ON THE VARIETY OF SPECIAL DIVISORS AND MODULI

by

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B.S., C.C.N.Y.
(1969)

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF
DOCTOR OF PHILOSOPHY
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
September 1973

Signature redacted
Signature of Author ....
Department of Mathematics, August 13, 1973

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Chairman, Departmental Committee
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Submitted to the Department of Mathematics on August 13, 1973
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Let S be an analytic space over $\mathbb{C}$ and let X be a family of curves over S. Let $X_S^{(n)}$ denote the n^{th} symmetric product of X over S and let $\mathfrak{J}_S$ denote the family of Jacobians over S. Suppose we are given a map $f : X_S^{(n)} \to \mathfrak{J}_S$ and let $u : f^*\Omega^1_{\mathfrak{J}_{S/S}} \to \Omega^1_{X^{(n)}_{S/S}}$ be the map induced by f. We define the analytic subspace $Z^r(u)$ of $X_S^{(n)}$ to be given by the vanishing of $\overset{n-r+1}{\Lambda} u$.

We study two cases:

(1) $S = \text{Spec}\ (\mathbb{C})$ and f is the classical map defined by integrating holomorphic differentials. We let $G_n^r(X)$ denote $Z^r(u)$ in this situation.

(2) $S = T_g$, the Teichmüller space, X is the universal family of Teichmüller surfaces of g, and f is the natural relativization over T_g of the map in case (1). We let $\mathfrak{D}_n^r$ denote $Z^r(u)$ in this situation.

Put $\tau = (r+1)(n-r) - rg$. We show that $\mathfrak{D}_n^1 - \mathfrak{D}_n^2$, if nonempty, is smooth of pure dimension $3g - 3 + \tau + 1$. From this result, we may conclude that for a generic curve X the analytic space $G_n^1(X) - G_n^2(X)$, if nonempty, is smooth of pure dimension $\tau + 1$.

Variational formulas due to Schiffer and Spencer and Rauch are employed in the study of $\mathfrak{D}_n^r$.

Thesis supervisor: Steven L. Kleiman
Title: Associate Professor of Mathematics

Dedication

In better words than I could say it:

I hear my father; I need never fear.

I hear my mother; I shall never be lonely, or want for love.

When I am hungry it is they who provide for me; when I am in dismay, it is they who fill me with comfort.

When I am astonished or bewildered, it is they who make the weak ground firm beneath my soul: it is in them that I put my trust.

When I am sick it is they who send for the doctor; when I am well and happy, it is in their eyes that I know best that I am loved; and it is towards the shining of their smiles that I lift up my heart and in their laughter that I know my best delight.

I hear my father and my mother and they are my giants, my king and my queen, beside whom there are no others so wise or worthy or honorable or brave or beautiful in this world.

I need never fear: nor ever shall I lack for loving kindness.

from "A Death in the Family,"
by James Agee

Acknowledgements

I am indebted to my advisor, Professor Steven Kleiman, who proposed the problem and who has given me much of his time and many useful suggestions. For their kind and generous help, I would like to thank Dr. Dan Laksov and Professors Michael Artin and Arthur Mattuck. For valuable discussions, I thank Professors G. Kempf, H. Rauch and A. Mayer, as well as my good friends Dr. Ridgway Scott and Professor John Morgan. I thank Phyllis Ruby for her excellent typing and, finally, express my gratitude to M.I.T. and the National Science Foundation for their financial support.

Table of Contents

Introduction

Let X be a complete, nonsingular curve of genus g over an algebraically closed field K. Let $X^{(n)}$ denote the n^{th} symmetric product of X. Let G_n^r denote the subvariety of $X^{(n)}$ of all divisors D such that $\dim |D| \geq r$. (In the literature, e.g. [16], G_n^r is more often used to denote the subvariety of the Jacobian of X of all linear systems of degree n and projective dimension at least r.)

Put τ equal to $(r+1)(n-r) - rg$. Brill and Noether [4] asserted that if τ were nonnegative and X were a generic curve, then G_n^r would have dimension $\tau + r$. The recent work of Kleiman and Laksov [14, 15] and Kempf [12] shows that for X any curve, if $\tau \geq 0$, then G_n^r has dimension at least $\tau + r$. We will show, in the case $K = \mathbb{C}$, that if X is a generic curve, then $G_n^1 - G_n^2$, if nonempty, has dimension $\tau + 1$.

We work in the category of analytic spaces over $\mathbb{C}$. We do this because we want to consider the Teichmüller space, an analytic, but not algebraic, variety ([7]).

Let Y be an analytic space over $\mathbb{C}$ and let E and F be locally free O_Y-modules of rank g and n respectively. Suppose we are given a map $u : E \to F$.

In Chapter I, we define the analytic space $Z^r(u)$ to be given by the vanishing of the map $\Lambda^{n-r+1} u$. Then we study the infinitesimal structure of $Z^r(u)$.

Let S be an analytic space over $\mathbb{C}$ and let X be a family of curves of genus g over S. Let $X_S^{(n)}$ denote the n^{th} symmetric product of X over S (cf. [11]) and let $\mathcal{J}_S$ denote the family of Jacobians over S. Suppose we are given a map $f : X_S^{(n)} \to \mathcal{J}_S$. Let $u : f^*\Omega^1_{\mathcal{J}_{S/S}} \to \Omega^1_{X^{(n)}_{S/S}}$ be the map induced by f. We study the analytic space $Z^r(u) \subseteq X_S^{(n)}$ in the following two situations:

(i) $S = \operatorname{Spec}(\mathbb{C})$ and f is the classical map defined by integrating holomorphic differentials. We let $G^r_n(X)$ denote $Z^r(u)$ in this situation. A $\mathbb{C}$-valued point of $G^r_n(X)$ is a divisor D of degree n and projective dimension at least r.

(ii) $S = T_g$, the Teichmüller space, X is the universal family of Teichmüller surfaces of genus g, and f is the natural relativization over T_g of the map in case (i). We let $\mathfrak{G}^r_n$ denote $Z^r(u)$ in this situation.

In Chapter I, we show how to compute the dimension of the tangent space at a point D of $G_n^r(X)$. This is done by seeing when a tangent vector to $X^{(n)}$ at D is a tangent vector to $G_n^r(X)$ at D. (Severi appears to employ a similar argument in [25], pp. 388-9, with his "Linearmantels".)

In Chapter II, we describe a variation of structure of Riemann surfaces due to Schiffer and Spencer [23]. We derive variational formulas similar to those in [23], but much closer in form to those in Rauch [22]. We then state a result due to C. Patt [21].

In the second section of Chapter II, we review the work of Meis [20], and give his examples of Riemann surfaces.

In Chapter III, we use Patt's Theorem and the variational formulas to help to analyze the structure of $\mathfrak{D}_n^r$. Our main result is

<u>Theorem</u>: Suppose $y \varepsilon \mathfrak{D}_n^1 - \mathfrak{D}_n^2$. Then the dimension of the tangent space to $\mathfrak{D}_n^1$ at y is $3g - 3 + \tau + 1$.

From this result, we can conclude that if X is a generic Riemann surface, then $G_n^1(X) - G_n^2(X)$, if nonempty, is smooth of pure dimension $\tau + 1$.

We then use Meis's examples and perform computations which show that if $\tau \geq 0$, then the analytic space $\mathfrak{G}_n^2$ (resp. $\mathfrak{G}_n^3$) has a component of dimension

$3g - 3 + \tau + 2$ (resp. $3g - 3 + \tau + 3$).

In Chapter IV, we discuss some open questions. In particular, we discuss the moduli of curves with "extra-special" (i.e. $\tau < 0$) divisors on them.

Chapter I

The Analytic Space of Special Divisors and its Infinitesimal Structure

§1.

We will work in the category of analytic spaces over $\mathbb{C}$. We take the Séminaire Cartan, 1960-61, as our foundational reference. In particular, we allow the structure sheaf of an analytic space to contain nilpotents. We work in this category because we will want to consider the Teichmüller space, an analytic, but not algebraic, variety [7].

Let S be an analytic space over $\mathbb{C}$. Denote by ((an/S)) the category of analytic spaces over S. Let Y be an analytic space over S and let E and F be locally free O_Y-modules of rank g and n respectively. Suppose we are given a map $u : E \to F$. Define the functor $Z^r(u) : ((an/S))^0 \to ((Sets))$ by

$$Z^r(u)(T) = \{g \in \operatorname{Hom}(T,Y) \,\Big|\, \overset{n-r+1}{\Lambda} g^* u = 0\}.$$

We wish to show that this functor is represented by an analytic subspace of Y.

Definition 1 [7]: Let S be an analytic space and let $G : ((an/S))^0 \to ((Sets))$ be a functor. We say that G is of a local nature if for every T the presheaf $U \to G(U)$, where U runs through the open sets of T, is a sheaf.

Remark: This is the analog to the notion of a Zariski sheaf in the category of contravariant functors from ((Schemes)) to ((Sets)).

Lemma 1: Let (S_i) be a covering of an analytic space S by open sets. Let $G : ((an/S))^0 \to ((Sets))$ be a functor. Then G is representable iff G is of a local nature and for every i, the functor $G/S_i : ((an/S_i))^0 \to ((Sets))$ is representable.

Proof: [7], Corollary 5.7 of Expose 7. ∎

Our functor $Z^r(u)$ is clearly of a local nature. Hence, by the lemma, its representability is a local question.

Let y be a point of Y. Since E and F are locally free of rank g and n respectively, the map u

is given locally at y by an $n \times g$ matrix $[f_{jk}]$ of functions regular at y. The functor $Z^r(u)$ is then locally represented by the analytic subspace defined by the vanishing of the minors of order $n - r + 1$ of the matrix $[f_{jk}]$. Thus we have

Proposition 1: $Z^r(u)$ is represented by an analytic subspace of Y. ▮

We will use $Z^r(u)$ to also denote this analytic subspace.

Put $\rho = \text{rank } (u \otimes \kappa(y))$. Locally at y, both E and F split off a direct summand of rank ρ, and u maps one summand isomorphically onto the other. The map that u induces on the other two summands is given by an $(n-\rho) \times (g-\rho)$ matrix $[e_{jk}]$ of functions regular at y. The analytic space $Z^r(u)$ is also defined locally at y by the vanishing of the minors of order $(n - r + 1 - \rho)$ of the matrix $[e_{jk}]$ (cf. [14]).

Proposition 2: Assume $r > 0$. Then the points of $Z^{r+1}(u)$ are singular points of $Z^r(u)$.

Proof: Suppose $y \in Z^{r+1}(u)$. Then we have $\rho < n - r$. By construction, the e_{jk} above vanish at y, hence are in the maximal ideal m of $O_{Y,y}$. The analytic space $Z^r(u)$ is defined locally at y by the vanishing of the minors of order $(n - r + 1 - \rho)$ of the matrix $[e_{jk}]$ and, since $\rho < n - r$, all these minors are of order at least 2, hence are in m^2. Thus y cannot be a smooth point of $Z^r(u)$. ∎

We want now to study the infinitesimal structure of $Z^r(u)$. Let ξ denote a tangent vector to Y at y. We will also use ξ to denote the comorphism, which is a $\mathbb{C}$-homomorphism of local rings

$$\xi : O_{Y,y} \to \mathbb{C}[\varepsilon]/(\varepsilon^2).$$

We are interested in seeing when ξ is a tangent vector to $Z^r(u)$ at y. By definition, this will be true if $\overset{n-r+1}{\Lambda} \xi^* u = 0$.

Proposition 3: ξ is a tangent vector to $Z^r(u)$ at y iff the minors of order $n - r + 1$ of the matrix $[\xi(f_{jk})]$ are all zero.

Proof: It is easy to see that the map $\xi^* u$ is given by the matrix $[\xi(f_{jk})]$. Thus we have $\overset{n-r+1}{\Lambda} \xi^* u = 0$

iff the minors of order $n - r + 1$ of $[\xi(f_{jk})]$ all vanish. ∎

We now assume that Y is smooth of dimension m over $\mathbb{C}$. Let $y \in Y$ and let $\sigma_1,\ldots,\sigma_m$ be local parameters on Y at y. Let s_ℓ in $\mathbb{C}$ be given by

$$\xi(\sigma_\ell) = s_\ell \varepsilon \qquad \ell = 1,2,\ldots,m$$

Then, by Taylor's Theorem, we have

$$\xi(f_{jk}) = f_{jk}(y) + \varepsilon \sum_{\ell=1}^{m} s_\ell \frac{\partial f_{jk}}{\partial \sigma_\ell}(y).$$

The vanishing of the minors of order $n - r + 1$ of the matrix $[\xi(f_{jk})]$ gives rise to linear equations in the s_ℓ. These equations must be satisfied for ξ to be a tangent vector to $Z^r(u)$ at y. If we view $s_1,\ldots,s_m$ as being unknowns, then <u>the dimension of the solution space of this system of equations is the dimension of the tangent space to $Z^r(u)$ at y</u>.

If $y \in Z^r(u) - Z^{r+1}(u)$, we will want to use the following lemma.

Lemma 2: Let A be a commutative ring (with unit). Let $M = [a_{jk}]$ be an $m \times n$ matrix over A. Suppose a minor μ of order r is a unit, and that every minor of order $r + 1$ containing μ vanishes. Then every minor of order $r + 1$ vanishes.

Proof: The following proof owes its brevity to D. Eisenbud.

Without loss of generality, we may assume that μ is the leading (i.e. upper left) minor of order r. Since μ is a unit, we may perform column operations using the first r columns to change M to the matrix

$$M' = \left|\begin{array}{cccc} & \mu & & O \\ \hline a_{r+1,1} & \cdots & a_{r+1,r} & \\ \vdots & & \vdots & N \\ a_{m,1} & \cdots & a_{m,r} & \end{array}\right|$$

where N is an $(m-r) \times (n-r)$ matrix.

Then, by row operations using the first r rows, we may change M' to the matrix

$$M'' = \left|\begin{array}{cc} \mu & 0 \\ \hline 0 & N \end{array}\right|$$

Now, no minor containing μ is affected by performing these row and column operations. Hence, the minors of order $r + 1$ of M'' which contain μ are all zero. Thus N is the zero matrix.

But this implies that every column of M is a linear combination of the first r columns of M. Hence, every minor of order $r + 1$ of M is zero. ∎

Suppose now that $y \varepsilon Z^r(u) - Z^{r+1}(u)$. Then the matrix $[f_{jk}]$ has rank $n - r$. We may thus assume that the leading minor of order $n - r$ of $[f_{jk}]$, call it μ, is nonzero. Let μ' denote the leading minor of order $n - r$ of $[\xi(f_{jk})]$. Then $\mu' = \mu + c\varepsilon$ for some $c \varepsilon \mathbb{C}$. Since μ is nonzero, μ' does not lie in the maximal ideal of $\mathbb{C}[\varepsilon]/(\varepsilon^2)$, hence is a unit. We then have, by Proposition 3 and lemma 2, that ξ is a tangent vector to $Z^r(u)$ at y iff the minors of order $n - r + 1$ of $[\xi(f_{jk})]$ which contain μ' all vanish. Obviously, there are $r \cdot (g - n + r)$ such minors. If the equations in the $\mathbb{S}_{\ell}$ given by the vanishing of these minors are linearly independent (over $\mathbb{C}$), then the dimension of the tangent space to $Z^r(u)$ at y is $m - r \cdot (g - n + r)$. We could then conclude that y is a smooth point of $Z^r(u)$ by virtue of the following proposition.

Proposition 4: Either $Z^r(u)$ is empty, or each component has codimension at most $r \cdot (g - n + r)$ in Y.

Proof: This is proved in [13] for Y a scheme. With the obvious modifications, the proof is valid for Y an analytic space. ▮

§2.

In this section, we describe the two situations to which we will apply the theory in §1.

