MAXIMUM LIKELIHOOD DETECTION FOR PROBABILISTIC MODELS OF OPTICAL CODE DIVISION MULTIPLE ACCESS CHANNELS

by

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Submitted in Partial Fulfillment
of the Requirements for the Degree
of
DOCTOR OF PHILOSOPHY
in
ELECTRICAL ENGINEERING AND COMPUTER SCIENCE
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
April 1988

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April 1, 1988

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The aim of this thesis is to investigate Maximum Likelihood detection for optical Direct Sequence Spread Spectrum signals in Code Division Multiple Access channels using incoherent optical detectors. We consider detection of one among several deterministic pulse patterns, or "signatures", in the presence of interference from a random number of other users. Four stochastic models for interfering signatures are explored under two different assumptions: that detectors only recognize ON-OFF signals, and that detectors can recognize multilevel signals. Also, three distinct timing situations for the signatures are considered.

For Discrete Time models the Maximum Likelihood detector for several situations of interest turns out to be a "Template detector". A Template detector essentially looks for a specific pulse pattern in the received signal. It has previously been proposed in the literature but without reference to its Maximum Likelihood properties. It can be implemented with Optical Delay lines. Two other interesting, rather simple, Maximum Likelihood detection rules apply to two signature models under one of the timing situations considered. Probability of error for the Template detectors are treated for several models. Bounds for the probability of error of Maximum Likelihood detectors for two models are also presented. For Continuous Time models, we derive the probability of error for one of the interference models and in one simple but important situation relate it to the Discrete Time model.

Sufficient conditions, in terms of minimum post detection integration time, for optical intensities of signals to add up are derived. This additivity property is needed for a detector to operate with multilevel signals. Another consideration discussed is the path dependent attenuation of optical intensity. This makes operation with multilevel signals infeasible in a passive network, in general. The results requiring multilevel signals serve mainly as a theoretical guideline for design.

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ACKNOWLEDGEMENTS

This acknowledgement is intended to express my deep gratitude to all the people that have contributed in some way towards my Doctoral dissertation.

The burden of steering me through graduate research fell on Professor Pierre Amedee Humblet, whom I am indebted to most of all. He has contributed to this work with ideas, guidance, supervision and encouragement. He has also challenged me with high standards. Professors Robert Gallager and Robert Kennedy, my thesis readers, also helped improve the quality of this dissertation with their supervision. The final responsibility for the correctness of this document, however, rests on me.

Belonging to the Local Communication Networks Group at MIT has been a pleasant experience, even considering the many frustrations of research. The opportunity to think creatively on a challenging problem has been one reason. Another reason has been the friendly atmosphere among students and faculty. I would like to thank them all for many interesting discussions on all topics and for their camaraderie.

I thank NYNEX, The Advanced Research Projects Agency (DoD) and The Organization of the American States for funding my graduate work since my Master's Degree.

My office at MIT has not been the same since Erdal Arikan, Isidro Castiñeyra and Edward Tiedemann committed graduation. Technical discussions do not last nearly as many hours as they used to.

Several special friends helped me retain my mental sanity. Trips to Cape Cod while my supervisor was not looking were great fun. Karen Adler is responsible for making it a summer tradition. Grace Chang introduced me to Buffalo wings and suggested I drop by the gym once in a while. I wish her well in her work.

It is an honour to be an Alumni of the *108 Gourmet Cooking Society of Ashdown House.* I am a contemporary of such luminaries as Maurice Borgeaud, Jolene Kiolbassa, Janina Masojada, the recent Carey Rappaport, and the great Stephen Evangelides. I would like to apologize once more for the Vinegar Steak, which enters history with this page.

Eduardo and Deborah Grimaldo shared with me the aprehension of difficult times for our beloved country. I wish them success in their future. Julie Anderson, Charlene Kuo, Jim Smelt and Mary Phillips are also thought of fondly.

I hope Stephen Evangelides has forgiven me, along with others, for his birthday surprise. Jolene Kiolbasa will have to show me one day the results of her own birthday present. Maurice Borgeaud has been a close friend for years. (I look forward to seeing you again; even if it means a return to pedestrian life.)
...And of course, Diane Gaylor. Thank you.

My parents and relatives have encouraged me through these years. I wish I could have shared more moments with them. This work is dedicated to them.

TABLE OF CONTENTS

FIGURES

CHAPTER 1
INTRODUCTION

This work focuses on Maximum Likelihood detection for probabilistic models of Code Division Multiple Access systems in Fiber Optic Local Area Networks.

Inspection of a Loss vs. Wavelength characteristic for optical fibers shows that bandwidths in excess of 100THz (1THz=1Terahertz=10^{12}Hz) are available for communication with losses of approximately 1db/Km [S I]. However, with foreseeable technology, electronic processing speed will limit the maximum rate at which a given user can operate on the network. Signal detection methods employing simple electronics and exploiting optical processing become relevant. Use of the available bandwidth must also be addressed. The bandwidth is so large that a single user could only make use of a small fraction of it. However, this bandwidth could be comfortably shared among users. Multiple Access systems are protocols that allow several users to share a common communication channel. Efficient bandwidth utilization, a major design principle in channel access protocols like "Aloha" [Abramson] or "tree algorithms" [Capet], stops being the main concern. Simple access protocols like Frequency Division Multiple Access systems and Code Division Multiple Access (CDMA) systems, which are not primarily concerned with the possibility of collision events and efficient recovery from them, are important possibilities. These two protocols need not require network switching functions other than end-to-end frequency, or code, selection at the receiver or transmitter nodes. This is important since network functions like active routing and/or store and forward operations are also likely to limit unnecessarily the maximum user rates due to the information processing and decision making involved. Another important consideration is the relative cost of present day active optical devices, e.g. optical sources and detectors, with respect to the rest of the hardware. Until coherent optical technology is mastered and becomes cost effective, simple Direct Detection technology may offer an attractive alternative in many situations.

Bearing the previous discussion in mind, our work focuses on analysing models for CDMA systems using Direct Sequence Spread Spectrum signals [C M^c] in a Local Area Network (LAN) using Direct Detection optical technology [N R]. These signals consist of optical pulse patterns. Because we have a multi-user environment we are interested in the design of good detectors in the presence of interuser interference. A commonly proposed detector for these systems is the Optical Delay Line Correlation Detector [P S F], [C S W]. This detector taps the optical medium with a fiber. The fiber is coupled into a detector at several points of its length. The distances between these coupling points correspond to the relative pulse delays of the pattern to be detected. The detector receives the superposition of the signals present at these coupling points and converts it to an electrical signal for threshold detection. With Light Emitting Diode sources, for example, the average optical intensity at the output is roughly proportional to the sum of the average intensities present at the tap points, so when a threshold corresponding to the number of pulses present in the signal pattern is reached or exceeded at the output the signal is detected.

Searching for a better performance detector we have introduced four probabilistic models to describe the interuser interference and asked what the Maximum Likelihood detection rules for these models are. The models aim at tractability since the precise description of interuser interference depends on the structure of each of the patterns employed by each user and requires consideration of all possible pattern combinations that could be jointly placed on the channels by all users. It turns out for these models that the Maximum Likelihood (M.L.) detection rule under many situations of interest can also be implemented using a tapped delay line arrangement and similar threshold detection technology as for the Optical Delay Line Correlation Detector above. The probability of decoding error however seems to improve appreciably. In the context of the previous discussion these results point towards a simple but effective detector design for these CDMA signals, contributing thus towards simplicity of electronic detection in a relatively simple access protocol.

In this chapter we will review the main results of this thesis. Section 1.1 discusses some of the assumptions implied in the models used for analysis. Section 1.2 contains an outline of the main results and their meaning.

1.1 SYSTEM MODEL

In the interest of analysis, we model a CDMA system probabilistically. We concentrate on the dynamics of interuser interference and hence our model neglects several important noise and signal distortion processes which will be discussed below. Several of these processes may not be very relevant since a LAN environment is expected to be benign. Other processes like path dependent signal attenuation due to couplers excess loss will merit more careful consideration.

The system we want to model is as follows. A "session" in the network consists of a transmitter sending information to a receiver. The receiver has initially achieved synchronization with the transmitter. The transmitter then encodes information bits into optical pulse patterns which we call "signatures" and broadcasts them on the optical channel. The receiver of a session i must detect which of the possible signatures in its session is sent at each transmission interval to reconstruct the bit stream. Other concurrent sessions use different sets of signatures and interfere with session i. We assume that only one transmitter transmits to a given receiver at any given time[1]. We assume that apart from the source and destination users no other information processing nodes, e.g. central controllers or relay nodes, intervene in a session.

[1] A new transmitter who tries to send information to an already busy receiver can be assumed out of synchronism with such receiver; in which case its signature is also treated as an interfering signature. We assume that if by chance new transmitters with a common destination receiver happen to be in synchronism with the busy receiver the receiver finds a way to silence one of the contending transmitters.

Chapter 2 contains the precise definitions to be used in later chapters. For the sake of discussing the results in this chapter some of these definitions will be informally given in the rest of this section.

Noise and Distortion. The fiber medium can be considered virtually noise free. For the relatively short distances in a Local Area Network, fibers will introduce negligible distortion. Coupling attenuation and coupler excess loss do present a problem. The optical intensity of a pulse is attenuated as it travels through different passive optical couplers [Personick]. We assume that, for any communicating transmitter receiver pair, the received optical power is within the operating range of the detector. Since coupling loss and excess loss vary from coupler to coupler, the attenuation of a pulse depends on the path taken through the network. Signature pulses from different transmitters will experience different attenuations according to the topology of the fiber medium and the uniformity of the coupling devices used. Some of the analysis in chapters 3 and 4 assume that all optical pulses arriving at a receiver have equal intensity regardless of their origin in the network. The limitations of this assumption are discussed in Chapter 5. Theoretically one could envision a system using optical amplifiers after each coupler stage such that the gain compensates for the loss through the stage. The amplifier will introduce noise in the signal however so such a system would pose its own problems. Rather than argue for the feasibility of a system where all received pulses have equal intensity we state this as an assumption and later consider its implications.

Most of the noise processes come from the optical receiver operation. A clear treatment of these processes can be found in [Shapiro]. Thermal noise at the receivers can be modeled as Additive White Gaussian noise. Dark current shot noise can be modeled as an Inhomogeneous Poisson process, where electron charges are emitted at the device output at instants generated by a Poisson process of time varying rate $\lambda_D(t)$. For Avalanche Photodetectors the device gain is subject to unpredictable fluctuations which can be modeled by a multiplicative noise process. It is an implicit assumption in this work that the Signal to Noise Ratio in the Local

Area Network considered is high enough so that these noise processes are not a major concern. Once more, we are trying to focus on interuser interference.

Direct Detection. Ignoring the noise, the output of optical detection devices due to incident light can also be appropriately modeled by an Inhomogeneous Poisson process of rate $\lambda_I(t)$ proportional to the instantaneous value of optical intensity, $I(t)$, incident on the device at time t. The arrivals represent electric charges generated. Ideally, assuming no receiver noise, a detector would fail to detect the presence of an optical pulse only if during the observation interval the Poisson process generates no charges. However, as the optical intensity of the pulse increases, the probability of such an event tends to zero. It will be an implicit assumption in our models that this probability can be assumed zero, unless stated otherwise.

Signatures. We model the interfering signatures as a sequence of pulses placed on a time "frame" consisting of n contiguous time intervals, or "slots", of equal duration. This aims at modeling a Direct Sequence Spread Spectrum signal. The way in which the pulses are placed on these slots obeys a probabilistic description. Two main descriptions will be used for signatures: "Bernoulli signatures" and "Roulette signatures". A Bernoulli signature is generated by placing, or not, a pulse with probability p, or $1 - p$, independently on each slot of the frame. A Roulette signature is generated by placing a total of a pulses in a frame; each of these pulses is placed on a slot chosen equiprobably out of the n slots, independently of all other pulse positions. One other special description that will be used will be called a "null signature." A null signature has no pulses in its frame. By contrast we refer to signatures whose probabilistic description allows pulses in the frame as non null signatures. The null signature will be used together with other non null signatures to form encoding systems as explained further below. A model in which non null interfering signatures are Bernoulli signatures will be called a "Bernoulli scheme". A "Roulette scheme" is defined similarly.

At each frame interval an active transmitter chooses one of α possible "letters"[2] and encodes it into a signature. We will be interested in two encoding schemes. In an "asymmetric scheme" one of the transmitter letters is encoded into a null signature; all other letters are encoded using a given signature scheme. In a "symmetric scheme" all transmitter letters are encoded using a given signature scheme and no null signature is used. The work in chapters 3 and 4 in fact considers only asymmetric schemes with two signatures (i.e. $\alpha = 2$) or symmetric schemes with integer $\alpha \geq 2$. This is because we expect asymmetric schemes to perform as symmetric schemes as α becomes large, provided that the probability of using the null signature is small compared to the sum of the probabilities for the non null signatures. We stress at this point that the pulse patterns for signatures of the given session i are considered deterministic.

Synchronism. Most of our work will be on Discrete Time models. By this we mean that slots for all users are assumed in perfect synchronism. We contemplate two possibilities: "Frame Synchronous" systems, in Chapter 3, and "Discrete Time Asynchronous" systems, in Chapter 4. The latter refers to a Discrete Time model where frames of different users may be asynchronous. Chapter 4 also contains a section on "Continuous Time Asynchronous" systems which assumes no synchronization.

In real life many systems will be "Continuous Time Asynchronous", where slots will not be in synchronism. In such cases our Discrete Time models are then a simplification of the situation for the sake of tractability. The section on Continous Time Asynchronous systems treats only one of the models under study but serves to give insight on the situation. In fact Discrete Time Asynchronous systems have been proposed [P S S]. Slot synchronism is maintained by broadcasting a clock signal on the network. This would present problems, however, for networks of high clock rates and many users, since the signal propagation delay varies depending on the transmitter receiver pair locations. Two sessions which are slot synchronous at one point in the network may not be at another.

[2] Letters can be thougth of as blocks of $m = log_2\alpha$ information bits when m is an integer.

Channel model. Assuming a noiseless, distortionless physical channel, with lossless coupler stages, and assuming that the probability of generating no charges when light is incident on the detector equals zero, no pulse erasures are possible and all pulses arrive at a receiver with equal intensity. A transmitter in a session sends signature frames to the session receiver. Superposed on this frame are the signature pulses placed by interferers, according to the encoding signature scheme chosen in the network. We will work on two types of channel descriptions: "OR channel" and "ADDER channel." By "OR channel" we mean that a receiver can decide, at every slot, whether the slot is empty or occupied. This definition attempts to describe a Direct Detection system which simply tries to detect photons in a slot. By "ADDER channel" we mean that the receiver is capable, at every slot, of knowing exactly how many pulses are present in the slot. —The meaning of these channel definitions in a Continuous Time Asynchronous model will be discussed in more detail in section 4.2.— The definition attempts to describe a Direct Detection system which compares the optical energy in a slot against a threshold scale to decide how many pulses are present. Notice that this implies that optical power can be added. In fact the fiber medium, save for small non linear effects [Abernathy], is linear in the electric field while optical intensity is proportional to the square of the field magnitude. However, the time average of the total optical intensity is, under certain conditions, well approximated by the sum of the individual intensities of the overlapping pulses. Conditions like frequency separation or broad linewidths (e.g. LED sources) of the overlapping optical sources permit an ADDER channel assumption to be valid. Yet another problem for this assumption is the path dependent pulse attenuation in passive optical media as discussed above. The Optical Delay Line Correlation Detectors mentioned above assume implementation of an ADDER channel. So do some of our results in chapters 3 and 4. For these reasons we discuss these optical channel issues in Chapter 5.

1.2 OUTLINE OF RESULTS

This section serves both as a thesis outline and as a review of the main results.

The different schemes explained in section 1.1 were combined to model interesting design alternatives to CDMA Direct Detection Optical systems. For each combination we tried to investigate the Maximum Likelihood detection rule. Table 1 contains a hierarchy of features that may be useful for future reference.

TABLE 1

- BERNOULLI SCHEMES
  - 2-signature asymmetric scheme
    - OR channel
    - ADDER channel
  - many signature symmetric scheme
    - OR channel
    - ADDER channel
- ROULETTE SCHEMES
  - 2-signature asymmetric scheme
    - OR channel
    - ADDER channel
  - many signature symmetric scheme
    - OR channel
    - ADDER channel

Chapter 1. This is an introductory chapter. It presents the motivation for this research, a discussion of the main assumptions made and a summary of the results.

Chapter 2. In this chapter the main definitions to be used are given and the models for the Code Division Multiple Access systems are introduced. Several of these definitions are more rigorous versions of the terms previously introduced in section 1.1.

Chapter 3. Chapter 3 treats the Frame Synchronous situation. The first result is Lemma 3.1 which treats 2-signature asymmetric schemes on the OR channel. Loosely speaking it states that no matter what model we choose for the non null interfering signatures, the M.L. detector for a session is a Template detector, defined presently.

A Template detector checks if all slots for which the non null signature of the session should

have pulses are occupied in a received frame. If all such slots are occupied the decision is for the non null signature. If any such slot is empty the decision is for the null signature.
Some reflection will reveal that an Optical Delay Line Correlation detector, with threshold equal to the number of pulses in the non null signature, and a Template detector have equal probability of error performance in an OR channel model. The result in Lemma 3.1 is rather a statement on optimality[3], bearing in mind the simplifying assumptions made. This result is not restricted to Bernoulli or Roulette schemes, and applies also to Discrete Time Asynchronism. For Frame Synchronous symmetric Bernoulli and symmetric Roulette schemes, OR channel, the M.L. detector also turns out to be a Template detector when all signatures of the session i of interest have an equal number of pulses. The definition of the Template detector for a symmetric scheme is a natural extension of the one stated. It considers as feasible signatures those for which all its pulse positions are occupied in the received signal. It then decides equiprobably among all feasible signatures.

Theorems 3.1 and 3.2 treat the 2-signature, asymmetric Bernoulli scheme and 2-signature, asymmetric Roulette scheme, respectively, for the ADDER channel model. Loosely speaking they state that under certain "load conditions" the M.L. detector for these schemes is a Template detector.

The load conditions are as follows. Let $M+1$ be the maximum possible number of network users. This implies that session i can experience a maximum of M "active" interfering sessions. Let p be the pulse probability for a Bernoulli scheme. The load condition in Theorem 3.1 is

$$(M+1)p \leq 1$$

The condition is also a necessary condition for the Template detector to be a M.L. detector if this equivalence is to apply regardless of assumed probability distribution for the number of active interfering sessions in the network and regardless of received pulse pattern. If the

[3] For properties of a Maximum Likelihood Detector see for example [Van Trees].

load condition above is not met then, in general, a M.L. detector would need to evaluate the likelihood function for one or more received pulse patterns.

For the Roulette scheme, let a be the number of pulses per frame and n be the number of slots per frame. We assume a Binomial distribution for the number of active interfering sessions and also assume that the probability that an active session sends the non null signature is at most one half. The load condition in Theorem 3.2 is

$$(M+1)\left(\frac{a}{n}\right) \leq 1$$

This condition comes close to being a necessary condition for the Template detector to be a M.L. detector in a sense similar to the condition above for the asymmetric Bernoulli scheme[4]. Notice however that a specific probability distribution for the number of active interfering sessions and for the signature probabilities has been assumed. The assumptions can be relaxed, but this does not lead to a simple characterization.

A Template detector operating on an ADDER channel would be expected to have lower probability of error than an Optical Delay Line Correlation detector since it limits the number of pulses detected per slot to at most one. Template detectors have been proposed in the literature in the context of optical direct detection systems. Joseph Hui [Hui] analyses the probability of error for one such system on the assumption that the number of interfering pulses per slot obeys a Poisson distribution[5]. Our results show that the Template detector is also a Maximum Likelihood detector for specific probabilistic models of CDMA systems in several situations of interest. Notice that if we considered detection of signatures in the presence of Additive White Gaussian noise and no interference then a Correlation detector would be the Maximum Likelihood detector.

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[4] See discussion in section 3.2.1. The bound is not as tight as before, so it turns out not to be, strictly speaking, a necessary condition. Moreover, the tightness of the bound depends on the value of a so, in this sense, it is somewhat less pleasing than before.

[5] Pulse arrivals over a frame obey a homogeneous Poisson Process.

These results suggests that under several situations of interest an ADDER channel is in fact not relevant and hence we can dispense with the problems involved in trying to implement it.

For Frame Synchronous symmetric schemes on an ADDER channel the results are basically as follows. For symmetric Bernoulli schemes the M.L. detection rule has a straightforward expression if the number of active sessions on the network is known to the detector. It involves detection of the number of pulses present in each of several selected slots and computing a metric based on these numbers. The operation is carried out for each feasible signature (i.e. those for whom no due pulses are missing) using a different set of slots for each. The decision is for the feasible signature with maximum metric. The expression for the rule is stated in Lemma 3.2. When the number of active sessions is not known, we have been unable to reduce the M.L. detection rule to a simple structure. We argue through an example that such a simplified structure is not likely to apply.
For a symmetric Roulette scheme a straightforward rule also applies and it is stated in Lemma 3.3. The rule is as for the Bernoulli scheme but this time the metric is just the product of the number of pulses present in the selected slots. The decision is again for the signature with minimum metric. Notice that detection of the number of pulses present in a slot is the same requirement needed in an Optical Delay Line Correlation Detector. The ability to form the product (or add logarithms) would need to be implemented.

This discussion summarizes the Maximum Likelihood detection results in Chapter 3. The chapter also presents numerical evaluations for probability of detection error expressions for the Template detector and Optical Delay Line Correlation detector — which in subsequent chapters will be called "Sum" detector —. For these evaluations we assumed equiprobable source letters and a Binomial probability distribution for the number of active interfering sessions. It was also possible to evaluate upperbounds for the probability of error of a Maximum Likelihood detector for the Roulette schemes on an ADDER channel.
We considered only the 2-signature case for the four schemes above. Let h denote the number

of pulses in the signatures of session i. For Bernoulli schemes we set the pulse probability parameter p for interference sessions to $p = h/n$, where n is the number of slots per frame. For Roulette schemes we set the number of pulses per signature a to $a = h$. These values are chosen for consistency between the session i of interest and the interfering sessions.

Numerically we explored the behaviour of the probability of detection error P_e for the Template detector and the Sum detector as a function of the number of pulses per signature h, the maximum number of interfering sessions M, the probability q of a session being active, and of the number of slots per frame n. On an ADDER channel, the Template detector shows lower values of P_e than the Sum detector for given h, M, q, n. The difference in values seems significant. For fixed M, q, n, an optimal value of h is observed. The behaviour of this optimal value was explored further. Note that a Template detector or a Sum detector operating on an OR channel are equivalent to a Template detector operating on an ADDER channel.
We also compare the performance of the four signature schemes on an ADDER channel when a Template detector is used. Roulette schemes show lower values of P_e than corresponding Bernoulli schemes. Asymmetric schemes show lower values of P_e than corresponding symmetric schemes.

Chapter 4. In this chapter we treat the Discrete Time Asynchronous and Continuous Time Asynchronous models.

In a Discrete Time Asynchronous model a signature frame of an interfering session can overlap two consecutive frames of the session i of interest. This implies that the channel is not memoryless, that there is possible dependence between slots of adjacent frames and that the probabilistic description of the interference may be different between consecutive slots in the same frame of session i.

For the OR channel we have been able to show that the Template detector and the M.L. detector are the same for three of the four schemes above. For the symmetric Roulette scheme we have not been able to reduce the M.L. rule to a simple expression.

For the ADDER channel the conclusions are as follows. The symmetric Bernoulli scheme was treated first because it is equivalent to the Frame Synchronous case if we assume steady state sessions. Thus the results of the Synchronous case apply. The main result of this chapter is for the asymmetric Bernoulli scheme and is stated in Theorem 4.1. It extends the result of Theorem 3.1 to the Asynchronous model. Writing down the probability of error however is a difficult task because the channel is not memoryless and different time shifts between frames of different sessions need to be taken into account. Since the possibilities are too numerous we have not included probability of error analysis.

For Roulette schemes we have been unable to derive a straight forward expression for the M.L. detection rules. We include a discussion of the main difficulties to be encountered in trying to derive these rules. The comment above regarding probability of error analysis also applies in these cases so we have omitted it again.

A Continous Time Asynchronous model is less tractable. It presents at least the same problems encountered in a Discrete Time Asynchronous model. We have not attempted to treat Maximum Likelihood detection in this case. Instead we tried to express the probability of error for a Template detector. The definition for a Template detector is as for the Discrete Time model but we have to be more precise on what detecting the presence of a pulse means. We have assumed that the optical power received is very high, a point which is explained in detail under the heading "High Power assumption" in section 4.4. In this case a pulse is detected present even if it only partially overlaps the detection slot. We have obtained results for the symmetric Bernoulli scheme. The exact probability of error expression is derived. Since the expression is somewhat involved we have considered the simple but interesting situation of signatures consisting of isolated pulses. In this case we can show that the probability of error, experienced by the Template detector, is as if the probability of placing a pulse in a slot for the active interfering sessions had roughly twice the value of the equivalent Discrete Time model. Better detector performance is obtained if we assume that after optical to electrical conversion, the slot detector hardlimits the output current and then integrates over a slot time to determine the presence of a pulse. In this case an interfering pulse must arrive at the slot detector in near synchronism for detection to be triggered. In this case the probability of error for the Template detector in the Discrete Time Asynchronous model serves as an upperbound for the probability of error of the Continuous Time Asynchronous situation.

Chapter 5. In this chapter we return to the assumptions underlying an ADDER channel model. These assumptions are, first, that the detector can distinguish how many optical sources are present in the received optical signal at a given time and, secondly, that all pulses received at a detector in the network arrive with equal intensity regardless of their source and of their path through the optical medium. In section 5.1 we discuss the conditions under which the first assumption is valid. We consider $N+1$ optical emissions on a linear medium, each of constant amplitude, constant center frequency and a random time varying phase. The spatial dependence of the waves is ignored for simplicity. The phase is a stochastic Wiener Levy process, an accepted model for optical sources [Lax], which implies a Lorentzian shape for the spectral density of the source. We assume that optical sources are drawn from a batch whose smallest possible linewidth is ω_L. We consider optical energy detectors with post detection integration over an interval of length T. The problem to be addressed is the fact that the medium is linear in the source electric field emitted and not in the field intensity, which is proportional to the square of the field amplitude.

The main result of this chapter appears in Theorem 5.3. It states that the integrator output will tend to the sum of the intensities of the sources if the integration interval satisfies

$$T \gg \frac{(N+1)N}{\omega_L}$$

This insures both that the expected value of the output is close to $N+1$ and that the standard deviation is much less than $1/2$, so that the number of sources present can be reliably obtained by comparing the output to a linear threshold scale. The result corresponds to a worst case situation where all center frequencies are chosen equal and all source linewidths have their smallest possible value ω_L. It will often be the case that the frequency separation of different

sources is such that the spectra of the sources are non overlapping, specially if their linewidth is very small, and the ADDER channel assumption is still valid on this account.
For an optical CDMA system the value of $N+1$ will typically be only a small fraction of $M+1$, the number of users in the network, since not all of them will typically be active and among those who are their pulses need not necessarily overlap. For Continuous Time Asynchronism pulses may only partially overlap so the duration of the overlap region would also need to be considered as a limiting factor for the intensity of this region to be registered by the detector as a sum of intensities.

In section 5.2 we discuss a problem presented by the Poisson process behaviour of the optical to electrical conversion of a detector. For $N+1$ large, the variance of the Poisson process that models the generation of electrical charges by incident photons becomes too large for effective operation of a linear threshold scale trying to detect this value. The probability that the total charge generated corresponds to the interval $[(N+1)-1/2\ ,\ (N+1)+1/2]$ decreases with $N+1$.

In section 5.3 we discuss the problem of path dependent intensity attenuation of the optical pulses. Since pulses coming from different transmitters arrive at a detector through different paths, they experience different coupling losses and excess losses in a passive optical broadcast medium. Thus the received intensity for detection is bound to be typically different for each of the pulses being added. Even if all pulses travel through an equal number of couplers the tolerance on the coupling loss and excess loss of each of the couplers would need to be extremely tight if the received intensity is to be a linear function of the number of pulses received. Thus implementation of an ADDER channel does not seem feasible in general even if the coherence of optical sources is not a problem. The results derived on the ADDER channel serve then rather as theoretical guides on the nature of the detection problem.

Chapter 6. This chapter contains conclusions based on the previous results. Since Chapter 1 contains a summary of the results, we do not summarize them again here. We have tried to look at them globally and put them into context. We have also taken the liberty of trying to conjecture the behaviour of the schemes under study in situations were analytical results have not been obtained. We argue these conjectures using the previous results. Finally, we make some recommendations for further research.

CHAPTER 2

CODE DIVISION MULTIPLE ACCESS MODEL

In this chapter we introduce the system that we propose to treat. Section 2.1 describes the CDMA system to be analysed and the structure of the models to be used. Section 2.2 defines most of the special notation necessary for later discussions and serves also as a symbol reference table. Other definitions needed for discussion of some specific topics will be introduced in later sections. Section 2.3 defines and discusses briefly three detectors which are relevant to our work: Sum detector, Template detector and Maximum Likelihood detector. In section 2.4 we introduce the four specific models whose Maximum Likelihood detectors we plan to investigate in chapters 3 and 4.

2.1 SYSTEM MODEL

Protocol. The CDMA system we have in mind is as follows. There are $M+1$ potential channel users in the network. Every user has a transmitter and a receiver for operation. A "session" consists of a transmitter sending information to a receiver. We assume here that only one transmitter is transmitting to a given receiver at any given time[1]. Several distinct sessions may be active on the network at any given time, with a maximum of $M+1$ active sessions. A receiver is assumed in synchronism with the transmitter in its session. The transmitter encodes information into a sequence of Spread Spectrum signals which we call "signatures." Each signature corresponds to a transmitter "letter" and consists of a fixed length pattern of pulses. In a real setting each session in the network would be assigned a distinct Direct

[1] A new transmitter who tries to send information to an already busy receiver can be assumed out of synchronism with such receiver; in which case its signature is also treated as an interfering signature.

Sequence Spread Spectrum code and so the signature patterns would be particular to each session. This can be done by requiring transmitters to use a different code depending on the destination of its transmission. As far as our models are concerned, signatures of interfering concurrent sessions will be modeled by one of four probabilistic models defined in section 2.4. For simplicity of terminology we reserve the index i for the session of interest. Inactive users send nothing. The receiver in session i tries to decode the sequence of transmitter letters out of the received pulse pattern resulting from pulses coming from the transmitter in session i and interfering pulses broadcast on the channel by interfering concurrent sessions.

Spread Spectrum Format. The spread spectrum signals in the CDMA system obey the following format:

A *frame* is a sequence of n *slots*. A signature in the Direct Sequence Spread Spectrum system is a possibly empty sequence of pulses placed in the slots of a frame. Physically a slot is a time interval (length β seconds say) where an optical pulse may be emitted by a light source. In Spread Spectrum terminology slots are commonly referred to as "chips" and n corresponds to the system processing gain. For Direct Detection systems only light intensity is detectable and signatures can be represented by an n-vector with integer coordinates, coordinate k corresponding to the number of pulses present in slot k. Some of our signature models will allow more than one pulse per slot so the corresponding vector component may be greater than one.

Signature schemes. A sequence of I.I.D. source letters, from an α-letter alphabet $\mathcal{A}$, arrives at each active transmitter; one letter per frame. The transmitter generates and transmits a sequence of signatures, one per letter, each of which depends on its session index j and on the letter which is being sent. For interfering sessions this sequence of signatures will be modeled as a time variant sequence of randomly chosen pulse patterns[2]. Thus we assume

[2] We make this assumption to prevent receivers from attempting to "learn" the signature patterns of interfering sessions. It is realistic to assume that, for reasons of simplicity, detectors will not want to use previously received frames to acquire knowledge of the interference pulse patterns and use it in decoding.

that interfering signatures are generated probabilistically following a prespecified process. We assume that, at each transmission from the transmitter of session i, the session receiver exactly knows the signatures that would be used by its session transmitter for each of its source letters —i.e. their pulse pattern is deterministic— but does not know which letter is sent. On the other hand the receiver of session i only knows the probabilistic process which generates the signatures for sessions $j \neq i$.

We will use the phrase *symmetric scheme* for a system where each user j is assigned a set of signatures $S^j = \{s^{j1}, \ldots, s^{j\alpha}\}$ and they are all generated using identical probabilistic processes (see Bernoulli, Roulette schemes below). For session i the receiver knows S^i exactly. We will use the phrase *asymmetric scheme* for a system where a "null signature", denoted by ϕ, represents one of the source letters and consists of a frame with no pulses. The other signatures are all generated using identical probabilistic processes. For session i the receiver knows the non null signature, s^i say, exactly. We will only treat asymmetric schemes with two letter alphabets. We refer to them as *2-signature asymmetric schemes.* For equiprobable source letters and large α the performance of the asymmetric scheme should approach that of an equivalent symmetric scheme. *2-signature symmetric schemes* refers to symmetric schemes with two letter alphabets.

Two types of probabilistic descriptions for non null signatures are considered.
Bernoulli signatures are generated by a sequence of independent Bernoulli trials over the slots of a frame. In each slot a pulse is placed with probability p or left empty with probability $1 - p$.
Roulette signatures are generated by placing exactly a pulses in a frame. Each pulse position is independent of the others and has a uniform distribution over the set of slots. For optical systems this model allows the uncommon case of signatures with more than one optical pulse per slot. It is adopted however for simplicity of analysis. For small values of a/n the model will be realistic since the probability of multiple pulses per slot will be low.

Synchronization. We contemplate three possible timing models for our systems:

Frame Synchronism: Frames for all active users start at the same instant, relative to the receiver for session i.

Discrete Time Asynchronism: Slots for all active users start at the same instant, relative to the receiver for session i; start of frames are integer random variables uniformely distributed over the set of slots modulo n.

Continuous Time Asynchronism: Start of frames are continuous random variables distributed over a time interval of length $n\beta$, where β is the slot duration.

Mathematical description of Physical Channel. The channel is assumed to be noiseless and the sole source of errors is interuser interference. Interference is assumed additive. In optical systems this implies that optical intensities are additive. This assumption, which is not always valid, will be discussed in Chapter 5. If $X^j(\tau)$ denotes the light intensity, at instant τ at the receiver of session i, due to the transmitted signal of session j then the signal at the receiver of session i is given by

$$Y(\tau) = \sum_{j=1}^{M+1} X^j(\tau)$$

$Y(\tau)$ describes the output of the physical optical channel as a function of time. We assume that the receiver in session i is completely synchronized to its transmitter and knows the set S^i of signatures of this transmitter. No other transmitter transmits to this receiver. The only uncertainty for the destination receiver is the source letter being decoded from the currently received frame. From the perspective of session i, the physical optical channel can then be modeled mathematically as follows. At each channel use, the channel has as input the signature sent by the transmitter of session i. Channel noise consists of interuser interference. It is additive noise and the channel output —input plus interference noise— is the sum of the inputs, denoted by the symbol X^j, from all active interfering sessions j over the frame of session i.

For the Frame Synchronous model X^j is an n-vector from the set S^j of signatures and the channel is memoryless. For the Discrete Time Asynchronous model X^j is also an n-vector.

The channel is not however memoryless in general since a signature of session j can overlap two consecutive frames of session i. X^j corresponds to the sequence of pulses corresponding to session j over the frame of interest of session i.

For Discrete Time models it will be convenient to define the "output" vector Y and the "interference" vector Z, with respect to user i, as

$$Y = \sum_{j=1}^{M+1} X^j \tag{2.1}$$

$$Z = \sum_{\substack{j=1 \\ j \neq i}}^{M+1} X^j \tag{2.2}$$

Let y denote the value of a received output vector. The physical channel is described by the transition probabilities

$$\Pr[Y = y | X^i = s^{i\sigma}] \qquad \forall\ y,\ \forall\ s^{i\sigma} \in \mathbf{S}^i$$

For the Continuous Time Asynchronous model the channel is not memoryless, for reasons similar to the Discrete Time Asynchronous model, and X^j is a time waveform corresponding to the input waveform $X^j(\tau)$ of session j over the frame of session i.

OR channel and ADDER channel. We choose to incorporate, into the mathematical description of the physical channel, two alternative assumptions on the detector devices being used. This will result in two mathematical channel models. It will be simpler to describe their meaning for Discrete Time models here. The meaning of these channel models for Continuous Time Asynchronous models will be discussed in Section 4.4.

An ADDER channel assumes that receivers can detect the number of pulses present in a slot. The output vector is a vector of non negative integer components and is given by equation (2.1) .

An "OR channel" model assumes that the receiver can only detect the presence or absence of pulses in a slot. In this case we can represent the output vector by a binary vector which specifies the non zero components. Mathematically the OR channel can be analysed by considering the same mathematical channel description that applies to the ADDER channel but noting that now all output vectors with identical set of occupied slots map into the same OR channel output vector.

These two models will be used for analysis in chapters 3 and 4.

2.2 SPECIAL NOTATION

Since a majority of the results refer to Discrete Time models we will explain most of our definitions as applies for a Discrete Time model and defer discussion of their extension to Continuous Time Asynchronous models to those sections that may require them. The following notations and definitions for Discrete Time models are presented here for reference:

q	probability of a session being active.
M	maximum possible number of interfering sessions.
n	number of slots in a frame.
σ, γ, ρ	encoder alphabet letters.
$s^{j\sigma} = (s_0^{j\sigma}, \ldots, s_{n-1}^{j\sigma})$	user j signature for letter σ.
$\mathbf{S}^j = \{s^{j1}, \ldots, s^{j\alpha}\}$	user j signature set.
$\widehat{X^i}$	receiver estimate of X^i.
$X^j = (X_0^j, \ldots, X_{n-1}^j)$	user j channel input, a random vector.
$x^j = (x_0^j, \ldots, x_{n-1}^j)$	experimental value of X^j.
$Y = (Y_0, \ldots, Y_{n-1})$	channel output, a random vector.
$y = (y_0, \ldots, y_{n-1})$	experimental value of Y.
$Z = (Z_0, \ldots, Z_{n-1})$	interference vector, a random vector.
$z = (z_0, \ldots, z_{n-1})$	experimental value of Z.

We reserve the two following symbols for 2-signature asymmetric schemes:

$\phi = (0, \ldots, 0)$	null signature.
$s^i = (s_0^i, \ldots, s_{n-1}^i)$	non null signature of user i.

