Remarks on Affine Springer Fibres

REMARKS ON AFFINE SPRINGER FIBRES

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Let G be a simply connected almost simple algebraic group over $\mathbf{C}$ and let $\mathfrak{g}$ be the Lie algebra of G. Let B be a Borel subgroup of G, let T be a maximal torus of B and let $\mathfrak{t},\mathfrak{b}$ be the Lie algebras of T,B. Let $\mathcal{B}$ be the variety of Borel subalgebras of $\mathfrak{g}$. For any nilpotent element $N\in\mathfrak{g}$ we set $\mathcal{B}_N=\{\mathfrak{b}\in\mathcal{B};N\in\mathfrak{b}\}$ (a Springer fibre). In [KL] an affine analogue of $\mathcal{B}_N$ ("affine Springer fibre") was introduced. Let $F=\mathbf{C}((\epsilon)$, $A=\mathbf{C}[[\epsilon]]$, where ϵ is an indeterminate and let $\mathfrak{g}(F)=F\otimes\mathfrak{g}$ (a Lie algebra over F), $L=A\otimes\mathfrak{g}$ (a Lie algebra over A). An element $\xi\in\mathfrak{g}(F)$ is said to be topologically nilpotent if $\lim_{n\to\infty}\text{ad}(\xi)^n=0$ in $\text{End}_F(\mathfrak{g}(F))$. Let $\tilde{X}$ be the set of all Iwahori subalgebras of $\mathfrak{g}(F)$; this is an increasing union of projective varieties over $\mathbf{C}$. According to [KL], for any regular semisimple, topologically nilpotent element $\xi\in\mathfrak{g}(F)$, the set $\tilde{X}_\xi=\{I\in\tilde{X};\xi\in I\}$ is a nonempty, locally finite union of projective varieties all of the same dimension, say b_ξ. Let $[\tilde{X}_\xi]$ be the set of irreducible components of $\tilde{X}_\xi$, a finite or countable set.

In the remainder of this paper, h denotes a fixed regular element in $\mathfrak{t}$. Then $\epsilon h\in\mathfrak{g}(F)$ is regular semisimple, topologically nilpotent so that the affine Springer fibre $\tilde{X}_{\epsilon h}=\{I\in\tilde{X};\epsilon h\in I\}$ is defined. From [KL,§5] we see that $b_{\epsilon h}=\nu$ where $\nu=\dim\mathcal{B}$. As in [KL, §3], there is a free abelian group Λ (see Sec.2) of rank equal to the rank of $\mathfrak{g}$ which acts freely on $\tilde{X}_{\epsilon h}$ in such a way that the induced Λ-action on $[\tilde{X}_{\epsilon h}]$ is also free and has only finitely many orbits. In this paper we will describe a fundamental domain for the Λ-action on $\tilde{X}_{\epsilon h}$. Namely, let $\mathfrak{S}'$ be the Steinberg variety of triples $(E,\mathfrak{b}_1,\mathfrak{b}_2)$ where $\mathfrak{b}_1\in\mathcal{B},\mathfrak{b}_2\in\mathcal{B}$ and $E\in\mathfrak{b}_1\cap\mathfrak{b}_2$ is nilpotent. Let $\mathfrak{S}$ be the fibre at $\mathfrak{b}$ of the projection $\mathfrak{S}'\to\mathcal{B}$, $(E,\mathfrak{b}_1,\mathfrak{b}_2)\mapsto\mathfrak{b}_2$. We can identify $\mathfrak{S}$ with $\{(E,\mathfrak{b}_1);\mathfrak{b}_1\in\mathcal{B},E\in\mathfrak{n}\cap\mathfrak{b}_1\}$. We state the following result.

Theorem. *There is a locally closed subvariety of $\tilde{X}_{\epsilon h}$ which is a fundamental domain for the Λ-action on $\tilde{X}_{\epsilon h}$ such that $\tilde{X}_{\epsilon h}$ is isomorphic to $\mathfrak{S}$.*

From the theorem one can deduce some information on the representation of the affine Weyl group on the vector space $\mathbf{C}[\tilde{X}_{\epsilon h}]$ with basis $[\tilde{X}_{\epsilon h}]$ defined in [L2], see Section 6.