Situation 1: A Riemann Surface

Let X be a compact Riemann surface of genus $g > 0$. Choose a point $P \varepsilon X$ and denote by $\pi_1(X,P)$ the fundamental group formed by the homotopy classes of closed curves from P. The group $\pi_1(X,P)$ can be generated by 2g generators $\gamma_1,\ldots,\gamma_g,\delta_1,\ldots,\delta_g$ which satisfy the single relation

$$\gamma_1\delta_1\gamma_1^{-1}\delta_1^{-1}\ldots\gamma_g\delta_g\gamma_g^{-1}\delta_g^{-1} = 1 \qquad [27].$$

Such an ordered system of generators is called a canonical homotopy basis.

We will also use the symbols $\gamma_1,\ldots,\gamma_g$, $\delta_1,\ldots,\delta_g$ to denote the classes of these elements in $H_1(X,Z)$. These classes form a canonical homology basis, meaning that the matrix of the intersection pairing on X with respect to this basis is

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

where I denotes the $g \times g$ identity and 0 the $g \times g$ zero matrix ([27], [8]).

Let $d\zeta_1, \ldots, d\zeta_g$ be a basis of the holomorphic differentials on X. Choose a canonical homology basis $\{\gamma_i, \delta_i\}_{i=1}^{g}$ and put

$$a_{ij} = \int_{\gamma_i} d\zeta_j$$

$$b_{ij} = \int_{\delta_i} d\zeta_j \qquad i,j = 1,\ldots,g.$$

One calls the $g \times 2g$ matrix $[a_{ij}, b_{ij}]$ the <u>period matrix</u> of X. The $2g$ columns of this matrix generate a maximal lattice subgroup $\mathcal{L}$ of $\mathbb{C}^g$ and the complex analytic torus $\mathbb{C}^g/\mathcal{L}$ is the <u>Jacobian variety</u> J of X ([8], §8).

Fix a point $P_0 \in X$. Consider the mapping $\psi : X \to J$ given by

$$\psi(P) = \left(\int_{P_0}^{P} d\zeta_1, \ldots, \int_{P_0}^{P} d\zeta_g\right) \text{ mod periods.}$$

Denote by $X^{(n)}$ the n^{th} symmetric product of X. Given a divisor D of degree n on X, we will use D to also denote the point of $X^{(n)}$ representing this divisor. For any n, we may extend ψ to a map $f : X^{(n)} \to J$ as follows. If D is the divisor $\sum_{i=1}^{n} P_i$ then

$$f(D) = \left(\sum_{i=1}^{n} \int_{P_0}^{P_i} d\zeta_1, \ldots, \sum_{i=1}^{n} \int_{P_0}^{P_i} d\zeta_g \right) \text{ mod periods.}$$

Let $u : f^*\Omega_J^1 \to \Omega^1_{X^{(n)}}$ be the map induced by f. Since $X^{(n)}$ and J are smooth of dimension n and g respectively, the sheaves $f^*\Omega_J^1$ and $\Omega^1_{X^{(n)}}$ are locally free $O_{X^{(n)}}$-modules of rank g and n respectively. Thus, we may consider the analytic subspace $Z^r(u)$ of $X^{(n)}$. We will denote by

$$G_n^r(X),$$

or just G_n^r if the reference to X is clear, the analytic space $Z^r(u)$ arising in this situation.

Proposition 5: $D \in G_n^r \Leftrightarrow \dim |D| \geq r$.

Proof: Let M denote the matrix of u evaluated at D. Then it is shown in [9] (also in [18] and [5]) that the rank of M is $n - \dim |D|$. But $D \in G_n^r$ iff the rank of M is at most $n - r$, hence $D \in G_n^r \Leftrightarrow \dim |D| \geq r$. ∎

Thus, the definition of G_n^r is consistent with the classical one (cf. the Introduction). In the next section, we will write out the matrix M explicitly.

Put τ equal to $(r+1)(n-r) - rg$. We know by [14] that if τ is at least zero, then G_n^r is nonempty. Hence, by Proposition 4, every component of G_n^r has dimension at least $n - r(g - n + r) = \tau + r$. Brill and Noether [4] asserted that for a generic curve X, if τ is nonnegative, then $G_n^r(X)$ has dimension equal to $\tau + r$. To prove something true for a generic Riemann surface, one is led to consider the following situation.

Situation 2: The Universal Family of Teichmüller Surfaces

We first give a definition of a Teichmüller surface. Our definition is from [1] (also see [21], [22]).

If P and Q are points of X, then we know that $\pi_1(X,P)$ and $\pi_1(X,Q)$ are isomorphic. Furthermore, we can associate a unique isomorphism with every arc σ from P to Q: it is the isomorphism T^{σ} which transforms the homotopy class of a closed curve C from P into the homotopy class of $\sigma^{-1}C\sigma$. Choose a canonical homotopy basis for $\pi_1(X,P)$ and denote it by (Γ,Δ). Let (Γ',Δ') be a canonical basis for $\pi_1(X,Q)$. We shall say that (Γ,Δ) and (Γ',Δ') are equivalent iff $(\Gamma',\Delta') = T^{\sigma}(\Gamma,\Delta)$ for some σ; i.e. each generator in (Γ,Δ) is transformed by T^{σ} into the corresponding generator in (Γ',Δ'). The conditions for an equivalence relation are obviously satisfied.

Suppose now that ϕ is a topological mapping of X onto another surface X'. Then any canonical basis (Γ,Δ) on X is transformed into a canonical basis $\phi(\Gamma,\Delta)$ on X' formed by the images of the generators. We shall say that ϕ maps $(X,(\Gamma,\Delta))$ onto $(X',(\Gamma',\Delta'))$ iff $\phi(\Gamma,\Delta)$ is equivalent to (Γ',Δ'). Finally, if ϕ is a conformal mapping, we say that ϕ maps $(X,(\Gamma,\Delta))$ conformally onto $(X',(\Gamma',\Delta'))$, and the two pairs are said to be conformally equivalent. Again, the conditions for an equivalence relation are trivially fulfilled.

Definition 2: A Teichmüller surface is a class of conformally equivalent pairs $(X,(\Gamma,\Delta))$.

Theorem 1: There exists an analytic space T_g and a family V of Teichmüller surfaces of genus g over T_g which is universal in the following sense: for every family X of Teichmüller surfaces of genus g over an analytic space S, there exists a unique map $\Phi : S \to T_g$, such that X is isomorphic (as a family of Teichmüller surfaces) to the pullback via Φ of V/T_g.

Proof: [7], Theorem 3.1 of Expose 7 and Expose 17. ▮

T_g is called the Teichmüller space (for Teichmüller surfaces of genus g). The Teichmüller space is a smooth, irreducible, and simply connected analytic space [7].

We will now relativize the map ψ of situation 1 over the Teichmüller space and obtain a map from the universal family of Teichmüller surfaces to the family of Jacobians, as is done in [19].

Let $h : V \to T_g$ denote the structural morphism. By well-known topological facts, since T_g is simply connected, the fiber bundle $R^1 h_* Z$ is trivial. Thus, there are sections of this bundle which give rise to cycles $\gamma_i(s), \delta_i(s)$, $i = 1,\dots,g$, which form a canonical homology basis of $H_1(V ,Z)$, $s \varepsilon T_g$ [19]

Consider the sheaf Ω^1_{V/T_g}. For all $s \varepsilon T_g$, we have

$$\dim H^0(V_s,\Omega^1_{V/T_g} \otimes \kappa(s)) = \dim H^0(V_s,\Omega^1_{V_s}) = g.$$

Hence, $h_*\Omega^1_{V/T_g}$ is a vector bundle of rank g over T_g and we have

$$h_*\Omega^1_{V/T_g} \otimes \kappa(s) \cong H^0(V_s,\Omega^1_{V_s})$$

by [6].

Choose holomorphic sections $d\zeta^*_i$, $i = 1,\ldots,g$, of $h_*\Omega^1_{V/T_g}$ such that $\{d\zeta^*_i(s)\}^g_{i=1}$ is a basis for $H^0(V_s,\Omega^1_{V_s})$, $s \varepsilon T_g$ (cf. [18]). Put

$$a_{ij}(s) = \int_{\gamma_i(s)} d\zeta^*_j(s)$$

$$b_{ij}(s) = \int_{\delta_i(s)} d\zeta^*_j(s) \qquad i,j = 1,\ldots,g.$$

For each $s \varepsilon T_g$, the matrix $[a_{ij}(s), b_{ij}(s)]$ is the period matrix of V_s. Recall that the columns of this matrix generate a maximal lattice subgroup of $\mathbb{C}^g$. Let $\mathcal{J}$ be the quotient of $T_g \times \mathbb{C}^g$ by this family of lattices. The induced projection $\mathcal{J} \to T_g$ gives a complex analytic family of complex tori, the fiber $\mathcal{J}_s$ being the Jacobian variety of the Teichmüller surface V_s [19].

Since our concern will only be local, we assume that there exist sections of $V \to T_g$. Let $P_0^*(s)$ be such a section. As in [19], define a map $\psi : V \to \mathcal{J}$ by

$$\psi(s,P) = (s, \int_{P_0^*(s)}^{P} d\zeta_1^*(s), \ldots, \int_{P_0^*(s)}^{P} d\zeta_g^*(s)) \text{ mod periods}$$

for $p \in V_s$.

Denote by $V_{T_g}^{(n)}$ the n^{th} symmetric product of V over T_g (cf. [11]). Extend ψ to a map $f : V_{T_g}^{(n)} \to \mathcal{J}$ as follows. If $s \in T_g$ and $D \in (V_{T_g}^{(n)})_s$ is the divisor $\sum_{i=1}^{n} P_i$ on V_s, then

$$f(s,D) = (s, \sum_{i=1}^{n} \int_{P_0^*(s)}^{P_i} d\zeta_1^*(s), \ldots, \sum_{i=1}^{n} \int_{P_0^*(s)}^{P_i} d\zeta_g^*(s)) \text{ mod periods}$$

Let $u : f^*\Omega^1_{\mathcal{J}/T_g} \to \Omega^1_{V^{(n)}_{T_g}/T_g}$ be the map induced by f. Since $\mathcal{J}$ and $V^{(n)}_{T_g}$ are smooth over T_g of relative dimension g and n respectively, the sheaves $f^*\Omega^1_{\mathcal{J}/T_g}$ and $\Omega^1_{V^{(n)}_{T_g}/T_g}$ are locally free of rank g and n respectively. Thus, we may consider the analytic subspace $Z^r(u) \subseteq V^{(n)}_{T_g}$ of §1. We will denote by

$$\mathcal{G}^r_n$$

the analytic space $Z^r(u)$ which arises in this situation.

§3.

We return to situation 1 of the preceding section. We will compute the matrix of the map $\xi^* u$, where ξ is a tangent vector at a point of $X^{(n)}$. As in [18], we will first direct our attention to a point nP on $X^{(n)}$.

Let $P \varepsilon X$ and let t be a local parameter on X at P. Consider the divisor nP. Let $t_1, \ldots, t_n$ be n copies of t. Then the elementary symmetric functions of the t_j, denoted $\sigma_1(t_1, \ldots, t_n), \ldots, \sigma_n(t_1, \ldots, t_n)$, form a system of local parameters on $X^{(n)}$ at the point nP [2].

To understand the map u, we need to study the space of holomorphic 1-forms on $X^{(n)}$. We have

<u>Proposition 6</u>: The space of holomorphic 1-forms on X is naturally isomorphic to the space of holomorphic 1-forms on $X^{(n)}$. Both these spaces are isomorphic to the space of holomorphic symmetric 1-forms on the Cartesian product X^n.

<u>Proof</u>: [18] (we give the correspondence below). ∎

We will identify the space of holomorphic symmetric 1-forms on X^n with the space of holomorphic 1-forms on $X^{(n)}$.

We now make explicit the correspondence between holomorphic 1-forms on X and holomorphic symmetric 1-forms on X^n. Let $d\zeta$ be a holomorphic 1-form on X, and write

$$d\zeta = \sum_{\ell=0}^{\infty} a_\ell t^\ell dt.$$

Put

$$\tau_j = t_1^j dt_1 + \ldots + t_n^j dt_n \qquad j = 0,1,2,\ldots$$

Then we have

<u>Proposition 7</u>: The symmetric 1-form $d\tilde{\zeta}$ on X^n which corresponds to $d\zeta$ may be written

$$d\tilde{\zeta} = \sum_{\ell=0}^{\infty} a_\ell \tau_\ell$$

<u>Proof</u>: [18] (also see [5], lemma 2). ∎

Now, we can express $d\tilde{\zeta}$ in terms of $d\sigma_1, \ldots, d\sigma_n$ by using the following identities [18]:

$$\sigma_k\tau_0 - \sigma_{k-1}\tau_1 + \ldots + (-1)^k\tau_k = d\sigma_{k+1}$$

(By convention, $\sigma_k = 0$ and $d\sigma_k = 0$ if $k > n$)

Inverting these identities, and writing out only the linear terms, we obtain

$$\tau_k = (-1)^k(d\sigma_{k+1} - \sigma_1 d\sigma_k - \sigma_2 d\sigma_{k-1} - \ldots - \sigma_k d\sigma_1)$$
$$+ \text{ higher order terms}$$

Thus we get

$$(*) \quad d\tilde{\zeta} = a_0 d\sigma_1 + a_1(-d\sigma_2 + \sigma_1 d\sigma_1) +$$
$$a_2(d\sigma_3 - \sigma_1 d\sigma_2 - \sigma_2 d\sigma_1 + \text{higher order terms})$$
$$+ \ldots$$

Now suppose $d\zeta_1, \ldots, d\zeta_g$ is a basis for the holomorphic Abelian differentials on X. Write

$$d\zeta_k = \sum_{\ell=0}^{\infty} a_{k,\ell} t^{\ell} dt \qquad k = 1,\ldots,g$$

and put ϕ_k equal to $\sum_{\ell=0}^{\infty} a_{k,\ell} t^{\ell}$.

Recall that the map $f : X^{(n)} \to J$ is given by

$$f(\sum_{i=1}^{n} P_i) = (\sum_{i=1}^{n} \int_{P_0}^{P_i} d\zeta_1,\ldots, \sum_{i=1}^{n} \int_{P_0}^{P_i} d\zeta_g) \text{ mod periods}$$

Let $\frac{\partial\tilde{\zeta}_k}{\partial\sigma_j}$ be given by

$$d\tilde{\zeta}_k = \sum_{j=1}^{n} \frac{\partial\tilde{\zeta}_k}{\partial\sigma_j} d\sigma_j .$$

Then we have

Proposition 8: The Jacobian matrix of f, i.e. the matrix of the map u, locally at nP is

$$\left[\frac{\partial\tilde{\zeta}_k}{\partial\sigma_j}\right] \qquad \begin{matrix} j = 1,\ldots,n \\ k = 1,\ldots,g \end{matrix}$$

Proof: This readily follows from the definitions (cf. [5], lemma 2 and [9] p. 80 ff.). ▮

It is easily seen from (*) that

$$(**)\quad \frac{\partial\tilde{\zeta}_k}{\partial\sigma_j} = (-1)^{j-1}(a_{k,j-1} + \sum_{\ell=1}^{n}(-1)^{\ell-1}a_{k,j+\ell-1}\sigma_\ell)$$

$$+ \text{ higher order terms}$$

Thus we have

$$\frac{\partial\tilde{\zeta}_k}{\partial\sigma_j}(nP) = (-1)^{j-1}a_{k,j-1}$$

Hence, the matrix of u evaluated at nP is just

$$M = \left[\frac{(-1)^{j-1}}{(j-1)!}\,\phi_k^{(j-1)}(P)\right] \qquad \begin{matrix} j = 1,\ldots,n \\ k = 1,\ldots,g. \end{matrix}$$

This matrix has rank $n - \dim|nP|$ by Proposition 5.

