Boolean-Valued Models and Their Applications

by

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Abstract

Boolean-valued models generalize classical two-valued models by allowing arbitrary complete Boolean algebras as value ranges. The goal of my dissertation is to study Boolean-valued models and explore their philosophical and mathematical applications.

In Chapter 1, I build a robust theory of first-order Boolean-valued models that parallels the existing theory of two-valued models. I develop essential model-theoretic notions like “Boolean-valuation", “diagram", “elementary diagram", and prove a series of theorems on Boolean-valued models, including the (strengthened) Soundness and Completeness Theorem, the Löwenheim-Skolem Theorems, the Elementary Chain Theorem, and many more.

Chapter 2 gives an example of a philosophical application of Boolean-valued models. I apply Boolean-valued models to the language of mereology to model indeterminacy in the parthood relation. I argue that Boolean-valued semantics is the best degree-theoretic semantics for the language of mereology. In particular, it trumps the well-known alternative - fuzzy-valued semantics. I also show that, contrary to what many have argued, indeterminacy in parthood entails neither indeterminacy in existence nor indeterminacy in identity, though being compatible with both.

Chapter 3 (joint work with Bokai Yao) gives an example of a mathematical application of Boolean-valued models. Scott and Solovay famously used Boolean-valued models on set theory to obtain relative consistency results. In Chapter 3, I investigate two ways of extending the Scott-Solovay construction to set theory with urelements. I argue that the standard way of extending the construction faces a serious problem, and offer a new way that is free from the problem.

Thesis Supervisor: Vann McGee
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Contents

Introduction **11**

1 A Theory of Boolean-Valued Models **15**

1.1 Introduction 15

1.2 Boolean Valued Models 19

1.2.1 Boolean Algebra 19

1.2.2 Boolean Valued Models 20

1.3 Two Important Constructions 22

1.3.1 Direct Products 22

1.3.2 Quotient Models 25

1.4 Theories and Boolean Valuations 32

1.4.1 Theories 32

1.4.2 Boolean Valuations 34

1.5 Relationships Between Models 47

1.5.1 Duplicate Resistant Models 47

1.5.2 Isomorphism, Submodel, and Diagram 50

1.5.3 Elementary Submodel and Downward Löwenheim-Skolem 53

1.5.4 Elementary Equivalence and Elementary Chain 57

1.6 Supervaluationism 62

1.7 Full Models 66

1.7.1 Full Models 66

1.7.2 Łoś Theorem on Σ_1^1 Formulas 69

1.8 True Identity Models 75

1.9 Löwenheim-Skolem Theorems 79

2 Boolean Mereology **83**

2.1 Introduction 83

2.2 Many Degrees: A Natural Start 85

2.3 What Are Boolean Degrees? 88

2.4 Why Boolean Degrees? 90

2.5 Atomic Boolean Models 95

2.5.1 Classical Mereology 97

2.5.2 Atomic Boolean Models 100

2.6 The Nature of Mereological Vagueness 106

2.6.1 Semantic Boolean Mereology 108

2.6.2 Ontic Boolean Mereology 113

2.7 Appendices 117

2.7.1 Preliminaries on Boolean Model Theory 117

2.7.2 Soundness and Completeness 120

2.7.3 Equivalence Between Systems 122

2.7.4 The *SE* Models 125

2.7.5 The VE Models 130

2.7.6 Identity and Anti-Symmetry 132

3 Boolean-Valued Models with Urelements
(with Bokai Yao) **135**

3.1 Introduction 135

3.2 The Model U^B 138

3.2.1 The Old Construction 138

3.2.2 U and U^B 140

3.2.3 The Easy Axioms 141

3.2.4 The Axiom of Replacement 143

3.2.5 U^B Can Recover Collection 149

3.3 Non-Witnessing U^B and Witnessing $\overline{U^B}$ 151

3.3.1 U^B Is Almost Never Witnessing 151
3.3.2 The New Construction $\overline{U^B}$ 154
3.3.3 $\overline{U^B}$ and the Mixing Lemma 156
3.3.4 U^B and $\overline{U^B}$ 158
3.4 Collection and Witnessing 162
3.4.1 Equivalence Between Collection and Witnessing 162
3.4.2 A Non-Atomic Example 166

Introduction

Classical models are bivalent: there are two truth values, 1 (the True) and 0 (the False). The set of classical truth values forms the smallest non-trivial complete Boolean algebra 2. Under classical semantics, the logical symbols are interpreted as algebraic operations on 2: conjunction as binary meet, disjunction as binary join, negation as Boolean complement, universal quantifier as infinite meet and existential quantifier as infinite join.

This algebraic conception of classical semantics suggests a natural way to generalize the classical models. What is really in need to carry out the semantic construction is the underlying algebraic structure of the truth values. The complete Boolean algebra 2 can in principle be replaced by any complete Boolean algebra. The resultant models, whose value range can be an arbitrary complete Boolean algebra, are called Boolean-valued models. The goal of this dissertation is to thoroughly investigate Boolean-valued models and explore their philosophical and mathematical applications.

Historically, Boolean-valued models have been widely employed in the context of set theory. Introduced by Dana Scott, Robert Solovay and others, Boolean-valued models for the language of set theory are used to give semantics to Paul Cohen's syntactic forcing, which is a method for obtaining independence results. Nevertheless, outside the domain of set theory, Boolean-valued models for arbitrary first order languages, as a subject on its own, have not been as well-studied. Although the definition of a Boolean-valued model has been mentioned at multiple occasions, little effort has been made to provide a full-fledged theory of Boolean-valued models that parallels that of classical two-valued models.

The primary goal of Chapter 1 of this dissertation is to fulfill this theoretical gap. I will build, step-by-step, a robust and detailed theory of Boolean-valued models. I will start with two useful model-theoretic constructions on Boolean-valued models: the direct product construction and the quotient model construction, with which I will prove the (generalized) Łoś Theorem. Then, I will introduce the key notion of "Boolean-valuations", and prove that Boolean-valued models are sound and complete with respect to Boolean-valuations. With the notion of "Boolean-valuations", I will then define notions like "diagram", "elementary diagram", "elementary chain", etc, and prove a number of results on them, including the equivalence theorems, the (generalized) Löwenheim-Skolem Theorems, the (generalized) Elementary Chain Theorem, and so on. I will also discuss, in Chapter 1, some important special kinds of Boolean-valued models that are particularly interesting: witnessing models, full models and true-identity models.

I believe that the theory of Boolean-valued models, as a species of model theory, is of tremendous interest on its own. But Boolean-valued models are worth studying for a variety of other reasons as well. From a logical perspective, a number of critical model-theoretic results on two-valued models can be shown to be special cases of more generalized theorems on Boolean-valued models, including the aforementioned Łoś Theorem, Löwenheim-Skolem Theorems, and many more. These generalized theorems shed light on which essential features of two-valued models are responsible for the holdings of these results on them. For example, we will see in Chapter 1 that both Łos' Theorem and Downward Löwenheim-Skolem Theorem are grounded in the property of being witnessing, and both the theorem that every countably incomplete ultraproduct is ω_1 saturated and the theorem that Σ^1_1 formulas are preserved under ultraproducts are grounded in the property of being full.

From a philosophical perspective, Boolean-valued models gives rise to a new intriguing degree-theoretic semantics that is both classical and non-bivalent. With the existence of non-extreme truth values, Boolean-valued models have fruitful applications to the general phenomenon of indeterminacy. We will see an example of this kind in Chapter 2, where Boolean-valued semantics is applied to the language of mereology to model indeterminacy in the parthood relation. I will argue, in Chapter 2, that Boolean-valued semantics is the best degree-theoretic semantics for the language of mereology. In particular, I will argue that it trumps the well-known alternative - fuzzy-valued semantics, for three main reasons: (a) it allows for incomparable degrees of parthood, (b) it enforces classical logic, and (c) it is compatible with all the axioms of classical mereology. Moreover, I will explore, under the framework of Boolean semantics, the connection between indeterminacy in parthood and indeterminacy in existence/identity. I will show that, contrary to many have argued, indeterminacy in parthood entails neither indeterminacy in existence nor indeterminacy in identity, although being compatible with both.

From a mathematical perspective, Boolean-valued models, as mentioned, have generated a fruitful theoretic enterprise when applied to ZFC, Zermelo–Fraenkel set theory. In Chapter 3 (joint work with Bokai Yao), I take a step further in this direction by investigating Boolean-valued models of ZFCU, set theory with (potentially class many) urelements. I will first show that the most direct, and also the most commonly adopted way of expanding a Boolean-valued universe of pure sets to a Boolean-valued universe that allows urelements has a serious problem. In particular, this traditional kind of Boolean-valued universe with urelements is not witnessing and hence is not in the scope of Łoś Theorem. This means that the quotient model method, which is the simplest, and the most powerful method of proving relative consistency results, cannot be used on the traditional kind of Boolean-valued universe with urelements. I will remedy this problem by developing a new way of constructing a Boolean-valued universe with urelements, one that satisfies the Mixing Lemma, and is witnessing given the Axiom of Collection in the background theory. I will also show that the Boolean-valued universe generated by old construction is an elementary submodel of the Boolean-valued universe generated by the new construction. Moreover, I will prove that over the background theory ZFCU_R, the Axiom of Collection is equivalent with the statement that for any complete Boolean algebra, every Boolean-valued universe with urelements generated by the new construction is witnessing.

Chapter 1

A Theory of Boolean-Valued Models

1.1 Introduction

Traditionally, a model of a first order language $\mathscr{L}$ has as its value range the complete Boolean algebra $2 = \{0,1\}$. Logical symbols in the language are interpreted as operations on the Boolean algebra: conjunction as binary meet, disjunction as binary join, negation as Boolean complement, universal quantifier as infinite meet and existential quantifier as infinite join. A natural way to generalize the traditional models, then, is to instead of just using the complete Boolean algebra 2 as the value range, use arbitrary complete Boolean algebra as value ranges.

Boolean-valued models are worth studying for a variety of reasons. To begin with, the supervaluation models, which are used in the standard approach to vagueness, can be shown to be a special type of Boolean-valued models (Theorem 1.6.1). In fact, we can show that there is a duality between the class of supervaluation models and a subclass of true identity Boolean-valued models (Theorem 1.6.3). Also, two important features of Boolean-valued models - that they are degree-theoretic and that they induce classical logic - let them give rise to attractive theories of different types of vagueness.[1] Moreover, since the logic of Boolean-valued models is both classical and non-bivalent, they are particularly useful in illustrating certain points in the philosophy of model theory. For example, it seems to serve

[1] We will see an example of this kind in Chapter 2, where we apply Boolean-valued models to mereological indeterminacy. For an application to the general phenomenon of vagueness, see McGee and McLaughlin [25].

as a strong case against the claim that our classical rules of inferences pin down uniquely the range of semantic values ([5]).

Moreover, many important model-theoretic theorems on two-valued models can be shown to be special cases of more generalized theorems on Boolean-valued models. These results will shed light on what essential features of two-valued models are responsible for the truth of these theorems on two-valued models. There are at least four examples of this kind throughout this chapter:

1. The property of being witnessing and Łoś Theorem. (Theorem 1.3.4)

2. The property of being witnessing and Downward Löwenheim-Skolem Theorem. (Theorem 1.5.6)

3. The property of being full and the theorem that every countably incomplete ultraproduct is ω_1-saturated. (Theorem 1.7.4)

4. The property of being full and the theorem that Σ_1^1 formulas are preserved under untraproducts. (Theorem 1.7.7)

Also, results on Boolean-valued models will also help us to have a better understanding of certain model-theoretic constructions on two-valued models. For example, as we will see in Section 1.3, the ultraproduct construction on two valued models can be reduced to a two-step construction of taking first a direct product and then a quotient model.

Nevertheless, Boolean-valued models, as a subject on their own, have not been well-studied, at least in comparison to the two-valued models. On two-valued models there exists a full-fledged and fruitful theory - the entirety of model theory, roughly speaking, that is based on important basic notions like "diagram", "submodel", "elementary", etc. Few of these notions, however, have been generalized to Boolean-valued models, and this is also the case with the many model-theoretic results based on these notions. There are a number of natural questions on the model-theoretic properties of Boolean-valued models that await answers: What is the diagram/elementary diagram of a Boolean-valued model? What does it mean for a Boolean-valued model to be a submodel/elementary submodel of another? Do Löwenheim-Skolem Theorems hold on all Boolean-valued models? etc. The primary goal of this chapter is to answer these questions, in order to develop, step by step, a robust and detailed theory on Boolean-valued models.

Although some of these questions are (relatively) straightforward to answer, like “What does it mean for a Boolean-valued model to be a submodel/elementary submodel of another?", some of these questions are quite the opposite. One of the most difficult notions is that of a diagram. When we only have two truth values, the diagram of a model is a set of sentences, and therefore a theory. But when there are more than two truth values, the “diagram” of a model, if we want it to be something close to what we have in the two-valued case, cannot be just a theory. The natural suggestion is that the diagram is a set of ordered pairs whose first component is a sentence and second component is a truth value. In this paper, we will call a set of this form a “Boolean-valuation". (First-order) Boolean valuations are natural generalizations of (first-order) theories. A question that turns out to be quite difficult and require a lot of efforts is “What does it mean for a Boolean-valuation to be consistent and are consistent Boolean-valuations precisely those that have models?". One of the major results of this chapter (Theorem 1.4.9.1) is that (under our definition of consistency), Boolean-valued models are sound and complete with respect to Boolean-valuations, which is a theorem that generalizes the known result that Boolean-valued models are sound and complete with respect to first-order theories. Corollaries to this theorem include the compactness theorem (Corollary 1.4.9.2) on Boolean valuations and the (weaker version of) Downward-Löwenheim-Skolem theorem on Boolean valuations (Corollary 1.4.9.3).

With the notion of “Boolean valuation", we are then able to define notions like “diagram"(Def 1.5.5), “elementary diagram"(Def 1.5.9), etc., and prove the equivalence theorems between diagrams and submodels (Theorem 1.5.2), elementary diagrams and elementary submodels (Theorem 1.5.5), etc. Another difficult question concerns the (stronger version of) Downward-Löwenheim-Skolem theorem. With the help of Boolean-valuations, we will prove that the (stronger version of) Downward-Löwenheim-Skolem theorem can be generalized to witnessing Boolean-valued models (Theorem 1.5.6), though not to Boolean-valued models in general (Theorem 1.5.7).

For the discussion of the Upward-Löwenheim-Skolem theorems to be non-trivial, we will have to look at a special type of Boolean-valued models, the ones that define identity in the standard, or true way (Def 1.8.1). We will investigate which kind of Boolean valuations corresponds to the "true identity" models. The next major result (Theorem 1.8.7) is that true identity Boolean-valued models are sound and complete with respect to Boolean valuations that "respect identity" (Def 1.8.2). From there, we will establish the Upward-Löwenheim-Skolem theorems on true identity Boolean-valued models (Theorem 1.9.7, 1.9.8).

We will also discuss another special type of Boolean-valued models - the full models (Def. 1.7.2). Full Booelan-valued models turn out to be equivalent to Boolean-valued models that remain witnessing no matter how they are expanded (Theorem 1.7.8). They are, therefore, even more akin to the two-valued models than witnessing Boolean-valued models. Two major results in this chapter are that two important corollaries of Łoś Theorem are generalizable to full Boolean-valued models: that every countably incomplete ultraproduct is ω_1-saturated (Theorem 1.7.4), and that Σ^1_1 formulas are preserved under untraproducts (Theorem 1.7.7).

We organize this chapter as follows. In Section 1.2 we cover some preliminaries on Boolean algebras and introduce Boolean-valued models. In Section 1.3, we introduce two important construction on Boolean-valued models that will be useful throughout the paper. In particular, we will introduce the quotient construction and prove the Generalized Łoś Theorem. In Section 1.4, we first review the proof of the theorem that Boolean-valued models are sound and complete with respect to first-order theorems, and then in 1.5.2, we introduce Boolean valuations, define their consistency condition, and prove that Boolean-valued models are sound and complete with respect to first-order Boolean valuations. In Section 1.5, with the help of Boolean valuations, we extend basic model theoretic notions like "diagram", "submodel", "elementary embedding" to Boolean-valued models, prove the equivalence theorems, and prove the (stronger version of) Downward-Löwenheim-Skolem theorem on witnessing Boolean-valued models. We will also study chains of models and generalize the Elementary Chain Theorem to the Boolean-valued case. In Section 1.6, we discuss the connection between supervaluation models and Boolean-valued models. We prove that supervaluation models are equivalent to a special type of Boolean-valued models. In Section 1.7, we will investigate the full Boolean-valued models. In Section 1.8, we will discuss the true identity Boolean-value models. Finally, in Section 1.9, we discuss the Upward-Löwenheim-Skolem theorems on Boolean-valued models.

1.2 Boolean Valued Models

1.2.1 Boolean Algebra

This chapter assumes that the reader already has some basic knowledge about Boolean algebras and model theory. The main purpose of this subsection is just to introduce the symbols that will be used for Boolean operations. For a more detailed introduction of Boolean algebras, see Givant and Halmos [14].

Definition 1.2.1. A lattice is a non-empty partially ordered set $< L, \leqslant >$ such that for any $x, y \in L$, $\{x, y\}$ has a supremum(join), $x \sqcup y$, which is the least element that is greater than or equal to both x and y, and a infimum(meet), $x \sqcap y$, which is the greatest element that is less than or equal to both x and y.

Definition 1.2.2. A lattice L is bounded just in case it has a top element 1_L such that $\forall x \in L(x \leqslant 1_L)$, and a bottom element 0_L such that $\forall x \in L(0_L \leqslant x)$.

Definition 1.2.3. A lattice L is distributive just in case for any $x, y, z \in L$,

$$x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$$
$$x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$$

Definition 1.2.4. Let L be a bounded lattice and $x, y \in L$. y is a complement of x just in case $x \sqcup y = 1_L$ and $x \sqcap y = 0_L$.

Definition 1.2.5. A bounded lattice L is complemented just in case for any $x \in L$, there exists some $y \in L$ such that y is a complement of x.

In a distributive lattice, it can be easily shown that if an element x has a complement, then it has a unique complement. We denote the complement of x, if it exists, as $-x$.

Definition 1.2.6. A *Boolean algebra* B is a bounded distributive complemented lattice.

Definition 1.2.7. A Boolean algebra B is κ-complete (where κ is a cardinal) just in case for any subset $D \subseteq B$ such that $|D| \leqslant \alpha$, both the supremum of D, $\bigsqcup D$, and the infimum of D, $\bigsqcap D$, exist in B. A Boolean algebra B is complete just in case for any κ, B is κ-complete.

1.2.2 Boolean Valued Models

Definition 1.2.8. Let $\mathscr{L}$ be an arbitrary first order language. For simplicity, we assume that $\mathscr{L}$ has no function symbols, but only relation symbols and constants.[2] Let B be a non-trivial complete Boolean algebra. A B-valued model[3] $\mathfrak{A}$ for the language $\mathscr{L}$ consists of:

1. A universe A of elements;
2. The B-value of the identity symbol: a function $[\![=]\!]^{\mathfrak{A}} : A^2 \to B$;
3. The B-values of the relation symbols: (let P be an n-ary relation) $[\![P]\!]^{\mathfrak{A}} : A^n \to B$;
4. The B-values of the constant symbols: (let c be a constant) $[\![c]\!]^{\mathfrak{A}} \in A$.

And it needs to satisfy:

1. For the B-value of the identity symbol[4]: for any $a_1, a_2, a_3 \in A$

$$[\![a_1 = a_1]\!]^{\mathfrak{A}} = 1_B \tag{1.1}$$

$$[\![a_1 = a_2]\!]^{\mathfrak{A}} = [\![a_2 = a_1]\!]^{\mathfrak{A}} \tag{1.2}$$

$$[\![a_1 = a_2]\!]^{\mathfrak{A}} \sqcap [\![a_2 = a_3]\!]^{\mathfrak{A}} \leqslant [\![a_1 = a_3]\!]^{\mathfrak{A}} \tag{1.3}$$

2. For the B-value of relation symbols: let P be an n-ary relation; for any $\langle a_1, \ldots, a_n \rangle, \langle b_1, \ldots, b_n \rangle$

[2]Our theory can be easily generalized to first order languages with function symbols, as functions can always be treated as relations that satisfy special conditions.

[3]Our definition of Boolean-valued models is the standard one. You can find the same definition in many other places, including, Bell [3], Button and Walsh [5], Hamkins and Seabold [16], etc.

[4]Here and in the following, when the context is clear, we use $[\![a_i = a_j]\!]^{\mathfrak{A}}$ to abbreviate $[\![=]\!]^{\mathfrak{A}}(a_i, a_j)$, and similarly for cases of the relation symbols.

$\in A^n$,

$$[\![P(a_1,...,a_n)]\!]^{\mathfrak{A}} \sqcap (\prod_{1\leqslant i\leqslant n} [\![a_i = b_i]\!]^{\mathfrak{A}}) \leqslant [\![P(b_1,...,b_n)]\!]^{\mathfrak{A}} \tag{1.4}$$

Given a B-valued model $\mathfrak{A}$ for $\mathscr{L}$, we define satisfaction in $\mathfrak{A}$ as follows:

Definition 1.2.9. Let *Var* be the set of all variables. (We will use $v_1, v_2, ...$ to range over variables.) An assignment on $\mathfrak{A}$ is a function from *Var* to A. Given a assignment x on $\mathfrak{A}$, we define the value of an open formula of $\mathscr{L}$ in $\mathfrak{A}$ under assignment x as follows.

1. We first define the value of terms in $\mathfrak{A}$:

   (a) Let v_i be a variable. Then $[\![v_i]\!]^{\mathfrak{A}}[x] = x(v_i) = x_i$[5].

   (b) Let c be a constant. Then $[\![c]\!]^{\mathfrak{A}}[x] = [\![c]\!]^{\mathfrak{A}}$.

2. We then define the value of atomic formulas in $\mathfrak{A}$:

   (a) Let t_1, t_2 be terms (a term is either a variable or a constant). Then $[\![t_1 = t_2]\!]^{\mathfrak{A}}[x] = [\![a_i = a_j]\!]^{\mathfrak{A}}$, where $a_i = [\![t_1]\!]^{\mathfrak{A}}[x]$ and $a_j = [\![t_2]\!]^{\mathfrak{A}}[x]$.

   (b) Let $t_1, ..., t_n$ be terms. Then $[\![P(t_1,...,t_n)]\!]^{\mathfrak{A}}[x] = [\![P(a_1,...,a_n)]\!]^{\mathfrak{A}}$, where $a_1 = [\![t_1]\!]^{\mathfrak{A}}[x]$, ..., $a_n = [\![t_n]\!]^{\mathfrak{A}}[x]$.

3. We finally define the value of complex formulas in $\mathfrak{A}$:

   (a) Let ϕ be a formula. Then $[\![\neg\phi]\!]^{\mathfrak{A}}[x] = -[\![\phi]\!]^{\mathfrak{A}}[x]$.

   (b) Let ϕ, ψ be formulas. Then $[\![\phi \wedge \psi]\!]^{\mathfrak{A}}[x] = [\![\phi]\!]^{\mathfrak{A}}[x] \sqcap [\![\psi]\!]^{\mathfrak{A}}[x]$.

   (c) Let ϕ, ψ be formulas. Then $[\![\phi \vee \psi]\!]^{\mathfrak{A}}[x] = [\![\phi]\!]^{\mathfrak{A}}[x] \sqcup [\![\psi]\!]^{\mathfrak{A}}[x]$.

   (d) Let ϕ be a formula. Then $[\![\exists v_i \phi]\!]^{\mathfrak{A}}[x] = \bigsqcup_{a\in A} [\![\phi]\!]^{\mathfrak{A}}[x(v_i/a)]$, where $x(v_i/a)$ is the assignment on $\mathfrak{A}$ that takes v_i to a and agrees with x everywhere else.

   (e) Let ϕ be a formula. Then $[\![\forall v_i \phi]\!]^{\mathfrak{A}}[x] = \prod_{a\in A} [\![\phi]\!]^{\mathfrak{A}}[x(v_i/a)]$, where $x(v_i/a)$ is the assignment on $\mathfrak{A}$ that takes v_i to a and agrees with x everywhere else.

[5] Here and in the following, given an assignment x, we will use x_i to abbreviate $x(v_i)$.

Clearly, both $[\![\exists v_i\phi]\!]^{\mathfrak{A}}[x]$ and $[\![\forall v_i\phi]\!]^{\mathfrak{A}}[x]$ are well-defined as B is assumed to be complete.

It is easy to see that traditional two-valued models for first order languages are just special cases of Boolean valued models, when we require B to be the two-element Boolean algebra 2 and that the interpretation of the identity symbol is the true identity function on the universe. [6]

In the following, like in the case of atomic formulas, when the context is clear, we will occasionally use $[\![\phi(x_1,...,x_n)]\!]^{\mathfrak{A}}$, instead of $[\![\phi(v_1,...,v_n)]\!]^{\mathfrak{A}}[x]$.

Theorem 1.2.1. Let $\mathfrak{A}$ be a B-valued model for $\mathscr{L}$. For any formula $\phi(v_1,...,v_n)$ in $\mathscr{L}$, any assignments x,y on $\mathfrak{A}$,

$$[\![\phi(x_1,...,x_n)]\!]^{\mathfrak{A}} \sqcap (\prod_{1\leqslant i\leqslant n}[\![x_i = y_i]\!]^{\mathfrak{A}}) \leqslant [\![\phi(y_1,...,y_n)]\!]^{\mathfrak{A}}$$

Proof. By a straightforward induction on the complexity of $\phi(v_1,...,v_n)$.

□

1.3 Two Important Constructions

Given one or more Boolean valued models, there are many ways to use them to construct new Boolean valued models. In this section, we will introduce two of these methods of construction that will be particularly useful for later purposes.

1.3.1 Direct Products

Definition 1.3.1. Let I be an index set. For each $i \in I$, let B_i be a Boolean algebra. The product algebra $\prod_{i\in I} B_i$ is defined as the algebra on the Cartesian product of all the B_i's

[6] Again, we assume that the reader has some basic knowledge of traditional two-valued models. For a detailed introduction on model theory, see Chang and Keisler [7], or Hodges [17].

with the following operations[7] (let $\langle p_i\rangle_{i\in I}, \langle q_i\rangle_{i\in I} \in \prod_{i\in I} B_i$)

$$\langle p_i\rangle_{i\in I} \sqcap \langle q_i\rangle_{i\in I} = \langle p_i \sqcap^i q_i\rangle_{i\in I}$$
$$\langle p_i\rangle_{i\in I} \sqcup \langle q_i\rangle_{i\in I} = \langle p_i \sqcup^i q_i\rangle_{i\in I}$$
$$-\langle p_i\rangle_{i\in I} = \langle -^i p_i\rangle_{i\in I}$$

Note that in a product algebra $\prod_{i\in I} B_i$, $\langle p_i\rangle_{i\in I} \leqslant \langle q_i\rangle_{i\in I}$ just in case for all $i \in I$, $p_i \leqslant^i q_i$. Also, $1_{\prod_{i\in I} B_i} = \langle 1_{B_i}\rangle_{i\in I}$ and similarly $0_{\prod_{i\in I} B_i} = \langle 0_{B_i}\rangle_{i\in I}$.

Proposition 1.3.1. Let I be an index set. For each $i \in I$, let B_i be a complete Boolean algebra. Then their product algebra $\prod_{i\in I} B_i$ is a complete Boolean algebra.

Definition 1.3.2. Let I be an index set. For each $i \in I$, let $\mathfrak{A}_\mathrm{i}$ be a B_i-valued model of the language $\mathscr{L}$. Then the direct product model, $\prod_{i\in I} \mathfrak{A}_\mathrm{i}$, of the $\mathfrak{A}_\mathrm{i}$'s, is defined as the following $\prod_{i\in I} B_i$-valued model of $\mathscr{L}$:

1. The universe is $\prod\limits_{i\in I} A_i$, where for each i, A_i is the universe of $\mathfrak{A}_\mathrm{i}$.
2. Let $\langle a_i\rangle_{i\in I}, \langle b_i\rangle_{i\in I} \in \prod\limits_{i\in I} \mathfrak{A}_\mathrm{i}$, $[\![\langle a_i\rangle_{i\in I} = \langle b_i\rangle_{i\in I}]\!]^{\prod\limits_{i\in I}\mathfrak{A}_\mathrm{i}} = \langle [\![a_i = b_i]\!]^{\mathfrak{A}_\mathrm{i}}\rangle_{i\in I}$.
3. Let $\langle a_i^1\rangle_{i\in I}, \langle a_i^2\rangle_{i\in I}, \ldots, \langle a_i^n\rangle_{i\in I} \in \prod\limits_{i\in I} \mathfrak{A}_\mathrm{i}$, $[\![P(\langle a_i^1\rangle_{i\in I}, \ldots, \langle a_i^n\rangle_{i\in I})]\!]^{\prod\limits_{i\in I}\mathfrak{A}_\mathrm{i}} = \langle [\![P(a_i^1, \ldots, a_i^3)]\!]^{\mathfrak{A}_\mathrm{i}}\rangle_{i\in I}$.
4. For any constant c in $\mathscr{L}$, $[\![c]\!]^{\prod\limits_{i\in I}\mathfrak{A}_\mathrm{i}} = \langle [\![c]\!]^{\mathfrak{A}_\mathrm{i}}\rangle_{i\in I}$.

Theorem 1.3.1. Let I be an index set. For each $i \in I$, let $\mathfrak{A}_\mathrm{i}$ be a B_i-valued model. Then the direct product model, $\prod_{i\in I} \mathfrak{A}_\mathrm{i}$, as defined in Def 1.3.2, is a $\prod_{i\in I} B_i$-valued model - that is, it satisfies Def 1.2.8.

Proof. We just need to check that requirements (1)-(4) in Def 1.2.8 are satisfied. This follows straightforwardly from Def 1.3.1 and Def 1.3.2. □

Theorem 1.3.2 (Direct Product Theorem)**.** Let I be an index set. For each $i \in I$, let $\mathfrak{A}_\mathrm{i}$ be a B_i-valued model. Let $\prod_{i\in I} \mathfrak{A}_\mathrm{i}$ be their direct product model. Given an assignment

[7] In the following, $\sqcap^i$ denotes the meet operation in B_i, and similarly for $\sqcup^i$, $-^i$, etc.

$x : Var \to \prod_{i\in I} A_i$ on $\prod_{i\in I} \mathfrak{A}_\mathrm{i}$, for each $i \in I$, let $y_i : Var \to A_i$ be the assignment on $\mathfrak{A}_\mathrm{i}$ such that for any $v_n \in Var$, $y_i(v_n) = proj_i(x(v_n))$, where $proj_i : \prod_{i\in I} A_i \to A_i$ is the ith projection function. Then, for any formula ϕ in $\mathscr{L}$,

$$[\![\phi]\!]^{\prod_{i\in I}\mathfrak{A}_\mathrm{i}}[x] = \langle [\![\phi]\!]^{\mathfrak{A}_\mathrm{i}}[y_i] \rangle_{i\in I}$$

Proof. By induction on the complexity of ϕ.

Let t_i be a term. Then $[\![t_i]\!]^{\prod_{i\in I}\mathfrak{A}_\mathrm{i}}[x] = \langle [\![t_i]\!]^{\mathfrak{A}_\mathrm{i}}[y_i] \rangle_{i\in I}$, by clause 4 of Def 1.3.2 and the relationship between x and y_i.

The case of the atomic formulas are covered by clause 2 and 3 of Def 1.3.2.

The cases of sentential connectives are straightforward. For example,

$$\begin{aligned}
[\![\phi \wedge \psi]\!]^{\prod_{i\in I}\mathfrak{A}_\mathrm{i}}[x] &= [\![\phi]\!]^{\prod_{i\in I}\mathfrak{A}_\mathrm{i}}[x] \sqcap [\![\psi]\!]^{\prod_{i\in I}\mathfrak{A}_\mathrm{i}}[x] \\
&= \langle [\![\phi]\!]^{\mathfrak{A}_\mathrm{i}}[y_i] \rangle_{i\in I} \sqcap \langle [\![\psi]\!]^{\mathfrak{A}_\mathrm{i}}[y_i] \rangle_{i\in I} \\
&= \langle [\![\phi]\!]^{\mathfrak{A}_\mathrm{i}}[y_i] \sqcap^i [\![\psi]\!]^{\mathfrak{A}_\mathrm{i}}[y_i] \rangle_{i\in I} \\
&= \langle [\![\phi \wedge \psi]\!]^{\mathfrak{A}_\mathrm{i}}[y_i] \rangle_{i\in I}
\end{aligned}$$

Finally, the case of quantified formulas:

$$\begin{aligned}
[\![\forall v_j \phi]\!]^{\prod_{i\in I}\mathfrak{A}_\mathrm{i}}[x] &= \prod_{\langle a_i \rangle_{i\in I} \in \prod_{i\in I} A_i} [\![\phi]\!]^{\prod_{i\in I}\mathfrak{A}_\mathrm{i}}[x(v_j/\langle a_i \rangle_{i\in I})] \\
&= \prod_{\langle a_i \rangle_{i\in I} \in \prod_{i\in I} A_i} \langle [\![\phi]\!]^{\mathfrak{A}_\mathrm{i}}[y_i(v_j/a_i)] \rangle_{i\in I} \\
&= \langle \prod_{a_i \in A_i} [\![\phi]\!]^{\mathfrak{A}_\mathrm{i}}[y_i(v_j/a_i)] \rangle_{i\in I} \\
&= \langle [\![\forall v_j \phi]\!]^{\mathfrak{A}_\mathrm{i}}[y_i] \rangle_{i\in I}
\end{aligned}$$

And similarly for existential formulas.

□

Corollary 1.3.2.1. For any sentence ϕ in $\mathscr{L}$, $[\![\phi]\!]^{\prod_{i\in I}\mathfrak{A}_\mathrm{i}} = 1$ if and only if for any $i \in I$, $[\![\phi]\!]^{\mathfrak{A}_\mathrm{i}} = 1$.

1.3.2 Quotient Models

Definition 1.3.3. Let B, C be Boolean algebras. Then a function $h : B \to C$ is a homomorphism just in case for any $p, q \in B$, $h(p \sqcap^B q) = h(p) \sqcap^C h(q)$ and $h(-^B p) = -^C h(p)$.

It is easy to show that when $h : B \to C$ is a homomorphism, for any $p, q \in B$, $h(p \sqcup^B q) = h(p) \sqcup^C h(q)$, $h(p) \leqslant^C h(q)$ if $p \leqslant^B q$, $h(1_B) = 1_C$, and $h(0_B) = 0_C$.

Definition 1.3.4. Let B, C be Boolean algebras. Let $h : B \to C$ be a homomorphism. Then h is a complete homomorphism just in case for any $D \subseteq B$ that has a supremum, $h[D]$ has a supremum in C and $h(\bigsqcup^B D) = \bigsqcup^C h[D]$.

Similarly, it is easy to show that when $h : B \to C$ is a complete homomorphism, for any $D \subseteq B$ that has a infimum, $h[D]$ has a infimum in C and $h(\bigsqcap^B D) = \bigsqcap^C h[D]$.

Definition 1.3.5. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. Let C be a complete Boolean algebra. Let $h : B \to C$ be a homomorphism. Then the C-valued quotient model $\mathfrak{A}^h$ of $\mathscr{L}$ is defined as follows:

1. Universe:

   Let $a_1, a_2 \in A$, define $a_1 \equiv_h a_2$ iff $h(\llbracket a_1 = a_2 \rrbracket^{\mathfrak{A}}) = 1_C$.

   It is easy to show that $\equiv_h$ is an equivalence relation on A^2, using Def 1.2.8.

   Given $a_i \in A$, let $[a_i]_h = \{a_j \in A \mid a_i \equiv_h a_j\}$. Let the universe of $\mathfrak{A}^h$ be $A^h = \{[a_i]_h \mid a_i \in A\}$.

2. $\llbracket = \rrbracket^{\mathfrak{A}^h} : A^h \times A^h \to C$ is the function such that for any $[a_1]_h, [a_2]_h \in A^h$,

   $$\llbracket [a_1]_h = [a_2]_h \rrbracket^{\mathfrak{A}^h} = h(\llbracket a_1 = a_2 \rrbracket^{\mathfrak{A}})$$

   It is easy to show that $\llbracket = \rrbracket^{\mathfrak{A}^h}$ is well-defined, using Def 1.2.8 and Def 1.3.3.

3. Let P be an n-ary relation in $\mathscr{L}$. $\llbracket P \rrbracket^{\mathfrak{A}^h} : (A^h)^n \to C$ is the function such that for any $\langle [a_1]_h, ..., [a_n]_h \rangle \in (A^h)^n$,

   $$\llbracket P([a_1]_h, ..., [a_n]_h) \rrbracket^{\mathfrak{A}^h} = h(\llbracket P(a_1, ..., a_n) \rrbracket^{\mathfrak{A}})$$

Similarly, it is easy to show that $[\![=]\!]^{\mathfrak{A}^h}$ is well-defined, using Def 1.2.8 and Def 1.3.3.

4. Let c be a constant in $\mathscr{L}$. $[\![c]\!]^{\mathfrak{A}^h} = [[\![c]\!]^{\mathfrak{A}}]_h$.

Lemma 1.3.2.1. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. Let $h : B \to C$ be a complete homomorphism. Let x, x' be assignments on $\mathfrak{A}$ such that for any $v_i \in Var$, $x(v_i) \equiv_h x'(v_i)$. Then, for any formula ϕ of $\mathscr{L}$,

$$h([\![\phi]\!]^{\mathfrak{A}}[x]) = h([\![\phi]\!]^{\mathfrak{A}}[x'])$$

Proof. By induction on the complexity of ϕ. The base cases hold by Def 1.2.8 and Def 1.3.3. The inductive cases for sentential connective are straightforward, again using Def 1.2.8 and Def 1.3.3. The inductive cases for quantified formulas make use of the fact that h is a complete homomorphism. For example,

$$\begin{aligned} h([\![\forall v_i \phi]\!]^{\mathfrak{A}}[x]) &= h(\prod_{a \in A} [\![\phi]\!]^{\mathfrak{A}}[x(v_i/a)]) \\ &= \prod_{a \in A} h([\![\phi]\!]^{\mathfrak{A}}[x(v_i/a)]) \\ &= \prod_{a \in A} h([\![\phi]\!]^{\mathfrak{A}}[x'(v_i/a)]) \\ &= h(\prod_{a \in A} [\![\phi]\!]^{\mathfrak{A}}[x'(v_i/a)]) \\ &= h([\![\forall v_i \phi]\!]^{\mathfrak{A}}[x']) \end{aligned}$$

□

Theorem 1.3.3. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. Let $h : B \to C$ be a complete homomorphism. Let $\mathfrak{A}^h$ be the C-valued quotient model as defined in Def 1.3.5. Given $x : Var \to A^h$ an arbitrary assignment on $\mathfrak{A}^h$, let $y : Var \to A$ be an assignment on $\mathfrak{A}$ such that for any $v_i \in Var$, $y(v_i) \in x(v_i)$. Then, for any formula ϕ in $\mathscr{L}$,

$$[\![\phi]\!]^{\mathfrak{A}^h}[x] = h([\![\phi]\!]^{\mathfrak{A}}[y])$$

Proof. By induction on the complexity of ϕ.

Terms: Let t be a term. By the relationship between x and y and Def 1.3.5, it is easy to show that $[\![t]\!]^{\mathfrak{A}}[y] \in [\![t]\!]^{\mathfrak{A}^h}[x]$.

Base cases: Let t_1, t_2 be terms. Suppose $[\![t_1]\!]^{\mathfrak{A}^h}[x] = [a_i]_h \in A^h$ and $[\![t_2]\!]^{\mathfrak{A}^h}[x] = [a_j]_h \in A^h$.

$$[\![t_1 = t_2]\!]^{\mathfrak{A}^h}[x] = [\![[a_i]_h = [a_j]_h]\!]^{\mathfrak{A}^h} = h([\![a_i = a_j]\!]^{\mathfrak{A}}) = h([\![t_1 = t_2]\!]^{\mathfrak{A}}[y])$$

The last equation holds because of Lemma 1.3.2.1 and the fact that $[\![t_1]\!]^{\mathfrak{A}}[y] \equiv_h a_i$ and $[\![t_2]\!]^{\mathfrak{A}}[y] \equiv_h a_j$.

The other base case is very similar.

Inductive cases: The cases of sentential connectives are straightforward. For example,

$$\begin{aligned}
[\![\phi \wedge \psi]\!]^{\mathfrak{A}^h}[x] &= [\![\phi]\!]^{\mathfrak{A}^h}[x] \sqcap^C [\![\psi]\!]^{\mathfrak{A}^h}[x] \\
&= h([\![\phi]\!]^{\mathfrak{A}}[y]) \sqcap^C h([\![\psi]\!]^{\mathfrak{A}}[y]) \\
&= h([\![\phi]\!]^{\mathfrak{A}}[y] \sqcap^B [\![\psi]\!]^{\mathfrak{A}}[y]) \\
&= h([\![\phi \wedge \psi]\!]^{\mathfrak{A}}[y])
\end{aligned}$$

Finally, the cases of quantified formulas. Again, we make use of the fact that h is a complete homomorphism.

$$\begin{aligned}
[\![\forall v_i \phi]\!]^{\mathfrak{A}^h}[x] &= \prod_{[a]_h \in A^h} [\![\phi]\!]^{\mathfrak{A}^h}[x(v_i/[a]_h)] \\
&= \prod_{[a]_h \in A^h} h([\![\phi]\!]^{\mathfrak{A}}[y(v_i/a)]) \\
&= \prod_{a \in A} h([\![\phi]\!]^{\mathfrak{A}}[y(v_i/a)]) \\
&= h(\prod_{a \in A} [\![\phi]\!]^{\mathfrak{A}}[y(v_i/a)]) \\
&= h([\![\forall v_i \phi]\!]^{\mathfrak{A}}[y])
\end{aligned}$$

The third equation holds because of Lemma 1.3.2.1. The fourth equation holds because h is a complete homomorphism. The case of existential formulas is very similar.

□

Boolean valued models and traditional two-valued models are similar in many important ways, but they also have some interesting differences. In particular, some significant features of two-valued models are not shared by all Boolean valued models. For example, two-valued models are "witnessing", in the following sense: if an existential sentence $\exists v_i \phi$ has value 1 in a two-valued model, then there has to some object a in the universe of the model such that $\phi[a]$ has value 1; similarly, if $\exists v_i \phi$ has value 0, then there has to be some object a in the universe of the model such that $\phi[a]$ has value 0. Nevertheless, there can be Boolean valued models that fail to have this feature. There can be, for example, a B-valued model in which the sentence $\exists v_i \phi$ has the value $p \in B$, but for no object a of the universe, $\phi[a]$ has value p.

It turns out many important results in traditional model theory essentially reply on the fact that two-valued models are witnessing. Therefore, these results cannot be generalized to all Boolean valued models, but only to those that are similar to two-valued models in this respect. To this end, we introduce the following definition on Boolean valued models.

Definition 1.3.6. Let $\mathfrak{A}$ be a B-valued model for the language $\mathscr{L}$. Then $\mathfrak{A}$ is *witnessing*[8] just in case for any formula $\phi(u, v_1, ..., v_n)$[9] of $\mathscr{L}$, any $a_1, ..., a_n \in A$, there is an $a \in A$ such that[10]

$$[\![\exists u \phi(u, v_1, ..., v_n)]\!]^{\mathfrak{A}}[a_1, ..., a_n] = [\![\phi(u, v_1, ..., v_n)]\!]^{\mathfrak{A}}[a, a_1, ..., a_n]$$

Lemma 1.3.3.1. Let $\mathfrak{A}$ be a witnessing B-valued model of $\mathscr{L}$. Let $h : B \to C$ be a homomorphism. Let x, x' be assignments on $\mathfrak{A}$ such that for any $v_i \in Var$, $x(v_i) \equiv_h x'(v_i)$. Then, for any formula ϕ of $\mathscr{L}$,

$$h([\![\phi]\!]^{\mathfrak{A}}[x]) = h([\![\phi]\!]^{\mathfrak{A}}[x'])$$

[8]Some people, including Hamkins and Seabold [16] and Jech [19], call witnessing models "full" models instead. In this dissertation, we will also discuss full models, where full models are defined as those in which there is an "upper" element for each antichain and each sequence of elements of the same length (see Section 1.7). This definition of full models is shared by Bell [3] and some others. A hidden misunderstanding on this subject seems to be that these two definition coincide. But in fact they do not. Full models, defined in terms of antichains, are all witnessing models, yet the converse does not hold, as we will show in Section 1.7.

[9]Here and in the following, when we write something like $\phi(v_1, ..., v_m)$, we mean that ϕ is a formula with at most $v_1, ..., v_m$ as its free variables.

[10]If ϕ is a formula with at most $v_1, ..., v_m$ as its free variables, and if x and y are two assignments on a B-valued model $\mathfrak{A}$ such that for any $1 \leqslant i \leqslant m$, $x_i = y_i$, then it is easy to show that $[\![\phi]\!]^{\mathfrak{A}}[x] = [\![\phi]\!]^{\mathfrak{A}}[y]$. Hence, when the context is clear, we can simply write $[\![\phi]\!]^{\mathfrak{A}}[x_1, ..., x_m]$ to denote $[\![\phi]\!]^{\mathfrak{A}}[x]$ (or equivalently, $[\![\phi]\!]^{\mathfrak{A}}[x]$).

Proof. By induction on the complexity of ϕ. The base cases and the cases for sentential connectives are exactly the same as in Lemma 1.3.2.1. The only cases worth mentioning are the cases for quantified formulas. Let $\phi = \exists v_i \psi$. Then, since $\mathfrak{A}$ is witnessing, for some $a \in A$, $[\![\exists v_i \psi]\!]^{\mathfrak{A}}[x] = [\![\psi]\!]^{\mathfrak{A}}[x(v_i/a)]$. Hence $h([\![\exists v_i \psi]\!]^{\mathfrak{A}}[x]) = h([\![\psi]\!]^{\mathfrak{A}}[x(v_i/a)])$. By inductive hypothesis,

$$h([\![\psi]\!]^{\mathfrak{A}}[x(v_i/a)]) = h([\![\psi]\!]^{\mathfrak{A}}[x'(v_i/a)]) \leqslant h([\![\exists v_i \psi]\!]^{\mathfrak{A}}[x'])$$

Similarly, $h([\![\exists v_i \psi]\!]^{\mathfrak{A}}[x']) \leqslant h([\![\exists v_i \psi]\!]^{\mathfrak{A}}[x])$. The case for universal formals is very alike.

□

Theorem 1.3.4 (Generalized Łoś Theorem)**.** Let $\mathfrak{A}$ be a witnessing B-valued model of $\mathscr{L}$. Let $h : B \to C$ be a homomorphism. Let $\mathfrak{A}^h$ be the C-valued quotient model as defined in Def 1.3.5. Given $x : Var \to A^h$ an arbitrary assignment on $\mathfrak{A}^h$, let $y : Var \to A$ be an assignment on $\mathfrak{A}$ such that for any $v_i \in Var$, $y(v_i) \in x(v_i)$. Then, for any formula ϕ in $\mathscr{L}$,

$$[\![\phi]\!]^{\mathfrak{A}^h}[x] = h([\![\phi]\!]^{\mathfrak{A}}[y])$$

Proof. By induction on the complexity of ϕ. Again, the base cases and the cases for sentential connectives are exactly the same as in Theorem 1.3.3. Let $\phi = \exists v_i \psi$.

$$\begin{aligned}
[\![\exists v_i \psi]\!]^{\mathfrak{A}^h}[x] &= \bigsqcup_{[a]_h \in A^h} [\![\psi]\!]^{\mathfrak{A}^h}[x(v_i/[a]_h)] \\
&= \bigsqcup_{[a]_h \in A^h} h([\![\psi]\!]^{\mathfrak{A}}[y(v_i/a)]) \\
&\leqslant \bigsqcup_{a \in A} h([\![\psi]\!]^{\mathfrak{A}}[y(v_i/a)]) \\
&\leqslant h(\bigsqcup_{a \in A} [\![\psi]\!]^{\mathfrak{A}}[y(v_i/a)]) \\
&= h([\![\exists v_i \psi]\!]^{\mathfrak{A}}[y])
\end{aligned}$$

The fourth inequation holds because since h is a homomorphism, for any subset $D \subseteq B$,

$\bigsqcup h[D] \leqslant h(\bigsqcup D)$.[11]

For the other direction, since $\mathfrak{A}$ is witnessing, for some $a \in A$, $[\![\exists v_i \psi]\!]^{\mathfrak{A}}[y] = [\![\psi]\!]^{\mathfrak{A}}[y(v_i/a)]$. Hence,

$$\begin{aligned} h([\![\exists v_i \psi]\!]^{\mathfrak{A}}[y]) &= h([\![\psi]\!]^{\mathfrak{A}}[y(v_i/a)]) \\ &= [\![\psi]\!]^{\mathfrak{A}^h}[x(v_i/[a]_h)] \\ &\leqslant [\![\exists v_i \psi]\!]^{\mathfrak{A}^h}[x] \end{aligned}$$

The case for universal formulas is very similar.

□

We conclude this section with some remarks on the two constuctions introduced in this section. A well-known key method of constructing new models from old ones in model theory is the ultraproduct construction. Given $\{\mathfrak{A}_i \mid i \in I\}$ a set of (two-valued) models indexed by I and D an ultrafilter on the Boolean algebra formed on $P(I)$, we can construct the ultraproduct $\prod_D \mathfrak{A}_{\mathrm{i}}$.[12] Now, the ultraproduct construction on two-valued models can be shown to be just a special case of our direct product construction and quotient construction on Boolean valued models. Two-valued models are special cases of Boolean valued models, and hence both kinds of constructions can be applied to them. Given a set of (two-valued) models $\{\mathfrak{A}_i \mid i \in I\}$ indexed by I, we can first take their direct product $\prod_{i\in I} \mathfrak{A}_{\mathrm{i}}$, as defined in Def 1.3.2, which is a $\prod_{i\in I} 2$ valued model, where $\prod_{i\in I} 2$ is the product algebra of $|I|$ many copies of the two element Boolean algebra 2.

Next, let D be an ultrafilter on $P(I)$. It is easy to show that the Boolean algebra $P(I)$ and $\prod_{i\in I} 2$ are isomorphic. (Each element of $\prod_{i\in I} 2$ can be thought of as the characteristic function of a subset of I.) Hence we can take D as a ultrafilter on $\prod_{i\in I} 2$. Let $h_D : \prod_{i\in I} 2 \to 2$ be the characteristic function of D. h is then, a homomorphism, by a well-known result on Boolean algebras.[13] We can then construct the quotient model $(\prod_{i\in I} \mathfrak{A}_{\mathrm{i}})^{h_D}$ as defined in Def 1.3.5. This quotient model, $(\prod_{i\in I} \mathfrak{A}_{\mathrm{i}})^{h_D}$, can be easily shown to be the same model as the ultraproduct $\prod_D \mathfrak{A}_{\mathrm{i}}$.

[11]Note that the other direction might not hold, if h is not a complete homomorphism.

[12]For a detailed definition, see, for example, Chang and Keisler [7, p. 215-216].

[13]See, for example, Givant and Halmos [14, p. 158].

One of the most important theorems on ultraproducts is the fundamental theorem of ultraproducts (or equivalently, Łoś Theorem), which states that given an ultraproduct $\prod_D \mathfrak{A}_i$, a formula $\phi(v_1, ..., v_n)$ of $\mathscr{L}$, and $[\langle a_i^1 \rangle_{i\in I}]_D, ..., [\langle a_i^n \rangle_{i\in I}]_D \in \prod_D A_i$, $\prod_D \mathfrak{A}_i \models \phi[[\langle a_i^1 \rangle_{i\in I}]_D$, ..., $[\langle a_i^n \rangle_{i\in I}]_D]$ iff $\{i \in I \,|\, \mathfrak{A}_i \models \phi[a_i^1, ..., a_i^n]\} \in D$.[14] This theorem, it turns out, is a special case of Theorem 1.3.4, the Generalized Łos's Theorem, as its name may suggest. This is because, firstly, $\prod_{i\in I} \mathfrak{A}_i$, the direct product of the $\mathfrak{A}_i$'s, is a witnessing model, as any direct product of a set of witnessing models is also witnessing:

Lemma 1.3.4.1. Let I be an index set. For each $i \in I$, let $\mathfrak{C}_i$ be a witnessing B_i-valued model of $\mathscr{L}$. Then their direct product $\prod\limits_{i\in I} \mathfrak{C}_i$ is a witnessing $\prod\limits_{i\in I} B_i$-valued model of $\mathscr{L}$.

Proof. For simplicity we ignore the parameters. Let $\phi(v_n)$ be a formula of $\mathscr{L}$. For any $i \in I$, since $\mathfrak{C}_i$ is witnessing, for some $a_i \in C_i$, $[\![\exists v_n \phi(v_n)]\!]^{\mathfrak{C}_i} = [\![\phi(v_n)]\!]^{\mathfrak{C}_i}[a_i]$. Then $\langle a_i \rangle_{i\in I} \in \prod\limits_{i\in I} C_i$ is a witness of $\exists v_n \phi(v_n)$ in $\prod\limits_{i\in I} \mathfrak{C}_i$, by Theorem 1.3.2.

