Age of information for broadcast and collection in spatially distributed wireless networks

by

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Abstract

We consider a wireless network with a base station broadcasting and collecting time-sensitive data to and from spatially distributed nodes in the presence of wireless interference. The Age of Information (AoI) is the time that has elapsed since the most-recently delivered packet was generated, and captures the freshness of information. In the context of broadcast and collection, we define the Age of Broadcast (AoB) to be the amount of time elapsed until all nodes receive a fresh update, and the Age of Collection (AoC) as the amount of time that elapses until the base station receives an update from all nodes. We quantify the average broadcast and collection ages in two scenarios: 1) instance-dependent, in which the locations of all nodes and interferers are known, and 2) instance-independent, in which they are not known but are located randomly, and expected age is characterized with respect to node locations. In the instance-independent case, we show that AoB and AoC scale super-exponentially with respect to the radius of the region surrounding the base station. Simulation results highlight how expected AoB and AoC are affected by network parameters such as network density, medium access probability, and the size of the coverage region.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

Collection and broadcast of fresh information over spatially-distributed wireless nodes is important for proper functioning of real-time systems, such as search-and-rescue drones or environmental monitoring using IoT sensors [30]. Dynamic environments and the lack of wired infrastructure necessitate deployment of highly-distributed, adhoc network of sensors to gather and send information updates wirelessly, where nodes must communicate with minimal coordination overhead using simple random access schemes.

Such networks must also operate under wireless communication constraints, including interference, fading, and path loss. Ensuring broadcast and collection of the freshest information possible in such a setting is a considerable challenge.

A popular paradigm for measuring the freshness of information observed from a process is the Age of Information (AoI) [23, 18, 17]. The literature addressing AoI and wireless networks is extensive. Average and peak AoI in wireless networks were first characterized in [2]. Optimal wireless link scheduling was studied in [8, 9, 10, 11], relying on a centralized scheduler that is able to coordinate link activations, and the authors of [12] considered scheduling policies that minimize AoI in wireless networks with packets randomly arriving and queueing at the base station. In addition, the authors of [13] studied scheduling with random packet arrivals in a random access setting. Several works have addressed AoI and broadcast. In particular, the authors of [16, 14] found optimal centralized scheduling policies for broadcast from a base station to a number of nodes, minimizing functions of AoI such as Expected Weighted Sum AoI. In [31] the authors investigated AoI in multicast and broadcast networks with i.i.d. exponential (continuous-time) inter-packet delivery times. Works such as [15] investigated network scheduling to minimize AoI under general wireless channel unreliability, while [11] studied scheduling policies with random arrivals, modeling the problem as a Markov Decision Process. From an information theoretic perspective, [1] explored the effect of coding on the AoI in two-user broadcast networks, and [25] addressed AoI for Broadcast in CSMA/CS wireless networks, assuming network connectivity follows the Protocol Model [6]. The authors of [4] explored AoI in all-to-all broadcast wireless networks, deriving average and peak AoI using fundamental properties of graphs.

More recently, AoI in spatially-distributed networks has been investigated. The authors of [24] investigated data dissemination and gathering, modeling spatial separation as edges on a mobility graph. The authors of [19, 20], deployed stochastic geometry analysis to capture the spatiotemporal statistics of AoI in networks where nodes are distributed as a homogeneous point process. The authors of [28, 29] optimized network parameters such as the medium access probability to minimize average and peak AoI, leveraging knowledge of the interference statistics of Poisson-distributed wireless networks. While AoI has been considered in spatially-distributed wireless networks, the important cases of wireless broadcast and collection in a spatially distributed network have not been addressed.

Our main contribution in this work is to introduce the notion of Age to the broadcast and collection of information. We define two metrics – the *age of broadcast* (AoB) and the *age of collection* (AoC) – that characterize the amount of time elapsed since all receivers successfully receive an update in the broadcast case or all transmitters successfully deliver a packet to the base station in the collection case. We consider both the instance-dependent, and the instance-independent AoB and AoC. In the instance-dependent scenario, the locations of all interferers, transmitters and receivers are fixed and known. In the instance-independent scenario, the positioning of nodes and interferers is unknown but is distributed according to a Poisson point process.

The rest of the thesis is organized as follows. In Section 1.1, we introduce the system model and define AoB and AoC. We then detail preliminaries in Section 1.2. In Chapter 2, we characterize the expected AoB, then characterize AoC in Section 3.1. Numerical results from simulation are presented in Section 4.1, and concluding remarks and future directions are stated in Section 5.1.

1.1 System Model

We now introduce the network model, the traffic model, as well as AoI before formally defining AoB and AoC.

Notation: Common notation can be found in Table 1.1. Whenever necessary for clarity, the expected value operator with respect to the distribution of some random element X will be denoted by $\mathbb{E}_X[\cdot]$. The spatial point process models in this work are simple point processes, meaning node positions are distinct almost everywhere. Therefore, the convention will be that a node located at position $y \in \mathbb{R}^2$ will simply be referred to as node y. The ℓ-2 norm will be denoted by $\|\cdot\|$. Random elements will generally be represented with an uppercase letter, a realization of which will be represented with a lowercase letter. For example, a realization of a point process Φ is ϕ. For some set $\mathcal{W}$, the operator $[\cdot]_k$ produces $[\mathcal{W}]_k = \{A \subseteq \mathcal{W} \,\text{s.t.}\, |A| = k\}$, the set of subsets of $\mathcal{W}$ with cardinality k.

1.1.1 Network Model

Consider a base station, denoted by $\mathcal{O}$, in the Euclidean plane situated at the origin, with a finite set of nodes randomly distributed in a disk $b_2(0, r)$ of finite radius r. The nodes are distributed within a disk as a homogeneous Poisson Point Process with intensity λ, denoted by Φ_N. Interferers are also distributed according to a homogeneous Poisson Point Process Φ_I that is distributed across $\mathbb{R}^2$ with intensity λ

Common notation	
Notation	Description
Φ	Poisson Point process in $\mathbb{R}^2$ composed of two independent processes $\Phi \triangleq \Phi_N \cup \Phi_I$
λ	Intensity of Φ_I and Φ_N
$b_2(x, r)$	Disk in $\mathbb{R}^2$ centered at x with radius r
H_{ij}	Channel fading coefficient between a transmitter i and receiver j
$\mu_{ji}^{\Phi_I}$	Probability of successful delivery of packet from j to i in the presence of interferers Φ_I
θ	SIR threshold value; $\theta > 1$
p	medium access probability common to all nodes and interferers, including the base station when broadcasting
β	Path loss exponent
$\mathcal{O}$	Base station situated at the origin; $\mathcal{O} = (0, 0)$
$X_i[t]$	Inter-packet reception duration for the packet reception process of receiver i at time t
$A_{ji}(k)$	AoI at receiver i transmitted from j
$\ell(x)$	Path loss function $\ell(x) = \|x\|^{-\beta},\ x \in \mathbb{R}^2$

Table 1.1

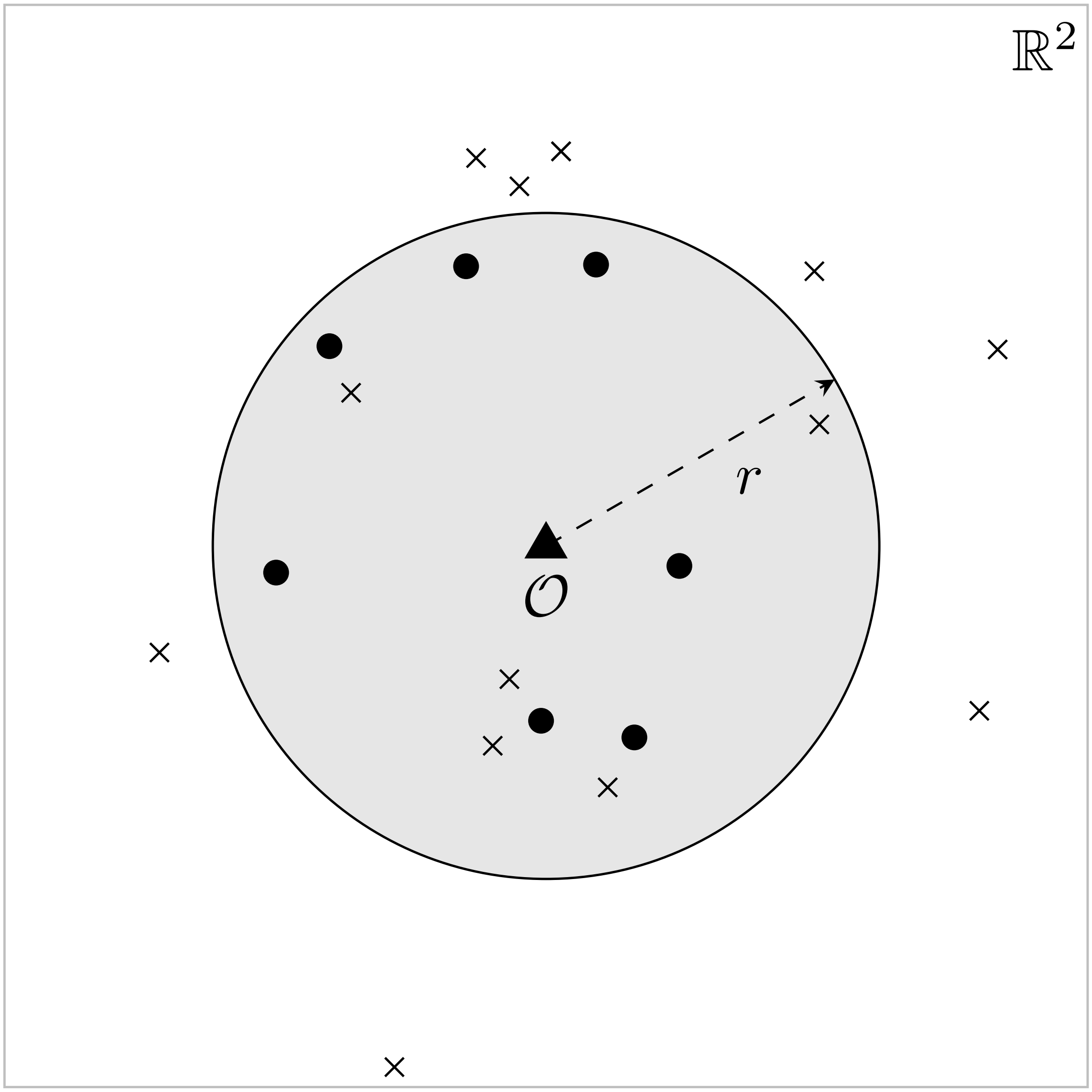

Figure 1-1: Example of a spatial realization ϕ_N of nodes (black circles), confined to a disk $b_2(0, r)$, and interferers (crosses) ϕ_I, distributed across the Euclidean plane, with the base station (black triangle) in the center

(see Figure 1-1). We denote the combined point process of nodes and interferers by

$$\Phi \triangleq \Phi_N \cup \Phi_I \,. \tag{1.1}$$

This spatial model captures a scenario in which the base station may be one of many broadcast and collection nodes in a spatially-large wireless network, and where the base station is only interested in communicating with nodes within its vicinity. Each information update consists of a single, timestamped packet. When broadcasting, the base station attempts transmission of a packet to all nodes in the disk; the packet is successfully received at a receiver if the signal-to-interference ratio (SIR) exceeds a fixed threshold $\theta > 1$. Similarly, during collection, the base station successfully receives a packet from a given transmitter when the SIR exceeds θ. All transmission attempts occur at the start of discrete time-slots, the packet duration and the slot length both normalized to 1. Therefore, time t is defined to be discrete, denoting the t^{th} slot. Medium access is granted to a transmitter – including the base station when transmitting– via an ALOHA-type random access scheme with a fixed common medium access probability (MAP) of p. That is, in any given time-slot the probability that a given transmitter attempts transmission is p, independent of all other time-slots and users in the network. In all subsequent sections we assume the packet delivery process is at steady state, having started at time $t = -\infty$.