Now let ξ denote a tangent vector to $X^{(n)}$ at nP and let s_j in $\mathbb{C}$ be given by $\xi(\sigma_j) = s_j\varepsilon$, $j = 1,\ldots,n$. Then we have, as in §1,

$$\xi(\frac{\partial\tilde{\zeta}_k}{\partial\sigma_j}) = \frac{\partial\tilde{\zeta}_k}{\partial\sigma_j}(nP) + \varepsilon\sum_{\ell=1}^{n} s_\ell\,\frac{\partial^2\tilde{\zeta}_k}{\partial\sigma_\ell\partial\sigma_j}(nP).$$

One sees from (**) that

$$\frac{\partial^2 \tilde{\zeta}_k}{\partial\sigma_\ell \partial\sigma_j}(nP) = (-1)^{j+\ell} a_{k,j+\ell-1} \hat{s}_\ell$$

for $\ell = 1,\ldots,n$.

To sum up, we have shown the following

Proposition 9: Let $P \in X$ and let t be a local parameter on X at P. Let $d\zeta_1,\ldots,d\zeta_g$ be a basis for the holomorphic differentials on X and write $d\zeta_k = \phi_k dt$. Let ξ be a tangent vector to $X^{(n)}$ at nP and let $\hat{s}_j$ in $\mathbb{C}$ be given by

$\xi(\sigma_j) = \hat{s}_j \varepsilon$, $j = 1,\ldots,n$, where $\sigma_1,\ldots,\sigma_n$ are local parameters on $X^{(n)}$ at nP. Then the matrix

$\left[\xi\left(\frac{\partial\tilde{\zeta}_k}{\partial\sigma_j}\right)\right]$, the matrix of $\xi^* u$, is

$$\left[\frac{(-1)^{j-1}}{(j-1)!}\phi_k^{(j-1)}(P) + \varepsilon \sum_{\ell=1}^{n} \frac{(-1)^{j+\ell}}{(j+\ell-1)!} \hat{s}_\ell \phi_k^{(j+\ell-1)}(P)\right]$$

$j = 1,\ldots,n$; $k = 1,\ldots,g$.

Now consider a divisor of degree n of the form $D = m_1P_1 + \ldots + m_dP_d$. The obvious map

$$X^{(m_1)} \times \ldots \times X^{(m_d)} \to X^{(n)}$$

is a local analytic isomorphism [18]. Thus, a local calculation on $X^{(n)}$ can be performed instead on $X^{(m_1)} \times \ldots \times X^{(m_d)}$, using d sets of elementary symmetric functions as local parameters. Also, the tangent space to $X^{(n)}$ at D is isomorphic to the direct sum of the tangent spaces to $X^{(m_j)}$ at m_jP_j, $j = 1,\ldots,n$.

Let $f_j : X^{(m_j)} \to J$ be the map defined in situation 1 of §2. Let $u_j : f_j^*\Omega_J^1 \to \Omega^1_{X^{(m_j)}}$ be the map induced by f_j. Locally, the map f is the one induced by the f_j and the map u is the one induced by the u_j, using the fact that $X^{(m_1)} \times \ldots \times X^{(m_d)} \to X^{(n)}$ is a local analytic isomorphism.

Thus, the matrix of u locally at D is obtained by "stacking" the matrices of the u_j locally at m_jP_j. Since a tangent vector ξ to $X^{(n)}$ at D is determined by tangent vectors ξ_j to $X^{(m_j)}$ at m_jP_j, the matrix of $\xi^* u$ is also obtained by "stacking" the matrices of the $\xi_j^* u_j$. Thus, by using Proposition 9 and the above discussion, one can write the matrix of $\xi^* u$ for any D. Let M' denote the matrix of $\xi^* u$.

In the simple case when all $m_j = 1$, i.e. $D = P_1 + \ldots + P_n$ with all points distinct, it is easy to be very explicit. Let t_j denote a local parameter on X at P_j and write $d\zeta_k = \phi_{jk} dt_j$. Then the matrix M' is just

$$M' = [\phi_{jk}(P_j) + \varepsilon s_j \phi'_{jk}(P_j)] \qquad \begin{array}{l} j = 1,\ldots,n \\ k = 1,\ldots,g\ . \end{array}$$

We can simplify M' somewhat, for any divisor D. Put i equal to $\dim H^1(X, O_X(D))$. One calls i the index of specialty of D and D is a special divisor if i is positive. With our notation, the Riemann-Roch Theorem is $n - \dim |D| = g - i$. Assume that i is positive, and choose a basis of the holomorphic differentials such that $d\zeta_{g-i+1}, \ldots, d\zeta_g$ vanish on D. Thus, the last i columns of the matrix of u evaluated at D (we have denoted this matrix by M) are identically zero.

Assume $D \varepsilon G_n^r - G_n^{r+1}$. Then $\dim |D| = r$ and M has rank $n - r$. By performing row permutations, if necessary, we may arrange to have the leading minor of order $n - r$ of M be nonzero. (this involves slightly altering the form of M from that specified earlier). Call this minor μ. Let μ' denote the leading minor of order $n - r$ of M', the matrix of $\xi^* u$. Recall that since μ is nonzero, μ' is a unit in $\mathbb{C}[\varepsilon]/(\varepsilon^2)$. Hence, by lemma 2, to have that every minor of order $n - r + 1$ of M' is zero, it is sufficient that the minors of order $n - r + 1$ containing μ' all vanish.

Now, the last i columns of M' will have "pure" ε-terms, i.e. elements of the maximal ideal of $\mathbb{C}[\varepsilon]/(\varepsilon^2)$. This is because, by our choice of a basis of differentials, the last i columns of M were identically zero. Thus, in computing a minor of order $n - r + 1$ containing μ', any ε's in the first $n - r$ columns will be "cancelled" by the ε in the last column of the minor of order $n - r + 1$. Therefore, we have established

<u>Lemma 3</u>: For purposes of computing the minors of order $n - r + 1$ containing μ', we may replace the first $n - r$ columns of M' by the first $n - r$ columns of M. ∎

Denote by $\Delta_D M$ the matrix obtained by replacing the first $n - r$ columns of M' by the first $n - r$ columns of M.

To illustrate this, we again turn to the case when D consists of n distinct points. Then we have

$$\Delta_D \acute{M} = \left[\begin{array}{cccc|c}
 & & & & \overbrace{\hphantom{k = g-i+1,\ldots,g}}^{\text{i columns}} \\
 & \phi_{1,1}(P_1) & \cdots & \phi_{1,n-r}(P_1) & \\
 & \vdots & \cdots & \vdots & \\
 & \phi_{n-r,1}(P_{n-r}) & \cdots & \phi_{n-r,n-r}(P_{n-r}) & \varepsilon S_j \phi'_{j,k}(P_j) \\
 & \phi_{n-r+1,1}(P_{n-r+1}) & \cdots & \phi_{n-r+1,n-r}(P_{n-r+1}) & j = 1,\ldots,n \\
\text{r rows}\left\{\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right. & \vdots & & \vdots & k = g-i+1,\ldots,g \\
 & \phi_{n,1}(P_n) & \cdots & \phi_{n,n-r}(P_n) &
\end{array}\right]$$

Now, for any D, there are $r \cdot (g - n + r) = r \cdot i$ minors of order $n - r + 1$ of $\Delta_D M$ which contain μ. The vanishing of each of these minors gives a linear equation in $\tilde{s}_1, \ldots, \tilde{s}_n$. One might expect that in a generic situation these $i \cdot r$ equations would be independent. In Chapter III, we will consider the generic case by analyzing situation 2 of the preceding section. To see what the functions f_{jk} of §1 are in situation 2, we will use variational formulas similar to those first derived by Schiffer and Spencer [23], and later refined by Rauch [22] to the form in which we use them. We derive these formulas in the next chapter.

Chapter II

The Analytic Theory

§1. Variation of Structure of Teichmüller Surfaces

We describe a variation of structure of Teichmüller surfaces due to Schiffer and Spencer [23]. We then derive variational formulas similar to those found in [22], [21] and [23]. For generalities on Riemann surface theory, we recommend [27], and for more on Teichmüller surfaces and the variational formulas, we recommend [22] and the references given there.

Let X be a compact Riemann surface of genus $g > 0$. Let $\Gamma = (\gamma_1,\ldots,\gamma_g)$ and $\Delta = (\delta_1,\ldots,\delta_g)$ be a canonical homotopy basis as in I - §2. Let Π be the simply connected surface obtained by the canonical dissection of X determined by Γ and Δ (cf. [27]). Let $\partial\Pi$ denote the boundary of Π. We have

$$\partial\Pi = \sum_{i=1}^{g} \gamma_i + \delta_i + \gamma_i^{-1} + \delta_i^{-1} .$$

Let α and β be Abelian integrals on X. (For a good classical treatment of Abelian integrals, see

Bliss [3], Chapter V.) Suppose that α and $d\beta$ have at most poles in the interior of Π and are analytic on $\partial\Pi$. (So, in particular, $d\alpha$ has no residues.) Then

$$\int_{\partial\Pi} \alpha d\beta = \sum_{i=1}^{g} \left(\int_{\gamma_i} \alpha d\beta + \int_{\delta_i} \alpha d\beta + \int_{\gamma_i^{-1}} \alpha d\beta + \int_{\delta_i^{-1}} \alpha d\beta\right)$$

$$= \sum_{i=1}^{g} \left(\int_{\gamma_i} (\alpha^{+} - \alpha^{-}) d\beta + \int_{\delta_i} (\alpha^{+} - \alpha^{-}) d\beta\right)$$

where α^{+} is the value of α on γ_i (resp. on δ_i in the second integral) and α^{-} is the value of α on γ_i^{-1} (resp. on δ_i^{-1}). But, since $d\alpha$ has no residues, the value of α on γ_i^{-1} differs from the value of α on γ_i by the period of $d\alpha$ around δ_i, and the value of α on δ_i differs from the value of α on δ_i^{-1} by the period of $d\alpha$ around γ_i. Hence we have

$$(1) \quad \int_{\partial\Pi} \alpha d\beta = \sum_{i=1}^{g} \left(\int_{\gamma_i} d\alpha \int_{\delta_i} d\beta - \int_{\delta_i} d\alpha \int_{\gamma_i} d\beta\right)$$

(cf. [3], Theorem 37.1).

We will derive several expressions from (1). Let ζ be an Abelian integral of the first kind. Let a_i, $i = 1,\dots,g$, denote the Γ-periods of $d\zeta$; that is,

$$a_i = \int_{\gamma_i} d\zeta$$

Let $w \varepsilon X$ and let

$$\tau_{w,\nu}(z)$$

denote the (normalized) elementary integral of the second kind with pole of order $\nu + 1$ at w and zero Γ-periods. In (1), put $\alpha = \zeta$ and $d\beta = d\tau_{w,\nu}$. Then, applying the Residue Theorem to the left side of (1), we get

$$(2) \qquad \zeta^{(\nu+1)}(w) = \frac{\nu!}{2\pi i} \sum_{j=1}^{g} a_j \int_{\delta_j} d\tau_{w,\nu}(z)$$

Now let

$$\eta_{xy}(w)$$

be the (normalized) elementary integral of the third kind with residues -1 at x and $+1$ at y and zero Γ-periods. In (1), putting $\alpha = \zeta$ and $\beta = \eta_{xw}$, and using the Residue Theorem, we obtain

$$(3) \quad \zeta(w) - \zeta(x) = \frac{1}{2\pi i} \sum_{j=1}^{g} a_j \int_{\delta_j} \frac{\partial \eta_{xw}(z)}{\partial z} dz$$

N.B.: By abuse of notation, we will use the letters t, x, y, w to represent local parameters at the points as well as the points themselves.

Differentiating (3) a total of $(\nu + 1)$ times with respect to w yields

$$(3') \quad \zeta^{(\nu+1)}(w) = \frac{1}{2\pi i} \sum_{j=1}^{g} a_j \int_{\delta_j} \frac{\partial^{\nu+2} \eta_{xw}(z)}{\partial w^{\nu+1} \partial z}$$

Remark: The expansion of η_{xw} near z is of the form

$$\eta_{xw}(z) = \log (z - w) - \log (z - x) + \text{regular terms}$$

so differentiation of $\eta_{xw}(z)$ with respect to x or w or z makes sense. For example,

$$\frac{\partial^2 \eta_{xw}(z)}{\partial w \partial z} = \frac{1}{(z - w)^2} + \text{regular terms}$$

(cf. [23], §4.1).

Again in (1), put $\alpha = \tau_{w,\nu}(z)$ and $d\beta = d\eta_{xy}(z)$, use the Residue Theorem, and get

$$(4) \qquad \tau_{w,\nu}(y) - \tau_{w,\nu}(x) = \frac{1}{\nu!} \frac{\partial^{\nu+1} \eta_{xy}(w)}{\partial w^{\nu+1}}$$

Finally, putting $\alpha = \tau_{t,0}(z)$ and $d\beta = d\tau_{y,\nu-1}(z)$, and applying the Residue Theorem, we obtain

$$(5) \qquad \frac{\partial^{\nu} \tau_{t,0}(y)}{\partial y^{\nu}} = (\nu - 1)! \frac{\partial \tau_{y,\nu-1}(t)}{\partial t}.$$

Suppose w occurs with multiplicity m in a special divisor D. Then the values $\zeta_k'(w), \ldots,$ $\frac{(-1)^{m-1}}{m!} \zeta_k^{(m)}(w)$ would be entries in the k^{th} column of the matrix M of I - 3. What we are interested in is how these values change as Teichmüller surface X is deformed. We will now define a new Teichmüller surface X^* by using a variation due to Schiffer and Spencer [23].

Let Q be in the interior of Π and assume Q is not a zero of $d\zeta$ and $Q \neq w$. Let t be a local parameter at Q and let D be a disk around Q, lying in the domain of t, completely contained in the interior of Π and containing neither w nor any zero of $d\zeta$. Let β be a Jordan curve contained in D, but not containing Q. Let

$$B$$

denote the interior of the region enclosed by β. Then B is topologically and conformally equivalent to the unit disk (Riemann Mapping Theorem).

A new Jordan curve β^* can be defined in D by replacing t by t^*, where t^* is given by

$$(7) \qquad t^*(t) = t + \frac{c}{t} ,$$

if c is sufficiently small.

The mapping defined by (7) will carry an annulus about β onto an annulus about β^*, both annuli contained in D. By taking c small enough, the annulus about β will contain β^*, and the annulus about β^* will contain β. Let

$$B^*$$

denote the interior of the region enclosed by β^*.

We now define the new Teichmüller surface X^*. As D was chosen so that it would not intersect any of the curves of Γ or Δ (recall that D was contained in the interior of Π), these same curves will be used as a canonical homotopy basis on X^*.

Now delete the disk B from X. This leaves a surface with boundary (namely β). Adjoin to this surface the closed disk $\overline{B^*}$ in such a manner that each point t_0 on β is identified with the point $t_0^* = (t_0 + \frac{c}{t_0})$ on β^*. On the complement of $\overline{B^*}$, the surfaces X and X^* have the same points and the same conformal structure.

Let ζ^* be the integral of the first kind on X^* such that

$$\int_{\gamma_i} d\zeta^* = a_i \qquad i = 1,\ldots,g.$$

Let $\tau^*_{w,\nu}$ be the normalized elementary integral of the second kind on X^* with pole of order $\nu + 1$ at w. Then we have, as in [2],

$$\zeta^{*(\nu+1)}(w) = \frac{1}{2\pi i} \sum_{j=1}^{g} a_j \int_{\delta_j} d\tau^*_{w,\nu}(z).$$

Our objective now is to compute

$$\Delta\zeta^{(\nu+1)}(w) = \zeta^{*(\nu+1)}(w) - \zeta^{(\nu+1)}(w).$$

Put $X_0 = X - \overline{B}$. Suppose $x,y \in X_0$ and define the integral of the third kind η^*_{xy} on X^* with the same normalization on X^* as η_{xy} on X. On X_0 slit along an arc from x to y, a determination of η_{xy} is an analytic function. If a determination of η^*_{xy} is also chosen on the same slit region, it will be a single-valued analytic function as well. The difference $\Delta\eta_{xy}(z) = \eta^*_{xy}(z) - \eta_{xy}(z)$, for these determinations, will be a single-valued analytic function which can be continued along the slit remaining single-valued. The singularities at x and y will cancel out, hence $\Delta\eta_{xy}(z)$ is a single-valued analytic function on X_0. By contour integration, we have

$$(8)\quad \Delta\eta_{xy}(w) - \Delta\eta_{xy}(z) = \frac{1}{2\pi i} \int_{\beta} \Delta\eta_{xy}(u) \frac{\partial}{\partial u} \eta_{zw}(u)du$$

$$+ \frac{1}{2\pi i} \int_{\partial\pi} \Delta\eta_{xy}(u) \frac{\partial}{\partial u} \eta_{zw}(u)du$$

where $z, w \in X_0$.

