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The Transfer of Spectral Energy in Non-Linear Dispersive Systems

by

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Abstract

In the first chapter, an analysis is made of the long time behavior of the spectral cumulants in a conservative system of random, weakly non-linear, gravity waves. The system of equations describing this behavior is found to be closed. In particular it is found that, to the first closure, the spectral energy is transferred by means of a resonance mechanism. The second chapter deals with a conservative system of random, weakly non-linear, surface tension waves on which a forcing mechanism is applied. Finally in the third chapter, the energy transfer in the local spectral neighborhood of a travelling wave is discussed.

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Biographical Note

Alan C. Newell was born in Dublin, Ireland on November 5, 1941. He attended Trinity College, Dublin, for undergraduate studies and received two bachelor degrees, one in Mathematics and the other in Physics, in October, 1962. Since then, he has been at the Massachusetts Institute of Technology where he received his S.M. degree in June, 1965. He is married and has two children.

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Note:

On account of not being able to obtain a binder clip with a sufficiently large bite, this thesis has been divided into two parts, Volume I pgs. 1 - 158 , Volume II pgs. 159 - 318. As the total thesis will be bound together by the library at some later date, this page will serve as an explanation for the temporary division and will not be numbered with the rest of the thesis.

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CHAPTER I

Introduction.

It is well known that a transfer of energy occurs between different wave components in a nonlinear system. In the case of fully developed turbulence, the expression for the energy transfer contains third order cumulants. This leads to a closure difficulty, for the rate of change of the third order spectral cumulant is given in terms of fourth order spectral cumulants, and so on. In order to obtain a finite closure for the system of equations describing the rates of change of the spectral cumulants, it is necessary to adopt some assumption. One such assumption is that of Heisenberg for the case of isotropic turbulence in which the energy transfer terms are replaced by an expression containing only energy densities. Another such assumption is the Quasi Gaussian hypothesis, which assumes that the fourth order spectral cumulant is zero.

In the following work a weakly nonlinear conservative system of random waves is considered. The model chosen is that of ocean waves including both surface tension and gravity forces. Unlike the case of fully developed turbulence, it is found that, without any assumptions as to the nature of the statistical distributions, the system of equations describing the long time behavior of the spectral cumulants is closed. A physical interpretation for the mechanism for energy transfer can also be given.

In a paper in the Journal of Fluid Mechanics [8], O.M. Phillips suggested a mechanism by which weakly interacting gravity waves could exchange energy. He showed that this was a resonance mechanism which can be represented as follows. If $(\tilde{k}_i, \omega(\tilde{k}_i))$, i = 1,2,3,4; are the wave number vectors and corresponding frequencies of four discrete gravity waves, and if when $\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 = \tilde{k}$, for some choice of the sign parameters, $\pm\omega(k_1) \pm \omega(k_2) \pm \omega(k_3) = \pm\omega(k)$, then it is possible for energy to be transferred between these four wave components. This is due to the nonlinear terms of the equation describing the system, as a linear system would allow the four waves to travel independently. The time scale associated with this energy exchange is of the order of ε^{-2}, where ε is the small parameter describing the relative magnitude of the nonlinear terms. Phillips examined the initial growth of a wave produced by this mechanism. Benney [1] developed equations describing the long time sharing of energy process between four such waves.

In [4], Hasselmann examined the spectral energy transfer between random gravity waves, in which the velocity potential and the function describing the surface elevation were homogeneous random quantities over the ocean surface. Using a perturbation technique and assuming that the statistical distribution of the random quantities was Gaussian, he obtained an expression for the spectral energy transfer.

The expression clearly demonstrated that the mechanism for energy transfer was identical to that suggested by Phillips. However the time scale on which the spectral energy was exchanged for the random problem was found to be of the order ε^{-4}.

Using a model equation, Benney and Saffman [2] showed that in the case when triad resonances are possible, the Gaussian assumption was not necessary as long as the zeroth order term in the asymptotic expansion of the higher cumulants remained continuous, and that a closure, at the ε^{-2} time scale, for the spectral energy was indeed possible. Triad resonances occur when the corresponding frequencies of three wave numbers $\underset{\sim}{k}_1$, $\underset{\sim}{k}_2$, and $\underset{\sim}{k}_3$ which are related by $\underset{\sim}{k}_1 + \underset{\sim}{k}_2 = \underset{\sim}{k}_3$, obey the relation $\pm\omega(k_1) \pm \omega(k_2) = \pm\omega(k_3)$, for some choice of the sign parameters. Benney and Saffman then conjectured that in the case of gravity waves, their analysis, if continued, would lead to the same result Hasselmann obtained; namely, that the equation describing the transfer of spectral energy would not contain any spectral cumulants other than energy densities and would therefore be closed.

It was in order to examine such a conjecture that the following analysis was undertaken. At first, the analysis included the effects of the surface tension and so, as shown by Phillips [8], it is possible that certain frequency triads (for example, $\pm\omega(k_1) \pm \omega(k_2) \pm \omega(k_3)$)

could vanish somewhere in the spectrum. A perturbation scheme is adopted and a multiple time scale device inserted in order to keep the asymptotic expansions for each of the spectral cumulants well ordered in time. It is to be emphasized that the ordering procedure must take place in physical space. It is found that there are two distinct types of terms in the asymptotic expansions. The first type begins with the order ε^0 terms, and each term in this series carries as a factor an exponential with an imaginary exponent (the sum of frequencies). These terms describe the first oscillation of the spectral cumulants arising from the linear balance in the governing equations for the system.

However, there are also terms in the spectral cumulants higher than the second, which are at least of order ε and which have the property that they can eliminate the fast oscillation and thus give rise to a type of quasi steady behavior for the corresponding cumulant in physical space. This means, if one looks at the long time behavior of the physical cumulants higher than the second, that one would find that the quasi steady terms remain; whereas the Riemann Lebesgue lemma shows that terms of the first type tend to zero. This implies that an initially Gaussian state does not remain Gaussian to all orders in ε . It is found that these terms, described as quasi steady terms, never appear in a secular manner.

The first "t" growth secularities arise when one investigates the second order ($O(\varepsilon^2)$) components of the asymptotic expansions for the spectral cumulants. In the case of the spectral energy, secular terms arise because of triad resonances.

In taking the long time limit for the order ε^2 component of the second order spectral cumulant (which represents the spectral energy) it is assumed that the zeroth order term in the perturbation expansion for the fourth order spectral cumulant is continuous. However, this can be shown a posteriori to be a consistent assumption. For when one removes the secular terms from the long time behavior of the ε^2 component in the perturbation expansion for the fourth order spectral cumulant, using the fact that the zeroth order terms in the perturbation expansions for the spectral cumulants are slowly varying functions of time, it is found that the zeroth order term for the fourth order spectral cumulant does in fact change continuously in time. This requires the assumption that the zeroth order terms in higher order spectral cumulants were continuous. However it can be shown that the assumption that the zeroth order term in the perturbation expansions for any of the spectral cumulants is continuous is also consistent.

Thus keeping the asymptotic expansions for the spectral cumulants well ordered in time to order ε^2, gives a system of equations describing the rates of change of the zeroth order terms in the asymptotic expansions for these spectral cumulants, with respect to the time scale $\varepsilon^2 t$. In particular, it is found that, if one neglects surface tension effects (thereby eliminating the possibility of frequency triads being zero anywhere in the spectrum), the zeroth order component of the spectral energy remains constant in time and that all of the other zeroth order spectral cumulants behave in an oscillatory manner on the $\varepsilon^2 t$ time scale.

In order to find the effects of gravity wave resonances on the behavior of the cumulants, the analysis is continued to the $\varepsilon^4 t$ time scale, neglecting surface tension. However on account of the first closure of the system at the $\varepsilon^2 t$ time scale, a modified approach is necessary in order to avoid spurious higher closures. This modification will be more fully explained in the analysis but in essence, it involves choosing free terms, which can be inserted as arbitrary functions when one integrates with respect to the fast time t , in order to ensure that all of the components in the asymptotic expansions except the first term tend to zero as the fast time t tends to infinity. However, as will be explained later, it is not necessary, nor indeed possible, to suppress the quasi steady terms. It is then possible to continue the analysis to the $\varepsilon^4 t$ time scale, where it is found that in order to suppress secularities, the rate of change of the zeroth order spectral energy has to be chosen. This gives an integro-differential equation identical to that obtained by Hasselmann. It is fairly clear to see at this stage, that the rates of change of the zeroth order components of the other spectral cumulants will be given by equations similar to those developed at the $\varepsilon^2 t$ time scale.

One concludes therefore, that it is possible to reach a first closure for the spectral energy independently of the statistical nature of the system. Whether this would be true for a second closure for the spectral energy is open to conjecture. For example, if one continued the problem from the $\varepsilon^2 t$ time scale, still permitting resonant triads to occur, could one in fact, reach a system of closed equations at the $\varepsilon^4 t$ time scale? It is possible that when triad resonances exist that the energy could tend to become localized in the spectrum before the $\varepsilon^4 t$ time scale. One might then expect some sort of sideband mechanism which Brooke Benjamin proposed and which is examined in Chapter III of this thesis, to become effective along with the resonant quartets. In fact, there are terms which in the present analysis are zero, which arise as derivatives of the spectral cumulants across surfaces given by a frequency triad being zero, and these may be the manifestation of the sidebend mechanism entering the statistical problem. The author and Dr. Benney intend to examine this possibility in a later paper.

Equations of Motion.

The equation of motion for an inviscid incompressible fluid free from vorticity is

$$\nabla^2 \varphi = 0, \tag{1}$$

where $\varphi(x,y,z,t)$ is the velocity potential. The boundary conditions are

$$\frac{\partial \varphi}{\partial z} = 0, \qquad z = -h. \tag{2}$$

z = -h , h constant, is the equation of the rigid bottom.

$$\frac{\partial \varphi}{\partial t} + g\xi - \frac{S}{\rho} \frac{\xi_{xx}(1+\xi_y^2) + \xi_{yy}(1+\xi_x^2) - 2\xi_{xy}\xi_x\xi_y}{(1+\xi_x^2+\xi_y^2)^{3/2}}$$

$$+ \frac{1}{2}|\nabla\varphi|^2 = 0, \qquad z = \xi(x,y,t), \tag{3}$$

$$\frac{\partial \xi}{\partial t} + \tilde{\nabla}\varphi . \tilde{\nabla}\xi = \frac{\partial \varphi}{\partial z}, \qquad z = \xi(x,y,t). \tag{4}$$

z = (x,y,t) is the equation prescribing the elevation of the surface above some equilibrium level, z = 0 . Equation (2) is the condition on the normal velocity at the rigid horizontal bottom. Equation (3) is Bernoulli's equation (including the effects of surface tension) applied at the free surface $z = \xi(x,y,t)$.

Equation (4) expresses the kinematic condition that a fluid element at the interface remains there.

One considers waves of small, but not infinitesimal amplitude, by setting

$$\phi = \varepsilon \psi, \tag{5}$$

$$\zeta = \varepsilon \eta,$$

ε being a measure of the wave slope.

Substituting (5) into equations (1), (2), (3) and (4) and expanding the boundary conditions at the surface around the equilibrium position, z = 0 one obtains

$$\nabla^2 \psi = 0. \tag{6}$$

$$\psi_z = 0, \qquad \text{at } z = -h. \tag{7}$$

$$\psi_t + g\eta - \frac{S}{\rho}\left(\eta_{xx} + \eta_{yy}\right) + \varepsilon\left(\eta\psi_{tz} + \frac{1}{2}|\nabla\psi|^2\right)$$
$$+ \varepsilon^2\left\{\frac{1}{2}\eta^2\psi_{tzz} + \eta\left(\psi_x\psi_{xz} + \psi_y\psi_{yz} + \psi_{zz}\psi_z\right)\right.$$
$$-\frac{S}{\rho}\left(\eta_{xx}\eta_y^2 + \eta_{yy}\eta_x^2 - 2\eta_{xy}\eta_x\eta_y - \frac{3}{2}\eta_{xx}\eta_x^2 - \frac{3}{2}\eta_{xx}\eta_y^2\right.$$
$$\left.\left.-\frac{3}{2}\eta_x^2\eta_{yy} - \frac{3}{2}\eta_y^2\eta_{yy}\right)\right\} + O(\varepsilon^3) = 0, \quad z = 0. \tag{8}$$

$$\eta_t - \psi_z + \varepsilon(\psi_x\eta_x + \psi_y\eta_y - \eta\psi_{zz}) + \varepsilon^2(\eta\eta_x\psi_{xz}$$
$$+ \eta\eta_y\psi_{yz} - \tfrac{1}{2}\eta^2\psi_{zzz}) + O(\varepsilon^3) = 0, \quad z = 0. \tag{9}$$

The expansions are taken up to $O(\varepsilon^2)$ only, as the higher order coefficients are not required for the final result.

The linear equation (6) together with the boundary condition (7) allow solutions of the form

$$\psi(x,y,z,t) = \int_{-\infty}^{\infty} B(\tilde{k},t)\,\frac{\cosh|\tilde{k}|(z+h)}{\cosh|\tilde{k}|h}\,e^{-i\tilde{k}\cdot\tilde{r}}\,d\tilde{k}. \tag{10}$$

$$\eta(x,y,t) = \int_{-\infty}^{\infty} A(\tilde{k},t)\,e^{-i\tilde{k}\cdot\tilde{r}}\,d\tilde{k}. \tag{11}$$

Since ψ and η are spatially random quantities, their transforms $B(\tilde{k},t)$ and $A(\tilde{k},t)$ must be regarded as generalized functions of $\tilde{k}$. The generalized function approach will be used as it is easier to manipulate than the Fourier-St integral.

Substituting (10) and (11) into (8) and (9) using the Fourier Convolution theorem one obtains

$$B_t + gA + \frac{S}{\rho}|\tilde{k}|^2 A + \varepsilon \int_{-\infty}^{\infty} \Big[|\tilde{k}_2| \tanh |\tilde{k}_2| h \, A(\tilde{k}_1) B_t(\tilde{k}_2)$$

$$+ \frac{1}{2} \left(|\tilde{k}_1||\tilde{k}_2| \tanh |\tilde{k}_1| h \tanh |\tilde{k}_2| h - \tilde{k}_1 \cdot \tilde{k}_2 \right) B(\tilde{k}_1) B(\tilde{k}_2) \Big] \delta(\tilde{k}_1 + \tilde{k}_2 - \tilde{k}) \, d\tilde{k}_1 \, d\tilde{k}_2$$

$$+ \varepsilon^2 \int_{-\infty}^{\infty} \Big[\frac{1}{2} |\tilde{k}_3|^2 A(\tilde{k}_1) A(\tilde{k}_2) B_t(\tilde{k}_3) + |\tilde{k}_3| \tanh |\tilde{k}_3| h \left(|\tilde{k}_2|^2 - \tilde{k}_2 \cdot \tilde{k}_3 \right)$$

$$A(\tilde{k}_1) B(\tilde{k}_2) B(\tilde{k}_3) - \frac{S}{\rho} \left\{ (\tilde{k}_1 \times \tilde{k}_2) \cdot (\tilde{k}_1 \times \tilde{k}_3) - \frac{3}{2} |\tilde{k}_1|^2 \tilde{k}_2 \cdot \tilde{k}_3 \right\}$$

$$A(\tilde{k}_1) A(\tilde{k}_2) A(\tilde{k}_3) \Big] \delta(\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 - \tilde{k}) \, d\tilde{k}_1 \, d\tilde{k}_2 \, d\tilde{k}_3 + O(\varepsilon^3) = 0, \qquad (12)$$

$$A_t = |\tilde{k}| \tanh |\tilde{k}| h \, B + \varepsilon \int_{-\infty}^{\infty} \left(|\tilde{k}_2|^2 + \tilde{k}_1 \cdot \tilde{k}_2 \right) A(\tilde{k}_1) B(\tilde{k}_2)$$

$$\delta(\tilde{k}_1 + \tilde{k}_2 - \tilde{k}) \, d\tilde{k}_1 \, d\tilde{k}_2 + \varepsilon^2 \int_{-\infty}^{\infty} |k_3| \tanh |k_3| h$$

$$\left(\frac{1}{2} |k_3|^2 + \tilde{k}_2 \cdot \tilde{k}_3 \right) A(\tilde{k}_1) A(\tilde{k}_2) B(\tilde{k}_3) \, \delta(\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 - \tilde{k}) \, d\tilde{k}_1 \, d\tilde{k}_2 \, d\tilde{k}_3$$

$$+ O(\varepsilon^3). \qquad (13)$$

To eliminate B_t from under the integrals in equation (12), one solves for B_t by successive approximations.

$$B_t = {}_0B_t + \varepsilon \, {}_1B_t + \varepsilon^2 \, {}_2B_t + \cdots$$

On setting

$$A(\tilde{k},t) = \sum_s a^s(\tilde{k},t)\, e^{isωt},$$

$$B(\tilde{k},t) = \sum_s \frac{i\nu^2(\tilde{k})}{s\omega}\, a^s(\tilde{k},t)\, e^{is\omega t},$$

$$\omega(\tilde{k}) = \sqrt[+]{|\tilde{k}|\tanh|\tilde{k}|h\left(g + \frac{S}{\rho}|\tilde{k}|^2\right)}$$

$$\nu^2(\tilde{k}) = g + \frac{S}{\rho}|\tilde{k}|^2. \qquad s = +,-.$$

one can then write equations (12) and (13) as

$$\sum_s \frac{i\nu^2}{s\omega} a_t^s(\tilde{k},t)\, e^{is\omega t} = \varepsilon \sum_{s_1 s_2} \int_{-\infty}^{\infty} g K_{k_1 k_2}^{s_1 s_2} a_1^{s_1} a_2^{s_2} e^{i(s_1\omega_1 + s_2\omega_2)t}$$

$$\delta(\tilde{k}_1 + \tilde{k}_2 - \tilde{k})\, d\tilde{k}_1\, d\tilde{k}_2 + \varepsilon^2 \sum_{s_1 s_2 s_3} \int_{-\infty}^{\infty} g K_{k_1 k_2 k_3}^{s_1 s_2 s_3} a_1^{s_1} a_2^{s_2} a_3^{s_3}$$

$$e^{i(s_1\omega_1 + s_2\omega_2 + s_3\omega_3)t}\, \delta(\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 - \tilde{k})\, d\tilde{k}_1\, d\tilde{k}_2\, d\tilde{k}_3$$

$$+ \cdots$$

$$\cdots + \varepsilon^{r-1} \sum_{s_1 \cdots s_r} \int_{-\infty}^{\infty} g K_{k_1 \cdots k_r}^{s_1 \cdots s_r} a_1^{s_1} a_2^{s_2} \cdots a_r^{s_r} e^{i(s_1\omega_1 + \cdots + s_r\omega_r)t}$$

$$\delta(\tilde{k}_1 + \tilde{k}_2 + \cdots + \tilde{k}_r - \tilde{k})\, d\tilde{k}_1 \cdots d\tilde{k}_r \qquad (12)'$$

$$+ \cdots,$$

and

$$\sum_{s} a_t^{s}(\tilde{k},t)\, e^{is\omega t} = \varepsilon \sum_{s_1 s_2} \int_{-\infty}^{\infty} i H_{k_1 k_2}^{s_1 s_2}\, a_1^{s_1} a_2^{s_2}\, e^{i(s_1\omega_1 + s_2\omega_2)t}$$

$$\delta(\tilde{k}_1 + \tilde{k}_2 - \tilde{k})\, d\tilde{k}_1\, d\tilde{k}_2 + \varepsilon^2 \sum_{s_1 s_2 s_3} \int_{-\infty}^{\infty} i H_{k_1 k_2 k_3}^{s_1 s_2 s_3}\, a_1^{s_1} a_2^{s_2} a_3^{s_3}$$

$$e^{i(s_1\omega_1 + s_2\omega_2 + s_3\omega_3)t}\, \delta(\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 - \tilde{k})\, d\tilde{k}_1\, d\tilde{k}_2\, d\tilde{k}_3$$

$$+ \ldots$$

$$\cdots + \varepsilon^{r-1} \sum_{s_1 \cdots s_r} \int_{-\infty}^{\infty} i H_{k_1 \cdots k_r}^{s_1 \cdots s_r}\, a_1^{s_1} \cdots a_r^{s_r}\, e^{i(s_1\omega_1 + \cdots + s_r\omega_r)t}$$

$$\delta(\tilde{k}_1 + \tilde{k}_2 + \cdots + \tilde{k}_r - \tilde{k})\, d\tilde{k}_1 \cdots d\tilde{k}_r + \cdots \quad (13)'$$

In the above equations the following notations have been used.

$$g K_{k_1 k_2}^{s_1 s_2} = \frac{1}{2}\left(\omega_1^2 + \omega_2^2 + s_1\omega_1 s_2\omega_2 - \tilde{k}_1 \cdot \tilde{k}_2\, \frac{\nu_1^2 \nu_2^2}{s_1\omega_1\, s_2\omega_2}\right),$$

$$g K_{k_1 k_2 k_3}^{s_1 s_2 s_3} = \frac{1}{3}\, g \overset{123}{P}\, \hat{K}_{k_1 k_2 k_3}^{s_1 s_2 s_3},$$

$\overset{123}{P}$ being the cyclic permutation over 1, 2, and 3 .

$$g \hat{K}_{k_1 k_2 k_3}^{s_1 s_2 s_3} = \frac{1}{4}\left(\nu_3^2 |\tilde{k}_3|^2 + \nu_2^2 |\tilde{k}_2|^2\right) + \frac{1}{2}\, \frac{\nu_2^2 \omega_3^2 \left(|\tilde{k}_2|^2 - \tilde{k}_2 \cdot \tilde{k}_3\right)}{s_2\omega_2\, s_3\omega_3}$$

$$+ \frac{1}{2}\, \frac{\nu_3^2 \omega_2^2 \left(|\tilde{k}_3|^2 - \tilde{k}_2 \cdot \tilde{k}_3\right)}{s_2\omega_2\, s_3\omega_3} - g K_{k_2 k_3}^{s_2 s_3} \sqrt[+]{|\tilde{k}_2 + \tilde{k}_3|^2}\, \tanh |\tilde{k}_2 + \tilde{k}_3| h$$

$$+ S/\rho\, \left[(\tilde{k}_1 \times \tilde{k}_2) \cdot (\tilde{k}_1 \times \tilde{k}_3) - \frac{3}{2} |\tilde{k}_1|^2\, \tilde{k}_2 \cdot \tilde{k}_3\right],$$

$$H^{\nu_1\nu_2}_{k_1k_2} = \frac{1}{2}\left[\frac{\nu_2^2}{\nu_2\omega_2}\left(|\tilde{k}_2|^2+\tilde{k}_1\cdot\tilde{k}_2\right) + \frac{\nu_1^2}{\nu_1\omega_1}\left(|\tilde{k}_1|^2+\tilde{k}_1\cdot\tilde{k}_2\right)\right],$$

$$H^{\nu_1\nu_2\nu_3}_{k_1k_2k_3} = \frac{1}{3}\overset{123}{P}\,\hat{H}^{\nu_1\nu_2\nu_3}_{k_1k_2k_3},$$

$$\hat{H}^{\nu_1\nu_2\nu_3}_{k_1k_2k_3} = \frac{1}{2}\left[\nu_3\omega_3\left(\frac{1}{2}|\tilde{k}_3|^2+\tilde{k}_2\cdot\tilde{k}_3\right)+\nu_2\omega_2\left(\frac{1}{2}|\tilde{k}_2|^2+\tilde{k}_2\cdot\tilde{k}_3\right)\right],$$

$$\omega_i = \omega(\tilde{k}_i), \qquad \nu_i = \nu(\tilde{k}_i).$$

Multiplying equation $(12)'$ by $-\frac{i\omega}{\nu^2}$ and adding and subtracting this to equation $(13')$ one obtains

$$a^\nu_t(\tilde{k},t) = \sum_r \varepsilon^{r-1}\sum_{\nu_1\cdots\nu_r}\int_{-\infty}^{\infty} L^{\nu\nu_1\cdots\nu_r}_{kk_1\cdots k_r}\, a_1^{\nu_1}\cdots a_r^{\nu_r}\, e^{i(s_1\omega_1+\cdots+s_r\omega_r)t}$$
$$\delta(\tilde{k}_1+\cdots+\tilde{k}_r-\tilde{k})\,d\tilde{k}_1\cdots d\tilde{k}_r, \quad (14)$$

where

$$L^{\nu\nu_1\cdots\nu_r}_{kk_1\cdots k_r} = \frac{i}{2}\left(H^{\nu_1\cdots\nu_r}_{k_1\cdots k_r} - \frac{\nu\omega g}{\nu^2}K^{\nu_1\cdots\nu_r}_{k_1\cdots k_r}\right).$$

Equation (14) is the governing equation of the system and would seem to be typical for all conservative systems of weakly coupled waves.

There are certain properties of the $L^{\nu\nu_1\cdots\nu_r}_{kk_1\cdots k_r}$ which will be needed later in the work. These are,

(i) $\mathcal{L}^{\nu\nu_1\cdots\nu_r}_{kk_1\cdots k_r}$ is symmetric to any interchange between the numbers (1...r). It is clear that this can be accomplished as shown in the construction of the first two terms and from the form of equation (14).

(ii) $\mathcal{L}^{\nu\nu_1\cdots\nu_r}_{kk_1\cdots k_r}$ is a purely imaginary quantity (the $\tilde{k}_i$ being real). This can be seen by inspection.

(iii) $\mathcal{L}^{\nu\nu_1\cdots\nu_r}_{kk_1\cdots k_r} = \overset{*}{\mathcal{L}}^{-\nu-\nu_1\cdots-\nu_r}_{-k-k_1\cdots-k_r} = -\mathcal{L}^{-\nu-\nu_1\cdots-\nu_r}_{-k-k_1\cdots-k_r}$.

To prove this one notes that since $\eta(\tilde{r},t)$ is real,

$$A(\tilde{k},t) = A^*(-\tilde{k},t),$$

which implies

$$a^{\nu}(\tilde{k},t) = \overset{*}{a}^{-\nu}(-\tilde{k},t). \tag{15}$$

From the governing equation (14), since t is real

$$\overset{*}{a}^{-\nu}_t(-\tilde{k},t) = \sum_r \varepsilon^{r-1} \sum_{\nu_1\cdots\nu_r} \int_{-\tilde{\infty}}^{\tilde{\infty}} \overset{*}{\mathcal{L}}^{-\nu+\nu_1\cdots+\nu_r}_{-k+k_1\cdots+k_r} \overset{*}{a}^{\nu_1}_1 \cdots \overset{*}{a}^{\nu_r}_r \, e^{-i(s_1\omega_1+\cdots+s_r\omega_r+s\omega)t} \delta(\tilde{k}_1+\cdots+\tilde{k}_r+\tilde{k})\, d\tilde{k}_1\cdots d\tilde{k}_r.$$

Setting $\begin{cases}\nu_i \\ \tilde{k}_i\end{cases} \longrightarrow \begin{cases}-\nu_i \\ -\tilde{k}_i\end{cases}$ $i = 1,\ldots,r$, on the right hand side of the above equation and using equation (15), one obtains

$$\overset{*}{a}^{-\nu}_t(-\tilde{k},t) = \sum_r \varepsilon^{r-1} \sum_{\nu_1\cdots\nu_r} \int_{-\tilde{\infty}}^{\tilde{\infty}} \overset{*}{\mathcal{L}}^{-\nu-\nu_1\cdots-\nu_r}_{-k-k_1\cdots-k_r} a^{\nu_1}_1 \cdots a^{\nu_r}_r \, e^{i(s_1\omega_1+\cdots+s_r\omega_r-s\omega)t} \delta(\tilde{k}_1+\cdots+\tilde{k}_r-\tilde{k})\, d\tilde{k}_1\cdots d\tilde{k}_r.$$

Comparing this result with equation (14) one sees that

$$\mathcal{L}_{k k_1 \cdots k_r}^{\nu \nu_1 \cdots \nu_r} = \mathcal{L}^{*\,-\nu -\nu_1 \cdots -\nu_r}_{-k -k_1 \cdots -k_r}$$

But $\mathcal{L}_{k k_1 \cdots k_r}^{\nu \nu_1 \cdots \nu_r}$ is an imaginary quantity, and so

$$\mathcal{L}_{k k_1 \cdots k_r}^{\nu \nu_1 \cdots \nu_r} = \mathcal{L}^{*\,-\nu -\nu_1 \cdots -\nu_r}_{-k -k_1 \cdots -k_r} = -\mathcal{L}^{-\nu -\nu_1 \cdots -\nu_r}_{-k -k_1 \cdots -k_r}.$$

(iv) $\delta(\tilde{k})\, \mathcal{L}_{k k_1 \cdots k_r}^{\nu \nu_1 \cdots \nu_r} = 0.$

This notation will be explained in the following proof:
Since all mean values are zero

$$\langle \eta \rangle = 0,$$

$$\langle A(\tilde{k},t) \rangle = 0.$$

This implies $\quad \langle a^{\nu}(\tilde{k},t) \rangle = 0, \quad$ for all time.

Therefore $\quad \langle a^{\nu}_t(\tilde{k},t) \rangle = 0.$

From equation (14)

$$\langle a^{\nu}_t(\tilde{k},t) \rangle = 0 = \sum_r \varepsilon^{r-1} \sum_{\nu_1 \cdots \nu_r} \int_{-\infty}^{\infty} \mathcal{L}_{k k_1 \cdots k_r}^{\nu \nu_1 \cdots \nu_r} \langle a_1^{\nu_1} \cdots a_r^{\nu_r} \rangle$$

$$e^{i(s_1\omega_1 + \cdots + s_r\omega_r - s\omega)t}\, \delta(\tilde{k}_1 + \cdots + \tilde{k}_r - \tilde{k})\, d\tilde{k}_1 \cdots d\tilde{k}_r .$$

Also the mean value of r Fourier components can be decomposed into cumulant transforms of the form

$$\langle a_1^{\nu_1} \cdots a_r^{\nu_r} \rangle = \delta(\tilde{k}_1 + \cdots + \tilde{k}_r) \left[\phi^{(r)\,\nu_1 \cdots \nu_r}_{(k_1 \cdots k_{r-1})} + \cdots \right]$$

and since $\delta(\tilde{k}_1 + \cdots + \tilde{k}_r)\, \delta(\tilde{k}_1 + \cdots + \tilde{k}_r - \tilde{k}) = \delta(\tilde{k})\, \delta(\tilde{k}_1 + \cdots + \tilde{k}_r)$,
it is found that

$$0 = \sum_r \varepsilon^{r-1} \sum_{\nu_1 \cdots \nu_r} \int_{-\infty}^{\infty} \delta(\tilde{k})\, L_{k k_1 \cdots k_r}^{\nu \nu_1 \cdots \nu_r} \langle a_1^{\nu_1} \cdots a_r^{\nu_r} \rangle\, e^{i(s_1\omega_1 + \cdots + s_r\omega_r - s\omega)t}\, d\tilde{k}_1\, dk_2 \cdots dk_r .$$

Therefore $L_{0 k_1 \cdots k_r}^{\nu \nu_1 \cdots \nu_r} = 0$, when $k + \ldots + k = 0$. Hence one uses the notation,

$$\delta(\tilde{k})\, L_{k k_1 \cdots k_r}^{\nu \nu_1 \cdots \nu_r} = 0 .$$

Statistics.

The primary interest in the following analysis is the behavior in time of the statistical properties of the wave motion. One assumes that $\psi(x,y,z,t)$ and $\eta(x,y,t)$ are random functions of the spatial variables x and y with spatial homogeneity. No further assumption is made as to the nature of the statistical distributions. All that is required is a knowledge that the cumulants are initially smooth.

Spatial homogeneity implies that the mean value (ensemble average) of $\langle \eta(\tilde{r})\, \eta(\tilde{r}+\tilde{s})\rangle$ is a function of $\tilde{s}$ only. The cumulants formed from these mean values have the necessary behavior at infinity to permit Fourier transforms. To be quite general let us suppose $\langle \eta \rangle = \overset{(1)}{R} \neq 0$, $\overset{(1)}{R}$ being a function of t only. Then

$$\langle \eta(\tilde{r}_1)\, \eta(\tilde{r}_2)\rangle = \overset{(2)}{R}(\tilde{r}_2-\tilde{r}_1, t) + \overset{(1)}{R}(t)\, \overset{(1)}{R}(t).$$

In general $\langle \eta_1 \cdots \eta_r \rangle$ has many possible decompositions; for example $2 \times (r-2)$; $2 \times 3 \times (r-5)$; etc. The number of terms in any particular decomposition

$$r = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_m p_m ,$$

is

$$\frac{{}_rC_{p_1}\; {}_{(r-p_1)}C_{p_1} \cdots {}_{(r-p_1(\alpha_1-1))}C_{p_1}}{\alpha_1!} \times \frac{{}_{(r-\alpha_1 p_1)}C_{p_2}\; {}_{(r-\alpha_1 p_1-p_2)}C_{p_2} \cdots}{\alpha_2!} \times \cdots .$$

where p_i is the size of the decomposition and α_i is the number of particular decompositions of this size which occur. For example if the mean value of the random quantity is zero (namely the order one cumulant) the possible decompositions of the mean value of six quantities are

$$0 \times 6 \ , \quad 2 \times 4 \ , \quad 3 \times 3 \ , \quad 2 \times 2 \times 2 \ .$$

There is $6C_6 = 1$ member in the 0×6 class; there are $\frac{6C_2}{1!}\frac{4C_4}{1!} = 15$ members in the (2×4) class; $\frac{6C_3 \; 3C_3}{2!} = 10$ members in the (3×3) class; and $\frac{6C_2 \; 4C_2 \; 2C_2}{3!} = 15$ members in the $(2 \times 2 \times 2)$ class.

The relation

$$\langle \eta(\tilde{r})\, \eta(\tilde{r}+\tilde{s}) \rangle = \overset{(2)}{R}(\tilde{s}),$$

implies that

$$\langle A(\tilde{k}_1)\, A(\tilde{k}_2) \rangle = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \langle \eta(\tilde{r})\, \eta(\tilde{s}) \rangle\, e^{i\tilde{k}_1 \cdot \tilde{r} + i\tilde{k}_2 \cdot \tilde{s}}\, d\tilde{r}\, d\tilde{s}$$

Let $\tilde{r} = \tilde{s} + \tilde{p}$, $\tilde{s} = \tilde{s}$; then

$$\langle A(\tilde{k}_1)\, A(\tilde{k}_2) \rangle = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \overset{(2)}{R}(\tilde{p})\, e^{i(\tilde{k}_1+\tilde{k}_2)\cdot \tilde{s} + i\tilde{k}_1 \cdot \tilde{p}}\, d\tilde{s}\, d\tilde{p},$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \overset{(2)}{R}(\tilde{p})\, e^{i\tilde{k}_1 \cdot \tilde{p}}\, d\tilde{p} \int_{-\infty}^{\infty} e^{i(\tilde{k}_1+\tilde{k}_2)\cdot \tilde{s}}\, d\tilde{s}$$

$$\langle A(\tilde{k}_1)\, A(\tilde{k}_2)\rangle = \frac{\delta(\tilde{k}_1+\tilde{k}_2)}{2\pi}\int_{-\infty}^{\infty} \overset{(2)}{R}(\tilde{p})\, e^{i\tilde{k}_1\cdot\tilde{p}}\, d\tilde{p},$$

$$= \delta(\tilde{k}_1+\tilde{k}_2)\ \overset{(2)}{Q}(\tilde{k}_1, t).$$

Therefore

$$\langle a^{\nu_1}(\tilde{k}_1)\, a^{\nu_2}(\tilde{k}_2)\rangle = \delta(\tilde{k}_1+\tilde{k}_2)\ \overset{(2)}{Q}{}^{\nu_1\nu_2}(\tilde{k}_1),$$

where

$$\overset{(2)}{Q}(\tilde{k}_1, t) = \sum_{\nu_1,\nu_2} \overset{(2)}{Q}{}^{\nu_1\nu_2}(\tilde{k}_1, t)\, e^{i(s_1\omega_1+s_2\omega_2)t}, \quad \tilde{k}_2 = -\tilde{k}_1$$

and $\omega_2 = \omega(\tilde{k}_2) = \omega(-\tilde{k}_1) = \omega_1.$

$Q(k,t)$ is an ordinary function of both $\tilde{k}$ and t. The corresponding spectral cumulant of $R(\tilde{p}_1, \tilde{p}_2, \ldots \tilde{p}_{r-1})$ is $\overset{(r)}{Q}(\tilde{k}_1 \cdots \tilde{k}_{r-1})$ and

$$\delta(\tilde{k}_1+\cdots+\tilde{k}_r)\ \overset{(r)}{Q}(\tilde{k}_1 \cdots \tilde{k}_{r-1})$$

$$= \delta(\tilde{k}_1+\cdots+\tilde{k}_r) \sum_{\nu_1\cdots\nu_r} \overset{(r)}{Q}{}^{\nu_1\cdots\nu_r}(\tilde{k}_1\cdots\tilde{k}_{r-1})\, e^{i(s_1\omega_1+\cdots+s_r\omega_r)t}.$$

The Energy Density.

The energy in a parallelepiped of unit cross section extending from the bottom of the ocean to the free surface is composed of three parts,

(a) The kinetic energy, the ensemble average of which, over the horizontal spatial variables x and y, is denoted by $\overline{E}_{K.E}$.

(b) The potential energy, the ensemble average of which, over the horizontal spatial variables x and y, is denoted by $\overline{E}_{Pot.}$.

and (c) The work done on the fluid by the restoring action of the surface tension at the surface. The ensemble average of this quantity over the horizontal spatial variables x and y is denoted by $\overline{E}_{S.T}$.

(a) $\overline{E}_{K.E} = \frac{1}{2}\int_V \rho\, \overline{|\nabla\varphi|^2}\, d\tau,$

where V is the volume of the parallelepiped described above, and $d\tau$ is the elemental volume dxdydz.

$$\overline{E}_{K.E} = \frac{1}{2}\int_V \rho\, \overline{\nabla\cdot\varphi(\nabla\varphi)}\, d\tau \qquad \text{since } \nabla^2\varphi = 0,$$

$$= \frac{1}{2}\int_S \rho\, \overline{\varphi\, \nabla\varphi\cdot\hat{n}}\, dS,$$

S being the surface circumscribing the above described volume, $\hat{n}$ being the unit outward normal to this surface and dS being an elemental area on this surface. Since $\overline{\varphi \cdot \nabla\varphi \cdot \hat{n}}$ is independent of x and y the only contributions come from the bottom and the top. At the bottom

$$\nabla\varphi \cdot \hat{n} = 0.$$

At the top surface

$$\nabla\varphi \cdot \hat{n} = \frac{-\varphi_x \xi_x}{\sqrt{1+\xi_x^2+\xi_y^2}} + \frac{-\varphi_y \xi_y}{\sqrt{1+\xi_x^2+\xi_y^2}} + \frac{\varphi_z}{\sqrt{1+\xi_x^2+\xi_y^2}}$$

$$= \frac{\xi_t}{\sqrt{1+\xi_x^2+\xi_y^2}} .$$

Noting that $dS = \sqrt{1 + \xi_x^2 + \xi_y^2}\ dA$, where dA is the projection of the perturbed surface onto the horizontal equilibrium surface z = 0 , one finds the average energy per unit area is

$$\frac{1}{2}\rho\,\overline{\varphi\,\xi_t} = \frac{1}{2}\rho\varepsilon^2\,\overline{\psi\,\eta_t}$$

$$= \frac{1}{2}\rho\varepsilon^2\int_{-\infty}^{\infty}\langle B(\breve{k}_1)A_t(\breve{k}_2)\rangle\, e^{-i(\breve{k}_1+\breve{k}_2)\cdot\breve{r}}\, d\breve{k}_1\, d\breve{k}_2 ,$$

$$= \frac{1}{2}\rho\varepsilon^2\sum_{s_1 s_2}\int_{-\infty}^{\infty}\frac{i\nu_1^2\,|\breve{k}_2|\tanh|\breve{k}_2|h\, i\nu_2^2}{s_1\omega_1\, s_2\omega_2}\langle a_1^{s_1} a_2^{s_2}\rangle\, e^{-i(\breve{k}_1+\breve{k}_2)\cdot\breve{r}}\, e^{i(s_1\omega_1+s_2\omega_2)t}\, d\breve{k}_1\, d\breve{k}_2$$

$$+\ O(\varepsilon^3)\ \text{stationary terms},$$

$$\overline{E}_{K.E} = \frac{1}{2}\rho\varepsilon^2 \sum_{\eta_1\eta_2} \int_{-\infty}^{\infty} -\frac{v^2}{\eta_1\eta_2} \overset{(2)\eta_1\eta_2}{\Phi}(\tilde{k})\, e^{i(s_1+s_2)\omega t} d\tilde{k} .$$

(b) $$\overline{E}_{pot} = \int^{\xi} \rho g z \, dz ,$$

$$= \frac{1}{2}\rho g \overline{\xi^2} + \text{Constant},$$

$$= \frac{1}{2}\rho g \varepsilon^2 \overline{\eta^2},$$

$$= \frac{1}{2}\rho g \varepsilon^2 \sum_{\eta_1\eta_2} \int_{-\infty}^{\infty} \langle a_1^{\eta_1} a_2^{\eta_2} \rangle e^{-i(\tilde{k}_1+\tilde{k}_2)\cdot\tilde{r}} e^{i(s_1\omega_1+s_2\omega_2)t} d\tilde{k}_1 d\tilde{k}_2 ,$$

$$= \frac{1}{2}\rho g \varepsilon^2 \sum_{\eta_1\eta_2} \int_{-\infty}^{\infty} \overset{(2)\eta_1\eta_2}{\Phi}(\tilde{k})\, e^{i(s_1+s_2)\omega t} d\tilde{k} .$$

(c) Energy due to surface tension. The energy gained by the fluid as the surface tension force S acts on the stretched area $(da)'$, with (da) as the elemental area of the unstretched surface,

$$\overline{E}_{S.T} = S \overline{\int (da)' - (da)} ,$$

$$= S \int \left[\left(1 + \xi_x^2 + \xi_y^2\right)^{1/2} - 1 \right] dA ,$$

$$\bar{E}_{S.T} = \frac{1}{2} S \varepsilon^2 \int \overline{(\eta_x^2 + \eta_y^2)} \, dA + O(\varepsilon^3).$$

Therefore the average energy per unit area is given by

$$\frac{1}{2} S \varepsilon^2 \langle \eta_x^2 + \eta_y^2 \rangle + O(\varepsilon^3) \text{ stationary terms},$$

$$= -\frac{1}{2} S \varepsilon^2 \int_{-\infty}^{\infty} \tilde{k}_1 \cdot \tilde{k}_2 \langle a_1^{s_1} a_2^{s_2} \rangle e^{-i(\tilde{k}_1 + \tilde{k}_2)\cdot \tilde{r}} e^{i(s_1\omega_1 + s_2\omega_2)t} \, d\tilde{k}_1 d\tilde{k}_2$$

$$+ O(\varepsilon^3),$$

$$= \frac{1}{2} \varepsilon^2 S \int_{-\infty}^{\infty} |\tilde{k}|^2 \overset{(2)}{Q}{}^{s_1 s_2}(\tilde{k}) \, e^{i(s_1+s_2)\omega t} \, d\tilde{k}.$$

Therefore the average total energy per unit area is

$$\frac{1}{2} \rho \varepsilon^2 \sum_{s_1 s_2} \int_{-\infty}^{\infty} \left(g + \frac{S|\tilde{k}|^2}{\rho} \right) \left(1 - \frac{1}{s_1 s_2} \right) \overset{(2)}{Q}{}^{s_1 s_2}(\tilde{k}) \, e^{i(s_1+s_2)\omega t} \, d\tilde{k}$$

$$+ O(\varepsilon^3) \text{ stationary terms},$$

$$= \rho \varepsilon^2 \sum_{s} \int_{-\infty}^{\infty} \left(g + \frac{S|\tilde{k}|^2}{\rho} \right) \overset{(2)}{Q}{}^{s\,-s}(\tilde{k}) \, d\tilde{k} + O(\varepsilon^3).$$

The spectral cumulant $Q^{(2)a-a}(k)$ is thus a measure of the spectral energy in the system.

Method of Approach.

The governing equation for the system of nonlinear waves is

$$a_t^{\nu}(\tilde{k},t) = \sum_r \varepsilon^{r-1} \sum_{\nu_1 \cdots \nu_r} \int_{-\infty}^{\infty} L_{k k_1 \cdots k_r}^{\nu \nu_1 \cdots \nu_r} a_1^{\nu_1} \cdots a_r^{\nu_r} e^{i(s_1\omega_1 + \cdots + s_r\omega_r - s\omega)t} \delta(\tilde{k}_1 + \cdots + \tilde{k}_r - \tilde{k})\, d\tilde{k}_1 \cdots d\tilde{k}_r ,$$

where the $a^{\nu}(\tilde{k},t)$ are generalized functions of $\tilde{k}$. If $\eta(x,y,t)$ were non-random and possessed a continuous Fourier decomposition then $a^{\nu}(\tilde{k},t)$ would represent the amplitude of a wave with wave number $\tilde{k}$ and with wave speed $\frac{\nu\omega}{|\tilde{k}|}$.

The perturbation expansion

$$a^{\nu}(\tilde{k},t) = a_0^{\nu}(\tilde{k},t) + \varepsilon a_1^{\nu}(\tilde{k},t) + \cdots + \varepsilon^r a_r^{\nu}(\tilde{k},t) + \cdots$$

is applied. Substituting this into the governing equation and equating powers of ε one obtains expressions for $a_0^{\nu}(k)$, $a_1^{\nu}(k)$, etc. From these quantities the perturbation expansions for the mean values are obtained

$$\langle a_1^{\nu_1} \cdots a_r^{\nu_r} \rangle = \langle a_{01}^{\nu_1} \cdots a_{0r}^{\nu_r} \rangle + \mathcal{P}^{1\cdots r} \langle a_{11}^{\nu_1} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle$$

$$+ \varepsilon^2 \mathcal{P} \left\{ \langle a_{21}^{\nu_1} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle + \langle a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} \cdots a_{0r}^{\nu_r} \rangle \right\}$$

$$+ O(\varepsilon^3) + \cdots ,$$

and the perturbation expansions for the spectral cumulants are obtained in turn from the perturbed mean values, as

$$\overset{(r)}{Q}{}^{\nu_1\cdots\nu_r}(\tilde{k}_1\cdots\tilde{k}_{r-1}) = \overset{(r)}{Q}{}_0^{\nu_1\cdots\nu_r}(k_1\cdots k_{r-1}) + \varepsilon\, \overset{(r)}{Q}{}_1^{\nu_1\cdots\nu_r}(k_1\cdots k_{r-1}) + \cdots .$$

$P\{\}$ is a permutation operator on $1\cdots r$ giving all the possible terms, a typical one being $\{\ \}$. For example,

$$P\left\{\langle a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3}\rangle\right\} = \langle a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3}\rangle + \langle a_{11}^{\nu_1} a_{02}^{\nu_2} a_{13}^{\nu_3}\rangle + \langle a_{01}^{\nu_1} a_{12}^{\nu_2} a_{13}^{\nu_3}\rangle .$$

The long time behavior of $\overset{(r)}{Q}{}_m^{\nu_1\cdots\nu_r}(\tilde{k}_1 \quad \tilde{k}_{r-1})$, $(m \geq 1)$, is examined with reference to the resulting behavior it gives to the corresponding cumulant in physical space. In most cases this will be equivalent to finding the long time behavior for $\overset{(r)}{Q}{}_m^{\nu_1\cdots\nu_r}(\tilde{k}_1,\cdots \tilde{k}_{r-1})$ which will then be denoted by $\overset{(r)}{\tilde{Q}}_m(\tilde{k}_1,\cdots \tilde{k}_{r-1})$ and substituted into the expression

$$\overset{(r)}{R}_m(\tilde{p}_1,\cdots \tilde{p}_{r-1}) = \sum_{\nu_1\cdots\nu_r} \int_{-\infty}^{\infty} \overset{(r)}{\tilde{Q}}{}_m^{\nu_1\cdots\nu_r}(\tilde{k}_1,\cdots \tilde{k}_{r-1})\, e^{-i\tilde{k}_1\cdot\tilde{p}_1 \cdots -i\tilde{k}_{r-1}\cdot\tilde{p}_{r-1}}\, e^{i(s_1\omega_1 + s_2\omega_2 + \cdots + s_r\omega_r)t}\, \delta(\tilde{k}_1 + \cdots + \tilde{k}_r)\, d\tilde{k}_1\cdots d\tilde{k}_r .$$

However if any $\overset{(r)}{Q}{}_m^{\nu_1\cdots\nu_r}(\tilde{k}_1,\cdots \tilde{k}_{r-1})$ should contain functions with arguments which are not independent of $(\nu_1\omega_1 + \cdots + \nu_r\omega_r)t$ then the asymptotics must be applied to the total function

$$\sum_{\nu_1\cdots\nu_r} \int_{-\infty}^{\infty} \overset{(r)}{Q}{}_m^{\nu_1\cdots\nu_r}(\tilde{k}_1\cdots \tilde{k}_{r-1})\, e^{i(s_1\omega_1 + \cdots + s_r\omega_r)t}\, e^{-i\tilde{k}_1\cdot\tilde{p}_1 \cdots -i\tilde{k}_{r-1}\cdot\tilde{p}_{r-1}}\, \delta(\tilde{k}_1 + \cdots + \tilde{k}_r)\, d\tilde{k}_1\cdots d\tilde{k}_r .$$

The latter type terms do not belong to the same ordering procedure as the former and give rise to a quasi steady behavior in the physical cumulant. (See Appendix III.)

If any "t" growths, or secular behaviors, should occur in the asymptotic expansions a device will be used to remove the troublesome terms so that the remaining terms form a well ordered asymptotic expansion in physical space for all time. This device will consist of introducing time scales,

$$T_0 = t, \quad T_1 = \varepsilon t, \quad T_2 = \varepsilon^2 t, \; \cdots \quad T_r = \varepsilon^r t, \; \cdots$$

and allowing $Q_0^{(r)\,\alpha_1 \cdots \alpha_r}(\tilde{k}_1 \cdots \tilde{k}_{r-1})$ to be a function of these longer time scales. With this multiple time scaling procedure, the operator $\frac{\partial}{\partial t}$ is replaced by $\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T_1} + \cdots + \varepsilon^r \frac{\partial}{\partial T_r} + \cdots$, where T_0 has been replaced by t for convenience. Any secularities occurring will be removed by choosing the long time behavior of $Q_0^{(r)\,\alpha_1 \cdots \alpha_r}(\tilde{k}_1 \cdots \tilde{k}_{r-1}, T_1, T_2 \cdots)$. In the present problem it will be shown that to the order to which we are interested only the time scales T_2 and T_4 are required.

An equation which will be used frequently is,

$$\frac{dx}{dt} = e^{i\omega t}, \quad x(0) = 0,$$

whose solution

$$x(t) = \frac{e^{i\omega t} - 1}{i\omega},$$

will be denoted by $\Delta(\omega)$. On account of the laborious and lengthy nature of the manipulations the following notational contractions have been adopted.

$$\Delta_{123\cdots r,0} = \Delta(\nu_1\omega_1 + \nu_2\omega_2 + \nu_3\omega_3 + \cdots + \nu_r\omega_r - \nu\omega),$$

$$W_{123\cdots r,0} = \nu_1\omega_1 + \nu_2\omega_2 + \nu_3\omega_3 + \cdots + \nu_r\omega_r - \nu\omega,$$

$$\delta_{12\cdots r,0} = \delta(\tilde{k}_1 + \tilde{k}_2 + \cdots + \tilde{k}_r - \tilde{k}),$$

$$\delta_{12,34} = \delta(\tilde{k}_1 + \tilde{k}_2 - \tilde{k}_3 - \tilde{k}_4),$$

$$\delta^{\nu}_{123,0} = \delta(\nu_1\omega_1 + \nu_2\omega_2 + \nu_3\omega_3 - \nu\omega).$$

$a_{01}^{\nu_1}$ will be taken to mean the zeroth order term in the perturbation expansion for $a^{\nu_1}(\tilde{k}_1)$. $a_{ij}^{\nu_j}$ will signify the i^{th} order term in the perturbation expansion for $a^{\nu_j}(\tilde{k}_j)$.

Analysis.

The order one balance of the governing equation (14) gives

$$\frac{\partial a_0^{\nu}(\tilde{k}, t, T_1, \cdots)}{\partial t} = 0.$$

Therefore one has

$$a_0^{\nu}(\tilde{k}) = a_0(\tilde{k}, T_1, T_2, \cdots). \tag{16}$$

Equation (16) implies that $\overset{(r)}{Q}{}_0^{\nu_1 \cdots \nu_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1})$ is a function of the longer time scales only.

The order ε balance in the governing equation (14) gives

$$a_1^{\nu}(\tilde{k}, t) = \sum_{\nu_1 \nu_2} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2} a_{01}^{\nu_1} a_{02}^{\nu_2} \Delta_{12,0} \, \delta_{12,0} \, d\tilde{k}_1 d\tilde{k}_2 ,$$

which may be written

$$a_{11}^{\nu_1} = \sum_{\nu_\ell \nu_m} \int_{-\infty}^{\infty} L_{k_1 k_\ell k_m}^{\nu_1 \nu_\ell \nu_m} a_{0\ell}^{\nu_\ell} a_{0m}^{\nu_m} \Delta_{\ell m,1} \, \delta_{\ell m,1} \, d\tilde{k}_\ell d\tilde{k}_m . \tag{17}$$

One is interested in the long time behavior of the order ε spectral cumulants formed from the perturbation expansion applied to the "generalized" amplitudes. In order to find the long time behavior of the $O(\varepsilon)$ component of the second order cumulants one examines

$$\langle a_{11}^{\nu_1} a_{02}^{\nu_2} + a_{01}^{\nu_1} a_{12}^{\nu_2} \rangle = \delta(\tilde{k}_1 + \tilde{k}_2) \overset{(2)}{Q}{}_1^{\nu_1 \nu_2}(\tilde{k}_1).$$

From equation (17)

$$\langle a_{11}^{\nu_1} a_{02}^{\nu_2} + a_{01}^{\nu_1} a_{12}^{\nu_2} \rangle = \overset{12}{P} \sum_{\nu_\ell \nu_m} \int_{-\infty}^{\infty} L_{k_1 k_\ell k_m}^{\nu_1 \nu_\ell \nu_m} \langle a_{0\ell}^{\nu_\ell} a_{0m}^{\nu_m} a_{02}^{\nu_2} \rangle \Delta_{em,1}\, \delta_{em,1}\, d\tilde{k}_\ell\, d\tilde{k}_m .$$

$\overset{12}{P}$ means the cyclic permutation over 1 and 2 .

$$\langle a_{0\ell}^{\nu_\ell} a_{0m}^{\nu_m} a_{02}^{\nu_2} \rangle = \delta(\tilde{k}_\ell + \tilde{k}_m + \tilde{k}_2)\, \overset{(3)}{Q}{}_0^{\nu_\ell \nu_m \nu_2}(k_\ell, k_m),$$

and one uses the fact that

$$\delta(\tilde{k}_\ell + \tilde{k}_m + \tilde{k}_2)\,\delta(\tilde{k}_\ell + \tilde{k}_m - \tilde{k}_1) = \delta(\tilde{k}_1 + \tilde{k}_2)\,\delta(\tilde{k}_\ell + \tilde{k}_m - \tilde{k}_1),$$

$$\overset{12}{P} \langle a_{11}^{\nu_1} a_{02}^{\nu_2} \rangle = \overset{12}{P}\, \delta(\tilde{k}_1 + \tilde{k}_2) \sum_{\nu_\ell \nu_m} \int_{-\infty}^{\infty} L_{k_1 k_\ell k_m}^{\nu_1 \nu_\ell \nu_m}\, \overset{(3)}{Q}{}_0^{\nu_\ell \nu_m \nu_2}(k_\ell, k_m)$$

$$\Delta(\nu_\ell \omega_\ell + \nu_m \omega_m - \nu_1 \omega_1)\, \delta(\tilde{k}_\ell + \tilde{k}_m - \tilde{k}_1)\, d\tilde{k}_\ell\, d\tilde{k}_m .$$

$\overset{(3)}{Q}{}_0^{\nu_\ell \nu_m \nu_2}(k_\ell, k_m)$ is independent of t (it will, in fact, depend on the higher time scales T_2, T_4, ...). Thus taking the limit $t \to \infty$, T_1 fixed, one has a limit of the type

$$\lim_{t \to \infty} \int_{-\infty}^{\infty} f(\omega)\, \Delta(\omega)\, d\omega$$

$$= \pi f(0) + i P \int_{-\infty}^{\infty} \frac{f(\omega)}{\omega}\, d\omega,$$

where P represents the Cauchy principal value of the integral. This can be written

$$= \int_{-\infty}^{\infty} f(\omega) \left(\pi \delta(\omega) + \frac{iP}{\omega} \right) d\omega,$$

$\delta(\omega)$ being the Dirac delta function. For a detailed proof, see Appendix II, page 158. Schematically therefore,

$$\lim_{t \to \infty} \Delta(\omega) = \pi \delta(\omega) + \frac{iP}{\omega}.$$

Hence one finds

$$\lim_{t \to \infty} \overset{12}{\rho} \langle \overset{\lambda_1}{a}_{11} \overset{\lambda_2}{a}_{02} \rangle = \overset{12}{\rho} \delta(\tilde{k}_1 + \tilde{k}_2) \sum_{\lambda_\ell \lambda_m} \int_{-\infty}^{\infty} L^{\lambda_1 \lambda_\ell \lambda_m}_{k_1 k_\ell k_m} \overset{(3)}{Q}{}_0^{\lambda_\ell \lambda_m \lambda_2}(k_\ell, k_m)$$

$$\left(\pi \delta(\lambda_\ell \omega_\ell + \lambda_m \omega_m - \lambda_1 \omega_1) + \frac{iP}{\lambda_\ell \omega_\ell + \lambda_m \omega_m - \lambda_1 \omega_1} \right)$$

$$\delta(\tilde{k}_\ell + \tilde{k}_m - \tilde{k}_1)\, d\tilde{k}_\ell\, d\tilde{k}_m,$$

and is therefore of order one. To obtain this result the assumption as to the smooth behavior of $\overset{(3)}{Q}{}_0^{\lambda_\ell \lambda_m \lambda_2}(k_\ell, k_m)$ has been made. This assumption will be shown to be consistent in the following analysis. It is then a matter of uniqueness. Namely if one finds a consistent continuous solution for the initial value problem with well behaved initial values, then one can say that the consistent solution which is reached, is in fact, the solution.

Examining the order ε component of the third order cumulant, one has

$$\overset{123}{P}\langle a_{11}^{\nu_1} a_{02}^{\nu_2} a_{03}^{\nu_3}\rangle = \overset{123}{P}\sum_{\nu_\ell \nu_m}\int_{-\infty}^{\infty} L_{k_1 k_\ell k_m}^{\nu_1 \nu_\ell \nu_m}\langle a_{0\ell}^{\nu_\ell} a_{0m}^{\nu_m} a_{02}^{\nu_2} a_{03}^{\nu_3}\rangle$$

$$\Delta_{\ell m,1}\,\delta_{\ell m,1}\,d\tilde{k}_\ell\,d\tilde{k}_m.$$

The mean value $\langle a_{0\ell}^{\nu_\ell} a_{0m}^{\nu_m} a_{02}^{\nu_2} a_{03}^{\nu_3}\rangle$ decomposes into

$$\delta_{\ell m 23}\overset{(4)}{\phi}{}_0^{\nu_\ell \nu_m \nu_2 \nu_3}(k_\ell, k_m, k_2) + \delta_{\ell m}\delta_{23}\overset{(2)}{\phi}{}_0^{\nu_\ell \nu_m}(k_\ell)\overset{(2)}{\phi}{}_0^{\nu_2 \nu_3}(k_2) + 2\delta_{\ell 2}\delta_{m3}\overset{(2)}{\phi}{}_0^{\nu_\ell \nu_2}(k_\ell)\overset{(2)}{\phi}{}_0^{\nu_m \nu_3}(k_m).$$

Again the fourth order cumulant term is an order one term in long time; the term $\delta(k_\ell+k_m)\,\delta(k_2+k_3)$ $\overset{(2)}{\phi}{}_0^{\nu_\ell \nu_m}(k_\ell)\ \overset{(2)}{\phi}{}_0^{\nu_2 \nu_3}(k_2)$ vanishes by reason of the mean value property: namely $\delta(\tilde{k}_\ell+\tilde{k}_m)$ reacts with $\delta(\tilde{k}_\ell+\tilde{k}_m-\tilde{k}_1)$ to give $\delta(\tilde{k}_1)$. The third term is in fact the interesting one, for here the above limiting process does not apply. Integrating over $\tilde{k}_\ell$ and $\tilde{k}_m$ one finds the contribution from this term in $\overset{123}{P}\langle a_{11}^{\nu_1} a_{02}^{\nu_2} a_{03}^{\nu_3}\rangle$ is

$$2\overset{123}{P}\,\delta(\tilde{k}_1+\tilde{k}_2+\tilde{k}_3)\sum_{\nu_\ell \nu_m} L_{k_1 k_\ell k_m}^{s_1 s_\ell s_m}\overset{(2)}{\phi}{}_0^{\nu_2 \nu_\ell}(k_2)\ \overset{(2)}{\phi}{}_0^{\nu_3 \nu_m}(k_3)\ \Delta(\nu_\ell\omega_2+\nu_m\omega_3-\nu_1\omega_1).$$

The corresponding third order cumulant is physical space is

$$\overset{(3)}{R}_1(\tilde{p}_1, \tilde{p}_2) = \sum_{\nu_1\nu_2\nu_3}\int_{-\infty}^{\infty}\overset{(3)}{\phi}{}_1^{\nu_1 \nu_2 \nu_3}(k_1, k_2)\, e^{-i\tilde{k}_1\cdot\tilde{p}_1 - i\tilde{k}_2\cdot\tilde{p}_2}\, e^{i(s_1\omega_1+s_2\omega_2+s_3\omega_{12})t}\, d\tilde{k}_1 d\tilde{k}_2$$

Since $\overset{(3)}{\phi}{}_1^{\nu_1 \nu_2 \nu_3}(\tilde{k}_1, \tilde{k}_2)\Big|_{\text{3rd part}}$

$$= 2\overset{123}{P} \sum_{\nu_\ell \nu_m} \alpha^{\nu_1 \; \nu_\ell \; \nu_m}_{k_1 \; -k_2 \; +k_1+k_2} \overset{(2)}{Q}{}^{\nu_2 \nu_\ell}_0(k_2) \overset{(2)}{Q}{}^{\nu_3 \nu_m}_0(-\tilde{k}_1-\tilde{k}_2) \, \Delta(\nu_\ell \omega_2 + \nu_m \omega_{12} - \nu_1 \omega_1),$$

if $\nu_\ell = -\nu_2$ and $\nu_m = -\nu_3$ one has that the time "t" dependent part of $\overset{(3)}{Q}{}^{\nu_1 \nu_2 \nu_3}_1(k_1, k_2)$ is, in this case,

$$2\overset{123}{P} \sum_{\substack{\nu_\ell = -\nu_2 \\ \nu_m = -\nu_3}} \alpha^{\nu_1 \; -\nu_2 \; -\nu_3}_{k_1 \; -k_2 \; k_1+k_2} \overset{(2)}{Q}{}^{\nu_2 -\nu_2}_0(k_2) \overset{(2)}{Q}{}^{\nu_3 -\nu_3}_0(k_3) \, \Delta(\nu_1 \omega_1 + \nu_2 \omega_2 + \nu_3 \omega_{12}).$$

The corresponding behavior of $\overset{(3)}{R}_1(\tilde{p}_1, \tilde{p}_2)$ is, {see also App. III},

$$\sum_{\nu_1 \nu_2 \nu_3} \int_{-\infty}^{\infty} 2\overset{123}{P} \alpha^{\nu_1 \; -\nu_2 \; -\nu_3}_{k_1 \; -k_2 \; k_1+k_2} \overset{(2)}{Q}{}^{\nu_2 -\nu_2}_0(\tilde{k}_2) \overset{(2)}{Q}{}^{\nu_3 -\nu_3}_0(-\tilde{k}_1-\tilde{k}_2) \, e^{-i\tilde{k}_1 \cdot \tilde{p}_1 - i\tilde{k}_2 \cdot \tilde{p}_2} \, \Delta(\nu_1 \omega_1 + \nu_2 \omega_2 + \nu_3 \omega_{12}) \, d\tilde{k}_1 d\tilde{k}_2,$$

which is zero at $t = 0$ and tends to a constant value as $t \to \infty$, namely,

$$\sum_{\nu_1 \nu_2 \nu_3} \int_{-\infty}^{\infty} 2\overset{123}{P} \alpha^{\nu_1 \; -\nu_2 \; -\nu_3}_{k_1 \; -k_2 \; k_1+k_3} \overset{(2)}{Q}{}^{\nu_2 -\nu_2}_0(\tilde{k}_2) \overset{(2)}{Q}{}^{\nu_3 -\nu_3}_0(-\tilde{k}_1-\tilde{k}_2) \, e^{-i\tilde{k}_1 \cdot \tilde{p}_1 - i\tilde{k}_2 \cdot \tilde{p}_2}$$

$$\left(\pi \, \delta(s_1 \omega_1 + s_2 \omega_2 + s_3 \omega_{12}) + \frac{i P}{\nu_1 \omega_1 + \nu_2 \omega_2 + \nu_3 \omega_{12}} \right) d\tilde{k}_1 d\tilde{k}_2,$$

for times $t \gg 1$ but less than that time scale on which $Q_0^{\nu-\nu}(k)$ (the energy) changes. One finds on examination of the remaining spectral cumulants that no $\overset{(r)}{Q}{}^{\nu_1 \cdots \nu_r}_1(\tilde{k}_1, \cdots \tilde{k}_{r-1})$ exhibits either a secular term or a quasi steady term. Therefore there is no need for a T_1 time scale in the problem. If the system had a non-zero mean then the spectral cumulants would exhibit a dependence on the T_1, time scale, as has been shown by Benney.

The ε^2 balance.

At this stage the order ε^2 components of the spectral cumulants are examined. Anticipating time growths one introduces the time scales

$$T_0 = t, \qquad T_2 = \varepsilon^2 t,$$

whereupon the ε^2 balance in the governing equation becomes

$$a_2^{\nu}(\tilde{k},t) = -t\,\frac{\partial a_0^{\nu}(\tilde{k})}{\partial T_2} + b_2^{\nu}(\tilde{k},t) + c_2^{\nu}(\tilde{k},t),$$

Or rewriting,

$$a_{21}^{\nu_1} = -t\,\frac{\partial a_{01}^{\nu_1}}{\partial T_2} + b_{21}^{\nu_1} + c_{21}^{\nu_1}. \qquad (18)$$

$$b_{21}^{\nu_1} = \sum_{\nu_\ell \nu_m \nu_n} \int_{-\infty}^{\infty} \mathcal{L}_{k_1\,k_\ell\,k_m\,k_n}^{\nu_1\,\nu_\ell\,\nu_m\,\nu_n}\, a_{0\ell}^{\nu_\ell}\, a_{0m}^{\nu_m}\, a_{0n}^{\nu_n}\, \Delta_{\ell m n,1}\, \delta_{\ell m n,1}\, d\tilde{k}_\ell\, d\tilde{k}_m\, d\tilde{k}_n.$$

$$c_{21}^{\nu_1} = 2\sum_{\nu_\ell \nu_m \nu_n \nu_p} \int_{-\infty}^{\infty} \mathcal{L}_{k_1\,k_\ell\,k_m}^{\nu_1\,\nu_\ell\,\nu_m}\, \mathcal{L}_{k_\ell\,k_n\,k_p}^{\nu_\ell\,\nu_n\,\nu_p}\, a_{0m}^{\nu_m}\, a_{0n}^{\nu_n}\, a_{0p}^{\nu_p}\, \frac{\Delta_{mnp,1} - \Delta_{\ell m,1}}{i\,w_{np,\ell}}\, \delta_{np,\ell}\, \delta_{\ell m,1}\, d\tilde{k}_\ell \cdots d\tilde{k}_p.$$

The general mean value expression expanded with the prescribed perturbation yields

$$\langle a_1^{\nu_1} \cdots a_r^{\nu_r} \rangle = \langle a_{01}^{\nu_1} \cdots a_{0r}^{\nu_r} \rangle + \varepsilon \overset{1\cdots r}{P} \langle a_{11}^{\nu_1} \cdots a_{0r}^{\nu_r} \rangle + \varepsilon^2 \Big[\overset{1\cdots r}{P} \langle a_{21}^{\nu_1} \cdots a_{0r}^{\nu_r} \rangle + P \langle a_{11}^{\nu_1} a_{12}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle \Big] + O(\varepsilon^3).$$

Lemma.

To find the behavior in time of the spectral cumulant $\overset{(r)}{Q}_2(\tilde{k}_1, \cdots \tilde{k}_{r-1})^{\nu_1 \cdots \nu_r}$ the following lemmas are used.

(a) $$\langle a_{21}^{\nu_1} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle \sim -t \left\langle \frac{\partial a_{01}^{\nu_1}}{\partial T_2} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \right\rangle + t \langle a_{01}^{\nu_1} \cdots a_{0r}^{\nu_r} \rangle F_{k_1}^{s_1} + O(1),$$

(b)

$$\langle a_{11}^{\nu_1} a_{12}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle \sim t \langle a_{03}^{\nu_3} \cdots a_{0r}^{\nu_r} \rangle \delta(\tilde{k}_1 + \tilde{k}_2) \delta_{-\nu_1}^{\nu_2} G_{k_1}^{s_1} + O(1),$$

$$F_{k_1}^{s_1} = \sum_{\nu_m} \int_{-\infty}^{\infty} \Big\{ 3 \, \mathcal{L}_{k_1 k_m -k_m k_1}^{\nu_1 \nu_m -\nu_m \nu_1} + 4 \Big(\sum_{\nu_e} \int_{-\infty}^{\infty} \mathcal{L}_{k_1 k_e k_m}^{\nu_1 \nu_e \nu_m} \mathcal{L}_{k_e -k_m k_1}^{\nu_e -\nu_m \nu_1} \Big(\pi \delta_{em,1}^{\nu} + \frac{iP}{W_{em,1}} \Big) \delta_{em,1} \, d\tilde{k}_e \Big) \Big\} \overset{(2)}{Q}_0^{\nu_m -\nu_m}(\tilde{k}_m) \, d\tilde{k}_m,$$

$$G_{k_1}^{s_1} = 4\pi \sum_{\nu_e \nu_m} \int_{-\infty}^{\infty} \mathcal{L}_{k_1 k_e k_m}^{\nu_1 \nu_e \nu_m} \mathcal{L}_{-k_1 -k_e -k_m}^{-\nu_1 -\nu_e -\nu_m} \overset{(2)}{Q}_0^{\nu_e -\nu_e}(k_e) \overset{(2)}{Q}_0^{\nu_m -\nu_m}(k_m) \, \delta_{em,1}^{\nu} \delta_{em,1} \, d\tilde{k}_e \, d\tilde{k}_m .$$

$\delta_{-\nu_1}^{\nu_2}$ is the Kronecker delta function, namely,

$$\delta_{-\nu_1}^{\nu_2} = \begin{cases} 1, & \nu_2 = -\nu_1 \\ 0, & \nu_2 \neq -\nu_1 . \end{cases}$$

Proof of (a).

Consider the mean value,

$$\langle b_{21}^{\nu_1} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle = \sum_{\nu_e \nu_m \nu_n} \int_{-\infty}^{\infty} \mathcal{L}_{k_1 k_e k_m k_n}^{\nu_1 \nu_e \nu_m \nu_n} \langle a_{0e}^{\nu_e} a_{0m}^{\nu_m} a_{0n}^{\nu_n} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle \, \Delta_{emn,1} \delta_{emn,1} \, d\tilde{k}_e \, d\tilde{k}_m \, d\tilde{k}_n .$$

The only way a "t" growth can occur is when $\langle a_{0\ell}^{\nu_\ell} a_{0m}^{\nu_m} a_{0n}^{\nu_n} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle$ is decomposed into $\delta(k_\ell + k_m) \, \phi_0^{(2)\nu_\ell \nu_m}(k_\ell)$ X Any possible decompositions of $\langle a_{0n}^{\nu_n} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle$. A "t" growth will only occur when

$$\nu_\ell \omega_\ell + \nu_m \omega_m + \nu_n \omega_n - \nu_1 \omega_1 \equiv 0.$$

On account of the symmetry in (ℓ,m,n) one may write

$$\langle b_{21}^{\nu_1} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle = 3 \sum_{\nu_\ell \nu_m \nu_n} \int_{-\infty}^{\infty} L_{k_1 k_\ell k_m k_n}^{\nu_1 \nu_\ell \nu_m \nu_n} \, \delta(\tilde{k}_\ell + \tilde{k}_m) \, \phi_0^{(2)\nu_\ell \nu_m}(k_\ell)$$

$$\times \langle a_{0n}^{\nu_n} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle \, \Delta_{\ell m n,1} \, \delta_{\ell m n,1} \, d\tilde{k}_\ell \, d\tilde{k}_m \, d\tilde{k}_n$$

$$+ O(1).$$

But $\delta(\tilde{k}_\ell + \tilde{k}_m) \, \delta(\tilde{k}_\ell + \tilde{k}_m + \tilde{k}_n - \tilde{k}_1) = \delta(\tilde{k}_\ell + \tilde{k}_m) \, \delta(\tilde{k}_n - \tilde{k}_1)$,

and the $\delta(\tilde{k}_n + \tilde{k}_2 + \ldots + \tilde{k}_r)$, which is a common "factor" in the expansion of $\langle a_{0n}^{\nu_n} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle$ into its spectral cumulants, becomes $\delta(k_1 + k_2 + \ldots + k_r)$. Thus, integrating over $\tilde{k}_m$ and $\tilde{k}_n$ one obtains

$$\langle b_{21}^{\nu_1} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle = 3 \sum_{\nu_\ell \nu_m \nu_n} \int_{-\infty}^{\infty} L_{k_1 k_\ell k_m k_n}^{\nu_1 \nu_\ell \nu_m \nu_n} \, \phi_0^{(2)\nu_\ell \nu_m}(\tilde{k}_\ell)$$

$$\langle a_{01}^{\nu_n} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle \, \Delta\big((\nu_\ell + \nu_m)\omega_\ell + (\nu_n - \nu_1)\omega_1\big) \, d\tilde{k}_\ell$$

$$+ O(1).$$

A "t" growth occurs when $\nu_n = \nu_1$, $\nu_m = -\nu_\ell$, and therefore the expression

$$\langle b_{21}^{\nu_1} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle \sim 3t \langle a_{01}^{\nu_1} \cdots a_{0r}^{\nu_r} \rangle \sum_{\nu_\ell} \int_{-\infty}^{\infty} L_{k_1 k_\ell - k_\ell k_1}^{\nu_1 \nu_\ell - \nu_\ell \nu_1} \Phi_0^{(2) \nu_\ell - \nu_\ell}(k_\ell) \, dk_\ell + O(1).$$

Considering the mean value,

$$\langle c_{21}^{\nu_1} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle = 2 \sum_{\nu_\ell \nu_m \nu_n \nu_p} \int_{-\infty}^{\infty} L_{k_1 k_\ell k_m}^{\nu_1 \nu_\ell \nu_m} L_{k_\ell k_n k_p}^{\nu_\ell \nu_n \nu_p} \langle a_{0m}^{\nu_m} a_{0n}^{\nu_n} a_{0p}^{\nu_p} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle \frac{\Delta_{mnp,1} - \Delta_{\ell m,1}}{i W_{np,\ell}}$$

$$\delta_{np,\ell} \, \delta_{\ell m,1} \, dk_\ell \, dk_m \, dk_n \, dk_p .$$

The only way a "t" growth can occur in this expansion is when

$$\nu_m \omega_m + \nu_n \omega_n + \nu_p \omega_p - \nu_1 \omega_1 \equiv 0 .$$

From Appendix II,

$$\lim_{t \to \infty} \int_{-\infty}^{\infty} f(\omega) \frac{\Delta(0) - \Delta(\omega)}{-i\omega} \, d\omega$$

$$= t \int_{-\infty}^{\infty} f(\omega) \left[\pi \delta(\omega) + \frac{iP}{\omega} \right] d\omega + O(1).$$

Schematically therefore, one may write

$$\frac{\Delta(0) - \Delta(\omega)}{-i\omega} \sim t \left(\pi \delta(\omega) + \frac{iP}{\omega} \right) + O(1).$$

This can only occur on decompositions of the type

$$\delta(\tilde{k}_m + \tilde{k}_p) \, \Phi_0^{(2) \nu_m \nu_p}(k_m) \langle a_{0n}^{\nu_n} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle ,$$

$$\delta(\tilde{k}_m + \tilde{k}_n)\ \overset{(2)}{\phi}{}_0^{\, \gamma_m \gamma_n}(\tilde{k}_m)\ \langle G_{0p}^{\gamma_p} G_{02}^{\gamma_2} \cdots G_{0r}^{\gamma_r} \rangle,$$

$$\delta(\tilde{k}_n + \tilde{k}_p)\ \overset{(2)}{\phi}{}_0^{\, \gamma_n \gamma_p}(\tilde{k}_n)\ \langle G_{0m}^{\gamma_m} G_{02}^{\gamma_2} \cdots G_{0r}^{\gamma_r} \rangle.$$

The last term gives zero as $\delta(\tilde{k}_n + \tilde{k}_p)$ reacts with $\delta(\tilde{k}_n + \tilde{k}_p - \tilde{k}_e)$ to give $\delta(k_e)$, and therefore vanishes by reason of the zero mean value property. On account of symmetry in n and p one may write the expression,

$$\langle C_{21}^{\gamma_1} G_{02}^{\gamma_2} \cdots G_{0r}^{\gamma_r} \rangle = 4 \sum_{\gamma_e \gamma_m \gamma_n \gamma_p} \int_{-\infty}^{\infty} \mathcal{L}_{k_1 k_e k_m}^{\gamma_1 \gamma_e \gamma_m}\ \mathcal{L}_{k_e k_n k_p}^{\gamma_e \gamma_n \gamma_p}$$

$$\delta_{mn}\ \overset{(2)}{\phi}{}_0^{\, \gamma_m \gamma_n}(k_m)\ \langle G_{0p}^{\gamma_p} G_{02}^{\gamma_2} \cdots G_{0r}^{\gamma_r} \rangle$$

$$\frac{\Delta_{mnp.1} - \Delta_{em.1}}{i W_{np.e}}\ \delta_{np.e}\ \delta_{mnp.1}\ d\tilde{k}_e \cdots d\tilde{k}_p .$$

$$+\ O(1).$$

Also, as $\delta_{mn}\ \delta_{np.e}\ \delta_{mnp.1} = \delta_{mn}\ \delta_{p.1}\ \delta_{em.1}$ integrating over $\tilde{k}_n$ and $\tilde{k}_p$, and setting $\gamma_n = -\gamma_m$, $\gamma_p = \gamma_1$, one obtains

$$\langle C_{21}^{\gamma_1} G_{02}^{\gamma_2} \cdots G_{0r}^{\gamma_r} \rangle = 4 \sum_{\gamma_e \gamma_m} \int_{-\infty}^{\infty} \mathcal{L}_{k_1 k_e k_m}^{\gamma_1 \gamma_e \gamma_m}\ \mathcal{L}_{k_e\, -k_m\, k_1}^{\gamma_e\, -\gamma_m\, \gamma_1}$$

$$\overset{(2)}{\phi}{}_0^{\, \gamma_m - \gamma_m}(k_m)\ \langle G_{01}^{\gamma} \cdots G_{0r}^{\gamma_r} \rangle\ \frac{\Delta(0) - \Delta_{em.1}}{-i W_{em.1}}$$

$$\delta_{em.1}\ d\tilde{k}_e\, d\tilde{k}_m\ +\ O(1).$$

Taking the limit $t \to \infty$, with T_2 fixed,

$$\langle G_{21}^{\nu_1} G_{02}^{\nu_2} \cdots G_{0r}^{\nu_r} \rangle \sim 4t \langle G_{01}^{\nu_1} \cdots G_{0r}^{\nu_r} \rangle \times \sum_{\nu_l \nu_m} \int_{-\infty}^{\infty} \mathcal{L}_{k_1 k_l k_m}^{\nu_1 \nu_l \nu_m} \mathcal{L}_{k_l - k_m\, k_1}^{\nu_l - \nu_m\, \nu_1} Q_0^{(2)\, \nu_m - \nu_m}(k_m) \left(\pi \delta_{elm,1} + \frac{iP}{W_{elm,1}} \right) \delta_{elm,1}\, dk_l\, dk_m + O(1).$$

Hence one finds that

$$\langle G_{21}^{\nu_1} G_{02}^{\nu_2} \cdots G_{0r}^{\nu_r} \rangle \sim -t \left\langle \frac{\partial G_{01}^{\nu_1}}{\partial T_2} G_{02}^{\nu_2} \cdots G_{0r}^{\nu_r} \right\rangle + t \langle G_{01}^{\nu_1} \cdots G_{0r}^{\nu_r} \rangle F_{k_1}^{s_1} + O(1).$$

Proof of (b).

Consider the expression for the following mean value,

$$\langle G_{11}^{\nu_1} G_{12}^{\nu_2} G_{03}^{\nu_3} \cdots G_{0r}^{\nu_r} \rangle = \sum_{\nu_l \nu_m \nu_n \nu_p} \int_{-\infty}^{\infty} \mathcal{L}_{k_1 k_l k_m}^{\nu_1 \nu_l \nu_m} \mathcal{L}_{k_2 k_n k_p}^{\nu_2 \nu_n \nu_p} \langle G_{0l}^{\nu_l} G_{0m}^{\nu_m} G_{0n}^{\nu_n} G_{0p}^{\nu_p} G_{03}^{\nu_3} \cdots G_{0r}^{\nu_r} \rangle \Delta_{elm,1} \Delta_{np,2}\, \delta_{elm,1}\, \delta_{np,2}\, dk_l \cdots dk_p.$$

Since $\nu_\ell \omega_\ell + \nu_m \omega_m - \nu_1 \omega_1$ is never zero identically the only possible "t" growth occurs when

$$\nu_\ell \omega_\ell + \nu_m \omega_m - \nu_1 \omega_1 \equiv - \nu_n \omega_n - \nu_p \omega_p + \nu_2 \omega_2$$

for then (see Appendix II) one has

$$\Delta(\omega)\ \Delta(-\omega) \sim 2\pi t\, \delta(\omega) + O(1).$$

The only decomposition which exhibits this behavior is of the form $\delta_{\ell n}\, \delta_{mp}\, \overset{(2)}{Q}{}_0^{\nu_\ell \nu_n}(k_\ell)\, \overset{(2)}{Q}{}_0^{\nu_m \nu_p}(k_m)$ x Any remaining decomposition or decompositions of the mean value $\langle a_{03}^{\nu_3} \cdots a_{0r}^{\nu_r} \rangle$. Therefore when one integrates over $\tilde{k}_n$ and $\tilde{k}_p$ one can reach the above situation only when $s_2 = -s_1$. Hence (since there are two possible terms $\delta_{\ell n}\, \delta_{mp}$ and $\delta_{\ell p}\, \delta_{mn}$) one has the result

$$\langle a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} \cdots a_{0r}^{\nu_r} \rangle \sim 4\pi t\, \delta(\tilde{k}_1 + \tilde{k}_2)\, \delta_{-\nu_1}^{\nu_2} \langle a_{03}^{\nu_3} \cdots a_{0r}^{\nu_r} \rangle$$

$$\times \sum_{\ell \nu_m} \int_{-\infty}^{\infty} L_{k_1\, k_\ell\, k_m}^{\nu_1\, \nu_\ell\, \nu_m}\, L_{-k_1\, -k_\ell\, -k_m}^{-\nu_1\, -\nu_\ell\, -\nu_m}\, \overset{(2)}{Q}{}_0^{\nu_\ell -\nu_\ell}(k_\ell)$$

$$\overset{(2)}{Q}{}_0^{\nu_m -\nu_m}(k_m)\, \delta_{\ell m,1}^{\nu}\, \delta_{\ell m,1}\, d\tilde{k}_\ell\, d\tilde{k}_m + O(1),$$

which completes the proofs of the above lemmas.

The second order cumulant.