The transmission power from every transmitter, including interferers, is fixed and normalized to 1. The wireless channel experiences Rayleigh fading and path loss attenuation. The fading loss random variable H is i.i.d. exponentially distributed with mean 1. For a transmission from a transmitter x to a receiver y, the path loss is defined to be

$$\ell(x-y) \triangleq \|x-y\|^{-\beta} \,.$$

The path loss exponent β is generally chosen to be in the interval $(2,4)$. At time-slot t the medium access indicator random variable $Z_x[t]$ is 1 if a transmitter x attempts transmission and 0 otherwise. Given the realization of node and interferer locations ϕ and including medium access probability, transmission power, fading, and path loss, we may represent the signal power observed at receiver y for a broadcast from the base station to be

$$S^{\phi_I}_{\mathcal{O}y}[t] = Z_{\mathcal{O}}[t] H_{\mathcal{O}y}[t] \ell(y).$$

Similarly, the interference observed at y is given by

$$I^{\phi_I}_{\mathcal{O}y}[t] = \sum_{x \in \phi_I} Z_x[t] H_{xy}[t] \ell(x-y)\,.$$

Therefore, the SIR is given by the ratio of $S^{\phi}_{\mathcal{O}y}[t]$ and $I^{\phi}_{\mathcal{O}y}[t]$,

$$SIR^{\phi_I}_{\mathcal{O}y}[t] = \frac{S^{\phi_I}_{\mathcal{O}y}[t]}{I^{\phi_I}_{\mathcal{O}y}[t]} = \frac{Z_{\mathcal{O}}[t] H_{\mathcal{O}y}[t] \ell(y)}{\sum_{x \in \phi_I} Z_x[t] H_{xy}[t] \ell(x-y)}\,. \tag{1.2}$$

For collection, the transmission signal from a transmitter x in ϕ_N is subject to interference from both the field of interferers as well as other transmitters in ϕ_N. Therefore, the SIR is

$$SIR^{\phi}_{x\mathcal{O}}[t] = \frac{Z_x[t] H_{x\mathcal{O}}[t] \ell(x)}{\sum_{y \in \phi \backslash x} Z_y[t] H_{y\mathcal{O}}[t] \ell(y)}\,.$$

In the following subsection, we formally define the AoI metric, which will then be used to define AoB and AoC.

1.1.2 Age of Information

AoI is denoted by $A[t]$. Let $G[t]$ be the time stamp of the most recent packet successfully received as of time t. The time evolution of AoI is then defined in Equation (1.3):

$$A[t+1] = \begin{cases} A[t]+1, & \text{if no reception} \\ \min\{t-G[t],\, A[t]\}+1, & \text{if reception} \end{cases}\,. \tag{1.3}$$

The AoI at a receiver x and at time t corresponding to information updates from some node y is denoted by $A_{yx}[t]$.

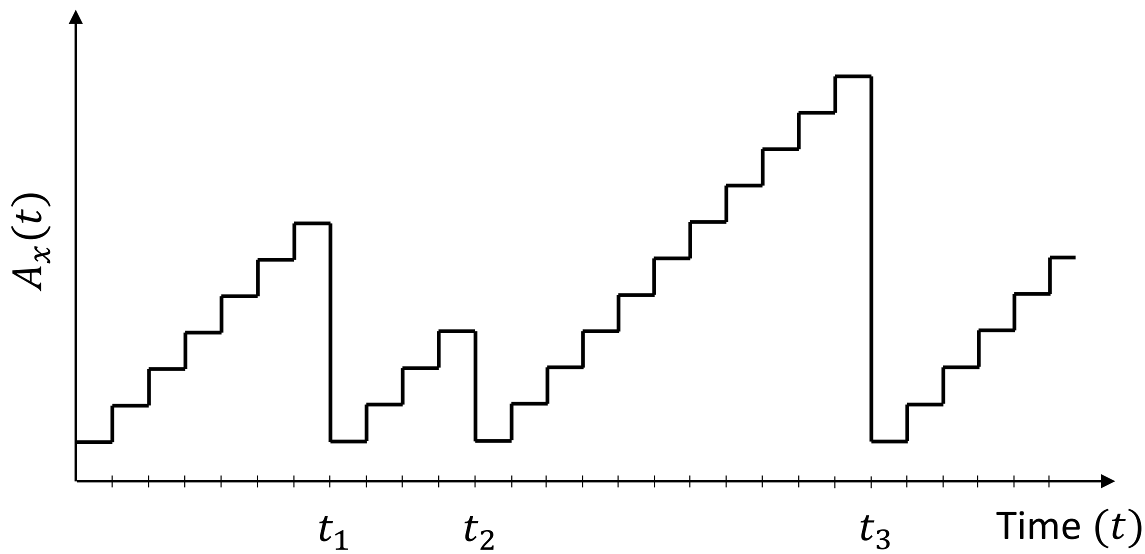

Figure 1-2: Age evolution over discrete time-slots

We assume any information source node can generate an update at-will, i.e. at each time t, an information update packet is instantaneously generated and transmitted with probability p. Therefore, the time stamp associated with a packet transmitted at time t will always be t, and Equation (1.3) becomes

$$A[t+1] = \begin{cases} A[t]+1, & \text{if no reception} \\ 1, & \text{if reception} \end{cases} . \tag{1.4}$$

An example of the age evolution over time is provided in Figure 1-2.

1.1.3 AoB and AoC

Having defined AoI and the network model, we now define Age of Broadcast and Age of Collection.

Age of Broadcast

The AoB $B_{\mathcal{O}}^{\phi_N}[t]$ with respect to some realization of the receiver locations ϕ_N and base station $\mathcal{O}$ at time-slot t is defined as

$$B_{\mathcal{O}}^{\phi}[t] \triangleq \max_{i \in \phi_N} A_{\mathcal{O}i}^{\phi_I}[t] \,. \tag{1.5}$$

Given the base station begins broadcasting at $t - B_{\mathcal{O}}^{\phi}[t]$, the time until all receivers get an update cannot be less than $B_{\mathcal{O}}^{\phi}[t]$. Moreover, at least one base station has an update that is no greater than $t - B_{\mathcal{O}}^{\phi}[t]$.

Age of Collection

The AoC $C_{\mathcal{O}}^{\phi}[t]$ with respect to the base station $\mathcal{O}$ and the realization of node and interferer locations ϕ at time t is defined as

$$C_{\mathcal{O}}^{\phi}[t] \triangleq \max_{j \in \phi_N} A_{j\mathcal{O}}^{\phi}[t] \,. \tag{1.6}$$

For a given time t, the base station will have received at least one update from all but one transmitter since $t - C_{\mathcal{O}}^{\phi}[t]$.

In both broadcast and collection settings, we adopt the convention that if $\phi_N = \emptyset$, then AoB and AoC is 0 for all time. Having defined AoB and AoC, we establish preliminary results that are used in subsequent sections to analyze AoB and AoC.

1.2 Preliminaries

When broadcasting, the probability the base station successfully delivers an update to an arbitrary receiver y given the locations of the interferer positions ϕ_I is determined by the medium access probability and the channel characteristics. Since transmission attempts from transmitters and interferers alike are i.i.d. and Bernoulli in each time slot, the probability of successful delivery to y is time-invariant and the time index t can be dropped. Given the spatial realization of interferers ϕ_I, the success probability is given by,

$$\mu_{\mathcal{O}y}^{\phi_I} = \mathbb{P}\left(SIR_{\mathcal{O}y} > \theta\right) . \tag{1.7}$$

Averaging over the channel fading and the random access, and given the spatial realization ϕ_I, the conditional reception success probability at a receiver y is given by

$$\mu_{\mathcal{O}y}^{\phi_I} = \mathbb{P}\left(SIR_{\mathcal{O}y}^{\phi_I} > \theta \,\middle|\, \Phi_I = \phi_I\right) = p \prod_{x \in \phi_I} \left(1 - \frac{p}{1 + \theta \frac{\ell(y)}{\ell(x-y)}}\right) . \tag{1.8}$$

The derivation of Equation (1.8) can be found in Appendix A.1 and is similar to the analysis in [27] Lemma 1. Through an identical line of reasoning, the conditional success probability during collection with respect to transmitter $x \in \phi_N$ is given by

$$\mu_{x\mathcal{O}}^{\phi} = p \prod_{y \in \phi \setminus x} \left(1 - \frac{p}{1 + \theta \frac{\ell(x)}{\ell(y)}}\right) . \tag{1.9}$$

where the sources of interference are both ϕ_I and $\phi_N \setminus x$; thus success probability is conditioned on ϕ instead of ϕ_I.

Note that $\mu_{\mathcal{O}y}^{\phi_I}$ and $\mu_{x\mathcal{O}}^{\phi}$ are dependent on the realization ϕ. Thus, when not given ϕ, the reception success probability is a random variable.

Next, we de-condition Equation (1.8) and Equation (1.9) on Φ_I by taking the average over all realizations of the interferer locations. The packet reception success probability from the base station $\mathcal{O}$ to a receiver y is then given by

$$\mu(\|y\|) = p \exp\left(-p\lambda\pi C\|y\|^2\right) , \tag{1.10}$$

where

$$C \triangleq \Gamma(1+\delta)\Gamma(1-\delta)\theta^{\delta} \tag{1.11}$$

and the gamma function $\Gamma(\cdot)$ is defined as $\Gamma(x) \triangleq \int_0^\infty t^{x-1}e^{-t}\, dt$, and $\delta = \frac{2}{\beta}$.