For symmetric signature schemes let y be a received output vector and $s^{i\sigma}$ be a signature being considered as hypothesis for X^i. We define the following sets of slots for convenience:

$\psi_+^{i\sigma} = \{k : s_k^{i\sigma} \neq 0\}$	non-zero slots of signature $s^{i\sigma}$.
$\psi_-^{i\sigma} = \{k : s_k^{i\sigma} = 0, y_k \neq 0\}$	non-zero slots of output not corresponding to $s^{i\sigma}$.
$\psi_0^{i\sigma} = \{k : y_k = 0\}$	empty slots of output vector.
$\Psi = \psi_+^{i\sigma} \bigcup \psi_-^{i\sigma} = \{k : y_k \neq 0\}$	non-zero slots of y.

For 2-signature asymmetric schemes let s^i be the non null signature. We define:

$\psi_+^i = \{k : s_k^i \neq 0\}$	non-zero slots of s^i.
$\psi_-^i = \{k : s_k^i \neq 0\}$	non-zero slots of output not corresponding to s^i.
$\psi_0^i = \{k : s_k^i \neq 0\}$	empty slots of output vector.
$\Psi = \{k : y_k \neq 0\}$	non-zero slots of y.

The number of non-zero slots in a signature is important. We define for the symmetric schemes and the 2-signature asymmetric schemes, respectively:

$h(\sigma) = \|\psi_+^{i\sigma}\|$	No. of non-zero slots of $s^{i\sigma}$. Symmetric scheme.
$h = \|\psi_+^i\|$	No. of non-zero slots of s^i. Asymmetric scheme.

In the two events below ψ denotes a set of slots.

$A(\psi)$	Event $y_k > 0, \forall k \in \psi$. i.e. ψ is fully occupied.
$\overline{A}(\psi)$	Event $y_k = 0$, for some $k \in \psi$. i.e. ψ not full.

2.3 DETECTORS

In subsequent chapters we are concerned with Maximum Likelihood Detection of our Spread Spectrum signal models. For some schemes in the Discrete Time models this detector takes the form of a "Template detector" defined below. For comparison purposes we also define and analyse the "Sum detector". These detectors are used by the session receiver to decide which transmitter letter is being received. For clarity, the definitions presented here correspond to Discrete Time models. We leave the explanation of their extension to a Continuous Time Asynchronous model to Section 4.4.

Sum detector. We will refer to two versions of a Sum detector; one for 2-signature asymmetric schemes and one for symmetric schemes.

Consider a 2-signature asymmetric scheme. Let y be a received output vector. Let ψ_+^i denote the set of non-zero slots in the non null signature s^i. A "Sum detector" of threshold T is defined for Discrete Time models as a detector implementing the rule

$$\left.\begin{array}{lll} \widehat{X^i} = s^i & \textit{if} & \sum_{k \in \psi_+^i} y_k \geq T \\ \widehat{X^i} = \phi & \textit{otherwise.} & \end{array}\right\} \quad (2.4)$$

For symmetric schemes let $\psi_+^{i\sigma}$ denote the set of slots in signature $s^{i\sigma}$ which contain pulses. A "Sum detector" of threshold T is defined for Discrete Time models as a detector implementing the rule

$$\left.\begin{array}{l} \widehat{X^i} = s^{i\sigma} \textit{ such that } \quad \sum_{k \in \psi_+^{i\sigma}} y_k \geq T, \sigma \in \mathcal{A} \\ \textit{If } \sigma \textit{ is not unique choose equiprobably among all such } \sigma. \\ \textit{If no } \sigma \textit{ satisfies the condition choose equiprobably among all } \sigma. \end{array}\right\} \quad (2.5)$$

In fact, better performance would be attained in an ADDER channel if instead of this rule we implemented a rule that decides for that letter for which the contents of the slots in $\psi_+^{i\sigma}$ add up to a maximum, among all letters. Analysis of the probability of error for this rule would require computing the probability that a random variable —the sum of the contents of the slots in $\psi_+^{i\sigma}$ — does not have the maximum value among several other random variables —the sum of the contents of the slots in $\psi_+^{i\gamma}$, for other letters γ— given that a signature $s^{i\sigma}$ is sent. Notice that all variables are not identically distributed. Computing this type of probability is a difficult task.

The rule in (2.5) is simpler to analyse. Moreover, for moderate interference levels its probability of error is expected to approximate that of the rule selecting the maximum sum value mentioned above. Results in Chapter 3 seem to indicate that omiting analysis of the latter rule does not compromise our conclusions.

For a signature $s^{i\sigma}$ with no more than one pulse per slot the Sum detector is equivalent to implementing the Correlation Detection

$$\sum_{k=0}^{n-1} y_k \cdot s_k^{i\sigma} \overset{?}{>} T \tag{2.6}$$

A similar comment applies to the non null signature of an asymmetric scheme.

In optical systems Sum detection, or its version that chooses the maximum sum value as mentioned above, is commonplace. It is achieved with optical delay lines and a threshold detector [Hui]. It thus simplifies electronic processing which may otherwise slow down decoding speed. These arrangements are normally referred to as Optical Delay Line Correlation Detectors.

Template detector. We will refer to two versions of a Template detector; one for 2-signature asymmetric schemes and one for symmetric schemes.

Let y be a received output vector. For an asymmetric scheme let ψ_+^i denote the set of non-zero slots in the non null signature s^i. Let $A(\psi_+^i)$ denote the event: " y has at least one pulse in each of the slots of ψ_+^i", and $\overline{A}(\psi_+^i)$ the event: "y has one pulse missing from at least one slot of ψ_+^i". A "Template detector" for 2-signature asymmetric schemes implements the detection rule

$$\left.\begin{array}{lll} \widehat{X^i} = \phi & if & \overline{A}(\psi_+^i) \\ \widehat{X^i} = s^i & if & A(\psi_+^i) \end{array}\right\} \tag{2.7}$$

In words: signature s^i is chosen **iff** y has a pulse in each of the slots that should be occupied in s^i.

Consider now a symmetric scheme. Let $\psi_+^{i\sigma}$ denote the set of non-zero slots in signature $s^{i\sigma}$. Let $A(\psi_+^{i\sigma})$ denote the event: " y has at least one pulse in each of the slots of $\psi_+^{i\sigma}$", and $\overline{A}(\psi_+^{i\sigma})$ the event: "y has one pulse missing from at least one slot of $\psi_+^{i\sigma}$". The Template detector for symmetric schemes implements the detection rule

$$\left.\begin{array}{l} \widehat{X^i} = s^{i\sigma} \textit{ such that } A(\psi_+^{i\sigma}) \\ \textit{If } \sigma \textit{ is not unique choose equiprobably among all such } \sigma. \end{array}\right\} \tag{2.8}$$

In words: the decision is equiprobable among all letters for which all pulses corresponding to their signatures are present in y.

A Template detector checks individually each slot of the hypothesized signature and considers this signature feasible if all such slots have pulses. Notice that a Template detector operating on an ADDER channel in fact reduces it to an OR channel since at each slot it only checks whether no pulses are present or whether one or more pulses are present.

For optical systems such a Template detector could be implemented by using a threshold detector at each individual delay line tap. The threshold detectors need not be different than for the optical Sum detector but several threshold detectors would be necessary. A AND circuit would be needed to detect if all pulses sought are present simultaneously. The following chapters however indicate that the improvement in performance may be significant with respect to a Sum detector and that the Template detector has some M.L. detection properties.

Maximum Likelihood Detection. The Maximum Likelihood detector [Van Trees], after observing an output vector y, implements the detection rule

$$\widehat{X^i} = s^{i\sigma} \qquad s.t. \qquad \Pr[Y = y | X^i = s^{i\sigma}] = \max_{s^{i\gamma} \in \mathbf{S}^i} \Pr[Y = y | X^i = s^{i\gamma}] \tag{2.9}$$

For 2-signature asymmetric schemes this reduces to

$$\widehat{X^i} = \begin{cases} s^i, & \text{if } \Pr[Y = y | X^i = s^i] \geq \Pr[Y = y | X^i = \phi]; \\ \phi, & \text{if } \Pr[Y = y | X^i = s^i] < \Pr[Y = y | X^i = \phi]. \end{cases} \tag{2.10}$$

When the likelihood probabilities are equal the decision on $\widehat{X^i}$ can be arbitrary.

The Maximum Likelihood receiver minimizes the probability of detection error $\Pr[\widehat{X^i} \neq X^i]$ when all source letters are equally likely. It is a convenient rule since knowledge of a priori letter probabilities is often not available.

2.4 PROBABILISTIC MODELS FOR INTERFERENCE

We will treat four specific probabilistic models for Spread Spectrum signals. By using these probabilistic models we are trying to reach a compromise between realism and tractability, the aim being to obtain conclusions useful for the design and analysis of real systems. The source alphabet $\mathcal{A}$ is the same for all users. The receiver in session i has perfect knowledge of the set of signatures $\mathbf{S}^i$ at each transmission in its session but only has a probabilistic knowledge of the patterns for signatures used by other concurrent sessions j.

2-signature, asymmetric, Bernoulli scheme. The source letter alphabet size is $\alpha = 2$ for all sessions. The signature set is $\mathbf{S}^j = \{\phi, s^j\}$ for all sessions j. s^j is an independent Bernoulli signature for all interfering sessions.

Symmetric, Bernoulli scheme. The source letter alphabet size α is arbitrary and equal for all sessions. All signatures in the set $\mathbf{S}^j$ are independent Bernoulli signatures for all interfering sessions.

2-signature, asymmetric, Roulette scheme. The alphabet size is $\alpha = 2$ for all sessions. $\mathbf{S}^j = \{\phi, s^j\}$ for all sessions. s^j is an independent Roulette signature for all interfering sessions.

Symmetric, Roulette scheme. The source letter alphabet size α is arbitrary and equal for all sessions. All signatures in the set $\mathbf{S}^j$ are independent Roulette signatures for all interfering sessions.

For probability of error calculations we assume that potential interfering sessions are active, with probability q, or inactive, with probability $1-q$, independently of each other. The Probability Mass Function for the number of active interfering sessions U is Binomial

$$F(U) = \binom{M}{U} q^U (1-q)^{M-U} \tag{2.11}$$

For Frame Synchronous schemes, ς will denote the number of interfering sessions which are both active and sending a non null signature. $\mathcal{F}(\varsigma)$ denotes its Probability Mass Function. If the scheme is asymmetric U and ς need not be equal since some active interfering sessions may be sending null signatures and are in fact not interfering with i. In such a case we define the interference parameter

$$Q \equiv \Pr[X^i = s^i] q$$

and the distribution for ς is also Binomial with interference parameter Q.

$$\mathcal{F}(\varsigma) = \binom{M}{\varsigma} Q^\varsigma (1-Q)^{M-\varsigma} \tag{2.12}$$

If the scheme is symmetric $\varsigma = U$ and

$$\mathcal{F}(\varsigma) = \binom{M}{\varsigma} q^\varsigma (1-q)^{M-\varsigma} \tag{2.13}$$

We assume, unless otherwise specified, that the source letters are equiprobable. Knowledge of the detection rule is necessary for probability of error calculations. We analyse the probability of error for the Template detector and Sum detector and compare their performance. For the Roulette schemes we are also able to evaluate upperbounds for the M.L. detector probability of error in the ADDER channel.

CHAPTER 3

FRAME SYNCHRONOUS MODEL

In this chapter we analyse the Frame Synchronous model.

We start by treating the OR channel in section 3.1. The lemma presented in that section will facilitate analysis of the OR channel for the schemes under study. Since the lemma in fact also allows Discrete Time Asynchronism it will also be used in Chapter 4 when we discuss asynchronous models.

In sections 3.2 and 3.3 we analyse the Bernoulli and the Roulette schemes respectively. Each section will present first the 2-signature asymmetric scheme and then the symmetric scheme. Thus we cover the four probabilistic models introduced in section 2.4. For each of these four schemes, we first analyse Maximum Likelihood detection for both the OR channel and the ADDER channel and then present the probability of error expressions for both the Template detector and the Sum detector. It is understood that the implementation of these detectors depends on whether the scheme under consideration is asymmetric or symmetric. We do not include derivations of the probability of error for the Maximum Likelihood detectors because, when they do not reduce to Template detectors, the M.L. rule calls for evaluation of the Likelihood Ratio as a function of the received vector y and analysis of its behaviour appears to be difficult. For Roulette schemes we have been able to evaluate upperbounds for the probability of error of Maximum Likelihood detectors. For Bernoulli schemes the expressions for these upperbounds could not be simplified and the number of computational steps required for their evaluation makes their evaluation typically impractical.

Section 3.4 presents numerical evaluation of the expressions derived for the probability of error of the different schemes. We have plots for the error probability of the Sum detector and Template detector of the different schemes as the level of interference in the network varies due to the change in system parameters. We have also included figures comparing the different schemes and figures showing the upperbounds for probability of error of M.L. detectors of Roulette schemes.

3.1 MAXIMUM LIKELIHOOD DETECTION, OR CHANNEL

Lemma 3.1 below does not require Frame Synchronism to be valid. We present it here since it will simplify treatment in this chapter.

<u>Lemma 3.1</u> Consider a Discrete Time, 2-signature, asymmetric scheme implemented over an OR channel. Let session j's signature, s^j, be independent of session i's signature, s^i, $\forall j \neq i$. Then the Maximum Likelihood Detector for session i is the Template detector.

<u>Proof:</u> See appendix AP-1.

∎

The template detector for 2-signature asymmetric schemes is given by equation (2.7). The general argument for the proof of the lemma is very simple: For any y such that all pulses of s^i are present in y —the event $A(\psi^i_+)$—, y is more likely given $X^i = s^i$ than given $X^i = \phi$ since in the former case "interfering" sessions need only interfere with the slots in the set $\psi^i_- \subseteq \Psi$ and not with all the slots of Ψ. On the other hand, for any y such that $\overline{A}(\psi^i_+)$ holds the transmitter in session i must have sent ϕ since the channel is by assumption physically noiseless and a pulse cannot disappear.

Notice that the distribution of active "interfering" transmitters can be arbitrary and that the signature scheme need not be either a Bernoulli scheme or a Roulette scheme, it need only be asymmetric. There can be dependence among interfering transmitters, including their source letter distribution. The source letter distribution for session i can also be arbitrary. The result holds as long as the behaviour of the transmitter of session i is independent of the other transmitters. For the result to apply to all users we would need the behaviour of all transmitters to be mutually independent.

We do not have a lemma of comparable generality for symmetric schemes implemented over an OR channel. In general, Maximum Likelihood detection would require observation of all slots in a frame and the rule would depend on the particular probabilistic description of interfering signatures that applies. We will treat the situations individually for each model.

3.2 BERNOULLI SCHEMES

3.2.1 2-SIGNATURE, ASYMMETRIC, BERNOULLI SCHEME

In this section we denote by h the number of slots with pulses in signature s^i. Recall that h is assumed known by the detector. The variable ς denotes the number of concurrent active interfering sessions j which are both active and sending the non null signature s^j. Let $\mathcal{F}(\varsigma)$ denote the Probability Mass Function for ς. Also, recall that Z denotes the vector representing inter user interference.

MAXIMUM LIKELIHOOD DETECTION.

OR channel. For the OR channel the M.L. detector for session i is a Template detector (Lemma 3.1).

ADDER channel. For the ADDER channel let us first derive the likelihood probabilities to then establish the M.L. detection rule. We present the derivation here instead of in an appendix since it will be relevant in Chapter 4.

For all y such that $\overline{A}(\psi_+^i)$ holds:

$$\Pr[Y = y | X^i = s^i] = 0 \tag{3.1}$$

since no pulse erasures are possible in our model.

Now consider the alternative: all y such that $A(\psi_+^i)$ holds.
Condition the channel output on having ς active interfering sessions[1] sending s^j.

We have:

$$\begin{aligned}
\Pr[Y = y | X^i = s^i, \varsigma] &= \prod_{l \notin \psi_+^i} \Pr[Z_l = y_l | \varsigma] \prod_{k \in \psi_+^i} \Pr[Z_k = y_l - 1 | \varsigma] \\
&= \left\{ \prod_{k \in \psi_+^i} \binom{\varsigma}{y_k - 1} \prod_{l \notin \psi_+^i} \binom{\varsigma}{y_l} \right\} p^{\sum_{i=1}^{n} y_i - h} (1-p)^{\varsigma n - \sum_{j=1}^{n} y_j + h} \\
\Pr[Y = y | X^i = \phi, \varsigma] &= \prod_{k=1}^{n} \Pr[Z_k = y_k | \varsigma] \\
&= \left\{ \prod_{k=1}^{n} \binom{\varsigma}{y_k} \right\} p^{\sum_{k=1}^{n} y_k} (1-p)^{\varsigma n - \sum_{k=1}^{n} y_k} \\
&= \Pr[Y = y | X^i = s^i, \varsigma] \left\{ \prod_{k \in \psi_+^i} \frac{\varsigma - y_k + 1}{y_k} \right\} p^h (1-p)^{-h} \\
&= \Pr[Y = y | X^i = s^i, \varsigma] \; \Theta_\varsigma(y)
\end{aligned}$$

with

$$\Theta_\varsigma(y) = \left\{ \prod_{k \in \psi_+^i} \frac{\varsigma - y_k + 1}{y_k} \right\} p^h (1-p)^{-h} \tag{3.2}$$

We let $\varsigma_i, \varsigma_\phi$ denote the minimum values for ς under the hypotheses $X^i = s^i, X^i = \phi$ respectively. Since we are treating the ADDER channel they are obtained from the vector y by the equations

$$\varsigma_i = \max_k (Z_k = y_k - s_k^i) \tag{3.3}$$

$$\varsigma_\phi = \max_k (y_k) \tag{3.4}$$

[1] Note that ς satisfies $y_k - 1 \leq \varsigma \leq M, \; \forall k$. Otherwise there could not possibly be enough pulses to attain y_k for at least some k.

It is understood that $\varsigma_i, \varsigma_\phi$ are functions of the output vector y but for simplicity of notation we will omit y as an argument. The dependence should be clear from the context. We have then the conditional probabilities

$$\Pr[Y = y|X^i = s^i] = \sum_{\varsigma=\varsigma_i}^{M} \mathcal{F}(\varsigma) \Pr[Y = y|X^i = s^i, \varsigma] \tag{3.5}$$

$$\Pr[Y = y|X^i = \phi] = \sum_{\varsigma=\varsigma_\phi}^{M} \mathcal{F}(\varsigma) \Pr[Y = y|X^i = s^i, \varsigma]\Theta_\varsigma(y) \tag{3.6}$$

Theorem 3.1 Consider a Frame Synchronous, 2-signature, asymmetric, Bernoulli scheme operating on an ADDER channel. Let M be the maximum number of interfering sessions and p the pulse placement probability for the Bernoulli signatures. If

$$(M+1)p \leq 1 \tag{3.7}$$

then the M.L. detector for session i is a Template detector.

Proof: We need to show that (3.7) implies (2.7). For all y such that $\overline{A}(\psi^i_+)$ holds we have $\widehat{X^i} = \phi$ as in (3.1) . It remains to show that if (3.7) holds then, for all y such that $A(\psi^i_+)$ holds,

$$\Pr[Y = y|X^i = s^i] \geq \Pr[Y = y|X^i = \phi] \tag{3.8}$$

Assume $A(\psi^i_+)$ holds. Since $\Theta_\varsigma(y)$ is maximized w.r.t. y setting $y_k = 1, \forall k \in \psi^i_+$, and w.r.t. ς by $\varsigma = M$, we can thus bound (3.2) by

$$\Theta_\varsigma(y) \leq \left[\frac{Mp}{1-p}\right]^h \qquad ;\forall\varsigma \tag{3.9}$$

and (3.7) implies

$$\Theta_\varsigma(y) \leq 1 \qquad ;\forall\varsigma, y \tag{3.10}$$

Since for a Bernoulli signature $s_k^i \in \{0,1\}$, $\forall k$, then from (3.3) , (3.4) it follows that either $\varsigma_\phi = \varsigma_i$ or $\varsigma_\phi = \varsigma_i + 1$. In either case (3.10) implies (3.8) and the theorem is proven.

∎

Inequality (3.7) has only been shown to be a sufficient condition for the M.L. detector to be a Template detector. However (3.7) is also a necessary condition in the following sense: if we wanted the Template detector to be a M.L. detector for all received vectors y and any assumed Probability Mass Function $\mathcal{F}(\varsigma)$ then (3.7) is a necessary condition. If the condition is not met we see from (3.2) that for $\mathcal{F}(M) = 1$ and received y such that $y_k = 1$ for all slots in ψ_+^i then the Maximum Likelihood decision should be $\widehat{X^i} = \phi$ instead of $\widehat{X^i} = s^i$ as for the Template detector. In fact, the bound in (3.7) insures that the Template detector is a M.L. detector for each possible value of ς and y.

(3.7) expresses a plausible network operating condition. Supposing that $p = h/n$, i.e. the fraction of occupied slots per signature which applies to session i, then we can rewrite the bound as $(M+1)h \leq n$, which states that the average number of pulses present in a frame, if all sessions are interfering, must not exceed the total number of slots per frame. We also note that observation of the slots k in ψ_+^i is sufficient to estimate X^i; this is also true for the OR channel.

PROBABILITY OF ERROR. We assume a Binomial distribution for the number of active interfering sessions as stated in (2.11). Hence $\mathcal{F}(\varsigma)$, defined above, is Binomial with interference parameter $Q \equiv \Pr[X^i = s^i]q$, where q denotes the probability of a given session being active and is given by

$$\mathcal{F}(\varsigma) = \binom{M}{\varsigma} Q^\varsigma (1-Q)^{M-\varsigma} \tag{2.12}$$

Template detector. A Template detector reduces the ADDER channel to an OR channel. The probability of error in both cases is the probability that the null signature is sent and yet there is at least one pulse in each of the h slots observed by the detector. The expression is

$$P_e = \Pr[X^i = \phi] \sum_{\varsigma=1}^{M} \binom{M}{\varsigma} Q^{\varsigma}(1-Q)^{M-\varsigma}(1-(1-p)^{\varsigma})^h \tag{3.11}$$

$$; \; 0 < Q < 1, \; 0 < p < 1.$$

Sum detector. For the Sum detector we let the value of the threshold be $T = h$. As the load increases, the threshold T should in fact be increased to minimize P_e. $T = h$ however is a sensible value for networks with no adaptive ability. This is a sensible value for networks not congested, so we use it for comparison. For the OR channel (3.11) applies. For the ADDER channel we have

$$P_e = \Pr[X^i = \phi] \sum_{\varsigma=1}^{M} \binom{M}{\varsigma} Q^{\varsigma}(1-Q)^{M-\varsigma} \left\{ \sum_{r=h}^{\varsigma h} \binom{\varsigma h}{r} (1-p)^{\varsigma h - r} (p)^r \right\} \tag{3.12}$$

$$; \; 0 < Q < 1, \; 0 < p < 1.$$

The expression within braces is the probability that the number of pulses present in ψ^i_+ is h or more, given $X^i = \phi, \varsigma$.

We can explore further the optimal value of threshold T. We assume that as the level of interference increases the optimal value of T increases in unit increments, starting from $T = h$. We want to investigate the value of p at which the optimal T first becomes $h+1$. Let $b(r; N, a)$ denote the Binomial probability of r successes in N trials with success probabillity a. Consider the probability of error for $T \geq h+1$.

$$\begin{aligned} P_e = &\Pr[X^i = \phi] \sum_{\varsigma=1}^{M} b(\varsigma; M, Q) \sum_{r=T}^{\varsigma h} b(r; \varsigma h, p) \\ &+ \Pr[X^i = s^i] \left[b(0; M, Q) + \sum_{\varsigma=1}^{M} b(\varsigma; M, Q) \sum_{r=0}^{T-h-1} b(r; \varsigma h, p) \right] \end{aligned}$$

Replacing $T = h+1$ and subtracting this expression from (3.12) we obtain

$$\begin{aligned} \Delta P_e = &\Pr[X^i = \phi] \sum_{\varsigma=1}^{M} b(\varsigma; M, Q) b(h; \varsigma h, p) \\ &- \Pr[X^i = s^i] \left[b(0; M, Q) + \sum_{\varsigma=1}^{M} b(\varsigma; M, Q) b(0; \varsigma h, p) \right] \end{aligned}$$

If $\Delta P_e \leq 0$ then $T = h$ is optimal.

We can upperbound ΔP_e by setting $b(0; M, Q) = 0$ in the RHS of this expression. In fact, for $M \to \infty$,

$$b(0; M, Q) = (1 - Q)^M \to 0$$

Also, we set $\Pr[X^i = \phi] = 1/2$. In a sense this corresponds to worst case interference over the possible values of $\Pr[X^i = \phi]$. It corresponds to maximizing $\Pr[X^i = s^i]$ subject to $\Pr[X^i = \phi] \geq 1/2$. We do not consider $\Pr[X^i = \phi] \leq 1/2$ since it would be a simple matter to interchange the roles of s^i, ϕ to achieve lower channel interference.

Setting $\Pr[X^i = \phi] = \Pr[X^i = s^i] = 1/2$, $b(0; M, Q) = 0$, ΔP_e becomes

$$\Delta P_e = \frac{1}{2} \sum_{\varsigma=1}^{M} b(\varsigma; M, Q) \left\{ b(h; \varsigma h, p) - b(0; \varsigma h, p) \right\}$$

A sufficient (but not necessary) condition for $\Delta P_e \leq 0$ is then

$$R(h, \varsigma, p) \equiv \frac{b(h; \varsigma h, p)}{b(0; \varsigma h, p)} = \binom{\varsigma h}{h} \left(\frac{p}{1-p} \right)^h \leq 1 \qquad ; \forall \varsigma$$

Since the value of $Q = (1/2)q$ in $b(\varsigma; M, Q)$ must be considered unknown, the condition $R(h, \varsigma, p) \leq 1 \ \forall \varsigma$ is only necessary in the sense that it is required for $\Delta P_e \leq 0$ to hold regardless of the value of Q.

Using Stirling's factorial approximation

$$n! \simeq \sqrt{2\pi n} \left(\frac{n}{e} \right)^n$$

we get

$$\begin{aligned} R(\varsigma, h, p) &\simeq \frac{1}{\sqrt{2\pi}} \left(\frac{\varsigma h}{(\varsigma h - h) h} \right)^{\frac{1}{2}} \frac{(\varsigma h)^{\varsigma h}}{(\varsigma h - h)^{\varsigma h - h} h^h} \left(\frac{p}{1-p} \right)^h \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{\varsigma}{(\varsigma - 1) h} \right)^{\frac{1}{2}} \left[\varsigma \left(\frac{\varsigma}{\varsigma - 1} \right)^{\varsigma - 1} \frac{p}{1-p} \right]^h \qquad ; \varsigma > 1 \end{aligned}$$

Notice that for $\varsigma = 1$ $b(h; \varsigma h, p) = p^h$ and $b(0; \varsigma h, p) = (1-p)^h$ so $p \leq 1/2$ implies $R(1, h, p) \leq 1$. We make several simple approximations at this point. We assume that Stirling's approximation is accurate enough. Since $h \geq 1, \varsigma/(\varsigma - 1) \leq 2\ \forall \varsigma \geq 2$, we can upperbound the expression under the square root to obtain the upperbound

$$R(\varsigma, h, p) \leq \frac{1}{\sqrt{\pi}} \left[\varsigma \left(\frac{\varsigma}{\varsigma - 1} \right)^{\varsigma - 1} \frac{p}{1 - p} \right]^h \qquad ; \varsigma > 1$$

and since $1/\sqrt{\pi} < 1$ then it is sufficient, for $R(\varsigma, h, p) \leq 1$ to hold, that the expression within brackets be less than or equal to one.

$$\left[\varsigma \left(\frac{\varsigma}{\varsigma - 1} \right)^{\varsigma - 1} + 1 \right] p \leq 1 \qquad ; \varsigma > 1$$

For $\varsigma \geq 2$, ς increases with ς and for large ς it converges to $e\varsigma$. Since $\varsigma \leq M$ the condition

$$(eM + 1)p \leq 1$$

is sufficient for the optimal value of T to be $T = h$ when M is reasonably large. The bound also accounts for the case $\varsigma = 1$.

Notice that this condition is similar to that in Theorem 3.1, even though the maximum value of p is smaller. However, the slack approximation on the factor

$$\frac{1}{\sqrt{2\pi}} \left(\frac{\varsigma}{(\varsigma - 1)h} \right)^{\frac{1}{2}}$$

above and having set $b(0; M, Q) = 0$ in the expression for ΔP_e means that for somewhat higher values of p the value $T = h$ is also optimal.

3.2.2 SYMMETRIC, BERNOULLI SCHEME

Define $h(\sigma) = |\psi_+^{i\sigma}|$ to be the number of pulses in the signature for letter σ. Let ς be as in section 3.2.1 which reduces in this case to the number of active interfering sessions since in this scheme they all send non null signatures. $\mathcal{F}(\varsigma)$ is its Probability Mass Function.

Treatment in this section will require the definition of the set of slots ψ_c^{jx} as follows: Let $B(i, y)$ be the set of feasible letters of i given output y, i.e. $B(i, y)$ is the set of letters γ for which all signature pulses are present in y. In the previous notation,

$$B(i, y) = \{\gamma : \text{ s.t. } A(\psi_+^{i\gamma}), \gamma \in \mathcal{A}\} \tag{3.13}$$

Let $C(i,y)$ be the set of all slots for which at least one feasible letter has a pulse present.

$$C(i,y) = \bigcup_{\gamma \in B(i,y)} \psi_+^{i\gamma} \tag{3.14}$$

Let σ be a feasible letter given y, i.e. $\sigma \in B(i,y)$. We then define $\psi_c^{i\sigma}$ to be those slots in which $s^{i\sigma}$ has no pulses but at least some other feasible letter has a pulse. In set notation:

$$\psi_c^{i\sigma} = C(i,y) \setminus \psi_+^{i\sigma} \tag{3.15}$$

MAXIMUM LIKELIHOOD DETECTION.

OR channel. Consider the OR channel first. Consider a received vector y. For all y such that $\overline{A}(\psi_+^{i\sigma})$ holds

$$\Pr[Y = y | X^i = s^{i\sigma}] = 0 \tag{3.16}$$

While for all y such that $A(\psi_+^{i\sigma})$ holds we have:

$$\begin{aligned} \Pr[Y = y | X^i = s^{i\sigma}, \varsigma] &= \Pr[Z_k = y_k;\ \forall k \notin \psi_+^{i\sigma} | \varsigma] \\ &= [1 - (1-p)^{\varsigma}]^{|\Psi| - h(\sigma)} (1-p)^{\varsigma(n - |\Psi|)} \end{aligned}$$

Ψ is the set of all occupied slots. The likelihood probabilities are then:

$$\Pr[Y = y | X^i = s^{i\sigma}] = \sum_{\varsigma=1}^{M} \mathcal{F}(\varsigma)[1 - (1-p)^{\varsigma}]^{|\Psi| - h(\sigma)} (1-p)^{\varsigma(n-|\Psi|)} \qquad ;\ \forall \sigma \text{ s.t. } \psi_+^{i\sigma}.$$

Since this expression is maximized over $\sigma \in \mathcal{A}$ with the maximum $h(\sigma)$ then the M.L. rule is

$$\widehat{X^i} = s^{i\sigma} \qquad \text{s.t.} \qquad h(\sigma) = \max_{\gamma \in B(i,y)} h(\gamma)$$

If $h(\sigma) = h,\ \forall \sigma$, then the choice is arbitrary among all feasible letters and (3.16) ,(3.17) imply the symmetric scheme Template detector of equation (2.8).

ADDER channel, known ς. For the ADDER channel case we have only been successful in reducing the M.L. detector rule to a simpler expression when the number of active interfering sessions ς is known to the decoder. We will also present the derivation of the likelihood probabilities when ς is a random variable and mention some of the difficulties in the analysis.

Consider again a received vector y. For any y and letter σ such that $\overline{A}(\psi_+^{i\sigma})$ holds

$$\Pr[Y = y | X^i = s^{i\sigma}] = 0 \tag{3.18}$$

Now condition the output vector Y on knowledge of how many interfering sessions, ς, were present in this channel use. Consider any y, σ such that $A(\psi_+^{i\sigma})$ holds[2].

$$\begin{aligned} &\Pr[Y = y | X^i = s^{i\sigma}, \varsigma] \\ &\quad = \Pr[Z = y - s^{i\sigma} | \varsigma] \\ &\quad = \prod_{k \in \psi_+^{i\sigma}} \binom{\varsigma}{(y_k - 1)} p^{(y_k - 1)} (1-p)^{\varsigma - (y_k - 1)} \prod_{l \notin \psi_+^{i\sigma}} \binom{\varsigma}{y_l} p^{y_l} (1-p)^{\varsigma - y_l} \end{aligned} \tag{3.19}$$

Note that

$$\binom{\varsigma}{y_k} = \binom{\varsigma}{y_k - 1} \frac{\varsigma - y_k + 1}{y_k} \quad ; \forall k \in \psi_c^{i\sigma}$$

All factors are well defined since, $\forall k \in \psi_c^{i\sigma}$, $\varsigma \geq y_k$ and $y_k \geq 1$. Hence rewrite (3.19) as

$$\begin{aligned} \Pr[Y = y | X^i = s^{i\sigma}, \varsigma] = &\left\{ \prod_{k \in C(i,y)} \binom{\varsigma}{(y_k - 1)} p^{(y_k - 1)} (1-p)^{\varsigma - (y_k - 1)} \right\} \left\{ \prod_{k \notin C(i,y)} \binom{\varsigma}{y_k} p^{y_k} (1-p)^{\varsigma - y_k} \right\} \\ &\times \left(\prod_{l \in \psi_c^{i\sigma}} \frac{\varsigma - y_l + 1}{y_l} \right) \left(\frac{p}{1-p} \right)^{|\psi_c^{i\sigma}|} \end{aligned} \tag{3.20}$$

[2] Note: $y_k - 1 \leq \varsigma, \forall k \in \psi_+^{i\sigma}$ and $y_k \leq \varsigma, \forall k \notin \psi_+^{i\sigma}$.

<u>Lemma 3.2</u> Consider a Frame Synchronous, symmetric, Bernoulli scheme operating on an ADDER channel. Let ς, the number of active interfering sessions on the channel, be known to the decoder. Then the M.L. detector for session i takes the form

$$\left.\begin{array}{l} \widehat{X^i} = s^{i\sigma} \textit{ such that } \sigma \textit{ maximizes, over } \gamma \in B(i,y), \textit{ the metric} \\ \\ m(i,\gamma) = \left(\prod_{l \in \psi_o^{i\gamma}} \frac{\varsigma - y_l + 1}{y_l} \right) \left(\frac{p}{1-p} \right)^{|\psi_o^{i\gamma}|} \end{array} \right\} \quad (3.21)$$

If σ is not unique choose arbitrarily among all such σ.

<u>Proof:</u> From (3.20) $m(i,\gamma)$ is the only factor in the likelihood probabilities which depends on the signatures. Maximizing the metric is equivalent to maximizing the likelihood probabilities. If the metric is maximized by more than one γ we can choose arbitrarily among all such γ since their likelihood probabilities will be equal.

∎

An interesting observation can be made in this situation where ς is known to the decoder. A symmetry exists between a system which works with pulses, as described, and one which works with "holes", where the holes are defined as the absence of a pulse. For each received slot k, the number u_k of holes is defined by

$$u_k = \varsigma + 1 - y_k$$

and can be computed by the detector. We can also define a signature of session i in terms of its "holes", of probability $1-p$, rather than its pulses. From these definitions it can be seen that there exists an equivalence between a system whose signatures $s^{i\sigma}$ for session i have $h(\sigma)$ pulses and whose interfering sessions place pulses with probability p and another system whose signatures for session i have $h(\sigma)$ holes and whose interfering sessions place holes with probability p. Both systems would have identical transition probabilities for corresponding signatures. One operates with pulses while the other with holes.

One conclusion can be derived from these observations. Let us assume that we have a system where all signatures of session i are equally likely, have h pulses and the value of p is set to $p = h/n$ for interfering sessions. The probability of error of a Maximum Likelihood detector as a function of h, as h varies from 1 to n (and p varies accordingly), would be symmetric around the value $n/2$. The reason is that, as $n/2$ is exceeded, a Maximum Likelihood detector operating with holes should be able to achieve the same probability of error as its equivalent pulse system, of less than $n/2$ pulses per signature, since their transition probabilities are the same and thus the communication channels are identical from an information theoretical point of view. This is evident from observing equation (3.21) where the role of pulses and holes are symmetric.

In fact as h exceeds $n/2$ the occurrence of a hole is less probable than the occurrence of a pulse and holes carry more information than pulses so observation of received holes simplifies detector operation without limiting the performance of a detector.

Observation of holes is only feasible for symmetric Bernoulli schemes for which ς is known. For asymmetric Bernoulli schemes the number of interfering sessions that send non null signatures is a random variable and computation of the number of holes present cannot be exact. For Roulette signatures one could define the number of holes to be n less the number of pulses placed at random over the set of slots; but the possibility of pulse overlap implies that the position of holes cannot be observed, which means that even for a symmetric Roulette scheme with known ς operation based on holes is unfeasible.

ADDER channel, random ς. Now let ς be a random variable with Probability Mass Function $\mathcal{F}(\varsigma)$. Let ς_σ denote the minimum value of ς under the hypothesis $X^i = s^{i\sigma}$. Since it is an ADDER channel

$$\varsigma_\sigma = \max_k (Z_k^{i\sigma} = y_k - s_k^{i\sigma}) \tag{3.22}$$

It is understood that ς_σ is a function of the output vector y. To simplify notation we omit y from the notation. The dependence should be clear from the context.

Consider all y, σ such that $A(\psi_{+}^{i\sigma})$ holds. Taking the expectation of (3.20) over all possible values of ς

$$\begin{aligned}
\Pr[Y = y | X^i = s^{i\sigma}] &= \sum_{\varsigma=\varsigma_\sigma}^{M} \mathcal{F}(\varsigma) \left\{ \prod_{k \in C(i,y)} \binom{\varsigma}{(y_k - 1)} p^{(y_k - 1)} (1-p)^{\varsigma - (y_k - 1)} \right\} \\
&\quad \times \left\{ \prod_{k \notin C(i,y)} \binom{\varsigma}{y_k} p^{y_k} (1-p)^{\varsigma - y_k} \right\} \\
&\quad \times \left(\prod_{l \in \psi_o^{i\sigma}} \frac{\varsigma - y_l + 1}{y_l} \right) \left(\frac{p}{1-p} \right)^{|\psi_c^{i\sigma}|} \\
&= \sum_{\varsigma=\varsigma_\sigma}^{M} G(\varsigma, y) \left(\prod_{l \in \psi_o^{i\sigma}} \frac{\varsigma - y_l + 1}{y_l} \right) \left(\frac{p}{1-p} \right)^{|\psi_o^{i\sigma}|} \qquad (3.23)
\end{aligned}$$

where $G(\varsigma, y)$ is not a function of σ. The values for ς_σ among different letters σ differ in at most one. This follows from (3.22) and the fact that for Bernoulli signatures $s_k^{i\sigma} \in \{0, 1\}$.