One first examines the long time behavior of the second order cumulant

$$\delta(\breve{k}_1+\breve{k}_2)\,\overset{(2)}{\phi}{}_2^{\nu_1\nu_2}(\breve{k}_1) = \langle a_{21}^{\nu_1} a_{02}^{\nu_2} + a_{11}^{\nu_1} a_{12}^{\nu_2} + a_{01}^{\nu_1} a_{22}^{\nu_2} \rangle .$$

Using the lemmas, one has that

$$\delta(\breve{k}_1+\breve{k}_2)\,\overset{(2)}{\phi}{}_2^{\nu_1\nu_2}(\breve{k}_1) \sim -t\,\langle \frac{\partial a_{01}^{\nu_1}}{\partial T_2} a_{02}^{\nu_2} \rangle + t\,\langle a_{01}^{\nu_1} a_{02}^{\nu_2} \rangle F_{k_1}^{s_1}$$
$$-t\,\langle a_{01}^{\nu_1} \frac{\partial a_{02}^{\nu_2}}{\partial T_2} \rangle + t\,\langle a_{01}^{\nu_1} a_{02}^{\nu_2} \rangle F_{k_2}^{s_2}$$
$$+\; t\,\delta(\breve{k}_1+\breve{k}_2)\,\delta_{-\nu_1}^{\nu_2}\, G_{k_1}^{s_1} + O(1).$$

Therefore one obtains

$$\overset{(2)}{\phi}{}_2^{\nu_1\nu_2}(\breve{k}_1) \sim t\left\{ -\frac{\partial \overset{(2)}{\phi}{}_0^{\nu_1\nu_2}(k_1)}{\partial T_2} + \overset{(2)}{\phi}{}_0^{\nu_1\nu_2}(k_1)\left(F_{k_1}^{s_1} + F_{-k_1}^{s_2}\right) + \delta_{-s_1}^{s_2}\, G_{k_1}^{s_1}\right\}$$
$$+\, O(1).$$

When $t = O(\varepsilon^{-2})$, $\overset{(2)}{\phi}{}_2^{\nu_1\nu_2}(\breve{k}_1)$ will be the same order as $\overset{(2)}{\phi}{}_0^{\nu_1\nu_2}(\breve{k}_1)$ unless

$$\frac{\partial \overset{(2)}{\phi}{}_0^{\nu_1\nu_2}(\breve{k}_1)}{\partial T_2} = \overset{(2)}{\phi}{}_0^{\nu_1\nu_2}(\breve{k}_1)\left(F_{k_1}^{s_1} + F_{-k_1}^{s_2}\right) + \delta_{-\nu_1}^{\nu_2}\, G_{k_1}^{s_1}. \quad (19)$$

The energy density is given by the above cumulant when $\nu_2 = -\nu_1$,

$$\frac{\partial \overset{(2)}{\phi}{}_0^{\nu_1 -\nu_1}(\breve{k}_1)}{\partial T_2} = \overset{(2)}{\phi}{}_0^{\nu_1 -\nu_1}(\breve{k}_1)\left(F_{k_1}^{-s_1} + F_{-k_1}^{-s_1}\right) + G_{k_1}^{s_1}.$$

From the properties of the $\mathcal{L}_{k k_1 k_2}^{s s_1 s_2}$, $\mathcal{L}_{k k_1 k_2 k_3}^{s s_1 s_2 s_3}$, previously mentioned, the principal part terms cancel each other when one adds $F_{k_1}^{s_1}$ and $F_{-k_1}^{-s_1}$. This corresponds to the fact that modal interactions between nonresonant waves change only the phase and not the amplitude of the waves. Thus one obtains the result,

$$\frac{\partial \overset{(2)}{\Phi}{}_0^{s-s}(k)}{\partial T_2} = 4\pi \sum_{s_\ell s_m} \int_{-\infty}^{\infty} \mathcal{L}_{k k_\ell k_m}^{s s_\ell s_m} \mathcal{L}_{k k_\ell k_m}^{* s s_\ell s_m} \overset{(2)}{\Phi}{}_0^{s_\ell - s_\ell}(k_\ell)$$

$$\overset{(2)}{\Phi}{}_0^{s_m - s_m}(k_m)\ \delta(s_\ell\omega_\ell + s_m\omega_m - s\omega)\ \delta(\tilde{k}_\ell + \tilde{k}_m - \tilde{k})\, d\tilde{k}_\ell\, d\tilde{k}_m$$

$$+ 8\pi\, \overset{(2)}{\Phi}{}_0^{s-s}(k) \sum_{s_\ell s_m} \int_{-\infty}^{\infty} \mathcal{L}_{k k_\ell k_m}^{s s_\ell s_m} \mathcal{L}_{k_\ell -k_m k}^{s_\ell -s_m s}$$

$$\overset{(2)}{\Phi}{}_0^{s_m - s_m}(k_m)\ \delta(s_\ell\omega_\ell + s_m\omega_m - s\omega)\ \delta(\tilde{k}_\ell + \tilde{k}_m - \tilde{k})\, d\tilde{k}_\ell\, d\tilde{k}_m. \tag{20}$$

This result was obtained by Benney and Saffman [2]. However if triad resonances are forbidden, (suppose one is looking at gravity waves only and neglecting surface tension), then it is clear that

$$\frac{\partial \overset{(2)}{\Phi}{}_0^{s-s}(k)}{\partial T_2} = 0. \tag{21}$$

It may be noted that equation (20) leads to a conservation of energy in the sense that

$$\frac{\partial}{\partial T_2} \sum_{s} \int_{-\infty}^{\infty} \nu^2(\tilde{k})\, \overset{(2)}{\Phi}{}_0^{s-s}(\tilde{k})\, d\tilde{k} = 0,$$

where $V^2(\tilde{k}) = g + S/\rho\,|\tilde{k}|^2$. In order to see this, the following lemma is proved.

<u>Lemma.</u>

Given that $\tilde{k}_1 + \tilde{k}_2 = \tilde{k}$

and $\sigma_1\omega_1 + \sigma_2\omega_2 = \sigma\omega$

then
$$L^{\sigma_1\ \sigma\ -\sigma_2}_{k_1\ k\ -k_2} = \frac{V^2}{\sigma\omega}\ \frac{\sigma_1\omega_1}{V_1^2}\ L^{\sigma\ \sigma_1\ \sigma_2}_{k\ k_1\ k_2}.$$

<u>Proof:</u> From the formulae on page 19 one has

$$\frac{4}{i}L^{\sigma_1\sigma-\sigma_2}_{k_1,k-k_2} = \frac{V_2^2\left(|k_2|^2-\tilde{k}\cdot\tilde{k}_2\right)}{-\sigma_2\omega_2} + \frac{V^2\left(|k|^2-\tilde{k}\cdot\tilde{k}_2\right)}{\sigma\omega} - \frac{\sigma_1\omega_1}{V_1^2}\left(\omega^2+\omega_2^2-\sigma\omega\,\sigma_2\omega_2-\tilde{k}\cdot\tilde{k}_2\,\frac{V^2V_2^2}{\sigma\omega\,\sigma_2\omega_2}\right)$$

Using the facts that $\tilde{k}_1 + \tilde{k}_2 = \tilde{k}$ and $\sigma_1\omega_1+\sigma_2\omega_2 = \sigma\omega$ and rearranging the above expression, one has

$$\frac{4}{i}L^{\sigma_1\sigma-\sigma_2}_{k_1,k-k_2} = \frac{\sigma_1\omega_1}{V_1^2}\ \frac{V^2}{\sigma\omega}\left\{\frac{\sigma\omega\,V_1^2}{\sigma_1\omega_1\,V^2}\ \frac{V_2^2}{\sigma_2\omega_2}\ \tilde{k}_2\cdot\tilde{k}_1 + \frac{V_1^2}{\sigma_1\omega_1}\ \tilde{k}_1\cdot\tilde{k}\right.$$

$$-\frac{\sigma\omega}{V^2}\left(\omega_1^2+\omega_2^2+2\sigma_1\omega_1\sigma_2\omega_2+\omega_2^2-\sigma_1\omega_1\sigma_2\omega_2-\omega_2^2\right)$$

$$\left.+\frac{\sigma\omega}{V^2}\ \tilde{k}\cdot\tilde{k}_2\ \frac{V^2V_2^2}{\sigma\omega\,\sigma_2\omega_2}\right\},$$

$$= \frac{\sigma_1\omega_1}{\sigma\omega}\ \frac{V^2}{V_1^2}\left\{\frac{V_2^2}{\sigma_2\omega_2}\left(|\tilde{k}_2|^2+\tilde{k}_1\cdot\tilde{k}_2\right)+\frac{V_1^2}{\sigma_1\omega_1}\left(|\tilde{k}_1|^2+\tilde{k}_1\cdot\tilde{k}_2\right)\right.$$

$$\left.-\frac{\sigma\omega}{V^2}\left(\omega_1^2+\omega_2^2+\sigma_1\omega_1-\tilde{k}_1\cdot\tilde{k}_2\ \frac{V_1^2V_2^2}{\sigma_1\omega_1\,\sigma_2\omega_2}\right)\right\}$$

$$= \frac{\sigma_1\omega_1}{\sigma\omega}\ \frac{V^2}{V_1^2}\ \frac{4}{i}\ L^{\sigma\ \sigma_1\ \sigma_2}_{k\ k_1\ k_2}.$$

From equation (20), one has

$$\frac{\partial \overset{(2)}{\Phi}{}_0^{s,-s}(k)}{\partial T_2} = 4\pi \sum_{s_1 s_2} \int_{-\infty}^{\infty} \mathcal{L}_{k\, k_1\, k_2}^{s\, s_1\, s_2}\, \mathcal{L}_{-k\, -k_1\, -k_2}^{-s\, -s_1\, -s_2}\, \overset{(2)}{\Phi}{}_0^{s_1,-s_1}(k_1)\, \overset{(2)}{\Phi}{}_0^{s_2,-s_2}(k_2)$$

$$\delta(s_1\omega_1 + s_2\omega_2 - s\omega)\, \delta(\tilde{k}_1 + \tilde{k}_2 - \tilde{k})\, d\tilde{k}_1\, d\tilde{k}_2$$

$$+ 4\pi \sum_{s_1 s_2} \int_{-\infty}^{\infty} \mathcal{L}_{k\, k_1\, k_2}^{s\, s_1\, s_2}\, \mathcal{L}_{k_1\, k\, -k_2}^{s_1\, s\, -s_2}\, \overset{(2)}{\Phi}{}_0^{s,-s}(k)\, \overset{(2)}{\Phi}{}_0^{s_2,-s_2}(k_2)$$

$$\delta(s_1\omega_1 + s_2\omega_2 - s\omega)\, \delta(\tilde{k}_1 + \tilde{k}_2 - \tilde{k})\, d\tilde{k}_1\, d\tilde{k}_2$$

$$+ 4\pi \sum_{s_1 s_2} \int_{-\infty}^{\infty} \mathcal{L}_{k\, k_1\, k_2}^{s\, s_1\, s_2}\, \mathcal{L}_{k_2\, k\, -k_1}^{s_2\, s\, -s_1}\, \overset{(2)}{\Phi}{}_0^{s,-s}(k)\, \overset{(2)}{\Phi}{}_0^{s_1,-s_1}(k_1)$$

$$\delta(s_1\omega_1 + s_2\omega_2 - s\omega)\, \delta(\tilde{k}_1 + \tilde{k}_2 - \tilde{k})\, d\tilde{k}_1\, d\tilde{k}_2$$

Therefore one sees that

$$\frac{\partial}{\partial T_2} \sum_s \int_{-\infty}^{\infty} V^2\, \overset{(2)}{\Phi}{}_0^{s,-s}(\tilde{k})\, d\tilde{k}$$

$$= 4\pi \sum_{s s_1 s_2} \int_{-\infty}^{\infty} \mathcal{L}_{k\, k_1\, k_2}^{s\, s_1\, s_2}\, \mathcal{L}_{-k\, -k_1\, -k_2}^{-s\, -s_1\, -s_2}\, \overset{(2)}{\Phi}{}_0^{s_1,-s_1}(k_1)\, \overset{(2)}{\Phi}{}_0^{s_2,-s_2}(k_2)\, \delta_{12,0}^{s}\, \delta_{12,0}\, d\tilde{k}_1\, d\tilde{k}_2\, d\tilde{k}$$

$$+ 4\pi \sum_{s s_1 s_2} \int_{-\infty}^{\infty} \mathcal{L}_{k\, k_1\, k_2}^{s\, s_1\, s_2}\, \mathcal{L}_{k_1\, k\, -k_2}^{s_1\, s\, -s_2}\, \overset{(2)}{\Phi}{}_0^{s,-s}(k)\, \overset{(2)}{\Phi}{}_0^{s_2,-s_2}(k_2)\, \delta_{12,0}^{s}\, \delta_{12,0}\, d\tilde{k}_1\, d\tilde{k}_2\, d\tilde{k}$$

$$+ 4\pi \sum_{s s_1 s_2} \int_{-\infty}^{\infty} \mathcal{L}_{k\, k_1\, k_2}^{s\, s_1\, s_2}\, \mathcal{L}_{k_2\, k\, -k_1}^{s_2\, s\, -s_1}\, \overset{(2)}{\Phi}{}_0^{s,-s}(k)\, \overset{(2)}{\Phi}{}_0^{s_1,-s_1}(k_1)\, \delta_{12,0}^{s}\, \delta_{12,0}\, d\tilde{k}_1\, d\tilde{k}_2\, d\tilde{k}.$$

In the second integral change $\begin{cases} \nu \longleftrightarrow -\nu_1 \\ \breve{k} \longleftrightarrow -\breve{k}_1 \end{cases}$,

and in the third integral interchange $\begin{cases} \nu \longleftrightarrow -\nu_2 \\ \breve{k} \longleftrightarrow -\breve{k}_2 \end{cases}$.

$$\frac{\partial}{\partial T_2} \sum_{\nu} \int_{-\infty}^{\infty} V^2 \overset{(2)}{Q}{}_0^{\nu,-\nu}(\breve{k})\, d\breve{k}$$

$$= 4\pi \sum_{\nu\nu_1\nu_2} \int_{-\infty}^{\infty} \Big\{ V^2 L_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2} L_{-k\,-k_1\,-k_2}^{-\nu\,-\nu_1\,-\nu_2}$$

$$+ V_1^2 L_{-k_1\,-k\,k_2}^{-\nu_1\,-\nu\,\nu_2} L_{-k\,-k_1\,-k_2}^{-\nu\,-\nu_1\,-\nu_2} + V_2^2 L_{-k_2\,k_1\,-k}^{-\nu_2\,\nu_1\,-\nu}$$

$$L_{-k\,-k_1\,-k_2}^{-\nu\,-\nu_1\,-\nu_2} \Big\} \overset{(2)}{Q}{}_0^{\nu_1,-\nu_1}(k_1) \overset{(2)}{Q}{}_0^{\nu_2,-\nu_2}(k_2)\, \delta_{12,0}\, \delta_{12,0}\, d\breve{k}_1\, d\breve{k}_2\, d\breve{k}.$$

Using the properties of the coefficients, namely

$$L_{-k\,-k_1\,-k_2}^{-\nu\,-\nu_1\,-\nu_2} = -L_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2},$$

and $$V_1^2 L_{k_1\,k\,-k_2}^{\nu_1\,\nu\,-\nu_2} = \frac{V^2 \nu_1\omega_1}{\nu\omega} L_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2}$$

one obtains

$$\frac{\partial}{\partial T_2} \sum_{\nu} \int_{-\infty}^{\infty} V^2 \overset{(2)}{Q}{}_0^{\nu,-\nu}(k)\, d\breve{k}$$

$$= 4\pi \sum_{\nu\nu_1\nu_2} \int_{-\infty}^{\infty} \frac{V^2}{\nu\omega} L_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2} L_{-k\,-k_1\,-k_2}^{-\nu\,-\nu_1\,-\nu_2} (\nu\omega - \nu_1\omega_1 - \nu_2\omega_2)$$

$$\overset{(2)}{Q}{}_0^{\nu_1,-\nu_1}(k_1) \overset{(2)}{Q}{}_0^{\nu_2,-\nu_2}(k_2)\, \delta(s_1\omega_1 + s_2\omega_2 - s\omega)\, \delta(\breve{k}_1 + \breve{k}_2 - \breve{k})\, d\breve{k}_1\, d\breve{k}_2\, d\breve{k}$$

$$= 0.$$

Therefore $\sum_{s} \int_{-\infty}^{\infty} v^2(\tilde{k}) \; \phi_0^{(2)\,r-2}(\tilde{k}) \, d\tilde{k}$ is a constant with respect to T_2 .

The long time behavior of $\overset{(r)}{\mathcal{Q}}_2{}^{\nu_1 \cdots \nu_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1})$.

Using the lemmas (a) and (b), one can show that the elimination of secular terms from $\overset{(r)}{\mathcal{Q}}_2{}^{\nu_1 \cdots \nu_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1})$ leads to the following long time, (T_2), behavior of $\overset{(r)}{\mathcal{Q}}_0{}^{\nu_1 \cdots \nu_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1})$

$$\frac{\partial \overset{(r)}{\mathcal{Q}}_0{}^{\nu_1 \cdots \nu_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1})}{\partial T_2} = \overset{(r)}{\mathcal{Q}}_0{}^{\nu_1 \cdots \nu_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1}) \left\{ F_{k_1}^{s_1} + F_{k_2}^{s_2} + \cdots + F_{-k_1 \cdots -\tilde{k}_{r-1}}^{s_r} \right\}. \quad (22)$$

In order to obtain this result, some further notations have been adopted. The ε^2 coefficient, under the prescribed perturbation on $a^{\nu}(k)$, of the mean value quantity $\langle a_1^{\nu_1} \cdots a_r^{\nu_r} \rangle$ is $\overset{1 \cdots r}{P} \langle a_{21}^{\nu_1} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle$ $+ P \langle a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} \cdots a_{0r}^{\nu_r} \rangle$. The mean value expression

$$\langle a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} \cdots a_{0r}^{\nu_r} \rangle = \sum \left\{ a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} \cdots a_{0m_1}^{\nu_{m_1}} \right\} \left\{ a_{0m_1+1}^{\nu_{m_1+1}} \cdots a_{0m_2}^{\nu_{m_2}} \right\} \{ \ \} \cdots$$
$$+ \sum \left\{ a_{11}^{\nu_1} a_{03}^{\nu_3} \cdots \right\} \left\{ a_{12}^{\nu_2} a_{0m}^{\nu_m} \cdots \right\} \{ \ \} \cdots ,$$

where the curly brackets stand for the cumulants formed from the mean values. It will, first of all, be shown that

$$G_t \left\{ a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} \cdots a_{0r}^{\nu_r} \right\} = 0 , \qquad r > 2 ,$$

where $G_t \{ \ \}$ stands for the "t" growth of the quantity $\{ \ \}$ in long time. For example, if one considers the case when $r = 3$, one obtains

$$G_t \left\{ a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} \right\} = \delta(\tilde{k}_1 + \tilde{k}_2)\, \delta_{-\nu_1}^{\nu_2}\, G_{k_1}^{s_1} \langle a_{03}^{\nu_3} \rangle , \text{ from lemma (b)}$$
$$= 0 , \qquad \langle a_{03}^{\nu_3} \rangle = 0.$$

Next one considers the case when $r = 4$.

$$G_t \{ a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} a_{04}^{\nu_4} \} = G_t \langle a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} a_{04}^{\nu_4} \rangle$$

$$- G_t \langle a_{11}^{\nu_1} a_{12}^{\nu_2} \rangle \langle a_{03}^{\nu_3} a_{04}^{\nu_4} \rangle$$

$$- G_t \langle a_{11}^{\nu_1} a_{03}^{\nu_3} \rangle \langle a_{12}^{\nu_2} a_{04}^{\nu_4} \rangle$$

$$- G_t \langle a_{11}^{\nu_1} a_{04}^{\nu_4} \rangle \langle a_{12}^{\nu_2} a_{03}^{\nu_3} \rangle .$$

One knows $G_t \langle a_{11}^{\nu_1} a_{03}^{\nu_3} \rangle = 0$, since $\langle a_{11}^{\nu_1} a_{02}^{\nu_2} \rangle \sim O(1)$.
Therefore using lemma (b) on the first two terms,

$$G_t \{ a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} a_{04}^{\nu_4} \} = \delta(\check{k}_1 + \check{k}_2)\, \delta_{-\nu_1}^{\nu_2}\, G_{k_1}^{s_1} \langle a_{03}^{\nu_3} a_{04}^{\nu_4} \rangle$$

$$- \delta(\check{k}_1 + \check{k}_2)\, \delta_{-\nu_1}^{\nu_2}\, G_{k_1}^{s_1} \langle a_{03}^{\nu_3} a_{04}^{\nu_4} \rangle$$

$$= 0.$$

Let it be assumed that

$$G_t \{ a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} \cdots a_{0m}^{\nu_m} \} = 0, \qquad m = 3, \cdots (r-1). \quad \cdots (c)$$

Then consider

$$G_t \langle a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} \cdots a_{0r}^{\nu_r} \rangle = G_t \{ a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} \cdots a_{0r}^{\nu_r} \}$$

$$+ \; G_t \sum \{ a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} \cdots a_{0m_1}^{\nu_{m_1}} \} \{ \; \} \{ \; \} \cdots \{ \; \}$$

$$+ \; G_t \sum \{ a_{11}^{\nu_1} a_{03}^{\nu_3} \cdots \} \{ a_{12}^{\nu_2} a_{04}^{\nu_4} \cdots \} \cdots \{ \; \} .$$

The last term on the right hand side is zero since $G_t \{ a_{11}^{\nu_1} a_{02}^{\nu_2} \cdots a_{0m}^{\nu_m} \} = 0$. From hypothesis (c), there is only one set of terms coming from the second part on the right hand side which can contribute, and this is

$$\{ a_{11}^{\nu_1} a_{12}^{\nu_2} \} \times \text{any decomposition of } \langle a_{03}^{\nu_3} \cdots a_{0r}^{\nu_r} \rangle ,$$

Since using the hypothesis,

$$G_t \; \{ a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} \cdots a_{0m}^{\nu_m} \} = 0, \quad m \geq 3 .$$

Therefore one has

$$\begin{aligned} G_t \; \{ a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} \cdots a_{0r}^{\nu_r} \} &= G_t \langle a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} \cdots a_{0r}^{\nu_r} \rangle \\ &\quad - G_t \langle a_{11}^{\nu_1} a_{12}^{\nu_2} \rangle \langle a_{03}^{\nu_3} \cdots a_{0r}^{\nu_r} \rangle \\ &= \delta(\breve{k}_1 + \breve{k}_2) \, \delta_{-\nu_1}^{\nu_2} \, G_{k_1}^{s_1} \langle a_{03}^{\nu_3} \cdots a_{0r}^{\nu_r} \rangle \\ &\quad - \delta(\breve{k}_1 + \breve{k}_2) \, \delta_{-\nu_1}^{\nu_2} \, G_{k_1}^{s_1} \langle a_{03}^{\nu_3} \cdots a_{0r}^{\nu_r} \rangle \\ &= 0 . \end{aligned}$$

By induction, this result is true for all $r > 2$.

Therefore for $r > 2$, one obtains that

$$G_t\ \delta(\tilde{k}_1 + \cdots + \tilde{k}_r)\ \overset{(r)}{\phi}{}_2^{\,\nu_1 \cdots \nu_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1})$$

$$= G_t\, \mathbb{P}\left\{ a_{21}^{\nu_1} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \right\} + \mathbb{P}\, G_t \left\{ a_{11}^{\nu_1} a_{12}^{\nu_2} a_{03}^{\nu_3} \cdots a_{0r}^{\nu_r} \right\}$$

$$= G_t\, \mathbb{P}^{1 \cdots r}\Big[\langle a_{21}^{\nu_1} a_{02}^{\nu_2} \cdots a_{0r}^{\nu_r} \rangle$$

$$- \sum \langle a_{21}^{\nu_1} a_{02}^{\nu_2} \cdots a_{0m}^{\nu_m} \rangle \langle a_{0m+1}^{\nu_{m+1}} \cdots \rangle \cdots \langle\ \rangle \Big]$$

$$= -\frac{\partial}{\partial T_2} \langle a_{01}^{\nu_1} \cdots a_{0r}^{\nu_r} \rangle + \langle a_{01}^{\nu_1} \cdots a_{0r}^{\nu_r} \rangle \left(F_{k_1}^{s_1} + \cdots + F_{k_r}^{s_r} \right)$$

$$+ \sum \left(\frac{\partial}{\partial T_2} \langle a_{01}^{\nu_1} \cdots a_{0m}^{\nu_m} \rangle \right) \langle a_{0m+1}^{\nu_{m+1}} \cdots \rangle \cdots \langle\ \rangle$$

$$- \left(\sum \langle a_{01}^{\nu_1} \cdots a_{0m}^{\nu_m} \rangle \langle a_{0m+1}^{\nu_{m+1}} \cdots \rangle - \right) \left(F_{k_1}^{s_1} + \cdots + F_{k_r}^{s_r} \right)$$

$$= -\delta(\tilde{k}_1 + \tilde{k}_2 + \cdots + \tilde{k}_r) \left[\frac{\partial \overset{(r)}{\phi}{}_0^{\,\nu_1 \cdots \nu_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1})}{\partial T_2} \right.$$

$$\left. - \overset{(r)}{\phi}{}_0^{\,\nu_1 \cdots \nu_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1}) \left(F_{k_1}^{s_1} + \cdots + F_{k_r}^{s_r} \right) \right].$$

In order that $G_t\ \overset{(r)}{\phi}{}_2^{\,\nu_1 \cdots \nu_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1}) = 0,$

one has

$$\frac{\partial \overset{(r)}{Q_0}{}^{\nu_1 \cdots \nu_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1})}{\partial T_2} = \overset{(r)}{Q_0}{}^{\nu_1 \cdots \nu_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1}) \left[F_{k_1}^{s_1} + \cdots + F_{-k_1 \cdots -k_{r-1}}^{s_r} \right]. \tag{22}$$

If triad resonances are possible (e.g., one is examining the high frequency part of the spectrum where surface tension plays a major role) then equations (20) and (22) give a closure for the system of spectral cumulants. Note that F_k^s depends only on the energy.

However, if there are no triad resonances, then the energy is independent of the time scale T_2 and equation (22) may be integrated to give

$$\overset{(r)}{Q_0}{}^{\nu_1 \cdots \nu_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1}) = \overset{(r)}{Q_0}{}^{\nu_1 \cdots \nu_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1}) \Big|_{t=0} \exp \left\{ F_{k_1}^{s_1} + \cdots + F_{-k_1 \cdots -k_{r-1}}^{s_r} \right\} T_2 . \tag{23}$$

It is to be stressed, however, that this equation is only valid for that portion of the time scale where T_2 is finite. When there are no triad resonances, only the principal part type terms remain in F_k^s and it may be seen by inspection that F_k^s is pure imaginary. Therefore all the spectral cumulants, in this case, excepting the energy, oscillate on a time scale T_2. This term in reality is producing a modified frequency

$$s\omega \longrightarrow s\omega + \epsilon^2 \frac{F_k^s}{i} ,$$

and is analogous to the Stokes frequency modulation obtained in a discrete analysis when no energy transfer has taken place.

The ε^3 balance.

If one wishes to look at the resonant mechanism developed by the gravity wave spectrum, one must continue the analysis to the ε^4 stage. It will be assumed in the following analysis that triads such as ($\nu_1\omega_1 + \nu_2\omega_2 + \nu_3\omega_3$) cannot vanish for any part of the spectrum.

The ε^3 balance of the governing equation gives

$$a_3^{\nu}(\tilde{k}) = -t \frac{\partial a_1^{\nu}}{\partial T_2} + b_3^{\nu} + c_3^{\nu} + \sum_{j=1}^{3} {}_j d_3^{\nu} . \tag{24}$$

The quantities on the right hand side are as follows.

$$b_3^{\nu} = \sum_{\nu_1 \cdots \nu_4} \int_{-\infty}^{\infty} L_{k k_1 k_2 k_3 k_4}^{\nu \nu_1 \nu_2 \nu_3 \nu_4} a_{01}^{\nu_1} a_{02}^{\nu_2} a_{03}^{\nu_3} a_{04}^{\nu_4} \Delta_{1234,0} \, \delta_{1234,0} \, d\tilde{k}_1 \cdots d\tilde{k}_4 ,$$

$$c_3^{\nu} = 3 \sum_{\nu_1 \cdots \nu_5} \int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\nu \nu_1 \nu_2 \nu_3} L_{k_1 k_4 k_5}^{\nu_1 \nu_4 \nu_5} a_{02}^{\nu_2} a_{03}^{\nu_3} a_{04}^{\nu_4} a_{05}^{\nu_5} \frac{\Delta_{2345,0} - \Delta_{123,0}}{i W_{45,1}} \delta_{45,1} \, \delta_{123,0} \, d\tilde{k}_1 \cdots d\tilde{k}_5 .$$

$${}_1 d_3^{\nu} = \sum_{\nu_1 \cdots \nu_6} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2} L_{k_1 k_3 k_4}^{\nu_1 \nu_3 \nu_4} L_{k_2 k_5 k_6}^{\nu_2 \nu_5 \nu_6} a_{03}^{\nu_3} a_{04}^{\nu_4} a_{05}^{\nu_5} a_{06}^{\nu_6} \frac{\Delta_{3456,0} - \Delta_{234,0} - \Delta_{156,0} + \Delta_{12,0}}{i W_{34,1} \, i W_{56,2}} \delta_{34,1} \delta_{56,2} \delta_{12,0} \, d\tilde{k}_1 \cdots d\tilde{k}_6 .$$

$${}_2 d_3^{\nu} = 4 \sum_{\nu_1 \cdots \nu_6} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2} L_{k_1 k_3 k_4}^{\nu_1 \nu_3 \nu_4} L_{k_3 k_5 k_6}^{\nu_3 \nu_5 \nu_6} a_{02}^{\nu_2} a_{04}^{\nu_4} a_{05}^{\nu_5} a_{06}^{\nu_6} \frac{1}{i W_{56,3}} \left\{ \frac{\Delta_{2456,0} - \Delta_{12,0}}{i W_{456,1}} - \frac{\Delta_{234,0} - \Delta_{12,0}}{i W_{34,1}} \right\} \delta_{56,3} \delta_{34,1} \delta_{12,0} \, d\tilde{k}_1 \cdots d\tilde{k}_6 .$$

$${}_3 d_3^{\nu} = 2 \sum_{\nu_1 \cdots \nu_5} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2} L_{k_1 k_3 k_4 k_5}^{\nu_1 \nu_3 \nu_4 \nu_5} a_{02}^{\nu_2} a_{03}^{\nu_3} a_{04}^{\nu_4} a_{05}^{\nu_5} \frac{\Delta_{2345,0} - \Delta_{12,0}}{i W_{345,1}} \delta_{345,1} \, \delta_{12,0} \, d\tilde{k}_1 \cdots d\tilde{k}_5 .$$

Examination of the ε^3 components of the spectral cumulants shows that there is no need for the T_3 time scale. However at a first examination, this is not clear. In fact, as will be shown, quite the reverse seems to be the case. But on closer examination and using an example to illustrate the point, it is found that a modified approach is necessary in order to continue the problem at this stage. From the modification, it will then be clear that the T_3 scale is, in fact, unnecessary.

One first examines the second order cumulant $\overset{(2)\,\alpha\alpha'}{\phi_3}(k)$,

$$\delta(k+k')\,\overset{(2)\,\alpha\alpha'}{\phi_3}(k) = \overset{00'}{P}\langle a_{0'}^{\alpha'} a_3^{\alpha}\rangle + \overset{00'}{P}\langle a_{1'}^{\alpha'} a_2^{\alpha}\rangle ,$$

where

$$\overset{00'}{P}\langle a_{0'}^{\alpha'} a_3^{\alpha}\rangle = \langle a_0^{\alpha'}(\tilde{k}')\, a_3^{\alpha}(\tilde{k}) + a_0^{\alpha}(\tilde{k})\, a_3^{\alpha'}(k')\rangle .$$

Since $s_1\omega_1 + s_2\omega_2 + s_3\omega_3 + s_4\omega_4 - s\omega$ is never zero identically, $\overset{00'}{P}\langle a_{0'}^{\alpha'} b_3^{\alpha}\rangle \sim O(1)$ for long time. One now examines

$$\langle a_{0'}^{\alpha'} c_3^{\alpha} + a_0^{\alpha} c_{3'}^{\alpha'}\rangle = 3\overset{00'}{P}\sum_{\alpha_1\cdots\alpha_5}\int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\alpha\alpha_1\alpha_2\alpha_3}\, L_{k_1 k_4 k_5}^{\alpha_1\alpha_4\alpha_5}\,\langle a_{0'}^{\alpha'} a_{02}^{\alpha_2} a_{03}^{\alpha_3} a_{04}^{\alpha_4} a_{05}^{\alpha_5}\rangle$$

$$\frac{\Delta_{2345,0} - \Delta_{123,0}}{i W_{45,1}}\,\delta_{45,1}\,\delta_{123,0}\, d\tilde{k}_1 \cdots d\tilde{k}_5 .$$

The fifth order mean value decomposes into products of second by third order spectral cumulants. The notation convention used, is that $\delta_{0'2}\,\delta_{345}$ stands for the break

$$\delta(\tilde{k}'+\tilde{k}_2)\,\delta(\tilde{k}_3+\tilde{k}_4+\tilde{k}_5)\,\overset{(2)}{\varphi}{}_0^{s's_2}(k')\,\overset{(3)}{\varphi}{}_0^{s_5 s_4 s_3}(k_5,k_4).$$

The array of cumulants into which the above mean value decomposes is

$$\overset{(1)}{\delta_{0'2}\,\delta_{345}} \qquad \overset{(1)}{\delta_{0'3}\,\delta_{245}} \qquad \overset{(2)}{\delta_{0'4}\,\delta_{235}} \qquad \overset{(2)}{\delta_{0'5}\,\delta_{234}}$$

$$\overset{(3)}{\delta_{23}\,\delta_{0'45}} \qquad \overset{(4)}{\delta_{24}\,\delta_{0'35}} \qquad \overset{(4)}{\delta_{25}\,\delta_{0'34}}$$

$$\overset{(4)}{\delta_{34}\,\delta_{0'25}} \qquad \overset{(4)}{\delta_{35}\,\delta_{0'24}}$$

$$\overset{(5)}{\delta_{45}\,\delta_{0'23}}\,.$$

Symmetry in (2,3) and (4,5) means that the behavior of some terms is the same as that for others. The only terms which can give "t" growths are S_1 and S_3. This occurs because in these decompositions, it is possible that $s_1\omega_1 + s_2\omega_2 + s_3\omega_3 - s\omega$ is identically zero. Then the time "t" dependent term in $\langle a_0^{s'}(k')\,c_3^{s}(k) + a_0^{s}(k)\,c_3^{s'}(k')\rangle$ takes on the form $\dfrac{\Delta(\mu)-\Delta(0)}{i\mu} \sim t\left(\pi\delta(\mu) + \dfrac{iP}{\mu}\right) + O(1)$. as shown in Appendix II page 258. If μ is a triad, and therefore does not vanish anywhere in the spectrum,

$\dfrac{\Delta(\mu)-\Delta(0)}{i\mu} \sim \dfrac{it}{\mu} + O(1)$. Therefore, one obtains that

$$2S_1 \sim 6t\,\delta_{00'}\,\overset{(2)}{\varphi}{}_0^{ss'}(k)\,P\sum_{s_2 s_5 s_6}\int_{-\infty}^{\infty} L_{kkk_2-k_2}^{ss s_2 -s_2}\, L_{k_2 k_5 k_6}^{s_2 s_5 s_6}\,\overset{(3)}{\varphi}{}_0^{s_5 s_6 -s_2}(k_5 k_6)\,\frac{i\delta_{56,2}}{w_{56,2}}\,d\tilde{k}_2\,d\tilde{k}_5\,d\tilde{k}_6$$

and

$$S_3 \sim 3t\,\delta_{00'}\,\overset{00'}{P}\sum_{\lambda_2}\int_{-\infty}^{\infty} L^{\lambda\lambda_2-\lambda_2}_{k k_2 -k_2}\,\overset{(2)\lambda_2-\lambda_2}{Q_0}(k_2)\,dk_2 \sum_{\lambda_5\lambda_6}\int_{-\infty}^{\infty} L^{\lambda\lambda_5\lambda_6}_{k k_5 k_6}\,\overset{(3)\lambda_5\lambda_6\lambda'}{Q_0}(k_5,k_6)\,\frac{i\delta_{56,0}}{W_{56,0}}\,d\tilde{k}_5\,d\tilde{k}_6 .$$

The next term is

$$\langle a_{0'}^{\lambda'}, a_3^{\lambda} + a_0^{\lambda}, a_3^{\lambda'}\rangle = \overset{00'}{P}\sum_{\lambda_1\cdots\lambda_6}\int_{-\infty}^{\infty} L^{\lambda\lambda_1\lambda_2}_{k k_1 k_2}\,L^{\lambda_1\lambda_3\lambda_4}_{k_1 k_3 k_4}\,L^{\lambda_2\lambda_5\lambda_6}_{k_2 k_5 k_6}$$

$$\langle a_{0'}^{\lambda'} a_{03}^{\lambda_3} a_{04}^{\lambda_4} a_{05}^{\lambda_5} a_{06}^{\lambda_6}\rangle \int_0^t \Delta_{34,1}\,\Delta_{56,2}\,e^{iW_{12,0}t}\,dt$$

$$\delta_{34,1}\,\delta_{56,2}\,\delta_{12,0}\,d\tilde{k}_1\cdots d\tilde{k}_6 .$$

The time "t" dependent term of this expression is $\int_0^t \Delta_{34,1}\,\Delta_{56,2}\,e^{iW_{12,0}t}\,dt$ which is a term of the type $\int_0^t \Delta(\omega)\,\Delta(\nu)\,e^{i\mu t}\,dt$ and can only exhibit a "t" growth when either (1) $\nu = -\mu$, or (2) $\omega = -\mu$. It is seen from Appendix II that $\int_0^t \Delta(\omega)\,\Delta(-\mu)\,e^{i\mu t}\,dt = \int_0^t \Delta(\omega)\,\Delta(\mu)\,dt$ $\sim t\,\frac{i}{\omega}\,\frac{i}{\mu} + O(1)$. This is so as ω and μ, being triads, cannot vanish anywhere in the spectrum. The decomposition of the above mean value denoted by $\delta_{0'3}\,\delta_{456}$, $\delta_{0'4}\,\delta_{356}$ can allow $\omega = -\mu$. The decompositions $\delta_{0'5}\,\delta_{346}$ and $\delta_{0'6}\,\delta_{345}$ allow case (1) where $\nu = -\mu$. Since there is complete symmetry between (3 4) and (5 6),

$$G_t\,\langle a_{0'}^{\lambda'}, a_3^{\lambda} + a_0^{\lambda}, a_3^{\lambda'}\rangle$$

$$= 4\overset{00'}{P}\,\delta_{00'}\,\overset{(2)\lambda\lambda'}{Q_0}(\tilde{k})\sum_{\lambda_1\lambda_2\lambda_5\lambda_6}\int_{-\infty}^{\infty} L^{\lambda\lambda_1\lambda_2}_{k k_1 k_2}\,L^{\lambda_1\lambda-\lambda_2}_{k_1 k -k_2}\,L^{\lambda_2\lambda_5\lambda_6}_{k_2 k_5 k_6}\,\overset{(3)\lambda_5\lambda_6-\lambda_2}{Q_0}(k_5,k_6)$$

$$\frac{i\delta_{12,0}}{W_{12,0}}\,\frac{i\delta_{56,2}}{W_{56,2}}\,d\tilde{k}_1\,d\tilde{k}_2\,d\tilde{k}_5\,d\tilde{k}_6 .$$

$G_t\langle\rangle$ stands for the coefficient of t in the long time behavior of $\langle\ \rangle$. The term

$$\langle a_{0'}^{\prime\,\lambda'}{}_2 a_3^{\lambda} + a_0^{\lambda}{}_2 a_{3'}^{\lambda'}\rangle = 4\,\overset{00'}{P}\sum_{\lambda_1\cdots\lambda_6}\int_{-\infty}^{\infty} L^{\lambda\,\lambda_1\,\lambda_2}_{k\,k_1\,k_2}\, L^{\lambda_1\,\lambda_3\,\lambda_4}_{k_1\,k_3\,k_4}\, L^{\lambda_3\,\lambda_5\,\lambda_6}_{k_3\,k_5\,k_6}$$

$$\langle a_{01}^{\lambda_1'} a_{02}^{\lambda_2} a_{04}^{\lambda_4} a_{05}^{\lambda_5} a_{06}^{\lambda_6}\rangle \int_0^t \frac{\Delta_{456,1}-\Delta_{34,1}}{iW_{56,3}}\, e^{iW_{12,0}t}\,dt$$

$$\delta_{56,3}\,\delta_{34,1}\,\delta_{12,0}\,d\tilde{k}_1\cdots d\tilde{k}_6 .$$

The time dependent term of this expression, which is of the form $\int_0^t \frac{\Delta(\omega)-\Delta(\mu)}{i(\omega-\mu)}\, e^{i\nu t}\,dt$ can only exhibit a "t" growth when $\mu = -\nu$, whereupon the above quantity behaves in long time in a manner shown schematically,

$$\int_0^t \frac{\Delta(\omega)-\Delta(-\nu)}{i(\omega+\nu)}\, e^{i\nu t}\,dt \sim t\,\frac{i}{\omega+\nu}\,\frac{i}{\nu} + O(1).$$

The quantities $\omega+\nu$, ν are triads and cannot vanish anywhere in the spectrum.

The triad

$$\lambda_3\omega_3 + \lambda_4\omega_4 - \lambda_1\omega_1 \equiv -(\lambda_1\omega_1 + \lambda_2\omega_2 - \lambda\omega)$$

only on the decompositions $\delta_{0'4}\,\delta_{256}$ and $\delta_{24}\,\delta_{0'56}$ when one makes $\lambda_4 = \lambda$ and $\lambda_3 = -\lambda_2$ in the former and $\lambda_4 = -\lambda_2$, $\lambda_3 = \lambda$ in the latter. This gives two "t" growth terms, the former being

$$4t\overset{00'}{P}\delta_{00'}\overset{(2)}{\mathcal{Q}}{}_0^{\nu\nu'}(k)\sum_{\nu_1\nu_2\nu_5\nu_6}\int_{-\infty}^{\infty}\mathcal{L}_{k k_1 k_2}^{\nu\nu_1\nu_2}\mathcal{L}_{k_1 k -k_2}^{\nu_1\nu -\nu_2}\mathcal{L}_{-k_2 k_5 k_6}^{-\nu_2\nu_5\nu_6}\overset{(3)}{\mathcal{Q}}{}_0^{\nu_5\nu_6\nu_2}(k_5,k_6)$$

$$\frac{i\delta_{56,2}}{W_{56,2}}\frac{i\delta_{12,0}}{W_{12,0}}\,d\tilde{k}_1\,d\tilde{k}_2\,d\tilde{k}_5\,d\tilde{k}_6,$$

and the latter is

$$4t\overset{00'}{P}\delta_{00'}\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty}\mathcal{L}_{k k_1 k_2}^{\nu\nu_1\nu_2}\mathcal{L}_{k_1 k -k_2}^{\nu_1\nu -\nu_2}\overset{(2)}{\mathcal{Q}}{}_0^{\nu_2 -\nu_2}(k_2)\frac{i\delta_{12,0}}{W_{12,0}}d\tilde{k}_1 d\tilde{k}_2$$

$$\times\sum_{\nu_5\nu_6}\int_{-\infty}^{\infty}\mathcal{L}_{k k_5 k_6}^{\nu\nu_5\nu_6}\overset{(3)}{\mathcal{Q}}{}_0^{\nu_5\nu_6\nu'}(k_5,k_6)\frac{i\delta_{56,0}}{W_{56,0}}d\tilde{k}_5 d\tilde{k}_6.$$

One now considers the term

$$\langle a_{0'3}^{\nu'}a_3^{\nu}+a_{03}^{\nu}a_{3'}^{\nu'}\rangle = 2\overset{00'}{P}\sum_{\nu_1\cdots\nu_5}\int_{-\infty}^{\infty}\mathcal{L}_{k k_1 k_2}^{\nu\nu_1\nu_2}\mathcal{L}_{k_1 k_3 k_4 k_5}^{\nu_1\nu_3\nu_4\nu_5}$$

$$a_{02}^{\nu_2}a_{03}^{\nu_3}a_{04}^{\nu_4}a_{05}^{\nu_5}\frac{\Delta_{2345,0}-\Delta_{12,0}}{iW_{345,1}}\delta_{345,1}\delta_{12,0}\,d\tilde{k}_1\cdots d\tilde{k}_5.$$

It is shown in the appendix that a "t" dependent term of the type $\frac{\Delta(\mu)-\Delta(\mu)}{i(\mu-\mu)}\sim O(1)$, and since neither $W_{2345,0}$ nor $W_{12,0}$ can be identically zero there are no further possibilities of "t" growths.

The next term one considers is

$$\langle a_{1'}^{\nu'}b_2^{\nu}+a_1^{\nu}b_{2'}^{\nu'}\rangle = \overset{00'}{P}\sum_{\nu_1\cdots\nu_5}\int_{-\infty}^{\infty}\mathcal{L}_{k' k_1 k_2}^{\nu'\nu_1\nu_2}\mathcal{L}_{k k_3 k_4 k_5}^{\nu\nu_3\nu_4\nu_5}\langle a_{01}^{\nu_1}a_{02}^{\nu_2}a_{03}^{\nu_3}a_{04}^{\nu_4}a_{05}^{\nu_5}\rangle$$

$$\Delta_{345,0}\Delta_{12,0'}\delta_{345,0}\delta_{12,0'}\,d\tilde{k}_1\cdots d\tilde{k}_5.$$

Secular "t" growths can only occur when

$$W_{345,0} \equiv \nu_3\omega_3 + \nu_4\omega_4 + \nu_5\omega_5 - \nu\omega \equiv 0,$$

that is, on the decompositions denoted by $\delta_{34}\ \delta_{125}$ in which we set $\nu_4 = -\nu_3$ and $\nu_5 = \nu$, $\delta_{35}\ \delta_{124}$ and $\delta_{45}\ \delta_{123}$. From symmetry in (1,2) and (3,4,5) one obtains the same result for each term. Hence one has

$$G_t \langle a_1^{\nu'} b_2^{\nu} + a_1^{\nu} b_2^{\nu'} \rangle = 3 \overset{00'}{P} \delta_{00'} \sum_{\nu_2} \int_{-\infty}^{\infty} L_{k\,k\,k_2-k_2}^{\nu\,\nu\,\nu_2-\nu_2} \overset{(2)}{\phi}{}_0^{\nu_2-\nu_2}(k_2)\, d\tilde{k}_2$$

$$\times \sum_{\nu_5\nu_6} \int_{-\infty}^{\infty} L_{k'\,k_5\,k_6}^{\nu'\,\nu_5\,\nu_6} \overset{(3)}{\phi}{}_0^{\nu_5\,\nu_6\,\nu}(k_5, k_6) \frac{i\,\delta_{56,0'}}{W_{56,0'}}\, d\tilde{k}_5\, d\tilde{k}_6 .$$

By similar reasoning,

$$G_t \langle a_1^{\nu'} c_2^{\nu} + a_1^{\nu} c_2^{\nu'} \rangle = 4 \overset{00'}{P} \delta_{00'} \sum_{\nu_1\nu_2} \int_{-\infty}^{\infty} L_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2} L_{k_1\,k\,-k_2}^{\nu_1\,\nu\,-\nu_2} \frac{i\,\delta_{12,0}}{W_{12,0}}\, d\tilde{k}_1 d\tilde{k}_2$$

$$\times \sum_{\nu_5\nu_6} \int_{-\infty}^{\infty} L_{k'\,k_5\,k_6}^{\nu'\,\nu_5\,\nu_6} \overset{(3)}{\phi}{}_0^{\nu_5\,\nu_6\,\nu}(k_5, k_6) \frac{i\,\delta_{56,0'}}{W_{56,0'}}\, d\tilde{k}_5\, d\tilde{k}_6 .$$

Before adding all these terms to find $G_t\, \delta(\tilde{k}+\tilde{k}')\, \overset{(2)}{\phi}{}_3^{\nu\nu'}(k)$, some notational definitions are introduced.

One defines

$$\mathcal{F}^{\nu\,\nu\,\nu_2-\nu_2}_{k\,k\,k_2-k_2} = 3\,\ell^{\nu\,\nu\,\nu_2-\nu_2}_{k\,k\,k_2-k_2} + 4\sum_{\nu_1}\int_{-\infty}^{\infty} \ell^{\nu\,\nu_1\,\nu_2}_{k\,k_1\,k_2}\,\ell^{\nu_1\,\nu-\nu_2}_{k_1\,k-k_2}\,\frac{i\delta_{12,0}}{W_{12,0}}\,d\tilde{k}_1,$$

and

$$\overset{(0)}{F}{}^{s}_{k} = \sum_{\nu_2}\int_{-\infty}^{\infty} \mathcal{F}^{\nu\,\nu\,\nu_2-\nu_2}_{k\,k\,k_2-k_2}\,\overset{(2)}{Q}{}_0^{\nu_2-\nu_2}(k_2)\,d\tilde{k}_2.$$

Let $\widetilde{\overset{(2)}{Q}}{}_1^{\nu\,\nu'}(k) = \lim_{t\to\infty} \overset{(2)}{Q}{}_1^{\nu\,\nu'}(k),$

that is, the order one term in the long time behavior of $\overset{(2)}{Q}{}_1^{\nu\,\nu'}(k)$ which is

$$\overset{oo'}{\rho}\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} \ell^{\nu\,\nu_1\,\nu_2}_{k\,k_1\,k_2}\,\overset{(3)}{Q}{}_0^{\nu_1\,\nu_2\,\nu'}(k_1,k_2)\,\frac{i\delta_{12,0}}{W_{12,0}}\,d\tilde{k}_1\,d\tilde{k}_2.$$

One uses the fact that $\Delta_{12,0} \sim \dfrac{i}{W_{12,0}}$,

when $\nu_1\omega_1+\nu_2\omega_2-\nu\omega$ does not vanish anywhere in the spectrum. One also defines

$$\overset{(1)}{F}{}^{s}_{k} = \sum_{\nu_2}\int_{-\infty}^{\infty} \mathcal{F}^{\nu\,\nu\,\nu_2-\nu_2}_{k\,k\,k_2-k_2}\,\widetilde{\overset{(2)}{Q}}{}_1^{\nu_2-\nu_2}(k_2)\,d\tilde{k}_2.$$

Adding terms (6) and (7) gives

$$t\,\overset{oo'}{\rho}\,\delta_{oo'}\,\overset{(0)}{F}{}^{s}_{k}\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} \ell^{\nu'\,\nu_1\,\nu_2}_{k'\,k_1\,k_2}\,\overset{(3)}{Q}{}_0^{\nu_1\,\nu_2\,\nu}(k_1,k_2)\,\frac{i\delta_{12,0'}}{W_{12,0'}}\,d\tilde{k}_1\,d\tilde{k}_2.$$

Adding terms (2) and (5) gives

$$t \overset{oo'}{P} \delta_{oo'} \overset{(0)}{F}{}^{s}_{k} \sum_{\nu_1\nu_2} \int_{-\infty}^{\infty} \ell^{\nu\nu_1\nu_2}_{k k_1 k_2} \overset{(3)}{Q}{}^{\nu_1\nu_2\nu'}_{0}(k_1,k_2) \frac{i\delta_{12,0}}{W_{12,0}} d\tilde{k}_1 d\tilde{k}_2 .$$

Adding these two latter terms one obtains

$$t \delta_{oo'} \overset{(2)}{\tilde{Q}}{}^{\nu\nu'}_{1}(k) \left(\overset{(0)}{F}{}^{s}_{k} + \overset{(0)}{F}{}^{s'}_{k'} \right).$$

Adding terms (1), (3), and (4), one obtains

$$t \overset{oo'}{P} \delta_{oo'} \overset{(2)}{Q}{}^{\nu\nu'}_{0}(k) \overset{(1)}{F}{}^{s}_{k}$$

$$= t \delta_{oo'} \overset{(2)}{Q}{}^{\nu\nu'}_{0}(k) \left(\overset{(1)}{F}{}^{s}_{k} + \overset{(1)}{F}{}^{s'}_{k'} \right).$$

Therefore one has that

$$\overset{(2)}{Q}{}^{\nu\nu'}_{3}(k) = -t \frac{\partial \overset{(2)}{\tilde{Q}}{}^{\nu\nu'}_{1}(k)}{\partial T_2} + t \overset{(2)}{\tilde{Q}}{}^{\nu\nu'}_{1}(k) \left(\overset{(0)}{F}{}^{s}_{k} + \overset{(0)}{F}{}^{s'}_{k'} \right) + t \overset{(2)}{Q}{}^{\nu\nu'}_{0}(k) \left(\overset{(1)}{F}{}^{s}_{k} + \overset{(1)}{F}{}^{s'}_{k'} \right) + O(1).$$

In particular when $\nu' = -\nu$,

$$\overset{(2)}{Q}{}^{\nu-\nu}_{3}(k) = -t \frac{\partial \overset{(2)}{\tilde{Q}}{}^{\nu-\nu}_{1}(k)}{\partial T_2} + O(1),$$

since $\overset{(i)}{F}{}^{s}_{k} + \overset{(i)}{F}{}^{-s}_{-k} = 0, \quad i = 0, 1$

from the properties of the coefficients $\ell^{\nu\nu_1\cdots\nu_r}_{k k_1\cdots k_r}$.

It would seem therefore that one requires a T_3 scale in order to suppress "t" growth secularities. One would then obtain

$$\frac{\partial \overset{(2)}{\phi}{}_0^{\,\jmath\,-\jmath}(k)}{\partial T_3} = -\frac{\partial \overset{(2)}{\tilde{\phi}}{}_1^{\,\jmath\,-\jmath}(k)}{\partial T_2},$$

where

$$\overset{(2)}{\tilde{\phi}}{}_1^{\,\jmath\,-\jmath}(k) = P\sum_{\jmath_1 \jmath_2}^{0-0}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\jmath\,\jmath_1\,\jmath_2}\,\overset{(3)}{\phi}{}_0^{\,\jmath_1\,\jmath_2\,-\jmath}(k_1 k_2)\,\frac{i\,\delta_{12,0}}{\lambda_{12,0}}\,d\tilde{k}_1\,d\tilde{k}_2 .$$

In general, therefore one obtains a closure at the T_3 time scale, which is an infinite closure as in general

$$\frac{\partial \overset{(r)}{\phi}{}_0^{\,\jmath_1 \cdots \jmath_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1})}{\partial T_3} = f\left(\overset{(r+1)}{\phi}_0, \overset{(r)}{\phi}_0, \cdots\right), \text{ for each } r.$$

However it may be noted, that if initially at $t = 0$,

$$\overset{(2)}{\phi}{}_0^{\,\jmath\,\jmath'}(k) = 0, \quad \jmath' \neq -\jmath,$$

and

$$\overset{(r)}{\phi}{}_0^{\,\jmath_1 \cdots \jmath_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1}) = 0,$$

then a consistent solution at the T_2 time scale would be

$$\overset{(r)}{\phi}{}_0^{\,\jmath_1 \cdots \jmath_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1}) = 0,$$

or

$$\frac{\partial \overset{(r)}{\phi}{}_0^{\,\jmath_1 \cdots \jmath_r}(\tilde{k}_1 \cdots \tilde{k}_{r-1})}{\partial T_2} = 0.$$

Since in particular

$$\frac{\partial \overset{(3)}{\phi}{}_0^{\,\jmath_1\,\jmath_2\,-\jmath}(k_1 k_2)}{\partial T_2} = 0,$$

then

$$\frac{\partial \overset{(2)}{\tilde{\phi}}{}_1^{\,\jmath\,-\jmath}(k)}{\partial T_2} = 0.$$

In fact, continuing the analysis on the assumption that the system was initially Gaussian leads to the result obtained by Hasselman. This is rather curious for it indicates that the energy transfer mechanism is radically different depending on whether the initial distribution is Gaussian or not. It therefore seems that the term $\frac{\partial \tilde{\phi}_1^{(2)}(\vec{k})}{\partial T_2}$ might be in some sense a spurious one.

The argument could be used that, since $\tilde{\phi}_1^{(2)}(\vec{k})$ contains integrals of the zeroth order spectral cumulants of the third order which have been shown to oscillate on the T_2 scale, the term $-\varepsilon^3 t \frac{\partial \tilde{\phi}_1^{(2)}(\vec{k})}{\partial T_2}$ by the Riemann Lebesgue lemma can be at most $O\left(\varepsilon^3 \times t \times \frac{1}{T_2}\right) = O(\varepsilon)$, and therefore does not affect the order one term of the asymptotic expansion. However, the objection raised to this line of reasoning is that the form of the solution

$$\phi_0^{(r)\, s_1 \cdots s_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1}) = \phi_0^{(r)\, s_1 \cdots s_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1})\Big|_{t=0} \operatorname{Exp}\left(F_{k_1}^{s_1} + \cdots + F_{-k_1 \cdots -\tilde{k}_{r-1}}^{s_r}\right) T_2$$

is only valid in ranges of finite T_2, as can be demonstrated by the example of the equation,

$$\ddot{x} + x = \varepsilon x^3 + \varepsilon^2 \dot{x}(1 - x^2).$$

The first closure of this equation at the T_1 time scale is indeed of an oscillatory nature, however this solution does not persist to times $O(\varepsilon^{-2})$.

The correct resolution of the spurious term is the fact that it is being produced by allowing free waves to occur in the solutions for all the perturbation terms in the original solution. This approach does not maximize the information which one can extract from a multiple time scaling analysis. This will be demonstrated by consideration of the following example.

Consider the problem

$$\ddot{x} + x = \varepsilon x^2. \qquad \ldots. \quad (25)$$

Let us treat this two ways. The first approach will be to expand

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots ,$$

and let $T_0 = t$, $T_1 = \varepsilon t$, $T_2 = \varepsilon^2 t$. The second will be to set

$$x = \sum_s a^s e^{ist}, \quad a^{*-s} = a^s$$

and set $\sum_s a_t^s e^{ist} = 0$,

thereby obtaining an equation for a_t^s, which is

$$a_t^s = \frac{-is}{2} \varepsilon \sum_{s_1, s_2} a^{s_1} a^{s_2} e^{i(s_1 + s_2 - s)t} .$$

One then expands

$$a^s = a_0^s + \varepsilon a_1^s + \cdots .$$

If one follows the first approach Equation (25) becomes

$$\left(\frac{\partial}{\partial t^2} + 2\varepsilon^2 \frac{\partial^2}{\partial t \partial T_2} + 2\varepsilon^3 \frac{\partial^2}{\partial t \partial T_3}\right)\left(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3\right)$$

$$+ \left(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3\right) = \varepsilon x_0^2 + 2\varepsilon^2 x_0 x_1 + \varepsilon^3\left(2x_0 x_2 + x_1^2\right) + \cdots.$$

The order one balance gives

$$\frac{\partial^2 x_0}{\partial t^2} + x_0 = 0.$$

Therefore $x_0 = a_0 e^{it} + a_0^* e^{-it}.$

The order ε balance gives

$$\frac{\partial^2 x_1}{\partial t^2} + x_1 = 2a_0 a_0^* + a_0^2 e^{2it} + a_0^{*2} e^{-2it}$$

The general solution of this problem is

$$x_1 = a_1 e^{it} + a_1^* e^{-it} + 2a_0 a_0^* + \left(-\tfrac{1}{3} a_0^2 e^{2it} - \tfrac{1}{3} a_0^{*2} e^{-2it}\right),$$

where the free wave components are included to satisfy the initial conditions. However since there are only two initial conditions to be satisfied, namely $x(0)$, $\frac{dx(0)}{dt}$, the complex amplitude a_0 would be sufficient to satisfy these, so a_1 is not really required. However if one were to treat the problem by the second approach one obtains as the governing equation

$$a^s_t = -\frac{is}{2} \varepsilon \sum_{s_1 s_2} a^{s_1} a^{s_2} e^{i(s_1+s_2-s)t}.$$

Applying the perturbation

$$a^s = a_0^s + \varepsilon a_1^s + \cdots$$

in this equation and equating powers of ε, one obtains upon integration that

$$a_0^s = a_0^s(T_2),$$

and $$a_1^s = -\frac{is}{2}\left(\sum_{s_1 s_2} a_0^{s_1} a_0^{s_2} \Delta(s_1+s_2-s)\right).$$

The equivalent x_1 is $\sum_s a_1^s e^{ist}$ which equals

$$-\frac{a_0^{+2}}{3} e^{2it} - \frac{a_0^{-2}}{3} e^{-2it} + 2a_0^+ a_0^-$$

$$+\left(\frac{1}{2} a_0^{+2} - a_0^+ a_0^- - \frac{1}{6} a_0^{-2}\right) e^{it} + \left(\frac{1}{2} a_0^{-2} - a_0^+ a_0^- - \frac{1}{6} a_0^{+2}\right) e^{-it}.$$

Calling $a_0^+ = a_0$, and $a_0^- = a_0^*$, one has that

and $$a_1 = \frac{1}{2} a_0^2 - a_0 a_0^* - \frac{1}{6} a_0^{*2}$$

which ensures that

$$x_1(0) = 0, \quad \frac{dx_1}{dt}(0) = 0.$$

One can see therefore that if one treats the problem in the second approach that the free waves are included in the order ε solution. One now continues the problem with the first approach keeping the free wave terms.

$$x_0 = a_0 e^{it} + a_0^* e^{-it}$$

$$x_1 = a_1 e^{it} + a_1^* e^{-it} - \frac{1}{3} a_0^2 e^{2it} - \frac{1}{3} a_0^{*2} e^{-2it} + 2a_0 a_0^*.$$

The $O(\varepsilon^2)$ balance from equation (25) gives

$$\left(\frac{\partial^2}{\partial t^2} + 1\right) x_2 = -2\frac{\partial^2 x_0}{\partial t\, \partial T_2} + 2x_0 x_1,$$

$$\left(\frac{\partial^2}{\partial t^2}+1\right)x_2 = -2\left(i\frac{\partial a_0}{\partial T_2}e^{it} - i\frac{\partial a_0^*}{\partial T_2}e^{-it}\right)$$
$$+2\left(a_0e^{it}+a_0^*e^{-it}\right)\left(a_1e^{it}+a_1^*e^{-it}+2a_0a_0^* - \frac{1}{3}a_0^2e^{2it}-\frac{1}{3}a_0^{*2}e^{-2it}\right).$$

To remove secularities, one chooses

$$-2i\frac{\partial a_0}{\partial T_2}+4a_0^2a_0^* - \frac{2}{3}a_0^2a_0^* = 0.$$

Therefore

$$\frac{\partial a_0}{\partial T_2} = -\frac{5i}{3}a_0^2a_0^*.$$

Then, integrating the remaining terms, one has

$$x_2 = a_2e^{it}+a_2^*e^{-it}-\frac{2}{3}a_0a_1e^{2it}-\frac{2}{3}a_0^*a_1^*e^{-2it}$$
$$+2\left(a_0a_1^*+a_0^*a_1\right)+\frac{1}{12}a_0^3e^{3it}+\frac{1}{12}a_0^{*3}e^{-3it}.$$

The ε^3 balance gives

$$\left(\frac{\partial^2}{\partial t^2}+1\right)x_3 = -2\frac{\partial^2 x_0}{\partial t\,\partial T_3}-2\frac{\partial^2 x_1}{\partial t\,\partial T_2}+2x_0x_2+x_1^2,$$

$$= -2\left(i\frac{\partial a_0}{\partial T_3}e^{it}-i\frac{\partial a_0^*}{\partial T_3}e^{-it}\right)-2\left(i\frac{\partial a_1}{\partial T_2}e^{it}-i\frac{\partial a_1^*}{\partial T_2}e^{-it}\right)$$
$$+2\left(a_0e^{it}+a_0^*e^{-it}\right)\left\{a_2e^{it}+a_2^*e^{-it}-\frac{2}{3}a_0a_1e^{2it}\right.$$
$$\left.-\frac{2}{3}a_0^*a_1^*e^{-2it}+2a_0a_1^*+2a_0^*a_1+\frac{1}{12}a_0^3e^{3it}+\frac{1}{12}a_0^{*3}e^{-3it}\right\}$$

$$+\left\{ a_1 e^{it} + a_1^* e^{-it} + 2a_0 a_0^* - \frac{1}{3} a_0^2 e^{2it} - \frac{1}{3} a_0^{*2} e^{-2it} \right\}^2 .$$

To remove secularities in x_3, one suppresses all coefficients of e^{it} and e^{-it} by setting

$$-2i \frac{\partial a_0}{\partial T_3} - 2i \frac{\partial a_1}{\partial T_2} + 4a_0 (a_0 a_1^* + a_0^* a_1) - \frac{4}{3} a_0 a_0^* a_1 + 4 a_0 a_0^* a_1 - \frac{2}{3} a_0^2 a_1^* = 0.$$

Therefore, by rearranging, one has that

$$\frac{\partial a_0}{\partial T_3} = - \frac{\partial a_1}{\partial T_2} - 2i a_0 (a_0 a_1^* + a_0^* a_1) + \frac{2i}{3} a_0 a_0^* a_1 - 2i a_0 a_0^* a_1 + \frac{i}{3} a_0^2 a_1^* ,$$

$$= - \frac{\partial a_1}{\partial T_2} - \frac{5i}{3} a_0 (a_0 a_1^* + a_0^* a_1) - \frac{5i}{3} a_0 a_0^* a_1 .$$

Examining the zeroth order energy one obtains

$$\frac{\partial a_0 a_0^*}{\partial T_3} = - \frac{\partial}{\partial T_2} (a_0 a_1^* + a_0^* a_1) .$$

The right hand side is exactly of the form of the $\frac{\partial \tilde{\Phi}_1^{(2),-)}(k)}{\partial T_2}$ in the general problem. If one lets

$$a_1 = \frac{1}{2} a_0^2 - a_0 a_0^* - \frac{1}{6} a_0^{*2} ,$$

and uses

$$\frac{\partial a_0}{\partial T_2} = -\frac{5i}{3} a_0^2 a_0^*,$$

one finds that

$$\frac{\partial}{\partial T_2}\left(a_0 a_1^* + a_0^* a_1\right) \neq 0.$$

It would seem therefore from the above equation A that the zeroth order energy does change. But one knows that in this problem the energy is conserved. Therefore in order to keep the zeroth order term in the energy a meaningful quantity one must take $a_1 = 0$; namely one must suppress all the free wave terms in the solutions for the perturbed quantities and throw all the initial conditions into a_0.

Essentially therefore one must use two devices for finding the maximum information in the higher closures. The first is to use the method of multiple time scales (or some equivalent technique) to remove all t growth secularities and the second is to add arbitrary functions of the higher time scales to the solution of the higher order perturbed quantity in order to suppress free waves. In the general problem one must have as the solution of

$$\frac{\partial a_1^{\lambda}}{\partial t} = \sum_{\lambda_1\lambda_2} \int_{-\infty}^{\infty} L_{kk_1k_2}^{\lambda\lambda_1\lambda_2}\, a_{01}^{\lambda_1} a_{02}^{\lambda_2}\, e^{i\omega_{12,0}t}\, \delta_{12,0}\, dk_1\, dk_2,$$

$$a_1^{\lambda} = b_1^{\lambda} + \alpha_1^{\lambda}(k, T_2)$$

This will mean that if one chooses $\dfrac{\partial \overset{(r)}{\phi}{}_0^{\,\nu_1 \cdots \nu_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1})}{\partial T_m}$

to eliminate "t" growths, and the free terms to eliminate the order one behavior of $\overset{(r)}{\phi}{}_m^{\,\nu_1 \cdots \nu_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1})$, that

$$\lim_{t \to \infty} \overset{(r)}{\phi}{}^{\nu_1 \cdots \nu_r}(\tilde{k}_1, \cdots \tilde{k}_{r-1}) = \overset{(r)}{\phi}{}_0^{\nu_1 \cdots \nu_r}(\tilde{k}_1 \cdots \tilde{k}_{r-1}) + O(\varepsilon^{r-2}) \text{ quasi-steady terms.}$$

The problem is now re-examined, this time adding in arbitrary functions to the solution of the perturbed amplitudes. The governing equation to the system was

$$a_t^{\nu} = \sum_r \varepsilon^{r-1} \sum_{\nu_1 \cdots \nu_r} \int_{-\infty}^{\infty} L_{k k_1 \cdots k_r}^{\nu \nu_1 \cdots \nu_r} a_1^{\nu_1} \cdots a_r^{\nu_r} e^{i w_{12 \cdots r,0} t} \delta_{12 \cdots r,0} \, d\tilde{k}_1 \cdots d\tilde{k}_r .$$

Proceeding as before, one has

$$a_0^{\nu} = a_0^{\nu}(\tilde{k}, T_2, \cdots)$$

and

$$a_1^{\nu} = b_1^{\nu} + \alpha_1^{\nu}(\tilde{k}, T_2, \cdots),$$

where

$$b_1^{\nu} = \sum_{\nu_1 \nu_2} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2} a_{01}^{\nu_1} a_{02}^{\nu_2} \Delta_{12,0} \, \delta_{12,0} \, d\tilde{k}_1 d\tilde{k}_2 .$$

The order ε^2 balance gives

$$a_2^{\nu} = -t \frac{\partial a_0^{\nu}}{\partial T_2} + b_2^{\nu} + c_2^{\nu} + \alpha_2^{\nu}(\tilde{k}, T_2, \cdots) + \beta_2^{\nu},$$

where

$$b_2^{\nu} = \sum_{\nu_1 \nu_2 \nu_3} \int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\nu \nu_1 \nu_2 \nu_3} a_{01}^{\nu_1} a_{02}^{\nu_2} a_{03}^{\nu_3} \Delta_{123,0} \, \delta_{123,0} \, d\tilde{k}_1 d\tilde{k}_2 d\tilde{k}_3 ,$$

$$C_2^{\nu} = 2\sum_{\nu_1 \cdots \nu_4} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2} L_{k_1 k_3 k_4}^{\nu_1 \nu_3 \nu_4} a_{02}^{\nu_2} a_{03}^{\nu_3} a_{04}^{\nu_4} \frac{\Delta_{234,0} - \Delta_{12,0}}{iW_{34,1}} \delta_{34,1}\, \delta_{12,0}\, d\breve{k}_1 \cdots d\breve{k}_4,$$

$$\beta_2^{\nu} = 2\sum_{\nu_1 \nu_2} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2} a_{01}^{\nu_1} \alpha_{12}^{\nu_2} \Delta_{12,0}\, \delta_{12,0}\, d\breve{k}_1 d\breve{k}_2$$

and $\alpha_2^{\nu}(\breve{k}, T_2)$ is the arbitrary function introduced from this integration.

Clearly the results are precisely the same to the T_2 time scale stage as the added quantities do not produce any long time "t" growth behavior. The effect of the added terms are first felt at the T_3 time scale. The ε^3 balance of the governing equation (14) is

$$\frac{\partial a_3^{\nu}}{\partial t} = -\frac{\partial a_1^{\nu}}{\partial T_2} + \sum_{\nu_1 \nu_2} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2} \left(a_{02}^{\nu_2} a_{21}^{\nu_1} + a_{11}^{\nu_1} a_{12}^{\nu_2} + a_{01}^{\nu_1} a_{22}^{\nu_2} \right) e^{iW_{12,0} t} \delta_{12,0}\, d\breve{k}_1 d\breve{k}_2$$

$$+ P^{123} \sum_{\nu_1 \nu_2 \nu_3} \int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\nu \nu_1 \nu_2 \nu_3} a_{01}^{\nu_1} a_{02}^{\nu_2} a_{13}^{\nu_3} e^{iW_{123,0} t} \delta_{123,0}\, d\breve{k}_1 d\breve{k}_2 d\breve{k}_3$$

$$+ \sum_{\nu_1 \cdots \nu_4} \int_{-\infty}^{\infty} L_{k k_1 k_2 k_3 k_4}^{\nu \nu_1 \nu_2 \nu_3 \nu_4} a_{01}^{\nu_1} a_{02}^{\nu_2} a_{03}^{\nu_3} a_{04}^{\nu_4} e^{iW_{1234,0} t} \delta_{1234,0}\, d\breve{k}_1 \cdots d\breve{k}_4 .$$

$$a_3^{\nu} = -t\frac{\partial a_1^{\nu}}{\partial T_2} + b_3^{\nu} + c_3^{\nu} + \sum_{i=1}^{3} {}_i d_3^{\nu} + \beta_3^{\nu} + \sum_{i=1}^{4} {}_i \gamma_3^{\nu}.$$

where b_3^{ν}, c_3^{ν} and ${}_i d_3^{\nu}$ $(i=1,2,3)$ are as defined previously,

and,

$$\beta_3^{\nu} = 3\sum_{\nu_1\nu_2\nu_3}\int_{-\infty}^{\infty} L_{k\,k_1\,k_2\,k_3}^{\nu\,\nu_1\,\nu_2\,\nu_3}\, \alpha_{11}^{\nu_1} a_{02}^{\nu_2} a_{03}^{\nu_3}\, \Delta_{123,0}\, \delta_{123,0}\, d\breve{k}_1 d\breve{k}_2 d\breve{k}_3.$$

$${}_1\gamma_3^{\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2}\, a_{01}^{\nu_1}\alpha_{22}^{\nu_2}\, \Delta_{12,0}\, \delta_{12,0}\, d\breve{k}_1 d\breve{k}_2.$$

$${}_2\gamma_3^{\nu} = 4\sum_{\nu_1\cdots\nu_4}\int_{-\infty}^{\infty} L_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2} L_{k_1\,k_3\,k_4}^{\nu_1\,\nu_3\,\nu_4}\, a_{02}^{\nu_2} a_{03}^{\nu_3}\alpha_{14}^{\nu_4}\, \frac{\Delta_{234,0}-\Delta_{12,0}}{i w_{34,1}}\, \delta_{34,1}\,\delta_{12,0}\, d\breve{k}_1\cdots d\breve{k}_4.$$

$${}_3\gamma_3^{\nu} = 2\sum_{\nu_1\cdots\nu_4}\int_{-\infty}^{\infty} L_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2} L_{k_1\,k_3\,k_4}^{\nu_1\,\nu_3\,\nu_4}\, \alpha_{12}^{\nu_2} a_{03}^{\nu_3} a_{04}^{\nu_4}\, \frac{\Delta_{234,0}-\Delta_{12,0}}{i w_{34,1}}\, \delta_{34,1}\,\delta_{12,0}\, d\breve{k}_1\cdots d\breve{k}_4.$$

$${}_4\gamma_3^{\nu} = \sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2}\, \alpha_{11}^{\nu_1}\alpha_{12}^{\nu_2}\, \Delta_{12,0}\, \delta_{12,0}\, d\breve{k}_1 d\breve{k}_2.$$

The notational definitions that need to be added are,

$$\langle a_0^{\nu}(k)\,\alpha_1^{\nu'}(k')\rangle = \delta(k+k')\, A_{a\alpha}^{\nu\nu'}(k) = \delta(k+k')\, A_{\alpha a}^{\nu'\nu}(k').$$

$$\langle \alpha_1^{\nu}(k)\,\alpha_1^{\nu'}(k')\rangle = \delta(k+k')\, \mathcal{A}(k).$$

$$\langle a_0^{\nu}(k)\, a_0^{\nu'}(k')\,\alpha_1^{\nu''}(k'')\rangle = \delta(k+k'+k'')\, A_{aa\alpha}^{\nu\nu'\nu''}(k,k') = \delta(k+k'+k'')\, A_{a\alpha a}^{\nu\nu''\nu'}(k,k'').$$

$$\langle a_0^{\nu}(k)\,\alpha_2^{\nu'}(k')\rangle = \delta(k+k')\, B_{a\alpha}^{\nu\nu'}(k) = \delta(k+k')\, B_{\alpha a}^{\nu'\nu}(k').$$

In looking at $\delta(k+k')\,\overset{(2)}{\varphi}{}_3^{\,\gamma\gamma'}(k)$ there are many new terms which now have to be included; namely

$$\langle \alpha_1^{\gamma} b_{2'}^{\gamma'} + \alpha_{1'}^{\gamma'} b_2^{\gamma} \rangle, \quad \langle \alpha_1^{\gamma} c_{2'}^{\gamma'} + \alpha_{1'}^{\gamma'} c_2^{\gamma} \rangle,$$

$$\langle \alpha_1^{\gamma} \beta_{2'}^{\gamma'} + \alpha_{1'}^{\gamma'} \beta_2^{\gamma} \rangle, \quad \langle \alpha_1^{\gamma} \alpha_{2'}^{\gamma'} + \alpha_{1'}^{\gamma'} \alpha_2^{\gamma} \rangle,$$

$$\langle a_{0'}^{\gamma'} \beta_3^{\gamma} + a_0^{\gamma} \beta_{3'}^{\gamma'} \rangle \text{ and } \sum_{j=1}^{4} \langle a_{0'}^{\gamma'}\, {}_j\gamma_3^{\gamma} + a_0^{\gamma}\, {}_j\gamma_{3'}^{\gamma'} \rangle .$$

Consider

$$\langle a_{0'}^{\gamma'} \beta_3^{\gamma} + a_0^{\gamma} \beta_{3'}^{\gamma'} \rangle = 3\overset{00'}{\rho} \sum_{\gamma_1\gamma_2\gamma_3} \int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\gamma \gamma_1 \gamma_2 \gamma_3} \langle a_{0'}^{\gamma'} \alpha_{11}^{\gamma_1} a_{02}^{\gamma_2} a_{03}^{\gamma_3} \rangle \, \Delta_{123,0}\, \delta_{123,0}\, d\tilde{k}_1 d\tilde{k}_2 d\tilde{k}_3 .$$

The "t" growths come from the (2 X 2) decompositions $\delta_{0'1}\delta_{23}$, $\delta_{0'2}\delta_{13}$ and $\delta_{0'3}\delta_{12}$. The final two terms give the same result so there are only two distinct "t" growth terms. One finds that

$$G_t \langle a_{0'}^{\gamma'} \beta_3^{\gamma} + a_0^{\gamma} \beta_{3'}^{\gamma'} \rangle$$
$$= 3\overset{00'}{\rho}\,\delta_{00'}\, A_{a\alpha}^{\gamma'\gamma}(k') \sum_{\gamma_2} \int_{-\infty}^{\infty} L_{k k k_2 -k_2}^{\gamma\gamma\gamma_2-\gamma_2}\, \overset{(2)}{\varphi}{}_0^{\,\gamma_2-\gamma_2}(k_2)\, d\tilde{k}_2$$

$$+ 6\overset{00'}{\rho}\,\delta_{00'}\, \overset{(2)}{\varphi}{}_0^{\,\gamma\gamma'}(k) \sum_{\gamma_2} \int_{-\infty}^{\infty} L_{k k k_2 -k_2}^{\gamma\gamma\gamma_2-\gamma_2}\, A_{\alpha a}^{\gamma_2-\gamma_2}(\tilde{k}_2)\, d\tilde{k}_2 .$$

Clearly

$$G_t \langle a_{0'}^{\eta'}\, {}_1\gamma_3^{\eta} + a_0^{\eta}\, {}_1\gamma_{3'}^{\eta'} \rangle = 0,$$

$$\text{as} \quad \Delta_{12,0} \sim \frac{1}{W_{12,0}} + O\left(\frac{1}{t}\right).$$

$$\langle a_{0'}^{\eta'}\, {}_2\gamma_3^{\eta} + a_0^{\eta}\, {}_2\gamma_{3'}^{\eta'} \rangle$$

$$= 4 \overset{00'}{P} \sum_{\eta_1 \cdots \eta_4} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\eta \eta_1 \eta_2} L_{k_1 k_3 k_4}^{\eta_1 \eta_3 \eta_4} \langle a_{0'}^{\eta'} a_{02}^{\eta_2} a_{03}^{\eta_3} a_{14}^{\eta_4} \rangle \frac{\Delta_{234,0} - \Delta_{12,0}}{i W_{34,1}} \delta_{34,1}\, \delta_{12,0}\, dk_1 \cdots dk_4.$$

"t" growths can occur when $\eta_2\omega_2 + \eta_3\omega_3 + \eta_4\omega_4 - \eta\omega = 0$, namely on the decompositions denoted by $\delta_{0'3}\,\delta_{24}$ and $\delta_{0'4}\,\delta_{23}$.

Note: There is no contribution from the decomposition denoted by $\delta_{0'2}\,\delta_{34}$ as $\delta_{34}\,\delta_{34,1} = \delta_{34}\,\delta(k_1)$.