I thank Peng Shan and Zhiwei Yun for discussions.

Supported in part by NSF grant DMS-1566618

2. Let U be the unipotent radical of B. Let $\mathfrak{n}$ be the Lie algebra of U. Let $G(F), U(F), T(F)$ be the group of F-points of G, U, F respectively. Let $G(F)$ be the group of F-points of G. Note that $G(F)$ acts naturally on $\mathfrak{g}(F)$ by the adjoint representation $g : x \mapsto \mathrm{Ad}(g)(x)$. Let Λ be the subgroup of $T(F)$ consisting of the elements $\chi(\epsilon)$ where χ runs over the one parameter subgroups $\mathbf{C}^* \to T$ (viewed as homomorphisms $F^* \to T(F)$). Let X be the set of A-Lie subalgebras of $\mathfrak{g}(F)$ of the form $\mathrm{Ad}(g)(L)$ for some $g \in G(F)$. We shall regard X as an increasing union of projective algebraic varieties over $\mathbf{C}$ as in [L1,§11]. For each $L' \in X$, $L'/\epsilon L'$ inherits from L' a bracket operation and becomes a simple Lie algebra over $\mathbf{C}$. Let $\pi_{L'} : L' \to L'/\epsilon L'$ be the obvious map. Let $\mathcal{B}_{L'}$ be the set of Borel subalgebras of $L'/\epsilon L'$. Now $\tilde{X}$ consists of all $\mathbf{C}$-Lie subalgebra of $\mathfrak{g}(F)$ of the form $\pi_{L'}^{-1}(\mathfrak{b}')$ for some $L' \in X$ and some $\mathfrak{b}' \in \mathcal{B}_{L'}$. We define $\pi : \tilde{X} \to X$ by $I \mapsto L'$ where $I \subset L'$. Note that $g : I \mapsto \mathrm{Ad}(g)I$ is a well defined action of $G(F)$ on $\tilde{X}$ which is transitive. According to [KL], $t : I \mapsto \mathrm{Ad}(t)I$ defines a free action of Λ on $\tilde{X}_{\epsilon h} = \{I \in \tilde{X}; \epsilon h \in I\}$ inducing a free action of Λ with finitely many orbits on $[\tilde{X}_{\epsilon h}]$. Let $X_{\epsilon h} = \{L' \in X; \epsilon h \in L'\}$.

If $\xi \in \mathfrak{n}(F) := F \otimes \mathfrak{n}$ then $\exp(\xi) \in U(F)$ is well defined. Let $\mathfrak{n}(F)' = \oplus_{i \in \mathbf{Z}; i<0} \epsilon^i \mathfrak{n} \subset \mathfrak{n}(F)$. Let $U(F)' = \{\exp(\xi); \xi \in \mathfrak{n}(F)'\} \subset U(F)$. It is well known that any $L' \in X$ can be written in the form $\mathrm{Ad}(t)\mathrm{Ad}(u)L$ where $t \in \Lambda, u \in U(F)'$ are uniquely determined. Hence we have a partition $X_{\epsilon h} = \sqcup_{t \in \Lambda} X_{\epsilon h, t}$ where $X_{\epsilon h, t} = \{\mathrm{Ad}(t)\mathrm{Ad}(u)L; u \in U(F)', \epsilon h \in \mathrm{Ad}(u)L\}$ is a locally closed subset of $X_{\epsilon h}$. Let $\tilde{X}_{\epsilon h, t} = \pi^{-1}(X_{\epsilon h, t})$. This is a locally closed subset of $\tilde{X}_{\epsilon h}$. Let $\Omega = X_{\epsilon h, 1}, \tilde{\Omega} = \tilde{X}_{\epsilon h, 1} = \pi^{-1}(\Omega)$. Note that

(a) $\tilde{X}_{\epsilon h} = \sqcup_{t \in \Lambda} \mathrm{Ad}(t)\tilde{\Omega}$

as a set. Thus, $\tilde{\Omega}$ is a fundamental domain for the Λ-action on $\tilde{X}_{\epsilon h}$. Let $\omega = \{\mathbf{E} \in \mathfrak{n}(F)'; \mathrm{Ad}(\exp(\mathbf{E}))(\epsilon h) \in L\}$. In preparation for the proof of the theorem we will prove the following result.