But, upon applying (1), we see that

$$\int_{\partial\pi} \Delta\eta_{xy}(u) \frac{\partial}{\partial u} \eta_{zw}(u)du = 0.$$

Hence we have

$$(9) \quad \Delta\eta_{xy}(w) - \Delta\eta_{xy}(z) = \frac{1}{2\pi i} \int_{\beta}(\eta^*_{xy}(u) - \eta_{xy}(u))\frac{\partial}{\partial u} \eta_{zw}(u)du$$

Now, by our choice of c, we have that $\overline{B}$ is contained in the union of $\overline{B^*}$ and an annulus about β^*. Hence, we may view η^*_{xy} as being a function on $\overline{B}$ (by restriction), as well as on $\overline{B^*}$. Thus, we can make sense of the expression $\eta^*_{xy}(t)$, as well as the expression $\eta^*_{xy}(t^*(t))$. In the term $(\eta^*_{xy}(u) - \eta_{xy}(u))$, which measures the variation of the integral of the third kind as the surface changes, we clearly want to view η^*_{xy} as a function on X^*. Hence, in replacing the dummy variable u by the variable of integration t, we must replace the u in the argument of η^*_{xy} by $t^*(t)$, obtaining

$$(9') \quad \Delta\eta_{xy}(w) - \Delta\eta_{xy}(z) = \frac{1}{2\pi i} \int_{\beta}(\eta^*_{xy}(t^*(t))-\eta_{xy}(t))\frac{\partial}{\partial t}\eta_{zw}(t)dt$$

On the other hand, since η^*_{xy} is analytic on the union of $\overline{B^*}$ and an annulus about β^*, and $\overline{B}$ is contained in this union, $\eta^*_{xy}(t)$, as well as $\eta_{xy}(t)$, is analytic on $\overline{B}$. Hence, by Cauchy's Theorem, the two integrals

$$\int_\beta \eta^*_{xy}(t)\frac{\partial}{\partial t}\eta_{zw}(t)dt \quad \text{and} \quad \int_\beta \eta_{xy}(t)\frac{\partial}{\partial t}\eta_{zw}(t)dt$$

both vanish. Thus, we may replace $\eta_{xy}(t)$ by $\eta^*_{xy}(t)$ in (9'), yielding

$$(10) \quad \Delta\eta_{xy}(w) - \Delta\eta_{xy}(z) = \frac{1}{2\pi i}\int_\beta (\eta^*_{xy}(t^*(t))-\eta^*_{xy}(t))\frac{\partial}{\partial t}\eta_{zw}(t)dt$$

This is <u>the basic formula</u> for computing the variation of periods of integrals (cf. [21]).

Now we can consider n points $Q_1,\ldots,Q_n$; disjoint disks $D_1,\ldots,D_n$ with curves $\beta_1,\ldots,\beta_n$ in each respectively; and $\beta^*_1,\ldots,\beta^*_n$ defined by

$$t^*_j(t_j) = t_j + \frac{c_j}{t_j} \qquad j = 1,\ldots,n$$

where t_j is a local parameter at Q_j. The variation from all the disks will be defined by adding the integrals on the right side of (10) for each of the β_j.

We proceed in the case $j = 1$. We have

$$\eta^*_{xy}(t^*(t)) - \eta^*_{xy}(t) = \eta^*_y(t + \frac{c}{t}) - \eta^*_{xy}(t)$$

$$= \frac{c}{t}\frac{\partial}{\partial t}\eta^*_{xy}(t) + O(c^2).$$

Thus, from (10), we get

$$(11) \quad \Delta\eta_{xy}(w) - \Delta\eta_{xy}(z) = \frac{1}{2\pi i}\int_\beta \frac{c}{t}\eta^*_{xy}(t)\frac{\partial}{\partial t}\eta_{zw}(t)dt + O(c^2).$$

Differentiating with respect to w yields

$$(11') \quad \frac{\partial}{\partial w}\Delta\eta_{xy}(w) = \frac{1}{2\pi i}\int_\beta \frac{c}{t}\frac{\partial}{\partial t}\eta^*_{xy}(t)\frac{\partial^2}{\partial w\partial t}\eta_{zw}(t)dt + O(c^2).$$

Now, the right side of (11') is obviously $O(c)$, so the left side is also $O(c)$. Hence, it can be seen that we may replace η^*_{xy} by η_{xy} in the integral on the right side of (11) and retain the $O(c^2)$. Thus we have

$$(12)\quad \Delta\eta_{xy}(w) - \Delta\eta_{xy}(z) = \frac{1}{2\pi i}\int_{\beta} \frac{c}{t}\frac{\partial}{\partial t}\eta_{xy}(t)\frac{\partial}{\partial t}\eta_{zw}(t)dt$$

$$+ O(c^2).$$

By (3'), we may write

$$(13)\quad \Delta\zeta^{(\nu+1)}(w) = \frac{1}{2\pi i}\sum_{j=1}^{g} a_j \int_{\delta_j} \frac{\partial^{\nu+2}}{\partial w^{\nu+1}\partial z}\Delta\eta_{xw}(z)dz.$$

In (12), interchange y and w and differentiate $(\nu+1)$-times with respect to w and once with respect to z, obtaining

$$(14)\quad \frac{\partial^{\nu+2}\Delta\eta_{xw}(z)}{\partial w^{\nu+1}\partial z} = -\frac{1}{2\pi i}\int_{\beta}\frac{c}{t}\frac{\partial^{\nu+2}\eta_{xw}(t)}{\partial w^{\nu+1}\partial t}\frac{\partial^2\eta_{zy}(t)}{\partial z\partial t}dt + O(c^2)$$

Now, replacing w by t and y by w in (4) and putting $\nu = 0$, we obtain

$$\tau_{t,0}(w) - \tau_{t,0}(x) = \frac{\partial}{\partial t}\eta_{xw}(t).$$

Then, by differentiating $(\nu+1)$ times with respect to w, we see that

$$\frac{\partial^{\nu+2}}{\partial w^{\nu+1}\partial t}\,\eta_{xw}(t) = \frac{\partial^{\nu+1}}{\partial w^{\nu+1}}\,\tau_{t,0}(w).$$

Again in (4), replace w by t and x by z, put $\nu = 0$ and then differentiate with respect to z, getting

$$\frac{\partial^2}{\partial z\partial t}\,\eta_{zy}(t) = -\frac{\partial}{\partial z}\,\tau_{t,0}(z).$$

<u>Remark</u>: In the derivation of (4), we needed that x was in the interior of Π, while z here is on $\partial\Pi$. However, since the functions in (4) are continuous in x as long as x is distinct from w (cf. the earlier Remark in this section), the equality (4) is also valid for x on the boundary of Π.

Now, by substituting the above equalities into (14), and then substituting into (13), we see that

$$\Delta\zeta^{(\nu+1)}(w) = \frac{1}{(2\pi i)^2}\sum_{j=1}^{g} a_j \int_{\beta}\int_{\delta_j} \frac{c}{t}\,\frac{\partial^{\nu+1}}{\partial w^{\nu+1}}\tau_{t,0}(w)\;\frac{\partial\tau_{t,0}}{\partial z}(z)\,dz\,dt$$

$$+\;O(c^2)$$

But, (2) with $\nu = 0$ is just

$$\zeta'(t) = \frac{1}{2\pi i} \sum_{j=1}^{g} a_j \int_{\delta_j} d\tau_{t,0}(z)$$

hence

$$\Delta\zeta^{(\nu+1)}(w) = \frac{1}{2\pi i} \int_{\beta} \frac{c}{t} \frac{\partial^{\nu+1}\tau_{t,0}(w)}{\partial w^{\nu+1}} \zeta'(t)dt + O(c^2)$$

Now, by (5), we have

$$\frac{\partial^{\nu+1}}{\partial w^{\nu+1}} \tau_{t,0}(w) = \nu! \frac{\partial\tau_{w,\nu}}{\partial t} (t) ,$$

so finally we obtain

$$\Delta\zeta^{(\nu+1)}(w) = \frac{\nu!}{2\pi i} \int_{\beta} \frac{c}{t} \frac{\partial}{\partial t} \tau_{w,\nu}(t)\zeta'(t)dt + O(c^2)$$

Applying the Residue Theorem, we see that

$$\Delta\zeta^{(\nu+1)}(w) = \nu! c \frac{\partial}{\partial t} \tau_{w,\nu}(Q)\zeta'(Q) + O(c^2).$$

Notation: Write $\tau'_{w,\nu}(Q)$ for $\frac{\partial}{\partial t} \tau_{w,\nu}(Q)$.

If we perform this variation at n points, $Q_1,\ldots,Q_n$, then we obtain

$$(15) \quad \Delta\zeta^{(\nu+1)}(w) = \nu! \sum_{k=1}^{n} c_k\tau'_{w,\nu}(Q_k)\zeta'(Q_k) + O(c^2)$$

where $c = \max_{1 \leq k \leq n} |c_k|$.

We will want to use the following theorem, due to Patt [21]:

Theorem 2: One may choose $3g - 3$ points $Q_1, \ldots, Q_{3g-3}$ on X such that, if c_m is the variation parameter at Q_m, then a neighborhood of the origin in the $c_1, \ldots, c_{3g-3}$ space describes a complex-analytic structure for a neighborhood of X in the Teichmüller space. Moreover, the set of collections of $3g - 3$ points with this property is open in X^{3g-3}.

Proof: The first assertion follows from Theorems 2 and 4 of [21]. Although Patt does not state the second assertion, his proofs demonstrate it, as was noted by Farkas [5], p. 885. ▮

§2. Meis's Work

In [20], Meis demonstrates the existence of special divisors for the case $r = 1$. He does this by considering the universal space of special divisors $\mathcal{D}^1_n$ over the Teichmüller space and explicitly exhibiting a special fiber of dimension $\tau + 1$ in the case in which n is the minimum integer such that $\tau \geq 0$. He may then conclude that a component of the universal space of special divisors has dimension $3g - 3 + \tau + 1$, and that this component maps surjectively down to the (irreducible) Teichmüller space. Hence, he shows that for an arbitrary Riemann surface X, the subspace G^1_n of $X^{(n)}$ is nonempty of dimension at least $\tau + 1$ if n is any integer such that $\tau \geq 0$. His methods also show that for a generic suface, G^1_n has a component of dimension $\tau + 1$ if n is the minimum integer such that $\tau \geq 0$.

We present Meis's examples below, and will make use of them in the next chapter. Suppose g is given and that $r = 1$. Then the minimum n such that $\tau \geq 0$ is

$$n = \begin{cases} \frac{g+2}{2} & \text{if } g \text{ even} \Rightarrow \tau = 0 \\ \frac{g+3}{2} & \text{if } g \text{ odd} \Rightarrow \tau = 1 \end{cases}$$

So, the $r = 1$ case breaks up naturally into even and odd genus subcases. Meis gives one class of even genus surfaces and one of odd genus surfaces.

Even genus case

Suppose $g = 2m$ and consider the Riemann surface of the algebraic function

$$y^{m+1} = (x-1)(x-2)(x-3)(x-4)^m(x-5)^m(x-6)^m.$$

This surface has $m + 1$ sheets and ramification points of order m over the points $x = 1,2,3,4,5,6$. By the Riemann-Hurwitz formula, the surface has genus $2m$. Meis shows that the following form a basis for the holomorphic differentials:

$$d\zeta_k = \frac{(x-4)^{k-1}(x-5)^{k-1}(x-6)^{k-1}dx}{y^k} \qquad k = 1,\dots,m$$

$$d\zeta_{k+m} = xd\zeta_k \qquad k = 1,\dots,m.$$

One can easily compute the order of vanishing of the differentials at the ramification points and at the points over $x = 0$ and $x = \infty$ (and these are the only points where the differentials might vanish). To do this, notice that a local parameter at the point $x = j$, for $j = 1,\ldots,6$, is $\sqrt[m+1]{x - j}$, a local parameter at the points over $x = 0$ is x, and a local parameter at the points over $x = \infty$ is $\frac{1}{x}$. Then express the $d\zeta_j$ as $f_j dt$, where t is a local parameter, and see what the order of vanishing of f_j is at $t = 0$. Meis gets the following table for the order of vanishing of the differentials at the point(s) over the given value of x:

x =	1	2	3	4	5	6	0	∞
$d\zeta_k$	m-k	m-k	m-k	k-1	k-1	k-1	0	1
$d\zeta_{k+m}$	m-k	m-k	m-k	k-1	k-1	k-1	1	0

for $k = 1,\ldots,m$.

Odd genus case

Supppose $g = 2m + 1$ and consider the Riemann surface of the algebraic function:

$$y^3 = \frac{\prod_{i=1}^{m+2} (x - i)}{\prod_{j=m+3}^{2m+2} (x - j)}$$

This surface has 3 sheets, and ramification points of order 2 over $x = 1,2,\ldots,2m+2$ and over $x = \infty$. Thus, $g = \frac{(4m+6)}{2} + 1 - 3 = 2m + 1$. Meis shows that a basis for the holomorphic differentials is given by:

$$d\zeta_k = \frac{x^{k-1}dx}{y^2 \prod\limits_{j=m+3}^{2m+2} (x - j)} \qquad k = 1,\ldots,m+1$$

$$d\zeta_{k+m+1} = y\,d\zeta_k \qquad k = 1,\ldots,m.$$

Meis gets the following table for the order of vanishing of the differentials:

	x =	1	m+3	0	∞
$1 \le k \le m+1$	$d\zeta_k$	0	1	$k-1$	$3(m+1-k)$
$1 \le k \le m$	$d\zeta_{k+m+1}$	1	0	$k-1$	$3(m+1-k)-2$

For examples of special divisors with $r = 1$ and n minimum so that $\tau \ge 0$, Meis takes:

g even: the $\frac{g+2}{2}$ points over 0.

g odd : the $\frac{g+3}{2}$ ramification points over

$x = 1,2,\ldots,m+2$ (note $m+2 = \frac{g+3}{2}$).

In Chapter III, we will combine the deformation theory of I and II - 1, and then will use Meis's examples for some explicit computations.

Chapter III

Determination of the Dimension of the Tangent Space to the Universal Analytic Space of Special Divisors

§1.

We return to situation 2 of I - 2. Recall that we let T_g denote the Teichmüller space for Teichmüller surfaces of genus g and we let V denote the universal family of Teichmüller surfaces of genus g over T_g.

Let X be a compact Riemann surface of genus g > 1. Let $\{\gamma_j, \delta_j\}_{j=1}^{g}$ be a canonical homotopy basis and let $\{d\zeta_k\}_{k=1}^{g}$ be a basis of the holomorphic differentials. Put

$$A_{jk} = \int_{\gamma_j} d\zeta_k \qquad j,k = 1,\ldots,g.$$

Let P be a point of X and let t be a local parameter on X at P. Write

$$d\zeta_k = \sum_{\ell=0}^{\infty} a_{k,\ell} t^{\ell} dt.$$

Fix a point P_0 different from P. Choose a point $(Q_1,\ldots,Q_{3g-3})$ from the open subset of X^{3g-3} in Patt's Theorem such that all the Q_m are different from P and P_0 and such that none of the Q_m is a zero of any $d\zeta_k$. Perform the variation described in II - 1, taking the disk about each Q_m sufficiently small so that no two disks intersect and no disk contains P, P_0, or any zero of any $d\zeta_k$. Let c_m be the variation parameter at Q_m, $m = 1,\ldots,3g-3$, as in II - 1. (Note: the choice of the point $(Q_1,\ldots,Q_{3g-3})$ will be further modified later.)