□

Since every $\mathfrak{A}_i$ is a two-valued model, every $\mathfrak{A}_i$ is witnessing. Hence their direct product $\prod_{i\in I} \mathfrak{A}_i$ is witnessing, by the above lemma. Hence it is in the scope of Theorem 1.3.4. Let $h_D : \prod_{i\in I} 2 \to 2$ be the characteristic function of the ultrafilter D. We have already argued that the quotient model $(\prod_{i\in I} \mathfrak{A}_i)^{h_D}$ is the same as the ultraproduct $\prod_D \mathfrak{A}_i$. Also it is easy to see that any $[\langle a_i \rangle_{i\in I}]_D \in \prod_D A_i$ is the same as $[\langle a_i \rangle_{i\in I}]_{h_D} \in (\prod_{i\in I} A_i)^{h_D}$. According to Theorem 1.3.4, then,

$$[\![\phi]\!]^{(\prod_{i\in I} \mathfrak{A}_i)^{h_D}}[[\langle a_i^1 \rangle_{i\in I}]_{h_D}, ..., [\langle a_i^n \rangle_{i\in I}]_{h_D}] = h_D([\![\phi]\!]^{\prod_{i\in I} \mathfrak{A}_i}[\langle a_i^1 \rangle_{i\in I}, ..., \langle a_i^n \rangle_{i\in I}])$$

But $[\![\phi]\!]^{\prod_{i\in I} \mathfrak{A}_i}[\langle a_i^1 \rangle_{i\in I}, ..., \langle a_i^n \rangle_{i\in I}] = \langle [\![\phi]\!]^{\mathfrak{A}_i}[a_i^1, ..., a_i^n] \rangle_{i\in I}$, by Theorem 1.3.2. And hence $h_D([\![\phi]\!]^{\prod_{i\in I} \mathfrak{A}_i}[\langle a_i^1 \rangle_{i\in I}, ..., \langle a_i^n \rangle_{i\in I}]) = 1$ just in case $h_D(\langle [\![\phi]\!]^{\mathfrak{A}_i}[a_i^1, ..., a_i^n] \rangle_{i\in I}) = 1$, just in case $\{i \in I \,|\, \mathfrak{A}_i \models \phi[a_i^1, ..., a_i^n]\} \in D$.

The two major theorems (Theorem 1.3.3 and Theorem 1.3.4) are interesting for the following reasons. First, they show that what essentially makes Łoś Theorem hold on ultraproducts is the fact that it is the quotient model of a *witnessing* model. We see that

[14]For a detailed proof of Łoś Theorem, see, for example, Chang and Keisler [7, p. 217-219].

Theorem 1.3.4 only holds on witnessing Boolean valued models, and if a B-valued model $\mathfrak{A}$ is not witnessing, then we only have the weakened theorem (Theorem 1.3.3) that requires $h : B \to C$ to be a complete homomorphism. In the special case when $h : B \to 2$, h is then the characteristic function of a complete ultrafilter on B. This is, indeed, a quite serious limitation, as complete ultrafilters are usually rare. Also, a number of interesting results on ultraproducts that follow from Łoś Theorem depend on the ultrafilters used to build these ultraproducts being incomplete to certain degrees. Hence, these results cannot be generalized to random Boolean valued models, but only to Boolean valued models that share certain important features with two-valued models, like the feature of being witnessing, as Łoś Theorem can only be generalized to the latter but not the former.

1.4 Theories and Boolean Valuations

1.4.1 Theories

We construct Boolean valued models as models for first order languages. It is natural, then, to wonder about the relationship between first-order theories and Boolean valued models. In this subsection we intend to answer these questions[15].

Definition 1.4.1. Let T be a theory in a first order language $\mathscr{L}$. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. $\mathfrak{A}$ is a model of T just in case for any $\phi \in T$, $[\![\phi]\!]^{\mathfrak{A}} = 1_B$.

Definition 1.4.2. Let T be a theory and ϕ be a sentence in a first order language $\mathscr{L}$. ϕ is a Boolean-consequence of T, in symbols, $T \models_B \phi$ just in case for any Boolean valued model $\mathfrak{A}$, if $\mathfrak{A}$ is a model of T, then $\mathfrak{A}$ is a model of ϕ.

Theorem 1.4.1. Let T be a theory and ϕ be a sentence in a first order language $\mathscr{L}$. If $T \vdash \phi$, then $T \models_B \phi$.

Proof. We can prove this by showing that all the axioms of first order logic have value 1 in every Boolean valued model, and that the rules of inference always preserve truth.

[15] To my knowledge, the theorem that Boolean-valued models are sound and complete with respect to first-order theories first appeared in [29].

The proof that all the sentential axioms have value 1 is straightforward. For example, (let $x : Var \to A$ be an assignment), $[\![(\phi \wedge \psi) \to \phi]\!]^{\mathfrak{A}}[x] = 1$ iff $[\![\phi \wedge \psi]\!]^{\mathfrak{A}}[x] \leqslant [\![\phi]\!]^{\mathfrak{A}}[x]$ iff $[\![\phi]\!]^{\mathfrak{A}}[x] \sqcap [\![\psi]\!]^{\mathfrak{A}}[x] \leqslant [\![\phi]\!]^{\mathfrak{A}}[x]$. But the latter is true in every Boolean algebra. The cases of the other sentential axioms are very similar.

That the identity axioms always have value 1 follows straightforwardly from the clauses on the identity symbol in Def 1.2.8 and Theorem 1.2.1.

For the quantifier axioms, let ϕ and ψ be formulas. For the first axiom, suppose v_i is a variable that is not free in ϕ, we want to show that for any assignment $x : Var \to A$, $[\![\forall v_i(\phi \to \psi) \to (\phi \to \forall v_i \psi)]\!]^{\mathfrak{A}}[x] = 1$. This is the case iff $[\![\forall v_i(\phi \to \psi)]\!]^{\mathfrak{A}}[x] \leqslant [\![\phi \to \forall v_i \psi]\!]^{\mathfrak{A}}[x]$. But

$$
\begin{aligned}
[\![\forall v_i(\phi \to \psi)]\!]^{\mathfrak{A}}[x] &= \prod_{a \in A} [\![\phi \to \psi]\!]^{\mathfrak{A}}[x(v_i/a)] \\
&= \prod_{a \in A} -[\![\phi]\!]^{\mathfrak{A}}[x(v_i/a)] \sqcup [\![\psi]\!]^{\mathfrak{A}}[x(v_i/a)] \\
&= \prod_{a \in A} -[\![\phi]\!]^{\mathfrak{A}}[x] \sqcup [\![\psi]\!]^{\mathfrak{A}}[x(v_i/a)] \\
&= -[\![\phi]\!]^{\mathfrak{A}}[x] \sqcup \prod_{a \in A} [\![\psi]\!]^{\mathfrak{A}}[x(v_i/a)] \\
&\leqslant [\![\phi \to \forall v_i \psi]\!]^{\mathfrak{A}}[x]
\end{aligned}
$$

The third equation holds as v_i is not free in ϕ. For the second quantifier axiom, let ψ be obtained from ϕ by freely substituting each free occurrence of v_i in ϕ by the term t, such that no variable v_j in t will occur bound in ψ at the place where it is introduced. We want to show that for any assignment $x : Var \to A$, $[\![\forall v_i \phi \to \psi]\!]^{\mathfrak{A}}[x] = 1$. This is just in case $[\![\forall v_i \phi]\!]^{\mathfrak{A}}[x] \leqslant [\![\psi]\!]^{\mathfrak{A}}[x]$, which is just in case $\prod_{a \in A} [\![\phi]\!]^{\mathfrak{A}}[x(v_i/a)] \leqslant [\![\psi]\!]^{\mathfrak{A}}[x]$. But the latter is always true, as $[\![\psi]\!]^{\mathfrak{A}}[x] = [\![\phi]\!]^{\mathfrak{A}}[x(v_i/a')]$, where $a' = [\![t]\!]^{\mathfrak{A}}[x] \in A$.

Moving on to the rules of inferences. We start with *Modus Ponens*. Suppose both $[\![\phi]\!]^{\mathfrak{A}}[x]$ and $[\![\phi \to \psi]\!]^{\mathfrak{A}}[x]$ are 1. The latter means that $[\![\phi]\!]^{\mathfrak{A}}[x] \leqslant [\![\psi]\!]^{\mathfrak{A}}[x]$, and since $[\![\phi]\!]^{\mathfrak{A}}[x] = 1$, $[\![\psi]\!]^{\mathfrak{A}}[x] = 1$.

For *Universal Generalization*, we suppose for any assignment x, $[\![\phi]\!]^{\mathfrak{A}}[x] = 1$. Then it follows straightforwardly that $[\![\forall v_i \phi]\!]^{\mathfrak{A}}[x] = \prod_{a \in A} [\![\phi]\!]^{\mathfrak{A}}[x(v_i/a)] = 1$.

□

Corollary 1.4.1.1. Let ϕ be a theorem of first order logic. Then in any Boolean valued model $\mathfrak{A}$, $[\![\phi]\!]^{\mathfrak{A}} = 1$.

Theorem 1.4.2. Let T be a theory in $\mathscr{L}$. T is consistent if and only if for some complete Boolean Algebra B, T has a B-valued model $\mathfrak{A}$.

Proof. For the left to right direction, if T is consistent, then by the Completeness Theorem on two-valued models, T has a two-valued model. But a two-valued model is a Boolean valued model.

For the right to left direction, suppose T is inconsistent. Then for some theorem ϕ of first order logic, $T \vdash \neg\phi$. Assume for reductio that T has a B-valued model $\mathfrak{A}$, then by Theorem 1.4.1, $[\![\neg\phi]\!]^{\mathfrak{A}} = 1$. Hence $[\![\phi]\!]^{\mathfrak{A}} = 0$, but this contradicts Corollary 1.4.1.1.

□

Corollary 1.4.2.1. Let B be any complete Boolean algebra. A theory T has a B-valued model just in case every finite subset of T has a B-valued model.

Theorem 1.4.3. Let T be a theory and ϕ be a sentence in a first order language $\mathscr{L}$. If $T \models_B \phi$, then $T \vdash \phi$.

Proof. Suppose $T \models_B \phi$, then for any two-valued model $\mathfrak{A}$, if $\mathfrak{A}$ is a model of T, then $\mathfrak{A}$ is a model of ϕ. By the soundness theorem on two-valued models[16], $T \vdash \phi$. □

Corollary 1.4.3.1. Let T be a theory and ϕ be a sentence in a first order language $\mathscr{L}$. $T \models_B \phi$ if and only if $T \vdash \phi$.

1.4.2 Boolean Valuations

When there are only two truth values, the notion of "theory" is sufficient for describing the relationship between models and sentences. Given a two-valued model of a language $\mathscr{L}$, the set of all sentences of $\mathscr{L}$ that are true in the model forms a complete theory in $\mathscr{L}$. This theory decides the value of all sentences of $\mathscr{L}$ in the model: if ϕ is a member of the theory, then ϕ has value 1 in the model, and if ϕ is not a member of the theory, then ϕ has value 0

[16]See, for example, Chang and Keisler [7, p. 66].

in the model. This theory, in a certain sense, provides a full description of the model given that our expressive power is limited to $\mathscr{L}$.

The situation is different, however, when we allow more than two truth values. Given a B-valued model of $\mathscr{L}$ where B is a proper extension of 2, the theory in $\mathscr{L}$ that consists of all sentences of $\mathscr{L}$ that are true in the model no longer decides the value of all sentences of $\mathscr{L}$ in the model. A simple example to illustrate this point is as follows: Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two B-valued models of $\mathscr{L}$, where B is the four element Boolean algebra $\{0, p, -p, 1\}$ and $\mathscr{L}$ is the language $\{P, c\}$ where P is a unary predicate and c is a constant. Let $A = \{a\}$ and $A' = \{a'\}$. Let $[\![c]\!]^{\mathfrak{A}} = a$ and $[\![c]\!]^{\mathfrak{A}'} = a'$. Let $[\![P]\!]^{\mathfrak{A}}(a) = p$ and $[\![P]\!]^{\mathfrak{A}'}(a') = -p$. Then it is easy to see that the set of sentences of $\mathscr{L}$ that have value 1 in $\mathfrak{A}$ is the same as the set of sentences of $\mathscr{L}$ that have value 1 in $\mathfrak{A}'$. But obviously not all sentences of $\mathscr{L}$ have the same value in $\mathfrak{A}$ and $\mathfrak{A}'$.

This result is hardly surprising. Knowing which sentences have the top value only allows us to know the values of those sentences that have extreme values. When we only have two values, this amounts to knowing the value of every sentence. But when we have more than two values, knowing the values of those that have extreme values is not enough: we still need to know the values of those that have intermediate values. And the latter is simply not decided by the former.

Therefore, in a Boolean-valued setting, we need a notion stronger than the notion of “theory", one that is sufficiently strong to fulfill the kind of jobs that the notion of “theory" does in the setting of two-valued models: one that is able to, for example, provide a full description of a model that decides the value of every sentence in the model. A natural candidate, as we will introduce right now, is the notion of “Boolean-valuations".

Definition 1.4.3. Let B be a complete Boolean algebra. Let $\mathscr{L}$ be a first order language. A Boolean-valuation S^B in $\mathscr{L}$ is a set of pairs of the form $\langle \phi, p \rangle$ such that ϕ is a sentence of $\mathscr{L}$ and p is an element of B. We say that B is the value range of the Boolean valuation S^B.

Definition 1.4.4. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. Let S^B be a $\mathscr{L}$-Boolean-valuation with value range B.[17] $\mathfrak{A}$ is a model of S^B just in case for any sentence $\phi \in \mathscr{L}$, for any $p \in B$, if

[17] Here and in the following, we use the superscript of a Boolean-valuation to indicate the value range of the Boolean-valuation.

$\langle \phi, p \rangle \in S^B$, then $[\![\phi]\!]^{\mathfrak{A}} = p$.

Intuitively, a Boolean-valuation assigns values of a Boolean algebra to certain sentences of a language. When a pair $\langle \phi, p \rangle$ is in the Boolean-valuation S^B, we can think of the Boolean-valuation as "saying" that the sentence ϕ has value p. If a model $\mathfrak{A}$ is a model of S^B, then figuratively, what S^B says about those sentences that are mentioned in S^B is what actually is the case in $\mathfrak{A}$. We can already see why the notion of Boolean-valuations will be useful for our purpose: a full description of a Boolean-valued model with respect to a particular language, intuitively, is simply an assignment of values to all the sentences in the language. But the latter, from a set-theoretic perspective, is just a collection of sentence-value pairs, which is simply a Boolean-valuation given our definition.

Also, theories, in a natural sense, can be understood as special cases of Boolean-valuations. Roughly, a theory T is a Boolean valuation $T^B = \{\langle \phi, 1 \rangle \mid \phi \in T\}$. A model $\mathfrak{A}$ is a model of T just in case $\mathfrak{A}$ is a model of T^B. The notion of "Boolean-valuation" is a natural generalization of the notion of "theory", in the context of Boolean valued models.

An important property of theories is consistency. Consistent theories, as we have seen, precisely correspond to theories that have Boolean valued models. This is a nice synergy between syntax and semantics. But what about Boolean-valuations? What does it mean for a Boolean-valuation to be "consistent"? Are consistent Boolean-valuations precisely those that have models? These are the questions that we will answer in the rest of the section.

Definition 1.4.5. Let S^B be a Boolean-valuation of $\mathscr{L}$. Let $h : B \to 2$ be a homomorphism. S^B_h is the following set of sentences: for any $\phi \in \mathscr{L}$, any $p \in B$,

1. If $\langle \phi, p \rangle \in S^B$ and $h(p) = 1$, then $\phi \in S^B_h$.

2. If $\langle \phi, p \rangle \in S^B$ and $h(p) = 0$, then $\neg\phi \in S^B_h$.

3. Nothing else is in S^B_h.

Definition 1.4.6. A Boolean-valuation S^B is consistent if and only if for any homomorphism $h : B \to 2$, S^B_h is a consistent theory.

Consistency of Boolean-valuations is thus defined in terms of consistency of theories. Let T be a theory and let T^B be the Boolean-valuation $\{\langle\phi,1\rangle \mid \phi \in T\}$. It follows straightforwardly from Def 1.4.5 and Def 1.4.6 that T is consistent just in case T^B is consistent in the sense of Def 1.4.6, as every homomorphism takes 1_B to 1_2.

The major result of this section will be that consistent Boolean-valuations are precisely those that have models. To reach that result, though, we will have to prove a series of subsidiary theorems first, which are also interesting on their own. In the following, whenever we mention a Boolean-valuation, we always assume that it is a Boolean-valuation of the language $\mathscr{L}$. Also, occasionally, we will call a Boolean-valuation S^B a B-valuation.

Definition 1.4.7. A Boolean-valuation S'^B is a sub-valuation of S^B if and only if $S'^B \subseteq S^B$ and the value range of S'^B is the same as that of S^B.

Theorem 1.4.4. If a Boolean-valuation S^B is consistent , then every sub-valuation of S^B is consistent.

Proof. Let S'^B be a sub-valuation of S^B . Then for every homomorphism $h : B \to 2$, $S'^B_h \subseteq S^B_h$. If S'^B is inconsistent, then S'^B_h is inconsistent for some homomorphism h, and then S^B_h will be inconsistent.

□

Proposition 1.4.1. Let S^B be a Boolean-valuation and let $h : B \to 2$ be a homomorphism. For any finite subset $\Delta \subseteq S^B_h$, for some finite sub-valuation S'^B of S^B, $S'^B_h = \Delta$.

Theorem 1.4.5. A Boolean-valuation S^B is consistent if and only if every finite sub-valuation of S^B is consistent.

Proof. The direction from left to right follows directly from Theorem 1.4.4.

For the other direction, let S^B be an inconsistent B-valuation. Then for some homomorphism $h : B \to 2$, S^B_h is inconsistent. Hence some finite subset T of S^B_h is inconsistent. By Prop 1.4.1, for some finite sub-valuation T^B of S^B, $T^B_h = T$. Hence T^B_h is inconsistent. Hence T^B is inconsistent. □

Theorem 1.4.6. Let S^B be a consistent B-valuation. For any sentence $\psi \in \mathscr{L}$, for some $r \in B$, $S^B \cup \{\langle\psi, r\rangle\}$ is consistent.

Proof. Let $X = \{h : B \to 2 \mid h \text{ is a homomorphism}\}$.

Let $K = \{\Delta^\beta \mid \Delta^\beta$ is a finite sub-valuation of $S^B\}$. Enumerate K by α where $\alpha = |K|$. For each $\beta < \alpha$, Δ^β is a finite sub-valuation of S^B, and $S^B = \bigcup_{\beta<\alpha} \Delta^\beta$.

For any $\beta < \alpha$, $h \in X$, we form Δ_h^β according to Def 1.4.5. For any $\beta < \alpha$, $h \in X$, $\Delta_h^\beta \subseteq S_h^B$. Also for any $h \in X$, $\{\Delta_h^\beta \mid \beta < \alpha\} = \{\Delta \mid \Delta$ is a finite subset of $S_h^B\}$.

Fix an $\beta < \alpha$. Let $\Delta^\beta = \{\langle\phi_1, p_1\rangle, ..., \langle\phi_k, p_k\rangle\}$ for some $k < \omega$. For any $h \in X$, let $q_\beta^h = q_1 \sqcap ... \sqcap q_k$, where for any $1 \leqslant i \leqslant k$, $q_i = p_i$ if $h(p_i) = 1$, and $q_i = -p_i$ if $h(p_i) = 0$.

To continue with the proof we need to prove two claims.

Claim 1.4.6.1. For any $\beta < \alpha$, $h \in X$, $h(q_\beta^h) = 1$.

Proof of the Claim. Let $q_\beta^h = q_1 \sqcap ... \sqcap q_k$ as defined above. Then for any $1 \leqslant i \leqslant k$, $h(q_i) = 1$. Hence $h(q_\beta^h) = 1$.

■

Let $J_\beta^+ = \{h_j \in X \mid \Delta_{h_j}^\beta \vdash \psi\}$ and $J_\beta^- = \{h_k \in X \mid \Delta_{h_k}^\beta \vdash \neg\psi\}$.
Let $q_\beta^+ = \bigsqcup_{h_j \in J_\beta^+} q_\beta^{h_j}$ and $q_\beta^- = \bigsqcup_{h_k \in J_\beta^-} q_\beta^{h_k}$.

Claim 1.4.6.2. For some $r \in B$, $r \geqslant \bigsqcup_{\beta<\alpha} q_\beta^+$ and $-r \geqslant \bigsqcup_{\beta<\alpha} q_\beta^-$.

Proof of the Claim. We only need to show that

$$\bigsqcup_{\beta<\alpha} q_\beta^+ \sqcap \bigsqcup_{\beta<\alpha} q_\beta^- = 0$$

By infinite distribution, this is equivalent to

$$\bigsqcup_{\beta,\gamma<\alpha} (q_\beta^+ \sqcap q_\gamma^-) = 0$$

That is, for any $\beta, \gamma < \alpha$, $q_\beta^+ \sqcap q_\gamma^- = 0$, i.e.

$$\bigsqcup_{h_j \in J_\beta^+} q_\beta^{h_j} \sqcap \bigsqcup_{h_k \in J_\gamma^-} q_\gamma^{h_k} = 0$$

Again by infinite distribution, this is equivalent to

$$\bigsqcup_{h_j \in J^+_\beta} \bigsqcup_{h_k \in J^-_\gamma} (q^{h_j}_\beta \sqcap q^{h_k}_\gamma) = 0$$

That is, for any $h_j \in J^+_\beta$, any $h_k \in J^-_\gamma$, $q^{h_j}_\beta \sqcap q^{h_k}_\gamma = 0$.

Suppose not, then for some $h_j \in J^+_\beta$, $h_k \in J^-_\gamma$, for some $p \neq 0 \in B$, $q^{h_j}_\beta \sqcap q^{h_k}_\gamma = p$.

Since $p \neq 0$, there is some $h \in X$ such that $h(p) = 1$. Hence $h(q^{h_j}_\beta) = 1$, $h(q^{h_k}_\gamma) = 1$.

But by definition of $q^{h_j}_\beta$, then, for any p_i such that some pair of the form $\langle \phi_i, p_i \rangle \in \Delta^\beta$, if $h_j(p_i) = 1$, then $q^{h_j}_\beta \leqslant p_i$, and hence $h(p_i) = 1$. And similarly, if $h_j(p_i) = 0$, then $q^{h_j}_\beta \leqslant -p_i$, and hence $h(-p_i) = 1$, $h(p_i) = 0$.

Hence for any p_i such that some pair of the form $\langle \phi_i, p_i \rangle \in \Delta^\beta$, $h_j(p_i) = h(p_i)$. Hence by Def 1.4.5, $\Delta^\beta_{h_j} = \Delta^\beta_h$. Similarly, $\Delta^\gamma_{h_k} = \Delta^\gamma_h$

But since $h_j \in J^+_\beta$, $\Delta^\beta_{h_j} \vdash \psi$; and since $h_k \in J^-_\gamma$, $\Delta^\gamma_{h_k} \vdash \neg\psi$. Hence $\Delta^\beta_h \vdash \psi$, $\Delta^\gamma_h \vdash \neg\psi$.

But $\Delta^\beta_h \subseteq S^B_h$, $\Delta^\gamma_h \subseteq S^B_h$. Hence $S^B_h \vdash \psi \wedge \neg\psi$. Hence S^B_h is inconsistent. But this is a contradiction as S^B is assumed to be consistent.

■

Pick an $r \in B$ that witnesses Claim 1.4.6.2. Finally, we will show that $S^B \cup \{\langle \psi, r \rangle\}$ is consistent.

Suppose it is not consistent. Then for some $h \in X$, one of the two following situations holds:

(a) $h(r) = 1$ and $S^B_h \cup \{\psi\}$ is inconsistent.

(b) $h(r) = 0$ and $S^B_h \cup \{\neg\psi\}$ is inconsistent.

We will show that both (a) and (b) lead to contradiction.

Assume (a). Since $S^B_h \cup \{\psi\}$ is inconsistent, $S^B_h \vdash \neg\psi$. Hence for some $\beta < \alpha$, $\Delta^\beta_h \vdash \neg\psi$. Hence $h \in J^-_\beta$.

Hence $-r \geqslant \bigsqcup_{\gamma<\alpha} q^-_\gamma \geqslant q^-_\beta = \bigsqcup_{h_k \in J^-_\beta} q^{h_k}_\beta \geqslant q^h_\beta$.

But by Claim 1.4.6.1, $h(q^h_\beta) = 1$. Hence $h(-r) = 1$, $h(r) = 0$. Contradiction.

Assume (b). Since $S_h^B \cup \{\neg\psi\}$ is inconsistent, $S_h^B \vdash \psi$. Hence for some $\beta < \alpha$, $\Delta_h^\beta \vdash \psi$. Hence $h \in J_\beta^+$.

Hence $r \geqslant \bigsqcup_{\gamma<\alpha} q_\gamma^+ \geqslant q_\beta^+ = \bigsqcup_{h_j \in J_\beta^+} q_\beta^{h_j} \geqslant q_\beta^h$.

But by Claim 1.4.6.1, $h(q_\beta^h) = 1$. Hence $h(r) = 1$. Contradiction.

□

Definition 1.4.8. A Boolean-valuation S^B is maximal if and only if for every sentence ϕ, there is some $p \in B$ such that $\langle \phi, p \rangle \in S^B$.

Theorem 1.4.7. Every consistent Boolean-valuation is contained in some maximal consistent Boolean-valuation.

Proof. Let S^B be a consistent B-valuation. Let $D = \{\langle \phi, p \rangle \mid \phi$ is a sentence of $\mathscr{L}, p \in B\}$. Arrange all the pairs in D in a list:

$$\langle \phi_0, p_0 \rangle, \langle \phi_1, p_1 \rangle, \ldots, \langle \phi_\alpha, p_\alpha \rangle, \ldots \quad \alpha < |D|$$

such that the list associates in a one-one fashion an ordinal with each pair.

We shall form an increasing chain of consistent B-valuations:

$$S^B = S_0^B \subseteq S_1^B \subseteq \ldots \subseteq S_\alpha^B \subseteq \ldots \quad \alpha < |D|$$

If $S^B \cup \{\langle \phi_0, p_0 \rangle\}$ is consistent, define $S_1^B = S^B \cup \{\langle \phi_0, p_0 \rangle\}$. Otherwise, define $S_1^B = S^B$.

At the α-th stage, if α is a successor ordinal, define

$$\begin{cases} S_\alpha^B = S_{\alpha-1}^B \cup \{\langle \phi_{\alpha-1}, p_{\alpha-1} \rangle\} & \text{if } S_{\alpha-1}^B \cup \{\langle \phi_{\alpha-1}, p_{\alpha-1} \rangle\} \text{ is consistent} \\ S_\alpha^B = S_{\alpha-1}^B & \text{if otherwise} \end{cases}$$

If α is a limit ordinal, define $S_\alpha^B = \bigcup_{\beta<\alpha} S_\beta^B$. Let T^B be the union of all the S_α^B's.

Claim 1.4.7.1. T^B is a consistent B-valuation.

Proof of the Claim. Suppose not. Then for some homomorphism $h : B \to 2$, T_h^B is inconsistent. Then for some finite subset $\{\psi_1, \psi_2, \ldots, \psi_k\} \subseteq T_h^B$, $\{\psi_1, \psi_2, \ldots, \psi_k\}$ is inconsistent.

By Prop 1.4.1 , for some finite sub-valuation Δ^B of T^B, $\Delta^B_h = \{\psi_1, \psi_2, ..., \psi_k\}$. Hence Δ^B is inconsistent. But since Δ^B is finite, for some $\alpha < |D|$, $\Delta^B \subseteq S^B_\alpha$. But then S^B_α is inconsistent. Contradiction.

■

Claim 1.4.7.2. T^B is maximal.

Proof of the Claim. Let ϕ be a sentence of $\mathscr{L}$. By Theorem 1.4.6, for some $p \in B$, $T^B \cup \{\langle\phi, p\rangle\}$ is consistent. But then $\{\langle\phi, p\rangle\}$ will be added to T^B at the stage when it is enumerated.

■

Hence S^B is contained in a maximal consistent B-valuation, namely T^B.

□

When S^B is a consistent B-valuation, it is easy to show that for any sentence ϕ, for any $p, q \in B$, if $\langle\phi, p\rangle$ and $\langle\phi, q\rangle$ are both in S^B, then $p = q$. This is because otherwise, there is some homomorphism $h : B \to 2$ such that $h(p) \neq h(q)$, and hence both ϕ and $\neg\phi$ will be in S^B_h, making S^B inconsistent. Hence, in the following, when the context is clear, we will use the term $[\![\phi]\!]^S$ to denote the unique p such that $\langle\phi, p\rangle \in S^B$.

With the help of Theorem 1.4.6 and Theorem 1.4.7 we are finally able to prove the completeness theorem on Boolean-valuations.[18]

Theorem 1.4.8. Let $\mathscr{L}$ be a countable language. Let S^B be a consistent Boolean-valuation of $\mathscr{L}$. Then S^B has a B-valued model that is witnessing.

Proof. Let $X = \{h : B \to 2 \mid h \text{ is a homomorphism}\}$.

Let S^B be a consistent scheme in $\mathscr{L}$. Let C be a countable set of new constants (not appearing in $\mathscr{L}$). Let $\mathscr{L}' = \mathscr{L} \cup C$.

Arrange all formulas with one free variable in $\mathscr{L}$ into a list:

$$\phi_0, \phi_1, ..., \phi_i, ... \quad i < \omega$$

[18] Our proof of Theorem 1.4.8 is in the same spirit as Henkin's proof of the completeness theorem on two-valued first-order models.

We now define an increasing sequence of B-valuations of $\mathscr{L}$:

$$S^B = S_0^B \subseteq S_1^B \subseteq \ldots \subseteq S_i^B \subseteq \ldots \quad i < \omega$$

and a sequence $d_0, \ldots, d_i, \ldots$, $i < \omega$, of constants from C, in the following way:

Suppose S_i^B has been defined. We first add to S_i^B a pair of the form $\langle \exists v_i \phi_i(v_i), p \rangle$ such that $S_i^B \cup \{\langle \exists v_i \phi_i(v_i), p \rangle\}$ is consistent. Theorem 1.4.6 guarantees the existence of such a pair. Then, we let d_i be the first constant in C that has not appeared in $S_i^B \cup \{\langle \exists v_i \phi_i(v_i), p \rangle\}$. Since until S_i^B we have only added finitely many pairs to S^B, which contains no constant in C, and each pair we have added at most contains finitely many new constants, there exists such a new constant in C that hasn't appeared. Then, we add to S_i^B the pair $\langle \phi_i(d_i), p \rangle$.

Claim 1.4.8.1. $S_{i+1}^B = S_i^B \cup \{\langle \exists v_i \phi_i(v_i), p \rangle, \langle \phi_i(d_i), p \rangle\}$ is consistent.

Proof of the Claim. Suppose not. Then for some $h \in X$, $(S_{i+1}^B)_h$ is inconsistent. There are two situations:

(a) $h(p) = 1$. Then $(S_i^B)_h \cup \{\exists v_i \phi_i(v_i), \phi_i(d_i)\}$ is inconsistent.

Then $(S_i^B)_h \cup \{\exists v_i \phi_i(v_i)\} \vdash \neg \phi_i(d_i)$.

Since d_i does not appear on the left hand side, $(S_i^B)_h \cup \{\exists v_i \phi_i(v_i)\} \vdash \forall v_i \neg \phi_i(v_i)$.

But then $(S_i^B)_h \cup \{\exists v_i \phi_i(v_i)\}$ is inconsistent, contradicting our choice of p.

(b) $h(p) = 0$. Then $(S_i^B)_h \cup \{\neg \exists v_i \phi_i(v_i), \neg \phi_i(d_i)\}$ is inconsistent.

Then $(S_i^B)_h \cup \{\neg \exists v_i \phi_i(v_i)\} \vdash \phi_i(d_i)$.

Since d_i does not appear on the left hand side, $(S_i^B)_h \cup \{\neg \exists v_i \phi_i(v_i)\} \vdash \forall v_i \phi_i(v_i)$.

But then $(S_i^B)_h \cup \{\neg \exists v_i \phi_i(v_i)\}$ is inconsistent, contradicting our choice of p.

■

Let $T'^B = \bigcup_{i \in \omega} S_i^B$. T'^B is consistent, as if not, then by Theorem 1.4.5, a finite subvaluation of T'^B will be in consistent, meaning that some S_i^B will be inconsistent.

Since T'^B is a consistent B-valuation of $\mathscr{L}'$, by Theorem 1.4.7 it is contained in some maximal consistent B-valuation of $\mathscr{L}'$. Let T^B be such a B-valuation.

Let $A = C$. We will construct a B-valued model $\mathfrak{A}$ of $\mathscr{L}'$ with universe A/C:

1. Let c be a constant in $\mathscr{L}'$. Then $[\![c]\!]^{\mathfrak{A}} = d_i$ such that $[\![c = d_i]\!]^T = [\![\exists v_i(v_i = c)]\!]^T$. (If there is more than one $d_i \in A$ that satisfies this, then just pick a random one.)

2. Let P be an n-nary relation. For any $\langle c_1, ..., c_n \rangle \in A^n$, let $[\![P(c_1, ..., c_n)]\!]^{\mathfrak{A}} = [\![P(c_1, ..., c_n)]\!]^T$.

3. For the identity symbol, for any $d_i, d_j \in A$, let $[\![d_i = d_j]\!]^{\mathfrak{A}} = [\![d_i = d_j]\!]^T$.

Claim 1.4.8.2. $\mathfrak{A}$ is a B-valued model.

Proof of the Claim. For any $d_i, d_j, d_k \in A$,

(1) $[\![d_i = d_i]\!]^{\mathfrak{A}} = 1$.

Suppose not. Then for some $h \in X$, $h([\![d_i = d_i]\!]^{\mathfrak{A}}) = 0$. Then $d_i \neq d_i \in T_h^B$, making T_h^B inconsistent.

(2) $[\![d_i = d_j]\!]^{\mathfrak{A}} = [\![d_j = d_i]\!]^{\mathfrak{A}}$

Suppose not. Then for some $h \in X$, $h([\![d_i = d_j]\!]^{\mathfrak{A}}) \neq h([\![d_j = d_i]\!]^{\mathfrak{A}})$. Then (without loss of generality) $d_i = d_j \in T_h^B$ and $d_j \neq d_i \in T_h^B$, making T_h^B inconsistent.

(3) $[\![d_i = d_j]\!]^{\mathfrak{A}} \sqcap [\![d_j = d_k]\!]^{\mathfrak{A}} \leqslant [\![d_i = d_k]\!]^{\mathfrak{A}}$

Suppose not. Then for some $h \in X$, $h([\![d_i = d_k]\!]^{\mathfrak{A}}) = 0$ but $h([\![d_i = d_j]\!]^{\mathfrak{A}} \sqcap [\![d_j = d_k]\!]^{\mathfrak{A}}) = 1$. Hence $h([\![d_i = d_j]\!]^{\mathfrak{A}}) = 1$ and $h([\![d_j = d_k]\!]^{\mathfrak{A}})) = 1$. Hence $d_i = d_j, d_j = d_k \in T_h^B$ but $d_i \neq d_k \in T_h^B$, making T_h^B inconsistent.

For any n-nary relation P, for any $\langle c_1, ..., c_n \rangle, \langle c'_1, ..., c'_n \rangle \in A^n$,

$$[\![P(c_1, ..., c_n)]\!]^{\mathfrak{A}} \sqcap (\prod_{1 \leqslant i \leqslant n} [\![c_i = c'_i]\!]^{\mathfrak{A}}) \leqslant [\![P(c'_1, ..., c'_n)]\!]^{\mathfrak{A}}$$

For simplicity we only prove for the case when $n = 1$. The proofs for the cases when $n > 1$ are very similar.

Suppose not. Then for some $h \in X$, $h([\![P(c'_1)]\!]^{\mathfrak{A}}) = 0$ but $h([\![c_1 = c'_1]\!]^{\mathfrak{A}} \sqcap [\![P(c_1)]\!]^{\mathfrak{A}}) = 1$. Hence $h([\![c_1 = c'_1]\!]^{\mathfrak{A}}) = 1$ and $h([\![P(c_1)]\!]^{\mathfrak{A}}) = 1$. Hence $c_1 = c'_1, P(c_1) \in T_h^B$ but $\neg P(c'_1) \in T_h^B$, making T_h^B inconsistent.

■

Finally we will show that $\mathfrak{A}$ is a model of T^B, i.e. for any ϕ of $\mathscr{L}'$, $[\![\phi]\!]^{\mathfrak{A}} = [\![\phi]\!]^T$. We prove this by an induction on the complexity of ϕ.

Atomic cases:

(a) $[\![c = c']\!]^{\mathfrak{A}} = [\![d_i = d_j]\!]^T$ where $[\![c = d_i]\!]^T = [\![\exists v_i(c = v_i)]\!]^T = 1$ and $[\![c' = d_j]\!]^T = [\![\exists v_i(c' = v_i)]\!]^T = 1$.

We just need to show that $p = [\![d_i = d_j]\!]^T = [\![c = c']\!]^T = q$.

Suppose not. Then for some $h \in X$, $h(p) \neq h(q)$. Hence (WLOG) $d_i = d_j \in T_h^B$, $c \neq c' \in T_h^B$. But $c = d_i, c' = d_j \in T_h^B$. T_h^B is inconsistent. Contradiction.

(b) For the atomic cases of relations, again, we just show it for unary relations. The cases of other n-nary relations where $n > 1$ are very similar.

$[\![P(c)]\!]^{\mathfrak{A}} = [\![P(d_i)]\!]^T$ where $[\![c = d_i]\!]^T = [\![\exists v_i(c = v_i)]\!]^T = 1$.

We just need to show that $p = [\![P(d_i)]\!]^T = [\![P(c)]\!]^T = q$.

Suppose not. Then for some $h \in X$, $h(p) \neq h(q)$. Hence (WLOG) $P(d_i) \in T_h^B$, $\neg P(c) \in T_h^B$. But $c = d_i \in T_h^B$. T_h^B is inconsistent. Contradiction.

Inductive cases:

(a) $\phi = \neg\psi$.

$$[\![\phi]\!]^{\mathfrak{A}} = [\![\neg\psi]\!]^{\mathfrak{A}} = -[\![\psi]\!]^{\mathfrak{A}} = -[\![\psi]\!]^T = [\![\neg\psi]\!]^T$$

The last equation holds for the following reason. Suppose not, and suppose $[\![\psi]\!]^T = p$ and $[\![\neg\psi]\!]^T = q \neq -p$. Then for some $h \in X$, $h(-p) \neq h(q)$. WLOG we can assume $h(-p) = 1$ and $h(q) = 0$. Then $h(p) = 0$. Then $\neg\psi \in T_h^B$ and $\neg\neg\psi \in T_h^B$, making T_h^B inconsistent. Contradiction.

(b) $\phi = \psi_1 \wedge \psi_2$.

$$[\![\psi_1 \wedge \psi_2]\!]^{\mathfrak{A}} = [\![\psi_1]\!]^{\mathfrak{A}} \sqcap [\![\psi_2]\!]^{\mathfrak{A}} = [\![\psi_1]\!]^T \sqcap [\![\psi_2]\!]^T = [\![\psi_1 \wedge \psi_2]\!]^T$$

The last equation holds for the following reasons. Suppose not, and suppose $[\![\psi_1]\!]^T \sqcap [\![\psi_2]\!]^T = p \neq q = [\![\psi_1 \wedge \psi_2]\!]^T$. Then for some $h \in X$, $h(p) = 1$ and $h(q) = 0$, or $h(p) = 0$ and $h(q) = 1$. Suppose $h(p) = 1$ and $h(q) = 0$. Then $\psi_1, \psi_2 \in T_h^B$, but $\neg(\psi_1 \wedge \psi_2) \in T_h^B$, making T_h^B inconsistent. On the other hand, suppose $h(p) = 0$ and $h(q) = 1$. Then $\psi_1 \wedge \psi_2 \in T_h^B$. Then both $h([\![\psi_1]\!]^T)$ and $h([\![\psi_2]\!]^T)$ have to be 1, as otherwise $\neg\psi_1$ or $\neg\psi_2$ would be in T_h^B, which would make T_h^B inconsistent. But then $h([\![\psi_1]\!]^T \sqcap [\![\psi_2]\!]^T) = h(p)$ has to be 1. Contradiction.

(c) $\phi = \psi_1 \vee \psi_2$.

$$[\![\psi_1 \vee \psi_2]\!]^{\mathfrak{A}} = [\![\psi_1]\!]^{\mathfrak{A}} \sqcup [\![\psi_2]\!]^{\mathfrak{A}} = [\![\psi_1]\!]^T \sqcup [\![\psi_2]\!]^T = [\![\psi_1 \vee \psi_2]\!]^T$$

The last equation holds for the following reasons. Suppose not, and suppose $[\![\psi_1]\!]^T \sqcup [\![\psi_2]\!]^T = p \neq q = [\![\psi_1 \vee \psi_2]\!]^T$. Then for some $h \in X$, $h(p) = 1$ and $h(q) = 0$, or $h(p) = 0$ and $h(q) = 1$. Suppose $h(p) = 1$ and $h(q) = 0$. Then $\neg(\psi_1 \vee \psi_2) \in T_h^B$, and hence both $h([\![\psi_1]\!]^T)$ and $h([\![\psi_2]\!]^T)$ have to be 0m as otherwise ψ_1 or ψ_2 would be in T_h^B, which would make T_h^B inconsistent. But then $h([\![\psi_1]\!]^T \sqcup [\![\psi_2]\!]^T) = h(p)$ has to be 0. Contradiction. On the other hand, suppose $h(p) = 0$ and $h(q) = 1$. Then $h([\![\psi_1]\!]^T) = 0, h([\![\psi_2]\!]^T) = 0$. Hence $\neg\psi_1, \neg\psi_2 \in T_h^B$, but $\psi_1 \vee \psi_2 \in T_h^B$, making T_h^B inconsistent.

(d) $\phi = \exists v_i \psi(v_i)$.

Let $\theta(v_i)$ be any formula with only v_i free. Then it is easy to show that for any $d_i \in A$, $[\![\theta(v_i)]\!]^{\mathfrak{A}}[d_i] = [\![\theta(d_i)]\!]^{\mathfrak{A}}$, as $[\![d_i]\!]^{\mathfrak{A}}$ is some $d_j \in A$ such that $[\![d_i = d_j]\!]^{\mathfrak{A}} = 1$. Hence,

$$[\![\exists v_i \psi(v_i)]\!]^{\mathfrak{A}} = \bigsqcup_{d_i \in A} [\![\psi(v_i)]\!]^{\mathfrak{A}}[d_i] = \bigsqcup_{d_i \in A} [\![\psi(d_i)]\!]^{\mathfrak{A}} = \bigsqcup_{d_i \in A} [\![\psi(d_i)]\!]^T$$

We need to show that $\bigsqcup_{d_i \in A} [\![\psi(d_i)]\!]^T = [\![\exists v_i \psi(v_i)]\!]^T$.

For the $\leqslant$ direction: We just need to show that for any $d_i \in A$, $[\![\psi(d_i)]\!]^T \leqslant [\![\exists v_i \psi(v_i)]\!]^T$. Suppose not, and suppose for some $d_i \in A$, $[\![\psi(d_i)]\!]^T = p$ and $[\![\exists v_i \psi(v_i)]\!]^T = q$ and

$p \nleqslant q$. Then $p \sqcap -q \neq 0$. Then for some $h \in X$, $h(p \sqcap -q) = 1$. Then $h(q) = 0$, and hence $\neg \exists v_i \psi(v_i) \in T_h^B$. But $\psi(d_i) \in T_h^B$, making T_h^B inconsistent.

For the $\geqslant$ direction: by the setup of T'^B (hence of T^B), at some stage of the sequence (say, the ith stage), both $\langle \exists v_i \psi(v_i), p\rangle$ and $\langle \psi(d_i), p\rangle$ are added to T'^B, for some $p \in B$. Hence for some $d_i \in A$, $[\![\exists v_i \psi(v_i)]\!]^T = [\![\psi(d_i)]\!]^T$.

Finally obviously $\mathfrak{A}$ is witnessing.

□

Corollary 1.4.8.1 (Completeness)**.** Let $\mathscr{L}$ be a countable language. Let S^B be a consistent Boolean-valuation of $\mathscr{L}$. Then S^B has a B-valued model.

Theorem 1.4.9 (Soundness)**.** Let S^B be a Boolean-valuation that has a B-valued model, then S^B is consistent.

Proof. Let $\mathfrak{A}$ be a B-valued model of S^B. Suppose S^B is inconsistent, then for some homomorphism $h : B \to 2$, S_h^B is inconsistent. Then, some finite subset $\Delta_h \subseteq S_h^B$ is inconsistent.

Let $\Delta_h = \{\phi_1, ..., \phi_n\}$. Let $\phi = \phi_1 \wedge ... \wedge \phi_n$. Clearly ϕ is a contradiction. Hence by Corollary 1.4.1.1, $[\![\phi]\!]^{\mathfrak{A}} = 0$.

Let $1 \leqslant i \leqslant n$. Consider ϕ_i. Since $\phi_i \in \Delta_h \subseteq S_h^B$, there are two possibilities:

(1) for some $p_i \in B$, $\langle \phi_i, p_i\rangle \in S^B$, and $h(p_i) = 1$;

(2) for some $p_i \in B$, $\langle \psi_i, p_i\rangle \in S^B$, and $h(p_i) = 0$, $\phi_i = \neg \psi_i$.

Suppose (1). Then since $\mathfrak{A}$ is a model of S^B, $[\![\phi_i]\!]^{\mathfrak{A}} = p_i$. $h([\![\phi_i]\!]^{\mathfrak{A}}) = h(p_i) = 1$.

Suppose (2). Then since $\mathfrak{A}$ is a model of S^B, $[\![\psi_i]\!]^{\mathfrak{A}} = p_i$. $[\![\phi_i]\!]^{\mathfrak{A}} = [\![\neg \psi_i]\!]^{\mathfrak{A}} = -p_i$. $h([\![\phi_i]\!]^{\mathfrak{A}}) = h(-p_i) = -h(p_i) = -0 = 1$.

In either case, $h([\![\phi_i]\!]^{\mathfrak{A}}) = 1$.

Hence $h([\![\phi]\!]^{\mathfrak{A}}) = h([\![\phi_1 \wedge ... \wedge \phi_n]\!]^{\mathfrak{A}}) = 1 \sqcap ... \sqcap 1 = 1$.

Hence $h([\![\phi]\!]^{\mathfrak{A}}) \neq 0$. Contradiction.

□

Corollary 1.4.9.1. Let $\mathscr{L}$ be a countable language. A Boolean-valuation S^B of $\mathscr{L}$ is consistent if and only if it has a B-valued model.

Corollary 1.4.9.2 (Compactness)**.** Let $\mathscr{L}$ be a countable language. A Boolean-valuation S^B of $\mathscr{L}$ has a B-valued model if and only if every finite sub-valuation of S^B has a B-valued model.

Corollary 1.4.9.3 (Downward-Löwenheim-Skolem)**.** Let $\mathscr{L}$ be a countable language. If a Boolean-valuation S^B of $\mathscr{L}$ has a B-valued model, then it has a countable witnessing B-valued model.

1.5 Relationships Between Models

Two-valued models can stand in different relationships with one another: for example, a model can be isomorphic to another, a model can be a submodel of another, a model can be a elementary submodel of another, etc. These concepts are the cornerstone of the theory of two-valued models. The primary goal of this section is to generalize these concepts to Boolean-valued models.

1.5.1 Duplicate Resistant Models

Before we move on to generalize these concepts, there is one important complication that I have to resolve first, which will be relevant to our later purposes. Astute readers might have already noticed that the identity symbol is interpreted somewhat abnormally in Boolean-valued models. The main abnormality, of course, is that a Boolean-valued model might “think" that two objects in its domain are identical to an intermediate degree between 0 and 1. We will postpone the discussion of identity in Boolean-valued models to Section 1.8. For current purposes, we will simply focus on the following minor yet interesting feature of Boolean-valued models: our definition of Boolean-valued models (Def 1.2.8) allows there to be “duplicates" in the models - that is, two different objects a,b in the domain such that the value of $a = b$ is 1 in the model.

The existence of duplicates in a model, in a natural sense, is both harmless and useless. To illustrate this point, we first introduce a new notion.

Definition 1.5.1. A B-valued mode $\mathfrak{A}$ of $\mathscr{L}$ is *duplicate resistant* just in case for any $a, b \in A$, if $[\![a = b]\!]^{\mathfrak{A}} = 1$, then a and b are the same element.

In other words, duplicate resistant models are those that disallow duplicates. The next results show that any Boolean-valued model is practically equivalent to a duplicate resistant model.

Definition 1.5.2. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. Let $h : B \to B$ be the identity function on B. The *duplicate resistant copy* of $\mathfrak{A}$, $\mathfrak{A}^d$, is the B-valued quotient model $\mathfrak{A}^h$ of $\mathscr{L}$.

Figuratively, the duplicate resistant copy of a model $\mathfrak{A}$ “collapses" all duplicates into one object and keeps the remaining of the model the same. The duplicate resistant copy of a model is practically equivalent to the original model, in the following sense.

Theorem 1.5.1. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$, and let $\mathfrak{A}^d$ be its duplicate resistant copy, as defined in Def 1.5.2. Given $x : Var \to A^d$ an arbitrary assignment on $\mathfrak{A}^d$, let $y : Var \to A$ be an assignment on $\mathfrak{A}$ such that for any $v_i \in Var$, $y(v_i) \in x(v_i)$. Then, for any formula ϕ,

$$[\![\phi]\!]^{\mathfrak{A}^d}[x] = [\![\phi]\!]^{\mathfrak{A}}[y]$$

Proof. The proof is a straightforward application of Theorem 1.3.3, since the identity function $h : B \to B$ is a complete homomorphism. □

In other words, the value of any formula under some assignment x in the original model is the same as the value of the formula in the duplicate resistant copy, when we assign instead of objects equivalence classes of objects to the variables. As a consequence, all sentences have the same value in the duplicate resistant copy.

We have argued that the existence of duplicates is harmless and useless, from a technical point of view[19]. This is mostly true, except that the existence of duplicates does create some technical difficulty when we intend to generalize concepts like isomorphism. Consider a

[19]The reason why I do not block the existence of duplicates in the definition of Boolean-valued models, as one does in the case of two-valued models, is that the possibility of having duplicates might have interesting applications to certain philosophical issues. Models are relative to languages. And sometimes the language under discussion might have limited expressive power in that it cannot distinguish between two potentially different objects. If we understand “=" as meaning “indistinguishable", then, we would want to allow there to be objects that are “duplicates" of each other, in the sense defined above.

model $\mathfrak{A}$ with a finite domain and consider adding to $\mathfrak{A}$ an new object b such that b is added as a duplicate of an original object a. Call the latter model $\mathfrak{A}'$. How are $\mathfrak{A}$ and $\mathfrak{A}'$ related? Intuitively, they should be practically the same model. For any $\mathfrak{A}$-assignment x, for any $\mathfrak{A}'$-assignment x' we get from x by replacing some occurrences of a with b, for any formula ϕ, $[\![\phi]\!]^{\mathfrak{A}}[x]$ will always be the same as $[\![\phi]\!]^{\mathfrak{A}'}[x']$. The addition of b is null in the sense that it makes no contribution to the evaluation of formulas. We would want our theory to indicate that the two models are isomorphic.

Nevertheless, if we generalize the concept of isomorphism in the most straightforward way, $\mathfrak{A}$ and $\mathfrak{A}'$ will not be isomorphic. This is because, in the two-valued framework, an isomorphism between models is a bijection, and there is simply no bijection between the domains of the two models.

Here's another example to illustrate the same point. Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two models that are completely identical, and let there be two duplicates a, b in their domains. Now add a new constant c to the language and let $[\![c]\!]^{\mathfrak{A}} = a$ and $[\![c]\!]^{\mathfrak{A}'} = b$. Intuitively, the expanded models $\mathfrak{A}$ and $\mathfrak{A}'$ are still practically the same model. But under the most straightforward generalization of the concept of submodel, $\mathfrak{A}$ will not be a submodel of $\mathfrak{A}'$ (or vice versa), simply because $[\![c]\!]^{\mathfrak{A}} \neq [\![c]\!]^{\mathfrak{A}'}$.

One natural solution to these difficulties is to first define the notions of isomorphism, submodel, etc. on duplicate resistant models, in the most straightforward way, and then define isomorphism, etc. on arbitrary Boolean-valued models using the former. For example, we can define two Boolean-valued models as isomorphic just in case their duplicate resistant copies are isomorphic. This is going to be the method that we will adopt in the following subsections, as I believe that under this method we have the most natural and simple definitions for concepts like isomorphism. Alternative methods are available, of course: for example, we can give a definition of isomorphism under which isomorphisms do not have to be bijections. In the end, which method we adopt is more of a matter of taste than a matter of mathematical significance.

1.5.2 Isomorphism, Submodel, and Diagram

In this and the next two subsections, for reasons we have given in the previous subsection, we will assume all Boolean-valued models are duplicate resistant. Also, whenever we do not specify otherwise, we assume all models are models of a first-order language $\mathscr{L}$.