The first decomposition above gives a "t" growth

$$4t \overset{00'}{P} \delta_{00'} \overset{(2)}{Q}{}_0^{\eta\eta'}(k) \sum_{\eta_1\eta_2} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\eta \eta_1 \eta_2} L_{k_1 k -k_2}^{\eta_1 \eta -\eta_2} A_{a\alpha}^{\eta_2 -\eta_2}(k_2) \frac{i\delta_{12,0}}{W_{12,0}}\, dk_1\, dk_2,$$

while the second gives

$$4t \overset{00'}{P} \delta_{00'} A_{a\alpha}^{\eta'\eta}(k') \sum_{\eta_1\eta_2} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\eta \eta_1 \eta_2} L_{k_1 k -k_2}^{\eta_1 \eta -\eta_2} \overset{(2)}{Q}{}_0^{\eta_2 -\eta_2}(k_2) \frac{i\delta_{12,0}}{W_{12,0}}\, dk_1\, dk_2.$$

One next considers the term

$$\langle a_{0'}^{s'}\, {}_3\gamma_3^{s} + a_0^{s}\, {}_3\gamma_{3'}^{s'} \rangle = 2 \overset{00'}{\rho} \sum_{s_1 \cdots s_4} \int_{-\tilde{\infty}}^{\tilde{\infty}} L_{k k_1 k_2}^{s s_1 s_2} L_{k_1 k_3 k_4}^{s_1 s_3 s_4} \langle a_{0'}^{s'} \alpha_{12}^{s_2} a_{03}^{s_3} a_{04}^{s_4} \rangle \frac{\Delta_{234,0} - \Delta_{12,0}}{i w_{34,1}} \delta_{34,1} \delta_{12,0}\, d\tilde{k}_1 \cdots d\tilde{k}_4 .$$

"t" growths can occur when $s_2\omega_2 + s_3\omega_3 + s_4\omega_4 - s\omega \equiv 0,$ for then the time t dependent terms of the expansion are of the form $\frac{\Delta(0)-\Delta(\mu)}{-i\mu} \sim \frac{it}{\mu} + O(1)$, when μ is a triad, See Appendix $\overline{II}$. This behavior can occur for the decompositions denoted by $\delta_{0'3}\, \delta_{24}$ and $\delta_{0'4}\, \delta_{23}$. From symmetry the second gives the same answer as the first. The decomposition $\delta_{0'2}\, \delta_{34}$ reacts with $\delta_{34,1}$ to give $\delta(\tilde{k}_1)$ and thus makes the expression zero by the mean value property. One finds that

$$G_t \langle a_{0'}^{s'}\, {}_3\gamma_3^{s} + a_0^{s}\, {}_3\gamma_{3'}^{s'} \rangle$$

$$= 4 \overset{00'}{\rho}\, \delta_{00'}\, \overset{(2)\, s s'}{Q_0}(k) \sum_{s_1 s_2} \int_{-\tilde{\infty}}^{\tilde{\infty}} L_{k k_1 k_2}^{s s_1 s_2} L_{k_1 k -k_2}^{s_1 s -s_2} A_{\alpha a}^{s_2 -s_2}(k_2) \frac{i \delta_{12,0}}{w_{12,0}}\, d\tilde{k}_1 d\tilde{k}_2 .$$

Clearly

$$G_t \langle a_{0'}^{s'}\, {}_4\gamma_3^{s} + a_0^{s}\, {}_4\gamma_{3'}^{s'} \rangle = 0, \text{ as } \Delta_{12,0} \sim O(1).$$

One now considers the term

One now considers the term

$$\langle \alpha_{1'}^{?'} b_2^{?} + \alpha_1^{?} b_{2'}^{?'} \rangle = \overset{00'}{\rho} \sum_{?_1 ?_2 ?_3} \int_{-\infty}^{\infty} \mathcal{L}_{k k_1 k_2 k_3}^{? ?_1 ?_2 ?_3} \langle \alpha_{1'}^{?'} a_{01}^{?_1} a_{02}^{?_2} a_{03}^{?_3} \rangle$$

$$\Delta_{123,0}\, \delta_{123,0}\, d\tilde{k}_1 d\tilde{k}_2 d\tilde{k}_3 .$$

The "t" growths occur on each of the (2 X 2) decompositions, giving

$$G_t \langle \alpha_{1'}^{?'} b_2^{?} + \alpha_1^{?} b_{2'}^{?'} \rangle$$

$$= 3 \overset{00'}{\rho} \delta_{00'} A_{\alpha\alpha}^{?'?}(k') \sum_{?_2} \int_{-\infty}^{\infty} \mathcal{L}_{k k k_2 -k_2}^{? ? ?_2 -?_2} \overset{(2)}{Q}{}_0^{?_2 -?_2}(k_2)\, d\tilde{k}_2 .$$

Similarly,

$$G_t \langle \alpha_{1'}^{?'} c_2^{?} + \alpha_1^{?} c_{2'}^{?'} \rangle$$

$$= 4 \overset{00'}{\rho} \delta_{00'} A_{\alpha\alpha}^{?'?}(k') \sum_{?_1 ?_2} \int_{-\infty}^{\infty} \mathcal{L}_{k k_1 k_2}^{? ?_1 ?_2} \mathcal{L}_{k_1 k -k_2}^{?_1 ? -?_2} \overset{(2)}{Q}{}_0^{?_2 -?_2}(k_2) \frac{i \delta_{12,0}}{W_{12,0}}\, d\tilde{k}_1 d\tilde{k}_2 .$$

Clearly, the expressions

$$\langle \alpha_1^{?} \beta_{2'}^{?'} + \alpha_{1'}^{?'} \beta_2^{?} \rangle \quad \text{and} \quad \langle \alpha_1^{?} \alpha_{2'}^{?'} + \alpha_{1'}^{?'} \alpha_2^{?} \rangle$$

are order one for long time. Adding these extra terms which contribute to the "t" growth behavior of one obtains

$$= t \overset{00'}{\rho} \delta_{00'} A_{\alpha\alpha}^{?'?}(k') \overset{(0)}{F}{}_k^{s} ,$$

$$= t \overset{00'}{\rho} \delta_{00'} A_{\alpha\alpha}^{?'?}(k') \overset{(0)}{F}{}_k^{s} ,$$

where $\overset{(0)}{F}{}^{s}_{k}$ and $\mathcal{F}^{\nu\nu\nu_2-\nu_2}_{kk\,k_2-k_2}$ are as defined previously.
If one adds these two latter terms one obtains

$$t\overset{oo'}{\rho}\,\overset{(0)}{F}{}^{s}_{k}\,\langle \alpha_1^{\nu'}(k')\,a_0^{\nu}(k) + a_0^{\nu'}(k')\,\alpha_1^{\nu}(k)\rangle$$

$$= t\overset{oo'}{\rho}\,\overset{(0)}{F}{}^{s}_{k}\,\delta_{oo'}\,{}_2\overset{(2)}{Q}{}_1^{\nu\nu'}(\tilde{k}),$$

where the last equation defines ${}_2\overset{(2)}{Q}{}_1^{\nu\nu'}(\tilde{k})$. The original $\delta_{oo'}\overset{(2)}{Q}{}_1^{\nu\nu'}(k)$ defined as $\langle a_{0'}^{\nu'} b_1^{\nu} + a_0^{\nu} b_{1'}^{\nu'}\rangle$ will now be called $\delta_{oo'}\,{}_1\overset{(2)}{Q}{}_1^{\nu\nu'}(k)$, and $\delta_{oo'}\,{}_1\overset{(2)}{Q}{}_1^{\nu\nu'}(k)$ will stand for its order one behavior for long times. Adding (2), (3) and (5) one obtains

$$3t\overset{oo'}{\rho}\,\delta_{oo'}\,\overset{(2)}{Q}{}_0^{\nu\nu'}(\tilde{k})\sum_{\nu_2}\int_{-\infty}^{\infty}\mathcal{L}^{\nu\nu\nu_2-\nu_2}_{kk\,k_2-k_2}\left(A^{\nu_2-\nu_2}_{\alpha a}(k_2)+A^{\nu_2-\nu_2}_{a\alpha}(k_2)\right)d\tilde{k}_2$$

$$+4t\overset{oo'}{\rho}\,\delta_{oo'}\,\overset{(2)}{Q}{}_0^{\nu\nu'}(\tilde{k})\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty}\mathcal{L}^{\nu\nu_1\nu_2}_{kk_1k_2}\,\mathcal{L}^{\nu_1\nu-\nu_2}_{k_1k-k_2}$$

$$\left(A^{\nu_2-\nu_2}_{a\alpha}(k_2)+A^{\nu_2-\nu_2}_{\alpha a}(k_2)\right)\frac{i\delta_{12,0}}{W_{12,0}}\,d\tilde{k}_1\,d\tilde{k}_2,$$

Since $$A^{\nu\nu'}_{\alpha a}(k) = A^{\nu'\nu}_{a\alpha}(k').$$

Therefore, as

$$\delta(\tilde{k}_2+\tilde{k}_2')\,{}_2\overset{(2)}{Q}{}_1^{\nu_2-\nu_2}(k_2) = \langle \alpha_1^{-\nu_2}(k_2')\,a_0^{\nu_2}(k_2) + a_0^{-\nu_2}(k_2')\,\alpha_1^{\nu_2}(k_2)\rangle$$

$$= \delta(\tilde{k}_2+\tilde{k}_2')\left[A^{\nu_2-\nu_2}_{a\alpha}(k_2) + A^{\nu_2-\nu_2}_{\alpha a}(k_2)\right],$$

One can write the above expression as

$$t\, P^{00'} f_{00'}\, \overset{(2)}{\phi}{}_0^{\nu\nu'}(k) \sum_{\nu_2} \int_{-\infty}^{\infty} J_{k\,k\,k_2\,-k_2}^{\nu\,\nu\,\nu_2\,-\nu_2}\; {}_2\overset{(2)}{\phi}{}_1^{\nu_2\,-\nu_2}(k_2)\, d\tilde{k}_2 .$$

If one adds these terms coming from the inclusion of free waves to the result obtained previously one obtains

$$\overset{(2)}{\phi}{}_3^{\nu\nu'}(k) = -t \frac{\partial}{\partial T_2}\left({}_1\overset{\widetilde{(2)}}{\phi}{}_1^{\nu\nu'}(\tilde{k}) + {}_2\overset{(2)}{\phi}{}_1^{\nu\nu'}(\tilde{k}) \right)$$

$$+\, t \left({}_1\overset{\widetilde{(2)}}{\phi}{}_1^{\nu\nu'}(\tilde{k}) + {}_2\overset{(2)}{\phi}{}_1^{\nu\nu'}(\tilde{k}) \right)\left(\overset{(0)}{F}{}_k^{s} + \overset{(0)}{F}{}_{-k}^{s'} \right)$$

$$+\, t\, \overset{(2)}{\phi}{}_0^{\nu\nu'}(\tilde{k}) \left\{ \sum_{\nu_2} \int_{-\infty}^{\infty} \left(J_{k\,k\,k_2\,-k_2}^{\nu\,\nu\,\nu_2\,-\nu_2} + J_{k'\,k'\,k_2\,-k_2}^{\nu'\,\nu'\,\nu_2\,-\nu_2} \right) \left({}_1\overset{\widetilde{(2)}}{\phi}{}_1^{\nu_2\,-\nu_2}(\tilde{k}_2) + {}_2\overset{(2)}{\phi}{}_1^{\nu_2\,-\nu_2}(\tilde{k}_2) \right) d\tilde{k}_2 \right\}$$

$$+\, O(1).$$

By choosing the cumulant ${}_2\overset{(2)}{\phi}{}_1^{\nu\nu'}(\tilde{k})$ defined by the free waves,

$${}_2\overset{(2)}{\phi}{}_1^{\nu\nu'}(\tilde{k}) = -\, {}_1\overset{\widetilde{(2)}}{\phi}{}_1^{\nu\nu'}(k),$$

one obtains that

$$\overset{(2)}{\phi}{}_3^{\nu\nu'}(\tilde{k}) = O(1).$$

The same procedure can be carried out for the remaining higher spectral cumulants and with similar choices

$$ {}_2\Phi_1^{(3)\nu\nu'\nu''}(\check{k},\check{k}') = -\,{}_1\tilde{\Phi}_1^{*(3)\nu\nu'\nu''}(\check{k},\check{k}') , $$

$$ {}_2\Phi_1^{(r)\nu_1\cdots\nu_r}(\check{k}_1\cdots\check{k}_{r-1}) = -\,{}_1\tilde{\Phi}_1^{*(r)\nu_1\cdots\nu_r}(\check{k}_1\cdots\check{k}_{r-1}) $$

one can show

$$ \Phi_3^{(r)\nu_1\cdots\nu_r}(\check{k}_1\cdots\check{k}_{r-1}) \sim O(1). $$

The star on ${}_1\tilde{\Phi}_1^{*(3)\nu\nu'\nu''}(\check{k},\check{k}')$ means the <u>continuous</u> (in Fourier space) order one component of $\Phi_1^{(3)\nu\nu'\nu''}(\check{k},\check{k}')$ is chosen to be zero. The cumulants formed from the free terms cannot eliminate the "live" functions which behave as Dirac delta functions as seen in Fourier space when the asymptotics are performed in physical space. (See Appendix III). However it is shown that the "live" terms in $\Phi_1^{(3)\nu\nu'\nu''}(\check{k},\check{k}')$ do not reoccur as secular growths in $\Phi_3^{(3)\nu\nu'\nu''}$. In the following pages we show that the most dangerous of the live terms (namely, the quasisteady terms) do not give a secular behavior in $\Phi_3^{(3)\nu\nu'\nu''}(\check{k},\check{k}')$. These occur in the terms $\overset{00'0''}{\rho}\langle a_{0'}^{\nu'} a_{0''}^{\nu''}({}_3d_3^{\nu} + {}_2d_3^{\nu})\rangle$ and $\overset{00'0''}{\rho}\langle a_0^{\nu}(b_{2'}^{\nu'} + c_{2'}^{\nu'})\, b_{1''}^{\nu''}\rangle$ which will be examined in the following pages.

Consider the term,

$$\langle a_{0'}^{\nu'} a_{0''}^{\nu''}\, {}_3d_3^{\nu}\rangle = 2\sum_{\nu_1 \cdots \nu_5}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2} L_{k_1 k_3 k_4 k_5}^{\nu_1 \nu_3 \nu_4 \nu_5} \langle a_{0'}^{\nu'} a_{0''}^{\nu''} a_{02}^{\nu_2} a_{03}^{\nu_3} a_{04}^{\nu_4} a_{05}^{\nu_5}\rangle$$

$$\frac{\Delta_{2345,0} - \Delta_{12,0}}{i\,W_{345,1}}\,\delta_{345,1}\,\delta_{12,0}\; d\tilde{k}_1 \cdots d\tilde{k}_5 .$$

A quasisteady behavior can be exhibited for the decompositions denoted by

$$\delta_{0'2}\ \delta_{0''3}\ \delta_{45} \qquad \delta_{0'2}\ \delta_{0''4}\ \delta_{35} \qquad \delta_{0'2}\ \delta_{0''5}\ \delta_{34}$$

$$\delta_{0''2}\ \delta_{0'3}\ \delta_{45} \qquad \delta_{0''2}\ \delta_{0'4}\ \delta_{35} \qquad \delta_{0''2}\ \delta_{0'5}\ \delta_{34} .$$

Since $\delta_{0'2}\,\delta_{0''3}\,\delta_{45}\,\delta_{345,1}\,\delta_{12,0} = \delta_{0'2}\,\delta_{0''3}\,\delta_{45}\,\delta_{10''}\,\delta_{00'0''}$; integrating over $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{k}_3$ and $\tilde{k}_5$ and setting $\nu_5 = -\nu_4$, $\nu_1 = \nu_3 = -\nu''$, $\nu_2 = -\nu'$, one obtains

$$\langle a_{0'}^{\nu'} a_{0''}^{\nu''}\, {}_3d_3^{\nu}\rangle_{q.s.} \cong 2\sum_{\nu_4}\int_{-\infty}^{\infty} L_{-k''-k''\, k_4 - k_4}^{-\nu''-\nu''\,\nu_4 -\nu_4}\, \overset{(2)}{\phi}{}_0^{\nu_4 -\nu_4}(k_4)\, d\tilde{k}_4$$

$$\delta_{00'0''}\, L_{k-k'-k''}^{\nu-\nu'-\nu''}\, \overset{(2)}{\phi}{}_0^{\nu'-\nu'}(k')\, \overset{(2)}{\phi}{}_0^{\nu''-\nu''}(k'')\, \frac{\Delta(-\omega)-\Delta(-\omega)}{i(-\omega+\omega)},$$

where $\omega = +\jmath\omega + \jmath'\omega' + \jmath''\omega''$. In Appendix II it is shown that

$$\frac{\Delta(-\omega) - \Delta(-\omega)}{i(-\omega+\omega)} \sim O(1),$$

but that

$$\frac{\Delta(-\omega) - \Delta(-\omega)}{i(-\omega+\omega)} e^{i\omega t} \sim O(t).$$

From the definition of $\dfrac{\Delta(-\omega) - \Delta(-\omega)}{i(-\omega+\omega)}$,

$$\frac{\Delta(-\omega) - \Delta(-\omega)}{i(-\omega+\omega)} = \int_0^t t e^{-i\omega t} dt$$

$$= \frac{t}{-i\omega} e^{-i\omega t} + \frac{\Delta(-\omega)}{i\omega}.$$

Therefore $\left(\int_0^t t e^{-i\omega t} dt\right) e^{i\omega t} = \dfrac{\Delta(\omega) - t}{i\omega} \sim t\left(\pi\delta(\omega) + \dfrac{iP}{\omega}\right) + O(1).$

The apparent t growth from this quasisteady term is

$$2t\,\delta_{\sigma\sigma'\sigma''}\, L_{k-k'-k''}^{\jmath\,-\jmath'\,-\jmath''}\, \overset{(2)}{\phi}{}_0^{\jmath'-\jmath'}(k')\, \overset{(2)}{\phi}{}_0^{\jmath''-\jmath''}(k'')\, \tilde{\Delta}(\jmath\omega+\jmath'\omega'+\jmath''\omega'') \sum_{\jmath_4} \int_{-\infty}^{\infty} L_{-k''-k''\,k_4-k_4}^{-\jmath''-\jmath''\,\jmath_4-\jmath_4}\, \overset{(2)}{\phi}{}_0^{\jmath_4-\jmath_4}(k_4)\, dk_4,$$

where

$$\tilde{\Delta}(\omega) = \pi\delta(\omega) + \frac{iP}{\omega}.$$

Therefore the total t growth quasisteady behavior of $\langle a_{0'}^{\gamma'} a_{0''}^{\gamma''}\, {}_3d_5^{\gamma} \rangle$ is

$$-2\,\delta_{00'0''}\, \overset{0'0''}{\rho} \left(G_{k-k'-k''}^{s-s'-s''} \; 3 \sum_{\gamma_4} \int_{-\infty}^{\infty} L_{k''k''k_4-k_4}^{\gamma''\gamma''\gamma_4-\gamma_4} \, \overset{(2)}{Q}_0{}^{\gamma_4-\gamma_4}(k_4)\, d\tilde{k}_4 \right),$$

where

$$G_{k-k'-k''}^{s-s'-s''} = L_{k-k'-k''}^{\gamma-\gamma'-\gamma''} \, \overset{(2)}{Q}_0{}^{\gamma'-\gamma'}(k') \, \overset{(2)}{Q}_0{}^{\gamma''-\gamma''}(k'') \, \tilde{\Delta}(\gamma\omega+\gamma'\omega'+\gamma''\omega'').$$

Similar terms from $\langle a_{0'}^{\gamma'} a_{0''}^{\gamma''}\, {}_2d_3^{\gamma} \rangle$ when added to those above gives $-2t\delta_{00'0''}\, \overset{0'0''}{\rho}\, G_{k-k'-k''}^{s-s'-s''}\, \overset{(0)}{F}{}_{k''}^{s''}$, where

$$\overset{(0)}{F}{}_k^{s} = \sum_{\gamma_2} \int \left\{ 3\, L_{kkk_2-k_2}^{\gamma\gamma\gamma_2-\gamma_2} + 2\overset{2-2}{\rho} \sum_{\gamma_1} \int_{-\infty}^{\infty} L_{kk_1k_2}^{\gamma\gamma_1\gamma_2}\, L_{k_1k-k_2}^{\gamma_1\gamma-\gamma_2}\, \frac{i\delta_{12,0}}{w_{12,0}}\, d\tilde{k}_1 \right\} \overset{(2)}{Q}_0{}^{\gamma_2-\gamma_2}(k_2)\, d\tilde{k}_2.$$

Consider the term

$$\langle a_0^{\gamma}\, b_{2'}^{\gamma'}\, b_{1''}^{\gamma''} \rangle = \sum_{\gamma_1\cdots\gamma_5} \int_{-\infty}^{\infty} L_{k'k_1k_2k_3}^{\gamma'\gamma_1\gamma_2\gamma_3}\, L_{k''k_4k_5}^{\gamma''\gamma_4\gamma_5} \langle a_0^{\gamma} a_{01}^{\gamma_1} a_{02}^{\gamma_2} a_{03}^{\gamma_3} a_{04}^{\gamma_4} a_{05}^{\gamma_5} \rangle$$

$$\Delta_{123,0'}\, \Delta_{45,0''}\, \delta_{123,0'}\, \delta_{45,0''}\, d\tilde{k}_1 \cdots d\tilde{k}_5 .$$

The "t" growths occur with quasisteady terms on the decompositions $\delta_{04}\,\delta_{12}\,\delta_{35}$, $\delta_{04}\,\delta_{13}\,\delta_{25}$, $\delta_{04}\,\delta_{23}\,\delta_{15}$, $\delta_{05}\,\delta_{12}\,\delta_{34}$, $\delta_{05}\,\delta_{13}\,\delta_{24}$, $\delta_{05}\,\delta_{23}\,\delta_{14}$. Consider

$$\delta_{04}\,\delta_{12}\,\delta_{35}\,\delta_{123,0'}\,\delta_{45,0''} = \delta_{04}\,\delta_{12}\,\delta_{3,0'}\,\delta_{50'}\,\delta_{00'0''}.$$

Upon integration over $\tilde{k}_2$, $\tilde{k}_4$, $\tilde{k}_3$ and $\tilde{k}_5$ one obtains upon setting $\nu_2 = -\nu_1$, $\nu_4 = -\nu$, $\nu_5 = -\nu'$, $\nu_3 = \nu'$,

$$\langle a_0^{\nu} b_{2'}^{\nu'} b_{1''}^{\nu''} \rangle_{q.s} = t \sum_{\nu_1} \int_{-\infty}^{\infty} L_{k' k_1 -k_1 \, k'}^{\nu' \nu_1 -\nu_1 \nu'} L_{k''-k-k'}^{\nu''-\nu-\nu'} \overset{(2)}{Q}{}_0^{\nu-\nu}(k) \overset{(2)}{Q}{}_0^{\nu'-\nu'}(k')$$

$$\overset{(2)}{Q}{}_0^{\nu_1-\nu_1}(k_1) \, \underline{\Delta}(-s\omega - s'\omega' - s''\omega'') \, d\tilde{k}_1 + O(1).$$

Taking the limit $t \to \infty$ in physical space, one obtains

$$t \, \delta_{00'0''} \, G_{k''-k-k'}^{s''-s-s'} \sum_{\nu_1} \int_{-\infty}^{\infty} L_{k' k' k_1 -k_1}^{\nu' \nu' \nu_1 -\nu_1} \overset{(2)}{Q}{}_0^{\nu_1-\nu_1}(k_1) \, d\tilde{k}_1 .$$

There are six of these terms, which when added to similar terms from $\langle a_0^{\nu} c_{2'}^{\nu'} b_{1''}^{\nu''} \rangle_{q.s}$ give

$$2t \, \delta_{00'0''} \, G_{k''-k-k'}^{s''-s-s'} \overset{(0)}{F}{}_{k'}^{s'} .$$

Therefore

$$\overset{00'0''}{P} \Big\{ G_t \langle a_{0'}^{\nu'} a_{0''}^{\nu''} ({}_1 d_3^{\nu} + {}_2 d_3^{\nu}) \rangle_{q.s} + G_t \langle a_0^{\nu} (b_{2'}^{\nu'} + c_{2'}^{\nu'}) b_{1''}^{\nu''}$$

$$+ a_0^{\nu} (b_{2''}^{\nu''} + c_{2''}^{\nu''}) b_{1'}^{\nu'} \rangle \Big\}$$

$$= -2 \, \delta_{00'0''} \overset{00'0''}{P} G_{k-k'-k''}^{s-s'-s''} \left(\overset{(0)}{F}{}_{k''}^{s''} + \overset{(0)}{F}{}_{k'}^{s'} \right)$$

$$+ 2 \, \delta_{00'0''} \overset{00'0''}{P} G_{k''-k-k'}^{s''-s-s'} \left(\overset{(0)}{F}{}_{k}^{s} + \overset{(0)}{F}{}_{k'}^{s'} \right),$$

$$= 0.$$

Therefore the choice

$$\delta_{oo'o''}\ {}_2\overset{(3)}{\phi}{}_1^{\,\gamma\gamma'\gamma''}(k,k') = \overset{oo'o''}{P}\ \langle a_o^{\gamma}\, a_{o'}^{\gamma'}\, a_{1''}^{\gamma''} \rangle ,$$

$$= - \delta_{oo'o''}\ {}_1\overset{\smile\,(3)}{\phi}{}_1^{\,\gamma\gamma'\gamma''}(k,k') ,$$

is sufficient to ensure that $\overset{(3)}{\phi}{}_3^{\,\gamma\gamma'\gamma''}(k,k') \sim O(1)$, where

$\delta_{oo'o''}\ {}_1\overset{\smile\,(3)}{\phi}{}_1^{\,\gamma\gamma'\gamma''}(k,k')$ is the long time order one behavior

of $\overset{oo'o''}{P}\ \langle a_o^{\gamma}\, a_{o'}^{\gamma'}\, b_{1''}^{\gamma''} \rangle$, not including the "live" terms.

The ε^4 Balance.

Since one is primarily interested in the long time behavior of the spectral energy, at this stage only the term $\overset{(2)}{\Phi}{}_4^{\nu-\nu}(\vec{k})$ will be examined. The ε^4 balance of the governing equation gives

$$a_4^{\nu} = -t\frac{\partial a_0^{\nu}}{\partial T_4} - \frac{t^2}{2}\frac{\partial^2 a_0^{\nu}}{\partial T_2^2} - t\frac{\partial a_2^{\nu}}{\partial T_2} + b_4^{\nu} + c_4^{\nu}$$
$$+ \sum_{i=1}^{3} {}_i d_4^{\nu} + \sum_{i=1}^{7} {}_i e_4^{\nu} + \beta_4^{\nu} + \sum_{j=1}^{4} {}_i\gamma_4^{\nu} + \sum_{i=1}^{11} {}_i\delta_4^{\nu}. \tag{26}$$

$$b_4^{\nu} = \sum_{\nu_1\cdots\nu_5}\int_{-\infty}^{\infty} L_{k\,k_1\,k_2\,k_3\,k_4\,k_5}^{\nu\,\nu_1\,\nu_2\,\nu_3\,\nu_4\,\nu_5}\, a_{01}^{\nu_1} a_{02}^{\nu_2} a_{03}^{\nu_3} a_{04}^{\nu_4} a_{05}^{\nu_5}\, \Delta_{12345,0}\, \delta_{12345,0}\, d\vec{k}_1 \cdots d\vec{k}_5 .$$

$$c_4^{\nu} = 4\sum_{\nu_1\cdots\nu_6}\int_{-\infty}^{\infty} L_{k\,k_1\,k_2\,k_3\,k_4}^{\nu\,\nu_1\,\nu_2\,\nu_3\,\nu_4}\, L_{k_1\,k_5\,k_6}^{\nu_1\,\nu_5\,\nu_6}\, a_{02}^{\nu_2} a_{03}^{\nu_3} a_{04}^{\nu_4} a_{05}^{\nu_5} a_{06}^{\nu_6}\, \frac{\Delta_{23456,0} - \Delta_{1234,0}}{i\,W_{56.1}}\, \delta_{1234,0}\, \delta_{56.1}\, d\vec{k}_1 \cdots d\vec{k}_6 .$$

$$ {}_1d_4^{\nu} = 3\sum_{\nu_1\nu_2\nu_3}\int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\nu\nu_1\nu_2\nu_3}\, a_{01}^{\nu_1}\int_0^t b_{12}^{\nu_2}\, b_{13}^{\nu_3}\, e^{iW_{123,0}t}\, dt\, \delta_{123,0}\, d\breve{k}_1 d\breve{k}_2 d\breve{k}_3 . $$

$$ {}_2d_4^{\nu} = 3\sum_{\nu_1\nu_2\nu_3}\int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\nu\nu_1\nu_2\nu_3}\, a_{02}^{\nu_2}\, a_{03}^{\nu_3}\int_0^t c_{21}^{\nu_1}\, e^{iW_{123,0}t}\, dt\, \delta_{123,0}\, d\breve{k}_1 d\breve{k}_2 d\breve{k}_3 . $$

$$ {}_3d_4^{\nu} = 3\sum_{\nu_1\nu_2\nu_3}\int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\nu\nu_1\nu_2\nu_3}\, a_{02}^{\nu_2}\, a_{03}^{\nu_3}\int_0^t b_{21}^{\nu_1}\, e^{iW_{123,0}t}\, dt\, \delta_{123,0}\, d\breve{k}_1 d\breve{k}_2 d\breve{k}_3 . $$

$$ {}_1l_4^{\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\int_0^t b_{11}^{\nu_1}\, b_{22}^{\nu_2}\, e^{iW_{12,0}t}\, dt\, \delta_{12,0}\, d\breve{k}_1 d\breve{k}_2 . $$

$$ {}_2l_4^{\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\int_0^t b_{11}^{\nu_1}\, c_{22}^{\nu_2}\, e^{iW_{12,0}t}\, dt\, \delta_{12,0}\, d\breve{k}_1 d\breve{k}_2 . $$

$$ {}_3l_4^{\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\int_0^t b_{31}^{\nu_1}\, a_{02}^{\nu_2}\, e^{iW_{12,0}t}\, dt\, \delta_{12,0}\, d\breve{k}_1 d\breve{k}_2 . $$

$$ {}_4l_4^{\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\, a_{02}^{\nu_2}\int_0^t c_{31}^{\nu_1}\, e^{iW_{12,0}t}\, dt\, \delta_{12,0}\, d\breve{k}_1 d\breve{k}_2 . $$

$$ {}_5l_4^{\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\, a_{02}^{\nu_2}\int_0^t {}_1d_{31}^{\nu_1}\, e^{iW_{12,0}t}\, dt\, \delta_{12,0}\, d\breve{k}_1 d\breve{k}_2 . $$

$$ {}_6l_4^{\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\, a_{02}^{\nu_2}\int_0^t {}_2d_{31}^{\nu_1}\, e^{iW_{12,0}t}\, dt\, \delta_{12,0}\, d\breve{k}_1 d\breve{k}_2 . $$

$$ {}_7\ell_4^{\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\, a_{02}^{\nu_2}\int_0^t {}_3a_{31}^{\nu_1}\, e^{i\omega_{12,0}t}\,dt\,\delta_{12,0}\,dk_1 dk_2. $$

$$ \beta_4^{\nu} = 4\sum_{\nu_1\cdots\nu_4}\int_{-\infty}^{\infty} L_{k k_1 k_2 k_3 k_4}^{\nu\nu_1\nu_2\nu_3\nu_4}\, \alpha_{11}^{\nu_1} a_{02}^{\nu_2} a_{03}^{\nu_3} a_{04}^{\nu_4}\, \Delta_{1234,0}\,\delta_{1234,0}\,dk_1\cdots dk_4. $$

$$ {}_1\gamma_4^{\nu} = 3\sum_{\nu_1\nu_2\nu_3}\int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\nu\nu_1\nu_2\nu_3}\, a_{02}^{\nu_2} a_{03}^{\nu_3}\int_0^t \beta_{21}^{\nu_1}\, e^{i\omega_{123,0}t}\,dt\,\delta_{123,0}\,dk_1 dk_2 dk_3. $$

$$ {}_2\gamma_4^{\nu} = 3\sum_{\nu_1\nu_2\nu_3}\int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\nu\nu_1\nu_2\nu_3}\, \alpha_{21}^{\nu_1} a_{02}^{\nu_2} a_{03}^{\nu_3}\, \Delta_{123,0}\,\delta_{123,0}\,dk_1 dk_2 dk_3. $$

$$ {}_3\gamma_4^{\nu} = 3\sum_{\nu_1\nu_2\nu_3}\int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\nu\nu_1\nu_2\nu_3}\, a_{01}^{\nu_1} P^{23}\alpha_{13}^{\nu_3}\int_0^t b_{12}^{\nu_2}\, e^{i\omega_{123,0}t}\,dt\,\delta_{123,0}\,dk_1 dk_2 dk_3. $$

$$ {}_4\gamma_4^{\nu} = 3\sum_{\nu_1\nu_2\nu_3}\int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\nu\nu_1\nu_2\nu_3}\, a_{01}^{\nu_1}\alpha_{12}^{\nu_2}\alpha_{13}^{\nu_3}\, \Delta_{123,0}\,\delta_{123,0}\,dk_1 dk_2 dk_3. $$

$$ {}_1\delta_4^{\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\int_0^t b_{11}^{\nu_1}\beta_{22}^{\nu_2}\, e^{i\omega_{12,0}t}\,dt\,\delta_{12,0}\,dk_1 dk_2. $$

$$ {}_2\delta_4^{\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\int_0^t b_{11}^{\nu_1}\alpha_{22}^{\nu_2}\, e^{i\omega_{12,0}t}\,dt\,\delta_{12,0}\,dk_1 dk_2. $$

$$ {}_3\delta_4^{\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\int_0^t b_{22}^{\nu_2}\alpha_{11}^{\nu_1}\, e^{i\omega_{12,0}t}\,dt\,\delta_{12,0}\,dk_1 dk_2. $$

$$ {}_4\delta_4^{\,\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2} \int_0^t C_{22}^{\nu_2}\alpha_{11}^{\nu_1} e^{iW_{12,0}t}\, \delta_{12,0}\, dk_1 dk_2. $$

$$ {}_5\delta_4^{\,\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2} \int_0^t \beta_{22}^{\nu_2}\alpha_{11}^{\nu_1} e^{iW_{12,0}t}\, \delta_{12,0}\, dk_1 dk_2. $$

$$ {}_6\delta_4^{\,\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\, \alpha_{11}^{\nu_1}\alpha_{22}^{\nu_2}\, \Delta_{12,0}\, \delta_{12,0}\, dk_1 dk_2. $$

$$ {}_7\delta_4^{\,\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\, a_{02}^{\nu_2} \int_0^t \beta_{31}^{\nu_1} e^{iW_{12,0}t}\, dt\, \delta_{12,0}\, dk_1 dk_2. $$

$$ {}_8\delta_4^{\,\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\, a_{02}^{\nu_2} \int_0^t {}_1\gamma_{31}^{\nu_1} e^{iW_{12,0}t}\, dt\, \delta_{12,0}\, dk_1 dk_2. $$

$$ {}_9\delta_4^{\,\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\, a_{02}^{\nu_2} \int_0^t {}_2\gamma_{31}^{\nu_1} e^{iW_{12,0}t}\, dt\, \delta_{12,0}\, dk_1 dk_2. $$

$$ {}_{10}\delta_4^{\,\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\, a_{02}^{\nu_2} \int_0^t {}_3\gamma_{31}^{\nu_1} e^{iW_{12,0}t}\, dt\, \delta_{12,0}\, dk_1 dk_2. $$

$$ {}_{11}\delta_4^{\,\nu} = 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\, a_{02}^{\nu_2} \int_0^t {}_4\gamma_{31}^{\nu_1} e^{iW_{12,0}t}\, dt\, \delta_{12,0}\, dk_1 dk_2. $$

One is interested in the long time behavior of

$$\langle a_0^{\nu'}(k') a_4^{\nu}(k) + a_1^{\nu'}(k') a_3^{\nu}(k) + a_2^{\nu'}(k) a_2^{\nu}(k) + a_1^{\nu}(k) a_3^{\nu'}(k') + a_0^{\nu}(k) a_4^{\nu'}(k') \rangle$$

in the particular case when $\nu' = -\nu$. A little manipulation, using previous expansions for $a_2^{\nu}(k)$ and $a_3^{\nu}(k)$ shows that the above expression equals

$$-t \frac{\partial}{\partial T_4} \langle a_0^{\nu'}(k') a_0^{\nu}(k) \rangle_{\nu'=-\nu} - \frac{t^2}{2} \frac{\partial^2}{\partial T_2^2} \langle a_0^{\nu'}(k') a_0^{\nu}(k) \rangle_{\nu'=-\nu}$$

$$-t \frac{\partial}{\partial T_2} \langle a_0^{\nu'}(k') a_2^{\nu}(k) + a_1^{\nu'}(k') a_1^{\nu}(k) + a_0^{\nu}(k) a_2^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$+ \langle a_0^{\nu'}(k') b_4^{\nu}(k) + a_0^{\nu}(k) b_4^{\nu'}(k') \rangle_{\nu'=-\nu} + \langle a_0^{\nu'}(k') c_4^{\nu}(k) + a_0^{\nu}(k) c_4^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$+ \sum_{j=1}^{3} \langle a_0^{\nu'}(k')\, {}_j d_4^{\nu}(k) + a_0^{\nu}(k)\, {}_j d_4^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$+ \sum_{j=1}^{7} \langle a_0^{\nu'}(k')\, {}_j e_4^{\nu}(k) + a_0^{\nu}(k)\, {}_j e_4^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$+ \langle b_1^{\nu'}(k') b_3^{\nu}(k) + b_1^{\nu}(k) b_3^{\nu'}(k') \rangle_{\nu'=-\nu} + \langle b_1^{\nu'}(k') c_3^{\nu}(k) + b_1^{\nu}(k) c_3^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$+ \sum_{j=1}^{3} \langle b_1^{\nu'}(k')\, {}_j d_3^{\nu}(k) + b_1^{\nu}(k)\, {}_j d_3^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$+ \langle b_2^{\nu'}(k') b_2^{\nu}(k) \rangle_{\nu'=-\nu} + \langle b_2^{\nu'}(k') c_2^{\nu}(k) + b_2^{\nu}(k) c_2^{\nu'}(k') \rangle_{\nu'=-\nu} + \langle c_2^{\nu}(k) c_2^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$+ \langle \beta_2^{\nu'}(k') b_2^{\nu}(k) + \beta_2^{\nu}(k) b_2^{\nu'}(k') \rangle_{\nu'=-\nu} + \langle \beta_2^{\nu'}(k') c_2^{\nu}(k) + \beta_2^{\nu}(k) c_2^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$+ \langle \alpha_2^{\eta'}(k')\, b_2^{\eta}(k) + \alpha_2^{\eta}(k)\, b_2^{\eta'}(k') \rangle_{\eta'=-\eta} + \langle \alpha_2^{\eta'}(k') c_2^{\eta}(k) + \alpha_2^{\eta}(k) c_2^{\eta'}(k') \rangle_{\eta'=-\eta}$$

$$+ \langle \beta_2^{\eta}(k)\, \beta_2^{\eta'}(k') \rangle_{\eta'=-\eta} + \langle \beta_2^{\eta'}(k')\, \alpha_2^{\eta}(k) + \beta_2^{\eta}(k)\, \alpha_2^{\eta'}(k') \rangle_{\eta'=-\eta} + \langle \alpha_2^{\eta'}(k') \alpha_2^{\eta}(k) \rangle_{\eta'=-\eta}$$

$$+ \langle \alpha_1^{\eta'}(k')\, b_3^{\eta}(k) + \alpha_1^{\eta}(k)\, b_3^{\eta'}(k') \rangle_{\eta'=-\eta} + \langle \alpha_1^{\eta'}(k')\, c_3^{\eta}(k) + \alpha_1^{\eta}(k)\, c_3^{\eta'}(k') \rangle_{\eta'=-\eta}$$

$$+ \sum_{j=1}^{3} \langle \alpha_1^{\eta'}(k')\, {}_j d_3^{\eta}(k) + \alpha_1^{\eta}(k)\, {}_j d_3^{\eta'}(k') \rangle_{\eta'=-\eta}$$

$$+ \langle b_1^{\eta'}(k')\, \beta_3^{\eta}(k) + b_1^{\eta}(k)\, \beta_3^{\eta'}(k') \rangle_{\eta'=-\eta} + \sum_{j=1}^{4} \langle b_1^{\eta'}(k')\, {}_j \gamma_3^{\eta}(k) + b_1^{\eta}(k)\, {}_j \gamma_3^{\eta'}(k') \rangle_{\eta'=-\eta}$$

$$+ \langle \alpha_1^{\eta'}(k')\, \beta_3^{\eta}(k) + \alpha_1^{\eta}(k)\, \beta_3^{\eta'}(k') \rangle_{\eta'=-\eta}$$

$$+ \sum_{j=1}^{4} \langle \alpha_1^{\eta'}(k')\, {}_j \gamma_3^{\eta}(k) + \alpha_1^{\eta}(k)\, {}_j \gamma_3^{\eta'}(k') \rangle_{\eta'=-\eta}$$

$$+ \langle c_0^{\eta'}(k')\, \beta_4^{\eta}(k) + c_0^{\eta}(k)\, \beta_4^{\eta'}(k') \rangle_{\eta'=-\eta}$$

$$+ \sum_{j=1}^{4} \langle c_0^{\eta'}(k')\, {}_j \gamma_4^{\eta}(k) + c_0^{\eta}(k)\, {}_j \gamma_4^{\eta'}(k') \rangle_{\eta'=-\eta}$$

$$+ \sum_{j=1}^{11} \langle c_0^{\eta'}(k')\, {}_j \delta_4^{\eta}(k) + c_0^{\eta}(k)\, {}_j \delta_4^{\eta'}(k') \rangle_{\eta'=-\eta} .$$

These terms are now examined individually for their possible secular growths. The first term considered is

$$\langle b_2^{\gamma}(k)\, b_2^{\gamma'}(k') \rangle_{\gamma' = -\gamma} = \sum_{\gamma_1 \cdots \gamma_6} \int_{-\infty}^{\infty} L^{\gamma\gamma_1\gamma_2\gamma_3}_{k k_1 k_2 k_3} L^{\gamma'\gamma_4\gamma_5\gamma_6}_{k' k_4 k_5 k_6}$$

$$\langle a_{01}^{\gamma_1} a_{02}^{\gamma_2} a_{03}^{\gamma_3} a_{04}^{\gamma_4} a_{05}^{\gamma_5} a_{06}^{\gamma_6} \rangle \Delta_{123,0} \Delta_{456,0'}$$

$$\delta_{123,0}\, \delta_{456,0'}\; d\tilde{k}_1 \cdots d\tilde{k}_6 \ldots \mathrm{I}$$

The mean value of six quantities decomposes into the spectral cumulants, 2×4, 0×6, 3×3, $2 \times 2 \times 2$. These decompositions will be shown in the following array.

$[2 \times 2 \times 2]$

$\delta_{12}\ \delta_{34}\ \delta_{56}$	$\delta_{12}\ \delta_{35}\ \delta_{46}$	$\delta_{12}\ \delta_{36}\ \delta_{45}$
$\delta_{13}\ \delta_{24}\ \delta_{56}$	$\delta_{13}\ \delta_{25}\ \delta_{46}$	$\delta_{13}\ \delta_{26}\ \delta_{45}$
$\delta_{14}\ \delta_{23}\ \delta_{56}$	$\delta_{14}\ \delta_{25}\ \delta_{36}$	$\delta_{14}\ \delta_{26}\ \delta_{35}$
$\delta_{15}\ \delta_{23}\ \delta_{46}$	$\delta_{15}\ \delta_{24}\ \delta_{36}$	$\delta_{15}\ \delta_{26}\ \delta_{34}$
$\delta_{16}\ \delta_{23}\ \delta_{45}$	$\delta_{16}\ \delta_{24}\ \delta_{35}$	$\delta_{16}\ \delta_{25}\ \delta_{34}$,

where $\delta_{12}\ \delta_{34}\ \delta_{56}$ is the notation for the decomposition

$$\delta(\breve{k}_1+\breve{k}_2)\,\delta(\breve{k}_3+\breve{k}_4)\,\delta(\breve{k}_5+\breve{k}_6)\ \overset{(2)\ \eta_1\eta_2}{\varphi_0}(\breve{k}_1)\ \overset{(2)\ \eta_3\eta_4}{\varphi_0}(\breve{k}_3)\ \overset{(2)\ \eta_5\eta_6}{\varphi_0}(\breve{k}_5).$$

From a previous formula it is shown that the number of independent decompositions of a sixth order mean value into products of second order cumulants is $\frac{{}_6C_4\ {}_4C_2\ {}_2C_2}{3!} = 15.$

From symmetry in (1,2,3) and (4,5,6) one only has two basic terms, S_1 and S_2, where S_1 is typified by $\delta_{12}\,\delta_{34}\,\delta_{56}$ and S_2 is typified by $\delta_{14}\ \delta_{25}\ \delta_{36}$.

S_1: Since $\delta_{12}\ \delta_{34}\ \delta_{56}\ \delta_{123,0}\ \delta_{456,0'}$

$$= \delta_{12}\ \delta_{34}\ \delta_{56}\ \delta_{3,0}\ \delta_{4,0'},$$

$$= \delta_{12}\ \delta_{00'}\ \delta_{56}\ \delta_{3,0}\ \delta_{40}.$$

If one integrates in I over $\breve{k}_2$, $\breve{k}_3$, $\breve{k}_4$ and $\breve{k}_6$ one obtains

$$S_1 = \delta(k+k')\sum_{\eta_1\cdots\eta_6}\int_{-\infty}^{\infty} \mathcal{L}^{\eta\,\eta_1\,\eta_2\,\eta_3}_{k\,k_1\,-k_1\,k}\ \mathcal{L}^{-\eta\,\eta_4\,\eta_5\,\eta_6}_{-k\,-k\,k_5\,-k_5}\ \overset{(2)\ \eta_1\eta_2}{\varphi_0}(\breve{k}_1)\ \overset{(2)\ \eta_3\eta_4}{\varphi_0}(\breve{k})$$

$$\overset{(2)\ \eta_5\eta_6}{\varphi_0}(\breve{k}_5)\ \Delta\big((\eta_1+\eta_2)\omega_1+(\eta_3-\eta)\omega\big)\ \Delta\big((\eta_4+\eta)\omega+(\eta_5+\eta_6)\omega_5\big)\ d\breve{k}_1 d\breve{k}_5.$$

There are possibilities of t^2, t and $O(1)$ terms here. The t^2 terms arise when one sets the arguments of both Δ functions zero by appropriately choosing the sign parameters. The second order cumulants connected with this term are energy densities, namely of the $Q_0^{(2)\nu-\nu}(\bar{k})$ type. However one may also have the case where the argument of one of the Δ functions is zero but the other is nonzero. At least one of the second order cumulants connected with this term is a $Q_0^{(2)++}(\bar{k})$ type. In fact neither of these terms persist. It will be shown that both the t^2 and the t growth terms cancel with similar terms from $\langle a_0^{-\nu}(k')\,{}_3d_4^{\nu}(k) + a_0^{\nu}(k)\,{}_3d_4^{-\nu}(k')\rangle$. One can represent

$$\Delta\left((\nu_1+\nu_2)\omega_1 + (\nu_3-\nu)\omega\right) = \delta_{-\nu_1}^{\nu_2}\,\delta_{\nu}^{\nu_3}\, t + \delta_{-\nu_1}^{\nu_2}\,\delta_{-\nu}^{\nu_3}\,\Delta(-2\nu\omega)$$
$$+ \delta_{\nu_1}^{\nu_2}\,\delta_{\nu}^{\nu_3}\,\Delta(2\nu_1\omega_1) + \delta_{\nu_1}^{\nu_2}\,\delta_{-\nu}^{\nu_3}\,\Delta(2\nu_1\omega_1 - 2\nu\omega),$$

and

$$\Delta\left((\nu_4+\nu)\omega + (\nu_5+\nu_6)\omega_5\right) = \delta_{-\nu}^{\nu_4}\,\delta_{-\nu_5}^{\nu_6}\, t + \delta_{-\nu_5}^{\nu_6}\,\delta_{\nu}^{\nu_4}\,\Delta(2\nu\omega)$$
$$+ \delta_{\nu_5}^{\nu_6}\,\delta_{-\nu}^{\nu_4}\,\Delta(2\nu_5\omega_5) + \delta_{\nu}^{\nu_4}\,\delta_{\nu_5}^{\nu_6}\,\Delta(2\nu_5\omega_5 + 2\nu\omega),$$

where $\delta^{\nu_i}_{\nu_j}$ is the Kronecker delta.

$$\delta^{\nu_i}_{\nu_j} = \begin{cases} 1, & \nu_i = \nu_j \\ 0, & \nu_i \neq \nu_j \end{cases} .$$

The product of these two Δ functions gives

$$\Delta\big((\nu_1+\nu_2)\omega_1 + (\nu_3-\nu)\omega\big)\, \Delta\big((\nu_4+\nu)\omega + (\nu_5+\nu_6)\omega_5\big)$$

$$= \overset{(1)}{\delta^{\nu_2}_{-\nu_1}\delta^{\nu_3}_{\nu}\delta^{\nu_4}_{-\nu}\delta^{\nu_6}_{-\nu_5}\, t^2} + t\Big[\overset{(2)}{\delta^{\nu_2}_{-\nu_1}\delta^{\nu_3}_{\nu}\delta^{\nu_6}_{-\nu_5}\delta^{\nu_4}_{\nu}\,\Delta(2\nu\omega)}$$

$$+ \delta^{\nu_4}_{-\nu_1}\delta^{\nu_6}_{-\nu_5}\delta^{\nu_2}_{-\nu_1}\delta^{\nu_3}_{-\nu}\,\Delta(-2\nu\omega)\Big]$$

$$\overset{(3)}{+\, t\Big[\delta^{\nu_2}_{-\nu_1}\delta^{\nu_3}_{\nu}\delta^{\nu_6}_{-\nu_5}\delta^{\nu_4}_{-\nu}\,\Delta(2\nu_5\omega_5) + \delta^{\nu_4}_{-\nu}\delta^{\nu_6}_{-\nu_5}\delta^{\nu_2}_{\nu_1}\delta^{\nu_3}_{\nu}\,\Delta(2\nu_1\omega_1)\Big]}$$

$$\overset{(4)}{+\, t\Big\{\delta^{\nu_2}_{-\nu_1}\delta^{\nu_3}_{\nu}\delta^{\nu_4}_{\nu}\delta^{\nu_6}_{\nu_5}\,\Delta(2\nu_5\omega_5 + 2\nu\omega)}$$

$$+ \delta^{\nu_4}_{-\nu}\delta^{\nu_6}_{-\nu_5}\delta^{\nu_2}_{\nu_1}\delta^{\nu_3}_{-\nu}\,\Delta(2\nu_1\omega_1 - 2\nu\omega)\Big]$$

$$\overset{(5)}{+\, \delta^{\nu_2}_{-\nu_1}\delta^{\nu_3}_{-\nu}\delta^{\nu_6}_{-\nu_5}\delta^{\nu_4}_{\nu}\,\Delta(2\nu\omega)\,\Delta(-2\nu\omega)} + O(1).$$

S_{15}: the final term in the above behaves in long time like $2\pi t\,\delta(\omega)$. Since $\omega = 0$ implies that $\check{k} = 0$, then from the mean value property there can be no contribution from this term. Let S_{11} denote the contribution from the first term in $\Delta\cdot\Delta$, then

$$q S_{11} = q t^2 \delta(k+k') \overset{(2)}{\phi}{}_0^{\nu-\nu}(\check{k}) \sum_{\nu_1\nu_5} \int_{-\infty}^{\infty} \mathcal{L}^{\nu\,\nu_1\,-\nu_1\,\nu}_{k\,k_1\,-k_1\,k}\, \mathcal{L}^{-\nu\,-\nu\,\nu_5\,-\nu_5}_{-k\,-k\,k_5\,-k_5}$$
$$\overset{(2)}{\phi}{}_0^{\nu_1-\nu_1}(\check{k}_1)\, \overset{(2)}{\phi}{}_0^{\nu_5-\nu_5}(k_5)\, d\check{k}_1 d\check{k}_5 .$$

Using

$$\mathcal{L}^{-\nu\,-\nu\,\nu_5\,-\nu_5}_{-k\,-k\,k_5\,-k_5} = -\mathcal{L}^{\nu\,\nu\,\nu_5\,-\nu_5}_{k\,k\,k_5\,-k_5},$$

we find that

$$q S_{11} = -q t^2 \delta(k+k') \overset{(2)}{\phi}{}_0^{\nu-\nu}(\check{k}) \left(\sum_{\nu_1} \int_{-\infty}^{\infty} \mathcal{L}^{\nu\,\nu\,\nu_1\,-\nu_1}_{k\,k\,k_1\,-k_1} \overset{(2)}{\phi}{}_0^{\nu_1-\nu_1}(\check{k}_1)\, d\check{k}_1 \right)^2 \quad \text{---}(1)_1$$

Let S_{13} denote the third term in the $\Delta\cdot\Delta$ expression.

$$q S_{13} = q t\, \delta(\check{k}+\check{k}') \sum_{\nu_1\nu_5} \int_{-\infty}^{\infty} \mathcal{L}^{\nu\,\nu_1\,-\nu_1\,\nu}_{k\,k_1\,-k_1\,k}\, \mathcal{L}^{-\nu\,-\nu\,\nu_5\,\nu_5}_{-k\,-k\,k_5\,-k_5}$$
$$\overset{(2)}{\phi}{}_0^{\nu_1-\nu_1}(k_1)\, \overset{(2)}{\phi}{}_0^{\nu-\nu}(k)\, \overset{(2)}{\phi}{}_0^{\nu_5\,\nu_5}(k_5)\, \Delta(2\nu_5\omega_5)\, d\check{k}_1 d\check{k}_5$$

$$+\; q t\, \delta(k+k') \sum_{\nu_1\nu_5} \int_{-\infty}^{\infty} \mathcal{L}^{\nu\,\nu_1\,\nu_1\,\nu}_{k\,k_1\,-k_1\,k}$$

$$\times \; \mathcal{L}^{-\nu\; -\nu\; \nu_5\; -\nu_5}_{-k\; -k\; k_5\; -k_5} \; \overset{(2)}{\Phi}{}^{\nu_1 -\nu_1}_0(\tilde{k}_1) \; \overset{(2)}{\Phi}{}^{\nu -\nu}_0(k) \; \overset{(2)}{\Phi}{}^{\nu_5 -\nu_5}_0(k_5) \; \Delta(2\nu_1\omega_1) \, d\tilde{k}_1 \, d\tilde{k}_5 .$$

In the second integral change $\nu_1 \to -\nu_5, \quad \nu_5 \to \nu_1$, $\tilde{k}_1 \to -\tilde{k}_5, \quad \tilde{k}_5 \to \tilde{k}_1$.

Then $q S_{13}$ becomes

$$q t \, \delta(k+k') \sum_{\nu_1\nu_5} \int_{-\infty}^{\infty} \mathcal{L}^{\nu\; \nu_1\; -\nu_1\; \nu}_{k\; k_1\; -k_1\; k} \; \mathcal{L}^{-\nu\; -\nu\; \nu_5\; \nu_5}_{-k\; -k\; k_5\; -k_5} \; \overset{(2)}{\Phi}{}^{\nu_1 -\nu_1}_0(\tilde{k}_1)$$

$$\overset{(2)}{\Phi}{}^{\nu -\nu}_0(k) \; \overset{(2)}{\Phi}{}^{\nu_5 \nu_5}_0(k_5) \; \Delta(2\nu_5\omega_5) \, d\tilde{k}_1 \, d\tilde{k}_5$$

$$+ \; q t \, \delta(k+k') \sum_{\nu_1\nu_5} \int_{-\infty}^{\infty} \mathcal{L}^{\nu\; -\nu_5\; -\nu_5\; \nu}_{k\; -k_5\; k_5\; k} \; \mathcal{L}^{-\nu\; -\nu\; \nu_1\; -\nu_1}_{-k\; -k\; k_1\; -k_1} \; \overset{(2)}{\Phi}{}^{-\nu_5 -\nu_5}_0(-\tilde{k}_5)$$

$$\overset{(2)}{\Phi}{}^{\nu -\nu}_0(k) \; \overset{(2)}{\Phi}{}^{\nu_1 -\nu_1}_0(\tilde{k}_1) \; \Delta(-2\nu_5\omega_5) \, d\tilde{k}_1 \, d\tilde{k}_5 ,$$

$$= q t \, \delta(k+k') \sum_{\nu_1\nu_5} \int_{-\infty}^{\infty} \mathcal{L}^{\nu\; \nu_1\; -\nu_1\; \nu}_{k\; k_1\; -k_1\; k} \; \mathcal{L}^{-\nu\; -\nu\; \nu_5\; \nu_5}_{-k\; -k\; k_5\; -k_5} \; \overset{(2)}{\Phi}{}^{\nu_1 -\nu_1}_0(k_1)$$

$$\overset{(2)}{\Phi}{}^{\nu -\nu}_0(k) \left[\overset{(2)}{\Phi}{}^{\nu_5 \nu_5}_0(k_5) \; \Delta(2\nu_5\omega_5) + \text{Comp. Conj.} \right] d\tilde{k}_1 \, d\tilde{k}_5 .$$

$$\text{---} \quad (1)_3$$

Similarly,

$$gS_{14} = gt\,\delta(k+k')\sum_{\lambda_1\lambda_5}\int_{-\infty}^{\infty} L^{\lambda\,\lambda\,\lambda_1\,-\lambda_1}_{k\,k\,k_1\,-k_1}\, L^{-\lambda\,\lambda\,\lambda_5\,\lambda_5}_{-k\,-k\,k_5\,-k_5}\, \Phi_0^{(2)\lambda_1-\lambda_1}(k_1)$$

$$\left[\Phi_0^{(1)\lambda\lambda}(k)\,\Phi_0^{(2)\lambda_5\lambda_5}(k_5)\,\Delta(2\lambda_5\omega_5+2\lambda\omega) + \text{Comp. Conj.}\right] d\breve{k}_1 d\breve{k}_5 .$$

$$--- (1)_4 .$$

Without any manipulation

$$gS_{12} = gt\,\delta(k+k')\sum_{\lambda_1\lambda_5}\int_{-\infty}^{\infty} L^{\lambda\,\lambda_1\,-\lambda_1\,\lambda}_{k\,k_1\,-k_1\,k}\, L^{-\lambda\,\lambda\,\lambda_5\,-\lambda_5}_{-k\,-k\,k_5\,-k_5}$$

$$\Phi_0^{(2)\lambda_1-\lambda_1}(k_1)\,\Phi_0^{(2)\lambda\lambda}(k)\,\Phi_0^{(2)\lambda_5-\lambda_5}(k_5)\,\Delta(2\lambda\omega)\, d\breve{k}_1 d\breve{k}_5$$

$$+\, gt\,\delta(k+k')\sum_{\lambda_1\lambda_5}\int_{-\infty}^{\infty} L^{\lambda\,\lambda_1\,-\lambda_1\,-\lambda}_{k\,k_1\,-k_1\,k}\, L^{-\lambda\,-\lambda\,\lambda_5\,-\lambda_5}_{-k\,-k\,k_5\,-k_5}$$

$$\Phi_0^{(2)\lambda_1-\lambda_1}(k_1)\,\Phi_0^{(2)\lambda_5-\lambda_5}(k_5)\,\Phi_0^{(2)-\lambda-\lambda}(k)\,\Delta(-2\lambda\omega)\, d\breve{k}_1 d\breve{k}_5 .$$

$$----- (1)_2$$

Note that as $\overset{(2)}{\phi}{}_0^{\,\nu\nu'}(k) = \overset{(2)}{\phi}{}_0^{\,\nu'\nu}(-k)$,

$$\overset{(2)}{\phi}{}_0^{\,-\nu-\nu}(k) = \overset{(2)}{\phi}{}_0^{\,-\nu-\nu}(-k) = \overset{(2)}{\phi}{}_0^{\,*\nu\nu}(k).$$

Consider the contribution from the second type of decomposition

$$S_2: \; \delta_{14}\,\delta_{25}\,\delta_{36}\,\delta_{123,0}\,\delta_{456,0'} = \delta_{00'}\,\delta_{14}\,\delta_{25}\,\delta_{36}\,\delta_{123,0}\,.$$

Therefore, integrating over $\breve{k}_4, \breve{k}_5, \breve{k}_0$,

$$S_2 = \delta(k+k') \sum_{\nu_1 \cdots \nu_6} \int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\nu\nu_1\nu_2\nu_3}\, L_{-k-k_1-k_2-k_3}^{-\nu\,\nu_4\nu_5\nu_6}\, \overset{(2)}{\phi}{}_0^{\,\nu_1\nu_4}(k_1)\, \overset{(2)}{\phi}{}_0^{\,\nu_2\nu_5}(k_2)\, \overset{(2)}{\phi}{}_0^{\,\nu_3\nu_6}(k_3)$$

$$\Delta(s_1\omega_1 + s_2\omega_2 + s_3\omega_3 - s\omega)\,\Delta(s_4\omega_1 + s_5\omega_2 + s_6\omega_3 + s\omega)\;\delta_{123,0}\; d\breve{k}_1\, d\breve{k}_2\, d\breve{k}_3\,.$$

The only way a "t" growth may occur is when

$$\nu_4\omega_1 + \nu_5\omega_2 + \nu_6\omega_3 + \nu\omega = -\nu_1\omega_1 - \nu_2\omega_2 - \nu_3\omega_3 + \nu\omega,$$

that is, when $\nu_4\nu_1 = \nu_5\nu_2 = \nu_6\nu_3 = -1$. Since $\Delta(\omega)\,\Delta(-\omega) \sim 2\pi t\,\delta(\omega) + O(1)$, (see Appendix II, page 258) one has

$$6S_2 \sim 12\pi t\,\delta(k+k') \sum_{\nu_1\nu_2\nu_3} \int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\nu\nu_1\nu_2\nu_3}\, L_{-k-k_1-k_2-k_3}^{-\nu-\nu_1-\nu_2-\nu_3}$$

$$\overset{(2)}{\phi}{}_0^{\,\nu_1-\nu_1}(k_1)\, \overset{(2)}{\phi}{}_0^{\,\nu_2-\nu_2}(k_2)\, \overset{(2)}{\phi}{}_0^{\,\nu_3-\nu_3}(k_3)\; \delta^1_{123,0}\, \delta_{123,0}\, d\breve{k}_1\, d\breve{k}_2\, d\breve{k}_3, \qquad \text{---- (2)}$$

where $\delta^1_{123,0} = \delta(s_1\omega_1 + s_2\omega_2 + s_3\omega_3 - s\omega)$.

The (2 X 4) decomposition has $\frac{{}^6C_4}{1!}\ \frac{{}^2C_2}{1!} = 15$ terms. Following the preceding notation, these may be laid out in the array,

$$\begin{array}{ccccc}
\delta_{12}\,\delta^{(1)}_{3456} & \delta_{13}\,\delta^{(1)}_{2456} & \delta_{14}\,\delta^{(2)}_{2356} & \delta_{15}\,\delta^{(2)}_{2346} & \delta_{16}\,\delta^{(2)}_{2345} \\
 & \delta_{23}\,\delta^{(1)}_{1456} & \delta_{24}\,\delta^{(2)}_{1356} & \delta_{25}\,\delta^{(2)}_{1346} & \delta_{26}\,\delta^{(2)}_{1345} \\
 & & \delta_{34}\,\delta^{(2)}_{1256} & \delta_{35}\,\delta^{(2)}_{1246} & \delta_{36}\,\delta^{(2)}_{1245} \\
 & & & \delta_{45}\,\delta^{(3)}_{1236} & \delta_{46}\,\delta^{(3)}_{1235} \\
 & & & & \delta_{56}\,\delta^{(3)}_{1234}.
\end{array}$$

S_1: Since $\delta_{12}\,\delta_{3456}\,\delta_{123,0}\,\delta_{456,0'}$

$$= \delta_{00'}\,\delta_{12}\,\delta_{3,0}\,\delta_{4560},$$

Upon integration over $\tilde{k}_2$ and $\tilde{k}_3$,

$$S_1 = \delta(\breve{k}+\breve{k}')\sum_{\nu_1\cdots\nu_6}\int_{-\infty}^{\infty} \mathcal{L}^{\nu\ \nu_1\ \nu_2\ \nu_3}_{k\ k_1\ -k_1\ k}\ \mathcal{L}^{-\nu\ \nu_4\ \nu_5\ \nu_6}_{-k\ k_4\ k_5\ k_6}$$

$$\overset{(2)\nu_1\nu_2}{Q_0}(k_1)\ \overset{(4)\nu_4\nu_5\nu_6\nu_3}{Q_0}(k_4,k_5,k_6)\ \Delta\big((s_1+s_2)\omega_1+(s_3-s)\omega\big)$$

$$\Delta(\nu_4\omega_4+\nu_5\omega_5+\nu_6\omega_6+\nu\omega)\ \delta_{4560}\ d\breve{k}_1\,d\breve{k}_4\,d\breve{k}_5\,d\breve{k}_6.$$

where

$$\delta_{4560} = \delta(\tilde{k}_4 + \tilde{k}_5 + \tilde{k}_6 + \tilde{k}).$$

One obtains a "t" growth when $\nu_2 = -\nu_1$, $\nu_3 = \nu$.

Therefore

$$S_1 = t\,\delta(k+k')\left(\sum_{\nu_1}\int_{-\infty}^{\infty} \mathcal{L}_{k\,k\,k_1\,-k_1}^{\nu\,\nu\,\nu_1\,-\nu_1}\, \phi_0^{(2)\,\nu_1-\nu_1}(k_1)\, d\tilde{k}_1\right)$$

$$\times \sum_{\nu_4\nu_5\nu_6}\int_{-\infty}^{\infty} \mathcal{L}_{-k\,k_4\,k_5\,k_6}^{-\nu\,\nu_4\,\nu_5\,\nu_6}\, \phi_0^{(4)\,\nu_4\nu_5\nu_6\nu}(k_4 k_5 k_6)\, \Delta_{4560}\, \delta_{4560}\, d\tilde{k}_4\, d\tilde{k}_5\, d\tilde{k}_6$$

$$+ O(1).$$

As yet, no asymptotics have been performed. On performing the asymptotics in t, it is found that

$$3S_1 \sim 3t\,\delta(k+k')\sum_{\nu_1}\int_{-\infty}^{\infty} \mathcal{L}_{k\,k\,k_1\,-k_1}^{\nu\,\nu\,\nu_1\,-\nu_1}\, \phi_0^{(2)\,\nu_1-\nu_1}(k_1)\, dk_1$$

$$\sum_{\nu_4\nu_5\nu_6}\int_{-\infty}^{\infty} \mathcal{L}_{-k\,k_4\,k_5\,k_6}^{-\nu\,\nu_4\,\nu_5\,\nu_6}\, \phi_0^{(4)\,\nu_4\nu_5\nu_6\nu}(k_4,k_5,k_6)\left[\pi\,\delta_{4560} + \frac{iP}{W_{4560}}\right]\delta_{4560}\, d\tilde{k}_4\, d\tilde{k}_5\, d\tilde{k}_6$$

$$+ O(1). \qquad \text{----} \; (3)_a$$

$$S_2 : \delta_{14}\,\delta_{2356}\,\delta_{123,0}\,\delta_{456,0'}$$

$$= \delta_{14}\,\delta_{00'}\,\delta_{123,0}\,\delta_{2356}\,.$$

S_2 in long time, is an order one term.

$$S_3 : \delta_{45}\,\delta_{1236}\,\delta_{123,0}\,\delta_{456,0'} = \delta_{00'}\,\delta_{45}\,\delta_{6,0'}\,\delta_{123,0}\,.$$

This term has the same behavior as S_1 with $1,2,3 \longrightarrow 4,5,6$ and $0 \longrightarrow -0$. Therefore, using the properties of $L_{k\,k_1\,k_2\,k_3}^{\,\gamma\,\gamma_1\,\gamma_2\,\gamma_3}$,

$$3(S_1 + S_3) \sim -3t\,\delta(k+k')\sum_{\gamma_1}\int_{-\infty}^{\infty} L_{k\,k_1\,-k_1\,k}^{\,\gamma\,\gamma_1\,-\gamma_1\,\gamma}\,\overset{(2)}{\phi}{}_0^{\gamma_1,-\gamma_1}(k_1)\,dk_1$$

$$\times \sum_{\gamma_4\gamma_5\gamma_6}\int_{-\infty}^{\infty} L_{k\,k_4\,k_5\,k_6}^{\,\gamma\,\gamma_4\,\gamma_5\,\gamma_6}\Big[\pi\,\delta_{456,0}^{\gamma}\Big(\overset{(4)}{\phi}{}_0^{\gamma_4\gamma_5\gamma_6-\gamma}(k_4k_5k_6) + \overset{(4)}{\phi}{}_0^{*\,\gamma_4\gamma_5\gamma_6-\gamma}(k_4k_5k_6)\Big)$$

$$+ \frac{iP}{W_{456,0}}\Big(\overset{(4)}{\phi}{}_0^{\gamma_4\gamma_5\gamma_6-\gamma}(k_4k_5k_6) - \overset{(4)}{\phi}{}_0^{*\,\gamma_4\gamma_5\gamma_6-\gamma}(k_4\,k_5\,k_6)\Big)\Big]\,\delta_{456,0}$$

$$dk_4\,dk_5\,dk_6 + O(1). \qquad \text{------}\quad (3).$$

The property that

$$\overset{(4)}{\varphi}{}_0^{\nu_4\nu_5\nu_6 -\nu}(k_4,k_5,k_6) = \overset{(4)}{\varphi}{}_0^{*\,-\nu_4 -\nu_5 -\nu_6\, \nu}(-k_4,-k_5,-k_6),$$

has also been used. The (3×3) decomposition has 10 terms given by the following array.

$$\begin{array}{llll}
\delta_{123}\overset{(1)}{\delta}_{456} & \delta_{124}\overset{(2)}{\delta}_{356} & \delta_{125}\overset{(2)}{\delta}_{346} & \delta_{126}\overset{(2)}{\delta}_{345} \\
 & \delta_{134}\overset{(2)}{\delta}_{256} & \delta_{135}\overset{(2)}{\delta}_{246} & \delta_{136}\overset{(2)}{\delta}_{245} \\
 & & \delta_{145}\overset{(3)}{\delta}_{236} & \delta_{146}\overset{(3)}{\delta}_{235} \\
 & & & \delta_{156}\overset{(3)}{\delta}_{234}.
\end{array}$$

S_1: δ_{123} reacts with $\delta_{123,0}$ to give $\delta(\breve{k}_1)$. Therefore from the mean value property, this term is zero. S_2 and S_3 are both order one quantities in long time. One next examines

$$\langle a_0^{\nu'}(k')\,{}_3a_4^{\nu}(k) + a_0^{\nu}(k)\,{}_3a_4^{\nu'}(k')\rangle_{\nu'=-\nu}$$

$$= 3\overset{00'}{P}\sum_{\nu_1\cdots\nu_6}\int_{-\infty}^{\infty} L_{kk_1k_2k_3}^{\nu\nu_1\nu_2\nu_3}\, L_{k_1k_4k_5k_6}^{\nu_1\nu_4\nu_5\nu_6}\,\langle a_{0'}^{\nu'} a_{02}^{\nu_2} a_{03}^{\nu_3} a_{04}^{\nu_4} a_{05}^{\nu_5} a_{06}^{\nu_6}\rangle$$

$$\frac{\Delta_{23456,0} - \Delta_{123,0}}{iW_{456,1}}\,\delta_{456,1}\,\delta_{123,0}\,d\breve{k}_1\cdots d\breve{k}_6.$$

The $(2 \times 2 \times 2)$ decomposition gives the array,

$$\delta_{0'2}\ \delta_{34}^{(1)}\ \delta_{56} \qquad \delta_{0'2}\ \delta_{35}^{(1)}\ \delta_{46} \qquad \delta_{0'2}\ \delta_{36}^{(1)}\ \delta_{45}$$

$$\delta_{0'3}\ \delta_{24}^{(1)}\ \delta_{56} \qquad \delta_{0'3}\ \delta_{25}^{(1)}\ \delta_{36} \qquad \delta_{0'3}\ \delta_{26}^{(1)}\ \delta_{45}$$

$$\delta_{0'4}\ \delta_{23}^{(2)}\ \delta_{56} \qquad \delta_{0'4}\ \delta_{25}^{(3)}\ \delta_{36} \qquad \delta_{0'4}\ \delta_{26}^{(3)}\ \delta_{35}$$

$$\delta_{0'5}\ \delta_{23}^{(2)}\ \delta_{46} \qquad \delta_{0'5}\ \delta_{24}^{(3)}\ \delta_{36} \qquad \delta_{0'5}\ \delta_{26}^{(3)}\ \delta_{34}$$

$$\delta_{0'6}\ \delta_{23}^{(2)}\ \delta_{45} \qquad \delta_{0'6}\ \delta_{24}^{(3)}\ \delta_{35} \qquad \delta_{0'6}\ \delta_{25}^{(3)}\ \delta_{34}\,.$$

Because of symmetry there are only 3 basic terms.

$$S_1 : \quad \delta_{0'2}\ \delta_{34}\ \delta_{56}\ .$$

$$\delta_{0'2}\ \delta_{34}\ \delta_{56}\ \delta_{123,0}\ \delta_{456,1}$$

$$= \delta_{2,0}\ \delta_{34}\ \delta_{56}\ \delta_{13}\ \delta_{00'},$$

$$= \delta_{00'}\ \delta_{2,0}\ \delta_{31}\ \delta_{4,1}\ \delta_{56}.$$

Integrating over $\breve{k}_2$, $\breve{k}_3$, $\breve{k}_4$ and $\breve{k}_6$, one obtains

$$S_1 = 3\overset{00'}{P}\delta_{00'} \sum_{\nu_1 \cdots \nu_6} \int_{-\infty}^{\infty} \mathcal{L}^{\nu\,\nu_1\,\nu_2\,\nu_3}_{k\,k_1\,k_5\,-k_1}\ \mathcal{L}^{\nu_1\,\nu_4\,\nu_5\,\nu_6}_{k_1\,k_4\,k_5\,-k_5}\ \overset{(2)}{\phi}{}^{-\nu\,\nu_2}_{0}(-k_2)\ \overset{(2)}{\phi}{}^{\nu_3\,\nu_4}_{0}(-k_1)$$

$$\overset{(2)}{\phi}{}^{\nu_5\,\nu_6}_{0}(k_5)\ \frac{\Delta\big((\nu_2-\nu)\omega + (\nu_3+\nu_4)\omega_1 + (\nu_5+\nu_6)\omega_5\big) - \Delta\big((\nu_1+\nu_3)\omega_1 + (\nu_2-\nu)\omega\big)}{i\big((\nu_4-\nu_1)\omega_1 + (\nu_5+\nu_6)\omega_5\big)}$$

$$d\breve{k}_1\, d\breve{k}_5 .$$

To obtain a "t" growth at least one of the arguments of the Δ functions must be zero identically. One sees therefore that $\nu_2 = \nu$ is a necessary condition.

$$S_1 = 3\overset{0-0}{P}\delta_{00'} \sum_{\nu_1\nu_3\cdots\nu_6} \int_{-\infty}^{\infty} \mathcal{L}^{\nu\,\nu_1\,\nu\,\nu_3}_{k\,k_1\,k\,-k_1}\ \mathcal{L}^{\nu_1\,\nu_4\,\nu_5\,\nu_6}_{k_1\,k_1\,k_5\,-k_5}\ \overset{(2)}{\phi}{}^{\nu-\nu}_{0}(k)$$

$$\overset{(2)}{\phi}{}^{\nu_3\,\nu_4}_{0}(-k_1)\ \overset{(2)}{\phi}{}^{\nu_5\,\nu_6}_{0}(k_5)\ \frac{\Delta\big((\nu_3+\nu_4)\omega_1 + (\nu_5+\nu_6)\omega_5\big) - \Delta\big((\nu_1+\nu_3)\omega_1\big)}{i\big((\nu_4-\nu_1)\omega_1 + (\nu_5+\nu_6)\omega_5\big)}\, d\breve{k}_1\, d\breve{k}_5$$

$$+ O(1).$$

There are two possible cases.

(1) $\nu_3 = -\nu_1$

(2) $\nu_3 = \nu_1$

If (1) $\nu_3 = -\nu_1$ then the permutation on ν and $\tilde{k}$ changes $\mathcal{L}_{k\,k_1\,k-k_1}^{\nu\,\nu_1\,\nu-\nu_1}\,\varphi_0^{(2)\,\nu-\nu}(k)$ into $\mathcal{L}_{-k\,k_1\,-k\,-k_1}^{-\nu\,\nu_1\,-\nu\,-\nu_1}$ $\times\,\varphi_0^{(2)\,\nu-\nu}(k)$ and leaves everything else unchanged.

Therefore since

$$\mathcal{L}_{k\,k_1\,k-k_1}^{\nu\,\nu_1\,\nu-\nu_1} + \mathcal{L}_{-k\,k_1\,-k\,-k_1}^{-\nu\,\nu_1\,-\nu\,-\nu_1} = 0,$$

there is no contribution from this term.

If (2) $\nu_3 = \nu$, then to obtain a "t" growth $\nu_4 = -\nu_3 = -\nu_1$ and $\nu_6 = -\nu_5$ is a necessary condition. The expression then becomes

$$3\beta^{0-0}\,\delta_{00'}\sum_{\nu_1\nu_5}\int_{-\infty}^{\infty}\mathcal{L}_{k\,k_1\,k-k_1}^{\nu\,\nu_1\,\nu\,\nu_1}\,\mathcal{L}_{k_1\,-k_1\,k_5\,-k_5}^{\nu_1\,-\nu_1\,\nu_5\,-\nu_5}\,\varphi_0^{(2)\,\nu-\nu}(k)\,\varphi_0^{(2)\,\nu_1-\nu_1}(-k_1)\,\varphi_0^{(2)\,\nu_5-\nu_5}(k_5)\,\frac{\Delta(0)-\Delta(2\nu_1\omega_1)}{-2is_1\omega_1}\,d\tilde{k}_1\,d\tilde{k}_5,$$

$$\approx 3t\,\delta_{00'} \sum_{\nu_1 \nu_5} \int_{-\infty}^{\infty} \mathcal{L}_{k\ k_1\ k\ -k_1}^{\nu\ \nu_1\ \nu\ \nu_1} \mathcal{L}_{k_1\ -k_1\ k_5\ -k_5}^{\nu_1\ -\nu_1\ \nu_5\ -\nu_5} \overset{(2)}{\varphi}{}_0^{\nu-\nu}(\check{k})$$

$$\overset{(2)}{\varphi}{}_0^{\nu_1-\nu_1}(-\check{k}_1)\ \overset{(2)}{\varphi}{}_0^{\nu_5-\nu_5}(k_5) \left[\pi\,\delta(2\nu_1\omega_1) + \frac{iP}{2\nu_1\omega_1}\right] d\check{k}_1\, d\check{k}_5$$

$$+ 3t\,\delta_{00'} \sum_{\nu_1 \nu_5} \int_{-\infty}^{\infty} \mathcal{L}_{-k\ k_1\ -k\ -k_1}^{-\nu\ \nu_1\ -\nu\ \nu_1} \mathcal{L}_{k_1\ -k_1\ k_5\ -k_5}^{\nu_1\ -\nu_1\ \nu_5\ -\nu_5} \overset{(2)}{\varphi}{}_0^{\nu-\nu}(\check{k})$$

$$\overset{(2)}{\varphi}{}_0^{\nu_1-\nu_1}(-\check{k}_1)\ \overset{(2)}{\varphi}{}_0^{\nu_5-\nu_5}(k_5) \left[\pi\,\delta(2\nu_1\omega_1) + \frac{iP}{2\nu_1\omega_1}\right] d\check{k}_1\, d\check{k}_5 .$$

$\delta(2\nu_1\omega_1)$ implies $\check{k}_1 = 0$, and therefore makes the expression zero. If one changes $\nu_1 \to -\nu_1$, $\check{k}_1 \to -\check{k}_1$ in the second integral one sees that

$$\mathcal{L}_{-k\ k_1\ -k\ -k_1}^{-\nu\ \nu_1\ -\nu\ \nu_1} \mathcal{L}_{k_1\ -k_1\ k_5\ -k_5}^{\nu_1\ -\nu_1\ \nu_5\ -\nu_5} \frac{iP}{2\nu_1\omega_1}$$

$$= \mathcal{L}_{-k\ -k_1\ -k\ k_1}^{-\nu\ -\nu_1\ -\nu\ -\nu_1} \mathcal{L}_{-k_1\ k_1\ k_5\ -k_5}^{-\nu_1\ \nu_1\ \nu_5\ -\nu_5} \frac{-iP}{2\nu_1\omega_1}$$

$$= - \mathcal{L}_{k\ k_1\ k\ -k_1}^{\nu\ \nu_1\ \nu\ +\nu_1} \mathcal{L}_{k_1\ -k_1\ k_5\ -k_5}^{\nu_1\ -\nu_1\ \nu_5\ -\nu_5} \frac{iP}{2\nu_1\omega_1} .$$

Therefore the two terms add to zero, and there is no secularity in S_1.

$$S_2: \ \delta_{0'4}\,\delta_{23}\,\delta_{56}\,\delta_{123,0}\,\delta_{456,1} = \delta_{23}\,\delta_{1,0}\,\delta_{4,0}\,\delta_{00'}\,\delta_{56}.$$

Integrating over $\breve{k}_1$, $\breve{k}_3$, $\breve{k}_4$ and $\breve{k}_6$, one obtains

$$3S_2 = 9\rho^{00'}\delta_{00'}\sum_{\nu_1-\nu_6}\int_{-\infty}^{\infty} L^{\nu\,\nu_1\,\nu_2\,\nu_3}_{k\,k\,k_2\,-k_2}\, L^{\nu_1\,\nu_4\,\nu_5\,\nu_6}_{k\,k\,k_5\,-k_5}\,\varphi_0^{(2)\nu_2\nu_3}(k_2)\,\varphi_0^{(2)\nu_4-\nu}(k)$$

$$\varphi_0^{(2)\nu_5\nu_6}(k_5)\,\frac{\Delta\big((\nu_2+\nu_3)\omega_2+(\nu_5+\nu_6)\omega_5+(\nu_4-\nu)\omega\big)-\Delta\big((\nu_1-\nu)\omega+(\nu_2+\nu_3)\omega_2\big)}{i\big((\nu_4-\nu_1)\omega+(\nu_5+\nu_6)\omega_5\big)}\,d\breve{k}_2\,d\breve{k}_5.$$

Let

$$\nu_2\to\nu_1,\quad \breve{k}_2\to\breve{k}_1,\quad \nu_1\to\nu_3,\quad \nu_3\to\nu_2,$$

$$3S_2 = 9\rho^{00'}\delta_{00'}\sum_{\nu_1-\nu_6}\int_{-\infty}^{\infty} L^{\nu\,\nu_3\,\nu_1\,\nu_2}_{k\,k\,k_1\,-k_1}\, L^{\nu_3\,\nu_4\,\nu_5\,\nu_6}_{k\,k\,k_5\,-k_5}\,\varphi_0^{(2)\nu_1\nu_2}(k_1)\,\varphi_0^{(2)\nu_4-\nu}(k)$$

$$\varphi_0^{(2)\nu_5\nu_6}(k_5)\,\frac{\Delta\big((\nu_1+\nu_2)\omega_1+(\nu_5+\nu_6)\omega_5+(\nu_4-\nu)\omega\big)-\Delta\big((\nu_3-\nu)\omega+(\nu_1+\nu_2)\omega_1\big)}{i\big((\nu_4-\nu_3)\omega+(\nu_5+\nu_6)\omega_5\big)}$$

Using the results proved in the appendix II, namely

$$E(\omega,\mu)\sim O(1),$$

$$E(\omega,0)\sim t\Big(\pi\delta(\omega)+\frac{iP}{\omega}\Big)+O(1),$$

$$\sim t\,\Delta(\omega)+O(1),$$

$$E(0,\mu) \sim t\left(\pi\delta(\mu) + \frac{iP}{\mu}\right) \sim t\,\Delta(\mu),$$

$$E(0,0) = \frac{t^2}{2},$$

one obtains that

$$\frac{\Delta\left((\eta_1+\eta_2)\omega_1 + (\eta_5+\eta_6)\omega_5 + (\eta_4-\eta)\omega\right) - \Delta\left((\eta_3-\eta)\omega + (\eta_1+\eta_2)\omega_1\right)}{i\left((\eta_4-\eta_3)\omega + (\eta_5+\eta_6)\omega_5\right)}$$

$$= \overset{(1)}{\frac{t^2}{2}\,\delta^{\eta_2}_{-\eta_1}\,\delta^{\eta_6}_{-\eta_5}\,\delta^{\eta_4}_{\eta}\,\delta^{\eta_3}_{\eta}} + \overset{(2)}{t\,\delta^{\eta_3}_{\eta}\,\delta^{\eta_2}_{-\eta_1}\,\delta^{\eta_6}_{-\eta_5}\,\delta^{\eta_4}_{-\eta}\,\Delta(-2\eta\omega)}$$

$$+ \overset{(3)}{t\,\delta^{\eta_3}_{\eta}\,\delta^{\eta_2}_{-\eta_1}\,\delta^{\eta_4}_{\eta}\,\delta^{\eta_6}_{\eta_5}\,\Delta(2\eta_5\omega_5)}$$

$$+ \overset{(4)}{t\,\delta^{\eta_3}_{\eta}\,\delta^{\eta_2}_{-\eta_1}\,\delta^{\eta_6}_{\eta_5}\,\delta^{\eta_4}_{-\eta}\,\Delta(2\eta_5\omega_5 - 2\eta\omega)}$$

$$+ \overset{(5)}{t\,\delta^{\eta_2}_{-\eta_1}\,\delta^{\eta_6}_{-\eta_5}\,\delta^{\eta_4}_{\eta}\,\delta^{\eta_3}_{-\eta}\,\Delta(-2\eta\omega)}.$$

$$\delta^{s_i}_{s_j} = \begin{cases} 1, & s_i = s_j \\ 0, & s_i \neq s_j \end{cases}.$$

Consider the contribution from the third term, which we denote by S_{23}.

$$3 S_{23} \sim 9t\, \delta(k+k') \sum_{\nu_1 \nu_5} \int_{-\infty}^{\infty} L^{\nu\,\nu\,\nu_1\,-\nu_1}_{k\,k\,k_1\,-k_1}\, L^{\nu\,\nu\,\nu_5\,\nu_5}_{k\,k\,k_5\,-k_5}$$

$$Q_0^{(2)\nu_1-\nu_1}(k_1)\, Q_0^{(2)\nu-\nu}(k)\, Q_0^{(2)\nu_5\nu_5}(k_5)\, \Delta(2\nu_5\omega_5)\, d\breve{k}_1\, d\breve{k}_5$$

$$+\, 9t\, \delta(k+k') \sum_{\nu_1 \nu_5} \int_{-\infty}^{\infty} L^{-\nu\,-\nu\,\nu_1\,-\nu_1}_{-k\,-k\,k_1\,-k_1}\, L^{-\nu\,-\nu\,\nu_5\,\nu_5}_{-k\,-k\,k_5\,-k_5}$$

$$Q_0^{(2)\nu_1-\nu_1}(k_1)\, Q_0^{(2)\nu-\nu}(k)\, Q_0^{(2)\nu_5\nu_5}(k_5)\, \Delta(2\nu_5\omega_5)\, d\breve{k}_1\, d\breve{k}_5$$

$$+\, O(1),$$

$$= 9t\, \delta(k+k') \sum_{\nu_1 \nu_5} \int_{-\infty}^{\infty} L^{-\nu\,\nu_1\,-\nu_1\,-\nu}_{-k\,k_1\,-k_1\,-k}\, L^{-\nu\,-\nu\,\nu_5\,\nu_5}_{-k\,-k\,k_5\,k_5}$$

$$Q_0^{(2)\nu_1-\nu_1}(k_1)\, Q_0^{(2)\nu-\nu}(k) \left[Q_0^{(2)\nu_5\nu_5}(k_5)\, \Delta(2\nu_5\omega_5) + \text{Comp Conj} \right] d\breve{k}_1 d\breve{k}_5$$

$$+\, O(1). \qquad \text{--- } (4)_3$$

Since $\quad L^{-\nu\,\nu_1\,-\nu_1\,-\nu}_{-k\,k_1\,-k_1\,-k} + L^{\nu\,\nu_1\,-\nu_1\,\nu}_{k\,k_1\,-k_1\,k} = 0,$

Terms $\;(1)_3 + (4)_3 = 0.$

With similar manipulations,

$$3S_{24} \sim gt\,\delta(k+k') \sum_{\nu_1\nu_5} \int_{-\infty}^{\infty} L^{-\nu\,\nu_1\,-\nu_1\,-\nu}_{-k\,k_1\,-k_1\,-k}\, L^{-\nu\,\nu\,\nu_5\,\nu_5}_{-k\,k\,k_5\,-k_5}$$

$$\overset{(2)}{Q}{}_0^{\nu_1-\nu_1}(k_1)\Big[\overset{(2)}{Q}{}_0^{\nu\nu}(-k)\,\overset{(2)}{Q}{}_0^{\nu_5\nu_5}(k_5)\,\Delta(2\nu_5\omega_5+2\nu\omega)\;+$$

$$\text{Comp. Conj.}\Big]\,d\breve{k}_1\,d\breve{k}_5 \qquad \text{---}\ (4)_4.$$

Using $\overset{(2)}{Q}{}_0^{\nu\nu}(-k) = \overset{(2)}{Q}{}_0^{\nu\nu}(k)$,

and $L^{-\nu\,\nu_1\,-\nu_1\,-\nu}_{-k\,k_1\,-k_1\,-k} = L^{\nu\,\nu_1\,-\nu_1\,\nu}_{k\,k_1\,-k_1\,k}$,

one sees that $(1)_4 + (4)_4 = 0.$

$$3S_{22} \sim gt\,\delta(k+k') \sum_{\nu_1\nu_5} \int_{-\infty}^{\infty} L^{\nu\,\nu_1\,-\nu_1\,\nu}_{k\,k_1\,-k_1\,k}\, L^{\nu\,-\nu\,\nu_5\,-\nu_5}_{k\,-k\,k_5\,-k_5}\, \overset{(2)}{Q}{}_0^{\nu_1-\nu_1}(k_1)$$

$$\overset{(2)}{Q}{}_0^{-\nu-\nu}(k)\,\overset{(2)}{Q}{}_0^{\nu_5-\nu_5}(k_5)\,\Delta(-2\nu\omega)\,d\breve{k}_1\,d\breve{k}_5$$

$$+\,gt\,\delta(k+k') \sum_{\nu_1\nu_5} \int_{-\infty}^{\infty} L^{-\nu\,\nu_1\,-\nu_1\,-\nu}_{-k\,k_1\,-k_1\,-k}\, L^{-\nu\,\nu\,\nu_5\,-\nu_5}_{-k\,-k\,k_5\,-k_5}\, \overset{(2)}{Q}{}_0^{\nu_1-\nu_1}(k_1)$$

$$\overset{(2)}{Q}{}_0^{\nu\nu}(-k)\,\overset{(2)}{Q}{}_0^{\nu_5-\nu_5}(k_5)\,\Delta(2\nu\omega)\,d\breve{k}_1\,d\breve{k}_5. \qquad \text{--}\ (4)_2$$

Equation $(4)_2$ cancels with $(1)_2$.

$$3S_{25} \sim 9t\,\delta(k+k')\left[\Delta(-2\nu\omega)+\Delta(2\nu\omega)\right]\sum_{\nu_1\nu_5}\int_{-\infty}^{\infty} L^{\nu\,\nu_1\,-\nu_1\,-\nu}_{k\,k_1\,-k_1\,k}$$

$$L^{\nu\,\nu\,\nu_5\,-\nu_5}_{k\,k\,k_5\,-k_5}\;\overset{(2)}{\phi}{}^{\nu_1-\nu_1}_0(k_1)\;\overset{(2)}{\phi}{}^{\nu-\nu}_0(k)\;\overset{(2)}{\phi}{}^{\nu_5-\nu_5}_0(k_5)\;d\tilde{k}_1\,d\tilde{k}_5 . \quad \text{---}(4)_6$$

Since $\Delta(-2\nu\omega)+\Delta(2\nu\omega)$ behaves in long time like

$\pi\delta(-2\nu\omega)+\dfrac{iP}{-2\nu\omega}+\pi\delta(2\nu\omega)+\dfrac{iP}{2\nu\omega}$ there is no contribution from this term as the principal parts cancel and $\delta(\omega)$ implies the expression is zero by the mean value property. Looking at the t^2 term, one obtains

$$3S_{21} = 9t^2\,\delta(k+k')\;\overset{(2)}{\phi}{}^{\nu-\nu}_0(k)\left(\sum_{\nu_2}\int_{-\infty}^{\infty} L^{\nu\,\nu_2\,-\nu_2\,\nu}_{k\,k_2\,-k_2\,k}\;\overset{(2)}{\phi}{}^{\nu_2-\nu_2}_0(k_2)\,d\tilde{k}_2\right)^2 . \quad \text{------}(4)_1 .$$

In this case the application of the permutation to the expression serves to double the result. Again one notes a cancellation, namely

$$\text{Terms } (4)_1 + (1)_1 = 0 .$$

One now examines S_3 . A typical term belonging to this array is

$$\delta_{0'4}\;\delta_{25}\;\delta_{36} .$$

$$S_3 : \quad \delta_{0'4}\ \delta_{25}\ \delta_{36}\ \delta_{123,0}\ \delta_{456,1}$$

$$= \delta_{00'}\ \delta_{4,0}\ \delta_{25}\ \delta_{36}\ \delta_{123,0}\,.$$

Integrating over $\tilde{k}_4$, $\tilde{k}_5$ and $\tilde{k}_6$, one obtains

$$6 S_3 = 18\,\delta(\tilde{k}+\tilde{k}')\ \overset{00'}{P} \sum_{\lambda_1 \cdots \lambda_6} \int_{-\infty}^{\infty} L^{\lambda\lambda_1\lambda_2\lambda_3}_{k k_1 k_2 k_3}\ L^{\lambda_1\lambda_4\lambda_5\lambda_6}_{k_1 k\, -k_2\, -k_3}\ Q_0^{(2)\lambda'\lambda_4}(k')$$

$$Q_0^{(2)\lambda_2\lambda_5}(k_2)\ Q_0^{(2)\lambda_3\lambda_6}(k_3)\ \frac{\Delta\big((\lambda_2+\lambda_5)\omega_2+(\lambda_3+\lambda_6)\omega_3+(\lambda_4-\lambda)\omega\big)-\Delta_{123,0}}{i(\lambda_4\omega+\lambda_5\omega_2+\lambda_6\omega_3-\lambda\omega_1)}$$

$$\delta_{123,0}\ d\tilde{k}_1\, d\tilde{k}_2\, d\tilde{k}_3\,.$$

A "t" growth may occur when $\lambda_2\lambda_5 = \lambda_3\lambda_6 = -\lambda_4\lambda = -1$, whereupon the time t dependent part of the expression becomes

$$\frac{\Delta(0) - \Delta(s_1\omega_1 + s_2\omega_2 + s_3\omega_3 - s\omega)}{-i W_{123,0}}\,.$$

As seen from the Appendix

$$\frac{\Delta(0) - \Delta(\omega)}{-i\omega} \sim t\left(\pi\delta(\omega) + \frac{iP}{\omega}\right) + O(1).$$

Therefore taking the limit and applying the permutation

$\nu \to -\nu$, $\breve{k} \to -\breve{k}$, one obtains

$$6S_3 \sim 18t\,\delta(k+k')\sum_{\nu_1\nu_2\nu_3}\int_{-\infty}^{\infty} L_{k\,k_1\,k_2\,k_3}^{\nu\,\nu_1\,\nu_2\,\nu_3}\, L_{k_1\,k\,-k_2\,-k_3}^{\nu_1\,\nu\,-\nu_2\,-\nu_3}\, \phi_0^{(2)\nu-\nu}(k)$$

$$\phi_0^{(2)\nu_2-\nu_2}(k_2)\,\phi_0^{(2)\nu_3-\nu_3}(k_3)\left(\pi\,\delta_{123,0}^{\nu} + \frac{iP}{W_{123,0}}\right)\delta_{123,0}\, d\breve{k}_1 d\breve{k}_2 d\breve{k}_3$$

$$+\,18\pi t\,\delta(k+k')\sum_{\nu_1\nu_2\nu_3}\int_{-\infty}^{\infty} L_{-k\,k_1\,k_2\,k_3}^{-\nu\,\nu_1\,\nu_2\,\nu_3}\, L_{k_1\,-k\,-k_2\,-k_3}^{\nu_1\,-\nu\,-\nu_2\,-\nu_3}\, \phi_0^{(2)\nu-\nu}(k)$$

$$\phi_0^{(2)\nu_2-\nu_2}(k_2)\,\phi_0^{(2)\nu_3-\nu_3}(k_3)\left(\pi\,\delta_{1230}^{\nu} + \frac{iP}{W_{1230}}\right)\delta_{1230}\, d\breve{k}_1 d\breve{k}_2 d\breve{k}_3$$

$$+\,O(1).$$

In the second integral set

$$\begin{cases}\nu_i\\ \breve{k}_i\end{cases} \longrightarrow \begin{cases}-\nu_i\\ -\breve{k}_i\end{cases} \qquad i = 1,2,3,$$

and use the properties of $L_{k\,k_1\,k_2\,k_3}^{\nu\,\nu_1\,\nu_2\,\nu_3}$. One sees that the principal part terms add to zero as there are three sign changes introduced, whereas the resonance terms add to double the expression. Therefore

$$6S_3 \sim 36\pi t\,\delta(k+k')\,\phi_0^{(2)\nu-\nu}(k)\sum_{\nu_1\nu_2\nu_3}\int_{-\infty}^{\infty} L_{k\,k_1\,k_2\,k_3}^{\nu\,\nu_1\,\nu_2\,\nu_3}\, L_{k_1\,k\,-k_2\,-k_3}^{\nu_1\,\nu\,-\nu_2\,-\nu_3}$$

$$\phi_0^{(2)\nu_2-\nu_2}(k_2)\,\phi_0^{(2)\nu_3-\nu_3}(k_3)\,\delta_{123,0}^{\nu}\,\delta_{123,0}\, d\breve{k}_1 d\breve{k}_2 d\breve{k}_3 + O(1). \quad \text{--(5)}$$

where $\delta^{2}_{123,0} = \delta(s_1\omega_1 + s_2\omega_2 + s_3\omega_3 - s\omega)$.