Lemma 3. *The map $\mathbf{E} = \epsilon^{-1}E_1 + \epsilon^{-2}E_2 + \epsilon^{-3}E_3 + \dots \to E_1$ is a bijection $\phi : \omega \xrightarrow{\sim} \mathfrak{n}$. (Here $E_1, E_2, E_3, \dots$ is a sequence of elements of $\mathfrak{n}$ with $E_i = 0$ for large i.)*

The equation defining ω is $\exp(\mathrm{ad}(\mathbf{E}))(\epsilon h) \in L$ that is

$$\epsilon h + \sum_{i \ge 1} \epsilon^{-i+1}[E_i, h] + (1/2) \sum_{i,j \ge 1} \epsilon^{-i-j+1}[E_i, [E_j, h]]$$
$$+ (1/6) \sum_{i,j,k \ge 1} \epsilon^{-i-j-k+1}[E_i, [E_j, [E_k, h]]] + \dots \in L,$$

that is

$$\sum_{i \ge 2} \epsilon^{-i+1}[E_i, h] + (1/2) \sum_{i,j \ge 1} \epsilon^{-i-j+1}[E_i, [E_j, h]]$$
$$+ (1/6) \sum_{i,j,k \ge 1} \epsilon^{-i-j-k+1}[E_i, [E_j, [E_k, h]]] + \dots \in L,$$

that is

$$[E_r, h] = -(1/2) \sum_{i,j\ge 1, i+j=r} [E_i, [E_j, h]]$$

$$\text{(a)} \qquad - (1/6) \sum_{i,j,k\ge 1, i+j+k=r} [E_i, [E_j, [E_k, h]]] + \dots$$

for $r = 2, 3, \dots$. In the right hand side we have $i < r, j < r, k < r$, etc. Hence if $E_{r'}$ is known for $r' < r$ then $[E_r, h]$ is a well defined element of $\mathfrak{n}$. Hence E_r is a well defined element of $\mathfrak{n}$. (Note that $E \mapsto [E, h]$ is a vector space isomorphism $\mathfrak{n} \xrightarrow{\sim} \mathfrak{n}$.)

It remains to show that $E_r = 0$ for large r. For $r \ge 1$ let $\mathfrak{n}^r$ be the subspace of $\mathfrak{n}$ spanned by all iterated brackets of r elements of $\mathfrak{n}$. (Thus, $\mathfrak{n}^1 = \mathfrak{n}$, $\mathfrak{n}^2$ is spanned by $[a, b]$ with a, b in $\mathfrak{n}$, $\mathfrak{n}^3$ is spanned by $[[a, b], c]]$ with a, b, c in $\mathfrak{n}$, etc.) Note that

(b) $E \mapsto [E, h]$ is an isomorphism $\mathfrak{n}^r \to \mathfrak{n}^r$ for any $r \ge 1$.

We show by induction on r that

$$\text{(c)} \qquad E_r \in \mathfrak{n}^r \text{ for } r = 1, 2, \dots$$

For $r = 1$ this is clear. Assume now that $r \ge 2$. From (a) and the induction hypothesis we deduce that $[E_r, h] \in \mathfrak{n}^r$. Using (b) we see that for some $E' \in \mathfrak{n}^r$ we have $[E_r, h] = [E', h]$, hence $[E_r - E', h] = 0$, hence $E_r = E'$. Thus $E_r \in \mathfrak{n}^r$, proving (c). Since $\mathfrak{n}^r = 0$ for large r we see that $E_r = 0$ for large r. This completes the proof of the lemma.