Definition 1.5.3 (Isomorphism)**.** Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two B-valued models. A bijection $f : A_1 \to A_2$ is an *isomorphism* just in case: (let t_i be a term)

1. For any $a_1, a_2 \in A_1$, $[\![t_1 = t_2]\!]^{\mathfrak{A}_1}[a_1, a_2] = [\![t_1 = t_2]\!]^{\mathfrak{A}_2}[f(a_1), f(a_2)]$.

2. Let P be an n-nary predicate. For any $\langle a_1, ..., a_n\rangle \in A_1^n$, $[\![P(t_1, ..., t_n)]\!]^{\mathfrak{A}_1}[a_1, ..., a_n]$ $= [\![P(t_1, ..., t_n)]\!]^{\mathfrak{A}_2}[f(a_1), ..., f(a_n)]$.

3. Let c be a constant. $[\![c]\!]^{\mathfrak{A}_2} = f([\![c]\!]^{\mathfrak{A}_1})$.

When there exists an isomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_2$, we say that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are *isomorphic*.

Definition 1.5.4 (Submodel)**.** Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two B-valued models. Let $A_1 \subseteq A_2$. $\mathfrak{A}_1$ is a *submodel* of $\mathfrak{A}_2$ just in case: (let t_i be a term)

1. For any $a_1, a_2 \in A_1$, $[\![t_1 = t_2]\!]^{\mathfrak{A}_1}[a_1, a_2] = [\![t_1 = t_2]\!]^{\mathfrak{A}_2}[a_1, a_2]$.

2. Let P be an n-nary predicate. For any $\langle a_1, ..., a_n\rangle \in A_1^n$, $[\![P(t_1, ..., t_n)]\!]^{\mathfrak{A}_1}[a_1, ..., a_n]$ $= [\![P(t_1, ..., t_n)]\!]^{\mathfrak{A}_2}[a_1, ..., a_n]$.

3. Let c be a constant. $[\![c]\!]^{\mathfrak{A}_2} = [\![c]\!]^{\mathfrak{A}_1}$.

Definition 1.5.5 (Diagram)**.** Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. Let $\mathscr{L}_{\mathfrak{A}} = \mathscr{L} \cup \{c_a \mid a \in A\}$, where $\{c_a \mid a \in A\}$ is a new set of constants, one for each $a \in A$. Expand $\mathfrak{A}$ to a model of $\mathscr{L}_{\mathfrak{A}}$ (call it $\mathfrak{A}^*$) such that for all $a \in A$, $[\![c_a]\!]^{\mathfrak{A}^*} = a$.

The *diagram* of $\mathfrak{A}$ is the B-valuation S^B which consists of all and only pairs of the form $\langle \phi, [\![\phi]\!]^{\mathfrak{A}^*}\rangle$ where ϕ is an atomic sentence or the negation of an atomic sentence of $\mathscr{L}_{\mathfrak{A}}$ and $[\![\phi]\!]^{\mathfrak{A}^*}$ is the value of ϕ in $\mathfrak{A}^*$.

Theorem 1.5.2. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two B-valued models. The following statements are equivalent:

(1) $\mathfrak{A}_1$ is isomorphic to a submodel of $\mathfrak{A}_2$.

(2) $\mathfrak{A}_2$ can be expanded to a model of the diagram of $\mathfrak{A}_1$.

Proof. (1) $\Rightarrow$ (2). Let $f : A_1 \to A_3 \subseteq A_2$ be an isomorphism, where $\mathfrak{A}_3$ is a submodel of $\mathfrak{A}_2$. Expand $\mathfrak{A}_2$ to a model of $\mathscr{L}_{\mathfrak{A}_1}$ (call it $\mathfrak{A}_2'$) as follows: for any $a \in A_1$, let $[\![c_a]\!]^{\mathfrak{A}_2} = f(a)$.

We will show that $\mathfrak{A}_2'$ is a model of the diagram of $\mathfrak{A}_1$. Let $\mathfrak{A}_1^*$ be the standard expansion of $\mathfrak{A}_1$ to $\mathscr{L}_{\mathfrak{A}_1}$.

Let $\phi(c_{a_1}, ..., c_{a_n})$ be an atomic sentence or the negation of an atomic sentence of $\mathscr{L}_{\mathfrak{A}_1}$, where $c_{a_1}, ..., c_{a_n}$ are all the constants of $\mathscr{L}_{\mathfrak{A}_1} \backslash \mathscr{L}$ that appear in ϕ. Let $\phi'(v_1, ..., v_n)$ be the formula of $\mathscr{L}$ that we get by substituting c_{a_1} in ϕ with v_1, ..., c_{a_n} in ϕ with v_n, and we assume that none of $v_1, ..., v_n$ appear in $\phi(c_{a_1}, ..., c_{a_n})$. Then

$$\begin{aligned}
[\![\phi(c_{a_1}, ..., c_{a_n})]\!]^{\mathfrak{A}_1^*} &= [\![\phi(v_1, ..., v_n)]\!]^{\mathfrak{A}_1}[a_1, ..., a_n] \\
&= [\![\phi(v_1, ..., v_n)]\!]^{\mathfrak{A}_3}[f(a_1), ..., f(a_n)] \\
&= [\![\phi(v_1, ..., v_n)]\!]^{\mathfrak{A}_2}[f(a_1), ..., f(a_n)] \\
&= [\![\phi(c_{a_1}, ..., c_{a_n})]\!]^{\mathfrak{A}_2'}
\end{aligned}$$

The second equation holds since $\mathfrak{A}_1$ is isomorphic to $\mathfrak{A}_3$. The third equation holds since $\mathfrak{A}_3$ is a submodel of $\mathfrak{A}_2$.

(2) $\Rightarrow$ (1). Let $\mathfrak{A}_2'$ be an expansion of $\mathfrak{A}_2$ to $\mathscr{L}_{\mathfrak{A}_1}$ such that $\mathfrak{A}_2'$ is a model of the diagram of $\mathfrak{A}_1$.

Construct $f : A_1 \to A_2$ as follows: for any $a \in A_1$, $f(a) = [\![c_a]\!]^{\mathfrak{A}_2'}$. Let $\mathfrak{A}_3$ be the submodel of $\mathfrak{A}_2$ whose domain is generated by $f[A_1]$.

We will show that the domain of $\mathfrak{A}_3$ is precisely $f[A_1]$. Let c be a constant in $\mathscr{L}$. And suppose $[\![c]\!]^{\mathfrak{A}_1} = a \in A_1$. Then $[\![c = c_a]\!]^{\mathfrak{A}_1^*} = 1$ and therefore $[\![c = c_a]\!]^{\mathfrak{A}_2'} = 1$. Since $\mathfrak{A}_2$ is duplicate resistant, $\mathfrak{A}_2'$ is also duplicate resistant. Hence $[\![c]\!]^{\mathfrak{A}_2'} = [\![c_a]\!]^{\mathfrak{A}_2'}$. Hence $[\![c]\!]^{\mathfrak{A}_3} = f(a) \in f[A_1]$.

We will next show that $f : A_1 \to A_3$ is an isomorphism. We first show that f is a bijection. Trivially it is surjective. Suppose $f(a_1) = f(a_2)$, then $[\![c_{a_1}]\!]^{\mathfrak{A}_2'} = [\![c_{a_2}]\!]^{\mathfrak{A}_2'}$ and therefore $[\![c_{a_1} = c_{a_2}]\!]^{\mathfrak{A}_2'} = 1$. Since $\mathfrak{A}_2'$ is a model of the diagram of $\mathfrak{A}_1$, $[\![c_{a_1} = c_{a_2}]\!]^{\mathfrak{A}_1^*} = 1$.

Hence $[\![a_1 = a_2]\!]^{\mathfrak{A}_1} = 1$. Since $\mathfrak{A}_1$ is duplicate resistant, $a_1 = a_2$. Hence f is injective.

Let $\phi(v_1,...,v_n)$ be an atomic formula of $\mathscr{L}$ with free variables $v_1,...,v_n$. Let $a_1,...,a_n \in A_1$. Then

$$\begin{aligned}
[\![\phi(v_1,...,v_n)]\!]^{\mathfrak{A}_1}[a_1,...,a_n] &= [\![\phi(c_{a_1},...,c_{a_n})]\!]^{\mathfrak{A}_1^*} \\
&= [\![\phi(c_{a_1},...,c_{a_n})]\!]^{\mathfrak{A}_2'} \\
&= [\![\phi(v_1,...,v_n)]\!]^{\mathfrak{A}_2}[f(a_1),...,f(a_n)] \\
&= [\![\phi(v_1,...,v_n)]\!]^{\mathfrak{A}_3}[f(a_1),...,f(a_n)]
\end{aligned}$$

The second equation holds because $\mathfrak{A}_2'$ is a model of the diagram of $\mathfrak{A}_1$. The third equation holds by the definition of f. The fourth equation holds because $\mathfrak{A}_3$ is a submodel of $\mathfrak{A}_2$.

Let c be a constant in $\mathscr{L}$. Then using the same reasoning as above, $[\![c]\!]^{\mathfrak{A}_3} = [\![c]\!]^{\mathfrak{A}_2'} = [\![c_a]\!]^{\mathfrak{A}_2'} = f(a) = f([\![c]\!]^{\mathfrak{A}_1})$, where $[\![c]\!]^{\mathfrak{A}_1} = a \in A_1$.

□

Definition 1.5.6 (Homomorphism)**.** Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two B-valued models. A surjection $f : A_1 \to A_2$ is a *homomorphism* just in case: (let t_i be a term)

1. For any $a_1, a_2 \in A_1$, if $[\![t_1 = t_2]\!]^{\mathfrak{A}_1}[a_1, a_2] = p$ (where $p \in B$), then $[\![t_1 = t_2]\!]^{\mathfrak{A}_2}[f(a_1), f(a_2)]$ = p.

2. Let P be an n-nary predicate. For any $\langle a_1,...,a_n\rangle \in A_1^n$, if $[\![P(t_1,...,t_n)]\!]^{\mathfrak{A}_1}[a_1,...,a_n] = p$, then = $[\![P(t_1,...,t_n)]\!]^{\mathfrak{A}_2}[f(a_1),...,f(a_n)] = p$.

3. Let c be a constant. $[\![c]\!]^{\mathfrak{A}_2} = f([\![c]\!]^{\mathfrak{A}_1})$.

When there exists a homomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_2$, we say that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are *homomorphic*.

Definition 1.5.7 (Positive Diagram)**.** Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. Let $\mathscr{L}_{\mathfrak{A}} = \mathscr{L} \cup \{c_a \mid a \in A\}$, where $\{c_a \mid a \in A\}$ is a new set of constants, one for each $a \in A$. Expand $\mathfrak{A}$ to a model of $\mathscr{L}_{\mathfrak{A}}$ (call it $\mathfrak{A}^*$) such that for all $a \in A$, $[\![c_a]\!]^{\mathfrak{A}^*} = a$.

The *positive diagram* of $\mathfrak{A}$ is the B-valuation S^B which consists of all and only pairs of the form $\langle \phi, [\![\phi]\!]^{\mathfrak{A}^*} \rangle$ where ϕ is an atomic sentence of $\mathscr{L}_{\mathfrak{A}}$ and $[\![\phi]\!]^{\mathfrak{A}^*}$ is the value of ϕ in $\mathfrak{A}^*$.

Theorem 1.5.3. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two B-valued models. The following statements are equivalent:

(1) $\mathfrak{A}_1$ is homomorphic to a submodel of $\mathfrak{A}_2$.

(2) $\mathfrak{A}_2$ can be expanded to a model of the positive diagram of $\mathfrak{A}_1$.

Proof. The same proof as that of Theorem 1.5.2 with minor adjustments.

□

1.5.3 Elementary Submodel and Downward Löwenheim-Skolem

Definition 1.5.8 (Elementary Submodel)**.** Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two B-valued models of $\mathscr{L}$. Let $A_1 \subseteq A_2$. $\mathfrak{A}_1$ is an *elementary submodel* of $\mathfrak{A}_2$ just in case: $\mathfrak{A}_1$ is a submodel of $\mathfrak{A}_2$, and for any formula $\phi(v_1, ..., v_n)$ of $\mathscr{L}$, any $a_1, ..., a_n \in A_1$,

$$[\![\phi(v_1, ..., v_n)]\!]^{\mathfrak{A}_1}[a_1, ..., a_n] = [\![\phi(v_1, ..., v_n)]\!]^{\mathfrak{A}_2}[a_1, ..., a_n]$$

Theorem 1.5.4. Let $\mathfrak{A}_1$ be a witnessing B-valued model and $\mathfrak{A}_2$ be a B-valued model. $\mathfrak{A}_1$is an elementary submodel $\mathfrak{A}_2$ if and only if $\mathfrak{A}_1$ is a submodel of $\mathfrak{A}_2$, and for any formula $\exists v \phi(v, v_1, ..., v_n)$ of $\mathscr{L}$, any $a_1, ..., a_n \in A_1$, for some $a \in A_1$,

$$[\![\exists v \phi(v, v_1, ..., v_n)]\!]^{\mathfrak{A}_1}[a_1, ..., a_n] = [\![\phi(v, v_1, ..., v_n)]\!]^{\mathfrak{A}_2}[a, a_1, ..., a_n]$$

Proof. The left to right direction is proved by directly applying Def 1.5.8 and the fact that $\mathfrak{A}_1$ is witnessing.

The right to left direction is proved by induction on the complexity of ϕ. The only non-trivial step is the inductive step on existential formulas. Consider $\exists v \phi(v, v_1, ..., v_n)$ and $a_1, ..., a_n \in A_1$. Obviously $[\![\exists v \phi]\!]^{\mathfrak{A}_1}[a_1, ..., a_n] \leqslant [\![\exists v \phi]\!]^{\mathfrak{A}_2}[a_1, ..., a_n]$. For the other direction, for some $a \in A_1$,

$$\begin{aligned}[\![\exists v_i \phi]\!]^{\mathfrak{A}_2}[a_1,\ldots,a_n] &= [\![\phi]\!]^{\mathfrak{A}_2}[a,a_1,\ldots,a_n] \\ &= [\![\phi]\!]^{\mathfrak{A}_1}[a,a_1,\ldots,a_n] \\ &\leqslant [\![\exists v_i \phi]\!]^{\mathfrak{A}_1}[a_1,\ldots,a_n]\end{aligned}$$

□

Definition 1.5.9 (Elementary Diagram)**.** Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. Let $\mathscr{L}_{\mathfrak{A}} = \mathscr{L} \cup \{c_a \mid a \in A\}$, where $\{c_a \mid a \in A\}$ is a new set of constants, one for each $a \in A$. Expand $\mathfrak{A}$ to a model of $\mathscr{L}_{\mathfrak{A}}$ (call it $\mathfrak{A}^*$) such that for all $a \in A$, $[\![c_a]\!]^{\mathfrak{A}^*} = a$.

The *elementary diagram* of $\mathfrak{A}$ is the B-valuation S^B which consists of and only of all pairs of the form $\langle \phi, [\![\phi]\!]^{\mathfrak{A}^*} \rangle$ where ϕ is a sentence of $\mathscr{L}_{\mathfrak{A}}$ and $[\![\phi]\!]^{\mathfrak{A}^*}$ is the value of ϕ in $\mathfrak{A}^*$.

Theorem 1.5.5. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two B-valued models. The following statements are equivalent:

(1) $\mathfrak{A}_1$ is isomorphic to an elementary submodel of $\mathfrak{A}_2$.

(2) $\mathfrak{A}_2$ can be expanded to a model of the elementary diagram of $\mathfrak{A}_1$.

Proof. (1) ⇒ (2). Similar proof as that of Theorem 1.5.2, with the minor adjustment that we now let ϕ be a random sentence instead of just an atomic sentence or the negation of an atomic sentence.

(2) ⇒ (1). The same construction we used in the proof of Theorem 1.5.2 gives us a submodel $\mathfrak{A}_3$ of $\mathfrak{A}_2$ that is isomorphic to $\mathfrak{A}_1$.

We only need to show that $\mathfrak{A}_3$ as so defined is an elementary submodel of $\mathfrak{A}_2$. Let $\phi(v_1,\ldots,v_n)$ be a formula of $\mathscr{L}$ with free variables $v_1,\ldots,v_n$. Let $f(a_1),\ldots,f(a_n) \in A_3$.

Then

$$\begin{aligned}
[\![\phi(v_1,...,v_n)]\!]^{\mathfrak{A}_3}[f(a_1),...,f(a_n)] &= [\![\phi(v_1,...,v_n)]\!]^{\mathfrak{A}_1}[a_1,...,a_n] \\
&= [\![\phi(c_{a_1},...,c_{a_n})]\!]^{\mathfrak{A}_1^*} \\
&= [\![\phi(c_{a_1},...,c_{a_n})]\!]^{\mathfrak{A}_2'} \\
&= [\![\phi(v_1,...,v_n)]\!]^{\mathfrak{A}_2}[f(a_1),...,f(a_n)]
\end{aligned}$$

The first equation holds because f is an isomorphism. The second equation holds by the definition of $\mathfrak{A}_1^*$. The third equation holds because $\mathfrak{A}_2'$ is a model of the elementary diagram of $\mathfrak{A}_1$. The fourth equation holds by the definition of f.

□

When $\mathfrak{A}_1$ is isomorphic to an elementary submodel of $\mathfrak{A}_2$, we say that $\mathfrak{A}_1$ is *elementarily embedded* in $\mathfrak{A}_2$.

In Section 1.4 we proved a weaker version of the generalized Downward-Löwenheim-Skolem Theorem (Corollary 1.4.9.3). With the notion of elementary submodels we can now prove a stronger version of this theorem. Again, we assume that $\mathscr{L}$ is a countable language.

Theorem 1.5.6 (Downward-Löwenheim-Skolem)**.** Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$ that is witnessing. Then $\mathfrak{A}$ has a countable elementary submodel.

Proof. Let ϕ be an arbitrary sentence of $\mathscr{L}$ that is of the form $\exists v\psi$. Since $\mathfrak{A}$ is witnessing, there is some $a \in A$ such that $[\![\exists v\psi]\!]^{\mathfrak{A}} = [\![\psi]\!]^{\mathfrak{A}}[a]$. Pick such a witness for each sentence of the form $\exists v\psi$. Let $X \subseteq A$ be the set of all picked witnesses. Construct an increasing sequence:

$$X = X_0 \subseteq X_1 \subseteq X_2 \subseteq ... \subseteq X_i \subseteq ...,\ i < \omega$$

Given X_i. Let $\exists v\psi(v,v_1,...,v_n)$ be a formula with $v_1,...,v_n$ free, and let $a_1,...,a_n \in X_i$. Again, since $\mathfrak{A}$ is witnessing, there is some $a \in A$ such that $[\![\exists v\psi]\!]^{\mathfrak{A}}[a_1,...,a_n] = [\![\psi]\!]^{\mathfrak{A}}[a,a_1,...,a_n]$. We pick a witness for each formula of the form $\exists v\psi(v,v_1,...,v_n)$ and $a_1,...,a_n \in X_i$. Let X_{i+1} be X union all the picked witnesses.

Let $A' = \bigcup_{i<\omega} X_i$. Since $\mathscr{L}$ is countable, X and each X_i is countable. Hence A' is also countable. Form a model $\mathfrak{A}'$ with universe A':

1. For any $a, b \in A'$, $[\![a = b]\!]^{\mathfrak{A}'} = [\![a = b]\!]^{\mathfrak{A}}$.

2. Let P be an n-ary relation. For any $a_1, ..., a_n \in A'$, $[\![P(a_1, ..., a_n)]\!]^{\mathfrak{A}'} = [\![P(a_1, ..., a_n)]\!]^{\mathfrak{A}}$.

3. Let c be a constant. Let $[\![c]\!]^{\mathfrak{A}'}$ be some $a \in A'$ such that $[\![v_i = c]\!]^{\mathfrak{A}}[a] = [\![\exists v_i v_i = c]\!]^{\mathfrak{A}}$. Such an a exists by the setup of $\mathfrak{A}'$.

It is easy to see that $\mathfrak{A}'$ is a submodel of $\mathfrak{A}$. For any constant c, $[\![[\![c]\!]^{\mathfrak{A}'} = [\![c]\!]^{\mathfrak{A}'}]\!]^{\mathfrak{A}} = 1$ by the choice of $[\![c]\!]^{\mathfrak{A}'}$, and since $\mathfrak{A}$ is duplicate resistant, $[\![c]\!]^{\mathfrak{A}'} = [\![c]\!]^{\mathfrak{A}}$.

We will show that $\mathfrak{A}'$ is also an elementary submodel of $\mathfrak{A}$. Let $\exists v \psi(v, v_1, ..., v_n)$ be a formula with $v_1, ..., v_n$ free, and let $a_1, ..., a_n \in A'$. Since $a_1, ..., a_n \in A' = \bigcup_{i<\omega} X_i$, for some $i < \omega$, $a_1, ..., a_n \in X_i$. Hence for some $a \in X_{i+1} \subseteq A'$, $[\![\exists v \psi]\!]^{\mathfrak{A}}[a_1, ..., a_n] = [\![\psi]\!]^{\mathfrak{A}}[a, a_1, ..., a_n]$. By Theorem 1.5.4, $\mathfrak{A}'$ is an elementary submodel.

□

Corollary 1.5.6.1. If a B-valuation S^B has a witnessing B-valued model $\mathfrak{A}$, then it has a countable witnessing B-valued model that is an elementary submodel of $\mathfrak{A}$.

The stronger Downward-Löwenheim-Skolem Theorem is a natural generalization of the homonymous theorem on two-valued models, as every two-valued model is witnessing. Interestingly, the requirement that $\mathfrak{A}$ is witnessing in the stronger Downward-Löwenheim-Skolem Theorem cannot be dropped, as the theorem no longer holds when $\mathfrak{A}$ is not necessarily witnessing. This result, I think, is another example of the fact that certain features of two-valued models can only be generalized to witnessing Boolean-valued models, but not to all Boolean-valued models.

Theorem 1.5.7. There exists a Boolean-valued model $\mathfrak{A}$ that does not have a countable elementary submodel.

Proof. Let B be a complete Boolean algebra such that from some $D \subseteq B$, $|D| = \omega_1$ and for any $C \subseteq D$ such that $|C| < \omega_1$, $\bigsqcup C \neq \bigsqcup D = p$. Let $D = \{p_\alpha \mid \alpha < \omega_1\}$.

Let $|A| = \omega_1$. Let $A = \{a_\alpha \mid \alpha < \omega_1\}$. Let P be a unary predicate. (Predicates of other arities can work as well.) Let $\mathfrak{A}$ be such that for any $\alpha < \omega_1$, $[\![P(a_\alpha)]\!]^{\mathfrak{A}} = p_\alpha$. The obviously $[\![\exists v P(v)]\!]^{\mathfrak{A}} = \bigsqcup_{\alpha<\omega_1} p_\alpha = \bigsqcup D = p$. And no countable submodel of $\mathfrak{A}$ is such that the value of $\exists v P(v)$ in it is p.

□

1.5.4 Elementary Equivalence and Elementary Chain

Definition 1.5.10 (Elementary Equivalence)**.** Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two B-valued models of $\mathscr{L}$. $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are *elementarily equivalent* just in case for any sentence ϕ in $\mathscr{L}$, $[\![\phi]\!]^{\mathfrak{A}_1} = [\![\phi]\!]^{\mathfrak{A}_2}$.

Theorem 1.5.8. Let $\{\mathfrak{A}_i \mid i \in I\}$ be a set of witnessing B-valued models such that for any $i, j \in I$, $\mathfrak{A}_i$ and $\mathfrak{A}_j$ are elementarily equivalent. Then there exists a B-valued model $\mathfrak{A}$ such that for any $i \in I$, $\mathfrak{A}_i$ is elementarily embedded in $\mathfrak{A}$.

Proof. For each $\mathfrak{A}_i$, let S_i^B be the elementary diagram of $\mathfrak{A}_i$. We assume that if $i \neq j$, then $\{c_a \mid a \in A_i\} \cap \{c_a \mid a \in A_j\} = \varnothing$. Let $\bigcup_{i\in I} S_i^B$ be the union of all the elementary diagrams.

Claim 1.5.8.1. $\bigcup\limits_{i\in I} S_i^B$ is a consistent B-valuation.

Proof of the Claim. By Theorem 1.4.5, we only need to show that every finite sub-valuation of $\bigcup_{i\in I} S_i^B$ is consistent. Let $\Delta^B = \{\langle\phi_1(c_1), p_1\rangle, ..., \langle\phi_n(c_n), p_n\rangle\}$ be a finite sub-valuation of $\bigcup_{i\in I} S_i^B$. WLOG we assume that for any $i \leqslant k \leqslant n$, $\langle\phi_k(c_k), p_k\rangle \in S_k^B$, and c_k is the only constant from $\{c_a \mid a \in A_k\}$ that appears in ϕ_k.

Assume for reductio that Δ^B is inconsistent. Then for some homomorphism $h : B \to 2$, Δ_h^B is inconsistent.

Suppose $\Delta_h^B = \{\theta_1(c_1), ..., \theta(c_n)\}$ such that $\theta_k = \phi_k$ if $h(p_k) = 1$ and $\theta_k = \neg\phi_k$ if $h(p_k) = 0$. Then $\theta_1(c_1) \vdash \neg\theta_2(c_2) \vee ... \vee \neg\theta_n(c_n)$.

Since $\langle\phi_1(c_1), p_1\rangle \in S_1^B$, $\theta(c_1) \in (S_1^B)_h$. Hence $(S_1^B)_h \vdash \neg\theta_2(c_2) \vee ... \vee \neg\theta_n(c_n)$. And by assumption $c_2, ..., c_n$ do not appear in $(S_1^B)_h$, hence $(S_1^B)_h \vdash \forall v_i \neg\theta_2(v_i) \vee ... \vee \forall v_i \neg\theta_n(v_i)$.

By assumption, $\forall v_i \neg\theta_2(v_i), ..., \forall v_i \neg\theta_n(v_i)$ are sentences of $\mathscr{L}$. Hence for each $2 \leqslant k \leqslant n$, for some $q_k \in B$, $\langle\forall v_i \neg\theta_k(v_i), q_k\rangle \in S_1^B$. Also since S_1^B is consistent (as it has a B-valued model, namely $\mathfrak{A}_1$), q_k is unique.

But all the $\mathfrak{A}_i$'s are elementarily equivalent. Hence for any $i \in I$, for any $2 \leqslant k \leqslant n$, $\langle \forall v_i \neg \theta_k(v_i), q_k \rangle \in S_i^B$. And as a result, for any $i \in I$, $\langle \forall v_i \neg \theta_2(v_i) \vee ... \vee \forall v_i \neg \theta_n(v_i), q_2 \sqcup ... \sqcup q_n \rangle \in S_i^B$.

Now since $(S_1^B)_h \vdash \forall v_i \neg \theta_2(v_i) \vee ... \vee \forall v_i \neg \theta_n(v_i)$, and since S_1^B is consistent, $h(q_2 \sqcup ... \sqcup q_n) = 1$. Hence for some $2 \leqslant k \leqslant n$, $h(q_k) = 1$.

Hence $\forall v_i \neg \theta_k(v_i) \in (S_k^B)_h$. But $\theta_k(c_k)$, by assumption, is also in $(S_k^B)_h$. Hence $(S_k^B)_h$ is consistent. But S_k^B is the elementary diagram of $\mathfrak{A}_\mathfrak{k}$, and therefore it has a B-valued model and should be consistent. Contradiction.

■

We showed that $\bigcup_{i \in I} S_i^B$ is consistent. By Corollary 1.4.8.1, it has a B-valued model $\mathfrak{A}'$. Let $\mathfrak{A}$ be the reduct of $\mathfrak{A}'$ to $\mathscr{L}$. By Theorem 1.5.5, for any $i \in I$, $\mathfrak{A}_i$ is elementarily embedded in $\mathfrak{A}$.

□

For the next theorem, we identify any Boolean algebra with its isomorphic copies.

Theorem 1.5.9. Let $\mathfrak{A}$ be a B-valued model. Let I be an arbitrary index set. Then $\mathfrak{A}$ is elementarily embedded in $\prod_{i \in I} \mathfrak{A}$.

Proof. Let $\mathfrak{A}'$ be the submodel of $\prod_{i \in I} \mathfrak{A}$ generated by $A' = \{\langle a \rangle_{i \in I} \mid a \in A\}$. It is easy to show that the domain of $\mathfrak{A}'$ is precisely A', since for any constant c, $[\![c]\!]^{\prod_{i \in I} \mathfrak{A}} = \langle [\![c]\!]^{\mathfrak{A}} \rangle_{i \in I} \in A'$.

We can show that $\mathfrak{A}'$ is an elementary submodel of $\prod_{i \in I} \mathfrak{A}$ by induction on the complexity of ϕ. The only non-trivial case is the inductive step on existential formulas. Let

$\phi(v, v_1, ..., v_n)$ be a formula with $v, v_1, ..., v_n$ free:

$$\begin{aligned}
[\![\exists v\phi]\!]^{\mathfrak{A}'}[\langle a_1\rangle_{i\in I}, ..., \langle a_n\rangle_{i\in I}] &= \bigsqcup_{\langle b\rangle_{i\in I}\in A'} [\![\phi]\!]^{\mathfrak{A}'}[\langle b\rangle_{i\in I}, \langle a_1\rangle_{i\in I}, ..., \langle a_n\rangle_{i\in I}] \\
&= \bigsqcup_{\langle b\rangle_{i\in I}\in A'} [\![\phi]\!]^{\prod_{i\in I}\mathfrak{A}}[\langle b\rangle_{i\in I}, \langle a_1\rangle_{i\in I}, ..., \langle a_n\rangle_{i\in I}] \\
&= \bigsqcup_{\langle b\rangle_{i\in I}\in A'} \langle [\![\phi]\!]^{\mathfrak{A}}[b, a_1, ..., a_n]\rangle_{i\in I} \\
&= \langle \bigsqcup_{b\in A} [\![\phi]\!]^{\mathfrak{A}}[b, a_1, ..., a_n]\rangle_{i\in I} \\
&= \langle [\![\exists v\phi]\!]^{\mathfrak{A}}[a_1, ..., a_n]\rangle_{i\in I} \\
&= [\![\exists v\phi]\!]^{\prod_{i\in I}\mathfrak{A}}[\langle a_1\rangle_{i\in I}, ..., \langle a_n\rangle_{i\in I}]
\end{aligned}$$

The second equation holds by inductive hypothesis. The third equation holds by Theorem 1.3.2. The fourth equation holds by Def 1.3.1.

Finally, by Def 1.3.1, it is easy to see that B is isomorphic to the Boolean algebra $B' = \{\langle p\rangle_{i\in I} \in \prod_{i\in I} B \mid p \in B\}$, and that the latter is a complete subalgebra of $\prod_{i\in I} B$.

Moreover, for any formula $\phi(v_1, ..., v_n)$, any $\langle a_1\rangle_{i\in I}, ..., \langle a_n\rangle_{i\in I}$,

$$\begin{aligned}
[\![\phi]\!]^{\mathfrak{A}'}[\langle a_1\rangle_{i\in I}, ..., \langle a_n\rangle_{i\in I}] &= [\![\phi]\!]^{\prod_{i\in I}\mathfrak{A}}[\langle a_1\rangle_{i\in I}, ..., \langle a_n\rangle_{i\in I}] \\
&= \langle [\![\phi]\!]^{\mathfrak{A}}[a_1, ..., a_n]\rangle_{i\in I} \in B'
\end{aligned}$$

And hence although the value range of $\mathfrak{A}'$ is officially $\prod_{i\in I} B$, only values from B' will actually be made used of. Hence $\mathfrak{A}'$, in a natural sense, really has B' as its value range. Let $f : A \to A'$ be such that for any $a \in A$, $f(a) = \langle a\rangle_{i\in I}$. It is easy to show that f is an isomorphism.

□

For the next theorem we need the following lemma.

Lemma 1.5.9.1. Let I be an index set. For any $i \in I$, let $\mathfrak{A}_i$ be a B_i-valued model that is witnessing. Then $\prod_{i\in I} \mathfrak{A}_i$ is a witnessing model.

Proof. For simplicity we ignore the parameters. Let $\phi(v_i)$ be a formula. Then $[\![\exists v_i\phi]\!]^{\prod_{i\in I}\mathfrak{A}_i} = \langle[\![\exists v_i\phi]\!]^{\mathfrak{A}_i}\rangle_{i\in I}$, by Theorem 1.3.2. Since for any $i \in I$, $\mathfrak{A}_i$ is witnessing, for some $a_i \in A_i$, $[\![\exists v_i\phi]\!]^{\mathfrak{A}_i} = [\![\phi]\!]^{\mathfrak{A}_i}[a_i]$. Pick such an a_i for each $\mathfrak{A}_i$. Then $\langle[\![\exists v_i\phi]\!]^{\mathfrak{A}_i}\rangle_{i\in I} = \langle[\![\phi]\!]^{\mathfrak{A}_i}[a_i]\rangle_{i\in I} = [\![\phi]\!]^{\prod_{i\in I}\mathfrak{A}_i}[\langle a_i\rangle_{i\in I}]$. □

Theorem 1.5.10. Let $\mathfrak{A}$ be a witnessing B-valued model. Let I be an arbitrary index set. Let $h : \prod_{i\in I} B \to B$ be a homomorphism such that for any $p \in B$, $h(\langle p\rangle_{i\in I}) = p$. Then $\mathfrak{A}$ and $(\prod_{i\in I}\mathfrak{A})^h$ are elementarily equivalent.

Proof. Let $\mathfrak{A}$ be a witnessing model and let $h : \prod_{i\in I} B \to B$ be a homomorphism such that for any $p \in B$, $h(\langle p\rangle_{i\in I}) = p$. Let ϕ be a sentence of $\mathscr{L}$. Let $[\![\phi]\!]^{\mathfrak{A}} = p \in B$.

By Lemma 1.5.9.1, $\prod_{i\in I}\mathfrak{A}$ is a witnessing model. Hence it is in the scope of Theorem 1.3.4. Hence $[\![\phi]\!]^{(\prod_{i\in I}\mathfrak{A})^h} = h([\![\phi]\!]^{\prod_{i\in I}\mathfrak{A}}) = h(\langle[\![\phi]\!]^{\mathfrak{A}}\rangle_{i\in I}) = h(\langle p\rangle_{i\in I}) = p$.

□

Definition 1.5.11 (Chain of Models)**.** Let α be an ordinal. For each $\beta < \alpha$, let $\mathfrak{A}_\beta$ be a B-valued model. A chain of models is an increasing sequence of models $\mathfrak{A}_0 \subset \mathfrak{A}_1 \subset \ldots \subset \mathfrak{A}_\beta \subset \ldots$, $\beta < \alpha$, where $\mathfrak{A}_0$ is a submodel of $\mathfrak{A}_1$, $\mathfrak{A}_1$ is a submodel of $\mathfrak{A}_2$, etc.

Definition 1.5.12 (Union of the Chain)**.** Given a chain of models $\mathfrak{A}_0 \subset \ldots \subset \mathfrak{A}_\beta \subset \ldots$, $\beta < \alpha$, the union of the chain is the B-valued model $\mathfrak{A} = \bigcup_{\beta<\alpha}\mathfrak{A}_\beta$ such that:

1. The universe of $\mathfrak{A}$ is $A = \bigcup_{\beta<\alpha} A_\beta$.
2. Let $a_1, a_2, \ldots, a_n \in A$. The for some $\beta < \alpha$, $a_1, \ldots, a_n \in A_\beta$.
   (a) Let $1 \leqslant i, j, \leqslant n$. $[\![a_i = a_j]\!]^{\mathfrak{A}} = [\![a_i = a_j]\!]^{\mathfrak{A}_\beta}$.
   (b) Let P be an n-ary relation. $[\![P(a_1, \ldots, a_n)]\!]^{\mathfrak{A}} = [\![P(a_1, \ldots, a_n)]\!]^{\mathfrak{A}_\beta}$.
   (c) Let c be a constant. $[\![c]\!]^{\mathfrak{A}} = [\![c]\!]^{\mathfrak{A}_\beta}$.

Proposition 1.5.1. The union of a chain is a B-valued model. Also, for every $\beta < \alpha$, $\mathfrak{A}_\beta$ is a submodel of $\bigcup_{\beta<\alpha}\mathfrak{A}_\beta$.

Theorem 1.5.11 (Generalized Elementary Chain Theorem)**.** Let $\{\mathfrak{A}_\beta \mid \beta < \alpha\}$ be an elementary chain of models. Then for any $\beta < \alpha$, $\mathfrak{A}_\beta$ is an elementary submodel of $\bigcup_{\beta<\alpha}\mathfrak{A}_\beta$.

Proof. Let $\mathfrak{A} = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta$. We need to show that for any $\beta < \alpha$, for any formula $\phi(v_1, ..., v_n)$, any $a_1, ..., a_n \in A_\beta$,

$$[\![\phi]\!]^{\mathfrak{A}}[a_1, ..., a_n] = [\![\phi]\!]^{\mathfrak{A}_{\mathfrak{beta}}}[a_1, ..., a_n]$$

The atomic cases are already covered by Proposition 1.5.1. The inductive cases on sentential connectives are straightforward. Let $\phi(v_1, ..., v_n) = \exists v\psi(v, v_1, ..., v_n)$.

Let $[\![\exists v\psi]\!]^{\mathfrak{A}}[a_1, ..., a_n] = \bigsqcup_{a\in A}[\![\psi]\!]^{\mathfrak{A}}[a, a_1, ..., a_n] = p_1 \in B$. Let $[\![\exists v\psi]\!]^{\mathfrak{A}_\beta}[a_1, ..., a_n] = \bigsqcup_{a\in A}[\![\psi]\!]^{\mathfrak{A}_\beta}[a, a_1, ..., a_n] = p_2 \in B$.

Since $\mathfrak{A} = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta$, $A_\beta \subseteq A$. By inductive hypothesis we have $p_2 \leqslant p_1$. Hence we only need to show that $p_1 \leqslant p_2$.

Suppose $p_1 \nleqslant p_2$. Then for some $a \in A$, $[\![\psi]\!]^{\mathfrak{A}}[a, a_1, ..., a_n] \nleqslant p_2$. Let $[\![\psi]\!]^{\mathfrak{A}}[a, a_1, ..., a_n]$ be p_3.

Since $a \in A = \bigcup_{\beta<\alpha} A_\beta$, for some $\eta < \alpha$, $a \in A_\eta$. Either $\eta \leqslant \beta$ or $\beta \leqslant \eta$. We will show that both possibilities lead to contradiction.

Suppose $\eta \leqslant \beta$. Then $a, a_1, ..., a_n \in A_\beta$. By inductive hypothesis, $[\![\psi]\!]^{\mathfrak{A}_\beta}[a, a_1, ..., a_n] = [\![\psi]\!]^{\mathfrak{A}}[a, a_1, ..., a_n] = p_3$. But then $p_3 \leqslant p_2 = [\![\exists v\psi]\!]^{\mathfrak{A}_\beta}[a_1, ..., a_n]$. Contradiction.

Suppose $\beta \leqslant \eta$. Then $a, a_1, ..., a_n \in A_\eta$. By inductive hypothesis, $[\![\psi]\!]^{\mathfrak{A}_\eta}[a, a_1, ..., a_n] = [\![\psi]\!]^{\mathfrak{A}}[a, a_1, ..., a_n] = p_3$. But since $a_1, ..., a_n \in A_\beta$, and $\mathfrak{A}_\beta$ is an elementary submodel of $\mathfrak{A}_\eta$ by the construction of the chain,

$$[\![\exists v\psi]\!]^{\mathfrak{A}_\eta}[a_1, ..., a_n] = [\![\exists v\psi]\!]^{\mathfrak{A}_\beta}[a_1, ..., a_n] = p_2$$

But then $p_3 \leqslant p_2$. Contradiction.

Hence $p_1 \leqslant p_2$. And therefore $p_1 = p_2$.

□

Corollary 1.5.11.1 (Robinson Consistency Theorem)**.** Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two languages and let $\mathscr{L} = \mathscr{L}_1 \cap \mathscr{L}_2$. Let B be a complete Boolean algebra. Suppose S^B is a maximal Boolean-valuation in $\mathscr{L}$ and $S_1^B \subseteq S^B$, $S_2^B \subseteq S^B$ are consistent Boolean-valuations in $\mathscr{L}_1$, $\mathscr{L}_2$ respectively. Then $S_1^B \cup S_2^B$ is consistent in the language $\mathscr{L}_1 \cup \mathscr{L}_2$.

Proof. Let $\mathfrak{A}_0 \models S_1^B$ and $\mathfrak{B}_0 \models S_2^B$. Let $\mathfrak{A}_0 \upharpoonright \mathscr{L}$ be the reduct of $\mathfrak{A}_0$ to $\mathscr{L}$ (and similarly for

$\mathfrak{B}_0 \restriction \mathscr{L}$) Then both $\mathfrak{A}_0 \restriction \mathscr{L}$ and $\mathfrak{B}_0 \restriction \mathscr{L}$ are models of S^B, and since S^B is maximal, $\mathfrak{A}_0 \restriction \mathscr{L}$ and $\mathfrak{B}_0 \restriction \mathscr{L}$ are elementarily equivalent. Then, by an argument similar to that in the proof of Theorem 1.5.8, we can show that the elementary diagram of $\mathfrak{A}_0 \restriction \mathscr{L}$ is consistent with the elementary diagram of $\mathfrak{B}_0$. Hence we there is some $\mathfrak{B}_1$ that is an elementary extension of $\mathfrak{B}_0$ and also models the elementary diagram of $\mathfrak{A}_0 \restriction \mathscr{L}$. Therefore there is an elementary embedding $f_1 : \mathfrak{A}_0 \restriction \mathscr{L}_{A_0} \to \mathfrak{B}_1 \restriction \mathscr{L}_{A_0}$, where $\mathscr{L}_{A_0} = \mathscr{L} \cup \{c_a \mid a \in A_0\}$, and $\mathfrak{A}_0 \restriction \mathscr{L}_{A_0}$ is the canonical expansion of $\mathfrak{A}_0 \restriction \mathscr{L}$.

For the same reason why the elementary diagram of $\mathfrak{A}_0 \restriction \mathscr{L}$ is consistent with the elementary diagram of $\mathfrak{B}_0$, the elementary diagram of $\mathfrak{B}_1 \restriction \mathscr{L}_{A_0}$ is consistent with the elementary diagram of the canonical expansion of $\mathfrak{A}_0$ to $\mathscr{L}_{A_0}$. Therefore there is an elementary extension $\mathfrak{A}_1$ of $\mathfrak{A}_0$ such that $\mathfrak{A}_1$ models the elementary diagram of $\mathfrak{B}_1 \restriction \mathscr{L}_{A_0}$. Hence there is an elementary embedding $g_1 : \mathfrak{B}_1 \restriction \mathscr{L}_{A_0} \to \mathfrak{A}_1 \restriction \mathscr{L}_{A_0}$, such that for any $a \in A_0$, $g_1(f_1(a)) = a$.

Repeating this construction method ad infinitum. Let $\mathfrak{A} = \bigcup_{n<\omega} \mathfrak{A}_n$ and $\mathfrak{B} = \bigcup_{n<\omega} \mathfrak{B}_n$. By Theorem 1.5.11, $\mathfrak{A} \models S_1^B$ and $\mathfrak{B} \models S_2^B$. Also, $f = \bigcup_{n<\omega} f_n$ is an isomorphism between $\mathfrak{A} \restriction \mathscr{L}$ and $\mathfrak{B} \restriction \mathscr{L}$. Putting $\mathfrak{A}$ and $\mathfrak{B}$ together we get a model for $S_1^B \cup S_2^B$.

□

1.6 Supervaluationism

In this section, we show that supervaluation models are special cases of Boolean-valued models. In particular, we show that every supervaluation model is equivalent to an elementary submodel of the direct product of the precisifications. Also, the class of supervaluation models is equivalent to a subclass of true-identity Boolean-valued models.

Definition 1.6.1. A *supervaluation* model $\mathfrak{S}$ for $\mathscr{L}$ is a pair $\langle A, \Sigma \rangle$ such that A is a domain of elements and $\Sigma = \{\sigma_i \mid i \in I\}$ is a collection of two-valued interpretation functions (indexed by I). In particular,[20]

[20] We assume here that a constant is always interpreted as the same individual in all precisifications. Although this is the default assumption in most standard formulations of supervaluationism (as in, for example, [12] or [32]), we are aware of the need for loosening this assumption in certain situations. The results we present below can be generated to more general definitions of supervaluation models, including ones in which constants can have different referents in different precisifications, and even ones in which the domains of different precisifications can be different. Due to the lack of space we will not present the details here. Roughly,

1. Let c be a constant in $\mathscr{L}$. For some $a \in A$, for any $i \in I$, $\sigma_i(c) = a$.

2. Let P be a n-ary relation in $\mathscr{L}$. For any $i \in I$, $\sigma_i(P) = R_i \subseteq A^n$.

For each $i \in I$, $\mathfrak{A}_\mathrm{i}$ is the two-valued model for $\mathscr{L}$ with domain A and interpretation function Σ_i. Every $\mathfrak{A}_\mathrm{i}$ is called a *precisification* in $\mathfrak{S}$.

For any formula $\phi(v_1,...,v_n)$ in $\mathscr{L}$, and any assignment function $x : Var \to A$,

$$[\![\phi]\!]^{\mathfrak{S}}[x] = \begin{cases} \textit{(super)true} \text{ if for every } i \in I, \mathfrak{A}_\mathrm{i} \models \phi[x]; \\ \textit{(super)false} \text{ if for every } i \in I, \mathfrak{A}_\mathrm{i} \models \neg\phi[x]; \\ \textit{undefined} \text{ if otherwise} \end{cases}$$

Definition 1.6.2. Given a supervaluation model $\mathfrak{S} = \langle A, \{\sigma_i \mid i \in I\}\rangle$, we construct a $P(I)$-valued model $\mathfrak{M}^{\mathfrak{S}}$ for $\mathscr{L}$ as follows (where $P(I)$ is the powerset of I endowed with the powerset algebra):

1. The domain of $\mathfrak{M}^{\mathfrak{S}}$ is A.

2. $[\![=]\!]^{\mathfrak{M}^{\mathfrak{S}}} : A^2 \to P(I)$ is such that for any $a, b \in A$, $[\![a = b]\!] = \varnothing$ if a and b are not the same element, and $[\![a = b]\!] = I$ if a and b are the same element.

3. Let c be a constant in $\mathscr{L}$, $[\![c]\!]^{\mathfrak{M}^{\mathfrak{S}}} = \sigma_i(c)$, for any $i \in I$.

4. Let P be a n-ary relation in $\mathscr{L}$. $[\![P]\!]^{\mathfrak{M}^{\mathfrak{S}}} : A^n \to P(I)$ is such that for any $a_1,...,a_n \in A$, $[\![P(a_1,...,a_n)]\!]^{\mathfrak{M}^{\mathfrak{S}}} = \{i \in I \mid \mathfrak{A}_\mathrm{i} \models P(a_1,...,a_n)\}$.

It is easy to check that $\mathfrak{M}^{\mathfrak{S}}$ satisfies Def 1.2.8.

Theorem 1.6.1. For any formula $\phi(v_1,...,v_n)$ in $\mathscr{L}$, and any assignment function $x : Var \to A$,

$$[\![\phi]\!]^{\mathfrak{M}^{\mathfrak{S}}}[x] = \{i \in I \mid \mathfrak{A}_\mathrm{i} \models \phi[x]\}$$

in cases where we have constants without a unvarying referent, we can simply regard a constant as a unary predicate that satisfies the special condition that its extension is a singleton. And in cases where we have precisifications with different domains, we can simply pretend that all precisifications have the union of all the domains as their domain, and have an existential predicate whose extension in each precisification is the actual domain of the precisification, and have the quantifiers be restricted to what satisfies the existential predicate in each precisification.

Proof. By induction on the complexity of ϕ. The atomic cases are covered by the definition of $\mathfrak{M}^{\mathfrak{S}}$. The cases for sentential connectives are straightforward. For the existential quantifier,

$$\begin{aligned}
[\![\exists v_j \phi]\!]^{\mathfrak{M}^{\mathfrak{S}}}[x] &= \bigcup_{a \in A} [\![\phi]\!]^{\mathfrak{M}^{\mathfrak{S}}}[x(v_j/a)] \\
&= \bigcup_{a \in A} \{i \in I \mid \mathfrak{A}_{\mathrm{i}} \models \phi[x(v_j/a)]\} \\
&= \{i \in I \mid \mathfrak{A}_{\mathrm{i}} \models \exists v_j \phi[x]\}
\end{aligned}$$

The case for the universal quantifier is similar.

□

$\mathfrak{M}^{\mathfrak{S}}$ is the Boolean counterpart of the supervaluation model $\mathfrak{S}$. They have the same domain, and for any ϕ in $\mathscr{L}$, the degree to which ϕ is true in $\mathfrak{M}^{\mathfrak{S}}$ is the set of all precisifications in $\mathfrak{S}$ in which ϕ is true. Therefore, ϕ is (super)true in $\mathfrak{S}$ iff $[\![\phi]\!]^{\mathfrak{M}^{\mathfrak{S}}} = I$, which is the top value in $P(I)$, and ϕ is (super)false in $\mathfrak{S}$ iff $[\![\phi]\!]^{\mathfrak{M}^{\mathfrak{S}}} = \varnothing$, which is the bottom value in $P(I)$. Since all classical tautologies have value 1 in every Boolean-valued model, all classical tautologies are (super)-true in every supervaluation model.

We next show that $\mathfrak{S}$ is an elementary submodel of the direct product of all the precisifications.

Theorem 1.6.2. Let $\mathfrak{S} = \langle A, \{\sigma_i \mid i \in I\}\rangle$ be a supervaluation model. Let $\{\mathfrak{A}_{\mathrm{i}} \mid i \in I\}$ be its set of precisifications. Let $\prod_{i \in I} \mathfrak{A}_{\mathrm{i}}$ be their direct product. $\mathfrak{M}^{\mathfrak{S}}$ is an elementary submodel of $\prod_{i \in I} \mathfrak{A}_{\mathrm{i}}$.

Proof. Clearly $P(I)$ and $\prod_{i \in I} 2$ are isomorphic. The elementary embedding is the function $f : A \to \prod_{i \in I} A_i$ that takes any $a \in A$ to $\langle a \rangle_{i \in I}$.

We just need to show that for any formula $\phi(v_1, \ldots, v_n)$ in $\mathscr{L}$ and any $a_1, \ldots, a_n \in A$,

$$[\![\phi(a_1, \ldots, a_n)]\!]^{\mathfrak{M}^{\mathfrak{S}}} = [\![\phi(\langle a_1 \rangle_{i \in I}, \ldots, \langle a_n \rangle_{i \in I})]\!]^{\prod_{i \in I} \mathfrak{A}_{\mathrm{i}}}$$

By the Direct Product Theorem (Theorem 1.3.2), $[\![\phi(\langle a_1 \rangle_{i \in I}, \ldots, \langle a_n \rangle_{i \in I})]\!]^{\prod_{i \in I} \mathfrak{A}_{\mathrm{i}}} = \{i \in I \mid \mathfrak{A}_{\mathrm{i}} \models \phi(a_1, \ldots, a_n)\} = [\![\phi(a_1, \ldots, a_n)]\!]^{\mathfrak{M}^{\mathfrak{S}}}$, by Theorem 1.6.1.

□

Observation 1.6.2.1. Let $\mathfrak{S} = \langle A, \{\sigma_i \mid i \in I\}\rangle$ be a supervaluation model. $\mathfrak{M}^{\mathfrak{S}}$ may not be a witnessing model, although $\prod_{i \in I} \mathfrak{A}_{\mathrm{i}}$ is always witnessing. The latter is because direct products always inherit the property of being witnessing, which follows from Theorem 1.3.2. It is easy to construct examples of the former. For example, we can let a unary predicate P be such that it has a non-empty extension in every $\mathfrak{A}_{\mathrm{i}}$ in $\mathfrak{S}$, yet there is no $a \in A$ that is in the extension of P in in every $\mathfrak{A}_{\mathrm{i}}$ in $\mathfrak{S}$. Then $\exists v_i P(v_i)$ will have value I in $\mathfrak{M}^{\mathfrak{S}}$ without a witness.

Corollary 1.6.2.1 (to Theorem 1.4.1 and Theorem 1.4.3)**.** Let T be a theory and ϕ be a sentence in a first order language $\mathscr{L}$. $T \vdash \phi$ if and only if for any supervaluation model $\mathfrak{S}$, if every member of T is (super)true in $\mathfrak{S}$, then ϕ is (super)true in $\mathfrak{S}$.

We have shown that every supervaluation model is equivalent to a true identity Boolean-valued model. Our next goal is to establish a duality between the class of supervaluation models and a subclass of true identity models.

Theorem 1.6.3. Let $\mathfrak{A}$ be a B-valued model. Then $\mathfrak{A}$ is the Boolean counterpart of a supervaluation model just in case $\mathfrak{A}$ is a true identity model and the value range of $\mathfrak{A}$ is isomorphic to a powerset algebra.