The (2 X 4) decomposition of $\langle a_{01}^{\tau_1} a_{02}^{\tau_2} a_{03}^{\tau_3} a_{04}^{\tau_4} a_{05}^{\tau_5} a_{06}^{\tau_6} \rangle$
gives the array,

$$
\begin{array}{lllll}
\delta_{0'2}\,\delta^{(1)}_{3456} & \delta_{0'3}\,\delta^{(1)}_{2456} & \delta_{0'4}\,\delta^{(2)}_{2356} & \delta_{0'5}\,\delta^{(2)}_{2346} & \delta_{0'6}\,\delta^{(2)}_{2345} \\
 & \delta_{23}\,\delta^{(3)}_{0'456} & \delta_{24}\,\delta^{(4)}_{0'356} & \delta_{25}\,\delta^{(4)}_{0'346} & \delta_{26}\,\delta^{(4)}_{0'345} \\
 & & \delta_{34}\,\delta^{(4)}_{0'256} & \delta_{35}\,\delta^{(4)}_{0'246} & \delta_{36}\,\delta^{(4)}_{0'245} \\
 & & & \delta_{45}\,\delta^{(5)}_{0'236} & \delta_{46}\,\delta^{(5)}_{0'235} \\
 & & & & \delta_{56}\,\delta^{(5)}_{0'234}\,.
\end{array}
$$

As a reminder $\delta_{23}\,\delta_{0'456}$ represents

$$\delta(\tilde{k}_2+\tilde{k}_3)\,\mathcal{Q}_0^{(2)\,\nu_2\nu_3}(k_2)\quad \delta(\tilde{k}'+\tilde{k}_4+\tilde{k}_5+\tilde{k}_6)\,\mathcal{Q}_0^{(4)\,\nu_4\nu_5\nu_6\nu'}(k_4,k_5,k_6).$$

The term S_1 given by the decompositions $\delta_{0'2}\,\delta_{3456}$ and $\delta_{0'3}\,\delta_{2456}$ can allow "t" growths but they are of the form

$$\overset{0-0}{P}\sum\int_{-\infty}^{\infty} \mathcal{L}_{k\,k\,k_1\,-k_1}^{\nu\,\nu\,\nu_1\,-\nu_1}\,\mathcal{Q}_0^{(2)\,\nu-\nu}(k)\;f(\tilde{k}_1,\tilde{k}_4,\tilde{k}_5,\tilde{k}_6)\,d\tilde{k}_1\,d\tilde{k}_4\,d\tilde{k}_5\,d\tilde{k}_6.$$

From the property that

$$\mathcal{L}_{k\;k_1\;-k_1\;k}^{\nu\;\nu_1\;-\nu_1\;\nu}+\mathcal{L}_{-k\;k_1\;-k_1\;-k}^{-\nu\;\nu_1\;-\nu_1\;-\nu}=0,$$

the two terms of the permutation clearly cancel. Neither S_2 nor S_4 can exhibit any "t" growth.

$$S_3:\quad \delta_{23}\,\delta_{0'456}\,\delta_{123,0}\,\delta_{456,1}$$

$$=\delta_{23}\,\delta_{1,0}\,\delta_{00'}\,\delta_{456,0}.$$

Integrating over $\tilde{k}_1$ and $\tilde{k}_3$,

$$S_3 = 3\delta(k+k')\overset{0-0}{P}\sum_{\nu_1\cdots\nu_6}\int_{-\infty}^{\infty}\mathcal{L}_{k\,k_1\,k_2\,-k_2}^{\nu\,\nu_1\,\nu_2\,\nu_3}\,\mathcal{L}_{k\,k_4\,k_5\,k_6}^{\nu_1\,\nu_4\,\nu_5\,\nu_6}$$

$$\times \overset{(2)\ \nu_2 \nu_3}{\varphi_0}(k_2) \overset{(4)\ \nu_4 \nu_5 \nu_6 -\nu}{\varphi_0}(k_4, k_5, k_6)$$

$$\frac{\Delta\left((\nu_2+\nu_3)\omega_2 + \nu_4\omega_4 + \nu_5\omega_5 + \nu_6\omega_6 - \nu\omega\right) - \Delta\left((\nu_1-\nu)\omega + (\nu_2+\nu_3)\omega_2\right)}{i(\nu_4\omega_4 + \nu_5\omega_5 + \nu_6\omega_6 - \nu_1\omega)}$$

$$\delta_{456,0}\, d\tilde{k}_2\, d\tilde{k}_4\, d\tilde{k}_5\, d\tilde{k}_6 .$$

A "t" growth is possible if $\nu_1 = \nu$ and $\nu_3 = -\nu_2$. Therefore,

$$S_3 \sim 3t\, \delta(\tilde{k}+\tilde{k}') \overset{0-0}{P} \sum_{\nu_2} \int_{-\infty}^{\infty} L^{\nu\nu\nu_2-\nu_2}_{kkk_2-k_2} \overset{(2)\ \nu_2-\nu_2}{\varphi_0}(k_2)\, d\tilde{k}_2$$

$$\times \sum_{\nu_4\nu_5\nu_6} \int_{-\infty}^{\infty} L^{\nu\nu_4\nu_5\nu_6}_{kk_4k_5k_6} \overset{(4)\ \nu_4\nu_5\nu_6-\nu}{\varphi_0}(k_4 k_5 k_6) \left[\pi\, \delta^{1}_{456,0} + \frac{iP}{W_{456,0}}\right]$$

$$\delta_{456,0}\, d\tilde{k}_4\, d\tilde{k}_5\, d\tilde{k}_6$$

$$+ O(1).$$

If one applies the permutation and uses the properties of $L^{\nu\nu_1\nu_2\nu_3}_{kk_1k_2k_3}$ in conjunction with the transformation in the second integral of

$$\begin{cases} \nu_i \\ \tilde{k}_i \end{cases} \to \begin{cases} -\nu_i \\ -\tilde{k}_i \end{cases}, \qquad i = 4,5,6 ,$$

one obtains

$$S_3 \sim 3t\,\delta(k+k') \sum_{\lambda_2} \int_{-\infty}^{\infty} L_{k\,k\,k_2\,-k_2}^{\lambda\,\lambda\,\lambda_2\,-\lambda_2} \overset{(2)}{Q}{}_0^{\lambda_2\,-\lambda}(k_2)\, dk_2$$

$$\sum_{\lambda_4\lambda_5\lambda_6} \int_{-\infty}^{\infty} L_{k\,k_4\,k_5\,k_6}^{\lambda\,\lambda_4\,\lambda_5\,\lambda_6} \Big[\pi\,\delta_{456,0}^{\lambda} \Big(\overset{(4)}{Q}{}_0^{\lambda_4\lambda_5\lambda_6-\lambda}(k_4,k_5,k_6) + \overset{(4)}{Q}{}_0^{*\,\lambda_4\lambda_5\lambda_6-\lambda}(k_4,k_5,k_6) \Big)$$

$$+ \frac{iP}{W_{456,0}} \Big(\overset{(4)}{Q}{}_0^{\lambda_4\lambda_5\lambda_6-\lambda}(k_4,k_5,k_6) - \overset{(4)}{Q}{}_0^{*\,\lambda_4\lambda_5\lambda_6-\lambda}(k_4,k_5,k_6) \Big) \Big] \delta_{456,0}\, dk_4\, dk_5\, dk_6$$

$$+ O(1) \qquad (6)$$

Clearly $\quad (6) + (3) = 0.$

$$S_5: \quad \delta_{45}\,\delta_{0'236}\,\delta_{123,0}\,\delta_{456,1}$$

$$= \delta_{45}\,\delta_{6,1}\,\delta_{123,0}\,\delta_{00'}.$$

This makes $\dfrac{\Delta_{23456,0} - \Delta_{123,0}}{i W_{456,1}}$ take on at worst the form $\dfrac{\Delta(\omega) - \Delta(\omega)}{i(\omega - \omega)}$ which is shown to be an order one term in the appendix. The (3 X 3) decomposition of the sixth order mean value into products of third order spectral cumulants cannot excite any secularities.

If one now adds (5) and (2), one sees the total "t" growth contribution from $\langle b_2^{-\nu}(k') b_2^{\nu}(k)\rangle$ and $\langle a_0^{-\nu}(k')\,{}_3d_4^{\nu}(k) + a_0^{\nu}(k)\,{}_3d_4^{-\nu}(k')\rangle$ is

$$12\pi t\,\delta_{00'} \sum_{\nu_1\nu_2\nu_3} \int_{-\infty}^{\infty} L^{\nu\,\nu_1\,\nu_2\,\nu_3}_{k\,k_1\,k_2\,k_3}\, L^{-\nu\,-\nu_1\,-\nu_2\,-\nu_3}_{-k\,-k_1\,-k_2\,-k_3}\, \Phi^{(2)\,\nu_1\,-\nu_1}_{0}(k_1)\, \Phi^{(2)\,\nu_2\,-\nu_2}_{0}(k_2)\, \Phi^{(2)\,\nu_3\,-\nu_3}_{0}(k_3)\, \delta^{\nu}_{123,0}\, \delta_{123,0}\, d\breve{k}_1\, d\breve{k}_2\, d\breve{k}_3$$

$$+\,36\pi t\,\delta_{00'}\, \Phi^{(2)\,\nu\,-\nu}_{0}(k) \sum_{\nu_1\nu_2\nu_3} \int_{-\infty}^{\infty} L^{\nu\,\nu_1\,\nu_2\,\nu_3}_{k\,k_1\,k_2\,k_3}\, L^{\nu_1\,\nu\,-\nu_2\,-\nu_3}_{k_1\,k\,-k_2\,-k_3}\, \Phi^{(2)\,\nu_2\,-\nu_2}_{0}(k_2)\, \Phi^{(2)\,\nu_3\,-\nu_3}_{0}(k_3)\, \delta^{\nu}_{123,0}\, \delta_{123,0}\, d\breve{k}_1\, d\breve{k}_2\, d\breve{k}_3 .$$

This is essentially the form of the final result. Since there is no possibility of a triad, such as $\nu_1\omega_1 + \nu_2\omega_2 + \nu_3\omega_3$, vanishing anywhere in the spectrum, the terms with coefficients $L^{\nu\,\nu_1\,\nu_2}_{k\,k_1\,k_2}$ do not play any different roles to the terms with coefficients $L^{\nu\,\nu_1\,\nu_2\,\nu_3}_{k\,k_1\,k_2\,k_3}$. In fact it is shown that the final result is the same as the above with

$$L^{\nu\,\nu_1\,\nu_2\,\nu_3}_{k\,k_1\,k_2\,k_3} \longrightarrow L^{\nu\,\nu_1\,\nu_2\,\nu_3}_{k\,k_1\,k_2\,k_3} - \frac{2i}{3}\, P^{123} \sum_{\nu_4} \frac{L^{\nu\,-\nu_4\,\nu_1}_{k\,k_2+k_3\,k_1}\, L^{-\nu_4\,\nu_2\,\nu_3}_{k_2+k_3\,k_2\,k_3}}{\nu_2\omega_2 + \nu_3\omega_3 + \nu_4\omega_{23}},$$

when $\overset{123}{P}$ is the cyclic permutation over

$1 \to 2 \to 3$.

Essentially the reason for this is that when one eliminates the possibility of triad resonances, one is still looking for the first closure for the energy density. Had one allowed triad resonances to remain, the energy density closure on the T_4 time scale would be a second closure and would be of a different form. In that case the terms with coefficients $L_{k k_1 k_2}^{\gamma \gamma_1 \gamma_2}$ could be expected to enter in a stronger way than the terms with coefficients $L_{k k_1 k_2 k_3}^{\gamma \gamma_1 \gamma_2 \gamma_3}$. This possibility will be examined at a later stage.

One now examines

$$\langle b_{2'}^{\gamma'} c_2^{\gamma} + b_2^{\gamma} c_{2'}^{\gamma'} \rangle_{\gamma'=-\gamma}$$

$$= 2\overset{00'}{P} \sum_{\gamma_1 \cdots \gamma_7} \int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\gamma \gamma_1 \gamma_2 \gamma_3} L_{k' k_4 k_5}^{\gamma' \gamma_4 \gamma_5} L_{k_4 k_6 k_7}^{\gamma_4 \gamma_6 \gamma_7} \langle a_{01}^{\gamma_1} a_{02}^{\gamma_2} a_{03}^{\gamma_3} a_{05}^{\gamma_5} a_{06}^{\gamma_6} a_{07}^{\gamma_7} \rangle$$

$$\Delta_{123,0} \frac{\Delta_{567,0'} - \Delta_{45,0'}}{i W_{67,4}} \delta_{67,4} \, \delta_{123,0} \, \delta_{45,0'} \, d\breve{k}_1 \cdots d\breve{k}_7 .$$

The

(2 X 2 X 2) decomposition is given by the following array,

$$\overset{(1)}{\delta_{12} \delta_{35} \delta_{67}} \qquad \overset{(2)}{\delta_{12} \delta_{36} \delta_{57}} \qquad \overset{(2)}{\delta_{12} \delta_{37} \delta_{56}}$$

$$\delta_{13}\overset{(1)}{\delta_{25}}\delta_{67} \quad \delta_{13}\overset{(2)}{\delta_{26}}\delta_{57} \quad \delta_{13}\overset{(2)}{\delta_{27}}\delta_{56}$$

$$\delta_{15}\overset{(3)}{\delta_{23}}\delta_{67} \quad \delta_{15}\overset{(4)}{\delta_{26}}\delta_{37} \quad \delta_{15}\overset{(4)}{\delta_{27}}\delta_{36}$$

$$\delta_{16}\overset{(2)}{\delta_{23}}\delta_{57} \quad \delta_{16}\overset{(5)}{\delta_{25}}\delta_{37} \quad \delta_{16}\overset{(5)}{\delta_{27}}\delta_{35}$$

$$\delta_{17}\overset{(2)}{\delta_{23}}\delta_{56} \quad \delta_{17}\overset{(5)}{\delta_{25}}\delta_{36} \quad \delta_{17}\overset{(5)}{\delta_{26}}\delta_{35}.$$

S_1, S_3: δ_{67} reacts with $\delta_{67.4}$ to give $\delta(\tilde{k}_4)$

Therefore the expression is zero by the mean value property.

S_2: $\delta_{12}\,\delta_{36}\,\delta_{57}$.

Since $\delta_{12}\,\delta_{36}\,\delta_{57}\,\delta_{67.4}\,\delta_{45.0}\,\delta_{123.0}$

$$= \delta_{12}\,\delta_{36}\,\delta_{57}\,\delta_{67.4}\,\delta_{450'}\,\delta_{3,0}$$

$$= \delta_{00'}\,\delta_{12}\,\delta_{3,0}\,\delta_{60}\,\delta_{450}\,\delta_{57},$$

Integrating over $\tilde{k}_2$, $\tilde{k}_3$, $\breve{k}_6$ and $\check{k}_7$, one obtains that

$$S_2 = 2\epsilon^{0-0} \sum_{\sigma_1 \cdots \sigma_7} \int_{-\infty}^{\infty} L^{\sigma\sigma_1\sigma_2\sigma_3}_{k k_1 -k_1 k} L^{-\sigma\sigma_4\sigma_5}_{-k k_4 k_5} L^{\sigma_4\sigma_6\sigma_7}_{k_4 -k -k_5} Q_0^{(2)\sigma_1\sigma_2}(\tilde{k}_1)$$

$$Q_0^{(2)\sigma_3\sigma_6}(k)\, Q_0^{(2)\sigma_5\sigma_7}(k_5)\ \Delta((\sigma_1+\sigma_2)\omega_1 + (\sigma_3-\sigma)\omega)$$

$$\frac{\Delta((\sigma_5+\sigma_7)\omega_5 + (\sigma_6+\sigma)\omega) - \Delta_{450}}{i(\sigma_6\omega + \sigma_7\omega_5 - \sigma_4\omega_4)} \delta_{450}\, d\tilde{k}_1\, d\tilde{k}_4\, d\tilde{k}_5 .$$

There are three types of secular terms in the above expression.

1) $0(t^2)$. This occurs when $(\sigma_1+\sigma_2)\omega_1 + (\sigma_3-\sigma)\omega \equiv 0$ and $(\sigma_5+\sigma_7)\omega_5 + (\sigma_6+\sigma)\omega \equiv 0$. The cumulants for this case are all of the energy type namely $Q_0^{(2)+-}(k)$.

2) $0(t)$ terms. These occur if $(\sigma_1+\sigma_2)\omega_1 + (\sigma_3-\sigma)\omega \equiv 0$ and $(\sigma_5+\sigma_7)\omega_5 + (\sigma_6+\sigma)\omega \equiv 0$ since

$$\frac{\Delta(0) - \Delta(\mu)}{-i\mu} \sim \left(\pi\delta(\mu) + \frac{iP}{\mu}\right)\left(t - i\frac{\partial}{\partial\mu}\right).$$

There are no triad resonances, therefore

$$\frac{\Delta(0) - \Delta_{450}}{-iW_{450}} \sim \frac{it}{W_{450}} + \frac{1}{W_{450}}\frac{\partial}{\partial W_{450}} .$$

The order one term will be of the form $\frac{1}{(W_{450})^2}$

On application of the permutation the $0(t^2)$ terms will add and the order $0(t)$ terms will cancel.

3) O(t) terms which arise when at least one of the expressions $(\nu_1+\nu_2)\omega_1+(\nu_3-\nu)\omega_2$, $(\nu_5+\nu_7)\omega_5+(\nu_6+\nu)\omega_7$ is zero identically. These occur with the second order cumulants $\overset{(2)}{\varphi}{}_0^{++}(k)$. These terms cancel with similar terms obtained from $\langle a_0^{\nu\prime\prime}{}_2 d_4^{\nu} + a_0^{\nu}{}_2 d_4^{\nu\prime\prime}\rangle_{\nu=-\nu}$ and $\langle a_0^{\nu\prime\prime}(k')_2 d_4^{\nu}(k) + a_0^{\nu}(k)_2 d_4^{\nu\prime\prime}(k')\rangle_{\nu=-\nu}$ in precisely the same manner as the same type terms cancelled when one added $\langle b_2^{-\nu}(k') b_2^{\nu}(k)\rangle$ and $\langle a_0^{-\nu}(k')_3 d_4^{\nu}(k) + a_0^{\nu}(k)_3 d_4^{-\nu}(k')\rangle$.

The $O(t^2)$ term from the above expression is

$$2t^2 \overset{0\text{-}0}{\rho}\,\delta(k+k')\sum_{\nu_1\nu_4\nu_5}\int_{-\infty}^{\infty} \mathcal{L}_{k\,k_1\,-k_1\,k}^{\nu\,\nu_1\,-\nu_1\,\nu}\,\mathcal{L}_{-k\,k_4\,k_5}^{-\nu\,\nu_4\,\nu_5}\,\mathcal{L}_{k_4\,-k\,-k_5}^{\nu_4\,-\nu\,-\nu_5}\,\overset{(2)}{\varphi}{}_0^{\nu_1,-\nu_1}(k_1)\,\overset{(2)}{\varphi}{}_0^{\nu,-\nu}(k)\,\overset{(2)}{\varphi}{}_0^{\nu_5,-\nu_5}(k_5)\,\frac{i\delta_{450}}{w_{450}}\,d\breve{k}_1\,d\breve{k}_4\,d\breve{k}_5.$$

The application of the permutation sends $\nu\to-\nu$, $\breve{k}\to-\breve{k}$. In the second integral, thus obtained, one puts $\nu_4\to-\nu_4$, $\breve{k}_4\to-\breve{k}_4$, $\nu_5\to-\nu_5$ and $\breve{k}_5\to-\breve{k}_5$.

Using the properties of the $\mathcal{L}_{k\,k_1\,k_2\cdots k_r\cdots}^{\nu\,\nu_1\,\nu_2\cdots\nu_r\cdots}$ one finds these two terms are the same and therefore add.

$$6S_2 \sim 24\, t^2\, \delta(\breve{k}+\breve{k}') \sum_{\lambda_1\lambda_4\lambda_5} \int_{-\infty}^{\infty} \mathcal{L}^{\lambda\,\lambda_1\,-\lambda_1\,\lambda}_{k\,k_1\,-k_1\,k}\; \mathcal{L}^{-\lambda\,\lambda_4\,\lambda_5}_{-k\,k_4\,k_5}$$

$$\mathcal{L}^{\lambda_4\,-\lambda\,-\lambda_5}_{k_4\,-k\,-k_5}\; \overset{(2)}{Q}{}^{\lambda_1-\lambda_1}_{0}(k_1)\; \overset{(2)}{Q}{}^{\lambda-\lambda}_{0}(k)\; \overset{(2)}{Q}{}^{\lambda_5-\lambda_5}_{0}(k_5)$$

$$\frac{i\delta_{450}}{W_{450}}\; d\breve{k}_1\, d\breve{k}_4\, d\breve{k}_5 \;+\; O(t), \quad --(7)$$

The exact expression for the order t terms of the type described in Case 3 has been omitted for the sake of brevity as these are unwieldy expressions, and similar in structure to those obtained from $\langle b_2^{-\lambda}(k')\; b_2^{\lambda}(k)\rangle$ and $\langle a_0^{\lambda'}(k')\;{}_3a_4^{\lambda}(k) + a_0^{\lambda}(k)\;{}_3a_4^{\lambda'}(k')\rangle_{\lambda'=-\lambda}$.

S_4: Since $\delta_{15}\;\delta_{26}\;\delta_{37}\;\delta_{67.4}\;\delta_{45.0'}\;\delta_{123.0}$

$$= \delta_{00'}\;\delta_{15}\;\delta_{26}\;\delta_{37}\;\delta_{123.0}\;\delta_{234},$$

Integration over $\breve{k}_5$, $\breve{k}_6$ and $\breve{k}_7$ yields

$$S_4 = 2\,\delta(k+k')\, \overset{0-0}{\Phi} \sum_{\lambda_1\cdots\lambda_7} \int_{-\infty}^{\infty} \mathcal{L}^{\lambda\,\lambda_1\,\lambda_2\,\lambda_3}_{k\,k_1\,k_2\,k_3}\; \mathcal{L}^{-\lambda\,\lambda_4\,\lambda_5}_{-k\,k_4\,-k_7}$$

$$\mathcal{L}^{\nu_4\ \nu_6\ \nu_7}_{k_4\ -k_2\ -k_3}\ \overset{(2)}{\phi}{}_0^{\nu_1\nu_5}(k_1)\ \overset{(2)}{\phi}{}_0^{\nu_2\nu_6}(k_2)\ \overset{(2)}{\phi}{}_0^{\nu_3\nu_7}(k_3)$$

$$\Delta_{123,0}\ \frac{\Delta(\nu_5\omega_1+\nu_6\omega_2+\nu_7\omega_3+\nu\omega)-\Delta_{450}}{i(\nu_6\omega_2+\nu_7\omega_3-\nu_4\omega_4)}$$

$$\delta_{123,0}\ \delta_{234}\ d\breve{k}_1\,d\breve{k}_2\,d\breve{k}_3\,d\breve{k}_4 .$$

The only way a "t" growth may occur is when

$$\nu_5\omega_1+\nu_6\omega_2+\nu_7\omega_3+\nu\omega \equiv -\nu_1\omega_1-\nu_2\omega_2-\nu_3\omega_3+\nu\omega$$

for then

$$\Delta(\omega)\,\Delta(-\omega) \sim 2\pi t\,\delta(\omega)+O(1).$$

This occurs when $\nu_5=-\nu_1$, $\nu_6=-\nu_2$ and $\nu_7=-\nu_3$.

If one takes the limit and then applies the permutation, one obtains

$$S_4 \sim -8\pi t\,\delta_{00'}\sum_{\nu_1 - \nu_4}\int_{-\infty}^{\infty}\mathcal{L}^{\nu\ \nu_1\ \nu_2\ \nu_3}_{k\ k_1\ k_2\ k_3}\ \frac{\mathcal{L}^{-\nu\ \nu_4\ -\nu_1}_{-k\ k_4\ -k_1}\ \mathcal{L}^{\nu_4\ -\nu_2\ -\nu_3}_{k_4\ -k_2\ -k_3}}{i(\nu_2\omega_2+\nu_3\omega_3+\nu_4\omega_4)}$$

$$\overset{(2)}{\phi}{}_0^{\nu_1-\nu_1}(k_1)\ \overset{(2)}{\phi}{}_0^{\nu_2-\nu_2}(k_2)\ \overset{(2)}{\phi}{}_0^{\nu_3-\nu_3}(k_3)\ \delta^{1}_{123,0}\ \delta_{123,0}\ \delta_{234}\ d\breve{k}_1\cdots d\breve{k}_4 .$$

Integrating over $\breve{k}_4$ and using the properties of the coefficients, one obtains

$$2S_4 \sim 24\bar{n}t\,\delta_{00'} \sum_{\gamma_1\gamma_2\gamma_3} \int_{-\infty}^{\infty} \mathcal{L}_{k\,k_1\,k_2\,k_3}^{\gamma\,\gamma_1\,\gamma_2\,\gamma_3} \frac{2i}{3} \sum_{\gamma_4}$$

$$\frac{\mathcal{L}_{k\,k_2+k_3\,k_1}^{\gamma\,-\gamma_4\,\gamma_1}\,\mathcal{L}_{k_2+k_3\,k_2\,k_3}^{-\gamma_4\,\gamma_2\,\gamma_3}}{\gamma_2\omega_2+\gamma_3\omega_3+\gamma_4\omega_{23}} \overset{(2)}{Q}{}_0^{\gamma_1-\gamma_1}(k_1)\,\overset{(2)}{Q}{}_0^{\gamma_2-\gamma_2}(k_2)\,\overset{(2)}{Q}{}_0^{\gamma_3-\gamma_3}(k_3)\,\delta^{1}_{123,0}\,\delta_{123,0}\,d\breve{k}_1\,d\breve{k}_2\,d\breve{k}_3 .$$

S_5 : This gives the same result as above except 1 and 2 have been interchanged. Noting that there are four terms in S_5 and that the expression is symmetric in 2 and 3 one may write,

$$2(S_4 + 2S_5)$$

$$\sim 24\bar{n}t\,\delta_{00'} \sum_{\gamma_1\gamma_2\gamma_3} \int_{-\infty}^{\infty} \mathcal{L}_{k\,k_1\,k_2\,k_3}^{\gamma\,\gamma_1\,\gamma_2\,\gamma_3} \left(\frac{2i}{3} P^{123} \sum_{\gamma_4} \frac{\mathcal{L}_{k\,k_2+k_3\,k_1}^{\gamma\,-\gamma_4\,\gamma_1}\,\mathcal{L}_{k_2+k_3\,k_2\,k_3}^{-\gamma_4\,\gamma_2\,\gamma_3}}{\gamma_2\omega_2+\gamma_3\omega_3+\gamma_4\omega_{23}} \right) \overset{(2)}{Q}{}_0^{\gamma_1-\gamma_1}(k_1)\,\overset{(2)}{Q}{}_0^{\gamma_2-\gamma_2}(k_2)\,\overset{(2)}{Q}{}_0^{\gamma_3-\gamma_3}(k_3)$$

$$\delta^{1}_{123,0}\,\delta_{123,0}\,d\breve{k}_1\,d\breve{k}_2\,d\breve{k}_3 + O(1). \quad (4\text{-}8).$$

The (2×4) decomposition has 15 terms given by the following array,

$$\delta_{12}\overset{(1)}{\delta}_{3567} \quad \delta_{13}\overset{(1)}{\delta}_{2567} \quad \delta_{15}\overset{(2)}{\delta}_{2367} \quad \delta_{16}\overset{(2)}{\delta}_{2357} \quad \delta_{17}\overset{(2)}{\delta}_{2356}$$

$$\delta_{23}\,\delta_{1567} \quad \delta_{25}\,\delta_{1367} \quad \delta_{26}\,\delta_{1357} \quad \delta_{27}\,\delta_{1356}$$

$$\delta_{35}\,\delta_{1267} \quad \delta_{36}\,\delta_{1257} \quad \delta_{37}\,\delta_{1256}$$

$$\delta_{56}\,\delta_{1237} \quad \delta_{57}\,\delta_{1236}$$

$$\delta_{67}\,\delta_{1235}.$$

S_4: $\delta_{67}\,\delta_{67.4} \rightarrow \delta(\breve{k}_4)$ which implies the expression is zero from the mean value property.

S_1: Since $\delta_{12}\,\delta_{3567}\,\delta_{67.4}\,\delta_{45.0'}\,\delta_{123.0}$

$$= \delta_{12}\,\delta_{3.0}\,\delta_{67.4}\,\delta_{00'}\,\delta_{450},$$

Integrating over $\breve{k}_3$ and $\breve{k}_2$, one obtains

$$S_1 = 2\,\beta\delta_{00'}\sum_{\nu_1\cdots\nu_7}^{0-0}\int_{-\infty}^{\infty} \mathcal{L}^{\nu\ \nu_1\ \nu_2\ \nu_3}_{k\ k_1\ -k_1\ k}\ \mathcal{L}^{-\nu\ \nu_4\ \nu_5}_{-k\ k_4\ k_5}\ \mathcal{L}^{\nu_4\ \nu_6\ \nu_7}_{k_4\ k_6\ k_7}\ Q_0^{(2)\,\nu_1\nu_2}(k_1)$$

$$Q_0^{(4)\,\nu_5\nu_6\nu_7\nu_3}(k_5,k_6,k_7)\ \Delta\big((\nu_1+\nu_2)\omega_1+(\nu_3-\nu)\omega\big)\ \frac{\Delta_{567.0'}-\Delta_{45.0'}}{iW_{67.4}}$$

$$\delta_{450}\,\delta_{67.4}\ d\breve{k}_1\,d\breve{k}_4\cdots d\breve{k}_7.$$

One obtains a "t" growth when $\nu_2=-\nu_1$, $\nu_3=\nu$. The long time behavior of the resulting expression is, after applying the permutation,

$$\sim 6t\,\delta(k+k')\sum_{\lambda_1}\int_{-\infty}^{\infty} L_{k k_1 -k_1 k}^{\lambda\,\lambda_1\,-\lambda_1\,\lambda}\,\overset{(2)\lambda_1-\lambda_1}{Q_0}(k_1)\,dk_1$$

$$\sum_{\lambda_4\lambda_5\lambda_6\lambda_7}\int_{-\infty}^{\infty} L_{-k\,k_4\,k_5}^{-\lambda\,\lambda_4\,\lambda_5}\,L_{k_4\,k_6\,k_7}^{\lambda_4\,\lambda_6\,\lambda_7}\left[\frac{\pi\,\delta_{5670}}{iW_{67.4}}\left(\overset{(4)\lambda_5\lambda_6\lambda_7\lambda}{Q_0}(k_5,k_6,k_7)\right.\right.$$

$$\left.\left.+\overset{(4)\lambda_5\lambda_6\lambda_7\lambda}{Q_0^{*}}(k_5,k_6,k_7)\right)-\frac{P}{W_{5670}W_{450}}\left(\overset{(4)\lambda_5\lambda_6\lambda_7\lambda}{Q_0}(k_5,k_6,k_7)-\overset{(4)\lambda_5\lambda_6\lambda_7\lambda}{Q_0^{*}}(k_5,k_6,k_7)\right)\right]$$

$$\delta_{67.4}\;\delta_{5670}\;d\check{k}_4\,d\check{k}_5\,d\check{k}_6\,d\check{k}_7$$

$$+\,O(1).\;\text{----}\;(9)$$

One may note that since the coefficient $L_{k k_1 k_2 \cdots}^{\lambda\lambda_1\lambda_2\cdots}$ is purely imaginary that the above expression is real.

$S_3:\ \delta_{56}\,\delta_{1237}\quad \delta_{67.4}\quad \delta_{45.0'}\quad \delta_{123.0}$
$\quad = \delta_{56}\,\delta_{70}\quad \delta_{123.0}\quad \delta_{00'}\quad \delta_{450}.$

This decomposition can make the argument $\lambda_5\omega_5+\lambda_6\omega_6+\lambda_7\omega_7+\lambda\omega\equiv 0$ when $\lambda_7=-\lambda$, and $\lambda_6=-\lambda_5$. The resulting "t" growth thus obtained is,

$$2S_3 \sim 4t\,\delta_{00'} \sum_{\nu_1\nu_2\nu_3} \int_{-\infty}^{\infty} L_{k\,k_1\,k_2\,k_3}^{\nu\,\nu_1\,\nu_2\,\nu_3} \Big[i\pi\,\delta^{1}_{123,0} \Big(\phi_0^{(4)\,\nu_1\nu_2\nu_3-\nu}(k_1,k_2,k_3) + \phi_0^{(4)\,\nu_1\nu_2\nu_3-\nu\,*}(k_1,k_2,k_3) \Big) - \frac{P}{W_{123,0}} \Big(\phi_0^{(4)\,\nu_1\nu_2\nu_3-\nu}(k_1,k_2,k_3) - \phi_0^{(4)\,\nu_1\nu_2\nu_3-\nu\,*}(k_1,k_2,k_3) \Big) \Big]$$

$$\delta_{123,0}\; d\breve{k}_1\, d\breve{k}_2\, d\breve{k}_3$$

$$\times \sum_{\nu_4\nu_5} \int_{-\infty}^{\infty} \frac{L_{-k\,k_4\,k_5}^{-\nu\,\nu_4\,\nu_5}\, L_{k_4\,-k_5\,-k}^{\nu_4\,-\nu_5\,-\nu}}{W_{450}}\, \phi_0^{(2)\,\nu_5-\nu_5}(k_5)\,\delta_{450}\, d\breve{k}_4\, d\breve{k}_5$$

$$+\; O(1). \qquad \text{---}\;(10).$$

Essentially the time dependent nature comes from the fact that

$$\frac{\Delta(0) - \Delta(\mu)}{-i\mu} \sim \frac{it}{\mu} + O(1),$$

when μ is a triad. The terms S_2 of the type $\delta_{15}\,\delta_{2367}$ do not contribute any "t" growths.

The (3 X 3) decomposition gives the following array,

$\delta_{123}\,\delta_{567}$	$\delta_{125}\,\delta_{367}$	$\delta_{126}\,\delta_{357}$	$\delta_{127}\,\delta_{356}$
	$\delta_{135}\,\delta_{267}$	$\delta_{136}\,\delta_{257}$	$\delta_{137}\,\delta_{256}$
		$\delta_{156}\,\delta_{237}$	$\delta_{157}\,\delta_{236}$
			$\delta_{167}\,\delta_{235}$.

The first term is zero as δ_{123} reacts with $\delta_{123,0}$ to give $\delta(\breve{k})$ which makes the expression zero by reason of the mean value property. None of the other terms allows an argument of any of the Δ functions to become identically zero. Hence there are no secularities exhibited by these particular decompositions. One now considers

$$\langle C_2^{\jmath} C_{2'}^{\jmath'} \rangle_{\jmath'=-\jmath} = 4 \sum_{\jmath_1 \cdots \jmath_8} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\jmath \jmath_1 \jmath_2} L_{k_1 k_3 k_4}^{\jmath_1 \jmath_3 \jmath_4} L_{k' k_5 k_6}^{\jmath' \jmath_5 \jmath_6} L_{k_5 k_7 k_8}^{\jmath_5 \jmath_7 \jmath_8}$$

$$\langle a_{02}^{\jmath_2} a_{03}^{\jmath_3} a_{04}^{\jmath_4} a_{06}^{\jmath_6} a_{07}^{\jmath_7} a_{08}^{\jmath_8} \rangle \frac{\Delta_{234,0} - \Delta_{12,0}}{i W_{34,1}}$$

$$\frac{\Delta_{678,0'} - \Delta_{56,0'}}{i W_{78,5}} \delta_{78,5}\, \delta_{34,1}\, \delta_{12,0}\, \delta_{56,0'}\, d\breve{k}_1 \cdots\cdots d\breve{k}_8 .$$

One may obtain secular growths,

when $\jmath_2\omega_2 + \jmath_3\omega_3 + \jmath_4\omega_4 - \jmath\omega \equiv 0$,

or $\jmath_6\omega_6 + \jmath_7\omega_7 + \jmath_8\omega_8 - \jmath'\omega' \equiv 0$,

or $\jmath_2\omega_2 + \jmath_3\omega_3 + \jmath_4\omega_4 - \jmath\omega \equiv -(\jmath_6\omega_6 + \jmath_7\omega_7 + \jmath_8\omega_8 - \jmath'\omega')$.

There are no other possibilities since no triad resonances are possible.

The (2 x 2 x 2) decomposition is given by the following array,

$$\overset{(1)}{\delta_{23}\, \delta_{46}\, \delta_{78}} \qquad \overset{(2)}{\delta_{23}\, \delta_{47}\, \delta_{68}} \qquad \overset{(2)}{\delta_{23}\, \delta_{48}\, \delta_{67}}$$

$$\delta_{24}\,\delta_{36}\,\delta_{78} \qquad \delta_{26}\,\delta_{37}\,\delta_{48} \qquad \delta_{26}\,\delta_{38}\,\delta_{67}$$

$$\delta_{26}\,\delta_{34}\,\delta_{78} \qquad \delta_{26}\,\delta_{37}\,\delta_{48} \qquad \delta_{26}\,\delta_{38}\,\delta_{47}$$

$$\delta_{27}\,\delta_{34}\,\delta_{68} \qquad \delta_{27}\,\delta_{36}\,\delta_{48} \qquad \delta_{27}\,\delta_{38}\,\delta_{46}$$

$$\delta_{28}\,\delta_{34}\,\delta_{67} \qquad \delta_{28}\,\delta_{36}\,\delta_{47} \qquad \delta_{28}\,\delta_{37}\,\delta_{46}.$$

The terms S_1, S_3 and S_5 are all zero by the zero mean value property.

S_2: Since $\delta_{23}\,\delta_{47}\,\delta_{68}\,\delta_{78.5}\,\delta_{34.1}\,\delta_{12.0}\,\delta_{56.0'}$

$$= \delta_{23}\,\delta_{4,0}\,\delta_{68}\,\delta_{7.0'}\,\delta_{00'}\,\delta_{56.0'}\,\delta_{12.0}.$$

Integrating over $\breve{k}_3$, $\breve{k}_4$, $\breve{k}_7$ and $\breve{k}_8$, one obtains that

$$S_2 = 4\,\delta_{00'} \sum_{\nu_1 \cdots \nu_8} \int_{-\infty}^{\infty} L^{\nu\,\nu_1\,\nu_2}_{k\,k_1\,k_2}\, L^{\nu_1\,\nu_3\,\nu_4}_{k_1\,-k_2\,k}\, L^{-\nu\,\nu_5\,\nu_6}_{-k\,k_5\,k_6}\, L^{\nu_5\,\nu_7\,\nu_8}_{k_5\,-k\,-k_6}$$

$$\overset{(2)|\nu_2\nu_3}{Q_0}(k_2)\, \overset{(2)|\nu_4\nu_7}{Q_0}(k)\, \overset{(2)|\nu_6\nu_8}{Q_0}(k_6)\, \frac{\Delta\big((\nu_2+\nu_3)\omega_2 + (\nu_4-\nu)\omega\big) - \Delta(s_1\omega_1 + s_2\omega_2 - s\omega)}{i(s_3\omega_2 + s_4\omega - s_1\omega_1)}$$

$$\frac{\Delta\big((\nu_6+\nu_8)\omega_6 + (\nu_7+\nu)\omega\big) - \Delta_{560}}{i(\nu_7\omega + \nu_8\omega_6 - \nu_5\omega_5)}\, \delta_{560}\,\delta_{12,0}\, d\breve{k}_1\, d\breve{k}_2\, d\breve{k}_5\, d\breve{k}_6.$$

For brevity, only the $O(t^2)$ terms will be written down. The $O(t)$ terms will cancel in a way which has been previously indicated.

$$4S_2 \sim 16t^2 \delta(k+k') \Phi_0^{(2)1,-1}(k)$$
$$\left(\sum_{\nu_1\nu_2} \int_{-\infty}^{\infty} L_{kk_1k_2}^{1\nu_1\nu_2} L_{k_1k-k_2}^{\nu_1 1-\nu_2} \Phi_0^{(2)\nu_2-\nu_2}(k_2) \frac{\delta_{12,0}}{W_{12,0}} dk_1 dk_2 \right)^2$$
$$+ O(t). \qquad --- \quad (11)$$

S_4: Since $\delta_{26}\,\delta_{37}\,\delta_{48}\,\delta_{78,5}\,\delta_{34,1}\,\delta_{12,0}\,\delta_{56,0'}$

$$= \delta_{15}\,\delta_{26}\,\delta_{37}\,\delta_{48}\,\delta_{34,1}\,\delta_{12,0}\,\delta_{00'},$$

Integrating over $\tilde{k}_5$, $\tilde{k}_6$, $\tilde{k}_7$ and $\tilde{k}_8$, the term containing t under the integral for $\langle C_2' C_2'' \rangle_{1,-1}$ becomes

$$\frac{\Delta(\nu_2\omega_2 + \nu_3\omega_3 + \nu_4\omega_4 - \nu\omega) - \Delta_{12,0}}{i(s_3\omega_3 + s_4\omega_4 - s_1\omega_1)} \times \frac{\Delta(\nu_6\omega_2 + \nu_7\omega_3 + \nu_8\omega_4 + \nu\omega) - \Delta(\nu_5\omega_1 + \nu_6\omega_2 + \nu\omega)}{i(\nu_7\omega_3 + \nu_8\omega_4 - \nu_5\omega_1)}$$

Since there are no triad resonances the only possible "t" growths occur when $\nu_6 = -\nu_2$, $\nu_7 = -\nu_3$ and $\nu_8 = -\nu_4$ for then

$$\Delta(\omega)\,\Delta(-\omega) \sim 2\pi t\,\delta(\omega) + O(1).$$

Therefore,

$$S_4 \sim 8\pi t \delta_{00'} \sum_{\nu_1 \cdots \nu_5} \int_{-\infty}^{\infty} \frac{L^{1\nu_1\nu_2}_{k k_1 k_2} L^{\nu_1 \nu_3 \nu_4}_{k_1 k_3 k_4}}{i(s_3\omega_3 + s_4\omega_4 - s_1\omega_1)} \frac{L^{-1\nu_5-\nu_2}_{-k-k_1-k_2} L^{\nu_5-\nu_3-\nu_4}_{-k_1-k_3-k_4}}{-i(\nu_3\omega_3+\nu_4\omega_4+\nu_5\omega_1)}$$

$$\overset{(2)}{\phi}{}^{\nu_2-\nu_2}_0(k_2)\, \overset{(2)}{\phi}{}^{\nu_3-\nu_3}_0(k_3)\, \overset{(2)}{\phi}{}^{\nu_4-\nu_4}_0(k_4)\, \delta_{34,1}\, \delta^1_{234,0}\, \delta_{234,0}\, d\tilde{k}_1 \cdots d\tilde{k}_4$$

$$+ O(1).$$

Integrating over $\tilde{k}_1$, and changing

$$4 \rightarrow 3 \rightarrow 2 \rightarrow 1 \text{ and } \nu_1 \rightarrow -\nu_4 ,$$

one obtains, by using the properties of the coefficients, that

$$2S_4 \sim -36\pi t \delta_{00'} \sum_{\nu_1\nu_2\nu_3} \int_{-\infty}^{\infty} -\frac{2i}{3} \sum_{\nu_4} \frac{L^{1\,-\nu_4\,\nu_1}_{k\, k_2+k_3\, k_1} L^{-\nu_4\,\nu_2\,\nu_3}_{k_2+k_3\, k_2\, k_3}}{\nu_2\omega_2 + \nu_3\omega_3 + \nu_4\omega_{23}}$$

$$\times -\frac{2i}{3} \sum_{\nu_5} \frac{L^{1\,-\nu_5\,\nu_1}_{k\, k_2+k_3\, k_1} L^{-\nu_5\,\nu_2\,\nu_3}_{k_2+k_3\, k_2\, k_3}}{\nu_2\omega_2 + \nu_3\omega_3 + \nu_5\omega_{23}} \overset{(2)}{\phi}{}^{\nu_1-\nu_1}_0(k_1)\, \overset{(2)}{\phi}{}^{\nu_2-\nu_2}_0(k_2)$$

$$\overset{(2)}{\phi}{}^{\nu_3-\nu_3}_0(k_3)\, \delta^1_{123,0}\, \delta_{123,0}\, d\tilde{k}_1\, d\tilde{k}_2\, d\tilde{k}_3 + O(1).$$

The terms S_6 and S_7 complete a permutation.

$$2(S_4 + S_6 + S_7)$$

$$\sim -36\bar{\pi} t\, \delta_{00'} \sum_{\nu_1\nu_2\nu_3} \int_{-\infty}^{\infty} -\frac{2i}{3} \sum_{\nu_4} \frac{L_{k\ k_2+k_3\ k_1}^{\nu\ -\nu_4\ \nu_1}\ L_{k_2+k_3\ k_2\ k_3}^{-\nu_4\ \nu_2\ \nu_3}}{\nu_2\omega_2 + \nu_3\omega_3 + \nu_4\omega_{23}}$$

$$\times -\frac{2i}{3} \beta^{123} \sum_{\nu_5} \frac{L_{k\ k_2+k_3\ k_1}^{\nu\ -\nu_5\ \nu_1}\ L_{k_2+k_3\ k_2\ k_3}^{-\nu_5\ \nu_2\ \nu_3}}{\nu_2\omega_2 + \nu_3\omega_3 + \nu_5\omega_{23}}\ \overset{(2)}{\varphi}{}_0^{\nu_1-\nu_1}(k_1)\ \overset{(2)}{\varphi}{}_0^{\nu_2-\nu_2}(k_2)$$

$$\overset{(2)}{\varphi}{}_0^{\nu_3-\nu_3}(k_3)\ \delta_{123,0}^{J}\ \delta_{123,0}\ d\breve{k}_1 d\breve{k}_2 d\breve{k}_3,$$

$$\sim -12\bar{\pi} t\, \delta_{00'} \sum_{\nu_1\nu_2\nu_3} \int_{-\infty}^{\infty} \left(\beta^{123} -\frac{2i}{3} \sum_{\nu_4} \frac{L_{k\ k_2+k_3\ k_1}^{\nu\ -\nu_4\ \nu_1}\ L_{k_2+k_3\ k_2\ k_3}^{-\nu_4\ \nu_2\ \nu_3}}{\nu_2\omega_2 + \nu_3\omega_3 + \nu_4\omega_{23}} \right)^2$$

$$\overset{(2)}{\varphi}{}_0^{\nu_1-\nu_1}(k_1)\ \overset{(2)}{\varphi}{}_0^{\nu_2-\nu_2}(k_2)\ \overset{(2)}{\varphi}{}_0^{\nu_3-\nu_3}(k_3)\ \delta_{123,0}^{J}\ \delta_{123,0}\ d\breve{k}_1 d\breve{k}_2 d\breve{k}_3$$

$$+ O(1). \qquad \text{---} (12)$$

The (2×4) decomposition of the mean value $\langle a_{02}^{\nu_2} a_{03}^{\nu_3} a_{04}^{\nu_4} a_{06}^{\nu_6} a_{07}^{\nu_7} a_{08}^{\nu_8} \rangle$ is given by the array,

$$\overset{(1)}{\delta_{23}\,\delta_{4678}} \quad \overset{(1)}{\delta_{24}\,\delta_{3678}} \quad \delta_{26}\,\delta_{3478} \quad \delta_{27}\,\delta_{3468} \quad \delta_{28}\,\delta_{3467}$$

$$\delta_{34}\,\delta_{2678} \quad \delta_{36}\,\delta_{2478} \quad \delta_{37}\,\delta_{2468} \quad \delta_{38}\,\delta_{2467}$$

$$\delta_{46}\,\delta_{2378} \quad \delta_{47}\,\delta_{2368} \quad \delta_{48}\,\delta_{2367}$$

$$\overset{(2)}{\delta_{67}\,\delta_{2348}} \quad \overset{(2)}{\delta_{68}\,\delta_{2347}}$$

$$\delta_{78}\,\delta_{2346}.$$

The only possible "t" growths occur in the term designated (1) and (2).

S_1: Since $\delta_{23}\,\delta_{4678}\,\delta_{78,5}\,\delta_{678,0'}\,\delta_{34,1}\,\delta_{234,0}$

$$= \delta_{23}\,\delta_{4,0}\,\delta_{6780}\,\delta_{78,5}\,\delta_{00'}\,\delta_{12,0},$$

Upon integration over $\tilde{k}_3$ and $\tilde{k}_4$, and putting

$J_3 = -J_2$, $J_4 = J$ the time dependent terms become

$$\frac{\Delta(0) - \Delta_{12,0}}{-iW_{12,0}} \; \frac{\Delta_{678,0'} - \Delta_{56,0'}}{iW_{78,5}} .$$ In the long time limit, these terms behave as

$$\frac{it}{W_{12,0}}\left[\pi\,\delta^{J}_{6780} + \frac{iP}{W_{6780}} - \frac{i}{W_{560}}\right]\frac{1}{iW_{78,5}} + O(1).$$

In S_2 if one relabels 5,6,7,8 by 1,2,3,4 one obtains the same result as S_1 with $\nu \to -\nu$, $\tilde{k} \to -\tilde{k}$. Therefore, adding these two terms, one obtains

$$2(S_1+S_2) \sim \delta t\, \delta_{00'} \left(\sum_{\nu_1\nu_2} \int_{-\infty}^{\infty} \frac{L_{k k_1 k_2}^{\nu \nu_1 \nu_2} L_{k_1 -k_2 k}^{\nu_1 -\nu_2 \nu}}{-i(\nu_1\omega_1+\nu_2\omega_2-\nu\omega)} \delta_{12,0}\, \overset{(2)}{Q}{}_0^{\nu_2-\nu_2}(k_2)\, dk_1 dk_2 \right)$$

$$\times \sum_{\nu_5\cdots\nu_8} \int_{-\infty}^{\infty} L_{-k\, k_5\, k_6}^{-\nu\, \nu_5\, \nu_6} L_{k_5\, k_7\, k_8}^{\nu_5\, \nu_7\, \nu_8}\, \delta_{78,5}\, \delta_{6780}$$

$$\left[\frac{\pi\, \delta_{6780}^{\nu}}{i W_{78,5}} \left(\overset{(4)}{Q}{}_0^{\nu_6\nu_7\nu_8\nu}(k_6,k_7,k_8) + \overset{(4)*}{Q}{}_0^{\nu_6\nu_7\nu_8\nu}(k_6,k_7,k_8) \right) - \frac{P}{W_{6780} W_{560}} \left(\overset{(4)}{Q}{}_0^{\nu_6\nu_7\nu_8\nu}(k_6,k_7,k_8) - \overset{(4)*}{Q}{}_0^{\nu_6\nu_7\nu_8\nu}(k_6,k_7,k_8) \right) \right] dk_5 \cdots dk_8 .$$

Changing $\begin{Bmatrix} \nu_i \\ \tilde{k}_i \end{Bmatrix} \longrightarrow \begin{Bmatrix} -\nu_i \\ -\tilde{k}_i \end{Bmatrix}$, $i = 5,6,7,8$,

one obtains, using that

$$\overset{(4)}{Q}{}_0^{-\nu_6\, -\nu_7\, -\nu_8\, \nu}(-k_6, -k_7, -k_8) = \overset{(4)*}{Q}{}_0^{\nu_6\, \nu_7\, \nu_8\, -\nu}(k_6, k_7, k_8)$$

and

$$L_{-k\, -k_1\, -k_2}^{-\nu\, -\nu_1\, -\nu_2} = -L_{k\, k_1\, k_2}^{\nu\, \nu_1\, \nu_2},$$

$$2(S_1+S_2) \sim \delta t\, \delta(k+k')$$

$$\sum_{\nu_1\nu_2} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2} L_{k_1\, k\, -k_2}^{\nu_1\, \nu\, -\nu_2}\, \overset{(2)}{Q}{}_0^{\nu_2-\nu_2}(k_2)\, \frac{i\delta_{12,0}}{W_{12,0}}\, dk_1 dk_2$$

$$\times \sum_{\nu_5 \cdots \nu_8} \int_{-\infty}^{\infty} \frac{L_{k\,k_5\,k_6}^{\nu\,\nu_5\,\nu_6} \, L_{k_5\,k_7\,k_8}^{\nu_5\,\nu_7\,\nu_8}}{i(\nu_5\omega_5 + \nu_6\omega_6 - \nu\omega)} \left[\pi^{\nu} \delta_{678,0} \left(Q_0^{(4)\,\nu_6\nu_7\nu_8 -\nu}(k_6,k_7,k_8) + Q_0^{(4)\,\nu_6\nu_7\nu_8 -\nu\,*}(k_6,k_7,k_8) \right) + \frac{iP}{W_{678,0}} \left(Q_0^{(4)\,\nu_6\nu_7\nu_8 -\nu}(k_6,k_7,k_8) - Q_0^{(4)\,\nu_6\nu_7\nu_8 -\nu\,*}(k_6,k_7,k_8) \right) \right]$$

$$\delta_{678,0}\, \delta_{56,0}\, dk_5 \cdots dk_8 . \qquad --- (13)$$

It may also be seen from the following array,

$$\delta_{234}\ \delta_{678} \qquad \delta_{236}\,\delta_{478} \qquad \delta_{237}\,\delta_{468} \qquad \delta_{238}\,\delta_{467}$$

$$\delta_{246}\ \delta_{378} \qquad \delta_{247}\,\delta_{368} \qquad \delta_{248}\,\delta_{367}$$

$$\delta_{267}\ \delta_{348} \qquad \delta_{268}\,\delta_{347}$$

$$\delta_{278}\ \delta_{346} ,$$

that the (3×3) decomposition does not give a "t" growth. Note that there is one term which is secular if triad resonances exist (the term $\delta_{278}\,\delta_{346}$), which implies that $\tilde{k}_5 = -\tilde{k}_2$ and $\breve{k}_6 = -\breve{k}_1$. This would allow

$$W_{56,0'} \equiv -W_{12,0}$$

when $\mathfrak{s}' = -\mathfrak{s}$, $\mathfrak{s}_6 = -\mathfrak{s}_1$, $\mathfrak{s}_5 = -\mathfrak{s}_2$.

One now examines

$$\langle a_0^{\mathfrak{s}'}(k')\,{}_2a_4^{\mathfrak{s}}(k) + a_0^{\mathfrak{s}}(k)\,{}_2a_4^{\mathfrak{s}'}(k')\rangle_{\mathfrak{s}'=-\mathfrak{s}}$$

$$= 6\,\rho^{00'}\sum_{\mathfrak{s}_1\ldots\mathfrak{s}_7}\int_{-\infty}^{\infty} L_{k\,k_1 k_2 k_3}^{\mathfrak{s}\,\mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3}\, L_{k_1 k_4 k_5}^{\mathfrak{s}_1\mathfrak{s}_4\mathfrak{s}_5}\, L_{k_4 k_6 k_7}^{\mathfrak{s}_4\mathfrak{s}_6\mathfrak{s}_7}\,\langle a_{01}^{\mathfrak{s}'} a_{02}^{\mathfrak{s}_2} a_{03}^{\mathfrak{s}_3} a_{05}^{\mathfrak{s}_5} a_{06}^{\mathfrak{s}_6} a_{07}^{\mathfrak{s}_7}\rangle$$

$$F(t)\,\delta_{67,4}\,\delta_{45,1}\,\delta_{123,0}\,d\breve{k}_1\ldots d\breve{k}_7$$

where

$$F(t) = \int_0^t \frac{\Delta_{567,1} - \Delta_{45,1}}{iW_{67,4}}\, e^{iW_{123,0}t}\,dt.$$

The (2 x 2 x 2) decomposition gives the array,

$$\overset{(1)}{\delta_{0'2}\,\delta_{35}\,\delta_{67}} \qquad \overset{(2)}{\delta_{0'2}\,\delta_{36}\,\delta_{57}} \qquad \overset{(2)}{\delta_{0'2}\,\delta_{37}\,\delta_{56}}$$

$$\overset{(1)}{\delta_{0'3}\,\delta_{25}\,\delta_{67}} \qquad \overset{(2)}{\delta_{0'3}\,\delta_{26}\,\delta_{57}} \qquad \overset{(2)}{\delta_{0'3}\,\delta_{27}\,\delta_{56}}$$

$$\overset{(3)}{\delta_{0'5}\,\delta_{23}\,\delta_{67}} \qquad \overset{(4)}{\delta_{0'5}\,\delta_{26}\,\delta_{37}} \qquad \overset{(4)}{\delta_{0'5}\,\delta_{27}\,\delta_{36}}$$

$$\overset{(5)}{\delta_{0'6}\,\delta_{23}\,\delta_{57}} \qquad \overset{(6)}{\delta_{0'6}\,\delta_{25}\,\delta_{37}} \qquad \overset{(6)}{\delta_{0'6}\,\delta_{27}\,\delta_{35}}$$

$$\overset{(5)}{\delta_{0'7}\,\delta_{23}\,\delta_{56}} \qquad \overset{(6)}{\delta_{0'7}\,\delta_{25}\,\delta_{36}} \qquad \overset{(6)}{\delta_{0'7}\,\delta_{26}\,\delta_{35}}.$$

In S_1 and S_3 : $\delta_{67}\ \delta_{67,4} \Rightarrow \delta(\breve{k}_4)\,\delta(\breve{k}_6+\breve{k}_7)$.
This makes the expression zero by the zero mean value property.

$$S_2 : \delta_{0'2}\ \delta_{36}\ \delta_{57}\ \delta_{67,4}\ \delta_{45,1}\ \delta_{123,0}$$

$$= \delta_{00'}\ \delta_{2,0}\ \delta_{13}\ \delta_{6,1}\ \delta_{75}\ \delta_{45,1}.$$

This term will give a "t^2" growth which cancels when one applies the permutation $\jmath \to -\jmath$, $\breve{k} \to -\breve{k}$ and adds the two terms.

$$S_5 : \delta_{0'6}\ \delta_{23}\ \delta_{57}\ \delta_{67,4}\ \delta_{567,1}\ \delta_{123,0}$$

$$= \delta_{00'}\ \delta_{1,0}\ \delta_{23}\ \delta_{6,0}\ \delta_{57}\ \delta_{45,0},$$

Upon integration over $\breve{k}_1$, $\breve{k}_3$, $\breve{k}_6$ and $\breve{k}_7$, this term can allow t^2 growths where the second order cumulants are all energy densities and "t" growths when at least one of the second order cumulants is of the $\phi_0^{(2)++}$ type. These latter terms will cancel in the same manner as shown previously in the case $\langle e_2^{\jmath}(k)\, e_2^{-\jmath}(k')\rangle$ and $\langle a_0^{-\jmath}(k')\,{}_3a_4^{\jmath}(k) + a_0^{\jmath}(k)\,{}_3a_4^{-\jmath}(k')\rangle$. The "$t^2$" term occurs when

$$J_1 = J, \quad J_6 = J, \quad J_7 = -J_5, \quad J_3 = -J_2,$$

whereupon F(t) becomes

$$\int_0^t \frac{\Delta(0) - \Delta_{45,0}}{-iW_{45,0}}\, dt.$$

Let $W_{45,0} = \mu$, and this term takes the form

$$\int_0^t \frac{\Delta(0)-\Delta(\mu)}{-i\mu}\, dt = \int_0^t \frac{t}{-i\mu}\, dt + \int_0^t \frac{e^{i\mu t}-1}{-\mu^2}\, dt,$$

$$= \frac{-t^2}{2i\mu} + \frac{t}{\mu^2} + \frac{\Delta(\mu)}{-\mu^2}.$$

Now, one considers

$$\int_{-\infty}^{\infty} f(\mu) \left\{ \frac{-t^2}{2i\mu} + \frac{t}{\mu^2} + \frac{\Delta(\mu)}{-\mu^2} \right\} d\mu$$

$$= \int_{-\infty}^{\infty} f(\mu) \cdot \frac{-t^2}{2i\mu} - t \int_{-\infty}^{\infty} f(\mu)\, d\left(\frac{1}{\mu}\right) + \frac{1}{2i} \int_{-\infty}^{\infty} f(\mu) \cdot (e^{i\mu t}-1)\, d\left(\frac{1}{\mu^2}\right),$$

Integration by parts gives

$$\int_{-\infty}^{\infty} \frac{t^2}{-2i\mu} f\, d\mu + \int_{-\infty}^{\infty} \frac{t}{\mu} f_\mu\, d\mu - \int \frac{1}{2i\mu^2} f_\mu (e^{i\mu t}-1)\, d\mu$$

$$- \int \frac{1}{2i\mu^2} f(\mu) \cdot it e^{i\mu t} d\mu,$$

$$= \int_{-\infty}^{\infty} \frac{t^2}{-2i\mu} f \, d\mu + \int_{-\infty}^{\infty} \frac{t}{\mu} f_\mu \, d\mu - \int_{-\infty}^{\infty} \frac{1}{2i\mu} f_{\mu\mu} (e^{i\mu t}-1) \, d\mu$$

$$- \int_{-\infty}^{\infty} \frac{1}{i\mu} f_\mu \, it \, e^{i\mu t} d\mu - \frac{1}{2i} \int_{-\infty}^{\infty} f \, (it)^2 \frac{e^{i\mu t}}{\mu} \, d\mu ,$$

$$= \frac{t^2}{2} \int_{-\infty}^{\infty} f(\mu) \cdot \frac{e^{i\mu t}-1}{i\mu} \, d\mu - it \int_{-\infty}^{\infty} f_\mu \cdot \frac{e^{i\mu t}-1}{i\mu} \, d\mu + O(1) .$$

If μ is a triad and cammot be zero anywhere in the spectrum, the above is equal to, in the limit $t \to \infty$

$$\sim \frac{t^2}{2} \int_{-\infty}^{\infty} f \cdot \frac{i}{\mu} \, d\mu - it \int_{-\infty}^{\infty} f_\mu \cdot \frac{i}{\mu} \, d\mu + O(1) .$$

The $O(t)$ terms can be written $\quad - \int_{-\infty}^{\infty} f \frac{1}{\mu^2} \, d\mu$

Since $\quad \int f_\mu \frac{1}{\mu} \, d\mu = \int f \frac{1}{\mu^2} \, d\mu ,$

when $\mu \neq 0$ anywhere in the spectrum . The application of the permutation $\jmath \to -\jmath$, $k \to -k$ cancels the $O(t)$ terms and doubles the $O(t^2)$ terms. Therefore,

$$2S_5 \sim 12t^2 \delta_{00'} \sum_{\nu_2\nu_4\nu_5} \int_{-\infty}^{\infty} L_{k k_2 k_2 -k_2}^{\nu\nu\nu_2-\nu_2} L_{k k_4 k_5}^{\nu\nu_4\nu_5} L_{k_4 k -k_5}^{\nu_4\nu-\nu_5}$$

$$\phi_0^{(2)\nu-\nu}(k)\,\phi_0^{(2)\nu_2-\nu_2}(k_2)\,\phi_0^{(2)\nu_5-\nu_5}(k_5)\,\frac{i\delta_{45,0}}{W_{45,0}}\,d\breve{k}_2\,d\breve{k}_4\,d\breve{k}_5$$

$$+\ O(1). \qquad ---(14)$$

If one changes $\nu_i \to -\nu_i$, $\breve{k}_i \to -\breve{k}_i$, $i = 4,5$ one sees (14) = - (7) .

$$S_4 : \quad \delta_{0'5}\,\delta_{26}\,\delta_{37}\,\delta_{67,4}\,\delta_{45,1}\,\delta_{123,0}$$

$$= \delta_{5,0}\,\delta_{26}\,\delta_{37}\,\delta_{123,0}\,\delta_{234}\,\delta_{00'}.$$

One integrates over $\breve{k}_5$, $\breve{k}_6$ and k_7 . A "t" growth is possible when $\nu_7\nu_3 = \nu_6\nu_2 = -\nu_5\nu = -1$, for then

$$F(t) = \int_0^t \frac{\Delta(s\omega - s_2\omega_2 - s_3\omega_3 - s_1\omega_1) - \Delta(\nu_4\omega_4 + \nu_5\omega_5 - \nu\omega_1)}{-i(\nu_2\omega_2 + \nu_3\omega_3 + \nu_4\omega_4)}\, e^{i(s_1\omega_1 + s_2\omega_2 + s_3\omega_3 - s\omega)t}\, dt,$$

$$= \int_0^t \left\{ \frac{\Delta(s_1\omega_1 + s_2\omega_2 + s_3\omega_3 - s\omega)}{-i(s_1\omega_2 + s_3\omega_3 + s_4\omega_4)} + \frac{\Delta(s_4\omega_4 + s_5\omega_5 - s_1\omega_1)\, e^{iW_{123,0}t}}{i(s_1\omega_2 + s_3\omega_3 + s_4\omega_4)} \right\} dt$$

$$\sim \frac{t}{-iW_{234}} \left[\pi \delta^{s}_{123,0} + \frac{iP}{W_{123,0}} \right] + O(1).$$

The permuataion $s \to -s$, $\breve{k} \to -\breve{k}$ will make the principal part terms cancel because of an odd number of sign changes, whereas the resonance terms add. After integration over $\breve{k}_4$, one obtains

$$2S_4 \sim 36\pi t \delta_{00'} \overset{(2)}{Q}{}^{s-s}_0(k) \sum_{s_1 s_2 s_3} \int_{-\infty}^{\infty} L^{s\, s_1\, s_2\, s_3}_{k\, k_1\, k_2\, k_3} \left[-\frac{2i}{3} \sum_{s_4} \frac{L^{s_1\, -s_4\, s}_{k_1\, -k_2-k_3\, k}\, L^{-s_4\, -s_2\, -s_3}_{k_2-k_3\, -k_2\, -k_3}}{-(s_2\omega_2 + s_3\omega_3 - s_4\omega_{23})} \right] \overset{(2)}{Q}{}^{s_2-s_2}_0(k_2)\, \overset{(2)}{Q}{}^{s_3-s_3}_0(k_3)\, \delta^{s}_{123,0}\, \delta_{123,0}\, d\breve{k}_1 d\breve{k}_2 d\breve{k}_3 + O(1).$$

S_6 behaves in a similar fashion.

$$2S_4 + 4S_6$$

$$\sim 36\pi t \delta_{00'} \overset{(2)}{Q}{}^{s-s}_0(k) \sum_{s_1 s_2 s_3} \int_{-\infty}^{\infty} L^{s\, s_1\, s_2\, s_3}_{k\, k_1\, k_2\, k_3}$$

$$\times \overset{0-2-3}{P} \left[-\frac{2i}{3} \sum_{s_4} \frac{L^{s_1\ -s_4\ s}_{k_1\ -k_2-k_3\ k}\ L^{-s_4\ -s_2\ -s_3}_{-k_2-k_3\ -k_2\ -k_3}}{-s_2\omega_2 - s_3\omega_3 + s_4\omega_{23}} \right]$$

$$\overset{(2)}{\varphi}{}_0^{s_2 -s_2}(k_2)\ \overset{(2)}{\varphi}{}_0^{s_3 -s_3}(k_3)\ \delta_{123,0}^{s}\ \delta_{123,0}\ d\breve{k}_1\, d\breve{k}_2\, d\breve{k}_3$$

$$+ O(1), \qquad \text{----(15)}$$

where $\overset{0-2-3}{P}$ is a cyclic permutation over (0,-2,-3) sending 0 → -2 → -3 → 0 . The (2 x 4) decomposition gives the following array,

$$\delta_{0'2}\ \overset{(1)}{\delta}_{3567} \quad \delta_{0'3}\ \overset{(1)}{\delta}_{2567} \quad \delta_{0'5}\ \overset{(2)}{\delta}_{2367} \quad \delta_{0'6}\ \overset{(3)}{\delta}_{2357} \quad \delta_{0'7}\ \overset{(3)}{\delta}_{2356}$$

$$\delta_{23}\ \overset{(4)}{\delta}_{0'567} \quad \delta_{25}\ \overset{(5)}{\delta}_{0'367} \quad \delta_{26}\ \overset{(6)}{\delta}_{0'357} \quad \delta_{27}\ \overset{(6)}{\delta}_{0'356}$$

$$\delta_{35}\ \overset{(5)}{\delta}_{0'267} \quad \delta_{36}\ \overset{(6)}{\delta}_{0'257} \quad \delta_{37}\ \overset{(6)}{\delta}_{0'256}$$

$$\delta_{56}\ \overset{(7)}{\delta}_{0'237} \quad \delta_{57}\ \overset{(7)}{\delta}_{0'236}$$

$$\delta_{67}\ \overset{(8)}{\delta}_{0'235}.$$

S_1: This term gives a "t" growth which cancels on application of the permutation. The terms S_2, S_3, S_5 and $S_6 \sim O(1)$.

$$S_4: \delta_{23}\,\delta_{0'567}\,\delta_{67,4}\,\delta_{45,1}\,\delta_{123,0}$$

$$= \delta_{23}\,\delta_{1,0}\,\delta_{00'}\,\delta_{67,4}\,\delta_{567,0}.$$

Upon integration over $\breve{k}_1$ and $\breve{k}_3$, and setting $\nu_1 = \nu$ and $\nu_3 = -\nu_2$, one obtains that $W_{123,0} \equiv 0$. Therefore

$$F(t) = \int_0^t \frac{\Delta_{567,0} - \Delta_{45,0}}{iW_{67,4}}\, dt,$$

and

$$F(t) \sim \frac{t}{iW_{67,4}}\left[\pi\delta^{\nu}_{567,0} + \frac{iP}{W_{567,0}} - \frac{i}{W_{45,0}}\right] + O(1).$$

Therefore, one obtains

$$S_4 \sim \left(6t\,\delta_{00'} \sum_{\nu_2} \int_{-\infty}^{\infty} L^{\nu\,\nu\;\;\nu_2\,-\nu_2}_{k\,k\;\;k_2\,-k_2}\, Q_0^{(2)\,\nu_2,-\nu_2}(k_2)\, d\breve{k}_2\right)$$

$$\times \sum_{\nu_4\cdots\nu_7} \int_{-\infty}^{\infty} \frac{L^{\nu\,\nu_4\,\nu_5}_{k\,k_4\,k_5}\, L^{\nu_4\,\nu_6\,\nu_7}_{k_4\,k_6\,k_7}}{iW_{67,4}}\, \delta_{67,4}$$

$$\times \left[\pi \, \delta_{567,0}^{1} \left(\overset{(4)}{\phi}_0{}^{\nu_5 \nu_6 \nu_7 -\nu}(k_5, k_6, k_7) + \overset{(4)}{\phi}_0{}^{*\,\nu_5 \nu_6 \nu_7 -\nu}(k_5, k_6, k_7) \right) \right.$$

$$\left. + \left(\frac{iP}{W_{567,0}} - \frac{i}{W_{45,0}} \right) \left(\overset{(4)}{\phi}_0{}^{\nu_5 \nu_6 \nu_7 -\nu}(k_5, k_6, k_7) - \overset{(4)}{\phi}_0{}^{*\,\nu_5 \nu_6 \nu_7 -\nu}(k_5, k_6, k_7) \right) \right] \delta_{567,0}$$

$$d\breve{k}_4 \cdots d\breve{k}_7 . \qquad \cdots (16)$$

Setting $\nu_i \to -\nu_i$ and $\breve{k}_i \to -\breve{k}_i$ (i = 4,5,6,7) one obtains the expression (9) with three sign changes. Therefore (16) = -(9) .

$$S_7 : \quad \delta_{56} \, \delta_{0'237} \, \delta_{67,4} \, \delta_{45,1} \, \delta_{123,0}$$

$$= \delta_{56} \, \delta_{7,1} \, \delta_{00'} \, \delta_{45,1} \, \delta_{123,0} .$$

Integrating over $\breve{k}_6$ and $\breve{k}_7$, and setting $\nu_7 = \nu_1$ and $\nu_6 = -\nu_5$,

$$F(t) = \int_0^t \frac{\Delta(0) - \Delta(\mu)}{-i\mu} \, e^{i\omega t} \, dt .$$

Let $\mu = \nu_4 \omega_4 + \nu_5 \omega_5 - \nu_1 \omega_1$ and $\omega = \nu_1 \omega_1 + \nu_2 \omega_2 + \nu_3 \omega_3 - \nu \omega$ and consider

$$\int f(\omega, \mu) \, F(t) \, d\omega \, d\mu$$

$$\sim \int f(\omega, \mu) \left(\pi \delta(\mu) + \frac{iP}{\mu} \right) \int_0^t t e^{i\omega t} dt \, d\omega \, d\mu ,$$

when one takes the limit $t \to \infty$ over the μ integration. There is no "t" growth in this term since

$$\int_0^t t e^{i\omega t} dt \sim O(1).$$

The (3 X 3) decomposition gives the array,

$$\begin{array}{llll}
\delta_{0'23}\,\delta_{567} & \delta_{0'25}\,\delta_{367} & \delta_{0'26}\,\delta_{357} & \delta_{0'27}\,\delta_{356} \\
 & \delta_{0'34}\,\delta_{267} & \delta_{0'36}\,\delta_{257} & \delta_{0'37}\,\delta_{256} \\
 & & \delta_{0'56}\,\delta_{237} & \delta_{0'57}\,\delta_{236} \\
 & & & \delta_{0'67}\,\delta_{235}.
\end{array}$$

Each of these terms gives an O(1) behavior in long time. Consider $\langle a_0^{1'}(\tilde{k}')_{,} e_4^{1}(\tilde{k}) + a_0^{1}(k)_{,} e_4^{1'}(k') \rangle_{1'=-1}$

$$= 2P^{00'} \sum_{1,\cdots,7} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{1 1_1 1_2} L_{k_2 k_5 k_6 k_7}^{1_2 1_5 1_6 1_7} L_{k_1 k_3 k_4}^{1_1 1_3 1_4} \langle a_{0'}^{1'} a_{03}^{1_3} a_{04}^{1_4} a_{05}^{1_5} a_{06}^{1_6} a_{07}^{1_7} \rangle$$

$$\int_0^t \Delta_{34,1} \Delta_{567,2}\, e^{iW_{12,0}t} dt\, \delta_{34,1}\, \delta_{567,2}\, \delta_{12,0}\, d\check{k}_1 \cdots d\check{k}_7 .$$

The (2 X 2 X 2) decomposition gives the following array.

$$\overset{(1)}{\delta_{0'3}\,\delta_{45}\,\delta_{67}} \qquad \overset{(1)}{\delta_{0'3}\,\delta_{46}\,\delta_{57}} \qquad \overset{(1)}{\delta_{0'3}\,\delta_{47}\,\delta_{56}}$$

$$\overset{(1)}{\delta_{0'4}\,\delta_{35}\,\delta_{67}} \qquad \overset{(1)}{\delta_{0'4}\,\delta_{36}\,\delta_{57}} \qquad \overset{(1)}{\delta_{0'4}\,\delta_{37}\,\delta_{56}}$$

$$\overset{(2)}{\delta_{0'5}\,\delta_{34}\,\delta_{67}} \qquad \overset{(3)}{\delta_{0'5}\,\delta_{36}\,\delta_{47}} \qquad \overset{(3)}{\delta_{0'5}\,\delta_{37}\,\delta_{46}}$$

$$\overset{(2)}{\delta_{0'6}\,\delta_{34}\,\delta_{57}} \qquad \overset{(3)}{\delta_{0'6}\,\delta_{35}\,\delta_{47}} \qquad \overset{(3)}{\delta_{0'6}\,\delta_{37}\,\delta_{45}}$$

$$\overset{(2)}{\delta_{0'7}\,\delta_{34}\,\delta_{56}} \qquad \overset{(3)}{\delta_{0'7}\,\delta_{35}\,\delta_{46}} \qquad \overset{(3)}{\delta_{0'7}\,\delta_{36}\,\delta_{45}}.$$

S_2: $\delta_{34}\,\delta_{34.1} = \delta(\tilde{k}_1)\,\delta(\tilde{k}_3 + \tilde{k}_4)$ which implies that the expression is zero by the zero mean value property.