Let $s_0 \in T_g$ be the module point of X (i.e. $V_{s_0} = X$). By definition of the variation in II - 1, there exists a complex-analytic neighborhood U of s_0 in T_g such that, for all $s' \in U$, the curves $\{\gamma_j,\delta_j\}_{j=1}^{g}$ are a canonical homotopy basis on $V_{s'}$, the points P_0 and P are on $V_{s'}$, and t is a local parameter on $V_{s'}$ at P. Choose holomorphic sections $d\zeta_k^*$, $k = 1,\ldots,g$, of $h_*\Omega^1_{V/T_g}$ such that

$$\int_{\gamma_j} d\zeta_k^*(s') = A_{jk} \qquad \begin{array}{l} s' \in U \\ j,k = 1,\ldots,g \end{array}$$

(cf. I - 2 and [19], §2).

Proposition 10: With notation as in II - 1 and above, if we define $a^*_{k,\ell}$ by

$$d\zeta^*_k = \sum_{\ell=0}^{\infty} a^*_{k,\ell} t^{\ell} dt,$$

then we have

$$a^*_{k,\ell} = a_{k,\ell} + \sum_{m=1}^{3g-3} c_m \tau'_{P,\ell}(Q_m)\zeta'_k(Q_m) + O(c^2).$$

Proof: The variational formula (specifically equation (15) of II - 1 with $n = 3g - 3$) shows that this equality holds in a complex-analytic neighborhood of (s_0, P) on V. This is the main import of the variational formula. ∎

In order to study the map $u : f^*\Omega^1_{\mathcal{Y}/T_g} \to \Omega^1_{V^{(n)}_{T_g}/T_g}$ of I - 2, we proceed in a similar manner to I - 3. We first consider the divisor nP on X. Let $t_1,\ldots,t_n$ be n copies of t and let $\sigma_1,\ldots,\sigma_n$ denote the n elementary symmetric functions in $t_1,\ldots,t_n$.

Proposition 11: Local parameters on $V_{T_g}^{(n)}$ at (s_0,nP) are given by $c_1,\ldots,c_{3g-3},\sigma_1,\ldots,\sigma_n$.

Proof: By Patt's Theorem, local parameters on T_g at s_0 are given by $c_1,\ldots,c_{3g-3}$. By [2], local parameters on $X^{(n)}$ at nP are given by $\sigma_1,\ldots,\sigma_n$. By the definition of the variation in II - 1, local parameters on $(V_{T_g}^{(n)})_{s'}$ at nP, for $s' \in U$, are also given by $\sigma_1,\ldots,\sigma_n$. Thus, local parameters on $V_{T_g}^{(n)}$ at (s_0,nP) are given by $c_1,\ldots,c_{3g-3}$, $\sigma_1,\ldots,\sigma_n$. ∎

The following proposition relativizes Proposition 6 to our present situation. It is proved by making the obvious modifications in the proof cited for Proposition 6. As in I - 3, we will make precise the correspondence in the proposition after we have stated it.

Proposition 12: The space of relative holomorphic 1-forms on $V_{T_g}^{(n)}$ over T_g and the space of relative holomorphic 1-forms on V over T_g are naturally isomorphic. Both spaces are isomorphic to the space of relative symmetric holomorphic 1-forms on $V^n_{T_g}$, the product over T_g of n copies of V, over T_g. ∎

Similarly to I - 3, we will identify the space of relative symmetric holomorphic 1-forms on $V^n_{T_g}$ over T_g with the space of relative holomorphic 1-forms on $V^{(n)}_{T_g}$ over T_g.

We will now make explicit the correspondence between relative holomorphic 1-forms on V over T_g and relative symmetric holomorphic 1-forms on $V^n_{T_g}$ over T_g. Let $d\tilde{\zeta}^*_k$ denote the relative symmetric holomorphic 1-form on $V^n_{T_g}$ over T_g corresponding to $d\zeta^*_k$. Recall from I - 3 that

$$\tau_j = t_1^j dt_1 + \dots + t_n^j dt_n \qquad j = 0,1,2,\dots$$

<u>Proposition 13</u>: We may write

$$d\tilde{\zeta}^*_k = \sum_{\ell=0}^{\infty} a^*_{k,\ell}\tau_\ell$$

<u>Proof</u>: Equality holds in a complex-analytic neighborhood of $(s_0,P,\ldots,P)$ on $V^n_{T_g}$ by Proposition 7 and Proposition 10. ∎

Now, as in I - 3, we use the identities

$$\sigma_k\tau_0 - \sigma_{k-1}\tau_1 + \ldots + (-1)^k\tau_k = d\sigma_{k+1}.$$

Thus, writing out $a^*_{k,\ell}$ and using the above identities to write $d\tilde{\zeta}^*_k$ in terms of $d\sigma_1,\ldots,d\sigma_n$, we may write

$$(\#) \quad d\tilde{\zeta}^*_k = \sum_{\ell=0}^{\infty}(-1)^\ell[(a_{k,\ell} + \sum_{m=1}^{3g-3} c_m\tau'_{P,\ell}(Q_m)\zeta'_k(Q_m))$$

$$(d\sigma_{\ell+1} - \sigma_1 d\sigma_\ell - \ldots - \sigma_\ell d\sigma_1)]$$

$$+ O(\sigma^2,c^2)$$

where $O(\sigma^2,c^2)$ denotes higher order terms in the σ_j and the c_m.

Now, by the definition of the map $f : V^{(n)}_{T_g} \rightarrow \mathcal{J}$ in I - 2, it is easy to see that f is given at (s_0,nP) by

$$f(s_0,nP) = \left(s_0, \int_{P_0}^{P} d\tilde{\zeta}_1^*(s_0), \ldots, \int_{P_0}^{P} d\tilde{\zeta}_g^*(s_0)\right) \text{ mod periods}$$

where the integrals $\int_{P_0}^{P} d\tilde{\zeta}_k^*(s_0)$ are evaluated by recalling that $t_1,\ldots,t_n$ are just copies of t.

Let $\dfrac{\partial\tilde{\zeta}_k^*}{\partial\sigma_j}$ be given by

$$d\zeta_k^* = \sum_{j=1}^{n} \frac{\partial\tilde{\zeta}_k^*}{\partial\sigma_j}\, d\sigma_j .$$

Then we have

<u>Proposition 14</u>: The map $u : f^*\Omega^1_{\mathcal{J}/T_g} \to \Omega^1_{V^{(n)}_{T_g}/T_g}$ is given locally at (s_0,nP) by the matrix

$$\left[\frac{\partial\tilde{\zeta}_k^*}{\partial\sigma_j}\right] \qquad \begin{array}{l} j = 1,\ldots,n \\ k = 1,\ldots,g. \end{array}$$

<u>Proof</u>: This follows easily from the definitions of f and $\dfrac{\partial\tilde{\zeta}_k^*}{\partial\sigma_j}$. (Compare with [5] and [9]). $\blacksquare$.

Note that

$$\left[\frac{\partial\tilde{\zeta}_k^*}{\partial\sigma_j}(s_0,nP)\right] \qquad \begin{array}{l} j = 1,\ldots,n \\ k = 1,\ldots,g \end{array}$$

is just the matrix M of I - 3.

Now let ξ be a tangent vector to $V_{T_g}^{(n)}$ at (s_0,nP). Let s_j and b_m in $\mathbb{C}$ be given by

$$\xi(\sigma_j) = s_j\varepsilon \quad j = 1,\ldots,n$$

$$\xi(c_m) = b_m\varepsilon \quad m = 1,\ldots,3g-3.$$

Then, using Taylor's Theorem as in I - 1, we have

$$(1) \quad \xi\left(\frac{\partial\tilde{\zeta}_k^*}{\partial\sigma_j}\right) = \frac{\partial\tilde{\zeta}_k^*}{\partial\sigma_j}(s_0,nP) + \varepsilon\sum_{\ell=1}^{n} s_\ell \frac{\partial^2\tilde{\zeta}_k^*}{\partial\sigma_\ell\partial\sigma_j}(s_0,nP)$$

$$+ \varepsilon\sum_{m=1}^{3g-3} b_m \frac{\partial^2\tilde{\zeta}_k^*}{\partial c_m\partial\sigma_j}(s_0,nP)$$

We now use (#) to compute the partial derivatives of the functions $\frac{\partial\tilde{\zeta}_k^*}{\partial\sigma_j}$ with respect to σ_ℓ and with respect to c_m. (We remind the reader that the functions σ_ℓ and c_m vanish at (s_0, nP).) We obtain

$$\frac{\partial^2\tilde{\zeta}_k^*}{\partial\sigma_\ell\partial\sigma_j}(s_0, nP) = (-1)^{j+\ell} a_{k,j+\ell-1}$$

and

$$\frac{\partial^2\tilde{\zeta}_k^*}{\partial c_m\partial\sigma_j}(s_0, nP) = \tau'_{P,j-1}(Q_m)\zeta'_k(Q_m).$$

Substituting these expressions for the partial derivatives into (1) gives us

Proposition 15:

$$\xi\left(\frac{\partial\tilde{\zeta}_k^*}{\partial\sigma_j}\right) = \frac{\partial\tilde{\zeta}_k^*}{\partial\sigma_j}(s_0, nP) + \varepsilon \sum_{\ell=1}^{n} (-1)^\ell s_\ell a_{k,j+\ell-1}$$

$$+ \varepsilon \sum_{m=1}^{3g-3} b_m \tau'_{P,j-1}(Q_m)\zeta'_k(Q_m). \quad \blacksquare$$

Now on to the general case. Consider a divisor D on X of the form $D = m_1P_1 + \dots + m_dP_d$. Assume D is in $G^r_n(X)$ and choose a basis $\{d\zeta_k\}^g_{k=1}$ of the holomorphic differentials on X such that the last $i = \dim H^1(X,O_X(D))$ of them vanish on D.

In performing the variation in II - 1, choose a point $(Q_1,\ldots,Q_{3g-3})$ from the open set X^{3g-3} in Patt's Theorem so that each Q_m is different from $P_0,P_1,\ldots,P_d$, and any other zero of any $d\zeta_k$. Take the disk about each Q_m sufficiently small so that no two disks intersect and such that no disk contains $P_0,P_1,\ldots,P_d$, or any other zero of any $d\zeta_k$.

Let $f_j : V^{(m_j)}_{T_g} \to \mathcal{Y}$ be the map defined in I - 2 and $u_j : f^*_j\Omega^1_{\mathcal{Y}/T_g} \to \Omega^1_{V^{(m_j)}_{T_g}/T_g}$ be the map induced by f_j. The obvious map $V^{(m_1)}_{T_g} \times_{T_g} \cdots \times_{T_g} V^{(m_d)}_{T_g} \to V^{(n)}_{T_g}$ is a local analytic isomorphism by an argument analogous to that in [18]. Let ξ be a tangent vector to $V^{(n)}_{T_g}$ at (s_0,D) and let ξ_j be the tangent vector to $V^{(m_j)}_{T_g}$ at (s_0,m_jP_j) induced by ξ, for $j = 1,\ldots,d$. Then, by the same reasoning as in I - 3, the matrix of $\xi^* u$ is obtained by "stacking" the matrices of the $\xi^* u_j$, for $j = 1,\ldots,d$.

Let $\mathcal{M}'$ denote the matrix of $\xi^* u$. By our choice of basis of the holomorphic differentials on X, the last i columns of M, the matrix $\left[\frac{\partial \zeta_k^*}{\partial \sigma_j}(s_0, D)\right]$, are identically zero, hence the last i columns of $\mathcal{M}'$ contain "pure" ε terms (i.e. members of the maximal ideal of $\mathbb{C}[\varepsilon]/(\varepsilon^2)$). Thus, as in lemma 3, we have

<u>Lemma 4</u>: For purposes of computing the minors of order $n - r + 1$ of $\mathcal{M}'$, we may replace the first $n - r$ columns of $\mathcal{M}'$ by the first $n - r$ columns of M. ∎

Let $\mathcal{M}$ denote the resulting matrix.

$\mathcal{M}$ has a particularly nice form in the case that $D = P_1 + \ldots + P_n$, with all points distinct. Let t_j be a local parameter at P_j and write $d\zeta_k = \phi_{j,k} dt_j$. Then we have

$$\mathcal{M} = \left[\begin{array}{l|l} \phi_{j,k}(P_j) & \varepsilon(s_j\phi'_{j,k}(P_j) + \sum\limits_{m=1}^{3g-3} b_m \tau'_{P_{j},0}(Q_m)\zeta'_k(Q_m)) \\ & \\ j = 1,\ldots,n & j = 1,\ldots,n \\ k = 1,\ldots,g-i & k = g-i+1,\ldots,g \end{array}\right]$$

Going back to the general case, recall that, by Proposition 1, ξ will be a tangent vector to $\mathcal{G}^r_n$ at (s_0,D) iff the minors of order $n - r + 1$ of the matrix $\mathcal{M}$ all vanish. Assume $D = m_1P_1 + \ldots + m_dP_d$ is in $G^r_n(X) - G^{r+1}_n(X)$. Then the matrix M has rank precisely $n - r$. Hence, by permuting the rows of M, if necessary, we end up with a matrix whose leading minor of order $n - r$, which we will denote by μ, is nonzero. We will continue to denote this matrix by M, although its form may differ slightly from that specified earlier.

Perform the same row permutations as above on the matrix $\mathcal{M}$ and denote the resulting matrix also by $\mathcal{M}$. Then μ is also the leading minor of order $n - r$. of $\mathcal{M}$, so we may apply Lemma 2. Thus, for all the minors of order $n - r + 1$ of $\mathcal{M}$ to vanish, it is sufficient that every minor of order $n - r + 1$ which contains μ vanishes. The vanishing of each of these minors gives rise to a linear equation in the s_j and the b_m.

Let $\mu_{j,k}$ denote the minor or order $n - r + 1$ of $\mathcal{M}$ obtained by adjoining to μ the first $n - r$ elements of the $n - r + j^{th}$ row of $\mathcal{M}$ and the first $n - r$ elements and the $n - r + j^{th}$ element of the $n - r + k^{th}$ column of $\mathcal{M}$ (thus j runs from 1 through r and k runs from 1 through i). The equation $\mu_{j,k} = 0$ is of the form

$$\varepsilon E_{j,k} = 0$$

where $E_{j,k}$ is a linear equation in the s_j and the b_m with coefficients in $\mathbb{C}$.

We will now view the s_j and the b_m as being unknowns (as in I - 1). Thus, $E_{j,k}$ is an equation in $3g - 3 + n$ unknowns. By the discussion after Proposition 1, the dimension of the tangent space to $\mathfrak{D}^{r}_{n}$ at (s_0,D) is

$$3g - 3 + n - (\text{the number of } E_{j,k} \text{ which are linearly independent}).$$

Consider the coefficient of b_m in $E_{j,k}$. This coefficient will be a linear combination of certain of the $\tau'_{P_{j,\nu}}(Q_m)\zeta'_k(Q_m)$. That is, the coefficient of b_m will be a certain quadratic differential (the above linear combination of certain of the $d\tau_{P_{j,\nu}}d\zeta_k$) evaluated at the point Q_m. It should be noted that, by the symmetry of the matrix $\mathfrak{M}$ in the b_m, this quadratic differential does not depend on m, but only on j and k. The coefficient of b_1 is the value of this quadratic differential at Q_1, the coefficient of b_2 the value at Q_2, etc. Put

$$\alpha_{j,k}$$

equal to the above linear combination of certain of the $d\tau_{P_{j,\nu}}d\zeta_k$. Then $\alpha_{j,k}$ is a (not necessarily finite) quadratic differential.

<u>Notation</u>: Choose a local parameter u_m on X at Q_m and write $\alpha_{j,k} = g(u_m)du_m^2$. Then we will write

$$\alpha_{j,k}(Q_m)$$

for $g(0)$.

Hence, by the above discussion, $\alpha_{j,k}(Q_m)$, the value of the quadratic differential at Q_m, *is the coefficient of* b_m *in* $E_{j,k}$.