Proof. The left to right direction holds by Theorem 1.6.1. For the other direction, let $\mathfrak{A}$ be a true identity B-valued model with value range $\mathscr{P}(I)$. For each $i \in I$, we construct a 2-valued model $\mathfrak{A}_{\mathrm{i}}$ with domain A as follows. For any constant c, let $[\![c]\!]^{\mathfrak{A}_{\mathrm{i}}} = [\![c]\!]^{\mathfrak{A}} \in A$. For any n-ary relation P, any $a_1, \ldots, a_n \in A$, let $\mathfrak{A}_{\mathrm{i}} \models P(a_1, \ldots, a_n)$ iff $i \in f([\![P(a_1, \ldots, a_n)]\!]^{\mathfrak{A}})$. Then it is easy to show that $\mathfrak{A}$ is the Boolean counterpart of the supervaluation model with precisifications $\{\mathfrak{A}_{\mathrm{i}} \mid i \in I\}$.

□

The duality we established above shows that Boolean-valued models generalize supervaluation models in two respects. First, Boolean-valued models allow identity clauses to take intermediate truth values, whereas supervaluation models require true identity. Second, Boolean-valued models allow the value range of a model to be any complete Boolean algebra, whereas supervaluation models require powerset algebras (or those embeddable in a powerset algebra in a complete way).

1.7 Full Models

1.7.1 Full Models

We have argued in both Section 1.3 and 1.5 that certain features of two-valued models are not shared by all Boolean-valued models but only those that are witnessing. As a step further, in this section, we will define another condition on Boolean-valued models that is even more demanding than being witnessing, and that some interesting features of two-valued models can only be generalized to full Boolean-valued models.

Definition 1.7.1 (Antichain)**.** Let B be a Boolean algebra. A subset $D \subseteq B$ is an *antichain* just in case for any $p, q \in D$, $p \sqcap q = 0$.

Definition 1.7.2 (Full Model)**.** Let $\mathfrak{A}$ be a B-valued model. $\mathfrak{A}$ is a *full* model just in case for any antichain $D \subseteq B$, and $\{a_d \mid d \in D\} \subseteq A$, there is an $a \in A$ such that for any $d \in D$, $d \leqslant [\![a = a_d]\!]^{\mathfrak{A}}$.

Proposition 1.7.1. Any two-valued model is full.

The definition of a full model might seem obscure to those who are not familiar with Boolean-valued models. In the next subsection, we will present an alternative characterization of full models that are much more accessible. In particular, we will show (Theorem 1.7.8) that full models are precisely those that are *highly witnessing* (Def. 1.7.5), in the sense that they remain witnessing no matter how they are expanded.

Theorem 1.7.1. Let I be an arbitrary index set. For any $i \in I$, let $\mathfrak{A}_\mathrm{i}$ be a full B_i-valued model. Let $\prod_{i \in I} \mathfrak{A}_\mathrm{i}$ be the $\mathfrak{A}_\mathrm{i}$'s product model. Then $\prod_{i \in I} \mathfrak{A}_\mathrm{i}$ is full.

Proof. Let $D \subseteq \prod_{i \in I} B_i$ be an antichain. Let $A = \{\langle a_i \rangle^d_{i \in I} \mid d \in D\} \subseteq \prod_{i \in I} A_i$. For any $i \in I$, let $pr_i : \prod_{i \in I} B_i \to B_i$ be the ith projection function on $\prod_{i \in I} B_i$. Let $pr_i[D] = \{pr_i(d) \mid d \in D\}$.

Since D is an antichain in $\prod_{i\in I} B_i$, for any $i \in I$, $pr_i[D]$ is an antichain in B_i. Similarly, let $pr'_i : \prod_{i\in I} A_i \to A_i$ be the ith projection function on $\prod_{i\in I} A_i$. $pr'_i[A] = \{pr_i(\langle a_i\rangle^d_{i\in I}) \mid \langle a_i\rangle^d_{i\in I} \in A\}$.

Since $\mathfrak{A}_i$ is a full model, there is an $b_i \in A_i$ such that for any $pr_i(d) \in pr_i[D]$, $pr_i(d) \leqslant [\![b_i = a^d_i]\!]^{\mathfrak{A}_i}$, where $a^d_i = pr'_i(\langle a_i\rangle^d_{i\in I})$. Pick such a b_i for each $i \in I$. Form $\langle b_i\rangle^d_{i\in I} \in \prod_{i\in I} A_i$. Hence, for any $d \in D$, $d \leqslant [\![\langle a_i\rangle^d_{i\in I} = \langle b_i\rangle^d_{i\in I}]\!]^{\prod_{i\in I}\mathfrak{A}_i}$.

Hence $\prod_{i\in I}\mathfrak{A}_i$ is a full model.

□

Theorem 1.7.2. Let I be an arbitrary index set. For any $i \in I$, let $\mathfrak{A}_i$ be a full B_i-valued model. Let $\prod_{i\in I}\mathfrak{A}_i$ be the $\mathfrak{A}_i$'s product model. If $\prod_{i\in I}\mathfrak{A}_i$ is a full model, then for any $i \in I$, $\mathfrak{A}_i$ is a full model.

Proof. Fix an $i \in I$. Let $D_i \subseteq B_i$ be an antichain. Let $\{a^c_i \mid c \in D_i\} \subseteq A_i$. We construct the following antichain D on $\prod_{i\in I} B_i$, with $pr_i[D] = D_i$: $d \in D$ iff (a) for some $c \in D_i$, $pr_i(d) = c$, and (b) for any $j \in I$, if $i \neq j$, then $pr_j(d) = 0_{B_j}$.

Let $d \in D$. We construct an element $\langle a_i\rangle^d_{i\in I} \in \prod_{i\in I} A_i$: let $pr'_i(\langle a_i\rangle^d_{i\in I}) = a^c_i$, where $c = pr_i(d)$. Then for any $j \in I$ that is different from i, let $pr'_j(\langle a_i\rangle^d_{i\in I})$ be an random element in A_j.

Since $\prod_{i\in I}\mathfrak{A}_i$ is full, for some $\langle b_i\rangle^d_{i\in I} \in \prod_{i\in I} A_i$, for any $d \in D$, $d \leqslant [\![\langle b_i\rangle^d_{i\in I} = \langle a_i\rangle^d_{i\in I}]\!]^{\mathfrak{A}}$. In particular, for any $d \in D$, $pr_i(d) \leqslant [\![a^d_i, b_i]\!]^{\mathfrak{A}_i}$. Hence for any $c \in D_i$, $c \leqslant [\![a^c_i, b_i]\!]^{\mathfrak{A}_i}$. Hence $\mathfrak{A}_i$ is full.

□

Theorem 1.7.3. Let $\mathfrak{A}$ be a full B-valued model. Then $\mathfrak{A}$ is witnessing.

Proof. For simplicity we ignore the parameters. Let $\phi(v)$ be a formula with only v free. Let $[\![\exists v\phi]\!]^{\mathfrak{A}} = p \in B$. We will show that for some $a \in A$, $[\![\phi(v)]\!]^{\mathfrak{A}}[a] = p$. If $p = 0$, then the statement is trivial. So we assume $p > 0$.

Let $D = \{d \in B\backslash\{0\} \mid \text{for some } a^d \in A, d \leqslant [\![\phi(a^d)]\!]^{\mathfrak{A}}\}$. Let Q be the set of all antichains made up of elements in D. By Zorn's lemma, Q has a maximal element. Call it C.

We can show that D is dense below p. Let $0 \neq p' \leqslant p$. Since $p = \bigsqcup_{a\in A}[\![\phi(a)]\!]^{\mathfrak{A}}$, for some $a \in A$, $p' \sqcap [\![\phi(a)]\!]^{\mathfrak{A}} \neq 0$. But $p' \sqcap [\![\phi(a)]\!]^{\mathfrak{A}} \in D$ and $p' \sqcap [\![\phi(a)]\!]^{\mathfrak{A}} \leqslant p'$.

Hence $p \leqslant \bigsqcup C$: suppose not, then $p \sqcap -(\bigsqcup C) \neq 0$. Since D is dense below p, for some $d \in D$, $d \leqslant p \sqcap -(\bigsqcup C) \leqslant -(\bigsqcup C)$. Then $C \cup \{d\}$ is an antichain in D that properly extends C. Contradiction.

For every $d \in C$, let a^d be some element in A such that $d \leqslant [\![\phi(a^d)]\!]^{\mathfrak{A}}$.

Since $\mathfrak{A}$ is full, there is some $a \in A$ such that for all $d \in C$, $d \leqslant [\![a = a^d]\!]^{\mathfrak{A}}$.

Since $d \leqslant [\![\phi(a^d)]\!]^{\mathfrak{A}}$ as well, $d \leqslant [\![a = a^d]\!]^{\mathfrak{A}} \sqcap [\![\phi(a^d)]\!]^{\mathfrak{A}} \leqslant [\![\phi(a)]\!]^{\mathfrak{A}}$. Hence $p = [\![\exists v \phi]\!]^{\mathfrak{A}} \leqslant \bigsqcup C \leqslant [\![\phi(a)]\!]^{\mathfrak{A}}$. And trivially $[\![\phi(a)]\!]^{\mathfrak{A}} \leqslant [\![\exists v \phi]\!]^{\mathfrak{A}}$. Hence $[\![\phi(a)]\!]^{\mathfrak{A}} = [\![\exists v \phi]\!]^{\mathfrak{A}}$.

□

We proved that full models are witnessing. But are witnessing models full? The answer, it turns out, is negative. There exist witnessing models that are not full. Being full is a condition that is properly stronger than being witnessing. An example of a witnessing but not full model will be given in the next section.

We began this section by claiming that more features of two-valued models can be generalized to full Boolean-valued models. In the rest of this section, we will illustrate this point by two examples. The first example is given by the following theorem, and the second example will be given in the next subsection.

Theorem 1.7.4. Let $\mathscr{L}$ be a countable language. Let $\mathfrak{A}$ be a full B-valued model of $\mathscr{L}$. Let $h : B \to 2$ be a countably incomplete homomorphism. Then the quotient model $\mathfrak{A}^h$ is an ω_1-saturated model of $\mathscr{L}$.

Proof. We want to show that $\mathfrak{A}^h$ is ω_1-saturated, i.e. for every countable sequence $\langle [a_n]_h \rangle_{n<\omega}$ that consists of elements in A^h, for every type $\Sigma(v)$ of $\mathscr{L} \cup \{c_i \mid i < \omega\}$, if $\Sigma(v)$ is consistent with $Th((\mathfrak{A}^h, [a_n]_h)_{n<\omega})$[21], then $\Sigma(v)$ is realized in $(\mathfrak{A}^h, [a_n]_h)_{n<\omega}$.

Let $\Sigma(v) = \{\sigma_1(v), \sigma_2(v), \ldots\}$ of $\mathscr{L} \cup \{c_i \mid i < \omega\}$ be consistent with $Th((\mathfrak{A}^h, [a_n]_h)_{n<\omega})$. Let $\Delta(v)$ be a finite subset of $\Sigma(v)$. Let $\phi(v)$ the formula that is the conjunction of all formulas in $\Delta(v)$

Then $\exists v \phi(v)$ is a sentence consistent with $Th((\mathfrak{A}^h, [a_n]_h)_{n<\omega})$. As a result, it is in $Th((\mathfrak{A}^h, [a_n]_h)_{n<\omega})$. Hence $\Delta(v)$ is realized in $(\mathfrak{A}^h, [a_n]_h)_{n<\omega}$.

[21]This is the set of all sentences true in the model $(\mathfrak{A}^h)^*$, where $(\mathfrak{A}^h)^*$ is the model resulting from expanding $\mathfrak{A}^h$ to the language $\mathscr{L} \cup \{c_i \mid i < \omega\}$ by interpreting each c_i with $[a_i]_h$. And we use $(\mathfrak{A}^h, [a_n]_h)_{n<\omega}$ to denote $(\mathfrak{A}^h)^*$.

Also $\mathscr{L}$ is arbitrary and $\mathscr{L} \cup \{c_i \mid i < \omega\}$ is countable. Hence to show what we want to show, we just need to show the following: Let $\Sigma(v)$ be a type of $\mathscr{L}$. If every finite subset of $\Sigma(v)$ is realized in $\mathfrak{A}^h$, then $\Sigma(v)$ is realized in $\mathfrak{A}^h$.

Assume the antecedent. Let $U \subseteq B$ be the subset of B whose characteristic function is h. Then U is a countably incomplete ultrafilter. Hence there is some $D \subseteq U$ such that $\bigsqcap D \notin U$. Hence we can find a countable descending chain in B: $1 = p_0 \geqslant p_1 \geqslant p_2 \geqslant \ldots$, such that for all $n < \omega$, $p_n \in U$, but $\bigsqcap_{n<\omega} p_n = 0$.

For each $n < \omega$, let $q_n = p_n \sqcap [\![\exists v(\sigma_1(v) \wedge \ldots \wedge \sigma_n(v))]\!]^{\mathfrak{A}}$.

By Theorem 1.7.3, $\mathfrak{A}$ is witnessing and hence it is in the scope of Theorem 1.3.4. Hence $[\![\exists v(\sigma_1(v) \wedge \ldots \wedge \sigma_n(v))]\!]^{\mathfrak{A}} \in U$ iff $\mathfrak{A}^h \models \exists v(\sigma_1(v) \wedge \ldots \wedge \sigma_n(v))$. But by assumption, $\{\sigma_1(v), \ldots, \sigma_n(v)\}$, a finite subset of $\Sigma(v)$, is realized in $\mathfrak{A}^h$. Hence for every $n < \omega$, $[\![\exists v(\sigma_1(v) \wedge \ldots \wedge \sigma_n(v))]\!]^{\mathfrak{A}} \in U$. Hence for every $n < \omega$, $q_n \in U$.

Also $\bigsqcap_{n<\omega} q_n \leqslant \bigsqcap_{n<\omega} p_n = 0$.

For all $n < \omega$, since $p_{n+1} \leqslant p_n$ and $[\![\exists v(\sigma_1(v) \wedge \ldots \wedge \sigma_{n+1}(v))]\!]^{\mathfrak{A}} \leqslant [\![\exists v(\sigma_1(v) \wedge \ldots \wedge \sigma_n(v))]\!]^{\mathfrak{A}}$, $q_{n+1} \leqslant q_n$. Hence $-q_n \leqslant -q_{n+1}$.

Consider $\{q_n \sqcap q_{n+1} \mid n < \omega\}$. This is an antichain: let $i \leqslant j < \omega$, $(q_i \sqcap -q_{i+1}) \sqcap (q_j \sqcap -q_{j+1}) = (q_i \sqcap q_j) \sqcap (-q_{i+1} \sqcap -q_{j+1}) = q_j \sqcap -q_{i+1} = 0$.

For any $n < \omega$, let $a_n \in A$ be such that $[\![\sigma_1(v) \wedge \ldots \wedge \sigma_n(v)]\!]^{\mathfrak{A}}[a_n] = [\![\exists v(\sigma_1(v) \wedge \ldots \wedge \sigma_n(v))]\!]^{\mathfrak{A}}$. The existence of such an a_n is guaranteed by Theorem 1.7.3.

Since $\mathfrak{A}$ is full, there is an $a \in A$ such that for all $n < \omega$, $q_n \sqcap -q_{n+1} \leqslant [\![a_n = a]\!]^{\mathfrak{A}}$.

We will show that $[a]_h \in \mathfrak{A}^h$ realizes $\Sigma(x)$. Let $i \leqslant n < \omega$. $[\![\sigma_i(a)]\!]^{\mathfrak{A}} \geqslant [\![\sigma_1(a) \wedge \ldots \wedge \sigma_n(a)]\!]^{\mathfrak{A}} \geqslant [\![\sigma_1(a_n) \wedge \ldots \wedge \sigma_n(a_n)]\!]^{\mathfrak{A}} \sqcap [\![a_n = a]\!]^{\mathfrak{A}} \geqslant q_n \sqcap -q_{n+1}$. Hence $[\![\sigma_i(a)]\!]^{\mathfrak{A}} \geqslant \bigsqcup_{i\leqslant n<\omega}(q_n \sqcap -q_{n+1}) = (\bigsqcup_{i\leqslant n<\omega} q_n) \sqcap (\bigsqcup_{i\leqslant n<\omega} -q_{n+1}) = q_i \sqcap (\bigsqcup_{j<\omega} -q_j) = q_i \sqcap -(\bigsqcup_{j<\omega} q_j) = q_i \sqcap 1 = q_i \in U$.

□

1.7.2 Łoś Theorem on Σ^1_1 Formulas

A well-know corollary of the Łoś Theorem on two-valued models is that Σ^1_1 formulas are preserved under ultraproducts.[22] In this subsection, we will show that this corollary can

[22] See, for example, Chang and Keisler [7, p. 221-222].

be generalized to full Boolean-valued models, yet not necessarily to witnessing Boolean-valued models. First we need some definitions.

Definition 1.7.3. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. For any $n \in \omega$, we define $X^n_{\mathfrak{A}}$ as the following set: $X^n_{\mathfrak{A}} = \{R : A^n \to B \mid \text{for any } \langle a_1, \dots, a_n\rangle, \langle b_1, \dots, b_n\rangle \in A^n, R(a_1, \dots, a_n) \sqcap (\bigsqcap_{1 \leqslant i \leqslant n} [\![a_i = b_i]\!]^{\mathfrak{A}}) \leqslant R(b_1, \dots, b_n)\}$. We call the $X^n_{\mathfrak{A}}$'s the *second-order domains* of $\mathfrak{A}$. For each $n \in \omega$, we call $X^n_{\mathfrak{A}}$ the n-ary second-order domain of $\mathfrak{A}$.

Definition 1.7.4. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. $\mathfrak{A}$ is *second-order full* just in case for any $i \in \omega$, if $D \subseteq B$ is an antichain and $\{R^d : A^n \to B \mid d \in D\} \subseteq X^n_{\mathfrak{A}}$, then there exists some $R \in X^n_{\mathfrak{A}}$ such that for any $d \in D$, $d \leqslant \bigsqcap_{\langle a_1, \dots, a_n\rangle \in A^n} (R(a_1, \dots, a_n) \leftrightarrow R^d(a_1, \dots, a_n))$.

Theorem 1.7.5. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. $\mathfrak{A}$ is second-order full.

Proof. For simplicity we only prove the case where $i = 1$. The cases where $i > 1$ are very similar. Let $D \subseteq B$ be an antichain and $\{R^d : A \to B \mid d \in D\} \subseteq X^1_{\mathfrak{A}} = X$.

We define $R : A \to B$ as follows: for any $a \in A$,

$$R(a) = \bigsqcup_{c \in D} (c \sqcap R^c(a))$$

We first prove that $R \in X$, i.e. for any $a, b \in A$, $R(a) \sqcap [\![a = b]\!]^{\mathfrak{A}} \leqslant R(b)$:

$$\begin{aligned}
R(a) \sqcap [\![a = b]\!]^{\mathfrak{A}} &= (\bigsqcup_{c \in D} (c \sqcap R^c(a))) \sqcap [\![a = b]\!]^{\mathfrak{A}} \\
&= \bigsqcup_{c \in D} (c \sqcap (R^c(a) \sqcap [\![a = b]\!]^{\mathfrak{A}})) \\
&\leqslant \bigsqcup_{c \in D} (c \sqcap R^c(b)) \\
&= R(b)
\end{aligned}$$

The third line holds since $R^c(a) \sqcap [\![a = b]\!]^{\mathfrak{A}}) \leqslant R^c(b)$, as $R^c \in X$.

Next, we prove that for any $d \in D$, $d \leqslant \bigsqcap_{a \in A} (R(a) \leftrightarrow R^d(a))$. Fix some $d \in D$, we just need to show that for any $a \in A$, $d \leqslant R(a) \leftrightarrow R^d(a)$. That is, $d \leqslant -R(a) \sqcup R^d(a)$ and $d \leqslant R(a) \sqcup -R^d(a)$.

This is equivalent to $d \sqcap R^d(a) \leqslant R(a) \leqslant -d \sqcup R^d(a)$, and by definition $R(a) = \bigsqcup_{c \in D}(c \sqcap R^c(a))$.

The first inequality holds trivially, as $d \in D$. For the second inequality, we just need to show that for any $c \in D$, $c \sqcap R^c(a) \leqslant -d \sqcup R^d(a)$.

There are two cases. First, if $c = d$, then $R^c(a) = R^d(a)$, and hence the inequality holds. Second, if $c \neq d$, then since $c, d \in D$ and D is an antichain, $c \sqcap d = 0$, and hence $c \leqslant -d$, and hence the inequality holds. Either way the inequality holds.

Hence we find a $R \in X$ that witnesses the existential claim, and therefore $\mathfrak{A}$ is second-order full.

□

Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. The next thing we will do is to define the value of a Σ^1_1 sentence in $\mathfrak{A}$. We first need some notation. Let P be an n-ary predicate that is not in $\mathscr{L}$. Let $R \in X^n_{\mathfrak{A}}$. Then we may expand $\mathfrak{A}$ to a model of $\mathscr{L} \cup \{P\}$ by interpreting the new predicate P as R. We use $(\mathfrak{A}, R)$ to denote the expanded model $\mathscr{L} \cup \{P\}$, where $[\![P]\!]^{(\mathfrak{A},R)} = R$.

A Σ^1_1 formula over $\mathscr{L}$ is a formula ψ of the following form:

$$\psi = \exists P_1, \ldots, P_m \phi$$

where for every $1 \leqslant i \leqslant m$, P_i is a new predicate symbol not occurring in $\mathscr{L}$, and ϕ is a formula in the expanded first-order language $\mathscr{L} \cup \{P_1, \ldots, P_m\}$.

Let $\psi = \exists P_1, \ldots, P_m \phi$ be a Σ^1_1 formula. For every $1 \leqslant i \leqslant m$, we use n_i to denote the arity of P_i. Then given an assignment $x : Var \to A$, we define the value of ψ in $\mathfrak{A}$ under assignment x as follows:

$$[\![\exists P_1, \ldots, P_m \phi]\!]^{\mathfrak{A}}[x] = \bigsqcup_{R^1 \in X^{n_1}_{\mathfrak{A}}} \bigsqcup_{R^2 \in X^{n_2}_{\mathfrak{A}}} \cdots \bigsqcup_{R^m \in X^{n_m}_{\mathfrak{A}}} [\![\phi]\!]^{(\mathfrak{A}, R_1, \ldots, R_m)}[x]$$

The next thing we will prove is that if $\mathfrak{A}$ is a second-order full model, then for any Σ^1_1 formula $\psi = \exists P_1, \ldots, P_m \phi$, any assignment $x : Var \to A$, there is some $R_1 \in X^{n_1}_{\mathfrak{A}}, R_2 \in$

$X^{n_2}_{\mathfrak{A}}, ..., R_m \in X^{n_m}_{\mathfrak{A}}$ such that $[\![\psi]\!]^{\mathfrak{A}}[x] = [\![\phi]\!]^{(\mathfrak{A},R_1,...,R_m)}[x]$. To this end we first need a lemma.

Lemma 1.7.5.1. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. Let $P_1, ..., P_m$ be new predicates that does not occur in $\mathscr{L}$. Let ϕ be a first-order formula of $\mathscr{L} \cup \{P_1, ..., P_m\}$. For every $1 \leqslant i \leqslant m$, let n_i be the arity of P_i. Let $R_1, R'_1 \in X^{n_1}_{\mathfrak{A}}, R_2, R'_2 \in X^{n_2}_{\mathfrak{A}}, ..., R_m, R'_m \in X^{n_m}_{\mathfrak{A}}$. Then for any assignment $x : Var \to A$,

$$[\![\phi]\!]^{(\mathfrak{A},R_1,...,R_m)}[x] \ \sqcap \ (\bigsqcap_{1\leqslant i\leqslant m} p_i) \leqslant [\![\phi]\!]^{(\mathfrak{A},R'_1,...,R'_m)}[x]$$

where $p_i = \bigsqcap_{\langle a_1,...,a_{n_i}\rangle \in A^{n_i}}(R_i(a_1, \dots, a_{n_i}) \leftrightarrow R'_i(a_1, \dots, a_{n_i}))$.

Proof. By a straightforward induction on the complexity of ϕ. The atomic cases are covered by the definition of $X^n_{\mathfrak{A}}$.

□

Theorem 1.7.6. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$ that is second-order full. Let $P_1, ..., P_m$ be new predicates that does not occur in $\mathscr{L}$. Let ϕ be a first-order formula of $\mathscr{L} \cup \{P_1, ..., P_m\}$. For every $1 \leqslant i \leqslant m$, let n_i be the arity of P_i. Then given any assignment $x : Var \to A$, there is some $R_1 \in X^{n_1}_{\mathfrak{A}}, R_2 \in X^{n_2}_{\mathfrak{A}}, ..., R_m \in X^{n_m}_{\mathfrak{A}}$ such that $[\![\exists P_1, ..., P_m\phi]\!]^{\mathfrak{A}}[x] = [\![\phi]\!]^{(\mathfrak{A},R_1,...,R_m)}[x]$.

Proof. For simplicity we assume $m = 1$ and $n_1 = 1$. The proof we are about to give can be easily generalized to the more general cases. Also for simplicity we ignore the parameters and assume that ϕ is a sentence.

Let $X^1_{\mathfrak{A}} = \{S : A \to B \mid \text{for any } a, b \in A, S(a) \sqcap [\![a = b]\!]^{\mathfrak{A}} \leqslant S(b)\} = X$. Let $[\![\exists P_1\phi]\!]^{\mathfrak{A}} = \bigsqcup_{S\in X}[\![\phi]\!]^{(\mathfrak{A},S)} = p \in B$. If $p = 0$, then the statement is trivial. So we assume $p > 0$.

Let $D = \{d \in B\backslash\{0\} \mid \text{for some } S \in X, d \leqslant [\![\phi]\!]^{(\mathfrak{A},S)}\}$. Let Q be the set of all antichains made up of elements in D. By Zorn's lemma, Q has a maximal element. Call it C.

We can show that D is dense below p. Let $0 \neq p' \leqslant p$. Since $p = \bigsqcup_{S\in X}[\![\phi]\!]^{(\mathfrak{A},S)}$, for some $S \in X$, $p' \sqcap [\![\phi]\!]^{(\mathfrak{A},S)} \neq 0$. But $p' \sqcap [\![\phi]\!]^{(\mathfrak{A},S)} \in D$ and $p' \sqcap [\![\phi]\!]^{(\mathfrak{A},S)} \leqslant p'$.

Hence $p \leqslant \bigsqcup C$: suppose not, then $p \sqcap -(\bigsqcup C) \neq 0$. Since D is dense below p, for some $d \in D$, $d \leqslant p \sqcap -(\bigsqcup C) \leqslant -(\bigsqcup C)$. Then $C \cup \{d\}$ is an antichain in D that properly extends C. Contradiction.

For every $d \in C$, let R^d be some element in X such that $d \leqslant [\![\phi]\!]^{(\mathfrak{A},R^d)}$.

Since $\mathfrak{A}$ is full, there is some $R \in X$ such that for all $d \in C$, $d \leqslant \bigsqcap_{a \in A}(R(a) \leftrightarrow R^d(a))$. By the choice of R^d, $d \leqslant [\![\phi]\!]^{(\mathfrak{A},R^d)} \sqcap \bigsqcap_{a \in A}(R(a) \leftrightarrow R^d(a)) \leqslant [\![\phi]\!]^{(\mathfrak{A},R)}$, by Lemma 1.7.5.1. Therefore, $[\![\exists P_1 \phi]\!]^{\mathfrak{A}} = p \leqslant \bigsqcup C \leqslant [\![\phi]\!]^{(\mathfrak{A},R)}$.

□

The moral of Theorem 1.7.6 is that every second-order full model is "second-order witnessing", and since every model is second-order full, every model is "second-order witnessing". This feature will be essential when we prove the generalized corollary of Łoś Theorem on full models. But to do so we first need to define the notion of a highly witnessing model. The plan, then, is to show that Σ^1_1 formulas are preserved under quotient models of highly witnessing models, and that highly witnessing models coincide with full models.

Definition 1.7.5. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. $\mathfrak{A}$ is *highly witnessing* just in case for any language $\mathscr{L}'$ that expands $\mathscr{L}$, for any expansion $\mathfrak{A}'$ of $\mathfrak{A}$ to $\mathscr{L}'$, $\mathfrak{A}$ is witnessing.

Theorem 1.7.7 (Σ^1_1 formulas are preserved under quotient models)**.** Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$ that is second-order full and highly witnessing. Let $h : B \to C$ be a homomorphism. Let $\mathfrak{A}^h$ be the C-valued quotient model. Given $x : Var \to A^h$ an arbitrary assignment on $\mathfrak{A}^h$, let $y : Var \to A$ be an assignment on $\mathfrak{A}$ such that for any $v_i \in Var$, $y(v_i) \in x(v_i)$. Then, for any Σ^1_1 formula ψ,

$$h([\![\psi]\!]^{\mathfrak{A}}[y]) \leqslant [\![\psi]\!]^{\mathfrak{A}^h}[x]$$

Proof. Let $\psi = \exists P_1, ..., P_m \phi$, where for every $1 \leqslant i \leqslant m$, the arity of P_i is n_i. Since $\mathfrak{A}$ is second-order full, by Theorem 1.7.6, for some $R_1 \in X^{n_1}_{\mathfrak{A}}, R_2 \in X^{n_2}_{\mathfrak{A}}, ..., R_m \in X^{n_m}_{\mathfrak{A}}$, $[\![\exists P_1, ..., P_m \phi]\!]^{\mathfrak{A}}[y]$ $= [\![\phi]\!]^{(\mathfrak{A},R_1,...,R_m)}[y]$. Hence $h([\![\exists P_1, ..., P_m \phi]\!]^{\mathfrak{A}}[y]) = h([\![\phi]\!]^{(\mathfrak{A},R_1,...,R_m)}[y])$.

Expand $\mathfrak{A}$ to a model of $\mathscr{L} \cup \{P_1, ..., P_m\}$, $\mathfrak{A}'$, by setting $[\![P_1]\!]^{\mathfrak{A}'} = R_1$, $[\![P_2]\!]^{\mathfrak{A}'} = R_2$, ..., $[\![P_m]\!]^{\mathfrak{A}'} = R_m$. Since $\mathfrak{A}$ is highly witnessing, $\mathfrak{A}'$ is witnessing.

Hence we can apply the Generalized Łoś Theorem to $\mathfrak{A}'$ and get $h([\![\phi]\!]^{\mathfrak{A}'}[y]) = [\![\phi]\!]^{(\mathfrak{A}')^h}[x]$. Also, for any $1 \leqslant i \leqslant m$, by definition of a quotient model, since $[\![P_i]\!]^{\mathfrak{A}'} = R_i$, $[\![P_i]\!]^{(\mathfrak{A}')^h} = R^h_i : (A^h)^{n_i} \to C$, where for any $\langle [a_1]_h, [a_2]_h, ..., [a_{n_i}]_h \rangle \in (A^h)^{n_i}$, $R^h_i([a_1]_h, [a_2]_h, ..., [a_{n_i}]_h) = h(R_i(a_1, a_2, ..., a_{n_i}))$. It is easy to see that $(\mathfrak{A}')^h = (\mathfrak{A}^h, R^h_1, R^h_2, ..., R^h_m)$.

Hence $h([\![\exists P_1, ..., P_m \phi]\!]^{\mathfrak{A}}[y]) = h([\![\phi]\!]^{\mathfrak{A}'}[y]) = [\![\phi]\!]^{(\mathfrak{A}')^h}[x] \leqslant [\![\exists P_1, ..., P_m \phi]\!]^{\mathfrak{A}^h}[x]$, as obviously $R^h_1 \in X^{n_1}_{\mathfrak{A}^h}$, $R^h_2 \in X^{n_2}_{\mathfrak{A}^h}$, ..., $R^h_m \in X^{n_m}_{\mathfrak{A}^h}$.

□

Theorem 1.7.8. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. $\mathfrak{A}$ is a full model if and only if $\mathfrak{A}$ is highly witnessing.

Proof. For the left to right direction. Let $\mathfrak{A}$ be a full model. Then any expansion of $\mathfrak{A}$ is a full model, since fullness is only determined by the domain and the values of the identity formulas. Since every full model is witnessing, every expansion of $\mathfrak{A}$ is witnessing.

For the right to left direction. We assume that $\mathfrak{A}$ is not full. Then for some antichain $D \subseteq B$, $\{a_d \mid d \in D\} \subseteq A$, every $a \in A$ is such that for some $d \in D$, $d \nleqslant [\![a = a_d]\!]^{\mathfrak{A}}$. Pick such an antichain $D \subseteq B$ and $\{a_d \mid d \in D\} \subseteq A$.

For every $d \in D$, let $R^d : A \to B$ be such that for any $b \in A$, $R^d(b) = [\![b = a_d]\!]^{\mathfrak{A}}$. It is easy to see that $R^d \in X^1_{\mathfrak{A}}$.

Let $R : A \to B$ be such that for any $b \in A$,

$$R(b) = \bigsqcup_{c \in D} (c \sqcap R^c(b))$$

By the proof of Theorem 1.7.5, for any $d \in D$, $d \leqslant \bigsqcap_{b \in A}(R(b) \leftrightarrow R^d(b))$.

Claim 1.7.8.1. R has no witness, i.e. no $a \in A$ is such that $\bigsqcup_{b \in A} R(b) = R(a)$.

Proof of the Claim.

Suppose otherwise and let a be a witness. Then

$$R(a) = \bigsqcup_{b \in A} R(b) = \bigsqcup_{b \in A} \bigsqcup_{c \in D} (c \sqcap R^c(b))$$

Then for any $d \in D$, $d \leqslant R(a)$, since $d = d \sqcap R^d(a^d) = d \sqcap [\![a^d = a^d]\!]^{\mathfrak{A}} = d \sqcap 1$.

Hence for $d \in D$, $d \leqslant R(a) \sqcap \bigsqcap_{b \in A}(R(b) \leftrightarrow R^d(b)) \leqslant R^d(a) = [\![a = a^d]\!]^{\mathfrak{A}}$, contradicting our assumption that $\mathfrak{A}$ is not a full model.

■

Let P be a new unary predicate not occurring in $\mathscr{L}$. Expand $\mathfrak{A}$ to a model of $\mathscr{L} \cup \{p\}$, $\mathfrak{A}'$, by setting $[\![P]\!]^{\mathfrak{A}'} = R$. Then since R has no witness, $\exists v P(v)$ has no witness in $\mathfrak{A}'$, and hence $\mathfrak{A}'$ is not witnessing. Hence $\mathfrak{A}$ is not highly witnessing.

□

Corollary 1.7.8.1. Let $\mathfrak{A}$ be a full B-valued model of $\mathscr{L}$. Let $h : B \to C$ be a homomorphism. Let $\mathfrak{A}^h$ be the C-valued quotient model. Given $x : Var \to A^h$ an arbitrary assignment on $\mathfrak{A}^h$, let $y : Var \to A$ be an assignment on $\mathfrak{A}$ such that for any $v_i \in Var$, $y(v_i) \in x(v_i)$. Then, for any Σ^1_1 formula ψ, $h(\llbracket \phi \rrbracket^{\mathfrak{A}}[y]) \leqslant \llbracket \psi \rrbracket^{\mathfrak{A}^h}[x]$.

Two remarks are in order. First, the theorem on two-valued models that Σ^1_1 formulas are preserved under ultraproducts is a special case of Corollary 1.7.8.1. As mentioned in Section 1.3, given a collection of two-valued models $\{\mathfrak{A}_i \mid i \in I\}$, and $D \subseteq P(I)$ a ultrafilter, we just let $\mathfrak{A}$ be the direct product of all the $\mathfrak{A}_i$'s and h be the characteristic function of D. Then the ultraproduct will be the same as the quotient model $\mathfrak{A}^h$, and applying the above corollary gives us the traditional result.

Second, Theorem 1.7.8 provides us another way to show that full models are witnessing. This is simply because obviously highly witnessing models are witnessing, and since every full model is highly witnessing, every full model is witnessing.

1.8 True Identity Models

The identity symbol in Boolean-valued models is interpreted in a non-standard way. When B is a complete Boolean algebra that properly extends 2, our definition of Boolean-valued models allows that in some B-valued model $\mathfrak{A}$, for some $a, b \in A$, $\llbracket a = b \rrbracket^{\mathfrak{A}} = p \in B$, where p is neither 1_B or 0_B. This is an interesting feature of Boolean-valued models, which I believe will give rise to attractive philosophical applications. But that is a topic for another paper. In this section, we will study a special type of Boolean-valued models: those in which the identity symbol is interpreted in a standard way.

Definition 1.8.1 (True Identity Model)**.** A B-valued model $\mathfrak{A}$ is a *true identity model* just in case $\llbracket = \rrbracket^{\mathfrak{A}} : A \times A \to B$ is the real identity function on $A \times A$, i.e. for any $a, b \in A$, if a and b are not the same element, then $\llbracket a = b \rrbracket^{\mathfrak{A}} = 0_B$.

Proposition 1.8.1. Let $\mathscr{L}$ be a first order language whose only non-logical symbols are constants. Let $\mathfrak{A}$ be a B-valued true identity model of $\mathscr{L}$. Then for any formula $\phi(v_1,...,v_n) \in \mathscr{L}$, any $a_1,...,a_n \in A$, $[\![\phi]\!]^{\mathfrak{A}}[a_1,...,a_n] \in \{0_B, 1_B\}$.

Theorem 1.8.1. Let $\{\mathfrak{A}_i \mid i \in I\}$ be a non-empty collection of Boolean-valued models. Suppose for some $i \in I$, $|A_i| > 1$, and for some $a_1, a_2 \in A_i$, $[\![a_1 = a_2]\!]^{\mathfrak{A}_i} \neq 1$. Then the product model $\prod_{i\in I} \mathfrak{A}_i$ is not a true identity model.

Proof. Just pick two elements $b_1, b_2 \in \prod_{i\in I} A_i$ such that the ith component of b_1 is a_1, the ith component of b_2 is a_2, and b_1 and b_2 have the same element at every other position. Then $[\![b_1 = b_2]\!]^{\prod_{i\in I}\mathfrak{A}_i}$ is an intermediate value.

□

Theorem 1.8.2. Let $\mathfrak{A}$ be a B-valued true identity model. Let $h : B \to C$ be a homomorphism. Then the quotient model $\mathfrak{A}^h$ is a C-valued true identity model. Moreover, $\mathfrak{A}$ and $\mathfrak{A}^h$ have the same domain.

Proof. $A = A^h$ because for any $a_1, a_2 \in A$, $a_1 \equiv_h a_2$ iff $h([\![a_1 = a_2]\!]^{\mathfrak{A}}) = 1$ iff $a_1 = a_2$, as $\mathfrak{A}$ is a true identity model. Also, if $[a_1]_h \neq [a_2]_h$, then $a_1 \neq a_2$, and then $[\![[a_1]_h = [a_2]_h]\!]^{\mathfrak{A}^h} = h([\![a_1 = a_2]\!]^{\mathfrak{A}}) = h(0_B) = 0_C$.

□

We've argued in Section 1.7 that not all witnessing models are full. The following results aim to provide an example for this claim.

Theorem 1.8.3. Let $\mathfrak{A}$ be a B-valued true identity model. If B is a proper Boolean extension of 2, and if $|A| > 1$, then $\mathfrak{A}$ is not a full model.

Proof. Since B is a proper extension of 2, there is some $p \in B$ such that $0 \neq p \neq 1$. Then $\{p, -p\}$ is an antichain. Let a_1, a_2 be any two different elements in A. Then for any $a \in A$, either $p \nleqslant [\![a = a_1]\!]^{\mathfrak{A}}$, or $-p \nleqslant [\![a = a_2]\!]^{\mathfrak{A}}$, as $\mathfrak{A}$ is a true identity model.

□

Theorem 1.8.4. Let $\mathscr{L}$ be an arbitrary first order language. Let B be a complete Boolean algebra that properly extends 2. Then there is a witnessing B-valued true identity model $\mathfrak{A}$ of $\mathscr{L}$, whose domain has more than one element.

Proof. Pick $p \in B$ such that $0 \neq p \neq 1$. For any n-ary relation P in $\mathscr{L}$, for any $a_1, ..., a_n \in A$, let $[\![P(a_1, ..., a_n)]\!]^{\mathfrak{A}} = p$. Also let $[\![=]\!]^{\mathfrak{A}}$ be the identity function on $A \times A$. It is easy to show that $\mathfrak{A}$ is witnessing.

□

Corollary 1.8.4.1. Let $\mathscr{L}$ be an arbitrary first order language. Let B be a complete Boolean algebra that properly extends 2. Then there is a witnessing B-valued true identity model of $\mathscr{L}$ that is not full.

In Section 1.4.1 we have proved a collection of results involving theories of first order languages and Boolean-valued models. In the following we will state a few theorems about theories and Boolean-valued true identity models. We will state the results without proofs as they are all very straightforward.

Theorem 1.8.5. Let T be a theory in $\mathscr{L}$. T is consistent if and only if for some complete Boolean Algebra B, T has a B-valued true identity model $\mathfrak{A}$.

Theorem 1.8.6. Let B be any complete Boolean algebra. A theory T has a B-valued true identity model just in case every finite subset of T has a B-valued true identity model.

Recall that in Section 1.4, we argue that the notion of Boolean-valuation is a natural generalization of the notion of theory. For the rest of this section we consider questions involving Boolean-valuations and true identity models. For example, what kind of Boolean-valuations correspond to true identity models? Does compactness holds on these Boolean-valuations? etc. Again, we assume that $\mathscr{L}$ is a countable language.

Definition 1.8.2. A B-valuation S^B *respects identity* just in case for any countable set of new constants D, S^B can be extended into a consistent B-valuation S'^B of $\mathscr{L} \cup D$ such that for any constants c, d in $\mathscr{L} \cup D$, either $\langle c = d, 1\rangle \in S'^B$ or $\langle c = d, 0\rangle \in S'^B$.

Theorem 1.8.7. A B-valuation S^B respects identity if and only if it has a true identity B-valued model.

Proof. For the right to left direction, we suppose S^B has a true identity B-valued model $\mathfrak{A}$. Let D be a countable set of new constants. Expand $\mathfrak{A}$ to a model of $\mathscr{L} \cup D$ arbitrarily:

for any $c \in D$, let $[\![c]\!]^{\mathfrak{A}}$ be a random element in A. Let S'^B be the set of all pairs of the form $\langle \phi, p \rangle$ where ϕ is a sentence of $\mathscr{L} \cup D$ and $p = [\![\phi]\!]^{\mathfrak{A}}$. Then S'^B is a consistent B-valuation that extends S^B such that for any constants c, d in $\mathscr{L} \cup D$, either $\langle c = d, 1 \rangle \in S'^B$ or $\langle c = d, 0 \rangle \in S'^B$.

The proof for the left to right direction is similar that that of Theorem 1.4.8. Let C be a new countable set of constants. Let $\mathscr{L}' = \mathscr{L} \cup D$. Enumerate all formulas with one free variable in $\mathscr{L}'$: $\phi_0(v), \phi_1(v), ...$

For any sentence ψ in $\mathscr{L}'$, for some $p \in B$, $S^B \cup \{\langle \psi, p \rangle\}$ is such that it is contained in some consistent B-valuation of $\mathscr{L}'$ that has either $\langle c = d, 1 \rangle$ or $\langle c = d, 0 \rangle$, for any constants $c, d \in \mathscr{L}'$, as S^B respects identity, and any consistent B-valuation is contained in some maximal consistent B-valuation.

Now form an increasing chain of B-valuations:

$$S^B = S^B_0 \subseteq S^B_1 \subseteq ... \subseteq S^B_i \subseteq ... \quad i < \omega$$

Given S^B_i, first add $\langle \exists v \phi_i(v), p \rangle$ to S^B_i, where $S^B_i \cup \{\langle \exists v \phi_i(v), p \rangle\}$ is such that it is contained in some consistent B-valuation of $\mathscr{L}'$ that has either $\langle c = d, 1 \rangle$ or $\langle c = d, 0 \rangle$, for any constants $c, d \in \mathscr{L}'$. Then add $\langle \phi_i(d_i), p \rangle$, where d_i is some new constant from C that has not appeared in $S^B_i \cup \{\langle \exists v \phi_i(v), p \rangle\}$. Such a new constant exists as there are only finitely many constants from C in $S^B_i \cup \{\langle \exists v \phi_i(v), p \rangle\}$.

It is easy to show aht $S^B_{i+1} = S^B_i \cup \{\langle \exists v \phi_i(v), p \rangle, \langle \phi_i(d_i), p \rangle\}$ is such that it is contained in some consistent B-valuation of $\mathscr{L}'$ that has either $\langle c = d, 1 \rangle$ or $\langle c = d, 0 \rangle$, for any constants $c, d \in \mathscr{L}'$.

Let $S'^B = \bigcup_{i<\omega} S^B_i$. It is also easy to show that S'^B is contained in some consistent B-valuation of $\mathscr{L}'$ that has either $\langle c = d, 1 \rangle$ or $\langle c = d, 0 \rangle$, for any constants $c, d \in \mathscr{L}'$. Extend S'^B to such a B-valuation, and then extend the latter to a maximal consistent B-valuation in $\mathscr{L}'$. Call it T^B.

We can construct a B-valued model for T^B using C as the domain in the same way as we do in the proof of Theorem 1.4.8, with the only the following change. For any $d_i \in C$, let $[d_i] = \{d_j \in C \mid [\![d_i = d_j]\!]^T = 1\}$. Let $A = \{[d_i] \mid d_i \in C\}$. For any constant c of $\mathscr{L}'$, let $[\![c]\!]^{\mathfrak{A}} = [d_i]$ such that $[\![c = d_i]\!]^T = 1$. And similar changes to the interpretation of other symbols of $\mathscr{L}'$.

In the same way as in the proof of Theorem 1.4.8, we can show that $\mathfrak{A}$ is a B-valued model of T^B that is witnessing. Also, it is very easy to show that $\mathfrak{A}$ is a true identity model.

□

Corollary 1.8.7.1. A B-valuation S^B respects identity if and only if it has a witnessing true identity B-valued model.

Theorem 1.8.8. A B-valuation S^B respects identity if and only if every finite sub-valuation of S^B respects identity.

Proof. The left to right direction is obvious.

For the right to left direction, suppose that S^B does not respect identity. Then for some countable set of new constants D, for some constants $c, d \in \mathscr{L} \cup D$, both $S^B \cup \{\langle c = d, 1\rangle\}$ and $S^B \cup \{\langle c = d, 0\rangle\}$ are inconsistent. By Theorem 1.4.5, for some finite sub-valuation $\Delta^B \subseteq S^B$, $\Delta^B \cup \{\langle c = d, 1\rangle\}$. Similarly, for some finite sub-valuation $\Delta'^B \subseteq S^B$, $\Delta'^B \cup \{\langle c = d, 0\rangle\}$. But then, $\Delta^B \cup \Delta'^B$, a finite sub-valuation of S^B, does not respect identity.

□

Corollary 1.8.8.1. A B-valuation S^B has a true identity model if and only if every finite sub-valuation of S^B has a true identity model.

1.9 Löwenheim-Skolem Theorems

In previous sections we proved two versions of the downward Löwenheim-Skolem Theorem:

Theorem 1.9.1. Let $\mathscr{L}$ be a countable language. If a Boolean-valuation S^B of $\mathscr{L}$ has a B-valued model, then it has a countable witnessing B-valued model.

Theorem 1.9.2. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$ that is witnessing. Then $\mathfrak{A}$ has a countable elementary submodel.

A natural question is: what about the upward Löwenheim-Skolem Theorem? Can it also be generalized to a Boolean-valued setting? In this section we investigate this question.

The case of the upward Löwenheim-Skolem is much more complicated than its downward counterpart. Recall that in Section 1.5 we observed that our definition of Boolean-valued models allow there to be "null" duplicates in a model. And with the existence of null duplicates it is boringly easy to add more objects to a domain of a model without changing which sentences are true in the model:

Theorem 1.9.3. Let T be a consistent theory of $\mathscr{L}$. Then for any complete Boolean algebra B, if T has a B-valued model of size α, it has B-valued models of arbitrary sizes larger than α.

Proof. Just pick some random element of the domain and add as many duplicates of the element to the domain as desired.

□

Note that the above theorem is much stronger than the normal upward Löwenheim-Skolem in the two-valued case. It says that any consistent theory can have models that are arbitrarily large, including, for example, a theory that says there are only two objects. This is a counter-intuitive result. Surely if a sentence saying that there are only two objects is true in a model, then we would want there to be only two objects in the domain of the model.

One might think that the culprit of this counter-intuitive result is the existence of duplicates. What if we require the models to be duplicate resistant (Def 1.5.1)? Will it still be the case that consistent theories can have arbitrarily large models? The answer, interestingly, is positive, as the following results show.

Theorem 1.9.4. If T has a duplicate resistant model $\mathfrak{A}$ with $|A| > 1$, then T has duplicate resistant models of arbitrary sizes larger than $|A|$.

Proof. We just make use of the direct product construction. Let I be an arbitrarily large index set. By Theorem 1.5.9, $\prod_{i \in I} \mathfrak{A}$ is a model of T. □

Also, adding the further requirement that models should be full does not help.

Corollary 1.9.4.1. If T has a duplicate resistant full model $\mathfrak{A}$ with $|A| > 1$, then T has duplicate resistant full models of arbitrary sizes larger than $|A|$.

Proof. By Theorem 1.7.1, direct powers inherit fullness. □

The real culprit of this (kind of) result is the fact that the identity symbol is interpreted in a non-standard way in Boolean-valued models. As a result, there can be, for example, some Boolean-valued model in which the sentence $\exists v_1 \exists v_2 \forall v_3 (v_3 = v_1 \vee v_3 = v_2)$ - that there are at most two things - is true but the domain of the model consists of way more than two things. Indeed, the only sentence that has control over the size of the domain of a model is the sentence saying that there is at most one thing.

Theorem 1.9.5. Let ϕ be the sentence $\exists v_1 \forall v_2 (v_1 = v_2)$. If $\mathfrak{A}$ is a duplicate resistant model of ϕ, then $|A| = 1$.

Proof. $[\![\exists v_1 \forall v_2 (v_1 = v_2)]\!]^{\mathfrak{A}} = \bigsqcup_{a \in A} \bigsqcap_{b \in A} [\![a = b]\!]^{\mathfrak{A}}$. Fix some $a \in A$. Consider $\bigsqcap_{b \in A} [\![a = b]\!]^{\mathfrak{A}}$. We will show that $\bigsqcap_{b \in A} [\![a = b]\!]^{\mathfrak{A}} = \bigsqcap_{c,d \in A} [\![c = d]\!]^{\mathfrak{A}}$. The $\geqslant$ direction holds trivially. The $\leqslant$ direction holds as for any $a, c, d \in A$, $[\![a = c]\!]^{\mathfrak{A}} \sqcap [\![a = d]\!]^{\mathfrak{A}} \leqslant [\![c = d]\!]^{\mathfrak{A}}$.

Hence $\bigsqcup_{a \in A} \bigsqcap_{b \in A} [\![a = b]\!]^{\mathfrak{A}} = \bigsqcup_{a \in A} \bigsqcap_{c,d \in A} [\![c = d]\!]^{\mathfrak{A}} = \bigsqcap_{c,d \in A} [\![c = d]\!]^{\mathfrak{A}} = 1$.

Hence for any $c, d \in A$, $[\![c = d]\!]^{\mathfrak{A}} = 1$. Since $\mathfrak{A}$ is duplicate resistant, c and d are the same element.

□

We have argued that the real reason why we have these counter-intuitive results is that the identity symbol is interpreted abnormally. Hence, in order to solve the problem, we should, instead of requiring the models to be duplicate resistant, require the models to be true identity models, as these are the Boolean-valued models in which identity is standard. Once we introduce this requirement, then, we can generalize the upward Löwenheim-Skolem theorem in the most natural way. We assume that $\mathscr{L}$ is countable.

Theorem 1.9.6. Let ϕ expresses the sentence "there are exactly n things", where $n < \omega$. Let $\mathfrak{A}$ be a true identity model of ϕ. Then $|A| = n$.

Proof. By appealing to Proposition 1.8.1.

□

Theorem 1.9.7. If a B-valuation S^B has an infinite B-valued true identity models, then it has infinite B-valued true identity models of any power $\alpha \geqslant \omega$.

Proof. Let $c_\beta, \beta < \alpha$ be a list of new constant. Consider the B-valuation $S'^B = S^B \cup \{\langle c_\gamma = c_\beta, 0\rangle \mid \gamma < \beta < \alpha\}$. By Theorem 1.8.7, S^B respects identity. And hence every finite sub-valuation of S'^B respects identity. By Theorem 1.8.7 again, every finite sub-valuation of S'^B has a B-valued true identity model. By Corollary 1.8.8.1, S'^B has a B-valued true identity model.

□

Theorem 1.9.8. If a B-valuation S^B has arbitrarily large finite B-valued true identity models, then it has an infinite B-valued true identity model.

Proof. The same proof as that of Theorem 1.9.7. □

Corollary 1.9.8.1. Every infinite true identity model has arbitrarily large elementary extensions.

As a special case of Theorem 1.9.7 and Theorem 1.9.8, we also have:

Theorem 1.9.9. If a theory T has arbitrarily large finite B-valued true identity models, then it has an infinite B-valued true identity model.

Theorem 1.9.10. If a theory T has an infinite B-valued true identity models, then it has infinite B-valued true identity models of any power $\alpha \geqslant \omega$.