The proof is in Appendix A.2 and is similar to the analysis in [7] (see Section

3.2.3). Note that μ no longer depends explicitly on node or interferer geometry and is instead only a function of the distance between $\mathcal{O}$ and y. Thus, in the instance-independent analysis we will express the success probability purely as a function of distance between the base station and the node.

Conditioned on Φ_N, we find the success probability in the collection case deconditioning on Φ_I to be

$$\mu_{y\mathcal{O}}^{\phi_N} = \mathbb{P}\left(H_{y\mathcal{O}} \geq \frac{\theta\left(I_{y\mathcal{O}}^{\phi_I} + I_{y\mathcal{O}}^{\phi_N\setminus\{y\}}\right)}{\ell(y)}\right) \tag{1.12}$$

$$= \mathbb{E}\left[\exp\left(-\frac{\theta I_{y\mathcal{O}}^{\phi_I}}{\ell(y)}\right)\right] \cdot \mathbb{E}\left[\exp\left(-\frac{\theta I_{y\mathcal{O}}^{\phi_N\setminus\{y\}}}{\ell(y)}\right)\right] \tag{1.13}$$

$$= p\exp\left(-p\lambda\pi C\|y\|^2\right) \cdot \prod_{j\in\phi_N\setminus\{y\}}\left(1 - \frac{p}{1+\theta\frac{\ell(y)}{\ell(j)}}\right) \tag{1.14}$$

$$= \mu(\|y\|) \cdot \prod_{j\in\phi_N\setminus\{y\}}\left(1 - \frac{p}{1+\theta\frac{\ell(y)}{\ell(j)}}\right), \tag{1.15}$$

where $I_{y\mathcal{O}}^{\phi_N\setminus\{y\}}$ denotes the interference induced by the transmission of the nodes in $\phi_N \setminus \{y\}$. Having established packet reception probabilities results in both instance-dependent and instance-independent cases, we leverage this insight in deriving AoB and AoC in the following chapters, starting with broadcast.

Chapter 2

Broadcast

In this section we characterize the Expected AoB (EAoB), the expectation taken with respect to the ALOHA network traffic. In the instance-dependent case, we analyze EAoB given perfect knowledge of node and interferer locations. In the instance-independent case, node and interferer locations are unknown, so we find EAoB in expectation over the node and interferer point processes.

2.1 Instance-dependent (BD)

We assume the locations of all interferers and receivers are known. Interferers are located according to a realization of Φ_I, denoted ϕ_I. Receivers are also distributed according to a realization of Φ_N and is denoted ϕ_N. Recall that the network is at steady state, having started at $t = -\infty$. Therefore, the AoB process is stationary and EAoB, defined as $\mathbb{E}\left[B^{\phi}_{\mathcal{O}}[t]\right]$, is the same for all finite t and the dependence on time can be dropped to give $\mathbb{E}\left[B^{\phi}_{\mathcal{O}}\right]$.

To determine the EAoB, it is helpful to connect average broadcast age to the average broadcast delay, a related-yet-distinct metric [26]. For each receiver $y \in \phi_N$, define $X^{\phi_I}_{\mathcal{O}y}[t]$ to be the time elapsed until successful reception of the next packet at receiver i since time t. Broadcast delay $D^{\phi}_{\mathcal{O}}[t]$ is then the time elapsed from time t

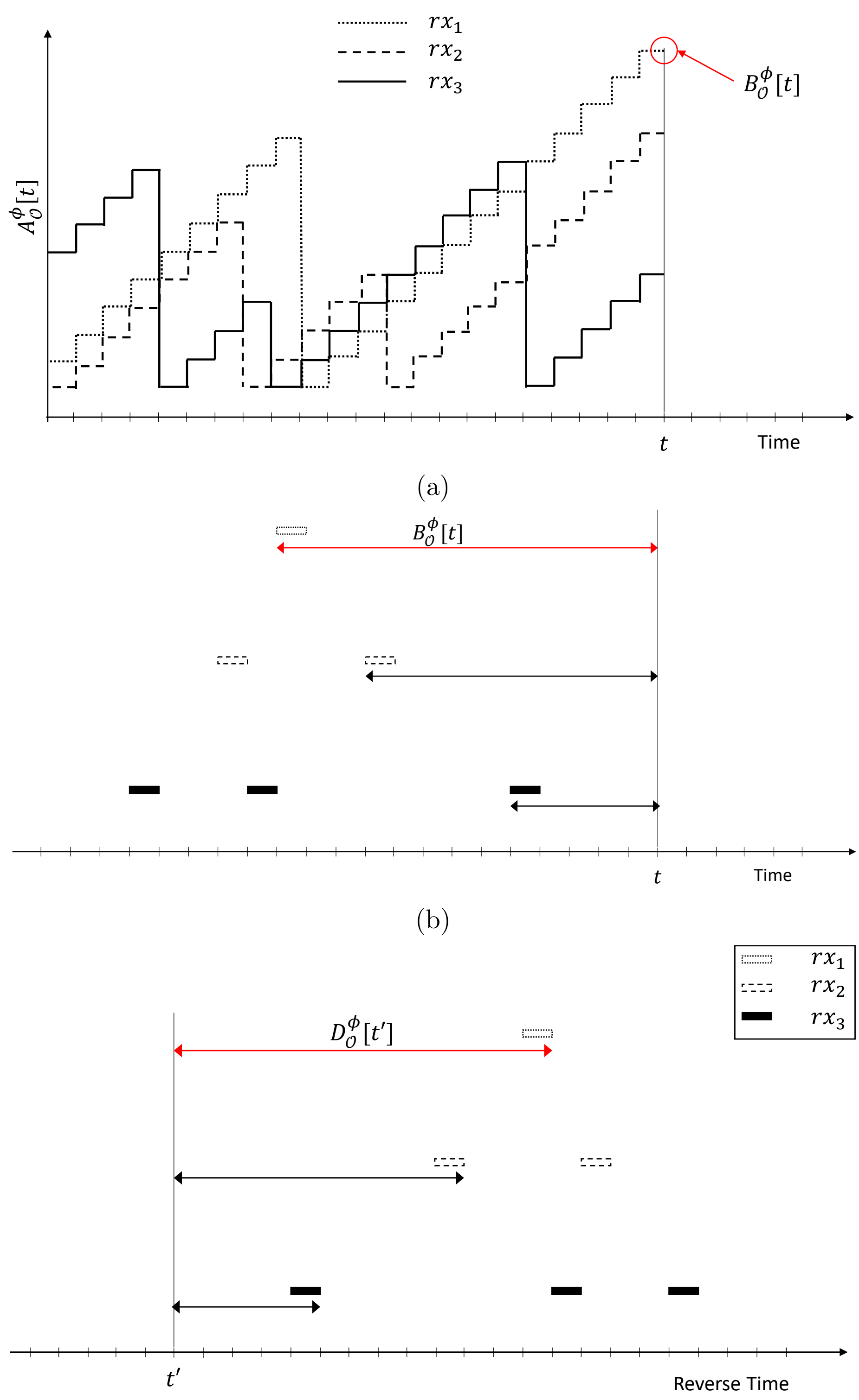

Figure 2-1: An illustrative example supporting Claim 1, where (a) is the age evolution of three receivers that form the set ϕ_N, (b) describes the packet arrival process up to time t, and (c) is the reverse of the process in (b).

until the time at which all receivers in ϕ_N have received the next packet. That is,

$$D_{\mathcal{O}}^{\phi}[t] \triangleq \max_{y \in \phi_N} X_{\mathcal{O}y}^{\phi_I}[t] \tag{2.1}$$

Since the packet reception process is stationary – by virtue of the ALOHA random access and i.i.d. fading – the average broadcast delay $\mathbb{E}\left[D_{\mathcal{O}}^{\phi}[t]\right]$ is the same for all finite t and the dependence on t can be dropped to give $\mathbb{E}\left[D_{\mathcal{O}}^{\phi}\right]$. We now reason that the average broadcast delay is equivalent to EAoB. This is evident by observation of the reverse packet reception process (see Figure 2-1). Consider the instantaneous AoB at time t, which is the maximum AoI in ϕ_N, and is exactly the time that elapsed between the current time point and the first time point in the past since which all receivers in the receiver set have successfully received at least one information update (see Figure 2-1b). Now consider the broadcast delay, given by Equation (2.1) and shown in Figure 2-1c. Since the packet reception process's evolution in the forward direction is identical in distribution to that of its reverse process looking back in time, we may conclude the average broadcast delay must be equal to the EAoB. The formal claim is outlined below.

Claim 1.

$$\mathbb{E}\left[B_{\mathcal{O}}^{\phi}\right] = \mathbb{E}\left[D_{\mathcal{O}}^{\phi}\right]. \tag{2.2}$$

Proof. For any time t the packet reception vector $\vec{\mathbb{1}}_{\mathcal{O}}^{\phi_I}[t] = \left\{\mathbb{1}_{\mathcal{O}i}^{\phi_I}\right\}_{i=1}^{|\phi_N|}$ can take values from a finite set of states $\mathcal{N}$ of cardinality $2^{|\phi_N|}$. Therefore, the packet reception process over time is a discrete-time Markov Chain. Since the packet reception process is stationary and time-invariant by virtue of the ALOHA transmissions and i.i.d fading, this Markov Chain is irreducible, aperiodic, and positive recurrent. We denote the probability that the process is in state $u \in \mathcal{N}$ to be p_u. WLOG, we denote the probability that the process transitions from state $j \in \mathcal{N}$ at time t to $k \in \mathcal{N}$ at time $t+1$ to be p_{jk} for any time $t \geq 0$. By Kolmogorov's criterion for Markov chain time-reversibility, the process is time-reversible if and only if the condition

$$p_{j_1 j_2} p_{j_2 j_3} \dots p_{j_{n-1} j_n} p_{j_n j_1} = p_{j_1 j_n} p_{j_n j_{n-1}} \dots p_{j_3 j_2} p_{j_2 j_1} \tag{2.3}$$

is satisfied for all finite sequences of states $j_1, j_2, \dots, j_n \in \mathcal{N}$.

Since the event the process is in state j at time t is independent of the event it is in state k at time $t+1$ for all time $t \geq 0$, the condition in Equation (2.3) holds, establishing that the packet reception process is time-reversible. As shown in Figure 2-1, for any time t, the AoB $B_{\mathcal{O}}^{\phi}[t]$ is shown to be equivalent to $D_{\mathcal{O}}^{\phi}(t')$, the broadcast delay of the time-reversed process starting from the same time slot. Since time-reversibility implies that expectations with respect to forward time are the same as that over reverse time, coupled with the fact that $\mathbb{E}\left[D_{\mathcal{O}}^{\phi}[t]\right] = \mathbb{E}\left[D_{\mathcal{O}}^{\phi}\right]$, we may conclude that

$$\mathbb{E}\left[B_{\mathcal{O}}^{\phi}\right] = \mathbb{E}\left[D_{\mathcal{O}}^{\phi}\right] .$$

□

This equivalence between the average broadcast delay and average AoB facilitates an analytical derivation of expected AoB. We begin by first defining the joint distribution of packet reception at each time t. Assuming knowledge of the locations of all nodes in ϕ_I and ϕ_N, at time slot t, the joint distribution of packet reception for all the receivers, i.e. the joint distribution of $\left\{\mathbb{1}_{\mathcal{O}i}^{\phi_I}[t]\right\}_{i\in\phi_N}$, the packet reception indicator random variables, can be obtained. Using this joint distribution, it is possible to determine the average broadcast age explicitly.

