_{\alpha,\beta=1}^{\dim\mathfrak{g}^\natural}\langle[a_\alpha,u],[v,a^\beta]\rangle:J^{\{a^\alpha\}}J^{\{a_\beta\}}:+2(k+1)(\partial+2\lambda)J^{\{[[e_\theta,u],v]^\natural\}}$$
$$+2\lambda\sum_{\alpha,\beta=1}^{\dim\mathfrak{g}^\natural}\langle[a_\alpha,u],[v,a^\beta]\rangle J^{\{[a^\alpha,a_\beta]\}}+2p(k)\lambda^2\langle u,v\rangle.$$

.

Here $\{a_\alpha\}$ (resp. $\{u_\gamma\}$) is a basis of $\mathfrak{g}^\natural$ (resp. $\mathfrak{g}_{1/2}$) and $\{a^\alpha\}$ (resp. $\{u^\gamma\}$) is the corresponding dual basis w.r.t. $(\,.\,|\,.\,)$ (resp w.r.t. $\langle\cdot,\cdot\rangle_{\rm ne}=(e_{-\theta}|[\cdot,\cdot])$), $a^\natural$ is the orthogonal projection of $a\in\mathfrak{g}_0$ on $\mathfrak{g}^\natural$, $a^\natural_i$ is the projection of $a^\natural$ on the ith minimal ideal $\mathfrak{g}^\natural_i$ of $\mathfrak{g}^\natural$, $k_i=k+\frac{1}{2}(h^\vee-h^\vee_{0,i})$, where $h^\vee_{0,i}$ is the dual Coxeter number of $\mathfrak{g}^\natural_i$ with respect to the restriction of the form $(\,.\,|\,.\,)$, and $p(k)$ is the monic quadratic polynomial given in Table 4 of [1]. See Appendix 8 for the derivation of formula (3) from the formulas given in [12].

Identify $\mathfrak{w}$ with $\mathfrak{g}^\natural\oplus\mathfrak{g}_{-1/2}\oplus\mathbb{C}L$ by identifying $\pi_Z J^{\{a\}}$ with a, $\pi_Z G^{\{v\}}$ with v and $\pi_z L$ with L. As in Remark 7.8, a basis of $Zhu_{L_0}(W^k(\mathfrak{g},\theta))$ is given by

$$\{u_{i_1}^{m_1}*\cdots*u_{i_t}^{m_t}*a_{j_1}^{n_1}*\cdots*a_{j_r}^{n_r}*L^k\,|\,i_1<\cdots i_t;\ j_1<\cdots<j_r;\ m_p,n_q\in\{0,1\}\text{ if }a_{i_p}\text{ or }u_{j_q}\text{ is odd}\}.$$

Moreover the commutation relations among the generators are as follows (here $[\cdot,\cdot]_{\mathfrak{g}}$ denotes the bracket in $\mathfrak{g}$, while $[\cdot,\cdot]$ is the bracket in $Zhu_{L_0}(W^k(\mathfrak{g},\theta))$.

(1) L is a central element,
(2) $[a,b]=[a,b]_{\mathfrak{g}}$ if $a,b\in\mathfrak{g}^\natural$,
(3) $[a,v]=[a,v]_{\mathfrak{g}}$ if $a\in\mathfrak{g}^\natural$ and $v\in\mathfrak{g}_{-1/2}$,
(4)

$$[u,v]=\langle u,v\rangle\left(\sum_{\alpha=1}^{\dim\mathfrak{g}^\natural}(a^\alpha*a_\alpha-[a^\alpha,a_\alpha]_{\mathfrak{g}})-2(k+h^\vee)L-\frac{1}{2}p(k)\right)$$
$$+\sum_{\alpha,\beta=1}^{\dim\mathfrak{g}^\natural}\langle[a_\alpha,u]_{\mathfrak{g}},[v,a^\beta]_{\mathfrak{g}}\rangle(2a^\alpha*a_\beta-[a^\alpha,a_\beta]_{\mathfrak{g}}).$$

By (2), (3) we can drop the subscript $\mathfrak{g}$ from the bracket. Moreover observe that

$$\sum_{\alpha=1}^{\dim \mathfrak{g}^\natural} [a^\alpha, a_\alpha]_\mathfrak{g} = 0$$

and that

$$2a^\alpha * a_\beta - [a^\alpha, a_\beta]_\mathfrak{g} = 2a^\alpha * a_\beta - [a^\alpha, a_\beta] = a^\alpha * a_\beta + p(a_\alpha, a_\beta) a_\beta * a^\alpha.$$