Chapter 2

Boolean Mereology

2.1 Introduction

When we look around and inspect the ordinary objects around us, we will find that many ordinary objects lack a precise mereological boundary, or at least they appear to do so. Many ordinary objects are such that in certain natural situations, we can find things that are neither definitely part of it nor definitely not part of it. Here are some typical examples:

Example One Consider Tibbles the cat. Suppose Tibbles has a whisker, call it W, that has loosened up and is about to fall off. Is W part of Tibbles?

Example Two Consider Mount Kilimanjaro, the tallest mountain in Africa. Suppose there is tree, call it T, that is located somewhere at the boundary of Kilimanjaro - say, somewhere in between Mweka Camp and Materuni Waterfall. Is T part of Kilimanjaro?

Example Three Consider Tim, an ordinary human being. Suppose there is a cell, call it C, in Tim's epidermis that has lost its nucleus and is about to be shed from the surface of Tim's skin. Is *C* part of Tim?

Example Four Consider Theseus the ship. Suppose there is an iron nail, call it N, that is in the process of being hammered into Theseus by a repairer. Is N part of Theseus?

There are countless other examples of this type, involving ordinary objects of almost all kinds, including animals, humans, artifacts, geographical areas, plants, buildings, and so on. If we describe the cases and ask the common man questions of the form "is W/T/C/N part of Tibbles/Kilimanjaro/Tim/Theseus?", the answer we would most likely get would be a hesitant "sort of/more or less/to some extent". These answers, I believe, are natural and intuitive. They indicate that an all-encompassing theory of the relation of parthood should have the ability to accommodate indeterminacy.

In this chapter I will present a novel degree-theoretic semantic framework that is able to handle mereological indeterminacy with ease. The semantic framework I am about to introduce is called *Boolean-valued semantics*, whose key feature is that degrees of truth form a Boolean ordering. I will argue that Boolean-valued semantics is the best *degree-theoretic* semantics for the language of mereology. In particular, I will argue that it trumps the well-known alternative - fuzzy-valued semantics, for three main reasons: (a) it allows for incomparable degrees of parthood, (b) it enforces classical logic, and (c) it is compatible with all the axioms of classical mereology. Moreover, I will explore, under the framework of Boolean semantics, the connection between vagueness in parthood and vagueness in existence/identity. I will show that, contrary to what many have argued, vagueness in parthood *entails* neither vagueness in existence nor vagueness in identity, although being compatible with both.

What I won't do in this chapter is to develop a full-fledged philosophical theory of mereological vagueness that has a decisive answer to every relevant question. The main goal of this chapter is to construct a superior semantic framework for indeterminacy of parthood, and I believe that it should never be the job of the semantics to take a stand on deeper philosophical questions like "What is the nature of mereological indeterminacy?". An ideal semantic framework should be flexible with regard to which philosophical viewpoints one further upholds. In the final section of this chapter, I will illustrate the neutrality and flexibility of Boolean semantics by sketching out two different philosophical theories of mereological vagueness, one coming from applying Boolean semantics to the view that mereological vagueness is linguistic, and the other coming from applying Boolean semantics to the view that mereological vagueness is ontic. Another issue that I won't discuss in this chapter is higher-order vagueness. In this chapter, I will adopt (without arguing) a McGee-style position that the issue of higher-order vagueness lies in the interpretation of the *meta-language*.[1] And since the the purpose of this chapter is to build a semantics, that is, an interpretation framework of the *object* language - the language of mereology, the issue of higher-order vagueness, on our assumption, lies outside of the scope of our discussion.

The plan of this chapter goes as follows. I will start in Section 2.2 by arguing that facing mereological vagueness, a natural, and good place to start is to adopt a degree-theoretic semantics. In Section 2.3, I will present in detail Boolean semantics, which is a degree-theoretic semantics whose key feature is that truth degrees form a Boolean structure. I will explain how Boolean semantics can be applied to cases of mereological indeterminacy. In Section 2.4, I will argue that Boolean semantics is the better degree-theoretic semantics for handling mereological indeterminacy, in comparison to the alternative. The goal of Section 2.5 is to investigate a special kind of Boolean models for the language of mereology that are of particular interest - the atomic Boolean models. Via these models I will also discuss the connection between mereological vagueness on the one hand and vagueness in existence and identity on the other hand. Finally, in Section 2.6, we end this chapter with a discussion on the nature of mereological vagueness. In particular, we show that Boolean mereology is neutral on the nature of mereological vagueness, and one can construct different theories of mereological vagueness by combining Boolean semantics with different views on the nature of mereological vagueness.

2.2 Many Degrees: A Natural Start

The language of mereology, depending on one's taste, is a first-order or second-order language whose only non-logical symbol is the binary relation symbol of parthood, $\precsim$. The classical semantics, for either first-order or second-order languages, has as its value range the two-valued Boolean algebra $\{0,1\}$. The classical semantics, therefore, leaves little if not no room for mereological indeterminacy, as, for example, W is either part of Tibbles to degree 0, meaning that it is not part of Tibbles, or it is part of Tibbles to degree 1, meaning that it is part of Tibbles. In order to accommodate mereological indeterminacy, therefore,

[1] See [23] and [25] for arguments for this viewpoint and replies to objects.

we at least need revision of some kind to the classical semantics [2].

A natural and straightforward move is to enlarge the range of truth degrees. If “yes" corresponds to degree 1 and “no" corresponds to degree 0, then we might want some intermediate degree between 0 and 1 to correspond to the common man’s hesitant “sort of", when responding to the question “is W part of Tibbles". If we have decided to add more degree of parthood, then, there seems to be no harm but only benefits if we add more than just one. Consider the case of Tibbles. It is certainly possible that there is a different whisker, call it W’, that has also loosened up and is about to fall off. But we can imagine that W’ is looser than W, and also has a stronger inclination to fall off. In this case, then, it seems quite intuitive to say that the extent to which W’ is part of Tibbles is lower than the extent to which W is part of Tibbles. If we want to transform this “extent talk" to “degrees talk", we will then want to have multiple intermediate degrees that are comparable to each other, so that we can assign a lower intermediate degree to “W’ is part of Tibbles" and a higher one to “W is part of Tibbles".

Let us call a semantic framework “degree-theoretic" if it allows for multiple degrees of truth in addition to the extreme ones. The semantic framework that I am about to develop, Boolean semantics, is a degree-theoretic one. There are, I believe, a number of advantages to use a degree-theoretic semantics on cases of mereological indeterminacy. First, under a degree-theoretic framework, the changes that need to be made to the classical semantics are quite unsubstantial and procedural. All we need to do is to replace the classical value range $\{0, 1\}$ with a value range of a larger size. The core idea behind the classical semantics story stays unchanged, including, for example, that constants in the language are interpreted by objects in the domain, that truth values are assigned to the atomic formulas by an assignment function that comes with the model, that complex formulas have their values calculated from the values of simpler formulas using certain algebraic operations, and so on[3].

[2]Although most people think that at least some change to classical semantics is needed for handling mereological indeterminacy, there are also exceptions. For epistemicists like Williamson [36], sentences like “W is part of Tibbles" do indeed have a definite truth value, and it is just impossible for us humans to know the truth values of these sentences. Mereological vagueness is explained, on this view, as a kind of ignorance that we cannot possibly overcome. Most people find this view highly counter-intuitive. Under this view, there will have to be basic mereological facts about ordinary objects in the world that are simply epistemically inaccessible to us, no matter how our cognitive abilities improve. It seems to me to be a heavy philosophical burden to postulate these unreachable facts about the mereological relations among ordinary objects.

[3]Admittedly it is of course theoretically possible for there to be degree-theoretic views of mereological

What we end up with is a natural generalization of the classical semantics theory, rather than a radical deviation. Second, a degree-theoretic semantics offers at least some level of explanation of what mereological indeterminacy is. Under a degree-theoretic framework, cases of mereological indeterminacy are cases of intermediate parthood degrees, that is, cases where some object is part of another to an intermediate degree between 0 and 1. The phenomenon of indeterminacy is explained in terms of non-extreme truth degrees. Of course, this does not answer all the questions we care about regarding mereological indeterminacy, such as, for example, “What is the nature of mereological indeterminacy?", or “Is mereological indeterminacy worldly or not?". But it is a decent first step. Last but not least, as we have already observed, our ordinary intuition about the relation of parthood involves that it is susceptible to comparison. Among the two loosened up whiskers the looser one is less a part of Tibbles than the tighter one. Among the two trees at the boundary the further one is less a part of Kilimanjaro than the closer one. So on and so forth. Such intuitions can be neatly captured by a degree-theoretic semantics as long as we have multiple comparable intermediate degrees.

The above discussion is not meant to be a decisive argument against using non-degree-theoretic semantics for cases mereological indeterminacy. There is a variety of different non-degree-theoretic semantics, and I do not believe there is a sufficiently strong objection against them all. Each one has its own problems, and I will postpone the discussion of some of them to the later sections[4]. The above discussion is only meant to point out some general advantages enjoyed by having a degree-theoretic semantics, and that the latter is a good place to start, if our goal is to develop a semantics for the relation of parthood that tolerates indeterminacy.

indeterminacy that are not truth functional. But to my knowledge in the current context this is not something worth of special discussion.

[4]For example, we will talk about supervaluation semantics and its connection to Boolean semantics in Section 2.6.

2.3 What Are Boolean Degrees?

The classical value range $\{0,1\}$ is the two-element complete Boolean algebra, and in classical semantics, logical terms like “and", “or", etc. are interpreted by the algebraic operations - meet, join, etc. - on the Boolean algebra. If our plan is to enlarge the classical value range while keeping the rest of classical semantics unchanged, then the natural suggestion is to use larger complete Boolean algebras as value range and still interpret logical terms using Boolean operations. Degrees of truth, then, form a complete Boolean algebra that has more than two elements.

Definition 2.3.1. A Boolean algebra[5] is a set B together with binary operations $\sqcap$ and $\sqcup$, unary operation $-$, and elements 0 and 1 that satisfies:

1. commutative and associative laws for $\sqcap$ and $\sqcup$;

2. distributive laws for $\sqcap$ over $\sqcup$ and $\sqcup$ over $\sqcap$;

3. for any $x, y \in B$, $x \sqcup (x \sqcap y) = x$; $x \sqcap (x \sqcup y) = x$; $x \sqcup -x = 1$; $x \sqcap -x = 0$.

In each Boolean algebra we can define an ordering $\leqslant$ as follows: for any $x, y \in B$, $x \leqslant y$ just in case $x \sqcap y = x$. We can show that this ordering is a partial order: in fact, it gives rise to a bounded distributive complemented lattice. 1 is the top element with respect to this ordering, and 0 is the bottom element with respect to this ordering[6].

Definition 2.3.2. A complete Boolean algebra B is a Boolean algebra where each subset of B has a supremum with respect to the ordering $\leqslant$.

In classical semantics, models are $\{0,1\}$-valued. In Boolean semantics, models are B-valued[7], where B can be any complete Boolean algebra. Just as in the classical case, a Boolean model $\mathfrak{A}$ comes with a pre-given set of objects, A, as its domain. Any constant in the language is interpreted by an object in the domain. The identity symbol is interpreted

[5] For a detailed introduction to Boolean algebras, see [14].

[6] In fact, an alternative characterization of a Boolean algebra is a bounded distributive complemented lattice.

[7] For a more formal definition of a Boolean-valued model, see Def. 2.7.1 in the Appendix.

by a function from A^2 to B that satisfies the following conditions: for any $a_1, a_2, a_3 \in A$[8],

$$\llbracket a_1 = a_1 \rrbracket^{\mathfrak{A}} = 1$$

$$\llbracket a_1 = a_2 \rrbracket^{\mathfrak{A}} = \llbracket a_2 = a_1 \rrbracket^{\mathfrak{A}}$$

$$\llbracket a_1 = a_2 \rrbracket^{\mathfrak{A}} \sqcap \llbracket a_2 = a_3 \rrbracket^{\mathfrak{A}} \leqslant \llbracket a_1 = a_3 \rrbracket^{\mathfrak{A}}$$

An n-ary relation symbol P is interpreted by a function from A^n to B that satisfies the following conditions: for any $a_1, ..., a_n, b_1, ..., b_n \in A$[9],

$$\llbracket P(a_1, ..., a_n) \rrbracket^{\mathfrak{A}} \sqcap \Big(\bigsqcap_{1 \leqslant i \leqslant n} \llbracket a_i = b_i \rrbracket^{\mathfrak{A}} \Big) \leqslant \llbracket P(b_1, ..., b_n) \rrbracket^{\mathfrak{A}}$$

Again, just as in the classical case, the sentential connectives and quantifiers are interpreted by algebraic operations on the Boolean algebra: conjunction by binary meet, disjunction by binary join, negation by complementation, universal quantifier by inifnite meet and existential quantifier by infinite join. In particular, given an assignment function x from the set of all variables to A, and suppose ϕ, ψ are formulas,

$$\llbracket \neg\phi \rrbracket^{\mathfrak{A}}[x] = -\llbracket \phi \rrbracket^{\mathfrak{A}}[x]$$

$$\llbracket \phi \wedge \psi \rrbracket^{\mathfrak{A}}[x] = \llbracket \phi \rrbracket^{\mathfrak{A}}[x] \sqcap \llbracket \psi \rrbracket^{\mathfrak{A}}[x]$$

$$\llbracket \phi \vee \psi \rrbracket^{\mathfrak{A}}[x] = \llbracket \phi \rrbracket^{\mathfrak{A}}[x] \sqcup \llbracket \psi \rrbracket^{\mathfrak{A}}[x]$$

$$\llbracket \exists v_i \phi \rrbracket^{\mathfrak{A}}[x] = \bigsqcup_{a \in A} \llbracket \phi \rrbracket^{\mathfrak{A}}[x(v_i/a)]$$

$$\llbracket \forall v_i \phi \rrbracket^{\mathfrak{A}}[x] = \bigsqcap_{a \in A} \llbracket \phi \rrbracket^{\mathfrak{A}}[x(v_i/a)]$$

where $x(v_i/a)$ is the assignment function that takes v_i to a and agrees with x at everywhere else.

Now we have shown that Boolean semantics arises from classical semantics simply

[8]Here and in the following, for any sentence ϕ and any Boolean model $\mathfrak{A}$, $\llbracket \phi \rrbracket^{\mathfrak{A}}$ means the value of ϕ in $\mathfrak{A}$. We might omit the superscript occasionally when the context is clear.

[9]In any complete Boolean algebra B, for any $D \subseteq B$, $\bigsqcap D$ is the infimum of D with respect to the ording $\leqslant$, whose existence is guaranteed by the definition of a complete Boolean algebra (with an easy derivation). Similarly, $\bigsqcup D$ is the supremum of D with respect to $\leqslant$.

by replacing the two-element complete Boolean algebra of classical truth degrees with a complete Boolean algebra of any size. After this change, we faithfully follow the classical procedure, step-by-step. The new value range can be as large as we want, as there can be arbitrarily large complete Boolean algebras. Therefore, there can be multiple intermediate degrees in between the top degree 1 and the bottom degree 0. Ordered by $\leqslant$, some of the intermediate degrees are higher/lower than some others. These Boolean degrees are perfect for modeling mereological indeterminacy. The whisker W_1 that is firmly attached to Tibbles is part of Tibbles to degree 1; the whisker W_2 that has already fallen off from Tibbles is part of Tibbles to degree 0; the whisker W that has loosened up and is inclined to fall off is part of Tibbles to degree *p*, where *p* is an intermediate degree between 0 and 1 in a complete Boolean algebra that is sufficiently large; the whisker W' that is just like W except that it is looser and has a greater inclination to fall off is part of Tibbles to degree *q*, where *q* is some intermediate degree between 0 and 1 that is strictly less than *p*. Boolean mereology centers around the simple idea that parthood comes in Boolean degrees. The basic thought behind the view is that while the classical picture does great in modeling the parthood relations among abstract mathematical objects like geometrical spheres or spacial-temporal regions that are perfectly precise, it is inadequate when we wish to further theorize about the parthood relations among ordinary objects like cats and mountains that have vague mereological boundaries. To deal with the ordinary objects we need a wider range of parthood degrees in addition to 0 and 1, and we will argue in the next section that the wider range should be a larger complete Boolean algebra.

2.4 Why Boolean Degrees?

In the literature on mereological indeterminacy, or the literature on vagueness in general, the most mainstream, or even perhaps the only currently available version of degree theory, is the one which changes the classical semantics by replacing the classical value range with the real interval $[0, 1]$, ordered in the standard way. Let us call a degree-theoretic semantics of this kind, or just a degree-theoretic semantics under which the degrees of truth are ordered linearly, a *fuzzy* semantics. Of course, my definition here of a fuzzy semantics is very general, and as it stands a cluster of views that differ from each other in bigger or smaller details satisfy this definition. But the points that I am about to make in the rest of this section should be applicable to them all.

Since any complete Boolean algebra larger than $\{0,1\}$ is not a linear order, Boolean semantics, in the sense that matters, is not a fuzzy semantics. Boolean semantics actually shares a lot in common with a fuzzy semantics. They both originate from the simple thought that the classical semantics is inadequate at modeling the mereological status of ordinary objects because it offers too few options. Therefore, they both plan to change the classical semantics by enlarging the value range while keeping the rest untouched. The key difference, of course, is which structure we should replace the classical value range with. It is interesting to note that the classical value range $\{0,1\}$ is the only non-degenerate ordering that is both linear and Boolean. So both Boolean semantics and fuzzy semantics agree in that we should generalize some algebraic property of the classical value range in order to build larger ranges, but they disagree on which algebraic property we should generalize: for the fuzzy semantics, it is the property of being linear; for Boolean semantics, it is the property of being Boolean.

Despite sharing commonalities, Boolean mereology and the fuzzy alternative differ in substantial ways. In the rest of this section, I will argue that Boolean semantics is the better degree-theoretic semantic framework when it comes to theorizing about mereological indeterminacy. The biggest motivation behind the fuzzy views is that our intuition that parthood among ordinary objects is not an all-or-nothing matter; rather, it seems to come in different degrees. Common sense confirms that the tighter whisker W is part of Tibbles to a greater extent than the looser whisker W', though neither of the two whiskers are definitely part of Tibbles, as they are both on the verge of falling off. The biggest selling point of the fuzzy views, I think, is that it is able to capture this intuition. Under a fuzzy view, we can, for example, say that W is part of Tibbles to degree 0.5 while W' is part of Tibbles to degree 0.4; or in general, the tighter a shaky whisker is, the higher the degree we assign to it being part of Tibbles. But we can do the same thing with a Boolean ordering of truth degrees. Complete Boolean algebras can be as large as we want, and therefore there can be as many intermediate parthood degrees as want. As long as the Boolean value range has more than four elements, there will be two intermediate degrees p, q between 0 and 1 such that q is strictly less than p, so that we can let p be the degree to which W is part of Tibbles and q be the degree to which W' is part of Tibbles.

Second, although sometimes we have borderline cases of parthood whose degrees of parthood seem comparable, sometimes we have borderline cases of parthood whose degrees of parthood seem *incomparable*. Consider, for example, the tree T that is boundary of Mount Kilimanjaro. It is indeterminate whether T is part of Kilimanjaro, meaning that the degree to which T is part of Kilimanjaro is an intermediate value between 0 and 1, just as the degree to which the whisker W is part of Tibbles. But should the former degree be higher than the latter, or should the latter be higher than the former, or should they be equivalent? How exactly should we compare the degree to which T is part of Kilimanjaro to the degree to which W is part of Tibbles? I think it is impossible to answer these questions. Unlike in the case of W and W', there is simply *no* sensible dimension by which we can compare the degree to which T is part of Kilimanjaro and the degree to which W is part of Tibbles. The two degrees should be simply *incomparable*. It is absurd to assert that T is more part of Kilimanjaro than W is part of Tibbles and equally absurd to assert the opposite. But under a fuzzy semantics we have no choice but to have the two degrees be comparable to each other, since a linear ordering of degrees is connected, meaning that for any two fuzzy degrees p, q, either $p \leqslant q$ or $q \leqslant p$. This is, I believe, a unfortunate consequence of using a fuzzy semantics on mereology. And we can avoid it by adopting a Boolean semantics instead. Any complete Boolean algebra that is larger than $\{0, 1\}$ is not connected, and therefore there will be elements p, q such that neither $p \leqslant q$ nor $q \leqslant p$. Boolean mereology thus has the resources to refrain from comparing the degree to which T is part of Kilimanjaro and the degree to which W is part of Tibbles. In short, under Boolean mereology, unlike under its fuzzy counterpart, we do not have to make incomparable comparisons.

Third, the most commonly held and perhaps the most powerful objection to the fuzzy views is that they are in tension with classical rules of reasoning.[10] Departing from classical logic, I believe, comes with great costs, for at least two reasons. First, classical rules and tautologies that are invalid under the fuzzy views - say, for example, the law of excluded

[10] See, for exmaple, [20].

middle - are widely endorsed and employed in almost all other areas in philosophy and in mathematics. Rejecting classical logic would mean that fuzzy mereology has to be an isolated, lonely bubble in the theory space. Second, the way in which the fuzzy views violate classical logic brings upon unwelcome consequences. For example, consider the sentence that W is part of Tibbles. The truth degree of this sentence has to be an intermediate value, since W is a borderline case. But by the same reasoning, the negation of this sentence - that W is not part of Tibbles - also has to have an intermediate truth value. And because the values are ordered linearly, the conjunction of the two sentences - that W is both part of and not part of Tibbles - has to have an intermediate truth value as well, at least under the standard form of the fuzzy view. But that sounds wrong: nothing can be both part of and not part of Tibbles. The conjunction has the form of a contradiction, and a contradiction should be outright false instead of being somewhere in between truth and falsity.

Boolean mereology, in contrary, avoids all these problems, as it not only is compatible with but also enforces classical logic. As we will prove in the Appendix, Boolean-valued models, for first-order languages, for example, are sound and complete with respect to first-order logic. This means that all the theorems of first-order logic are true to degree 1 in every Boolean-valued model. Therefore, sentences saying that W is both part of and not part of Tibbles always have degree 0 in Boolean-valued models. Similarly, sentences saying that W is either part of Tibbles or not part of Tibbles always have degree 1. With Boolean truth degrees, we can have a many-degree truth-functional semantics with classical rules of inferences satisfied.

Last but not least, under Boolean mereology, not only can we have theorems of classical logic satisfied, we can also have principles of classical mereology satisfied. This point will be exemplified in the next section where we discuss a special kind of Boolean models for the language of mereology - the atomic Boolean models. Basically, we can have Boolean-valued models of mereology where all the principles of classical mereology have value 1. In contrast, this is something that is incredibly difficult, if not utterly impossible, to achieve, under the fuzzy approach.

For example, consider the case of Tibbles, of which W is a vague part. Under fuzzy semantics, the sentence that W is part of Tibbles should be a real number in $(0,1)$. Let's say that W is part of Tibbles to degree 0.5. Now, clearly Tibbles is distinct from the whole mereological universe (whose existence is guaranteed by classical mereology): lots of things, the Eiffel Tower, for example, are part of Tibbles to degree 0. A consequence of classical mereology - the principle of strong complementation[11] - says that everything that is distinct from the universe has a (mereological) complement. Since Tibbles is distinct from the universe to degree 1, there has to be an object, call it Complement, such that it is the complement of Tibbles to degree 1. This means that (1) Complement overlaps with Tibbles to degree 0, and (2) the fusion of Tibbles and Complement is identical to the entire universe to degree 1. But, then, what should be the degree to which W is part of Complement? In order for the degree to which Complement overlaps with Tibbles to be 0, the degree to which W is part of Tibbles and is part of Complement has to be 0, which means that the degree to which W is part of Complement can only be 0. But then the fusion of Tibbles and Complement is such that W is part of it to degree 0.5, whereas the universe is such that W is part of it to degree 1. So the fusion of Tibbles and Complement is not identical to the entire universe to degree 1. Contradiction.

For similar but slightly more difficult reasons, we can see that even the principle of weak supplementation[12] is going to fail under fuzzy semantics. And it is not hard to see that the failure of these classical mereological principles under fuzzy semantics is essentially due to the linear ordering of the truth values. In the case of Tibbles and Complement, in order for the principle of strong complementation to be true, we need the degree x to which W is part of Complement to be such that the supremum of x and 0.5 is 1 and the infimum of x and 0.5 is 0. Nevertheless, when the truth values are linearly ordered, there simply is no such value. When the truth values form a Boolean ordering, on the other hand, such a value does exist, as we will see shortly below. In any case, the general point here is simply that by adopting a fuzzy semantics we will have to sacrifice part of classical mereology, and this is a sacrifice that cannot be ignored, as classical mereology is well-understood and deeply intertwined with other areas in contemporary metaphysics. We can avoid this sacrifice by

[11]Formally, the principle of strong complementation is the following sentence in $\mathscr{L}_M$: $\forall v_1(\neg U(v_1) \to \exists v_2(\neg v_1 \circ v_2 \wedge \forall v_3(Fu(v_3,\{v_1,v_2\}) \to U(v_3)))))$, where $U(v_1) := \forall v_2(v_2 \lesssim v_1)$. For the definition of $\mathscr{L}_M$ and other defined notions, see Def. 2.5.1.

[12]Formally, the principle of weak supplementation is the following sentence in $\mathscr{L}_M$: $\forall v_1 \forall v_2(v_1 \lnsim v_2 \to \exists v_3(v_3 \lesssim v_2 \wedge \neg v_1 \circ v_3))$. For the definition of $\mathscr{L}_M$ and other defined notions, see Def. 2.5.1.

adopting Boolean semantics instead.

2.5 Atomic Boolean Models

The goal of this section is to investigate a special kind of Boolean model for mereology, which we will call the atomic Boolean models. These models arise from a simple and natural idea. We start with a pre-given set of mereological atoms S. Then, taking a complete Boolean algebra B as value range, we let domains of the models consist of functions from S to B. Intuitively, any function $f : S \to B$ corresponds to an object composed of the mereological atoms. For any $a \in S$, $f(a)$ is the degree to which the atom a is part of (the object represented by) f.

The atomic Boolean models[13] are particularly interesting and worth studying for multiple reasons. First, as mentioned above, atomic Boolean models are intuitively motivated. If the world is built up from mereological atoms, and if mereological relations comes in degrees, then the natural picture is that every object in the world is composed of the atoms to certain degrees. That is, it should be the case that every object in the world can be represented by a function from the set of all atoms to Boolean degrees, which is exactly what atomic Boolean models are like. Second, as argued above, Boolean mereology, unlike the fuzzy views, is easily compatible with axioms of classical mereology. Below we will exemplify this point by showing that a special case of the atomic Boolean models - the $SEVI$ models - are models of the system CM, which is equivalent to classical mereology. So with Boolean semantics we can have a degree-theoretic semantics of mereology with all axioms of classical mereology satisfied.

Third, in the literature on vague mereology, there has been a fair amount of discussion on the relationship between vague parthood on the one hand, and vague existence and vague identity on the other hand.[14] Many, for example, have either argued or tacitly assumed that

[13]The atomic Boolean models, as we will see in a moment, are models of the axiom of Atomicity. This does not mean, however, that Boolean semantics are stuck with atomic mereology. Note that atomic Boolean models are a *special kind* of Boolean-valued models for mereology that naturally arise *on the assumption that* the world is atomic. There can certainly be other types of Boolean-valued models for mereology that model, for example, some kind of gunky mereology. We focus on atomic Boolean models here simply because of their simplicity and their effectiveness in illustrating our points, as will be listed below.

[14]Here's a non-comprehensive list of articles that have touched on these equestions: Evans [11], Weather-

vague parthood entails vague existence, and therefore proponents of mereological vagueness are also stuck with existential vagueness. A study of atomic Boolean models, as I will show below, will shed light on how, under Boolean semantics, vague parthood is connected with vague existence and vague identity. In particular, I will show that their connection neither takes the form of entailment nor takes the form of exclusion, as there can be atomic Boolean models, though being models of vagueness, that disallows vagueness in existence/identity, and atomic Boolean models that allows vagueness in existence/identity.

Last but not least, I believe that atomic Boolean models are mathematically interesting to study as well. This is because atomic Boolean models are similar in multiple aspects to the standard Boolean-valued models of set theory, as presented in, say, Bell [3]. For example, the definition of the values of the atomic clauses on parthood in the atomic Boolean models is similar to the definition of the values of the clauses on subsethood in the Boolean models for set theory: the former is defined in terms of the degree to which every atom that is part of the first object is part of the second object, while the latter is defined in terms of the degree to which every element that is a member of the first set is a member of the second set. Another example is that when proving the axiom of Fusion holds in atomic Boolean models, we construct a fusion in the same way as we construct a mixture of a collection of Boolean-valued sets. These commonalities in techniques perhaps hint towards a deeper connection between Boolean-valued parthood and Boolean-valued membership, which seems to be worth of further study.

We will divide the rest of this section into two subsections. We will devote the first subsection to presenting a version of the formal theory of mereology that is tailored specifically to our needs. In the second subsection, we will define properly different kinds of the atomic Boolean models, use them to explore the relationship between vagueness in parthood and vagueness in existence/identity, and discuss which axioms of classical mereology hold in these different kinds of atomic Boolean models.

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son [35], Barnes and Williams [2] have argued that vague parthood entails vague identity; Cook [8], Sainsbury [30], and some others have argued for the opposite; van Inwagen [34], Lewis [21], Smith [31], Merricks [26] and many others hold that vague parthood entails vague existence; Morreau [27] and Donnelly [9] hold the opposite view.

2.5.1 Classical Mereology

As mentioned above, one of the primary goals of studying atomic Boolean models is to investigate the relation between vague parthood and vague existence/identity. We will also investigate how, given the presence of vague parthood, different axioms of classical mereology are connected with the presence/absence of vague existence/identity. But to meet these needs we will have to deviate from the standard formulation of classical mereology to some extent, for reasons I will explain in a moment. In particular, the deviation will come in two parts: (a) we will alter, in minor but important details, the way in which some non-primitive mereological notions are defined in terms of the notion of parthood, and (b) we will present and group the axioms of classical mereology in a way that is slightly more complicated and cumbersome than the standard.

Part (a) of the deviation further consists of two changes. The first, and the most important change we will make is that we will define an “existence" predicate and restrict quantification to objects that satisfy this predicate at certain places (for example, when defining “overlap", “fusion", etc.). The reason why we need this change is because the standard formulation of classical mereology tacitly assumes that everything in the domain of quantification fully exists, and therefore leaves no room for vague existence at all. In order to be able to discuss the *possibility* of vague existence, therefore, we have to define this “existence" predicate that serves the purpose of measuring the degree to which an object exists, and have it impact the domain of quantification at places that matter. The second change we will make is less non-trivial and is mostly just for convenience: we will define the notion of proper part without using the identity symbol. Later we will see that this small change allows all the axioms of atomic classical mereology except Anti-Symmetry to be formulated without the identity symbol. Therefore, it will follow directly from the formulation of these axioms that the truth/falsity of these axioms in a Boolean model is not affected by how identity is defined in the model, or in other words, whether we have vague identity or not.

Now we introduce the language of mereology and the defined notions:

Definition 2.5.1. The language of mereology, $\mathscr{L}_M$, is the second order language[15] whose signature contains a single binary relation $\precsim$ (parthood). We further define the following relations in this language:

1. $v_1 \precnsim v_2 := v_1 \precsim v_2 \wedge \neg v_2 \precsim v_1$.

2. $E(v_1) := \exists v_2(\neg v_1 \precsim v_2)$.

3. $v_1 \circ v_2 := \exists v_3(E(v_3) \wedge v_3 \precsim v_1 \wedge v_3 \precsim v_2)$.

4. $At(v_1) := E(v_1) \wedge \forall v_2(E(v_2) \to \neg v_2 \precnsim v_1)$.

5. $FU(v_1, X_1) = \forall v_2(X_1(v_2) \to v_2 \precsim v_1) \wedge \forall v_3(v_3 \precsim v_1 \wedge E(v_3) \to \exists v_4(X_1(v_4) \wedge v_3 \circ v_4))$.

Intuitively, $v_1 \precsim v_2$ means that v_1 is a part of v_2. $v_1 \precnsim v_2$ means that v_1 is a proper part of v_2. $E(v_1)$ means that v_1 exists, or that v_1 is not zero, in the sense that v_1 is not a part of everything. $v_1 \circ v_2$ means that v_1 and v_2 overlap. $At(v_1)$ means that v_1 is a mereological atom. $FU(v_1, X_1)$ means that v_1 fuses the X_1's.

We now move on to axioms of mereology, which are sentences in $\mathscr{L}_M$. We divide these axioms into four groups, for purposes we will explain in a moment:

Definition 2.5.2. The *minimal* theory of Classical Mereology (*MCM*) contains the following three axioms:

(Transitivity)	$\forall v_1 \forall v_2 \forall v_3(v_1 \precsim v_2 \wedge v_2 \precsim v_3 \to v_1 \precsim v_3)$
(Supplementation)	$\forall v_1 \forall v_2(v_2 \not\precsim v_1 \to \exists v_3(E(v_3) \wedge v_3 \precsim v_2 \wedge \neg v_1 \circ v_3))$
(Fusion)	$\forall X_1(\exists v_1 X_1(v_1) \to \exists v_2(FU(v_2, X_1))$

The theory of Classical Mereology without Identity (CM^-) contains *MCM* and the follow-

[15] Whether classical mereology should be formulated as a first-order or second-order theory is not a trivial issue, and one might have different preferences based on their other theoretical commitments. For example, a nominalist might want to avoid quantifying over second-order entities. But none of these concerns, I think, matter to our discussion of mereological indeterminacy. In this chapter I define the theory of classical mereology as a second-order theory simply because this is the more demanding option, and all the Boolean constructions we have laid out in this chapter can be easily carried over to the first-order case.

ing extra axiom:

$$\text{(NoZero)} \qquad \exists v_1 \exists v_2 (v_1 \not\lesssim v_2) \rightarrow \neg \exists v_3 \neg (E(v_3))$$

The theory of Classical Mereology (*CM*) contains CM^- and the following extra axiom:

$$\text{(Anti-Symmetry)} \qquad \forall v_1 \forall v_2 (v_1 \lesssim v_2 \wedge v_2 \lesssim v_1 \rightarrow v_1 = v_2)$$

The minimal theory of Atomic Classical Mereology (*MACM*) / the theory of Atomic Classical Mereology without Identity (ACM^-) / the theory of Atomic Classical Mereology (*ACM*) contains *MCM*/CM^-/*CM* and the following extra axiom:

$$\text{(Atomicity)} \qquad \forall v_1 (E(v_1) \rightarrow \exists v_2 (At(v_2) \wedge v_2 \lesssim v_1))$$

We have the minimal theory consisting of Transitivity, Supplementation and Fusion because these, as we will show in the next subsection, will be the core axioms that will be satisfied no matter whether we have vague existence, vague identity, or not, as we will show in the next subsection. The axioms NoZero and Anti-Symmetry are listed separately because these are the ones that do take a stand on whether there is vague existence/identity or not: the former disallows vague existence and the latter requires vague identity. An interesting observation is that the minimal theory *MCM* together with Anti-Symmetry forms a neutral system that is in between the classical theory of mereology[16] and the (second-order) theory of complete Boolean algebras, in the following sense:

Theorem 2.5.1. *CM* is equivalent to Tarski's system, which is the theory closed under the following two axioms:

$$\text{(Transitivity)} \qquad \forall v_1 \forall v_2 \forall v_3 (v_1 \lesssim v_2 \wedge v_2 \lesssim v_3 \rightarrow v_1 \lesssim v_3)$$

$$\text{(UniqueFusionExistence)} \qquad \forall X_1 (\exists v_2 X_1(v_2) \rightarrow \exists ! v_1 (FU'(v_1, X_1)))$$

[16] By "the classical theory of mereology" I mean the theory that originates from Tarski's paper [33]. For a full development of Tarski's system, see [15].

where $FU'(v_1,X_1)$ is a slight variation of $FU(v_1,X_1)$, and is defined as follows:

$$FU'(v_1,X_1) = \forall v_2(X_1(v_2) \to v_2 \precsim v_1) \wedge \forall v_3(v_3 \precsim v_1 \to \exists v_4(X_1(v_4) \wedge \exists v_5(v_5 \precsim v_3 \wedge v_5 \precsim v_4))).$$

Theorem 2.5.2. The (second-order) theory of complete Boolean algebras is equivalent to *MCM* plus Anti-symmetry plus the following axiom:

$$\text{(ZeroExistence)} \qquad \exists v_1 \neg E(v_1)$$

The proofs of these theorems are in the Appendix.

2.5.2 Atomic Boolean Models

We shall now define the atomic Boolean models. As we mentioned above, the domain of these models consists of functions from a pre-given set of mereological atoms S to a complete Boolean algebra B. But which of these functions shall we include in the domain exactly? For reasons I will explain in a moment there are at least two collections of functions from S to B that may reasonably form the domain of a model:

1. $M = \{f : S \to B \mid \bigsqcup_{a \in S} f(a) = 1\}$.
2. $N = \{f : S \to B \mid \bigsqcup_{a \in S} f(a) > 0\}$.

In the Appendix (Lemma 2.7.8.1 and Lemma 2.7.13.1) we will prove that in any atomic Boolean model, for any $f : S \to B$ in the domain, $\bigsqcup_{a \in S} f(a) = [\![E(f)]\!]$, the degree to which f exists. So the set M consists of functions that correspond to objects that exist to degree 1. In our setting, to exist vaguely means to satisfy the existence predicate E to a degree that is in between 0 and 1. Therefore, atomic Boolean models with domain M has no room for vague existence at all. They will be used to show that under Boolean mereology, vague parthood does not entail vague existence, contrary to what many have argued, as there are Boolean models of vague parthood that are not models of vague existence. On the other hand, the set N consists of functions that correspond to objects that exist to any positive degree. Atomic Boolean models with domain N, therefore, have objects in their domains that exist vaguely. Under Boolean mereology, mereological vagueness can co-occur with existential vagueness, although not necessarily.

Definition 2.5.3. Let S be a set (of mereological atoms). Let B be a complete Boolean algebra. A B-valued SE ("Sharp-Existence") model on S, $\mathfrak{S}_S^B$, is a B-valued model for $\mathscr{L}_M$ with:

1. The domain $M = \{f : S \to B \mid \bigsqcup_{a \in S} f(a) = 1\}$.

2. For any $f_1, f_2 \in M$, $[\![f_1 \precsim f_2]\!]^{\mathfrak{S}_S^B} = \bigsqcap_{a \in S} f_1(a) \Rightarrow f_2(a)$[17].

A B-valued VE ("Vague-Existence") model on S, $\mathfrak{S}_V^B$, is a B-valued model for $\mathscr{L}_M$ with:

1. The domain $N = \{f : S \to B \mid \bigsqcup_{a \in S} f(a) > 0\}$.

2. For any $f_1, f_2 \in N$, $[\![f_1 \precsim f_2]\!]^{\mathfrak{S}_V^B} = \bigsqcap_{a \in S} f_1(a) \Rightarrow f_2(a)$.

In both kinds of models the values of parthood clauses are defined in the same way. Roughly, the degree to which an object is a part of another is defined as the degree of the sentence that every atom that is a part of the former is also a part of the latter.

Note that in defining these models we have omitted the definition of the values of identity clauses. This is because, depending on whether we want vague identity in our models or not, there are two different ways of defining identity in atomic Boolean models. The first way, which is given under the label "Vague-Identity", is to define identity in terms of the degree to which two objects share the same atoms. This is the way that is friendly to vague identity: it allows objects to be identical to each other to an intermediate degree. The second way, which is given under the label "Sharp-Identity", is to define identity "in the sharp way", that is, to define the degree to which two objects are identical as 1 when the corresponding functions are the same, and as 0 when the corresponding functions are different. This is the way, as you may expect, that is hostile to vague identity. Given two

[17] For any p, q in a Boolean algebra B, $p \Rightarrow q = -p \sqcup q$.

functions $f_1, f_2 : S \to B$:

(Vague-Identity) $$[\![f_1 = f_2]\!] = \prod_{a \in S} f_1(a) \Leftrightarrow f_2(a).$$

(Sharp-Identity) If f_1 and f_2 are not the same, then $[\![f_1 = f_2]\!] = 0$.

We can freely combine Vague/Sharp-Identity with SE/VE models and get four different kinds of models, as listed in the following:

Definition 2.5.4. Let S be a set (of mereological atoms). Let B be a complete Boolean algebra. The B-valued $SEVI$ ("Sharp-Existence Vague-Identity") model on S, $\mathfrak{S}^B_{SV}$, is the B-valued model for $\mathscr{L}_M$ with:

1. The domain $M = \{f : S \to B \mid \bigsqcup_{a \in S} f(a) = 1\}$.
2. For any $f_1, f_2 \in M$, $[\![f_1 \precsim f_2]\!]^{\mathfrak{S}^B_{SV}} = \prod_{a \in S} f_1(a) \Rightarrow f_2(a)$.
3. For any $f_1, f_2 \in M$, $[\![f_1 = f_2]\!]^{\mathfrak{S}^B_{SV}} = \prod_{a \in S} f_1(a) \Leftrightarrow f_2(a)$.

The B-valued $SESI$ ("Sharp-Existence Sharp-Identity") model on S, $\mathfrak{S}^B_{SS}$, is the B-valued model for $\mathscr{L}_M$ with:

1. The domain $M = \{f : S \to B \mid \bigsqcup_{a \in S} f(a) = 1\}$.
2. For any $f_1, f_2 \in M$, $[\![f_1 \precsim f_2]\!]^{\mathfrak{S}^B_{SS}} = \prod_{a \in S} f_1(a) \Rightarrow f_2(a)$.
3. For any $f_1, f_2 \in M$, if f_1 and f_2 are not the same, then $[\![f_1 \precsim f_2]\!]^{\mathfrak{S}^B_{SS}} = 0$.

The B-valued $VEVI$ ("Vague-Existence Vague-Identity") model on S, $\mathfrak{S}^B_{VV}$, is the B-valued model for $\mathscr{L}_M$ with:

1. The domain $N = \{f : S \to B \mid \bigsqcup_{a \in S} f(a) > 0\}$.
2. For any $f_1, f_2 \in N$, $[\![f_1 \precsim f_2]\!]^{\mathfrak{S}^B_{VV}} = \prod_{a \in S} f_1(a) \Rightarrow f_2(a)$.
3. For any $f_1, f_2 \in N$, $[\![f_1 = f_2]\!]^{\mathfrak{S}^B_{VV}} = \prod_{a \in S} f_1(a) \Leftrightarrow f_2(a)$.

The B-valued $VESI$ ("Vague-Existence Sharp-Identity") model on S, $\mathfrak{S}^B_{VS}$, is the B-valued model for $\mathscr{L}_M$ with:

1. The domain $N = \{f : S \to B \mid \bigsqcup_{a \in S} f(a) > 0\}$.
2. For any $f_1, f_2 \in N$, $[\![f_1 \precsim f_2]\!]^{\mathfrak{S}^B_{VS}} = \prod_{a \in S} f_1(a) \Rightarrow f_2(a)$.
3. For any $f_1, f_2 \in N$, if f_1 and f_2 are not the same, then $[\![f_1 \precsim f_2]\!]^{\mathfrak{S}^B_{VS}} = 0$.

Assuming that B is larger than $\{0, 1\}$, all of the four different kinds of models are models of mereological vagueness, as it is easy to see that in all of the models there are objects that are part of one another to an intermediate degree. But they deliver different answers on whether there is vagueness in existence and/or on whether there is vagueness in identity. Just as in the case of existential vagueness, mereological vagueness can co-occur with vagueness in identity, but not necessarily.

In the rest of this section we will investigate which axioms of mereology hold in these four kinds of models. Most results will be simply stated here with the proofs in the Appendix.

As we have mentioned before, we formulate most axioms of mereology (all except Anti-Symmetry) without using the identity symbol. And hence whether these axioms hold or not in these models do not depend upon whether they are VI or SI. In fact,

Theorem 2.5.3. In any SE model, Transitivity, Supplementation, Fusion, Atomicity and NoZero all have value 1.

Theorem 2.5.4. In any VE model, Transitivity, Supplementation, Fusion and Atomicity all have value 1, but NoZero has value 0.

So the core theory of atomic classical mereology - and by that I mean the system *MACM* - is satisfied by all four kinds of model discussed here. Therefore, all four models can be legitimately considered models of atomic classical mereology. The difference between the VE and the SE models, of course, is that the axiom of NoZero does not hold in the VE models. This is, I believe, a somewhat unfortunate result for the supporters of vague existence. It means that if we allow objects that exist vaguely, then we will have to have the model believe that there is an empty object that is part of everything, even when there is more than one object. Under the standard conception of classical mereology, such an empty object is disallowed, because it is normally considered as philosophically unmotivated[18]. Nevertheless, it is not hard to see why there has to be tension between existential vagueness and the axiom of NoZero, in the current context. Assuming there is more than one object, then the axiom of NoZero has value 1 just in case every object f in the domain satisfies the existence predicate to degree 1. So the axiom of NoZero literally leaves no room for objects that exist to intermediate degrees. Proponents of existential vagueness have to sacrifice the axiom of NoZero.

Luckily, proponents of existential vagueness could argue that although the axiom of NoZero, in its current form, cannot be satisfied by models in which objects may exist vaguely, there is a satisfiable weaker meta-principle that is in the same spirit. The latter is the principle that there cannot be in the domain any object that is truly empty - that is, any object that satisfies the existence predicate to degree 0. This has to be a principle in the meta-language because we simply do not have the expressive resources to state something of the form "x satisfies F to degree p" in the object language. As it is easy to see, all VE models satisfy this meta-principle straightforwardly according to the definition of their domain N. Proponents of VE models could argue that although the VE models believe that there is an empty object, there isn't really an empty object in the domain of these models, and the latter is all we care about.[19]

Moving on to the only axiom left - the axiom of Anti-Symmetry. As the readers might have expected, the holding or not of Anti-Symmetry in an atomic Boolean model is only associated with whether identity is defined in the vague way or in the sharp way in the model. Let us call a model a VI model if it is $SEVI$ or $VEVI$, and similarly a model a SI model if it is $SESI$ or $VESI$. It can be shown that:

Theorem 2.5.5. In any VI model, Anti-Symmetry has value 1.

[18] Although most people find the existence of an empty object philosophically unmotivated, there are some people who have provided ways to justify the existence of an empty object. Giraud [13] has construed it as a Meinongian object lacking all nuclear properties. Priest [28] has construed it as an Heideggerian nothing.

[19] This is an example of an intriguing and perhaps weird feature of Boolean-valued models. Some models could be such that an existential sentence is true in the model without there being a witness.

Theorem 2.5.6. In any *SI* model, Anti-Symmetry has value 0.

The opponents of vagueness in identity, therefore, has to sacrifice part of the standard package of classical mereology, just as the proponents of existential vagueness. In this case the sacrifice is the axiom of Anti-Symmetry. It is not hard to see why “Sharp-Identity" makes trouble for the holding of Anti-Symmetry: since there is mereological vagueness, there can be objects that are part of each other to an intermediate degree. Since their corresponding functions has to be different, “Sharp-Identity" insists that they are identical to degree 0, and hence the degree to which they are part of each other is strictly greater than the degree to which they are identical, which causes the failure of Anti-Symmetry.

Just as the proponents of existential vagueness, there are, I believe, some ways for the opponents of vagueness in identity to argue back. They could say that, for example, in the context of mereology, there should really be two different notions of identity: one is the notion of mereological coincidence, and the other is the notion of strict/real identity. Two objects mereologically coincide - that is, are identical in the former sense - just in case they are indistinguishable in terms of mereological relations. On the other hand, two objects are strictly identical just in case they are indistinguishable in terms of *any* kind of properties or relations, mereological or not. And the key idea is that the equality symbol in the axiom of Anti-Symmetry should be interpreted as mereological coincidence instead of as strict identity: if two objects are part of one another, then they should be indistinguishable in terms of mereological relations, but saying that they should also be indistinguishable in terms of any relations seems like overkill. In an atomic Boolean model, the degree to which two objects mereologically coincide should be defined according to “Vague-Identity", that is, as the degree to which two objects share the same atoms, and the degree to which two objects are strictly identical should be defined according to “Sharp-Identity", such that it can only be an extreme value. Since the relation that plays a role in Anti-Symmetry is mereological coincidence, we will have Anti-Symmetry holding in the models, and since strict identity is still defined traditionally, we also avoid the controversies surrounding vagueness in identity.[20]

Below is a chart summarizing which axioms hold in each of the four kinds of atomic

[20] A standard argument against vagueness in identity is Evans' Argument. See [11].

Boolean model:

	MACM	*MACM*+NoZero	*MACM*+Anti-Symmetry	*ACM*
SEVI	✓	✓	✓	✓
SESI	✓	✓	✗	✗
VEVI	✓	✗	✓	✗
VESI	✓	✗	✗	✗

Here's a summary of what we have achieved in this section. First, we have introduced a special kind of Boolean-valued models for mereology - the atomic Boolean models, and argued that they are intuitively motivated, given that the world is atomic. Second, we have used the *SEVI* models to illustrate our previous point that with Boolean degrees, we can have a degree-theoretic semantics that is compatible with the whole package of atomic classical mereology. Finally, we have used the atomic Boolean models to investigate the connection between mereological vagueness on the one hand and vagueness in existence and identity on the other hand. We have shown that contrary to what many have argued, mereological vagueness entails neither existential vagueness nor vagueness in identity. With the four different kinds of atomic Boolean models, proponents of mereological vagueness can freely choose between having and not having vagueness in existence or identity: *SEVI* models for sharp existence plus vague identity, *SESI* models for sharp existence plus sharp identity, *VEVI* models for vague existence plus vague identity, and *VESI* models for vague existence plus sharp identity. There are, nevertheless, prices to be paid. Although all four models are models for the core theory of atomic classical mereology, the axiom of NoZero does not hold in the “Vague-Existence" models and the axiom of Anti-Symmetry does not hold in the “Sharp-Identity" models.

2.6 The Nature of Mereological Vagueness

Our investigation of Boolean mereology so far has been fruitful, but not all important questions about mereological vagueness have been properly addressed. One essential question is: given that there is mereological vagueness, what is the source, or the nature of it? Is mereological vagueness a pure linguistic phenomenon, or is the world itself vague? Does the picture of Boolean mereology entail that mereological vagueness is semantic or ontological? In this section I intend to discuss these questions.

There are, I believe, two most commonly held answers to the question “What is the nature of mereological vagueness?". One option, which I will call “the semantic thesis" in the following, is to say that mereological vagueness has a semantic nature. The phenomenon exists because our linguistic practices are indeterminate, in the sense that they do not pin down the exact meanings of certain terms, including, perhaps, singular names like “Tibbles". The linguistic rules that we have governing the name “Tibbles" do not pick out a unique referent for it. The world in itself, on the other hand, is perfectly precise, mereologically speaking: there is no indeterminacy in the mereological organization of the underlying reality. Mereological indeterminacy happens when we try to represent what the world is like using natural languages: if there were no language, or if natural languages were perfectly precise, there would be no indeterminacy in the parthood relation.

The other option, which I will call “the ontic thesis", is to say that mereological vagueness has an ontic, or worldly, nature. There is indeed indeterminacy in the mereological organization of reality. Regardless of the terms we use to represent them, ordinary objects in the world, like for example Tibbles the cat, are themselves vague, in the sense that their mereological constitution is indeterminate. Mereological vagueness is a feature of the world itself, not a feature of our languages.

Which one of the two theses should we adopt, as Boolean mereologists? I believe that Boolean mereology, as the thesis that the relation of parthood should be modeled by Boolean degrees, is compatible with either thesis. Boolean mereology only says that sentences like “W is part of Tibbles" are true to an intermediate Boolean degree; it does not specify *why* these sentences are true to an intermediate Boolean degree. I will show below that the model-theoretic framework of Boolean-valued semantics can be applied to both theses and give rise to two distinctive views that have their unique advantages and disadvantages. I will call the view we get by combining the semantic thesis and Boolean-valued semantics “semantic Boolean mereology" and the view we get by combining the ontic thesis and Boolean-valued semantics “ontic Boolean mereology", and discuss them in turn in the following two subsections.

2.6.1 Semantic Boolean Mereology

The semantic thesis explains mereological indeterminacy in terms of linguistic indeterminacy and denies worldly indeterminacy. The most standard and commonly-held version of the view locates the indeterminacy in singular names like “Tibbles" or “Kilimanjaro". On this view, all there is in the world are objects with precise mereological boundaries. Names like “Tibbles" do not pick out a unique referent among the precise objects. Rather, there are multiple precise objects, located roughly where Tibbles is, that are equally qualified candidates for the role of being the referent of “Tibbles".