We partition the receivers into $\Xi[t] = \{i \in \phi_N \mid \mathbb{1}_{\mathcal{O}i}[t] = 1\}$ and $\Psi[t] = \{j \in \phi_N \mid \mathbb{1}_{\mathcal{O}j}[t] = 0\}$, the set of receivers that successfully received a packet at time t and the set that did not receive a packet, respectively. Defining the probability of the set of receivers $\mathcal{R}$ all successfully receiving a packet at time slot t as the following,

$$\mu_{\mathcal{O}\mathcal{R}}^{\phi_I}[t] = \mathbb{P}\left(\cap_{i\in\mathcal{R}}\left\{SIR_{\mathcal{O}i}^{\phi_I}[t] > \theta\right\}\right) = \mathbb{P}\left(\cap_{i\in\mathcal{R}}\left\{\mathbb{1}_{\mathcal{O}i}^{\phi_I}[t] = 1\right\}\right) , \tag{2.4}$$

and conversely $w_{\mathcal{O}\mathcal{R}}^{\phi_I}[t]$ as the probability of the set of receivers $\mathcal{R}$ NOT receiving a packet at time slot t as

$$w^{\phi_I}_{\mathcal{OR}}[t] = \mathbb{P}\left(\cap_{i\in\mathcal{R}}\left\{SIR^{\phi_I}_{\mathcal{O}i}[t] \leq \theta\right\}\right) = \mathbb{P}\left(\cap_{i\in\mathcal{R}}\left\{\mathbb{1}^{\phi_I}_{\mathcal{O}i}[t] = 0\right\}\right). \tag{2.5}$$

We next determine the probability rule of $\left\{\mathbb{1}^{\phi_I}_{\mathcal{O}i}[t]\right\}_{i\in\phi_N}$, which is equivalent to finding the probability rule for $\Xi[t]$ without loss of generality. By the inclusion-exclusion property, the probability rule for $\Xi[t]$ can be established.

The inclusion-exclusion principle formula represents the probability of the union of a set of events $A = \{A_1, A_2, \ldots, A_n\}$ by the probabilities of intersections of its subsets:

$$\mathbb{P}\left(\cup_{i=1}^{n} A_i\right) = \sum_{j=1}^{n}(-1)^{n-1}\sum_{L\in[A]_k}\mathbb{P}\left(\cap_{i\in L}A_i\right), \tag{2.6}$$

where $[A]_k$ denotes the set of subsets of A with cardinality k. Therefore, the probability rule for $\Xi[t]$ is

$$\begin{aligned}
p\left(\Xi[t]\right) &= \mathbb{P}\left(\left\{\cap_{i\in\Xi}\mathbb{1}^{\phi_I}_{\mathcal{O}i}[t] = 1\right\}\bigcap\left\{\cap_{j\in\Psi}\mathbb{1}^{\phi_I}_{\mathcal{O}j}[t] = 0\right\}\right) && (2.7)\\
&= \mathbb{P}\left(\cap_{i\in\Xi}\left\{\mathbb{1}^{\phi_I}_{\mathcal{O}i}[t] = 1\right\}\right) && (2.8)\\
&\quad - \mathbb{P}\left(\cup_{j\in\Psi}\left\{\mathbb{1}^{\phi_I}_{\mathcal{O}j}[t] = 1\right\}\cap\left\{\cap_{i\in\Xi}\mathbb{1}^{\phi_I}_{\mathcal{O}i}[t] = 1\right\}\right) && (2.9)\\
&= \mu^{\phi_I}_{\mathcal{O}\Xi}[t] + \sum_{k=1}^{|\Psi|}(-1)^k\sum_{L\in[\Psi]_k}\mu^{\phi_I}_{\mathcal{O}\{\Xi\cup L\}}[t]\,. && (2.10)
\end{aligned}$$

Without loss of generality, we may define the probability rule starting at time $t = 0$ for broadcast delay $D^{\phi}_{\mathcal{O}}[0]$. For the broadcast delay to be τ, at least one receiver must receive a packet at time $t = \tau - 1$. Moreover, of all the receivers that received a packet at time $\tau - 1$, at least one must have received no packets for times $t \in \{0, \ldots, \tau-2\}$, i.e. given non-empty $\Xi[\tau-1]$, there exists some non-empty subset $J \subseteq \Xi[\tau-1]$ such that $J \subseteq \cap_{i=0}^{\tau-2}\Psi[i]$. Therefore, $p^{\phi}_{D}(\tau)$, the probability that the broadcast delay is equal to τ, is given by

$$p_D^{\phi}(\tau) = \sum_{\Xi \in 2^{\Phi_N} \setminus \{\emptyset\}} p(\Xi[\tau-1]) \cdot \underbrace{\mathbb{P}(\cup_{J \in 2^{\Xi[\tau-1]} \setminus \{\emptyset\}} \{J \subseteq \cap_{i=0}^{\tau-2} \Psi[i]\})}_{(*)} \tag{2.11}$$

$$= \sum_{\Xi \in 2^{\Phi_N} \setminus \{\emptyset\}} p(\Xi[\tau-1]) \cdot \underbrace{\left(\sum_{n=1}^{|\Xi|} (-1)^{n+1} \sum_{J \in [\Xi]_n} \prod_{u=0}^{\tau-2} w_{\mathcal{O}J}^{\phi_I}[u] \right)}_{(\dagger)}, \tag{2.12}$$

where $2^{\{\cdot\}}$ denotes the power set of some set $\{\cdot\}$, and $(\dagger)$ is the inclusion-exclusion formula applied to $(*)$. Finally, we use Claim 1 to find EAoB given by

$$\mathbb{E}\left[B_{\mathcal{O}}^{\phi}[t]\right] = \mathbb{E}\left[D_{\mathcal{O}}^{\phi}\right] = \mathbb{E}\left[D_{\mathcal{O}}^{\phi}[0]\right] = \sum_{k=1}^{\infty} k \cdot p_D^{\phi}(k). \tag{2.13}$$

While this representation of average broadcast age is complete, a more intuitive characterization can be developed in the form of bounds on the average broadcast delay. We begin with an empirical observation. Based on simulation results, we observe that the average broadcast age is bounded above by the average broadcast age of an alternate packet reception process in which all packet reception indicators were independent random variables, albeit preserving the same distribution (see Figure 2-2). Recall that $X_{\mathcal{O}i}^{\phi_I}[t]$ denotes the time elapsed since time t until the next packet reception. The conjecture based on this observation is formalized as follows.

Conjecture. The EAoB is bounded above by

$$\mathbb{E}\left[B_{\mathcal{O}}^{\phi}\right] \leq \mathbb{E}\left[\tilde{D}_{\mathcal{O}}^{\phi}\right] = \mathbb{E}\left[\max_{i \in \phi_N} \tilde{X}_{\mathcal{O}i}^{\phi_I}[0]\right],$$

where $\tilde{X}_{\mathcal{O}i}^{\phi_I}[0] \overset{d}{=} X_{\mathcal{O}i}^{\phi_I}[0]$ and $\left\{\tilde{X}_{\mathcal{O}i}^{\phi_I}[0]\right\}_{i \in \phi_N}$ are independent.

The conjecture would hold if $\left\{X_{\mathcal{O}i}^{\phi_I}\right\}_{i \in \phi_N}$ are associated random variables [3]. Formally, two sets of random variables $\mathbf{S}$ and $\mathbf{T}$ are associated random variables if for all non-decreasing pairs of functions f, g,

$$\mathrm{Cov}\left(f(\mathbf{S},\mathbf{T}),\, g(\mathbf{S},\mathbf{T})\right) \geq 0\,.$$

While in general it is difficult to determine association, for the case of $|\phi_N| = 2$ it is readily established.

Claim 2. Given $|\phi_N| = 2$ with receivers x and y,

$$\max\left(X_{\mathcal{O}x}^{\phi_I},\, X_{\mathcal{O}y}^{\phi_I}\right) \leq \max\left(\tilde{X}_{\mathcal{O}x}^{\phi_I},\, \tilde{X}_{\mathcal{O}y}^{\phi_I}\right) \tag{2.14}$$

Proof. We begin proof of the above claim by determining association of the packet reception indicator random variables of each receiver. Consider $\mathbb{1}_{\mathcal{O}x}^{\phi_I}[k]$ and $\mathbb{1}_{\mathcal{O}y}^{\phi_I}[j]$:

1. *Case 1:* $k \neq j$ Since packet reception at different time slots k and j are independent, $\mathbb{1}_{\mathcal{O}x}^{\phi_I}[k]$ and $\mathbb{1}_{\mathcal{O}y}^{\phi_I}[j]$ are independent and are thus associated random variables [3].

2. *Case 2:* $k = j$ As shown in [3], if the covariance of two binary random variables is non-negative, then they are associated. In the following the covariance is derived for $k = j$. The time index as well as the dependence on ϕ_I and $\mathcal{O}$ is dropped for convenience :

$$\begin{aligned}
&\mathrm{Cov}(\mathbb{1}_x, \mathbb{1}_y) = \mathbb{E}\left[\mathbb{1}_x \mathbb{1}_y\right] - \mathbb{E}\left[\mathbb{1}_x\right]\mathbb{E}\left[\mathbb{1}_y\right] \\
&= \underbrace{\prod_{i\in\phi_I}\left[\frac{p}{\left(1+\theta\frac{\|x\|^\beta}{\|i-x\|^\beta}\right)\left(1+\theta\frac{\|y\|^\beta}{\|i-y\|^\beta}\right)} + 1 - p\right]}_{A} - \\
&\underbrace{\prod_{i\in\phi_I}\left(\frac{p}{1+\theta\frac{\|x\|^\beta}{\|i-x\|^\beta}} + 1 - p\right)\left(\frac{p}{1+\theta\frac{\|y\|^\beta}{\|i-y\|^\beta}} + 1 - p\right)}_{B}
\end{aligned}$$

We now compare a factor of A and a factor of B with respect to a single common interferer i to determine inductively if $A > B$, resulting in positive covariance.