Equation (3.23) presents some problems. We are after a simple letter detection test so that full computation of the likelihood probabilities not be required. Consider for simplicity a system with $|\psi_{+}^{i\sigma}| = h, \ \forall \sigma$. (3.23) simplifies to

$$\Pr[Y = y | X^i = s^{i\sigma}] = \sum_{\varsigma=\varsigma_\sigma}^{M} G'(\varsigma, y) \left(\prod_{l \in \psi_c^{i\sigma}} \frac{\varsigma - y_l + 1}{y_l} \right) \qquad (3.24)$$

$$G'(\varsigma, y) = G(\varsigma, y)(p/1-p)^{|\psi_c^{i\sigma}|}$$

Again $G'(\varsigma, y)$ is independent of σ.

For each possible output vector y define, for each feasible letter σ and $\varsigma_\sigma \leq \varsigma \leq M$, the metric

$$m_\varsigma(i, \sigma) = \left(\prod_{l \in \psi_c^{i\sigma}} \frac{\varsigma - y_l + 1}{y_l} \right)$$

which is a function of the vector y, of the letter σ and of the assumed number of interfering sessions ς. We will illustrate with an example the problems encountered in simplifying (3.24)

.

We consider a 2-signature scheme with $M = 10$. The signatures and a sample output vector are

$$s^{i\sigma} = (0,0,0,0,1,1,1)$$

$$s^{i\gamma} = (1,1,1,0,0,0,0)$$

$$y \;\; = (4,4,5,0,3,5,5)$$

We number slots from 0 to 6. Thus $C(i,y) = \{0,1,2,4,5,6\}$, $\psi_c^{i\sigma} = \{0,1,2\}, \psi_c^{i\gamma} = \{4,5,6\}$. Also, $\varsigma_\sigma = \varsigma_\gamma = 5$.

For $\varsigma = 5$ we have

$$m_5(i,\sigma) = \left(\frac{5-4+1}{4}\right)\left(\frac{5-4+1}{4}\right)\left(\frac{5-5+1}{5}\right) = \frac{1}{20}$$

$$m_5(i,\gamma) = \left(\frac{5-3+1}{3}\right)\left(\frac{5-5+1}{5}\right)\left(\frac{5-5+1}{5}\right) = \frac{1}{25}$$

and hence

$$m_5(i,\sigma) > m_5(i,\gamma)$$

It can be verified that $m_6(i,\sigma) > m_6(i,\gamma)$, $m_7(i,\sigma) = m_7(i,\gamma)$.

For $\varsigma = 8$

$$m_8(i,\sigma) = \left(\frac{8-4+1}{4}\right)\left(\frac{8-4+1}{4}\right)\left(\frac{8-5+1}{5}\right) = \frac{5}{4}$$

$$m_8(i,\gamma) = \left(\frac{8-3+1}{3}\right)\left(\frac{8-5+1}{5}\right)\left(\frac{8-5+1}{5}\right) = \frac{32}{25}$$

and hence

$$m_8(i,\sigma) < m_8(i,\gamma)$$

This also holds for $\varsigma = 9, 10$.

$m_\varsigma(i,\sigma)$ is the only factor in (3.24) dependent on σ. From this example we see that, even in instances for which all feasible signatures σ have equal value of ς_σ, computation of the Maximum Likelihood decision requires evaluation of the summation in (3.24) since $m_\varsigma(i,\sigma)$ may be maximum over some range of ς but not over another, as in this example. Moreover, since $\mathcal{F}(\varsigma)$ is a factor in $G'(\varsigma, y)$ in (3.24) (c.f. derivation of (3.23)), the range of values of ς which is most relevant depends on the probability distribution for ς. In general thus, there seems to be no simple way to compute the M.L. decision and we are left with evaluating the likelihood function (3.23) for all feasible letters.

PROBABILITY OF ERROR. The general expression for the probability of error is

$$P_e = \sum_{\mathbf{S}^i} \Pr[X^i = s^{i\sigma}] \Pr[\widehat{X^i} \neq s^{i\sigma} | X^i = s^{i\sigma}] \tag{3.25}$$

For the OR channel the M.L. detector turned out to be a Template detector and hence its probability of error will be treated below. For the ADDER channel, given that ς is known to the detector, the M.L. detector evaluates the metric in (3.21) for all feasible σ and chooses the signature with minimum metric; in case of a tie the choice is arbitrary. Note that this is not a Template detector. Computation of the probability of error, even for a 2-signature case, involves calculation of the probability that the product in (3.21) for some σ be less than or be equal to a similar product — with $y_l - 1$ replaced for y_l, $l \in \psi_c^{i\gamma}$ — given that γ was sent, say. This is typically a difficult calculation and we have not been successful in solving it. When ς is not known to the detector the situation seems even worst.

We derive probabilities of detection error for the Template and Sum detectors. We consider a 2-signature scheme and let $h(\sigma) = h(\gamma) = h$ and assume $\mathcal{F}(\varsigma)$ is Binomial as in section 3.2.1, but with interference parameter q. Define H as the number of pulse positions in one session signature which are not common to the other signatures of the session. H is only equal to the number of pulses h in a signature of session i if signatures of session i have no overlapping pulse positions. H is equal for both signatures and equals half the Hamming distance[3].

Template detector. The Template detector in the OR and ADDER channel yields

$$\begin{aligned} P_e &= \frac{1}{2} \Pr[A(\psi_c^{i\sigma}) | X^i = s^{i\sigma}] \\ &= \frac{1}{2} \sum_{\varsigma=1}^{M} \binom{M}{\varsigma} q^{\varsigma} (1-q)^{M-\varsigma} (1 - (1-p)^{\varsigma})^H \end{aligned} \tag{3.26}$$

$; \ 0 < q < 1, \ 0 < p < 1.$

[3] The definition of H can be generalized to a many signature scheme. It would be the number of slots for which a given feasible signature has pulses but for which no other feasible signature has pulses.

Sum detector. For the Sum detector (2.5) let $T = h$ as in section 3.2.1. For the OR channel (3.26) applies. For the ADDER channel the derivation is similar to that of (3.12) replacing q for Q and H for h.

$$P_e = \frac{1}{2}\sum_{\varsigma=1}^{M}\binom{M}{\varsigma} q^{\varsigma}(1-q)^{M-\varsigma}\left\{\sum_{r=H}^{\varsigma H}\binom{\varsigma H}{r}(1-p)^{\varsigma H - r}p^r\right\} \tag{3.27}$$

$; \; 0 < q < 1, \; 0 < p < 1.$

3.3 ROULETTE SCHEMES

3.3.1 2-SIGNATURE, ASYMMETRIC, ROULETTE SCHEME

ς and $\mathcal{F}(\varsigma)$ are defined as in section 3.2.1. In this section a stands for the fixed number of pulses per Roulette signature; equal for all sessions.

MAXIMUM LIKELIHOOD DETECTION.

OR channel. For the OR channel the M.L. detector for session i is a Template detector (Lemma 3.1).

ADDER channel. For the ADDER channel we define ς_i to be the number of active interfering sessions sending non null signatures under the hypothesis $X^i = s^i$.

$$\varsigma_i + 1 = (1/a)\sum_{k\in\Psi} y_k \tag{3.28}$$

ς_i is a function of y but for simplicity of notation we will omit y as an argument.

Appendix AP-2 shows that the M.L. detector takes the form:

$$\widehat{X^i} = \begin{cases} \phi, & \text{if } \Theta(y) > 1; \\ s^i, & \text{if } \Theta(y) \le 1. \end{cases} \tag{3.29}$$

with $\Theta(y)$ defined by

$$\Theta(y) = \frac{\mathcal{F}(\varsigma_i + 1)}{\mathcal{F}(\varsigma_i)}\left\{\frac{\prod_{j=1}^{a}(\varsigma_i a + j)}{\prod_{k\in\psi_+^i} y_k}\right\}\left(\frac{1}{n}\right)^a \tag{3.30}$$

We allow $\Theta(y) = \infty$ in (3.29) which is the case when any of the signature pulses is missing. Notice that the detector could compute ς_i from (3.28) . Notice also that observation of the slots k in ψ_+^i alone is not sufficient since computation of ς_i requires knowledge of the number of pulses in y and this will require observation of all slots in general.

<u>Theorem 3.2</u> Consider a Frame Synchronous, 2-signature, asymmetric, Roulette scheme operating on an ADDER channel. Let M be the maximum number of interfering sessions. The probability distribution for the number of active interfering sessions is assumed Binomial, each session being active with probability q. Let a be the fixed number of pulses in the Roulette signatures. Let $\Pr[X^i = s^i] = \Pr[X^j = s^j] \leq 1/2$ for all active sessions. Let the signature s^i of session i have at most one pulse per slot. If

$$(M+1)\left(\frac{a}{n}\right) \leq 1 \tag{3.31}$$

then the M.L. detector for session i is a Template detector.

<u>Proof:</u> We need to show that (3.31) implies that the M.L. detection rule (3.29) reduces to the Template detector of (2.7). If the event $\overline{A}(\psi_+^i)$ occurs we have $\Theta(y) = \infty$ and $\widehat{X^i} = \phi$ from (3.29) . It remains to consider the event $A(\psi_+^i)$. From the theorem statement, $\mathcal{F}(\varsigma)$ in (3.30) is Binomial with success probability $Q = \Pr[X^i = s^i]q, \forall j$. We can replace into (3.30) the bound $a(\varsigma_i + 1) \geq (\varsigma_i a + j), \forall \varsigma_i, j$; the values $y_k = 1\ \forall k \in \psi_+^i$ and the ratio

$$\frac{\mathcal{F}(\varsigma_i + 1)}{\mathcal{F}(\varsigma_i)} = \frac{M - \varsigma_i}{\varsigma_i + 1}\frac{Q}{1 - Q}$$

to obtain the upperbound

$$\Theta(y) \leq \frac{M - \varsigma_i}{\varsigma_i + 1}\frac{Q}{1 - Q}\left[\left(\frac{a}{n}\right)(\varsigma_i + 1)\right]^a \tag{3.32}$$

Assume first that $a > 1$. Considering ς_i as a continuous variable, the RHS of (3.32) is maximized w.r.t ς_i at

$$\varsigma_i^* = \frac{(a-1)M - 1}{a}$$

For $a > 1$, $0 < \varsigma_i^* < M$.

Replacing into the RHS of (3.32)

$$\Theta(y) \leq \frac{(a-1)^{a-1}}{a^a} \frac{Q}{1-Q} \left[\left(\frac{a}{n}\right)(M+1) \right]^a \tag{3.33}$$

From the theorem $\Pr[X^i = s^i] \leq 1/2$; hence, $Q = \Pr[X^i = s^i]q \leq 1 - Q,\ \forall q,\ 0 < q < 1$ so

$$\frac{(a-1)^{a-1}}{a^a} \frac{Q}{1-Q} \leq 1 \qquad ;\ a > 1.$$

We finally arrive at the upperbound

$$\Theta(y) \leq \left[\left(\frac{a}{n}\right)(M+1) \right]^a \qquad ;\ a > 1. \tag{3.34}$$

But we have left out the case $a = 1$. For $a = 1$ the bound in (3.32) is maximized, over the range $\varsigma_i \in [0, M]$, at $\varsigma_i^* = 0$, yielding a value

$$\frac{M}{n} \frac{Q}{1-Q}$$

so (3.34) is still a valid upperbound for $a > 0$.

Thus, if $(a/n)(M+1) \leq 1$ then $\Theta(y) \leq 1\ \forall \varsigma_i,\ \forall y$ such that $A(\psi_+^i)$ holds and the M.L. detector reduces to a Template detector.

∎

Note that the restriction of having at most one pulse per slot does not apply to interfering signatures and also that now observation of the slots in ψ_+^i is sufficient for detection. (3.31) is again a plausible network operating condition as for the asymmetric Bernoulli scheme.

In contrast with Theorem 3.1, Theorem 3.2 made use of a specific distribution for the set of active sessions. With the binomial distribution chosen, the condition in (3.31) is close to being a necessary condition for the Template detector to be a M.L. detector, for all received vectors y, all values of q, $0 < q < 1$, and all values of the parameter a.

To see this, consider the situation $a = 1, \varsigma_i = 0,\ y_k = 1\ \forall k \in \psi_+^i$. The exact value of $\Theta(y)$, from (3.30) , would be

$$\Theta(y) = \frac{M}{n} \frac{Q}{1-Q}$$

If $\Pr[X^i = s^i] = 1/2$ and $q \to 1$ then $Q/(1-Q) \to 1$ and $\Theta(y) \to M/n$. So unless $M \leq n$, the Template detector is not an M.L. detector, in view of (3.29) . But for $a = 1$ the bound in (3.31) is $M + 1 \leq n$. The difference is of little importance for M, n large. In this sense we can say that (3.31) is almost a necessary condition for a Template detector to be a Maximum Likelihood detector regardless of the values of a, q, for all received vectors y.

The value of $a = 1$ is realistic for a pulse position modulation system. However, for systems with $a > 1$ (3.31) could be relaxed to allow higher values of Ma/n since (3.34) would not be tight in these cases. In this sense the condition in (3.31) is less pleasing than in Theorem 3.1. Moreover, Theorem 3.2 specifies the probability distribution for the number of active interfering sessions and also constrains the source letter probabilities.

If Theorem 3.2 were to be extended to all distributions $\mathcal{F}(\varsigma)$ the condition to be satisfied instead of (3.31) would be (c.f. (3.30))

$$\frac{\mathcal{F}(\varsigma_i + 1)}{\mathcal{F}(\varsigma_i)} \leq \frac{n^a}{\prod_{j=1}^{a} (\varsigma_i a + j)} \qquad ; \forall \varsigma_i. \tag{3.35}$$

PROBABILITY OF ERROR. Let $\mathcal{F}(\varsigma)$ be Binomial as for P_e in section 3.2.1. The interference parameter is again $Q \equiv \Pr[X^i = s^i]q$.

Template detector. The probability of error expression[4] for a Template detector in both OR and ADDER channels is (appendix AP-3):

$$P_e = \Pr[X^i = \phi] \sum_{\varsigma=1}^{M} \binom{M}{\varsigma} Q^{\varsigma}(1-Q)^{M-\varsigma} \left\{ \sum_{r=h}^{\varsigma a} A(r,h) \binom{\varsigma a}{r} \left(\frac{n-h}{n}\right)^{\varsigma a - r} \left(\frac{h}{n}\right)^{r} \right\}$$
$$; \ 0 < Q < 1, \ 0 < h < n. \tag{3.36}$$

where $A(r, h)$ is the probability that all h slots are occupied when r pulses fall in these h slots and is given by

$$A(r,h) = \sum_{\nu=0}^{h} (-1)^{\nu} \binom{h}{\nu} \left(1 - \frac{\nu}{h}\right)^{r} \qquad ; \ r \geq h. \tag{3.37}$$

[4] In the expression, if $h > \varsigma a$ the summand is considered zero. This accounts for cases where $a < h$.

Sum detector. For the Sum detector probability of error we let $T = h$ as in section 3.2.1. For the OR channel (3.36) , (3.37) hold. For the ADDER channel the expression[4] is

$$P_e = \Pr[X^i = \phi] \sum_{\varsigma=1}^{M} \binom{M}{\varsigma} Q^{\varsigma}(1-Q)^{M-\varsigma} \left\{ \sum_{r=h}^{\varsigma a} \binom{\varsigma a}{r} \left(\frac{n-h}{n}\right)^{\varsigma a - r} \left(\frac{h}{n}\right)^{r} \right\} \quad (3.38)$$

$$; \ 0 < Q < 1, \ 0 < h < n.$$

The term within braces is the probability that the number of pulses present in ψ^i_+ is h or more, given $\varsigma, X^i = \phi$.

M.L. detector Upperbound, ADDER channel. We now borrow bounds for M.L. detection found in [Gallager]. The bounds are used there to prove the channel coding theorem. Here we use it as a reference to compare with the performance of the Template detector.

Let $P_{e,\phi}, P_{e,s^i}$ denote the probability of error for a Maximum Likelihood detector given channel inputs ϕ, s^i respectively. The probability of error P_e for a M.L. detector can be expressed as

$$P_e = \Pr[X^i = \phi] P_{e,\phi} + \Pr[X^i = s^i] P_{e,s^i} \quad (3.39)$$

From [Gallager], equation (5.3.4):

$$P_{e,\phi} \leq \sum_{y \in Y^*} \Pr[Y = y | X^i = \phi]^{1-s} \Pr[Y = y | X^i = s^i]^s \qquad ; \ 0 < s < 1. \quad (3.40)$$

Y^* is the set of vectors y such that $\Pr[Y = y|X^i = \phi] \neq 0$ and $\Pr[Y = y|X^i = s^i] \neq 0$. The exact same bound also applies to P_{e,s^i} so, factoring the bound and since $\Pr[X^i = \phi] + \Pr[X^i = s^i] = 1$, the same bound applies for P_e. The bound can be minimized with respect to the exponent s. An analytical minimization over s has not been obtained so instead we have used numerical evaluations of the bound to observe its behaviour as a function of s, c.f. Figure 3.12 on page 85. For the numerical example treated it seems that the optimal value of s is $s \to 0$ for all values of h. The correct interpretation of (3.40) as $s \to 0$ is a summation of

$\Pr[Y = y|X^i = \phi]$ over all outputs such that neither transition probability equals zero, c.f. [Gallager, equation (5.3.9)]. From (3.39) , (3.40) this means that this bound tends to

$$P_e \leq \sum_{y \in Y^*} \Pr[Y = y|X^i = \phi]$$

where the summation is as for (3.40) . Some reflection will reveal that, in fact, the expression for the probability of error of a Template detector can be written as

$$P_e = \Pr[X^i = \phi] \sum_{y \in Y^*} \Pr[Y = y|X^i = \phi]$$

so this latter expression is an even better bound for this example. The numerical example presented uses values of $M = 3, q = 0.3, n = 30$. For high inerference levels the Template detector will only be a M.L. detector for low values of a/n. For higher values, the performance of the M.L. detector is expected to diverge from that of the Template detector and if the bound reflects this relative decrease in P_e then $s = 0$ will probably not be optimal. s should be optimized for each value of a/n.

Evaluation of (3.40) is made difficult by the large number of possible output vectors y over which the summation must be made. We have simplified the expression for the bound to that in (3.41) . The detailed derivation appears in Appendix AP-4.

$$\begin{aligned} P_e \leq & \sum_{\varsigma_i=0}^{M-1} \binom{M}{\varsigma_i} Q^{\varsigma_i}(1-Q)^{M-\varsigma_i} \left[\frac{M-\varsigma_i}{\varsigma_i+1} \frac{Q}{1-Q} \frac{\prod_{j=1}^{a}(\varsigma_i a + j)}{n^a} \right]^{1-s} \\ & \times \sum_{k=0}^{\varsigma_i a} \binom{\varsigma_i a}{k} \left(\frac{h}{n}\right)^k \left(1-\frac{h}{n}\right)^{\varsigma_i a - k} \\ & \times \sum_{Y_+^i} \frac{k!}{\prod_{k \in \psi_+^i} (y_k - 1)!} \left(\frac{1}{h}\right)^k \left(\frac{1}{\prod_{k \in \psi_+^i} y_k} \right)^{1-s} \end{aligned} \tag{3.41}$$

The outer summation is over all values of ς_i, the number of interfering sessions under hypothesis $X^i = s^i$, such that neither transition probability equals zero. Given this value of ς_i, the second summation is over all values of the number k of "extra" pulses placed in the set of slots ψ_+^i.

By "extra" pulses we mean those pulses beyond the h pulses needed for $A(\psi_+^i)$ to hold, where h is the number of slots in ψ_+^i. Given these values of ς_i, k, the inner summation is over the set Y_+^i of all h-vector configurations corresponding to $h+k$ pulses in ψ_+^i and at least one pulse in each of its slots.

The bound is simpler to evaluate in this form since, instead of a summation over all possible n-vectors called for in (3.40) , we now have a summation only over those h-vector configurations compatible with ς_i, k. Even considering the outer and middle summations over ς_i, k respectively, the number of computations can be considerably reduced[5]. The numerical evaluations are discussed in section 3.4.

3.3.2 SYMMETRIC, ROULETTE SCHEME

The rule for Maximum Likelihood detection in this section will require the definitions of $B(i,y), C(i,y), \psi_c^{i\sigma}$ given in (3.13) , (3.14) , (3.15) respectively. ς is equal to the number of active interfering sessions. It can easily be computed from the total number of pulses received in y. Its dependence on y is omitted to simplify notation.

MAXIMUM LIKELIHOOD DETECTION.

OR channel. It will be best to consider directly the case when all signatures in session i have equal number of pulses. Let $|\psi_+^{i\sigma}| = h(\sigma) = h,\ \forall\sigma \in \mathcal{A}$. For any y, σ such that $\overline{A}(\psi_+^{i\sigma})$ holds

$$\Pr[Y = y | X^i = s^{i\sigma}] = 0$$

so the M.L. decision must be among the set of feasible signatures in $B(i,y)$. y is the received output vector. For each feasible $s^{i\sigma}$ we can divide the frame into 3 sets of slots: $\psi_+^{i\sigma}$ for which $s^{i\sigma}$ has pulses, $\psi_-^{i\sigma}$ for which y has pulses but $s^{i\sigma}$ has no pulses and $\psi_0^{i\sigma}$ for which y has no

[5] A further computational simplification is attained by noting that all permutations of a given h-vector configuration in the expression yields the same result.

pulses. For simplicity of notation denote the sizes of these sets by $\psi_{+}^{i\sigma} = h$, $\psi_{-}^{i\sigma} = u$, $\psi_{0}^{i\sigma} = v$. $n = h + u + v$.

For feasible signatures the likelihood probabilities, given ς, are

$$\Pr[Y = y | X^i = s^{i\sigma}, \varsigma] = \Pr[Z_k \neq 0 \ \forall k \in \psi_{-}^{i\sigma}, \ Z_k = 0 \ \forall k \in \psi_{0}^{i\sigma}, \varsigma]$$

There are a total of ςa pulses. Since all pulses have uniform distribution over the slots, the probability that no pulses are present in $\psi_0^{i\sigma}$ is

$$\left(1 - \frac{v}{n}\right)^{\varsigma a}$$

Let $b(r; \varsigma a, f)$ denote the Binomial probability of having r successes in ςa trials when the success probability is f. Then given that there are no pulses in $\psi_0^{i\sigma}$ the probability that all slots in $\psi_{-}^{i\sigma}$ are occupied can be written as

$$\sum_{r=u}^{\varsigma a} b(r; \varsigma a, \frac{u}{h+u}) A(r, u)$$

where $A(r, u)$ is given by equation (3.37) .

Taking the expectation over all ς we obtain, for all feasible letters σ

$$\Pr[Y = y | X^i = s^{i\sigma}] = \sum_{\varsigma=1}^{M} \mathcal{F}(\varsigma) \left(1 - \frac{v}{n}\right)^{\varsigma a} \sum_{r=u}^{\varsigma a} b(r; \varsigma a, \frac{u}{h+u}) A(r, u) \qquad (3.42)$$

In other words, the likelihood probabilities are equal for all feasible signatures. The M.L. detector for session i can choose one arbitrarily. This is equivalent to a Template detector.

ADDER channel.

Lemma 3.3 Consider a Frame Synchronous, symmetric, Roulette scheme operating on an ADDER channel. Let the signature set $\mathbf{S}^i$ of session i be such that signatures have at most one pulse per slot. Then the M.L. detector for session i takes the form

$$\left.\begin{array}{l} \widehat{X^i} = s^{i\sigma} \textit{ such that } \sigma \textit{ minimizes, over } \gamma \in B(i,y), \textit{ the metric} \\ \qquad m(i,\gamma) = \prod_{k\in\psi_c^{i\gamma}} y_k \\ \textit{If } \sigma \textit{ is not unique choose arbitrarily among all such } \sigma. \end{array}\right\} \tag{3.43}$$

Proof: See Appendix AP-5.

∎

Note that the restriction of having at most one pulse per slot does not apply to interfering signatures. Also note that if σ is not unique, an arbitrary choice among all letters of minimum metric does not alter the performance of the M.L. receiver. Finally notice that since $\psi_c^{i\gamma} \subseteq C(i,y)$, $\forall\gamma \in B(i,y)$, observation of the slots in $C(i,y) \subseteq \Psi$ is sufficient for M.L. detection.

PROBABILITY OF ERROR. The general expression for the probability of error is

$$P_e = \sum_{\mathbf{S}^i} \Pr[X^i = s^{i\sigma}]\Pr[\widehat{X^i} \neq s^{i\sigma} | X^i = s^{i\sigma}] \tag{3.44}$$

The situation is similar to the one encountered for the symmetric Bernoulli scheme. For the OR channel the M.L. detector turned out to be a Template detector and hence its probability of error will be treated below. For the ADDER channel the M.L. detector is not a Template detector. Computation of the probability of error, even for a 2-signature case, involves calculation of the probability that a product of multinomially distributed numbers —y_k, $k \in \psi_c^{i\sigma}$— be less than or be equal to the product of another set of multinomially distributed numbers —$y_k - 1$, $k \in \psi_c^{i\gamma}$— given that γ was sent, say. This is typically a difficult calculation and we have not been successful in solving it. We will be able to find a useful expression for upperbounding the probability of error of this Maximum Likelihood detector. The expression is numerically evaluated and discussed in section 3.4.

Template detector. An expression for the Template detector probability of error is simple to derive under the conditions

1- $\alpha = |\mathcal{A}| = 2$, i.e. 2-signature case.

2- $s_k^{i\sigma} \in \{0,1\}$, $\forall \sigma, k$, i.e. at most one pulse per slot in s^i.

Let the source letters be σ, γ. Define H as the number of pulse positions in one signature which are not common to the other signature as in section 3.2.2. The expression, explained further below, is

$$\begin{aligned} P_e &= \sum_{\sigma \in \mathcal{A}} \Pr[X^i = s^{i\sigma}] \Pr[\widehat{X^i} \neq s^{i\sigma} | X^i = s^{i\sigma}] \\ &= \Pr[\widehat{X^i} \neq s^{i\sigma} | X^i = s^{i\sigma}] \qquad ; \text{ due to symmetry} \\ &= \left(\frac{1}{2}\right) \sum_{\varsigma=1}^{M} \binom{M}{\varsigma} q^{\varsigma} (1-q)^{M-\varsigma} \left\{ \sum_{r=H}^{\varsigma a} A(r,H) \binom{\varsigma a}{r} \left(\frac{n-H}{n}\right)^{\varsigma a - r} \left(\frac{H}{n}\right)^{r} \right\} \qquad (3.45) \\ & \qquad ; \ 0 < a, h < n, \ 0 < q < 1, \ 0 < H < \lfloor n/2 \rfloor . \end{aligned}$$

where $A(r, H)$ is given by (3.37) . For $H = 0$ $P_e = 1/2$ since both signatures are equal.

The first step leading to (3.45) is due to the symmetric situation for the two signatures, which implies

$$\Pr[\widehat{X^i} \neq s^{i\sigma} | X^i = s^{i\sigma}] = \Pr[\widehat{X^i} \neq s^{i\gamma} | X^i = s^{i\gamma}]$$

This is so since from conditions 1 and 2 $h = h(\sigma) = h(\gamma)$, $H = |\psi_c^{i\sigma}| = |\psi_c^{i\gamma}|$ and also all interfering pulses have independent uniform distributions over the set of all slots.
The second step is similar to that in the derivation of (3.36) but replacing q for Q and H for h.

Sum detector. For the Sum detector again we let $T = h$ in (2.5). For the OR channel (3.45) applies. For the ADDER channel, and conditions 1 and 2 above, we obtain, as in (3.38) but replacing q and H,

$$P_e = \frac{1}{2}\sum_{\varsigma=1}^{M}\binom{M}{\varsigma}q^{\varsigma}(1-q)^{M-\varsigma}\left\{\sum_{r=H}^{\varsigma a}\binom{\varsigma a}{r}\left(\frac{n-H}{n}\right)^{\varsigma a-r}\left(\frac{H}{n}\right)^{r}\right\} \tag{3.46}$$
$$; \ 0 < a, h < n, \ 0 < q < 1, \ 0 < H \leq \lfloor n/2 \rfloor.$$

For $H = 0$ $\quad P_e = 1/2$ since both signatures are equal.

M.L. detector Upperbound, ADDER channel. Again we borrow bounds for M.L. detection from [Gallager].

Let $P_{e,\sigma}, P_{e,\gamma}$ denote the probability of error for a Maximum Likelihood detector given that the channel inputs were $s^{i\sigma}, s^{i\gamma}$ respectively. The probability of error P_e for a M.L. detector can be expressed as

$$P_e = \Pr[X^i = s^{i\sigma}]P_{e,\sigma} + \Pr[X^i = s^{i\gamma}]P_{e,\gamma} \tag{3.47}$$

From [Gallager, equation (5.3.4)]:

$$P_{e,\sigma} \leq \sum_{y \in Y^*} \Pr[Y = y | X^i = s^{i\sigma}]^{1-s} \Pr[Y = y | X^i = s^{i\gamma}]^{s} \qquad ; \ 0 < s < 1. \tag{3.48}$$

Y^* is the set of vectors y such that $\Pr[Y = y|X^i = s^{i\sigma}] \neq 0$ and $\Pr[Y = y|X^i = s^{i\gamma}] \neq 0$. The exact same bound also applies to $P_{e,\gamma}$ so, factoring the bound and since $\Pr[X^i = s^{i\sigma}] + \Pr[X^i = s^{i\gamma}] = 1$, the same bound applies for P_e.

For simplicity let signatures σ, γ be such that they have no common pulse positions. i.e. $\psi_+^{i\sigma} \bigcap \psi_+^{i\gamma} = \phi$ where ϕ denotes the empty set. Let $|\psi_+^{i\sigma}| = |\psi_+^{i\gamma}| = h$. Thus

$$C(i, y) \equiv \psi_+^{i\sigma} \bigcup \psi_+^{i\gamma}$$
$$|C(i, y)| = 2h$$

The bound can be minimized with respect to the exponent s. We can show by symmetry arguments that the optimal value of s is $s = 1/2$. The detailed derivation appears in Appendix AP-6. Again we have simplified the expression for the bound to that in (3.49) .

$$P_e \leq \sum_{\varsigma=1}^{M} \binom{M}{\varsigma} q^{\varsigma}(1-q)^{M-\varsigma} \times \sum_{k=a}^{\varsigma a} \binom{\varsigma a}{k} \left(\frac{2h}{n}\right)^{k} \left(1-\frac{2h}{n}\right)^{\varsigma a-k} \times \sum_{Y_C} \frac{k!}{\prod_{k\in C(i,y)} (y_k-1)!} \left(\frac{1}{2h}\right)^{k} \left(\frac{1}{\prod_{k\in C(i,y)} y_k}\right)^{1/2} \tag{3.49}$$

The outer summation is over all values of ς, the number of active interfering sessions. Given this value of ς, the second summation is over all values of the number k of interfering pulses placed in the set of slots $C(i,y)$. This value must be at least $a = h$ so that $C(i,y)$ can have at least one pulse for every slot. Given these values of ς_i, k, the inner summation is over the set Y_C of all possible $2h$-vector configurations consisting of k pulses in $C(i,y)$. As for the bound in the asymmetric Roulette scheme the number of computations is shortened since now we do not sum over all possible n-vectors in (3.48) [6]. Numerical evaluations and discussions appear in section 3.4.

[6] A further computational simplification is attained by noting that all permutations of a given $2h$-vector configuration in the expression yields the same result.

3.4 NUMERICAL EVALUATION OF PERFORMANCE

We have used here the probability of error expressions, P_e, derived for the Frame Synchronous model and evaluated them numerically to observe their behaviour on some examples. We observe their behaviour as the number of pulses per signature h is varied, as the maximum number of interfering sessions M varies, as the number of slots per frame n varies, and also compare the models. The results shown correspond to all four models presented in sections 3.2 and 3.3.

3.4.1 PARAMETERS

For consistency we set the pulse placement parameter $p = h/n$ in the Bernoulli schemes and $a = h$ in the Roulette schemes. For the Bernoulli scheme this ensures that the expected number of pulses per interfering signature equals the number of pulses in the signatures of session i being detected. For the Roulette schemes it insures that the number of pulses in the interfering signatures is equal to the number of pulses in the signatures of session i being detected.

Rate. A meaningful rate parameter for these models is the number of source bits per channel slot. For 2-signature systems the rate is $R = 1/n$. For a system with α source letters $R = \log_2 \alpha/n$.

Average Power. Let $\overline{\mathrm{P}}$ be the expected value of the number of pulses per signature for a given session. $\overline{\mathrm{P}}$ equals the average power (in pulses/signature) of the transmitters if the system is stationary frame to frame; i.e. if the probabilistic description of the signatures remains the same at every frame. This would correspond to a steady state situation in our network models.

For the four models we have:

$\overline{\mathrm{P}} = \Pr[X^i = s^i] \;\; np,$ 2-signature asymmetric Bernoulli.

$\overline{\mathrm{P}} = \sum_{\sigma \in \mathcal{A}} \Pr[X^i = s^{i\sigma}] np = np$ 2-signature symmetric Bernoulli.

$\overline{\mathrm{P}} = \Pr[X^i = s^i] a,$ 2-signature asymmetric Roulette.

$\overline{\mathrm{P}} = \sum_{\sigma \in \mathcal{A}} \Pr[X^i = s^{i\sigma}] a = a$ 2-signature symmetric Roulette.

We assume in all models that source letters are equiprobable to simplify the discussions. For the symmetric schemes, $\overline{\mathrm{P}}$ does not depend on the source letter probabilities but the assumption simplifies the expressions for the probability of error. For asymmetric schemes, the probability of error performance would improve if $\Pr[X^i = s^i]$ are reduced from $1/2$.

Notice that for $\alpha = 2$ and a fixed value of a/n, or the same value for $p = h/n$, the rates for all schemes are equal but the average power for the asymmetric schemes is half of that for the symmetric schemes when equiprobable source letters are used.

3.4.2 NUMERICAL EVALUATION

We will illustrate the behaviour of the probability of error expressions derived with some examples. We are interested in the value of probability of error P_e as different system parameters are altered. We have expressions only for 2-signature systems. We have plotted, in examples presented, expressions for the Template detectors and for the Sum detectors derived in sections 3.2 and 3.3. They correspond to the ADDER channel model. Recall however that the expressions for the Template detector apply also to the OR channel model; and that the P_e expression for a Sum detector with threshold h in an OR channel is also identical to the Template detector expression. Thus the curves plotted cover both detectors and both channel models.

Curves for the Template detector of the asymmetric Bernoulli scheme and the asymmetric Roulette scheme have been shown to represent a M.L. detector only if $(M+1)h \leq n$ for the Bernoulli scheme and if $(M+1)a \leq n$ for the Roulette scheme (c.f. Theorems 3.1, 3.2). In the graphs presented we indicate these regions. These conditions essentially mean that if the channel interference is low enough the M.L. receiver is a Template detector. The curves for the 2-signature, symmetric Bernoulli and Roulette schemes correspond to Template and Sum detectors. For Roulette schemes we will also show the upperbounds stated in subsections 3.3.1 and 3.3.2 for M.L. detectors.

P_e as a function of h. For clarity the graphs for P_e shown in this section appear as continuous lines even though only points corresponding to integer values of the variable parameter h are meaningful.

Figures 3.1,3.3,3.5,3.7 show curves of P_e vs h for the different models operating on the ADDER channel. Two curves appear per figure, one for the Template detector and one for the Sum detector. The expressions used are as follows:

	Template	Sum	Model
Figure 3.1	(3.11)	(3.12)	2-sig. asym. Bernoulli.
Figure 3.3	(3.26)	(3.27)	2-sig. sym. Bernoulli.
Figure 3.5	(3.36)	(3.38)	2-sig. asym. Roulette.
Figure 3.7	(3.45)	(3.46)	2-sig. sym. Roulette.

The system parameters used are: number of slots per frame $n = 100$, maximum number of interfering sessions $M = 10$ and probability of interfering session being active $q = 0.3$. Since the models are idealized these values of P_e should not be considered realistic for an equivalent synchronous physical system unless other noise components (e.g. receiver thermal noise) are negligible. Here we focus only on the effects of interuser interference. For the Bernoulli and Roulette asymmetric schemes the Template detector curves represent Maximum Likelihood detection only up to $h = 9$. The Template curves for the asymmetric schemes have an optimal value of h which falls outside the M.L. detection region.