How does Boolean-valued semantics accommodate this view? To simplify our discussion, let us assume that the world is atomic and nothing exists but (sharp) fusions of atoms. Let S be the collection of all atoms. Since everything that exists is a (sharp) fusion of atoms, the domain of our Boolean-valued model has to be the collection M' of all functions from S to $\{0,1\}$ except the one that takes all atoms to 0, where each function represents a fusion of atoms by being its characteristic function. As there are only precise objects in the domain, the identity symbol in the model can simply be interpreted as the sharp identity function on these objects. Now, since we want “Tibbles" to have no unique referent, “Tibbles" cannot be treated as an ordinary constant in the model. Rather, we need it to be the case that “Tibbles" indeterminately refer to multiple objects in the domain. In the context of Boolean-valued semantics, indeterminacy means having an intermediate truth value. So we want “Tibbles" to be interpreted in the model as a function from M' to B, which maps each object in the domain to degree to which the name “Tibbles" refers to it. In other words, we will treat “Tibbles" semantically as if it were a unary predicate. Of course, “Tibbles" cannot be treated as if it were an arbitrary unary predicate: there are further constraints that the interpretation of “Tibbles" has to satisfy. In particular, the interpretation of “Tibbles" has to be such that the sentence $\exists! v(Tibbles(v))$ - there is exactly one Tibbles - has value 1. As a result, the values attributed to the objects by (the interpretation of) “Tibbles" has to form a maximal antichain in the Boolean algebra.

Let me spell out the above picture in more details, by constructing a concrete B-valued[21] model for the language consisting of “Tibbles", “W", and “is part of", $\mathfrak{M}'$, tailored to the

[21] Here we assume B is an arbitrary complete Boolean algebra.

needs of the standard semantic approach. Again, we assume that the world is atomic and all that exist are (sharp) fusions of atoms. Also, we assume, just for simplicity, that the name “W" picks out, instead of a whisker, an atom in the whisker that is about to fall off from the cat. The domain of the model, M', consists of all functions from S to $\{0,1\}$, except the one that takes all $a \in S$ to 0. That is, $M' = \{g : S \to B \mid \text{for any } a \in S, g(a) = 0 \text{ or } 1, \text{ and for some } b \in S, g(b) \neq 0\}$, which is equivalent to $\mathscr{P}(S)$ (the powerset of S) minus the empty set. The language $\mathscr{L}' = \{t, w, \lesssim\}$, where $\lesssim$ is the binary relation of parthood, w is a constant playing the role of “W", and t is a unary predicate playing the role of “Tibbles". Since w is supposed to name an atom, the interpretation of w in $\mathfrak{M}'$ will be the characteristic function of a singleton subset $\{a\}$ of S. In other worlds, $[\![w]\!]^{\mathfrak{M}'} = g^a : S \to B$, where $a \in S$ and g^a takes a to 1 and every $b \neq a \in S$ to 0. The interpretation of $\lesssim$ in $\mathfrak{M}'$ will be the function from $M' \times M' \to 2$ that corresponds to the subset relationship on $\mathscr{P}(S)\backslash\varnothing$. The interpretation of $=$ in $\mathfrak{M}'$ will be the “real" identity relation on M': for any $g, g' \in M'$, $[\![g = g']\!] = 1$ if g and g' are the same and $[\![g = g']\!] = 0$ if g and g' are not the same. Finally, the interpretation of t in $\mathfrak{M}'$, $[\![t]\!]^{\mathfrak{M}'}$, will be a function from M' to B that satisfies the following conditions:

1. For any $g \neq g' \in M'$, $[\![t(g)]\!]^{\mathfrak{M}'} \sqcap [\![t(g')]\!]^{\mathfrak{M}'} = 0$.
2. $\bigsqcup_{g \in M'} [\![t(g)]\!]^{\mathfrak{M}'} = 1$.
3. For some $g \in M$ such that $[\![t(g)]\!]^{\mathfrak{M}'} \neq 0$, $g(a) = 1$, and for some $g' \in M$ such that $[\![t(g')]\!]^{\mathfrak{M}'} \neq 0$, $g(a) = 0$.

For every $g \in M'$, $[\![t(g)]\!]^{\mathfrak{M}'}$ is the degree to which t “refers to" g. $[\![t(g)]\!]^{\mathfrak{M}'} \neq 0$ means that g is a *possible*, or *permissible* referent of t. The third condition serves many purposes: first, it guarantees that there are more than one permissible referent of t; second, it means that w is part of some permissible referent of t yet is not part of some other permissible referent of t; and third, together with the first two conditions, it ensures that no g is the *determinate* referent of t, in the sense that $[\![t(g)]\!]^{\mathfrak{M}'} = 1$. The first two conditions also guarantee that that $[\![\exists! v(t(v))]\!]^{\mathfrak{M}'} = 1$: it is true in $\mathfrak{M}'$ that there is exactly one t.

What is the degree to which w is part of t in $\mathfrak{M}'$? We want it to be an intermediate degree between 0 and 1, capturing the fact that it is indeterminate whether w is part of

t. And the conditions we impose on the interpretation of t in $\mathfrak{M}'$ can indeed guarantee that. But there is a small complication. The sentence $w \lesssim t$ contains t syntactically as a constant, yet our model $\mathfrak{M}'$ treats t as a unary predicate. So we need to find some way to translate this sentence, or any sentence that contains t syntactically as a constant, to a sentence that contains t syntactically as a unary predicate. The trick we will use here is to translate any sentence of the form $\phi(t)$, which has t as a constant, to the sentence $\exists! v_i(t(v_i)) \wedge \forall v_j(t(v_j) \to \phi(v_j))$. It is easy to check that this translation recipe always preserves truth values for sentences involving constants. Moving on to the sentence under concern: (let $T' = \{g \in M' \mid [\![t(g)]\!]^{\mathfrak{M}'} \neq 0 \text{ and } g(a) = 0\}$)

$$
\begin{aligned}
[\![w \lesssim t]\!]^{\mathfrak{M}'} &= [\![\exists! v_i(t(v_i)) \wedge \forall v_j(t(v_j) \to (w \lesssim v_j))]\!]^{\mathfrak{M}'} \\
&= [\![\forall v_j(t(v_j) \to (w \lesssim v_j))]\!]^{\mathfrak{M}'} \\
&= \prod_{g \in M'} [\![t(g)]\!]^{\mathfrak{M}'} \Rightarrow g(a) \\
&= \prod_{g \in T'} -[\![t(g)]\!]^{\mathfrak{M}'} = -\bigsqcup_{g \in T'} [\![t(g)]\!]^{\mathfrak{M}'}
\end{aligned}
$$

The three conditions we impose on the interpretation of t guarantees that $0 < \bigsqcup_{g \in T'} [\![t(g)]\!]^{\mathfrak{M}'} < 1$. Therefore, $0 < [\![w \lesssim t]\!]^{\mathfrak{M}'} < 1$, which is exactly what we want.

Since $\mathfrak{M}'$ is a Boolean-valued model, all principles of classical logic will hold in it. Also, as it is easy to see that $\mathfrak{M}'$ restricted to the language of mereology $\mathscr{L}_M$ is isomorphic to the powerset model on S, the whole package of atomic classical mereology, by which I mean the system ACM, will hold in $\mathfrak{M}'$. One feature of $\mathfrak{M}'$ worth mentioning is that $\mathfrak{M}$ is not a “witnessing" model, in the sense there are existential sentences whose truth value is strictly greater than that of any of its instances. For example, the sentence “something is Tibbles" will have value 1 in the model without any of its instances having value 1. But this is exactly what supporters of the semantic thesis would want: although they would agree that “Tibbles exists" is true, they would not identity any (sharp) object in the domain as uniquely identical to Tibbles.

Therefore, Boolean-valued semantics, as shown above, provides an elegant model theory for the semantic thesis. Under semantic Boolean mereology, the actual world that we live in is just like to the model $\mathfrak{M}$ we constructed above. All there is are sharp objects, and the parthood relation that holds between them is also sharp. Mereological indeterminacy is grounded in the linguistic indeterminacy of terms like “Tibbles", which is further explained in terms of there being multiple objects in the domain to which the term applies to a degree larger than 0.

The standard model-theoretic framework that accompanies the semantic thesis is supervaluation semantics[22]. A supervaluation model consists of a fixed domain of objects[23] and multiple permissible precisifications. Each precisification can be understood as a two-valued model with the given domain. A sentence is (super)true if it is true in all precisifications, (super)false if false in all precisifications, and neither (super)true nor (super)false if otherwise. On cases like Tibbles, each permissible precisification assigns to “Tibbles" a different object in the domain as its referent. “W is part of Tibbles", in the intended model, will be a sentence that is neither (super)true nor (super)false. A supervaluation model is actually a special case of a Boolean-valued model like $\mathfrak{M}'$. Let $\mathfrak{S}$ be a supervaluation model for $\mathscr{L}'$ with domain D and precisifications $\{\mathfrak{A}_\mathrm{i} \mid i \in I\}$, where in each $\mathfrak{A}_\mathrm{i}$, $[\![w]\!]^{\mathfrak{A}_\mathrm{i}} = a \in D$ and $[\![t]\!]^{\mathfrak{A}_\mathrm{i}} = a_i \in D$. We can transform $\mathfrak{S}$ to a $\mathscr{P}(I)$-valued Boolean model $\mathfrak{M}^{\mathfrak{S}}$ with domain D as follows:

1. $[\![w]\!]^{\mathfrak{M}^{\mathfrak{S}}} = a$.

2. For any $b \in D$, $[\![t(b)]\!]^{\mathfrak{M}^{\mathfrak{S}}} = \{i \in I \mid \mathfrak{M}_\mathrm{i} \models t = b\}$.

3. For any $b, c \in D$, $[\![b \precsim c]\!]^{\mathfrak{M}^{\mathfrak{S}}} = \{i \in I \mid \mathfrak{M}_\mathrm{i} \models b \precsim c]\!]$.

Using the translation recipe we introduced above, $[\![w \precsim t]\!]^{\mathfrak{M}^{\mathfrak{S}}} = \{i \in I \mid \mathfrak{A}_\mathrm{i} \models w \precsim t\}$ will be a proper non-empty subset of I, as there are precisifications in which w is part of t and ones in which w is not part of t. That $w \precsim t$, therefore has an intermediate truth value in $\mathfrak{M}^{\mathfrak{S}}$, which corresponds to the fact that it is neither (super)true nor (super)false in $\mathfrak{S}$.

[22] As in, for example, [12].

[23] Sometimes supervaluationism is used on cases where it is indeterminate what the domain of quantification is. One example are cases of quantum indeterminacy (see [6] or [22]). On cases of mereological indeterminacy, nevertheless, it is usually safe to assume that the domain of quantification is determinate.

Although mathematically speaking, transforming a supervaluation model into a Boolean-valued model makes no significant difference, from a philosophical perspective such a transformation brings a number of benefits. Since supervaluation models now become Boolean-degree-theoretic, they enjoy all the advantages that the Boolean semantics has, as discussed in the previous sections: being truth-functional, having distinct comparable borderline statuses, having incomparable borderline statuses etc.

Semantic Boolean mereology is an attractive story and enjoys many theoretical advantages. For example, some people ([10]) have argued that vagueness should be a uniform phenomenon, in the sense that different types of vagueness should have the same nature: they are either all semantic or all ontic. Since there are strong arguments for vagueness in properties (like the property of being bald) being a semantic phenomenon, mereological vagueness should be theorized as a semantic phenomenon as well. To me, the biggest advantage of semantic Boolean mereology is that it naturally comes with a solution to the notorious problem of the many (see, for example, [24]). As long as we accept classical mereology, the principle of fusion existence will generate a great number of distinct objects that heavily overlap with each other, all located at where Tibbles is. There are, then, two seemingly contradictory intuitions. The first intuition is that there should be only one referent of “Tibbles", instead of many. The second intuition is that since these objects only have minute differences - say, only in whether it has an atom on the periphery of Tibbles like w, no one among them seems to have a better claim to be the referent of “Tibbles" than others. It is not hard to see how semantic Boolean mereology resolves this apparent contradiction. Under semantic Boolean mereology, the candidate referents are all such that it is indeterminate whether they are the referent of “Tibbles", in the sense that “Tibbles" refer to them to an intermediate degree, and none of these degrees are strictly higher or lower than any one of the others. This captures the second intuition. Meanwhile, “there is only one Tibbles" always has value 1 in the intended models, which corresponds to the first intuition.

Despite its advantages, semantic Boolean mereology also has some problems. An immediate consequence of semantic Boolean mereology is that the majority of names of ordinary objects - “Tibbles", “Kilimanjaro", “Marie Curie", “Earth", “Eiffel Tower", etc. - do not refer successfully, in the sense that they do not fix a unique referent. This is a bizarre consequence. It means that our ordinary methods about identifying and naming objects almost always fail, even under the best possible circumstances. The level of referential ambiguity displayed in the scenario in which I point to the only furry creature in the room and say “this is Tibbles", is the same as that displayed in the scenario in which I point to a corner where there are three men and say “this is John". If the foundation of our theory of meaning, as many have proposed, is that names designate objects, then that foundation is based on an impossible idealization.

Also, although semantic Boolean mereology is not completely incompatible with the existence of ordinary objects, ordinary objects under semantic Boolean mereology, in some sense, are ontologically shallow. Let us consider Tibbles the cat. Under this theory, the sentence that “Tibbles exists" is true to degree 1, and in this sense ordinary objects like Tibbles do exist. But since all there is in the domain of the intended models are objects with precise mereological boundaries, there is no existing object that is really, or determinately, identical to Tibbles. In other words, there is no object x in the domain such that “x is Tibbles" is true to degree 1. So Tibbles, in a certain sense, does not really exist. This is, I believe, not quite in line with our common-sense conception of Tibbles’ existence: normally we would think that there exists a cat in the world that truly is Tibbles.

2.6.2 Ontic Boolean Mereology

Unlike the semantic thesis, the ontic thesis holds that there are indeed objects in the world that are vague in their mereological organization, and names of these objects refer to them in the standard, determinate way. In the context of Boolean semantics, this is to say that there are objects in the domain such that they stand in the parthood relation with other objects to intermediate Boolean degrees; these objects are the (unique) referents of certain constants. The intended models for ontic Boolean mereology, then, are along the lines of our atomic Boolean models. Take, for example, the $SEVI$ model $\mathfrak{S}^B_{SV}$ for $\mathscr{L}_M$. We may extend $\mathfrak{S}^B_{SV}$ to a model for $\mathscr{L}'$ by letting w denote some $g^a \in M$ such that $a \in S$ and g^a takes a to 1 and every $b \neq a \in S$ to 0, and t denote some $f \in M \backslash M'$ such that $f(a)$

is some intermediate value between 0 and 1. In other words, w (determinately) denotes (the characteristic function of) some atom and t (determinately) denotes (the characteristic function of) a vague object whose value distribution on atoms, especially on w, involves intermediate values.

There are, then, two core differences between semantic Boolean mereology and ontic Boolean mereology. The first difference is that the domain of an intended model for semantic Boolean mereology contains only sharp objects, whereas the domain of an intended model for ontic Boolean mereology contains both sharp objects and vague objects. The second difference is that simple names like "Tibbles" under ontic Boolean mereology are interpreted normally as constants and have determinate referents, whereas under semantic Boolean mereology they are interpreted syntactically as unary predicates and have multiple indeterminate referents.

It is interesting to note that given a model intended by the semantic Boolean mereologists, it is possible for us to convert it into a model intended by the ontic Boolean mereologists. For example, consider $\mathfrak{M}'$ that we construct in the previous subsection. Let $\{g_i \mid i \in I\} \subseteq M'$ be the set of all elements in M' such that $[\![t(g_i)]\!]^{\mathfrak{M}'} \neq 0$, or in other words, the set of all permissible referents of t in $\mathfrak{M}'$. Then, construct a model $\mathfrak{M}^+$ for the language $\mathscr{L}'$ as follows. First we construct the direct power $\prod_{i\in I} \mathfrak{M}'$ of $|I|$ many copies of $\mathfrak{M}'$.[24] Then let $\mathfrak{M}^-$ be reduct of $\prod_{i\in I} \mathfrak{M}'$ to $\mathscr{L}^- = \mathscr{L}' \backslash \{t\}$. Let $\mathscr{L}^* = \mathscr{L}^- \cup \{t^*\}$, where t^* is a new constant, which is to play the role of "Tibbles". Then we let $\mathfrak{M}^+$ be the expansion of $\mathfrak{M}^-$ to $\mathscr{L}^*$ such that $[\![t^*]\!]^{\mathfrak{M}^+} = \langle g_i \rangle_{i\in I}$. By the conditions we impose on the permissible referents of t in $\mathfrak{M}'$, it is easy to see that $[\![w \lesssim t^*]\!]^{\mathfrak{M}^+}$ is an intermediate value between 0 and 1. Also, since $\mathfrak{M}'$ satisfies atomic classical mereology, $\mathfrak{M}^+$ will also satisfy atomic classical mereology.

In my opinion, Boolean-valued semantics provides the best model-theoretic framework for proponents of the ontic thesis. The two alternative semantic frameworks, in comparison to Boolean-valued semantics, both have serious problems. The first alternative is the fuzzy-valued model theory, and in section 2.4 I have already argued at length why it is less suitable than Boolean model theory, to the task of interpreting mereological indeterminacy.

[24] See Chapter 1 Def. 1.3.2 for a definition.

The second alternative is supervaluational model theory. But unlike in the semantic case, the combination of supervaluation semantics and the ontic thesis (see [1]) yields, in my opinion, an awkward theory (I call it “ontic supervaluationism"). Under ontic supervaluationism, there are multiple distinct “precisificatons" *of the underlying reality* that are used to explain mereological vagueness. Although the model-theoretic techniques employed in this view is basically identical to that in the semantic case, from the philosophical perspective ontic supervaluationism feels much more unnatural and faces more difficult questions, compared to its semantic counterpart. For example, in the case of semantic supervaluationism, we have a fairly good understanding of what a “precisification" is: it is a total interpretation function that is consistent with how we use terms like “Tibbles" in languages. But what is, or can be, a “precisification" *of the reality*, in the case of ontic supervaluationism? It cannot be language or mind dependent, as it is supposed to capture a feature of the world, so is it something that exists out there? What is its ontological status? If it is like a possible world that exists along side our world, why is the vagueness of the objects in our world grounded in these things? Also, following the ontic thesis, the referent of the name “Tibbles" needs to be an object that exists in the actual world, but somehow it also has to be a different object in each of these precisifications - how exactly can we reconcile these claims? I do not see an easy answer for any of these questions, and therefore I think that supervaluation semantics is not really a viable option for supporters of the ontic thesis.

Just like semantic Boolean mereology, or perhaps any philosophical theory, ontic Boolean mereology has its advantages and disadvantages. Its biggest advantage is that it overcomes the two difficulties held by semantic Boolean mereology, as presented in the previous subsection. Under ontic Boolean mereology, we are not stuck with a vast scale of referential failure. Also for ordinary objects like Tibbles, we will have something existing in the domain that is determinately Tibbles, so the existence of Tibbles is not ontologically shallow. The biggest problem plaguing ontic Boolean mereology, on the other hand, is the problem of the many. Again, if we accept the principle of fusion existence, there will be a number of distinct vague objects with minute differences, all located where Tibbles is. Now, ontic semantic mereology claims that there is among them a unique referent for “Tibbles", but which one of these objects should be the unique referent? Consider, for example, the model

$\mathfrak{S}^B_{SV}$. Every function in M corresponds to an object in the world, and as long as B is large enough, there can be many functions f in M that (1) has the same value on every other atom except $a \in S$, and (2) has an intermediate value on a (let a be the referent of w in $\mathfrak{S}^B_{SV}$). The difficult question seems to be: what makes one of them a better candidate for being the referent of t than others?

Note that this is a problem that troubles all supporters of the ontic thesis, not just supporters of ontic Boolean mereology. Ontic fuzzy mereology and ontic supervaluationism face this problem to roughly the same extent. In my opinion, the simplest and best way for the ontic Boolean mereologist to respond is to reject the principle of fusion existence and embrace an ontology that is less well-populated. In a model like $\mathfrak{S}^B_{SV}$, for example, they could say that not all functions in M correspond to an object existing in the world. Rather, only one of the many possible profiles of value distribution on the atoms relevant to Tibbles actually corresponds to an existing (ordinary) object - a cat, in particular, and that is the unique referent of “Tibbles". The difficult question they would face then, which I will call the “special condition question", is “What’s special about this particular value profile, compared to the others, that makes it a profile of an object?". At this point, there are two kinds of responses on the table. The first response is to suggest that there is some kind of naturalness condition satisfied by this value profile, perhaps in terms of contact and adhesion, that is responsible for its “objecthood" . The second response is to claim that it is simply a piece of brute fact that this particular value profile corresponds to an object. And in general, there are just brute facts of the world we live in to the effect that some Boolean value profiles correspond to actually existing (ordinary) objects whereas others do not.[25]

Does this mean that ontic Boolean mereologists have to completely forsake classical mereology? Not necessarily. What they have to deny is that classical mereology - the principle of fusion existence, in particular, holds of *ordinary objects* like cats. But they could still say that it holds on more fundamental and abstract entities like spatio-temporal regions. They could hold that, for example, any Boolean profile on spatio-temporal points(atoms)

[25]Note that when facing a similar many-valued version of the problem of many, the ontic fuzzy mereologists also typically tend to choose one of the two possible responses discussed here to the special condition question. Nicholas Smith, for example, uses the first kind of response in [31]. Peter van Inwagen uses the second kind of response in [34].

corresponds to a spacial-temporal region that is part of the ontology, but only one of the (relevant) special-temporal regions is occupied by a cat-like entity, which is Tibbles the cat. Of course, what they would have to answer, then, is a slightly different version of the special condition question, perhaps along the lines of “What’s special about this particular value profile, compared to the others, that makes it a profile of an ordinary, cat-like object?", and they could again adopt one of the two potential responses. The point here is just that ontic Boolean mereologists do have the freedom to choose between a sparse ontology and a sparser ontology, and between completely and partially denying the principle of fusion existence.

2.7 Appendices

2.7.1 Preliminaries on Boolean Model Theory

Definition 2.7.1. Let $\mathscr{L}$ be an arbitrary first-order/second-order language. For simplicity, we assume that $\mathscr{L}$ has no function symbols/variables, but only relation symbols/variables, individual constants/variables.[26] Let B be a complete Boolean algebra. A B-valued model $\mathfrak{A}$ for the language $\mathscr{L}$ consists of:

1. A universe A of elements;

2. The B-value of the identity symbol: a function $[\![=]\!]^{\mathfrak{A}} : A^2 \to B$;

3. The B-values of the relation symbols: (let P be a n-ary relation) $[\![P]\!]^{\mathfrak{A}} : A^n \to B$;

4. The B-values of the constant symbols: (let c be a constant) $[\![c]\!]^{\mathfrak{A}} \in A$.

And it needs to satisfy:

[26] Our theory can be easily generalized to first order languages with function symbols, as functions can always be treated as relations that satisfy special conditions.

1. For the B-value of the identity symbol[27]: for any $a_1, a_2, a_3 \in A$

$$[\![a_1 = a_1]\!]^{\mathfrak{A}} = 1_B \tag{2.1}$$

$$[\![a_1 = a_2]\!]^{\mathfrak{A}} = [\![a_2 = a_1]\!]^{\mathfrak{A}} \tag{2.2}$$

$$[\![a_1 = a_2]\!]^{\mathfrak{A}} \sqcap [\![a_2 = a_3]\!]^{\mathfrak{A}} \leqslant [\![a_1 = a_3]\!]^{\mathfrak{A}} \tag{2.3}$$

2. For the B-value of relation symbols: let P be an n-ary relation; for any $\langle a_1, ..., a_n \rangle$, $\langle b_1, ..., b_n \rangle \in A^n$,

$$[\![P(a_1, ..., a_n)]\!]^{\mathfrak{A}} \sqcap (\prod_{1 \leqslant i \leqslant n} [\![a_i = b_i]\!]^{\mathfrak{A}}) \leqslant [\![P(b_1, ..., b_n)]\!]^{\mathfrak{A}} \tag{2.4}$$

Definition 2.7.2. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. For any $n \in \omega$, we define D_A^n as the following set: $D_A^n = \{R : A^n \to B \mid \text{for any } \langle a_1, ..., a_n \rangle, \langle b_1, ..., b_n \rangle \in A^n, R(a_1, ..., a_n) \sqcap (\prod_{1 \leqslant i \leqslant n} [\![a_i = b_i]\!]^{\mathfrak{A}}) \leqslant R(b_1, ..., b_n)\}$. We call the D_A^n's the *second-order domains* of $\mathfrak{A}$. For each $n \in \omega$, we call D_A^n the n-ary second-order domain of $\mathfrak{A}$.

Given a B-valued model $\mathfrak{A}$ for $\mathscr{L}$, we define satisfaction in $\mathfrak{A}$ as follows:

Definition 2.7.3. Let *Var* be the set of all variables. (We will use $v_1, v_2, ...$ to range over individual variables, and $X_1, X_2, ...$ to range over relation variables.) An assignment s on $\mathfrak{A}$ is a function with domain *Var* such that:

1. For any individual variable v_i, $s(v_i) \in A$.

2. For any relation variable X_i of arity n, $s(X_i) \in D_A^n$[28].

Given a assignment s on $\mathfrak{A}$, we define the value of an open formula of $\mathscr{L}$ in $\mathfrak{A}$ under assignment x as follows.

1. We first define the value of terms in $\mathfrak{A}$:

[27]Here and in the following, when the context is clear, we use $[\![a_i = a_j]\!]^{\mathfrak{A}}$ to abbreviate $[\![=]\!]^{\mathfrak{A}}(a_i, a_j)$, and similarly for cases of the relation symbols.

[28]In the case when $\mathscr{L}$ is a first-order language, this line can simply be ignored, for obvious reasons. And similarly for 2(c), 3(f) and 3(g) below.

(a) Let v_i be an individual variable. Then $[\![v_i]\!]^{\mathfrak{A}}[s] = s(v_i)$.

(b) Let c be an individual constant. Then $[\![c]\!]^{\mathfrak{A}}[s] = [\![c]\!]^{\mathfrak{A}}$.

2. We then define the value of atomic formulas in $\mathfrak{A}$:

(a) Let t_1, t_2 be terms (a term is either an individual variable or an individual constant). Then $[\![t_1 = t_2]\!]^{\mathfrak{A}}[s] = [\![s(t_1) = s(t_2)]\!]^{\mathfrak{A}}$, where $s(t_1) = [\![t_1]\!]^{\mathfrak{A}}[s]$ and $s(t_2) = [\![t_2]\!]^{\mathfrak{A}}[s]$.

(b) Let $t_1, ..., t_n$ be terms. Then $[\![P(t_1, ..., t_n)]\!]^{\mathfrak{A}}[s] = [\![P(s(t_1), ..., s(t_n))]\!]^{\mathfrak{A}}$, where $s(t_1) = [\![t_1]\!]^{\mathfrak{A}}[s]$, ..., $s(t_n) = [\![t_n]\!]^{\mathfrak{A}}[s]$.

(c) Let $t_1, ..., t_n$ be terms. Then $[\![X_i(t_1, ..., t_n)]\!]^{\mathfrak{A}}[s] = [\![s(X_i)(s(t_1), ..., s(t_n))]\!]^{\mathfrak{A}}$, where $s(t_1) = [\![t_1]\!]^{\mathfrak{A}}[s]$, ..., $s(t_n) = [\![t_n]\!]^{\mathfrak{A}}[s]$.

3. We finally define the value of complex formulas in $\mathfrak{A}$:

(a) Let ϕ be a formula. Then $[\![\neg\phi]\!]^{\mathfrak{A}}[s] = -[\![\phi]\!]^{\mathfrak{A}}[s]$.

(b) Let ϕ, ψ be formulas. Then $[\![\phi \wedge \psi]\!]^{\mathfrak{A}}[s] = [\![\phi]\!]^{\mathfrak{A}}[s] \sqcap [\![\psi]\!]^{\mathfrak{A}}[s]$.

(c) Let ϕ, ψ be formulas. Then $[\![\phi \vee \psi]\!]^{\mathfrak{A}}[s] = [\![\phi]\!]^{\mathfrak{A}}[s] \sqcup [\![\psi]\!]^{\mathfrak{A}}[s]$.

(d) Let ϕ be a formula. Then $[\![\exists v_i\phi]\!]^{\mathfrak{A}}[s] = \bigsqcup_{a\in A} [\![\phi]\!]^{\mathfrak{A}}[s(v_i/a)]$, where $s(v_i/a)$ is the assignment on $\mathfrak{A}$ that takes v_i to a and agrees with s everywhere else.

(e) Let ϕ be a formula. Then $[\![\forall v_i\phi]\!]^{\mathfrak{A}}[s] = \bigsqcap_{a\in A} [\![\phi]\!]^{\mathfrak{A}}[s(v_i/a)]$, where $s(v_i/a)$ is the assignment on $\mathfrak{A}$ that takes v_i to a and agrees with s everywhere else.

(f) Let ϕ be a formula. Then $[\![\exists X_i\phi]\!]^{\mathfrak{A}}[s] = \bigsqcup_{R\in D_A^n} [\![\phi]\!]^{\mathfrak{A}}[s(X_i/R)]$, where n is the arity of X_i, and $s(X_i/R)$ is the assignment on $\mathfrak{A}$ that takes X_i to R and agrees with s everywhere else.

(g) Let ϕ be a formula. Then $[\![\forall X_i\phi]\!]^{\mathfrak{A}}[s] = \bigsqcap_{R\in D_A^n} [\![\phi]\!]^{\mathfrak{A}}[s(X_i/R)]$, where n is the arity of X_i, and $s(X_i/R)$ is the assignment on $\mathfrak{A}$ that takes X_i to R and agrees with s everywhere else.

The values of the quantified formulas are well-defined as B is complete. We say that ϕ is a first-order formula when ϕ has no second order variables.

Theorem 2.7.1. Let $\mathfrak{A}$ be a B-valued model for $\mathscr{L}$. For any formula $\phi(v_1,...,v_n)$ in $\mathscr{L}$, any assignments s, s' on $\mathfrak{A}$,

$$[\![\phi(s(v_1),...,s(v_n))]\!]^{\mathfrak{A}} \sqcap (\prod_{1 \leqslant i \leqslant n} [\![s(v_i) = s'(v_i)]\!]^{\mathfrak{A}}) \leqslant [\![\phi(s'(v_1),...,s'(v_n))]\!]^{\mathfrak{A}}$$

Proof. By a straightforward induction on the complexity of $\phi(v_1,...,v_n)$.

□

2.7.2 Soundness and Completeness

Definition 2.7.4. Let T be a theory in a language $\mathscr{L}$. Let $\mathfrak{A}$ be a B-valued model of $\mathscr{L}$. $\mathfrak{A}$ is a model of T just in case for any $\phi \in T$, $[\![\phi]\!]^{\mathfrak{A}} = 1_B$.

Definition 2.7.5. Let T be a theory and ϕ be a sentence in a language $\mathscr{L}$. ϕ is a Boolean-consequence of T, in symbols, $T \models_B \phi$ just in case for any Boolean valued model $\mathfrak{A}$, if $\mathfrak{A}$ is a model of T, then $\mathfrak{A}$ is a model of ϕ.

In the rest of this section we assume that $\mathscr{L}$ is a first-order language.

Theorem 2.7.2. Let T be a theory and ϕ be a sentence in $\mathscr{L}$. If $T \vdash \phi$, then $T \models_B \phi$.

Proof. We can prove this by showing that all the axioms of first order logic have value 1 in every Boolean valued model, and that the rules of inference always preserve truth.

The proof that all the sentential axioms have value 1 is straightforward. For example, (let $x : Var \to A$ be an assignment), $[\![(\phi \wedge \psi) \to \phi]\!]^{\mathfrak{A}}[x] = 1$ iff $[\![\phi \wedge \psi]\!]^{\mathfrak{A}}[x] \leqslant [\![\phi]\!]^{\mathfrak{A}}[x]$ iff $[\![\phi]\!]^{\mathfrak{A}}[x] \sqcap [\![\psi]\!]^{\mathfrak{A}}[x] \leqslant [\![\phi]\!]^{\mathfrak{A}}[x]$. But the latter is true in every Boolean algebra. The cases of the other sentential axioms are very similar.

That the identity axioms always have value 1 follows straightforwardly from the clauses on the identity symbol in Def 2.7.1 and Theorem 2.7.1.

For the quantifier axioms, let ϕ and ψ be formulas. For the first axiom, suppose v_i is a variable that is not free in ϕ, we want to show that for any assignment $x : Var \to A$, $[\![\forall v_i(\phi \to \psi) \to (\phi \to \forall v_i \psi)]\!]^{\mathfrak{A}}[x] = 1$. This is the case iff $[\![\forall v_i(\phi \to \psi)]\!]^{\mathfrak{A}}[x] \leqslant [\![\phi \to \forall v_i \psi]\!]^{\mathfrak{A}}[x]$. But

$$\begin{aligned}
[\![\forall v_i(\phi \to \psi)]\!]^{\mathfrak{A}}[x] &= \prod_{a \in A} [\![\phi \to \psi]\!]^{\mathfrak{A}}[x(v_i/a)] \\
&= \prod_{a \in A} -[\![\phi]\!]^{\mathfrak{A}}[x(v_i/a)] \sqcup [\![\psi]\!]^{\mathfrak{A}}[x(v_i/a)] \\
&= \prod_{a \in A} -[\![\phi]\!]^{\mathfrak{A}}[x] \sqcup [\![\psi]\!]^{\mathfrak{A}}[x(v_i/a)] \\
&= -[\![\phi]\!]^{\mathfrak{A}}[x] \sqcup \prod_{a \in A} [\![\psi]\!]^{\mathfrak{A}}[x(v_i/a)] \\
&\leqslant [\![\phi \to \forall v_i \psi]\!]^{\mathfrak{A}}[x]
\end{aligned}$$

The third equation holds as v_i is not free in ϕ. For the second quantifier axiom, let ψ be obtained from ϕ by freely substituting each free occurrence of v_i in ϕ by the term t, such that no variable v_j in t will occur bound in ψ at the place where it is introduced. We want to show that for any assignment $x : Var \to A$, $[\![\forall v_i \phi \to \psi]\!]^{\mathfrak{A}}[x] = 1$. This is just in case $[\![\forall v_i \phi]\!]^{\mathfrak{A}}[x] \leqslant [\![\psi]\!]^{\mathfrak{A}}[x]$, which is just in case $\prod_{a \in A} [\![\phi]\!]^{\mathfrak{A}}[x(v_i/a)] \leqslant [\![\psi]\!]^{\mathfrak{A}}[x]$. But the latter is always true, as $[\![\psi]\!]^{\mathfrak{A}}[x] = [\![\phi]\!]^{\mathfrak{A}}[x(v_i/a')]$, where $a' = [\![t]\!]^{\mathfrak{A}}[x] \in A$.

Moving on to the rules of inferences. We start with *Modus Ponens*. Suppose both $[\![\phi]\!]^{\mathfrak{A}}[x]$ and $[\![\phi \to \psi]\!]^{\mathfrak{A}}[x]$ are 1. The latter means that $[\![\phi]\!]^{\mathfrak{A}}[x] \leqslant [\![\psi]\!]^{\mathfrak{A}}[x]$, and since $[\![\phi]\!]^{\mathfrak{A}}[x] = 1$, $[\![\psi]\!]^{\mathfrak{A}}[x] = 1$.

For *Universal Generalization*, we suppose for any assignment x, $[\![\phi]\!]^{\mathfrak{A}}[x] = 1$. Then it follows straightforwardly that $[\![\forall v_i \phi]\!]^{\mathfrak{A}}[x] = \prod_{a \in A} [\![\phi]\!]^{\mathfrak{A}}[x(v_i/a)] = 1$.

□

Corollary 2.7.2.1. Let ϕ be a theorem of first order logic. Then in any Boolean valued model $\mathfrak{A}$, $[\![\phi]\!]^{\mathfrak{A}} = 1$.

Theorem 2.7.3. Let T be a theory in $\mathscr{L}$. T is consistent if and only if for some complete Boolean Algebra B, T has a B-valued model $\mathfrak{A}$.

Proof. For the left to right direction, if T is consistent, then by the Completeness Theorem on two-valued models, T has a two-valued model. But a two-valued model is a Boolean valued model.

For the right to left direction, suppose T is inconsistent. Then for some theorem ϕ of first order logic, $T \vdash \neg\phi$. Assume for reductio that T has a B-valued model $\mathfrak{A}$, then by Theorem 2.7.2, $[\![\neg\phi]\!]^{\mathfrak{A}} = 1$. Hence $[\![\phi]\!]^{\mathfrak{A}} = 0$, but this contradicts Corollary 2.7.2.1.

□

Corollary 2.7.3.1. Let B be any complete Boolean algebra. A theory T has a B-valued model just in case every finite subset of T has a B-valued model.

Theorem 2.7.4. Let T be a theory and ϕ be a sentence in a first order language $\mathscr{L}$. If $T \models_B \phi$, then $T \vdash \phi$.

Proof. Suppose $T \models_B \phi$, then for any two-valued model $\mathfrak{A}$, if $\mathfrak{A}$ is a model of T, then $\mathfrak{A}$ is a model of ϕ. By the soundness theorem on two-valued models[29], $T \vdash \phi$. □

Corollary 2.7.4.1. Let T be a theory and ϕ be a sentence in a first order language $\mathscr{L}$. $T \models_B \phi$ if and only if $T \vdash \phi$.

2.7.3 Equivalence Between Systems

In this section we prove the two promised theorems in Section 2.5 (Theorem 2.5.1 and Theorem 2.5.2).

Theorem 2.7.5. *CM* is equivalent to Tarski's system, which is the theory closed under the following two axioms:

(Transitivity)	$\forall v_1 \forall v_2 \forall v_3 (v_1 \precsim v_2 \wedge v_2 \precsim v_3 \rightarrow v_1 \precsim v_3)$
(UniqueFusionExistence)	$\forall X_1 (\exists v_1 X_1(v_1) \rightarrow \exists! v_2 (FU'(v_2, X_1))$

Proof. We first show that *CM* entails Tarski's system. (Transitivity) is already in *CM*. For (UniqueFusionExistence), let X_1 be such that $\exists v_1 X_1(v_1)$. By (Fusion), $\exists v_2 (FU(v_2, X_1))$.

[29] See, for example, Chang and Keisler [7, p. 66].

Let $v_3 \leqslant v_2$. If $E(v_3)$, then we are done. Suppose $\neg E(v_3)$. By (NoZero), $\forall v_4 \forall v_5 (v_4 \precsim v_5)$. Hence trivially $v_3 \precsim v_3$ and $v_3 \precsim v_1$.

For the other direction, we can just use the standard argument that these axioms are all theorems of Tarski's system. See, for example, [18].

□

Theorem 2.7.6. The (second-order) theory of complete Boolean algebra (*CBA*) is equivalent to *MCM* plus Anti-symmetry plus the following axiom:

$$\text{(ZeroExistence)} \qquad \exists v_1 \neg E(v_1)$$

Proof. We first show that the latter system entails *CBA*. In particular, we show that (Reflexivity), (SupremumExistence), (Complementation) and (Distribution) are all theorems of the latter system. We start with (Reflexivity):

$$\text{(Reflexivity)} \qquad \forall v_1 (v_1 \precsim v_1)$$

It is easy to check that for any v_1, $FU(v_1, \{v_2 \mid v_2 \precsim v_1\})$. Suppose $v_1 \not\precsim v_1$. Then by (Supplementation), some v_3 is such that $E(v_3) \wedge v_3 \precsim v_1 \wedge \neg v_1 \circ v_3$. Since $FU(v_1, \{v_2 \mid v_2 \precsim v_1\})$, $\exists v_4 (v_4 \precsim v_1 \wedge v_4 \circ v_3)$. But then $\exists v_5 (E(v_5) \wedge v_5 \precsim v_3 \wedge v_5 \precsim v_4)$. By (Transitivity), $v_5 \precsim v_1$. Hence $v_1 \circ v_3$. Contradiction.

We next define the notion of "v_1 is the supremum of the X_1's":

$$Sup(v_1, X_1) = \forall v_2 (X_1(v_2) \to v_2 \precsim v_1) \wedge forall v_3 (\forall v_4 (X_1(v_4) \to v_4 \precsim v_3) \to (v_1 \precsim v_3))$$

The axiom of (SupremumExistence) says that every X_1 has a supremum:

$$\text{(SupremumExistence)} \qquad \forall X_1 \exists v_1 (Sup(v_1, X_1))$$

We now show that (SupremumExistence) is a theorem of the latter system. By, (ZeroExistence), there is some object that is part of anything. By (Anti-Symmetry), this object is unique. From now on we will name it 0. Suppose $\neg \exists v_2 (X_1(v_2))$. Then it is easy to see that everything is a upper bound of X_1, and hence 0 is the supremum of X_1. Suppose $\exists v_2(X_1(v_2))$. By (Fusion), there is a v_1 such that $FU(v_1,X_1)$. We will show that $Sup(v_1,X_1)$. Obviously v_1 is a upper bound of X_1. Let v_3 be a upper bound of X_1. Assume for reductio that $v_1 \not\lesssim v_3$. By (Supplementation), some v_4 is such that $E(v_4) \wedge v_4 \lesssim v_1 \wedge \neg v_4 \circ v_3$. Since $FU(v_1,X_1)$, there is some v_5 such that $X_1(v_5) \wedge v_5 \circ v_4$. Hence $v_5 \lesssim v_3$, but then $v_4 \circ v_3$, contradiction.

For the next axiom we first define the dual notion to “supremum" - “infimum":

$$Inf(v_1,X_1) = Sup(v_1,\{v_2 \mid \forall v_3(X_1(v_3) \rightarrow v_2 \lesssim v_3)\})$$

(Fusion) guarantees that there is a “maximum" object that fuses all things. By (Anti-Symmetry), this object is unique. Henceforth we will name it 1. The axiom of (Complementation) says that:

$$\text{(Complementation)} \qquad \forall v_1 \exists v_2(Sup(1,\{v_1,v_2\}) \wedge Inf(0,\{v_1,v_2\}))$$

We show that this is also a theorem of the latter system. Given v_1, define $-v_1$ as the object that fuses $\{v_2 \mid \neg v_1 \circ v_2\}$. First we show that $Sup(1,\{v_1,-v_1\})$. Assume for reductio that $Sup(w,\{v_1,-v_1\})$ and $w \neq 1$. Then some u is such that $u \neq w$. Hence there is a x such that $E(x) \wedge x \lesssim u \wedge \neg x \circ w$. Hence $\neg x \circ v_1$. But then $x \lesssim -v_1$ and hence $x \circ w$. Contradiction. Next we show that $Inf(0,\{v_1,-v_1\})$. Let v_2 be such that $v_2 \lesssim v_1$ and $v_2 \lesssim -v_1$. Assume for reductio that $v_2 \neq 0$. Then there is some y such that $\neg y \circ v_1 \wedge y \circ v_2$. Contradiction.

For the next axiom we need two functional notions:

$$v_1 \sqcup v_2 := \text{ the unique } v \text{ such that } Sup(v,\{v_1,v_2\})$$
$$v_1 \sqcap v_2 := \text{ the unique } v \text{ such that } Inf(v,\{v_1,v_2\})$$

The axiom of (Distribution) says that:

$$\text{(Distribution)} \qquad \forall v_1 \forall v_2 \forall v_3(v_1 \sqcup (v_2 \sqcap v_3) = (v_1 \sqcup v_2) \sqcap (v_1 \sqcup v_3))$$

We show that this is again a theorem of the latter system. Let $u = v_1 \sqcup (v_2 \sqcap v_3)$ and $w = (v_1 \sqcup v_2) \sqcap (v_1 \sqcup v_3)$. We first show that $u \precsim w$, that is, $u \precsim v_1 \sqcup v_2$ and $u \precsim v_1 \sqcup v_3$. Assume for reductio $u \not\precsim v_1 \sqcup v_2$. Then there is some x such that $E(x) \wedge x \precsim u \wedge \neg x \circ (v_1 \sqcup v_2)$. Since $x \precsim u$, either $x \circ v_1$ or $x \circ (v_2 \sqcap v_3)$. Either way we have a contradiction. We next show that $w \precsim u$. Assume for reductio otherwise. Then there is some y such that $E(y) \wedge y \precsim w \wedge \neg y \circ u$. Hence $y \precsim v_1 \sqcup v_2$ and $y \precsim v_1 \sqcup v_3$. Suppose $v_1 \circ y$, then $w \circ y$ and we have a contradiction. Hence $\neg v_1 \circ y$. But since $y \precsim v_1 \sqcup v_2$. Hence $y \precsim v_2$. Similarly $y \precsim v_3$. Hence $u \precsim v_2 \sqcap v_3 \precsim u$. Contradiction.

It is (fairly) common knowledge that *CBA* is equivalent to (Transitivity), (Anti-Symmetry) plus the four axioms discussed above. Since the other two are already axioms of the latter system, we are done with this direction.

We move on to show the other direction: *CBA* entails the latter system. The only axiom worth mentioning is (Fusion). Other axioms either are already an axiom of *CBA* or are a theorem of *CBA* by a standard argument (any Boolean complement is a supplement, for example). For (Fusion), we will show that if $Sup(v_1, X_1)$, then $FU(v_1, X_1)$. That v_1 is a upper bound of X_1 is obviously the case. We only need to show that $\forall v_2(v_2 \precsim v_1 \wedge E(v_2) \rightarrow \exists v_3(X_1(v_3) \wedge v_3 \circ v_2))$. Suppose the antecedent. Assume for reductio that $\forall v_4(X(v_4) \rightarrow v_2 \sqcap v_4 = 0)$. Then by infinite distribution, $v_2 \sqcap v_1 = 0$. Since $E(v_2)$, $v_2 \neq 0$. Hence $v_2 \not\precsim v_1$. Contradiction.

□

2.7.4 The *SE* Models

In this section we prove the following result:

Theorem 2.7.7. In any *SE* model, Transitivity, Supplementation, Fusion, Atomicity and NoZero all have value 1.

Theorem 2.7.8 (Transitivity)**.** $\mathfrak{S}_S^B \models \forall v_1 \forall v_2 \forall v_3 (v_1 \precsim v_2 \wedge v_2 \precsim v_3 \rightarrow v_1 \precsim v_3)$.

Proof. For any $f_1, f_2, f_3 \in M$,

$$(\prod_{a \in S} f_1(a) \Rightarrow f_2(a)) \sqcap (\prod_{b \in S} f_2(b) \Rightarrow f_3(b)) \leqslant \prod_{c \in S} f_1(c) \Rightarrow f_3(c)$$

$\square$

Lemma 2.7.8.1. For any $f \in M$, $[\![E(f)]\!] = \bigsqcup_{a\in S} f(a) = 1$[30].

Proof. $[\![E(f)]\!] = [\![\exists v_2(\neg f \lesssim v_2)]\!] = \bigsqcup_{g\in M}\bigsqcup_{a\in S} f(a) \sqcap -g(a)$. We want to show that $\bigsqcup_{g\in M}\bigsqcup_{a\in S} f(a) \sqcap -g(a) = \bigsqcup_{a\in S} f(a)$. For any $a \in S$, let g^a be the function from S to B that takes a to 1 and every $b \neq a$ to 0. Obviously $g^a \in M$. Pick some $a \in S$, then it is easy to see for any $b \neq a \in S$, $f(a) \leqslant \bigsqcup_{c\in S} f(c) \sqcap -g^b(c)$. Hence $f(a) \leqslant [\![E(f)]\!]$. For the other direction, pick some $g \in M$. Obviously $\bigsqcup_{a\in S} f(a) \sqcap -g(a) \leqslant \bigsqcup_{a\in S} f(a)$.

$\square$

Lemma 2.7.8.2. For any $f_1, f_2 \in M$, $[\![f_1 \circ f_2]\!] = \bigsqcup_{a\in S} f_1(a) \sqcap f_2(a)$.

Proof. By definition, $[\![f_1 \circ f_2]\!] = [\![\exists v_3(E(v_3) \wedge v_3 \lesssim f_1 \wedge v_3 \lesssim f_2)]\!]$. Since every $g \in M$ is such that $[\![E(g)]\!] = 1$, $[\![f_1 \circ f_2]\!] = \bigsqcup_{g\in M} [\![g \lesssim f_1]\!] \sqcap [\![g \lesssim f_2]\!] = \bigsqcup_{g\in M}\bigsqcap_{a\in S} g(a) \Rightarrow (f_1(a) \sqcap f_2(a))$. We will show that this is equal to $\bigsqcup_{a\in S} f_1(a) \sqcap f_2(a) = p$.

For the $\leqslant$ direction: Fix $g \in M$. Since $\bigsqcup_{a\in S} g(a) = 1$, $\bigsqcap_{a\in S} g(a) \Rightarrow (f_1(a) \sqcap f_2(a)) = \bigsqcap_{a\in S} -g(a) \sqcup (f_1(a) \sqcap f_2(a)) \leqslant \bigsqcap_{a\in S} -g(a) \sqcup p = 0 \sqcup p = p$.

For the $\geqslant$ direction: Fix $a \in S$. Then it is easy to see that $f_1(a) = [\![g^a \lesssim f_1]\!]$, and similarly $f_2(a) = [\![g^a \lesssim f_2]\!]$. Hence $f_1(a) \sqcap f_1(a) \leqslant [\![f_1 \circ f_2]\!]$.

$\square$

Theorem 2.7.9 (Supplementation)**.** $\mathfrak{S}_S^B \models \forall v_1 \forall v_2(v_2 \not\lesssim v_1 \rightarrow \exists v_3(E(v_3) \wedge v_3 \lesssim v_2 \wedge \neg v_1 \circ v_3))$.

Proof. Let $f_1, f_2 \in M$. Since every $g \in M$ is such that $[\![E(g)]\!] = 1$, we just need to show that $-[\![f_2 \lesssim f_1[\![\leqslant \bigsqcup_{g\in M} [\![g \lesssim f_2]\!] \sqcap -[\![g \circ f_1]\!]$. $-[\![f_2 \lesssim f_1[\![= \bigsqcup_{a\in S} f_2(a) \sqcap -f_1(a)$. Fix some $a \in S$. $[\![g^a \lesssim f_1]\!] = f_2(a)$. By the previous lemma, $-[\![g^a \circ f_1]\!] = -(\bigsqcup_{b\in S} g^a(b) \sqcap f_1(b)) = -f_1(b)$.

$\square$

Theorem 2.7.10 (Fusion)**.** $\mathfrak{S}_S^B \models \forall X_1(\exists v_1 X_1(v_1) \rightarrow \exists v_2(FU(v_2, X_1))$.

[30]We omit the superscripts when the context is clear.

Proof. We will show that for any $R \in D^1_M$, $[\![\exists v_1 R(v_1) \to \exists v_2(FU(v_2,R))]\!] = 1$. That is, $q = \bigsqcup_{t\in M} R(t) \leqslant [\![\exists v_1(\forall v_2(R(v_2) \to v_2 \precsim v_1) \wedge \forall v_3(v_3 \precsim v_1 \wedge E(v_3) \to \exists v_4(R(v_4) \wedge v_3 \circ v_4)))]\!] = \bigsqcup_{f\in M}((\bigsqcap_{g\in M} R(g) \Rightarrow [\![g \precsim f]\!]) \sqcap (\bigsqcap_{h\in M}([\![h \precsim f]\!] \Rightarrow (\bigsqcup_{s\in M} R(s) \sqcap [\![h\circ s]\!])))).$

We define $f^R \in M$ as follows: pick some particular $a \in S$, let $f^R(a) = (\bigsqcup_{g\in M} R(g) \sqcap g(a)) \sqcup -q$. For any $b \neq a \in S$, let $f^R(b) = \bigsqcup_{g\in M} R(g) \sqcap g(b)$.

We first show that f^R is indeed in M, i.e. $\bigsqcup_{c\in S} f^R(c) = 1$:

$$\begin{aligned}
\bigsqcup_{c\in S} f^R(c) &= (\bigsqcup_{b\neq a\in S} f^R(b)) \sqcup f^R(a)\\
&= (\bigsqcup_{b\neq a\in S}\bigsqcup_{g\in M} R(g) \sqcap g(b)) \sqcup ((\bigsqcup_{g\in M} R(g) \sqcap g(a)) \sqcup -q)\\
&= (\bigsqcup_{c\in S}\bigsqcup_{g\in M} R(g) \sqcap g(c)) \sqcup -q\\
&= (\bigsqcup_{g\in M} R(g) \sqcap \bigsqcup_{c\in S} g(c)) \sqcup -q = (q \sqcap 1) \sqcup -q = 1
\end{aligned}$$

Now we show that $\bigsqcap_{g\in M} R(g) \Rightarrow [\![g \precsim f^R]\!] = \bigsqcap_{g\in M} R(g) \Rightarrow (\bigsqcap_{c\in S} g(c) \Rightarrow f^R(c)) = 1$. Pick any $g \in M$. $R(g) \Rightarrow (\bigsqcap_{c\in S} g(c) \Rightarrow f^R(c)) = -R(g) \sqcup ((\bigsqcap_{c\neq a} -g(c) \sqcup f^R(c)) \sqcap (-g(a) \sqcup f^R(a))) = (\bigsqcap_{c\neq a} -R(g) \sqcup -g(c) \sqcup f^R(c)) \sqcap (-R(g) \sqcup -g(a) \sqcup f^R(a))$. $\bigsqcap_{c\neq a} -R(g) \sqcup -g(c) \sqcup f^R(c) = \bigsqcap_{c\neq a} -R(g) \sqcup -g(c) \sqcup (\bigsqcup_{h\in M} R(h) \sqcap h(c)) \geqslant \bigsqcap_{c\neq a} -R(g) \sqcup -g(c) \sqcup (-R(g) \sqcap g(c)) = 1$. $-R(g) \sqcup -g(a) \sqcup f^R(a) = -R(g) \sqcup -g(a) \sqcup (\bigsqcup_{h\in M} R(h) \sqcap h(c)) \sqcup -q = 1$.