We define c to be

$$c \triangleq 1 + \theta \frac{\|x\|^\beta}{\|i\|^\beta}$$

and d to be

$$d \triangleq 1 + \theta \frac{\|x\|^\beta}{\|i\|^\beta}$$

for some $i \in \phi_I$. We may then represent the factor associated with interferer i in A as

$$\frac{p}{cd} + 1 - p\,,$$

and in B as

$$\frac{p^2}{cd} + \frac{p((1-p)}{c} + \frac{p(1-p)}{d} - (1-p)^2\,.$$

We may determine if the factor in A associated with i is greater than the factor in B associated with i by subtracting the two factors and see if the result is non-negative. If so, we may conclude that the factor in A associated with i is always at least as large as the factor in B associated with i, and therefore A is larger than B. Note that each factor of A is non-negative and lies in the interval $[0, 1]$. This is also the case for each factor of B.

Subtracting the latter from the former,

$$\frac{p}{cd} + 1 - p - \frac{p^2}{cd} - \frac{p((1-p)}{c} - \frac{p(1-p)}{d} - (1-p)^2 \tag{2.15}$$

$$= (1-c)(1-d)p - (1-c)(1-d)p^2 \geq 0\,, \tag{2.16}$$

since the MAP parameter p is always less than or equal to 1. This implies that for any $i \in \phi_I$, its contribution to A is at least as large as the corresponding factor in B. Therefore,

$$Cov(\mathbb{1}_{\mathcal{O}x}^{\phi_I}, \mathbb{1}_{\mathcal{O}y}^{\phi_I}) \geq 0\,,$$

leading us to conclude that $\mathbb{1}_{\mathcal{O}x}^{\phi_I}[k]$ and $\mathbb{1}_{\mathcal{O}x}^{\phi_I}[j]$ are associated random variables for all $k, j \geq 0$.

Having established association for the packet reception indicators associated with x and y, we may determine association of $X^{\phi_I}_{\mathcal{O}x}$ and $X^{\phi_I}_{\mathcal{O}y}$.

The inverse of the packet reception indicators $\mathbb{O}^{\phi_I}_{\mathcal{O}x} = 1 - \mathbb{1}^{\phi_I}_{\mathcal{O}x}$ and $\mathbb{O}^{\phi_I}_{\mathcal{O}y} = 1 - \mathbb{1}^{\phi_I}_{\mathcal{O}y}$ are a set of associated random variables as well [3].

We may then represent $X^{\phi_I}_{\mathcal{O}x}$ as a function of the inverse packet reception indicator random variables of receiver x as

$$X^{\phi_I}_{\mathcal{O}x}[t] = \min_k \left\{ 1 + \sum_{i=t}^{t+k-1} (\mathbb{O}^{\phi_I}_{\mathcal{O}x}[i]) \text{ s.t. } \mathbb{O}^{\phi_I}_{\mathcal{O}x}[i] = 1 \; \forall \, i \in \{t, \ldots, t+k-1\} \right\}, \quad (2.17)$$

and $X^{\phi_I}_{\mathcal{O}y}[t]$ can be represented similarly. The right-hand expression in Equation (2.17) is a non-decreasing function of the indicator random variables $\mathbb{O}^{\phi_I}_{\mathcal{O}x}[i]$. Since $X^{\phi_I}_{\mathcal{O}x}[t]$ and $X^{\phi_I}_{\mathcal{O}y}[t]$ can be represented as non-decreasing functions of a set of associated random variables, they must be associated (See [3] (P_4)).

A useful property of associated random variables, which we state without proof (see [22] Appendix A), is that the maximum over a set of associated random variables $X_1, \ldots, X_n$ is bounded above by the following,

$$\max \{X_1, \ldots, X_n\} \leq \max \left\{ \tilde{X}_1, \ldots, \tilde{X}_n \right\},$$

where $X_i \stackrel{d}{=} \tilde{X}_i$ and $\left\{ \tilde{X}_1, \ldots, \tilde{X}_n \right\}$ are independent random variables. Since we have already proven $X^{\phi_I}_{\mathcal{O}x}[t]$ and $X^{\phi_I}_{\mathcal{O}y}[t]$ to be associated, we may invoke this property and conclude the proof.

□

Figure 2-2 Illustrates the conjecture stated, plotting the simulated broadcast delay and the broadcast delay expected if all packet reception processes were independent,

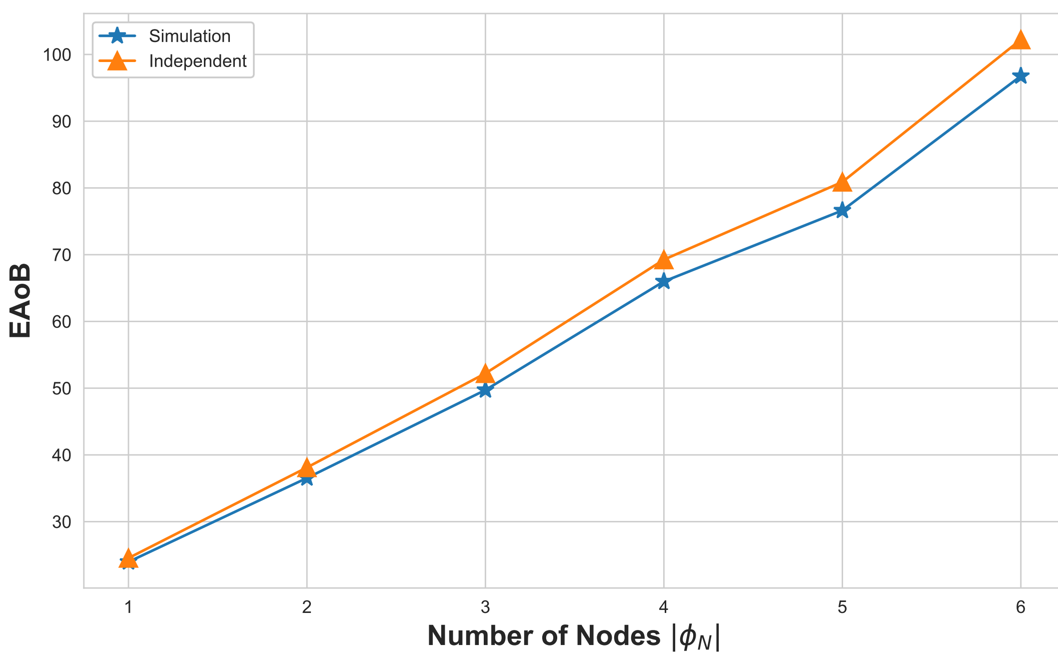

Figure 2-2: Simulation of Broadcast age compared against the maximum of independent random variables. The actual simulation's EAoB is consistently less than the independent random variable counterpart.

maintaining the same packet reception distributions.

Having established an explicit formulation of EAoB as well as a conjectured upper bound, we may turn to the more general, instance-independent regime. Upper bounds are pursued in the instance-independent scenario, as outlined in the following section.

2.2 Instance-independent (BI)

In Section 2.1 we considered a particular instantiation of the nodes. Here, we take the expectation with respect to node and interferer positions. An upper bound can be found with a differential equation approach. Consider the squared ordered distances of the i nearest receivers to $\mathcal{O}$ denoted $R_1^2 \leq R_2^2 \leq \ldots \leq R_i^2$. In two-dimensional Poisson Point Processes such as Φ_N, the squared ordered distances have the same distribution as that of the arrival times in a one-dimensional Poisson Process $\Phi'_N \subset \mathbb{R}^+$ of intensity $\lambda' = \lambda\pi$ [21]

Focusing on this one-dimensional point process, consider a small interval in $\mathbb{R}^+$ given by $(x, x+\Delta]$ for very small Δ. A receiver $y \in \Phi_N$ exists in the point process Φ'_N in the interval $(x, x+\Delta] \subset \mathbb{R}^+$ with probability $\lambda'\Delta$. We define the EAoB over the set of receivers in Φ_N that map to $(0, u] \in \mathbb{R}^+$ to be $\overline{B}(u)$. If a receiver does not exist in $(x, x+\Delta] \subset \mathbb{R}^+$, then the EAoB $\overline{B}(x+\Delta)$ would be the same as $\overline{B}(x)$. If a receiver y does exist in the interval, either y receives a packet after all the other receivers in $(0, x]$ with probability ς or it does not with probability $1-\varsigma$. By setting the probability of y getting a packet after the rest of the receivers to $\varsigma = 1$, we upper bound the time to broadcast to all receivers that are in $(0, x+\Delta]$, as shown in the following:

$$\overline{B}(x+\Delta) = (1-\lambda'\Delta)\overline{B}(x) + (\lambda'\Delta)\left(\overline{B}(x) + \varsigma\frac{1}{\mu\left(\sqrt{x+\Delta}\right)}\right) \tag{2.18}$$

$$= \overline{B}(x) + (\lambda'\Delta)\varsigma\frac{1}{\mu\left(\sqrt{x+\Delta}\right)} \tag{2.19}$$

$$\leq \overline{B}(x) + (\lambda'\Delta)\frac{1}{\mu\left(\sqrt{x+\Delta}\right)}\ , \tag{2.20}$$

where $\mu(\cdot)$ is given by Equation (1.10). As mentioned before, we upper bound in Equation (2.20) by setting ς to 1. The average packet reception delay $\frac{1}{\mu\left(\sqrt{x+\Delta}\right)}$ is readily found by taking the reciprocal of $\mu(\sqrt{x+\Delta})$ as given by Equation (1.10) since the packet reception process is i.i.d Bernoulli. By bringing $\overline{B}(x)$ over to the left-hand side of the equation, dividing both sides by Δ and taking the limit as $\Delta \to 0$, we arrive at the following:

$$\lim_{\Delta \to 0} \frac{\overline{B}(x+\Delta) - \overline{B}(x)}{\Delta} \leq \lim_{\Delta \to 0} \frac{\lambda'}{\mu\left(\sqrt{x+\Delta}\right)} \tag{2.21}$$

$$\frac{d\overline{B}}{dx} \leq \frac{\lambda'}{\mu\left(\sqrt{x}\right)} = \frac{\lambda\pi}{p} \exp\left(p\lambda\pi C x\right) \tag{2.22}$$

$$\frac{d\overline{B}}{dr} = \frac{d\overline{B}}{dx}\frac{dx}{dr} \leq \frac{\lambda'}{\mu\left(r\right)} \cdot 2r = \frac{2\lambda\pi r}{p} \exp\left(p\lambda\pi C r^2\right) \tag{2.23}$$

Solving the differential equation in Equation (2.23) with the initial condition $B(0) = 0$, we obtain

$$B(r) \leq \frac{1}{p^2 C}\left(e^{p\lambda\pi C r^2} - 1\right) \tag{2.24}$$

In addition to the differential approach, a worst-case upper bound on EAoB can also be found. Analyzing broadcast with a common source of random interference is analytically complicated and motivates an approximating assumption that will be made through the remainder of this section. In particular, we assume that the packet reception process at each receiver is independent of all other receivers. This assumption serves to de-couple dependencies between nodes that make it difficult to calculate AoB but still preserves the dependence of AoB on node positions, and is similar to the mean-field approximation in [27].