4. For $E \in \mathfrak{n}$ we set $u_E = \exp(\mathbf{E}) \in U(F)'$ where $\mathbf{E} = \phi^{-1}(E)$ (see Lemma 3). Note that $\mathrm{Ad}(u_E)(\epsilon h) \in L$. Now $\mathbf{E} \mapsto \mathrm{Ad}(\exp(-\mathbf{E}))L$ is a bijection $\psi : \omega \xrightarrow{\sim} \vee$. Hence $\psi' := \psi\phi^{-1} : \mathfrak{n} \to \Omega$ is a bijection. We have $\psi'(E) = \mathrm{Ad}(u_E^{-1})L$. We show:

(a) *Let $E \in \mathfrak{n}$ and let $L_E = \mathrm{Ad}(u_E^{-1})L \in X$. Note that $\epsilon h \in L_E$. Then $\pi_{L_E}(\epsilon h) \in L_E/\epsilon L_E$ and $\pi_L(-[E, h]) \in L/\epsilon L$ correspond to each other under the Lie algebra isomorphism $\tau_E : L/\epsilon L \xrightarrow{\sim} L_E/\epsilon L_E$ induced by $\mathrm{Ad}(u_E^{-1}) : L \xrightarrow{\sim} L_E$.*

We must show that $Ad(u_E)(\epsilon h) = -[E, h] \mod \epsilon L$ or that $\mathrm{Ad}(\exp(\mathbf{E}))(\epsilon h) = -[E, h] \mod \epsilon L$ where $\mathbf{E} = \epsilon^{-1}E_1 + \epsilon^{-2}E_2 + \epsilon^{-3}E_3 + \dots$ corresponds to $E = E_1$ as in Lemma 3. Thus we must show that

$$\epsilon h + \sum_{i\ge 1} \epsilon^{-i+1}[E_i, h] + (1/2) \sum_{i,j\ge 1} \epsilon^{-i-j+1}[E_i, [E_j, h]]$$
$$+ (1/6) \sum_{i,j,k\ge 1} \epsilon^{-i-j-k+1}[E_i, [E_j, [E_k, h]]] + \dots = -[E_1, h] \mod \epsilon L,$$

or that

$$\sum_{i\ge 2} \epsilon^{-i+1}[E_i, h] + (1/2) \sum_{i,j\ge 1} \epsilon^{-i-j+1}[E_i, [E_j, h]]$$
$$+ (1/6) \sum_{i,j,k\ge 1} \epsilon^{-i-j-k+1}[E_i, [E_j, [E_k, h]]] + \dots \in \epsilon L.$$

But the left hand side is actually zero, by the proof of Lemma 3. This proves (a).

From (a) we deduce:

(b) *the map* $\beta \mapsto \tau_E(\beta)$ *is a bijection* $\{\beta \in \mathcal{B}_L; \pi_L(-[E,h]) \in \beta\} \to \{\beta' \in \mathcal{B}_{L_E}; \pi_{L_E}(\epsilon h) \in \beta'\}$.

Taking union over all $E \in \mathfrak{n}$ and using the bijection $\psi' : \mathfrak{n} \to \Omega$ we deduce

(c) *the map* $(E,\beta) \mapsto \pi_{L_E}^{-1}(\tau_E(\beta))$ *is a bijection* $\{(E,\beta) \in \mathfrak{n} \times \mathcal{B}_L; \pi_L(-[E,h]) \in \beta\} \xrightarrow{\sim} \tilde{\Omega}$.

We consider the bijection

(d) $\{(E,\beta) \in \mathfrak{n} \times \mathcal{B}_L; \pi_L(-[E,h]) \in \beta\} \to \mathfrak{S}$

given by $(E,\beta) \mapsto (-[E,h], \mathfrak{b}_1)$ where $\mathfrak{b}_1 \in \mathcal{B}$ is defined by $\pi_L(\mathfrak{b}_1) = \beta$. The composition of the inverse of (d) with the bijection (c) is a bijection

(e) $\mathfrak{S} \xrightarrow{\sim} \tilde{\Omega}$.

From the proof we see that the bijection (e) is an isomorphism of algebraic varieties. This proves the theorem.