S_1: $\delta_{0'3}\,\delta_{45}\,\delta_{67}$.

$$\delta_{0'3}\,\delta_{45}\,\delta_{67}\,\delta_{34.1}\,\delta_{567.2}\,\delta_{12.0}$$

$$= \delta_{0'3}\,\delta_{42}\,\delta_{67}\,\delta_{5,2}\,\delta_{12.0}\,\delta_{00'}.$$

Integration over $\breve{k}_3$, $\breve{k}_4$, $\breve{k}_5$ and $\breve{k}_7$ makes

$$F(t) = \int_0^t \Delta(\nu_3\omega + \nu_4\omega_2 - \nu_1\omega_1)\, \Delta(0)\, e^{i(s_1\omega_1 + s_2\omega_2 - s\omega)t}\, dt$$

when $\nu_7 = -\nu_6$, $\nu_5 = \nu_2$. Only the t^2 term will be retained. This occurs when $\nu_3 = \nu$, $\nu_4 = -\nu_2$ whereupon

$$\int_0^t t\, \Delta(s\omega - s_1\omega_1 - s_2\omega_2)\, e^{i(s_1\omega_1 + s_2\omega_2 - s\omega)t}\, dt$$

$$= \int_0^t t\, \Delta(s_1\omega_1 + s_2\omega_2 - s\omega)\, dt$$

$$\sim \frac{t^2}{2}\frac{i}{\mu} + O(t)\frac{1}{\mu^2} + O(1), \text{ when } \mu = s_1\omega_1 + s_2\omega_2 - s\omega.$$

The $O(t^2)$ terms add on application of the permutation, whereas the $O(t)$ terms cancel

$$6S_1 \sim 12t^2 \delta_{00'} \overset{(1)}{\phi}{}_0^{\nu -\nu}(k) \sum_{\nu_1\nu_2\nu_6} \int_{-\infty}^{\infty} i\, \frac{L_{k k_1 k_2}^{\nu \nu_1 \nu_2}\, L_{k_1 k -k_2}^{\nu_1 \nu -\nu_2}}{\nu_1\omega_1 + \nu_2\omega_2 - \nu\omega}\, L_{k_1 k_2 k_6 -k_6}^{\nu_1 \nu_2 \nu_6 -\nu_6}$$

$$\overset{(2)}{\phi}{}_0^{\nu_2 -\nu_2}(k_2)\, \overset{(1)}{\phi}{}_0^{\nu_6 -\nu_6}(k_6)\, \delta_{12,0}\, d\breve{k}_1\, d\breve{k}_2\, d\breve{k}_6 \, . \qquad ---(17)$$

The $O(t)$ terms remaining occur with at least one of the second order spectral cumulants of the $\overset{(2)}{\phi}{}_0^{++}$ type. These terms cancel with similar terms occurring in $\langle G_0^{\nu'}(k')_2\, e_4^{\nu}(k) + G_0^{\nu}(k)_2\, e_4^{\nu'}(k')\rangle_{\nu'=-\nu}$ which term also supplies the "t" term to cancel the above "t" term, in exactly the same manner. This is illustrated with $\langle e_2^{\nu}(k)\, e_2^{\nu}(k')\rangle$ and $\langle G_0^{-\nu}(k')_3 d_4^{\nu}(k) + G_0^{\nu}(k)_3 d_4^{-\nu}(k')\rangle$.

$\delta_3 : \quad \delta_{0'5}\ \delta_{36}\ \delta_{47}.$

Since $\delta_{0'5}\ \delta_{36}\ \delta_{47}\ \delta_{34,1}\ \delta_{234,0}\ \delta_{234\,0'}$

$$= \delta_{00'}\ \delta_{5,0}\ \delta_{36}\ \delta_{47}\ \delta_{34,1}\ \delta_{234,0},$$

Integrating over $\breve{k}_5$, $\breve{k}_6$ and $\tilde{k}_7$, the time dependent term in the expression for $\langle a_0^{-1}(k')\, {}_1e_4^{\,1}(k) + a_0^{1}(k)\, {}_1e_4^{-1}(k')\rangle$ becomes

$$F(t) = \int_0^t \Delta(\lambda_3\omega_3 + \lambda_4\omega_4 - \lambda_1\omega_1)\, \Delta(\lambda_5\omega + \lambda_6\omega_3 + \lambda_7\omega_4 - \lambda_1\omega_1)\, e^{i\omega_{1\dots,0}t}\, dt.$$

A "t" growth occurs when $\lambda_5 = \lambda,\ \lambda_6 = -\lambda_3$ and $\lambda_7 = -\lambda_4$.
If one calls

$$\lambda_2\omega_2 + \lambda_3\omega_3 + \lambda_4\omega_4 - \lambda\omega = -\omega$$

$$\lambda_3\omega_3 + \lambda_4\omega_4 - \lambda_1\omega_1 = -\mu,$$

$$F(t) = \int_0^t \Delta(-\mu)\, \Delta(\omega)\, e^{-i(-\omega+\mu)t}\, dt,$$

$$= \int_0^t \Delta(\omega)\, \frac{e^{-i\mu t} - 1}{-i\mu}\, e^{i(\mu-\omega)t}\, dt,$$

$$= \int_0^t \frac{\Delta(\omega)}{-i\mu} \left\{ e^{-i\omega t} - e^{-i(\mu-\omega)t} \right\} dt$$

$$= -\frac{1}{i\mu} \frac{\Delta(0) - \Delta(-\omega)}{i\omega} + \frac{1}{i\mu} E(\mu, \mu-\omega),$$

$$\sim -\frac{t}{i\mu} \left[\pi \delta(\omega) - \frac{iP}{\omega} \right] + O(1).$$

In S_3 when the permutation is applied, the principal part terms cancel, whereas the quartet resonance terms add.

$$6 S_3 \sim 24 \pi t\, \delta_{00'} \sum_{\nu_1 \cdots \nu_4} \int_{-\infty}^{\infty} \frac{L_{k k_1 k_2}^{\nu \nu_1 \nu_2} L_{k_1 k_3 k_4}^{\nu_1 \nu_3 \nu_4}}{i W_{34,1}} L_{k_2 k -k_3 -k_4}^{\nu_2 \nu -\nu_3 -\nu_4}$$

$$\overset{(2)}{Q}{}_0^{\nu -\nu}(k)\, \overset{(2)}{Q}{}_0^{\nu_3 -\nu_3}(k_3)\, \overset{(2)}{Q}{}_0^{\nu_4 -\nu_4}(k_4)\, \delta_{34,1}\, \hat{\delta}_{234,0}\, \delta_{234,0}\, d\tilde{k}_1 \cdots d\tilde{k}_4 + O(1).$$

Integrating over $\tilde{k}_1$ and setting

$$\frac{\nu_4}{\tilde{k}_4} \longrightarrow \frac{\nu_3}{\tilde{k}_3} \longrightarrow \frac{\nu_2}{\tilde{k}_2} \longrightarrow \frac{\nu_1}{\tilde{k}_1}, \quad \nu_1 \longrightarrow -\nu_4$$

one obtains

$$6 S_3 \sim 36 \pi t \, \delta_{00'} \, \overset{(2)}{\Phi}{}_0^{\lambda\,-\lambda}(k) \sum_{\lambda_1 \lambda_2 \lambda_3} \int_{-\infty}^{\infty} -\frac{2i}{3} \sum_{\lambda_4} \frac{\alpha_{k\; k_2+k_3\; k_1}^{\lambda\; -\lambda_4\; \lambda_1} \, \alpha_{k_2+k_3\; k_2\; k_3}^{-\lambda_4\; \lambda_2\; \lambda_3}}{\lambda_2 \omega_2 + \lambda_3 \omega_3 + \lambda_4 \omega_{23}}$$

$$\alpha_{k_1\; k\; -k_2\; -k_3}^{\lambda_1\; \lambda\; -\lambda_2\; -\lambda_3} \, \overset{(2)}{\Phi}{}_0^{\lambda_2 -\lambda_2}(k_2) \, \overset{(2)}{\Phi}{}_0^{\lambda_3 -\lambda_3}(k_3) \, \delta_{123,0}^{\lambda} \, \delta_{123,0} \, d\tilde{k}_1 \, d\tilde{k}_2 \, d\tilde{k}_3$$

$$+ \; O(1). \qquad \text{-----} \; (18)$$

The (2 X 4) decomposition gives the following array.

$$\delta_{0'3} \overset{(1)}{\delta_{4567}} \qquad \delta_{0'4} \overset{(1)}{\delta_{3567}} \qquad \delta_{0'5} \overset{(2)}{\delta_{3467}} \qquad \delta_{0'6} \overset{(2)}{\delta_{3457}} \qquad \delta_{0'7} \overset{(2)}{\delta_{3456}}$$

$$\delta_{34} \overset{(3)}{\delta_{0'567}} \qquad \delta_{35} \overset{(4)}{\delta_{0'467}} \qquad \delta_{36} \overset{(4)}{\delta_{0'457}} \qquad \delta_{37} \overset{(4)}{\delta_{0'456}}$$

$$\delta_{45} \overset{(4)}{\delta_{0'357}} \qquad \delta_{46} \overset{(4)}{\delta_{0'357}} \qquad \delta_{47} \overset{(4)}{\delta_{0'356}}$$

$$\delta_{56} \overset{(5)}{\delta_{0'347}} \qquad \delta_{57} \overset{(5)}{\delta_{0'346}}$$

$$\delta_{67} \overset{(5)}{\delta_{0'345}} .$$

S_3: δ_{34} reacts with $\delta_{34,1}$ to give $\delta(\breve{k}_1)$ and thus makes the expression zero by reason of the zero mean value property.

S_1: $\delta_{0'3}\ \delta_{4567}\ \delta_{34,1}\ \delta_{567,2}\ \delta_{234,0}$

$= \delta_{00'}\ \delta_{3,0}\ \delta_{42}\ \delta_{567,2}\ \delta_{12,0}.$

If one integrates over $\breve{k}_3$ and $\breve{k}_4$ and sets $\nu_4 = -\nu_2$ and $\nu_3 = \nu$ the time dependent term becomes

$$F(t) = \int_0^t \Delta(\nu\omega - \nu_1\omega_1 - \nu_2\omega_2)\, \Delta_{567,2}\, e^{i(\nu_1\omega_1 + \nu_2\omega_2 - \nu\omega)t}\, dt,$$

$$= \int_0^t \Delta(\nu_1\omega_1 + \nu_2\omega_2 - \nu\omega)\, \Delta(\nu_5\omega_5 + \nu_6\omega_6 + \nu_7\omega_7 - \nu_2\omega_2)\, dt,$$

$$\sim \frac{it}{\nu_1\omega_1 + \nu_2\omega_2 - \nu\omega} \left\{ \pi\delta_{567,2}^{\nu} + \frac{iP}{W_{567,2}} \right\} + O(1).$$

Thus,

$$2S_1 \sim \overset{0-0}{P}\ 4t\,\delta(\tilde{k}+\tilde{k}')\,\overset{(2)}{\phi}{}_0^{1-1}(k) \sum_{\nu_1\nu_2\nu_5\nu_6\nu_7} \int_{-\infty}^{\infty} \frac{i\,L_{k k_1 k_2}^{1\,\nu_1\nu_2}\,L_{k_1 k -k_2}^{\nu_1\,1\,-\nu_2}}{\nu_1\omega_1+\nu_2\omega_2-\nu\omega}$$

$$L_{k_2 k_5 k_6 k_7}^{\nu_2\nu_5\nu_6\nu_7}\,\overset{(4)}{\phi}{}_0^{\nu_5-\nu_6\nu_7-\nu_2}(k_5,k_6,k_7)\left\{\pi\,\delta_{567,2}+\frac{iP}{W_{567,2}}\right\}$$

$$\delta_{567,2}\ \delta_{12,0}\ d\tilde{k}_1\,d\tilde{k}_2\,d\tilde{k}_5\,d\tilde{k}_6\,d\tilde{k}_7\ +O(1).$$

One applies the permutation, and sets

$$\begin{matrix}\nu_i \\ \tilde{k}_i\end{matrix} \longrightarrow \begin{matrix}-\nu_i \\ -\tilde{k}_i\end{matrix}, \qquad i = 1,2,5,6,7,$$

in the second integral, and one obtains

$$2S_1 \sim 4t\,\delta_{00'}\,\overset{(2)}{\phi}{}_0^{\nu-\nu}(k) \sum_{\nu_1\nu_2\nu_5\nu_6\nu_7} \int_{-\infty}^{\infty} \frac{i\,L_{k k_1 k_2}^{\nu\,\nu_1\nu_2}\,L_{k_1 k -k_2}^{\nu_1\,\nu\,-\nu_2}}{\nu_1\omega_1+\nu_2\omega_2-\nu\omega}$$

$$L_{k_2 k_5 k_6 k_7}^{\nu_2\nu_5\nu_6\nu_7}\left\{\pi\,\delta_{567,2}\left(\overset{(4)}{\phi}{}_0^{\nu_5\nu_6\nu_7-\nu_2}(k_5,k_6,k_7)+\overset{(4)}{\phi}{}_0^{\nu_5\nu_6\nu_7-\nu_2\,*}(k_5,k_6,k_7)\right)\right.$$

$$\left.+\frac{iP}{W_{567,2}}\left(\overset{(4)}{\phi}{}_0^{\nu_5\nu_6\nu_7-\nu_2}(k_5,k_6,k_7)-\overset{(4)}{\phi}{}_0^{\nu_5\nu_6\nu_7-\nu_2\,*}(k_5,k_6,k_7)\right)\right\}$$

$$\delta_{567,2}\ \delta_{12,0}\ d\tilde{k}_1\,d\tilde{k}_2\,d\tilde{k}_5\,d\tilde{k}_6\,d\tilde{k}_7\ +O(1). \qquad ---(19).$$

There are no further "t" growth terms in S_2, S_4 or S_5. Neither do any "t" growths ensue from the following array giving the (3 X 3) decomposition.

$$\begin{array}{llll}
\delta_{0'34}\,\delta_{567} & \delta_{0'35}\,\delta_{467} & \delta_{0'36}\,\delta_{457} & \delta_{0'37}\,\delta_{456} \\
 & \delta_{0'45}\,\delta_{367} & \delta_{0'46}\,\delta_{367} & \delta_{0'47}\,\delta_{356} \\
 & & \delta_{0'56}\,\delta_{347} & \delta_{0'57}\,\delta_{346} \\
 & & & \delta_{0'67}\,\delta_{345}.
\end{array}$$

One now examines

$$\langle a_0^{\lambda'}(k')_7\, \ell_4^{\lambda}(k) + a_0^{\lambda}(k)_7\, \ell_4^{\lambda'}(k') \rangle_{\lambda'=-\lambda}$$

$$= 4 \sum_{\lambda_1\cdots\lambda_7}^{00'} \int_{-\infty}^{\infty} \mathcal{L}_{k k_1 k_2}^{\lambda \lambda_1 \lambda_2} \mathcal{L}_{k_1 k_3 k_4}^{\lambda_1 \lambda_3 \lambda_4} \mathcal{L}_{k_3 k_5 k_6 k_7}^{\lambda_3 \lambda_5 \lambda_6 \lambda_7} \langle a_{0'}^{\lambda'} a_{02}^{\lambda_2} a_{04}^{\lambda_4} a_{05}^{\lambda_5} a_{06}^{\lambda_6} a_{07}^{\lambda_7} \rangle$$

$$F(t)\, \delta_{567.3}\, \delta_{34.1}\, \delta_{12.0}\, d\tilde{k}_1 \cdots\cdots d\tilde{k}_7 .$$

$$F(t) = \int_0^t \frac{\Delta_{4567.1} - \Delta_{34.1}}{i\omega_{567.3}}\, e^{i\omega_{12.0} t}\, dt.$$

The (2 X 2 X 2) decomposition gives the following array.

$$\delta_{0'2}^{(1)}\,\delta_{45}\,\delta_{67} \qquad \delta_{0'2}^{(1)}\,\delta_{46}\,\delta_{57} \qquad \delta_{0'2}^{(1)}\,\delta_{47}\,\delta_{56}$$

$$\delta_{0'4}^{(2)}\,\delta_{25}\,\delta_{67} \qquad \delta_{0'4}^{(2)}\,\delta_{26}\,\delta_{57} \qquad \delta_{0'4}^{(2)}\,\delta_{27}\,\delta_{56}$$

$$\delta_{0'5}^{(3)}\,\delta_{24}\,\delta_{67} \qquad \delta_{0'5}^{(4)}\,\delta_{26}\,\delta_{47} \qquad \delta_{0'5}^{(4)}\,\delta_{27}\,\delta_{46}$$

$$\delta_{0'6}^{(3)}\,\delta_{24}\,\delta_{57} \qquad \delta_{0'6}^{(4)}\,\delta_{25}\,\delta_{47} \qquad \delta_{0'6}^{(4)}\,\delta_{27}\,\delta_{45}$$

$$\delta_{0'7}^{(3)}\,\delta_{24}\,\delta_{56} \qquad \delta_{0'7}^{(4)}\,\delta_{25}\,\delta_{46} \qquad \delta_{0'7}^{(4)}\,\delta_{26}\,\delta_{45}\,.$$

$$S_1:\quad \delta_{0'2}\,\delta_{45}\,\delta_{67}\,\delta_{567,3}\,\delta_{34,1}\,\delta_{12,0}$$

$$= \delta_{0'2}\,\delta_{00'}\,\delta(\breve{k}_1)\,\delta_{45}\,\delta_{67}\,\delta_{5,3}\,\delta_{34}\,.$$

$\delta(\breve{k}_1)$ makes the expression zero.

S_2: Since $\delta_{0'4}\, \delta_{25}\, \delta_{67}\, \delta_{567,3}\, \delta_{34,1}\, \delta_{12,0}$

$$= \delta_{00'}\, \delta_{4,0}\, \delta_{25}\, \delta_{32}\, \delta_{67}\, \delta_{12,0},$$

Upon integration over $\breve{k}_3$, $\breve{k}_4$, $\breve{k}_5$ and $\breve{k}_7$, F(t) becomes

$$\int_0^t \frac{\Delta(\nu_4\omega + \nu_5\omega_2 + (\nu_6+\nu_7)\omega_6 - \nu_1\omega_1) - \Delta(\nu_3\omega_2 + \nu_4\omega - \nu_1\omega_1)}{i((\nu_5-\nu_3)\omega_2 + (\nu_6+\nu_7)\omega_6)}\, e^{i(\nu_1\omega_1 + \nu_2\omega_2 - \nu\omega)t}\, dt.$$

There will be $O(t^2)$ terms with pure energy density cumulants and $O(t)$ terms with at least one second order cumulant of the type $\mathcal{C}_0^{(2)++}$. The latter terms will not be written on account of their cumbersome nature, but will cancel in exactly the manner illustrated earlier. The "t^2" term occurs when

$$\nu_7 = -\nu_6, \quad \nu_4 = \nu, \quad \nu_5 = -\nu_2, \quad \nu_3 = -\nu_2,$$

whereupon

$$F(t) = \int_0^t \frac{\Delta(\mu) - \Delta(\mu)}{i(\mu-\mu)}\, e^{-i\mu t}\, dt,$$

$$\sim \frac{t^2}{2i\mu} + O(t)\,\frac{1}{\mu^2} + O(1).$$ (See Appendix II Page 258).

Applying the permutation and adding, one obtains

$$3S_2 \sim 12t^2\,\delta(k+k')\sum_{\nu_1\nu_2\nu_6}\int_{-\infty}^{\infty}\frac{i\,L^{\nu\,\nu_1\,\nu_2}_{k\,k_1\,k_2}\,L^{\nu_1\,-\nu_2\,\nu}_{k_1\,-k_2\,k}}{\nu_1\omega_1+\nu_2\omega_2-\nu\omega}\,L^{-\nu_2\,-\nu_2\,\nu_6\,-\nu_6}_{-k_2\,-k_2\,k_6\,-k_6}$$

$$\overset{(2)}{\varphi}{}_0^{\nu-\nu}(k)\,\overset{(2)}{\varphi}{}_0^{\nu_2-\nu_2}(k_2)\,\overset{(2)}{\varphi}{}_0^{\nu_6-\nu_6}(k_6)\,\delta_{12,0}\,d\breve{k}_1\,d\breve{k}_2\,d\breve{k}_6. \qquad \text{--- (20)}$$

Since $L^{\nu_2\,\nu_2\,\nu_6\,-\nu_6}_{k_2\,k_2\,k_6\,-k_6} + L^{-\nu_2\,-\nu_2\,\nu_6\,-\nu_6}_{-k_2\,-k_2\,k_6\,-k_6} = 0,$

$$(20) + (17) = 0.$$

The term S_3 gives a similar structure for the time dependent terms.

$$3S_3 \sim 12t^2\,\delta(k+k')\sum_{\nu_1\nu_4\nu_5}\int_{-\infty}^{\infty}\frac{i\,L^{\nu\,-\nu_4\,-\nu_5}_{k\,-k_4\,-k_5}\,L^{-\nu_4\,\nu\,\nu_5}_{-k_4\,k\,k_5}}{-\nu_4\omega_4-\nu_5\omega_5-\nu\omega}\,L^{\nu\,\nu\,\nu_1\,-\nu_1}_{k\,k\,k_1\,-k_1}$$

$$\overset{(2)}{\varphi}{}_0^{\nu-\nu}(k)\,\overset{(2)}{\varphi}{}_0^{\nu_1-\nu_1}(k_1)\,\overset{(2)}{\varphi}{}_0^{\nu_5-\nu_5}(k_5)\,\delta_{450}\,d\breve{k}_1\,d\breve{k}_4\,d\breve{k}_5. \qquad \text{--- (21)}$$

It is clear that $(21) = -\frac{1}{2}(7)$,

Since $L^{\nu\,-\nu_4\,-\nu_5}_{k\,-k_4\,-k_5} = -L^{-\nu\,\nu_4\,\nu_5}_{-k\,k_4\,k_5}$,

and $L^{-\nu_4\,\nu\,\nu_5}_{-k_4\,k\,k_5} = -L^{\nu_4\,-\nu\,-\nu_5}_{k_4\,-k\,-k_5}$.

$$S_4: \quad \delta_{0'5}\,\delta_{26}\,\delta_{47}\,\delta_{567,3}\,\delta_{34,1}\,\delta_{12,0}$$

$$= \delta_{5,0}\,\delta_{26}\,\delta_{47}\,\delta_{00'}\,\delta_{34,1}\,\delta_{234,0}.$$

Upon integration over $\tilde{k}_5$, $\tilde{k}_6$ and $\tilde{k}_7$

$$F(t) = \int_0^t \frac{\Delta\left((s_4+s_7)\omega_4 + s_5\omega + s_6\omega_2 - s_1\omega_1\right) - \Delta\left(s_3\omega_3 + s_4\omega_4 - s_1\omega_1\right)}{i\left(s_5\omega + s_6\omega_2 + s_7\omega_4 - s_3\omega_3\right)}\, e^{i(s_1\omega_1 + s_2\omega_2 - s\omega)t}\, dt.$$

The only possibility of a "t" growth , see Appendix is when $s_7 = -s_4$, $s_5 = s$ and $s_6 = -s_2$.

$$F(t) \sim \frac{t}{iW_{34,1}}\left\{\pi\,\delta(s_2\omega_2 + s_3\omega_3 + s_4\omega_4 - s\omega) + \frac{iP}{W_{234,0}}\right\} + O(1).$$

On account of an odd number of sign changes when one applies the permutation, the principal part terms cancel, whereas the resonance terms add.

$$6S_4 \sim 48\pi t\,\delta(k+k')\sum_{s_1 s_2 s_3 s_4}\int_{-\infty}^{\infty}\frac{L^{s\,s_1\,s_2}_{k\,k_1\,k_2}\,L^{s_1\,s_3\,s_4}_{k_1\,k_3\,k_4}}{i(s_3\omega_3 + s_4\omega_4 - s_1\omega_1)}$$

$$L^{s_3\,s\,-s_2\,-s_4}_{k_3\,k\,-k_2\,-k_4}\,Q_0^{(2)\,s\,-s}(k)\,Q_0^{(2)\,s_2\,-s_2}(k_2)\,Q_0^{(2)\,s_4\,-s_4}(k_4)\,\delta^{s}_{234,0}\,\delta_{34,1}\,\delta_{234,0}$$

$$d\tilde{k}_1 \cdots d\tilde{k}_4 + O(1).$$

When the integration over $\tilde{k}_1$ is performed, one sets

$$\begin{Bmatrix} s_3 \\ \tilde{k}_3 \end{Bmatrix} \rightarrow \begin{Bmatrix} s_1 \\ \tilde{k}_1 \end{Bmatrix} \quad\text{and}\quad \begin{Bmatrix} s_4 \\ \tilde{k}_4 \end{Bmatrix} \rightarrow \begin{Bmatrix} s_3 \\ \tilde{k}_3 \end{Bmatrix} \quad\text{and } s_1 \rightarrow -s_4,$$

whereupon one obtains, using symmetry properties in 2 and 3 -

$$6S_4 \sim 36\pi t\, \delta(k+k')\, \overset{(2)}{\varphi}{}_0^{\eta_1 \cdot \eta}(k) \sum_{\eta_1\eta_2\eta_3} \int_{-\infty}^{\infty} \frac{-2i}{3}\, \rho^{23} \sum_{\eta_4} \frac{L_{k\;k_2+k_3\;k_2}^{\eta\;-\eta_4\;\eta_2}\, L_{k_2+k_3\;k_2\;k_3}^{-\eta_4\;\eta_2\;\eta_3}}{\eta_2\omega_1+\eta_3\omega_3+\eta_4\omega_{13}}$$

$$L_{k_1\,k\,-k_2\,-k_3}^{\eta_1\;\eta\;-\eta_2\;-\eta_3}\, \overset{(2)}{\varphi}{}_0^{\eta_2-\eta_2}(k_2)\, \overset{(2)}{\varphi}{}_0^{\eta_3-\eta_3}(k_3)\, \delta^{\eta}_{123,0}\, \delta_{123,0}\, d\breve{k}_1 d\breve{k}_2 d\breve{k}_3$$

$$+ O(1). \quad \text{----} \quad (22)$$

The (2 X 4) decomposition gives the array

$$\overset{(1)}{\delta_{0'2}\,\delta_{4567}} \quad \overset{(2)}{\delta_{0'4}\,\delta_{2567}} \quad \overset{(3)}{\delta_{0'5}\,\delta_{2467}} \quad \overset{(3)}{\delta_{0'6}\,\delta_{2457}} \quad \overset{(3)}{\delta_{0'7}\,\delta_{2456}}$$

$$\overset{(4)}{\delta_{24}\,\delta_{0'567}} \quad \overset{(5)}{\delta_{25}\,\delta_{0'467}} \quad \overset{(5)}{\delta_{26}\,\delta_{0'457}} \quad \overset{(5)}{\delta_{27}\,\delta_{0'456}}$$

$$\overset{(6)}{\delta_{45}\,\delta_{0'267}} \quad \overset{(6)}{\delta_{46}\,\delta_{0'257}} \quad \overset{(6)}{\delta_{47}\,\delta_{0'256}}$$

$$\overset{(7)}{\delta_{56}\,\delta_{0'247}} \quad \overset{(7)}{\delta_{57}\,\delta_{0'246}}$$

$$\overset{(7)}{\delta_{67}\,\delta_{0'245}}.$$

The only terms which exhibit a "t" growth are S_4 and S_2.

S_2: Since $\delta_{00'4}\,\delta_{2567}\,\delta_{567,3}\,\delta_{34,1}\,\delta_{12,0}$

$$= \delta_{00'}\,\delta_{4,0}\,\delta_{23}\,\delta_{5672}\,\delta_{12,0},$$

Integration over $\breve{k}_3$ and $\breve{k}_4$ and setting $s_4 = s$, $s_3 = -s_2$ makes

$$F(t) = \int_0^t \frac{\Delta(s\omega + s_5\omega_5 + s_6\omega_6 + s_7\omega_7 - s_1\omega_1) - \Delta(s\omega - s_1\omega_1 - s_2\omega_2)}{i(s_5\omega_5 + s_6\omega_6 + s_7\omega_7 + s_2\omega_2)}\, e^{i(s_1\omega_1 + s_2\omega_2 - s\omega)t}\, dt,$$

which behaves in the long time limit,

$$\sim \frac{it}{W_{12,0}}\left\{\pi\,\delta^{s}_{5672} + \frac{iP}{W_{5672}}\right\} + O(1).$$

Therefore,

$$S_2 \sim 4t\,\delta(k+k')\,\overset{(2)\,s-s}{Q_0}(k)\,\overset{0-0}{P}\sum_{s_1 s_2 s_5 s_6 s_7}\int_{-\infty}^{\infty}\frac{L^{s\,s_1\,s_2}_{k\,k_1\,k_2}\,L^{s_1\,-s_2\,s}_{k_1\,-k_2\,k}}{-i(s_1\omega_1 + s_2\omega_2 - s\omega)}$$

$$L^{-s_2\,s_5\,s_6\,s_7}_{-k_2\,k_5\,k_6\,k_7}\,\overset{(4)\,s_5\,s_6\,s_7\,s_2}{Q_0}(k_5, k_6, k_7)\left(\pi\,\delta^{s}_{5672} + \frac{iP}{W_{5672}}\right)$$

$$\delta_{5672}\,\delta_{12,0}\,d\breve{k}_1\,d\breve{k}_2\,d\breve{k}_5\,d\breve{k}_6\,d\breve{k}_7 + O(1).$$

The permutation gives the same integral with $\nu \to -\nu$ and $\breve{k} \to -\breve{k}$. In the second integral, one sets

$$\begin{cases} \nu_i \\ \breve{k}_i \end{cases} \to \begin{cases} -\nu_i \\ -k_i \end{cases}, \qquad i = 1,2,5,6,7$$

and one obtains

$$S_2 \sim 4t\,\delta(\breve{k}+\breve{k}')\,\overset{(2)}{\phi_0}{}^{\nu,-\nu}(k) \sum_{\nu_1\nu_2\nu_5\nu_6\nu_7} \int_{-\infty}^{\infty} \frac{L_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2}\, L_{k_1\,k-k_2}^{\nu_1\,\nu\,-\nu_2}}{i(s_1\omega_1+s_2\omega_2-s\omega)}$$

$$L_{-k_2\,k_5\,k_6\,k_7}^{-\nu_2\,\nu_5\,\nu_6\,\nu_7} \Big\{ \pi\,\delta_{5672}^{\nu} \Big(\overset{(4)}{\phi_0}{}^{\nu_5\nu_6\nu_7\nu_2}(k_5,k_6,k_7) + \overset{(4)*}{\phi_0}{}^{\nu_5\nu_6\nu_7\nu_2}(k_5,k_6,k_7) \Big)$$

$$+ \frac{iP}{W_{5672}} \Big(\overset{(4)}{\phi_0}{}^{\nu_5\nu_6\nu_7\nu_2}(k_5,k_6,k_7) - \overset{(4)*}{\phi_0}{}^{\nu_5\nu_6\nu_7\nu_2}(k_5,k_6,k_7) \Big) \Big\}\, \delta_{5672}\,\delta_{12,0}$$

$$d\breve{k}_1\,d\breve{k}_2\,d\breve{k}_5\,d\breve{k}_6\,d\breve{k}_7 + O(1). \qquad \cdots (23).$$

If one sets $\begin{cases} \nu_i \\ \breve{k}_i \end{cases} \longrightarrow \begin{cases} -\nu_i \\ -\breve{k}_i \end{cases}, \quad i = 5,6,7$

one sees

$$(23) = -(19)$$

Similarly,

$$S_4 \sim 4t\,\delta_{00'} \sum_{\nu_1\nu_2} \int_{-\infty}^{\infty} \frac{i\,L_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2}\, L_{k_1\,k-k_2}^{\nu_1\,\nu\,-\nu_2}}{s_1\omega_1+s_2\omega_2-s\omega}\, \overset{(2)}{\phi_0}{}^{\nu_2,-\nu_2}(k_2)\,\delta_{12,0}\,d\breve{k}_1\,d\breve{k}_2$$

$$\times \sum_{\nu_5\nu_6\nu_7} \int_{-\infty}^{\infty} \mathcal{L}_{k\,k_5\,k_6\,k_7}^{\nu\,\nu_5\,\nu_6\,\nu_7} \Big[\bar{\Lambda}\, \delta_{567,0}^{1} \Big(\overset{(4)}{\varphi}{}_0^{\nu_5\nu_6\nu_7-\nu}(k_5,k_6,k_7) + \overset{(4)}{\varphi}{}_0^{*\,\nu_5\nu_6\nu_7-\nu}(k_5,k_6,k_7) \Big)$$

$$+ \frac{iP}{W_{567,0}} \Big(\overset{(4)}{\varphi}{}_0^{\nu_5\nu_6\nu_7-\nu}(k_5,k_6,k_7) - \overset{(4)}{\varphi}{}_0^{\nu_5\nu_6\nu_7-\nu}(k_5,k_6,k_7) \Big) \Big] \delta_{567,0}$$

$$d\breve{k}_5\, d\breve{k}_6\, d\breve{k}_7 + O(1).$$

$$\text{----} \quad (24).$$

If one sets

$$\begin{Bmatrix} \nu_5 \\ \breve{k}_5 \end{Bmatrix} \to \begin{Bmatrix} \nu_1 \\ \breve{k}_1 \end{Bmatrix}, \quad \begin{Bmatrix} \nu_6 \\ \breve{k}_6 \end{Bmatrix} \to \begin{Bmatrix} \nu_2 \\ \breve{k}_2 \end{Bmatrix}, \quad \begin{Bmatrix} \nu_7 \\ \breve{k}_7 \end{Bmatrix} \to \begin{matrix} \nu_3 \\ \breve{k}_3 \end{matrix},$$

$$\begin{Bmatrix} \nu_1 \\ \breve{k}_1 \end{Bmatrix} \to \begin{Bmatrix} -\nu_4 \\ -\breve{k}_4 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \nu_2 \\ \breve{k}_2 \end{Bmatrix} \to \begin{Bmatrix} -\nu_5 \\ -\breve{k}_5 \end{Bmatrix},$$

one sees $(24) = -(10).$

The (3 X 3) decomposition given by the array,

$$\delta_{0'24}\,\delta_{567} \qquad \delta_{0'25}\,\delta_{467} \qquad \delta_{0'26}\,\delta_{457} \qquad \delta_{0'27}\,\delta_{456}$$

$$\delta_{0'45}\,\delta_{267} \qquad \delta_{0'46}\,\delta_{257} \qquad \delta_{0'47}\,\delta_{256}$$

$$\delta_{0'56}\,\delta_{247} \qquad \delta_{0'57}\,\delta_{246}$$

$$\delta_{0'67}\,\delta_{245}.$$

does not contribute any secular growth.

One next examines

$$\langle a_0^{\gamma'}(k')\,{}_2e_4^{\gamma}(k) + a_0^{\gamma}(k)\,{}_2e_4^{\gamma'}(k')\rangle_{\gamma'=-\gamma}$$

$$= 4\sum_{\gamma_1\cdots\gamma_8}\int_{-\infty}^{\infty} L_{kk_1k_2}^{\gamma\gamma_1\gamma_2}\, L_{k_1k_3k_4}^{\gamma_1\gamma_3\gamma_4}\, L_{k_2k_5k_6}^{\gamma_2\gamma_5\gamma_6}\, L_{k_5k_7k_8}^{\gamma_5\gamma_7\gamma_8}$$

$$\langle a_{0'}^{\gamma'} a_{03}^{\gamma_3} a_{04}^{\gamma_4} a_{06}^{\gamma_6} a_{07}^{\gamma_7} a_{08}^{\gamma_8}\rangle\, F(t)\,\delta_{78,5}\,\delta_{56,2}\,\delta_{34,1}\,\delta_{12,0}\, dk_1\cdots\cdots dk_8.$$

$$F(t) = \int_0^t \Delta_{34,1}\,\frac{\Delta_{678,2} - \Delta_{56,2}}{iW_{78,5}}\, e^{iW_{12,0}t}\, dt.$$

The $(2 \times 2 \times 2)$ decomposition is given by the following array.

$$\delta_{0'3}\ \overset{(1)}{\delta_{46}}\ \delta_{78} \qquad \delta_{0'3}\ \overset{(2)}{\delta_{47}}\ \delta_{68} \qquad \delta_{0'3}\ \overset{(2)}{\delta_{48}}\ \delta_{67}$$

$$\delta_{0'4}\ \overset{(1)}{\delta_{36}}\ \delta_{78} \qquad \delta_{0'4}\ \overset{(2)}{\delta_{37}}\ \delta_{68} \qquad \delta_{0'4}\ \overset{(2)}{\delta_{38}}\ \delta_{67}$$

$$\delta_{0'6}\ \overset{(3)}{\delta_{34}}\ \delta_{78} \qquad \delta_{0'6}\ \overset{(4)}{\delta_{37}}\ \delta_{48} \qquad \delta_{0'6}\ \overset{(4)}{\delta_{38}}\ \delta_{47}$$

$$\delta_{0'7}\ \overset{(5)}{\delta_{34}}\ \delta_{68} \qquad \delta_{0'7}\ \overset{(6)}{\delta_{36}}\ \delta_{48} \qquad \delta_{0'7}\ \overset{(6)}{\delta_{38}}\ \delta_{46}$$

$$\delta_{0'8}\ \overset{(5)}{\delta_{34}}\ \delta_{67} \qquad \delta_{0'8}\ \overset{(6)}{\delta_{36}}\ \delta_{47} \qquad \delta_{0'8}\ \overset{(6)}{\delta_{37}}\ \delta_{46}.$$

The terms S_1, S_3 and S_5 are zero.

S_1: $\delta_{78}\ \delta_{78,5}$ implies $\delta(\breve{k}_5)$ which makes the expression zero by the zero mean value property.

Similarly for S_3 and S_5.

$$S_2:\ \delta_{0'3}\ \delta_{47}\ \delta_{68}\ \delta_{78,5}\ \delta_{56,2}\ \delta_{34,1}\ \delta_{12,0}$$

$$= \delta_{00'}\ \delta_{3,0}\ \delta_{7,2}\ \delta_{86}\ \delta_{56,2}\ \delta_{42}\ \delta_{12,0}.$$

Integrating over $\tilde{k}_3$, $\tilde{k}_4$, $\tilde{k}_7$ and $\tilde{k}_8$

$$F(t) = \int_0^t \Delta(\nu_3\omega + \nu_4\omega_2 - \nu_1\omega_1) \; \frac{\Delta((\nu_6+\nu_8)\omega_6 + (\nu_7-\nu_2)\omega_2) - \Delta_{56,2}}{i(\nu_7\omega_2 + \nu_8\omega_6 - \nu_5\omega_5)} \; e^{iW_{12,0}t} \, dt,$$

To obtain a secular growth at all, one must have at least $\nu_3 = \nu$ and $\nu_4 = -\nu_2$. A "t^2" growth occurs if in addition $\nu_8 = -\nu_6$, $\nu_7 = +\nu_2$.

The cumulants connected with the "t^2" growth are all of the energy density type $\Phi_0^{(2)+-}(k)$. A "t" growth occurs for any other combination with at least one of the cumulants of the type $\Phi_0^{(2)++}(k)$. Only the "t^2" terms will be retained. The O(t) terms will vanish with the same term with which the O(t) terms cancel in a manner analogous to the illustrated case $\langle b_2^{\nu}(k)\; b_2^{-\nu}(k')\rangle$ and $\langle a_0^{-\nu}(k')\; {}_3d_4^{\nu}(k) + a_0^{\nu}(k)\; {}_3d_4^{-\nu}(k')\rangle$. When $\nu_3 = \nu$, $\nu_8 = -\nu_6$ $\nu_4 = -\nu_2$ and $\nu_7 = \nu_2$

$$F(t) = \int_0^t \Delta(\mu) \; \frac{t - \Delta(\nu)}{-i\nu} \; dt,$$

where $\mu = \nu_1\omega_1 + \nu_2\omega_2 - \nu\omega$,

$\nu = \nu_5\omega_5 + \nu_6\omega_6 - \nu_2\omega_2$,

$$F(t) \sim \frac{t^2}{2}\frac{i}{\mu}\cdot\frac{i}{\nu} + O(t)\frac{1}{\mu\nu^2} + O(1).$$

When one adds the term arising from the permutation to the original term, the above $O(t)$ terms cancel, giving the result,

$$4S_2 \sim -16t^2 \delta_{00'} \overset{(2)}{\phi}{}_0^{\lambda-\lambda}(k) \sum_{\lambda_1\lambda_2\lambda_5\lambda_6} \int_{-\infty}^{\infty} \frac{L_{k k_1 k_2}^{\lambda \lambda_1 \lambda_2} L_{k_1 k -k_2}^{\lambda_1 \lambda -\lambda_2}}{\lambda_1\omega_1 + \lambda_2\omega_2 - \lambda\omega}$$

$$\frac{L_{k_2 k_5 k_6}^{\lambda_2\lambda_5\lambda_6} L_{k_5 k_2 -k_6}^{\lambda_5\lambda_2-\lambda_6}}{\lambda_5\omega_5 + \lambda_6\omega_6 - \lambda_2\omega_2} \overset{(2)}{\phi}{}_0^{\lambda_2-\lambda_2}(k_2) \overset{(2)}{\phi}{}_0^{\lambda_6-\lambda_6}(k_6) \delta_{12,0} \delta_{56,2}$$

$$d\breve{k}_1 d\breve{k}_2 d\breve{k}_5 d\breve{k}_6$$

$$+ O(1). \quad ---- (25)$$

$$S_4: \delta_{0'6} \delta_{37} \delta_{48} \delta_{78,5} \delta_{56,2} \delta_{34,1} \delta_{12,0}$$

$$= \delta_{6,0} \delta_{37} \delta_{48} \delta_{51} \delta_{00'} \delta_{34,1} \delta_{12,0}.$$

Upon integration over $\breve{k}_5$, $\breve{k}_6$, $\breve{k}_7$ and $\breve{k}_8$,

$$F(t) = \int_0^t \Delta(\lambda_3\omega_3 + \lambda_4\omega_4 - \lambda_1\omega_1) \frac{\Delta(\lambda_6\omega + \lambda_7\omega_3 + \lambda_8\omega_4 - \lambda_2\omega_2) - \Delta(\lambda_5\omega_1 + \lambda_6\omega - \lambda_2\omega_2)}{i(\lambda_7\omega_3 + \lambda_8\omega_4 - \lambda_5\omega_1)} \times e^{i\omega_{12,0}t} dt.$$

A "t" growth may occur (see Appendix II) when $\lambda_7 = -\lambda_3$, $\lambda_6 = \lambda$ and $\lambda_8 = -\lambda_4$, whereupon

$$F(t) \sim \frac{t}{i(\eta_1\omega_1 - \eta_3\omega_3 - \eta_4\omega_4)} \quad \frac{1}{i(\eta_3\omega_3 + \eta_4\omega_4 + \eta_5\omega_1)}$$

$$\times \left\{ \pi\, \delta^{\eta}_{234,0} + \frac{iP}{\omega_{234,0}} \right\} + O(1).$$

The principal part terms cancel when one applies the permutation $\eta \to -\eta$ and $\tilde{k} \to -\tilde{k}$ and the result is,

$$2S_4 \sim 16\pi t\, \delta_{00'}\, \overset{(2)}{Q}{}^{\eta-\eta}_{0}(k) \sum_{\eta_1\eta_2\eta_3\eta_4\eta_5} \int_{-\infty}^{\infty} \frac{L^{\eta\,\eta_1\,\eta_2}_{k\,k_1\,k_2}\, L^{\eta_1\,\eta_3\,\eta_4}_{k_1\,k_3\,k_4}}{-i(\eta_3\omega_3 + \eta_4\omega_4 - \eta_1\omega_1)}$$

$$\frac{L^{\eta_2\,\eta_5\,\eta}_{k_2\,-k_1\,k}\, L^{\eta_5\,-\eta_3\,-\eta_4}_{-k_1\,-k_3\,-k_4}}{i(\eta_3\omega_3 + \eta_4\omega_4 + \eta_5\omega_1)}\, \overset{(2)}{Q}{}^{\eta_3-\eta_3}_{0}(k_3)\, \overset{(2)}{Q}{}^{\eta_4-\eta_4}_{0}(k_4)\, \delta^{\eta}_{234,0}\, \delta_{234,0}\, \delta_{34,1}\, dk_1 \cdots dk_4.$$

Integrating over $\tilde{k}_1$ and letting

$$\begin{Bmatrix}\eta_4\\ \tilde{k}_4\end{Bmatrix} \to \begin{Bmatrix}\eta_3\\ \tilde{k}_3\end{Bmatrix} \to \begin{Bmatrix}\eta_2\\ \tilde{k}_2\end{Bmatrix} \to \begin{Bmatrix}\eta_1\\ \tilde{k}_1\end{Bmatrix}, \text{ and } s_1 \to -s_4,\ s_5 \to -s_5$$

one obtains,

$$2S_4 \sim 36\pi t\, \delta_{00'}\, \overset{(2)}{Q}{}^{\eta-\eta}_{0}(k) \sum_{\eta_1\eta_2\eta_3} \int_{-\infty}^{\infty} -\frac{2i}{3} \sum_{\eta_4} \frac{L^{\eta\,-\eta_4\,\eta_1}_{k\,k_2+k_3\,k_1}\, L^{-\eta_4\,\eta_2\,\eta_3}_{k_2+k_3\,k_2\,k_3}}{\eta_2\omega_2 + \eta_3\omega_3 + \eta_4\omega_{23}}$$

$$\times -\frac{2i}{3} \sum_{\eta_5} \frac{L^{\eta_1\,-\eta_5\,\eta}_{k_1\,-k_2-k_3\,+k}\, L^{-\eta_5\,-\eta_2\,-\eta_3}_{-k_2-k_3\,-k_2\,-k_3}}{(-\eta_2\omega_2 - \eta_3\omega_3 + \eta_5\omega_{23})}\, \overset{(2)}{Q}{}^{\eta_2-\eta_2}_{0}(k_2)$$

$$\overset{(2)}{Q}{}^{\eta_3-\eta_3}_{0}(k_3)\, \delta^{\eta}_{123,0}\, \delta_{123,0}\, d\tilde{k}_1\, d\tilde{k}_2\, d\tilde{k}_3 + O(1).$$

S_6 behaves in a similar way and applies a permutation $0 \to -2 \to -3 \to 0$ in the second part of the integral. Therefore

$$2S_4 + 4S_6 \sim 36\bar{a}t\,\delta(k+k')\,\overset{(2)\,\gamma\,-\gamma}{\phi_0}(k) \sum_{\gamma_1\gamma_2\gamma_3} \int_{-\tilde{\omega}}^{\tilde{\omega}} \frac{-2i}{3} \sum_{\gamma_4} \frac{L^{\gamma\;-\gamma_4\;\gamma_1}_{k\;k_2+k_3\;k_1}\;L^{-\gamma_4\;\gamma_2\;\gamma_3}_{k_2+k_3\;k_2\;k_3}}{\gamma_2\omega_2+\gamma_3\omega_3+\gamma_4\omega_{23}}$$

$$\times \overset{0-2-3}{P}\;\frac{-2i}{3}\sum_{\gamma_5}\frac{L^{\gamma_1\;-\gamma_5\;\gamma}_{k_1\;-k_2-k_3\;k}\;L^{-\gamma_5\;-\gamma_2\;-\gamma_3}_{-k_2-k_3\;-k_2\;-k_3}}{(-\gamma_2\omega_2-\gamma_3\omega_3+\gamma_5\omega_{23})}$$

$$\overset{(2)\,\gamma_2-\gamma_2}{\phi_0}(k_2)\;\overset{(2)\,\gamma_3-\gamma_3}{\phi_0}(k_3)\;\delta^{a}_{123,0}\;\delta_{123,0}\;dk_1\,dk_2\,dk_3 + O(1) \cdots (26)$$

The (2×4) decomposition gives the following array.

$$\delta_{0'3}\overset{(1)}{\delta_{4678}} \quad \delta_{0'4}\overset{(1)}{\delta_{3678}} \quad \delta_{0'6}\overset{(2)}{\delta_{3478}} \quad \delta_{0'7}\overset{(3)}{\delta_{3468}} \quad \delta_{0'8}\overset{(3)}{\delta_{3467}}$$

$$\delta_{34}\overset{(4)}{\delta_{0'678}} \quad \delta_{36}\overset{(5)}{\delta_{0'478}} \quad \delta_{37}\overset{(6)}{\delta_{0'468}} \quad \delta_{38}\overset{(6)}{\delta_{0'467}}$$

$$\delta_{46}\overset{(5)}{\delta_{0'378}} \quad \delta_{47}\overset{(6)}{\delta_{0'368}} \quad \delta_{48}\overset{(6)}{\delta_{0'367}}$$

$$\delta_{67}\overset{(7)}{\delta_{0'348}} \quad \delta_{68}\overset{(7)}{\delta_{0'347}}$$

$$\delta_{78}\overset{(8)}{\delta_{0'346}}.$$

The only terms which exhibit a secular behavior are S_1 and S_2.

$$S_1: \quad \delta_{0'3}\,\delta_{4678}\,\delta_{78,5}\,\delta_{56,2}\,\delta_{34,1}\,\delta_{12,0}$$

$$= \delta_{0'3}\,\delta_{4678}\,\delta_{42}\,\delta_{00'}\,\delta_{12,0}\,\delta_{78,5}$$

$$= \delta_{00'}\,\delta_{3,0}\,\delta_{42}\,\delta_{678,2}\,\delta_{56,2}\,\delta_{12,0}.$$

When one integrates over $\tilde{k}_3$ and $\tilde{k}_4$, F(t) becomes

$$\int_0^t \Delta(\eta_3\omega+\eta_4\omega_2-\eta_1\omega_1)\ \frac{\Delta_{678,2}-\Delta_{56,2}}{i\,W_{78,5}}\ e^{i(\eta_1\omega_1+\eta_2\omega_2-s\omega)t}\,dt.$$

A "t" growth occurs when

$$\eta_4 = -\eta_2\,, \quad \eta_3 = \eta.$$

After applying the permutation and adding, one obtains

$$2S_1 \sim -8t\,\delta(k+k')\sum_{\eta_1\eta_2\eta_5\eta_6\eta_7\eta_8}\int_{-\infty}^{\infty}\frac{L^{\eta\,\eta_1\,\eta_2}_{k\,k_1\,k_2}\,L^{\eta_1\,\eta\,-\eta_2}_{k_1\,k\,-k_2}}{\eta_1\omega_1+\eta_2\omega_2-\eta\omega}\ \frac{L^{-\eta_2\,\eta_5\,\eta_6}_{-k_2\,k_5\,k_6}\,L^{\eta_5\,\eta_7\,\eta_8}_{k_5\,k_7\,k_8}}{-(\eta_5\omega_5+\eta_6\omega_6+\eta_2\omega_2)}$$

$$\overset{(2)}{\phi}{}_0^{\,\eta\,-\eta}(k)\left\{\pi\,\delta^{\eta}_{6782}\left(\overset{(4)}{\phi}{}_0^{\,\eta_6\eta_7\eta_8\eta_2}(k_6,k_7,k_8)+\overset{(4)}{\phi}{}_0^{*\,\eta_6\eta_7\eta_8\eta_2}(k_6,k_7,k_8)\right)\right.$$

$$\left.+\frac{iP}{W_{6782}}\left(\overset{(4)}{\phi}{}_0^{\,\eta_6\eta_7\eta_8\eta_2}(k_6,k_7,k_8)-\overset{(4)}{\phi}{}_0^{*\,\eta_6\eta_7\eta_8\eta_2}(k_6,k_7,k_8)\right)\right\}\delta_{6782}\,\delta_{562}\,\delta_{12,0}\,d\tilde{k}_1\,d\tilde{k}_2\,d\tilde{k}_5\cdots d\tilde{k}_8. \qquad \cdots(27).$$

S_2: Since $\delta_{0'6}\,\delta_{3478}\,\delta_{78,5}\,\delta_{56,2}\,\delta_{34,1}\,\delta_{12,0}$

$$= \delta_{6,0}\,\delta_{51}\,\delta_{3478}\,\delta_{781}\,\delta_{00'}\,\delta_{12,0},$$

Integrating over $\breve{k}_5$ and $\breve{k}_6$ and setting $\lambda_6 = \lambda$ and $\lambda_5 = -\lambda_1$,

$$f(t) = \int_0^t \Delta_{34,1} \frac{\Delta(s\omega + s_7\omega_7 + s_8\omega_8 - \lambda_1\omega_2) - \Delta(\lambda\omega - \lambda_1\omega_1 - \lambda_2\omega_2)}{i(\lambda_7\omega_7 + \lambda_8\omega_8 + \lambda_1\omega_1)} e^{iW_{12,0}t}\, dt,$$

$$\sim \frac{it}{W_{34,1}} \frac{1}{iW_{781}} \frac{-i}{W_{12,0}} + O(1).$$

Hence,

$$S_2 \sim 4t\,\delta_{00'} \sum_{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_7\lambda_8} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\lambda \lambda_1 \lambda_2} L_{k_1 k_3 k_4}^{\lambda_1 \lambda_3 \lambda_4} L_{k_2 -k_1 k}^{\lambda_2 -\lambda_1 \lambda} L_{-k_1 k_7 k_8}^{-\lambda_1 \lambda_7 \lambda_8}$$

$$\times \frac{-i}{W_{34,1} W_{781} W_{12,0}} Q_0^{(2)\,\lambda-\lambda}(k)\, Q_0^{(4)\,\lambda_3\lambda_4\lambda_7\lambda_8}(k_3, k_4, k_7)\, \delta_{78,1}\,\delta_{34,1}\,\delta_{12,0}\, d\breve{k}_1 \cdots d\breve{k}_4\, d\breve{k}_7\, d\breve{k}_8$$

$$+ 4t\,\delta_{00'} \sum_{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_7\lambda_8} \int_{-\infty}^{\infty} L_{-k k_1 k_2}^{-\lambda \lambda_1 \lambda_2} L_{k_1 k_3 k_4}^{\lambda_1 \lambda_3 \lambda_4} L_{k_2 -k_1 -k}^{\lambda_2 -\lambda_1 -\lambda} L_{-k_1 k_7 k_8}^{-\lambda_1 \lambda_7 \lambda_8}$$

$$\times \frac{-1}{W_{34.1} W_{781} W_{120}} \overset{(2)}{Q}{}_0^{\,1\,-1}(k) \overset{(4)}{Q}{}_0^{\,1_3 1_4 1_7 1_8}(k_3, k_4, k_7) \, \delta_{781} \delta_{34.1} \delta_{120} \, d\tilde{k}_1 \cdots d\tilde{k}_4 \, d\tilde{k}_7 \, d\tilde{k}_8$$

$$+ O(1). \qquad \text{----(28)}$$

In the second integral, change

$$\begin{cases} 1_1 \\ \tilde{k}_1 \end{cases} \longrightarrow \begin{cases} -1_1 \\ -\tilde{k}_1 \end{cases}, \quad \begin{cases} 1_2 \\ \tilde{k}_2 \end{cases} \longrightarrow \begin{cases} -1_2 \\ -\tilde{k}_2 \end{cases},$$

$$\begin{cases} 1_3 \\ \tilde{k}_3 \end{cases} \longleftrightarrow \begin{cases} 1_7 \\ \tilde{k}_7 \end{cases}, \quad \begin{cases} 1_4 \\ \tilde{k}_4 \end{cases} \longleftrightarrow \begin{cases} 1_8 \\ \tilde{k}_8 \end{cases}.$$

The second integral becomes

$$4t\, \delta_{00'} \sum_{1_1 1_2 1_3 1_4 1_7 1_8} \int_{-\tilde{\omega}}^{\tilde{\omega}} (-) L_{k k_1 k_2}^{1\, 1_1 1_2} L_{-k_1 k_7 k_8}^{-1_1 1_7 1_8} (-) L_{k_2 -k_1 k}^{1_2 -1_1 1} L_{k_1 k_3 k_4}^{1_1 1_3 1_4}$$

$$\times \frac{-1}{W_{781} W_{34.1} (-) W_{12.0}} \overset{(2)}{Q}{}_0^{\,1\,-1}(k) \overset{(4)}{Q}{}_0^{\,1_3 1_4 1_7 1_8}(k_3, k_4, k_7) \, \delta_{781} \delta_{34.1} \delta_{12.0}$$

$$d\tilde{k}_1 \cdots d\tilde{k}_4 \, d\tilde{k}_7 \, d\tilde{k}_8 .$$

This is the negative of the first integral and therefore

$$S_2 \sim O(1).$$

The (3×3) decomposition gives the array,

$$\delta_{0'34}\,\delta_{678} \quad \delta_{0'36}\,\delta_{478} \quad \delta_{0'37}\,\delta_{468} \quad \delta_{0'38}\,\delta_{467}$$

$$\delta_{0'46}\,\delta_{378} \quad \delta_{0'47}\,\delta_{368} \quad \delta_{0'48}\,\delta_{367}$$

$$\delta_{0'67}\,\delta_{348} \quad \delta_{0'68}\,\delta_{347}$$

$$\delta_{0'78}\,\delta_{346}\,.$$

The last term $\delta_{0'78}\,\delta_{346}$ gives a "t" growth.

$$S:\ \delta_{0'78}\,\delta_{346}\,\delta_{78,5}\,\delta_{34,1}\,\delta_{56,2}\,\delta_{12,0}$$

$$= \delta_{50'}\,\delta_{61}\,\delta_{78,0}\,\delta_{34,1}\,\delta_{00'}\,\delta_{12,0}\,.$$

Integrating over $\tilde{k}_5$ and $\tilde{k}_6$ and setting $\lambda_5 = \lambda$, $\lambda_6 = -\lambda_1$ one obtains

$$F(t) = \int_0^t \Delta_{34,1} \frac{\Delta(-\lambda_1\omega_1 + \lambda_7\omega_7 + \lambda_8\omega_8 - \lambda_2\omega_2) - \Delta(\lambda\omega - \lambda_1\omega_1 - \lambda_2\omega_2)}{i(\lambda_7\omega_7 + \lambda_8\omega_8 - \lambda\omega)} e^{iW_{12,0}t}\, dt,$$

$$\sim \frac{it}{W_{34,1}} \frac{1}{iW_{78,0}} \frac{-i}{W_{12,0}} + O(1)\,.$$

Hence, applying the permutation and adding,

$$\int \sim 4t\,\delta_{00'} \sum_{\nu_1\nu_2\nu_3\nu_4\nu_7\nu_8} \int_{-\infty}^{\infty} \alpha^{\nu\,\nu_1\,\nu_2}_{k\,k_1\,k_2}\ \alpha^{\nu_1\,\nu_3\,\nu_4}_{k_1\,k_3\,k_4}\ \alpha^{\nu_2\,\nu\,-\nu_1}_{k_2\,k\,-k_1}\ \alpha^{\nu\,\nu_7\,\nu_8}_{k\,k_7\,k_8}$$

$$\left[\overset{(3)}{\phi}{}_0^{\nu_7\nu_8-\nu}(k_7,k_8)\ \overset{(3)}{\phi}{}_0^{\nu_3\nu_4-\nu_1}(k_3,k_4) - \overset{(3)}{\phi}{}_0^{*\,\nu_7\nu_8-\nu}(k_7,k_8)\ \overset{(3)}{\phi}{}_0^{*\,\nu_3\nu_4-\nu_1}(k_3,k_4) \right]$$

$$\frac{-i\,\delta_{78,0}\,\delta_{34,1}\,\delta_{12,0}}{W_{78,0}\,W_{34,1}\,W_{12,0}}\, d\check{k}_1\, d\check{k}_2\, d\check{k}_3\, d\check{k}_4\, d\check{k}_7\, d\check{k}_8 + O(1). \quad \cdots(29)$$

This term meets its negative mate in the long time behavior of $\langle b_1^{\nu'}(k')\,{}_2d_3^{\nu}(k) + b_1^{\nu}(k)\,{}_2d_3^{\nu'}(k')\rangle_{\nu'=-\nu}$.

Consider $\langle a_0^{\nu'}(k')\,{}_6e_4^{\nu}(k) + a_0^{\nu}(k)\,{}_6e_4^{\nu'}(k')\rangle_{\nu'=-\nu}$

$$= 8\sum_{\nu_1\cdots\nu_8}\int_{-\infty}^{\infty} \alpha^{\nu\,\nu_1\,\nu_2}_{k\,k_1\,k_2}\ \alpha^{\nu_1\,\nu_3\,\nu_4}_{k_1\,k_3\,k_4}\ \alpha^{\nu_3\,\nu_5\,\nu_6}_{k_3\,k_5\,k_6}\ \alpha^{\nu_5\,\nu_7\,\nu_8}_{k_5\,k_7\,k_8}$$

$$\langle a_{0'}^{\nu'}\, a_{02}^{\nu_2}\, a_{04}^{\nu_4}\, a_{06}^{\nu_6}\, a_{07}^{\nu_7}\, a_{08}^{\nu_8}\rangle\, F(t)\,\delta_{78,5}\,\delta_{56,3}\,\delta_{34,1}\,\delta_{12,0}\ d\check{k}_1 \cdots\cdots d\check{k}_8 .$$

$$F(t) = \int_0^t \frac{1}{iW_{78,5}}\left(\frac{\Delta_{4678,1}-\Delta_{34,1}}{iW_{678,3}} - \frac{\Delta_{456,1}-\Delta_{34,1}}{iW_{56,3}} \right) e^{iW_{12,0}t}\, dt.$$

The $(2 \times 2 \times 2)$ decomposition gives the following array.

$$
\begin{array}{lll}
\delta_{0'2}\,\overset{(1)}{\delta_{46}}\,\delta_{78} & \delta_{0'2}\,\overset{(4)}{\delta_{47}}\,\delta_{68} & \delta_{0'2}\,\overset{(4)}{\delta_{48}}\,\delta_{67} \\
\delta_{0'4}\,\overset{(2)}{\delta_{26}}\,\delta_{78} & \delta_{0'4}\,\overset{(5)}{\delta_{27}}\,\delta_{68} & \delta_{0'4}\,\overset{(5)}{\delta_{28}}\,\delta_{67} \\
\delta_{0'6}\,\overset{(3)}{\delta_{24}}\,\delta_{78} & \delta_{0'6}\,\overset{(6)}{\delta_{27}}\,\delta_{48} & \delta_{0'6}\,\overset{(6)}{\delta_{28}}\,\delta_{47} \\
\delta_{0'7}\,\overset{(7)}{\delta_{24}}\,\delta_{68} & \delta_{0'7}\,\overset{(8)}{\delta_{26}}\,\delta_{48} & \delta_{0'7}\,\overset{(9)}{\delta_{28}}\,\delta_{46} \\
\delta_{0'8}\,\overset{(7)}{\delta_{24}}\,\delta_{67} & \delta_{0'8}\,\overset{(8)}{\delta_{26}}\,\delta_{47} & \delta_{0'8}\,\overset{(9)}{\delta_{27}}\,\delta_{46}\,.
\end{array}
$$

S_1, S_2, S_3 and S_4 are identically zero because of the zero mean value property.

$$S_5 : \ \delta_{0'4}\,\delta_{27}\,\delta_{68}\,\delta_{78,5}\,\delta_{56,3}\,\delta_{34,1}\,\delta_{12,0}$$

$$= \delta_{00'}\,\delta_{4,0}\,\delta_{27}\,\delta_{68}\,\delta_{32}\,\delta_{562}\,\delta_{12,0}\,.$$

Integrating over $\breve{k}_3$, $\breve{k}_4$, $\breve{k}_7$ and k_8, one obtains

$$F(t) = \int_0^t \frac{1}{i(s_7\omega_2 + s_8\omega_6 - s_5\omega_5)} \left[\frac{\Delta(s_4\omega + (s_6+s_8)\omega_6 + s_7\omega_2 - s_1\omega_1) - \Delta(s_3\omega_2 + s_4\omega - s_1\omega_1)}{i((s_6+s_8)\omega_6 + (s_7 - s_3)\omega_2]} \right.$$

$$\left. - \frac{\Delta(s_4\omega + s_5\omega_5 + s_6\omega_6 - s_1\omega_1) - \Delta(s_3\omega_2 + s_4\omega - s_1\omega_1)}{i(s_5\omega_5 + s_6\omega_6 - s_3\omega_2)} \right] e^{iW_{12,0}t}\, dt.$$

There are three types of possible secular terms.

(1) $0(t^2)$, with the second order cumulants all of the energy density type.

(2) $0(t)$ terms with the second order cumulants all of the energy density type. These cancel on application of the permutation as they contain an odd number of quantities which change sign.

(3) $0(t)$ terms with at least one of the second order cumulants of the $Q_0^{(2)++}$ type. These will cancel with the same term which cancels the $0(t^2)$ term in the manner illustrated previously in the case of $\langle b_2^{s}(k)\, b_2^{-s}(k')\rangle$ and $\langle a_0^{s}(k)\, {}_3d_4^{-s}(k') + a_0^{-s}(k')\, {}_3d_4^{s}(k)\rangle$.

The "t^2" term occurs when $s_7 = -s_2$, $s_8 = -s_6$, $s_3 = -s_2$ and $s_4 = s$.

$$F(t) \sim -\frac{t^2}{2}\, \frac{1}{W_{12,0}}\, \frac{1}{W_{562}} + 0(t)\, \frac{1}{(W_{ijk})^3}\, .$$

The order t terms cancel as stated previously on application of the permutation, whereas the "t^2" terms add.

$$2S_5 \sim -16t^2\delta_{00'}\overset{(2)}{\phi}{}_0^{\eta-\eta}(k)\sum_{\eta_1\eta_2\eta_5\eta_6}\int_{-\infty}^{\infty}\frac{\mathcal{L}^{\eta\eta_1\eta_2}_{kk_1k_2}\mathcal{L}^{\eta_1\eta-\eta_2}_{k_1k-k_2}}{\eta_1\omega_1+\eta_2\omega_2-\eta\omega}$$

$$\frac{\mathcal{L}^{-\eta_2\eta_5\eta_6}_{-k_2k_5k_6}\mathcal{L}^{\eta_5-\eta_2-\eta_6}_{k_5-k_2-k_6}}{\eta_5\omega_5+\eta_6\omega_6+\eta_2\omega_2}\overset{(2)}{\phi}{}_0^{\eta_2-\eta_2}(k_2)\overset{(2)}{\phi}{}_0^{\eta_6-\eta_6}(k_6)\,\delta_{562}\,\delta_{12,0}\,d\breve{k}_1\,d\breve{k}_2\,d\breve{k}_5\,d\breve{k}_6$$

$$+\,O(1). \qquad \text{---}(30)$$

It may be seen

$$(30) = -(25).$$

In a similar way it may be seen that

$$2S_7 \sim 16t^2\delta_{00'}\overset{(2)}{\phi}{}_0^{\eta-\eta}(k)\,\times$$

$$\left(\sum_{\eta_1\eta_2}\int_{-\infty}^{\infty}\frac{i\,\mathcal{L}^{\eta\eta_1\eta_2}_{kk_1k_2}\mathcal{L}^{\eta_1\eta-\eta_2}_{k_1k-k_2}}{W_{12,0}}\,\delta_{12,0}\,\overset{(2)}{\phi}{}_0^{\eta_2-\eta_2}(k_2)\,d\breve{k}_1\,d\breve{k}_2\right)^2. \qquad \text{--}(31)$$

Again, it is seen that

$$(31) = -(11).$$

$$S_6: \ \delta_{0'6}\ \delta_{27}\ \delta_{48}\ \delta_{78,5}\ \delta_{56,3}\ \delta_{34,1}\ \delta_{12,0}$$

$$= \delta_{00'}\ \delta_{6,0}\ \delta_{27}\ \delta_{48}\ \delta_{245}\ \delta_{34,1}\ \delta_{12,0}.$$

Upon integration over $\tilde{k}_6$, $\tilde{k}_7$ and $\tilde{k}_8$, F(t) becomes

$$\int_0^t \frac{1}{i(\eta_7\omega_2+\eta_8\omega_4-\eta_5\omega_5)}\left[\frac{\Delta(\eta_7\omega_2+(\eta_4+\eta_8)\omega_4+\eta_6\omega-\eta_1\omega_1)-\Delta_{34,1}}{i(\eta_6\omega+\eta_2\omega_2+\eta_8\omega_4-\eta_3\omega_3)}\right.$$

$$\left.-\frac{\Delta(\eta_4\omega_4+\eta_5\omega_5+\eta_6\omega-\eta_1\omega_1)-\Delta_{34,1}}{iW_{56,3}}\right]e^{iW_{12,0}t}\,dt.$$

A "t" growth occurs from the first part of the integral when $\eta_7 = -\eta_2$, $\eta_8 = -\eta_4$ and $\eta_6 = \eta$.

It is shown in Appendix that

$$F(t) \sim \frac{it}{W_{452}}\ \frac{i}{W_{12,0}}\left(\pi\,\delta_{234,0}^{\,?} + \frac{iP}{W_{234,0}}\right) + O(1).$$

The principal part terms cancel on application of the permutation and the resonance terms add.

$$2S_6 \sim 32\pi t\,\delta_{00'}\,\overset{(2)}{\phi}{}_0^{\lambda-\lambda}(k) \sum_{\lambda_1\ldots\lambda_5} \int_{-\infty}^{\infty} \frac{L_{k\,k_1\,k_2}^{\lambda\,\lambda_1\,\lambda_2}\, L_{k_1\,k_3\,k_4}^{\lambda_1\,\lambda_3\,\lambda_4}}{i(\lambda_3\omega_3+\lambda_4\omega_4-\lambda_1\omega_1)}$$

$$\times \frac{L_{k_3\,k_5\,k}^{\lambda_3\,\lambda_5\,\lambda}\, L_{k_5\,-k_2\,-k_4}^{\lambda_5\,-\lambda_2\,-\lambda_4}}{-i(\lambda_2\omega_2+\lambda_4\omega_4+\lambda_5\omega_5)}\, \overset{(2)}{\phi}{}_0^{\lambda_2-\lambda_2}(k_2)\, \overset{(2)}{\phi}{}_0^{\lambda_4-\lambda_4}(k_4)\, \delta_{234,0}^{\lambda}$$

$$\delta_{234,0}\,\delta_{34,1}\,\delta_{24,5}\, d\tilde{k}_1 \cdots d\tilde{k}_5 + O(1),$$

Since if $\lambda_2\omega_2+\lambda_3\omega_3+\lambda_4\omega_4-\lambda\omega=0$

$$\lambda_3\omega_3+\lambda_4\omega_4-\lambda_1\omega_1 = -\lambda_1\omega_1-\lambda_2\omega_2+\lambda\omega .$$

One integrates over $\tilde{k}_1$ and $\tilde{k}_5$ and sets

$$\begin{cases} s_4 \\ \tilde{k}_4 \end{cases} \to \begin{cases} s_3 \\ \tilde{k}_3 \end{cases} \to \begin{cases} s_1 \\ \tilde{k}_1 \end{cases}, \text{ and } s_1 \to -s_4 .$$

Using also the fact that part of the expression is symmetric in 2 and 3 , one obtains

$$2S_6 \sim 36\pi t\,\delta_{00'}\,\overset{(2)}{\phi}{}_0^{\lambda-\lambda}(k) \sum_{\lambda\lambda_2\lambda_3} \int_{-\infty}^{\infty} \left[-\frac{2i}{3} \sum_{\lambda_5} \frac{L_{k_1-k_1-k_3\,k}^{\lambda_1\,-\lambda_5\,\lambda}\, L_{-k_2-k_3\,-k_2\,-k_3}^{-\lambda_5\,-\lambda\,-\lambda_3}}{(-\lambda_2\omega_2-\lambda_3\omega_3+\lambda_5\omega_{23})} \right.$$

$$\times \overset{23}{\rho} - \frac{2i}{3}\sum_{\eta_4} \frac{\mathcal{L}_{k\;k_1+k_3\;k_2}^{\eta\;-\eta_4\;\eta_2}\,\mathcal{L}_{k_1+k_3\;k_1\;k_3}^{-\eta_4\;\eta_1\;\eta_3}}{(\eta_1\omega_1 + \eta_3\omega_3 + \eta_4\omega_{13})}\Bigg] \overset{(2)}{\phi}{}_0^{\eta_1-\eta_1}(k_1)\,\overset{(2)}{\phi}{}_0^{\eta_3-\eta_3}(k_3)$$

$$\delta_{123,0}^{\eta}\,\delta_{123,0}\,d\breve{k}_1 d\breve{k}_2 d\breve{k}_3 + O(1). \quad \text{---(32)}$$

S_8 and S_9 exhibit similar character;

$$2S_8 \sim 36\bar{\alpha}t\,\delta_{00'}\,\overset{(2)}{\phi}{}_0^{\eta-\eta}(k)\sum_{\eta_1\eta_2\eta_3}\int_{-\tilde{\omega}}^{\tilde{\omega}}\overset{23}{\rho}\Bigg[-\frac{2i}{3}\sum_{\eta_4}\frac{\mathcal{L}_{k\;k_1+k_3\;k_2}^{\eta\;-\eta_4\;\eta_2}\,\mathcal{L}_{k_1+k_3\;k_1\;k_3}^{-\eta_4\;\eta_1\;\eta_3}}{\eta_1\omega_1 + \eta_3\omega_3 + \eta_4\omega_{13}}$$

$$\times -\frac{2i}{3}\sum_{\eta_5}\frac{\mathcal{L}_{k_1\;k-k_3\;-k_2}^{\eta_1\;-\eta_5\;-\eta_2}\,\mathcal{L}_{k-k_3\;k\;-k_3}^{-\eta_5\;\eta\;-\eta_3}}{(\eta\omega - \eta_3\omega_3 + \eta_5\omega_{0-3})}\Bigg]\overset{(2)}{\phi}{}_0^{\eta_1-\eta_1}(k_1)\,\overset{(2)}{\phi}{}_0^{\eta_3-\eta_3}(k_3)$$

$$\delta_{123,0}^{\eta}\,\delta_{123,0}\,d\breve{k}_1 d\breve{k}_2 d\breve{k}_3 + O(1). \quad \text{--(33)}$$

$$2S_9 \sim 36\bar{\alpha}t\,\delta_{00'}\,\overset{(2)}{\phi}{}_0^{\eta-\eta}(k)\sum_{\eta_1\eta_2\eta_3}\int_{-\tilde{\omega}}^{\tilde{\omega}}\overset{13}{\rho}\Bigg[-\frac{2i}{3}\sum_{\eta_4}\frac{\mathcal{L}_{k\;k_1+k_3\;k_2}^{\eta\;-\eta_4\;\eta_2}\,\mathcal{L}_{k_1+k_3\;k_1\;k_3}^{-\eta_4\;\eta_1\;\eta_3}}{\eta_1\omega_1 + \eta_3\omega_3 + \eta_4\omega_{13}}$$

$$\times -\frac{2i}{3}\sum_{\eta_5}\frac{\mathcal{L}_{k_1\;k-k_2\;-k_3}^{\eta_1\;-\eta_5\;-\eta_3}\,\mathcal{L}_{k-k_2\;k\;-k_2}^{-\eta_5\;\eta\;-\eta_2}}{(\eta\omega - \eta_2\omega_2 + \eta_5\omega_{0-2})}\Bigg]\overset{(2)}{\phi}{}_0^{\eta_1-\eta_1}(k_1)\,\overset{(2)}{\phi}{}_0^{\eta_3-\eta_3}(k_3)$$

$$\delta_{123,0}^{\eta}\,\delta_{123,0}\,d\breve{k}_1 d\breve{k}_2 d\breve{k}_3 + O(1). \quad \text{--(34)}$$

The (2 X 4) decomposition is given by the following array.

$$
\begin{array}{lllll}
\delta_{0'2}\,\overset{(1)}{\delta_{4678}} & \delta_{0'4}\,\overset{(2)}{\delta_{2678}} & \delta_{0'6}\,\overset{(3)}{\delta_{2478}} & \delta_{0'7}\,\overset{(4)}{\delta_{2468}} & \delta_{0'8}\,\overset{(4)}{\delta_{2467}} \\
 & \delta_{24}\,\overset{(5)}{\delta_{0'678}} & \delta_{26}\,\overset{(6)}{\delta_{0'478}} & \delta_{27}\,\overset{(7)}{\delta_{0'468}} & \delta_{28}\,\overset{(7)}{\delta_{0'467}} \\
 & & \delta_{46}\,\overset{(8)}{\delta_{0'278}} & \delta_{47}\,\overset{(9)}{\delta_{0'268}} & \delta_{48}\,\overset{(9)}{\delta_{0'267}} \\
 & & & \delta_{67}\,\overset{(10)}{\delta_{0'248}} & \delta_{68}\,\overset{(10)}{\delta_{0'247}} \\
 & & & & \delta_{78}\,\overset{(11)}{\delta_{0'246}}.
\end{array}
$$

S_2 and S_5 are the only "t" growths.

S_2: Since $\delta_{0'4}\,\delta_{2678}\,\delta_{78,5}\,\delta_{56,3}\,\delta_{34,1}\,\delta_{12,0}$

$$= \delta_{4,0}\,\delta_{256}\,\delta_{32}\,\delta_{00'}\,\delta_{78,5}\,\delta_{12,0},$$

Upon integration over $\check{k}_3$ and $\check{k}_4$ and setting $\sigma_3 = -\sigma_2$ and $\sigma_4 = \sigma$ one obtains

$$F(t) = \int_0^t \frac{1}{iW_{78,5}} \left[\frac{\Delta(s\omega + s_6\omega_6 + s_7\omega_7 + s_8\omega_8 - \sigma_1\omega_1) - \Delta(\sigma\omega - \sigma_1\omega_1 - \sigma_2\omega_2)}{i(\sigma_6\omega_6 + \sigma_7\omega_7 + \sigma_8\omega_8 + \sigma_2\omega_2)} \right.$$

$$\left. - \frac{\Delta(\sigma\omega + \sigma_5\omega_5 + \sigma_6\omega_6 - \sigma_1\omega_1) - \Delta(\sigma\omega - \sigma_1\omega_1 - \sigma_2\omega_2)}{i(\sigma_5\omega_5 + \sigma_6\omega_6 + \sigma_2\omega_2)} \right] e^{iW_{12,0}t}\, dt,$$

It is shown in the Appendix II that for long time,

$$F(t) \sim \frac{-t}{W_{12,0}\,W_{562}} \left(\pi\,\delta^{\sigma}_{6782} + \frac{iP}{W_{6782}} \right) + O(1).$$

Therefore,

$$S_2 \sim -\delta t\,\delta_{00'}\,\overset{(2)\,\sigma-\sigma}{Q_0}(k) \sum_{\sigma_1\sigma_2\sigma_5\cdots\sigma_8} \int_{-\infty}^{\infty} \frac{L^{\sigma\,\sigma_1\,\sigma_2}_{k\,k_1\,k_2}\; L^{\sigma_1\,-\sigma_2\,\sigma}_{k_1\,-k_2\,k}}{\sigma_1\omega_1 + \sigma_2\omega_2 - \sigma\omega}$$

$$\times \frac{\mathcal{L}^{-\nu_2\ \nu_5\ \nu_6}_{-k_2\ k_5\ k_6}\ \mathcal{L}^{\nu_5\ \nu_7\ \nu_8}_{k_5\ k_7\ k_8}}{\nu_5\omega_5 + \nu_6\omega_6 + \nu_2\omega_2} \Big\{ \pi\, \delta^2_{6782} \Big(\overset{(4)}{\varphi}{}^{\nu_6\nu_7\nu_8\nu_2}_0(k_6,k_7,k_8) + \overset{(4)*}{\varphi}{}^{\nu_6\nu_7\nu_8\nu_2}_0(k_6,k_7,k_8) \Big)$$

$$+ \frac{iP}{W_{6782}} \Big(\overset{(4)}{\varphi}{}^{\nu_6\nu_7\nu_8\nu_2}_0(k_6,k_7,k_8) - \overset{(4)*}{\varphi}{}^{\nu_6\nu_7\nu_8\nu_2}_0(k_6,k_7,k_8) \Big) \Big\} \delta_{6782}\, \delta_{78,5}\, \delta_{12,0}\, d\tilde{k}_1 d\tilde{k}_2 d\tilde{k}_5 \cdots d\tilde{k}_8$$

$$+ O(1). \qquad \text{--- (35)}$$

One may note

$$(35) = -(27).$$

Similarly,

$$S_5 \sim 8t\,\delta_{00'} \sum_{\nu_1\nu_2\nu_5\cdots\nu_8} \int_{-\infty}^{\infty} \frac{\mathcal{L}^{\nu\ \nu_1\ \nu_2}_{k\ k_1\ k_2}\ \mathcal{L}^{\nu_1\ \nu\ -\nu_2}_{k_1\ k\ -k_2}}{-(\nu_1\omega_1 + \nu_2\omega_2 - \nu\omega)}\ \frac{\mathcal{L}^{\nu\ \nu_5\ \nu_6}_{k\ k_5\ k_6}\ \mathcal{L}^{\nu_5\ \nu_7\ \nu_8}_{k_5\ k_7\ k_8}}{\nu_5\omega_5 + \nu_6\omega_6 - \nu\omega}$$

$$\overset{(2)}{\varphi}{}^{\nu_2-\nu_2}_0(k_2) \Big[\pi\, \delta^1_{678,0} \Big(\overset{(4)}{\varphi}{}^{\nu_6\nu_7\nu_8-\nu}_0(k_6,k_7,k_8) + \overset{(4)*}{\varphi}{}^{\nu_6\nu_7\nu_8-\nu}_0(k_6,k_7,k_8) \Big)$$

$$+ \frac{iP}{W_{678,0}} \Big(\overset{(4)}{\varphi}{}^{\nu_6\nu_7\nu_8-\nu}_0(k_6,k_7,k_8) - \overset{(4)*}{\varphi}{}^{\nu_6\nu_7\nu_8-\nu}_0(k_6,k_7,k_8) \Big) \Big] \delta_{678,0}\, \delta_{78,5}$$

$$\delta_{12,0}\, d\tilde{k}_1 d\tilde{k}_2 d\tilde{k}_5 \cdots d\tilde{k}_8 + O(1).$$

Again one may note.

$$(36) = -(13).$$

The (3×3) decomposition given by the array,

$$\delta_{0'24}\,\delta_{678} \quad \delta_{0'26}\,\delta_{478} \quad \delta_{0'27}\,\delta_{468} \quad \delta_{0'28}\,\delta_{467}$$

$$\delta_{0'46}\,\delta_{278} \quad \delta_{0'47}\,\delta_{268} \quad \delta_{0'48}\,\delta_{267}$$

$$\delta_{0'67}\,\delta_{248} \quad \delta_{0'68}\,\delta_{247}$$

$$\delta_{0'78}\,\delta_{246}.$$

None of these terms exhibit any secular growth.