Setting $L' = 2(k+h^\vee)L + \frac{1}{2}p(k)$, a new generating space is $\mathfrak{g}^\natural \oplus \mathfrak{g}_{-1/2} \oplus \mathbb{C}L'$ and the commutation relations are (1) with L' in place of L, (2), (3) and

(4')

$$[u,v] = \langle u,v\rangle \left(\sum_{\alpha=1}^{\dim \mathfrak{g}^\natural} a^\alpha * a_\alpha - L' \right) + \sum_{\alpha,\beta=1}^{\dim \mathfrak{g}^\natural} \langle [a_\alpha, u]_\mathfrak{g}, [v, a^\beta]_\mathfrak{g} \rangle (a^\alpha * a_\beta + p(a_\alpha, a_\beta) a_\beta * a^\alpha).$$

It is then clear that $Zhu_{L_0}(W^k(\mathfrak{g}, \theta/2))$ does not depend on k if $k \neq -h^\vee$.

The involution ω is easily computed: since the generators are quasiprimary, we have by (6.4): $\omega(J^{\{a\}}) = g(J^{\{a\}})$, hence

$$\begin{aligned} &\omega(L') = L', \\ &\omega(a) = (-1)^{p(a)+1} (\sqrt{-1})^{p(a)} \phi(a), \ a \in \mathfrak{g}^\natural, \\ &\omega(v) = (-1)^{p(v)} (\sqrt{-1})^{p(v)+1} \phi(v), \ v \in \mathfrak{g}_{-1/2}. \end{aligned}$$

Recall that, if $k + h^\vee \neq 0$, then $W^k(\mathfrak{g}, \theta/2)$ has a unique simple quotient $W_k(\mathfrak{g}, \theta/2)$. Remark that the maximal proper ideal I^k of $W^k(\mathfrak{g}, \theta/2)$ is the kernel of the invariant Hermitian form on $W^k(\mathfrak{g}, \theta/2)$, hence one can induce a invariant Hermitian form on $W_k(\mathfrak{g}, \theta/2)$. The latter vertex algebra is unitary if and only if the invariant form on $W^k(\mathfrak{g}, \theta/2)$ is positive semidefinite. Recall from [1] that a level k is *collapsing* for $W^k(\mathfrak{g}, \theta/2)$ if $W_k(\mathfrak{g}, \theta/2)$ is contained in its affine vertex algebra part.

Theorem 7.9. *Assume that $W_k(\mathfrak{g}, \theta/2)$ is unitary.*

1. *If $\mathfrak{g} \neq sl(2)$ is a Lie algebra then k is a collapsing level.*
2. *If $\mathfrak{g}^\natural$ is not a Lie algebra then k is a collapsing level.*

In particular, if $W_k(\mathfrak{g}, \theta/2)$ unitary for three different values of k, then either $\mathfrak{g} = sl(2)$ or $\mathfrak{g}$ is not a Lie algebra and $\mathfrak{g}^\natural$ is a Lie algebra.

Proof. (1). By assumption $\mathfrak{g}_{-1/2} \neq 0$, take a nonzero $u \in \mathfrak{g}_{-1/2}$ such that $\phi(u) = u$ and compute using (7.6) with $m_1 = m_1' = 1$:

$$(G^{\{u\}}, G^{\{u\}}) = (G^{\{u\}}_{-3/2}\mathbf{1}, G^{\{u\}}_{-3/2}\mathbf{1}) = \sqrt{-1} \left\langle G^{\{u\}}_{3/2} G^{\{u\}}_{-3/2} \mathbf{1} \right\rangle = 4p(k)\langle u,u\rangle = 0.$$

If the form on $W^k(\mathfrak{g}, \theta/2)$ is positive semidefinite then $G^{\{u\}} \in I^k$, hence k is a collapsing level.

(2). Take $a \in \mathfrak{g}^\natural$ such that $p(a) = 1$, $\phi(a) = a$. Compute using (7.6) with $m_1 = m_1' = 1$

$$(J^{\{a\}}, J^{\{a\}}) = (J^{\{a\}}_{-1}\mathbf{1}, J^{\{a\}}_{-1}\mathbf{1}) = \sqrt{-1} \left\langle J^{\{a\}}_1 J^{\{a\}}_{-1} \mathbf{1} \right\rangle = 0,$$

hence $J^{\{a\}} \in I^k$. Assume that $\mathfrak{g}^\natural$ is simple; since $I^k \cap \mathfrak{g}^\natural$ is and ideal of $\mathfrak{g}^\natural$, then $\mathfrak{g}^\natural \subset I^k$. Since $\mathfrak{g}_{-1/2}$ is not the trivial representation of $\mathfrak{g}^\natural$, there exist $b \in \mathfrak{g}^\natural$ and $u \in \mathfrak{g}_{-1/2}$ such that $[b,u] \neq 0$. Since $[J^{\{b\}}{}_\lambda G^{\{u\}}] = G^{\{[b,u]\}}$, [1, Prop. 3.2] implies that k is collapsing.