We now state the worst-case upper bound, outlined in Claim 3.

Claim 3. The average AoB for a set of receivers Φ_N situated in $b_2(\mathcal{O}, r)$ is bounded above by

$$\mathbb{E}_\Phi\left[B_\mathcal{O}^\Phi\right] \leq \frac{\left(1 - e^{-\lambda\pi r^2}\right)\left(\Gamma(0, \lambda\pi r^2) + \log(\lambda\pi r^2) + \gamma\right)}{-\log(1 - \exp\left(-p\lambda\pi C r^2\right))} + 1$$

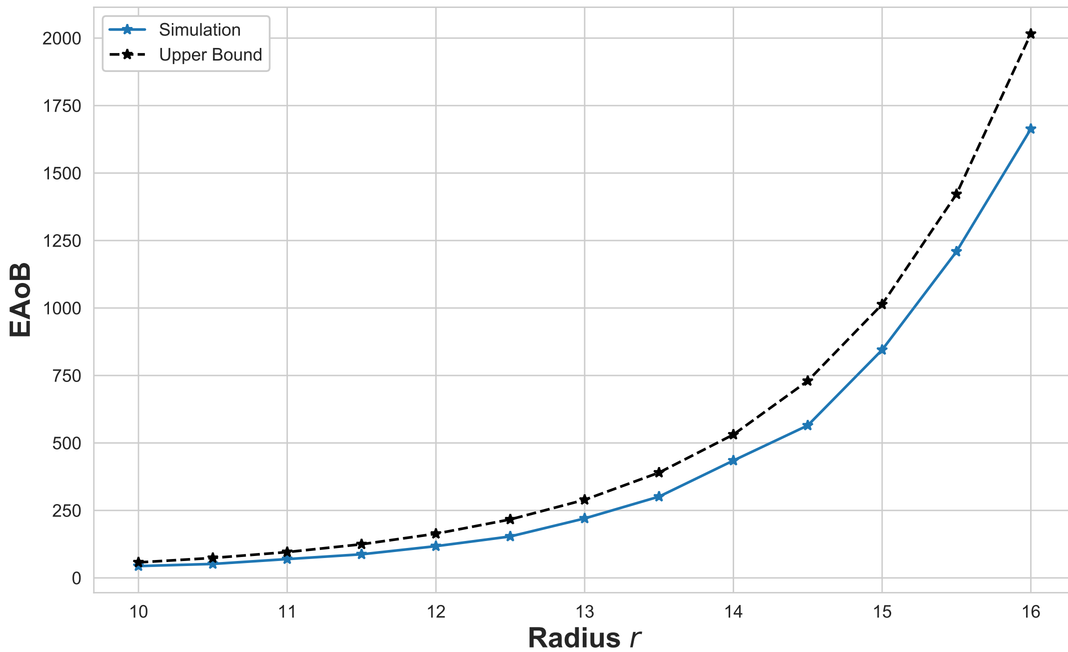

Figure 2-3: Instance-Independent AoB scaling

where C is as defined in Equation (1.11), $\Gamma(s,x)$ is the upper incomplete gamma function defined as $\Gamma(s,x) \triangleq \int_x^\infty t^{s-1}e^{-t}\,dt$, and γ is the Euler-Mascheroni constant.

To begin the proof, we first need the following lemma:

Lemma 2.2.1. $\mathbb{E}_N[H_N] = \Gamma(0,\zeta) + \log(\zeta) + \gamma$ for $N \stackrel{d}{=} \text{Pois}(\zeta)$

Proof. We first establish a convention that $H_0 = 0$, since $\sum_{k=1}^{0} \frac{1}{k}$ is a vacuous summation. H_N is generally represented to be

$$H_N = \sum_{k=1}^{N} \frac{1}{k}\,.$$

An alternate representation is

$$H_N = -\sum_{k=1}^{\infty} \binom{N}{k} \frac{(-1)^k}{k}\,.$$

Using this representation,

$$\mathbb{E}_N\left[H_N\right] = -\sum_{n=0}^{\infty}\sum_{k=1}^{n}\frac{e^{-\zeta}\zeta^n}{n!}(-1)^k\binom{n}{k}\frac{1}{k} = -1\sum_{k=1}^{\infty}\frac{(-\zeta^k)}{k!\cdot k}\sum_{n=1}^{\infty}\frac{e^{-\zeta}\zeta^{n-k}}{(n-k)!}$$
$$\overset{(*)}{=}\Gamma(0,\zeta)+\log(\zeta)+\gamma\,,$$

where $(*)$ comes from the fact that $\sum_{n=1}^{\infty}\frac{e^{-\zeta}\zeta^{n-k}}{(n-k)!} = 1$ and $-1\sum_{k=1}^{\infty}\frac{(-\zeta^k)}{k!\cdot k}$ is equivalent to the entire exponential integral function defined as $\mathrm{Ein}(x) = \int_0^x \frac{\left(1-e^{-t}\right)}{t}\,dt = \gamma + \ln(x)+\Gamma(0,x)$ for $x>0$. □

Using Lemma 2.2.1 and conditioning on the the number of receivers in the disk and the positions, we proceed to prove Claim 3.

$$\mathbb{E}\left[B_{\mathcal{O}}^{\Phi}\,|\,|\Phi_N| = n\right] = \sum_{k=1}^{n}(-1)^{k+1}\sum_{A\in[A]_k}\left(1-\prod_{j=1}^{k}\left(1-\mu(\|j\|)\right)\right)^{-1}$$

This equation stems from the broadcast delay equivalence outlined in Section 2.1. The expression for average broadcast delay can be found in [26], where the maximum of a set of random variables can be expressed using the maximum-minimums identity (See Equation (3.3)). Now, replacing $\mu(\|j\|)$ with the actual success probability, and taking the distance of all receivers to be the farthest possible, i.e. the perimeter of the disk of radius r, we get the following inequality

$$\leq \sum_{k=1}^{n}(-1)^{k+1}\sum_{A\in[\Phi_N]_k}\left(1-\prod_{j=1}^{k}\left(1-\mu(r)\right)\right)^{-1} = \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\left(1-\prod_{j=1}^{k}\left(1-\mu(r)\right)\right)^{-1}$$
$$\overset{(a)}{\leq} H_n\frac{1}{-\log(1-\mu(r))}+1 \implies \mathbb{E}_{|\Phi_N|}\left[\mathbb{E}\left[B_{\mathcal{O}}^{\Phi}\,|\,|\Phi_N|=n\right]\right]$$
$$\overset{(a)}{\leq} \mathbb{E}_{|\Phi_N|}\left[H_N\frac{1}{-\log(1-\mu(r))}+1\right]$$
$$= \frac{\left(1-e^{-\lambda\pi r^2}\right)\left(\Gamma(0,\lambda\pi r^2)+\log(\lambda\pi r^2)+\gamma\right)}{-\log(1-\mu)}+1$$

Where (a), once gain, is an upper bound for broadcast delay given geometric

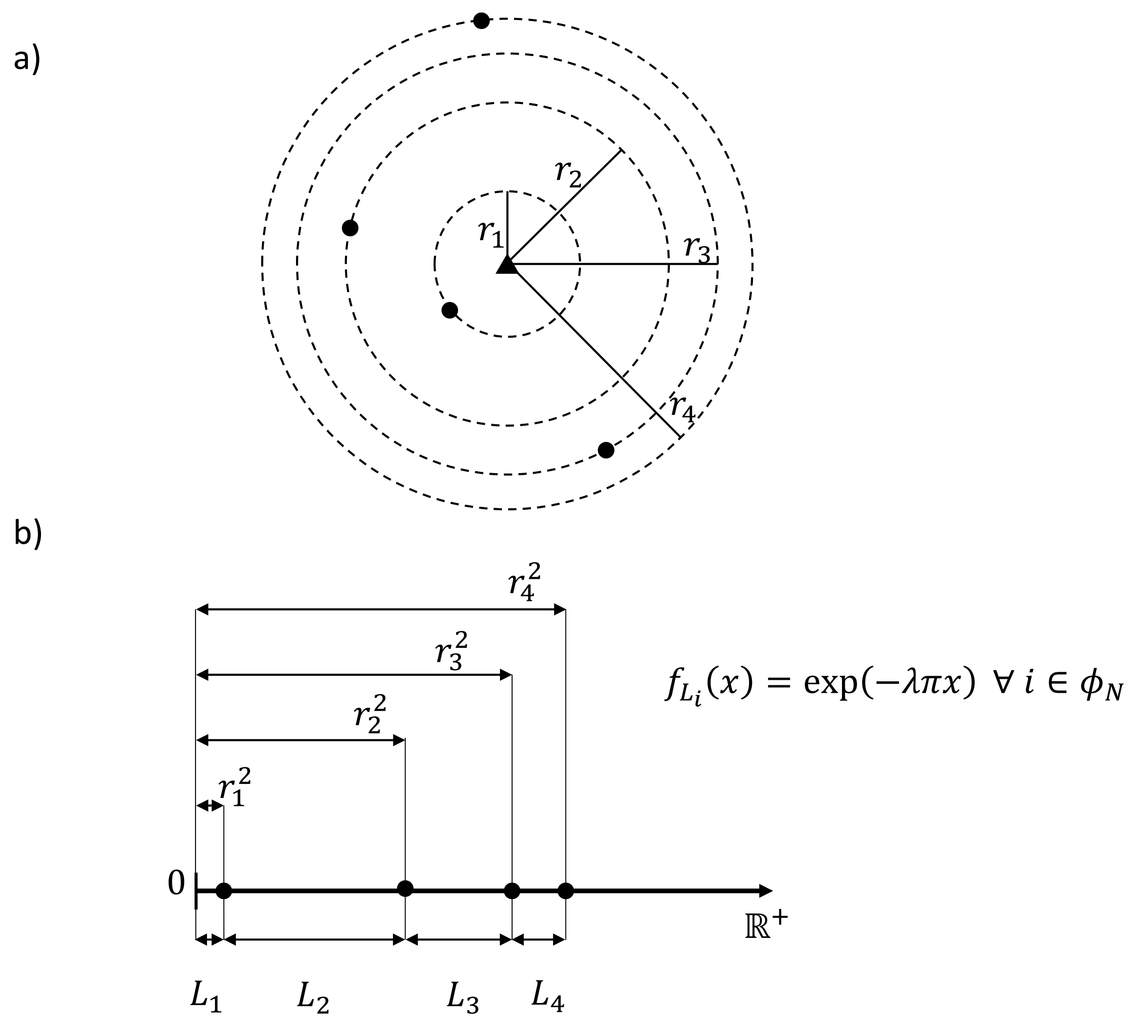

Figure 2-4: The squared ordered distances of receivers from the base station (shown in (a)) form a 1D Poisson Process. In (b), the "inter-distances" $L_1, \ldots, L_4$ of the squared distances is distributed according to an exponentially-distributed random variable with parameter $\lambda' = \lambda\pi$.

inter-packet reception times at each receiver [26]. The final equality is a result of finding the expectation $\mathbb{E}_{|\Phi_N|}[H_n]$, the derivation of which can be found in the Appendix, Lemma 2.2.1.