As h increases from 1 we have two contending effects: the expected number of interfering pulses increases yet, on the other hand, the probability of interfering pulses falling into each

2-SIGNATURE, ASYMMETRIC, BERNOULLI SCHEME
PROBABILITY OF ERROR
ADDER CHANNEL

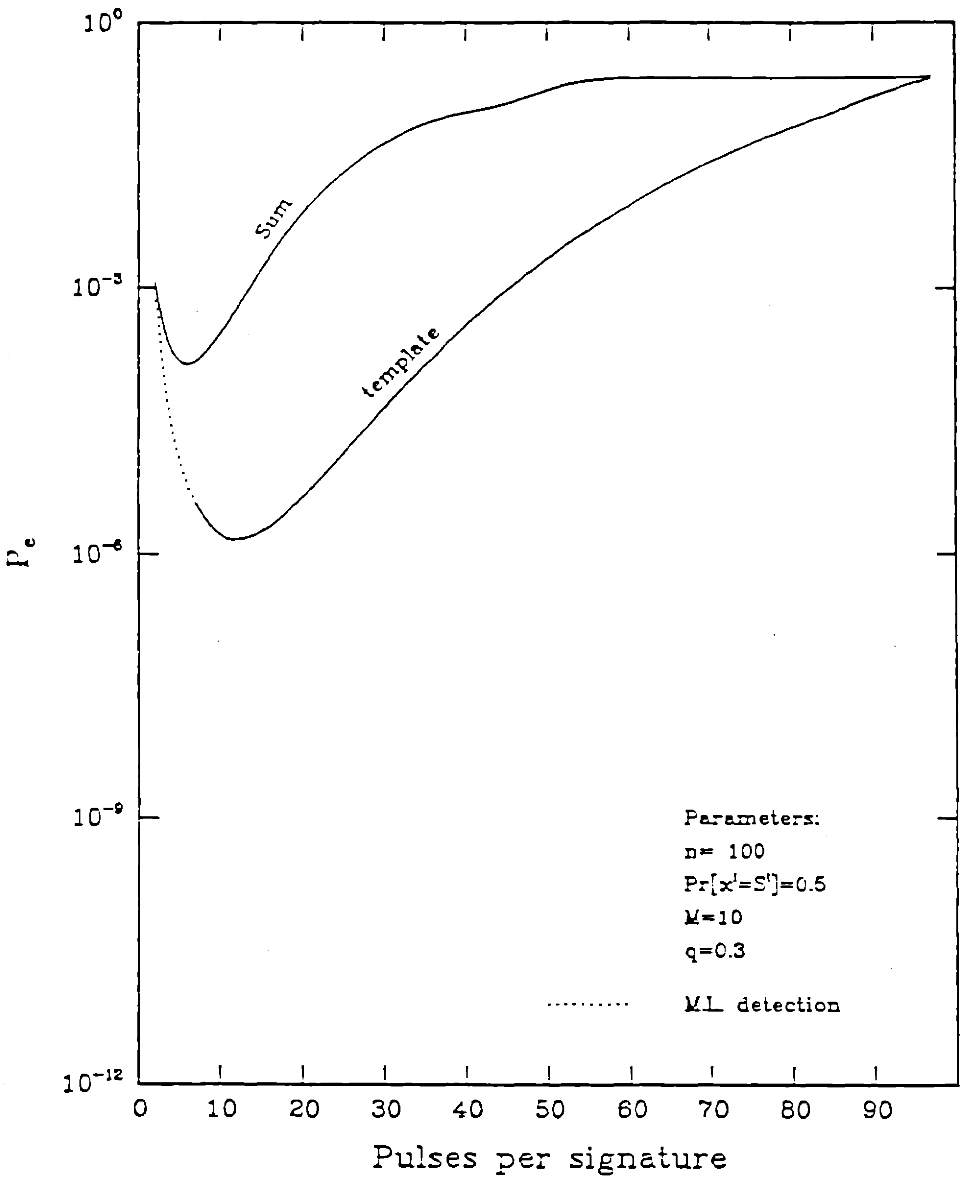

Figure 3.1

2-SIGNATURE, ASYMMETRIC, BERNOULLI SCHEME

ADDER CHANNEL

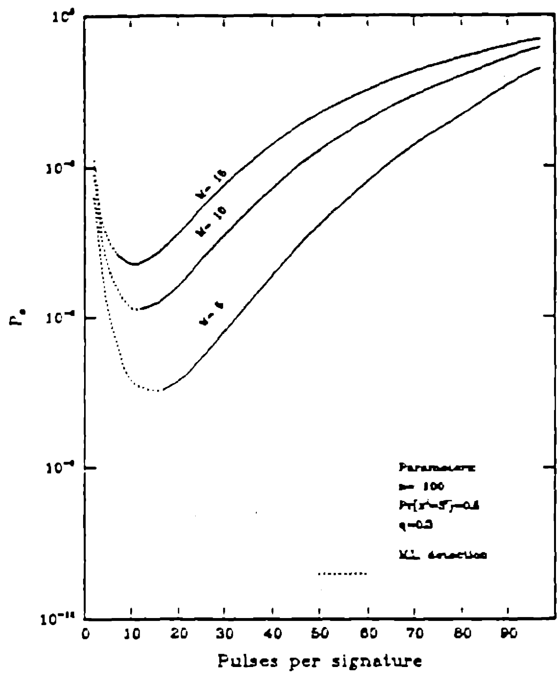

Figure 3.2.a

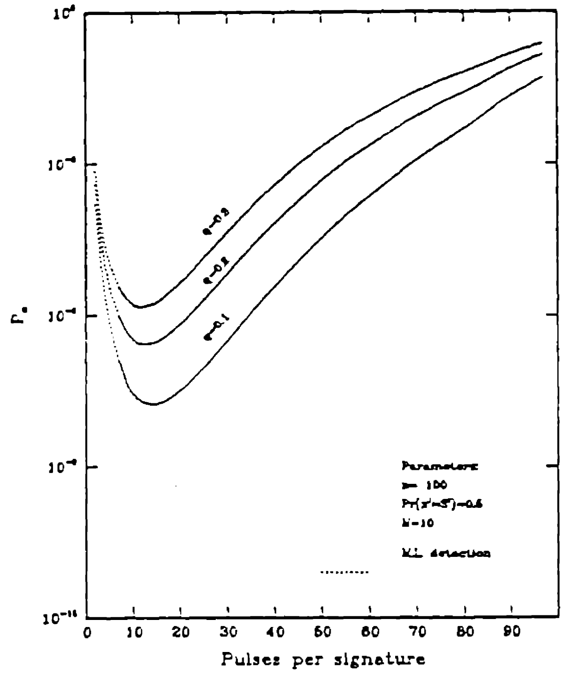

Figure 3.2.b

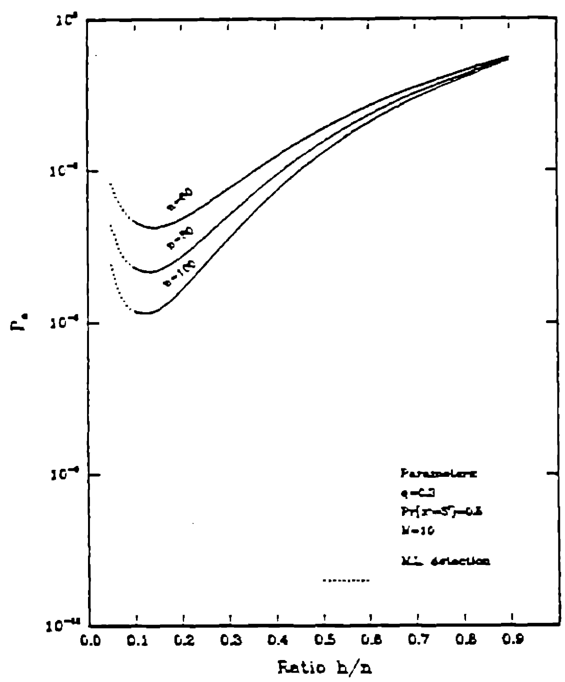

Figure 3.2.c

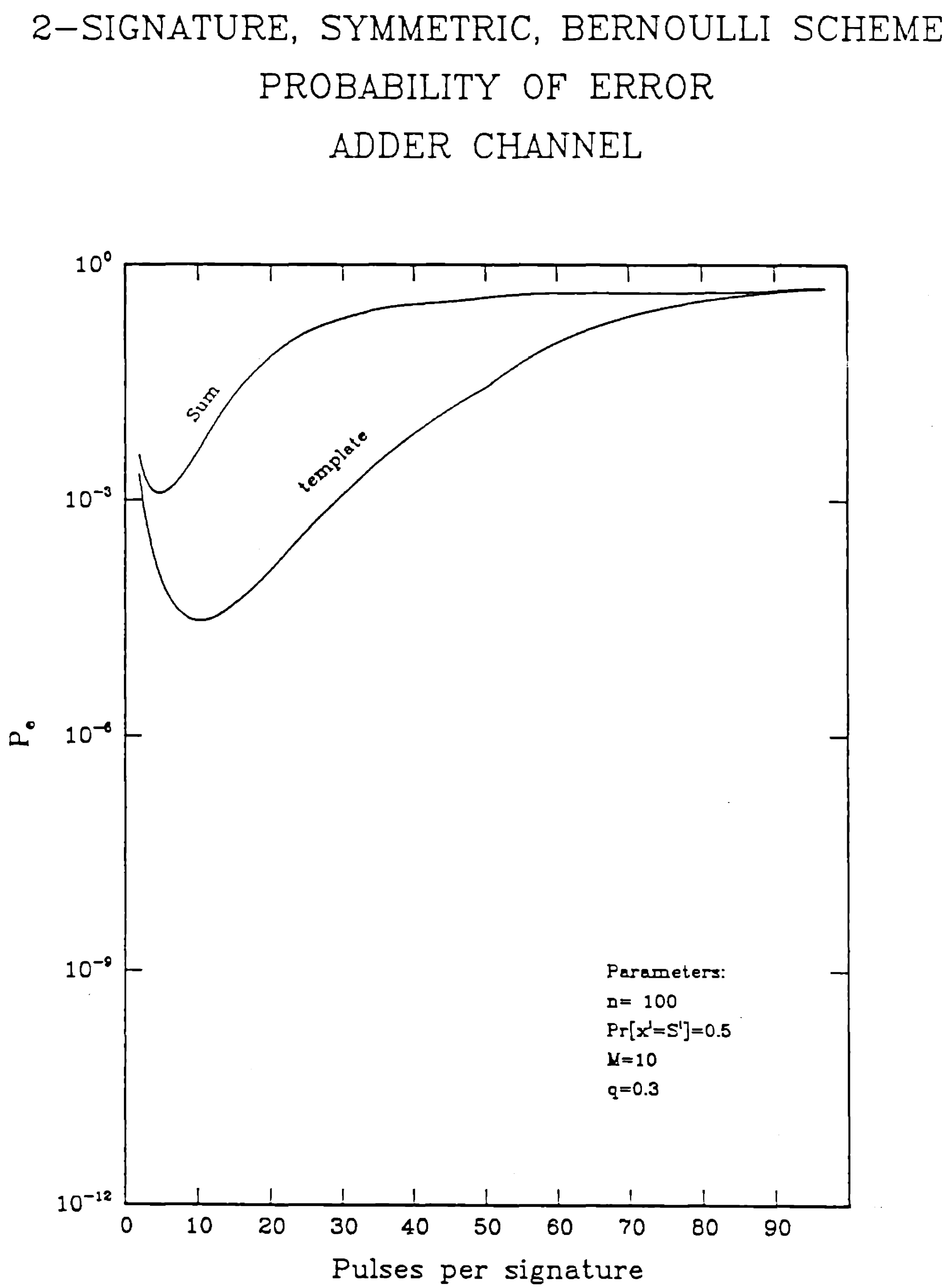
2–SIGNATURE, SYMMETRIC, BERNOULLI SCHEME
PROBABILITY OF ERROR
ADDER CHANNEL
Sum
template
Parameters:
n= 100
Pr[x'=S']=0.5
M=10
q=0.3
P_e
10^0
10^-3
10^-6
10^-9
10^-12
0
10
20
30
40
50
60
70
80
90
Pulses per signature

Figure 3.3

2-SIGNATURE, SYMMETRIC, BERNOULLI SCHEME

ADDER CHANNEL

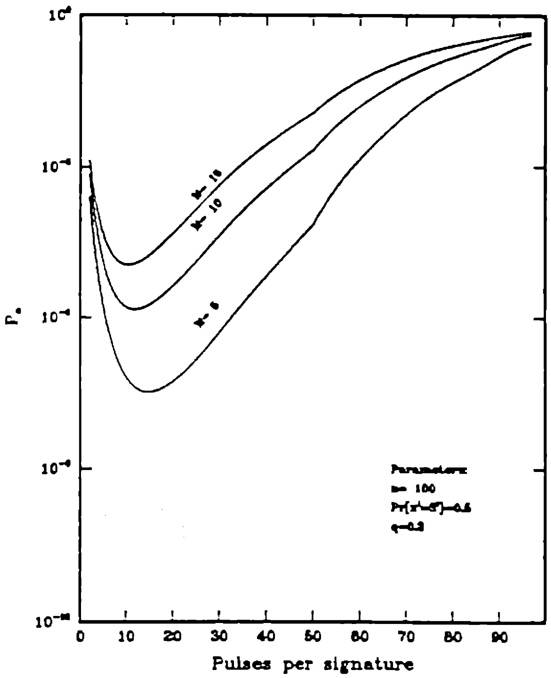

Figure 3.4.a

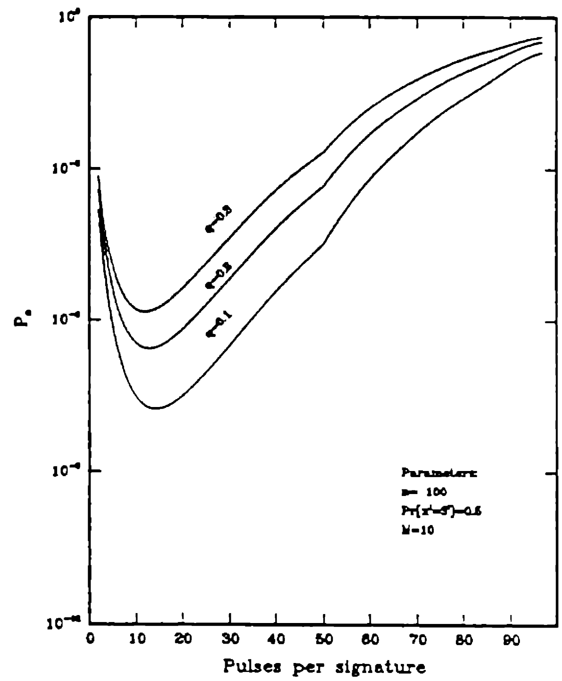

Figure 3.4.b

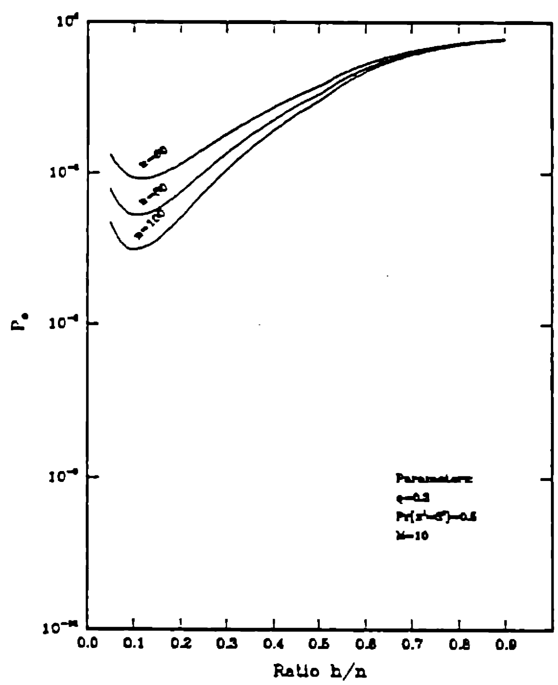

Figure 3.4.c

2–SIGNATURE, ASYMMETRIC, ROULETTE SCHEME
PROBABILITY OF ERROR
ADDER CHANNEL

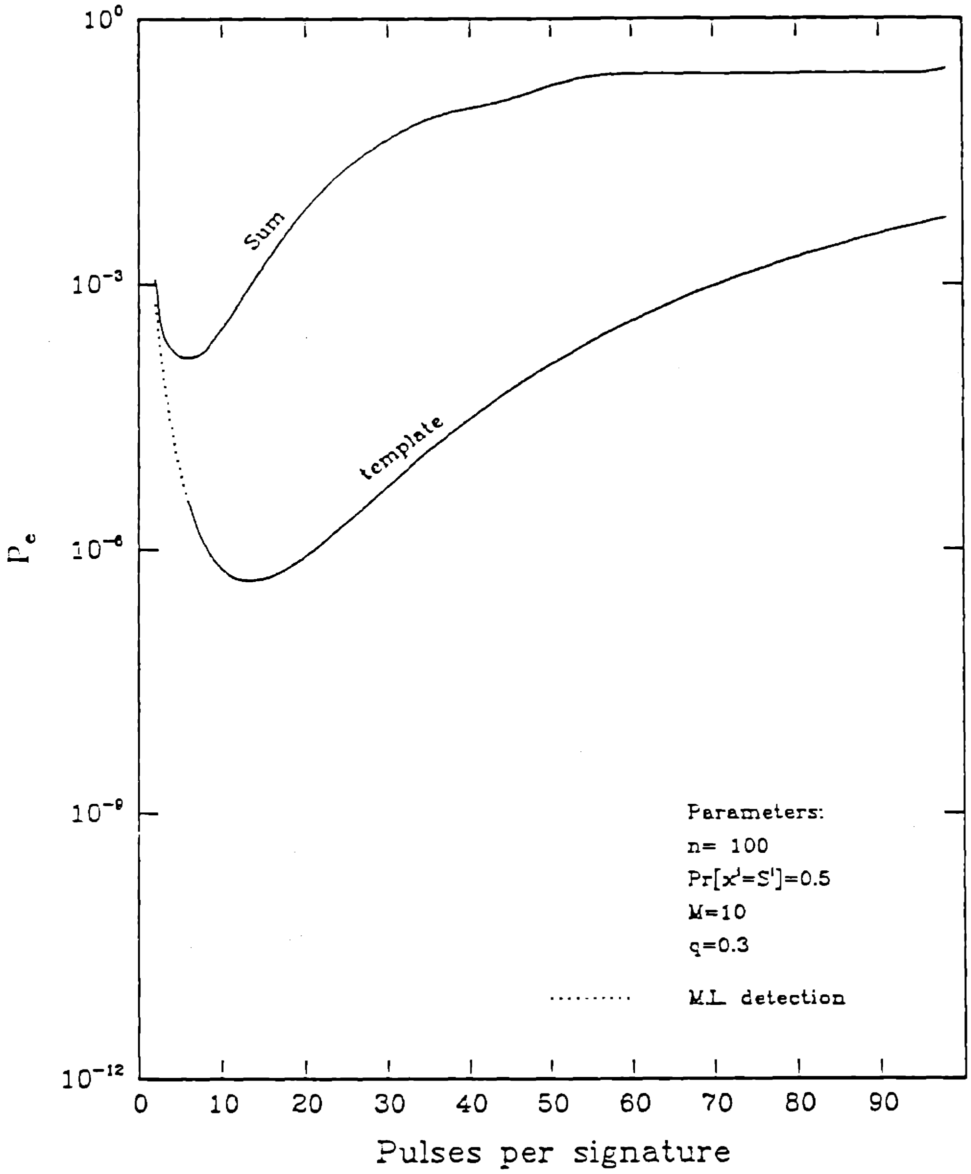

Figure 3.5

2-SIGNATURE, ASYMMETRIC, ROULETTE SCHEME

ADDER CHANNEL

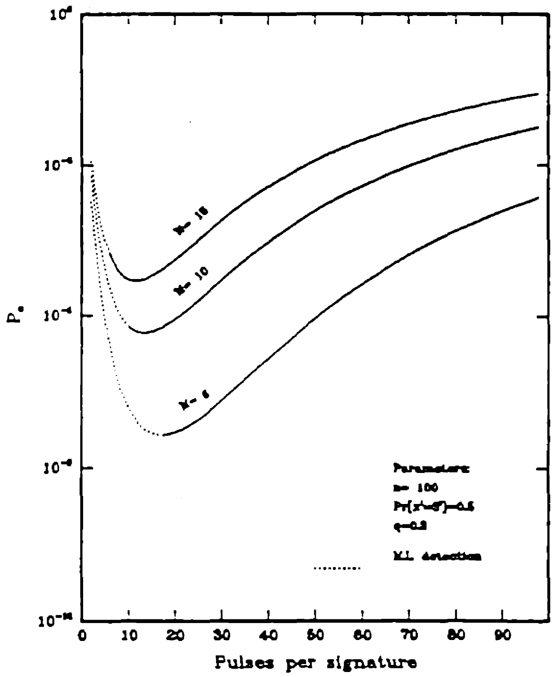

Figure 3.6.a

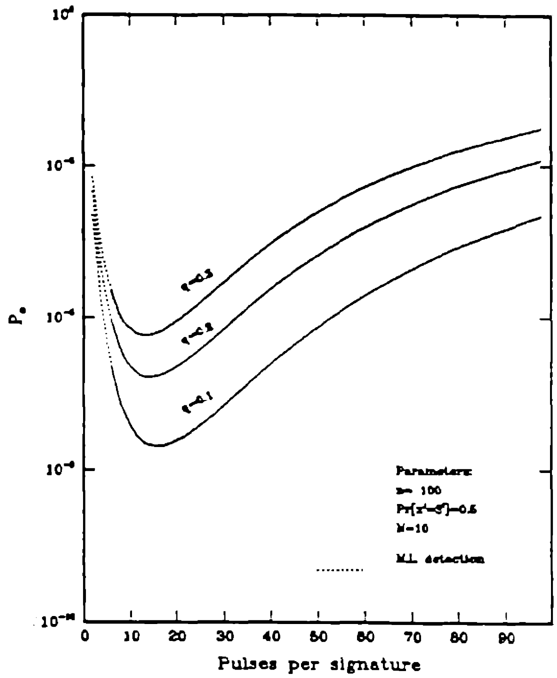

Figure 3.6.b

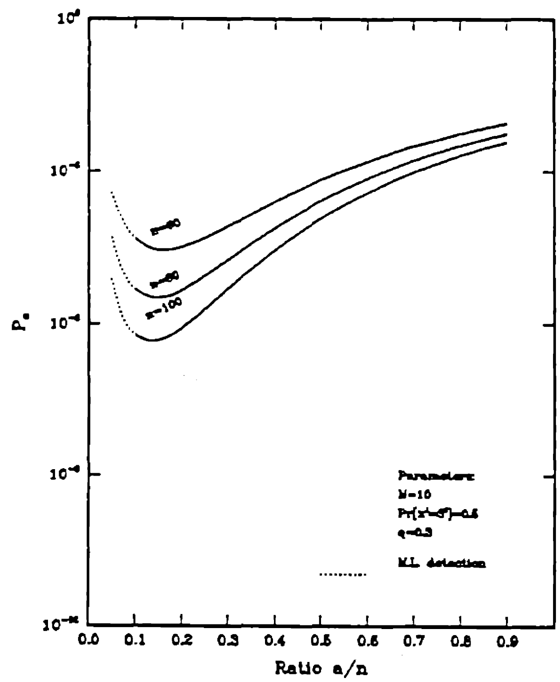

Figure 3.6.c

2-SIGNATURE, SYMMETRIC, ROULETTE SCHEME
PROBABILITY OF ERROR
ADDER CHANNEL

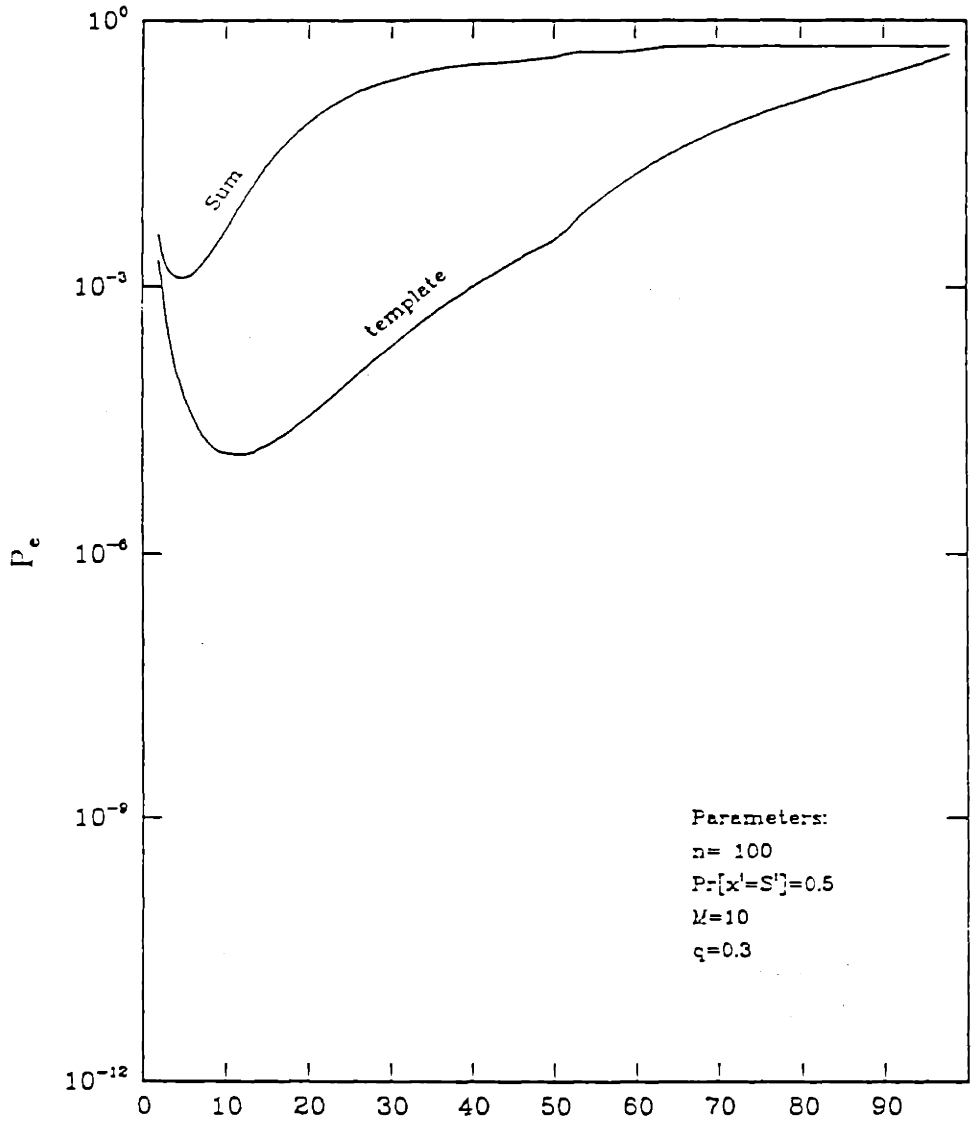

Figure 3.7

2-SIGNATURE, SYMMETRIC, ROULETTE SCHEME

ADDER CHANNEL

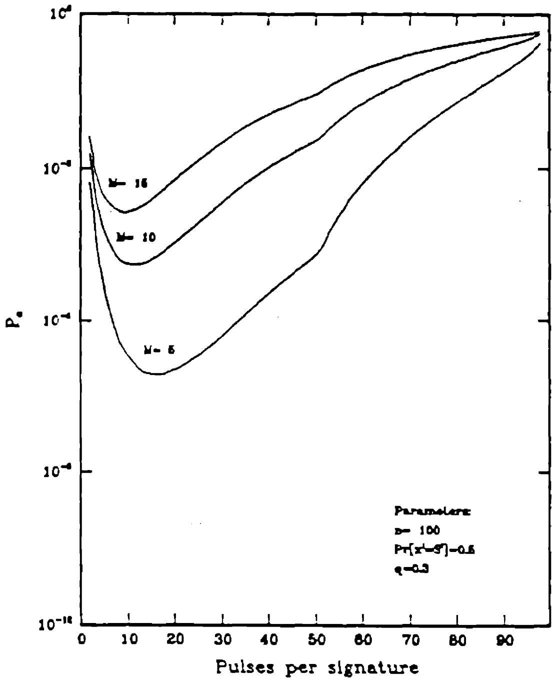

Figure 3.8.a

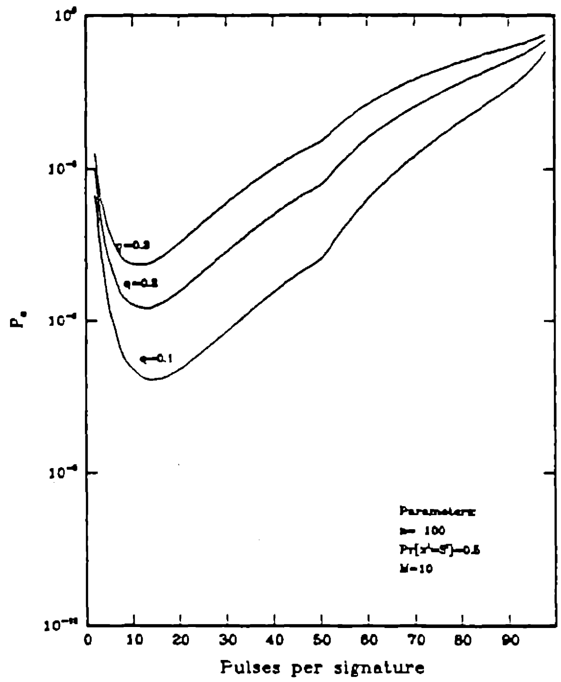

Figure 3.8.b

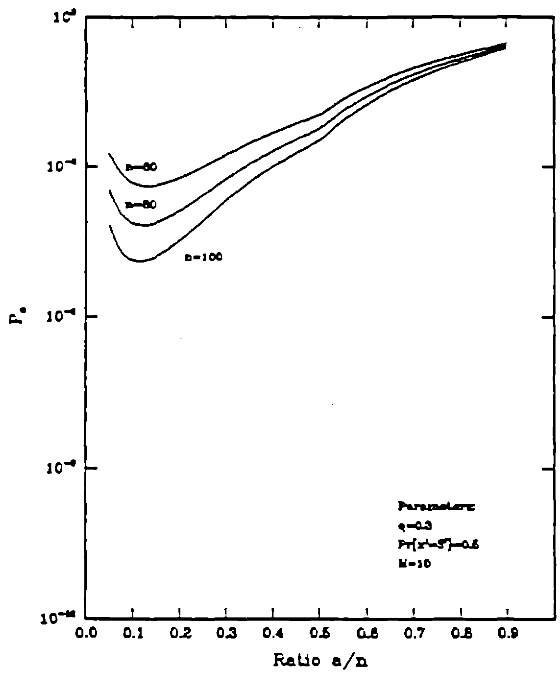

Figure 3.8.c

of the h pulse positions of s^i, so as to cause an error, decreases. The optimal point probably reflects this trade off.

For the symmetric schemes we have assumed that as h increases up to $\lfloor n/2 \rfloor$ the two signatures of session i are chosen so that no pulse positions are common between them; hence H, the number of pulse positions of one signature which are not present in the other signature, is given by $H = h$. As h increases from $\lfloor n/2 \rfloor$ towards n, overlap is inevitable and $H = n - h$ in P_e. Thus in this last region the increase in P_e with h is worse than over the region $h < \lfloor n/2 \rfloor$ since H decreases but interference proceeds increasing. This accounts for the "kink" observed in the graphs for symmetric schemes.

Finally we note that for given h the average power for the asymmetric schemes is half of that for the symmetric scheme.

Figures 3.1,3.3, 3.5, 3.7 also show P_e for the Sum detector with threshold h. In all cases we would expect the performance to be worse than for the Template detector, *it is important to note however that the difference seems significant.*

Expressions for P_e when $\alpha > 2$ in the symmetric schemes could also be tediously written down exactly and evaluated but we have chosen not to.

P_e as a function of M, q, n. Figures 3.2.a,3.2.b,3.2.c show P_e curves for a Template detector as M, q, n are varied respectively for the asymmetric Bernoulli scheme. The regions corresponding to Maximum Likelihood detection are also shown. Notice that the axis for figure 3.2.c is labeled with the average power ratio h/n instead of with h as in 3.2.a,3.2.b. Figures 3.4.a-3.4.c,3.6.a-3.6.c and 3.8.a-3.8.c show similar results for the symmetric Bernoulli scheme and the asymmetric and symmetric Roulette schemes respectively. We observe that P_e increases with M and q and with decreasing n. The dependence seems to be approximately exponential in all three cases. A point of interest in figures 3.2.a, 3.2.b is the sensitivity of the optimal value of h as M or q vary. Since around the optimal value of h the value of P_e varies more slowly w.r.t h system performance seems not to be affected so much by small errors in the assumed values of M and q. For large errors the increase in P_e with respect to its optimal value for the true M, q, is approximately exponential in the error.

Comparison among models. In Figure 3.9 we compare Template detector P_e curves for all schemes for the ADDER channel. The comparison assumes equal processing gain n and equal channel load M, q. Recall however that, for equiprobable source letters, a given value of pulses per signature implies half the average power for asymmetric schemes than those for symmetric schemes. We have altered the range of the abscisa to a maximum of 50 pulses per signature so that near the optimal h regions the curves are discernible. The Roulette schemes show lower probability of error P_e for given h than corresponding Bernoulli schemes. Asymmetric schemes show lower P_e for given h than corresponding symmetric schemes.

Supposing that the value of average power used were an important consideration. Figure 3.10 plots Template detector P_e curves for the ADDER channel case using instead as abscisa the value of average power, in pulses per signature. Again we have altered the abscisa range to a maximum of 30 pulses per signature for clarity of presentation. In this case the curves for asymmetric schemes, C and D, have lower P_e for low values of h than symmetric schemes, A and B. Then, at a point beyond the optimal $\overline{P}$ value in these curves, C intersects A and the Bernoulli symmetric scheme has lower P_e than the asymmetric scheme; similarly, D intersects B and the Roulette symmetric scheme has lower P_e than the asymmetric scheme.

One observation is due. For Roulette schemes, as the number of pulses per frame a increases the probability of position overlap for pulses of a given interfering non null signature increases. Hence as a tends to n the number of slots with pulses for such signatures is not as large as it would be for a scheme not allowing signatures with multiple pulses per slots. However this overlap effect does not affect the signatures being decoded since we let the number of occupied slots in these signatures,h or H, approach[7] n. For this reason the increase

[7] We do not take the expectation over values of h since we are modeling the interference probabilistically while assuming signatures of session i to be fixed.

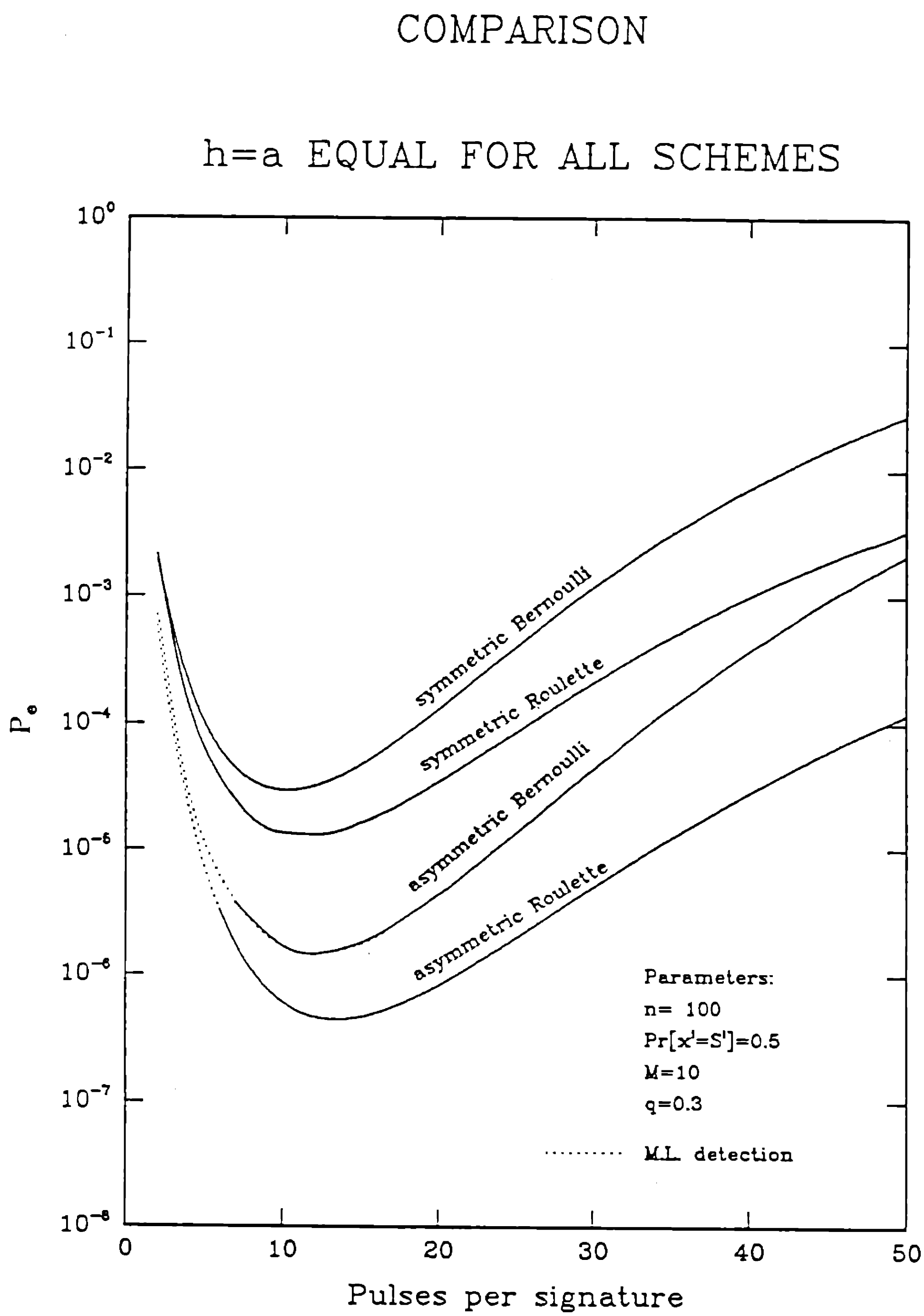
COMPARISON
h=a EQUAL FOR ALL SCHEMES
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
P_e
symmetric Bernoulli
symmetric Roulette
asymmetric Bernoulli
asymmetric Roulette
Parameters:
n= 100
Pr[x^i=S^i]=0.5
M=10
q=0.3
M.L. detection
0
10
20
30
40
50
Pulses per signature

Figure 3.9

COMPARISON

AVERAGE POWER EQUAL FOR ALL SCHEMES

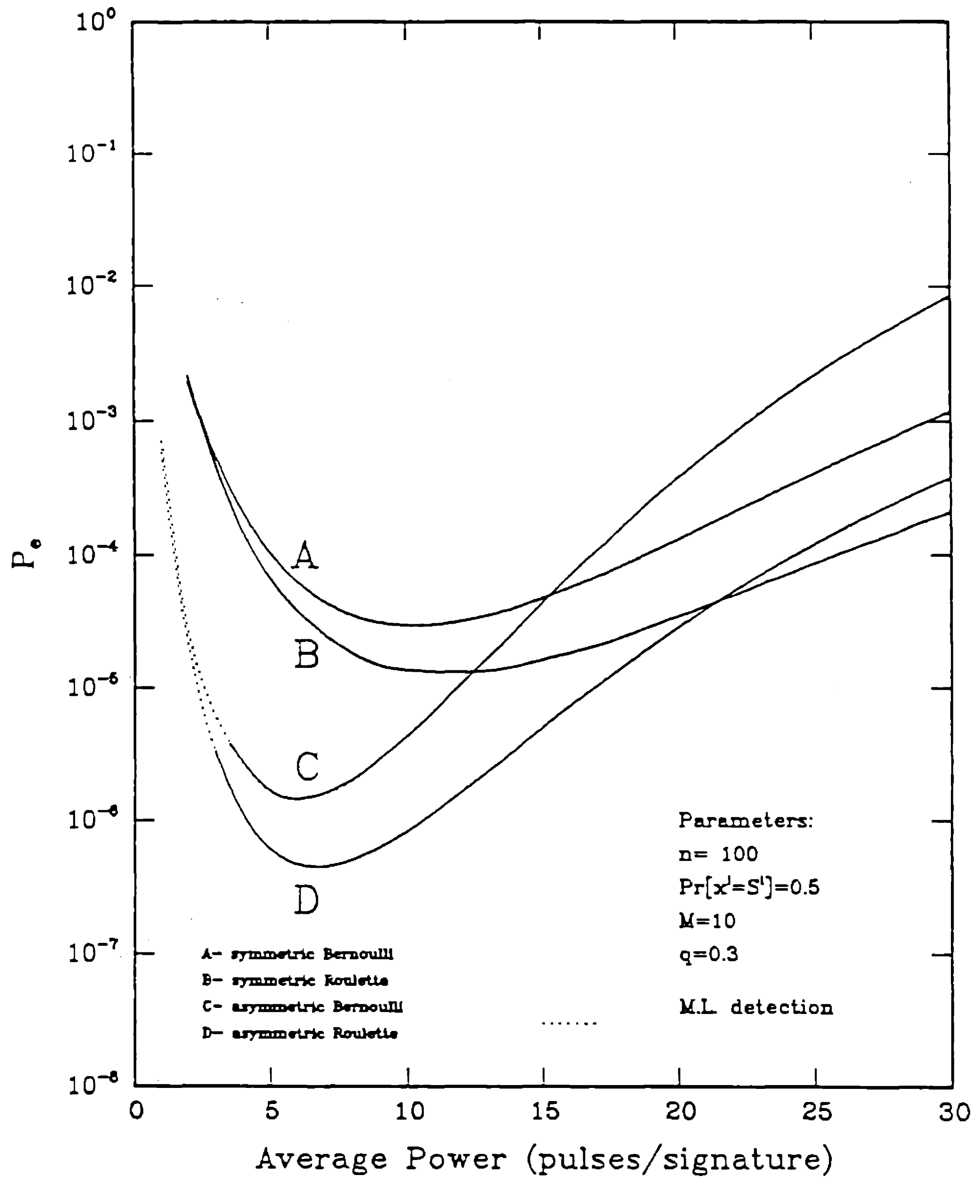

Figure 3.10

ROULETTE SIGNATURE SCHEME

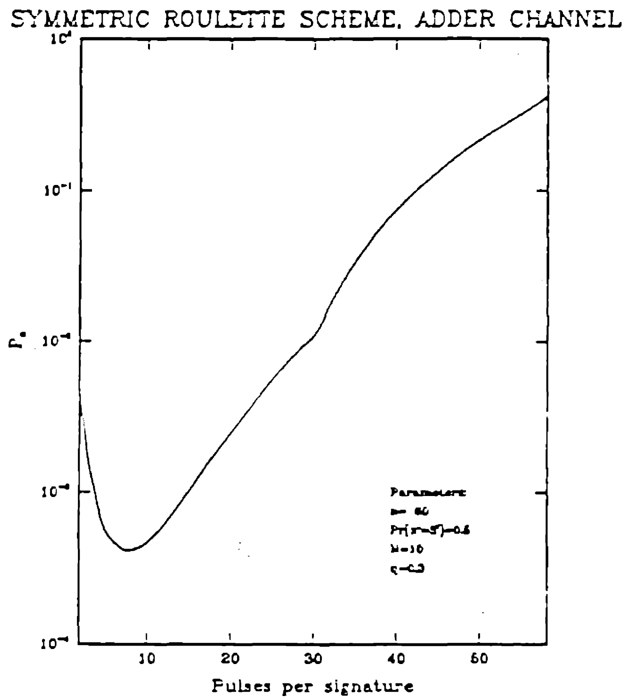

Figure 3.11.a

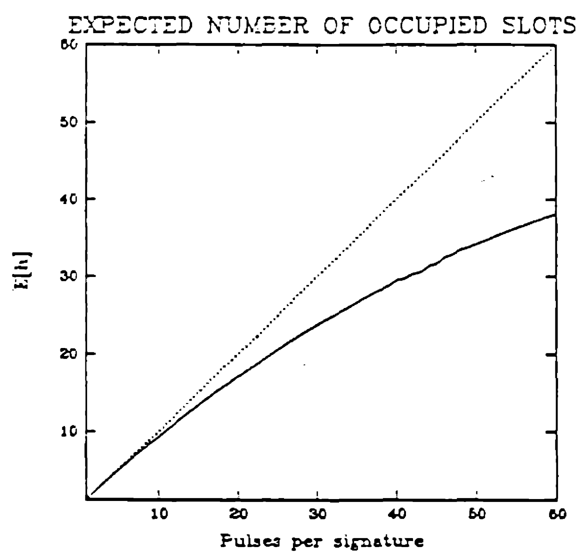

Figure 3.11.b

in P_e as the number of pulses per signature increases is slower for Roulette schemes than for Bernoulli schemes. Figure 3.11.b shows the expected value of the number of occupied slots l in a Roulette signature of n slots as a function of the total number of pulses per signature a, $1 \leq a \leq n$. The expression is (c.f. [Feller, section II.11, problem 11.9])

$$E[l] = \sum_{l=1}^{a} l n^{-a} \binom{n}{l} A(a,l) \tag{3.50}$$

where $A(a,l)$ is given by (3.37) . (3.50) is very sensitive to numerical round off error due to the alternating signs of the summands in $A(a,l)$. The value of $n = 60$, less than the value of 100 previously used, was chosen to avoid this condition. Figure 3.11.a is included for comparison with the Template detector performance.