The only remaining case, according to [1, Table 3], is $\mathfrak{g} = osp(m|n), m \geq 5$. In this case $\mathfrak{g}^\natural = osp(m-4|n) \oplus sl(2)$ and $\mathfrak{g}_{-1/2} = \mathbb{C}^{m-4|n} \otimes \mathbb{C}^2$, and the previous argument applies to $osp(m-4|n)$ acting on $\mathbb{C}^{m-4|n}$. □

Remark 7.10. The proof of Theorem 7.9 shows more generally that if there exists an odd (resp. even) element of integer (resp. half-integer) conformal weight in a W-algebra $W^k(\mathfrak{g}, x)$, which does not lie in the kernel of its homomorphism to $W_k(\mathfrak{g}, f)$, then the latter W-algebra is not unitary.

In general, even at collapsing levels, the simple vertex algebra $W_k(\mathfrak{g}, \theta/2)$ might not be unitary. It is clear that if $W^k(\mathfrak{g}, \theta/2)$ collapses to $\mathbb{C}$ then $W_k(\mathfrak{g}, \theta/2)$ is unitary. The list of such cases is given in Proposition 3.4 of [1].

In the next proposition we deal with other collapsing levels allowing unitarity.

Proposition 7.11. *Assume* $W_k(\mathfrak{g}, \theta/2) \neq \mathbb{C}$. *If* k *is a collapsing level and there is a conjugate linear involution* ϕ *on* $W_k(\mathfrak{g}, \theta/2)$ *such that the corresponding* ϕ*–invariant form is unitary, then the pair* $(\mathfrak{g}, k)$ *is one in the following list*

$$\mathfrak{g} = sl(m|n),\ m \neq n, n+1, n+2, m \geq 2, \qquad k = -1, \tag{7.7}$$

$$\mathfrak{g} = G_2, \qquad k = -4/3, \tag{7.8}$$

$$\mathfrak{g} = osp(m|n),\ m - n \geq 10,\ m - n \text{ even}, \qquad k = -2, \tag{7.9}$$

$$\mathfrak{g} = spo(2|3), \qquad k = -3/4, \tag{7.10}$$

$$\mathfrak{g} = D(2, 1; -\frac{1+n}{n+2}),\ n \in \mathbb{N}, \qquad k = -\frac{1+n}{n+2}. \tag{7.11}$$

Proof. Looking at [2, Table 5] one gets that in the cases listed in the statement there is a conjugate linear involution ϕ such that the ϕ–invariant Hermitian form on $W_k(\mathfrak{g}, \theta/2)$ is positive definite. In case (7.7) $W_k(\mathfrak{g}, \theta/2)$ is $M(\mathbb{C})$ (Heisenberg vertex algebra) and its unitarity is shown in Subsection 5.2. In cases (7.8), (7.9), (7.10), (7.11), $W_k(\mathfrak{g}, \theta/2)$ is a simple affine vertex algebra at positive integral level, hence unitarity follows from Subsection 5.3.

It remains only to check that the cases in the statement are the only cases where one can have unitarity at a collapsing level k, but, as explained in the discussion at the end of Subsection 5.3, a simple affine vertex algebra $V_k(\mathfrak{g})$ can be unitary if and only if $\mathfrak{g}$ is even and k is a positive integer. □

Corollary 7.12. *The following simple minimal W-algebras are unitary:*

1. $W_{-1}(sl(m|n), \theta/2) \cong M(\mathbb{C})$, $m \neq n, n+1, n+2, m \geq 2$, *where* $M(\mathbb{C})$ *is the Heisenberg vertex algebra with central charge* $c = 1$;
2. $W_{-4/3}(G_2, \theta/2) \cong V_1(sl(2))$ *with central charge* $c = 1$;
3. $W_{-2}(osp(m|n), \theta/2) \cong V_{\frac{m-n-8}{2}}(sl(2)), m - n \geq 10, m$ *and* n *even, with central charge* $c = \frac{3(m-n-8)}{m-n-4}$;
4. $W_{-3/4}(spo(2|3), \theta/2) \cong V_1(sl(2))$ *with central charge* $c = 1$;
5. $W_{-\frac{1+n}{n+2}}(D(2, 1; -\frac{1+n}{n+2}), \theta/2) \cong V_n(sl(2))$ *with central charge* $c = \frac{3n}{2+n}$, $n \in \mathbb{Z}_+$.