An informal sketch of the proof is as follows. In comparison to any instantiation of the receiver set ϕ_N, the worst case placement of the receivers is such that all receivers are the maximum distance away from the origin, i.e. $\|i\| = r\ \forall\ i \in \phi_N$. The mean broadcast age is readily found for this worst-case scenario, thereby producing an upper bound on the average broadcast age.

Claim 3 is intuitive and provides a worst-case bound on the EAoB.

Figure 2-3 compares the upper bound through the differential analysis above against simulation.

In the next section we characterize the collection problem and find bounds on the performance.

Chapter 3

Collection

3.1 Collection

In the collection problem, the base station acts as a receiver situated at the origin, with a set of transmitters ϕ_N positioned within $b_2(0, r)$ sending updates to the base station. We characterize the expected age of collection (EAoC), defined as $\mathbb{E}\left[C_{\mathcal{O}}^{\phi}[t]\right]$. As in the broadcast section, the collection age is characterized in both instance-dependent and instance-independent regimes.

3.1.1 Instance-dependent (CD)

We begin the instance-dependent analysis by connecting AoC with a metric we denote as collection delay. For each transmitter $i \in \phi_N$, define $Y_{i\mathcal{O}}^{\phi}[t]$ to be the time elapsed between the current time t and the successful reception of the next packet transmitted by i to the base station after time t. The collection delay $\mathcal{K}_{\mathcal{O}}^{\phi}[t]$ is defined as

$$\mathcal{K}_{\mathcal{O}}^{\phi}[t] \triangleq \max_{i \in \phi_N} Y_{i\mathcal{O}}^{\phi}[t]$$

Since the packet reception process is i.i.d. over time and the AoC process is stationary since the network began at $t = -\infty$, the time index can be dropped. By an identical line of reasoning as that in Claim 1, we conclude that

$$\mathbb{E}\left[C_{\mathcal{O}}^{\phi}\right] = \mathbb{E}\left[\mathcal{K}_{\mathcal{O}}^{\phi}\right] . \tag{3.1}$$

Since $\theta > 1$, the event of a packet reception at time t at the base station from transmitter x is disjoint from the event of a packet reception in the same time slot from transmitter $j \neq x$. That is, due to the threshold setting being larger than 1, only a single packet can be received at the receiver in a single time slot. When packet reception events from different transmitters are disjoint and the packet reception process is time invariant, we observe that the update collection process resembles a coupon collection process.

In the classical Coupon Collector Problem, there are n distinct coupons that are to be collected. Coupons are drawn randomly at each time step. At any time, the probability of drawing any one of n coupons is uniformly $\frac{1}{n}$, independent of all other time steps, and so the resulting average time it takes to draw all n distinct coupons at least once is nH_n, where H_n denotes the n^{th} harmonic number $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

The variant of the CCP in the collection scenario is one in which $|\phi_N|$ distinct coupons need to be drawn but do not have a uniform probability of being drawn by the base station [5]. Additionally, there is the possibility of drawing an unwanted NULL coupon – the event where no packet is successfully received – which occurs if either no transmitter attempts a transmission or all attempted transmissions failed to exceed θ.

The expression for the average collection can be found, expressed in Claim 4.

Claim 4. Given knowledge of the node and interferer locations ϕ the EAoC is given by

$$\mathbb{E}\left[C_{\mathcal{O}}^{\phi}\right] = \mathbb{E}\left[\max_{i \in \phi_N} Y_{i\mathcal{O}}^{\phi}[0]\right] = \sum_{i=1}^{n} (-1)^{i+1} \sum_{A \in [\phi_N]_k} \frac{1}{\sum_{u \in A} \mu_{u\mathcal{O}}^{\phi}} . \tag{3.2}$$

Proof. Since we have established the equivalence of EAoC and expected collection delay in Equation (3.1), we focus on the expected collection delay. We invoke the maximum-minimums identity to represent $\max_{i \in \phi_N} Y_{i\mathcal{O}}^{\phi}$ as a sum of the minima of the non-empty subsets of ϕ_N. The maximum-minimums identity states that for a finite set of numbers A with cardinality n,

$$\max A = \sum_{k=1}^{n} (-1)^{k+1} \sum_{L \in [A]_k} \min L \,. \tag{3.3}$$

Applying the identity to $\max_{i \in \phi_N} Y_{i\mathcal{O}}^{\phi}$,

$$\mathbb{E}\left[\max_{i \in \phi_N} Y_{i\mathcal{O}}^{\phi}\right] = \sum_{k=1}^{|\phi_N|} (-1)^{k+1} \sum_{A \in [\phi_N]_k} \mathbb{E}\left[\min_{j \in A} Y_{j\mathcal{O}}^{\phi}\right] . \tag{3.4}$$

Due to the disjointedness of packet reception events between any transmitters in a time slot t, the random variable $\min_{j \in A} Y_{j\mathcal{O}}^{\phi}$ is a geometric random variable with parameter $\sum_{j \in A} \mu_{j\mathcal{O}}^{\phi}$. Thus,

$$\mathbb{E}[\max_{i \in \phi_N} Y_{i\mathcal{O}}^{\phi}] = \sum_{k=1}^{|\phi_N|} (-1)^{k+1} \sum_{A \in [\phi_N]_k} \frac{1}{\sum_{j \in A} \mu_{j\mathcal{O}}^{\phi}} \,, \tag{3.5}$$

and the proof is complete. □

We proceed to find bounds on the EAoC in the instance-independent case in the following subsection.

3.1.2 Instance-independent (CI)

In this scenario, the locations of the transmitters are no longer assumed to be known, distributed according to the Poisson Point Process Φ_N. We begin with an upper bound on EAoC. Conditioning on the size of Φ_N to be n, the nodes are distributed i.i.d. uniform in the disk $b_2(\mathcal{O}, r)$. Based on this conditioning, and assuming no nodes are present within a small distance ϵ of the base station, an upper bound for EAoC is outlined in the following claim:

Claim 5. Conditioned on the number of transmitters $|\Phi_N| = n$ in the disk $b_2(\mathcal{O}, r)$

the EAoC is bounded above as given by

$$\mathbb{E}\left[C_{\mathcal{O}}^{\phi} \,\middle|\, |\Phi_N| = n\right] \leq (\overline{\mu})^{-1} H_n\,, \tag{3.6}$$

where

$$\overline{\mu} = \left(1 - \frac{p}{1 + \theta\left(\frac{\epsilon}{r}\right)^{\beta}}\right)^{n-1} \cdot \mu(r) \tag{3.7}$$

Proof. The most disadvantaged transmitter in terms of successful delivery probability is one situated at a distance r from the base station, while the remaining $n-1$ transmitters are close enough to the base station to observe no path loss, i.e. at distance ϵ. The success probability of this disadvantaged transmitter is

$$\left(1 - \frac{p}{1 + \theta\left(\frac{\epsilon}{r}\right)^{\beta}}\right)^{n-1} \cdot \mu(r)\,, \tag{3.8}$$

since each of the other transmitters contribute equally to the interference observed at the base station.

If all transmitters have this same pessimistic delivery probability, the EAoC would be that of a classical CCP, the resulting EAoC given in Claim 5. □

We now present numerical simulations that highlight the interplay between broadcast and collection.

Chapter 4

Numerical Analysis

4.1 Numerical Results

We examine EAoB and EAoC with different network parameter settings using numerical simulation. Unless stated otherwise, the default network parameter settings for simulation are provided in Table 4.1. The EAoB and EAoC for each parameter settings is determined using Monte Carlo simulation, simulating 250000 time slots per trial.

In Figure 4-1, the scaling behavior of EAoB and EAoC combined is shown as the radius of $b_2(\mathcal{O}, r)$ is increased. The scaling behavior is super-exponential in both cases, with EAoC consistently larger than EAoB for the same radius. This is expected, since in the collection scenario at most one packet can be delivered to the base station in

Simulation Parameter Settings	
Parameter	Default Value
λ	1.0e−2
θ	5
r	10
β	4.0
p	0.2

Table 4.1: Table of default parameter values when held constant as part of the numerical simulation