5. Let NT be the normalizer of T in G and let $W = NT/T$ be the Weyl group. For any $w \in W$ let $\mathcal{B}_w$ be the variety consisting of all $\mathfrak{b}_1 \in \mathcal{B}$ such that $(\mathfrak{b}, \mathfrak{b}_1)$ are in relative position w. Note that $\mathcal{B}_w$ is isomorphic to $\mathbf{C}^{|w|}$ where $|w| \in \mathbf{N}$ is the length of w. Let $\mathfrak{S}_w = \{(E, \mathfrak{b}_1) \in \mathfrak{S}; \mathfrak{b}_1 \in \mathcal{B}_w\}$. The second projection $\mathfrak{S}_w \to \mathcal{B}_w$ makes $\mathfrak{S}_w$ into a vector bundle with fibres of dimension $\nu - |w|$. Hence $\mathfrak{S}_w$ is isomorphic to $\mathbf{C}^\nu$ as an algebraic variety. We have a partition $\mathfrak{S} = \sqcup_{w \in W} \mathfrak{S}_w$ (as a set) with $\mathfrak{S}_w$ locally closed in $\mathfrak{S}$ (the closure of $\mathfrak{S}_w$ in $\mathfrak{S}$ is denoted by $\overline{\mathfrak{S}_w}$). Hence we have a partition $\tilde{\Omega} = \sqcup_{w \in W} \tilde{\Omega}_w$ (as a set) where $\tilde{\Omega}_w$ corresponds to $\mathfrak{S}_w$ under 4(e). Note that $\tilde{\Omega}_w$ is isomorphic to $\mathbf{C}^\nu$ as an algebraic variety and that $\tilde{\Omega}_w$ is locally closed in $\tilde{\Omega}$. For $w \in W, t \in \Lambda$ we set $\tilde{\Omega}_{w,t} = \mathrm{Ad}(t)\tilde{\Omega}_w$. Using 2(a) we see that

(a) $\tilde{X}_{\epsilon h} = \sqcup_{(w,t) \in W \times \Lambda} \tilde{\Omega}_{w,t}$

as a set, where $\tilde{\Omega}_{w,t}$ is locally closed in $\tilde{X}_{\epsilon h}$ and is isomorphic to $\mathbf{C}^\nu$. Let $\overline{\tilde{\Omega}_{w,t}}$ be the closure of $\tilde{\Omega}_{w,t}$ in $\tilde{X}_{\epsilon h}$. Note that $\tilde{\Omega}_{w,t}$ is open dense in $\overline{\tilde{\Omega}_{w,t}}$. Since $\tilde{X}_{\epsilon h}$ is of pure dimension ν, we see that

(b) $(w,t) \mapsto \overline{\tilde{\Omega}_{w,t}}$ *is a bijection* $W \times \Lambda \xrightarrow{\sim} [\tilde{X}_{\epsilon h}]$.

In particular,

(c) *the number of* Λ*-orbits on* $[\tilde{X}_{\epsilon h}]$ *is equal to the order of* W.

A result closely related to (c) (but not (c) itself) appears in [TS].

6. Let $[\mathfrak{S}]$ be the set of irreducible components of $\mathfrak{S}$ (a finite set naturally indexed by W by $w \mapsto \overline{\mathfrak{S}_w}$). The bijection 5(b) gives rise to an imbedding $[\mathfrak{S}] \to [\tilde{X}_{\epsilon h}]$, $\overline{\mathfrak{S}_w} \mapsto \overline{\tilde{\Omega}_{w,1}}$ hence to an imbedding of vector spaces

(a) $\mathbf{C}[\mathfrak{S}] \to \mathbf{C}[\tilde{X}_{\epsilon h}]$

with bases $[\mathfrak{S}], [\tilde{X}_{\epsilon h}]$. Springer has shown that W acts naturally on $\mathbf{C}[\mathfrak{S}]$ (this is known to be the regular representation of W in a nonstandard basis). In [L2] it is shown that the affine Weyl group of G acts naturally on $\mathbf{C}[\tilde{X}_{\epsilon h}]$. Hence, by restriction, W acts on $\mathbf{C}[\tilde{X}_{\epsilon h}]$. From the definitions we see that the imbedding (a) is compatible with the W-actions.

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