We next show that $q \leqslant \bigsqcap_{h\in M}([\![h \precsim f^R]\!] \Rightarrow (\bigsqcup_{s\in M} R(s) \sqcap [\![h \circ s]\!]))$. Fix any $h \in M$. We want to show that $q \leqslant (\bigsqcup_{c\in S} h(c) \sqcap -f^R(c)) \sqcup (\bigsqcup_{d\in S}\bigsqcup_{s\in M} R(s) \sqcap s(d) \sqcap h(d)) = p$. Now it is easy to see that $p = p_1 \sqcup p_2$, where $p_1 = (\bigsqcup_{c\neq a} h(c) \sqcap -f^R(c)) \sqcup (\bigsqcup_{d\neq a}\bigsqcup_{s\in M} R(s) \sqcap s(d) \sqcap h(d))$ and $p_2 = (h(a) \sqcap -f^R(a)) \sqcup (\bigsqcup_{s\in M} R(s) \sqcap s(a) \sqcap h(a))$. But $p_1 = (\bigsqcup_{c\neq a} h(c) \sqcap -f^R(c)) \sqcup (\bigsqcup_{d\neq a} f^R(d) \sqcap h(d)) = \bigsqcup_{c\neq a}(h(c) \sqcap -f^R(c)) \sqcup (f^R(c) \sqcap h(c)) = \bigsqcup_{c\neq a} h(c) \geqslant -h(a)$, as $\bigsqcup_{b\in S} h(b) = 1$.

On the other hand, let $\bigsqcup_{s\in M} R(s) \sqcap s(a) = p_3$. Then $p_2 = (h(a) \sqcap -f^R(a)) \sqcup (p_3 \sqcap h(a)) = (h(a) \sqcap -p_3 \sqcap q) \sqcup (p_3 \sqcap h(a)) = (h(a) \sqcap q) \sqcup (h(a) \sqcap p_3)$. Hence $p = p_1 \sqcup p_2 \geqslant -h(a) \sqcup (h(a) \sqcap q) \sqcup (h(a) \sqcap p_3) \geqslant q$.

□

To prove Atomicity we need some more lemmas.

Lemma 2.7.10.1. Let $f \in M$. $\mathfrak{S}_S^B \models \forall v(E(v) \to \neg v \precsim\!\!\!\!/\; f)$ just in case $\{f(a) \mid a \in S\}$ is an antichain in B.

Proof. Right to left direction. Let $f \in M$ be such that $\{f(a) \mid a \in S\}$ is an antichain. Fix some random $g \in M$. We will show that $[\![E(g) \to \neg g \precsim\!\!\!\!/\; f]\!] = 1$. That is,

$$\bigsqcup_{a \in S} g(a) \leqslant (\bigsqcup_{b \in S} g(b) \sqcap -f(b)) \sqcup (\bigsqcap_{c \in S} g(c) \sqcup -f(c))$$

Fix some random $a \in S$. It is easy to see that

$$g(a) \sqcap -f(a) \leqslant \bigsqcup_{b \in S} g(b) \sqcap -f(b)$$
$$g(a) \sqcap (\bigsqcap_{c \in S \setminus \{a\}} -f(c)) \leqslant \bigsqcap_{c \in S} g(c) \sqcup -f(c)$$

Since $\{f(a) \mid a \in S\}$ is an antichain, $f(a) \leqslant (\bigsqcap_{c \in S \setminus \{a\}} -f(c))$. Hence, $g(a) \sqcap f(a) \leqslant \bigsqcap_{c \in S} g(c) \sqcup -f(c)$. Therefore,

$$g(a) = (g(a) \sqcap -f(a)) \sqcup (g(a) \sqcap f(a)) \leqslant (\bigsqcup_{b \in S} g(b) \sqcap -f(b)) \sqcup (\bigsqcap_{c \in S} g(c) \sqcup -f(c))$$

Left to right direction. Let $f \in M$ be such that for some $a, b \in S$, $f(a) \sqcap f(b) > 0$. Define $g \in M$ as follows: for any $c \in S$,

$$g(c) = \begin{cases} f(a) \sqcap -f(b) & \text{if } c = a; \\ f(c) & \text{if otherwise.} \end{cases}$$

It is easy to see that $[\![E(g)]\!] =]\!]E(f)]\!]$. And hence g is indeed in M. We will show that $[\![E(g) \to \neg g \precsim\!\!\!\!/\; f]\!] < 1$. That is,

$$\bigsqcap_{a \in S} -g(a) \sqcup (\bigsqcup_{b \in S} g(b) \sqcap -f(b)) \sqcup (\bigsqcap_{c \in S} g(c) \sqcup -f(c)) < 1$$

Observe that $\prod_{a \in S} -g(a) = 0$, as $g \in M$. Also $\bigsqcup_{b \in S} g(b) \sqcap -f(b) = 0$. And $\prod_{c \in S} g(c) \sqcup -f(c) = g(a) \sqcup -f(a) = (f(a) \sqcap -f(b)) \sqcup -f(a) = -f(a) \sqcup -f(b) < 1$, as $f(a) \sqcap f(b) > 0$. Hence the whole thing is less than 1.

□

Lemma 2.7.10.2. Let $f \in M$. $\mathfrak{S}^S_A \models At(f)$ just in case $\{f(a) \mid a \in S\}$ is a maximal antichain in B.

Proof. Recall that $At(f) = E(f) \wedge \forall v(E(v) \to \neg v \precnsim f)$. The result follows from the previous lemma as for any $f \in M$, $[\![E(f)]\!] = 1$.

□

Theorem 2.7.11 (Atomicity)**.** $\mathfrak{S}^B_S \models \forall v_1(E(v_1) \to \exists v_2(At(v_2) \wedge v_2 \precsim v_1))$.

Proof. Fix some random $f \in M$. Since $[\![E(f)]\!] = 1$, we need to show that $[\![\exists v_2(At(v_2) \wedge v_2 \precsim f)]\!] = 1$. Let $C = \{a \in S \mid f(a) \neq 0\}$. Enumerate C by $\alpha = |C|$: $C = \{a_1, ..., a_\beta, ... \mid \beta < \alpha\}$. Define $g \in M$ as follows: for any $c \in S$,

$$g(c) = \begin{cases} f(a_\beta) \sqcap (\prod_{\gamma < \beta}) - f(a_\gamma) & \text{if } c = a_\beta \in C; \\ \\ f(c) = 0 & \text{if } c \notin C. \end{cases}$$

Hence $[\![E(g)]\!] = \bigsqcup_{a \in C} g(a) = \bigsqcup_{a \in C} f(a) = [\![E(f)]\!] = 1$. Also, $[\![g \precsim f]\!] = 1$. Since $\{g(a) \mid a \in S\}$ is an antichain, by Lemma 2.7.10.1, $[\![\forall v(E(v) \to \neg v \precnsim g)]\!] = 1$. Hence $[\![At(g)]\!] = [\![E(f)]\!] = 1$.

□

Theorem 2.7.12 (NoZero)**.** $\mathfrak{S}^B_S \models \exists v_1 \exists v_2(v_1 \not\precsim v_2) \to \neg\exists v_3 \neg(E(v_3))$

Proof. This can be proven simply by showing that $[\![\neg\exists v_3 \neg(E(v_3))]\!] = 1$, as for any $f \in M$, $[\![E(f)]\!] = 1$.

□

Corollary 2.7.12.1. $\mathfrak{S}^B_S$ is a model of ACM^-.

2.7.5 The VE Models

In this section we prove the following result:

Theorem 2.7.13. In any VE model, Transitivity, Supplementation, Fusion and Atomicity all have value 1, but NoZero has value 0.

Transitivity is proven in the same way as before.

Lemma 2.7.13.1. For any $f \in N$, $[\![E(f)]\!] = \bigsqcup_{a \in S} f(a)$.

Proof. The same proof as in that of Lemma 2.7.8.1.

□

Lemma 2.7.13.2. For any $f_1, f_2 \in N$, $[\![f_1 \circ f_2]\!] = \bigsqcup_{a \in S} f_1(a) \sqcap f_2(a)$.

Proof. For this proof and many followings, we need to consider two cases. Case one is when $\bigsqcup_{a \in S} f_1(a) \sqcap f_2(a) = 0$. Then for any $a \in S$, $f_1(a) \sqcap f_2(a) = 0$. Then $[\![f_1 \circ f_2]\!] = \bigsqcup_{g \in N} \bigsqcup_{a \in S} g(a) \sqcap \bigsqcap_{b \in S} g(b) \Rightarrow (f_1(b) \sqcap f_2(b)) = \bigsqcup_{g \in N} \bigsqcup_{a \in S} g(a) \sqcap \bigsqcap_{b \in S} -g(b) = 0$.

Case two is when $\bigsqcup_{a \in S} f_1(a) \sqcap f_2(a) > 0$. Then define $f \in N$ such that for any $a \in S$, $f(a) = f_1(a) \sqcap f_2(a)$. It is easy to see that $[\![f \precsim f_1]\!] = [\![f \precsim f_2]\!] = 1$.

$[\![f_1 \circ f_2]\!] = [\![\exists v(E(v) \wedge v \precsim f_1 \wedge v \precsim f_2)]\!] = \bigsqcup_{g \in S^B} [\![E(g) \wedge g \precsim f_1 \wedge g \precsim f_2]\!]$.

Fix some random $g \in S^B$, $[\![E(g) \wedge g \precsim f_1 \wedge g \precsim f_2]\!] = \bigsqcup_{a \in S} g(a) \sqcap \bigsqcap_{b \in S} (g(b) \Rightarrow (f_1(b) \sqcap f_2(b))) \leqslant \bigsqcup_{a \in S} g(a) \sqcap (g(a) \Rightarrow (f_1(a) \sqcap f_2(a))) \leqslant \bigsqcup_{a \in S} f_1(a) \sqcap f_2(a) = [\![E(f)]\!] = [\![E(f) \wedge f \precsim f_1 \wedge f \precsim f_2]\!]$. Hence $\bigsqcup_{g \in S^B} [\![E(g) \wedge g \precsim f_1 \wedge g \precsim f_2]\!] = [\![E(f) \wedge f \precsim f_1 \wedge f \precsim f_2]\!] = [\![E(f)]\!] = \bigsqcup_{a \in S} f_1(a) \sqcap f_2(a)$.

□

Theorem 2.7.14 (Supplementation)**.** $\mathfrak{S}_V^B \models \forall v_1 \forall v_2 (v_2 \not\precsim v_1 \rightarrow \exists v_3 (E(v_3) \wedge v_3 \precsim v_2 \wedge \neg v_1 \circ v_3))$.

Proof. Let $f_1, f_2 \in N$. We want to show that $[\![f_2 \not\precsim f_1]\!] \leqslant [\![\exists v(E(v) \wedge v \precsim f_2 \wedge \neg f_1 \circ v)]\!]$. $[\![f_2 \not\precsim f_1]\!] = \bigsqcup_{a \in S} -f_1(a) \sqcap f_2(a)$.

Again, there are two cases. If $[\![f_2 \not\precsim f_1]\!] = 0$, then we are done. If $[\![f_2 \not\precsim f_1]\!] > 0$, then define $f \in N$ such that for any $a \in S$, $f(a) = -f_1(a) \sqcap f_2(a)$. We can easily show that

$[\![f \lesssim f_2]\!] = 1$. Also, $[\![E(f)]\!] = \bigsqcup_{a \in S} -f_1(a) \sqcap f_2(a)$, by Lemma 2.7.13.1, and $[\![\neg f_1 \circ f]\!] = -(\bigsqcup_{a \in S} f_1(a) \sqcap -(f_1(a) \sqcap f_2(a))) = 1$, by Lemma 2.7.13.2. Hence $[\![\exists v(E(v) \lesssim f_2 \wedge \neg f_1 \circ v)]\!] \geqslant [\![E(f) \wedge f \lesssim f_2 \wedge \neg f_1 \circ f]\!] = \bigsqcup_{a \in S} -f_1(a) \sqcap f_2(a) = [\![f_2 \not\lesssim f_1]\!]$.

□

Theorem 2.7.15 (Fusion)**.** $\mathfrak{S}_V^B \models \forall X_1(\exists v_1 X_1(v_1) \to \exists v_2(FU(v_2, X_1))$.

Proof. Let $R \in D_M^1$. Again, there are two cases. Case one: $\bigsqcup_{a \in S} \bigsqcup_{g \in N} R(g) \sqcap g(a) = 0$. Then for any $g \in N, a \in S$, $R(g) \sqcap g(a) = 0$. This case can be proven easily by unpacking the definitions. Case two: $\bigsqcup_{a \in S} \bigsqcup_{g \in N} R(g) \sqcap g(a) > 0$. Then define $f^R \in N$: for any $a \in S$, let $f(a) = \bigsqcup_{g \in S^B} R(g) \sqcap g(a)$. We will show that $[\![FU(f^R, R)]\!] = [\![\forall v_2(R(v_2) \to v_2 \lesssim f^R) \wedge \forall v_3(v_3 \lesssim f^R \wedge E(v_3) \to \exists v_4(R(v_4) \wedge v_3 \circ v_4))]\!] = 1$.

$[\![\forall v_2(R(v_2) \to v_2 \lesssim f^R)]\!] = \prod_{h \in S^B} R(h) \Rightarrow (\prod_{a \in S} h(a) \Rightarrow f^R(a))$. Fix some $h \in N$. Then $-R(h) \sqcup (\prod_{a \in S} -h(a) \sqcup (\bigsqcup_{g \in S^B} R(g) \sqcap g(a))) = \prod_{a \in S} -(R(h) \sqcap h(a)) \sqcup \bigsqcup_{g \in S^B} R(g) \sqcap g(a)) \geqslant \prod_{a \in S} -(R(h) \sqcap h(a)) \sqcup (R(h) \sqcap g(h)) = 1$.

$[\![\forall v_3(v_3 \lesssim f^R \wedge E(v_3) \to \exists v_4(R(v_4) \wedge v_3 \circ v_4))]\!] = \prod_{g \in S^B} ([\![g \lesssim f^R]\!] \sqcap [\![E(g)]\!]) \Rightarrow (\bigsqcup_{h \in S^B} (R(h) \sqcap [\![h \circ g]\!]))$. Fix some $g \in N$. $\bigsqcup_{h \in S^B} (R(h) \sqcap [\![h \circ g]\!]) = \bigsqcup_{h \in S^B} R(h) \sqcap \bigsqcup_{a \in S} h(a) \sqcap g(a) = \bigsqcup_{a \in S} \bigsqcup_{h \in S^B} R(h) \sqcap h(a) \sqcap g(a) = \bigsqcup_{a \in S} f^R(a) \sqcap g(a) = [\![f^R \circ g]\!]$. But $[\![f^R \circ g]\!] = [\![\exists v_1(E(v_1) \wedge v_1 \lesssim f^R \wedge v_1 \lesssim g)]\!] = \bigsqcup_{t \in S^B} [\![E(t)]\!] \sqcap [\![t \lesssim f^R]\!] \sqcap [\![t \lesssim g]\!] \geqslant [\![E(g)]\!] \sqcap [\![g \lesssim f^R]\!]$.

□

Lemma 2.7.15.1. Let $f \in N$. $\mathfrak{S}_V^S \models At(f)$ just in case $\{f(a) \mid a \in S\}$ is a maximal antichain in B.

Proof. Using the same proof as in Lemma 2.7.10.1 we can show that for any $f \in N$, $\mathfrak{S}_V^B \models \forall v(E(v) \to \neg v \not\lessapprox f)$ just in case $\{f(a) \mid a \in S\}$ is an antichain in B.

Recall that $At(f) = E(f) \wedge \forall v(E(v) \to \neg v \not\lessapprox f)$. Suppose $[\![At(f)]\!] = 1$. Then $\{f(a) \mid a \in S\}$ is an antichain. Also, since $\bigsqcup_{a \in S} f(a) = [\![E(f)]\!] = 1$, $\{f(a) \mid a \in S\}$ is a maximal antichain. Similarly, suppose $\{f(a) \mid a \in S\}$ is a maximal antichain, then $[\![E(f)]\!] = \bigsqcup_{a \in S} f(a) = 1$. Also, $[\![\forall v(E(v) \to \neg v \not\lessapprox f)]\!] = 1$. Hence $[\![At(f)]\!] = 1$.

□

Theorem 2.7.16 (Atomicity)**.** $\mathfrak{S}_V^B \models \forall v_1(E(v_1) \to \exists v_2(At(v_2) \wedge v_2 \precsim v_1))$.

Proof. The same proof as in Theorem 2.7.11, using the previous lemma.

□

Theorem 2.7.17 (NoZero is false.)**.** $\mathfrak{S}_V^B \models \neg(\exists v_1 \exists v_2(v_1 \not\precsim v_2) \to \neg\exists v_3 \neg(E(v_3)))$

Proof. This can be proven by showing two things. First, $[\![\exists v_1 \exists v_2(v_1 \not\precsim v_2)]\!]$ has value 1. $[\![\exists v_1 \exists v_2(v_1 \not\precsim v_2)]\!] = \bigsqcup_{f_1, f_2 \in N} \bigsqcup_{a \in S} f_1(a) \sqcap -f_2(a)$. Let $f_1, f_2 \in N$ be such that for some $a \in S$, $f_1(a) = 1$ and $f_2(a) = 0$. Then $f_1(a) \sqcap -f_2(a) = 1$. Second, $[\![\neg\exists v_3 \neg(E(v_3))]\!]$ has value 0. Define $f^p \in S^B$ to be the constant function that takes every $a \in S$ to p, where $0 < p < 1$, and $f^{-p} \in N$ to be the constant function that takes every $a \in S$ to $-p$. Then $[\![E(f^p)]\!] = p$ and $[\![E(f^{-p})]\!] = -p$. Hence $[\![\neg\exists v_3 \neg(E(v_3))]\!] \geqslant [\![E(f^p)]\!] \sqcap [\![E(f^{-p})]\!] = 0$.

□

Corollary 2.7.17.1. $\mathfrak{S}_V^B$ is a model of $MACM$, but not a model of ACM^-.

2.7.6 Identity and Anti-Symmetry

Recall that an atomic Boolean model is a VI model if it is $SEVI$ or $VEVI$, and similarly is a TI model if it is $SETI$ or $VETI$.

Proposition 2.7.1. In any VI model, Anti-Symmetry has value 1.

Proof. Directly follows from Vague-Identity: for any $f_1, f_2 \in M/N$, $[\![f_1 = f_2]\!] = \prod_{a \in S} f_1(a) \Leftrightarrow f_2(a) = [\![f_1 \precsim f_2]\!] \sqcap [\![f_2 \precsim f_1]\!]$.

□

Proposition 2.7.2. In any TI model, Anti-Symmetry has value 0.

Proof. Define $f_1 : S \to B$ as follows: for some $a \in S$, $f_1(a) = p$, where $0 < p < 1$; for any $b \neq a \in S$, $f_1(b) = 1$. Define $f_2 : S \to B$ as follows: $f_2(a) = -p$ and for any $b \neq a \in S$, $f_2(b) = 1$. Define $f : S \to B$ as follows: for any $c \in S$, $f(c) = 1$. It is easy to see that $f_1, f_2 \in M \subseteq N$.

It is also easy to see that $[\![f_1 \precsim f]\!] = [\![f_2 \precsim f]\!] = 1$. And $[\![f \precsim f_1]\!] = 1 \Rightarrow p = p$, $[\![f \precsim f_2]\!] = 1 \Rightarrow -p = -p$. Also, since f, f_1, f_2 are different functions, $[\![f_1 = f]\!] = [\![f_1 = f]\!] = 0$.

Hence $[\![f_1 \precsim f \wedge f \precsim f_1 \rightarrow f = f_1]\!] = (1 \sqcap p) \Rightarrow 0 = -p$. And $[\![f_2 \precsim f \wedge f \precsim f_2 \rightarrow f = f_2]\!] = (1 \sqcap -p) \Rightarrow 0 = p$. Hence $[\![\forall v_1 \forall v_2 (v_1 \precsim v_2 \wedge v_2 \precsim v_1 \rightarrow v_1 = v_2)]\!] \leqslant p \sqcap -p = 0$.

□

Corollary 2.7.17.2. In any $SEVI$ model, Transitivity, Supplementation, Fusion, Atomicity, NoZero and Anti-Symmetry all have value 1.

Corollary 2.7.17.3. In any $SETI$ model, Transitivity, Supplementation, Fusion, Atomicity and NoZero all have value 1, but Anti-Symmetry has value 0.

Corollary 2.7.17.4. In any $VEVI$ model, Transitivity, Supplementation, Fusion, Atomicity, and Anti-Symmetry all have value 1, but NoZero has value 0.

Corollary 2.7.17.5. In any $VETI$ model, Transitivity, Supplementation, Fusion and Atomicity all have value 1, but NoZero and Anti-Symmetry have value 0.

Chapter 3

Boolean-Valued Models with Urelements

(with Bokai Yao)

3.1 Introduction

Boolean-valued models have a long history in set theory, to the extent that it is not unfair to say that Boolean-valued models were birthed within set theory. In 1965, Solovay discovered the idea of using Boolean-valued models to describe forcing, which greatly simplifies Cohen's syntactic method of using forcing conditions. Let V be a class model, or a universe of ZFC, and let B be a complete Boolean algebra in V. Roughly, we are able to construct, via transfinite recursion, a Boolean-valued universe V^B of ZFC within V , whose elements are usually called B-names.[1] In order to show that a sentence ϕ is consistent with ZFC, we just need to find a B such that $[\![\phi]\!]^{V^B} \neq 0$. Since V^B is always witnessing, the quotient model[2] $(V^B)^h$, where $h: B \to 2$ is a homomorphism that takes $[\![\phi]\!]^{V^B}$ to 1, satisfies ZFC and ϕ, by Łoś Theorem. Hence we have found a classical model of ZFC + ϕ and have proven the relative consistency of ϕ.

The intended model of *ZFC* is a universe of *pure* sets. But if the material world is not a total vacuum, there are objects in the world that are not sets: people, tables, planets,

[1] For details, see [3].

[2] For the definition of "witnessing" and the quotient model construction, see Def 1.3.5 and Def 1.3.6 in Chapter 1.

and so forth. Let us call an object that is not a set a *urelement*. Just as there can be sets of sets, it should be case that there can be sets of urelements as well. Just as there are principles governing the behavior of pure sets - the axioms of ZFC - there should be principles govering the behavior of sets and urelements - the axioms of a theory of sets and urelements.

Combining the two trains of thought leaves us with a series of interesting questions: can we construct a Boolean-valued universe of sets and urelements, in the same way as we construct a Boolean-valued universe of (pure) sets? Will the construction give us a model of the background theory of sets and urelements, just as V^B is model of ZFC? Will the construction give us a witnessing model?

In this chapter we will discuss these questions in turn. We will work in two different axiomatic systems - ZFCU_R and ZFCU - in the first-order language of set theory with urelements $\{\in, \mathscr{A}\}$, where $\mathscr{A}$ is a unary predicate for urelements. In both systems it is an axiom that urelements have no members, and we allow the urelements to form a proper class. In addition,

Definition 3.1.1. ZCU is the urelement set theory which includes the following axioms: Extensionality, Foundation, Pairing, Union, Powerset, Infinity, Separation and Choice.

$$\mathrm{ZFCU}_R = \mathrm{ZCU} + \text{Replacement}.$$
$$\mathrm{ZFCU} = \mathrm{ZCU} + \text{Collection}.$$

ZFCU is known to be strictly stronger than ZFCU_R.[3] For example, there can be models of ZFCU_R where the urelements form a proper class but every set of urelements is finite. In models as such, the axiom of Collection fails. And many ZFC theorems, such as the Reflection Principle, are provable in ZFCU but not in ZFCU_R. For these reasons, one might consider ZFCU_R as an inadequate set theory with urelements. However, since ZFCU_R proves transfinite recursion, the basic construction of Boolean-valued models can still be carried out. It is thus natural to study Boolean-valued models with urelements over this weaker theory.

[3]See [37] for a richer hierarchy of axioms in urelement set theory.

In this chapter we will start by presenting an old, standard construction of a Boolean-valued universe with urelements over a class model U of ZFCU$_R$, which we call U^B. We will show that this old construction is legit, in the sense that over the background theory ZFCU$_R$, U^B is a model of ZFCU$_R$, and if Collection holds in U, Collection holds in U^B (Theorem 3.2.6). Nevertheless, the old construction U^B comes with a big problem: as we will show in Section 3.3.1, U^B does not satisfy the Mixing Lemma and is not witnessing, which means that Solovay's forcing method cannot be applied to it. We will remedy this problem by presenting a new construction of a Boolean-valued universe with urelements, which we call $\overline{U^B}$. Unlike U^B, $\overline{U^B}$ can be proven to always satisfy the Mixing Lemma (Theorem 3.3.3), and hence can be shown to be witnessing given that U satisfies Collection. In fact, we will prove that over ZFCU$_R$, Collection is equivalent to the claim that $\overline{U^B}$ is witnessing ,for every complete Boolean algebra B (Theorem 3.4.1). We will also prove that the old construction U^B is an elementary submodel of the new costruction $\overline{U^B}$ (Theorem 3.3.3.4), and therefore $\overline{U^B}$ is also legit.

We organize this chapter as follows. In Section 3.2, we present the old construction U^B, discuss how it is related to U, and prove that all the axioms of ZFCU$_R$ have value 1 in U^B, and that given that U satisfies Collection, Collection also has value 1 in U^B. In section 3.3, we will first argue that U^B is almost never witnessing, which is partially because U^B does not satisfy the Mixing Lemma. Then, we will present the new construction $\overline{U^B}$, show that $\overline{U^B}$ always satisfies the Mixing Lemma, and prove that U^B is an elementary submodel of $\overline{U^B}$. Finally, in Section 3.4, we will prove the major result that over ZFCU$_R$, Collection is equivalent to the claim that $\overline{U^B}$ is witnessing, for every complete Boolean algebra B. We end this chapter with a few conjectures.

Before we move on to the next section, let us introduce some notations and basic facts about ZFCU$_R$ that will be useful for this chapter. We use U to denote the universe of everything, V to denote the universe of pure sets, $\mathscr{A}$ to denote the unary predicate of urelements as well as the class of all urelements. We let $\tau, \sigma, \eta, ...$ stand for B-names in a Boolean-valued universe with urelements and $p, q, ...$ stand for elements of a Boolean algebra.

ZFCU$_R$ proves that every set x has a transitive closure, denoted by $TC(x)$, which is the smallest transitive set t such that $x \subseteq t$. For every x, the kernel of x, denoted by $ker(x)$, is the set of urelements in $TC(\{x\})$. For any set A of urelements, we define

$V_0(A) = A$;

$V_{\alpha+1}(A) = P(V_\alpha(A)) \cup V_\alpha(A)$;

$V_\gamma(A) = \bigcup_{\alpha<\gamma} V_\alpha(A)$, where γ is a limit ordinal;

$V(A) = \bigcup_{\alpha\in Ord} V_\alpha(A)$.

Namely, $V(A)$ is the accumulative hierarchy built from A. For every x, $ker(x) \subseteq A$ iff $x \in V(A)$. The universe of ZFCU$_R$ is non-rigid: for any definable permutation i of $\mathscr{A}$, i can be extended to an automorphism of the universe by letting $i(x) = \{i(y) : y \in x\}$ for every x. And $i(x) = x$ whenever i point-wise fixes $ker(x)$ (i.e., for all $a \in ker(x), i(a) = a$).

3.2 The Model U^B

3.2.1 The Old Construction

In this section, we review the standard way of constructing a Boolean-Valued model with urelements with the background theory ZFCU$_R$,[4] which is the most straightforward generalization of the construction of V^B with the background theory ZFC (See [3]). We call this model U^B.

Definition 3.2.1. Let $\tau \in U$. τ is a B-name iff τ is a urelement, or τ is a function from a set of B-names to B.

Definition 3.2.2. The B-valued model U^B is defined as follows:

1. The domain of U^B is the class of B-names.

2. Let τ, σ be B-names, we first define:

$$[\![\tau \subseteq \sigma]\!] = \prod_{\eta\in dom(\tau)} \tau(\eta) \Rightarrow [\![\eta \in \sigma]\!]$$

[4]For example, such a construction is used and discussed in [4].

We then define:

$$[\![\tau = \sigma]\!] = \begin{cases} 1 & \text{if } \tau, \sigma \in \mathscr{A} \text{ and } \tau = \sigma \\ 0 & \text{if } \tau \in \mathscr{A} \text{ or } \sigma \in \mathscr{A}, \text{ and } \tau \neq \sigma \\ [\![\tau \subseteq \sigma]\!] \sqcap [\![\sigma \subseteq \tau]\!] & \text{if } \tau, \sigma \notin \mathscr{A} \end{cases}$$

We next define:

$$[\![\tau \in \sigma]\!] = \bigsqcup_{\eta \in dom(\sigma)} [\![\eta = \tau]\!] \sqcap \sigma(\eta)$$

We finally define:

$$[\![\mathscr{A}(\tau)]\!] = \begin{cases} 1 & \text{if } \tau \in \mathscr{A} \\ 0 & \text{if } \tau \notin \mathscr{A} \end{cases}$$

We next state without proving some basic facts about U^B. The proofs of these facts are minimally different from the proofs of the similar facts about V^B, the Boolean-valued universe with pure sets only (see, for example, [3]).

Theorem 3.2.1 (Induction Principle for U^B)**.** For any formula $\phi(x)$,

$$\forall \tau \in U^B (\forall \eta \in dom(\tau) \phi(\eta) \to \phi(\tau)) \to \forall \tau \in U^B (\phi(\tau))$$

Proposition 3.2.1. For any τ, σ, π in U^B,

(i) $[\![\tau = \tau]\!] = 1$.

(ii) $\tau(\eta) \leqslant [\![\eta \in \tau]\!]$, for any $\eta \in dom(\tau)$.

(iii) $[\![\tau = \sigma]\!] = [\![\sigma = \tau]\!]$.

(iv) $[\![\tau = \sigma]\!] \sqcap [\![\sigma = \pi]\!] \leqslant [\![\tau = \pi]\!]$.

(v) $[\![\tau = \sigma]\!] \sqcap [\![\tau \in \pi]\!] \leqslant [\![\sigma \in \pi]\!]$.

(vi) $[\![\tau = \sigma]\!] \sqcap [\![\pi \in \tau]\!] \leqslant [\![\pi \in \sigma]\!]$.

(vii) $[\![\tau = \sigma]\!] \sqcap [\![\mathscr{A}(\tau)]\!] \leqslant [\![\mathscr{A}(\sigma)]\!]$

Corollary 3.2.1.1. U^B is a Boolean-valued model[5]. Hence all the axioms of the first-order predicate calculus have value 1 in U^B, and all of its rules of inferences are valid in U^B.

Proposition 3.2.2. For any formula $\phi(x)$ and any τ in U^B,

1. $[\![\exists x \in \tau(\phi(x))]\!] = \bigsqcup_{\eta \in dom(\tau)} (\tau(\eta) \sqcap [\![\phi(\eta)]\!])$.
2. $[\![\forall x \in \tau(\phi(x))]\!] = \bigsqcap_{\eta \in dom(\tau)} (\tau(\eta) \Rightarrow [\![\phi(\eta)]\!])$.

3.2.2 U and U^B

In this section we show that the universe U in a certain sense "sits inside" U^B. In particular, we will find a representative for each element $x \in U$ in U^B, and we will show that the representatives preserve the values of restricted formulas. Also, we show that U is in a certain sense isomorphic to U^2, which can be viewed as a submodel of U^B, for any (non-trivial) complete Boolean algebra B.

Definition 3.2.3. For each $x \in U$, we define:

$$\check{x} = \begin{cases} \{\langle x, 1 \rangle\} & \text{if } x \in \mathscr{A} \\ \{\langle \check{y}, 1 \rangle \mid y \in x\} & \text{if } x \notin \mathscr{A} \end{cases}$$

Proposition 3.2.3. (i) For any $x \in U$, $\tau \in U^B$, $[\![\tau \in \check{x}]\!] = \bigsqcup_{y \in x} [\![\tau = \check{y}]\!]$.

(ii) For any $x, y \in U$,

$$x \in y \leftrightarrow U^B \models \check{x} \in \check{y}$$
$$x = y \leftrightarrow U^B \models \check{x} = \check{y}$$

(iii) The map $x \mapsto \check{x}$ is one-one from U to U^B.

[5] In the sense of Def 1.2.8 in Chapter 1.

(iv) For any formula $\phi(v_1,\dots,v_n)$, any $x_1,\dots x_n \in U$,

$$\phi(x_1,\dots,x_n) \leftrightarrow U^2 \models \phi(\check{x}_1,\dots,\check{x}_n)$$

And if ϕ is restricted then

$$\phi(x_1,\dots,x_n) \leftrightarrow U^B \models \phi(\check{x}_1,\dots,\check{x}_n)$$

Proof. The proofs of these statements are all the same as the proofs of the similar statements on V^B, with minor adjustments. For proofs of the similar statements on V^B, see [].

□

Corollary 3.2.1.2. For any Σ_1 formula $\phi(v_1,\dots,v_n)$, any $x_1,\dots x_n \in U$,

$$\phi(x_1,\dots,x_n) \rightarrow U^B \models \phi(\check{x}_1,\dots,\check{x}_n)$$

3.2.3 The Easy Axioms

In this subsection and the next, we will show that all the axioms of ZFCU_R have value 1 in U^B, given that $U \models \mathrm{ZFCU}_R$. The tricky axiom is Replacement, which will be the topic of the next subsection. The proof that all the other axioms of ZFCU_R (all except Replacement) are true in U^B is standard: again, similar to the proof that these axioms have value 1 in V^B (see for example [16]). We show as examples that the Axiom of Separation and the Powerset Axiom have value 1 in U^B.

Theorem 3.2.2 (Separation)**.** Let $\phi(v)$ be a formula. $U^B \models \forall x \exists y \forall z(z \in y \leftrightarrow z \in x \wedge \phi(z))$.

Proof. Let τ be a B-name. Define $\sigma \in U^B$ such that $dom(\sigma) = dom(\tau)$, and for any $\eta \in dom(\sigma)$, $\sigma(\eta) = \tau(\eta) \sqcap [\![\phi(\eta)]\!]$. Using 3.2.2, it is easy to check that

$$[\![\forall z(z \in \sigma \leftrightarrow z \in \tau \wedge \phi(z))]\!] = 1$$

□

Theorem 3.2.3 (Powerset). $U^B \models \forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x \wedge \neg \mathscr{A}(z))$.

Proof. Let τ be a B-name. Define $\sigma \in U^B$ such that $dom(\sigma) = B^{dom(\tau)}$, and for any $\mu \in dom(\sigma)$, $\sigma(\mu) = \llbracket \mu \subseteq \tau \rrbracket$. Obviously $\llbracket \neg \mathscr{A}(\mu) \rrbracket = 1$, for any $\mu \in dom(\sigma)$.

We will show that U^B “thinks" σ is the powerset of τ, in the sense that

$$U^B \models \forall z (z \in \sigma \leftrightarrow z \subseteq \tau \wedge \neg \mathscr{A}(z))$$

The left to right direction holds by the definition of σ and 3.2.2. We are left to show that for any $\pi \in \mathscr{U}^B$,

$$\llbracket \pi \subseteq \tau \wedge \neg \mathscr{A}(\pi) \rrbracket \leqslant \llbracket \pi \in \sigma \rrbracket \tag{3.1}$$

For any $\pi \in U^B$, we define π^* as follows: $dom(\pi^*) = dom(\tau)$, and for any $\eta \in dom(\pi^*)$, $\pi^*(\eta) = \llbracket \eta \in \pi \rrbracket$. Hence $\pi^* \in dom(\sigma)$. We will show first that

$$\llbracket \pi \subseteq \tau \wedge \neg \mathscr{A}(\pi) \rrbracket \leqslant \llbracket \pi = \pi^* \rrbracket \tag{3.2}$$

If π is such that $\llbracket \mathscr{A}(\pi) \rrbracket = 1$, then the inequality trivially holds. Suppose $\llbracket \mathscr{A}(\pi) \rrbracket = 0$. We just need to show that $\llbracket \pi \subseteq \tau \rrbracket \leqslant \llbracket \pi = \pi^* \rrbracket$.

It is easy to see that by the definition of π^*, $\llbracket \pi^* \subseteq \pi \rrbracket = 1$. For any $\rho \in U^B$, $\llbracket \rho \in \tau \wedge \rho \in \pi \rrbracket = \bigsqcup_{\eta \in dom(\tau)} \tau(\eta) \sqcap \llbracket \eta = \rho \rrbracket \sqcap \rrbracket \rho \in \pi \rrbracket \leqslant \bigsqcup_{\eta \in dom(\pi^*)} \llbracket \eta = \rho \rrbracket \sqcap \rrbracket \eta \in \pi \rrbracket = \llbracket \rho \in \pi^* \rrbracket$. Hence $\llbracket \pi \subseteq \tau \rrbracket \leqslant \llbracket \pi \subseteq \pi^* \rrbracket = \llbracket \pi \subseteq \pi^* \rrbracket \sqcap \llbracket \pi^* \subseteq \pi \rrbracket = \llbracket \pi = \pi^* \rrbracket$. We next show that

$$\llbracket \pi \subseteq \tau \rrbracket \leqslant \llbracket \pi^* \in \sigma \rrbracket \tag{3.3}$$

This is because $\llbracket \pi \subseteq \tau \rrbracket = \llbracket \forall x (x \in \pi \rightarrow x \in \tau) \rrbracket = \bigsqcap_{\eta \in U^B} \llbracket \eta \in \pi \rrbracket \Rightarrow \rrbracket \eta \in \tau \rrbracket \leqslant \bigsqcap_{\eta \in dom(\pi^*)} \pi^*(\eta) \Rightarrow \llbracket \eta \in \tau \rrbracket = \llbracket \forall x \in \pi^* (x \in \tau) \rrbracket = \llbracket \pi^* \subseteq \tau \rrbracket = \sigma(\pi^*) \leqslant \llbracket \pi^* \in \sigma \rrbracket$. Finally, (3.1) follows from combining (3.2) and (3.3).

□

As mentioned in Section 3.1, the Axiom of Collection is not a theorem of ZFCU_R. Therefore, it would be irrational to expect that the axiom of collection always has value

1 in U^B, when U is merely a model of ZFCU$_R$. What we do have, nevertheless, is that Collection has value 1 in U^B *given that* U satisfies Collection.

Theorem 3.2.4 (Collection)**.** If $U \models$ Collection, then $U^B \models$ Collection.

Proof. It suffices to show[6] that for every $\tau \in U^B$, there is a $\rho \in U^B$ such that

$$[\![\forall x \in \tau \exists y \phi(x,y)]\!] \leqslant [\![\forall x \in \tau \exists y \in \rho \phi(x,y)]\!].$$

Now fix $\tau \in U^B$. For any $\sigma \in dom(\tau)$, let $X_\sigma = \{p \in B \mid \exists \pi \in U^B(p = [\![\phi(\sigma,\pi)]\!])\}$. By Collection and Separation in U, it follows that there is a $Y_\sigma \subseteq U^B$ such that $\forall p \in X_\sigma \exists \pi \in Y_\sigma(p = [\![\phi(\sigma,\pi)]\!])$. Then $[\![\exists y \phi(\sigma,x)]\!] = \bigsqcup_{\pi \in Y_\sigma} [\![\phi(\sigma,\pi)]\!]$. This shows that for every $\sigma \in dom(\tau)$, there is a $Y_\sigma \subseteq U^B$ such that $[\![\exists y \phi(\sigma,x)]\!] = \bigsqcup_{\pi \in Y_\sigma} [\![\phi(\sigma,\pi)]\!]$. By Collection again, we can collect those Y_σ into a set $\bar{Y}$. Now let ρ be $((\bigcup \bar{Y}) \cap U^B) \times \{1\}$. For any $\sigma \in dom(\tau)$, $[\![\exists y \phi(\sigma,x)]\!] = \bigsqcup_{\pi \in \bigcup \bar{Y}} [\![\phi(\sigma,\pi)]\!] = [\![\exists y \in \rho \ \varphi(x,y)]\!]$. Thus, ρ is as desired.

□

3.2.4 The Axiom of Replacement

The case with the Axiom of Replacement, unlike the other axioms, is much trickier. This is because the standard proof of the fact that Replacement has value 1 in V^B makes use of Collection in the background theory, which is ZFC. Nevertheless, since our background theory is ZFCU$_R$ instead, and as we have mentioned, Collection is not a theorem of ZFCU$_R$, the standard proof cannot be adjusted into a proof of the fact that the Replacement has value 1 in U^B. We have to use some entirely different techniques.

In this subsection, we prove that it is indeed the case that with the background theory ZFCU$_R$, Replacement has value 1 in U^B. The key method we use here is what we call "purification". Given a set of urelements A and a B-name τ, we can construct the A-purification of τ, $\overset{A}{\tau}$, which is also a B-name. Intuitively, $\overset{A}{\tau}$ is what we get by "purifying off" the urelements in τ that are not in A. Through a long series of lemmas, we manage to prove that as long as A is big enough, the degree to which $\tau = \overset{A}{\tau}$ is always greater than or equal to the

[6]For simplicity we ignore the parameters here. Our proofs can be easily transformed to include parameters. Similarly for Theorem 3.2.5 and Theorem 3.4.1.

degree to which τ is the unique thing that ϕ's with σ, for any σ in the domain of a fixed π. With that, we can then easily show that Replacement has value 1 in U^B.

Definition 3.2.4 (Purification)**.** Let A be a set of urelements. For any urelement $a \in \mathscr{A}$, we define $\overset{A}{a}$ as a. Let τ be a B-name. We define $\overset{A}{\tau} \in U^B$ recursively as follows:

$$dom(\overset{A}{\tau}) = \{\overset{A}{\eta} \mid \eta \in dom(\tau) \cap A\}$$

Let $\mu \in dom(\overset{A}{\tau})$. We define $X_\mu = \{\eta \in dom(\tau) \mid \overset{A}{\eta} = \mu\}$, and

$$\overset{A}{\tau}(\mu) = \bigsqcup_{\eta \in X_\mu} \tau(\eta)$$

Definition 3.2.5. Let a, b be two urelements. We define $i^a_b : U \to U$ as the automorphism generated by the permutation of $\mathscr{A}$ which only swaps a and b.

Proposition 3.2.4. Let μ be a B-name. Let c be a urelement such that $c \notin ker(\mu)$. Then for any urelement b, $i^b_c(\mu)$ is a B-name. Also, $dom(i^b_c(\mu)) = \{i^b_c(\eta) \mid \eta \in dom(\mu)\}$, and for any $i^b_c(\eta) \in dom(i^b_c(\mu))$, $i^b_c(\mu)(i^b_c(\eta)) = \mu(\eta)$.

Lemma 3.2.4.1. Let c be a urelement. Let η, μ be B-names such that $c \notin ker(\eta) \cup ker(\mu)$. Then, for any urelement b,

$$[\![\eta = i^b_c(\mu)]\!] \leqslant [\![\eta = \mu]\!]$$

Proof. We use the induction principle on μ. Since $c \notin ker(\eta) \cup ker(\mu)$, for any $\nu \in dom(\mu), \gamma \in dom(\eta)$, $c \notin ker(\nu) \cup ker(\gamma)$. By inductive hypothesis, then, for any $\nu \in dom(\mu), \gamma \in dom(\eta)$, any urelement b,

$$[\![\gamma = i^b_c(\nu)]\!] \leqslant [\![\gamma = \nu]\!]$$

We first show that $[\![\eta \subseteq i^b_c(\mu)]\!] \leqslant [\![\eta \subseteq \mu]\!]$. That is,

$$\prod_{\gamma \in dom(\eta)} \eta(\gamma) \Rightarrow [\![\gamma \in i^b_c(\mu)]\!] \leqslant \prod_{\gamma \in dom(\eta)} \eta(\gamma) \Rightarrow [\![\gamma \in \mu]\!]$$

It suffices to show that for any $\gamma \in dom(\eta)$, $[\![\gamma \in i_c^b(\mu)]\!] \leqslant [\![\gamma \in \mu]\!]$. By 3.2.4, $[\![\gamma \in i_c^b(\mu)]\!] = \bigsqcup_{\nu \in dom(\eta)} [\![\gamma = i_c^b(\nu)]\!] \sqcap \mu(\nu)$. Hence $[\![\gamma \in i_c^b(\mu)]\!] \leqslant [\![\gamma \in \mu]\!]$ by inductive hypothesis.

By similar reasoning we also have $[\![i_c^b(\mu) \subseteq \eta]\!] \leqslant [\![\mu \subseteq \eta]\!]$.

□

Lemma 3.2.4.2. Let A be a set of urelements, τ be a B-name. Then,

$$[\![\overset{A}{\tau} \subseteq \tau]\!] = \bigsqcap_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow [\![\overset{A}{\eta} \in \tau]\!]$$

Similarly, for any urelement $c \notin ker(\tau)$, and any urelement b,

$$[\![i_c^b(\tau) \subseteq \tau]\!] = \bigsqcap_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow [\![i_c^b(\eta) \in \tau]\!]$$

Proof. We need to show that

$$\bigsqcap_{\mu \in dom(\overset{A}{\tau})} \overset{A}{\tau}(\mu) \Rightarrow [\![\mu \in \tau]\!] = \bigsqcap_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow [\![\overset{A}{\eta} \in \tau]\!]$$

Recall that for each $\mu \in dom(\overset{A}{\tau})$, we let $X_\mu = \{\eta \in dom(\tau) \mid \overset{A}{\eta} = \mu\}$. Hence the above equation holds because for any $\mu \in dom^B(\overset{A}{\tau})$,

$$\begin{aligned}\overset{A}{\tau}(\mu) \Rightarrow [\![\mu \in \tau]\!] &= \bigsqcap_{\eta \in X_\mu} \tau(\eta) \Rightarrow [\![\mu \in \tau]\!] \\ &= \bigsqcap_{\eta \in X_\mu} \tau(\eta) \Rightarrow [\![\overset{A}{\eta} \in \tau]\!]\end{aligned}$$

The second statement holds for similar reasons.

□

Lemma 3.2.4.3. Let A be a set of urelements, τ be a B-name. Then, for any B-name σ,

$$[\![\sigma \in \overset{A}{\tau}]\!] = \bigsqcup_{\eta \in dom(\tau)} \tau(\eta) \sqcap [\![\sigma = \overset{A}{\eta}]\!]$$

Similarly, for any urelement $c \notin ker(\tau)$, any urelement b,

$$\llbracket \sigma \in i_c^b(\tau) \rrbracket = \bigsqcup_{\eta \in dom(\tau)} \tau(\eta) \sqcap \llbracket \sigma = i_c^b(\eta) \rrbracket$$

Proof. We need to show that

$$\bigsqcup_{\mu \in dom(\overset{A}{\tau})} \overset{A}{\tau}(\mu) \sqcap \llbracket \sigma = \mu \rrbracket = \bigsqcup_{\eta \in dom(\tau)} \tau(\eta) \sqcap \llbracket \sigma = \overset{A}{\eta} \rrbracket$$

It holds because for any $\mu \in dom(\overset{A}{\tau})$, $\overset{A}{\tau}(\mu) \sqcap \llbracket \sigma = \mu \rrbracket = \bigsqcup_{\eta \in X_\mu} \tau(\eta) \sqcap \llbracket \sigma = \overset{A}{\eta} \rrbracket$. The second statement holds for similar reasons.

$\square$

Lemma 3.2.4.4. Let A be a set of urelements, τ be a B-name. Let $C_\tau = ker(\tau) \backslash A$. Let c be a urelement such that $c \notin ker(\tau) \cup A$. Then,

$$\bigsqcap_{b \in C_\tau} \llbracket \tau = i_c^b(\tau) \rrbracket \leqslant \llbracket \tau = \overset{A}{\tau} \rrbracket$$

Proof. We use the induction principle on τ. Since $c \notin ker(\tau) \cup A$, for any $\eta \in dom(\tau)$, $c \notin ker(\eta) \cup A$. Assume as inductive hypothesis that for any $\eta \in dom(\tau)$,

$$\bigsqcap_{b \in C_\eta} \llbracket \eta = i_c^b(\eta) \rrbracket \leqslant \llbracket \eta = \overset{A}{\eta} \rrbracket$$

Our goal is to show that

$$\bigsqcap_{b \in C_\tau} \llbracket \tau \subseteq i_c^b(\tau) \rrbracket \sqcap \llbracket i_c^b(\tau) \subseteq \tau \rrbracket \leqslant \llbracket \tau \subseteq \overset{A}{\tau} \rrbracket \sqcap \llbracket \overset{A}{\tau} \subseteq \tau \rrbracket \tag{3.4}$$

Observe that

$$
\begin{aligned}
[\![\tau \subseteq \overset{A}{\tau}]\!] \sqcap [\![\overset{A}{\tau} \subseteq \tau]\!] &= \prod_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow ([\![\eta \in \overset{A}{\tau}]\!] \sqcap [\![\overset{A}{\eta} \in \tau]\!]) \\
&= \prod_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow (\bigsqcup_{\mu \in dom(\tau)} \tau(\mu) \sqcap [\![\eta = \overset{A}{\mu}]\!] \sqcap [\![\overset{A}{\eta} = \mu]\!]) \\
&\geqslant \prod_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow (\tau(\eta) \sqcap [\![\eta = \overset{A}{\eta}]\!] \sqcap [\![\overset{A}{\eta} = \eta]\!]) \\
&= \prod_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow [\![\eta = \overset{A}{\eta}]\!] \\
&\geqslant \prod_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow \prod_{b \in C_\eta} [\![\eta = i_c^b(\eta)]\!] \\
&= \prod_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow \prod_{b \in C_\tau} [\![\eta = i_c^b(\eta)]\!]
\end{aligned}
$$

where the first line holds by 3.2.4.2. The second holds by 3.2.4.3. The second to the last line holds by the inductive hypothesis. The last line holds because for any $b \in C_\tau \backslash C_\eta$, $i_c^b(\eta) = \eta$.

Also observe that

$$
\begin{aligned}
\prod_{b \in C_\tau} [\![\tau \subseteq i_c^b(\tau)]\!] \sqcap [\![i_c^b(\tau) \subseteq \tau]\!] &= \prod_{b \in C_\tau} \prod_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow [\![\eta \in i_c^b(\tau)]\!] \sqcap [\![i_c^b(\eta) \in \tau]\!] \\
&= \prod_{\eta \in dom(\tau)} \tau(\eta) \Rightarrow \prod_{b \in C_\tau} [\![\eta \in i_c^b(\tau)]\!] \sqcap [\![i_c^b(\eta) \in \tau]\!]
\end{aligned}
$$

where the first line holds by 3.2.4.2.

Therefore, to show (3.4), we just need to show that for any $\eta \in dom(\tau)$, any $b \in C_\tau$, (second line holds by 3.2.4.3)

$$
\begin{aligned}
[\![\eta = i_c^b(\eta)]\!] &\geqslant [\![\eta \in i_c^b(\tau)]\!] \sqcap [\![i_c^b(\eta) \in \tau]\!] \\
&= \bigsqcup_{\mu \in dom(\tau)} \tau(\mu) \sqcap [\![\eta = i_c^b(\mu)]\!] \sqcap [\![\mu = i_c^b(\eta)]\!]
\end{aligned}
$$

But this holds because for any $\mu \in dom(\tau)$,

$$\begin{aligned}\tau(\mu) \sqcap [\![\eta = i_c^b(\mu)]\!] \sqcap [\![\mu = i_c^b(\eta)]\!] &\leqslant [\![\eta = i_c^b(\mu)]\!] \sqcap [\![\mu = i_c^b(\eta)]\!] \\ &\leqslant [\![\eta = \mu]\!] \sqcap [\![\mu = i_c^b(\eta)]\!] \\ &\leqslant [\![\eta = i_c^b(\eta)]\!]\end{aligned}$$

where the second line holds by 3.2.4.1, since $c \notin ker(\eta) \cup ker(\mu)$, as $c \notin ker(\tau)$ and $\eta, \mu \in dom(\tau)$.

□

Theorem 3.2.5 (Replacement)**.** Let $\phi(v_1, v_2)$ be a formula. $U^B \models \forall u(\forall x \in u \exists! y \phi(x,y) \rightarrow \exists w \forall x \in u \exists y \in w \phi(x,y))$.

Proof. We may assume Collection does not hold in U, otherwise $U^B \models$ Replacement as $U^B \models$ Collection by Theorem 3.2.4. So we may assume there is a proper class of urelements in U.

It suffices to show that for every $\pi \in U^B$, there is a $\rho \in U^B$ such that for every $\sigma \in dom(\pi)$,

$$[\![\exists! y \phi(\sigma, y)]\!] \leqslant [\![\exists y \in \rho\ \phi(\sigma, y)]\!] \tag{3.5}$$

Consider any π. Let $A = ker(B) \cup ker(\pi)$.

Claim 3.2.5.1. For every $\sigma \in dom(\pi)$ and $\tau \in U^B$, there is a $\tau^* \in U^B$ such that $ker(\tau^*) \subseteq A$ and $[\![\phi(\sigma, \tau) \wedge \forall z(\phi(\sigma, z) \rightarrow z = \tau)]\!] \leqslant [\![\phi(\sigma, \tau^*)]\!]$.