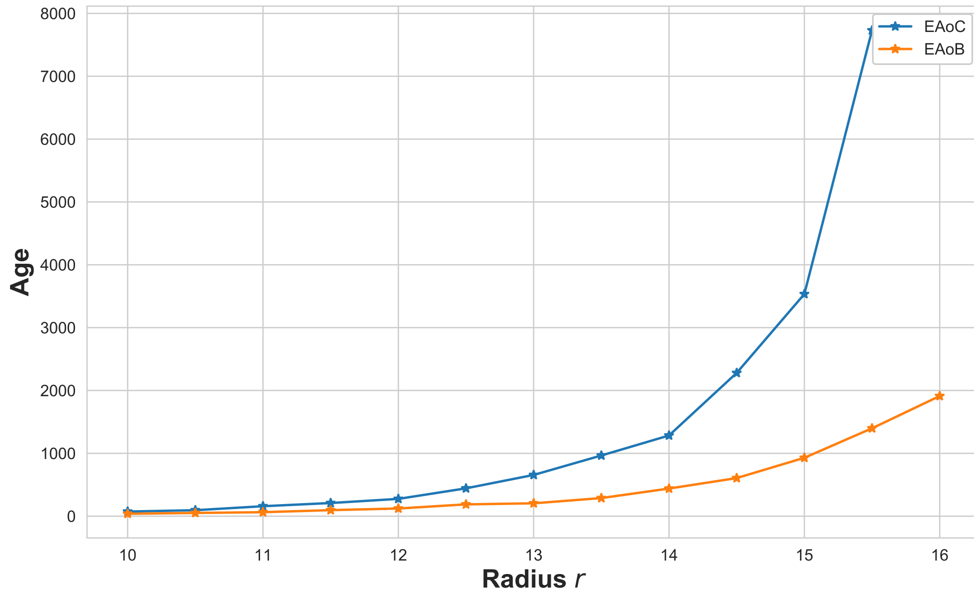

Figure 4-1: Instance-independent Age scaling with radius r going from 10 to 15.5

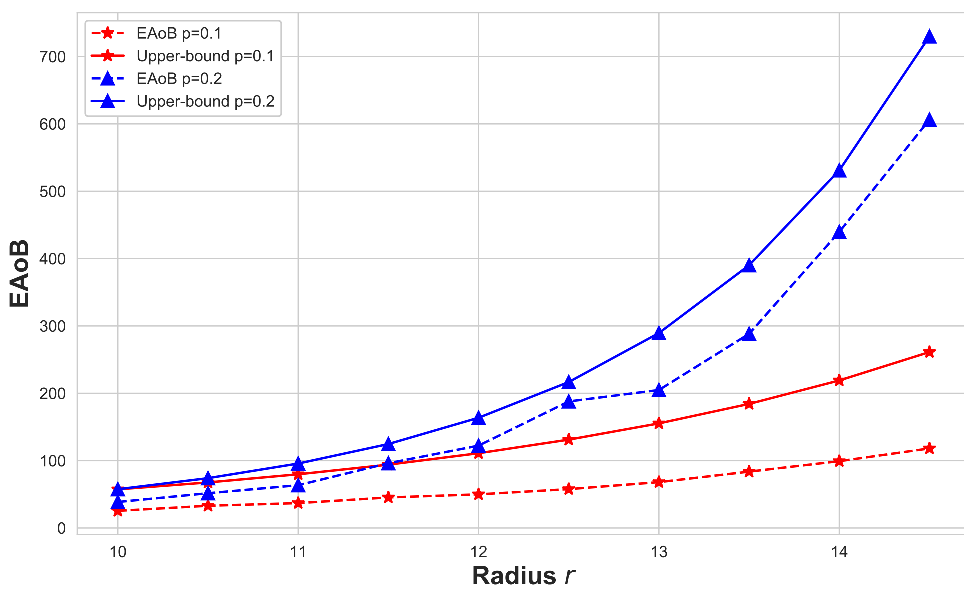

Figure 4-2: Instance-independent EAoB scaling with radius r going from 10 to 14

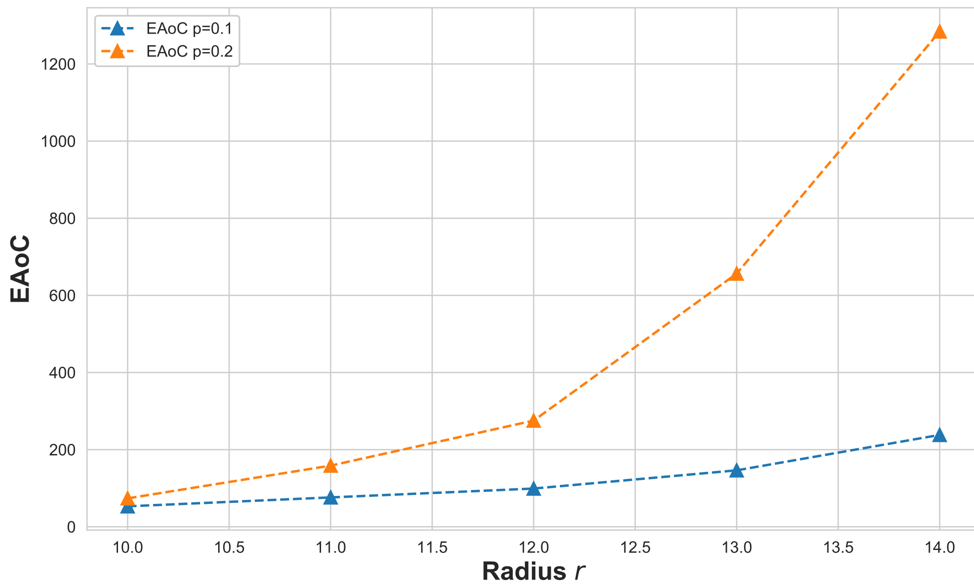

Figure 4-3: Instance-independent EAoC scaling with radius r going from 10 to 14

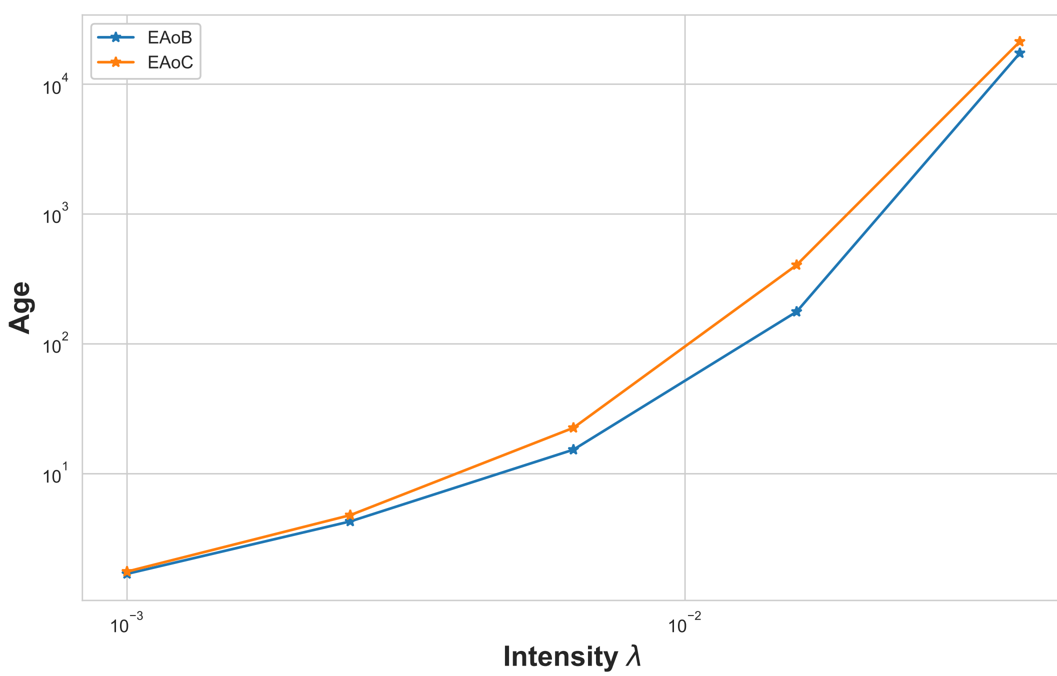

Figure 4-4: Instance-independent EAoB and EAoC scaling, with intensity λ going from $1e-3$ to $1e-1$.

a given time slot, whereas when broadcasting, it is possible for multiple receivers to get a packet simultaneously. Moreover, the interference observed at the base station in the collection case is, in expectation, larger since both Φ_I and Φ_N are sources of interference when collecting. Figures 4-2 and 4-3 plot EAoB and EAoC versus radius for different values of the medium access probability, showing that a greater value of p result in larger age for these parameter settings. In Figure 4-4, the density is varied on a logarithmic scale. The figure depicts the exponential growth of EAoB and EAoC with respect to node and interferer intensity λ.

Chapter 5

Conclusion and future work

5.1 Conclusion

We defined AoB and AoC as information freshness metrics suitable for the cases of broadcast and collection, respectively, in spatially-distributed wireless networks. We characterized the expected AoB and AoC when the locations of nodes and interferers are known and unknown. When the locations are known and the packet transmission process is stationary, we showed that expected AoB and AoC were equivalent to the expected broadcast delay and collection delay, respectively. Upper-bounds were found in the instance-independent scenario: the AoB upper-bound is a solution to a differential equation, and the AoC upper-bound uses the solution to the worst-case packet delivery success probability given a small exclusion radius ϵ. We demonstrated through numerical simulation the relation between AoB and AoC and network parameters such as density and medium access probability. Future work could introduce mobility by allowing nodes in the disk to change positions each time slot. Simultaneous broadcast and collection could also be investigated, where the base station may act as a transceiver and switch randomly between broadcast and collection in each time slot.

Appendix A

Proofs

A.1 Conditional Probability of Successful Packet Reception

Proof. Recall that the SIR conditioned on Φ_I is defined in Section 1.1.1 Equation (1.2). Substituting Equation (1.2) for the SIR,

$$
\begin{aligned}
\mu_{\mathcal{O}y}^{\phi_I} &\overset{(a)}{=} p \cdot \mathbb{P}\left(H_{\mathcal{O}y} > \theta \frac{\sum_{x\in\phi_I} Z_x H_{xy} \ell(x-y)}{\ell(y)}\right) && \text{(A.1)}\\
&\overset{(b)}{=} p \cdot \mathbb{E}\left[\exp\left(-\theta \frac{\sum_{x\in\phi_I} Z_x H_{xy} \ell(x-y)}{\ell(y)}\right)\right] && \text{(A.2)}\\
&\overset{(c)}{=} p \prod_{x\in\phi_I} \mathbb{E}_H\left[p \cdot \exp\left(-\theta \frac{H_{xy}\ell(x-y)}{\ell(y)}\right) + 1 - p\right] && \text{(A.3)}\\
&\overset{(d)}{=} p \prod_{x\in\phi_I} \left(\frac{p}{1+\theta\frac{\ell(x-y)}{\ell(y)}} + 1 - p\right), && \text{(A.4)}
\end{aligned}
$$

where (a) is a result of taking expectation with respect to $Z_{\mathcal{O}}$, (b) follows from the complementary cumulative distribution function (CCDF) of $H_{\mathcal{O}y}$ which is $1 - F_{H_{\mathcal{O}y}}(h) = e^{-h}$, (c) follows from the independence and identical distribution of the fading variables and the expectation with respect to Z_x, and (d) is the result of taking expectation with respect to the fading variables H_{xy}. Equation (1.8) is an algebraic simplification of Equation (A.4). □

A.2 Proof of the success probability calculation

Proof. Given the conditional success probability is defined as the probability that the SIR exceeds θ, the success probability averaged over Φ_I is given by

$$\mu(\|y\|) = \mathbb{E}_{\Phi_I}\left[\mu_{\mathcal{O}y}^{\Phi_I}\right] = p\mathbb{E}_{\Phi_I}\left[\mathbb{P}\left(H_{\mathcal{O}y} > \frac{\theta I_{\mathcal{O}y}^{\Phi_I}}{\ell(y)}\right)\right] \tag{A.5}$$

$$= p\,\underbrace{\mathbb{E}_{\Phi_I}\left[\exp\left(-\frac{\theta I_{\mathcal{O}y}^{\Phi_I}}{\ell(y)}\right)\right]}_{A}. \tag{A.6}$$

Note that the multiplicative term A is the Laplace transform of the interference $I_{\mathcal{O}y}^{\Phi_I}$ evaluated at $s = \theta/\ell(y)$. The Laplace transform for interference given Rayleigh fading and power-law pathloss is given in [7], Equation (3.20), to be

$$\mathcal{L}_I(s) = \exp\left(-p\lambda\pi\Gamma(1+\delta)\Gamma(1-\delta)s^{\delta}\right) \tag{A.7}$$

substituting θr^{β} provides the desired result. □

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