One now considers

$$\langle a_0^{\lambda'}(k')\, b_4^{\lambda}(k) + a_0^{\lambda}(k)\, b_4^{\lambda'}(k') \rangle_{\lambda,\lambda'=-\lambda}$$

$$= \rho^{00'} \sum_{\lambda_1\cdots\lambda_5} \int_{-\infty}^{\infty} L_{k k_1 k_2 k_3 k_4 k_5}^{\lambda\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5} \langle a_{01}^{\lambda_1} a_{02}^{\lambda_2} a_{03}^{\lambda_3} a_{04}^{\lambda_4} a_{05}^{\lambda_5} a_{0'}^{\lambda'} \rangle \Delta_{12345,0}$$

$$\delta_{12345,0}\; dk_1 \cdots dk_5 .$$

The only way a "t" growth can occur in this term is when

$$\lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3 + \lambda_4\omega_4 + \lambda_5\omega_5 = \lambda\omega .$$

This can occur only for a $(2 \times 2 \times 2)$ decomposition of the type $\delta_{12}\,\delta_{34}\,\delta_{50'}$. When the integration is performed over $\breve{k}_2$, $\breve{k}_4$ and $\breve{k}_5$, one sees that a "t" growth is possible when

$$s_2 = -s_1, \quad s_4 = -s_3 \quad s_5 = s.$$

However on application of the permutation $s \to -s$ $\tilde{k} \to -\breve{k}$, the "t" growth vanishes because

$$L^{s\; s_1\; -s_1\; s_3\; -s_3\; s}_{k\; k_1\; -k_1\; k_3\; -k_3\; k} + L^{-s\; s_1\; -s_1\; s_3\; -s_3\; -s}_{-k\; k_1\; -k_1\; k_3\; -k_3\; -k} = 0.$$

Similar reasoning and the fact that there are no triad resonances implies that

$$\langle a_0^{s'}(k'), a_4^{s}(k) + a_0^{s}(k)\, c_4^{s'}(k') \rangle_{s'=-s} \sim O(1).$$

$$\langle a_0^{s'}(k'), d_4^{s}(k) + a_0^{s}(k), d_4^{s'}(k') \rangle_{s'=-s}$$

$$= 3 \rho^{00'} \sum_{s_1 \cdots s_7} \int_{-\breve{\infty}}^{\breve{\infty}} L^{s\; s_1\; s_2\; s_3}_{k\; k_1\; k_2\; k_3} L^{s_2\; s_4\; s_5}_{k_2\; k_4\; k_5} L^{s_3\; s_6\; s_7}_{k_3\; k_6\; k_7} \langle a_{0'}^{s'} a_{01}^{s_1} a_{04}^{s_4} a_{05}^{s_5} a_{06}^{s_6} a_{07}^{s_7} \rangle$$

$$\int_0^t \Delta_{45,2}\, \Delta_{67,3}\, e^{iW_{123,0}t}\, dt \;\; \delta_{123,0}$$

$$\delta_{45,2}\, \delta_{67,3}\, d\breve{k}_1 \cdots d\breve{k}_7.$$

The only "t" growths in this term come from the decompositions

$$\delta(\breve{k}_1+\breve{k}_4+\breve{k}_5)\,\delta(\breve{k}'+\breve{k}_6+\breve{k}_7)\,\overset{(3)}{\phi}{}_0^{\,s_4 s_5 s_1}(k_4,k_5)\,\overset{(3)}{\phi}{}_0^{\,s_6 s_7 s'}(k_6,k_7)$$

and

$$\delta(\breve{k}_1+\breve{k}_6+\breve{k}_7)\,\delta(\breve{k}'+\breve{k}_4+\breve{k}_5)\,\overset{(3)}{\phi}{}_0^{\,s_6 s_7 s_1}(k_6,k_7)\,\overset{(3)}{\phi}{}_0^{\,s_4 s_5 s'}(k_4,k_5).$$

The reason a "t" growth is possible is that

$$s_1\omega_1 + s_2\omega_2 + s_3\omega_3 - s\omega \equiv 0$$

after integration over $\breve{k}_2$ and $\breve{k}_3$ and when $s_3 = s$ and $s_2 = -s_1$. On account of symmetry of the above expression in 2 and 3, these terms both give the same result. Therefore,

$$\langle a_0^{s'}(k')\, a_4^{s}(k) + a_0^{s}(k)\, a_4^{s'}(k')\rangle_{s'=-s}$$

$$\sim 6t\,\delta_{00'}\,\overset{s-s}{P}\sum_{s_1 s_4 s_5 s_6 s_7}\int_{-\infty}^{\infty} L_{k\,k_1\,-k_1\,k}^{s\,s_1\,-s_1\,s}\, L_{-k_1\,k_6\,k_7}^{-s_1\,s_6\,s_7}\, L_{k\,k_4\,k_5}^{s\,s_4\,s_5}$$

$$\overset{(3)}{\phi}{}_0^{\,s_6 s_7 s_1}(k_6,k_7)\,\overset{(3)}{\phi}{}_0^{\,s_4 s_5 -s_1}(k_4,k_5)\,\frac{i\delta_{45,0}}{W_{45,0}}\,\frac{i\delta_{67,1}}{W_{67,1}}\, d\breve{k}_1\, d\breve{k}_4 \cdots d\breve{k}_7$$

$$+\, O(1). \qquad \cdots (37)$$

The negative mate for this term comes from

$$\langle b_1^{s'}(k')\, C_3^{s}(k) + b_1^{s}(k)\, C_3^{s'}(k')\rangle_{s'=-s},$$

$$= 3 \overset{00'}{\rho} \sum_{\jmath_1 \cdots \jmath_7} \int_{-\infty}^{\infty} L_{k' k_1 k_2}^{\jmath' \jmath_1 \jmath_2} L_{k k_3 k_4 k_5}^{\jmath \jmath_3 \jmath_4 \jmath_5} L_{k_3 k_6 k_7}^{\jmath_3 \jmath_6 \jmath_7} \langle a_{01}^{\jmath_1} a_{02}^{\jmath_2} a_{04}^{\jmath_4} a_{05}^{\jmath_5} a_{06}^{\jmath_6} a_{07}^{\jmath_7} \rangle$$

$$\frac{\Delta_{4567,0} - \Delta_{345,0}}{i W_{67,3}} \Delta_{12,0'} \, \delta_{67,3} \, \delta_{345,0} \, \delta_{12,0'} \, d\tilde{k}_1 \cdots d\tilde{k}_7 .$$

The decompositions $\delta(\tilde{k}_1+\tilde{k}_2+\tilde{k}_4)\,\delta(\tilde{k}_5+\tilde{k}_6+\tilde{k}_7)\,\overset{(3)}{\phi}{}_0^{\jmath_1 \jmath_2 \jmath_4}(k_1,k_2)\,\overset{(3)}{\phi}{}_0^{\jmath_6 \jmath_7 \jmath_5}(k_6,k_7)$

and $\delta(\tilde{k}_1+\tilde{k}_2+\tilde{k}_5)\,\delta(\tilde{k}_6+\tilde{k}_7+\tilde{k}_4)\,\overset{(3)}{\phi}{}_0^{\jmath_1 \jmath_2 \jmath_5}(k_1,k_2)\,\overset{(3)}{\phi}{}_0^{\jmath_6 \jmath_7 \jmath_4}(k_6,k_7)$

can make

$$\jmath_3 \omega_3 + \jmath_4 \omega_4 + \jmath_5 \omega_5 - \jmath \omega = 0 ,$$

with certain choices of $\jmath_4$ and $\jmath_5$. Thus

$$\langle b_1^{\jmath'}(k')\, C_3^{\jmath}(k) + b_1^{\jmath}(k)\, C_3^{\jmath'}(k') \rangle_{\jmath' = -\jmath}$$

$$\sim 6t\, \delta_{00'} \overset{0-0}{\rho} \sum_{\jmath_1 \jmath_2 \jmath_3 \jmath_6 \jmath_7} \int_{-\infty}^{\infty} L_{-k\, k_1\, k_2}^{-\jmath\, \jmath_1\, \jmath_2} L_{k\, k_3\, k-k_3}^{\jmath\, \jmath_3\, \jmath-\jmath_3} L_{k_3 k_6 k_7}^{\jmath_3 \jmath_6 \jmath_7}$$

$$\overset{(3)}{\phi}{}_0^{\jmath_1 \jmath_2 \jmath_3}(k_1,k_2)\, \overset{(3)}{\phi}{}_0^{\jmath_6 \jmath_7 -\jmath_3}(k_6,k_7)\, \frac{i\delta_{67,3}}{W_{67,3}}\, \frac{i\delta_{12\ominus}}{W_{12\ominus}}\, d\tilde{k}_1 d\tilde{k}_2 d\tilde{k}_3 d\tilde{k}_6 d\tilde{k}_7$$

$$+ O(1) . \quad \cdots (38)$$

If one sets

$$\begin{cases} \jmath_3 \\ \tilde{k}_3 \end{cases} \rightarrow \begin{cases} -\jmath_1 \\ -\tilde{k}_1 \end{cases}, \quad \begin{cases} \jmath_1 \\ k_1 \end{cases} \rightarrow \begin{cases} -\jmath_4 \\ -\tilde{k}_4 \end{cases} \text{ and } \begin{cases} \jmath_2 \\ k_2 \end{cases} \rightarrow \begin{cases} -\jmath_5 \\ -\tilde{k}_5 \end{cases},$$

one obtains $\langle b_1^{j'}(k')\, c_3^{j}(k) + b_1^{j}(k)\, c_3^{j'}(k') \rangle_{1,-1}$

$$\sim 6t\, \overset{0-0}{P}\, \delta_{00'} \sum_{j_1 j_4 \cdots j_7} \int_{-\infty}^{\infty} L^{1\, j_4\, j_5}_{k\, k_4\, k_5}\, L^{1\, j_1\, j-j_1}_{k\, k_1\, k-k_1}\, L^{-j_1\, j_6\, j_7}_{-k_1\, k_6\, k_7}$$

$$\overset{(3)}{\Phi}{}_0^{*\, j_4\, j_5\, -j_1}(k_4,k_5)\, \overset{(3)}{\Phi}{}_0^{j_6\, j_7\, j_1}(k_6,k_7)\, \frac{i\delta_{671}}{W_{671}}\, \frac{i\delta_{45,0}}{W_{45,0}}\, d\breve{k}_1\, d\breve{k}_4 \cdots d\breve{k}_7$$

$$+ O(1). \qquad --- (39)$$

Adding (37) and (39), one obtains

$$6t\, \delta_{00'}\, \overset{0-0}{P} \sum_{j_4 j_5} \int_{-\infty}^{\infty} L^{1\, j_4\, j_5}_{k\, k_4\, k_5} \left\{ \overset{(3)}{\Phi}{}_0^{j_4\, j_5\, -j}(k_4,k_5) + \overset{(3)}{\Phi}{}_0^{*\, j_4\, j_5\, -j}(k_4,k_5) \right\} \frac{i\delta_{45,0}}{W_{45,0}}\, d\breve{k}_4\, d\breve{k}_5$$

$$\times \sum_{j_1 j_6 j_7} \int_{-\infty}^{\infty} L^{1\, j_1\, j-j_1}_{k\, k_1\, k-k_1}\, L^{-j_1\, j_6\, j_7}_{-k_1\, k_6\, k_7}\, \overset{(3)}{\Phi}{}_0^{j_6\, j_7\, j_1}(k_6,k_7)\, \frac{i\delta_{671}}{W_{671}}\, d\breve{k}_1\, d\breve{k}_6\, d\breve{k}_7$$

$$+ O(1).$$

The permutation $\overset{0-0}{P}$ puts $j \to -j$ and $\breve{k} \to -\breve{k}$. In the permuted term let $j_4 \to -j_4$, $\breve{k}_4 \to -\breve{k}_4$, $j_5 \to -j_5$, $\breve{k}_5 \to -\breve{k}_5$, whereupon one sees that because of two sign changes the first integral above stays the same. However the second integral changes sign because

$$\mathcal{L}^{-\lambda\ \lambda_1\ -\lambda_1\ -\lambda}_{-k\ k_1\ -k_1\ -k} + \mathcal{L}^{\lambda\ \lambda_1\ -\lambda_1\ \lambda}_{k\ k_1\ -k_1\ k} = 0.$$

Therefore

$$(39) + (37) \sim O(1).$$

It can be seen that both $\langle a_0^{\lambda'}(k')\,{}_3e_4^{\lambda}(k) + a_0^{\lambda}(k)\,{}_3e_4^{\lambda'}(k')\rangle_{\lambda'=-\lambda}$ and $\langle a_0^{\lambda'}(k')\,{}_4e_4^{\lambda}(k) + a_0^{\lambda}(k)\,{}_4e_4^{\lambda'}(k')\rangle_{\lambda'=-\lambda}$ * behave like order one quantities in long time.

Now consider

$$\langle e_1^{\lambda'}(k')\,d_3^{\lambda}(k) + e_1^{\lambda}(k)\,d_3^{\lambda'}(k')\rangle_{\lambda'=-\lambda}$$

$$= \rho^{00'} \sum_{\lambda_1\cdots\lambda_8} \int_{-\infty}^{\infty} \mathcal{L}^{\lambda\lambda_1\lambda_2}_{kk_1k_2}\,\mathcal{L}^{\lambda_1\lambda_3\lambda_4}_{k_1k_3k_4}\,\mathcal{L}^{\lambda_2\lambda_5\lambda_6}_{k_2k_5k_6}\,\mathcal{L}^{\lambda'\lambda_7\lambda_8}_{k'k_7k_8}\,\langle a_{03}^{\lambda_3} a_{04}^{\lambda_4}$$

$$a_{05}^{\lambda_5} a_{06}^{\lambda_6} a_{07}^{\lambda_7} a_{08}^{\lambda_8}\rangle\,\Delta_{78,0'} \int_0^t \Delta_{34,1}\Delta_{56,2}\, e^{iW_{12,0}t}\,dt$$

$$\delta_{34,1}\,\delta_{56,2}\,\delta_{12,0}\,\delta_{78,0'}\,d\check{k}_1 \cdots\cdots d\check{k}_8.$$

A "t" growth arises from the decompositions

$$\delta(\check{k}_3+\check{k}_4+\check{k}_5)\,\delta(\check{k}_6+\check{k}_7+\check{k}_8)\,\overset{(3)}{\varphi}{}_0^{\lambda_3\lambda_4\lambda_5}(k_3,k_4)\,\overset{(3)}{\varphi}{}_0^{\lambda_7\lambda_8\lambda_6}(k_7,k_8),$$

$$\delta(\check{k}_3+\check{k}_4+\check{k}_6)\,\delta(\check{k}_5+\check{k}_7+\check{k}_8)\,\overset{(3)}{\varphi}{}_0^{\lambda_3\lambda_4\lambda_6}(k_3,k_4)\,\overset{(3)}{\varphi}{}_0^{\lambda_7\lambda_8\lambda_5}(k_7,k_8),$$

$$\delta(\check{k}_3+\check{k}_4+\check{k}_6)\,\delta(\check{k}_4+\check{k}_7+\check{k}_8)\,\overset{(3)}{\varphi}{}_0^{\lambda_5\lambda_6\lambda_3}(k_5,k_6)\,\overset{(3)}{\varphi}{}_0^{\lambda_7\lambda_8\lambda_4}(k_7,k_8)$$

and

$$\delta(\check{k}_4+\check{k}_5+\check{k}_6)\,\delta(\check{k}_3+\check{k}_7+\check{k}_8)\,\overset{(3)}{\varphi}{}_0^{\lambda_5\lambda_6\lambda_4}(k_5,k_6)\,\overset{(3)}{\varphi}{}_0^{\lambda_7\lambda_8\lambda_3}(k_7,k_8).$$

Each of these after certain integrations and choices of the sign parameters make the time t dependent part of the integrand of the form,

$$\Delta(\omega)\int_0^t \Delta(-\mu)\,\Delta(\nu)\,e^{i\mu t}\,dt,$$

$$= \Delta(\omega)\int_0^t \Delta(\mu)\,\Delta(\nu)\,dt$$

$$\sim t\,\frac{i}{\omega}\,\frac{i}{\mu}\,\frac{i}{\nu} + O(1),$$

when ω, μ and ν are all triads. Therefore

$$\langle b_1^{\eta'}(k'), d_3^{\eta}(k) + b_1^{\eta}(k), d_3^{\eta'}(k')\rangle_{\eta'=-\eta}$$

$$\sim 4t\,\delta_{00'}\sum_{\eta_1\eta_2\eta_5\eta_6\eta_7\eta_8}\int_{-\infty}^{\infty} \alpha_{k\,k_1\,k_2}^{\eta\,\eta_1\,\eta_2}\,\alpha_{k_2\,k_5\,k_6}^{\eta_2\,\eta_5\,\eta_6}\,\alpha_{k_1\,-k_2\,k}^{\eta_1\,-\eta_2\,\eta}\,\alpha_{-k\,k_7\,k_8}^{-\eta\,\eta_7\,\eta_8}$$

$$\left\{ \overset{(3)}{\phi}{}_0^{\eta_5\eta_6-\eta_2}(k_5,k_6)\,\overset{(3)}{\phi}{}_0^{\eta_7\eta_8\eta}(k_7,k_8) - \overset{(3)}{\phi}{}_0^{\eta_5\eta_6-\eta_2\,*}(k_5,k_6)\,\overset{(3)}{\phi}{}_0^{\eta_7\eta_8\eta\,*}(k_7,k_8)\right\}\frac{i\delta_{780}}{W_{780}}$$

$$\frac{i\delta_{56,2}}{W_{56,2}}\,\frac{i\delta_{12,0}}{W_{12,0}}\,d\breve{k}_1\,d\breve{k}_2\,d\breve{k}_5\cdots d\breve{k}_8 + O(1). \quad\cdots(40)$$

This term meets its negative mate in

$$\langle a_0^{j'}(k')\, 5\, \ell_4^{j}(k) + a_0^{j}(k)\, 5\, \ell_4^{j'}(k') \rangle_{j'=-j}$$

$$= 2^8 \sum_{j_1 \cdots j_8}^{00'} \int_{-\infty}^{\infty} \mathcal{L}_{k k_1 k_2}^{j j_1 j_2} \mathcal{L}_{k_1 k_3 k_4}^{j_1 j_3 j_4} \mathcal{L}_{k_3 k_5 k_6}^{j_3 j_5 j_6} \mathcal{L}_{k_4 k_7 k_8}^{j_4 j_7 j_8}$$

$$\langle a_0^{j'} a_{02}^{j_2} a_{05}^{j_5} a_{06}^{j_6} a_{07}^{j_7} a_{08}^{j_8} \rangle \int_0^t \left\{ \int_0^t \Delta_{56,3} \Delta_{78,4}\, e^{i\omega_{34,1} t}\, dt \right\}$$

$$e^{i\omega_{12,0} t}\, dt \;\; \delta_{56,3}\, \delta_{78,4}\, \delta_{34,1}\, \delta_{12,0}\, d\breve{k}_1 \cdots d\breve{k}_8 .$$

The "t" growths arise from the decompositions

$$\delta(\breve{k}' + \breve{k}_5 + \breve{k}_6)\, \delta(\breve{k}_2 + \breve{k}_7 + \breve{k}_8)\, \overset{(3)}{\phi_0}{}^{j_5 j_6 j'}(k_5, k_6)\, \overset{(3)}{\phi_0}{}^{j_7 j_8 j_2}(k_7, k_8),$$

$$\delta(\breve{k}' + \breve{k}_7 + \breve{k}_8)\, \delta(\breve{k}_2 + \breve{k}_5 + \breve{k}_6)\, \overset{(3)}{\phi_0}{}^{j_7 j_8 j'}(k_7, k_8)\, \overset{(3)}{\phi_0}{}^{j_5 j_6 j_2}(k_5, k_6)$$

From symmetry between (5,6) and (7,8) these terms give the same result. Upon certain choices of the sign parameters (after the integrations have been performed), the time t dependent term in the integrand takes on the form

$$\int_0^t \left\{ \int_0^t \Delta(\omega)\, \Delta(\mu)\, e^{-i\nu t}\, dt \right\} e^{i\nu t}\, dt$$

$$\sim \frac{-it}{\omega \mu \nu} + O(1),$$

where ω, μ and ν are triads. Therefore,

$$\langle a_0^{\nu'}(k')_5 \ell_4^{\nu}(k) + a_0^{\nu}(k)_5 \ell_4^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$\sim 4t\,\delta_{00'} \sum_{\nu_1\nu_2\nu_5\cdots\nu_8} \int_{-\infty}^{\infty} \alpha_{k k_1 k_2}^{\nu \nu_1 \nu_2} \alpha_{k_1\, k-k_2}^{\nu_1\, \nu\, -\nu_2} \alpha_{k\, k_5\, k_6}^{\nu\, \nu_5\, \nu_6}$$

$$\alpha_{-k_2\, k_7\, k_8}^{-\nu_2\, \nu_7\, \nu_8} \left\{ \overset{(3)}{Q}{}_0^{\nu_5 \nu_6 -\nu}(k_5, k_6)\, \overset{(3)}{Q}{}_0^{\nu_7 \nu_8 \nu_2}(k_7, k_8) - \overset{(3)}{Q}{}_0^{\nu_5 \nu_6 -\nu\,*}(k_5, k_6)\, \overset{(3)}{Q}{}_0^{\nu_7 \nu_8 \nu_2\,*}(k_7, k_8) \right\}$$

$$\frac{i\delta_{56,0}}{W_{56,0}}\; \frac{i\delta_{782}}{W_{782}}\; \frac{i\delta_{12,0}}{W_{12,0}}\; d\breve{k}_1 d\breve{k}_2 d\breve{k}_5 \cdots d\breve{k}_8 + O(1).$$

If one changes

$$\begin{Bmatrix}\nu_5 \\ \breve{k}_5\end{Bmatrix} \leftrightarrow \begin{Bmatrix}-\nu_7 \\ -\breve{k}_7\end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix}\nu_6 \\ \breve{k}_6\end{Bmatrix} \longleftrightarrow \begin{Bmatrix}-\nu_8 \\ -\breve{k}_8\end{Bmatrix},$$

one obtains

$$4t\,\delta_{00'} \sum_{\nu_1\nu_2\nu_5\cdots\nu_8} \int_{-\infty}^{\infty} \alpha_{k k_1 k_2}^{\nu \nu_1 \nu_2} \alpha_{k_1\, k-k_2}^{\nu_1\, \nu\, -\nu_2} \alpha_{k\, -k_7\, -k_8}^{\nu\, -\nu_7\, -\nu_8} \alpha_{-k_2\, -k_5\, -k_6}^{-\nu_2\, -\nu_5\, -\nu_6}$$

$$\left[\overset{(3)}{Q}{}_0^{\nu_7 \nu_8 \nu\,*}(k_7, k_8)\, \overset{(3)}{Q}{}_0^{\nu_5 \nu_6 -\nu_2\,*}(k_5, k_6) - \overset{(3)}{Q}{}_0^{\nu_7 \nu_8 \nu}(k_7, k_8)\, \overset{(3)}{Q}{}_0^{\nu_5 \nu_6 -\nu_2}(k_5, k_6) \right]$$

$$\frac{i\delta_{56,2}}{-W_{56,2}} \frac{i\delta_{78,0}}{-W_{78,0}} \frac{i\delta_{12,0}}{W_{12,0}} d\check{k}_1 d\check{k}_2 d\check{k}_5 \cdots d\check{k}_8 + O(1) \quad \cdots (41)$$

Using the property that

$$\ell_{-k\,-k_1\,-k_2}^{-\nu\,-\nu_1\,-\nu_2} = -\ell_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2},$$

one sees

$$(41) = -(40) + O(1).$$

Let us now consider

$$\langle b_1^{\nu'}(k')\,{}_2a_3^{\nu}(k) + b_1^{\nu}(k)\,{}_2a_3^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$= 4\rho^{00'} \sum_{\nu_1 \cdots \nu_8} \int_{-\infty}^{\infty} \ell_{k'\,k_7\,k_8}^{\nu'\,\nu_7\,\nu_8} \ell_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2} \ell_{k_1\,k_3\,k_4}^{\nu_1\,\nu_3\,\nu_4} \ell_{k_3\,k_5\,k_6}^{\nu_3\,\nu_5\,\nu_6}$$

$$\langle a_{02}^{\nu_2} a_{03}^{\nu_3} a_{05}^{\nu_5} a_{06}^{\nu_6} a_{07}^{\nu_7} a_{08}^{\nu_8} \rangle \; F(t) \; \delta_{56,3}\, \delta_{34,1}\, \delta_{12,0}$$

$$\delta_{78,0'} \; d\check{k}_1 \cdots \cdots d\check{k}_8 \cdot$$

$$F(t) = \frac{\Delta_{78,0'}}{iW_{56,3}} \left[\frac{\Delta_{2456,0} - \Delta_{12,0}}{iW_{456,1}} - \frac{\Delta_{234,0} - \Delta_{12,0}}{iW_{34,1}} \right].$$

A "t" growth occurs from the decomposition $\delta(\breve{k}_2+\breve{k}_5+\breve{k}_6)$ $\delta(\breve{k}_4+\breve{k}_7+\breve{k}_8)\,\overset{(3)}{\varphi}{}_0^{\nu_5\nu_6\nu_2}(k_5,k_6)\,\overset{(3)}{\varphi}{}_0^{\nu_7\nu_8\nu_4}(k_7,k_8)$.

After integration over $\breve{k}_3$ and $\breve{k}_4$ and setting $\nu_3=-\nu_2$ and $\nu_4=\nu$ one obtains that

$$\nu_2\omega_2+\nu_3\omega_3+\nu_4\omega_4-\nu\omega\equiv 0.$$

Hence

$$\langle b_1^{\nu'}(k')\,{}_2a_3^{\nu}(k)+b_1^{\nu}(k)\,{}_2a_3^{\nu'}(k')\rangle_{\nu'=-\nu}$$

$$\sim 4t\,\delta_{00'}\,\overset{0-0}{P}\sum_{\nu_1\nu_2\nu_5\cdots\nu_8}\int_{-\infty}^{\infty}L_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2}\,L_{k_1\,k-k_2}^{\nu_1\,\nu-\nu_2}\,L_{-k_2\,k_5\,k_6}^{-\nu_2\,\nu_5\,\nu_6}$$

$$L_{-k\,k_7\,k_8}^{-\nu\,\nu_7\,\nu_8}\,\overset{(3)}{\varphi}{}_0^{\nu_7\nu_8\nu}(k_7,k_8)\,\overset{(3)}{\varphi}{}_0^{\nu_5\nu_6\nu_2}(k_5,k_6)\,\frac{i\delta_{562}}{W_{562}}\,\frac{i\delta_{780}}{W_{780}}\,\frac{i\delta_{12,0}}{W_{12,0}}$$

$$d\breve{k}_1\,d\breve{k}_2\,d\breve{k}_5\cdots d\breve{k}_8+O(1).$$

Changing $\begin{cases}\nu_1\\ \breve{k}_1\end{cases}\longleftrightarrow\begin{cases}\nu_2\\ \breve{k}_2\end{cases}$, $\begin{cases}\nu_5\\ \breve{k}_5\end{cases}\longrightarrow\begin{cases}-\nu_3\\ -\breve{k}_3\end{cases}$,

$\begin{cases}\nu_6\\ \breve{k}_6\end{cases}\longrightarrow\begin{cases}-\nu_4\\ -\breve{k}_4\end{cases}$, $\begin{cases}\nu_7\\ \breve{k}_7\end{cases}\longrightarrow\begin{cases}-\nu_7\\ -\breve{k}_7\end{cases}$ and $\begin{cases}\nu_8\\ \breve{k}_8\end{cases}\longrightarrow\begin{cases}-\nu_8\\ -\breve{k}_8\end{cases}$,

one obtains

$$\langle b_1^{\nu'}(k')\,{}_2a_3^{\nu}(k)+b_1^{\nu}(k)\,{}_2a_3^{\nu'}(k')\rangle_{\nu'=-\nu}$$

$$\sim 4t\,\delta_{00'}\sum_{\nu_1\cdots\nu_4\nu_7\nu_8}\int_{-\infty}^{\infty}L_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2}\,L_{k_2-k_1\,k}^{\nu_2-\nu_1\,\nu}\,L_{k_1\,k_3\,k_4}^{\nu_1\,\nu_3\,\nu_4}\,L_{k\,k_7\,k_8}^{\nu\,\nu_7\,\nu_8}$$

$$\left\{ \phi_0^{(3)\lambda_7\lambda_8-\lambda}(k_7,k_8)^{*}\, \phi_0^{(3)\lambda_3\lambda_4-\lambda_1}(k_3,k_4)^{*} - \phi_0^{(3)\lambda_7\lambda_8-\lambda}(k_7,k_8)\, \phi_0^{(3)\lambda_3\lambda_4-\lambda_1}(k_3,k_4) \right\}$$

$$\frac{i\delta_{34,1}}{W_{34,1}} \; \frac{i\delta_{78,0}}{W_{78,0}} \; \frac{i\delta_{12,0}}{W_{12,0}} \; d\tilde{k}_1 \cdots d\tilde{k}_4 \, d\tilde{k}_7 \, d\tilde{k}_8 + O(1). \cdots (42)$$

It is clear that

$$(42) = -(29) + O(1).$$

If one considers

$$\left\langle b_1^{\lambda'}(k')\, {}_3a_3^{\lambda}(k) + b_1^{\lambda}(k)\, {}_3a_3^{\lambda'}(k') \right\rangle_{\lambda'=-\lambda}$$

$$= 2\beta \sum_{\lambda_1 \cdots \lambda_7}^{00'} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\lambda\lambda_1\lambda_2} \, L_{k_1 k_3 k_4 k_5}^{\lambda_1\lambda_3\lambda_4\lambda_5} \, L_{k' k_7 k_8}^{\lambda'\lambda_7\lambda_8} \left\langle G_{02}^{\lambda_2} G_{03}^{\lambda_3} G_{04}^{\lambda_4} G_{05}^{\lambda_5} G_{07}^{\lambda_7} G_{08}^{\lambda_8} \right\rangle$$

$$\frac{\Delta_{2345,0} - \Delta_{12,0}}{iW_{345,1}} \; \Delta_{78,0'} \, \delta_{78,0'} \, \delta_{345,1} \, \delta_{12,0} \, d\tilde{k}_1 \cdots d\tilde{k}_7 .$$

The only "t" growths which can occur would come from a $(2 \times 2 \times 2)$ decomposition of the type $\delta_{27}\,\delta_{35}\,\delta_{48}$ $\phi_0^{(2)\lambda_2\lambda_7}(k_2)\, \phi_0^{(2)\lambda_3\lambda_5}(k_3)\, \phi_0^{(2)\lambda_4\lambda_8}(k_4)$. After integration the above integrand contains a time "t" dependent term of the form

$$\frac{\Delta(\omega) - \Delta(\omega)}{-i(\omega - \omega)} \, \Delta(\omega) \sim -\frac{t}{\omega^2} + O(1),$$

with $\omega = \lambda_1\omega_1 + \lambda_2\omega_2 - \lambda\omega$. See Appendix II.

If triad resonances were allowed there would also be a "t^2" term. The fact that $\lambda_1\omega_1 + \lambda_2\omega_2 - \lambda\omega$ is never zero in the spectrum allows one to write

$$\int_{-\infty}^{\infty} f_\omega \frac{1}{\omega}\, d\omega = - \int_{-\infty}^{\infty} f' \frac{1}{\omega^2}\, d\omega .$$

However on application of the permutation $\lambda \to -\lambda$ and $\tilde{k} \to -\tilde{k}$ and adding the two terms the "t" growths cancel.

One now examines the contributions which arise from the inclusion of the free terms.

$$\langle \beta_2^{\lambda'}(k')\, b_2^{\lambda}(k) + \beta_2^{\lambda}(k)\, b_2^{\lambda'}(k') \rangle_{\lambda'=-\lambda}$$

$$= 2\rho^{00'} \sum_{\lambda_1 \cdots \lambda_5} \int_{-\infty}^{\infty} L_{kk_1k_2k_3}^{\lambda\lambda_1\lambda_2\lambda_3}\, L_{k'k_4k_5}^{\lambda'\lambda_4\lambda_5} \langle a_{01}^{\lambda_1} a_{02}^{\lambda_2} a_{03}^{\lambda_3} a_{04}^{\lambda_4} \alpha_{15}^{\lambda_5} \rangle$$

$$\Delta_{123,0}\, \Delta_{45,0'}\, \delta_{123,0}\, \delta_{45,0'}\, d\tilde{k}_1 \cdots d\tilde{k}_5 .$$

The fifth order mean value decomposes into products of (2 X 3) zeroth order spectral cumulants. Clearly the only "t" growths which occur come from the breaks which make

$$\lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3 - \lambda\omega \equiv 0 .$$

These decompositions are

$$\delta(\breve{k}_1+\breve{k}_2)\, \overset{(2)}{\Phi}{}_0^{\nu_1\nu_2}(k_1)\ \delta(\breve{k}_3+\breve{k}_4+\breve{k}_5)\ A_{aa\alpha}^{\nu_3\nu_4\nu_5}(k_3,k_4),$$

$$\delta(\breve{k}_1+\breve{k}_3)\, \overset{(2)}{\Phi}{}_0^{\nu_1\nu_3}(k_1)\ \delta(\breve{k}_2+\breve{k}_4+\breve{k}_5)\ A_{aa\alpha}^{\nu_2\nu_4\nu_5}(k_2,k_4)$$

and

$$\delta(\breve{k}_2+\breve{k}_3)\, \overset{(2)}{\Phi}{}_0^{\nu_2\nu_3}(k_2)\ \delta(\breve{k}_1+\breve{k}_4+\breve{k}_5)\ A_{aa\alpha}^{\nu_1\nu_4\nu_5}(k_1,k_4).$$

Let $G\{f\}$ denote the coefficient of t in the long time behavior of $\{f\}$.

$$G_t \left< \beta_2^{\nu'}(k')\, b_2^{\nu}(k) + \beta_2^{\nu}(k)\, b_2^{\nu'}(k') \right>_{\nu'=-\nu}$$

$$= 6\,\delta_{oo'}\, \overset{oo'}{P} \sum_{\nu_1\nu_4\nu_5} \int_{-\infty}^{\infty} \mathcal{L}_{k\,k_1\,-k_1\,k}^{\nu\,\nu_1\,-\nu_1\,\nu}\ \mathcal{L}_{-k\,k_4\,k_5}^{-\nu\,\nu_4\,\nu_5}\ \overset{(2)}{\Phi}{}_0^{\nu_1-\nu_1}(k_1)$$

$$A_{aa\alpha}^{\nu\,\nu_4\nu_5}(k,k_4)\ \frac{i\Delta_{450}}{W_{450}}\ d\breve{k}_1\, d\breve{k}_4\, d\breve{k}_5,$$

Since $\Delta_{450} \sim \dfrac{i}{W_{450}}$,

$$= 6\,\delta_{oo'}\, \overset{o-o}{P} \sum_{\nu_1} \int_{-\infty}^{\infty} \mathcal{L}_{k\,k_1\,-k_1\,k}^{\nu\,\nu_1\,-\nu_1\,\nu}\ \overset{(2)}{\Phi}{}_0^{\nu_1-\nu_1}(k_1)\, d\breve{k}_1$$

$$\times \sum_{\nu_4\nu_5} \int_{-\infty}^{\infty} \mathcal{L}_{-k\,k_4\,k_5}^{-\nu\,\nu_4\,\nu_5}\ A_{aa\alpha}^{\nu\,\nu_4\nu_5}(k,k_4)\ \frac{i\Delta_{450}}{W_{450}}\ d\breve{k}_4\, d\breve{k}_5. \quad \cdots(43)$$

$$\delta(\bar{k}+\bar{k}_4+\bar{k}_5)\, A^{\lambda\,\lambda_4\,\lambda_5}_{aa\alpha\,(k,k_4)} = \langle a_0^{\lambda}(k)\, a_0^{\lambda_4}(k_4)\, \alpha_1^{\lambda_5}(k_5)\rangle .$$

$$\langle \beta_2^{\lambda'}(k')\, c_2^{\lambda}(k) + \beta_2^{\lambda}(k)\, c_2^{\lambda'}(k')\rangle_{1'=-\lambda}$$

$$= 4 \overset{00'}{P} \sum_{\lambda_1 \cdots \lambda_6} \int_{-\infty}^{\infty} L^{\lambda\,\lambda_1\,\lambda_2}_{k\,k_1\,k_2}\, L^{\lambda_1\,\lambda_3\,\lambda_4}_{k_1\,k_3\,k_4}\, L^{\lambda'\,\lambda_5\,\lambda_6}_{k'\,k_5\,k_6} \langle a_{02}^{\lambda_2} a_{03}^{\lambda_3} a_{04}^{\lambda_4} a_{05}^{\lambda_5} \alpha_{16}^{\lambda_6}\rangle$$

$$\frac{\Delta_{234,0} - \Delta_{12,0}}{i W_{34,1}}\, \Delta_{45,0'}\, \delta_{34,1}\, \delta_{12,0}\, \delta_{45,0'}$$

$$d\bar{k}_1 \cdots d\bar{k}_6 .$$

From similar reasoning

$$G_t \langle \beta_2^{\lambda'}(k')\, c_2^{\lambda}(k) + \beta_2^{\lambda}(k)\, c_2^{\lambda'}(k')\rangle_{1'=-\lambda}$$

$$= \delta\, \delta_{00'} \overset{0-0}{P} \sum_{\lambda_1\lambda_2} \int_{-\infty}^{\infty} L^{\lambda\,\lambda_1\,\lambda_2}_{k\,k_1\,k_2}\, L^{\lambda_1\,\lambda\,-\lambda_2}_{k_1\,k-k_2}\, \overset{(2)}{Q}{}_0^{\lambda_2-\lambda_2}(k_2)\, \frac{i\delta_{12,0}}{W_{12,0}}\, d\bar{k}_1 d\bar{k}_2$$

$$\times \sum_{\lambda_4\lambda_5} \int_{-\infty}^{\infty} L^{-\lambda\,\lambda_4\,\lambda_5}_{k\,k_4\,k_5}\, A^{\lambda\,\lambda_4\,\lambda_5}_{aa\alpha\,(k,k_4)}\, \frac{i\delta_{450}}{W_{450}}\, d\bar{k}_4 d\bar{k}_5 .$$

---- (44).

$$G_t \langle \alpha_2^{\nu'}(k')\, b_2^{\nu}(k) + \alpha_2^{\nu}(k)\, b_2^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$= G_t \overset{00'}{P} \sum_{\nu_1\nu_2\nu_3} \int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\nu \nu_1 \nu_2 \nu_3} \langle \alpha_{2'}^{\nu'} a_{01}^{\nu_1} a_{02}^{\nu_2} a_{03}^{\nu_3} \rangle \Delta_{123,0}\, \delta_{123,0}\, dk_1 dk_2 dk_3 .$$

The only "t" growths which occur arise from the breaks which can make (with suitable choices of the sign parameters) the expression

$$\nu_1\omega_1 + \nu_2\omega_2 + \nu_3\omega_3 - \nu\omega \equiv 0 .$$

Hence

$$G_t \langle \alpha_2^{\nu'}(k')\, b_2^{\nu}(k) + \alpha_2^{\nu}(k)\, b_2^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$= 3\delta_{00'} \overset{00'}{P} B_{\alpha a}^{\nu'\nu}(k') \sum_{\nu_2} \int_{-\infty}^{\infty} L_{k k k_2 -k_2}^{\nu \nu \nu_2 -\nu_2} Q_0^{(2)\,\nu_2 -\nu_2}(k_2)\, dk_2 ,$$

when

$$\delta(k+k')\, B_{\alpha a}^{\nu'\nu}(k') = \langle \alpha_2^{\nu'}(k')\, a_0^{\nu}(k) \rangle = \delta(k+k')\, B_{a\alpha}^{\nu\nu'}(k).$$

Similarly,

$$G_t \langle \alpha_2^{\nu'}(k')\, c_2^{\nu}(k) + \alpha_2^{\nu}(k)\, c_2^{\nu'}(k') \rangle$$

$$= 4\,\delta_{oo'}\,\overset{oo'}{P}\,\mathcal{B}^{\nu'\nu}_{\alpha a}(k')\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L^{\nu\nu_1\nu_2}_{k k_1 k_2}\, L^{\nu_1\,\nu-\nu_2}_{k_1 k -k_2}\,\overset{(2)}{\phi}{}^{\nu_2-\nu_2}_{0}(k_2)\,\frac{i\delta_{12,0}}{w_{12,0}}\,d\breve{k}_1 d\breve{k}_2.$$

Adding the two latter contributions, one obtains

$$G_t \left\langle \alpha_2^{\nu'}(k')\left(b_2^{\nu}(k)+c_2^{\nu}(k)\right)+\alpha_2^{\nu}(k)\left(b_2^{\nu'}(k')+c_2^{\nu'}(k')\right)\right\rangle_{\nu'=-\nu}$$

$$= \delta_{oo'}\,\overset{oo'}{P}\,\mathcal{B}^{\nu'\nu}_{\alpha a}(k')\sum_{\nu_2}\int_{-\infty}^{\infty} F^{\nu\nu\,\nu_2-\nu_2}_{k k\, k_2 -k_2}\,\overset{(2)}{\phi}{}^{\nu_2-\nu_2}_{0}(k_2)\,d\breve{k}_2, \quad \cdots (45)$$

where

$$F^{\nu\nu\,\nu_2-\nu_2}_{k k\, k_2 -k_2} = 3\, L^{\nu\nu\,\nu_2-\nu_2}_{k k\, k_2 -k_2} + 4\sum_{\nu_1}\int_{-\infty}^{\infty} L^{\nu\nu_1\nu_2}_{k k_1 k_2}\, L^{\nu_1\,\nu-\nu_2}_{k_1 k -k_2}\,\frac{i\delta_{12,0}}{w_{12,0}}\,d\breve{k}_1.$$

$$\langle \beta_2^{\nu}(k)\,\beta_2^{\nu'}(k')\rangle_{\nu'=-\nu} \sim O(1),$$ as no triad resonances exist.

$$\langle \beta_2^{\nu}(k)\,\alpha_2^{\nu'}(k') + \beta_2^{\nu'}(k')\,\alpha_2^{\nu}(k)\rangle \sim O(1).$$

$$\langle \alpha_2^{\nu'}(k')\,\alpha_2^{\nu}(k)\rangle \sim O(1).$$

$$\langle \alpha_1^{\nu'}(k')\, b_3^{\nu}(k) + \alpha_1^{\nu}(k)\, b_3^{\nu'}(k') \rangle_{\nu'=-\nu} \sim O(1),$$

Since $\nu_1\omega_1 + \nu_2\omega_2 + \nu_3\omega_3 + \nu_4\omega_4 - \nu\omega \not\equiv 0.$

$$\langle \alpha_1^{\nu'}(k')\, C_3^{\nu}(k) + \alpha_1^{\nu}(k)\, C_3^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$= 3 P^{00'} \sum_{\nu_1\cdots\nu_5} \int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\nu \nu_1 \nu_2 \nu_3} L_{k_1 k_4 k_5}^{\nu_1 \nu_4 \nu_5} \langle \alpha_1^{\nu'} a_{02}^{\nu_2} a_{03}^{\nu_3} a_{04}^{\nu_4} a_{05}^{\nu_5} \rangle$$

$$\frac{\Delta_{2345,0} - \Delta_{123,0}}{i W_{45,1}} \delta_{45,1}\, \delta_{123,0}\, dk_1 \cdots dk_5 .$$

There are two types of "t" growths which are possible.

(i) The decompositions $\delta(k'+k_2)\,\delta(k_3+k_4+k_5)\, A_{\alpha a}^{\nu'\nu_2}(k')\, \overset{(3)}{Q}_0^{\nu_4\nu_5\nu_3}(k_4,k_5)$ and $\delta(k'+k_3)\,\delta(k_2+k_4+k_5)\, A_{\alpha a}^{\nu'\nu_3}(k')\, \overset{(3)}{Q}_0^{\nu_4\nu_5\nu_2}(k_4,k_5)$.

(ii) The decomposition $\delta(k_2+k_3)\,\delta(k'+k_4+k_5)\, \overset{(2)}{Q}_0^{\nu_2\nu_3}(k_2)\, A_{\alpha a a}^{\nu'\nu_4\nu_5}(k',k_4)$.

Both these decompositions, after integration and suitable choice of the sign parameters, reduce the time "t" dependent terms to a form

$$\frac{\Delta(\omega) - \Delta(0)}{i\omega} \sim \frac{it}{\omega} + O(1),$$

when ω is a triad and therefore cannot vanish anywhere in the spectrum. Hence

$$G_t \langle \alpha_1^{\nu'} C_3^{\nu} + \alpha_1^{\nu} C_3^{\nu'} \rangle_{\nu'=-\nu}$$

$$= 6\delta_{00'} \overset{00'}{P} A_{\alpha a}^{\nu'\nu}(k') \sum_{\nu_1\nu_4\nu_5} \int_{-\infty}^{\infty} L_{k k_1 k-k_1}^{\nu \nu_1 \nu-\nu_1} L_{k_1 k_4 k_5}^{\nu_1 \nu_4 \nu_5} \overset{(3)}{\phi_0}{}^{\nu_4 \nu_5 -\nu_1}(k_4,k_5) \frac{i\delta_{45,1}}{W_{45,1}} d\check{k}_1 d\check{k}_4 d\check{k}_5$$

$$+ 3\delta_{00'} \overset{00'}{P} \sum_{\nu_2} \int_{-\infty}^{\infty} L_{k k_2 k_2-k_2}^{\nu \nu \nu_2 -\nu_2} \overset{(2)}{\phi_0}{}^{\nu_2 -\nu_2}(k_2) d\check{k}_2 \sum_{\nu_4\nu_5} \int_{-\infty}^{\infty} L_{k k_4 k_5}^{\nu \nu_4 \nu_5} A_{\alpha a a}^{-\nu \nu_4 \nu_5}(-k, k_4) \frac{i\delta_{45,0}}{W_{45,0}} d\check{k}_4 d\check{k}_5.$$

$$\text{-- --- } (46) + (47).$$

Next, one considers,

$$\langle \alpha_1^{\nu'}(k')\, {}_2 a_3^{\nu}(k) + \alpha_1^{\nu}(k)\, {}_2 a_3^{\nu'}(k') \rangle$$

$$= 4\overset{00'}{P} \sum_{\nu_1 \cdots \nu_6} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2} L_{k_1 k_3 k_4}^{\nu_1 \nu_3 \nu_4} L_{k_3 k_5 k_6}^{\nu_3 \nu_5 \nu_6} \langle \alpha_1^{\nu'} a_{02}^{\nu_2} a_{04}^{\nu_4} a_{05}^{\nu_5} a_{06}^{\nu_6} \rangle$$

$$\int_0^t \frac{\Delta_{456,1} - \Delta_{34,1}}{iW_{56,3}} e^{iW_{12,0}t} dt\; \delta_{56,3}\, \delta_{34,1}\, \delta_{12,0}\, d\check{k}_1 \cdots d\check{k}_6.$$

The time "t" dependent term in the integrand is of the form

$$\int_0^t \frac{\Delta(\omega) - \Delta(\mu)}{i(\omega-\mu)} e^{i\nu t} dt.$$

This gives a "t" growth only when $\mu = -\nu$. This occurs on the decompositions

$$\delta(k'+k_2)\,\delta(k_4+k_5+k_6)\,A_{\alpha a}^{\gamma'\gamma_2}(k')\,\phi_0^{(3)\,\gamma_5\gamma_6\gamma_4}(k_5,k_6)$$

and $$\delta(k_2+k_4)\,\delta(k'+k_5+k_6)\,\phi_0^{(2)\,\gamma_2\gamma_4}(k_2)\,A_{\alpha a a}^{\gamma'\gamma_5\gamma_6}(k',k_5).$$

The "t" growth contribution of the first decomposition is

$$4t\,\delta_{00'}\,\rho^{00'}\,A_{\alpha a}^{\gamma'\gamma}(k')\sum_{\gamma_1\gamma_2\gamma_5\gamma_6}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\gamma\gamma_1\gamma_2}\,L_{k_1 -k_2 k}^{\gamma_1 -\gamma_2 \gamma}\,L_{-k_2 k_5 k_6}^{-\gamma_2\gamma_5\gamma_6}$$

$$\phi_0^{(3)\,\gamma_5\gamma_6-\gamma_2}(k_5,k_6)\,\frac{i\delta_{562}}{W_{562}}\,\frac{i\delta_{12,0}}{W_{12,0}}\,d\breve{k}_1\,d\breve{k}_2\,d\breve{k}_5\,d\breve{k}_6. \qquad ---(48)$$

The second contribution is

$$4t\,\delta_{00'}\,\rho^{00'}\sum_{\gamma_1\gamma_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\gamma\gamma_1\gamma_2}\,L_{k_1 -k_2 k}^{\gamma_1 -\gamma_2 \gamma}\,\phi_0^{(2)\,\gamma_2-\gamma_2}(k_2)\,\frac{i\delta_{12,0}}{W_{12,0}}\,d\breve{k}_1\,d\breve{k}_2$$

$$\times\sum_{\gamma_5\gamma_6}\int_{-\infty}^{\infty} L_{k k_5 k_6}^{\gamma\gamma_5\gamma_6}\,A_{\alpha a a}^{-\gamma\gamma_5\gamma_6}(-k,k_5)\,\frac{i\delta_{56,0}}{W_{56,0}}\,d\breve{k}_5\,d\breve{k}_6. \qquad --(49)$$

$$\langle \alpha_1^{\nu'}(k'),\, \alpha_3^{\nu}(k) + \alpha_1^{\nu}(k),\, \alpha_3^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$= \rho^{00'} \sum_{\nu_1 \cdots \nu_6} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2} L_{k_1 k_3 k_4}^{\nu_1 \nu_3 \nu_4} L_{k_2 k_5 k_6}^{\nu_2 \nu_5 \nu_6} \langle \alpha_{1'}^{\nu'} a_{03}^{\nu_3} a_{04}^{\nu_4} a_{05}^{\nu_5} a_{06}^{\nu_6} \rangle$$

$$\int_0^t \Delta_{34,1} \Delta_{56,2} e^{i\omega_{12,0} t} dt \, \delta_{34,1} \delta_{56,2} \delta_{12,0} \, dk_1 \cdots dk_6 .$$

The term in the integrand which is a function of the fast time "t" is of the form $\int_0^t \Delta(\omega) \Delta(\mu) e^{i\nu t} dt$, and only gives a "t" growth when either ω or $\mu = -\nu$. This occurs for the decomposition

$$\delta_{0'3} \delta_{456} A_{\alpha a}^{\nu' \nu_3}(k') Q_0^{(3)\, \nu_5 \nu_6 \nu_4}(k_5, k_6), \quad \delta_{0'4} \delta_{356} A_{\alpha a}^{\nu' \nu_4}(k') Q_0^{(3)\, \nu_5 \nu_6 \nu_3}(k_5, k_6),$$

$$\delta_{0'5} \delta_{346} A_{\alpha a}^{\nu' \nu_5}(k') Q_0^{(3)\, \nu_3 \nu_4 \nu_6}(k_3, k_4) \text{ and } \delta_{0'6} \delta_{345} A_{\alpha a}^{\nu' \nu_6}(k') Q_0^{(3)\, \nu_3 \nu_4 \nu_5}(k_3, k_4).$$

Since there is symmetry between (3,4) and (5,6) , the four terms give the same result.

$$G_t \langle \alpha_{1'}^{\nu'},\, \alpha_3^{\nu} + \alpha_1^{\nu},\, \alpha_{3'}^{\nu'} \rangle_{\nu'=-\nu}$$

$$= 4 \rho^{00'} \delta_{00'} A_{\alpha a}^{\nu' \nu}(k') \sum_{\nu_1 \nu_2 \nu_5 \nu_6} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2} L_{k_1 k -k_2}^{\nu_1 \nu -\nu_2}$$

$$\mathcal{L}_{k_2\, k_5\, k_6}^{\lambda_2\, \lambda_5\, \lambda_6}\ \Phi_0^{(3)\, \lambda_5\, \lambda_6\, -\lambda_2}(k_5, k_6)\ \frac{i\delta_{56,2}}{W_{56,2}}\ \frac{i\delta_{12,0}}{W_{12,0}}\ d\tilde{k}_1\, d\tilde{k}_2\, d\tilde{k}_5\, d\tilde{k}_6.$$

------ (50).

$$\langle \alpha_1^{\lambda'}(k')\, {}_3\alpha_3^{\lambda}(k) + \alpha_1^{\lambda}(k)\, {}_3\alpha_3^{\lambda'}(k') \rangle \sim O(1),$$

as $E(\omega, \omega) \sim O(1)$.

Next, one considers

$$\langle b_1^{\lambda'}(k')\, \beta_3^{\lambda}(k) + b_1^{\lambda}(k)\, \beta_3^{\lambda'}(k') \rangle_{1\frac{1}{2}-\lambda}$$

$$= 3P^{00'} \sum_{\lambda_1 \cdots \lambda_5} \int_{-\infty}^{\infty} \mathcal{L}_{k\, k_1\, k_2\, k_3}^{\lambda\, \lambda_1\, \lambda_2\, \lambda_3}\ \mathcal{L}_{k'\, k_4\, k_5}^{\lambda'\, \lambda_4\, \lambda_5}\ \langle a_{11}^{\lambda_1}\, a_{02}^{\lambda_2}\, a_{03}^{\lambda_3}\, a_{04}^{\lambda_4}\, a_{05}^{\lambda_5} \rangle$$

$$\Delta_{123,0}\ \Delta_{45,0'}\ \delta_{123,0}\ \delta_{45,0'}\ d\tilde{k}_1 \cdots d\tilde{k}_5.$$

A "t" growth can occur only when

$$\lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3 - \lambda\omega = 0.$$

This happens on the decompositions

$$\delta_{12}\ \delta_{345}\ A_{\alpha\alpha}^{\lambda_1\, \lambda_2}(k_1)\ \Phi_0^{(3)\, \lambda_4\, \lambda_5\, \lambda_3}(k_4, k_5),\ \delta_{13}\ \delta_{245}\ A_{\alpha\alpha}^{\lambda_1\, \lambda_3}(k_1)\ \Phi_0^{(3)\, \lambda_4\, \lambda_5\, \lambda_2}(k_4, k_5)$$

and $\delta_{23}\ \delta_{145}\ \Phi_0^{(2)\, \lambda_2\, \lambda_3}(k_2)\ A_{\alpha\alpha\alpha}^{\lambda_4\, \lambda_5\, \lambda_1}(k_4, k_5).$

The first two terms give a "t" growth contribution which is

$$6t\,\rho^{00'}\delta_{00'}\sum_{\nu_1\nu_4\nu_5}\int_{-\infty}^{\infty} L^{\nu\,\nu_1\,-\nu_1\,\nu}_{k\,k_1\,-k_1\,k}\, L^{\nu'\,\nu_4\,\nu_5}_{k'\,k_4\,k_5}\, A^{\nu_1\,-\nu_1}_{\alpha\alpha}(k_1)\, Q^{(3)\,\nu_4\,\nu_5\,\nu}_{0}(k_4,k_5)\,\frac{i\delta_{450}}{w_{450}}\,dk_1\,dk_4\,dk_5. \quad \cdots (51).$$

The second gives a contribution

$$3t\,\rho^{00'}\delta_{00'}\sum_{\nu_2}\int_{-\infty}^{\infty} L^{\nu\,\nu\,\nu_2\,-\nu_2}_{k\,k\,k_2\,-k_2}\, Q^{(2)\,\nu_2\,-\nu_2}_{0}(k_2)\,dk_2$$

$$\times\sum_{\nu_4\nu_5}\int_{-\infty}^{\infty} L^{\nu'\,\nu_4\,\nu_5}_{k'\,k_4\,k_5}\, A^{\nu\,\nu_4\,\nu_5}_{\alpha\alpha\alpha}(k,k_4)\,\frac{i\delta_{450}}{w_{450}}\,dk_4\,dk_5. \quad \cdots (52)$$

$$\langle l_1^{\nu'}(k')\,{}_1\gamma_3^{\nu}(k) + l_1^{\nu}(k)\,{}_1\gamma_3^{\nu'}(k')\rangle_{\nu'=-\nu} \sim O(1),$$

as there are no triad resonances.

$$\langle l_1^{\nu'}(k')\,{}_2\gamma_3^{\nu}(k) + l_1^{\nu}(k)\,{}_2\gamma_3^{\nu'}(k')\rangle_{\nu'=-\nu}$$

$$= 4\,\rho^{00'}\sum_{\nu_1-\nu_6}\int_{-\infty}^{\infty} L^{\nu\,\nu_1\,\nu_2}_{k\,k_1\,k_2}\, L^{\nu_1\,\nu_3\,\nu_4}_{k_1\,k_3\,k_4}\, L^{\nu'\,\nu_5\,\nu_6}_{k'\,k_5\,k_6}\,\langle a_{02}^{\nu_2}\,a_{03}^{\nu_3}\,\alpha_{14}^{\nu_4}\,a_{05}^{\nu_5}\,a_{06}^{\nu_6}\rangle$$

$$\frac{\Delta_{234,0}-\Delta_{12,0}}{iw_{34,1}}\,\Delta_{56,0'}\,\delta_{34,1}\,\delta_{56,0'}\,\delta_{12,0}\,dk_1\cdots dk_6.$$

"t" growths can only occur when

$$\nu_2\omega_2 + \nu_3\omega_3 + \nu_4\omega_4 - \nu\omega = 0.$$

This arises on the decompositions

$$\delta(k_2+k_3)\;\delta(k_4+k_5+k_6)\;\overset{(2)}{\phi}{}_0^{\nu_2\nu_3}(k_2)\;A_{\alpha aa}^{\nu_4\nu_5\nu_6}(k_4,k_5)$$

and

$$\delta(k_2+k_4)\;\delta(k_3+k_5+k_6)\;\overset{(3)}{\phi}{}_0^{\nu_5\nu_6\nu_3}(k_5,k_6)\;A_{a\alpha}^{\nu_2\nu_4}(k_2).$$

The first term leads to a "t" growth contribution,

$$4t\,\overset{00'}{\rho}\,\delta_{00'}\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\,L_{k_1 k-k_2}^{\nu_1\nu-\nu_2}\,\overset{(2)}{\phi}{}_0^{\nu_2-\nu_1}(k_2)\,\frac{i\delta_{12,0}}{w_{12,0}}\,d\tilde{k}_1\cdot d\tilde{k}_2$$

$$\times\sum_{\nu_5\nu_6}\int_{-\infty}^{\infty} L_{k' k_5 k_6}^{\nu'\nu_5\nu_6}\,A_{\alpha aa}^{\nu\nu_5\nu_6}(k,k_5)\,\frac{i\delta_{560}}{w_{560}}\,d\tilde{k}_5\,d\tilde{k}_6. \quad ---(53)$$

The second gives the contribution,

$$4t\,\overset{00'}{\rho}\,\delta_{00'}\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu\nu_1\nu_2}\,L_{k_1 k-k_2}^{\nu_1\nu-\nu_2}\,A_{a\alpha}^{\nu_2-\nu_2}(k_2)\,\frac{i\delta_{12,0}}{w_{12,0}}\,d\tilde{k}_1\,d\tilde{k}_2$$

$$\times\sum_{\nu_5\nu_6}\int_{-\infty}^{\infty} L_{k' k_5 k_6}^{\nu'\nu_5\nu_6}\,\overset{(3)}{\phi}{}_0^{\nu_5\nu_6\nu_3}(k_5,k_6)\,\frac{i\delta_{560}}{w_{560}}\,d\tilde{k}_5\,d\tilde{k}_6. \quad --(54).$$

A similar analysis shows that

$$G_t \langle b_1^{\nu'}(k')\,{}_3\gamma_3^{\nu}(k) + b_1^{\nu}(k)\,{}_3\gamma_3^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$= 4\,\rho^{00'}\,\delta_{00'} \sum_{\nu_1\nu_2} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2}\, L_{k_1 k - k_2}^{\nu_1 \nu - \nu_2}\, A_{\alpha\alpha}^{\nu_2 - \nu_2}(k_2)\, \frac{i\delta_{12,0}}{W_{12,0}}\, dk_1\, dk_2$$

$$\times \sum_{\nu_5\nu_6} \int_{-\infty}^{\infty} L_{k' k_5 k_6}^{\nu' \nu_5 \nu_6}\, Q_0^{(3)\,\nu_5 \nu_6 \nu}(k_5, k_6)\, \frac{i\delta_{560}}{W_{560}}\, dk_5\, dk_6. \quad \cdots (55).$$

$$\langle b_1^{\nu'}(k')\,{}_4\gamma_3^{\nu}(k) + b_1^{\nu}(k)\,{}_4\gamma_3^{\nu'}(k') \rangle_{\nu'=-\nu} \approx O(1),$$

as no triad resonances exist.

$$\langle \alpha_1^{\nu'}(k')\,\beta_3^{\nu}(k) + \alpha_1^{\nu}(k)\,\beta_3^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$= 3\rho^{00'} \sum_{\nu_1\nu_2\nu_3} \int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\nu \nu_1 \nu_2 \nu_3} \langle \alpha_{1'}^{\nu'}\, \alpha_{11}^{\nu_1}\, a_{02}^{\nu_2}\, a_{03}^{\nu_3} \rangle\, \Delta_{123,0}\, \delta_{123,0}\, dk_1\, dk_2\, dk_3.$$

A "t" growth occurs when

$$\nu_1\omega_1 + \nu_2\omega_2 + \nu_3\omega_3 - \nu\omega \equiv 0.$$

If one defines

$$\langle \alpha_1^{\nu}(k)\, \alpha_1^{\nu'}(k') \rangle = \delta(k+k')\, a^{\nu\nu'}(k),$$

one obtains

$$G_t \langle \alpha_1^{\gamma'}(k')\beta_3^{\gamma}(k) + \alpha_1^{\gamma}(k)\beta_3^{\gamma'}(k')\rangle_{\gamma'=-\gamma}$$

$$= 3\rho^{00'}\delta_{00'}\,Q^{\gamma'\gamma}(k')\sum_{\gamma_2}\int_{-\infty}^{\infty} L_{k\,k\,k_2\,-k_2}^{\gamma\,\gamma\,\gamma_2\,-\gamma_2}\,Q_0^{(2)\,\gamma_2-\gamma_2}(k_2)\,d\tilde{k}_2 \qquad (56)$$

$$+ 6\rho^{00'}\delta_{00'}\,A_{\alpha\alpha}^{\gamma'\gamma}(k')\sum_{\gamma_2}\int_{-\infty}^{\infty} L_{k\,k\,k_2\,-k_2}^{\gamma\,\gamma\,\gamma_2\,-\gamma_2}\,A_{\alpha\alpha}^{\gamma_2-\gamma_2}(k_2)\,d\tilde{k}_2. \qquad (57)$$

Similarly,

$$G_t \langle \alpha_1^{\gamma'}(k')\left({}_2\gamma_3^{\gamma}(k) + {}_3\gamma_3^{\gamma}(k)\right) + \alpha_1^{\gamma}(k)\left({}_2\gamma_3^{\gamma'}(k') + {}_3\gamma_3^{\gamma'}(k')\right)\rangle_{\gamma'=-\gamma}$$

$$= 4\rho^{00'}\delta_{00'}\,Q^{\gamma'\gamma}(k')\sum_{\gamma_1\gamma_2}\int_{-\infty}^{\infty} L_{k\,k_1\,k_2}^{\gamma\,\gamma_1\,\gamma_2}\,L_{k_1\,k\,-k_2}^{\gamma_1\,\gamma\,-\gamma_2}\,Q_0^{(2)\,\gamma_2-\gamma_2}(k_2)\,\frac{i\delta_{12,0}}{W_{12,0}}\,d\tilde{k}_1 d\tilde{k}_2 \qquad (58).$$

$$+ 4\rho^{00'}\delta_{00'}\,A_{\alpha\alpha}^{\gamma'\gamma}(k')\sum_{\gamma_1\gamma_2}\int_{-\infty}^{\infty} L_{k\,k_1\,k_2}^{\gamma\,\gamma_1\,\gamma_2}\,L_{k_1\,k\,-k_2}^{\gamma_1\,\gamma\,-\gamma_2}\left(A_{\alpha\alpha}^{\gamma_2-\gamma_2}(k_2) + A_{\alpha\alpha}^{\gamma_2-\gamma_2}(k_2)\right)$$

$$\frac{i\delta_{12,0}}{W_{12,0}}\,d\tilde{k}_1 d\tilde{k}_2. \qquad (59).$$

Neither $\langle \alpha_1^{\nu'}(k'), \gamma_3^{\nu}(k) + \alpha_1^{\nu}(k), \gamma_3^{\nu'}(k') \rangle_{\nu'=-\nu}$

nor $\langle \alpha_1^{\nu'}(k')\, {}_4\gamma_3^{\nu}(k) + \alpha_1^{\nu}(k)\, {}_4\gamma_3^{\nu'}(k') \rangle_{\nu'=-\nu}$

exhibit any "t" growth.

Next one considers the term

$$\langle a_0^{\nu'}(k')\beta_4^{\nu}(k) + a_0^{\nu}(k)\beta_4^{\nu'}(k') \rangle \sim O(1)$$

since $s_1\omega_1 + s_2\omega_2 + s_3\omega_3 + s_4\omega_4 - s\omega \neq 0.$

One then considers

$$\langle a_0^{\nu'}(k'), \gamma_4^{\nu}(k) + a_0^{\nu}(k), \gamma_4^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$= 6\,\rho^{00'} \sum_{\nu_1 \ldots \nu_5} \int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\nu \nu_1 \nu_2 \nu_3} L_{k_1 k_4 k_5}^{\nu_1 \nu_4 \nu_5} \langle a_{0'}^{\nu'} a_{01}^{\nu_2} a_{03}^{\nu_3} a_{04}^{\nu_4} \alpha_{15}^{\nu_5} \rangle$$

$$\int_0^t \Delta_{45,1}\, e^{i w_{123,0} t}\, dt\; \delta_{123,0}\, \delta_{45,1}\, dk_1 \cdots dk_5.$$

"t" growths can only occur when $\nu_1\omega_1 + \nu_2\omega_2 + \nu_3\omega_3 - \nu\omega = 0.$

This occurs on the cumulant decompositions,

(1) $\delta(k'+k_2)\,\delta(k_3+k_4+k_5)$ and $\delta(k'+k_3)\,\delta(k_2+k_4+k_5)$

(2) $\delta(k'+k_4+k_5)\,\delta(k_1+k_3)$.

The first type of decomposition gives a term

$$12\,t\,\rho^{00'}\delta_{00'}\,\overset{(2)}{Q}{}_{\nu}^{\nu\nu'}(k') \sum_{\nu_1\nu_4\nu_5} \int_{-\infty}^{\infty} L_{k k_1 k -k_1}^{\nu \nu_1 \nu -\nu_1} L_{k_1 k_4 k_5}^{\nu_1 \nu_4 \nu_5} A_{aak}^{-\nu_1 \nu_4 \nu_5}(-k_1, k_4)$$

$$\frac{i\,\delta_{45,1}}{w_{45,1}}\, dk_1\, dk_4\, dk_5 + O(1). \qquad \ldots (60)$$

The second gives a term

$$6t\,\overset{00'}{\rho}\,\delta_{00'}\sum_{\nu_2}\int_{-\infty}^{\infty} L_{k\,k\,k_2\,-k_2}^{\nu\,\nu\,\nu_2\,-\nu_2}\,\overset{(2)}{\Phi}{}_0^{\nu_2-\nu_2}(k_2)\,d\breve{k}_2$$

$$\times\sum_{\nu_4\nu_5}\int_{-\infty}^{\infty} L_{k\,k_4\,k_5}^{\nu\,\nu_4\,\nu_5}\,A_{\alpha\alpha}^{-\nu\,\nu_4\,\nu_5}(-k,k_4)\,\frac{i\delta_{45,0}}{W_{45,0}}\,d\breve{k}_4\,d\breve{k}_5 + O(1). \quad (61)$$

One now considers $\langle a_0^{\nu'}(k')\,{}_2\gamma_4^{\nu}(k) + a_0^{\nu}(k)\,{}_2\gamma_4^{\nu'}(k')\rangle_{\nu'=-\nu}$

$$= 3\overset{00'}{\rho}\sum_{\nu_1\nu_2\nu_3}\int_{-\infty}^{\infty} L_{k\,k_1\,k_2\,k_3}^{\nu\,\nu_1\,\nu_2\,\nu_3}\,\langle a_{0'}^{\nu'}\,a_{21}^{\nu_1}\,a_{02}^{\nu_2}\,a_{03}^{\nu_3}\rangle\,\Delta_{123,0}\,\delta_{123,0}\,d\breve{k}_1 d\breve{k}_2 d\breve{k}_3$$

$$\sim 3t\,\overset{00'}{\rho}\,\delta_{00'}\,B_{\alpha\alpha}^{\nu'\nu}(k')\sum_{\nu_2}\int_{-\infty}^{\infty} L_{k\,k\,k_2\,-k_2}^{\nu\,\nu\,\nu_2\,-\nu_2}\,\overset{(2)}{\Phi}{}_0^{\nu_2-\nu_2}(k_2)\,d\breve{k}_2 \quad (62)$$

$$+ 6t\,\overset{00'}{\rho}\,\delta_{00'}\,\overset{(2)}{\Phi}{}_0^{\nu\nu'}(k)\sum_{\nu_2}\int_{-\infty}^{\infty} L_{k\,k\,k_2\,-k_2}^{\nu\,\nu\,\nu_2\,-\nu_2}\,B_{\alpha\alpha}^{\nu_2-\nu_2}(k_2)\,d\breve{k}_2 \quad (63)$$

$$+ O(1).$$

When one considers the next term one finds that

$$\langle a_0^{\nu'}(k')\, {}_3\gamma_4^{\nu}(k) + a_0^{\nu}(k)\, {}_3\gamma_4^{\nu'}(k') \rangle_{\nu'=-\nu}$$

$$= 6 P^{00'} \sum_{\nu_1 \cdots \nu_5} \int_{-\infty}^{\infty} L_{k k_1 k_2 k_3}^{\nu \nu_1 \nu_2 \nu_3} L_{k_2 k_4 k_5}^{\nu_2 \nu_4 \nu_5} \langle a_0^{\nu'} a_{01}^{\nu_1} a_{04}^{\nu_4} a_{05}^{\nu_5} \alpha_{13}^{\nu_3} \rangle$$

$$\int_0^t \Delta_{45,2}\, e^{i W_{123,0} t}\, dt\, \delta_{123,0}\, \delta_{45,2}\, dk_1 \cdots dk_5,$$

$$\sim 6t P^{00'} \delta_{00'} Q_0^{(2)\nu\nu'}(k) \sum_{\nu_2 \nu_4 \nu_5} \int_{-\infty}^{\infty} L_{k k k_2 -k_2}^{\nu \nu \nu_2 -\nu_2} L_{k_2 k_4 k_5}^{\nu_2 \nu_4 \nu_5} A_{aaa}^{\nu_4 \nu_5 -\nu_2}(k_4, k_5)$$

$$\frac{i \delta_{45,2}}{W_{45,2}}\, d\breve{k}_2\, d\breve{k}_4\, d\breve{k}_5 \qquad (64)$$

$$+ 6t P^{00'} \delta_{00'} A_{aa}^{\nu'\nu}(k') \sum_{\nu_2 \nu_4 \nu_5} \int_{-\infty}^{\infty} L_{k k k_2 -k_2}^{\nu \nu \nu_2 -\nu_2} L_{k_2 k_4 k_5}^{\nu_2 \nu_4 \nu_5} Q_0^{(3)\nu_4 \nu_5 -\nu_2}(k_4, k_5)$$

$$\frac{i \delta_{45,2}}{W_{45,2}}\, d\breve{k}_2\, d\breve{k}_4\, d\breve{k}_5 \qquad (65)$$

$$+ 6t P^{00'} \delta_{00'} \sum_{\nu_2} \int_{-\infty}^{\infty} L_{k k k_2 -k_2}^{\nu \nu \nu_2 -\nu_2} A_{aa}^{\nu_2 -\nu_2}(k_2)\, d\breve{k}_2$$

$$\times \sum_{\nu_4 \nu_5} \int_{-\infty}^{\infty} L_{k k_4 k_5}^{\nu \nu_4 \nu_5} Q_0^{(3)\nu_4 \nu_5 -\nu}(k_4, k_5)\, \frac{i \delta_{45,0}}{W_{45,0}}\, d\breve{k}_4\, d\breve{k}_5 \qquad (66)$$

$$+ O(1).$$

The next term $\langle a_0^{\nu'}(k')\,{}_4\gamma_4^{\nu}(k) + a_0^{\nu}(k)\,{}_4\gamma_4^{\nu'}(k')\rangle_{\nu'=-\nu}$

$$= 3\rho^{00'} \sum_{\nu_1\nu_2\nu_3} \int_{-\infty}^{\infty} L_{kk_1k_2k_3}^{\nu\nu_1\nu_2\nu_3} \langle a_{0'}^{\nu'} a_{01}^{\nu_1} \alpha_{12}^{\nu_2}\alpha_{13}^{\nu_3}\rangle \Delta_{123,0}\,\delta_{123,0}\, dk_1\, dk_2\, dk_3$$

$$\sim 3t\rho^{00'}\delta_{00'}\, Q_0^{(2)\nu\nu'}(k) \sum_{\nu_2}\int_{-\infty}^{\infty} L_{kkk_2-k_2}^{\nu\nu\nu_2-\nu_2}\, Q^{\nu_2-\nu_2}(k_2)\, dk_2 \qquad (67)$$

$$+6t\rho^{00'}\delta_{00'}\, A_{\alpha\alpha}^{\nu'\nu}(k') \sum_{\nu_1}\int_{-\infty}^{\infty} L_{kk_1k-k_1}^{\nu\nu_1\nu-\nu_1}\, A_{\alpha\alpha}^{\nu_1-\nu_1}(k_1)\, dk_1 \qquad (68)$$

$$+ O(1).$$

The term $\langle a_0^{\nu'}(k')\,{}_1\delta_4^{\nu}(k) + a_0^{\nu}(k)\,{}_1\delta_4^{\nu'}(k')\rangle_{\nu'=-\nu}$

$$= 2\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{kk_1k_2}^{\nu\nu_1\nu_2}\langle a_{0'}^{\nu'}\int_0^t b_{11}^{\nu_1}\beta_{22}^{\nu_2}\, e^{i\omega_{12,0}t}\,dt\rangle\, \delta_{12,0}\,dk_1\,dk_2,$$

$$= 4\sum_{\nu_1\cdots\nu_6}\int_{-\infty}^{\infty} L_{kk_1k_2}^{\nu\nu_1\nu_2}\, L_{k_1k_3k_4}^{\nu_1\nu_3\nu_4}\, L_{k_2k_5k_6}^{\nu_2\nu_5\nu_6}\langle a_{0'}^{\nu'}a_{03}^{\nu_3}a_{04}^{\nu_4}a_{05}^{\nu_5}\alpha_{16}^{\nu_6}\rangle$$

$$\int_0^t \Delta_{34,1}\,\Delta_{56,2}\, e^{i\omega_{12,0}t}\,dt\;\delta_{34,1}\,\delta_{56,2}$$

$$\delta_{12,0}\,dk_1\cdots dk_6.$$

The time dependent terms in this expression are of the form $\int_0^t \Delta(\omega)\,\Delta(\mu)\,e^{i\nu t}\,dt$ where ω, μ, ν are fixed and can only exhibit a "t" growth when either ω of $\mu \equiv -\nu$. This occurs on the decompositions $\delta_{0'3}\,\delta_{456}$, $\delta_{0'4}\,\delta_{356}$, $\delta_{0'5}\,\delta_{346}$ and $\delta_{0'6}\,\delta_{345}$.
Therefore, with a little manipulation taking the long time limit,

$$\left\langle a_0^{\eta'}(k'),\, \delta_4^{\eta}(k) + a_0^{\eta}(k),\, \delta_4^{\eta'}(k') \right\rangle_{\eta'=-\eta}$$

$$\sim 8t\,\delta_{00'}\, P^{00'}\, Q_0^{(2)\eta\eta'}(k) \sum_{\eta_1\eta_2\eta_5\eta_6} \int_{-\infty}^{\infty} L^{\eta\eta_1\eta_2}_{k k_1 k_2}\, L^{\eta_1\eta-\eta_2}_{k_1 k-k_2}\, L^{\eta_2\eta_5\eta_6}_{k_2 k_5 k_6}\, A^{\eta_5\eta_6-\eta_2}_{a\alpha a}(k_5, k_6)$$

$$\frac{i\delta_{12,0}}{w_{12,0}}\;\frac{i\delta_{56,2}}{w_{56,2}}\; dk_1\, dk_2\, dk_5\, dk_6 \qquad (69)$$

$$+\,4t\,\delta_{00'}\, P^{00'}\, Q_0^{(2)\eta\eta'}(k) \sum_{\eta_1\eta_2\eta_3\eta_4} \int_{-\infty}^{\infty} L^{\eta\eta_1\eta_2}_{k k_1 k_2}\, L^{\eta_1\eta_3\eta_4}_{k_1 k_3 k_4}\, L^{\eta_2\eta-\eta_1}_{k_2 k-k_1}$$

$$A^{\eta_3\eta_4-\eta_1}_{a a \alpha}(k_3, k_4)\;\frac{i\delta_{12,0}}{w_{12,0}}\;\frac{i\delta_{34,1}}{w_{34,1}}\; dk_1 \cdots dk_4 \qquad (70)$$

$$+\,4t\,P^{00'}\,\delta_{00'}\, A^{\eta'\eta}_{a\alpha}(k') \sum_{\eta_1\eta_2\eta_3\eta_4} \int_{-\infty}^{\infty} L^{\eta\eta_1\eta_2}_{k k_1 k_2}\, L^{\eta_1\eta_3\eta_4}_{k_1 k_3 k_4}\, L^{\eta_2\eta-\eta_1}_{k_2 k-k_1}$$

$$Q_0^{(3)\eta_3\eta_4-\eta_1}(k_3, k_4)\;\frac{i\delta_{12,0}}{w_{12,0}}\;\frac{i\delta_{34,1}}{w_{34,1}}\; dk_1 \cdots dk_4 \qquad (71)$$

$+ O(1).$

The term,

$$\langle a_0^{\gamma'}(k')\,{}_2\delta_4^{\gamma}(k) + a_0^{\gamma}(k)\,{}_2\delta_4^{\gamma'}(k')\rangle_{\gamma'=-\gamma}$$

$$= 2\rho^{00'} \sum_{\gamma_1 \cdots \gamma_4} \int_{-\infty}^{\infty} L_{kk_1k_2}^{\gamma\gamma_1\gamma_2} \langle a_{0'}^{\gamma'} \int_0^t b_{11}^{\gamma_1} a_{22}^{\gamma_2} e^{i\omega_{12,0}t}\,dt\rangle\, \delta_{12,0}\, dk_1 dk_2$$

$$= 2\rho^{00'} \sum_{\gamma_1 - \gamma_4} \int_{-\infty}^{\infty} L_{kk_1k_2}^{\gamma\gamma_1\gamma_2} L_{k_1k_3k_4}^{\gamma_1\gamma_3\gamma_4} \langle a_{0'}^{\gamma'} a_{22}^{\gamma_2} a_{03}^{\gamma_3} a_{04}^{\gamma_4}\rangle \frac{\Delta_{234,0} - \Delta_{12,0}}{i\omega_{34,1}}\, \delta_{34,1}\,\delta_{12,0}\, dk_1 \cdots dk_4,$$

$$\sim 4t\rho^{00'} f_{00'}\, Q_0^{(2)\gamma\gamma'}(k) \sum_{\gamma_1\gamma_2} \int_{-\infty}^{\infty} L_{kk_1k_2}^{\gamma\gamma_1\gamma_2} L_{k_1,k,-k_2}^{\gamma_1\gamma-\gamma_2} B_{ac}^{\gamma_2-\gamma_2}(k_2) \frac{\partial_{12,0}}{W_{12,0}}\, dk_1 dk_2 \quad (72)$$

$+ O(1).$

Clearly $\langle a_0^{\lambda'}(k')_3 \delta_4^{\lambda}(k) + a_0^{\lambda}(k)_3 \delta_4^{\lambda'}(k') \rangle_{\lambda'=-\lambda} \sim O(1)$,
since the time "t" dependent terms are $\dfrac{\Delta_{1345,0} - \Delta_{12,0}}{i\omega_{345,2}}$
which as shown in Appendix II has no "t" growth.

Consider next the term

$$\langle a_0^{\lambda'}(k')_4 \delta_4^{\lambda}(k) + a_0^{\lambda}(k)_4 \delta_4^{\lambda'}(k') \rangle_{\lambda'=-\lambda}$$

$$= 2 P^{00'} \sum_{\lambda_1\lambda_2} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\lambda \lambda_1 \lambda_2} \langle a_{0'}^{\lambda'} \int_0^t c_{22}^{\lambda_2} d_{11}^{\lambda_1} e^{i\omega_{12,0} t} dt \rangle \delta_{12,0}\, dk_1 dk_2,$$

$$= 4 P^{00'} \sum_{\lambda_1 \cdots \lambda_6} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\lambda \lambda_1 \lambda_2} L_{k_2 k_3 k_4}^{\lambda_2 \lambda_3 \lambda_4} L_{k_3 k_5 k_6}^{\lambda_3 \lambda_5 \lambda_6} \langle a_{0'}^{\lambda'} d_{11}^{\lambda_1} a_{04}^{\lambda_4} a_{05}^{\lambda_5} a_{06}^{\lambda_6} \rangle$$

$$\int_0^t \frac{\Delta_{456,2} - \Delta_{34,2}}{i\omega_{56,3}} e^{i\omega_{12,0} t} dt \; \delta_{456,2}\, \delta_{34,2}$$

$$\delta_{12,0}\, dk_1 \cdots dk_6 .$$

The time t dependent terms of this expression are
$\int_0^t \dfrac{\Delta(\omega) - \Delta(\mu)}{i(\omega-\mu)} e^{i\nu t} dt$ which can only give a "t" growth
when $\mu = -\nu$, since the terms when $\omega = 0$ is of the form
$\int \dfrac{t - \Delta(\mu)}{-i\mu} e^{i\nu t} dt$ which behaves like $\int_0^t t e^{i\nu t} dt$
and thus in an order one quantity for long time. The "t"
growth arising from the decomposition $\delta_{0'4}\, \delta_{156}$ gives

the term

$$4t\rho^{00'}\delta_{00'}\overset{(2)}{\phi}{}_{0}^{\gamma\gamma'}(k)\sum_{\gamma_1\gamma_2\gamma_5\gamma_6}\int_{-\infty}^{\infty}\mathcal{L}_{k\,k_1\,k_2}^{\gamma\,\gamma_1\,\gamma_2}\,\mathcal{L}_{k_2\,k-k_1}^{\gamma_2\,\gamma\,-\gamma_1}\,\mathcal{L}_{-k_1\,k_5\,k_6}^{-\gamma_1\,\gamma_5\,\gamma_6}$$

$$A_{\alpha\alpha}^{\gamma_1\gamma_5\gamma_6}(k_1,k_5)\ \frac{i\delta_{561}}{W_{561}}\ \frac{i\delta_{12,0}}{W_{12,0}}\ d\breve{k}_1 d\breve{k}_2 d\breve{k}_5 d\breve{k}_6 . \qquad (73)$$

The "t" growth associated with the decomposition $\delta_{14}\delta_{0'56}$ gives the term

$$4t\rho^{00'}\delta_{00'}\sum_{\gamma_1\gamma_2}\int_{-\infty}^{\infty}\mathcal{L}_{k\,k_1\,k_2}^{\gamma\,\gamma_1\,\gamma_2}\,\mathcal{L}_{k_2\,k-k_1}^{\gamma_2\,\gamma\,-\gamma_1}\,A_{\alpha\alpha}^{\gamma_1-\gamma_1}(k_1)\ \frac{i\delta_{12,0}}{W_{12,0}}\,d\breve{k}_1 d\breve{k}_2$$

$$\times\sum_{\gamma_5\gamma_6}\int_{-\infty}^{\infty}\mathcal{L}_{k\,k_5\,k_6}^{\gamma\,\gamma_5\,\gamma_6}\,\overset{(3)}{\phi}{}_{0}^{\gamma_5\gamma_6-\gamma}(k_5,k_6)\ \frac{i\delta_{56,0}}{W_{56,0}}\,d\breve{k}_5 d\breve{k}_6 . \qquad (74)$$

The term $\langle a_0^{\gamma'}(k')\,{}_5\delta_4^{\gamma}(k) + a_0^{\gamma}(k)\,{}_5\delta_4^{\gamma'}(k')\rangle_{\gamma'=-\gamma}$

$$= 2\rho^{00'}\sum_{\gamma_1\gamma_2}\int_{-\infty}^{\infty}\mathcal{L}_{k\,k_1\,k_2}^{\gamma\,\gamma_1\,\gamma_2}\int_0^t\langle a_{0'}^{\gamma'}\alpha_{11}^{\gamma_1}\beta_{22}^{\gamma_2}\rangle e^{iW_{12,0}t}\,dt\ \delta_{12,0}\,d\breve{k}_1 d\breve{k}_2,$$

$$= 4\rho^{00'}\sum_{\gamma_1\cdots\gamma_4}\int_{-\infty}^{\infty}\mathcal{L}_{k\,k_1\,k_2}^{\gamma\,\gamma_1\,\gamma_2}\,\mathcal{L}_{k_2\,k_3\,k_4}^{\gamma_2\,\gamma_3\,\gamma_4}\langle a_{0'}^{\gamma'}\alpha_{11}^{\gamma_1}a_{03}^{\gamma_3}\alpha_{14}^{\gamma_4}\rangle$$

$$\frac{\Delta_{134,0}-\Delta_{12,0}}{iW_{34,2}}\,\delta_{34,2}\,\delta_{12,0}\,d\breve{k}_1\cdots d\breve{k}_4,$$

$$\sim 4t \overset{00'}{\rho}\, \delta_{00'}\, \overset{(2)\nu\nu'}{Q_0}(k) \sum_{\nu_1\nu_2} \int_{-\infty}^{\infty} L^{\nu\nu_1\nu_2}_{kk_1k_2}\, L^{\nu_2\,\nu-\nu_1}_{k_2\,k-k_1}\, Q^{\nu_1-\nu_1}(k_1)\, \frac{i\delta_{12,0}}{w_{12,0}}\, dk_1\, dk_2 \qquad \cdots (75)$$

$$+ 4t \overset{00'}{\rho}\, \delta_{00'}\, A^{\nu'\nu}_{\alpha\alpha}(k') \sum_{\nu_1\nu_2} \int_{-\infty}^{\infty} L^{\nu\nu_1\nu_2}_{kk_1k_2}\, L^{\nu_2\,\nu-\nu_1}_{k_2\,k-k_1}\, A^{\nu_1-\nu_1}_{\alpha\alpha}(k_1)\, \frac{i\delta_{12,0}}{w_{12,0}}\, dk_1\, dk_2 \qquad -- (76)$$

$$+ \; O(1).$$

Clearly $\langle a_0^{\nu'}(k')\, {}_6\delta_4^{\nu}(k) + a_0^{\nu}(k)\, {}_6\delta_4^{\nu'}(k') \rangle_{\nu'=-\nu} \sim O(1)$,

since $\nu_1\omega_1 + \nu_2\omega_2 - \nu\omega \not\equiv 0$.