Remark 7.13. Case (4) of Corollary 7.12 is of special interest since $W_k(spo(2|3))$, tensored with one fermion, is the $N = 3$ superconformal algebra. The collapsing level corresponds to the central charge 1 of the simple W-algebra, isomorphic to $V_1(sl(2))$, hence to the central charge $c = 3/2$ of the $N = 3$ superconformal algebra, which is therefore unitary. This has been already observed in [16].

Remark 7.14. Another interesting case of Corollary 7.12 is (5). Recall that $W_k(D(2,1;a)$, tensored with four fermions and one boson, is the big $N=4$ superconformal algebra [13]. It follows from Corollary 7.12 that this algebra is unitary when $a=-\frac{1+n}{n+2}, n\in\mathbb{Z}_+$, the central charge being $-6a$.

8. Appendix: λ-brackets in minimal $W-$algebras

If $u\in\mathfrak{g}_{-1/2}$ and $v\in\mathfrak{g}_{1/2}$, then a direct computation shows that

$$[u,v]=\sum_\alpha([u,v]|a^\alpha)a_\alpha+\tfrac{([u,v]|x)}{(x|x)}x=\sum_\alpha(a_\alpha|[u,v])a^\alpha+\tfrac{(x|[u,v])}{(x|x)}x,$$

so

$$[u_\gamma,v]^\natural=\sum_\alpha([u_\gamma,v]|a^\alpha)a_\alpha=\sum_\alpha(u_\gamma|[v,a^\alpha])a_\alpha,$$
$$[u,u^\gamma]^\natural=\sum_\alpha(a_\alpha|[u,u^\gamma])a^\alpha=\sum_\alpha([a_\alpha,u]|u^\gamma)a^\alpha.$$

Moreover,

$$[[u,u^\gamma],[u_\gamma,v]]^\natural=\sum_{\alpha,\beta}([a_\alpha,u]|u^\gamma)(u_\gamma|[v,a^\beta])[a^\alpha,a_\beta].$$

Since, if $v\in\mathfrak{g}_{-1/2}$, $v=\sum_\gamma(v|u^\gamma)[e_{-\theta},u_\gamma]$, we obtain

$$2[e_\theta,v]=2\sum_\gamma(v|u^\gamma)[e_\theta,[e_{-\theta},u_\gamma]]=2\sum_\gamma(v|u^\gamma)[x,u_\gamma]=\sum_\gamma(v|u^\gamma)u_\gamma.$$

Substituting we find

$$\sum_\gamma([a_\alpha,u]|u^\gamma)(u_\gamma|[v,a^\beta])=(\sum_\gamma([a_\alpha,u]|u^\gamma)u_\gamma|[v,a^\beta])$$
$$=2([e_\theta,[a_\alpha,u]]|[v,a^\beta])=2\langle[a_\alpha,u],[v,a^\beta]\rangle.$$

Recall from [1], [14] that

$$[G^{\{u\}}{}_\lambda G^{\{v\}}]=-2(k+h^\vee)\langle u,v\rangle L+\langle u,v\rangle\sum_{\alpha=1}^{\dim\mathfrak{g}^\natural}:J^{\{a^\alpha\}}J^{\{a_\alpha\}}:+ \sum_{\gamma=1}^{\dim\mathfrak{g}_{1/2}}:J^{\{[u,u^\gamma]^\natural\}}J^{\{[u_\gamma,v]^\natural\}}:+2(k+1)(\partial+2\lambda)J^{\{[[e_\theta,u],v]^\natural\}} +\lambda\sum_{\gamma\in S_{1/2}}J^{\{[[u,u^\gamma],[u_\gamma,v]]^\natural\}}+2p(k)\lambda^2\langle u,v\rangle, \tag{8.1}$$

where $p(k)$ is a monic quadratic polynomial in k, listed in [1, Table 4]. Using the above formulas we can rewrite (8.1) as

$$
\begin{aligned}
(8.2)\qquad [G^{\{u\}}{}_\lambda G^{\{v\}}] = -2(k+h^\vee)\langle u,v\rangle L + \langle u,v\rangle \sum_{\alpha=1}^{\dim \mathfrak{g}^\natural} :J^{\{a^\alpha\}}J^{\{a_\alpha\}}: +\\
2\sum_{\alpha,\beta}\langle [a_\alpha,u],[v,a^\beta]\rangle :J^{\{a^\alpha\}}J^{\{a_\beta\}}: +2(k+1)(\partial+2\lambda)J^{\{[[e_\theta,u],v]^\natural\}}\\
+2\lambda\sum_{\alpha,\beta}\langle [a_\alpha,u],[v,a^\beta]\rangle J^{\{[a^\alpha,a_\beta]\}} + 2p(k)\lambda^2\langle u,v\rangle.
\end{aligned}
$$

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