Another comparison among schemes is obtained by considering the work of Gurantz, Gardner, Viterbi and Zbik [G G V Z]. They treat the single session capacity and Cut-off rate for a signature scheme called Pulse Interval Modulation. Signatures are single pulse signatures placed in a frame of n slots and there are a total of n signatures, one for each slot position. Once the signature pulse is transmitted, rather than wait for the end of its frame, a session waits a fixed time interval before the next signature frame starts. The minimum time interval between succesive signature frames is built in to allow control over the level of interference in the channel.

Capacity C and Cut-off rate R_0 for this system are computed assuming a Discrete Time Asynchronous model. Interference is modeled as follows. Given an input signature, over a signature frame of session i, output vectors of equal number of pulses m, $1 \leq m \leq n$, are equally probable, with probability

$$\begin{cases} \binom{n-1}{m-1}, & \text{if input signature pulse present in output vector;} \\ 0, & \text{otherwise.} \end{cases}$$

m is considered a random variable. Notice that this model precludes overlap among interference pulses of different sessions. The model, however, would be expected to hold well for low interference levels. The model could equally be applied to a Pulse Position Modulation system with no wait time interval between signature frames so the expressions below for C, R_0 would also hold. These expressions are

$$C = \log_2 n - E[\log_2 m] \qquad \text{bits/frame}$$

$$R_0 = \log_2 n - \log_2 E[m] \qquad \text{bits/frame}$$

where the expectation is over the number of pulses m received over a signature frame of session i. (m includes the input signature pulse.) Both capacity and Cut-off rate are achieved with equiprobable input signatures. Notice that the description of interference above can also be applied to a Bernoulli signature scheme or to a Roulette signature scheme as long as pulse overlap events are neglected. For a Bernoulli scheme this implies a low value of pulse probability p. For a Roulette scheme this implies a high value of number of slots n. In these two schemes signatures of session i would be single pulse; while in fact interfering signatures need not be single pulse.

The expressions for capacity and Cut-off rate are equal when m is a constant. R_0 is often interpreted as a good measure of the highest information rates achievable in practice in a point to point channel. If coding over frames of a session is allowed in a Frame Synchronous model and systems for which ς, the number of active interfering sessions sending non null signatures, is fixed then this interpretation of R_0 suggests that information rates close to capacity are easier to achieve for a Roulette scheme with one pulse per signature and large n than for an equivalent Bernoulli scheme, since for the former scheme m is a constant over a frame of session i while for the latter it is a random variable.

Finally, the proper information theoretical characterization of the information rates of a multi-user communication channel is the Capacity Region, which specifies the joint set of rates of all users for which reliable communication can be achieved. Single session capacity and Cut-off rates are still meaningful measures since cooperation among users may often be impractical.

Upperbounds for M.L. detection of Roulette schemes. For an asymmetric Roulette scheme operating on an ADDER channel Lemma 3.3 shows that the Template detector is an M.L. detector up to some value of h. For higher values of h eventually the M.L. detector calls for individually checking the Likelihood Ratio for each received channel output. Inequality (3.41) shows an upperbound for the probability of error of this M.L. detector. Figure 3.12 shows the bound in (3.41) for several values of s, suggesting that $s \to 0$ is optimal for this example. As discussed in the derivation of this bound in subsection 3.3.1, this may well be a consequence of the low values chosen for M, q, and $s = 0$ may not be optimal for high interference levels. The values of the parameters have been chosen smaller than in previous examples to further facilitate its numerical evaluation. The figure also shows this P_e curve for a Template detector and the region over which Lemma 3.3 shows it to be an M.L. detector. Since the bound obtained is above the expression for probability of error of the Template detector, as explained at the end of subsection 3.3.1, no further conclusions on the performance of the M.L. detector or the Template detector can be drawn.

For a symmetric Roulette scheme operating on an ADDER channel, Figure 3.13 shows the upperbound in (3.49) for a Maximum Likelihood detector. It also compares this bound to the probability of error of the Template detector. As before smaller parameters than in previous examples have been used to facilitate evaluation. We observe that for low values of a the bound is above but close to the curve for the Template detector. As a increases toward $n/2$ the bound intersects the curve for the Template detector and proceeds below it. The bound is so close however to the Template detector curve that no new conclusion can be drawn about the effectiveness of either the Template detector or the M.L. detector.

2-SIGNATURE, ASYMMETRIC, ROULETTE SCHEME
UPPERBOUND FOR M.L. DETECTION P_e
ADDER CHANNEL

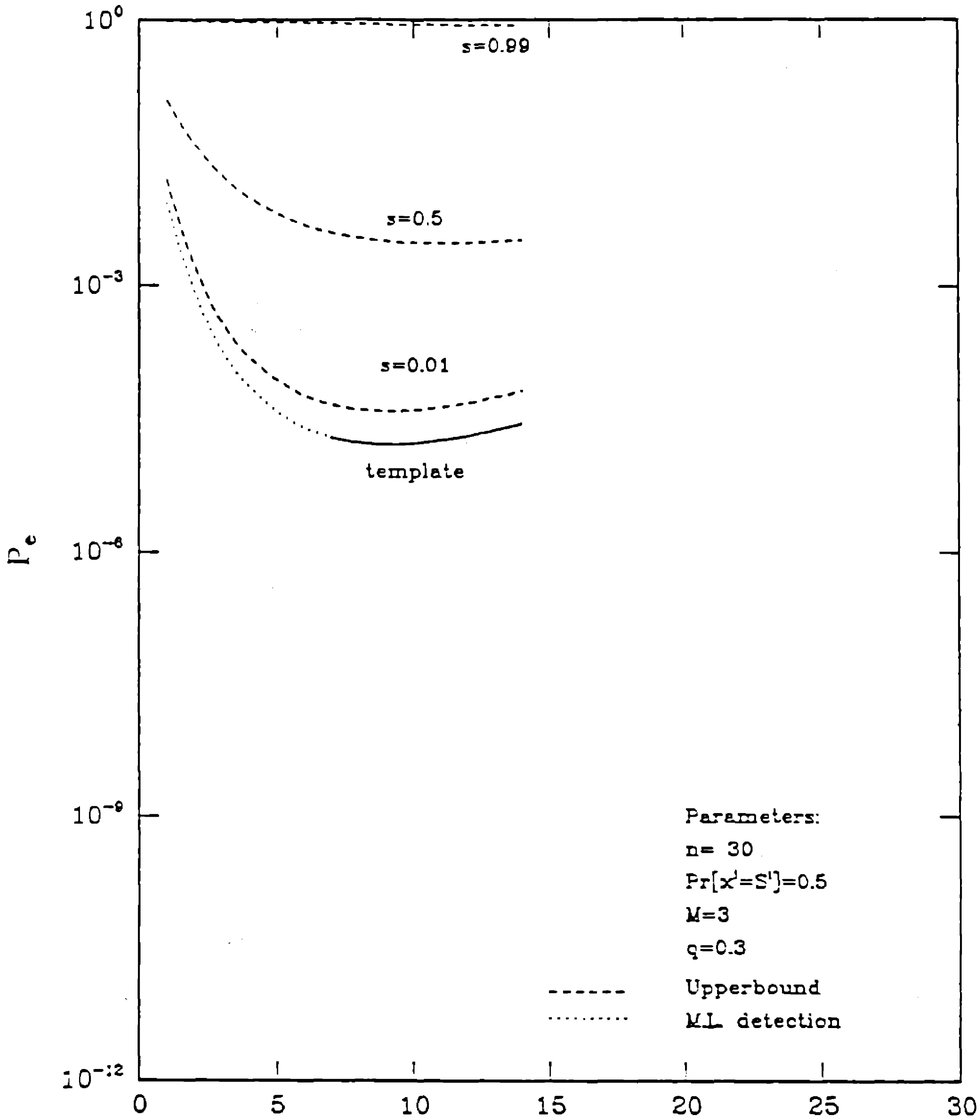

Figure 3.12

2–SIGNATURE, SYMMETRIC, ROULETTE SCHEME
UPPERBOUND FOR M.L. DETECTION P_e
ADDER CHANNEL

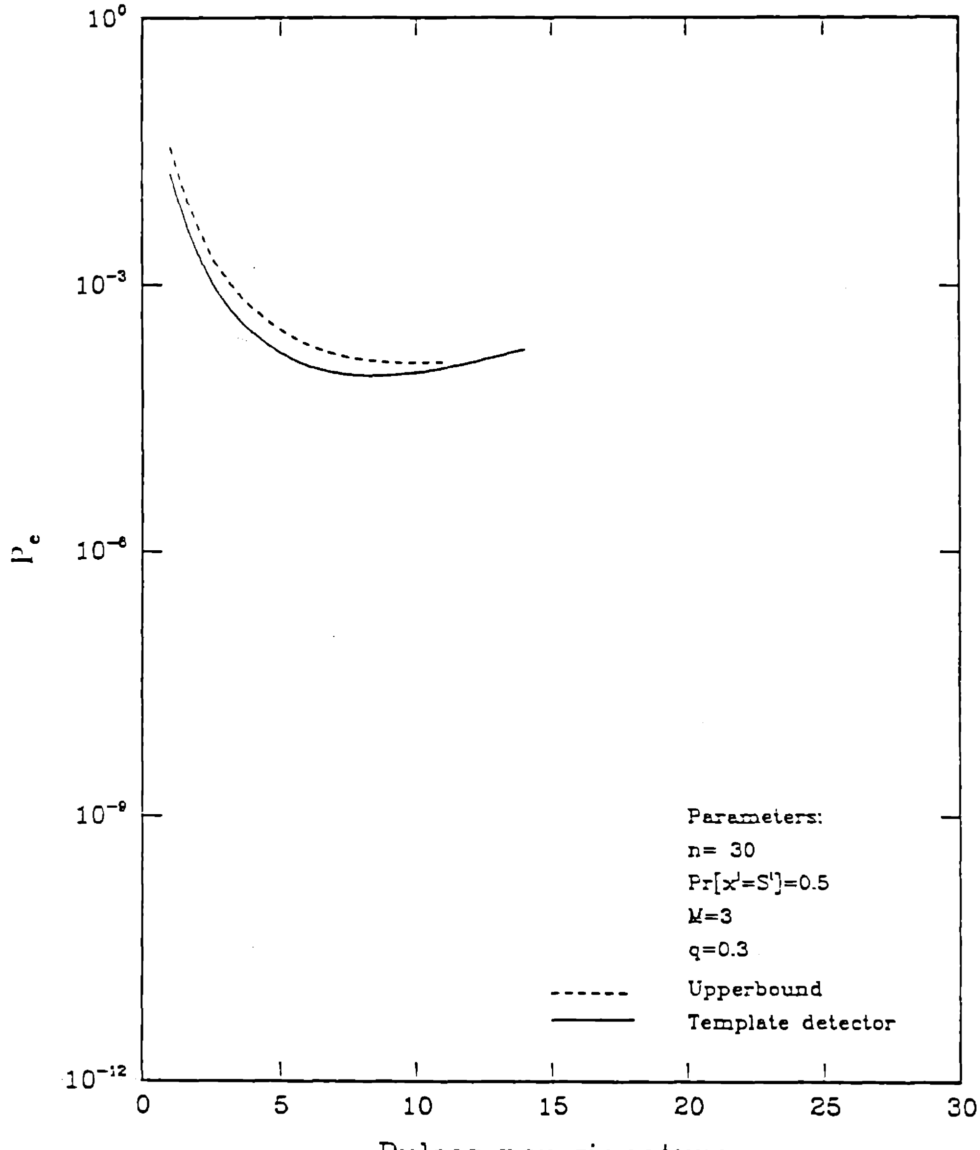

Figure 3.13

Optimal value of pulses per signature. We will now explore further the behaviour of the optimal number of pulses per signature. To illustrate the behaviour we choose the 2-signature, symmetric, Roulette scheme. In figures 3.8.a and 3.8.b we observed that the optimal value of pulses per signature a decreases as M or q increase. In Figure 3.14 we illustrate the optimal a, which minimizes P_e, as M and q are varied but the expected number of interfering sessions, $E[\varsigma] = Mq$, is kept constant. Four values of $E[\varsigma]$ are chosen. The idea is to explore if a characterization of the optimal a based on $E[\varsigma]$ alone is possible. From the figures we see that in general the optimal value depends on the pair M, q rather than its product. The figures also suggest that M is a stronger interference parameter than q if it is desired to hold Mq constant. In the logarithmic scale of figure 3.14.d the plots for $M = 20, q = 0.9$ and $M = 40, q = 0.45$ appear too close to be distinguished. In fact the plot for $M = 20$ achieves the minimum of the two. The cases illustrated in this last figure correspond to large interference for the given value n of slots per frame.

In Figure 3.15 we illustrate the behaviour of the optimal value of the ratio a/n as a function of the number of slots per frame. We observe that this optimal ratio decreases with n. Computation of the corresponding value of a, however, will reveal that the optimal a in fact increases. The figure thus suggests that the optimal value of a increases, more slowly than linearly, with increasing n.

At this point it is worth noting that [Hui] treats a situation with similar characteristics. The system is symmetric, Continuous Time Asynchronous, with a constant number ς of interfering sessions, and the detector is a template detector —c.f. [Hui, section III.C, Hardlimiting and Filtering]—. Square interfering pulses appear in a slot according to a Poisson process of rate $\lambda = \varsigma a/n$. The Poisson assumption allows computation of the probability of false detection of a pattern and of the "sum capacity" for the channel. This expression can be optimized over ς, a, n. It is found that the value of a which achieves capacity is proportional to the ratio n/ς. Thus the ratio a/n that achieves capacity would be independent of n. This situation is different to that of Figure 3.15.

OPTIMAL VALUE OF a, SYMMETRIC ROULETTE

Mq=0.1

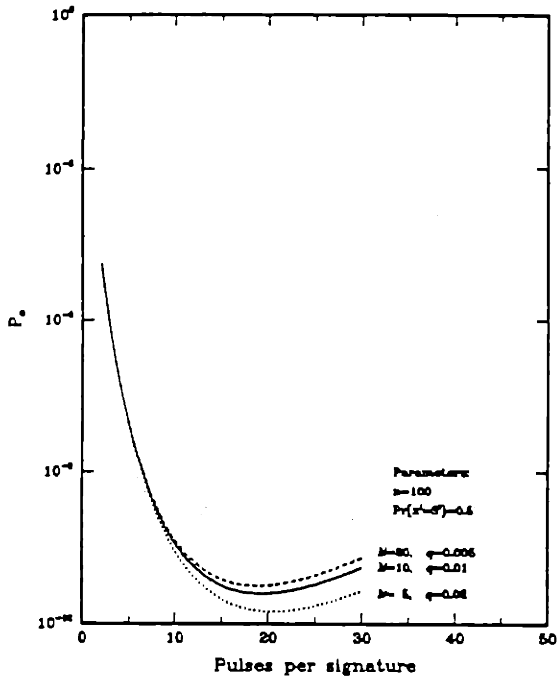

Figure 3.14.a

Mq=2.0

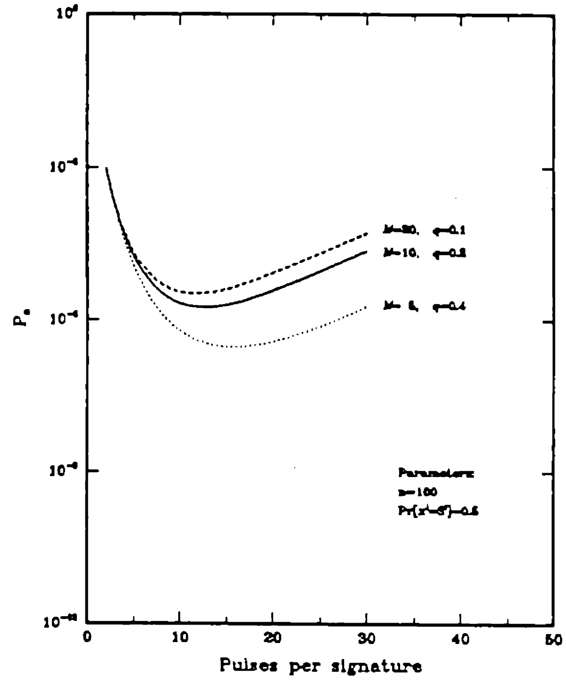

Figure 3.14.b

Mq=4.0

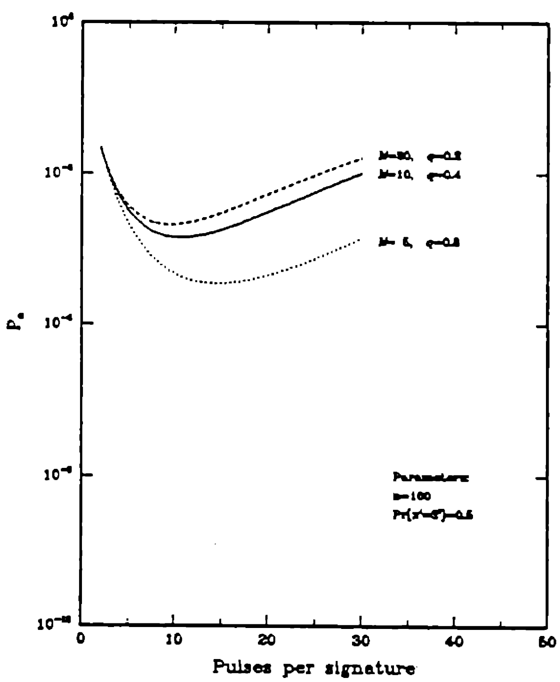

Figure 3.14.c

Mq=18.0

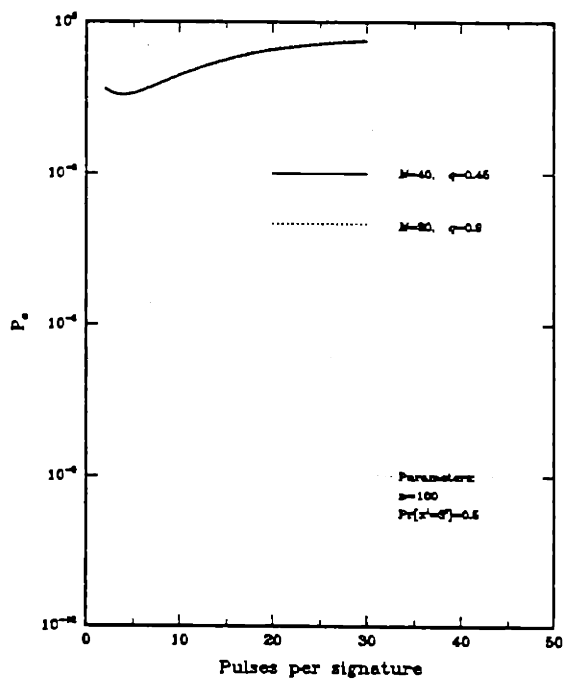

Figure 3.14.d

2-SIGNATURE, SYMMETRIC, ROULETTE SCHEME
OPTIMAL RATIO a/n FOR TEMPLATE DETECTOR
ADDER CHANNEL

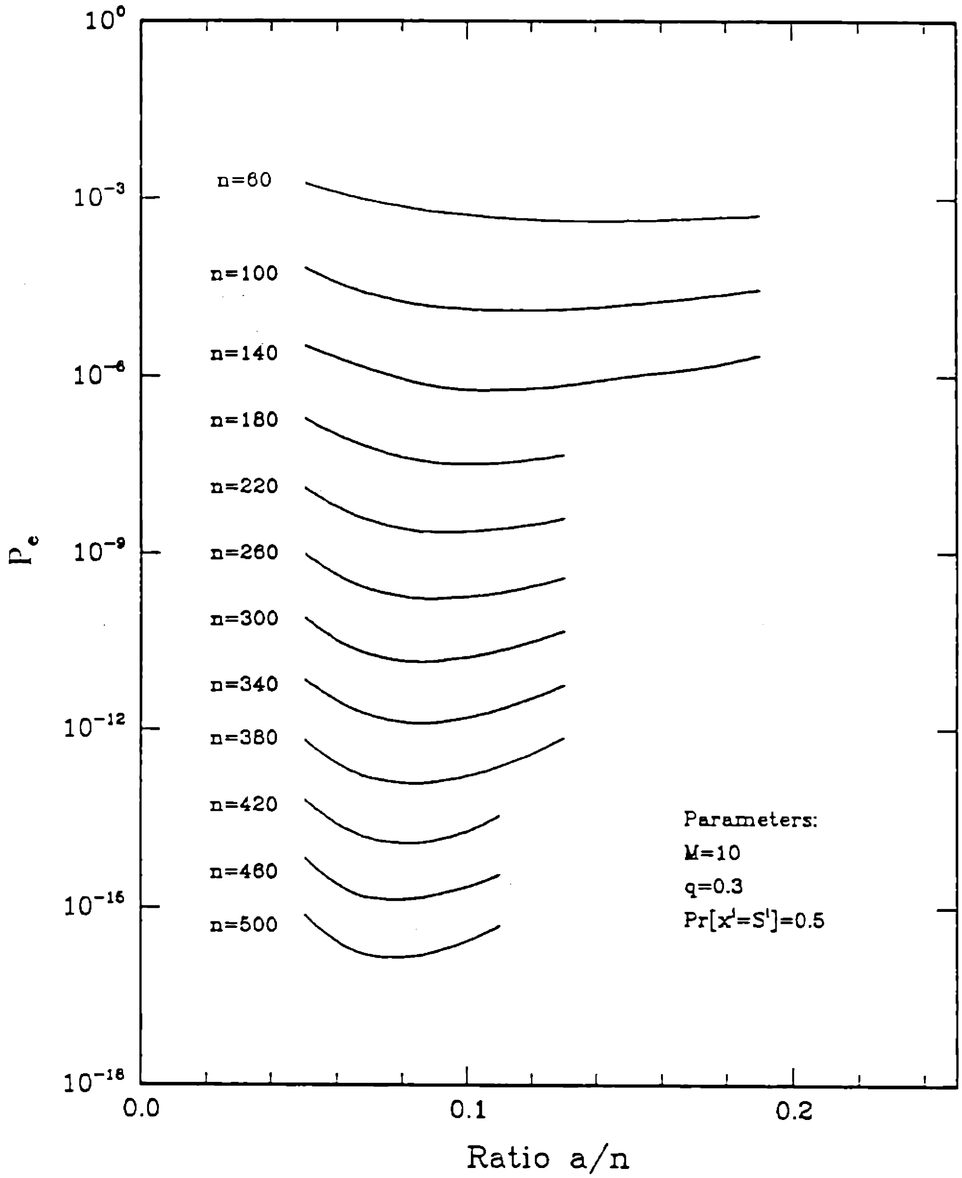

Figure 3.15

Apart from the fact that the probabilistic models of interference differ from Hui's to ours, there is another important difference in the systems being treated. Capacity calculations implicitly assume coding over several frames while our models do not. Thus, the value of a which achieves capacity, and minimum probability of error if the system is operating at a rate equal to capacity, need not be the same than the value of a which achieves minimum probability of detection error when no coding is used.

3.4.3 COMMENTS

Limitations. The comments in this section have the following limitations:

- Our models are mathematical abstractions that neglect several processes like physical noise and attenuation through the optical path.

- The models are not an exact representation of the statistics of the multiuser interference channel because:

  - Roulette schemes allow signatures having more than one optical pulse per slot.

  - P_e corresponds to an ensemble average over several networks with the parameters given.

  - Our codes are time varying, with randomly chosen signatures. In fact, since it would seem realistic to assume that detection is carried out at each frame independently of previous frames, a time varying code still seems an appropriate model.

- The comparisons we are making are based on numerical evaluations of the P_e expressions over a restricted domain of the relevant parameters.

- The results evaluated correspond to a Frame Synchronous system so they can only be considered an approximation to the Discrete Time Asynchronous and Continuous Time Asynchronous systems.

Nevertheless, these discussions should contribute to understanding the dynamics of the interference process.

Summary of observations on figures presented.

- Use of a Template detector on an ADDER channel seems to afford significant decrease in P_e over use of a Sum detector. This is apparent by comparing the values of P_e at the optimal value of h when using each detector.

- For all schemes, an optimal h exists and depends on M and q. This h probably reflects the trade off between increased information in a signature (more pulses) and increased interuser interference.

- For all models P_e seems to have an approximately exponential dependence on M, q, n, h. Sensitivity of P_e degradation with respect to its value at the optimal h is small for small errors in assumed values of M, q, n and large for large errors in assumed M, q, n.

- Roulette models show lower values of P_e than Bernoulli models for a given value of number of pulses per signature.

- The asymmetric schemes show lower values of P_e than the symmetric schemes for a given value of pulses per signature. For the 2-signature symmetric schemes, at some point overlap of pulse positions among signatures of the same session starts to occur. P_e starts increasing faster with increasing values of pulses per signature. This problem does not apply to 2-signature asymmetric schemes.

Discussion. The effect of multiple pulses per slot in Roulette signatures for high values of pulses per signatures, a, must be borne in mind when considering Roulette schemes. However it seems that this effect is not an important consideration over the region where the optimal value of a is to be found.

If, for symmetric schemes, instead of the Sum detector described in (2.5) we used a detector which adds the contents of the slots in $\psi_{+}^{i\gamma}$ for all signatures $s^{i\gamma}$, and decides for the one giving the maximum value of the sum, the probability of error for this modified sum detector would be lower than for the Sum detector. However, from the P_e curves plotted for symmetric schemes we see the pronounced way in which increasing the number of pulses per signature increases the probability of error for the Sum detector with respect to the Template detector. This seems to suggest that this effect will also be pronounced for the modified sum detector mentioned above and the Template detector probably still affords a significant improvement in terms of probability of error.

Since asymmetric schemes use null signatures for one of the letters the expected interference is less than for symmetric schemes. We can observe in the graphs that asymmetric schemes exhibit lower probability of error for a given value of pulses per signature. Asymmetric and symmetric Roulette schemes exhibit lower probability of error for a given value a/n than asymmetric and symmetric Bernoulli schemes, respectively, with an equal value of p. The fact that in a Bernoulli scheme the number of pulses per interfering signature is a random variable of finite variance $np(1-p)$ while the Roulette schemes have a constant number a of pulses per signature may be responsible for this observation. For Bernoulli schemes, events corresponding to signatures with a number of pulses greater than the expected value would be more relevant in determining the probability of error than equivalent events with less pulses than the expected value.

When P_e is plotted versus average power $\overline{P}$, defined above, rather than versus the parameters a, h, the number of pulses in the non null signature of asymmetric schemes is twice the

number in the signatures of symmetric schemes. Thus, asymmetric schemes both reach their minimum value of P_e, and also start saturating with interference, at earlier values of $\overline{P}$ than their corresponding symmetric schemes.
The optimal values of P_e remain the same for all four schemes.

The best choice of model in a real situation would depend on the specific physical system being treated. A system using a fixed spread spectrum pattern with equal number of pulses for all users would probably best be represented by a Roulette scheme. On the other hand, a system in which signatures are generated by choosing the separation between subsequent pulses to be pseudo-random and independent, but keeping a fixed number of slots per frame, would probably best be represented by a Bernoulli scheme. This would certainly be the case if the separation is chosen according to a geometric distribution. We note however that the qualitative behaviour of the curve P_e vs h of both models is similar, which would seem to indicate that this behaviour is in a certain sense robust with respect to the detailed model used for the system.

In the examples presented, P_e decreases with h over the M.L. region of operation. So does its relative improvement with respect to a Sum detector. Both, the M.L. region and the optimal h decrease with M and n. Increase of the value of n or decrease in the values M, q, all decrease P_e. The dependence of P_e on these parameters seems approximately exponential. Accurate assesment of the optimal h for the asymmetric Bernoulli scheme is important and likely to be a design difficulty in real networks because this value is a function of q and M. However, performance around this region does not seem to be so sensitive to the exact value of h. For non adaptive networks, design would have to be made for the highest expected values of number of sessions.

CHAPTER 4
ASYNCHRONISM
DISCRETE AND CONTINUOUS

This chapter treats both Discrete and Continuous Time asynchronous schemes. The main result in the Discrete Time Asynchronous case appears in Theorem 4.1. For Continuous Time Asynchronism we give up the attempt to find a Maximum Likelihood detector, as will be explained below, and instead ask for the probability of error for a Template detector. We only have results for the symmetric Bernoulli scheme.

In treating the Discrete Time Asynchronous models we will try to follow a structure similar to that of the previous chapter. However we have made some modifications which we hope will make the treatment of the chapter more natural. Section 4.2 for Bernoulli schemes treats first the symmetric scheme since it simply reduces to the Frame Synchronous model and the results from Chapter 3 apply. Then we treat the asymmetric scheme. Lemma 3.1 is used to treat Maximum Likelihood detection for asymmetric schemes operating on the OR channel. Theorem 4.1 is an extension of Theorem 3.1, which treats the M.L. detection on the ADDER channel, to the Discrete Time Asynchronous case. In section 4.3 we treat both the asymmetric and symmetric Roulette schemes together. The M.L. detector for the asymmetric Roulette scheme on an OR channel is a Template detector, from Lemma 3.1. For the symmetric Roulette scheme the expression for likelihood probabilities are complicated and we have not succeeded in finding an equivalent straigthforward rule. For the ADDER channel we are unable to find simple M.L. detection rules for these Roulette schemes and we mention some of the main difficulties encountered.

Save for the symmetric Bernoulli scheme, derivation of probability of error expressions is a difficult task because of all the possible interference situations one would need to consider.

We only mention the expression for the symmetric Bernoulli scheme which is as for the Frame Synchronous case.

For Continuous Time Asynchronism we have the same problems as with Discrete Time Asynchronism as far as Maximum Likelihood detection for schemes other than symmetric Bernoulli are concerned. There is another important problem. We are dealing now with a waveform channel and finding an appropriate function basis and vector space representation for the channel is likely to be a difficult task. Moreover, even if White Gaussian Noise is incorporated in the channel model (e.g. receiver noise) so that receiver performance is limited by noise, there is still a non-Gaussian noise component which corresponds to interuser interference. Hence manipulation of the vector space representation of waveforms is also likely to be intractable as vector components are not independent random variables.

These considerations lead us to give up the attempt to address the Maximum Likelihood receiver structure for the Continuous Time Asynchronous model and focus instead on the probability of error for a Template detector. The results refer to the symmetric Bernoulli scheme and are presented in subsections 4.4.1 and 4.4.2. Subsection 4.4.3 considers the situation of signatures with isolated pulses — i.e. no two succesive slots are occupied — which is simpler to evaluate numerically. Subsection 4.4.5 considers a situation equivalent to limiting the optical intensity received over a slot.

4.1 GENERAL DEFINITIONS, DISCRETE TIME MODELS

Refer to Figure 4.1. We denote the frame of session i being decoded by the symbol F_0^i. This frame occupies a discrete time interval $[T_0, T_\omega]$. The preceeding frame will be denoted by the symbol F_{-1}^i. The symbol F_0^j denotes the frame of an interfering session j which starts within the discrete time interval $[T_0, T_\omega]$, and F_{-1}^j its preceeding frame.

The following definitions will also be useful. Again please refer to Figure 4.1.

$\mathcal{M}$	set of active interfering sessions over $[T_0, T_\omega]$.
$\varsigma = \lvert\mathcal{M}\rvert$	number of active interfering sessions.

DISCRETE TIME ASYNCHRONISM
(ASYMMETRIC SCHEME EXAMPLE)

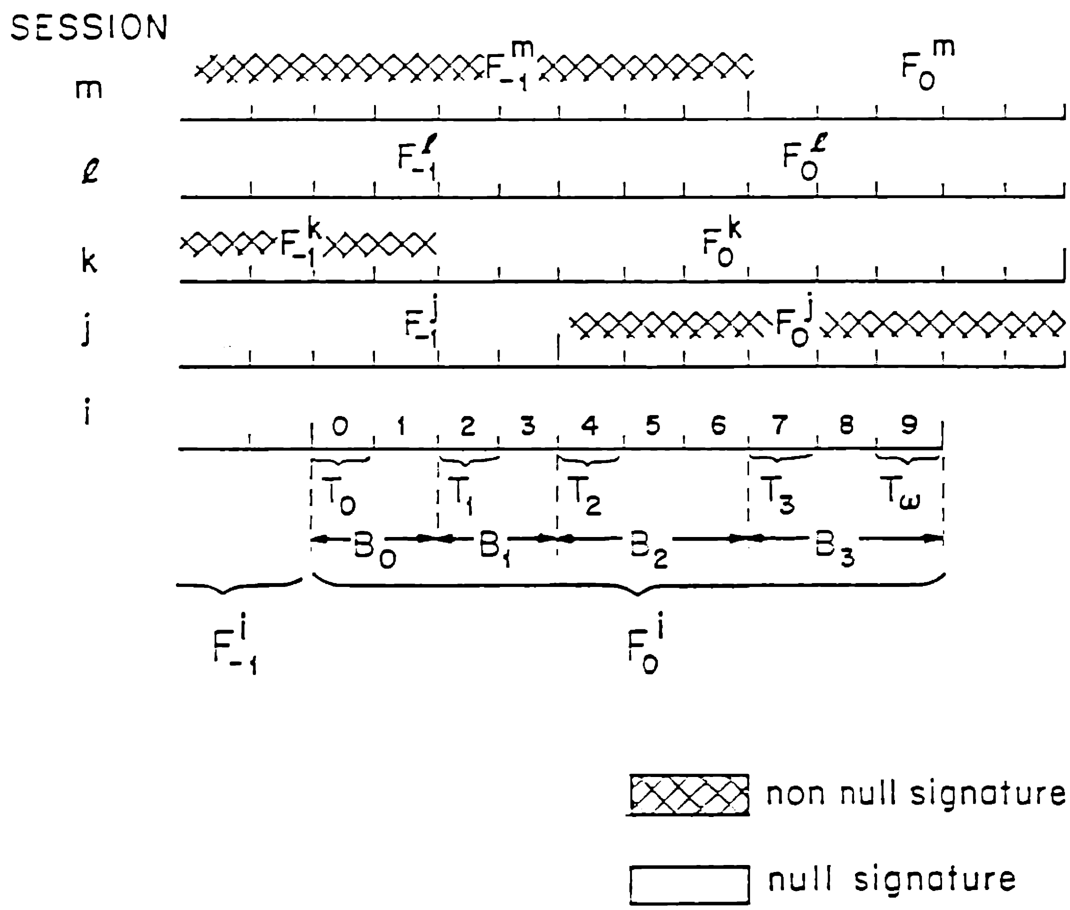

Figure 4.1

t^j	slot at which $F_0^j, j \in \mathcal{M}$, starts.
$\sigma(F_{-1}^j), \sigma(F_0^j)$	letters encoded by the frames F_{-1}^j, F_0^j.
$T_k, k = 0, \ldots, \eta$	slots, chronologically, where some F_0^j starts. $\eta \leq \varsigma$.
$T = (T_0, \ldots, T_\eta)$	notation: vector of slot instants T_k defined above.
$B_k, k = 0, \ldots, \eta$	slot region: $B_k = [T_k, T_{k+1} - 1],\ k < \eta.\ B_\eta \equiv [T_\eta, T_\omega]$.
$b_k = T_{k+1} - T_k$	Number of slots in region B_k. $b_\eta \equiv T_\omega - T_\eta$.
$y(k)$	vector: sub-vector, corresponding to B_k, of output vector Y.