Also $\langle a_0^{\nu'}(k')\, {}_7\delta_4^{\nu}(k) + a_0^{\nu}(k)\, {}_7\delta_4^{\nu'}(k') \rangle_{\nu'=-\nu} \sim O(1)$,

since the time "t" dependence of this expression is

$\int_0^t \Delta_{345,1}\, e^{i w_{12,0} t}\, dt$ which is order one for long time. Now consider

Now consider $\langle a_0^{\nu'}(k')\, {}_8\delta_4^{\nu}(k) + a_0^{\nu}(k)\, {}_8\delta_4^{\nu'}(k') \rangle_{\nu'=-\nu}$

$$= 2 \sum_{\nu_1\nu_2} \int_{-\infty}^{\infty} L^{\nu\nu_1\nu_2}_{kk_1k_2} \langle a_0^{\nu'} \int_0^t {}_1\delta_3^{\nu_1}\, a_{02}^{\nu_2}\, e^{i w_{12,0} t}\, dt \rangle\, \delta_{12,0}\, dk_1\, dk_2,$$

$$= 4 \sum_{\lambda_1 \cdots \lambda_4} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\lambda \lambda_1 \lambda_2} L_{k_1 k_3 k_4}^{\lambda_1 \lambda_3 \lambda_4} \langle a_{01}^{\lambda'} a_{02}^{\lambda_2} a_{03}^{\lambda_3} \alpha_{24}^{\lambda_4} \rangle \frac{\Delta_{234,0} - \Delta_{12,0}}{i w_{34,1}}$$

$$\delta_{34,1} \, \delta_{12,0} \, d\breve{k}_1 \cdots d\breve{k}_4 ,$$

$$\sim 4 t P^{00'} \delta_{00'} Q_0^{(2) \lambda \lambda'}(k) \sum_{\lambda_1 \lambda_2} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\lambda \lambda_1 \lambda_2} L_{k_1 k - k_2}^{\lambda_1 \lambda - \lambda_2} B_{\alpha\alpha}^{\lambda_2 - \lambda}(k_2) \frac{i \delta_{12,0}}{w_{12,0}} d\breve{k}_1 d\breve{k}_2 \quad (77)$$

$$+ 4 t P^{00'} \delta_{00'} B_{\alpha\alpha}^{\lambda' \lambda}(k') \sum_{\lambda_1 \lambda_2} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\lambda \lambda_1 \lambda_2} L_{k_1 k - k_2}^{\lambda_1 \lambda - \lambda_2} Q_0^{(2) \lambda_2 - \lambda}(k_2) \frac{i \delta_{12,0}}{w_{12,0}} d\breve{k}_1 d\breve{k}_2 \quad (78)$$

$$+ O(1).$$

One next considers $\langle a_0^{\lambda'}(k') \, q \delta_4^{\lambda}(k) + a_0^{\lambda}(k) \, q \delta_4^{\lambda'}(k') \rangle_{\lambda' = -\lambda}$

$$= 2 \sum_{\lambda_1 \lambda_2} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\lambda \lambda_1 \lambda_2} \langle a_0^{\lambda'} \int_0^t {}_2\delta_{31}^{\lambda_1} a_{02}^{\lambda_2} e^{i w_{12,0} t} dt \rangle \delta_{12,0} \, d\breve{k}_1 d\breve{k}_2 ,$$

$$= 8 \sum_{\lambda_1 \cdots \lambda_6} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\lambda \lambda_1 \lambda_2} L_{k_1 k_3 k_4}^{\lambda_1 \lambda_3 \lambda_4} L_{k_3 k_5 k_6}^{\lambda_3 \lambda_5 \lambda_6} \langle a_0^{\lambda'} a_{02}^{\lambda_2} a_{04}^{\lambda_4} a_{05}^{\lambda_5} \alpha_{16}^{\lambda_6} \rangle$$

$$\int_0^t \frac{\Delta_{456,1} - \Delta_{34,1}}{i w_{56,3}} e^{i w_{12,0} t} dt \; \delta_{56,3} \, \delta_{34,1}$$

$$\delta_{12,0} \, d\breve{k}_1 \cdots d\breve{k}_6 ,$$

$$\sim \delta t \, \delta_{00'} \overset{00'}{P} \sum_{\lambda_1 \lambda_2} \int_{-\infty}^{\infty} \overset{\lambda \lambda_1 \lambda_2}{L}_{k k_1 k_2} \overset{\lambda_1 \lambda -\lambda_2}{L}_{k_1 k -k_2} \overset{(2)\lambda_2 -\lambda_2}{Q}_0(k_1) \frac{i\delta_{12,0}}{W_{12,0}} d\breve{k}_1 d\breve{k}_2$$

$$\times \sum_{\lambda_5 \lambda_6} \int_{-\infty}^{\infty} \overset{\lambda \lambda_5 \lambda_6}{L}_{k k_5 k_6} \overset{\lambda' \lambda_5 \lambda_6}{A}_{\alpha\alpha\alpha}(k', k_5) \frac{i\delta_{56,0}}{W_{56,0}} d\breve{k}_5 d\breve{k}_6 . \qquad (79)$$

$$+ \delta t \, \delta_{00'} \overset{00'}{P} \overset{(2)\lambda\lambda'}{Q}_0(k) \sum_{\lambda_1 \lambda_2 \lambda_5 \lambda_6} \int_{-\infty}^{\infty} \overset{\lambda \lambda_1 \lambda_2}{L}_{k k_1 k_2} \overset{\lambda_1 \lambda -\lambda_2}{L}_{k_1 k -k_2}$$

$$\overset{-\lambda_2 \lambda_5 \lambda_6}{L}_{-k_2 k_5 - k_6} \overset{\lambda_2 \lambda_5 \lambda_6}{A}_{\alpha\alpha\alpha}(k_2, k_5) \frac{i\delta_{562}}{W_{562}} \frac{i\delta_{12,0}}{W_{12,0}} d\breve{k}_1 d\breve{k}_2 d\breve{k}_5 d\breve{k}_6$$

$$\text{-- } (80)$$

$$+ \; O(1).$$

Considering the term

$$\langle a_0^{\nu'}(k')\,{}_{10}\delta_4^{\nu}(k) + a_0^{\nu}(k)\,{}_{10}\delta_4^{\nu'}(k')\rangle_{\nu'=-\nu}$$

$$= 2\overset{00'}{\rho}\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} \mathcal{L}_{k k_1 k_2}^{\nu\nu_1\nu_2}\langle a_{0'}^{\nu'}\int_0^t {}_3\delta_{31}^{\nu_1} a_{02}^{\nu_2} e^{i\omega_{12,0}t}\,dt\rangle\,\delta_{12,0}\,d\tilde{k}_1 d\tilde{k}_2,$$

$$= 4\overset{00'}{\rho}\sum_{\nu_1\cdots\nu_6}\int_{-\infty}^{\infty} \mathcal{L}_{k k_1 k_2}^{\nu\nu_1\nu_2}\mathcal{L}_{k_1 k_3 k_4}^{\nu_1\nu_3\nu_4}\mathcal{L}_{k_3 k_5 k_6}^{\nu_3\nu_5\nu_6}\langle a_{0'}^{\nu'} a_{02}^{\nu_2} a_{14}^{\nu_4} a_{05}^{\nu_5} a_{06}^{\nu_6}\rangle$$

$$\int_0^t \frac{\Delta_{456,1}-\Delta_{34,1}}{i\omega_{56,3}} e^{i\omega_{12,0}t}\,dt\,\delta_{12,0}\,\delta_{56,3}\,\delta_{34,1}\,d\tilde{k}_1\cdots d\tilde{k}_6,$$

$$\sim 4t\overset{00'}{\rho}\delta_{00'}A_{\alpha\alpha}^{\nu'\nu}(k')\sum_{\nu_1\nu_2\nu_5\nu_6}\int_{-\infty}^{\infty}\mathcal{L}_{k k_1 k_2}^{\nu\nu_1\nu_2}\mathcal{L}_{k_1 k -k_2}^{\nu_1\nu -\nu_2}\mathcal{L}_{-k_2 k_5 k_6}^{-\nu_2\nu_5\nu_6}$$

$$\overset{(3)}{Q}{}_0^{\nu_5\nu_6\nu_2}(k_5,k_6)\,\frac{i\delta_{562}}{\omega_{562}}\,\frac{i\delta_{12,0}}{\omega_{12,0}}\,d\tilde{k}_1 d\tilde{k}_2 d\tilde{k}_5 d\tilde{k}_6 \times \qquad (81)$$

$$+ 4t\overset{00'}{\rho}\delta_{00'}\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty}\mathcal{L}_{k k_1 k_2}^{\nu\nu_1\nu_2}\mathcal{L}_{k_1 k -k_2}^{\nu_1\nu -\nu_2}A_{\alpha\alpha}^{\nu_2 -\nu_2}(k_2)\frac{i\delta_{12,0}}{\omega_{12,0}}\,d\tilde{k}_1 d\tilde{k}_2$$

$$\times\sum_{\nu_5\nu_6}\int_{-\infty}^{\infty}\mathcal{L}_{k k_5 k_6}^{\nu\nu_5\nu_6}\overset{(3)}{Q}{}_0^{\nu_5\nu_6\nu'}(k_5,k_6)\,\frac{i\delta_{56,0}}{\omega_{56,0}}\,d\tilde{k}_5 d\tilde{k}_6 \qquad (82)$$

$$+ O(1).$$

Finally, the term $\langle a_0^{\lambda'}(k')\,{}_1\delta_4^{\lambda}(k) + a_0^{\lambda}(k)\,{}_1\delta_4^{\lambda'}(k')\rangle_{\lambda'=-\lambda}$

$$= 2\overset{00'}{\rho}\sum_{\lambda_1\lambda_2}\int_{-\infty}^{\infty} L_{k\,k_1\,k_2}^{\lambda\,\lambda_1\,\lambda_2}\,\langle a_{01}^{\lambda'}\int_0^t {}_4\delta_{31}^{\lambda_1}\,a_{02}^{\lambda_2}\,e^{i\omega_{12,0}t}\,dt\rangle\,\delta_{12,0}\,d\breve{k}_1 d\breve{k}_2,$$

$$= 2\sum_{\lambda_1\cdots\lambda_4}\int_{-\infty}^{\infty} L_{k\,k_1\,k_2}^{\lambda\,\lambda_1\,\lambda_2}\,L_{k_1\,k_3\,k_4}^{\lambda_1\,\lambda_3\,\lambda_4}\,\langle a_0^{\lambda'}\,a_{02}^{\lambda_2}\,\alpha_{13}^{\lambda_3}\,\alpha_{14}^{\lambda_4}\rangle\,\frac{\Delta_{234,0}-\Delta_{12,0}}{i\omega_{34,1}}$$

$$\delta_{34,1}\,\delta_{12,0}\,d\breve{k}_1\cdots d\breve{k}_4,$$

$$\sim 4t\,\overset{00'}{\rho}\,\delta_{00'}\,A_{\alpha\alpha}^{\lambda'\lambda}(k')\sum_{\lambda_1\lambda_2}\int_{-\infty}^{\infty} L_{k\,k_1\,k_2}^{\lambda\,\lambda_1\,\lambda_2}\,L_{k_1\,k\,-k_2}^{\lambda_1\,\lambda\,-\lambda_2}\,A_{\alpha\alpha}^{\lambda_2\,-\lambda_2}(k_2)\,\frac{i\delta_{12,0}}{\omega_{12,0}}$$

$$d\breve{k}_1 d\breve{k}_2 \qquad (83)$$

$$+\,O(1).$$

One now adds terms (43) through (83). Consider

$(43) + (44) + (61) + (79)$

$$= 2t\,\overset{0-0}{\rho}\,\delta_{00'}\,\overset{(0)}{F}{}_k^{s}\left[\sum_{\lambda_4\lambda_5}\int_{-\infty}^{\infty} L_{-k\,k_4\,k_5}^{-\lambda\,\lambda_4\,\lambda_5}\,A_{\alpha\alpha}^{\lambda\,\lambda_4}(k,k_4)\,\frac{i\delta_{450}}{\omega_{450}}\,d\breve{k}_4 d\breve{k}_5\right.$$

$$+ \sum_{\lambda_4 \lambda_5} \int_{-\infty}^{\infty} \mathcal{L}_{k\, k_4\, k_5}^{\lambda\, \lambda_4\, \lambda_5} A_{\alpha \alpha_4 \alpha}^{-\lambda\, \lambda_4\, \lambda_5}(-k, k_4) \frac{i\delta_{45,0}}{W_{45,0}} d\tilde{k}_4 d\tilde{k}_5 \Bigg],$$

$$= 2t\,\delta_{00'} \Bigg[\overset{0-0}{\rho} \sum_{\lambda_4 \lambda_5} \int_{-\infty}^{\infty} \mathcal{L}_{-k\, k_4\, k_5}^{-\lambda\, \lambda_4\, \lambda_5} A_{\alpha \alpha_4 \alpha}^{\lambda\, \lambda_4\, \lambda_5}(k, k_4) \frac{i\delta_{45 0}}{W_{45 0}} d\tilde{k}_4 d\tilde{k}_5 \Bigg]$$

$$\times \left\{ \overset{(0)\lambda}{F}_k + \overset{(0)-\lambda}{F}_{-k} \right\},$$

$$= 0, \quad \text{since} \quad \overset{(0)\lambda}{F}_k + \overset{(0)-\lambda}{F}_{-k} = 0.$$

$$\overset{(0)\lambda}{F}_k = \sum_{\lambda_2} \int_{-\infty}^{\infty} \Bigg[3\mathcal{L}_{k\, k\, k_2\, -k_2}^{\lambda\, \lambda\, \lambda_2\, -\lambda_2} + 4 \sum_{\lambda_1} \int_{-\infty}^{\infty} \mathcal{L}_{k\, k_1\, k_2}^{\lambda\, \lambda_1\, \lambda_2} \mathcal{L}_{k_1\, k\, -k_2}^{\lambda_1\, \lambda\, -\lambda_2} \frac{i\delta_{12,0}}{W_{12,0}} d\tilde{k}_1 \Bigg]$$

$$\overset{(1)}{Q}{}_0^{\lambda_2 - \lambda_2}(k_2)\, d\tilde{k}_2 .$$

Now consider (47) + (49) + (52) + (53)

$$= t\,\delta_{00'} \overset{0-0}{\rho} \overset{(0)\lambda}{F}_k \Bigg[\overset{0-0}{\rho} \sum_{\lambda_4 \lambda_5} \int_{-\infty}^{\infty} \mathcal{L}_{k\, k_4\, k_5}^{\lambda\, \lambda_4\, \lambda_5} A_{\alpha \alpha_4}^{-\lambda\, \lambda_4\, \lambda_5}(-k, k_4) \frac{i\delta_{45,0}}{W_{45,0}} d\tilde{k}_4 d\tilde{k}_5 \Bigg]$$

$$= 0, \quad \text{since} \quad \overset{(0)\lambda}{F}_k + \overset{(0)-\lambda}{F}_{-k} = 0.$$

Now consider terms (45) + (62) + (78)

$$= t\,\delta_{00'}\overset{0-0}{P}\overset{-1\,1}{B}_{\alpha\alpha}(-k)\overset{(0)1\,5}{F}_{k} + t\,\delta_{00'}\overset{0-0}{P}\overset{-1\,1}{B}_{\alpha\alpha}(-k)\overset{(0)1\,5}{F}_{k}$$

$$= t\,\delta_{00'}\overset{0-0}{P}\overset{(0)1\,5}{F}_{k}\left(\overset{-1\,1}{B}_{\alpha\alpha}(-k) + \overset{1-1}{B}_{\alpha\alpha}(k)\right),$$

using the fact that $\overset{-1\,1}{B}_{\alpha\alpha}(-k) = \overset{1-1}{B}_{\alpha\alpha}(k)$,

$$= 0, \text{ since } \overset{(0)1\,5}{F}_{k} + \overset{(0)1-5}{F}_{-k}.$$

Now consider terms (46) + (65),

$$= 6t\overset{0-0}{P}\delta_{00'}\left(\overset{-1\,1}{A}_{\alpha\alpha}(-k) + \overset{-1\,1}{A}_{\alpha\alpha}(-k)\right)$$

$$\left[\sum_{\lambda_2\lambda_4\lambda_5}\int_{-\infty}^{\infty}\mathcal{L}^{1\,1\,\lambda_2-\lambda_2}_{k\,k\,k_2-k_2}\,\mathcal{L}^{\lambda_2\lambda_4\lambda_5}_{k_2\,k_4\,k_5}\,\overset{(3)\lambda_4\lambda_5-\lambda_2}{\Phi_0}(k_4,k_5)\,\frac{i\,\delta_{45,2}}{W_{45,2}}\right.$$

$$\left.d\tilde{k}_2\,d\tilde{k}_4\,d\tilde{k}_5\right].$$

Since $\delta_{00'} A^{-\nu\,\nu}_{\alpha\alpha}(-k) = \langle a_0^{-\nu}(k')\, a_1^{\nu}(k) \rangle = \langle a_1^{\nu}(k)\, a_0^{-\nu}(k') \rangle$

$$= \delta_{00'} A^{\nu\,-\nu}_{\alpha\alpha}(k),$$

therefore $A^{-\nu\,\nu}_{\alpha\alpha}(-k) = A^{\nu\,-\nu}_{\alpha\alpha}(k)$.

Therefore applying the permutation $\overset{0-0}{P}$ in the above expression leaves the first bracket unchanged but inserts one negative sign into the second bracket since

$\mathcal{L}^{-\nu\,-\nu\,\nu_2\,-\nu_2}_{-k\,-k\,k_2\,-k_2} = -\mathcal{L}^{\nu\,\nu\,\nu_2\,-\nu_2}_{k\,k\,k_2\,-k_2}$. Therefore the two terms in the permutation cancel and (46) + (65) = 0 .

One now adds the terms

$$(48) + (50) + (71) + (81)$$

$$= 4t\overset{0-0}{P}\delta_{00'}\left(A^{-\nu\,\nu}_{\alpha\alpha}(-k) + A^{-\nu\,\nu}_{\alpha\alpha}(-k)\right)\left\{\sum_{\nu_1\nu_2\nu_4\nu_5}\int_{-\infty}^{\infty}\mathcal{L}^{\nu\,\nu_1\,\nu_2}_{k\,k_1\,k_2}\right.$$

$$\mathcal{L}^{\nu_1\,\nu\,-\nu_2}_{k_1\,k\,-k_2}\,\frac{i\delta_{12,0}}{W_{12,0}}\left[\mathcal{L}^{\nu_2\,\nu_4\,\nu_5}_{k_2\,k_4\,k_5}\,\overset{(3)}{\Phi}{}_0^{\nu_4\,\nu_5\,-\nu_2}(k_4,k_5)\,\frac{i\delta_{45,2}}{W_{45,2}}\right.$$

$$\left.\left.+\,\mathcal{L}^{-\nu_2\,\nu_4\,\nu_5}_{-k_2\,k_4\,k_5}\,\overset{(3)}{\Phi}{}_0^{\nu_4\,\nu_5\,\nu_2}(k_4,k_5)\,\frac{i\delta_{452}}{W_{452}}\right]dk_1\,dk_2\,dk_4\,dk_5.\right.$$

Applying the permutation leaves the first bracket unchanged, but makes the second bracket negative as can be seen by putting $\begin{Bmatrix} \lambda_i \\ \tilde{h}_i \end{Bmatrix} \rightarrow \begin{Bmatrix} -\lambda_i \\ -h_i \end{Bmatrix}, \quad i = 1, 2$ and using the properties of the $\mathcal{L}^{\lambda\lambda_1-\lambda_2}_{k h_1 h_2}$. Therefore (48) + (50) + (71) + (81) = 0.

Now add the terms (51) + (66)

$$= 6t \overset{0-0}{P}\delta_{00'} \sum_{\lambda_2} \int_{-\infty}^{\infty} \mathcal{L}^{\lambda\lambda\lambda_2-\lambda_2}_{kk h_2 -h_2} A^{\lambda_2-\lambda}_{\alpha\alpha}(h_2)\, d\breve{h}_2$$

$$\times \sum_{\lambda_4\lambda_5} \int_{-\infty}^{\infty} \left\{ \mathcal{L}^{-\lambda\lambda_4\lambda_5}_{-k h_4 h_5} \overset{(3)}{Q}{}^{\lambda_4\lambda_5\lambda}_{0}(h_4,h_5) \frac{i\delta_{450}}{W_{450}} + \mathcal{L}^{\lambda\lambda_4\lambda_5}_{k h_4 h_5} \overset{(3)}{Q}{}^{\lambda_4\lambda_5-\lambda}_{0}(h_4,h_5) \frac{i\delta_{45,0}}{W_{45,0}} \right\} d\breve{h}_4 d\breve{h}_5.$$

The permutation $\lambda \rightarrow -\lambda$ and $\breve{k} \rightarrow -\breve{k}$, leaves the second bracket unchanged but introduces one sign change in the first. Therefore, since $\mathcal{L}^{\lambda\lambda\lambda_2-\lambda_2}_{kk h_2-h_2} + \mathcal{L}^{-\lambda-\lambda\lambda_2-\lambda_2}_{-k-k h_2-h_2} = 0$,

(51) + (66) = 0.

Now one adds

(54) + (55) + (74) + (82)

$$\equiv 8t\overset{0-0}{P}\delta_{00'} \left\{ \sum_{\lambda_1\lambda_2} \int_{-\infty}^{\infty} \mathcal{L}^{\lambda\lambda_1\lambda_2}_{k h_1 h_2} \mathcal{L}^{\lambda_1\lambda-\lambda_2}_{h_1 k -h_2} A^{\lambda_2-\lambda_2}_{\alpha\alpha}(h_2) \frac{i\delta_{12,0}}{W_{12,0}} dh_1 dh_2 \right\}$$

$$\times \left\{ \sum_{\nu_4 \nu_5} \int_{-\infty}^{\infty} \left[\mathcal{L}^{-\nu\, \nu_4\, \nu_5}_{-k\, k_4\, k_5} \overset{(3)}{\Phi}{}^{\nu_4 \nu_5 \nu}_{0}(k_4, k_5) \frac{i\delta_{450}}{W_{450}} + \mathcal{L}^{\nu\, \nu_4\, \nu_5}_{k\, k_4\, k_5} \overset{(3)}{\Phi}{}^{\nu_4 \nu_5 -\nu}_{0}(k_4, k_5) \frac{i\delta_{4510}}{W_{4510}} \right] dk_4 dk_5 \right\}.$$

The permutation changes the sign of the first bracket but leaves the sign of the second bracket unchanged. Therefore $(54) + (55) + (74) + (82) = 0.$

Now consider term $[60]$,

$$= 12 t P \overset{(2)}{\Phi}{}^{0-\nu\, \nu-\nu}_{0}(k) \sum_{\nu_1 \nu_4 \nu_5} \int_{-\infty}^{\infty} \mathcal{L}^{\nu\, \nu_1\, \nu-\nu_1}_{k\, k_1\, k-k_1} \mathcal{L}^{\nu_1\, \nu_4\, \nu_5}_{k_1\, k_4\, k_5} A^{-\nu_1\, \nu_4\, \nu_5}_{\alpha\, \alpha\, \alpha}(-k_1, k_4) \frac{i\delta_{451}}{W_{451}} dk_4 dk_5.$$

The permutation changes $\mathcal{L}^{\nu\, \nu_1\, \nu-\nu_1}_{k\, k_1\, k-k_1}$ into $\mathcal{L}^{-\nu\, \nu_1\, -\nu-\nu_1}_{-k\, k_1\, -k\, -k_1}$ which by the properties of these coefficients is $-\mathcal{L}^{\nu\, \nu_1\, \nu-\nu_1}_{k\, k_1\, k-k_1}$. Therefore the two terms in the permutation cancel.

Therefore $(66) = 0$

Similarly, $(64) = 0$

Adding $(69) + (80)$

$$= 8t\,\overset{0-0}{\rho}\,\delta_{00'}\,\overset{(2)\,\lambda-\lambda}{Q_0}(k)\sum_{\lambda_1\lambda_2\lambda_4\lambda_5}\int_{-\infty}^{\infty} \mathcal{L}_{k\,k_1\,k_2}^{\lambda\,\lambda_1\,\lambda_2}\,\mathcal{L}_{k_1\,k\,-k_2}^{\lambda_1\,\lambda\,-\lambda_2}\,\frac{i\delta_{12,0}}{W_{12,0}}$$

$$\left\{\mathcal{L}_{k_2\,k_4\,k_5}^{\lambda_2\,\lambda_4\,\lambda_5}\,\underset{\alpha\alpha\alpha}{A}^{\lambda_4\,\lambda_5\,-\lambda_2}(k_4,k_5)\,\frac{i\delta_{45,2}}{W_{45,2}} + \mathcal{L}_{-k_2\,k_4\,k_5}^{-\lambda_2\,\lambda_4\,\lambda_5}\,\underset{\alpha\alpha\alpha}{A}^{\lambda_4\,\lambda_5\,\lambda_2}(k_4,k_5)\,\frac{i\delta_{452}}{W_{452}}\right\}$$

$$d\tilde{k}_1\,d\tilde{k}_2\,d\tilde{k}_4\,d\tilde{k}_5.$$

Applying the permutation and setting $\begin{cases}\lambda_i \\ k_i\end{cases} \rightarrow \begin{cases}-\lambda_i \\ -k_i\end{cases}, \; i=1,2$

in the second expression one obtains a cancellation.

Therefore $(69) + (80) = 0.$ (because of three sign changes).

Consider terms $(70) + (73)$

$$= 4t\,\overset{0-0}{\rho}\,\delta_{00'}\,\overset{(2)\,\lambda-\lambda}{Q_0}(k)\sum_{\lambda_1\lambda_2\lambda_4\lambda_5}\int_{-\infty}^{\infty} \mathcal{L}_{k\,k_1\,k_2}^{\lambda\,\lambda_1\,\lambda_2}\,\mathcal{L}_{k_1\,k\,-k_2}^{\lambda_1\,\lambda\,-\lambda_2}$$

$$\frac{i\delta_{12,0}}{W_{12,0}}\left\{\mathcal{L}_{k_2\,k_4\,k_5}^{\lambda_2\,\lambda_4\,\lambda_5}\,\underset{\alpha\alpha\alpha}{A}^{\lambda_4\,\lambda_5\,-\lambda_2}(k_4,k_5)\,\frac{i\delta_{45,2}}{W_{45,2}} + \mathcal{L}_{-k_2\,k_4\,k_5}^{-\lambda_2\,\lambda_4\,\lambda_5}\right.$$

$$\left.\underset{\alpha\alpha\alpha}{A}^{\lambda_4\,\lambda_5\,\lambda_2}(k_4,k_5)\,\frac{i\delta_{452}}{W_{452}}\right\} d\tilde{k}_1\,d\tilde{k}_2\,d\tilde{k}_4\,d\tilde{k}_5,$$

= 0 , from similar reasoning as above.

Consider

$$(63) + (72) + (77)$$

$$= 2t \overset{0-0}{\rho}\, \delta_{00'}\, \overset{(2)\,s-s}{Q_0}(k) \sum_{\lambda_2} \int_{-\infty}^{\infty} \left\{ \tfrac{1}{2} \overset{2-2}{\rho}\, f^{\lambda\lambda\lambda_2-\lambda_2}_{kk\,k_2-k_2} \right\} B^{\lambda_2-\lambda_2}_{\alpha\alpha}(k_2)\, d\breve{k}_2$$

$$= 0, \quad \text{since} \left\{ \tfrac{1}{2} \overset{2-2}{\rho}\, f^{-\lambda-\lambda\,\lambda_2-\lambda_2}_{-k-k\,k_2-k_2} \right\} = -\left\{ \tfrac{1}{2} \overset{2-2}{\rho}\, f^{\lambda\lambda\lambda_2-\lambda_2}_{kk\,k_2-k_2} \right\}.$$

Consider $(67) + (75)$

$$= \overset{0-0}{\rho}\, \delta_{00'}\, \overset{(2)\,\lambda-\lambda}{Q_0}(k) \sum_{\lambda_2} \int_{-\infty}^{\infty} \left\{ \tfrac{1}{2} \overset{2-2}{\rho}\, f^{\lambda\lambda\lambda_2-\lambda_2}_{kk\,k_2-k_2} \right\} a^{\lambda_2-\lambda_2}(k_2)\, d\breve{k}_2$$

$$= 0, \quad \text{since} \left\{ \tfrac{1}{2} \overset{2-2}{\rho}\, f^{-\lambda-\lambda\,\lambda_2-\lambda_2}_{-k-k\,k_2-k_2} \right\} = -\left\{ \tfrac{1}{2} \overset{2-2}{\rho}\, f^{\lambda\lambda\lambda_2-\lambda_2}_{kk\,k_2-k_2} \right\}.$$

Now consider $(68) + (57)$

$$= 6t \overset{0-0}{\rho} \left(\delta_{00'} A^{-\lambda\lambda}_{\alpha\alpha}(-k) + \delta_{00'} A^{\lambda-\lambda}_{\alpha\alpha}(k) \right)$$

$$\times \sum_{\lambda_2} \int_{-\infty}^{\infty} L^{\lambda\lambda\lambda_2-\lambda_2}_{kk\,k_2-k_2}\, A^{\lambda_2-\lambda_2}_{\alpha\alpha}(k_2)\, d\breve{k}_2,$$

$$= 0, \quad \text{since } L^{-\lambda-\lambda\,\lambda_2-\lambda_2}_{-k-k\,k_2-k_2} = -L^{\lambda\lambda\lambda_2-\lambda_2}_{kk\,k_2-k_2}.$$

Next consider (59) + (76) + (83)

$$= 4t\overset{0-0}{P}\delta_{00'} A_{\alpha a}^{-\nu\nu}(-k)\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} \ell_{k k_1 k_2}^{\nu\nu_1\nu_2}\, \ell_{k_1 k-k_2}^{\nu_1 \nu-\nu_2}\left(A_{a\alpha}^{\nu_2-\nu_2}(k_2)\right.$$

$$\left. + A_{\alpha a}^{\nu_2-\nu_2}(k_2)\right)\frac{i\delta_{12,0}}{W_{12,0}}\, d\breve{k}_1\, d\breve{k}_2,$$

$$+ 4t\overset{0-0}{P}\delta_{00'} A_{\alpha a}^{\nu-\nu}(k)\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} \ell_{k k_1 k_2}^{\nu\nu_1\nu_2}\, \ell_{k_1 k-k_2}^{\nu_1 \nu-\nu_2}\left(A_{a\alpha}^{\nu_2-\nu_2}(k_2)\right.$$

$$\left. + A_{\alpha a}^{\nu_2-\nu_2}(k_2)\right)\frac{i\delta_{12,0}}{W_{12,0}}\, d\breve{k}_1\, d\breve{k}_2,$$

$$= 4t\overset{0-0}{P}\delta_{00'}\left(A_{\alpha a}^{-\nu\nu}(-k) + A_{\alpha a}^{\nu-\nu}(k)\right)\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} \ell_{k k_1 k_2}^{\nu\nu_1\nu_2}\, \ell_{k_1 k-k_2}^{\nu_1 \nu-\nu_2}$$

$$\left(A_{a\alpha}^{\nu_2-\nu_2}(k_2) + A_{\alpha a}^{\nu_2-\nu_2}(k_2)\right)\frac{i\delta_{12,0}}{W_{12,0}}\, d\breve{k}_1\, d\breve{k}_2.$$

Apply the permutation and set $\begin{cases}\nu_i \\ \breve{k}_i\end{cases} \rightarrow \begin{cases}-\nu_i \\ -\breve{k}_i\end{cases}, \quad i = 1, 2$

in the second expression, which then from the properties of the coefficients becomes the negative of the first.

Therefore $(59) + (76) + (83) = 0.$

Finally, adding

$$(56) + (58)$$

$$= t \, \overset{0-0}{\rho} \, \delta_{00'} \, \overset{\lambda-\lambda}{a}(k) \, \overset{(0)s}{F}_{k}$$

$$= 0, \quad \text{since} \quad \overset{(0)s}{\bar{F}}_{k} + \overset{(0)-s}{F}_{-k} = 0.$$

Hence there are no "t" growth contributions from terms (43) through (83).

One now can write down the asymptotic form for

$$\delta(\tilde{k} + \tilde{k}') \, \overset{(2)\,\lambda-\lambda}{\Phi_4}(k).$$

$$\delta(\breve{k} + \breve{k}') \, \overset{(2)\,\lambda-\lambda}{\Phi_4}(\breve{k}) = - t \, \delta(\breve{k} + \breve{k}') \, \frac{\partial \overset{(2)\,\lambda-\lambda}{\Phi_0}(k)}{\partial T_4}$$

$$- t \, \delta(\breve{k} + \breve{k}') \, \frac{\partial \overset{(2)\,\lambda-\lambda}{\tilde{\Phi}_2}(k)}{\partial T_2}$$

$$+ 12\pi t \, \delta(\tilde{k} + \tilde{k}') \sum_{\lambda_1 \lambda_2 \lambda_3} \int_{-\infty}^{\infty} g^{\lambda \lambda_1 \lambda_2 \lambda_3}_{k k_1 k_2 k_3}$$

$$\overset{*}{g}^{\lambda \lambda_1 \lambda_2 \lambda_3}_{k k_1 k_2 k_3} \, \overset{(2)\,\lambda_1-\lambda_1}{\Phi_0}(k_1) \, \overset{(2)\,\lambda_2-\lambda_2}{\Phi_0}(k_2) \, \overset{(2)\,\lambda_3-\lambda_3}{\Phi_0}(k_3)$$

$$\delta^{\lambda}_{123,0} \, \delta_{123,0} \, d\breve{k}_1 \, dk_2 \, d\breve{k}_3$$

$$+ 36\pi t\, \delta(\tilde{k}+\tilde{k}')\, \overset{(2)}{\phi}{}_0^{\nu-\nu}(k) \sum_{\nu_1\nu_2\nu_3} \int_{-\infty}^{\infty} g_{k k_1 k_2 k_3}^{\nu\nu_1\nu_2\nu_3}$$

$$g_{k_2 k -k_2 -k_3}^{\nu_1 \nu -\nu_2 -\nu_3}\, \overset{(2)}{\phi}{}_0^{\nu_2-\nu_2}(k_2)\, \overset{(2)}{\phi}{}_0^{\nu_3-\nu_3}(k_3)\, \delta_{123,0}\, \delta_{123,0}$$

$$d\tilde{k}_1\, d\tilde{k}_2\, d\tilde{k}_3$$

$$+ O(1),$$

where

$$g_{k k_1 k_2 k_3}^{\nu\nu_1\nu_2\nu_3} = L_{k k_1 k_2 k_3}^{\nu\nu_1\nu_2\nu_3} - \frac{2i}{3} P^{123} \sum_{\nu_4} \frac{L_{k\, k_2+k_3\, k_1}^{\nu\, -\nu_4\, \nu_1}\, L_{k_2+k_3\, k_2\, k_3}^{-\nu_4\, \nu_2\, \nu_3}}{\nu_2\omega_2 + \nu_3\omega_3 + \nu_4\omega_{23}},$$

and $\overset{(2)}{\tilde{\phi}}{}_2^{\nu-\nu}(k)$ is the order one term in the long time behavior of $\overset{(2)}{\phi}{}_2^{\nu-\nu}(k)$. By choosing the cumulants generated from mean values of the free term products, one can eliminate the order one behavior of $\overset{(2)}{\phi}{}_2^{\nu-\nu}(k)$ in the same manner as already demonstrated at T_3 time scale. Therefore in order to suppress the remaining secular terms, one chooses

$$\frac{\partial \overset{(2)}{\phi}{}_0^{\nu-\nu}(\tilde{k})}{\partial T_4} = 12\pi \sum_{\nu_1\nu_2\nu_3} \int_{-\infty}^{\infty} g_{k k_1 k_2 k_3}^{\nu\nu_1\nu_2\nu_3}\, g_{k k_1 k_2 k_3}^{*\,\nu\nu_1\nu_2\nu_3}\, \overset{(2)}{\phi}{}_0^{\nu_1-\nu_1}(k_1)$$

$$\overset{(2)}{\phi}_0{}^{\lambda_2-\lambda_2}(k_2)\ \overset{(2)}{\phi}_0{}^{\lambda_3-\lambda_3}(k_3)\ \delta^{\lambda}_{123,0}\ \delta_{123,0}\ d\breve{k}_1\, d\breve{k}_2\, d\breve{k}_3$$

$$+\ 12\pi\ \overset{(2)}{\phi}_0{}^{\lambda-\lambda}(k)\ \overset{123}{P} \sum_{\lambda_1\lambda_2\lambda_3} \int_{-\infty}^{\infty} g^{\lambda\lambda_1\lambda_2\lambda_3}_{kk_1k_2k_3}\ g^{\lambda_1\lambda-\lambda_2-\lambda_3}_{k_1k-k_2-k_3}$$

$$\overset{(2)}{\phi}_0{}^{\lambda_2-\lambda_2}(k_2)\ \overset{(2)}{\phi}_0{}^{\lambda_3-\lambda_3}(k_3)\ \delta^{\lambda}_{123,0}\ \delta_{123,0}\ d\breve{k}_1\, d\breve{k}_2\, d\breve{k}_3 .$$

------- [27] .

It is proved algebraically in Appendix I that for the case $h = \infty$, $\frac{2\lambda_1\omega_1\lambda_2\omega_2\lambda_3\omega_3}{ig^2} g^{\lambda\lambda_1\lambda_2\lambda_3}_{kk_1k_2k_3} = D^{\lambda_1\lambda_2\lambda_3}_{k_1k_2k_3}$, where $D^{\lambda_1\lambda_2\lambda_3}_{k_1k_2k_3}$ is the coefficient Hasselmann obtained. However there is a difference in the result for the case of finite depth. In the case of infinite depth, the resulting equation [27] is exactly that which Hasselmann obtained, by using the Gaussian assumption.

The conservation of energy property can be proved in a similar manner as before using the fact that

$g^{\lambda_1\lambda-\lambda_2-\lambda_3}_{k_1k-k_2-k_3} = \frac{\lambda_1\omega_1}{\lambda\omega} g^{\lambda\lambda_1\lambda_2\lambda_3}_{kk_1k_2k_3}$ when $\breve{k}_1 + \breve{k}_2 + \breve{k}_3 = \breve{k}$ and

$\lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3 = \lambda\omega .$

Consider

$$\frac{\partial}{\partial T_4} \sum_{\nu} \int_{-\infty}^{\infty} \overset{(2)}{Q}{}_0^{\nu -\nu}(\breve{k})\, d\breve{k}$$

$$= 12\pi \sum_{\nu\nu_1\nu_2\nu_3} \int_{-\infty}^{\infty} g^{\nu\,\nu_1\,\nu_2\,\nu_3}_{k k_1 k_2 k_3}\; g^{-\nu\, -\nu_1\, -\nu_2\, -\nu_3}_{-k -k_1 -k_2 -k_3}\; \overset{(2)}{Q}{}_0^{\nu_1-\nu_1}(k_1)\, \overset{(2)}{Q}{}_0^{\nu_2-\nu_2}(k_2)\, \overset{(2)}{Q}{}_0^{\nu_3-\nu_3}(k_3)$$

$$\delta^{\nu}_{123,0}\; \delta_{123,0}\; d\breve{k}_1\, d\breve{k}_2\, d\breve{k}_3\, d\breve{k}$$

$$+ 12\pi P \sum^{123}_{\nu\nu_1\nu_2\nu_3} \int_{-\infty}^{\infty} g^{\nu\,\nu_1\,\nu_2\,\nu_3}_{k k_1 k_2 k_3}\; g^{\nu_1\, \nu\, -\nu_2\, -\nu_3}_{k_1\, k\, -k_2\, -k_3}\; \overset{(2)}{Q}{}_0^{\nu-\nu}(k)\, \overset{(2)}{Q}{}_0^{\nu_2-\nu_2}(k_2)\, \overset{(2)}{Q}{}_0^{\nu_3-\nu_3}(k_3)$$

$$\delta^{1}_{123,0}\; \delta_{123,0}\; d\breve{k}_1\, d\breve{k}_2\, d\breve{k}_3\, d\breve{k}.$$

In the second integral replace $\begin{cases} \nu \\ \breve{k} \end{cases}$ by $\begin{cases} -\nu_1 \\ -k_1 \end{cases}$ and $\begin{cases} \nu_1 \\ \breve{k}_1 \end{cases}$ by $\begin{cases} -\nu \\ -\breve{k} \end{cases}$, and obtain that

$$\frac{\partial}{\partial T_4} \sum_{\nu} \int_{-\infty}^{\infty} \overset{(2)}{Q}{}_0^{\nu-\nu}(\breve{k})\, d\breve{k}$$

$$= 12\pi \sum_{\nu\nu_1\nu_2\nu_3} \int_{-\infty}^{\infty} g^{-\nu\, -\nu_1\, -\nu_2\, -\nu_3}_{-k -k_1 -k_2 -k_3} \left\{ g^{\nu\,\nu_1\,\nu_2\,\nu_3}_{k k_1 k_2 k_3} + P^{123}\, g^{-\nu_1,\, -\nu\, +\nu_2\, +\nu_3}_{-k_1 - k + k_2 + k_3} \right\}$$

$$\overset{(2)}{Q}{}_0^{\nu_1-\nu_1}(k_1)\, \overset{(2)}{Q}{}_0^{\nu_2-\nu_2}(k_2)\, \overset{(2)}{Q}{}_0^{\nu_3-\nu_3}(k_3)\; \delta^{1}_{123,0}\, \delta_{123,0}\, d\breve{k}_1\, d\breve{k}_2\, d\breve{k}_3\, d\breve{k}.$$

But $g^{-\nu_1 \, -\nu \, \nu_2 \, \nu_3}_{-k_1 \, -k \, k_2 \, k_3} = - g^{\nu_1 \, \nu \, -\nu_2 \, -\nu_3}_{k_1 \, k \, -k_2 \, -k_3} = - \frac{\nu_1 \omega_1}{\nu \omega} g^{\nu \, \nu_1 \, \nu_2 \, \nu_3}_{k \, k_1 \, k_2 \, k_3}$.

Therefore

$$\frac{\partial}{\partial T_4} \sum_{\nu} \int_{-\infty}^{\infty} \overset{(2)}{Q}{}_0^{\nu \, -\nu}(\tilde{k}) \, d\tilde{k}$$

$$= 12\pi \sum_{\nu \nu_1 \nu_2 \nu_3} \int_{-\infty}^{\infty} g^{\nu \, \nu_1 \, \nu_2 \, \nu_3}_{k \, k_1 \, k_2 \, k_3} \, g^{*\nu \, \nu_1 \, \nu_2 \, \nu_3}_{k \, k_1 \, k_2 \, k_3} \left(1 - P^{123} \frac{\nu_1 \omega_1}{\nu \omega}\right) \overset{(2)}{Q}{}_0^{\nu_1 \, -\nu_1}(k_1) \, \overset{(2)}{Q}{}_0^{\nu_2 \, -\nu_2}(k_2) \, \overset{(2)}{Q}{}_0^{\nu_3 \, -\nu_3}(k_3) \, \delta(\nu_1 \omega_1 + \nu_2 \omega_2 + \nu_3 \omega_3 - \nu \omega) \, \delta_{123,0} \, d\tilde{k}_1 \, d\tilde{k}_2 \, d\tilde{k}_3 ,$$

$= 0$, since $1 - P^{123} \frac{\nu_1 \omega_1}{\nu \omega} = \frac{1}{\nu \omega} (\nu \omega - \nu_1 \omega_1 - \nu_2 \omega_2 - \nu_3 \omega_3) = 0$.

One now considers the behavior of the general cumulant $\overset{(r)}{\Phi}{}_0^{\gamma_1 \cdots \gamma_r}(\breve{k}_1, \cdots \breve{k}_{r-1})$ on the T_4 time scale. One chooses the cumulants formed from the free terms,

$$\left\{a_0 \alpha_{2'}^{\gamma} + \alpha_1^{\gamma} \alpha_{1'}^{\gamma'} + a_{0'}^{\gamma'} \alpha_2^{\gamma}\right\}, \ \overset{00'0''}{P}\left\{a_{0'}^{\gamma'} a_{0''}^{\gamma''} \alpha_2^{\gamma} + a_0^{\gamma} \alpha_{1'}^{\gamma'} \alpha_{1''}^{\gamma''}\right\}, \ \cdots \text{etc.}$$

which one may denote as $\delta_{00'}\,{}_2\overset{(2)}{\Phi}{}_2^{\gamma\gamma'}(k)$, $\delta_{00'0''}\,{}_2\overset{(3)}{\Phi}{}_2^{\gamma\gamma'\gamma''}(k,k')$ etc. in order to eliminate the order one continuous behavior of the ε^2 components of the spectral cumulants. It is shown in Appendix III, that the "live" terms belonging to $\overset{(4)}{\Phi}{}_2^{\gamma\gamma'\gamma''\gamma'''}(k,k',k'')$ (and $\overset{(3)}{\Phi}{}_2^{\gamma\gamma'\gamma''}(k,k')$) do not return as secular growths in $\overset{(4)}{\Phi}{}_4^{\gamma\gamma'\gamma''\gamma'''}(k,k',k'')$. because of the choices made for the zeroth order spectral cumulant change on the T_2 time scale, and therefore these "live" terms do not have to be suppressed by the cumulants formed from the free terms.

Using similar manipulations to those employed at the T_2 time scale, and using equations (22), one may readily obtain

$$\frac{\partial \overset{(r)}{\Phi}{}_0^{\gamma_1 \cdots \gamma_r}(\breve{k}_1, \cdots \breve{k}_{r-1})}{\partial T_4} = \overset{(r)}{\Phi}{}_0^{\gamma_1 \cdots \gamma_r}(\breve{k}_1 \cdots \breve{k}_{r-1})\left(\overset{(0)}{H}{}_{k_1}^{s_1} + \cdots + \overset{(0)}{H}{}_{-k_1 \cdots -k_{r-1}}^{s_r}\right), \cdots [28].$$

where

$$\overset{(0)}{H}{}_k^{s} = 18 \sum_{\gamma_1\gamma_2\gamma_3} \int_{-\infty}^{\infty} g_{kk_1k_2k_3}^{\gamma\gamma_1\gamma_2\gamma_3}\, g_{k_1 k -k_2 -k_3}^{\gamma_1 \gamma -\gamma_2 -\gamma_3}\, \overset{(2)}{\Phi}{}_0^{\gamma_2 -\gamma_2}(k_2)\, \overset{(2)}{\Phi}{}_0^{\gamma_3 -\gamma_3}(k_3) \left(\pi \delta_{123,0}^{1} + \frac{iP}{W_{123,0}}\right) \delta_{123,0}\, dk_1 dk_2 dk_3.$$

Conclusion.

It has been found that the long time behavior of the spectral energy for any statistical distribution is exactly the same as the behavior which would be obtained assuming the system were Gaussian. The equation describing this long time behavior, equation (27), shows that energy can be fed into a wave vector k by the resonance action of these other wave vectors, while simultaneously the wave vector k can lose (gain) energy by itself interacting with two other wave vectors.

It has also been shown that the higher spectral cumulants change in a continuous fashion as given by equations (22) and (28). In the case of gravity waves the higher spectral cumulants change on a shorter time scale (T_2) than the energy density (T_4).

Appendix I

In order to compare the result [27] with that obtained by Hasselmann, it is desirable to show that

$$\frac{2\nu_1\omega_1\nu_2\omega_2\nu_3\omega_3}{ig^2}\, g^{\nu\,\nu_1\,\nu_2\,\nu_3}_{k\,k_1\,k_2\,k_3} = D^{\nu_1\,\nu_2\,\nu_3}_{k_1\,k_2\,k_3},$$

when $\nu_1\omega_1+\nu_2\omega_2+\nu_3\omega_3 = \nu\omega$, $\tilde{k}_1+\tilde{k}_2+\tilde{k}_3 = \tilde{k}$.

$D^{\nu_1\nu_2\nu_3}_{k_1k_2k_3}$ is the coefficient used by Hasselmann and is defined in his paper [4] as

$$D^{\nu_1\,\nu_2\,\nu_3}_{k_1\,k_2\,k_3} = \frac{1}{3}\left(\hat{D}^{\nu_1\,\nu_2\,\nu_3}_{k_1\,k_2\,k_3} + \hat{D}^{\nu_2\,\nu_3\,\nu_1}_{k_2\,k_3\,k_1} + \hat{D}^{\nu_3\,\nu_1\,\nu_2}_{k_3\,k_1\,k_2}\right),$$

where

$$\hat{D}^{\nu_1\,\nu_2\,\nu_3}_{k_1\,k_2\,k_3} = A + B + C + D.$$

$$A = \frac{i\,D^{\nu_2\,\nu_3}_{k_2\,k_3}}{\omega_{23}^2-(\nu_2\omega_2+\nu_3\omega_3)^2}\left\{2\nu\omega\left(\frac{\omega_1^2\omega_{23}^2}{g^2}-\tilde{k}_1\cdot(\tilde{k}_2+\tilde{k}_3)\right) - \frac{\nu_1\omega_1\,(\tilde{k}_2+\tilde{k}_3)^2}{\cosh^2|k_2+k_3|h}\right\}.$$

$$B = -i\,D^{\nu_2\,\nu_3}_{k_2\,k_3}\,\frac{\nu_1\omega_1}{g^2}\left(\omega_1^2+\omega_{23}^2\right).$$

$$C = E^{s_2\,s_3}_{k_2\,k_3}\left\{\frac{(\nu_1\omega_1)^3(\nu_2\omega_2+\nu_3\omega_3)}{g} - g\,\tilde{k}_1\cdot(\tilde{k}_2+\tilde{k}_3)\right\}.$$

$$D = \frac{s_1\omega_1}{2g^2}\ \check{k}_2\cdot\check{k}_3\left\{ s\omega\left(\omega_2^2+\omega_3^2\right) + s_2\omega_2\, s_3\omega_3\left(s_2\omega_2 + s_3\omega_3\right)\right\}$$

$$- \frac{s_1\omega_1}{2g^2}\ \omega_2^2 k_3^2\left(s\omega + s_3\omega_3\right) - \frac{s_1\omega_1}{2g^2}\ \omega_3^2 k_2^2\left(s\omega + s_2\omega_2\right).$$

$$D^{s_2 s_3}_{k_2 k_3} = i\left(s_2\omega_2 + s_3\omega_3\right)\left(|\check{k}_2||\check{k}_3| \tanh|k_2|h \tanh|\check{k}_3|h - \check{k}_2\cdot\check{k}_3\right)$$

$$-\frac{i}{2}\left\{\frac{s_2\omega_2 k_3^2}{\cosh^2|k_3|h} + \frac{s_3\omega_3 k_2^2}{\cosh^2|k_2|h}\right\}.$$

and

$$E^{s_2 s_3}_{k_2 k_3} = \frac{1}{2g}\left\{\check{k}_2\cdot\check{k}_3 - \frac{s_2\omega_2\, s_3\omega_3}{g^2}\left(\omega_2^2+\omega_3^2 + s_2\omega_2\, s_3\omega_3\right)\right\}.$$

It will be sufficient to show

$$\frac{2 s_1\omega_1 s_2\omega_2 s_3\omega_3}{ig^2}\left(\hat{L}^{s\, s_1 s_2 s_3}_{k\, k_1 k_2 k_3} - 2i\sum_{s_4}\frac{L^{s\, -s_4\, s_1}_{k\, k_2+k_3\, k_1}\, L^{-s_4\, s_2\, s_3}_{k_2+k_3\, k_2\, k_3}}{s_2\omega_2 + s_3\omega_3 + s_4\omega_{23}}\right)$$

$$= \hat{D}^{s_1 s_2 s_3}_{k_1 k_2 k_3}.$$

Consider $\dfrac{2 s_1\omega_1 s_2\omega_2 s_3\omega_3}{ig^2}\ \dfrac{i}{2}\left(\hat{H}^{s_1 s_2 s_3}_{k_1 k_2 k_3} - s\omega\, \hat{K}^{s_1 s_2 s_3}_{k_1 k_2 k_3}\right)$

$$= \frac{s_1\omega_1}{g^2}\left\{ \frac{s_2\omega_2\omega_3^2}{2}\left(\frac{1}{2}k_3^2 + \tilde{k}_2\cdot\tilde{k}_3\right) + \frac{s_3\omega_3\omega_2^2}{2}\left(\frac{1}{2}k_2^2 + \tilde{k}_2\cdot\tilde{k}_3\right)\right.$$

$$- \frac{s\omega\, s_2\omega_2 s_3\omega_3}{4}\left(k_2^2+k_3^2\right) - \frac{s\omega}{2}\Big(\omega_3^2 k_2^2 + \omega_2^2 k_3^2$$

$$\left. - \left(\omega_2^2+\omega_3^2\right)\tilde{k}_2\cdot\tilde{k}_3\Big) + \frac{s\omega\, s_2\omega_2 s_3\omega_3}{g}\,\omega_{23}^2\, K^{s_2 s_3}_{k_2 k_3}\right\}$$

$$= \frac{s_1\omega_1}{2g^2}\Big[\tilde{k}_2\cdot\tilde{k}_3\left(s\omega\left(\omega_2^2+\omega_3^2\right) + s_2\omega_2 s_3\omega_3\left(s_2\omega_2 + s_3\omega_3\right)\right)$$

$$+ \frac{1}{2}s_2\omega_2\omega_3^2 k_2^2 + \frac{1}{2}s_3\omega_3\omega_2^2 k_2^2 - \frac{1}{2}s\omega\, s_2\omega_2 s_3\omega_3\left(k_2^2+k_3^2\right)$$

$$- s\omega\left(\omega_3^2 k_2^2 + \omega_2^2 k_3^2\right)\Big] + \frac{s\omega\, s_1\omega_1 s_2\omega_2 s_3\omega_3}{g^3}\,\omega_{23}^2\, K^{s_2 s_3}_{k_2 k_3}$$

$$= D + \frac{s\omega\, s_1\omega_1 s_2\omega_2 s_3\omega_3}{g^3}\,\omega_{23}^2\, K^{s_2 s_3}_{k_2 k_3},$$

since,

$$P^{123}\, s_1\omega_1\left\{\frac{1}{2}s_2\omega_2\omega_3^2 k_3^2 + \frac{1}{2}s_3\omega_3\omega_2^2 k_2^2 - \frac{1}{2}s\omega\, s_2\omega_2 s_3\omega_3\left(k_2^2+k_3^2\right)\right.$$

$$\left. - s\omega\left(\omega_3^2 k_2^2 + \omega_2^2 k_3^2\right) + \omega_2^2 k_3^2\left(s\omega + s_3\omega_3\right) + \omega_3^2 k_2^2\left(s\omega + s_2\omega_2\right)\right\}$$

$$= 0.$$

The last step was obtained by showing that

$$\overset{123}{P}\ s_1\omega_1 \Big\{ -\frac{s_2\omega_2\ s_3\omega_3}{2}\ k_3^2\left(s\omega - s_3\omega_3\right) - \frac{s_2\omega_2 s_3\omega_3}{2}\ k_2^2\left(s\omega - s_2\omega_2\right)$$

$$+\ s_3\omega_3\,\omega_2^2 k_3^2 + \omega_3^2 k_2^2\, s_2\omega_2 \Big\}$$

$$= \overset{123}{P}\ s_1\omega_1 \Big\{ -\frac{s_2\omega_2 s_3\omega_3}{2}\Big[k_3^2\left(s\omega - s_3\omega_3 - 2s_2\omega_2\right) + k_2^2\left(s\omega - s_2\omega_2 - 2s_3\omega_3\right)\Big]\Big\}$$

$$= -\frac{s_1\omega_1 s_2\omega_2\ s_3\omega_3}{2}\ \overset{123}{P}\Big\{k_3^2\left(s\omega - s_3\omega_3 - 2s_2\omega_2\right) + k_2^2\left(s\omega - s_2\omega_2 - 2s_3\omega_3\right)\Big\}$$

$$= -\frac{s_1\omega_1 s_2\omega_2\ s_3\omega_3}{2}\ \overset{123}{P}\Big\{ k_3^2\left(s\omega - s_3\omega_3 - 2s_2\omega_2 + s\omega - s_3\omega_3 - 2s_1\omega_1\right)\Big\}$$

$$= -s_1\omega_1 s_2\omega_2\ s_3\omega_3\left(s\omega - s_1\omega_1 - s_2\omega_2 - s_3\omega_3\right)\left(k_1^2 + k_2^2 + k_3^2\right)$$

$$= 0, \quad \text{since} \quad s_1\omega_1 + s_2\omega_2 + s_3\omega_3 = s\omega.$$

Next one considers

$$I = -\frac{4 s_1\omega_1 s_2\omega_2 S_3\omega_3}{g^2}\sum_{s_4}\frac{L_{k\ \ k_2+k_3\ \ k_1}^{s\ \ -s_4\ \ s_1}\ L_{k_2+k_3\ \ k_2\ \ k_3}^{-s_4\ \ s_2\ \ s_3}}{s_2\omega_2 + s_3\omega_3 + s_4\omega_{23}}$$

$$= -\frac{4 s_1\omega_1 s_2\omega_2 s_3\omega_3}{g^2}\left\{\frac{L^{\nu - \nu_1}_{k\, k_2+k_3\, k_1} L^{-\nu_2\nu_3}_{k_2+k_3\, k_2\, k_3}}{\nu_2\omega_2+\nu_3\omega_3+\omega_{23}} + \frac{L^{\nu + \nu_1}_{k\, k_2+k_3\, k_1} L^{+\nu_2\nu_3}_{k_2+k_3\, k_2\, k_3}}{\nu_2\omega_2+\nu_3\omega_3-\omega_{23}}\right\},$$

$$= \frac{\nu_1\omega_1\nu_2\omega_2\nu_3\omega_3}{g^2\left[(\nu_2\omega_2+\nu_3\omega_3)^2-\omega_{23}^2\right]}\left\{(\nu_2\omega_2+\nu_3\omega_3-\omega_{23})\left(H^{-\nu_1}_{k_2+k_3\, k_1} - \nu\omega\, K^{-\nu_1}_{k_2+k_3\, k_1}\right)\right.$$

$$\left(H^{\nu_2\nu_3}_{k_2\, k_3} + \omega_{23} K^{\nu_2\nu_3}_{k_2\, k_3}\right) + (\nu_2\omega_2+\nu_3\omega_3+\omega_{23})\left(H^{+\nu_1}_{k_2+k_3\, k_1} - \nu\omega\, K^{+\nu_1}_{k_2+k_3\, k_1}\right)$$

$$\left.\left(H^{\nu_2\nu_3}_{k_2\, k_3} - \omega_{23} K^{\nu_2\nu_3}_{k_2\, k_3}\right)\right\},$$

$$= \frac{\nu_1\omega_1\nu_2\omega_2\nu_3\omega_3}{g^2\left[(\nu_2\omega_2+\nu_3\omega_3)^2-\omega_{23}^2\right]}\left\{\left[(\nu_2\omega_2+\nu_3\omega_3)H^{\nu_2\nu_3}_{k_2\, k_3} - \omega_{23}^2 K^{\nu_2\nu_3}_{k_2\, k_3}\right]\right.$$

$$\left[H^{-\nu_1}_{k_2+k_3\, k_1} - \nu\omega\, K^{-\nu_1}_{k_2+k_3\, k_1} + H^{+\nu_1}_{k_2+k_3\, k_1} - \nu\omega\, K^{+\nu_1}_{k_2+k_3\, k_1}\right]$$

$$-\omega_{23}\left[H^{\nu_2\nu_3}_{k_2\, k_3} - (\nu_2\omega_2+\nu_3\omega_3) K^{\nu_2\nu_3}_{k_2\, k_3}\right]$$

$$\left.\left[H^{-\nu_1}_{k_2+k_3\, k_1} - \nu\omega\, K^{-\nu_1}_{k_2+k_3\, k_1} - H^{+\nu_1}_{k_2+k_3\, k_1} + \nu\omega\, K^{+\nu_1}_{k_2+k_3\, k_1}\right]\right\}.$$

Consider $\qquad (\nu_2\omega_2 + \nu_3\omega_3)\, K^{\nu_2\,\nu_3}_{k_2\,k_3} - H^{\nu_2\,\nu_3}_{k_2\,k_3}$

$$= \frac{\nu_2\omega_2 + \nu_3\omega_3}{2g}\left(\omega_2^2 + \omega_3^2 + \frac{1}{\nu_2\omega_2\nu_3\omega_3}\left(\omega_2^2\omega_3^2 - g^2\tilde{k}_2\cdot\tilde{k}_3\right)\right)$$

$$- \frac{g}{2\nu_2\omega_2\nu_3\omega_3}\left(\nu_2\omega_2\left(k_3^2 + \tilde{k}_2\cdot\tilde{k}_3\right) + \nu_3\omega_3\left(k_2^2 + \tilde{k}_2\cdot\tilde{k}_3\right)\right),$$

$$= \frac{1}{2g\,\nu_2\omega_2\nu_3\omega_3}\Big[(\nu_2\omega_2 + \nu_3\omega_3)\left(\nu_2\omega_2\nu_3\omega_3(\omega_2^2+\omega_3^2) + \omega_2^2\omega_3^2 - g^2\tilde{k}_2\cdot\tilde{k}_3\right)$$
$$- g^2\nu_2\omega_2 k_3^2 - g^2\nu_3\omega_3 k_2^2 - g^2(\nu_2\omega_2 + \nu_3\omega_3)\tilde{k}_2\cdot\tilde{k}_3\Big].$$

Now one uses

$$\omega_i^2 = g k_i \tanh k_i h,$$

Define $\qquad \tanh|\tilde{k}_i|h = T_i, \quad \operatorname{sech}|\tilde{k}_i|h = S_i.$

$$(\nu_2\omega_2 + \nu_3\omega_3)\, K^{\nu_2\,\nu_3}_{k_2\,k_3} - H^{\nu_2\,\nu_3}_{k_2\,k_3}$$

$$= \frac{1}{2g\,\nu_2\omega_2\nu_3\omega_3}\Big[-2g^2\tilde{k}_2\cdot\tilde{k}_3(\nu_2\omega_2 + \nu_3\omega_3) - g^2\nu_2\omega_2 k_3^2 S_3^2$$
$$- \nu_2\omega_2\omega_3^4 - g^2\nu_3\omega_3 k_2^2 S_2^2 - \nu_3\omega_3\,\omega_2^4$$
$$+ \nu_3\omega_3\,\omega_2^4 + \nu_2\omega_2\,\omega_3^4 + 2(\nu_2\omega_2)^2(\nu_3\omega_3)^3$$
$$+ 2(\nu_3\omega_3)^2(\nu_2\omega_2)^3\Big]$$

$$= \frac{1}{2g\,\lambda_2\omega_2\lambda_3\omega_3}\Big[-2g^2\,\tilde{k}_2\cdot\tilde{k}_3\,(\lambda_2\omega_2+\lambda_3\omega_3) - g^2\lambda_2\omega_2\,k_3^2 S_3^2$$

$$- g^2\lambda_3\omega_3\,k_2^2 S_2^2 + 2\omega_2^2\omega_3^2(\lambda_2\omega_2+\lambda_3\omega_3)\Big]$$

$$= \frac{g}{\lambda_2\omega_2\,\lambda_3\omega_3}\Bigg[(\lambda_2\omega_2+\lambda_3\omega_3)(k_2T_2\,k_3T_3 - \tilde{k}_2\cdot\tilde{k}_3) - \frac{\lambda_2\omega_2\,k_3^2}{2\cosh^2 k_3 h}$$

$$- \frac{\lambda_3\omega_3\,k_2^2}{2\cosh^2 k_2 h}\Bigg]$$

$$= \frac{-ig}{\lambda_2\omega_2\lambda_3\omega_3}\,D^{\lambda_2\lambda_3}_{k_2k_3}\,. \qquad \text{--- (1)}$$

By inspection

$$K^{\lambda_2\lambda_3}_{k_2k_3} = \frac{-g^2}{\lambda_2\omega_2\lambda_3\omega_3}\,E^{\lambda_2\lambda_3}_{k_2k_3}\,. \qquad \text{--- (2)}$$

Solving (1) and (2) for $K^{\lambda_2\lambda_3}_{k_2k_3}$ and $H^{\lambda_2\lambda_3}_{k_2k_3}$ in terms of $D^{\lambda_2\lambda_3}_{k_2k_3}$ and $E^{\lambda_2\lambda_3}_{k_2k_3}$ one obtains

$$K^{\lambda_2\lambda_3}_{k_2k_3} = -\frac{g^2}{\lambda_2\omega_2\lambda_3\omega_3}\,E^{\lambda_2\lambda_3}_{k_2k_3}$$

$$H^{\lambda_2\lambda_3}_{k_2k_3} = \frac{ig}{\lambda_2\omega_2\lambda_3\omega_3}\,D^{\lambda_2\lambda_3}_{k_2k_3} - \frac{\lambda_2\omega_2+\lambda_3\omega_3}{\lambda_2\omega_2\lambda_3\omega_3}\,g^2 E^{\lambda_2\lambda_3}_{k_2k_3}\,.$$

One then finds,

$$H^{-\ \lambda_1}_{k_2+k_3\ k_1} + H^{+\ \lambda_1}_{k_2+k_3\ k_1} = \frac{g}{\lambda_1\omega_1}\check{k}\cdot\check{k}_1 ,$$

$$H^{-\ \lambda_1}_{k_2+k_3\ k_1} - H^{+\ \lambda_1}_{k_2+k_3\ k_1} = -\frac{g}{\omega_{23}}\check{k}\cdot(\check{k}_2+\check{k}_3),$$

$$-\lambda\omega\, K^{-\ \lambda_1}_{k_2+k_3\ k_1} - \lambda\omega\, K^{+\ \lambda_1}_{k_2+k_3\ k_1} = -\frac{\lambda\omega}{g}\left(\omega_1^2+\omega_{23}^2\right),$$

$$-\lambda\omega\, K^{-\ \lambda_1}_{k_2+k_3\ k_1} + \lambda\omega\, K^{+\ \lambda_1}_{k_2+k_3\ k_1} = \frac{\lambda\omega}{g\,\lambda_1\omega_1\,\omega_{23}}\left(\omega_1^2\omega_{23}^2 - g^2\check{k}_1\cdot(\check{k}_2+\check{k}_3)\right).$$

One now considers

$$\frac{\lambda\omega\ \lambda_2\omega_2\ \lambda_3\omega_3}{g^3}\,\omega_{23}^2\,K^{\lambda_2\lambda_3}_{k_2k_3} + I$$

$$= -\frac{\lambda\omega}{g}\,\omega_{23}^2\,E^{\lambda_2\lambda_3}_{k_2k_3} + \frac{\lambda_1\omega_1\ \lambda_2\omega_2\ \lambda_3\omega_3}{g^2\left[(\lambda_2\omega_2+\lambda_3\omega_3)^2-\omega_{23}^2\right]}\left[\left\{\frac{\lambda_2\omega_2+\lambda_3\omega_3}{\lambda_2\omega_2\lambda_3\omega_3}\,ig\,D^{\lambda_2\lambda_3}_{k_2k_3}\right.\right.$$

$$\left.-\frac{(\lambda_2\omega_2+\lambda_3\omega_3)^2}{\lambda_2\omega_2\ \lambda_3\omega_3}\,g^2\,E^{\lambda_2\lambda_3}_{k_2k_3} + \frac{g^2\omega_{23}^2}{\lambda_2\omega_2\ \lambda_3\omega_3}\,E^{\lambda_2\lambda_3}_{k_2k_3}\right\}\left(\frac{-\lambda\omega}{g}\left(\omega_1^2+\omega_{23}^2\right)+\frac{g}{\lambda_1\omega_1}\check{k}_1\cdot\check{k}\right)$$

$$+ \frac{-ig\,\omega_{23}\,D^{\sigma_2\sigma_3}_{k_2k_3}}{\sigma_2\omega_2\,\sigma_3\omega_3}\left(\frac{-g}{\omega_{23}}\,\breve{k}_1\cdot(\breve{k}_2+\breve{k}_3)+\frac{\sigma\omega}{g\,\sigma_1\omega_1\omega_{23}}\left(\omega_1^2\omega_{23}^2-g^2\breve{k}_1\cdot(\breve{k}_2+\breve{k}_3)\right)\right)\Bigg],$$

$$= -\frac{\sigma\omega}{g}\,\omega_{23}^2\,E^{\sigma_2\sigma_3}_{k_2k_3} + \frac{\sigma_1\omega_1\,\sigma_2\omega_2\,\sigma_3\omega_3}{g^2\left[(\sigma_2\omega_2+\sigma_3\omega_3)^2-\omega_{23}^2\right]}\,E^{\sigma_2\sigma_3}_{k_2k_3}\ \times$$

$$\left[\left(-g^2\,\frac{(\sigma_2\omega_2+\sigma_3\omega_3)^2-\omega_{23}^2}{\sigma_2\omega_2\,\sigma_3\omega_3}\right)\left(-\frac{\sigma\omega}{g}\left(\omega_1^2+\omega_{23}^2+\frac{g}{\sigma\omega_1}\,\breve{k}_1\cdot\breve{k}\right)\right)\right]$$

$$+ \frac{\sigma_1\omega_1\,\sigma_2\omega_2\,\sigma_3\omega_3}{g^2\left[(\sigma_2\omega_2+\sigma_3\omega_3)^2-\omega_{23}^2\right]}\,D^{\sigma_2\sigma_3}_{k_2k_3}\left[\frac{ig\,(\sigma_2\omega_2+\sigma_3\omega_3)}{\sigma_2\omega_2\,\sigma_3\omega_3}\left(-\frac{\sigma\omega}{g}\left(\omega_1^2+\omega_{23}^2\right)\right.\right.$$

$$\left.+\frac{g}{\sigma_1\omega_1}\,\breve{k}\cdot\breve{k}_1\right) - \frac{ig\,\omega_{23}}{\sigma_2\omega_2\,\sigma_3\omega_3}\left(\frac{-g}{\omega_{23}}\,\breve{k}_1\cdot(\breve{k}_2+\breve{k}_3)+\frac{\sigma\omega}{g\,\sigma_1\omega_1\omega_{23}}\left(\omega_1^2\omega_{23}^2-g^2\breve{k}_1\cdot(\breve{k}_2+\breve{k}_3)\right)\right)\Bigg].$$

The first two terms add to give

$$E^{\sigma_2\sigma_3}_{k_2k_3}\left[\frac{(\sigma_1\omega_1)^3\,\sigma\omega}{g} - g\,\breve{k}_1\cdot\breve{k}\right]$$

$$= E^{\sigma_2\sigma_3}_{k_2k_3}\left[\frac{(\sigma_1\omega_1)^3(\sigma_2\omega_2+\sigma_3\omega_3)}{g}+\frac{\omega_1^4}{g}-g\,k_1^2-g\,\breve{k}_1\cdot(\breve{k}_2+\breve{k}_3)\right]$$

$$= E^{\eta_2 \eta_3}_{k_2 k_3} \left[\frac{(\eta_1\omega_1)^3(\eta_2\omega_2+\eta_3\omega_3)}{g} - g\, \breve{k}_1 (\breve{k}_2+\breve{k}_3) \right]$$

$$- g\, k_1^2 \operatorname{Sech}^2 |\breve{k}_1| h \; E^{\eta_2 \eta_3}_{k_2 k_3} ,$$

$$= C - g\, k_1^2 S_1^2 E^{\eta_2 \eta_3}_{k_2 k_3} .$$

The third term gives

$$\frac{i\,\eta_1\omega_1 D^{\eta_2\eta_3}_{k_2 k_3}}{g^2\left[(\eta_2\omega_2+\eta_3\omega_3)^2-\omega_{23}^2\right]} \Big\{ (\eta_2\omega_2+\eta_3\omega_3)\left(-\eta\omega(\omega_1^2+\omega_{23}^2) + \frac{g^2}{\eta_1\omega_1} \breve{k}\cdot\breve{k}_1\right)$$

$$+ g^2 \breve{k}\cdot(\breve{k}_2+\breve{k}_3) - \frac{\eta\omega}{\eta_1\omega_1}\left(\omega_1^2\omega_{23}^2 - g^2 \breve{k}_1\cdot(\breve{k}_2+\breve{k}_3)\right) \Big\},$$

$$= \frac{-i\, D^{\eta_2\eta_3}_{k_2 k_3}}{\left[(\eta_2\omega_2+\eta_3\omega_3)^2-\omega_{23}^2\right]} \eta\omega \left(\frac{\omega_1^2\omega_{23}^2}{g^2} - \breve{k}_1\cdot(\breve{k}_2+\breve{k}_3)\right)$$

$$+ \frac{i s_1\omega_1 D^{\eta_2\eta_3}_{k_2 k_3}}{g^2\left[(\eta_2\omega_2+\eta_3\omega_3)^2-\omega_{23}^2\right]} \Big[-(\eta_2\omega_2+\eta_3\omega_3)^2(\omega_1^2+\omega_{23}^2) + \omega_{23}^2(\omega_1^2+\omega_{23}^2)$$

$$-\omega_{23}^2\left(\omega_1^2+\omega_{23}^2\right)-\eta_1\omega_1\left(\eta_2\omega_2+\eta_3\omega_3\right)\left(\omega_1^2+\omega_{23}^2\right)\Big],$$

$$=\frac{A}{2}+\frac{i\,D^{\eta_2\eta_3}_{k_2k_3}\,\eta_1\omega_1\,k_{23}^2\,S_{23}^2}{2\left[\omega_{23}^2-\left(\eta_2\omega_2+\eta_3\omega_3\right)^2\right]}-\frac{iS_1\omega_1}{g^2}D^{\eta_2\eta_3}_{k_2k_3}\left(\omega_1^2+\omega_{23}^2\right)$$

$$-\frac{iS_1\omega_1\,D^{\eta_2\eta_3}_{k_2k_3}\left(\omega_1^2+\omega_{23}^2\right)}{g^2\left[\left(\eta_2\omega_2+\eta_3\omega_3\right)^2-\omega_{23}^2\right]}\left[\omega_{23}^2+\eta_1\omega_1\left(\eta_2\omega_2+\eta_3\omega_3\right)\right]$$

$$+\frac{i\,D^{\eta_2\eta_3}_{k_2k_3}}{\left(\eta_2\omega_2+\eta_3\omega_3\right)^2-\omega_{23}^2}\left[\left(\eta_2\omega_2+\eta_3\omega_3\right)\check{k}\cdot\check{k}_1+\eta_1\omega_1\,\check{k}\cdot\left(\check{k}_2+\check{k}_3\right)\right],$$

$$=\frac{A}{2}+B-\frac{i\left(S_1\omega_1+S_2\omega_2+S_3\omega_3\right)}{g^2\left[\left(\eta_2\omega_2+\eta_3\omega_3\right)^2-\omega_{23}^2\right]}\left(\omega_1^2+\omega_{23}^2\right)D^{\eta_2\eta_3}_{k_2k_3}$$

$$\times\left(\omega_{23}^2+\eta_1\omega_1\left(\eta_2\omega_2+\eta_3\omega_3\right)\right)+\frac{i\left(\eta_2\omega_2+\eta_3\omega_3\right)D^{\eta_2\eta_3}_{k_2k_3}}{g^2\left[\left(\eta_2\omega_2+\eta_3\omega_3\right)^2-\omega_{23}^2\right]}\left(\omega_1^2+\omega_{23}^2\right)$$

$$\times\left(\omega_{23}^2+\eta_1\omega_1\left(\eta_2\omega_2+\eta_3\omega_3\right)\right)+\frac{i\,D^{\eta_2\eta_3}_{k_2k_3}}{\left(\eta_2\omega_2+\eta_3\omega_3\right)^2-\omega_{23}^2}\Big(\left(\eta_2\omega_2+\eta_3\omega_3\right)\check{k}\cdot\check{k}_1$$

$$+\eta\omega\check{k}\left(\check{k}_2+\check{k}_3\right)-\left(\eta_2\omega_2+\eta_3\omega_3\right)\check{k}\cdot\left(\check{k}_2+\check{k}_3\right)\Big)+\frac{i\,D^{\eta_2\eta_3}_{k_2k_3}}{2\left(\omega_{23}^2-\left(\eta_2\omega_2+\eta_3\omega_3\right)^2\right)}\eta_1\omega_1\,k_{23}^2\,S_{23}^2,$$

$$= \frac{A}{2} + B + \frac{i s \omega D^{\eta_2 \eta_3}_{h_2 h_3}}{(\eta_2\omega_2 + \eta_3\omega_3)^2 - \omega_{23}^2} \left(- \frac{\omega_1^2 \omega_{23}^2}{q^2} + \tilde{k}_1 \cdot (\tilde{k}_2 + \tilde{k}_3) \right)$$

$$- \frac{i s \omega D^{\eta_2 \eta_3}_{h_2 h_3}}{q^2 \left[(\eta_2\omega_2 + \eta_3\omega_3)^2 - \omega_{23}^2 \right]} \left((\eta_1\omega_1)^3 (\eta_2\omega_2 + \eta_3\omega_3) + \omega_{23}^4 + \eta_1\omega_1 \omega_{23}^2 (\eta_2\omega_2 + \eta_3\omega_3) \right)$$

$$+ \frac{i (s_2\omega_2 + s_3\omega_3) D^{\eta_2 \eta_3}_{h_2 h_3}}{q^2 \left[(\eta_2\omega_2 + \eta_3\omega_3)^2 - \omega_{23}^2 \right]} \left(\omega_1^2 + \omega_{23}^2 \right) \left(\omega_{23}^2 + \eta_1\omega_1 (\eta_2\omega_2 + \eta_3\omega_3) \right)$$

$$+ \frac{i D^{\eta_2 \eta_3}_{h_2 h_3} \eta_1\omega_1 k_{23}^2 S_{23}^2}{2 \left(\omega_{23}^2 - (\eta_2\omega_2 + \eta_3\omega_3)^2 \right)} ,$$

$$= A + B - \frac{i s \omega D^{\eta_2 \eta_3}_{h_2 h_3}}{q^2 \left[(\eta_2\omega_2 + \eta_3\omega_3)^2 - \omega_{23}^2 \right]} \left((\eta_1\omega_1)^3 (\eta_2\omega_2 + \eta_3\omega_3) \right.$$

$$\left. + \omega_{23}^4 + \eta_1\omega_1 \omega_{23}^2 (\eta_2\omega_2 + \eta_3\omega_3) \right) + \frac{i (\eta_2\omega_2 + \eta_3\omega_3) D^{\eta_2 \eta_3}_{h_2 h_3}}{q^2 \left[(\eta_2\omega_2 + \eta_3\omega_3)^2 - \omega_{23}^2 \right]}$$

$$\times \left(\omega_1^2 + \omega_{23}^2 \right) \left(\omega_{23}^2 + \eta_1\omega_1 (\eta_2\omega_2 + \eta_3\omega_3) \right) \Big) \Big)$$

$$+\frac{iD^{\eta_2\eta_3}_{h_2h_3}}{(\eta_2\omega_2+\eta_3\omega_3)^2-\omega_{23}^2}\Big((\eta_2\omega_2+\eta_3\omega_3)\,\breve{k}\cdot\breve{h}_1+\eta\omega\,k_{23}^2-(\eta_2\omega_2+\eta_3\omega_3)\,\breve{k}(\breve{h}_1+\breve{h}_3)\Big)$$

$$+\frac{iD^{\eta_2\eta_3}_{h_2h_3}\;\eta_1\omega_1\;k_{23}^2S_{23}^2}{\omega_{23}^2-(\eta_2\omega_2+\eta_3\omega_3)^2},$$

$$=A+B-\frac{iD^{\eta_2\eta_3}_{h_2h_3}}{g^2\left[(\eta_2\omega_2+\eta_3\omega_3)^2-\omega_{23}^2\right]}\Big\{\quad\Big\}+\frac{iD^{\eta_2\eta_3}_{h_2h_3}\,\eta_1\omega_1\,h_{23}^2S_{23}^2}{(\eta_2\omega_2+\eta_3\omega_3)^2-\omega_{23}^2}.$$

$$\Big\{\quad\Big\}=\eta\omega(\eta_1\omega_1)^3(\eta_2\omega_2+\eta_3\omega_3)+\eta\omega\,\omega_{23}^4+\eta\omega\,\eta_1\omega_1(\eta_2\omega_2+\eta_3\omega_3)\omega_{23}^2$$

$$-(\eta_2\omega_2+\eta_3\omega_3)(\omega_1^2+\omega_{23}^2)\big(\omega_{23}^2+\eta_1\omega_1(\eta_2\omega_2+\eta_3\omega_3)\big)$$

$$-g^2\Big[(\eta_2\omega_2+\eta_3\omega_3)\,\breve{k}\cdot\breve{h}_1+\eta\omega(\breve{k}-\breve{h}_1)(\breve{h}_2+\breve{h}_3)$$

$$-(\eta_2\omega_2+\eta_3\omega_3)\,\breve{k}\cdot(\breve{h}_2+\breve{h}_3)\Big].$$

$$\Big[\quad\Big]=(\eta_2\omega_2+\eta_3\omega_3)\Big\{\breve{k}\cdot\breve{h}_1+(\breve{k}-\breve{h}_1)(\breve{h}_2+\breve{h}_3)-\breve{k}(\breve{h}_2+\breve{h}_3)\Big\}$$

$$+\eta_1\omega_1(\breve{h}_2+\breve{h}_3)^2,$$

$$=(\eta_2\omega_2+\eta_3\omega_3)\,k_1^2+\eta_1\omega_1(\breve{h}_2+\breve{h}_3)^2.$$

$$-g^2[\;\;] = -(s_2\omega_2 + s_3\omega_3)(\omega_1^4 + g^2 k_1^2 S_1^2)$$

$$- s_1\omega_1\,(\omega_{23}^4 + g^2 k_{23}^2 S_{23}^2).$$

Therefore

$$\{\;\;\} = s\omega(s_1\omega_1)^3(s_2\omega_2 + s_3\omega_3) + s\omega\,\omega_{23}^4$$

$$+ s\omega\, s_1\omega_1(s_2\omega_2 + s_3\omega_3)\omega_{23}^2$$

$$- (s_2\omega_2 + s_3\omega_3)\left(\omega_1^2\omega_{23}^2 + \omega_{23}^4 + (s_1\omega_1)^3(s_2\omega_2 + s_3\omega_3)\right)$$

$$- (s_2\omega_2 + s_3\omega_3)\,\omega_{23}^2\, s_1\omega_1\,(s_2\omega_2 + s_3\omega_3)$$

$$- (s_2\omega_2 + s_3\omega_3)(\omega_1^4 + g^2k_1^2S_1^2) - s_1\omega_1\,(\omega_{23}^4 + g^2k_{23}^2S_{23}^2),$$

$$= \omega_{23}^4\,(s\omega - s_1\omega_1 - s_2\omega_2 - s_3\omega_3)$$

$$+ \omega_{23}^2\,(s_2\omega_2 + s_3\omega_3)\left(s\omega\, s_1\omega_1 - \omega_1^2 - s_1\omega_1(s_2\omega_2 + s_3\omega_3)\right)$$

$$- s_1\omega_1\, g^2 k_{23}^2 S_{23}^2 + (s_2\omega_2 + s_3\omega_3)(s_1\omega_1)^3(s\omega - s_1\omega_1 - s_2\omega_2 - s_3\omega_3)$$

$$- g^2 k_1^2 S_1^2 (s_2\omega_2 + s_3\omega_3),$$

$$= - s_1\omega_1\, g^2 k_{23}^2 S_{23}^2 - g^2 k_1^2 S_1^2 (s_2\omega_2 + s_3\omega_3),$$

since $s_1\omega_1 + s_2\omega_2 + s_3\omega_3 = s\omega$.

Thus

$$\frac{s\omega \; s_2\omega_2 \; s_3\omega_3}{g^3}\,\omega_{23}^2 K^{s_2 s_3}_{k_2 k_3} + I$$

$$= A + B + C - g k_1^2 S_1^2 E^{s_2 s_3}_{k_2 k_3}$$

$$- \frac{i D^{s_2 s_3}_{k_2 k_3}}{g^2\left[(s_2\omega_2+s_3\omega_3)^2-\omega_{23}^2\right]}\left\{-s_1\omega_1 g^2 k_{23}^2 S_{23}^2 - g^2 k_1^2 S_1^2 (s_2\omega_2+s_3\omega_3)\right\}$$

$$+ \frac{i D^{s_2 s_3}_{k_2 k_3} \; s_1\omega_1 k_{23}^2 S_{23}^2}{\omega_{23}^2 - (s_2\omega_2+s_3\omega_3)^2} ,$$

$$= A + B + C - g k_1^2 S_1^2 E^{s_2 s_3}_{k_2 k_3}$$

$$+ \frac{i D^{s_2 s_3}_{k_2 k_3} \; k_1^2 S_1^2 (s_2\omega_2 + s_3\omega_3)}{(s_2\omega_2 + s_3\omega_3)^2 - \omega_{23}^2} .$$

$$\frac{2 s_1\omega_1 \, s_2\omega_2 \, s_3\omega_3}{i g^2} \, G^{s\, s_1\, s_2\, s_3}_{k\, k_1\, k_2\, k_3} - D^{s_1\, s_2\, s_3}_{k_1\, k_2\, k_3}$$

$$= \overset{123}{P} \; k_1^2 \,\mathrm{sech}^2|k_1|h \left[-g E^{s_2 s_3}_{k_2 k_3} + \frac{i(s_2\omega_2+s_3\omega_3)\, D^{s_2 s_3}_{k_2 k_3}}{(s_2\omega_2+s_3\omega_3)^2-\omega_{23}^2}\right].$$

This expression is not zero. However for the case when $h = \infty$ the two expressions agree, as $\operatorname{sech} \infty = 0$.

APPENDIX II

In this section the long time behavior of the Δ functions is examined. Consider

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} f(\omega)\,\Delta(\omega)\,d\omega.$$

It will be assumed that $f(\omega)$ is a continuously differentiable function of ω which vanishes sufficiently rapidly at $\pm\infty$.