Our aim now is to show that, in certain situations, by suitably choosing the point $(Q_1,\ldots,Q_{3g-3})$, we may conclude that the $E_{j,k}$ are linearly independent. By elementary linear algebra, to conclude that the $E_{j,k}$ are linearly independent, it is sufficient to show that the matrix of coefficients

$$A = [\alpha_{j,k}(Q_m)] \qquad \begin{array}{l} j = 1,\ldots,r \\ k = 1,\ldots,i \\ m = 1,\ldots,ri \end{array}$$

is nonsingular.

Lemma 5: Assume that the $\alpha_{j,k}$, for $j = 1,\ldots,r$ and $k = 1,\ldots,i$, are linearly independent. Then we may choose a point $(Q_1,\ldots,Q_{3g-3})$ from the open set in X^{3g-3} in Patt's Theorem such that each Q_m is different from P_0 and the zeros of $d\zeta_1,\ldots,d\zeta_g$ and such that the above matrix A is nonsingular.

Proof: The lemma will follow readily from the following

Sublemma: Let $\beta_1, \ldots, \beta_n$ be n linearly independent quadratic differentials on X. Let U be an open set contained in X^n. Then we may choose a point $(P_1, \ldots, P_n) \in U$ such that each P_m is different from a finite set of points of X and such that the matrix

$$[\beta_j(P_k)] \qquad \begin{array}{l} j = 1, \ldots, n \\ k = 1, \ldots, n. \end{array}$$

is nonsingular.

Proof: By induction on n. If $n = 1$, then β_1 is a non-trivial quadratic differential. Hence, β_1 is non-zero and finite on a dense open set of X. So, given any open set in X, there exists a point in that set satisfying the requirements of the sublemma.

Now suppose U is an open set contained in X^n. Let V be the projection of U onto $X^{n=1}$. Then V is open and, by induction, we may choose a point $(P_0, \ldots, P_{n-1}) \in V$ such that each P_m is different from a finite set of points of X and such that the leading subdeterminant of order $n - 1$ of the determinant

$$\begin{vmatrix} \beta_1(P_1) & \dots & \beta_1(P_{n-1}) & \beta_1 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \beta_n(P_1) & \dots & \beta_n(P_{n-1}) & \beta_n \end{vmatrix}$$

is nonzero. Expanding the full determinant by the last column, we obtain a non-trivial linear combination of $\beta_1,\dots,\beta_n$. By the linear independence of these quadratic differentials, this sum is a non-trivial quadratic differential, hence is nonzero and finite on an open dense set W contained in X. Since U is open in X^n and W is dense in X, we may choose a point in the intersection of U and $\{(P_0,\dots,P_{n-1})\} \times W$ which satisfies the requirements of our sublemma.

Now, since the set of points in X^{3g-3} in Patt's Theorem is open, it is easy to see that we may choose a point $(Q_1,\dots,Q_{3g-3})$ in this set such that each Q_m is different from P_0 and the zeros of $d\zeta_1,\dots,d\zeta_g$

and so that $Q_1, \ldots, Q_{ri}$ make the matrix A nonsingular. This completes the proof of the lemma. ▮

We then have

<u>Proposition 16</u>: Suppose D is in $G_n^r(X) - G_n^{r+1}(X)$. Then, if all the $\alpha_{j,k}$ are linearly independent, the dimension of the tangent space to $\mathfrak{G}_n^r$ at (s_0, D) is $3g - 3 + \tau + r$.

<u>Proof</u>: By Lemma 5, we may choose a point $(Q_1, \ldots, Q_{3g-3})$ from the open set in Patt's Theorem such that each Q_m is different from P_0 and the zeros of $d\zeta_1, \ldots, d\zeta_g$ (note that this latter set includes the points of D), and such that the equations $E_{j,k}$ are linearly independent. Thus, the dimension of the tangent space to $\mathfrak{G}_n^r$ at (s_0, D) is $3g - 3 + n - ir = 3g - 3 + \tau + r$. ▮

In the subsequent sections in this cahpter, we consider the real work of showing that the $\alpha_{j,k}$ are linearly independent.

§2. The cases $r = 1$ and $i = 1$

The case $r = 1$

For simplicity, we will first treat a divisor consisting of n distinct points. So assume $D = P_1 + \dots + P_n$, all points distinct, is in $G_n^1(X) - G_n^2(X)$. Recall that the matrix $\mathcal{M}$ is

$$\mathcal{M} = \left[\begin{array}{ll} \phi_{j,k}(P_j) & \varepsilon(s_j\phi'_{j,k}(P_j) + \sum_{m=1}^{3g-3} b_m\tau'_{P_{j,0}}(Q_m)\zeta'_k(Q_m)) \\ & \\ j = 1,\dots,n & j = 1,\dots,n \\ k = 1,\dots,g-i & k = g - i + 1,\dots,g \end{array}\right]$$

Let $|\hat{j}|$ denote the minor of order $n - 1$ obtained by omitting the j^{th} row from the matrix

$$[\phi_{j,k}(P_j)] \qquad \begin{array}{l} j = 1,\dots,n \\ k = 1,\dots,\underline{g-i} \end{array}$$

Then we have

$$\alpha_{1,k}(Q) = \sum_{j=1}^{n} (-1)^{j-1}|\hat{j}|\tau'_{P_{j,0}}(Q)\zeta'_{n+k-1}(Q)$$

for $k = 1,2,\dots,i$.

Suppose we had a linear relation of the form $\sum_{k=1}^{i} a_k \alpha_{1,k} = 0$ with some a_ℓ nonzero. Then this would imply

$$(*) \quad \left(\sum_{j=1}^{n} (-1)^{j-1} |\hat{j}| \tau'_{P_{j,0}}(Q)\right)\left(\sum_{k=1}^{i} a_k \zeta'_{n+k-1}(Q)\right) = 0.$$

But the $d\tau_{P_{j,0}}$, $j = 1,\ldots,n$, are linearly independent since they have poles at different points. This, together with the fact that $|\hat{n}| \neq 0$, implies that there is a dense open set of points of X where the expression $\sum_{j=1}^{n} (-1)^{j-1} |\hat{j}| \tau'_{P_{j,0}}(Q)$ is nonzero.

And the linear independence of $d\zeta_n, \ldots, d\zeta_g$, together with the fact that some a_ℓ is nonzero, implies that the other expression in parentheses in (*) is nonzero on a dense open set of points of X. Hence, we may choose a point Q such that (*) is nonzero, contradicting the assumption that $\alpha_{1,1}, \ldots, \alpha_{1,i}$ are linearly dependent.

Now suppose $D = m_1 P_1 + \ldots + m_d P_d$ is in

$G_n^1(X) - G_n^2(X)$. Then we have

$$\alpha_{1,k}(Q) = \zeta'_{n+k-1}(Q)(|\hat{1}|\tau'_{P_{1,0}}(Q) + \ldots + (-1)^{n-1}|\hat{n}|\tau'_{P_d,m_{d-1}}(Q)) = 0.$$

Hence, if there existed a linear relation

$\sum_{k=1}^{i} a_k\alpha_{1,k} = 0$, we would have

$$(\sum_{k=1}^{i} a_k\zeta'_{n+k-1}(Q))(|\hat{1}|\tau'_{P_{1,0}}(Q) + \ldots + (-1)^{n-1}|\hat{n}|\tau'_{P_d,m_{d-1}}(Q)) = 0.$$

The same reasoning as in the case of simple points applies, since the $d\tau_{P_{1,0}},\ldots,d\tau_{P_d,n_{d-1}}$ are easily seen to be linearly independent (they have either poles at different points or poles of differing orders at the same point).

Remark: The above reasoning shows that if $D \varepsilon G_n^r - G_n^{r+1}$, then the $\alpha_{j,k}$ for a fixed j are linearly independent.

The case $i = 1$

Again, first suppose that $D = P_1 + \ldots + P_n$, with all points distinct. Assume that $i = \dim H^1(X,O_X(D)) = 1$. For the sake of notation, write τ_j instead of $\tau_{P_{j,0}}$, $j = 1,\ldots,n$.

Our matrix $\mathcal{M}$ is

$$\mathcal{M} = \left[\begin{array}{ll} \phi_{j,k}(P_j) & \varepsilon(s_1\phi'_{1,g}(P_1) + \sum_{m=1}^{3g-3} b_m\tau'_1(Q_m)\zeta'_g(Q_m)) \\ & \\ j = 1,\ldots,n & \cdot \\ & \cdot \\ k = 1,\ldots,g-1 & \cdot \\ & \\ & \varepsilon(s_n\phi'_{n,g}(P_n) + \sum_{m=1}^{3g-3} b_m\tau'_n(Q_m)\zeta'_g(Q_m)) \end{array}\right]$$

Let R_q denote the q^{th} row of the matrix

$$[\phi_{j,k}(P_j)] \qquad \begin{array}{l} j = 1,\ldots,n \\ k = 1,\ldots,g-1 \end{array}$$

Then we have

$$\alpha_{j,1}(Q) = \zeta_g'(Q) \begin{vmatrix} R_2 \\ R_3 \\ \cdot \\ \cdot \\ \cdot \\ R_{n-r} \\ R_{n-r+j} \end{vmatrix} \tau_1'(Q) - \begin{vmatrix} R_1 \\ R_3 \\ \cdot \\ \cdot \\ \cdot \\ R_{n-r} \\ R_{n-r+j} \end{vmatrix} \tau_2'(Q)$$

$$+ \ldots + (-1)^{n-1} \begin{vmatrix} R_1 \\ \cdot \\ \cdot \\ \cdot \\ R_{n-r} \end{vmatrix} \tau_{n-r+j}'(Q)$$

for $j = 1,\ldots,r$.

Now suppose we had a linear relation of the form $\sum_{j=1}^{r} a_j\alpha_{j,1} = 0$ with some $a_\ell \neq 0$. Then this would imply the existence of a relationship of the form

$$(**) \qquad \zeta_g'(Q)\left(\sum_{j=1}^{n} b_j\tau_j'(Q)\right) = 0$$

with $b_{n-r+\ell} = a_\ell(-1)^{n-1} \begin{vmatrix} R_1 \\ \cdot \\ \cdot \\ \cdot \\ R_{n-r} \end{vmatrix} \neq 0.$

But again, by linear independence of $d\tau_1, \ldots, d\tau_n$, we may choose a point Q, not a zero of $d\zeta_g$, such that (**) is nonzero. Hence $\alpha_{1,1}, \ldots, \alpha_{r,1}$ are linearly independent.

The case when $i = 1$ and D has multiple points follows by the same reasoning, with a modification completely analogous to that used in the $r = 1$ and multiple point case (i.e. all the differentials of the second kind will still be linearly independent).

Thus we have established our main result:

Theorem 3: Suppose D is in $G_n^r(X) - G_n^{r+1}(X)$ and assume either $r = 1$ or $\dim H^1(X, \mathcal{O}_X(D)) = 1$. Then the dimension of the tangent space to $\mathfrak{D}_n^r$ at (s_0, D) is $3g - 3 + \tau + r$. Furthermore, $\mathfrak{D}_n^r$ is smooth at (s_0, D).

Proof: The first assertion follows from Proposition 16 and the work of this section. By Proposition 4, the dimension of $\mathfrak{D}_n^r$ at (s_0, D) is at least $3g - 3 + \tau + r$, hence $\mathfrak{D}_n^r$ is smooth at (s_0, D) by the first assertion.

Remarks:

1) Theorem 3 does not depend on τ being non-negative. This has implications about the moduli of curves with "extra-special" divisors (i.e. $\tau < 0$). We will come back to this in IV - 1.

2) It was already known that $\dim \mathfrak{D}_n^r = 3g - 3 + \tau + r$ in the case $i = 1$. Indeed, let

$$\overline{G_n^r}(X)$$

denote the image of $G_n^r(X)$ in the Jacobian J of X under the map $f : X^{(n)} \to J$ of I - 2. Let K denote the point of J which is the image of the canonical divisor. Then the map

$$f(D) \to K - f(D)$$

is an automorphism of J which, if $r = n - g + 1$, carries $\overline{G}_n^r$ onto $\overline{G}_{2g-2-n}^0$ ([16]). It is well-known that $\dim \overline{G}_{2g-2-n}^0 = 2g - 2 - n$ and that the singular points of $\overline{G}^0_{2g-2-n}$ are the points of $\overline{G}^1_{2g-2-n}$. It follows that $\dim \overline{G}^r_n = 2g - 2 - n = \tau$ and that the singular points of $\overline{G}^r_n$ are the points of $\overline{G}^{r+1}_n$. Thus we have that $G^r_n(X)$, if nonempty, has dimension $\tau + r$ (since if $f(D)$ is in $\overline{G}^r_n - \overline{G}^{r+1}_n$, then the fiber $f^{-1}f(D)$ has dimension r).

Now, if $r = n - g + 1$, then $\tau = n - 2r$ which, by Clifford's Theorem, must be nonnegative. Hence, $G^r_n(X)$ is nonempty for every X by [14]. Therefore, we have $\dim \mathfrak{G}^r_n = 3g - 3 + \tau + r$.

As consequences of Theorem 3 we have the following two important results.

Theorem 4: $\mathfrak{G}^1_n - \mathfrak{G}^2_n$, if nonempty, is smooth of pure dimension $3g - 3 + \tau + 1$.

Proof: This follows immediately from Theorem 3. ▮

Theorem 5: Suppose that for a generic curve X, we have $G^1_n(X) - G^2_n(X)$ is nonempty. Then $G^1_n(X) - G^2_n(X)$, for a generic X, is smooth of pure dimension $\tau + 1$.

Proof: Under our assumption, the image of $\mathcal{G}^1_n - \mathcal{G}^2_n$ in T_g would be a dense open subspace U. By Sard's Theorem, since $\mathcal{G}^1_n - \mathcal{G}^2_n$ is smooth, the generic fiber of the map $\mathcal{G}^1_n - \mathcal{G}^2_n \to U$ is smooth and since U has dimension $3g - 3$ and $\mathcal{G}^1_n - \mathcal{G}^2_n$ has dimension $3g - 3 + \tau + 1$, the generic fiber has dimension $\tau + 1$. Thus, for a generic curve, $G^1_n(X) - G^2_n(X)$ is smooth of dimension $\tau + 1$. ▮

Remarks:

(1) If $\tau \geq 0$, then by [14] we know that $G^r_n(X)$ is nonempty. If we knew that $G^r_n(X)$ were reduced for a generic X, then, since the points of G^{r+1}_n are singular points of G^r_n, we could conclude that $G^r_n(X) - G^{r+1}_n(X)$ is nonempty for generic X.

(2) Martens proved in [17], using Farkas's results [5], that $\overline{G}^1_n(X) - \overline{G}^2_n(X)$ is smooth for a generic curve X.

§3. The case $r = 2$

We now show that the hypotheses of Lemma 5 are satisfied in the case of explicit special divisors which we construct from Meis's examples. We assume throughout this section that $r = 2$.

Even genus case:

Suppose $g = 2m$ and n are given such that τ is nonnegative. Consider Meis's Riemann surface of genus g (see II - 2):

$$y^{m+1} = (x-1)(x-2)(x-3)(x-4)^{m}(x-5)^{m}(x-6)^{m}$$

Our divisor D will consist of the following:

(1) the $\frac{g+2}{2}$ $(= m+1)$ points over $x = 0$, denoted $P_1,\dots,P_{m+1}$.

(2) the (ramification) point over $x = 6$, denoted P_{m+2}, with multiplicity $m - i$ (where $i = 2 + g - n$), and

(3) the point over $x = 5$, denoted P_{m+3}.

Let $d\zeta_k$, $k = 1,\ldots,g$, be Meis's basis of holomorphic differentials (see II - 2). Let t_j be a local parameter at P_j, $j = 1,\ldots,m+3$, and write $d\zeta_k = \phi_{j,k}\, dt_j$.