Steady state. For the sake of explanation we will consider first, in the sections below, the detection of an isolated frame of session i in the presence of several active interfering sessions in steady state. By steady state here we meant that, over the frame being detected, no interfering user either becomes active if inactive or becomes inactive if active. Then we discuss extension of the results to detection of a sequence of successive frames of session i. Note that Asynchronism implies that an interfering frame may overlap two successive frames of session i and hence the communication channel should be considered a memory channel.

4.2 BERNOULLI SCHEMES

4.2.1 SYMMETRIC, BERNOULLI SCHEME

Consider a steady state situation and detection of a frame of session i. Because of the nature of the symmetric Bernoulli scheme, the Frame Synchronous model and the Discrete Time Asynchronous model are indistinguishable since all signatures for an interfering session j consist of independent identically distributed Bernoulli trials at every slot. Hence the analysis of section 3.2.2 is still valid. What follows is a summary of the results that apply:

MAXIMUM LIKELIHOOD DETECTION.

OR channel. Equation (3.17) gives the M.L. detection rule for the OR channel.

ADDER channel. Lemma 3.2 gives the M.L. detection rule for the ADDER channel when the number of active sessions is known by the detector. The same example illustrates the difficulty in finding a simple M.L. rule if the number of active sessions is not known by the detector.

PROBABILITY OF ERROR.

Template detector. Equation (3.26) gives the probability of error for a Template detector operating on either an OR or an ADDER channel. It assumes two equiprobable source letters and that both signatures for user i have equal number of occupied slots.

Sum detector. Equation (3.26) also applies to the probability of error for a Sum detector of threshold h operating on the OR channel, under the same assumptions. Equation (3.27) stating the probability of error for a Sum detector of threshold h operating on the ADDER channel.

4.2.2 2-SIGNATURE, ASYMMETRIC, BERNOULLI SCHEME

MAXIMUM LIKELIHOOD DETECTION.

OR channel. From Lemma 3.1, the M.L. detector for session i is a Template detector.

ADDER channel.

Theorem 4.1 Consider a Discrete Time Asynchronous, 2-signature, asymmetric, Bernoulli scheme operating on an ADDER channel. Let M be the maximum number of interfering sessions and p the pulse placement probability for the Bernoulli signatures. If

$$(M+1)p \leq 1 \tag{4.1}$$

then the M.L. detector for session i is a Template detector.

This theorem supersedes Theorem 3.1.

Method. The proof appears at the end of this section. We develop first Lemma 4.1, which treats detection of an isolated frame of session i and is the core of the proof, and then treat the memory channel where we consider a succession of frames of session i being detected.

The approach followed to treat the 2-signature, asymmetric, Bernoulli scheme can be sketched as follows. Interfering sessions are assumed to be operating in steady state. We consider a frame of session i being detected. It lasts a known discrete time interval $[T_0, T_\omega]$. Within this interval we note the instants at which any given interfering session j starts a new frame as in Figure 4.1. These instants divide the interval $[T_0, T_\omega]$ into smaller subintervals which we call *regions*. The rationale for focusing on these regions is that the statistics describing the interference process remains unchanged within a region and if we condition appropriately the event space describing the interference we can find the likelihood probabilities over such regions. This conditioning will lead to statistical independence among regions which will allow a product form solution for the frame. Then we remove the conditioning events for the final analysis.

Particular definitions. We will need special definitions for the asymmetric Bernoulli scheme. We first state the symbols to be defined and below we explain their meaning.

$h(k)$	number of pulses of s^i present in region B_k.
$\xi(k)$	number of non null interfering signatures over B_k.
$\psi_+^i(k)$	set of non-zero slots of s^i , over B_k.
Ω	Set of "interference events".
$\mathcal{P} = \Omega \times Y$	a Joint Event Space.

The variables above are defined over the frame F_0^i being decoded. The same definitions can be applied to the preceeding frame F_{-1}^i and, if needed, we will use the subscript $_{-1}$ to denote such a variable. e.g. Ω_{-1} denotes the "interference event" (to be defined below) which applies over the frame F_{-1}^i. Similarly for the succeeding frame F_{+1}^i.

Because concurrent sessions are assumed in steady state, the set of active interfering sessions $\mathcal{M}$, and the set of instants t^j, $\forall j \in \mathcal{M}$, remain constant from frame to frame. We define an interference event Ω to be the set of 4-tuples

$$\Omega = \{\ (\ j,\ t^j, \sigma(F_{-1}^j), \sigma(F_0^j)\ )\ :\ j \in \mathcal{M}\} \tag{4.2}$$

In words, Ω records each active interfering session, its timing and the source letters encoded in the frames which overlap the present and previous frames of i. Ω is a full description of the channel interference but omitting the interference pulse patterns. As an example let $\mathcal{A} = \{0, 1\}$ be the source alphabet in Figure 4.1 and let the slots be indexed 0 through 9. For the particular event illustrated the shaded frames are transmitting 1 and the clear ones 0. We get

$$\Omega = \{\ (m, 7, 1, 0), (l, 4, 0, 0), (k, 2, 1, 0), (j, 4, 0, 1)\ \}$$

$\boldsymbol{\Omega}$ is simply the set of all possible Ω events for the received vector y. T_k, B_k follow from Ω.

Finally we define the Joint Event Space $\mathcal{P}$ as the cartesian product $\mathcal{P} = \boldsymbol{\Omega} \times Y$, where Y stands for the set of possible channel output vectors y. The events in this space are well defined and their probability measure can be found from the description of the model as long as a probability measure is assigned to the events in the set $\boldsymbol{\Omega}$ and to the probabilistic description of the signatures. We see that knowledge of the Joint Probability Distribution for the set of interfering sessions, their timing, their source letter probabilities and of the probabilistic description of signatures enables to compute this measure. Hence we assume the measure for $\mathcal{P}$ to be well defined. The arguments leading to the proof of Lemma 4.1 below do not require explicit knowledge of the space $\mathcal{P}$ or its measure; it is just required that they be well defined.

Isolated frame of i. Consider detection of an isolated frame F_0^i of session i. Condition $\mathcal{P}$ on a given event Ω. In this conditioned space the variables $y(k), k = 0, \ldots, \eta$ are mutually independent. This is so because knowledge of Ω implies knowledge of $\xi(k)$ for each of the regions B_k and $y(k)$ is only dependent on $\xi(k)$ and the Bernoulli trials over B_k for each of these $\xi(k)$ signatures. Consider a vector y such that $A(\psi_+^i)$ holds so both $X^i = \phi$ or $X^i = s^i$ are possible. For a given region B_k the problem reduces to that of having $\xi(k)$ interfering users placing pulses over a frame of length b_k as described by the Frame Synchronous model of section 3.2.1. The derivation of the likelihood probabilities for $y(k)$ in region B_k follows the steps leading to (3.2) in Chapter 3. Throughout this section the variables Θ, θ, and their subscripted versions, are understood to be functions of the vector y being considered. The dependence should be clear from the context. y is omitted from the notation for clarity.

$$
\begin{aligned}
&\Pr[y(k) = y(k) | X^i = s^i, \Omega] \\
&\quad = \prod_{m \in \psi_+^i(k)} \Pr[Z_m = y_m - 1] \prod_{n \notin \psi_+^i(k)} \Pr[Z_n = y_n] \\
&\quad = \left\{ \prod_{m \in \psi_+^i(k)} \binom{\xi(k)}{y_m - 1} \prod_{n \notin \psi_+^i(k)} \binom{\xi(k)}{y_n} \right\} p^{\sum\limits_{l \in B_k} y_l - h(k)} (1-p)^{\xi(k) b_k - \sum\limits_{l \in B_k} y_l + h(k)} \\
&\Pr[y(k) = y(k) | X^i = \phi, \Omega] \\
&\quad = \prod_{m \in B_k} \Pr[Z_m = y_m] \\
&\quad = \left\{ \prod_{m \in B_k} \binom{\xi(k)}{y_m} \right\} p^{\sum\limits_{k \in B_k} y_m} (1-p)^{\xi(k) b_k - \sum\limits_{k \in B_k} y_m} \\
&\quad = \Pr[y(k) = y(k) | X^i = s^i, \Omega] \left\{ \prod_{m \in \psi_+^i(k)} \frac{\xi(k) - y_m + 1}{y_m} \right\} p^{h(k)} (1-p)^{-h(k)} \\
&\quad = \Pr[y(k) = y(k) | X^i = s^i, \Omega] \; \Theta_{\xi(k)}
\end{aligned}
$$

where the one but last step is algebraic manipulation and

$$
\Theta_{\xi(k)} = \left\{ \prod_{m \in \psi_+^i(k)} \frac{\xi(k) - y_m + 1}{y_m} \right\} p^{h(k)} (1-p)^{-h(k)} \tag{4.3}
$$

$$
y_m - 1 \leq \xi(k) \leq M \qquad ; \forall k, \; \forall m \in \psi_+^i(k)
$$

Because of the independence among the $y(k)$ vectors in the conditioned space, the likelihood probability expression for the frame has a product form solution of the form

$$\Pr[y = y|X^i = s^i, \Omega]$$
$$= \left\{ \prod_{m \in \psi_+^i(0)} \binom{\xi(0)}{y_m - 1} \prod_{n \notin \psi_+^i(0)} \binom{\xi(0)}{y_n} \times \cdots \times \prod_{m \in \psi_+^i(\eta)} \binom{\xi(\eta)}{y_m - 1} \prod_{n \notin \psi_+^i(\eta)} \binom{\xi(\eta)}{y_n} \right\}$$
$$\times \quad p^{\sum_{l=1}^{n} y_l - h} \, (1-p)^{\sum_{k=0}^{\eta} \xi(k) b_k - \sum_{l=1} y_l + h}$$

$$\Pr[y = y|X^i = \phi, \Omega]$$
$$= \Pr[Y = y|X^i = s^i, \Omega]$$
$$\times \left\{ \prod_{m \in \psi_+^i(0)} \frac{\xi(0) - y_m + 1}{y_m} \times \cdots \times \prod_{m \in \psi_+^i(\eta)} \frac{\xi(\eta) - y_m + 1}{y_m} \right\}$$
$$\times \quad p^h (1-p)^{-h}$$
$$= \Pr[Y = y|X^i = s^i, \Omega] \; \theta_{\xi(0)} \cdots \theta_{\xi(\eta)} p^h (1-p)^{-h}$$
$$= \Pr[Y = y|X^i = s^i, \Omega] \; \Theta_{\text{asynch}} \tag{4.4}$$

where

$$\Theta_{\text{asynch}} = \theta_{\xi(0)} \cdots \theta_{\xi(\eta)} p^h (1-p)^{-h} \tag{4.5}$$
$$\theta_{\xi(k)} = \prod_{m \in \psi_+^i(k)} \frac{\xi(k) - y_m + 1}{y_m} \tag{4.6}$$

Lemma 4.1 Consider the detection of an isolated frame of session i in a Discrete Time Asynchronous, 2-signature, asymmetric, Bernoulli scheme operating on an ADDER channel. Let M be the maximum number of interfering sessions and p the pulse placement probability for the Bernoulli signatures. Let $\Omega, A(\psi_+^i)$ be as defined above. If

(3.7) $$(M+1)p \leq 1$$

then, for all y such that $A(\psi_+^i)$ holds,

$$\Pr[Y = y|X^i = \phi, \Omega] \leq \Pr[Y = y|X^i = s^i, \Omega] \tag{4.7}$$

for all $\Omega \in \mathbf{\Omega}$.

Proof: Consider all y such that $A(\psi^i_+)$ holds. Let the event Ω be given. By inspection of (4.6) , the maximum possible value of $\theta_{\xi(k)}$, $\forall k$, corresponds to $\xi(k) = M$. Hence we can upperbound Θ_{asynch} as follows:

$$\Theta_{\text{asynch}} \leq \left\{ \prod_{m \in \psi^i_+} \frac{M - y_m + 1}{y_m} \right\} p^h (1-p)^{-h} = \Theta_M \tag{4.8}$$

The quantity Θ_M is the same factor we had for the Frame Synchronous case in (3.2). As in the proof of Theorem 3.1, it is maximized with $y_m = 1$, $\forall m \in \psi^i_+$, so the bound

$$\Theta_M \leq \left[\frac{Mp}{1-p}\right]^h \tag{3.9}$$

applies. Since (3.7) in Lemma 4.1 implies $\Theta_M \leq 1$ then, from (4.8) and (4.4) ,

$$\Pr[Y = y | X^i = \phi, \Omega] \leq \Pr[Y = y | X^i = s^i, \Omega]$$

for all $\Omega \in \mathbf{\Omega}$.

∎

If we make a weighted summation over the set $\mathbf{\Omega}$ we obtain

$$\Pr[Y = y | X^i = \phi] \leq \Pr[Y = y | X^i = s^i]; \quad \forall y \text{ s.t. } A(\psi^i_+) \tag{4.9}$$

This and

$$\Pr[Y = y | X^i = s^i] = 0 \qquad \forall y \text{ s.t. } \overline{A}(\psi^i_+) \tag{4.10}$$

taken together imply that the M.L. detector reduces to the Template detector (2.7). This is a corollary of Lemma 4.1.

Memory channel. This result still holds when we consider F^i_0 within a sequence of frames of session i. As pointed out in the introduction, Asynchronism implies the channel model is not memoryless anymore. For the 2-signature, asymmetric, Bernoulli scheme however, this dependence simply conditions (restricts) the set of events in Ω which apply[1] over F_0^i. Hence (4.7) in Lemma 4.1 can also be applied to the memory channel.

We first show that given Ω the likelihood probabilities are independent of past and future frames of session i. Let y_{-1}, y_{+1} denote respectively the preceeding and succeeding output vectors of the channel relative to the vector y. Similarly define y_{-l}, y_{+l}, $l = 1, 2, \ldots$ Let x^i denote any one of the two possible signatures. For any y such that $\overline{A}(\psi_+^i)$ holds (4.10) is true regardless of past or future. We are left with those y such that $A(\psi_+^i)$ holds.

<u>**Lemma 4.2**</u> For all y such that $A(\psi_+^i)$ holds

$$\Pr[Y = y | X^i = x^i, \ldots, y_{-2}, y_{-1}, \Omega, y_{-1}, y_{-2}, \ldots] = \Pr[Y = y | X^i = x^i, \Omega] \qquad (4.11)$$

<u>**Proof:**</u> Let $A(\psi_+^i)$ hold. Consider any component of Y. Y_k for any slot k in the frame F_0^i is the sum of all pulses placed in that slot by interfering sessions. Because we are treating a Bernoulli scheme, interfering signatures place pulses or not in a slot with probabilities p and $1-p$ respectively, independently of any other slot. So Y_k only depends on the number of interfering signatures over that slot and on x_k^i. But given Ω we know the number of interfering signatures for all slots in F_0^i so Y is independent of $\ldots, y_{-2}, y_{-1}, y_{-1}, y_{-2}, \ldots$

∎

<u>**Proof of Theorem 4.1:**</u> Since no pulses can be erased (4.10) holds. We need worry only about y such that $A(\psi_+^i)$ holds. Consider the Discrete Time Asynchronous steady state situation. Using Lemma 4.2, the likelihood probabilities over F_0^i can be written then as:

$$\Pr[Y = y | X^i = x^i] = \sum_{\Omega} \Pr[\Omega] \Pr[Y = y | X^i = x^i, \ldots, y_{-2}, y_{-1}, \Omega, y_{-1}, y_{-2}, \ldots]$$

[1] Esentially, Ω must be compatible with the previous and following frames of session i.

$$= \sum_{\Omega} \Pr[\Omega] \Pr[Y = y | X^i = x^i, \Omega] \tag{4.12}$$

where the summations are over events Ω compatible with Ω_{-1} and Ω_{+1}. From Lemma 4.1 and (4.12) we arrive again at (4.9) and the M.L. detector reduces to a Template detector. It remains to remove the steady state condition. This is simple since a transient in the set $\mathcal{M}$ of active interfering sessions can be reduced to a corresponding spell of ϕ signatures from that session and the arguments above hold intact.

∎

4.3 ROULETTE SCHEMES

Both the asymmetric and symmetric schemes will be treated together since many of the definitions and arguments used apply to both.

MAXIMUM LIKELIHOOD DETECTION.

OR channel. From Lemma 3.1, the M.L. detector for session i for the asymmetric scheme is a Template detector.

For the symmetric Roulette scheme, writing down the transition probabilities in a useful form is a difficult task. The problems involved will be best illustrated in the ADDER channel case. For the OR channel the transition probabilities would be the sum, over all output vectors which correspond to a given binary pattern denoting the occupied slots, of the transition probabilities which apply to those vectors in the ADDER channel. Transition probabilities for the ADDER channel present their problems as we will see below.

ADDER channel. We have not been able to derive a simple M.L. detection rule in this case. The Roulette schemes present certain difficulties for the ADDER channel case which we now discuss.

The natural question is: can we write down the likelihood probability expressions for the Asynchronous case of the Roulette schemes? Once we address this we could then attempt to derive a simplified rule for Maximum Likelihood decoding. In view of the treatment for the asymmetric Bernoulli scheme we define the regions B_k as before and adopt the same notation, but altering those definitions which are specific to the signature schemes.

Particular definitions. The following definitions, further explained below, are specific to Roulette schemes:

$h^{i\sigma}(k)$	Number of pulses of $s^{i\sigma}$ present in region B_k.
$a(j,k)$	Number of pulses in B_k due to $j \in \mathcal{M}$.
$a(k)$	Total number of interfering pulses in B_k.
$\mathbf{\Omega}$	Set of "interference events".
$\mathcal{P} = \mathbf{\Omega} \times Y$	a Joint Probability Space.

Let $a(j,k)$ be the number of optical pulses placed in region B_k by a session $j \in \mathcal{M}$ and let A^j denote the set of values $a(j,k)$ of session j for all regions $B_k, k = 0, \ldots, \eta$. An interference event Ω is defined to be the set of triplets

$$\Omega = \{ \, (\, j, \; t^j, A^j) : \; j \in \mathcal{M} \} \tag{4.13}$$

Given an event Ω we can compute the total number of interfering pulses for each region B_k as:

$$a(k) = \sum_{j \in \mathcal{M}} a(j,k) \tag{4.14}$$

The set $\mathbf{\Omega}$ is defined to be the set of all possible Ω. Notice that instead of source letters $\sigma(F^j_\bullet)$, as in the asymmetric Bernoulli scheme, we are interested in the number of pulses per region $a(j,k)$. This is because the nature of a Roulette signature implies that given these numbers the pulses have uniform distributions within the regions defined which makes for tractability.

As before, the Joint Probability Space $\mathcal{P} = \mathbf{\Omega} \times Y$ and its measure is assumed to be well defined.

Discussion. Consider an output vector y such that signature $s^{i\sigma}$ is one of several feasible hypothesis. For convenience this time allow ϕ to be one such possible signature, $s^{i\lambda}$ say, in which case $h^{i\lambda} = 0$. This will allow treatment of both the symmetric and asymmetric cases together. Condition $\mathcal{P}$ on an event $\Omega \in \mathbf{\Omega}^\sigma \subseteq \mathbf{\Omega}$, where $\mathbf{\Omega}^\sigma$ is the subset of interference events which are compatible with $y, s^{i\sigma}$. i.e. Ω is such that

$$a(k) + h^{i\sigma}(k) = \sum_{m \in B_k} y_m \qquad ; \forall k \tag{4.15}$$

In this conditioned space the variables $y(k), k = 0, \ldots, \eta$ are mutually independent. This is so since, given Ω, $a(k)$ is known for all B_k and over each region B_k the problem reduces to interfering sessions placing a total of $a(k)$ pulses over b_k slots and the pulse positions are independent, uniformly distributed over the b_k slots. The analysis for each region is similar to the Frame Synchronous model. We have (c.f. Appendix AP-4):

$$\Pr[y(k) = y(k) | X^i = s^{i\sigma}, \Omega] = \frac{a(k)!}{\prod_{m \in \psi_+^{i\sigma}(k)} (y_m - 1)! \prod_{n \in \psi_-^{i\sigma}(k)} (y_n)!} \left(\frac{1}{b_k}\right)^{a(k)}$$

Because of independence among the vectors $y(k)$, the expression for the frame has a product form solution, yielding:

$$\Pr[Y = y | X^i = s^{i\sigma}, \Omega] = \frac{a(0)!a(1)!\ldots a(\eta)!}{\prod_{m \in \psi_+^{i\sigma}} (y_m - 1)! \prod_{n \in \psi_-^{i\sigma}} (y_n)!} \left(\frac{1}{b_0}\right)^{a(0)} \left(\frac{1}{b_1}\right)^{a(1)} \cdots \left(\frac{1}{b_\eta}\right)^{a(\eta)}$$

The factorial denominator has the same expression for all events $\Omega \in \mathbf{\Omega}^\sigma$. Hence making a weighted summation over the set $\mathbf{\Omega}^\sigma$ in the space $\mathcal{P}$ we obtain the expression

$$\Pr[Y = y | X^i = s^{i\sigma}] = \frac{A(y, \sigma)}{\prod_{m \in \psi_+^{i\sigma}} (y_m - 1)! \prod_{n \in \psi_-^{i\sigma}} (y_n)!}$$

$$\text{where} \qquad A(y, \sigma) = \sum_{\Omega \in \mathbf{\Omega}^\sigma} \Pr[\Omega] \frac{a(0)!a(1)!\ldots a(\eta)!}{b_0^{a(0)} b_1^{a(1)} \ldots b_\eta^{a(\eta)}}$$

Here we see the difficulty with this approach. The numerator $A(y, \sigma)$ is a weighted summation which depends on the signature being considered. It is not evident if the minimization of the metric $m(i, \sigma)$ in Lemma 3.3 (c.f. Appendix AP-4) will imply maximization of the likelihood probability. We note though that should this be the case an analysis similar to that of section 4.2.4 is possible for the memory channel.

4.4 CONTINUOUS TIME ASYNCHRONISM, SYMMETRIC BERNOULLI SCHEME

As mentioned in the introduction to this chapter, we do not try to derive the Maximum Likelihood detection rule since this would probably call for a waveform channel analysis and such approach does not look promising. Instead we seek to find the probability of error for a Template detector. We only treat de symmetric Bernoulli scheme because for the other three schemes presented the dependence among contents of their slots remains an obstacle to deriving an expression for their probability of error.

Channel and detectors. Extension of the definition of OR channel to Continuous Time is straightforward. If the optical energy received over the slot detection interval exeeds a certain threshold the slot is declared occupied.

To extend the ADDER channel to Continuous Time we assume that the slot detector can quantize the energy received over the slot detection interval. The quanta are calibrated to the energy corresponding to one received optical pulse. The detector thus declares how many of these quanta are received over the slot detection interval. This models a detector which compares the received energy to a threshold scale and rounds off to determine how many pulses have been received.

If a waveform channel description is considered for the Continuous Time Asynchronous model the definition of the Maximum Likelihood detector can be made rigorous but finding the Maximum Likelihood detection rule is not promising.

A Sum detector is achieved by quantizing the total energy received over all observed slots and comparing against a threshold. Probability of error calculations for the Sum detector present a problem. They require computation of the probability that the total energy over the slot detection in an optical intensity waveform exceeds a given threshold. The waveform consists of the summation

$$\sum_{k \in \psi_+^{i\sigma}} X(\tau - T_k)$$

where T_k is the delay of slot k, in the hypothesized signature $s^{i\sigma}$, relative to the first slot in the signature, and $X(\tau)$ is the broadcast channel waveform received at time τ. This is a typically difficult task and we will not pursue it.

A Template detector is achieved by checking for the simultaneous occupation of all slots under observation an the detection over each such slot is as for the OR channel. We will be able to derive results in this case by making the "High Power assumption", below, on the intensity of the optical pulses.

Asynchronism assumption. In a Continuous Time Asynchronous model we assume that slots of any two session cannot perfectly align in time. This implies that each slot of each interfering session overlaps, in time, two slots of the session i being detected. We also assume that a signature pulse lasts the entire duration of a slot.

High Power assumption. Consider a model where an interfering pulse which partially overlaps a slot triggers detection of a pulse over that slot. This, together with the previous Asynchronism assumption, implies that any interfering pulse received triggers pulse detection in two contiguous slots, i.e. two contiguous slots will register the presence of one pulse each. The probability of error is thus increased with respect to the Discrete Time model since an interfering pulse implies greater interference. The assumption is equivalent to a high power received optical pulse since enough energy, or photons, is received for detection even if a pulse is present at the slot detector for a short amount of time. Since we are mostly concerned with a Local Area Network environment, this seems a reasonable assumption to make. Our analysis is premised on this assumption.

Low Power scenario. If the received optical pulses have low enough power the situation changes. From the Poisson Process model for optical detectors, the probability of an optical beam of intensity $I(t)$ generating no electrical charges when incident on an optical detector, over an interval of time T, is given by a Poisson distribution (c.f. section 1.1). For an Inhomogeneous rate process with zero arrivals over the interval T, this probability is [Shapiro]

$$P[0] = e^{-\int_T I(t)dt}$$

The exponential nature of the function implies that, for low intensity light, a linear decrease in the length of the interval T results in an exponential increase in the probability of detecting no pulse at all. This implies that, unless interfering pulses are almost synchronized with the slots of session i, then two consecutive interfering pulses are probably needed to trigger detection of a pulse over a slot under observation. In this case the asynchronous nature of the interference translates into lower probability of error for a given number of interfering pulses on the channel than for a Discrete Time model.

Probability of error calculations would require computing the probability that the waveform describing the intensity of the superposition of several optical pulses delivers enough energy to a slot for it to be declared occupied. The High Power assumption previously defined does not only seem more representative but is also more tractable.

4.4.1 TWO SESSION ANALYSIS

Consider a two session situation. Let session i be the session whose signatures are being decoded and j be the interfering session. By the phrase "l-run" we mean a sequence of l consecutive slots. "l-run of i" refers to an l-run in session i. "l-run of $s^{i\sigma}$" refers to an l-run in signature $s^{i\sigma}$.

Consider an l-run of $s^{i\sigma}$. From the Asynchronous assumption $l+1$ contiguous slots of session j are needed to completely overlap the l-run of i. At most $l+1$ pulses of session j can interfere with an l-run of i. Let ν denote the minimum number of occupied slots of session j

which, taken together, can overlap all slots in the l-run of i. ν can take two values: for l even $\nu = l/2$; for l odd $\nu = (l+1)/2$.

Consider a situation where exactly k slots, $\nu \leq k \leq l+1$, in the $l+1$-run of j are occupied. Placing k pulses in $l+1$ slots leaves $l+1-k$ empty slots. In deriving the probability of error expression for a Template detector we first determine the possible ways in which these k pulses can interfere with all slots in the l-run of i. In this context we refer to such an event as a "k-pulse error event." Equivalently we can determine the possible ways in which k pulses and $l+1-k$ unoccupied slots can be arranged such that no two consecutive empty slots appear in the $l+1$-run of j. (Any two empty consecutive slots in the $l+1$-run of j will leave a slot of i free of interference.)

Conceptually, the k occupied slots allow $k-1$ "spaces" among them plus two more "spaces" at either side, one before the first occupied slot and one after the last occupied slot. By "space" we mean a run of slots, including the 0-run. We consider empty slots as combinatorial objects to be placed in the "spaces" among the occupied slots. It is permissible for some of these "spaces" to be assigned no empty slots, since this results in a run of occupied slots of session j, but it is not permissible for any "space" to be assigned more than one empty slot, since this would result in two or more consecutive empty slots and thus not all slots in the l-run of i would be interfered with. Hence the number of arrangements with no two consecutive empty slots is equal to the number of ways in which one can place $l+1-k$ objects out of $(k-1)+2 = k+1$ spaces.

$$\text{number of } k\text{-pulse error events} = \binom{k+1}{l+1-k} \tag{4.16}$$

Let $\mathcal{P}(l)$ denote the probability of session j interfering with all slots in an l-run of i. We have:

$$\mathcal{P}(l) = p^{l+1} + \binom{l+1}{1} p^{l}(1-p)^{1} + \binom{l}{2} p^{l-1}(1-p)^{2} + \binom{l-1}{3} p^{l-2}(1-p)^{3} + \dots$$
$$\dots + \binom{\nu+1}{l+1-\nu} p^{\nu}(1-p)^{l+1-\nu} \tag{4.17}$$

The coefficients are obtained from (4.16) . For l even this is expressed as

$$\mathcal{P}(l) = p^{l+1} + \binom{l+1}{1}p^l(1-p)^1 + \binom{l}{2}p^{l-1}(1-p)^2 + \binom{l-1}{3}p^{l-2}(1-p)^3 + \dots$$
$$\dots + p^{\frac{l}{2}}(1-p)^{\frac{l}{2}+1} \qquad ; \; l \text{ even} \qquad (4.18)$$

while for l odd it is expressed as

$$\mathcal{P}(l) = p^{l+1} + \binom{l+1}{1}p^l(1-p)^1 + \binom{l}{2}p^{l-1}(1-p)^2 + \binom{l-1}{3}p^{l-2}(1-p)^3 + \dots$$
$$\dots + \left(\frac{l+1}{2}+1\right)p^{\frac{l+1}{2}}(1-p)^{\frac{l+1}{2}} \qquad ; \; l \text{ odd} \qquad (4.19)$$

Finally consider a given signature $s^{i\sigma}$ and assume that no other signature of session i has pulse positions in common with this one. Let $s^{i\sigma}$ contain exactly m distinct runs of occupied slots, denoted $l_i, i = 1, \dots, m$. Then the probability that all occupied slots of this signature are interfered with, in an OR or ADDER channel using a symmetric Bernoulli scheme, given that there is only one interfering session is

$$P_{\text{interf}} = \prod_1^m \mathcal{P}(l_i) \qquad (4.20)$$

with $\mathcal{P}(l_i)$ given in (4.17) .

Having derived this equation it would be possible to evaluate the error probability for any such scheme. An analytically more useful expression would be that of an expected probability of error, where the expectation is taken over all possible $s^{i\sigma}$ signatures. The problem with this approach is the great number of possible configurations. Moreover, since (4.20) is nonlinear, it is not clear that we can approximate, as $n \to \infty$, the value of the expected probability of error by computing the probability of error for a typical signature using (4.20) . It is not clear whether small runs of occupied slots in a signature $s^{i\sigma}$ contribute most to the probability of error, since they may be more typical, or large runs contribute most, since these are more easily interfered with than several, disjoint, shorter runs adding up to the same total number of occupied slots.

Instead of trying to proceed from (4.20) we will treat in subsection 4.4.3 a simpler but still interesting situation. Before that, we generalize (4.20) to the case with ς interfering sessions.

4.4.2 $\varsigma + 1$ SESSIONS ANALYSIS

Consider now a session i of interest received in the presence of ς other interfering sessions. Again we assume that the Asynchronism assumption and the High Power assumption apply. The approach will be to reduce this case to the situation in the two session analysis.

Lemma 4.3 Consider a Continuous Time Asynchronous, symmetric, Bernoulli scheme implemented on an OR or ADDER channel where $\varsigma + 1$ sessions are active. Let the Asynchronism assumption and High Power assumptions above hold. Then the probability P_{interf} that all pulse positions in a given signature pattern $s^{i\sigma}$ are interfered with by the remaining ς sessions is given by (4.20) , where $\mathcal{P}(l_i)$ is given by (4.17) but replacing $1 - (1 - p)^{\varsigma}$ for p. In (4.20) $l_i, i = 1, \ldots, m$ is the size of each of the m runs of occupied slots in $s^{i\sigma}$; while in (4.16) , ν denotes the minimum number of pulses in an interfering session needed to completely interfere with an l-run of session i.

Proof: Consider a 2-run of i. For each of the ς interfering sessions there exists exactly one slot which overlaps both of these two consecutive slots of session i. With probability $(1 - p)^{\varsigma}$ none of these ς slots will interfere with the 2-run of i, and with probability $1 - (1 - p)^{\varsigma}$ one or more of these ς slots will be occupied and interfere with the 2-run of i. The interference for an l-run of i is thus the same as in a "two session" case with $1 - (1 - p)^{\varsigma}$ replaced for p. Hence (4.20) gives the interference probability for a signature by replacing $1 - (1 - p)^{\varsigma}$ for p in (4.17) .

∎

4.4.3 ISOLATED PULSE SIGNATURES

One interesting situation can be treated with relative simplicity. Consider a signature of session i where no two occupied slots are consecutive, i.e. the signature consists of only 1-runs of occupied slots. Let ς be the number of active interfering sessions. Rather than using Lemma 4.3 to compute the probability of error consider first the case $\varsigma = 1$. Let session j be the interfering session. The probability of a 1-run of i being interfered with can be found by using (4.19) .

$$\mathcal{P}(1) = (2-p)p \tag{4.21}$$

This result can easily be checked by simply computing the probability that either or both slots, in the 2-run of j that overlap a 1-run of i, are occupied. For an arbitrary positive value of ς we can argue as follows: the probabililty that a given interfering session interferes with a 1-run of i is given by (4.21) . The situation is then equivalent to a Discrete Time model for symmetric Bernoulli schemes where $(2-p)p$ is replaced for the Bernoulli pulse probability p of interfering signatures. The probability of error for the scheme is thus given by equation (3.26) but substituting $(2-p)p$ for p. Figure 4.2 shows the probability of error curves for Discrete Time and Continuous Time Asynchronous symmetric Bernoulli schemes. As expected, the probability of error is higher for the Continuous Time Asynchronous model.

4.4.4 COMMENTS

The "isolated pulse signature" case is interesting since it corresponds to a symmetric Bernoulli scheme with low value of Bernoulli parameter p. In such a case signatures are likely to consist mostly of isolated occupied slots. Notice also that $(2-p)p \rightarrow 2p$ as $p \rightarrow 0$ so, under the Asynchronism assumption and High Power assumption, a Template detector for session i, operating on a Continuous Time Asynchronous, symmetric, Bernoulli scheme of pulse probability p, experiences a channel interference almost like that in a Discrete Time model whose interfering sessions have a pulse probability twice as large as that of the session i (the latter probability defined as $p \equiv h/n$).

2–SIGNATURE, SYMMETRIC, BERNOULLI SCHEME
ISOLATED PULSES SIGNATURE, CONTINUOUS TIME
TEMPLATE DETECTOR P_e

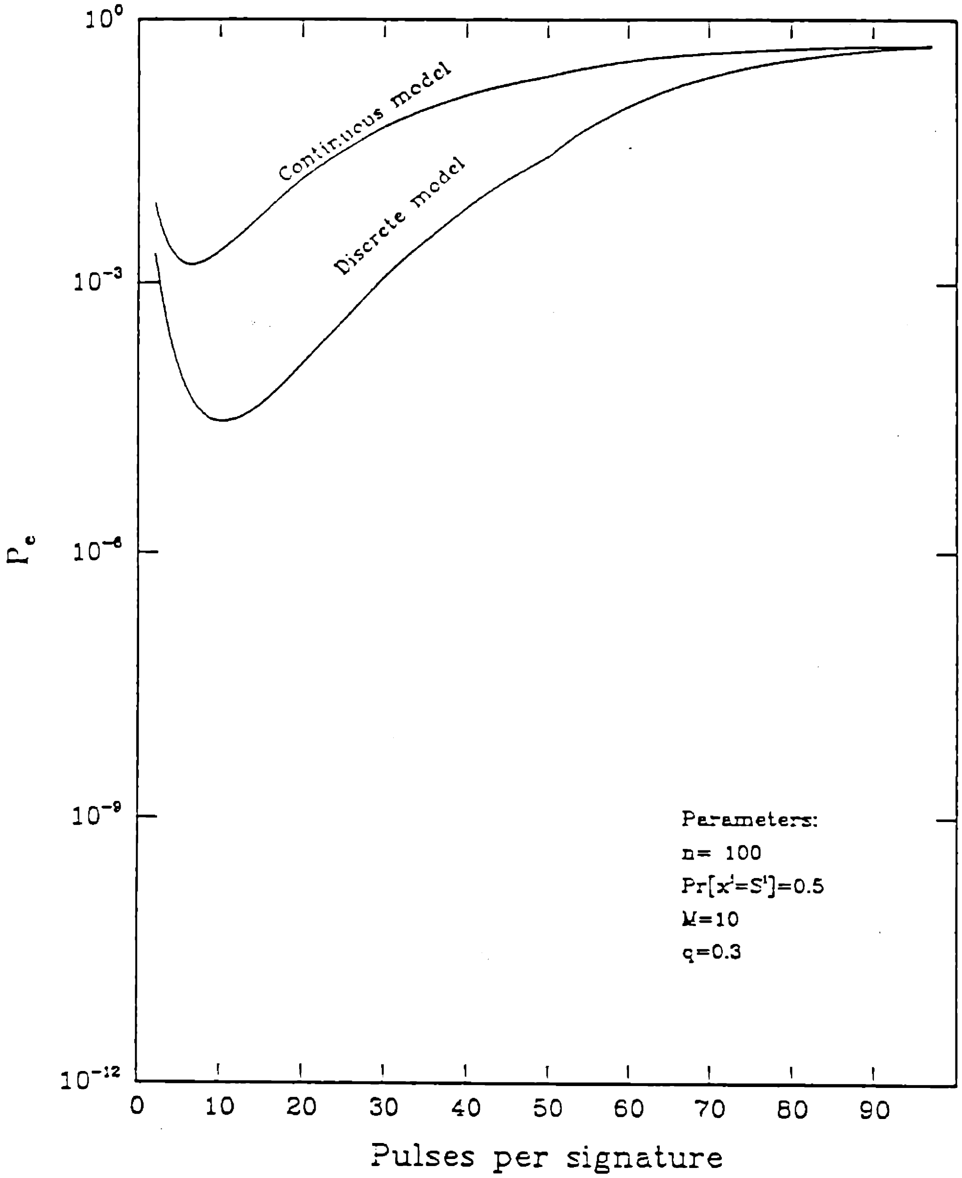

Figure 4.2

We have focused on the probability of error for a Template detector. The High Power assumption applied to the Sum detector would imply that the partial overlap of even a single pulse in any of the observed slots during the slot detection interval would yield enough energy to exceed the threshold and trigger detection of the signature. We would be better off using the Template detector.