Proof of the Claim. Let $p = [\![\phi(\sigma, \tau) \wedge \forall z(\phi(\sigma, z) \rightarrow z = \tau)]\!]$ and c be a urelement such that $c \notin ker(\tau) \cup A$, which exists by our assumption. Observe that $[\![\phi(\sigma, \tau)]\!] = [\![\phi(\sigma, i_c^b(\tau))]\!]$ for every $b \in ker(\tau) \backslash A$. Moreover, for each $b \in ker(\tau) \backslash A$,

$$
\begin{aligned}
p &\leqslant [\![\phi(\sigma,\tau)]\!] \sqcap ([\![\phi(\sigma,i^b_c(\tau))]\!] \Rightarrow [\![\tau = i^b_c(\tau)]\!]) \\
&\leqslant [\![\phi(\sigma,\tau)]\!] \sqcap [\![\tau = i^b_c(\tau)]\!] \\
&\leqslant [\![\tau = i^b_c(\tau)]\!]
\end{aligned}
$$

It follows that

$$
\begin{aligned}
p &\leqslant [\![\phi(\sigma,\tau)]\!] \sqcap \prod_{b \in ker(\tau)\setminus A} [\![\tau = i^b_c(\tau)]\!] \\
&\leqslant [\![\varphi(\sigma,\tau)]\!] \sqcap [\![\tau = \overset{A}{\tau}]\!] && \text{(by Lemma 3.2.4.4)} \\
&\leqslant [\![\varphi(\sigma,\overset{A}{\tau})]\!]
\end{aligned}
$$

As the kernel of $\overset{A}{\tau}$ is contained in A, this proves the claim.

■

Now for every $\sigma \in dom(\pi)$ and $p \in B$ such that there is some $\tau \in U^B$ with $p = [\![\phi(\sigma,\tau) \wedge \forall z(\phi(\sigma,z) \rightarrow z = \tau)]\!]$, let $\alpha_{\sigma,p}$ be the least α such that $\exists \tau^* \in V_\alpha(A)$ with $p \leqslant [\![\phi(\sigma,\tau^*)]\!]$. Such α exists by the claim. Let $\gamma = \bigcup_{\langle \sigma,p\rangle \in dom(\pi)\times B} \alpha_{\sigma,p}$. Let $\rho = (V_\gamma(A) \cap U^B) \times \{1\}$. It is easy to check that for every $\sigma \in dom(\pi)$, $[\![\exists! y\phi(\sigma,y)]\!] \leqslant [\![\exists y \in \rho\ \phi(\sigma,y)]\!]$, which completes the proof.

□

Theorem 3.2.6 (The Fundamental Theorem of U^B)**.** $U^B \models$ ZFCU$_R$. If Collection holds in U, then $U^B \models$ ZFCU.

3.2.5 U^B Can Recover Collection

In the previous subsection we proved that U^B preserves Collection. In this subsection, we prove that U^B can also recover Collection: there is some universe U such that $U \not\models$ Collection, and for some complete Boolean algebra B, $U^B \models$ Collection. To this end, we first introduce some new axioms.

(DC$_\omega$ Scheme) If for every x there is a y such that $\varphi(x,y)$, then there is an ω-sequence $\langle x_n : n < \omega \rangle$ such that $\varphi(x_n, x_{n+1})$ for every n.

(Plenitude) For every cardinal κ, there is some set of urelements of size κ.

(Tail) For every set of urelements A, there is a greatest cardinal κ such that there is $B \subseteq \mathscr{A}$ of size κ with $B \cap A = \varnothing$.

The following is proved in [37].

Theorem 3.2.7. Over ZFCU$_R$, the following implications hold and none of them can be reserved.

1. Plenitude → Collection;

2. Collection → DC$_\omega$ Scheme;

3. Tail → Collection. □

Theorem 3.2.8. Let λ and κ be infinite cardinals with $\lambda \leqslant \kappa$. If there is a double sequence $\{p_{\delta\xi} \mid \delta < \lambda, \xi < \kappa\}$ such that for any $\xi < \kappa$, $\bigsqcup_{\delta \in \lambda} p_{\delta\xi} = 1$, and for any $\delta < \lambda$, $\{p_{\delta\xi} \mid \xi < \kappa\}$ is an antichain, then $U^B \models |\check{\kappa}| = |\check{\lambda}|$.

Proof. Same proof as in Bell [3], p.109.

□

Corollary 3.2.8.1. Let $B = RO(\kappa^\omega)$, where κ is an infinite cardinal. Then, $U^B \models \check{\kappa}$ is countable.

Proof. For each $n < \omega$ and $\xi \in \kappa$, let $p_{n\xi} = \{g \in \kappa^\omega \mid g(n) = \xi\}$. Then, for any $\xi < \kappa$,

$$\bigsqcup_{n<\omega} p_{n\xi} = (\overline{\{g \in \kappa^\omega \mid \text{for some } n < \omega, g(n) = \xi\}})^\circ = \kappa^\omega$$

Also for any $n < \omega$, $\xi_1 \neq \xi_2 < \kappa$,

$$p_{n\xi_1} \sqcap p_{n\xi_2} = \{g \in \kappa^\omega \mid g(n) = \xi_1\} \cap \{g \in \kappa^\omega \mid g(n) = \xi_2\} = \varnothing$$

□

Theorem 3.2.9. If $U \models$ for every set of urelements, there is another infinite set of urelements disjoint from it, then for some complete Boolean algebra B, $U^B \models ZFCU$.

Proof. By Theorem 3.2.6 and Theorem 3.2.7, we may assume that in U, there is a least cardinal κ such that there is no set of urelements of size κ. Let $B = RO(\kappa^\omega)$. We will show that $U^B \models$ Collection.

By Corollary 3.2.8.1, $U^B \models |\check{\omega}| = |\check{\kappa}|$. Let $\tau \in U^B$ be such that $U^B \models \forall v \in \tau(\mathscr{A}(v))$. We can assume without loss of generality that every $\eta \in dom(\tau)$ is such that $\tau(\eta) > 0$.[7] Then, by definition of U^B, every $\eta \in dom(\tau)$ has to be a urelement. Let $A = dom(\tau)$. By assumption, $U \models |A| = \lambda < \kappa$. Hence $U^B \models |\check{A}| = |\check{\lambda}| \leqslant |\check{\kappa}| = |\check{\omega}|$. Also, it is easy to show that $U^B \models |\tau| \leqslant |\check{A}|$. Hence $U^B \models |\tau| \leqslant |\check{\omega}|$.

Since A is a set of urelements, there is an infinite set of urelements $B \in U$ such that $A \cap B = \varnothing$. Since $U \models |\omega| \leqslant |B|$, $U^B \models |\check{\omega}| \leqslant |\check{B}|$. But $U^B \models$ every set of urelements is countable. Hence $U^B \models$ Tail. By Theorem 3.2.7, $U^B \models$ Collection.

□

3.3 Non-Witnessing U^B and Witnessing $\overline{U^B}$

3.3.1 U^B Is Almost Never Witnessing

An important property of Boolean-valued models is the property of being witnessing. It is defined as follows:

Definition 3.3.1. Let $\mathfrak{M}$ be a Boolean-valued model for the languague $\mathscr{L}$. $\mathfrak{M}$ is *witnessing* iff for any formula $\phi(v, v_1, ..., v_n)$ in $\mathscr{L}$, any $x_1, ..., x_n \in M$, there is some $x \in M$ such that

$$[\![\exists v \phi(v, x_1, ..., x_n)]\!]^{\mathfrak{M}} = [\![\phi(x, x_1, ..., x_n)]\!]^{\mathfrak{M}}$$

In other words, a Boolean-valued model is witnessing just in case there exists a "witness" in the model for every existential sentence. Note that classical two-valued models are

[7] This is because for every $\tau \in U^B$, there is a τ' such that $[\![\tau' = \tau]\!] = 1$ and every $\eta \in dom(\tau')$ is such that $\tau'(\eta) > 0$.

trivially witnessing. The property of being witnessing is especially important for Boolean-valued models. This is mainly because witnessing models are precisely those that satisfy the generalized Łoś theorem (see Chapter 1 for details). Witnessing models are therefore well-behaved in the sense that the values of formulas in these models correspond neatly with the values of formulas in their two-valued quotient models. Hence, in the context of set theory, witnessing Boolean-valued universes are special in that they are the ones by which we prove relative consistency results.[8] Given a sentence ϕ that we want to prove to be consistent with ZFC, we simply find a complete Boolean algebra B such that $[\![\phi]\!]^{V^B} \neq 0$. Since V^B is witnessing, we can then apply Łoś theorem and obtain the two-valued quotient model generated by a ultrafilter that contains the value of the sentence, which will be a model of ϕ together with all the axioms of ZFC.

The old/standard Boolean-valued construction U^B, as introduced in the previous section, comes with a major problem: it is never witnessing given that B is a proper extension of 2. Consider, for example, the B-name $\tau = \{\langle a_1, p\rangle, \langle a_2, \neg p\rangle\}$, where a_1, a_2 are two different urelements[9], and p is an intermediate Boolean value. Consider, then the sentence $\phi(\tau) = \exists v(\mathscr{A}(v) \wedge v \in \tau)$. Since both a_1 and a_2 are urelements, $[\![\mathscr{A}(a_1)]\!]$ and $[\![\mathscr{A}(a_2)]\!]$ are both 1. Also, by the identity clauses on urelements, $[\![a_1 \in \tau]\!] = p$ and similarly $[\![a_2 \in \tau]\!] = \neg p$. Hence $[\![\phi(\tau)]\!] \geqslant p \sqcup \neg p = 1$.

Assume for reductio that $\phi(\tau)$ has a witness σ in U^B. Hence $[\![\mathscr{A}(\sigma)]\!] = 1$ and $[\![\sigma \in \tau]\!] = 1$. That $[\![\mathscr{A}(\sigma)]\!] = 1$ entails that σ is a urelement. But by the the identity clauses on urelements again, for any urelement a, $[\![a \in \tau]\!] = 0$ if a is neither a_1 nor a_2. Hence τ can only be a_1 or a_2. But neither $[\![a_1 \in \tau]\!]$ nor $[\![a_2 \in \tau]\!]$ is 1, as we have already observed. Hence $\phi(\tau)$ has no witness in U^B.

There are countless other counter-examples of this form, which raise an interesting question. It is well known that V^B is a witnessing model. So why is U^B not witnessing, when it is constructed under the same guiding idea as V^B?

That V^B is witnessing is a corollary to an important lemma on V^B: the Mixing Lemma. In V^B, given an antichain $\{p_i \mid i \in I\} \subseteq B$, and a sequence of names $\{x_i \mid i \in I\} \in V^B$, we may

[8]See [16] for more details.
[9]U^B is not witnessing even if there is only one urelement (assuming that B properly extends 2). Example: $\tau' = \{\langle a, p\rangle, \langle \varnothing, \neg p\rangle\}$ where a is a urelement. Then $[\![\exists v(v \in \tau')]\!]$ does not have a witness.

construct their *mixture* $u \in V^B$ as follows:

$$dom(u) = \bigcup_{i \in I} dom(x_i)$$

and, for $y \in dom(u)$,

$$u(y) = \bigsqcup_{i \in I} p_i \sqcap [\![y \in x_i]\!]^{V^B}$$

Using the mixture construction, it can be proven that

Theorem 3.3.1 (The Mixing Lemma in V^B)**.** For any antichain $\{p_i \mid i \in I\} \subseteq B$, any sequence of names $\{x_i \mid i \in I\} \in V^B$, there is some $u \in V^B$ (for example, their mixture) such that for any $i \in I$,

$$p_i \leqslant [\![u = x_i]\!]^{V^B}$$

The reason why U^B is not witnessing is because it does not satisfy the Mixing Lemma either. And the reason why the Mixing Lemma is violated in U^B is essentially due to the "strict" definitive clauses on urelements in U^B. Indeed, we can show that

Lemma 3.3.1.1. Let $\{p_i \mid i \in I\} \subseteq B$ be an antichain and $\{\tau_i \mid i \in I\} \in U^B$ be a sequence of B-names. If there is some $i \in I$ such that τ_i is an urelement, and there is some $j \in I$ such that $\tau_j \neq \tau_i$, then there is no $\sigma \in U^B$ such that for any $k \in I$, $p_k \leqslant [\![\sigma = \tau_k]\!]$.

Proof. Assume for reductio that there is such a $\sigma \in U^B$. Note that by 3.2.2, for any urelement $a \in U^B$, for any $\tau \in U^B$, $[\![\tau = a]\!]$ is either 0 or 1. Since $0 < p_i \leqslant [\![\sigma = \tau_i]\!]$ and τ_i is a urelement, $[\![\sigma = \tau_i]\!] = 1$. Hence σ is τ_i. But then $[\![\sigma =]\!]\tau_j]\!] = 0$, and hence it is not the case that $0 < p_j \leqslant]\!]\tau_j]\!]$. Contradiction.

□

The above observation gives rise to the question: Is there a different construction of Boolean-valued universe with urelements that satisfies the Mixing Lemma? Is there one that is witnessing? The answers to both questions, interestingly, are positive. The key for a Boolean-valued universe with urelements to satisfy the Mixing Lemma, as illustrated by the previous lemma, is that there has to be *mixtures* of different urelements, and similarly *mixtures* of urelements and sets. In the next subsection, we present a new construction of a Boolean-valued universe with urelements of this kind. We will show that this new construction, which we call $\overline{U^B}$, is closed under mixtures and therefore satisfies the Mixing Lemma.

3.3.2 The New Construction $\overline{U^B}$

We begin with the definition of the Boolean-valued universe $\overline{U^B}$.

Definition 3.3.2. $\tau : A \to B$ is a $\overline{B}$-urelement iff A is a set of urelements and for any $a \neq b \in A$, $\tau(a) \sqcap \tau(b) = 0$.

Definition 3.3.3. $\tau : X \to B$ is a $\overline{B}$-name iff for any $x \in X$, x is either a urelement or a $\overline{B}$-name, and for any urelement $a \in X$, any $x \neq a \in X$, $\tau(a) \sqcap \tau(x) = 0$.

Proposition 3.3.1. Every $\overline{B}$-urelement is a $\overline{B}$-name.

Definition 3.3.4. Let τ be a $\overline{B}$-name. $dom^{\mathscr{A}}(\tau) = \{a \in dom(\tau) \mid a \in \mathscr{A}\}$. $dom^{B}(\tau) = \{\eta \in dom(\tau) \mid \eta$ is a $\overline{B}$-name$\}$.

Definition 3.3.5. Let τ be a $\overline{B}$-name. For any urelement $a \notin dom^{\mathscr{A}}(\tau)$, $\tau(a) = 0$.

Definition 3.3.6. The B-valued model $\overline{U^B}$ is defined as follows:

1. The domain of $\overline{U^B}$ is the class of $\overline{B}$-names.

2. For any $\overline{B}$-names τ, σ, we first define:

$$[\![\mathscr{A}(\tau)]\!]^{\overline{U^B}} = \bigsqcup_{a \in \mathscr{A}} \tau(a)$$
$$[\![\tau \overset{\mathscr{A}}{=} \sigma]\!]^{\overline{U^B}} = \bigsqcap_{a \in \mathscr{A}} \tau(a) \Leftrightarrow \sigma(a)$$

We then define by double recursion:

$$\llbracket \tau \in \sigma \rrbracket^{\overline{U^B}} = \bigsqcup_{\mu \in dom^B(\sigma)} \llbracket \tau = \mu \rrbracket^{\overline{U^B}} \sqcap \sigma(\mu)$$

$$\llbracket \tau \subseteq \sigma \rrbracket^{\overline{U^B}} = \bigsqcap_{\eta \in dom^B(\tau)} \tau(\eta) \Rightarrow \llbracket \eta \in \sigma \rrbracket^{\overline{U^B}}$$

$$\llbracket \tau = \sigma \rrbracket^{\overline{U^B}} = \llbracket \tau \subseteq \sigma \rrbracket^{\overline{U^B}} \sqcap \llbracket \sigma \subseteq \tau \rrbracket^{\overline{U^B}} \sqcap \llbracket \tau \stackrel{\mathscr{A}}{=} \sigma \rrbracket^{\overline{U^B}}$$

We then state without proving some useful and elementary results on $\overline{B}$-names. Again, the proofs are similar to those in the case of V^B.

Theorem 3.3.2 (Induction Principle for $\overline{U^B}$)**.** For any formula $\phi(x)$,

$$\forall \tau \in \overline{U^B}(\forall \eta \in dom^B(\tau)\phi(\eta) \to \phi(\tau)) \to \forall \tau \in \overline{U^B}(\phi(\tau))$$

Proposition 3.3.2. For any τ, σ, π in $\overline{U^B}$,

(i) $\llbracket \tau = \tau \rrbracket^{\overline{U^B}} = 1$.

(ii) $\tau(\eta) \leqslant \llbracket \eta \in \tau \rrbracket^{\overline{U^B}}$, for any $\eta \in dom^B(\tau)$.

(iii) $\llbracket \tau = \sigma \rrbracket^{\overline{U^B}} = \llbracket \sigma = \tau \rrbracket^{\overline{U^B}}$.

(iv) $\llbracket \tau = \sigma \rrbracket^{\overline{U^B}} \sqcap \llbracket \sigma = \pi \rrbracket^{\overline{U^B}} \leqslant \llbracket \tau = \pi \rrbracket^{\overline{U^B}}$.

(v) $\llbracket \tau = \sigma \rrbracket^{\overline{U^B}} \sqcap \llbracket \tau \in \pi \rrbracket^{\overline{U^B}} \leqslant \llbracket \sigma \in \pi \rrbracket^{\overline{U^B}}$.

(vi) $\llbracket \tau = \sigma \rrbracket^{\overline{U^B}} \sqcap \llbracket \pi \in \tau \rrbracket^{\overline{U^B}} \leqslant \llbracket \pi \in \sigma \rrbracket^{\overline{U^B}}$.

(vii) $\llbracket \tau = \sigma \rrbracket^{\overline{U^B}} \sqcap \llbracket \mathscr{A}(\tau) \rrbracket^{\overline{U^B}} \leqslant \llbracket \mathscr{A}(\sigma) \rrbracket^{\overline{U^B}}$

Corollary 3.3.2.1. $\overline{U^B}$ is a Boolean-valued model. Hence all the axioms of the first-order predicate calculus have value 1 in $\overline{U^B}$, and all of its rules of inferences are valid in $\overline{U^B}$.

Proposition 3.3.3. For any formula $\phi(x)$ and any τ in $\overline{U^B}$,

1. $\llbracket \exists x \in \tau(\phi(x)) \rrbracket^{\overline{U^B}} = \bigsqcup_{\eta \in dom^B(\tau)} (\tau(\eta) \sqcap \llbracket \phi(\eta) \rrbracket^{\overline{U^B}})$.

2. $\llbracket \forall x \in \tau(\phi(x)) \rrbracket^{\overline{U^B}} = \bigsqcap_{\eta \in dom^B(\tau)} (\tau(\eta) \Rightarrow \llbracket \phi(\eta) \rrbracket^{\overline{U^B}})$.

3.3.3 $\overline{U^B}$ and the Mixing Lemma

In this subsection, we show that $\overline{U^B}$, unlike U^B, does satisfy the Mixing Lemma.

Definition 3.3.7. Let $\{\tau_i \mid i \in I\} \subseteq \overline{U^B}$ and $\{p_i \mid i \in I\} \subseteq B$ be an antichain. We define the *mixture* of $\{p_i \mid i \in I\}$ with respect to $\{\tau_i \mid i \in I\}$ to be the $\overline{B}$-name τ such that

$$dom(\tau) = \bigcup_{i \in I} dom(\tau_i)$$

For any $x \in dom(\tau)$, we define J_x as $\{i \in I \mid x \in dom(\tau_i)\}$, and define

$$\tau(x) = \bigsqcup_{i \in J_x} p_i \sqcap \tau_i(x)$$

Proposition 3.3.4. Let $\{\tau_i \mid i \in I\} \subseteq \overline{U^B}$ and $\{p_i \mid i \in I\} \subseteq B$ be an antichain. Their mixture τ is a $\overline{B}$-name.

Proof. Let $a \in dom^{\mathscr{A}}(\tau), x \neq a \in dom(\tau)$. Then $\tau(a) \sqcap \tau(v) = \bigsqcup_{i \in J_a} p_i \sqcap \tau_i(a) \sqcap \bigsqcup_{j \in J_x} p_j \sqcap \tau_j(x)$. We need to show that for any $i \in J_a, j \in J_x$,

$$p_i \sqcap \tau_i(a) \sqcap p_j \sqcap \tau_j(x) = 0$$

If $i \neq j$, then $p_i \sqcap p_j = 0$ as $\{p_i \mid i \in I\}$ is an antichain. If $i = j$, then $a, x \in dom(\tau_i)$, and hence $\tau_i(a) \sqcap \tau_i(x) = 0$ as τ_i is a $\overline{B}$-name.

□

Theorem 3.3.3 (The Mixing Lemma in $\overline{U^B}$)**.** Let $\{\tau_i \mid i \in I\} \subseteq \overline{U^B}$ and $\{p_i \mid i \in I\} \subseteq B$ be an antichain. Let τ be their mixture. Then, for any $i \in I$,

$$p_i \leqslant [\![\tau = \tau_i]\!]^{\overline{U^B}}$$

Proof. Let $i \in I$. We first show that

$$p_i \leqslant [\![\tau \subseteq \tau_i]\!]^{\overline{U^B}} = \prod_{\eta \in dom^B(\tau)} \tau(\eta) \Rightarrow [\![\eta \in \tau_i]\!]^{\overline{U^B}} \tag{3.6}$$

Since $\tau(\eta) = \bigsqcup_{j \in J_\eta} p_i \sqcap \tau_j(\eta)$, we just need to show that for any $j \in J_\eta$,

$$p_i \leqslant \neg p_j \sqcup \neg \tau_j(\eta) \sqcup [\![\eta \in \tau_i]\!]^{\overline{U^B}}$$

If $i \neq j$, then $p_i \leqslant \neg p_j$ as $\{p_i \mid i \in I\}$ is an antichain. If $i = j$, then $\eta \in dom(\tau_i)$. Hence $\tau_i(\eta) \leqslant [\![\eta \in \tau_i]\!]^{\overline{U^B}}$ by 3.3.2(ii). Hence $\neg \tau_j(\eta) \sqcup [\![\eta \in \tau_i]\!]^{\overline{U^B}} = 1$.

We next show that

$$p_i \leqslant [\![\tau_i \subseteq \tau]\!]^{\overline{U^B}} = \bigsqcap_{\eta \in dom^B(\tau_i)} \tau_i(\eta) \Rightarrow [\![\eta \in \tau]\!]^{\overline{U^B}} \tag{3.7}$$

Let $\eta \in dom^B(\tau_i)$. Then $\eta \in dom^B(\tau)$. Hence

$$\begin{aligned} p_i &\leqslant \tau_i(\eta) \Rightarrow (p_i \sqcap \tau_i(\eta)) \\ &= \tau_i(\eta) \Rightarrow \tau(\eta) \\ &\leqslant \tau_i(\eta) \Rightarrow [\![\eta \in \tau]\!]^{\overline{U^B}} \end{aligned}$$

We finally show that

$$p_i \leqslant [\![\tau \overset{\mathscr{A}}{=} \tau_i]\!]^{\overline{U^B}} = \bigsqcap_{a \in \mathscr{A}} \tau(a) \Leftrightarrow \tau_i(a) \tag{3.8}$$

Let $a \in dom^{\mathscr{A}}(\tau)$. Since $\tau(a) = \bigsqcup_{j \in J_a} p_j \sqcap \tau_j(a)$, we just need to show that the following two both hold:

$$\begin{aligned} p_i &\leqslant \bigsqcap_{j \in J_a} \neg p_j \sqcup \neg \tau_j(a) \sqcup \tau_i(a) \\ p_i &\leqslant \neg \tau_i(a) \sqcup \bigsqcup_{i \in J_a} p_j \sqcap \tau_j(a) \end{aligned}$$

For the first statement, if $i \neq j$, then $p_i \leqslant \neg p_j$; if $i = j$, then $\neg \tau_i(a) \sqcup \tau_i(a) = 1$. For the second statement, if $a \notin dom^(\tau_i)$, then $\neg \tau_i(a) = 1$. If $a \in dom^(\tau_i)$, then $i \in J_a$. Hence $RHS \geqslant \neg \tau_i(a) \sqcup (p_i \sqcap \tau_i(a)) = \neg \tau_i(a) \sqcup p_i \geqslant p_i$.

Combining (3.6) and (3.7) and (3.8) gives us what we want.

□

3.3.4 U^B and $\overline{U^B}$

We have constructed two different Boolean-valued universes with urelements: U^B and $\overline{U^B}$. We have shown that the latter satisfies the Mixing Lemma, whereas the former does not. Curious readers may have wondered how the two constructions are connected. The big result of this subsection is that U^B is, up to isomorphism, an elementary submodel of $\overline{U^B}$.

The plan to reach this result is as follows. We will first create an isomorphic copy of U^B within $\overline{U^B}$. consisting of the “sharp" $\overline{B}$-names. Then, we will show that this isomorphic copy is actually an elementary submodel of $\overline{U^B}$.

Definition 3.3.8. Let $\tau \in \overline{U^B}$. τ is a *sharp* $\overline{B}$-name iff $\tau = \{\langle a, 1\rangle\}$, for some $a \in \mathscr{A}$, or for any $x \in dom(\tau)$, x is a sharp $\overline{B}$-name.

Definition 3.3.9. $\overline{U^B_S}$ is the submodel of $\overline{U^B}$ whose domain is the class of all the sharp $\overline{B}$-names.

Proposition 3.3.5. Let τ, σ be sharp B-names. Then[10],

(i) $$[\![\mathscr{A}(\tau)]\!] = \begin{cases} 1 & \text{if for some urelement } a, \tau = \{\langle a, 1\rangle\}. \\ 0 & \text{if otherwise.} \end{cases}$$

(ii) $$[\![\tau = \sigma]\!] = \begin{cases} 1 & \text{if } [\![\mathscr{A}(\tau)]\!] = 1, [\![\mathscr{A}(\sigma)]\!] = 1, \text{ and } \tau = \sigma. \\ 0 & \text{if } [\![\mathscr{A}(\tau)]\!] = 1 \text{ or } [\![\mathscr{A}(\sigma)]\!] = 1, \text{ and } \tau \neq \sigma. \\ [\![\tau \subseteq \sigma]\!] \sqcap [\![\sigma \subseteq \tau]\!] & \text{if } [\![\mathscr{A}(\tau)]\!] = 0 \text{ and } [\![\mathscr{A}(\sigma)]\!] = 0. \end{cases}$$

(iii) $$[\![\tau \in \sigma]\!] = \begin{cases} 0 & \text{if } [\![\mathscr{A}(\sigma)]\!] = 1. \\ \bigsqcup_{\eta \in dom(\sigma)} [\![\tau = \eta]\!] \sqcap \sigma(\eta) & \text{if } [\![\mathscr{A}(\sigma)]\!] = 0. \end{cases}$$

Corollary 3.3.3.1. $\overline{U^B_S}$ and U^B are isomorphic.

[10]Here, for readability, we ignore the “model" superscript on Boolean values of formulas. $[\![\mathscr{A}(\tau)]\!]$ means, for example, $[\![\mathscr{A}(\tau)]\!]^{\overline{U^B}}$, or equivalently, $[\![\mathscr{A}(\tau)]\!]^{\overline{U^B_S}}$.

Proof. The isomorphism is witnessed by $f : \overline{U^B_S} \to U^B$ defined recursively as follows: (let τ be a sharp $\overline{B}$-name)

$$f(\tau) = \begin{cases} a & \text{if for some urelement } a, \tau = \{\langle a, 1\rangle\}. \\ \{\langle f(\eta), \tau(\eta)\rangle \mid \eta \in dom(\tau)\} & \text{if otherwise.} \end{cases}$$

That f is an isomorphism follows easily from 3.2.2 and 3.3.5.

□

We now show that $\overline{U^B_S}$ is an elementary submodel of $\overline{U^B}$. We first need some lemmas.

Lemma 3.3.3.1. Let $\tau \in \overline{U^B}$. Let $A_\tau = ker(\tau) \cup ker(B)$. Then, for some $\{\sigma_i \mid i \in I\} \subseteq \overline{U^B_S} \cap V(A_\tau)$, some maximal antichain $\{p_i \mid i \in I\} \subseteq B$,

$$p_i \leqslant [\![\tau = \sigma_i]\!]^{\overline{U^B}} \text{ for any } i \in I.$$

Proof. We use the induction principle on τ. The inductive hypothesis is that for any $\eta \in dom^B(\tau)$, for some $\{\mu^\eta_j \mid j \in J_\eta\} \subseteq \overline{U^B_S} \cap V(A_\eta)$, some maximal antichain $\{q^\eta_j \mid j \in J_\eta\} \subseteq B$,

$$q^\eta_j \leqslant [\![\eta = \mu^\eta_j]\!]^{\overline{U^B}} \text{ for any } j \in J_\eta.$$

We define $\pi \in \overline{U^B_S}$ as follows:

$$dom(\pi) = \{\mu^\eta_j \mid \eta \in dom^B(\tau), j \in J_\eta\}$$

For any $\nu \in dom(\pi)$, let $X_\nu = \{\langle \eta, j\rangle \mid \nu = \mu^\eta_j, \eta \in dom^B(\tau), j \in J_\eta\}$. Then,

$$\pi(\nu) = \bigsqcup_{\langle \eta, j\rangle \in X_\nu} \tau(\eta) \sqcap q^\eta_j$$

It is easy to check that $\pi \in V(A_\tau)$ and π is a sharp $\overline{B}$-name.

We now show that $[\![\tau \subseteq \pi]\!]^{\overline{U^B}} = 1$. That is, for any $\eta \in dom^B(\tau)$, $\tau(\eta) \leqslant [\![\eta \in \pi]\!]^{\overline{U^B}}$.

This is because

$$\begin{aligned}
[\![\eta \in \pi]\!]^{\overline{U^B}} &= \bigsqcup_{\nu \in dom(x)} [\![\eta = \nu]\!]^{\overline{U^B}} \sqcap x(\nu) \\
&\geqslant \bigsqcup_{j \in J_\eta} [\![\eta = \mu_j^\eta]\!]^{\overline{U^B}} \sqcap x(\mu_j^\eta) \\
&\geqslant \bigsqcup_{j \in J_\eta} [\![\eta = \mu_j^\eta]\!]^{\overline{U^B}} \sqcap \tau(\eta) \sqcap q_j^\eta \\
&= \bigsqcup_{j \in J_\eta} q_j^\eta \sqcap \tau(\eta) = \tau(\eta)
\end{aligned}$$

We next show that $[\![\pi \subseteq \tau]\!]^{\overline{U^B}} = 1$. That is, for any $\nu \in dom(\pi)$, $\pi(\nu) \leqslant [\![\nu \in \tau]\!]^{\overline{U^B}}$. This is because

$$\begin{aligned}
[\![\nu \in \tau]\!]^{\overline{U^B}} &= \bigsqcup_{\eta \in dom^B(\tau)} \tau(\eta) \sqcap [\![\nu = \eta]\!]^{\overline{U^B}} \\
&\geqslant \bigsqcup_{\langle \eta, j \rangle \in X_\nu} \tau(\eta) \sqcap [\![\mu_j^\eta = \eta]\!]^{\overline{U^B}} \\
&\geqslant \bigsqcup_{\langle \eta, j \rangle \in X_\nu} \tau(\eta) \sqcap q_j^\eta = \pi(\nu)
\end{aligned}$$

Finally we observe that $[\![\tau \overset{\mathscr{A}}{=} \pi]\!]^{\overline{U^B}} = \bigsqcap_{a \in \mathscr{A}} \neg\tau(a)$, since π is sharp. Hence $[\![\tau = \pi]\!]^{\overline{U^B}} = \bigsqcap_{a \in \mathscr{A}} \neg\tau(a)$. Also, it is easy to check that for any $a_i \in dom^{\mathscr{A}}(\tau)$, $[\![\tau = \{\langle a_i, 1 \rangle\}]\!] = \tau(a_i)$. Hence the statement holds as $\{\tau(a_i) \mid a_i \in dom^{\mathscr{A}}(\tau)\} \cup \{\bigsqcap_{a \in \mathscr{A}} \neg\tau(a)\}$ is a maximal antichain in B, by the definition of $\overline{B}$-names.

□

Corollary 3.3.3.2. Let $\tau \in U^B$. Then,

$$\bigsqcup_{\sigma \in \overline{U_S^B}} [\![\tau = \sigma]\!]^{\overline{U^B}} = 1$$

Proof. By the previous lemma, for some $\{\sigma_i \mid i \in I\} \subseteq \overline{U_S^B}$, some maximal antichain $\{p_i \mid i \in$

$I\} \subseteq B$, $p_i \leqslant [\![\tau = \sigma_i]\!]^{\overline{U^B}}$ for any $i \in I$. Hence,

$$\bigsqcup_{\sigma \in \overline{U^B_S}} [\![\tau = \sigma]\!]^{\overline{U^B}} \geqslant \bigsqcup_{i \in I} [\![\tau = \sigma_i]\!]^{\overline{U^B}} \geqslant \bigsqcup_{i \in I} p_i = 1$$

□

Corollary 3.3.3.3. $\overline{U^B_S}$ is an elemantary submodel of U^B. That is, for any formula $\phi(v_1, ..., v_n)$, any $\sigma_1, ..., \sigma_n \in \overline{U^B_S}$,

$$[\![\phi(\sigma_1, ..., \sigma_n)]\!]^{\overline{U^B_S}} = [\![\phi(\sigma_1, ..., \sigma_n)]\!]^{\overline{U^B}}$$

Proof. By induction on the complexity of ϕ. The atomic cases are already covered since $\overline{U^B_S}$ is a submodel of U^B. The cases for connectives are straightforward. Let $\phi(v_1, ..., v_n) = \exists v \psi(v, v_1, ..., v_n)$.

$$[\![\exists v \psi(v, \sigma_1, ..., \sigma_n)]\!]^{\overline{U^B_S}} = \bigsqcup_{\sigma \in \overline{U^B_S}} [\![\psi(\sigma, \sigma_1, ..., \sigma_n)]\!]^{\overline{U^B_S}}$$

On the other hand,

$$\begin{aligned} [\![\exists v \psi(v, \sigma_1, ..., \sigma_n)]\!]^{\overline{U^B}} &= \bigsqcup_{\tau \in U^B} [\![\psi(\tau, \sigma_1, ..., \sigma_n)]\!]^{\overline{U^B}} \\ &= \bigsqcup_{\tau \in U^B} \bigsqcup_{\sigma \in \overline{U^B_S}} [\![\psi(\tau, \sigma_1, ..., \sigma_n)]\!]^{\overline{U^B}} \sqcap [\![\tau = \sigma]\!]^{\overline{U^B}} \qquad \text{(by 3.3.3.2)} \\ &= \bigsqcup_{\sigma \in \overline{U^B_S}} [\![\psi(\sigma, \sigma_1, ..., \sigma_n)]\!]^{\overline{U^B}} \end{aligned}$$

Hence the case for quantifiers holds by inductive hypothesis.

□

Corollary 3.3.3.4. U^B is elementarily embedded in $\overline{U^B}$.

Corollary 3.3.3.5. $\overline{U^B} \models \mathrm{ZFCU}_R$. If Collection holds in U, then $\overline{U^B} \models$ ZFCU.

Corollary 3.3.3.6. For each $x \in \mathscr{U}$, define:

$$\check{x} = \begin{cases} \{\langle x, 1\rangle\} & \text{if } x \in \mathscr{A} \\ \{\langle \check{y}, 1\rangle \mid y \in x\} & \text{if } x \notin \mathscr{A} \end{cases}$$

Then all the clauses in 3.2.3 and 3.2.1.2 hold with respect to $\overline{U^B}$, meaning that $\check{x}$ is defined as above[11] and all the occurrences of U^B are replaced with $\overline{U^B}$.

3.4 Collection and Witnessing

In the previous section, we have shown that $\overline{U^B}$ always satisfies the Mixing Lemma. So is $\overline{U^B}$ always witnessing, then? Those who have read Chapter 1 of this dissertation might find this question trivial: after all, there is a theorem in Chapter 1 that says any full Boolean-valued model is witnessing (Theorem 1.7.3), which is equivalent to saying that any Boolean-valued model that satisfies the Mixing Lemma is witnessing.

Nevertheless, the situation here is actually much more complicated. The proof of the theorem that any full Boolean-valued model is witnessing actually makes use of the Axiom of Collection. But recall that our background theory here is ZFCU$_R$, and as mentioned above, Collection is not a theorem of ZFCU$_R$. It turns out that within ZFCU$_R$, given a special universe and complete Boolean algebra, it is possible for $\overline{U^B}$ to satisfy the Mixing Lemma without being witnessing. We will construct counter-examples of this kind in Subsection 3.4.2. But before we do that, we will first prove that over ZFCU$_R$, the Axiom of Collection is actually equivalent to the statement that $\overline{U^B}$ is always witnessing, which is going to be the goal of the upcoming subsection.

3.4.1 Equivalence Between Collection and Witnessing

The major result of this subsection is that:

Theorem 3.4.1. Over ZFCU$_R$, the following are equivalent:

[11] We have given two different definitions of $\check{x}$ in 3.2.3 and 3.3.3.6. But the ambiguity shouldn't matter since the $\check{x}$ in 3.3.3.6 is simply the isomorphic image of $\check{x}$ in 3.2.3 under f.

1. (Collection) $\forall u(\forall x \in u \exists y \phi(x,y) \rightarrow \exists v \forall x \in u \exists y \in v(\phi(x,y)))$.

2. ($\overline{U^B}$-Always-Witnessing) For any complete Boolean algebra B, $\overline{U^B}$ is witnessing.

Theorem 3.4.2. Collection implies $\overline{U^B}$-Always-Witnessing. That is, if $U \models ZFCU$, then for any complete Boolean algebra B, $\overline{U^B}$ is witnessing.

Proof. The same argument as in Chapter 1 Theorem 1.7.3, using the Axiom of Collection and the fact that $\overline{U^B}$ satisfies the Mixing Lemma.

□

Lemma 3.4.2.1. Let B be an atomic complete Boolean algebra and $\tau \in \overline{U^B}$. Let $A_\tau = ker(\tau) \cup ker(B)$. Then,[12]

$$\bigsqcup_{x \in V(A_\tau)} [\![\tau = \check{x}]\!] = 1$$

Proof. Since B is an atomic complete Boolean algebra, it is isomorphic to some powerset algebra $\mathscr{P}(I)$ ordered by $\subseteq$. We prove the lemma using the induction principle on τ. The inductive hypothesis is that for any $\eta \in dom^B(\tau)$, $\bigsqcup_{x \in V(A_\eta)} [\![\eta = \check{x}]\!] = 1$. Therefore, for any $i \in I$, there is a unique $v^i_\eta \in V(A_\eta)$ such that $i \in [\![\eta = \check{v}^i_\eta]\!]$.

For any $i \in I$, we define x_i as follows:

$$\begin{cases} x_i = a & \text{if } a \in dom^{\mathscr{A}}(\tau) \text{ and } i \in \tau(a) \\ x_i = \{v^i_\eta \mid i \in \tau(\eta)\} & \text{if otherwise} \end{cases}$$

That x_i is well-defined follows from the definition of B-names. Since $A_\tau = \bigcup_{\eta \in dom^B(\tau)} A_\eta \cup dom^{\mathscr{A}}(\tau)$, $x_i \in V(A_\tau)$.

We now show that for any $i \in I$,

$$i \in [\![\tau = \check{x}_i]\!] = [\![\tau \subseteq \check{x}_i]\!] \sqcap [\![\check{x}_i \subseteq \tau]\!] \sqcap [\![\tau \stackrel{\mathscr{A}}{=} \check{x}_i]\!]$$

[12] In this subsection and the next, $[\![\phi]\!]$ always means $[\![\phi]\!]^{\overline{U^B}}$.

We first show that

$$i \in [\![\tau \subseteq \check{x_i}]\!] = \prod_{\eta \in dom^B(\tau)} \tau(\eta) \Rightarrow [\![\eta \in \check{x_i}]\!]$$

Let $\eta \in dom^B(\tau)$. If $i \notin \tau(\eta)$, then $i \in \neg\tau(\eta) \subseteq \tau(\eta) \Rightarrow [\![\eta \in \check{x_i}]\!]$. If $i \in \tau(\eta)$, then $v^i_\eta \in x_i$, and hence $i \in [\![\eta = \check{v^i_\eta}]\!] \subseteq [\![\eta \in \check{x_i}]\!] \subseteq \tau(\eta) \Rightarrow [\![\eta \in \check{x_i}]\!]$.

We next show that

$$i \in [\![\check{x_i} \subseteq \tau]\!] = \prod_{v^i_\eta \in x_i} [\![\check{v^i_\eta} \in \tau]\!]$$

Let $v^i_\eta \in x_i$. Then $i \in \tau(\eta)$. Hence $i \in \tau(\eta) \sqcap [\![\eta = \check{v^i_\eta}]\!] \subseteq [\![\check{v^i_\eta} \in \tau]\!]$.

We finally show that

$$i \in [\![\tau \stackrel{\mathscr{A}}{=} \check{x_i}]\!] = \prod_{a \in \mathscr{A}} \tau(a) \Leftrightarrow \check{x_i}(a)$$

Let $a \in \mathscr{A}$. $i \in \tau(a)$ iff $x_i = a$ iff $i \in \check{x_i}(a)$. Hence $i \in \tau(a) \Leftrightarrow \check{x_i}(a)$.

Since for any $i \in I$, $i \in [\![\tau = \check{x_i}]\!]$,

$$\bigsqcup_{i \in I} [\![\tau = \check{x_i}]\!] = 1$$

Hence the lemma is proven as $x_i \in V(A_\tau)$ for any $i \in I$.

□

Corollary 3.4.2.1. Let B be an atomic complete Boolean Algebra. Let $\phi(v_1, ..., v_n)$ be a formula and $x_1, ..., x_n \in U$. Let $[\![\phi(x_1, ..., x_n)]\!]^2 = 1$ iff $U \models \phi(x_1, ..., x_n)$ and $[\![\phi(x_1, ..., x_n)]\!]^2 = 0$ iff $U \models \neg\phi(x_1, ..., x_n)$. Then,

$$[\![\phi(x_1, ..., x_n)]\!]^2 = [\![\phi(\check{x_1}, ..., \check{x_n})]\!]^{\overline{U^B}}$$

Proof. By induction on the complexity of ϕ. The atomic cases are already covered by 3.3.3.6. The cases for connectives are straightforward. Let $\phi(v_1, ..., v_n) = \exists v \psi(v, v_1, ..., v_n)$.

Then,

$$[\![\exists v\psi(v,x_1,\dots,x_n)]\!]^2 = \bigsqcup_{x\in U}[\![\psi(x,x_1,\dots,x_n)]\!]^2$$

On the other hand, since, by 3.4.2.1, every $\tau \in \overline{U^B}$ is such that $\bigsqcup_{x\in U}[\![\tau = \check{x}]\!]^B = 1$,

$$\begin{aligned}
[\![\exists v\psi(v,x_1,\dots,x_n)]\!]^{\overline{U^B}} &= \bigsqcup_{\tau\in\overline{U^B}}[\![\psi(\tau,\check{x}_1,\dots,\check{x}_n)]\!]^{\overline{U^B}} \\
&= \bigsqcup_{\tau\in\overline{U^B}}[\![\psi(\tau,\check{x}_1,\dots,\check{x}_n)]\!]^{\overline{U^B}} \sqcap \bigsqcup_{x\in U}[\![\tau = \check{x}]\!]^{\overline{U^B}} \\
&= \bigsqcup_{x\in U}[\![\psi(\check{x},\check{x}_1,\dots,\check{x}_n)]\!]^{\overline{U^B}} \\
&= \bigsqcup_{x\in U}[\![\psi(\check{x},\check{x}_1,\dots,\check{x}_n)]\!]^2 \\
&= [\![\exists v\psi(v,x_1,\dots,x_n)]\!]^2
\end{aligned}$$

The second to the last equation holds by inductive hypothesis.

□

Theorem 3.4.3. $\overline{U^B}$-Always-Witnessing implies Collection. That is, for any U that satisfies $ZFCU_R$, if for any complete Boolean algebra B, $\overline{U^B}$ is witnessing, then $U \models ZFCU$.

Proof. Assume the antecedent of Collection, that is, $U \models \forall x \in u\exists y\phi(x,y)$. Let $B = \mathscr{P}(u)$. By 3.4.2.1,

$$\overline{U^B} \models \forall x \in \check{u}\exists y\phi(x,y)$$

Clearly $\{\{x\} \mid x \in u\} \subseteq \mathscr{P}(u)$ is an antichain. Let μ be the mixture of $\{\{x\} \mid x \in u\}$ with respect to $\{\check{x} \mid x \in u\}$. By the Mixing Lemma, then, for any $x \in u$,

$$x \in [\![\mu = \check{x}]\!]$$

Hence, $[\![\mu \in \check{u}]\!] = \bigsqcup_{x\in u}[\![\mu = \check{x}]\!] = 1$. Therefore, $[\![\exists y\phi(\mu,y)]\!] = 1$.

Since $\overline{U^B}$ is witnessing, there is some $\tau \in \overline{U^B}$ such that $[\![\phi(\mu,\tau)]\!] = 1$. By 3.4.2.1,

$$\bigsqcup_{y \in V(A_\tau)} [\![\tau = \check{y}]\!] = 1$$

Let $x \in u$. Then for some $y \in V(A_\tau)$, $x \in [\![\tau = \check{y}]\!]$. Hence

$$x \in [\![\mu = \check{x}]\!] \sqcap [\![\phi(\mu,\tau)]\!] \sqcap [\![\tau = \check{y}]\!] \subseteq [\![\phi(\check{x},\check{y})]\!] \tag{3.9}$$

But $[\![\phi(\check{x},\check{y})]\!]$ is either 0 or 1 by 3.4.2.1. Hence $[\![\phi(\check{x},\check{y})]\!] = 1$. By 3.4.2.1 again, $U \models \phi(x,y)$. Hence,

$$U \models \forall x \in u \exists y (ker(y) \subseteq ker(\tau) \cup \mathrm{ker}(B) \wedge \phi(x,y))$$

By Replacement, it follows that

$$U \models \exists v \forall x \in u \exists y \in v (\phi(x,y))$$

□

3.4.2 A Non-Atomic Example

The proof of 3.4.3 suggests that if the Axiom of Collection fails in U, then there is an atomic complete Boolean algebra B such that $\overline{U^B}$ is not witnessing, though satisfying the Mixing Lemma. To see this, suppose for some formula ϕ, there is some $u \in U$ such that $U \models \forall x \in u \exists y \phi(x,y)$, but $U \models \neg \exists v \forall x \in u \exists y \in v \phi(x,y)$. Then we take $B = \mathscr{P}(u)$. Let μ, again, be the mixture of $\{\{x\} \mid x \in u\}$ with respect to $\{\check{x} \mid x \in u\}$. By the same reasoning as in 3.4.3, $[\![\exists y \phi(\mu,y)]\!] = 1$. But there cannot be any $\tau \in \overline{U^B}$ such that $[\![\phi(\mu,\tau)]\!] = 1$, since otherwise $U \models \exists v \forall x \in u \exists y \in v \phi(x,y)$, using the same argument in 3.4.3.

A unresolved question is: can the failure of being witnessing happen for some non-atomic complete Boolean algebra B? That is, can there be some universe U of ZFCU_R such that for a non-atomic complete Boolean albebra B, $\overline{U^B}$ is not witnessing? The answer turns out to be positive. In this subsection, we give such an example.

For the rest of this subsection we assume U is a universe that satisfies ZFCU$_R$, that there is no infinite set of urelements, and that for any $n \in \omega$, there is a set of urelements of size n.[13] We also let $B = RO(2^\omega)$. The main result of this subsection is that $\overline{U^B}$ is not witnessing, despite satisfying the Mixing Lemma.

Definition 3.4.1. We define $\gamma \in \overline{U^B}$ as follows:

$$dom(\gamma) = \{\check{n} \mid n \in \omega\}$$

and for any $n \in \omega$,

$$\gamma(\check{n}) = \{f \in 2^\omega \mid f(n) = 1\}$$

Proposition 3.4.1. Let $n \in \omega$, $f \in 2^\omega$. Then,

$$[\![\check{n} \in \gamma]\!] = \{f \in 2^\omega \mid f(n) = 1\}$$

$$[\![\check{n} \notin \gamma]\!] = \{f \in 2^\omega \mid f(n) = 0\}$$

Lemma 3.4.3.1. $\overline{U^B} \models \exists x(\forall y \in x(\mathscr{A}(y)) \wedge \exists n \in \check{\omega}(|x| = n \wedge n \notin \gamma))$

Proof. Since the sentence that for any $n \in \omega$, there is a set of urelements of size n is Σ_1, $\overline{U^B}$ "thinks" that for any $n \in \check{\omega}$, there is a set of urelements of size n. Hence we only need to show that

$$\overline{U^B} \models \exists n \in \check{\omega}(n \notin \gamma)$$

That is, $\bigsqcup_{n\in\omega} [\![\check{n} \notin \gamma]\!] = 2^\omega$. Let $f \in 2^\omega$. If $f(n) = 0$ for some $n \in \omega$, then $f \in [\![\check{n} \notin \gamma]\!]$. Hence,

$$\bigsqcup_{n\in\omega} [\![\check{n} \notin \gamma]\!] = (\overline{\{f \in 2^\omega \mid \text{for some } n \in \omega, f(n) = 0\}})^\circ = 2^\omega$$

□

Theorem 3.4.4. $\overline{U^B}$ is not witnessing.

[13]The existence of such a universe is relatively well-known. For details, see, for example, [37].

Proof. Assume for reductio that $\overline{U^B}$ is witnessing. By the previous lemma, if suffices to show that there is no $\tau \in \overline{U^B}$ such that

$$\overline{U^B} \models \forall y \in \tau(\mathscr{A}(y)) \wedge \exists n \in \check{\omega}(|\tau| = n \wedge n \notin \gamma)$$

Suppose otherwise. Let $\tau \in \overline{U^B}$ be such that $\overline{U^B} \models \forall y \in \tau(\mathscr{A}(y)) \wedge \exists n \in \check{\omega}(|\tau| = n \wedge n \notin \gamma)$. Since there are no infinite sets of urelements in U, $ker(\tau) = m$ for some $m \in \omega$. Let $ker(\tau) = \{a_1, ..., a_m\}$ where $a_1, ..., a_m \in \mathscr{A}$.

Claim 3.4.4.1. $\overline{U^B} \models \forall x \in \tau(y = \check{a}_1 \vee ... \vee y = \check{a}_m)$.

Proof of the Claim. Since $\overline{U^B} \models \forall y \in \tau(\mathscr{A}(y))$, for any $\eta \in dom^B(\tau)$, $\tau(\eta) \leqslant [\![\mathscr{A}(\eta)]\!] = \bigsqcup_{a \in \mathscr{A}} \eta(a) = \bigsqcup_{a \in \mathscr{A}} [\![\eta = \check{a}]\!]$, by the definition of B-names. Let $b \in \mathscr{A}$ be such that $b \neq a_i$ for any $i \leqslant m$. Then $[\![\eta = \check{b}]\!] = \eta(b)$ has to be 0, as otherwise $b \in ker(\eta) \subseteq ker(\tau)$.

Hence $\tau(\eta) \leqslant [\![\mathscr{A}(\eta)]\!] \leqslant \bigsqcup_{i \leqslant m} [\![\eta = \check{a}_i]\!]$. Hence $\overline{U^B} \models \forall x \in \tau(y = \check{a}_1 \vee ... \vee y = \check{a}_m)$. ■

Therefore, $\overline{U^B} \models |\tau| \leqslant \check{m}$. Let $M = \{f \in 2^\omega \mid \text{for any } i \leqslant m, f(m) = 1\} \in RO(2^\omega)$. Obviously $M \neq \varnothing$. Hence there is some ultrafilter $D \subseteq RO(2^\omega)$ such that $M \in D$. For any $i \leqslant m$,

$$[\![\check{i} \in \gamma]\!] = \{f \in 2^\omega \mid f(i) = 1\} \supseteq M \in D$$

Since $\overline{U^B} \models \exists n \in \check{\omega}(|\tau| = n \wedge n \notin \gamma)$, and since $\overline{U^B}$ is witnessing, for some $\sigma \in \overline{U^B}$, $[\![|\tau| = \sigma \wedge \sigma \leqslant \check{m} \wedge \sigma \notin \gamma]\!] = 1$. Since $\bigsqcup_{i \leqslant m} [\![\sigma = \check{i}]\!] = [\![\sigma \leqslant \check{m}]\!] = 1 \in D$, for some $i \leqslant m$, $[\![\sigma = \check{i}]\!] \in D$.

Hence $[\![\sigma \in \gamma]\!] \in D$ as $[\![\sigma \in \gamma]\!] \geqslant [\![\sigma = \check{i} \wedge \check{i} \in \gamma]\!]$. But $[\![\sigma \notin \gamma]\!] = 1$. Contradiction. □

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