The divisor $P_1 + \ldots + P_{m+1}$, Meis's example of a divisor with $r = 1$, is in $G^1_{m+1} - G^2_{m+1}$ (its index of specialty is easily seen to be m from the table in II - 2). Hence, the $(m+1) \times g$ matrix

$$N = [\phi_{j,k}(P_j)] \qquad \begin{array}{l} j = 1,\ldots,m+1 \\ k = 1,\ldots,g \end{array}$$

has rank m. By renumbering two of these points, if necessary, we may assume that the leading minor of order m of N is nonzero.

In the matrix $\mathfrak{M}$ of III - 1, we will take the row arising from P_{m+1} and move it to the next to the last row. This is done just to have the leading minor of order $n - 2$ of $\mathfrak{M}$ be nonzero. Thus $\mathfrak{M}$ has the following form:

$$
\mathfrak{M} = \begin{array}{c|cccccccccc:c|}
 & & & & & & & & & & & \overbrace{\qquad\qquad}^{\text{i columns}} \\
 & * & * & \cdots & & * & & & & & & \\
\text{m rows} & * & * & \cdots & & * & & & \bigcirc & & & \\
 & \vdots & \vdots & \cdots & & \vdots & & & & & & \\
 & * & * & \cdots & & * & & & & & & \\
 & * & 0 & 0 & \cdots & 0 & * & 0 & 0 & \cdots & 0 & \\
\text{m-i} & \dagger & * & 0 & \cdots & 0 & \dagger & * & 0 & \cdots & 0 & \bigcirc \\
\text{rows} & \dagger & \dagger & * & \cdots & 0 & \dagger & \dagger & * & \cdots & 0 & \\
 & \vdots & \vdots & & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
 & \dagger & \dagger & & \cdots & \cdot & \dagger & \dagger & \dagger & \cdots & * & \\
\hdashline
 & * & * & \cdots & & * & 0 & \cdots & & & 0 & \\
 & * & 0 & \cdots & & 0 & * & 0 & \cdots & & 0 &
\end{array}
$$

where a "*" is nonzero and a "†" may be nonzero. (The table in II - 2 of order of vanishing of the differentials is what is used to see that $\mathfrak{M}$ has the above form.) The last two rows are those which arise from P_{m+1} and P_{m+3}.

Note (from the table in II - 2) that the m differentials which vanish at P_{m+1}, vanish only simply there. Thus, the quadratic differentials $d\tau_{P_{m+1,0}} d\zeta_k$, for $k = g-i+1,\ldots,g$, will each have a simple pole at P_{m+1} (since $d\tau_{P_{m+1,0}}$ has a pole of order 2 there).

Now suppose there existed a linear dependence relation among the $\alpha_{j,k}$, say

$$(*)\quad a_1\alpha_{1,1} + a_2\alpha_{1,2} + \cdots + a_i\alpha_{1,i} + a_{i+1}\alpha_{2,1} + \cdots + a_{2i}\alpha_{2,i} = 0.$$

By definition of the $\alpha_{j,k}$ we have

$$\alpha_{1,k} = d\zeta_{g-i+k}\Big(\sum_{j=1}^{m} \mu_j d\tau_{P_{j,0}} + \sum_{\nu=0}^{m-i+1} \overline{\mu}_\nu d\tau_{P_{m+2,\nu}} + \mu d\tau_{P_{m+1,0}}\Big)$$

where μ_j, $\overline{\mu}_\nu$ and μ are $\pm$ minors of order $n - 2$ of (and μ is nonzero). Thus, the $\alpha_{1,k}$, for $k = 1,\ldots,i$, will all have a simple pole at P_{m+1} (since they contain $d\zeta_{g-i+k}d\tau_{P_{m+1,0}}$ with nonzero coefficient and all other terms are regular at P_{m+1}). But the $\alpha_{2,k}$ will all be finite at P_{m+1} (since they don't contain $d\tau_{P_{m+1,0}}$ at all).

Therefore, the relation (*) will imply the existence of a linear dependence relation among the $\alpha_{1,k}$, for $k = 1,\ldots,i$, and among the $\alpha_{2,k}$, for $k = 1,\ldots,i$. If (*) is nontrivial, then at least one of these relations will be nontrivial. But the $\alpha_{j,k}$ for fixed j are linearly independent (cf. the Remark after the $r = 1$ case). This contradiction shows that our divisor D satisfies the hypotheses of Lemma 5. Hence, by prudently choosing the points $Q_1,\ldots,Q_{3g-3}$ on our surface, we are assured that the tangent space to $\mathcal{G}^2_n$ at (s_0,D), where s_0 is the module point of Meis's Riemann surface, has dimension $3g - 3 + \tau + 2$ (by Proposition 16).

Example

We have the feeling that the reader might like an example here. We will oblige the reader, but he or she will probably regret it.

Consider the case $r = 2$, $g = 8$, $n = 8$, hence $i = 2$. Our surface is given by the algebraic function

$$y^5 = (x-1)(x-2)(x-3)(x-4)^4(x-5)^4(x-6)^4$$

The table of order of vanishing of the differentials is:

	$x = 1$	2	3	4	5	6	0	∞
$d\zeta_1$	3	3	3	0	0	0	0	1
$d\zeta_2$	2	2	2	1	1	1	0	1
$d\zeta_3$	1	1	1	2	2	2	0	1
$d\zeta_4$	0	0	0	3	3	3	0	1
$d\zeta_5$	3	3	3	0	0	0	1	0
$d\zeta_6$	2	2	2	1	1	1	1	0
$d\zeta_7$	1	1	1	2	2	2	1	0
$d\zeta_8$	0	0	0	3	3	3	1	0

We take for our divisor D the 5 points over 0, denoted $P_1,\ldots,P_s$; the point over $x = 6$ (denoted P_6) with multiplicity 2; and the point over $x = 5$, denoted P_7.

The matrix M has the form

$$M = \begin{bmatrix} \phi_{1,1}(P_1) & \phi_{1,2}(P_1) & \phi_{1,3}(P_1) & \phi_{1,4}(P_1) & 0 & 0 & 0 & 0 \\ \phi_{2,1}(P_2) & \phi_{2,2}(P_2) & \phi_{2,3}(P_2) & \phi_{2,4}(P_2) & 0 & 0 & 0 & 0 \\ \phi_{3,1}(P_3) & \phi_{3,2}(P_3) & \phi_{3,3}(P_3) & \phi_{3,4}(P_3) & 0 & 0 & 0 & 0 \\ \phi_{4,1}(P_4) & \phi_{4,2}(P_4) & \phi_{4,3}(P_4) & \phi_{4,4}(P_4) & 0 & 0 & 0 & 0 \\ \phi_{6,1}(P_6) & 0 & 0 & 0 & \phi_{6,5}(P_6) & 0 & 0 & 0 \\ -\frac{1}{2}\phi'_{6,1}(P_6) & -\frac{1}{2}\phi'_{6,2}(P_6) & 0 & 0 & -\frac{1}{2}\phi'_{6,5}(P_6) & -\frac{1}{2}\phi'_{6,6}(P_6) & 0 & 0 \\ \phi_{5,1}(P_5) & \phi_{5,2}(P_5) & \phi_{5,3}(P_5) & \phi_{5,4}(P_5) & 0 & 0 & 0 & 0 \\ \phi_{7,1}(P_7) & 0 & 0 & 0 & \phi_{7,5}(P_7) & 0 & 0 & 0 \end{bmatrix}$$

where we have moved the fifth row to the seventh row in order to make the leading minor of order 6, denoted μ, nonzero.

By lemma 4, we may assume that the "deformed" matrix $\mathfrak{M}$ has the same first 6 columns as M. The last two columns of $\mathfrak{M}$ are given by ε times:

Col. 7	Col. 8
$s_1\phi'_{1,7}(P_1) + \sum_{m=1}^{3g-3} b_m\tau'_{P_{1,0}}(Q_m)\zeta'_7(Q_m)$	$s_1\phi'_{1,8}(P_1) + \sum_{m=1}^{3g-3} b_m\tau'_{P_{1,0}}(Q_m)\zeta'_8(Q_m)$
$s_2\phi'_{2,7}(P_2) + \sum_{m=1}^{3g-3} b_m\tau'_{P_{2,0}}(Q_m)\zeta'_7(Q_m)$	$s_2\phi'_{2,8}(P_2) + \sum_{m=1}^{3g-3} b_m\tau'_{P_{2,0}})Q_m)\zeta'_8(Q_m)$
$s_3\phi'_{3,7}(P_3) + \sum_{m=1}^{3g-3} b_m\tau'_{P_{3,0}}(Q_m)\zeta'_7(Q_m)$	$s_3\phi'_{3,8}(P_3) + \sum_{m=1}^{3g-3} b_m\tau'_{P_{3,0}}(Q_m)\zeta'_8(Q_m)$
$s_4\phi'_{4,7}(P_r) + \sum_{m=1}^{3g-3} b_m\tau'_{P_{4,0}}(Q_m)\zeta'_7(Q_m)$	$s_4\phi'_{4,8}(P_4) + \sum_{m=1}^{3g-3} b_m\tau'_{P_{4,0}}(Q_m)\zeta'_8(Q_m)$
$\frac{1}{2}s_7\phi''_{6,7}(P_6) + \sum_{m=1}^{3g-3} b_m\tau'_{P_{6,0}}(Q_m)\zeta'_7(Q_m)$	$\sum_{m=1}^{3g-3} b_m\tau'_{P_{6,0}}(Q_m)\zeta'_8(Q_m)$
$\frac{1}{2}s_6\phi''_{6,7}(P_6) - \frac{1}{6}s_7\phi^{(3)}_{6,7}(P_6) + \sum_{m=1}^{3g-3} b_m\tau'_{P_{6,1}}(Q_m)\zeta'_7(Q_m)$	$-\frac{1}{6}s_7\phi^{(3)}_{6,8}(P_6) + \sum_{m=1}^{3g-3} b_m\tau'_{P_{6,1}}(Q_m)\zeta'_8(Q_m)$
$s_5\phi'_{5,7}(P_5) + \sum_{m=1}^{3g-3} b_m\tau'_{P_{5,0}}(Q_m)\zeta'_7(Q_m)$	$s_5\phi'_{5,8}(P_5) + \sum_{m=1}^{3g-3} b_m\tau'_{P_{5,0}}(Q_m)\zeta'_8(Q_m)$
$\sum_{m=1}^{3g-3} b_m\tau'_{P_{7,0}}(Q_m)\zeta'_7(Q_m)$	$\sum_{m=1}^{3g-3} b_m\tau'_{P_{7,0}}(Q_m)\zeta'_8(Q_m)$

Notice that s_8 does not appear (it would be in the last row) since $d\zeta_7$ and $d\zeta_8$ both vanish with multiplicity greater than 1 on the point P_7 (which is a simple point of D). This should have implications about the tangent space to $G_8^2(X)$ at D, but we have not been able to see just what those implications are.

Now, $\alpha_{1,1}$ and $\alpha_{1,2}$ will contain the terms $-\mu d\tau_{P_{5,0}} d\zeta_7$ and $-\mu d\tau_{P_{5,0}} d\zeta_8$ respectively. Hence, these quadratic differentials will have a pole at P_5. But $\alpha_{2,1}$ and $\alpha_{2,2}$ will be finite at P_5.

Odd genus case:

This case is quite similar. Suppose that $g = 2m + 1$ and n are given such that τ is non-negative. Consider Meis's surface:

$$y^3 = \frac{\prod_{j=1}^{m+2} (x - j)}{\prod_{k=m+3}^{2m+3} (x - k)}$$

Let P_0 and P_0' denote two of the three points over $x = 0$. Our divisor D will consist of:

(1) the $m + 2$ ramification points over $x = 1,2,\ldots,m+2$ which we will denote by $P_1,\ldots,P_{m+2}$

(2) P_0' and

(3) the point P_0 with multiplicity $m - i$.

The divisor $P_1 + \ldots + P_{m+2}$ was Meis's sample divisor. Similarly to the even genus case, we may assume that one of the two last rows in $\mathfrak{M}$ arises from P_{m+2}. The other of the two last rows arises from P_0'.

Now, the last m differentials in Meis's basis vanish simply at P_{m+2}, hence the quadratic differentials $d\tau_{P_{m+2,0}} d\zeta_{g-i+k}$, for $k = 1,\ldots,i$, will have a pole at P_{m+2}. We may apply the same reasoning as in the even genus case to conclude that a linear dependency relation among all the $\alpha_{j,k}$ would imply a linear dependency among those arising from a fixed row, a contradiction.

Thus, we have established

<u>Proposition 18</u>: $\mathcal{G}_n^2$ has a component of dimension $3g - 3 + \tau + 2$ for any n and g such that τ is nonnegative.

§4. The case $r = 3$

We assume throughout this section that $r = 3$.

We construct a special divisor from Meis's examples and then show that the $\alpha_{j,k}$ are linearly independent by considering the order of vanishing of the $\alpha_{j,k}$ at some of the points of the divisor.

Even genus case

Suppose $g = 2m$ and n are given such that τ is nonnegative. Consider Meis's Riemann surface of genus g in II - 2. Our divisor D will consist of:

(1) the $m + 1$ points over $x = 0$, denoted $P_1,\dots,P_{m+1}$

(2) the point over $x = 6$, denoted P_{m+2}, with multiplicity $(m - i)$ and

(3) the point over $x = 5$, denoted P_{m+3}, and the point over $x = 4$, denoted P_{m+4}.

Then our divisor has degree equal to $m + 1 + (m-i) + 2 = 2m - i + 3 = n$. Its index of specialty is i since only the last i differentials in Meis's basis vanish to order $m - i$ at the point over $x = 6$. More explicitly, recall from II - 2 that the order of vanishing of $d\zeta_{g-i+k}$ at the points P_{m+2}, P_{m+3} and P_{m+4} is $m - (i - k + 1)$, for $k = 1,\ldots,i$.

Let t_j be a local parameter at P_j, $j = 1,\ldots,m+4$, and write $d\zeta_k = \phi_{j,k}dt_j$. Since Meis's divisor $P_1 + \ldots + P_{m+1}$ is in $G^1_{m+1}(\mathbf{X}) - G^2_{m+1}(\mathbf{X})$, the $(m+1) \times g$ matrix

$$[\phi_{j,k}(P_j)] \qquad \begin{array}{l} j = 1,\ldots,m+1 \\ k = 1,\ldots,g \end{array}$$

has rank m. By renumbering these points, if necessary, we may assume that the leading minor of order m is nonzero. In the matrix $\mathcal{M}$, take the row arising from P_{m+1} and put it just above the row arising from P_{m+3}. This insures that the leading minor μ of order $n - 3$ of $\mathcal{M}$ is nonzero. ($\mathcal{M}$ has a form completely analogous to that described in the previous section.) The three last rows of $\mathcal{M}$ are those arising from P_{m+1}, P_{m+3} and P_{m+4}.

Now suppose there existed a linear relation of the form

$$(*) \quad a_1\alpha_{1,1} + \ldots + a_i\alpha_{1,i} + a_{i+1}\alpha_{2,1} + \ldots + a_{2i}\alpha_{2,i} + \ldots + a_{3i}\alpha_{3,i} = 0$$

We will show that all the a_j are 0 by considering the order of $\alpha_{j,k}$ at P_{m+1}, P_{m+3} and P_{m+4}.

Now, those $\alpha_{j,k}$ with $j = 1$ (and only those) will contain $d\tau_{P_{m+1,0}} d\zeta_{g--i+k}$ with a nonzero coefficient (namely $\pm\mu$). But $d\zeta_{g-i+k}$, $k = 1,\ldots,i$, vanishes simply at P_{m+1} (see II - 2). Hence, as in the previous section, the $\alpha_{1,k}$ will have a pole at P_{m+1}, while the $\alpha_{2,k}$ and the $\alpha_{3,k}$ will be regular there. This implies, by (*), that we must have

$$a_1\alpha_{1,1} + \ldots + a_i\alpha_{1,i} = 0.$$

But the $\alpha_{1,k}$ are linearly independent by III - 2, hence we must have $a_1 = a_2 = \ldots = a_i = 0$.