We can see that a Template detector is not optimal under the High Power assumption. Consider a signature consisting of two consecutive occupied slots in a frame. If detection over these two slots is triggered by a received signal, a Template detector would consider this signature feasible. However, a detector which also observes the two slots neighbouring the pair of slots that correspond to the signature would be able to determine events in which only a single interfering asynchronous pulse triggered the detection in the Template detector, and avoid the error.

4.4.5 LIMITING RECEIVED INTENSITY

An improvement to prevent that a single interfering pulse trigger detection in two consecutive slots under observation would be as follows. After the optical to electrical conversion, in the slot detectors of a Template detector, a non linear device that limits the value of output current is added in series. (c.f. [Hui, hardlimiting].) Ideally the current limit is calibrated so that, when a single pulse is partially incident on the detector, integration over the slot time exceeds the detection threshold only if the pulse overlaps the entire slot. This corresponds to perfect slot synchronism and in practice a certain uncertainty time would need to be incorporated, e.g. by lowering the detection threshold by some small amount. The value at which the current is limited should be low enough so that session pulses which are received with the weakest optical intensity can still trigger detection.

In a system as described above the Template detector probability of error expression for the Discrete Time, symmetric, Bernoulli scheme is an upperbound for the probability of error of the Template detector in a Continuous Time Asynchronous situation. For schemes other than the symmetric Bernoulli we have no probability of error expressions for Discrete Time Asynchronism, but the current limiting circuit described above would also mean a decrease in probability of error with respect to Discrete Time.

CHAPTER 5
OPTICAL ADDER CHANNEL

In this chapter we discuss the assumptions underlying an ADDER channel model for Direct Detection optical systems. Section 5.1 discusses the effect of coherence among the optical sources. Section 5.2 discusses problems presented by the Poisson process behaviour of the optical to electrical conversion in a detector. Section 5.3 discusses the problem of path dependent attenuation of optical signals.

5.1 COHERENCE AMONG OPTICAL SOURCES

We model a unit amplitude optical beam as a stochastic process. Using exponential notation the optical carrier electric field for a source k is

$$\mathcal{E}_k(t) = e^{j(\omega_k t + \alpha_k + \theta_k(t))} \tag{5.1}$$

where ω_k is the center angular frequency, α_k is an initial phase, and the set $\{\theta_k(t) : \forall k\}$ is a set of mutually independent identically distributed stochastic processes describing the time dependent phase change. Each phase process in fact corresponds to a Random Walk motion described by a Wiener-Levy process (Browninan motion process) discussed below. The phase noise model is a standard, succesful characterization for optical source phase noise when center angular frequency drift of the optical source is ignored [1]. We ignore the drift in the treatment below. We will see that this will lead us to worst case results. We consider ω_k, α_k, $\forall k$, to be unknown but constant parameters. The spatial dependence and polarization of the electric field are neglected.

[1] See [Lax],[Henry 82],[Henry 83], [Moslehi], [T C], among others.

Channel Model. We consider a steady state situation where $N+1$ sources are transmitting, $N \geq 1$, corresponding to as many different sessions on the network. We use subscripts $k, i, m, n, q, r \in \{1, \ldots, N+1\}$ to denote channel sources. The physical channel is linear in the electric field $\mathcal{E}(t)$. Its output is

$$\mathcal{E}(t) = \sum_k e^{j(\omega_k t + \alpha_k + \theta_k(t))} \tag{5.2}$$

This output is fed to a power detector whose output is the instantaneous optical intensity[2]

$$I(t) = |\mathcal{E}(t)|^2 = \sum_{k,i} e^{j(f_{k,i}(t) + \theta_k(t) - \theta_i(t))} \tag{5.3}$$

where $f_{k,i}(t)$ is a non-random part given by

$$f_{k,i}(t) = (\omega_k - \omega_i)t + (\alpha_k - \alpha_i)$$

(5.3) assumes a unit gain, perfect optical to electrical conversion. This assumption will be discussed appropriately further below.

We are ultimately interested in the time average

$$< I(t) > \equiv \frac{1}{T} \int_T I(t) \; dt \tag{5.4}$$

This corresponds to post detection integration of the photodetector current.

Phase noise. For $t \gg 1/\omega_k$ the observed behaviour of phase noise [Lax] can be accurately described by the stochastic differential equation

$$\frac{d}{dt}\theta_k(t) = n_k(t) \tag{5.5}$$

$\theta_k(t)$ is a random variable, $n_k(t)$ is a zero mean White Gaussian noise process of spectral density $S_{n_k}(\omega) = \gamma_k$. The solution for $\theta_k(t)$ corresponds to the stochastic integral of $n_k(t)$,

[2] $I(t)$ as defined here is not directly the optical intensity that would be measured by an optical device. The intensity measured by a unit gain photodetector would be $I(t)/2$.

i.e. a Wiener-Levy process (random walk), and hence is a Gaussian process. The condition $t \gg 1/\omega_k$ is readily met in operating optical sources. Using as initial conditions[3]

$$\theta_k(0) = 0 \qquad ;\forall k \tag{5.6}$$

the expected value and autocorrelation function of $\theta_k(t)$ are [Papoulis]

$$E[\theta_k(t)] = 0 \tag{5.7}$$

$$R_{\theta_k,\theta_k}(t_2,t_1) = \gamma_k t \qquad ;t \equiv \min(t_2,t_1) \tag{5.8}$$

and the expected value and autocorrelation function of the electric field phase are

$$E[\omega_k t + \alpha_k + \theta_k(t)] = \omega_k t + \alpha_k \tag{5.9}$$

$$R_{k,k}(t_2,t_1) = \omega_k^2 t_2 t_1 + \alpha_k^2 + \omega_k \alpha_k (t_2 + t_1) + \gamma_k t \qquad ;t \equiv \min(t_2,t_1) \tag{5.10}$$

Electric field power spectrum. The autocorrelation function for source k's optical emission is

$$\begin{aligned} \mathcal{R}_{k,k}(t_2,t_1) =& E[e^{j[\omega_k(t_2-t_1)+\theta_k(t_2)-\theta_k(t_1)]}] \\ =& e^{j\omega_k(t_2-t_1)} e^{-\frac{1}{2}\sigma^2} \qquad ;\ \sigma^2 \equiv E[(\theta_k(t_2) - \theta_k(t_1))^2] \end{aligned} \tag{5.11}$$

The last equation follows from the fact that the expression for $\mathcal{R}_{k,k}(t_2,t_1)$ is the generating function of a Gaussian random variable. Taking the square in the expression for σ^2, using equation 5.8, and defining $\tau = t_2 - t_1$ one obtains

$$\mathcal{R}_{k,k}(t_2,t_1) = e^{j\omega_k \tau} e^{-\frac{1}{2}\gamma_k|\tau|} \qquad ;\tau = t_2 - t_1 \tag{5.12}$$

The Fourier transform gives the power spectrum

$$S_{I_k}(\omega) = \frac{2\gamma_k}{\gamma_k^2 + (\omega - \omega_k)^2} \tag{5.13}$$

[3] Notice that the parameter α_k in equation (5.2) can account for any non zero initial condition.

w_k is the center angular frequency as mentioned before, and the "half-power" angular bandwidth is

$$\omega_{L_k} = \gamma_k \tag{5.14}$$

which is also known as the "Lorentzian" angular linewidth. We assume that $\gamma_k, \forall k$ can take values in a finite range and let ω_L denote the minimum.

Expected value of output $I(t)$. Define the "mixing" of sources k and i optical beams to be the process

$$\varphi_{k,i}(t) = \mathcal{E}_k(t)\mathcal{E}_i^*(t) = e^{j(f_{k,i}(t)+\theta_k(t)-\theta_i(t))} \tag{5.15}$$

Then the output intensity is

$$I(t) = \sum_{k,i} \varphi_{k,i}(t) \tag{5.16}$$

To compute the expected value of $I(t)$ we will need the cross-correlation at times $t_2 = t_1 = t$ for the electric field terms:

$$\begin{aligned} \mathcal{R}_{k,i}(t,t) =& E[\varphi_{k,i}(t)] \\ =& e^{jf_{k,i}(t)}e^{-\frac{1}{2}\sigma^2} \qquad ;\sigma^2 = E[(\theta_k(t) - \theta_i(t))^2] \end{aligned} \tag{5.17}$$

The expression is obtained as in equation (5.11) . Taking the square, noting that $\theta_k(t), \theta_i(t)$ are independent, with variance γ_k, γ_i, we get

$$\mathcal{R}_{k,i}(t,t) = \begin{cases} 1, & k = i; \\ e^{jf_{k,i}(t)}e^{-\frac{1}{2}(\gamma_k+\gamma_i)}, & k \neq i. \end{cases} \tag{5.18}$$

and the expected value of the output is

$$E[I(t)] = \sum_{k,i} E[\varphi_{k,i}(t)] = (N+1) + \sum_{\substack{k,i \\ k\neq i}} e^{jf_{k,i}(t)}e^{-\frac{1}{2}(\gamma_k+\gamma_i)t}$$

which can be bounded as

$$(N+1) - (N+1)Ne^{-\omega_L t} \leq E[I(t)] \leq (N+1) + (N+1)Ne^{-\omega_L t} \tag{5.19}$$

The bounds correspond to worst case expected values. Equality is achieved in the upperbound when $\omega_k = \omega_i, \alpha_k = \alpha_i;\ \forall k, i$ and $\gamma_k = \omega_L;\ \forall k$. We do not worry about achievability of the lower bound in a general case but note that for a two source situation if we have $\omega_k = \omega_i, \alpha_k = \alpha_i + (2m+1)\pi$, where m is an integer, the bound is met. (5.19) says that the expected value equals the number of sources present plus a real valued interuser interference term whose magnitude is maximized when there is maximum initial coherence among optical beams at the receiver device. This worst case corresponds to all having equal center angular frequency, all beams having equal initial phase as measured at the receiver, and all having the narrowest possible linewidth. Since the initial phases are confined to an interval of 2π and there exists the posssibility that they all lie within a small interval ϵ of each other we have chosen the worst case situation. Had we considered these phases as random variables and taken instead an expectation over the joint values of these phases we would obtain a result akin to an ensemble average but this would not describe the behaviour of particular instances of networks.

In the ADDER channel model, the detector tries to determine how many pulses have been received. We consider implementation of an optical ADDER channel by using a linear threshold scale. It is desired thus to have

$$(N+1) - \frac{1}{2} \le E[I(t)] < (N+1) + \frac{1}{2}$$

so that the expected value of intensity lies in the range corresponding to $N+1$ in the scale. From (5.19) this implies

$$(N+1)Ne^{-\omega_L t} < \frac{1}{2} \tag{5.20}$$

Thus we define, for convenience, the "Mixing Interference Level", expressed in dBs, to be

$$\begin{aligned} F_{\mathrm{dB}} &= 10\log_{10}\left[2(N+1)Ne^{-\omega_L t}\right] \\ &= 10(\ln 10)^{-1}(\ln[2(N+1)N] - \omega_L t) \end{aligned} \tag{5.21}$$

Theorem 5.1 Let the transmission system be as described in this section. Let ω_L be the minimum beam linewidth the optical sources could have. Let $N+1 \geq 2$ be the number of optical emissions present and let $I(t)$ denote the combined intensity as in (5.2) .

$$\text{If} \qquad t \gg \frac{\ln[2(N+1)N]}{\omega_L} \qquad \text{then} \qquad E[I(t)] \simeq N+1 \tag{5.22}$$

in the sense that

$$|E[I(t)] - (N+1)| \ll \frac{1}{2}$$

Proof: Follows from (5.21) .

∎

(5.22) expresses only a sufficient condition since it is based on the bounds in (5.19) . If some linewidths are greater than the minimum posssible and/or if not all frequencies or initial phases are equal then smaller t values would apply in (5.22) .
Note that if we consider sources of equal non unit electric field amplitude A in (5.1) and an optical to electrical conversion of non unit gain g in (5.3) , a constant equal to $A^2 g$ would appear on both sides of (5.21) and Theorem 5.1 would still hold, with the expected value of the detector output tending to $A^2 g(N+1)$.

$I(t)$ convergence to $N+1$. We assume that the optical detector output is a time average of the intensity $I(t)$, defined by (5.4) . The question of practical interest now is under what conditions, if any at all, the time average $< I(t) > \rightarrow N+1$ as $T \rightarrow \infty$. We can show that, if the time over which the time average is taken exceeds a simple function of ω_L and of the number of sources $N+1$, then the time average converges in probability to $N+1$. The importance of the result is to give an idea of the minimum detection integration time such that an ADDER channel can be implemented.

We will sometimes use the symbol T to denote the interval $[0,T]$. The meaning should be clear from the context. From equations (5.16) , (5.15) and interchanging the order of linear operators

$$
\begin{aligned}
E[< I(t) >] &= \sum_{k,i} \frac{1}{T} \int_T E\left[\varphi_{k,i}(t)\right] \; dt \\
&= (N+1) + \sum_{\substack{k,i \\ k \neq i}} \frac{1}{T} \int_T e^{j f_{k,i}(t) - \frac{1}{2}(\gamma_k + \gamma_i)t} \; dt \qquad ; \text{by } (5.18) \\
&= (N+1) + \sum_{\substack{k,i \\ k \neq i}} \frac{e^{j(\alpha_k - \alpha_i)}}{T[\frac{1}{2}(\gamma_k + \gamma_i) - j(\omega_k - \omega_i)]} \left(1 - e^{[\frac{1}{2}(\gamma_k + \gamma_i) - j(\omega_k - \omega_i)]T}\right)
\end{aligned}
$$

The expression can be bounded from above and below by

$$E[< I(t) >] \geq (N+1) - \frac{(N+1)N}{\omega_L T}\left(1 - e^{-\omega_L T}\right) \tag{5.23}$$

$$E[< I(t) >] \leq (N+1) + \frac{(N+1)N}{\omega_L T}\left(1 - e^{-\omega_L T}\right) \tag{5.24}$$

by an analysis similar to that for (5.19) . Again the magnitude of the interuser interference term is maximum for maximum initial coherence of the optical beams at the receiver.

Lemma 5.1

$$\lim_{T \to \infty} E[< I(t) >] = N + 1$$

Proof: Follows from (5.24) , (5.23) .

∎

As in (5.21) we also define the Mixing Interference Level for the average intensity:

$$
\begin{aligned}
F_{\mathrm{dB}} &= 10 \log_{10} \left[\frac{2(N+1)N\left(1 - e^{-\omega_L T}\right)}{\omega_L T}\right] \\
&= 10(\ln 10)^{-1}\left(\ln[2(N+1)N] - \ln(\omega_L T) + \ln(1 - e^{-\omega_L T})\right)
\end{aligned} \tag{5.25}
$$

Theorem 5.2

$$\text{If} \qquad T \gg \frac{2(N+1)N}{\omega_L} \qquad \text{then} \qquad E[< I(t) >] \simeq N+1 \tag{5.26}$$

In the sense that

$$|E[< I(t) >] - (N+1)| \ll \frac{1}{2}$$

Proof: By inspection of (5.23) , (5.24) , if the Mixing Interference Level tends to zero relationship (5.26) is satisfied. The only positive term in (5.25) is $\ln[2(N+1)N]$. If we neglect the last term the result follows.

∎

Again, with sources of electric field amplitude A and with optical to electrical conversion of gain g, Theorem 5.2 holds, and the integrator output tends to $A^2 g(N+1)$. Notice also the difference between the conditions for T in theorems 5.1 and 5.2. This is due to the fact that since an integrator acts as a memory device the effect of phase initial conditions lasts longer in this device than in the simple observation of $I(t)$.

We now show that the variance of $< I(t) >$ converges to zero with T. This, together with Chebishev's inequality

$$\Pr[\ |x - E[x]| \geq \epsilon] \leq \frac{\sigma_x^2}{\epsilon^2} \qquad ;\ \forall \epsilon > 0$$

will show that $< I(t) >$ converges in probability to $N+1$. By convergence in probability we mean that

$$\lim_{T\to\infty} \Pr[\ | < I(t) > -(N+1)| \geq \epsilon] = 0 \qquad \forall\ \epsilon > 0$$

In fact, since we are really interested in how large T needs to be so that $< I(t) > \simeq N+1$, we will seek the value of T for which the standard deviation of $< I(t) >$ is less than 1/2, so that $< I(t) >$ is typically less than half a unit away from $N+1$. We first prove Lemma 5.2 and then use it to lead up to Theorem 5.3 which contains the final result.

We bound the variance of $< I(t) >$ as follows. From (5.23) we have

$$E^2[< I(t) >] \geq (N+1)^2 - 2(N+1)N\frac{(1-e^{-\omega_L T})}{\omega_L T} + (N+1)^2 N^2 \left\{\frac{(1-e^{-\omega_L T})}{\omega_L T}\right\}^2 \tag{5.27}$$

Also

$$\begin{aligned} E[< I(t) >^2] &= \frac{1}{T^2}\int_T\int_T \sum_{k,i,m,n} E\left[e^{j(\theta_k(t_a)-\theta_i(t_a)+\theta_m(t_b)-\theta_n(t_b))}\right] dt_a dt_b \\ &= \frac{1}{T^2}\int_T\int_T \sum_{k,i,m,n} e^{-\frac{1}{2}\sigma^2}\ dt_a dt_b \end{aligned} \tag{5.28}$$

where t_a, t_b are dummy integration variables and σ^2, a function of k, i, m, n, denotes the variance of the gaussian random variable $\theta_k(t_a) - \theta_i(t_a) + \theta_m(t_b) - \theta_n(t_b)$, c.f. equation (5.11)

<u>Lemma 5.2</u> For equation (5.28)

$$\sigma^2 = \begin{cases} 0, & \text{if } k = i \text{ and } m = n; \\ \Gamma_b t_b, & \text{if } k = i \text{ and } m \neq n; \\ \Gamma_a t_a, & \text{if } k \neq i \text{ and } m = n; \\ \Gamma_a t_a + \Gamma_b t_b, & \text{if } k \neq i \text{ and } m \neq n. \end{cases} \tag{5.29}$$

where

$$\Gamma_a = \gamma_k + \gamma_i; \qquad \Gamma_b = \gamma_m + \gamma_n$$

and $\Gamma_a, \Gamma_b \geq 2\omega_L$.

<u>Proof:</u>

$$\sigma^2 = E\left[(\theta_k(t_a) - \theta_i(t_a) + \theta_m(t_b) - \theta_n(t_b))^2\right] \tag{5.30}$$

Taking the square in (5.30)

$$\begin{aligned} \sigma^2 =& E[\theta_k^2(t_a)] + E[\theta_i^2(t_a)] - 2E[\theta_k(t_a)\theta_i(t_a)] \\ &+ 2E[\theta_k(t_a)\theta_m(t_b)] - 2E[\theta_k(t_a)\theta_n(t_b)] + 2E[\theta_i(t_a)\theta_n(t_b)] - 2E[\theta_i(t_a)\theta_m(t_b)] \\ &E[\theta_m^2(t_b)] + E[\theta_n^2(t_b)] - 2E[\theta_m(t_b)\theta_n(t_b)] \end{aligned} \tag{5.31}$$

Let $t \equiv \min(s_1, s_2)$. From independence among sources, and from (5.7) , (5.8) , $E[\theta_r(s_1)\theta_q(s_2)]$ can only take two values

$$E[\theta_r(s_1)\theta_q(s_2)] = \begin{cases} 0, & \text{if } r \neq q; \\ \gamma_r t, & \text{if } r = q. \end{cases} \tag{5.32}$$

Thus the expression for σ^2 can be simplified. Four situations are possible: A) $k = i, m = n$; B) $k = i, m \neq n$; C) $k \neq i, m = n$; D) $k \neq i, m \neq n$.

A) $\underline{k = i, m = n}$: If $k = i, m = n$ then, by inspection of (5.30) , $\sigma^2 = 0$, proving the first part of (5.29) .

B) $\underline{k = i, m \neq n}$: By inspection of (5.31) , bearing (5.32) in mind, and eliminating terms of equal magnitude but of opposite sign

$$\sigma^2 = E[\theta_m^2(t_b)] + E[\theta_n^2(t_b)] = (\gamma_m + \gamma_n)t_b$$

C) $\underline{k \neq i, m = n}$: By inspection of (5.31) , similarly,

$$\sigma^2 = E[\theta_k^2(t_a)] + E[\theta_i^2(t_a)] = (\gamma_k + \gamma_i)t_a$$

D) $\underline{k \neq i, m \neq n}$: By inspection of (5.31) , similarly

$$\sigma^2 = E[\theta_k^2(t_a)] + E[\theta_i^2(t_a)] + E[\theta_m^2(t_b)] + E[\theta_n^2(t_b)] = (\gamma_k + \gamma_i)t_a + (\gamma_m + \gamma_n)t_b$$

Recalling that $\gamma_r \geq \omega_L, \forall r$, we arrive at Lemma 5.2.

∎

Replacing (5.29) into (5.28) we can write

$$E[< I(t) >^2] = (N+1)^2 + \sum_{\substack{k,i,m,n \\ k=i, m\neq n}} \frac{1}{T^2} \int_0^T dt_a \int_0^T dt_b e^{-\frac{1}{2}\Gamma_b t_b} + \sum_{\substack{k,i,m,n \\ k\neq i, m=n}} \frac{1}{T^2} \int_0^T dt_b \int_0^T dt_a e^{-\frac{1}{2}\Gamma_a t_a} + \sum_{\substack{k,i,m,n \\ k\neq i, m\neq n}} \frac{1}{T^2} \int_0^T dt_a \int_0^T dt_b e^{-\frac{1}{2}[\Gamma_a t_a + \Gamma_b t_b]}$$

$$\leq (N+1)^2 + 2(N+1)^2 N \left\{ \frac{\left(1 - e^{-\omega_L T}\right)}{\omega_L T} \right\}$$

$$+ (N+1)^2 N^2 \left\{ \frac{\left(1 - e^{-\omega_L T}\right)}{\omega_L T} \right\}^2 \qquad (5.33)$$

The last step is obtained by noting that $E[< I(t) >^2]$ is a maximum for $\Gamma_a = \Gamma_b = 2\omega_L$. This corresponds to $\gamma_r = \omega_L, \forall r$.

Replacing (5.27) , (5.33) into the definition of variance we obtain the upperbound

$$\begin{aligned} \text{var} \ < I(t) > &= E[< I(t) >^2] - E^2[< I(t) >] \\ &\leq 2N(N+1) \left\{ \frac{\left(1 - e^{-\omega_L T}\right)}{\omega_L T} \right\} \end{aligned} \qquad (5.34)$$

Lemma 5.3

$$\lim_{T \to \infty} \quad \text{var} \ < I(t) >= 0$$

Proof: Follows from (5.34) .

∎

Lemma 5.4 $< I(t) >$ converges in probability to $N+1$ in the limit $T \to \infty$.

Proof: From lemmas 5.1 and 5.3

$$\lim_{T \to \infty} E[< I(t) >] = N+1; \qquad \lim_{T \to \infty} \quad \text{var} \ < I(t) >= 0$$

From Chebishev's inequality this implies

$$\lim_{T \to \infty} \Pr[\ | < I(t) > -(N+1)| \geq \epsilon] = 0 \qquad ;\forall \ \epsilon > 0$$

∎

Let $\Lambda(T)$ denote the standard deviation of the time average intensity.

$$\Lambda(T) \equiv \sqrt{\text{var} < I(t) >}$$

Lemma 5.5

$$\text{If} \qquad T \gg \frac{8(N+1)N}{\omega_L} \qquad \text{then} \qquad \Lambda(T) \ll \frac{1}{2} \tag{5.35}$$

Proof:

$$\text{var} \ < I(t) > \ll \frac{1}{4} \Rightarrow \Lambda(T) \ll \frac{1}{2}$$

Replacing the upperbound in (5.34) for the variance of $< I(t) >$, these two inequalities are equivalent to

$$8(N+1)N \left\{ \frac{\left(1 - e^{-\omega_L T}\right)}{\omega_L T} \right\} \ll 1 \tag{5.36}$$

so $T \gg 8(N+1)N/\omega_L$ implies $\Lambda(T) \ll 1/2$

■

Theorem 5.3

$$\text{If } T \gg 8(N+1)N/\omega_L \qquad \text{then} \qquad < I(t) > \simeq N+1 \tag{5.37}$$

in the sense that

$$|E[< I(t) > -(N+1)| \ll \frac{1}{2} \qquad \text{and} \qquad \Lambda(T) \ll \frac{1}{2}$$

Proof: Follows from Theorem 5.2 and lemma 5.5.

■

Note again that if sources of electric field intensity A and optical to electrical conversion of gain g are assumed, a factor of $A^2 g$ would appear on both sides of (5.36) so Theorem 5.3 would still hold, with $< I(t) >$ tending to $A^2 g(N+1)$. From this theorem we see that the parameter $(N+1)N/\omega_L$ is important in expressing a sufficiently large value of T for the average intensity observed to be close to $N+1$.

Discussion. The results above show that for large enough post detection integration time the time average, i.e. the average number of photons detected per unit time, corresponds to the sum total of the average number of photons of each of the sources present at the detector input even when maximum coherence is assumed among these sources. The probability that the value of the time average is some quantity ϵ away from $N+1$ becomes arbitrarily small as T becomes larger.

Coherence among optical sources as treated above does not present as great a problem for an OR channel since detectors in this case need only determine the presence of light, regardless of its intensity. One problem that would exist is the possibility that coherence among sources so weakens this intensity that its presence is not detected. Since this effect is related to the generation of charges in an optical detector we discuss it in the following section.

5.2 POISSON PROCESS MODEL FOR DETECTION

As for section 5.1 we assume $< I(t) >= N+1$, where $N+1$ is the number of optical sources that make up $I(t)$.

Miss probability. As mentioned in section 1.1 optical photodedection can be modeled as an inhomogeneous Poisson process of rate $\lambda(t)$ proportional to the instantaneous optical intensity $I(t)$ incident on the detector. For simplicity we assume $\lambda(t) = I(t)$. There exists then a non-zero probability that no output charges are generated by the detector over the whole interval T of integration even if light is incident on the detector[4]. This is commonly referred to in detection theory as "miss probability." The larger the intensity the lower this probability.

[4] In practice a non zero current threshold would have to be exceeded before a detector declares a pulse present, so that dark current and thermal noise can be tolerated.

Since the superposition of several nearly coherent sources will also affect the intensity incident on a detector we are interested in the behaviour of this miss probability as a function of the coherence of the sources. It will be easy to lowerbound the value for miss probability.

From [Shapiro], the miss probability P_0 is given by

$$P_0 = e^{-\int_T \lambda(t)dt} = e^{-\int_T I(t)dt} = e^{-T<I(t)>} \tag{5.38}$$

$T < I(t) >$ is a real valued random variable and takes values in the open interval $(0, \infty)$. P_0 is a convex $\cup$ function in $T < I(t) >$ over the interval. Thus we can lowerbound the expected value of the miss probability by

$$E[P_0] \geq e^{-T\overline{<I(t)>}} \tag{5.39}$$

where the overbar denotes expectation. Replacing (5.24) into (5.39)

$$E[P_0] \geq e^{-T(N+1) - \frac{(N+1)N}{\omega_L}\left(1 - e^{-\omega_L T}\right)} \tag{5.40}$$

For values of $T \gg N/\omega_L$ the RHS of (5.40) can be approximated by $e^{-T(N+1)}$.

For an OR channel, the miss probability is an appropriate characterization of the detection error over a given time slot. For an ADDER channel we need to determine the number of pulses present over a slot. In this case the miss probability is not the appropriate characterization, as we discuss below.

Optical ADDER channel accuracy. In an optical ADDER channel, apart from the problem of coherence discussed in Section 5.1 and of miss probability, the Poisson process nature of the optical to electrical conversion presents drawbacks in the receiver to a linear threshold scale for detection of the number of pulses. Let $I(t)$ be the incident optical intensity. The charges generated by this intensity at the optical detector can be modeled as a Poisson process of rate $\lambda(t) = gI(t)$, where g is a proportionality constant[5]. The total number of charges K generated over a detection integration time T is given by the Poisson distribution

$$\Pr[K = k] = \frac{(gT < I(t) >)^k}{k!} e^{-gT<I(t)>} \tag{5.41}$$

Let $< I(t) >= N + 1$, [6]then the expected value and variance of K are

$$E[K] = g(N+1)T \tag{5.42}$$

$$\sigma_K^2 = g(N+1)T \tag{5.43}$$

For a linear threshold detection scale we calibrate $K = gT$ to correspond to one pulse received. We want to investigate the conditional probability that, given $< I(t) >= N + 1$, the total number of charges generated lies in the range $[g(N+1)T - (1/2)gT \ , \ g(N+1)T + (1/2)gT]$.

$$P_{N+1}(g,T) \equiv \Pr\left[|K - g(N+1)T| \leq \frac{1}{2}gT \;\middle|\; \lambda(t) = g(N+1)\right] \tag{5.44}$$

For $g(N+1)T$ large, we can use a Central Limit Theorem to approximate $P_{N+1}(g,T)$ by integrating a Gaussian PDF, of mean and variance given by (5.42) , (5.43) , over the range $[g(N+1)T - (1/2)gT \ , \ g(N+1)T + (1/2)gT]$.

$$\begin{aligned} P_{N+1}(g,T) &\simeq 1 - 2Q\left(\frac{\frac{1}{2}gT}{\sqrt{g(N+1)T}}\right) \\ &= 1 - 2Q\left(\frac{1}{2}\sqrt{\frac{gT}{N+1}}\right) \end{aligned} \tag{5.45}$$

where $Q(\cdot)$ denotes the Gaussian Q function and $Q(\infty) = 0$. Since $Q(\cdot)$ is monotonically decreasing in its argument, (5.45) says that for large $N+1$ the probability $P_{N+1}(g,T)$ tends to zero. Thus, just by virtue of the randomness of a Poisson process, aside from the problems due to coherence of the optical sources that make up $I(t)$, a linear threshold scale is inappropriate for too large number of pulses if the product gT is fixed. The proportionality constant g and the integration time T can be increased to address this problem (at the expense of lower pulse

[5] This constant corresponds to the optical to electrical conversion gain. It can also be used to account for non unit intensity sources.

[6] We use $N + 1$ instead of N for consistency with the previous section.

rates in the systems). For Direct Detection with no coding the Quantum Limit for detection of a pulse is approximately 10 photons/pulse. In our case we would want to work well above this limit so that thermal noise is not a problem. Assume conservatively 10^3 photons/pulse. Quantum Efficiency is very nearly 1 electron-hole pair/photon. This gives a gT product of the order of 10^3 electrons per pulse. Assuming $N+1=50$ for the ADDER channel we obtain

$$P_{N+1}(g,T) \simeq 1 - 2Q(2.24) \simeq 0.975$$

It is not clear what values of $P_{N+1}(g,T)$ would be required for a detector like the one in Lemma 3.3, which relies on an ADDER channel implementation, to have acceptably low probability of error. However it would seem from this example that the number of photons per pulse required would have to be much greater than 10^3 so that P_{N+1} is much closer to one.

g is rather a property of the device being used while T is a design parameter. Assuming a standard deviations in the argument for $Q(\cdot)$ are necessary for an acceptable value of $P_{N+1}(g,T)$ then the condition $T > (2a)^2(N+1)/g$ would be required, where $N+1$ would stand for the maximum number of pulses to be detected. For systems where coherence among optical sources is also a problem and the condition $T \gg 8(N+1)N/\omega_L$ in Theorem 5.3 is required, the minimum value of T satisfying these two conditions would determine the maximum slot rate which such an optical system can support.

5.3 PATH DEPENDENT ATTENUATION

We consider a passive optical broadcast channel. As a pulse in session j travels from transmitter to receiver it travels through several optical coupling devices. At each of them it will encounter coupling loss and excess loss. The fiber attenuation loss is neglected here. Only a fraction of the transmitted optical intensity arrives at the receiver. Since losses would probably be different for different paths, pulses of different sessions arrive at the receiver of session i with different intensity levels. This constitutes another impediment to the implementation of an optical ADDER channel.

For this discussion we assume that a slot detector declares m pulses detected if the energy received, over the slot detection interval, is within the range $[(m-1/2)E,(m+1/2)E]$. E is the energy of one received pulse to which the detector is calibrated. The detector is assumed perfect in the sense that it exactly detects the optical energy received. We are interested in the conditions under which unequal pulse intensities affect operation of this ADDER channel detection rule.

We consider the ideal situation of square pulses, all received nominally with constant intensity I over a slot interval β. We assume all received pulses to be synchronous over the detection slot at the receiver of session i. With perfect detectors the energy detected is proportional to the received intensity so detection of m pulses occurs when the received optical intensity over the slot is within the range $[(m-1/2)I,(m+1/2)I]$. Let $K_i(j)\ I$ be the power received at the slot detector in session i due to a pulse from the transmitter of session j. Let $\mathcal{L}_i$ be a set of m paths connecting the transmitters of m different sessions to the slot detector in session i. When only pulses from these m sessions arrive at the slot detector, m pulses will be detected if and only if

$$(m-1/2)I < I\sum_{\mathcal{L}_i} K_i(j) < (m+1/2)I$$

which can be rewritten as

$$|\sum_{\mathcal{L}_i}(K_i(j)-1)| < 1/2 \tag{5.46}$$

The summation is over all paths corresponding to sessions j in the set $\mathcal{L}_i$.
For a network to properly implement ADDER channel detection in this idealized situation (5.46) would have to be satisfied for all possible sets $\mathcal{L}_i$, for all sessions i, and for all values of $m \in [1, M+1]$.

Even assuming a star topology with equal number of optical couplers for all transmitter-receiver paths for the optical medium, as the number of users becomes large the maximum size of the sets $\mathcal{L}_i$ to be considered in (5.46) is so large that just the manufacturing tolerance of the optical coupler losses becomes a problem. From (5.46) , the total sum of loss deviations from their nominal values must remain within 1/2 for all choices of $\mathcal{L}_i$. This results in extremely tight tolerances for coupling loss and excess loss of the devices.

From the previous discussion it is apparent that only the simplest and smallest of networks could properly implement ADDER channel detection even if the coherence of optical sources is not a problem as described in section 5.1. Networks for which the number of couplers vary from path to path are very unlikely to meet the inequality (5.46) .

CHAPTER 6
CONCLUSIONS

The main results in this thesis have been summarized in Chapter 1, section 1.2. This chapter assumes familiarity with that section. In this chapter we try to deal with the significance of these results as a whole, put them in perspective, and attempt to extrapolate some of the results to the situations where we have no analytical results. We will also include suggestions for further research on the problem.

Nothing in the discussions in this section regarding the performance of the different schemes and their comparison should be mistaken for proofs. They are rather attempts to predict the performance of these schemes in the absence of further analytical results covering situations of interest. In doing so we have tried, however, to use the knowledge obtained from previous analytical results.

6.1 MAXIMUM LIKELIHOOD DETECTOR STRUCTURE

Optimality of Template detector. Lemma 3.1 and theorems 4.1, 3.2, which treat asymmetric schemes in Discrete Time, together with the derivations of sections 3.2.2 and 3.3.2 treating symmetric schemes in Discrete Time, are results which show optimality properties of the Template detector. They state that under several circumstances (c.f. section 1.2) the Template detector is a Maximum Likelihood detector. Even though the results are derived based on probabilistic models using a Discrete Time analysis, they suggest that in real life systems a Template detector may be a good performance detector. A Template detector is also attractive from the point of view of implementation since its technology is probably little more sophisticated than for a Sum detector (Optical Delay Line Correlation detector). From numerical evaluations of the probability of error expressions derived for the Frame Synchronous model, c.f. section 3.4, we expect the Template detector to exhibit an appreciably lower probability of error than the Sum detector.

ADDER channel. Theorems 4.1 and 3.2 state conditions under which a Template detector implemented on an ADDER channel is a Maximum Likelihood detector for asymmetric Bernoulli and Roulette schemes respectively. The conditions essentially relate to the level of interference in the channel. Since a Template detector reduces an ADDER channel to an OR channel, this seems to imply that the structure of an ADDER channel is of little significance for these schemes at low levels of interference. It is likely that a similar comment can be applied to symmetric schemes since for low enough levels of interference the events resulting in more than one pulse per slot are rather the exception than the norm. Under such conditions the potential of an ADDER channel is not required.

Template detector in Continuous Time Asynchronism. It seems difficult to derive the equivalent of the results above, on the optimality of a Template detector, for the Continuous Time Asynchronous model. Nevertheless, the performance of the Template detector is largely a consequence of its structure, which detects the presence of pulses individually for selected slots. It seems reasonable to expect the Template detector to offer a good performance-simplicity trade off even in the Continuous Time Asynchronous model. If detectors which implement the equivalent to limiting the received optical intensity over a slot, as described in section 4.4.5, are used then the probability of error experienced is lower than for Discrete Time models. This is an important observation. It means that probability of error results applying for Discrete Time Asynchronism serve as upperbounds for Continuous Time situations. More importantly, it is a way to overcome the unfavorably high probability of error observed in Figure 4.2 for the Template detector in Continuous Time.

Other Maximum Likelihood detectors derived. In the Discrete Time models, when the Maximum Likelihood detector does not reduce to a Template detector, we are generally left with computation of the Likelihood Ratio for each received frame if we want to reach a

Maximum Likelihood detection decision. Computation of this Likelihood Ratio is often not a simple operation. An interesting exception is that of Lemma 3.3 for Frame Synchronous, symmetric, Roulette schemes. The Maximum Likelihood detection rule for this scheme is fairly straightforward, requiring the computation of products of integers representing the contents of selected slots. A similar type of detection rule for a Frame synchronous, symmetric, Bernoulli scheme is expressed in Lemma 3.2; however, this one requires knowledge of the number of active interfering sessions, which is expected not to be a known parameter to the detector. Heuristic applications of the rule in Lemma 3.2 should not be ruled out however.

6.2 TEMPLATE DETECTOR PERFORMANCE

The curves for probability of error of a Template detector presented in Chapter 3 correspond to the Frame Synchronous model. However, many traits in the behaviour of these curves are expected to carry over to the Discrete Time Asynchronous model and to the Continuous Time Asynchronous model, in which case the curves plotted are a helpful guide on the behaviour to be expected. In extrapolating to the Continuous Time model one must be careful to consider the received power of the signals, as explained in section 4.4. If, as was assumed in our analysis, the power levels are sufficiently high, the probability of error will tend to be higher, in Continuous Time Asynchronous models, than for Discrete Time models. If the power levels were low enough, the probability of error would tend to be smaller than computed for Discrete Time models.