It is quite a bit more complicated to show that the other coefficients in (*) are zero. Recall that the order of $d\zeta_{g-i+k}$ at P_{m+3} and P_{m+4} is $m - (i-k+1)$, for $k = 1,\ldots,i$. The $\alpha_{2,k}$ will contain $d\tau_{P_{m+3,0}} d\zeta_{g-i+k}$ with a nonzero coefficient $(\pm\mu)'$, hence will have order $m - (i-k+1) - 2$ at P_{m+3} (all other terms in $\alpha_{2,k}$ are regular at P_{m+3}). The $\alpha_{3,k}$ will have order at least $m - (i-k+1)$ at P_{m+3} (since $\alpha_{3,k}$ does not contain $d\tau_{P_{m+3,0}}$ and every other differential of the second kind is regular at P_{m+3}). The converse situation will hold at P_{m+4}.

Recall that (*) has become

$$(*) \quad a_{i+1}\alpha_{2,1} + \dots + a_{2i}\alpha_{2,i} + a_{2i+1}\alpha_{3,1} + \dots + a_{3i}\alpha_{3,i} = 0.$$

Consider the following table of order of vanishing of the $\alpha_{j,k}$ at the points P_{m+3} and P_{m+4}:

	at P_{m+3}	P_{m+4}		at P_{m+3}	P_{m+4}
$\alpha_{2,1}$	$m-i-2$	$\geq m-i$	$\alpha_{3,1}$	$\geq m-i$	$m-i-2$
$\alpha_{2,2}$	$m-i-1$	$\geq m-i+1$	$\alpha_{3,2}$	$\geq m-i+1$	$m-i-1$
$\alpha_{2,3}$	$m-i$	$\geq m-i+2$	$\alpha_{3,3}$	$\geq m-i+2$	$m-i$
$\alpha_{2,4}$	$m-i+1$	$\geq m-i+3$	$\alpha_{3,4}$	$\geq m-i+3$	$m-i+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\alpha_{2,k}$	$m-(i-k+3)$	$\geq m-(i-k+1)$	$\alpha_{3,k}$	$\geq m-(i-k+1)$	$m-(i-k+3)$

Observing the orders at P_{m+3}, we see that we must have $a_{i+1} = a_{i+2} = 0$ (since $\alpha_{2,1}$ and $\alpha_{2,2}$ have lower order at P_{m+3} than any of the other $\alpha_{j,k}$). We needn't have that $a_{i+3} = 0$ since we may have that the order of $\alpha_{3,1}$ is $m - i$ at P_{m+3} and $\alpha_{3,1}$ and $\alpha_{2,3}$ may "cancel" each other.

However, now consider the orders at P_{m+4}. Since $\alpha_{3,1}$ and $\alpha_{3,2}$ have lower order at P_{m+4} than any of the other $\alpha_{j,k}$, we must have $a_{2i+1} = a_{2i+2} = 0$. But, going back to the situation at P_{m+3}, this implies that $a_{i+3} = a_{i+4} = 0$. And this in turn implies that $a_{2i+3} = a_{2i+4} = 0$ (going back to P_{m+4}). By continuing to go back and forth in this manner, we can show that all the a_ℓ are 0.

Thus, invoking Lemma 5 and Proposition 16, we may conclude that the dimension of the tangent space to $\mathfrak{D}_n^3$ at (s_0, D) is $3g - 3 + \tau + 3$.

Odd genus case

Suppose $g = 2m + 1$ and n are given such that τ is nonnegative. Consider Meis's Riemann surface of genus g

in II - 2. Our divisor D will consist of:

(1) the $m + 2$ ramification points over $x = 1,\dots,m+2$, which we denote by $P_1,\dots,P_{m+2}$

(2) one of the three points over $x = 0$, denoted P_0, with multiplicity $m - i$ and

(3) the other two points over $x = 0$, denoted P_0' and P_0''.

By a completely analogous argument to that in the even genus case, one can show that the dimension of the tangent space to $\mathfrak{D}^3_n$ at (s_0,D) is $3g - 3 + \tau + 3$. Thus, we have established

Proposition 19: $\mathfrak{D}^3_n$ has a component of dimension $3g - 3 + \tau + 3$ for any n and g such that τ is nonnegative.

This is as far as Meis's examples will carry us, perhaps due to our own limitations.

Chapter IV

Open Problems and Conjectures

§1. Moduli of Extra-Special Divisors

We return to Remark (1) after Theorem 3 of III - 2. Recall that we proved that $\mathcal{G}_n^1 - \mathcal{G}_n^2$, if nonempty, has pure dimension $3g - 3 + \tau + 1$, whether or not τ is nonnegative.

Suppose that τ is negative and that $\mathcal{G}_n^1 - \mathcal{G}_n^2$ is nonempty. Let θ be the map $\mathcal{G}_n^1 - \mathcal{G}_n^2 \to T_g$ and let W be an irreducible component of the image of $\mathcal{G}_n^1 - \mathcal{G}_n^2$ under this map. We can say what the dimension of W is if we know the dimension of a generic fiber of the map $\theta^{-1}(W) \to W$. Let d denote the dimension of such a generic fiber. Then we have

$$\dim W = 3g - 3 + \tau + 1 - d.$$

Example 1: Hyperelliptic Curves

A hyperelliptic Riemann surface of genus g is defined by the algebraic function

$$y^2 = A(x - \alpha_1) \dots (x - \alpha_{2(g+1)}) \qquad [3].$$

Hyperelliptic Riemann surfaces are characterized by having a nonempty G_2^1 or, equivalently, by a hyperelliptic Weierstrass point ([8]). Let P be a hyperelliptic Weierstrass point on a Riemann surface X. Then the gap sequence at P is $1, 3, 5, 7, \dots$. By general theory of Weierstrass points (cf. [8]), we may choose a basis of the holomorphic differentials $\{d\zeta_k\}$ on X such that the order of $d\zeta_k$ at P is $2(k-1)$, $k = 1,\dots,g$. The matrix $\Delta_D M$ of I - 3 for the divisor $D = 2P$ is

$$\begin{bmatrix} \phi_1(P) - \varepsilon \frac{s_2}{2} \phi_2^{(2)}(P) & 0 & \dots & 0 \\ -\frac{1}{2}\phi_1'(P) - \varepsilon \frac{s_1}{2}\phi_2^{(2)}(P) + \varepsilon \frac{s_2}{6} \phi_2^{(3)}(P) & 0 & \dots & 0 \end{bmatrix}$$

For all of the minors of order 2 of this matrix to vanish, s_1 and s_2 must satisfy one equation. Thus, the dimension of the tangent space to $G_2^1(X)$ at D is $2 - 1 = 1$. We can conclude that the dimension of G_2^1 at D is 1, since we always have $\dim G_n^r \geq r$ (cf. Remark (2) after Theorem 3). This serves to illustrate the methods of I - 3. We can independently conclude that <u>every</u> component of G_2^1 has dimension 1 by virtue of the following lemma.

Lemma 6: Every member of $G_2^1(X)$ is linearly equivalent to $2P_0$, where P_0 is a hyperelliptic Weierstrass point.

Proof: Suppose E is in $G_2^1(X)$. Then there is a function h whose poles are the points of E and by means of which X is displayed as a two-sheeted branched covering of the Riemann sphere, the branch points being hyperelliptic Weierstrass points. Let P_0 be a branch point and suppose P_0 occurs over $x = \alpha$. Then the function $x - \alpha$ has as its divisor $2P_0 - E$, hence E and $2P_0$ are linearly equivalent. ▮

Thus, each component of G_2^1 maps to a point of J, hence is 1-dimensional.

Now, we have $\tau = 2(2-1) - g = 2 - g$. Thus, the subvariety of T_g of all hyperelliptic surfaces has dimension $3g - 3 + (2-g) + 1 - 1 = 2g - 1$. This is very well-known and, in fact, our methods are very close to those of Farkas [5].

Example 2:

Suppose we wanted to compute the moduli of curves with nonempty G^1_3. By Clifford's Theorem, $\mathfrak{G}^2_3$ is empty hence, by Theorem 4, $\mathfrak{G}^1_3$, if nonempty, is smooth of pure dimension $3g - 3 + \tau + 1$. Now, $\tau = 2(3-1) - g = 4 - g$. So, $\mathfrak{G}^1_3$, if nonempty, has dimension $2g + 2$.

By Theorem 1 of [16], we have that, for $g \geq 4$, if $G^1_3(X)$ is nonempty, then every component has dimension at least $5 - g$ and at most dimension 2, with the upper bound occurring if and only if X is hyperelliptic. But each component of G^1_3 must have dimension at least 1. So, if there exists a non-hyperelliptic curve of genus g with nonempty G^1_3, then we must have that the dimension of the generic fiber of the map $\mathfrak{G}^1_3 \to T_g$ is 1. Hence, we would have that the dimension of the subvariety of T_g of curves with nonempty G^1_3 is $2g + 2 - 1 = 2g + 1$. This agrees with the number which appears in Segre [24] and Severi [26].

To be more concrete, for odd genus Meis's examples have branch points of order 2 over $x = 1,2,3,\ldots,2m+2$ and ∞. These branch points are Weierstrass points whose gap sequence is

$$1,\ 2,\ \hat{3},\ 4,\ 5,\ \hat{6},\ \ldots,\ 3(\tfrac{g-1}{2}) + 1.$$

Thus, 3 times one of these points is in G_3^1. Also note that these curves are not hyperelliptic, since hyperelliptic curves have only hyperelliptic Weierstrass points. The reader may convince himself that if P is one of the above branch points and $D = 3P$, then by the methods in I - 3, the dimension of the tangent space to $G_3^1(X)$ at D is 1.

We have established

<u>Proposition 17</u>: Suppose there exists a non-hyperelliptic curve of genus g with a nonempty G_3^1 (which condition is certainly true for g odd). Then the subvariety of T_g of curves with nonempty G_3^1 has dimension $2g + 1$. ▌

It is hoped that the machinery developed here will help to solve similar "moduli questions."

§2.

Suppose there exists a compact Riemann surface X of genus g and a divisor D on X such that the following two properties hold:

(1) $D = \sum_{j=1}^{n} P_j$, all points distinct

(2) There exists a basis $\{d\zeta_k\}_{k=1}^{g}$ of the holomorphic differentials on X such that

$$\operatorname{order}_{P_j} d\zeta_k = \begin{cases} 0 & 1 \leq k \leq g - i \quad \forall j \\ 1 & g - i + 1 \leq k \leq g \quad \forall j \end{cases}$$

where $i = \dim H^1(X, O_X(D))$.

Let $\alpha_{1,1}, \ldots, \alpha_{r,i}$ be the quadratic differentials of III - 1. Suppose there existed a linear relation

(*) $$\sum_{\substack{j=1,\ldots,r \\ k=1,\ldots,i}} b_{j,k}\alpha_{j,k} = 0$$

Put $d\tau_j = d\tau_{P_{j,0}}$. By definition of $\alpha_{j,k}$, we have

$$\alpha_{j,k} = d\sigma_{n+k-1}(e_1 d\tau_1 + \ldots + (-1)^{n-r-1} e_{n-r} d\tau_{n-r} + (-1)^{n-r} \mu d\tau_{n-r+j})$$

where the e_ℓ are minors of the matrix $\mathcal{M}$ of III - 1 and μ is the (nonzero) leading minor of order $n - r$ of $\mathcal{M}$. Computing orders, we see that

$$\text{order}_{P_\ell}\ \alpha_{j,k} = \begin{cases} -1 & j = 1, \forall k \\ \geq 1 & j \neq \ell, \forall k \end{cases}$$

Hence, a linear relationship such as (*) implies that the $\alpha_{j,k}$ for j fixed are linearly dependent. This contradicts what was shown in III - 2 (cf. the Remark there). Hence $\alpha_{1,1}, \ldots, \alpha_{r,i}$ are linearly independent. By Lemma 5 and Proposition 16, we may then conclude that the dimension of the tangent space to $\mathcal{G}_n^r$ at (s_0, D), where s_0 is the module point of X, is $3g - 3 + \tau + r$.

Question 1: For what values of g, r and n does there exist such an X and D?

Producing such an example would show that $\mathfrak{D}^r_n$ has a component of dimension $3g - 3 + \tau + r$.

Question 2: If g, r and n are such that τ is nonnegative, then is the situation described above generic; i.e., is there a dense open set of $\mathfrak{D}^r_n$ consisting of divisors on Riemann surfaces satisfying conditions (1) and (2)?

This would show that if $\tau \geq 0$, then for a generic Riemann surface X the analytic space $G^r_n(X)$ has dimension $\tau + r$, the result which we originally set out to prove.

Bibliography

[1] L. Ahlfors, The complex analytic structore of the space of closed Riemann surfaces, in *Analytic Functions*, Princeton Univ. Press, 1960; pp. 45-66.

[2] A. Andreotti, On a theorem of Torelli, Amer. Jour. of Math. 80(1958), pp. 801-828.

[3] G. A. Bliss, *Algebraic Functions*, Dover, New York 1966 (reprint).

[4] A. Brill and M. Noether, Uber die algebraischen Functionen und ihre Anwendung in der Geometrie, Math. Annalen 7(1874).

[5] H. Farkas, Special divisors and analytic subloci of the Teichmüller space, Amer. Jour. of Math. 88(1966), pp. 881-901.

[6] G. Grauert, Ein Theorem der analytischen Geometrie und die Modulräume komplexer Strukturen, Publ. Math. I.H.E.S. 5 (1960).

[7] A. Grothendieck, Exposes in *Séminaire Cartan*, 1960-61, Secretariat mathematique, Paris.

[8] R. C. Gunning, *Lectures on Riemann Surfaces*, Princeton Univ. Press 1966.

[9] ____________, *Jacobi Varieties*, Princeton Univ. Press 1972.

[10] K. Hensel and G. Landsberg, Theorie der algebraischen Functionen, Chelsea, New York 1965 (reprint).

[11] B. Iversen, Linear determinants with applications to the Picard scheme of a family of algebraic curves, Springer Lecture Notes 174, Berlin 1970.

[12] G. Kempf, Schubert methods with an application to algebraic curves, Stichting Mathematisch Centrum, Amsterday 1971.

[13] ________ and D. Laksov, The determinantal formula of Schubert calculus, to appear.

[14] S. L. Kleiman and D. Laksov, On the existence of special divisors, Amer. Jour. of Math. 94(1972); pp. 431-436.

[15] ____________________________, Another proof of the existence of special divisors, preprint.

[16] H. Martens, On the varieties of special divisors on a curve I, Jour. für reine und angew. Math. 227(1967); pp. 111-120.

[17] __________, Varieties of special divisors on a curve II, Jour. für reine und angew.Math. 233(1968); pp. 89-100.

[18] A. Mattuck and A. Mayer, The Riemann-Roch Theorem for algebraic curves, Annali Scuola Norm. Sup. Pisa 17(1963); pp. 223-237.

[19] A. Mayer, Rauch's variational formula and the heat equation, Math. Annalen 181.

[20] T. Meis, Die minimale Blätterzahl der Konkretisierung einer kompakten Riemannschen Fläche, Schr. Math. Inst., Heft 16, Univ. Münster (1960).

[21] C. Patt, Variations of Teichmüller and Torelli surfaces, Jour. d'Analyse Math. 11(1963); pp. 221-248.

[22] H. E. Rauch, Weierstrass points, branch points and moduli of Riemann surfaces, Comm. Pure and Appl. Math. 12(1959); pp. 543-560.

[23] M. Schiffer and D. C. Spencer, Functionals of finite Riemann surfaces, Princeton Univ. Press 1964.

[24] B. Segre, Sui moduli delle curve poligonali..., Math. Annalen 100(1928); pp. 537-551.

[25] F. Severi (and E. Löffler), Vorlesungen über algebraischen Geometrie, Tuebner, Liepzig 1921.

[26] F. Severi, Sul teorema di esistenza di Riemann, Rendic. Circ. Mat. di Palermo 46(1922).

[27] G. Springer, Introduction to Riemann surfaces, Addison-Wesley, Reading 1957.

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