The comments that follow assume that the optical signal power received, identical for all optical sources, is a fixed design parameter and that we want to compare the merits of different schemes. The discussion in section 3.4.3, however, also includes the case where the average optical energy received is fixed instead (Figure 3.10). In this case a fair comparison would need to assign more pulses to asymmetric schemes than to the corresponding symmetric schemes so that the average power for all schemes are the same.

Asymmetric vs. symmetric schemes. The reason for asymmetric schemes to exhibit lower values of probability of error than corresponding symmetric schemes seems to be the lower interference levels in the channel due to the use of a null signature. This is noticeable for 2-signature schemes and is expected to be less and less relevant as the size of the signature set increases and one null signature is used. As we consider Discrete and Continuous Time Asynchronous models, the use of a null signature is expected to have the same effect and hence asymmetric schemes should still perform better than symmetric schemes. The importance of symmetric schemes however is that they permit to increase the channel rate by using several signatures. In theory the use of several signatures does not change the channel interference level[1]. So we can increase the number of signatures used up to the point where signatures of the same session start having too many pulse positions in common.

Bernoulli vs. Roulette schemes. In the discussion under subsection 3.4.3 we argued that the possible reason for the Bernoulli schemes showing higher probability of error than corresponding Roulette schemes may be related to its higher variance for the number of pulses per interfering signature. If this is the case then, as we move from a Frame Synchronous to a Discrete Time Asynchronous model, we can expect Roulette schemes to exhibit higher values of probability of error than for the Frame Synchronous model, since now the number of pulses placed by any given interfering session over the frame being observed becomes a random variable. For the symmetric Roulette scheme this number can vary from 0 to $2a$ pulses, where a is the number of pulses per interfering signature; for the asymmetric Roulette scheme this is also the case but the distribution for this number is now affected by the possible ocurrence of null signatures.

[1] In practice, as the number of signatures α approaches n/h, where n is the number of slots/frame and h the number of pulses/signature, then to avoid common pulse position among signatures a more structured arrangement of pulses is needed and our models may not be realistic. The correlation among pulse positions of different signatures may become uncomfortably high.

Consider thus a symmetric Bernoulli scheme versus a symmetric Roulette scheme. As we move from a Frame Synchronous to a Discrete Time Asynchronous model, the performance of the Template detector for the Bernoulli scheme would remain the same while for the Roulette scheme, from the arguments above, the Template detector will exhibit higher probability of error. This may bring the probability of error curves closer together than they appear in the figures for Frame Synchronism (c.f. Figure 3.9).

A guide to the effect experienced by moving from Discrete to Continuous Time Asynchronism, assuming high enough received optical power levels, would be the analysis in subsection 4.4.3 (c.f. Figure 4.2) and the discussion in subsection 4.4.5. Under a High Power assumption as in subsection 4.4.3, the apparent interference level in the channel would increase and both schemes would experience even higher probabilities of error than predicted by a Discrete Time Asynchronous analysis. If instead we assume a postdetection current limiter as in subsection 4.4.5, the apparent interference level in the channel would decrease.

Consider now an asymmetric Bernoulli scheme versus an asymmetric Roulette scheme as we move from a Frame Synchronous to a Discrete Time Asynchronous model. The ocurrence of null signatures probably affects both schemes similarly, most likely increasing the probabilities of error for the Template detector. For the Roulette scheme however an added increase in the probability of error may be expected, as in the previous argument for symmetric schemes, which is not experienced by the Bernoulli scheme. This may bring the performance of the Roulette scheme closer to that of the Bernoulli scheme.

Moving now to a Continuous Time Asynchronous model, assuming high received optical power levels, we would expect both schemes to show even higher probabilities of error. Assuming a postdetection current limiter, we would expect lower probabilities of error.

Bandwidth expansion factor. In section 3.4 we observed how the probability of error expressions for the Template detector seem to behave as approximately exponential functions of the maximum number M of interfering sessions possible, the probability q of an interfering session being active, and the number n of slots per frame. In spread spectrum terminology n

is referred to as "bandwidth expansion factor." This dependence opens up the possibility of compensating for an increase in the possible number of interfering sessions in a network, or an increase in the probability of sessions being active, by a suitable increase in the bandwidth expansion factor. The fact that the probability of error dependence on all three parameters seems to be exponential makes for a favourable relationship. The exact form of the dependence, however, is not known and can only be speculated upon. Recall also that we are using observations based on specific values of the parameters mentioned above and one should be careful when extrapolating the behaviour to a grater range of parameter values.

Optimal number of pulses per signature. Both the Bernoulli scheme and the Roulette scheme show an optimal value of pulses per signature in the Frame Asynchronous model. As mentioned in section 3.4, this is probably due to a trade off between the increase in the number of pulses in the signature used to transmit the information and a corresponding increase in the interference level in the channel. If this is the case then the same effect is expected to hold true for Discrete and Continuous Time Asynchronous models. The fact that this effect is present in both types of schemes in a similar way also seems to imply that the trade off mentioned is somewhat insensitive to the detailed structure of the signatures being used as long as there is no strong correlation between signatures of different users and no essential change in their structure as the number of pulses per signature increases.

6.3 OTHER CONSIDERATIONS

Coherence among optical sources. Chapter 5 studied the effect of coherence of the optical sources on the implementation of an optical ADDER channel. It indicates that the coherence of the sources, measured in terms of their spectral bandwidth, limits the slot rate at which the schemes studied can operate. Broadly speaking, the minimum slot duration is limited by the product of the reciprocal of the spectral bandwidth times the number of optical pulses simultaneously present at the detector. For a detailed discussion refer to section 5.1. This analysis assumed all optical sources to have an equal optical center frequency which may not be realistic; if drift in center frequency were considered the minimum slot duration would be reduced. Light Emitting Diode sources, which have a broad spectral bandwidth, would present little problem in this respect. When much more coherent sources are used, like laser sources for single mode fiber, the possibility of implementing ADDER channel operations disappears since the assumption that optical intensities of different signals can be added together is no longer valid. OR channels only detect the presence or absence of light so they would be less sensitive to this problem of coherence, the only problem being the possibility that the intensity resulting from the superposition of optical signals is so weak that detection is missed.

Poisson process model for detection. The generation of electrical charges by photons incident on a photodetector is well described by a Poisson process model. Because of the variance in the total charge generated, as the number of pulses being detected becomes large a linear threshold scale to decide for this number becomes inappropriate since the spread of the total charge becomes larger and larger. Implementation of a non linear scale, where the spacing between thresholds corresponding to the number of pulses received becomes progressively larger as this number increases, may be more appropriate.

Path dependent attenuation. Another important problem for the implementation of an ADDER channel is the path dependent attenuation of optical signals in the network due to the different number of couplers and loss values which apply to each distinct path. This would present a serious problem for most networks attempting to use the optical medium as an ADDER channel.

6.4 DIRECTIONS FOR FURTHER RESEARCH

- A simple characterization of the optimal value for the number of pulses per signature for the different schemes when a Template detector is in use would be very useful, at least from a design point of view. Finding simple analytical approximations for the probability of error expressions derived would be of even more fundamental significance. A characterization for the optimal value of pulses per signature may have to wait for such an approximation to be available first. Such an approximation would also lead to a simpler relationship between the number of sessions M, their probability of being active q and the bandwidth expansion factor n, when the probability of error is held constant.

- It would be of theoretical significance to find results regarding the structure of a Maximum Likelihood detector for Discrete Time Asynchronous models in the Roulette schemes.

- An approximate expression for the probability of error of the M.L. detector stated in Lemma 3.2 for a symmetric Roulette scheme implemented on an ADDER channel would help clarify how effective the Template detector is in this situation.

- Analysis of the performance of many signature symmetric schemes is important since they can increase the number of bits transmitted per channel use. As the number of signature used increases, models like the ones in this thesis would probably have to be altered to account for the greater structure in pulse patterns required for signatures of a given session not to have too many common pulse positions.

- Many of the doubts concerning the behaviour of the schemes presented when a Discrete Time Asynchronous model is used could be resolved through Montecarlo simulation, even if no analytical results are available. For the behaviour in a Continuous Time Asynchronous model, experimental data may be the best characterization.

- Experimental data is also of great importance in deciding how critical some of the simplifying assumptions, like neglecting receiver thermal and shot noise.

- Research on better schemes for choosing signatures is relevant since it could such schemes could serve as guidelines for design of real systems. Schemes with a fixed number of pulses per signature and with pulses positions assigned to be approximately uniformly distributed look promising. Care would be needed to avoid pulse position overlap but still retain uniformity in the positions assigned.

APPENDIX AP-1

PROOF OF LEMMA 3.1

<u>Lemma 3.1</u> Consider a Discrete Time, 2-signature, asymmetric scheme implemented over an OR channel. Let session j's signature, s^j, be independent of session i's signature, s^i, $\forall j \neq i$. Then the Maximum Likelihood Detector for session i is the Template detector.

<u>Proof:</u> We show the equivalent statements:

a- $\forall\, y$ s.t. $\overline{A}(\psi^i_+)$ then $\Pr[Y = y|X^i = s^i] \leq \Pr[Y = y|X^i = \phi]$

b- $\forall\, y$ s.t. $A(\psi^i_+)$ then $\Pr[Y = y|X^i = s^i] \geq \Pr[Y = y|X^i = \phi]$

Session i is assumed active.

<u>Proof of a:</u>

Consider any y such that $\overline{A}(\psi^i_+)$ holds. Since the channel is physically noiseless:

$$\Pr[Y = y|X^i = \phi] > \Pr[Y = y|X^i = s^i] = 0.$$

◇

<u>Proof of b:</u>

Consider any y such that $A(\psi^i_+)$ holds. Define the following two complementary events:

E_1 Event: Interference places at least one <u>interfering</u> pulse in every slot of ψ^i_+.

E_0 Event: Interference such that at least one slot of ψ^i_+ has no <u>interfering</u> pulse.

Then, given E_1 in an <u>OR channel</u>:

$$\Pr[Y = y|X^i = s^i, E_1] = \Pr[Y = y|X^i = \phi, E_1]$$

while given E_0 in an OR channel:

$$\Pr[Y = y|X^i = s^i, E_0] \geq 0; \qquad \Pr[Y = y|X^i = \phi, E_0] = 0$$

so taking expectations over the two events E_1, E_0 we arrive at

$$\Pr[Y = y|X^i = s^i] \geq \Pr[Y = y|X^i = \phi]$$

◇

In words: Assume the event $A(\psi_+^i)$ holds. If there are interference pulses in every slot of ψ_+^i then whether s^i or ϕ were sent is irrelevant to the probability of getting an output y as far as the OR channel is concerned. But if not all slots of ψ_+^i have interference pulses then ϕ could not have been sent or $A(\psi_+^i)$ would not hold. s^i would have had to be sent. Hence $\Pr[Y = y|X^i = s^i]$ is at least as large as $\Pr[Y = y|X^i = \phi]$, if not more.

Lemma 3.1 is proven.

∎

APPENDIX AP-2

ADDER CHANNEL M.L. DETECTOR

ASYMMETRIC ROULETTE SCHEME

DERIVATION OF EQUATIONS (3.29), (3.30)

For all y such that $\overline{A}(\psi_+^i)$ holds

$$\Pr[Y = y | X^i = \phi] > \Pr[Y = y | X^i = s^i] = 0$$

since no erasures possible. While for all y such that $A(\psi_+^i)$ holds

$$\Pr[Y_k = y_k, \forall k | X^i = s^i] = \Pr[Z_k = y_k - 1 \;\forall k \in \psi_+^i, Z_l = y_l \;\forall l \notin \psi_+^i]$$

$$\Pr[Y_k = y_k, \forall k | X^i = \phi] = \Pr[Z_k = y_k \;\forall k]$$

The joint values of the components of the vector Z obey a Multinomial probability distribution. The probability of a pulse occupying any given slot is $1/n$ and this becomes the probability parameter in the distribution. Define ς_i to be the number of active interfering sessions sending non-null signatures under the hypothesis $X^i = s^i$.

$$\varsigma_i + 1 = (1/a) \sum_{k \in \Psi} y_k \tag{3.28}$$

ς_i is a function of y. We do not include y explicitly in the notation ς_i for clarity in later expressions. We have

$$
\begin{aligned}
\Pr[Y = y|X^i = s^i] &= \Pr[Z_k = y_k - 1 \ \forall k \in \psi^i_+ \ , \ Z_l = y_l \ \forall l \notin \psi^i_+] \\
&= \mathcal{F}(\varsigma_i) \left\{ \frac{\varsigma_i a!}{\prod\limits_{k \in \psi^i_+} (y_k - 1)! \prod\limits_{l \in \psi^i_-} y_l!} \right\} \left(\frac{1}{n}\right)^{\varsigma_i a} \\
\Pr[Y = y|X^i = \phi] &= \Pr[Z_k = y_k, \forall k] \\
&= \mathcal{F}(\varsigma_i + 1) \left\{ \frac{(\varsigma_i + 1)a!}{\prod\limits_{k \in \Psi} (y_k)!} \right\} \left(\frac{1}{n}\right)^{(\varsigma_i+1)a} \\
&= \Pr[Y = y|X^i = s^i] \frac{\mathcal{F}(\varsigma_i + 1)}{\mathcal{F}(\varsigma_i)} \left\{ \frac{\prod\limits_{j=1}^{a} (\varsigma_i a + j)}{\prod\limits_{k \in \psi^i_+} y_k} \right\} \left(\frac{1}{n}\right)^{a} \\
&= \Pr[Y = y|X^i = s^i] \ \Theta(y)
\end{aligned}
$$

where the last step is algebraic manipulation and $\Theta(y)$ is given by

$$
\Theta(y) = \frac{\mathcal{F}(\varsigma_i + 1)}{\mathcal{F}(\varsigma_i)} \left\{ \frac{\prod\limits_{j=1}^{a} (\varsigma_i a + j)}{\prod\limits_{k \in \psi^i_+} y_k} \right\} \left(\frac{1}{n}\right)^{a} \tag{3.30}
$$

From these results we obtain the M.L. rule:

$$
\widehat{X^i} = \begin{cases} \phi, & \text{if } \overline{A}(\psi^i_+); \\ \phi, & \text{if } A(\psi^i_+), \Theta(y) > 1; \\ s^i, & \text{if } A(\psi^i_+), \Theta(y) \le 1. \end{cases}
$$

Notice now that the event $\overline{A}(\psi^i_+)$ implies $y_k = 0$ for some $k \in \psi^i_+$. So if we allow division by zero in (3.30), which would occur iff $\overline{A}(\psi^i_+)$ holds, this rule can be written in the form of equation (3.29)

$$
\widehat{X^i} = \begin{cases} \phi, & \text{if } \Theta(y) > 1; \\ s^i, & \text{if } \Theta(y) \le 1. \end{cases} \tag{3.29}
$$

APPENDIX AP-3

TEMPLATE DETECTOR PROBABILITY OF ERROR ASYMMETRIC ROULETTE SCHEME, OR & ADDER CHANNEL

DERIVATION OF EQUATION (3.36)

Let h be the number of slots with pulses for the non null signature. $h = |\psi_+^i|$. We define and evaluate for convenience the following probabilities:

$$A(r,h) \stackrel{\text{def}}{=} \Pr[A(\psi_+^i) | r \text{ pulses in } \psi_+^i]$$

$$= \sum_{\nu=0}^{h} (-1)^\nu \binom{h}{\nu} \left(1 - \frac{\nu}{h}\right)^r \qquad ; \; r \geq h. \qquad (3.37)$$

$$F(r|\gamma) \stackrel{\text{def}}{=} \Pr[r \text{ pulses in } \psi_+^i | \gamma \text{ pulses in frame}]$$

$$= \binom{\gamma}{r} \left(\frac{n-h}{n}\right)^{\gamma - r} \left(\frac{h}{n}\right)^r$$

$$\mathcal{F}(\varsigma) \stackrel{\text{def}}{=} \Pr[\varsigma \text{ non null interfering signatures out of } M \text{ possible sessions}]$$

The expression for $A(r,h)$ is borrowed from combinatorial "occupancy" problems (c.f. [Feller, chapter IV, equation (2.3)]). Since each pulse has a position uniformely distributed over the n slots of the frame $F(r|\gamma)$ has a Binomial distribution.

For the probability of error P_e we note that each interfering session contributes a pulses to the frame, that at least h pulses are required for the event $A(\psi_+^i)$ and that, given $X^i = \phi$ (the only situation which could result in error), $\varsigma = 1$ is the minimum number of non null interfering signatures which could lead to a detection error in session i. The expression is[1]

[1] In the expression, if $h > \varsigma a$ the summand is considered zero. This accounts for cases where $a < h$.

then

$$
\begin{aligned}
P_e &= \Pr[X^i = \phi] \sum_{\varsigma=1}^{M} \mathcal{F}(\varsigma) \sum_{r=h}^{\varsigma a} A(r,h) F(r|\varsigma a) \\
&= \Pr[X^i = \phi] \sum_{\varsigma=1}^{M} \mathcal{F}(\varsigma) \left\{ \sum_{r=h}^{\varsigma a} A(r,h) \binom{\varsigma a}{r} \left(\frac{n-h}{n}\right)^{\varsigma a - r} \left(\frac{h}{n}\right)^{r} \right\} \\
&\qquad ; \; 0 < Q < 1, \; 0 < h < n. \qquad (3.36)
\end{aligned}
$$

where $A(r,h)$ is given by (3.37) shown at the begining of this appendix.

APPENDIX AP-4

UPPERBOUND, M.L. DETECTOR PROBABILITY OF ERROR ASYMMETRIC ROULETTE SCHEME, ADDER CHANNEL

DERIVATION OF EQUATION (3.41)

From [Gallager], equation (5.3.4), the bound for the probability of error $P_{e,\phi}$ for a Maximum Likelihood detector for this scheme, given that the null (or non null) signature is sent, is

$$P_{e,\phi} \leq \sum_{y \in Y^*} \Pr[Y = y | X^i = \phi]^{1-s} \Pr[Y = y | X^i = s^i]^s \qquad ; 0 < s < 1. \tag{3.40}$$

The summation is over the set Y^* of all y such that $\Pr[Y = y|X^i = \phi] \neq 0$ and $\Pr[Y = y|X^i = s^i] \neq 0$. The bound can be minimized with respect to the exponent s. The transition probabilities are given in appendix AP-2 and are repeated below. $\mathcal{F}(\varsigma_i)$ is binomial with parameter $Q = \Pr[X^i = S^i]q$. Notice that for all $y \in Y^*$ $A(\psi^i_+)$ holds. Thus the transition probabilities required are:

$$\Pr[Y = y|X^i = s^i] = \mathcal{F}(\varsigma_i) \left\{ \frac{\varsigma_i a!}{\prod\limits_{k \in \psi^i_+} (y_k - 1)! \prod\limits_{l \in \psi^i_-} y_l!} \right\} \left(\frac{1}{n}\right)^{\varsigma_i a}$$

$$\Pr[Y = y|X^i = \phi] = \Pr[Y = y|X^i = s^i] \frac{\mathcal{F}(\varsigma_i + 1)}{\mathcal{F}(\varsigma_i)} \left\{ \frac{\prod\limits_{j=1}^{a} (\varsigma_i a + j)}{\prod\limits_{k \in \psi^i_+} y_k} \right\} \left(\frac{1}{n}\right)^a$$

Replacing into (3.40) we obtain:

$$P_{e,\phi} \leq \sum_{y \in Y^*} \left[\binom{M}{\varsigma_i} Q^{\varsigma_i} (1-Q)^{M-\varsigma_i} \frac{\varsigma_i a!}{\prod\limits_{k \in \psi^i_+} (y_k - 1)! \prod\limits_{l \in \psi^i_-} y_l!} \left(\frac{1}{n}\right)^{\varsigma_i a} \right]$$

$$\times \left[\frac{M - \varsigma_i}{\varsigma_i + 1} \frac{Q}{1-Q} \quad \frac{\prod\limits_{j=1}^{a} (\varsigma_i a + j)}{\prod\limits_{k \in \psi^i_+} y_k} \left(\frac{1}{n}\right)^a \right]^{1-s} \tag{AP-4.1}$$

This summation can be rearranged as three nested summations as shown below. The outer summation is over all values of ς_i, the number of interfering sessions under hypothesis $X^i = s^i$, such that neither transition probability equals zero. Given this value of ς_i, the middle summation is over all values of the number k of "extra" pulses placed in the set of slots ψ_+^i. By "extra" pulses we mean those pulses beyond the h pulses needed for $A(\psi_+^i)$ to hold, where h is the number of slots in ψ_+^i. Given these values of ς_i, k, the inner summation is over the set Y' of all output vectors y such that: 1- neither transition probability equals zero for this y, 2- the number of pulses in y is $(\varsigma_i + 1)a$, 3- y has exactly k "extra" pulses present in the set of slots ψ_+^i.

$$P_{e,\phi} \leq \sum_{\varsigma_i=0}^{M-1} \binom{M}{\varsigma_i} Q^{\varsigma_i}(1-Q)^{M-\varsigma_i} \left[\frac{M-\varsigma_i}{\varsigma_i+1} \frac{Q}{1-Q} \quad \frac{\prod_{j=1}^{a}(\varsigma_i a + j)}{n^a} \right]^{1-s}$$
$$\times \sum_{k=0}^{\varsigma_i a} \binom{\varsigma_i a}{k} \left(\frac{h}{n}\right)^k \left(1-\frac{h}{n}\right)^{\varsigma_i a - k}$$
$$\times \sum_{Y'} \frac{k!}{\prod_{k\in\psi_+^i}(y_k-1)!} \left(\frac{1}{h}\right)^k \left(\frac{1}{\prod_{k\in\psi_+^i} y_k}\right)^{1-s} \frac{(\varsigma_i a - k)!}{\prod_{l\in\psi_-^i} y_l!} \left(\frac{1}{n-h}\right)^{\varsigma_i a - k} \qquad \text{(AP-4.2)}$$

ς_i runs from 0 to $M-1$. If $\varsigma_i = M$ then $\Pr[Y = y | X^i = \phi, \varsigma_i = M] = 0$ since the only way that an output vector (Frame Synchronous model) can have $(M+1)a$ pulses is if $X^i = s^i$. The middle and inner summations are obtained by re-expressing the multinomial probability, for each value of ς_i, as a summation over the events corresponding to all partitions of the $\varsigma_i a$ pulses into two populations: k pulses in ψ_+^i and $\varsigma_i a - k$ pulses in ψ_-^i.

The term inside the inner summation can be decomposed into a set of factors which depends only on the contents of ψ_+^i and another which depends only on the contents of ψ_-^i. As shown below, the inner summation can be expressed as two nested summations: one summing over the set Y'_+ of all possible h-vector configurations consisting of $h+k$ pulses in ψ_+^i and the second one summing over the set $\overline{Y}'_+$ of all possible configurations consisting of $\varsigma_i a - k$ pulses in ψ^i_-.

$$P_{e,\phi} \leq \sum_{\varsigma_i=0}^{M-1} \binom{M}{\varsigma_i} Q^{\varsigma_i}(1-Q)^{M-\varsigma_i} \left[\frac{M-\varsigma_i}{\varsigma_i+1} \frac{Q}{1-Q} \frac{\prod_{j=1}^{a}(\varsigma_i a + j)}{n^a} \right]^{1-s}$$
$$\times \sum_{k=0}^{\varsigma_i a} \binom{\varsigma_i a}{k} \left(\frac{h}{n}\right)^k \left(1-\frac{h}{n}\right)^{\varsigma_i a - k}$$
$$\times \sum_{Y'_+} \frac{k!}{\prod_{k\in\psi^i_+}(y_k-1)!} \left(\frac{1}{h}\right)^k \left(\frac{1}{\prod_{k\in\psi^i_+} y_k}\right)^{1-s} \sum_{\overline{Y}'_+} \frac{(\varsigma_i a - k)!}{\prod_{l\in\psi^i_-} y_l!} \left(\frac{1}{n-h}\right)^{\varsigma_i a - k} \tag{AP−4.3}$$

The summation over the set $\overline{Y}'_+$ adds up to 1 since it is the summation of a multinomial probability over all possible events. We arrive at a simplified expression for $P_{e,\phi}$:

$$P_{e,\phi} \leq \sum_{\varsigma_i=0}^{M-1} \binom{M}{\varsigma_i} Q^{\varsigma_i}(1-Q)^{M-\varsigma_i} \left[\frac{M-\varsigma_i}{\varsigma_i+1} \frac{Q}{1-Q} \frac{\prod_{j=1}^{a}(\varsigma_i a + j)}{n^a} \right]^{1-s}$$
$$\times \sum_{k=0}^{\varsigma_i a} \binom{\varsigma_i a}{k} \left(\frac{h}{n}\right)^k \left(1-\frac{h}{n}\right)^{\varsigma_i a - k}$$
$$\times \sum_{Y'_+} \frac{k!}{\prod_{k\in\psi^i_+}(y_k-1)!} \left(\frac{1}{h}\right)^k \left(\frac{1}{\prod_{k\in\psi^i_+} y_k}\right)^{1-s} \tag{3.41}$$

As explained at the end of subsection 3.3.1, this is also a bound for P_e, the probability of error for the Maximum Likelihood detector of the situation being analysed.

APPENDIX AP-5

ADDER CHANNEL M.L. DETECTOR

SYMMETRIC ROULETTE SCHEME

PROOF OF LEMMA 3.3

Lemma 3.3 Consider a Frame Synchronous, symmetric, Roulette scheme operating on an ADDER channel. Let the signature set S^i of session i be such that signatures have at most one pulse per slot. Then the M.L. detector for session i takes the form

$$\left.\begin{array}{l} \widehat{X^i} = s^{i\sigma} \textit{ such that } \sigma \textit{ minimizes, over } \gamma \in B(i,y), \textit{ the metric} \\ \qquad m(i,\gamma) = \prod_{k \in \psi_o^{i\gamma}} y_k \\ \textit{If } \sigma \textit{ is not unique choose arbitrarily among all such } \sigma. \end{array}\right\} \tag{3.43}$$

Definitions. Let Ψ be the set of non zero slots of the output vector y. Let $\mathcal{F}(\varsigma)$ be the Probability Mass Function for the number of active interfering sessions present, ς. Define the set of slots ψ_c^{ix} as follows:

Let $B(i,y)$ be the set of feasible letters of i given output y, i.e. $B(i,y)$ is the set of letters γ for which all its pulses are present in y

$$B(i,y) = \{\gamma : \text{ s.t. } A(\psi_+^{i\gamma}), \gamma \in \mathcal{A}\} \tag{3.13}$$

Let $C(i,y)$ be the set of all slots for which at least one feasible letter has a pulse present.

$$C(i,y) = \bigcup_{\gamma \in B(i,y)} \psi_+^{j\gamma} \tag{3.14}$$

Let σ be a feasible letter given y, i.e. $\sigma \in B(i,y)$. We then define $\psi_c^{i\sigma}$ to be those slots for which $s^{i\sigma}$ has no pulses but at least some other feasible letter has a pulse. In set notation:

$$\psi_c^{i\sigma} = C(i,y) \setminus \psi_+^{j\sigma} \tag{3.15}$$

Proof of Lemma 3.3. For all $\gamma \notin B(i,y)$ we have $\Pr[Y = y|X^i = s^{iy}] = 0$.
For all $\gamma \in B(i,y)$ we have

$$\Pr[Y = y|X^i = s^{i\gamma}] = \mathcal{T}(\varsigma) \left\{ \frac{\varsigma a!}{\prod\limits_{k \in \psi_+^{i\gamma}} (y_k - 1)! \prod\limits_{l \in \psi_-^{i\gamma}} y_l!} \right\} \left(\frac{1}{n}\right)^{\varsigma a} \tag{AP–5.1}$$

Recall that Z denotes the interference vector. The term within braces is the coefficient of a multinomial distribution term corresponding to the event $Z = y - s^{i\gamma}$. $\psi_+^{i\gamma}, \psi_+^{i\gamma}$ were defined in Section 2.2. They correspond respectively to the non-zero slots of the signature and to those non-zero slots of the output vector that do not correspond to the signature. Denominator factors corresponding to the set of empty output slots $\psi_0^{i\gamma}$ yield a product of 1 and have been omitted. $(1/n)$ is the probability that any pulse occupies a given slot. The set $B(i,y)$ cannot be empty.

The M.L. receiver chooses the signature corresponding to the letter for which this conditional probability is maximum. The only dissimilar factors among the expressions for the feasible letters are in the denominator of the multinomial coefficient (within braces). Finding the maximum among the probabilities is equivalent to finding the minimum among these denominators.

Notice that the factorial terms

$$y_k! \qquad ; \; k \notin C(i,y)$$

$$(y_k - 1)! \qquad ; \; k \in C(i,y)$$

are common to all feasible letters. After these factors are cancelled the denominator expression reduces, for a given feasible letter γ to

$$\prod_{k\in\psi_{+}^{i\gamma}} 1 \prod_{k\in\psi_{c}^{i\gamma}} y_k = \prod_{k\in\psi_{c}^{i\gamma}} y_k$$

Thus the metric to be minimized is given by (3.43)

$$m(i,\gamma) = \prod_{k\in\psi_{c}^{i\gamma}} y_k \tag{3.43}$$

among $\gamma \in B(i,y)$. If more than one letter achieves the minimum the likelihood probabilities for all such letters are equal and the decision can be arbitrary.

∎

APPENDIX AP-6

UPPERBOUND, M.L. DETECTOR PROBABILITY OF ERROR SYMMETRIC ROULETTE SCHEME, ADDER CHANNEL

DERIVATION OF EQUATION (3.49)

As for appendix AP-4 the bound is based on [Gallager], equation (5.3.4). Let $P_{e,\sigma}$ denote the probability of error of a Maximum Likelihood detector for the present scheme, given that signature $s^{i\sigma}$ (or $s^{i\gamma}$) is sent, the bound is

$$P_{e,\sigma} \leq \sum_{y \in Y^*} \Pr[Y = y | X^i = s^{i\sigma}]^{1-s} \Pr[Y = y | X^i = s^{i\gamma}]^s \qquad ;\ 0 < s < 1. \qquad (3.48)$$

The summation is over the set Y^* of all y such that $\Pr[Y = y|X^i = s^{i\sigma}] \neq 0$ and $\Pr[Y = y|X^i = s^{i\gamma}] \neq 0$. The transition probabilities for all feasible letters are given in appendix AP-5. $\mathcal{F}(\varsigma)$ is binomial with parameter q. From equation AP-5.1, for all $y, s^{i\gamma}$ such that $A(\psi_+^{i\gamma})$ holds:

$$\Pr[Y = y|X^i = s^{i\gamma}] = \mathcal{F}(\varsigma) \left\{ \frac{\varsigma a!}{\prod_{k \in \psi_+^{i\gamma}} (y_k - 1)! \prod_{l \in \psi_-^{i\gamma}} y_l!} \right\} \left(\frac{1}{n}\right)^{\varsigma a} \qquad (AP-5.1)$$

We assume that signatures σ, γ are such that they have no common pulse positions. i.e. $\psi_+^{i\sigma} \cap \psi_+^{i\gamma} = \phi$ where ϕ denotes the empty set. Let $|\psi_+^{i\sigma}| = |\psi_+^{i\gamma}| = h$. Let $C(i, y)$ be the set of slots for which either signature has pulses. Let $\overline{C}(i, y)$ be the complement of $C(i, y)$ in the set of slots of a frame. Thus

$$C(i, y) \equiv \psi_+^{i\sigma} \cup \psi_+^{i\gamma}$$

$$|C(i, y)| = 2h$$

$$|\overline{C}(i, y)| = n - 2h$$

The bound can be minimized with respect to the exponent s. We can show by symmetry arguments that the optimal value of s is $s = 1/2$.

To every output vector y there corresponds another vector y' resulting from exchanging the components in $\psi_+^{i\sigma}$ and $\psi_+^{i\gamma}$. We assume that the slots in both sets are ordered and that the exchange respects this ordering. We divide all vectors y of (3.48) above into three distinct sets A, B_1, B_2, as follows:

A: Place all vectors such that $y = y'$.

B_1, B_2: Divide the set of vectors y such that $y \neq y'$ into sets B_1, B_2 such that each vector in B_2 is the corresponding y' vector of one and only one vector y in B_1.

The bound in (3.48) can then be expressed as

$$\begin{aligned} P_{e,\sigma} \leq \sum_A \Pr[Y = y|X^i = s^{i\sigma}] \\ + \sum_{B_1} \Pr[Y = y|X^i = s^{i\sigma}]^{1-s} \Pr[Y = y|X^i = s^{i\gamma}]^s \\ + \sum_{B_2} \Pr[Y = y'|X^i = s^{i\sigma}]^{1-s} \Pr[Y = y'|X^i = s^{i\gamma}]^s \end{aligned}$$

But for all $y \in B_1$ and corresponding $y' \in B_2$

$$\Pr[Y = y|X^i = s^{i\sigma}] = \Pr[Y = y'|X^i = s^{i\gamma}]$$

$$\Pr[Y = y|X^i = s^{i\gamma}] = \Pr[Y = y'|X^i = s^{i\sigma}]$$

This follows from the expression for the likelihood probabilities —see (AP-5.1) above— and the correspondence between y and y'. Hence we can write, in terms of the vectors in B_1,

$$\begin{aligned} P_{e,\sigma} \leq \sum_A \Pr[Y = y|X^i = s^{i\sigma}] \\ + \sum_{B_1} \Pr[Y = y|X^i = s^{i\sigma}]^{1-s} \Pr[Y = y|X^i = s^{i\gamma}]^s \\ \times \Pr[Y = y|X^i = s^{i\gamma}]^{1-s} \Pr[Y = y|X^i = s^{i\sigma}]^s \end{aligned} \quad \text{(AP-6.1)}$$

The expression $a^{1-s}b^s + b^{1-s}a^s$ has a first derivative w.r.t. s equal to zero at $s = 1/2$. Hence the bound in (AP-6.1) also has a first derivative w.r.t. s equal to zero at $s = 1/2$. Since we know that this bound is convex $\cup$ over $s \in [0, 1]$, [Gallager], then the bound must be a minimum at $s = 1/2$.

By substituting $s = 1/2$ and the appropriate probabilities into the bound expression in (3.48) , at the begining of this appendix, we obtain:

$$P_{e,\sigma} \le \sum_{y \in Y^*} \binom{M}{\varsigma} q^{\varsigma}(1-q)^{M-\varsigma} \left(\frac{\varsigma_i a!}{n^{\varsigma_i a}}\right) \left[\frac{1}{\prod_{k\in\psi_+^{i\sigma}} (y_k - 1)! \prod_{l\in\psi_-^{i\sigma}} y_l! \prod_{k\in\psi_+^{i\gamma}} (y_k - 1)! \prod_{l\in\psi_-^{i\gamma}} y_l!}\right]^{1/2} \tag{AP-6.2}$$

The summation in the bound is over all possible vectors such that all slots in $C(i, y)$ have at least one pulse present. We can re-express the product in the denominator of (AP-6.2) in terms of the set of slots $C(i, y)$ as

$$P_{e,\sigma} \le \sum_{y \in Y^*} \binom{M}{\varsigma} q^{\varsigma}(1-q)^{M-\varsigma} \left(\frac{\varsigma_i a!}{n^{\varsigma_i a}}\right) \left[\frac{1}{\prod_{k\in C(i,y)} (y_k - 1)! \prod_{l\in \overline{C}(i,y)} y_l!}\right] \left[\frac{1}{\prod_{k\in C(i,y)} (y_k - 1)}\right]^{1/2} \tag{AP-6.3}$$

As for appendix AP-4, we can re-express the summation as four nested summations: The outer summation is over all values of ς, the number of active interfering sessions. Given this value of ς, the second summation is over all values of the number k of interfering pulses placed in the set of slots $C(i, y)$. This value must be at least $a = h$ so that $C(i, y)$ can have at least one pulse for every slot. Given these values of ς_i, k, the third and fourth summations –the fourth being the innermost summation– are over all possible configurations consisting of $h + k$ pulses in $C(i, y)$ –the third– and over all possible configurations consisting of $\varsigma a - k$ pulses in $\overline{C}(i, y)$ –the fourth–.

$$P_{e,\sigma} \le \sum_{\varsigma=1}^{M} \binom{M}{\varsigma} q^{\varsigma}(1-q)^{M-\varsigma}$$
$$\times \sum_{k=a}^{\varsigma a} \binom{\varsigma a}{k} \left(\frac{2h}{n}\right)^k \left(1 - \frac{2h}{n}\right)^{\varsigma a - k}$$
$$\times \sum_{Y_\sigma} \frac{k!}{\prod_{k\in C(i,y)} (y_k - 1)!} \left(\frac{1}{2h}\right)^k \left(\frac{1}{\prod_{k\in C(i,y)} y_k}\right)^{1/2} \sum_{Y_{\overline{\sigma}}} \frac{(\varsigma a - k)!}{\prod_{k\in \overline{C}(i,y)} y_k!} \left(\frac{1}{n-2h}\right)^{\varsigma a - k} \tag{AP-6.4}$$

ς runs from 1 to M. If $\varsigma = 0$ then either $\Pr[Y = y | X^i = s^{i\sigma}, \varsigma = 0] = 0$ or $\Pr[Y = y | X^i = s^{i\gamma}, \varsigma = 0] = 0$, whichever is not compatible with y. The rest of the summations are obtained by re-expressing the multinomial probability, for each value of ς, as a summation over the events corresponding to all partitions of the ςa pulses into two populations: k pulses in $C(i,y)$, $a \leq k \leq \varsigma a$, and $\varsigma a - k$ pulses in $\overline{C}(i,y)$. As for Appendix AP-4, we sum separetely over $C(i,y)$ and its complement. The innermost summation, over $\overline{C}(i,y)$, sums to 1 since it is the summation of a multinomial probability over all possible events. We arrive at a simplified expression for $P_{e,\sigma}$:

$$P_{e,\sigma} \leq \sum_{\varsigma=1}^{M} \binom{M}{\varsigma} q^{\varsigma}(1-q)^{M-\varsigma} \times \sum_{k=a}^{\varsigma a} \binom{\varsigma a}{k} \left(\frac{2h}{n}\right)^{k} \left(1 - \frac{2h}{n}\right)^{\varsigma a - k} \times \sum_{Y_\sigma} \frac{k!}{\prod_{k \in C(i,y)} (y_k - 1)!} \left(\frac{1}{2h}\right)^{k} \left(\frac{1}{\prod_{k \in C(i,y)} y_k}\right)^{1/2} \tag{3.49}$$

As explained at the end of subsection 3.3.2, this is also a bound for P_e, the probability of error for the Maximum Likelihood detector of the situation being analysed.

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