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} f(\omega)\,t\,\Delta(\omega)\,d\omega$$

$$= \lim_{t\to\infty}\int_{-\infty}^{\infty} f(\omega)\,\frac{e^{i\omega t}-1}{i\omega}\,d\omega,$$

$$= \lim_{t\to\infty}\int_{-\infty}^{\infty} f(\omega)\left(\frac{\cos\omega t-1}{i\omega} + \frac{\sin\omega t}{\omega}\right)d\omega,$$

$$= \lim_{t\to\infty}\left[\int_{-\infty}^{-\varepsilon} f(\omega)\,\frac{\cos\omega t}{i\omega} + \int_{\varepsilon}^{\infty} f(\omega)\,\frac{\cos\omega t}{i\omega}\,d\omega\right.$$

$$+\left\{i\int_{-\infty}^{-\varepsilon} f(\omega)\,\frac{1}{\omega}\,d\omega + i\int_{-\infty}^{\varepsilon} f(\omega)\,\frac{1}{\omega}\,d\omega\right\}$$

$$\left.+\int_{-\varepsilon}^{\varepsilon} f(\omega)\,\frac{\cos\omega t-1}{i\omega}\,d\omega + \int_{-\infty}^{\infty} f(\omega)\,\frac{\sin\omega t}{\omega}\,d\omega\right].$$

In the integrals containing t, set $\omega t = x$.

$$\lim_{t \to \infty} \int_{-\infty}^{\infty} f(\omega)\, \Delta(\omega)\, d\omega ,$$

$$= \lim_{t \to \infty} \Big[\int_{-\infty}^{-\varepsilon} f\left(\tfrac{x}{t}\right) \frac{\cos x}{ix}\, dx + \int_{\varepsilon}^{\infty} f\left(\tfrac{x}{t}\right) \frac{\cos x}{ix}\, dx$$

$$+\, i \int_{-\infty}^{-\varepsilon} f(\omega)\, \frac{1}{\omega}\, d\omega + i \int_{+\varepsilon}^{\infty} f(\omega)\, \frac{1}{\omega}\, d\omega$$

$$+ \int_{-\varepsilon}^{\varepsilon} f\left(\tfrac{x}{t}\right) \frac{\cos x - 1}{ix}\, dx + \int_{-\infty}^{\infty} f\left(\tfrac{x}{t}\right) \frac{\sin x}{x}\, dx \Big].$$

The order one contributions in an asymptotic expansion in $\frac{1}{t}$ come from the third, fourth, and sixth terms. The first two cancel as $\frac{\cos x}{x}$ is an odd function. The fifth term gives an order ε term as $\frac{\cos x - 1}{x} = O(x)$ for small x. Using the result

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi ,$$

and taking the limit , one obtains

$$\lim_{t \to \infty} \int_{-\infty}^{\infty} f(\omega)\, \Delta(\omega)\, d\omega = \pi f(0) + iP \int_{-\infty}^{\infty} f(\omega) \frac{1}{\omega}\, d\omega .$$

We can therefore write schematically,

$$\Delta(\omega) \sim \pi \delta(\omega) + \frac{iP}{\omega} + O\left(\tfrac{1}{t}\right) + \cdots$$

If $f(\omega)$ is continuously differentiable it may be shown

$$\Delta(\omega) \sim \pi\delta(\omega) + \frac{iP}{\omega} + O(t^{-n}) + \cdots .$$

- - - - - - - - - - for any n .

Consider the function $F(t)$ where

$$F(t) = \int_0^t \Delta(\omega) e^{i\mu t} dt = \frac{\Delta(\omega+\mu) - \Delta(\mu)}{i\omega} = E(\omega+\mu, \mu).$$

When ω and μ are nonzero and independent

$$F(t) \sim O(1).$$

When either ω or μ is identically zero

$$F(t) = \frac{\Delta(0) - \Delta(\mu)}{-i\mu} \quad \left(E(0,\mu)\right)$$

$$\text{or} \quad \frac{\Delta(\omega) - \Delta(0)}{i\omega} \quad \left(E(\omega, 0)\right).$$

Consider

$$\int_{-\infty}^{\infty} f(\omega) \frac{\Delta(\omega) - \Delta(0)}{i\omega} d\omega,$$

$$= \int_{-\infty}^{\infty} f(\omega) \, \frac{e^{i\omega t} - 1 - i\omega t}{(i\omega)^2} \, d\omega,$$

$$= \int_{-\infty}^{\infty} f(\omega) \left(e^{i\omega t} - 1 - i\omega t\right) \, d\left(\tfrac{1}{\omega}\right),$$

$$= -\int_{-\infty}^{\infty} f(\omega) \, \frac{ite^{i\omega t} - it}{\omega} \, d\omega - \int_{-\infty}^{\infty} \frac{\partial f}{\partial \omega} \, \frac{e^{i\omega t} - 1 - i\omega t}{\omega} \, d\omega,$$

$$= t \int_{-\infty}^{\infty} f(\omega) \, \frac{e^{i\omega t} - 1}{i\omega} \, d\omega - i \int_{-\infty}^{\infty} f_\omega(\omega) \, \frac{e^{i\omega t} - 1}{i\omega} \, d\omega$$

$$+ it \int_{-\infty}^{\infty} f_\omega \, d\omega.$$

The final integral is zero as $f(\omega)$ vanishes at the end points of the interval.

Hence, schematically

$$\frac{\Delta(\omega) - \Delta(0)}{i\omega} = E(\omega, 0) \sim \left(\dot{\pi}\,\delta(\omega) + \frac{iP}{\omega}\right)\left(t - i\frac{\partial}{\partial \omega}\right) + \cdots.$$

If ω is never zero in the spectrum

$$\delta(\omega) = 0,$$

$$E(\omega, 0) \sim \frac{i}{\omega}\left(t - i\frac{\partial}{\partial \omega}\right).$$

Similarly,

$$\frac{\Delta(0)-\Delta(\mu)}{-i\mu} \sim \left(\pi\delta(\mu)+\frac{iP}{\mu}\right)\left(t-i\frac{\partial}{\partial\mu}\right)+\cdots$$

When $\omega = 0$, there is no "t" growth, for then

$$F(t) = E(\mu,\mu) = \int_0^t t e^{i\mu t} dt,$$

$$= \frac{t e^{i\mu t}}{i\mu} - \frac{\Delta(\mu)}{i\mu}.$$

Consider

$$\int_{-\infty}^{\infty} f(\mu) \frac{i\mu t e^{i\mu t} - e^{i\mu t} + 1}{(i\mu)^2} d\mu$$

$$= \int_{-\infty}^{\infty} f(\mu) \left(i\mu t e^{i\mu t} - e^{i\mu t} + 1\right) d\left(\frac{1}{\mu}\right),$$

$$= -\int_{-\infty}^{\infty} f_\mu \frac{i\mu t e^{i\mu t} - e^{i\mu t} + 1}{\mu} d\mu - \int_{-\infty}^{\infty} f(\mu) \frac{it e^{i\mu t} - \mu t^2 e^{i\mu t} - it e^{i\mu t}}{\mu} d\mu,$$

$$= -it \int_{-\infty}^{\infty} f_\mu(\mu)\, e^{i\mu r} d\mu + i \int_{-\infty}^{\infty} f_\mu(\mu)\, \frac{e^{i\mu r}-1}{i\mu}\, d\mu$$

$$+ t^2 \int_{-\infty}^{\infty} f(\mu)\, e^{i\mu r} d\mu ,$$

$$= -t^2 \int_{-\infty}^{\infty} f(\mu)\, e^{i\mu r} d\mu + t^2 \int_{-\infty}^{\infty} f(\mu)\, e^{i\mu r} d\mu$$

$$+ i \int_{-\infty}^{\infty} f_\mu(\mu)\, \frac{e^{i\mu r}-1}{i\mu}\, d\mu .$$

Therefore, schematically

$$E(\mu,\mu) \sim \left(\pi \delta(\mu) + \frac{iP}{\mu}\right) i \frac{\partial}{\partial\mu} .$$

The following are a set of relevant properties of the $\Delta(\omega)$ and $E(\omega,\mu)$ functions.

$$E(\omega,0) \sim \left(\pi\delta(\omega) + \frac{iP}{\omega}\right)\left(t - i\frac{\partial}{\partial\omega}\right).$$

$$E(\omega,\omega) \sim \left(\pi\delta(\omega) + \frac{iP}{\omega}\right) i \frac{\partial}{\partial\omega}$$

$$E(0,0) = \int_0^t t\, dt = \frac{t^2}{2} .$$

$$E(-\omega, 0)\, e^{i\omega t} = E(\omega, \omega).$$

$$E(\omega, \omega) = t\, \underline{\Delta}(\omega) - E(\omega, 0).$$

$$\frac{E(\nu+\mu, \nu)}{i\mu}\, e^{-i\nu t} = E(\mu, -\nu).$$

$$\underline{\Delta}(-\omega)\, e^{i\omega t} = \underline{\Delta}(\omega).$$

Consider

$$\int_{-\infty}^{\infty} f(\omega)\, \Delta(\omega)\, \Delta(-\omega)\, d\omega,$$

$$= \int_{-\infty}^{\infty} f(\omega)\, \frac{e^{i\omega t}-1}{i\omega}\, \frac{e^{-i\omega t}-1}{-i\omega}\, d\omega,$$

$$= -2 \int_{-\infty}^{\infty} f(\omega)\, (1-\cos\omega t)\, d\left(\frac{1}{\omega}\right),$$

$$= 2\int_{-\infty}^{\infty} f_{\omega}(\omega)\, \frac{1-\cos\omega t}{\omega}\, d\omega + 2t\int_{-\infty}^{\infty} f(\omega)\, \frac{\sin\omega t}{\omega}\, d\omega,$$

In the limit t

$$= 2P\int_{-\infty}^{\infty} f_{\omega}(\omega)\, \frac{1}{\omega}\, d\omega + 2t\int_{-\infty}^{\infty} f(\omega)\, \pi\delta(\omega)\, d\omega.$$

Hence schematically,

$$\Delta(\omega)\, \Delta(-\omega) \sim 2\pi t\, \delta(\omega) + 2P.\frac{1}{\omega}\frac{\partial}{\partial\omega}.$$

Consider $F(t) = \int_0^t \Delta(\omega)\,\Delta(\mu)\,dt$, μ a Triad,

$$= \int_0^t \frac{1}{i\mu}\,\Delta(\omega)\left(e^{i\mu t}-1\right)dt,$$

$$= \frac{1}{i\mu}\Big(E(\omega+\mu,\mu) - E(\omega,0)\Big)\,dt$$

$$\sim \frac{it}{\mu}\left(\pi\delta(\omega) + \frac{iP}{\omega}\right) + O(1).$$

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Consider $F(t) = \int_0^t \frac{\Delta(-\mu)-\Delta(-\omega-\mu)}{-i\omega}\,e^{i\mu t}\,dt,$ ω a quartet, μ a triad, $\omega+\mu$ a Triad,

$$= \int_0^t -E(-\mu,-\omega-\mu)\,e^{i\mu t}\,dt,$$

$$= -\int_0^t E(\mu,-\omega)\,dt,$$

$$= -\int_0^t \frac{\Delta(\mu)-\Delta(-\omega)}{i(\mu+\omega)}\,dt,$$

$$= -\frac{1}{i(\omega+\mu)}\Big(E(\mu,0) - E(-\omega,0)\Big),$$

$$\sim -\frac{t}{i(\omega+\mu)}\left(\frac{i}{\mu} - \pi\delta(\omega) + \frac{iP}{\omega}\right) + O(1),$$

$$\sim \frac{t}{i\mu}\left(\pi\delta(\omega) - \frac{iP}{\omega}\right) + O(1).$$

-- -- -- --

$$F(t) = -\frac{1}{i\nu}\int_0^t \Delta(-\mu)\left[\Delta(-\omega) - \Delta(\nu-\omega)\right] e^{i(\omega+\mu)t}\,dt, \quad \mu, \nu \text{ Friends, } \omega \text{ quartet}$$

$$= -\frac{1}{i\nu}\int_0^t \frac{e^{i\omega t} - e^{i(\omega+\mu)t}}{-i\mu}\left(\Delta(-\omega) - \Delta(\nu-\omega)\right) dt,$$

$$= \frac{1}{i\nu\, i\mu}\left[\int_0^t \left(\Delta(\omega) - \Delta(-\omega)\, e^{i(\omega+\mu)t} - \Delta(\nu-\omega) e^{i\omega t} + \Delta(\nu-\omega) e^{i(\omega+\mu)t}\right) dt\right],$$

$$= \frac{1}{i\nu\, i\mu}\left[E(\omega,0) - E(\mu,\omega+\mu) - E(\nu,\omega) + E(\nu+\mu,\omega+\mu)\right]$$

$$\sim \frac{t}{i\nu\, i\mu}\left(\pi\delta(\omega) + \frac{iP}{\omega}\right) + O(1).$$

-- -- -- --

$$F(t) = \int_0^t \frac{1}{i(\omega-\nu)}\left(E(\omega-\mu,-\mu) - E(\nu-\mu,-\mu)\right) e^{i\mu t}\,dt,$$

$$= \int_0^t \frac{1}{i(\omega-\nu)}\left(E(\mu,\omega) - E(\mu,\nu)\right) dt, \text{ only } \omega \text{ a quartet,}$$

since $E(\omega-\mu,-\mu)\, e^{i\mu t} = E(\omega,\mu)$.

$$= \frac{1}{i(\omega-\nu)}\left[\frac{1}{i(\mu-\omega)}\left(E(\mu,0) - E(\omega,0)\right) - \frac{1}{i(\mu-\nu)}\left(E(\mu,0) - E(\nu,0)\right)\right],$$

$$\sim \frac{t}{(i(\omega-\nu)\, i(\mu-\omega))} \left\{ \frac{i}{\mu} - \pi\delta(\omega) - \frac{iP}{\omega} \right\}$$

$$- \frac{t}{i(\omega-\nu)\, i(\mu-\nu)} \left\{ \frac{i}{\mu} - \frac{i}{\nu} \right\} + O(1),$$

$$\sim \frac{t\,\pi\,\delta(\omega)}{-\mu\nu} + \frac{t}{i(\omega-\nu)\, i(\mu-\omega)} \left\{ \frac{iP.(\omega-\mu)}{\omega\mu} \right\}$$

$$- \frac{t}{i(\omega-\nu)\, i(\mu-\nu)} \left\{ \frac{i(\nu-\mu)}{\mu\nu} \right\} + O(1),$$

$$\sim \frac{t}{-\mu\nu}\,\pi\delta(\omega) - \frac{t \cdot P}{i(\omega-\nu)\,\omega\mu} + \frac{t}{i(\omega-\nu)\,\mu\nu} + O(1),$$

$$\sim \frac{t}{-\mu\nu}\,\pi\delta(\omega) - \frac{t}{i(\omega-\nu)\mu}\, P. \left\{ \frac{1}{\omega} - \frac{1}{\nu} \right\} + O(1),$$

$$\sim \frac{t}{-\mu\nu}\,\pi\delta(\omega) + \frac{t \cdot P}{i\omega\mu\nu} + O(1),$$

$$\sim \frac{t}{-\mu\nu} \left(\pi\delta(\omega) + \frac{iP}{\omega} \right) + O(1).$$

$$F(t) = -\frac{1}{i\mu}\int_0^t \Big(E(\nu,\nu) - E(\nu+\mu,\nu)\Big)\, e^{-i\nu t}\, dt,$$

$\nu,\ \mu,\ \nu+\mu$ Friuds,

$$= -\frac{1}{i\mu}\int_0^t \Big(E(-\nu,0) - E(-\nu,\mu)\Big)\, dt,$$

$$= -\frac{1}{i\mu}\left[\frac{1}{-i\nu}\left\{E(-\nu,0) - \frac{t^2}{2}\right\} - \frac{1}{i(-\nu-\mu)}\left\{E(-\nu,0) - E(\mu,0)\right\}\right]$$

$$\sim \frac{t^2}{2\mu\nu} + \frac{t(\nu-\mu)}{i\mu^2\nu^2} + O(1).$$

A relation which is frequently used, is

$$E(-\mu, -\omega-\mu)\, e^{i\mu t} = E(\mu, -\omega).$$

Proof

$$\text{L.H.S} = \frac{\Delta(-\mu) - \Delta(-\omega-\mu)}{i\omega}\, e^{i\mu t}$$

$$= \frac{\Delta(\mu)}{i\omega} + \frac{1}{i\omega}\,\frac{e^{-i\omega t} - e^{i\mu t}}{i(\omega+\mu)},$$

$$= \frac{e^{i\mu t}-1}{i\omega}\left\{\frac{1}{i\mu} - \frac{1}{i(\omega+\mu)}\right\} + \frac{1}{i(\omega+\mu)}\,\frac{e^{-i\omega t}-1}{i\omega},$$

$$= \frac{\dfrac{e^{i\mu t}-1}{i\mu} - \dfrac{e^{-i\omega t}-1}{-i\omega}}{i(\mu+\omega)},$$

$$= \frac{\Delta(\mu) - \Delta(-\omega)}{i(\mu+\omega)},$$

$$= E(\mu, -\omega).$$

Appendix III

The Ordering Procedure.

If one considers $\overset{(3)}{\varphi}_1{}^{\nu\nu'\nu''}(k,k')$ and looks for its long time behavior, one finds that one cannot do all of the asymptotics in Fourier space.

$$\delta(k+k'+k'')\overset{(3)}{\varphi}_1{}^{\nu\nu'\nu''}(k,k') = \rho^{00'0''}\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{kk_1k_2}^{\nu\nu_1\nu_2}\langle a_{0'}^{\nu'} a_{0''}^{\nu''} a_{01}^{\nu_1} a_{02}^{\nu_2}\rangle \Delta_{12,0}\, d\breve{k}_1 d\breve{k}_2.$$

The mean value $\langle a_{0'}^{\nu'} a_{0''}^{\nu''} a_{01}^{\nu_1} a_{02}^{\nu_2}\rangle$ decomposes into

$$\delta_{0'0''12}\overset{(4)}{\varphi}_0{}^{\nu'\nu''\nu_1\nu_2}(k',k'',k_1) + \delta_{0'0''}\delta_{12}\overset{(2)}{\varphi}_0{}^{\nu'\nu''}(k')\overset{(2)}{\varphi}_0{}^{\nu_1\nu_2}(k_1) + 2\delta_{0'1}\delta_{0''2}\overset{(2)}{\varphi}_0{}^{\nu'\nu_1}(k')\overset{(2)}{\varphi}_0{}^{\nu''\nu_2}(k'').$$

The first term gives the following behavior in long time

$$\rho^{00'0''}\delta_{00'0''}\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{kk_1k_2}^{\nu\nu_1\nu_2}\overset{(4)}{\varphi}_0{}^{\nu'\nu''\nu_1\nu_2}(k',k'',k_1)\left(\pi\delta_{12,0}^{\nu} + \frac{iP}{W_{12,0}}\right)\delta_{12,0}\, d\breve{k}_1 d\breve{k}_2,$$

which is a continuous function in Fourier space. One may therefore choose the corresponding cumulant formed from the free terms in such a way that it removes this O(1) term.

The second term in the above decomposition is zero identically as δ_{12} reacts with $\delta_{12,0}$ to give $\delta(\breve{k})$ and thus the expression is zero due to the zero mean value property.

However, when one integrates over $\breve{k}_1$ and $\breve{k}_2$ the third term becomes $\rho^{00'0''}2\delta_{00'0''}\sum_{\nu_1\nu_2} L_{k-k'-k''}^{\nu\nu_1\nu_2}\overset{(2)}{\varphi}_0{}^{\nu'\nu_1}(k')\overset{(2)}{\varphi}_0{}^{\nu''\nu_2}(k'')\,\Delta(s_1\omega'+s_2\omega''-s\omega)$.

In this term the asymptotics must be performed in physical space, and in general one cannot choose the cumulants formed from the free terms in order to suppress this expression. The resulting behavior in physical space is

$$\overset{(3)}{R}_1(\tilde{p},\tilde{p}') = P\sum_{ss's''}^{00'0''}\int_{-\infty}^{\infty} 2\sum_{s_1s_2} \ell_{k-k'-k''}^{s\,s_1\,s_2}\,\overset{(2)}{Q}_0^{s's_1}(k')\,\overset{(2)}{Q}_0^{s''s_2}(k'')\,e^{-ik\cdot\tilde{p}-ik'\cdot\tilde{p}'}$$

$$\Delta(s_1\omega'+s_2\omega''-s\omega)\,e^{i(s\omega+s'\omega'+s''\omega'')t}\,\delta(k+k'+k'')\,dk\,dk'\,dk''.$$

The time "t" dependent terms $\Delta(s_1\omega'+s_2\omega''-s\omega)\,e^{i(s\omega+s'\omega'+s''\omega'')t}$ may be written

$$\Delta(-s\omega+s_1\omega'+s_2\omega'')\,e^{i(s\omega-s_1\omega'-s_2\omega'')t}\,e^{i((s'+s_1)\omega'+(s''+s_2)\omega'')t}$$

which is

$$\Delta(s\omega-s_1\omega'-s_2\omega'')\,e^{i((s_1+s')\omega'+(s_2+s'')\omega'')t}.$$

Since the arguments of both the Δ function and the exponential are now independent for all choices of the sign parameters, one may perform the asymptotics in physical space using $s\omega-s_1\omega'-s_2\omega''$ as one variable and $(s'+s_1)\omega'+(s''+s_2)\omega''$ as the other. Performing the asymptotics over $s\omega-s_1\omega'-s_2\omega''$ first, one obtains

$$\overset{(3)}{R}_1(\tilde{p},\tilde{p}') \sim P\sum_{ss's''}^{00'0''}\int_{-\infty}^{\infty} 2\sum_{s_1s_2} \ell_{k\,k_1\,k_2}^{s\,s_1\,s_2}\,\overset{(2)}{Q}_0^{s's_1}(k')\,\overset{(2)}{Q}_0^{s''s_2}(k'')\,e^{-ik\cdot\tilde{p}-ik'\cdot\tilde{p}'}$$

$$\left(\pi\delta(s\omega-s_1\omega'-s_2\omega'')+\frac{iP}{s\omega-s_1\omega'-s_2\omega''}\right)e^{i((s_1+s')\omega'+(s_2+s'')\omega'')t}\,\delta(k+k'+k'')\,dk\,dk'\,dk''.$$

One sees now that there are a special class of these "live" functions which can remove the fast oscillation, namely

$\nu_1 = -\nu'$, $\nu_2 = -\nu''$, whereupon one obtains that

$$\overset{(3)}{R_1}(\tilde{p}, \tilde{p}') \sim \rho \sum_{\nu\nu'\nu''}^{00'0''} \int_{-\infty}^{\infty} \lambda\, \mathcal{L}_{k\ -k'\ -k''}^{\nu\ -\nu'\ -\nu''}\ \overset{(2)}{\varphi}_0^{\nu'-\nu'}(k')\ \overset{(2)}{\varphi}_0^{\nu''-\nu''}(k'')$$

$$e^{-i\tilde{k}\cdot\tilde{p} - i\tilde{k}'\cdot\tilde{p}'} \Big(\pi\, \delta(s\omega + s'\omega' + s''\omega'')$$

$$+ \frac{i P}{s\omega + s'\omega' + s''\omega''} \Big)\, \delta(\tilde{k} + \tilde{k}' + \tilde{k}'')\, d\tilde{k}\, d\tilde{k}'\, d\tilde{k}''$$

$$+ O\left(\frac{1}{t}\right),$$

As, when $\nu_1 = -\nu'$, $\nu_2 = -\nu''$ does not occur simultaneously, the Riemann Lebesgue lemma gives that the remaining terms behave in physical space like $\frac{1}{t}$ at the very most.

Since these live terms if viewed from Fourier space have a cusp like behavior the cumulants found from the free terms cannot be used to suppress them. It is shown that in fact it is not necessary to suppress them as they never reoccur as secular growths. This was demonstrated for the case of the quasisteady terms in $Q_3^{(3)\,\nu\nu'\nu''}(k,k')$. The following analysis shows that they do not reoccur as "t" growths in $Q_4^{(4)\,\nu\nu'\nu''\nu'''}(k,k',k'')$.

It is clear from the preceding analysis that the same results would be found if one had replaced $L_{kk_1k_2k_3}^{\nu\nu_1\nu_2\nu_3}$ by $G_{kk_1k_2k_3}^{\nu\nu_1\nu_2\nu_3}$ and neglected the $O(\varepsilon^2)$ terms in the governing equation. This is valid for the case when triads cannot vanish anywhere in the spectrum. Thus, one considers the following governing equation

$$a_t^{\nu} = \varepsilon^2 \sum_{\nu_1\nu_2\nu_3} \int_{-\infty}^{\infty} G_{kk_1k_2k_3}^{\nu\nu_1\nu_2\nu_3} a_1^{\nu_1} a_2^{\nu_2} a_3^{\nu_3} e^{iW_{123,0}t} \delta_{123,0}\, dk_1 dk_2 dk_3 .$$

By setting $T_0 = t$, $T_2 = \varepsilon^2 t$ and matching powers of in the above expression, one obtains

$$a_0^{\nu} = a_0^{\nu}(k, T_2),$$

$$a_2^{\nu} = -t\,\frac{\partial a_0^{\nu}}{\partial T_2} + b_2^{\nu} + \alpha_2^{\nu}(k, T_2),$$

$$a_4^{\nu} = -t\,\frac{\partial a_0^{\nu}}{\partial T_4} - t\,\frac{\partial a_2^{\nu}}{\partial T_2} - \frac{t^2}{2}\,\frac{\partial^2 a_0^{\nu}}{\partial T_2^2} + {}_3d_4^{\nu} + {}_2\delta_4^{\nu},$$

where b_2^{γ}, d_2^{γ}, ${}_3d_4^{\gamma}$ and ${}_2\delta_4^{\gamma}$ are as defined previously with $L_{kk_1k_2k_3}^{\gamma\gamma_1\gamma_2\gamma_3}$ replaced by $g_{kk_1k_2k_3}^{\gamma\gamma_1\gamma_2\gamma_3}$.

One now examines the occurence and possible reoccurence as "t" growths of the live time dependent terms in Fourier space. These first occur in $\overset{(4)}{\phi}{}_2^{\gamma\gamma'\gamma''\gamma'''}(k,k',k'')$. One considers therefore

$$\langle a_{0'}^{\gamma'} a_{0''}^{\gamma''} a_{0'''}^{\gamma'''} b_2^{\gamma} \rangle = \sum_{\gamma_1\gamma_2\gamma_3} \int_{-\infty}^{\infty} g_{kk_1k_2k_3}^{\gamma\gamma_1\gamma_2\gamma_3} \langle a_{0'}^{\gamma'} a_{0''}^{\gamma''} a_{0'''}^{\gamma'''} a_{01}^{\gamma_1} a_{02}^{\gamma_2} a_{03}^{\gamma_3} \rangle \Delta_{123,0}\, \delta_{123,0}\, dk_1\, dk_2\, dk_3.$$

The "live" terms occur on the decompositions $\delta_{0'1}\delta_{0''2}\delta_{0'''3}$ and are typified by the term

$$\sum_{\gamma_1\gamma_2\gamma_3} g_{k-k'-k''-k'''}^{\gamma\gamma_1\gamma_2\gamma_3} \overset{(2)}{\phi}{}_0^{\gamma'\gamma_1}(k') \overset{(2)}{\phi}{}_0^{\gamma''\gamma_2}(k'') \overset{(2)}{\phi}{}_0^{\gamma'''\gamma_3}(k''') \Delta(s_1\omega'+s_2\omega''+s_3\omega'''-s\omega).$$

In order to suppress "t" growths in the remaining continuous terms in Fourier space the familiar choices given on page have been made. These are

$$\frac{\partial \overset{(2)}{\phi}{}_0^{\gamma-\gamma}(k)}{\partial T_2} = 0,$$

$$\frac{\partial \overset{(2)}{\phi}{}_0^{\gamma\gamma'}(k)}{\partial T_2} = \overset{(2)}{\phi}{}_0^{\gamma\gamma'}(k)\left(\overset{(0)}{F}{}_k^{s} + \overset{(0)}{F}{}_{-k}^{s'}\right),$$

$$\frac{\partial \overset{(r)}{Q}_0{}^{\gamma_1 \cdots \gamma_r}(\breve{k}_1, \cdots \breve{k}_{r-1})}{\partial T_2} = \overset{(r)}{Q}_0{}^{\gamma_1 \cdots \gamma_r}(\breve{k}_1 \cdots \breve{k}_{r-1})\left(\overset{(0)}{F}{}^{s_1}_{k_1} + \cdots + \overset{(0)}{F}{}^{s_r}_{-k_1 \cdots k_{r-1}}\right),$$

$$\text{when} \quad \overset{(0)}{F}{}^{s}_{k} = 3\sum_{\gamma_2}\int_{-\tilde{\infty}}^{\tilde{\infty}} g^{\gamma\gamma\gamma_2-\gamma_2}_{kk\,k_2-k_2}\, \overset{(2)}{Q}_0{}^{\gamma_2-\gamma_2}(k_2)\, d\breve{k}_2 .$$

The cumulants formed from the free terms which one may denote as $\{\alpha_2^{\gamma}\, a_{0'}^{\gamma'}\, a_{0''}^{\gamma''} \cdots \}$ have been chosen to suppress the order one continuous behavior of $\overset{(r)}{Q}_2{}^{\gamma_1 \cdots \gamma_r}(\breve{k}_1, \cdots \breve{k}_{r-1})$, and do not include the "live" terms. It is therefore required to show that these live terms do not reoccur as "t" growths. These terms can reoccur from the following terms in $\delta_{00'0''0'''}\, \overset{(4)}{Q}_4{}^{\gamma\gamma'\gamma''\gamma'''}(k, k', k'')$,

$$-t\,\delta_{00'0''0'''}\frac{\partial \overset{(4)}{Q}_2{}^{\gamma\gamma'\gamma''\gamma'''}(k, k', k'')}{\partial T_2}, \quad P\{a_{0''}^{\gamma''} a_{0'''}^{\gamma'''} b_2^{\gamma} b_{2'}^{\gamma'}\}, \quad P\{a_{0'}^{\gamma'} a_{0''}^{\gamma''} a_{0'''}^{\gamma'''}, d_4^{\gamma}\}.$$

The live terms which are possible "t" growths from these three expressions can be shown to be respectively

$$-t\,\delta_{00'0''0'''}\frac{\partial}{\partial T_2} P \sum_{\gamma_1\gamma_2\gamma_3}^{00'0''0'''} g^{\gamma\gamma_1\gamma_2\gamma_3}_{k-k'-k''-k'''}\, \overset{(2)}{Q}_0{}^{\gamma'\gamma_1}(k')\, \overset{(2)}{Q}_0{}^{\gamma''\gamma_2}(k'')\, \overset{(2)}{Q}_0{}^{\gamma'''\gamma_3}(k''')\, \Delta(s_1\omega' + s_2\omega'' + s_3\omega''' - s\omega),$$

$$E(x, y) = \frac{\Delta(x) - \Delta(y)}{i(x-y)}.$$

$$\delta_{00'0''0'''} P \sum_{\gamma_1\gamma_2\gamma_3}^{00'0''0'''} g^{\gamma\gamma_1\gamma_2\gamma_3}_{k-k'-k''-k'''}\, \overset{(2)}{Q}_0{}^{\gamma'\gamma_1}(k')\, \overset{(2)}{Q}_0{}^{\gamma''\gamma_2}(k'')\, \overset{(2)}{Q}_0{}^{\gamma'''\gamma_3}(k''')\, E(s_1\omega' + s_2\omega'' + s_3\omega''' - s\omega,\ \gamma_1\omega' + s_2\omega'' + s_3\omega''' - s\omega) \times \left(\overset{(0)}{F}{}^{s_1}_{-k'} + \overset{(0)}{F}{}^{s_2}_{-k''} + \overset{(0)}{F}{}^{s_3}_{-k'''}\right),$$

$$t\,\delta_{00'0''0'''} P \sum_{\gamma_1\gamma_2\gamma_3}^{00'0''0'''} g^{\gamma\gamma_1\gamma_2\gamma_3}_{k-k'-k''-k'''}\, \overset{(2)}{Q}_0{}^{\gamma'\gamma_1}(k')\, \overset{(2)}{Q}_0{}^{\gamma''\gamma_2}(k'')\, \overset{(2)}{Q}_0{}^{\gamma'''\gamma_3}(k''')\, \Delta(s_1\omega' + s_2\omega'' + s_3\omega''' - s\omega)\left(F^{s'}_{k'} + F^{s''}_{k''} + F^{s'''}_{k'''}\right).$$

One multiplies each of these terms by the fast oscillation

$$e^{i(s\omega + s'\omega' + s''\omega'' + s'''\omega''')t}$$

and writes,

$$\Delta(s_1\omega' + s_2\omega'' + s_3\omega''' - s\omega)\, e^{i(s\omega + s'\omega' + s''\omega'' + s'''\omega''')t}$$

as

$$\Delta(s\omega - s_1\omega' - s_2\omega'' - s_3\omega''')\, e^{i((s'+s_1)\omega' + (s_2+s'')\omega'' + (s_3+s''')\omega''')t},$$

and

$$E(s_1\omega' + s_2\omega'' + s_3\omega''' - s\omega,\ s_1\omega' + s_2\omega'' + s_3\omega''' - s\omega)\, e^{i(s\omega + s'\omega' + s''\omega'' + s'''\omega''')t}$$

as

$$E(s_1\omega' + s_2\omega'' + s_3\omega''' - s\omega,\ s_1\omega' + s_2\omega'' + s_3\omega''' - s\omega)\, e^{i(s\omega - s_1\omega' - s_2\omega'' - s_3\omega''')t}\, e^{i((s_1+s')\omega' + (s_2+s'')\omega'' + (s_3+s''')\omega''')t},$$

which, by using $E(\omega,\omega)\, e^{-i\omega t}$ and $E(-\omega,0) = t\,\Delta(-\omega) - E(-\omega,-\omega)$,

{See Appendix II} is $\left(t\,\Delta(s\omega - s_1\omega' - s_2\omega'' - s_3\omega''') + O(1)\right) e^{i((s'+s_1)\omega' + (s''+s_2)\omega'' + (s'''+s_3)\omega''')t}$.

Using the fact that

$$\frac{\partial \overset{(2)\,\gamma' s_1}{\varphi_0}(k')}{\partial T_2} = \overset{(2)\,\gamma' s_1}{\varphi_0}(k') \left(\overset{(0)s'}{F}_{k'} + \overset{(0)s_1}{F}_{-k'} \right),$$

one sees that

$$\frac{\partial\, \overset{(2)\,\gamma' s_1}{\varphi_0}(k')\, \overset{(2)\,\gamma'' s_2}{\varphi_0}(k'')\, \overset{(2)\,\gamma''' s_3}{\varphi_0}(k''')}{\partial T_2} = \overset{(2)\,\gamma' s_1}{\varphi_0}(k')\, \overset{(2)\,\gamma'' s_2}{\varphi_0}(k'')\, \overset{(2)\,\gamma''' s_3}{\varphi_0}(k''')$$

$$\left(\overset{(0)s'}{F}_{k'} + \overset{(0)s_1}{F}_{-k'} + \overset{(0)s''}{F}_{k''} + \overset{(0)s_2}{F}_{-k''} + \overset{(0)s'''}{F}_{k'''} + \overset{(0)s_3}{F}_{-k'''} \right).$$

Adding the secular contributions from these three "live" terms one obtains

$$t\, P^{\nu 0' 0'' 0'''}_{\delta 0 0' 0'' 0'''} \sum_{\nu_1 \nu_2 \nu_3} g^{\nu \nu_1 \nu_2 \nu_3}_{k-k'-k''-k'''} \, Q^{(2)|\nu' \nu_1}_0(k')\, Q^{(2)|\nu'' \nu_2}_0(k'')\, Q^{(2)|\nu''' \nu_3}_0(k''')$$

$$\Delta(s\omega - s_1\omega' - s_2\omega'' - s_3\omega''')\, e^{i((s_1+s')\omega' + (s_2+s'')\omega'' + (s_3+s''')\omega''')t}$$

$$\Big[-F^{(0)|s'}_{k'} - F^{(0)|s''}_{k''} - F^{(0)|s'''}_{k'''} - F^{(0)|s_1}_{-k'} - F^{(0)|s_2}_{-k''} - F^{(0)|s_3}_{-k'''}$$

$$+ F^{(0)|s'}_{k'} + F^{(0)|s''}_{k''} + F^{(0)|s'''}_{k'''} + F^{(0)|s_1}_{-k'} + F^{(0)|s_2}_{-k''} + F^{(0)|s_3}_{-k'''}\Big]$$

$$= 0.$$

Therefore it is shown that the live terms do not return as secular growths, and that the ordering procedure remains valid in physical space with the preceding choices of $\dfrac{\partial Q^{(r)|\nu_1 \cdots \nu_r}_0(k_1 \cdots k_{r-1})}{\partial T_m}$, $m = 2, 4$ and $\{ a_2^{\nu}\, a_0^{\nu'}\, a_0^{\nu''} \cdots \}$.

CHAPTER II

Introduction.

Many efforts have been made in recent years to explain the generation of surface waves on the sea by a movement of air across the water. Jeffries [1925] proposed a sheltering mechanism which took account of the fact that if waves were already present on the sea, the pressure varies around the contour of the wave. Miles [7] proposed that instabilities (caused by Reynolds stress phase shifts) occurred because of viscous layers associated with the shear.

In 1957 Phillips [9], suggested the mechanism of direct resonance as a possible way of exciting free waves. This phenomenon is readily explained by the fact that the frequency generated by taking the Fourier transform of a convecting pressure distribution can be the same, for a certain class of wave vectors $\tilde{k}$ as the frequencies exhibited by the free wave problem. Phillips considered the excitation of these waves from an initially tranquil sea and showed that the ensemble average of the mean square elevation grew linearly with time.

Here we present a preliminary analysis extending the mechanism suggested by Phillips to the state where nonlinear terms in the inviscid model become important. It has already been shown in Chapter I that these nonlinear terms can introduce an internal resonance mechanism by which energy density can be transferred between different wave components.

The model used is that of random (spatially homogeneous) sea over which a random (and again, spatially homogeneous) normal pressure distribution is moving with constant velocity $\tilde{U}$. The analysis is carried to a stage where a balance exists between the energy being fed into the sea by the external pressure distribution and the energy being shared by different wave components in the sea due to the surface tension resonance mechanism. Equations are found which illustrate the way in which external energy can be redistributed throughout the spectrum by means of the latter mechanism.

Analysis.

The equations describing the motion of an irrotational, incompressible fluid, over whose surface a convected random pressure distribution is moving, are the same as those equations derived in Chapter I, with the exception of Bernoulli's equation at the free surface. Thus,

$$\nabla^2\varphi = 0. \qquad \text{--(1)}$$

$$\frac{\partial\varphi}{\partial z} = 0, \qquad z = -h. \qquad \text{--(2)}$$

$$\frac{\partial\varphi}{\partial t} + g\zeta + \frac{1}{2}|\nabla\varphi|^2 - \frac{S}{\rho}\,\frac{\zeta_{xx}(1+\zeta_y^2) + \zeta_{yy}(1+\zeta_x^2) - 2\zeta_{xy}\zeta_x\zeta_y}{(1+\zeta_x^2+\zeta_y^2)^{3/2}}$$

$$= -P(\bar{r} - \bar{U}t), \qquad z = \zeta(x,y,t) \qquad \text{--(3)}$$

$$\frac{\partial\zeta}{\partial t} + \bar{\nabla}\varphi\cdot\bar{\nabla}\zeta = \frac{\partial\varphi}{\partial z}, \qquad z = \zeta(x,y,t). \qquad \text{--(4).}$$

$P(\tilde{r} - \breve{U}t)$ is a spatially homogeneous random function over the surface of the sea.

One sets

$$\varphi(x, y, z, t) = \varepsilon \int_{-\infty}^{\infty} B(\tilde{k}, t) \frac{\cosh|\tilde{k}|(z+h)}{\cosh|\tilde{k}|h} e^{-i\tilde{k}\cdot\tilde{r}} d\tilde{k},$$

$$\xi(x, y, t) = \varepsilon \int_{-\infty}^{\infty} A(\tilde{k}, t)\, e^{-i\tilde{k}\cdot\tilde{r}} d\tilde{k},$$

$$P(\tilde{r} - \tilde{U}t) = \varepsilon^2 \int_{-\infty}^{\infty} p(\tilde{k})\, e^{-i\tilde{k}\cdot(\tilde{r}-\tilde{U}t)} d\tilde{k},$$

in equations (3) and (4) . The external pressure term has been taken to be $0(\varepsilon^2)$ as this choice provides a physical balance of interest. If, in addition, one writes

$$A(\tilde{k}, t) = \sum_{s} a^s(\tilde{k}, t)\, e^{is\omega t},$$

and

$$B(\tilde{k}, t) = \sum_{s} \frac{i\nu^2}{s\omega} a^s(\tilde{k}, t)\, e^{is\omega t}, \quad \nu^2 = g + \frac{S}{\rho}|\tilde{k}|^2,$$

and uses similar manipulations to those employed in Chapter I, the following governing equation is obtained.

$$a_t^{s} = \varepsilon\left\{ \sum_{s_1 s_2} \int_{-\infty}^{\infty} \ell_{k\,k_1\,k_2}^{s\,s_1\,s_2}\, a_1^{s_1} a_2^{s_2}\, e^{iW_{12,0}t}\, \delta_{12,0}\, d\tilde{k}_1 d\tilde{k}_2 \right.$$

$$\left. + \ \ell_k^{s}\, p(\tilde{k})\, e^{i(\tilde{k}\cdot\tilde{U} - s\omega)t} \right\}$$

$$+ \varepsilon^2 \left\{ \sum_{s_1 s_2 s_3} \int_{-\infty}^{\infty} \ell_{k\,k_1\,k_2\,k_3}^{s\,s_1\,s_2\,s_3}\, a_1^{s_1} a_2^{s_2} a_3^{s_3}\, e^{iW_{123,0}t}\, \delta_{123,0}\, d\tilde{k}_1 d\tilde{k}_2 d\tilde{k}_3 \right.$$

$$\left. - \frac{1}{2}\,\overset{12}{\rho} \sum_{s_1} \int_{-\infty}^{\infty} |\tilde{k}_2|\, a_1^{s_1}\, p(\tilde{k}_2)\, e^{i(s_1\omega_1 + \tilde{k}_2\cdot U - s\omega)t}\, \delta_{12,0}\, d\tilde{k}_1 d\tilde{k}_2 \right\}.$$

$$+ \ O(\varepsilon^3). \qquad ---(5)$$

The coefficients $\ell_{k\,k_1\,k_2}^{s\,s_1\,s_2}$ and $\ell_{k\,k_1\,k_2\,k_3}^{s\,s_1\,s_2\,s_3}$ are the same as previously defined in Chapter I, and $\ell_k^{s} = \frac{is\omega}{2\nu^2}$.

Before proceeding with the perturbation analysis, some relevant properties of the generalized function p(k) are discussed. $P(\tilde{r} - \tilde{U}t)$ is a spatially homogeneous random function over the surface of the sea. One can therefore show, in a similar manner as was used to show $\langle A(\tilde{k})\, A(\tilde{k}')\rangle = \delta(\tilde{k}+\tilde{k}')\, \overset{(2)}{\Phi}(\tilde{k})$, that $\langle p(\tilde{k})\, p(\tilde{k}')\rangle = \delta(\tilde{k}+\tilde{k}')\, \Pi(k)$ where

$$\langle P(\tilde{s})\, P(\tilde{s}+\tilde{r})\rangle = \varepsilon^4 \int_{-\infty}^{\infty} \Pi(\tilde{k})\, e^{-i\tilde{k}\cdot\tilde{r}}\, d\tilde{k}.$$

Since $P(\tilde{r}-\tilde{U}t)$ is real, $\pi(\tilde{k}) = \pi^*(-\tilde{k})$. ...(6)
However, if one also considers

$$\langle P(\tilde{s})\, P(\tilde{s}+\tilde{r})\rangle = \langle P(\tilde{s}')\, P(\tilde{s}'+\tilde{r})\rangle = \langle P(\tilde{s}-\tilde{r})\, P(\tilde{s})\rangle,$$

one sees that $\pi(\tilde{k}) = \pi(-\tilde{k})$. ---- (7).
Equations (6) and (7) imply that $\pi(k)$ is real. In this problem $|\mathcal{L}_0^{\gamma_1\gamma_2}|\pi(0)$ will be taken to be zero.

One now introduces the perturbation expansion

$$a^{\nu}(\tilde{k},t) = a_0^{\nu}(\tilde{k},t) + \varepsilon a_1^{\nu}(\tilde{k},t) + \varepsilon^2 a_2^{\nu}(\tilde{k},t) + \cdots, \quad \text{---(8)}$$

on $a^{\nu}(k,t)$ into equation (5), and anticipating secular behavior, one uses the multiple time scale technique, setting $T_0 = t$, $T_2 = \varepsilon^2 t$.

The order one balance of the governing equation (5) gives

$$a^{\nu}_{0_t} = 0,$$

which implies

$$a_0^{\nu}(\tilde{k}) = a_0^{\nu}(\tilde{k}, T_2). \quad \text{--- (9)}$$

The order ε balance gives

$$a_1^{\nu} = \mathcal{L}_{12}^{\nu}\, p(\tilde{k})\, \Delta(\tilde{k}\cdot\tilde{U} - \nu\omega) + b_1^{\nu},$$

where $$b_1^{\nu} = \sum_{\nu_1\nu_2} \int_{-\infty}^{\infty} \mathcal{L}_{kk_1k_2}^{\nu\nu_1\nu_2}\, a_{01}^{\nu_1} a_{02}^{\nu_2}\, \Delta_{12,0}\, \delta_{12,0}\, d\tilde{k}_1\, d\tilde{k}_2.$$

As seen previously in Chapter I, there are no secular terms arising from the long time behavior of $\{b_1^{\nu} a_0^{\nu'} a_0^{\nu''}\cdots\}$.

As no "t" growths occur in the long time behavior (with asymptotics performed in physical space) of $\Delta(\tilde{k}\cdot\tilde{U}-s\omega)\,\mathcal{L}_k^{s}$ $\{\,p(\tilde{k})\,a_0^{s''}a_0^{s'''}\cdots\,\}$ there is no need for a T_1 time scale.

The order ε^2 balance of the governing equation gives,

$$a_2^{s} = -t\,\frac{\partial a_0^{s}}{\partial T_2} + b_2^{s} + c_2^{s} + d_2^{s} + f_2^{s}, \qquad \cdots (11)$$

where

$$b_2^{s} = \sum_{s_1 s_2 s_3}\int_{-\infty}^{\infty} \mathcal{L}_{k k_1 k_2 k_3}^{s s_1 s_2 s_3}\, a_{01}^{s_1} a_{02}^{s_2} a_{03}^{s_3}\, \Delta_{123,0}\,\delta_{123,0}\, d\tilde{k}_1 d\tilde{k}_2 d\tilde{k}_3,$$

$$c_2^{s} = 2\sum_{s_1\cdots s_4}\int_{-\infty}^{\infty} \mathcal{L}_{k k_1 k_2}^{s s_1 s_2}\,\mathcal{L}_{k_1 k_3 k_4}^{s_1 s_3 s_4}\, a_{02}^{s_2} a_{03}^{s_3} a_{04}^{s_4}\,\frac{\Delta_{234,0}-\Delta_{12,0}}{i\,W_{34,1}}$$

$$\delta_{34,1}\,\delta_{12,0}\, d\tilde{k}_1\cdots d\tilde{k}_4,$$

$$d_2^{s} = -\frac{1}{2}\,P^{12}\,\mathcal{L}_k^{s}\sum_{s_1}\int_{-\infty}^{\infty} |\tilde{k}_2|\, p(\tilde{k}_2)\, a_{01}^{s_1}\,\Delta(s_1\omega_1 + \tilde{k}_2\cdot U - s\omega)\,\delta_{12,0}\, d\tilde{k}_1 d\tilde{k}_2,$$

$$f_2^{s} = 2\sum_{s_1 s_2}\int_{-\infty}^{\infty} \mathcal{L}_{k k_1 k_2}^{s s_1 s_2}\,\mathcal{L}_{k_1}^{s_1}\, a_{02}^{s_2}\, p(\tilde{k}_1)\int_0^t \Delta(\tilde{k}_1\cdot U - s_1\omega_1)\, e^{i(s_1\omega_1 + s_2\omega_2 - s\omega)t}\, dt$$

$$\delta_{12,0}\, d\tilde{k}_1 d\tilde{k}_2.$$

One now examines the long time behavior (with the asymptotics being performed in physical space, if necessary) of the second order cumulant

$$\langle a_0^{\nu'}(k')\, a_2^{\nu}(k) + a_1^{\nu'}(k')\, a_1^{\nu}(k) + a_0^{\nu}(k)\, a_2^{\nu'}(k') \rangle_{\nu'=-\nu},$$

which represents the Order ε^2 component in the prescribed perturbation on the energy density. The only secular growths come from the terms $\langle -t\, a_0^{\nu'} \frac{\partial a_0^{\nu}}{\partial T_2} - t\, a_0^{\nu} \frac{\partial a_0^{\nu'}}{\partial T_2} \rangle_{\nu'=-\nu}$,

$\langle \ell_1^{\nu} \ell_1^{\nu'} \rangle_{\nu'=-\nu},\ \{\langle p(k)\, p(k') \rangle\, \ell_k^{\nu} \ell_{k'}^{\nu'}\, \Delta(k\cdot U - s\omega)\, \Delta(k'\cdot U - s'\omega')\}_{\nu'=-\nu},$

and $\langle a_0^{\nu'}(k')\, a_2^{\nu}(k) + a_0^{\nu}(k)\, a_2^{\nu'}(k') \rangle_{\nu'=-\nu}$.

One first considers the term

$$\ell_k^{\nu}\, \ell_{k'}^{-\nu}\, \Delta(k\cdot U - \nu\omega)\, \Delta(k'\cdot U + \nu\omega')\, \langle p(k)\, p(k') \rangle$$

$$= \delta(k+k')\, \ell_k^{\nu}\, \ell_{-k}^{-\nu}\, \pi(k)\, \Delta(k\cdot U - s\omega)\, \Delta(\nu\omega - k\cdot U).$$

This behaves in the long time limit as $2\pi t\, |\ell_k^{\nu}|^2\, \pi(k)\, \delta(k\cdot U - \nu\omega)$. Thus it is seen that secular growths occur over a "discrete" continuum of wave vectors k, given by the relation $\underset{\sim}{k}\cdot \underset{\sim}{U} = \pm\, \omega(\underset{\sim}{k})$. This class of wave vectors $\underset{\sim}{k}$ will be denoted by K. One therefore must write the zeroth order energy of the sea as the sum of two integrals

$$\sum_{j} \int_{\infty-K-0} \psi_0^{j-j}(\breve{k})\, d\breve{k} + \sum_{j} \int_{K} \Phi_0^{j-j}(\breve{k})\, d\breve{k},$$

which can also be written

$$\sum_{j} \int_{-\breve{\infty}}^{\breve{\infty}} \left\{ \psi_0^{j-j}(\breve{k}) + \delta(\breve{k}\cdot\breve{U} - j\omega)\, \Phi_0^{j-j}(\breve{k}) \right\} d\breve{k}. \quad \text{-- (12)}$$

It is only in the zeroth order energy density, and not in the zeroth order components of the higher spectral cumulants, that this breakdown must be used.

One now considers the term $\langle b_1^{j}(\breve{k})\, b_1^{j'}(\breve{k}') \rangle_{j'=-j}$

$$= \sum_{j_1 \cdots j_4} \int_{-\breve{\infty}}^{\breve{\infty}} \mathcal{L}_{k k_1 k_2}^{j j_1 j_2}\, \mathcal{L}_{k' k_3 k_4}^{-j j_3 j_4} \langle a_{01}^{j_1} a_{02}^{j_2} a_{03}^{j_3} a_{04}^{j_4} \rangle$$

$$\Delta(j_1\omega_1 + j_2\omega_2 - j\omega)\, \Delta(j_3\omega_3 + j_4\omega_4 + j\omega')\, \delta_{12,0}\, d\breve{k}_1 d\breve{k}_2,$$

$$= \sum_{j_1 \cdots j_4} \int_{-\breve{\infty}}^{\breve{\infty}} \mathcal{L}_{k k_1 k_2}^{j j_1 j_2}\, \mathcal{L}_{k' k_3 k_4}^{-j j_3 j_4} \Big\{ \delta(\breve{k}_1 + \breve{k}_2 + \breve{k}_3 + \breve{k}_4)\, \Phi_0^{(4)\, j_1 j_2 j_3 j_4}(k_1, k_2, k_3)$$

$$+ \delta(\breve{k}_1 + \breve{k}_2)\, \delta(\breve{k}_3 + \breve{k}_4)\, \Phi_0^{(2)\, j_1 j_2}(\breve{k}_1)\, \Phi_0^{(2)\, j_3 j_4}(k_3)$$

$$+ 2\, \delta(\breve{k}_1 + \breve{k}_3)\, \delta(\breve{k}_2 + \breve{k}_4)\, \Phi_0^{(2)\, j_1 j_3}(\breve{k}_1)\, \Phi_0^{(2)\, j_2 j_4}(\breve{k}_2) \Big\}$$

$$\Delta(\nu_1\omega_1+\nu_2\omega_2-\nu\omega)\,\Delta(\nu_3\omega_3+\nu_4\omega_4+s\omega')\,\delta_{12,0}\,\delta_{34,0'}\,d\tilde{k}_1\,d\tilde{k}_2\,d\tilde{k}_3\,d\tilde{k}_4 .$$

Since $\overset{(4)}{Q}{}_0^{\nu_1\nu_2\nu_3\nu_4}(\tilde{k}_1,\tilde{k}_2,\tilde{k}_3)$ is a smooth function, the behavior in long time of the first term on the right hand side is order one. The second term is identically zero by the mean value property. In the third term, it is seen that after integrating over $\breve{k}_3$ and $\breve{k}_4$, secular growths can only arise when $\nu_3=-\nu_1$, $\nu_4=-\nu_2$. This term then becomes

$$2\,\delta(\tilde{k}+\tilde{k}')\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2}\, L_{-k\,-k_1\,-k_2}^{-\nu\,-\nu_1\,-\nu_2}\,\overset{(2)}{Q}{}_0^{\nu_2-\nu_2}(\tilde{k}_2)\,\overset{(2)}{Q}{}_0^{\nu_1-\nu_1}(\tilde{k}_1)$$

$$\Delta(s_1\omega_1+s_2\omega_2-s\omega)\,\Delta(-s_1\omega_1-s_2\omega_2+s\omega)\,\delta_{12,0}\,d\tilde{k}_1\,d\tilde{k}_2 .$$

Before any asymptotics are performed, one must replace

$\overset{(2)}{Q}{}_0^{\nu-\nu}(k)$ by $\Psi_0^{\nu-\nu}(\breve{k}) + \delta(\breve{k}\cdot\breve{U}-\nu\omega)\,\Phi_0^{\nu-\nu}(\breve{k})$.

The above expression then becomes the sum of three integrals,

$$2\,\delta(k+k')\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty} L_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2}\, L_{-k\,-k_1\,-k_2}^{-\nu\,-\nu_1\,-\nu_2}\,\Psi_0^{\nu_1-\nu_1}(\breve{k}_1)\,\Psi_0^{\nu_2-\nu_2}(k_2)$$

$$\Delta(s_1\omega_1+s_2\omega_2-s\omega)\,\Delta(-s_1\omega_1-s_2\omega_2+s\omega)\,\delta_{12,0}\,d\breve{k}_1\,d\breve{k}_2$$

$$+\,4\,\delta(k+k')\sum_{s_1 s_2}\int_{-\infty}^{\infty} \mathcal{L}^{s\,s_1\,s_2}_{k\,k_1\,k_2}\,\mathcal{L}^{-s\,-s_1\,-s_2}_{-k\,-k_1\,-k_2}\,\bar{\Psi}_0^{s_1 -s_1}(k_1)\,\bar{\Phi}_0^{s_2 -s_2}(k_2)$$

$$\delta(\breve{k}_2\cdot\breve{U}-s_2\omega_2)\,\Delta(s_1\omega_1+s_2\omega_2-s\omega)\,\Delta(-s_1\omega_1-s_2\omega_2+s\omega)\,\delta_{12,0}\,d\breve{k}_1\,d\breve{k}_2$$

$$+\,2\,\delta(\breve{k}+k')\sum_{s_1 s_2}\int_{-\infty}^{\infty} \mathcal{L}^{s\,s_1\,s_2}_{k\,k_1\,k_2}\,\mathcal{L}^{-s\,-s_1\,-s_2}_{-k\,-k_1\,-k_2}\,\bar{\Phi}_0^{s_1 -s_1}(k_1)\,\bar{\Phi}_0^{s_2 -s_2}(k_2)$$

$$\delta(\breve{k}_1\cdot\breve{U}-s_1\omega_1)\,\delta(\breve{k}_2\cdot\breve{U}-s_2\omega_2)\,\Delta(s_1\omega_1+s_2\omega_2-s\omega)\,\Delta(-s_1\omega_1-s_2\omega_2+s\omega)$$

$$\delta_{12,0}\,d\breve{k}_1\,d\breve{k}_2 .$$

Since $\bar{\Psi}_0^{s-s}(k)$ is a smooth function, the asymptotics may be performed on the first integral, and one obtains

$$4\pi t\,\delta(\breve{k}+\breve{k}')\sum_{s_1 s_2}\int_{-\infty}^{\infty} \mathcal{L}^{s\,s_1\,s_2}_{k\,k_1\,k_2}\,\mathcal{L}^{-s\,-s_1\,-s_2}_{-k\,-k_1\,-k_2}\,\bar{\Psi}_0^{s_1 -s_1}(k_1)\,\bar{\Psi}_0^{s_2 -s_2}(k_2)\,\delta^{s}_{12,0}\,\delta_{12,0}\,d\breve{k}_1\,d\breve{k}_2 .$$

The asymptotics can also be performed on the second term as there is one integration remaining when one integrates over $\breve{k}_1+\breve{k}_2-\breve{k}=0$, and $\breve{k}_2\cdot\breve{U}-s_2\omega_2=0$. One obtains that this term behaves in long time as

$$8\pi t\,\delta(k+k')\sum_{s_1 s_2}\int_{-\infty}^{\infty} \mathcal{L}^{s\,s_1\,s_2}_{k\,k_1\,k_2}\,\mathcal{L}^{-s\,-s_1\,-s_2}_{-k\,-k_1\,-k_2}\,\bar{\Psi}_0^{s_1 -s_1}(k_1)\,\bar{\Phi}_0^{s_2 -s_2}(k_2)\,\delta(\breve{k}_2\cdot\breve{U}-s_2\omega_2)\,\delta(s_1\omega_1+s_2\omega_2-s\omega)$$

$$\delta(\breve{k}_1+\breve{k}_2-\breve{k})\,d\breve{k}_1\,d\breve{k}_2 .$$

The asymptotics cannot be performed on the third term since $\delta(\breve{k}_1\cdot\breve{U}-\nu_1\omega_1)$, $\delta(\breve{k}_2\cdot\breve{U}-\nu_2\omega_2)$, $\delta(\breve{k}_1+\breve{k}_2-\breve{k})$ imply that the argument of the Δ function $\nu_1\omega_1+\nu_2\omega_2-\nu\omega$ becomes $\tilde{k}\cdot\breve{U}-\nu\omega$.

One obtains therefore that the third term is

$$2\delta(k+k')\,\Delta(\breve{k}\cdot\breve{U}-\nu\omega)\,\Delta(-\breve{k}\cdot\breve{U}+\nu\omega)\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty}L_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2}\,L_{-k\,-k_1\,-k_2}^{-\nu\,-\nu_1\,-\nu_2}$$

$$\bar{\Phi}_0^{\nu_1-\nu_1}(k_1)\,\bar{\Phi}_0^{\nu_2-\nu_2}(k_2)\,\delta(\breve{k}_1\cdot\breve{U}-\nu_1\omega_1)\,\delta(\breve{k}_2\cdot\breve{U}-\nu_2\omega_2)\,\delta_{12,0}\,d\breve{k}_1 d\breve{k}_2 .$$

The long time behavior of this term (the asymptotics being done in physical space) can be written

$$4\pi t\,\delta(k+k')\,\delta(\breve{k}\cdot\breve{U}-\nu\omega)\sum_{\nu_1\nu_2}\int_{-\infty}^{\infty}L_{k\,k_1\,k_2}^{\nu\,\nu_1\,\nu_2}\,L_{-k\,-k_1\,-k_2}^{-\nu\,-\nu_1\,-\nu_2}$$

$$\bar{\Phi}_0^{\nu_1-\nu_1}(k_1)\,\bar{\Phi}_0^{\nu_2-\nu_2}(k_2)\,\delta(\breve{k}_1\cdot\breve{U}-\nu_1\omega_1)\,\delta(\breve{k}_2\cdot\breve{U}-\nu_2\omega_2)$$

$$\delta(\breve{k}_1+\breve{k}_2-\breve{k})\,d\breve{k}_1 d\breve{k}_2 .$$

A similar analysis on

$\langle a_0^{\nu'}(k')\,c_2^{\nu}(k)+a_0^{\nu}(k)\,c_2^{\nu'}(k')\rangle_{\nu'=-\nu}$ yields that ,

$\langle a_0^{\nu'}(k')\,c_2^{\nu}(k)+a_0^{\nu}(k)\,c_2^{\nu'}(k')\rangle_{\nu'=-\nu}\sim$

$$8\pi t\,\delta(k+k')\,\bar{\Psi}_0^{\,s-s}(k)\sum_{s_1 s_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{s s_1 s_2}\, L_{k_1 k -k_2}^{s_1\, s\, -s_2}$$

$$\left(\bar{\Psi}_0^{\,s_2-s_2}(k_2) + \delta(\tilde{k}_2\cdot\tilde{U} - s_2\omega_2)\,\bar{\Phi}_0^{\,s_2-s_2}(k_2)\right)\delta(s_1\omega_1 + s_2\omega_2 - s\omega)$$

$$\delta(\tilde{k}_1+\tilde{k}_2-\tilde{k})\;d\tilde{k}_1\,d\tilde{k}_2$$

$$+\,8\pi t\,\delta(k+k')\,\delta(\tilde{k}\cdot\tilde{U}-s\omega)\,\bar{\Phi}_0^{\,s-s}(k)\sum_{s_1 s_2}\int_{-\infty}^{\infty} L_{k k_1 k_2}^{s s_1 s_2}\, L_{k_1 k -k_2}^{s_1\, s\, -s_2}$$

$$\left(\bar{\Psi}_0^{\,s_2-s_2}(k_2) + \delta(\tilde{k}_2\cdot\tilde{U} - s_2\omega_2)\,\bar{\Phi}_0^{\,s_2-s_2}(k_2)\right)\delta(s_1\omega_1 + s_2\omega_2 - s\omega)$$

$$\delta(\tilde{k}_1+\tilde{k}_2-\tilde{k})\;d\tilde{k}_1\,d\tilde{k}_2.$$

The term $-t\,\dfrac{\partial \bar{\Phi}_0^{(2)\,s-s}(k)}{\partial T_2}$ becomes

$-t\,\dfrac{\partial \bar{\Psi}_0^{\,s-s}(k)}{\partial T_2} - t\,\delta(\tilde{k}\cdot\tilde{U}-s\omega)\,\dfrac{\partial \bar{\Phi}_0^{\,s-s}(k)}{\partial T_2}$ and one can

use $\dfrac{\partial \bar{\Psi}_0^{\,s-s}(k)}{\partial T_2}$ and $\dfrac{\partial \bar{\Phi}_0^{\,s-s}(k)}{\partial T_2}$ to eliminate

secular growths from $\Phi_2^{(2)\,s-s}(k)$.

Thus, one obtains

$$\frac{\partial \bar{\Psi}_0^{\nu-\nu}(k)}{\partial T_2} = 4\pi \sum_{\nu_1\nu_2} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2} L_{-k -k_1 -k_2}^{-\nu -\nu_1 -\nu_2} \Psi_0^{\nu_1-\nu_1}(k_1) \Big(\bar{\Psi}_0^{\nu_2-\nu_2}(k_2)$$

$$+ 2\delta(\breve{k}_2\cdot\breve{U} - \nu_2\omega_2)\, \bar{\Phi}_0^{\nu_2-\nu_2}(k_2) \Big)\, \delta(s_1\omega_1 + s_2\omega_2 - s\omega)$$

$$\delta(\breve{k}_1 + \breve{k}_2 - \breve{k})\, d\breve{k}_1\, d\breve{k}_2$$

$$+ 8\pi \sum_{\nu_1\nu_2} \Psi_0^{\nu-\nu}(k) \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2} L_{k_1 k -k_2}^{\nu_1 \nu -\nu_2} \Big(\Psi_0^{\nu_2-\nu_2}(k_2) + \delta(\breve{k}_2\cdot\breve{U} - \nu_2\omega_2)\, \bar{\Phi}_0^{\nu_2-\nu_2}(k_2) \Big)$$

$$\delta(s_1\omega_1 + s_2\omega_2 - s\omega)\, \delta(\breve{k}_1 + \breve{k}_2 - \breve{k})\, d\breve{k}_1\, d\breve{k}_2, \qquad \breve{k} \in \infty - K$$

----- (13)

$$\frac{\partial \bar{\Phi}_0^{\nu-\nu}(k)}{\partial T_2} = 4\pi \sum_{\nu_1\nu_2} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2} L_{-k -k_1 -k_2}^{-\nu -\nu_1 -\nu_2} \bar{\Phi}_0^{\nu_1-\nu_1}(k_1)\, \bar{\Phi}_0^{\nu_2-\nu_2}(k_2)$$

$$\delta(\breve{k}_1\cdot\breve{U} - \nu_1\omega_1)\, \delta(\breve{k}_2\cdot\breve{U} - \nu_2\omega_2)\, \delta(\breve{k}_1 + \breve{k}_2 - \breve{k})\, d\breve{k}_1\, d\breve{k}_2$$

$$+ 8\pi\, \bar{\Phi}_0^{\nu-\nu}(\breve{k}) \sum_{\nu_1\nu_2} \int_{-\infty}^{\infty} L_{k k_1 k_2}^{\nu \nu_1 \nu_2} L_{k_1 k -k_2}^{\nu_1 \nu -\nu_2}$$

$$\Big(\Psi_0^{\nu_2-\nu_2}(k_2) + \delta(\breve{k}_2\cdot\breve{U} - \nu_2\omega_2)\, \bar{\Phi}_0^{\nu_2-\nu_2}(k_2) \Big)\, \delta(s_1\omega_1 + s_2\omega_2 - s\omega)$$

$$\delta(\breve{k}_1 + \breve{k}_2 - \breve{k})\, d\breve{k}_1\, d\breve{k}_2$$

$$+ 2\pi\, L_k^{\nu}\, L_{-k}^{-\nu}\, \pi(\breve{k}), \qquad \breve{k} \in K. \quad \cdots (14)$$

Equations (13) and (14) can be readily interpreted with the aid of the following lemmas.

Lemma 1. If $\breve{k}_1$, $\breve{k}_2 \in K$, $\breve{k}_1 + \breve{k}_2 = \breve{k}$,

and $s_1\omega_1 + s_2\omega_2 = s\omega$, then $\breve{k} \in K$.

Proof. $\breve{k}_1, \breve{k}_2 \in K \Rightarrow \breve{k}_1 \cdot \breve{U} = s_1\omega_1$, $\breve{k}_2 \cdot \breve{U} = s_2\omega_2$.

Therefore $(\breve{k}_1 + \breve{k}_2) \cdot \breve{U} = s_1\omega_1 + s_2\omega_2$,

and thus $\breve{k} \cdot \breve{U} = s\omega$.

Therefore $\breve{k} \in K$.

Lemma 2. If $\breve{k}$, $\breve{k}_1$, $\breve{k}_2 \in K$, and $\breve{k}_1 + \breve{k}_2 = \breve{k}$ then $s_1\omega_1 + s_2\omega_2 = s\omega$

Proof. $\breve{k}, \breve{k}_1, \breve{k}_2 \in K \Rightarrow \breve{k}_1 \cdot \breve{U} = s_1\omega_1$, $\breve{k}_2 \cdot \breve{U} = s_2\omega_2$

and $-\breve{k} \cdot \breve{U} = -s\omega$.

Adding, one obtains

$$(\breve{k}_1 + \breve{k}_2 - \breve{k}) \cdot \breve{U} = s_1\omega_1 + s_2\omega_2 - s\omega.$$

Hence $s_1\omega_1 + s_2\omega_2 - s\omega = 0$.

Equation (13) represents the long time change of the energy density $\bar{\Psi}_0^{\,\supset-\supset}(k)$, $\breve{k} \notin K$. The first term represents a feeding of energy to $\tilde{k}$ by two other wave vectors $\tilde{k}_1$, $\tilde{k}_2$ when $\tilde{k}$, $\tilde{k}_1$ and $\tilde{k}_2$ form a resonant triad. Only one of the wave vectors $\tilde{k}_1$ and $\tilde{k}_2$ can belong to K. Otherwise Lemma 1 would be violated. The second term on the right hand side of equation (13) represents the loss (gain) of energy density from the wave vector $\tilde{k}$, by its forming a resonant triad with two other wave vectors $\tilde{k}_1$ and $\tilde{k}_2$ only one of which can belong to K.

Equation (14) represents the long time change of the energy density of a wave vector $\tilde{k} \in K$. The first term represents a feeding of energy from two other wave vectors $\tilde{k}_1$ and $\tilde{k}_2$, both bolonging to K. The second term represents a loss (gain) of energy from $\tilde{k}$ by its forming a resonant triad with two wave vectors $\tilde{k}_1$ and $\tilde{k}_2$ neither of which belongs to K. The third term represents a loss (gain) of energy due to k forming a resonant triad with two wave vectors $\tilde{k}_1$, $\tilde{k}_2$ each belonging to K. The fourth term represents the feeding of the wave components belonging to K by the external pressure distribution.

It is clear that a consistent solution of equations (13) and (14) is $\bar{\Psi}_0^{\,\supset-\supset}(k) = 0$, if it was so initially. However this would provide a very unstable state as there is no way that the wave components $\tilde{k} \in K$ can lose energy unless one included viscosity to damp out the energy fed into the system by the external pressure distribution.

It can also be seen from equation (14) that if no internal resonance mechanism exists that the energy density would grow linearly with time which is consistent with the results Phillips obtained.

Conclusion.

It is stressed that the preceding analysis is only valid for the particular model which was chosen. In a more realistic treatment of air moving over water, or the effects of free stream turbulence on boundary layer stability, the models would have to include the effects of the vertical structure.

CHAPTER III

The Sideband Mechanism.

Recently, it has been pointed out by Brooke-Benjamin [3] that energy can be interchanged in a weakly non-linear system between a primary travelling wave, with wave number k, and corresponding frequency $\omega(k)$, and its neighboring wave components, with wave numbers $k \pm \varepsilon$ and corresponding frequencies $\omega(k \pm \varepsilon)$, where ε is the small parameter describing the relative magnitude of the non-linear terms. The basic reason for this is that the frequencies generated by the three wave components obey the relation

$$\omega(k+\lambda\varepsilon) + \omega(k-\lambda\varepsilon) - 2\omega(k) = O(\varepsilon^2),$$

which is the order at which the non-linear terms first affect the motion. Brooke-Benjamin found that the initial growth of the sidebands, in the case where their amplitudes were initially small, could be exponential.

In the following analysis, this phenomenon is examined using a model equation. One looks for a solution in the form of a travelling wave whose amplitude is a slowly varying function of space and time. When this form of solution is substituted into the model equation, a fully non-linear partial differential equation for the slowly varying amplitude is obtained. It is clear therefore, that if one looks for a solution of this equation in the form of a Fourier series, that all harmonics will be generated and that their amplitudes will be all of the same order after a short period of time has elapsed.

However, one could pose the question that if one had initially small sidebands, under what conditions would these sidebands become unstable? It is found that it is indeed possible for the energy in the sidebands initially to grow in an exponential fashion. Clearly the exponential growth cannot exist for all time as the energy required for the initial unstable growth must come from the primary wave. If one assumes that the truncated Fourier series solution (see page 301) is a good approximation for the slowly varying amplitude, then one can see that it is possible for the energy in the sidebands to take on a periodic structure when it becomes comparable with the energy in the primary wave.

Analysis.

The following model equation is typical of a conservative system of weakly non-linear dispersive waves.

$$u_{tt} + L_x u = -\varepsilon^2 (u^2 u_x)_x , \qquad \cdots (1)$$

where L_x is an even differential operator in x such that $L_x e^{ikx} = \omega^2(k) e^{ikx}$.

One compensates for the fact that there are two length scales in the problem by looking for a travelling wave solution whose amplitude is a slowly varying function of both x and t. Let

$$u(x,t) = v(X,T) e^{i(kx-\omega t)} + v^*(X,T) e^{-i(kx-\omega t)} , \qquad \cdots (2)$$

where $X = \mu x$, $T = \mu t$, $\mu \ll 1$. One substitutes the assumed form (2) into equation (1), and obtains the following equation,

$$-2i\omega\mu \left(\frac{\partial v}{\partial T} + \frac{d\omega}{dk} \frac{\partial v}{\partial X} \right) + \mu^2 \left(\frac{\partial^2 v}{\partial T^2} - \left\{ \omega \frac{d^2\omega}{dk^2} + \left(\frac{d\omega}{dk}\right)^2 \right\} \frac{\partial^2 v}{\partial X^2} \right)$$

$$+ O(\mu^3) = 2\varepsilon^2 \beta \omega v^2 v^* , \qquad \cdots (3)$$

where β is now a constant depending on the wave number of the primary and $\frac{d\omega}{dk}$ is the group velocity, which will henceforth be denoted as ω'.

If $\mu = O(\varepsilon^2)$, the first balance in the above equation (3), occurs between the nondispersive and nonlinear terms.

$$\frac{\partial v}{\partial T} + \omega' \frac{\partial v}{\partial X} = i\beta v^2 v^*. \qquad \text{-- (4)}$$

By a change of variables, $\xi = X - \omega' T$ and $\eta = T$, the above equation assumes the form,

$$\frac{\partial v}{\partial \eta} = i\beta v^2 v^*. \qquad \text{-- (5)}$$

This has the solution, see Benney [ ],

$$v(\xi, \eta) = f(\xi)\, e^{i\beta f(\xi) f^*(\xi) \eta}. \qquad \text{-- (6)}$$

If there were no nonlinear term, $\beta = 0$, then this solution shows that the amplitude is a function of $X - \omega' T$ only. Thus the locus of points at which the amplitude is constant moves with the group velocity, which locus describes the envelope. The nonlinear term serves to apply a Stokes type frequency modulation to the envelope. However the above solution is only valid for ranges of time in which $\varepsilon^2 t$ is finite.

A more interesting balance occurs when $\mu = O(\varepsilon)$ for then the dispersive term is important. If one now sets

$$\xi = X - \omega' T,$$

and $\eta = \mu T,$

one obtains the following fully nonlinear equation,

$$\frac{\partial v}{\partial \eta} = i\gamma \frac{\partial^2 v}{\partial \xi^2} + i\beta v^2 v^*, \qquad \gamma = \frac{1}{2}\omega''. \quad \text{--}(7)$$

For convenience, one replaces η by t and ξ by x and obtains the equivalent equation

$$\frac{\partial v}{\partial t} = i\gamma \frac{\partial^2 v}{\partial x^2} + i\beta v^2 v^*, \quad \text{--}(8).$$

Clearly a Fourier series approach in x would allow all the harmonics to be generated to potentially the same order, as the nonlinear term is now of order one. However, it is natural to first consider the stability of the nonlinear Stokes wave. One therefore looks for the behavior in time of a solution,

$$v(x,t) = a_0(t) + \varepsilon a_1(t) e^{ikx} + \varepsilon a_2(t) e^{-ikx} + O(\varepsilon^2). \quad \text{--}(9)$$

The small parameter ε is now the order of magnitude describing the relative size of the harmonics and is unrelated to the ε described previously. Substituting the assumed form (9) into equation (8), one obtains the following equations for $a_0(t)$, $a_1(t)$ and $a_2(t)$.

$$\frac{da_0}{dt} = i\beta a_0^2 a_0^* + 2i\beta\varepsilon^2 \left(a_0^* a_1 a_2 + a_0 a_1 a_1^* + a_0 a_2 a_2^* \right) + O(\varepsilon^4), \quad \text{--}(10)$$

$$\frac{da_1}{dt} = -i\gamma k^2 a_1 + i\beta \left(a_0^2 a_2^* + 2a_0 a_0^* a_1 \right) + O(\varepsilon^2), \quad \text{--}(11)$$

$$\frac{da_2}{dt} = -i\gamma k^2 a_2 + i\beta \left(a_0^2 a_1^* + 2a_0 a_0^* a_2 \right) + O(\varepsilon^2), \quad \text{--}(12)$$

It is important that the order ε^2 term be retained in equation (10), as otherwise, to a first closure, the energy in the mean would remain constant. This clearly cannot be the case if one expects a transfer of energy to occur. Setting

$$a_i = b_i \, e^{i\beta \int^t b_0 b_0^* \, dt} , \qquad i = 0, 1, 2,$$

equations (10), (11) and (12) become

$$\frac{db_0}{dt} = 2i\beta\varepsilon^2 \left(b_0^* b_1 b_2 + b_0 b_1 b_1^* + b_0 b_2 b_2^* \right), \qquad \text{---(13)}$$

$$\frac{db_1}{dt} = i\left(\beta b_0 b_0^* - \gamma k^2\right) b_1 + i\beta b_0^2 b_2^* , \qquad \text{--(14)}$$

$$\frac{db_2}{dt} = i\left(\beta b_0 b_0^* - \gamma k^2\right) b_2 + i\beta b_0^2 b_1^* , \qquad \text{--(15)}$$

where the higher order terms have now been dropped. Multiplying (13) by b_0^* and taking the complex conjugate and adding, one obtains

$$\frac{d\, b_0 b_0^*}{dt} = 2i\beta\varepsilon^2 \left(b_0^{*2} b_1 b_2 - b_0^2 b_1^* b_2^* \right). \qquad \text{--(16)}$$

By similar manipulations, it can be found,

$$\frac{d\, b_1 b_1^*}{dt} = i\beta\left(b_0^2 b_1^* b_2^* - b_0^{*2} b_1 b_2\right), \qquad \text{---(17)}$$

$$\frac{d\, b_2 b_2^*}{dt} = i\beta\left(b_0^2 b_1^* b_2^* - b_0^{*2} b_1 b_2\right), \qquad \text{---(18)}$$

$$\frac{d\, b_0^2}{dt} = 4i\beta\varepsilon^2\left(b_0 b_0^* b_1 b_2 + b_0^2\left(b_1 b_1^* + b_2 b_2^*\right)\right). \qquad \text{---(19)}$$

Multiplying (14) by b_2 and (15) by b_1, and adding, gives

$$\frac{d\, b_1 b_2}{dt} = 2i\left(\beta b_0 b_0^* - \gamma h^2\right) b_1 b_2 + i\beta b_0^2\left(b_1 b_1^* + b_2 b_2^*\right). \qquad \text{---(20)}$$

When one adds (17) and (18), one obtains

$$\frac{d\, b_1 b_1^* + b_2 b_2^*}{dt} = 2i\beta\left(b_0^2 b_1^* b_2^* - b_0^{*2} b_1 b_2\right). \qquad \text{---(21)}$$

If one sets $b_0 b_0^* = x$, $b_1 b_1^* + b_2 b_2^* = y$, $b_0^2 = u$ and $b_1 b_2 = \omega$, equations (19), (21), (16) and (20) respectively become

$$\frac{du}{dt} = 4i\beta\varepsilon^2\left(x\omega + uy\right), \qquad \text{---(22)}$$

$$\frac{dy}{dt} = 2i\beta\left(u\omega^* - u^*\omega\right), \qquad \text{---(23)}$$

$$\frac{dx}{dt} = 2i\beta\varepsilon^2(u^*w - uw^*), \qquad \text{-- (24)}$$

$$\frac{dw}{dt} = 2i(\beta x - \gamma\lambda^2)w + i\beta uy. \qquad \text{-- (25)}$$

Equations (23) and (24) imply that $x + \varepsilon^2 y = E$ = constant, which means that the total energy is conserved. Since

$$x = E - \varepsilon^2 y \qquad \text{-- (26)}$$

equations (22), (23) and (25) become

$$\frac{dy}{dt} = -2i\beta(u^*w - uw^*), \qquad \text{--- (27)}$$

$$\frac{du}{dt} = 4i\beta\varepsilon^2\left((E-\varepsilon^2 y)w + uy\right), \qquad \text{-- (28)}$$

$$\frac{dw}{dt} = 2i\left(\beta(E-\varepsilon^2 y) - \gamma\lambda^2\right)w + i\beta uy. \qquad \text{--- (29)}$$

With a little manipulation on equations (27), (28) and (29), the following equations can be found.

$$\frac{d(u^*w - uw^*)}{dt} = 2i(\beta E - \gamma\lambda^2 - 3\beta\varepsilon^2 y)(u^*w + uw^*) + 2i\beta uu^* y$$
$$- 8i\beta\varepsilon^2(E - \varepsilon^2 y)ww^*. \qquad \text{--- (30)}$$

$$\frac{d(u^*w + uw^*)}{dt} = 2i\left(\beta E - \gamma \lambda^2 - 3\beta\varepsilon^2 y\right)\left(u^*w - uw^*\right). \quad --(31)$$

$$\frac{dy}{dt} = -2i\beta\left(u^*w - uw^*\right). \quad --(32)$$

$$\frac{duu^*}{dt} = 4i\beta\varepsilon^2\left(E - \varepsilon^2 y\right)\left(u^*w - uw^*\right). \quad --(33)$$

$$\frac{dww^*}{dt} = -i\beta y\left(u^*w - uw^*\right). \quad --(34)$$

From equation (31),

$$u^*w - uw^* = \frac{i}{2\beta}\frac{dy}{dt}. \quad --(35)$$

If one substitutes equation (35) into (33), one obtains

$$\frac{duu^*}{dt} = -2\varepsilon^2\left(E - \varepsilon^2 y\right)\frac{dy}{dt},$$

which integrates to give

$$uu^* = \varepsilon^4 y^2 - 2\varepsilon^2 E y + C_1 .$$

Since $uu^* = (b_o b_o^*)^2 = x^2 = (E - \varepsilon^2 y)^2$, it is clear that $C_1 = E^2$. Therefore,

$$uu^* = \left(E - \varepsilon^2 y\right)^2. \quad --(36)$$

If one substitutes (35) into (34), one obtains

$$\frac{d\omega\omega^*}{dt} = \frac{1}{2} y \frac{dy}{dt},$$

which integrates to give

$$\omega\omega^* = y^2/_4 + C_2 . \quad --(37)$$

Since $\omega\omega^* = b_1 b_1^* b_2 b_2^*$, equation (37) implies

$$b_1 b_1^* b_2 b_2^* = \left(\frac{b_1 b_1^* + b_2 b_2^*}{2}\right)^2 + C_2 .$$

If one calls $b_1 b_1^* + b_2 b_2^* = E_1$, $b_1 b_1^* = E_{11}$, and $b_2 b_2^* = E_{12}$, one sees that

$$C_2 = E_{11} E_{12} - \left(\frac{E_1}{2}\right)^2 . \quad --(38)$$

If $b_1 b_1^* = b_2 b_2^*$ at $t = 0$, then $C_2 = 0$. If one now substitutes (35) into (31) and integrates, obe obtains

$$u^*\omega + u\omega^* = -\frac{1}{\beta}\left(\beta E - \gamma h^2 - {}^3/_2 \beta \varepsilon^2 y\right) y + C_3 \cdot \quad --(39)$$

One now substitutes (36), (37) and (39) into equation (30), and obtains the equation

$$\frac{d^2 y}{dt^2} = \alpha_0 + \left(l + \varepsilon^2 \alpha_1\right) y + \varepsilon^2 \alpha_2 y^2 - 10\beta^2 \varepsilon^4 y^3, \quad --(40)$$

where

$$\alpha_1 = -12\beta^2 C_3 + 16\beta^2 \varepsilon^2 C_2 ,$$

$$\alpha_2 = 6\beta^2 E - 18\beta\gamma k^2,$$

$$\alpha_0 = 4\beta C_3 (\beta E - \gamma k^2)$$

and

$$\ell = 4\gamma^2 k^2 \left(\frac{2\beta}{\gamma} E - k^2\right).$$

By choosing $C_2, C_3 = 0$, the equation (40) becomes

$$y'' = \ell y + \varepsilon^2 \alpha_2 y^2 - 10\beta^2 \varepsilon^4 y^3, \quad \text{--(41)}$$

where y is the energy in the sidebands. There are two cases to consider.

Case (1). $\ell > 0$. This leads to an initial exponential growth of the energy in the sidebands. If the truncated series $v = a_0 + a_1 e^{ikx} + a_2 e^{-ikx}$ were a reasonable approximation to the total solution one can see from the sign of coefficient of y^3 in equation (41), that a periodic structure for the energy is reached after a time scale ε^{-2}. However, it is clear when $y = O(\varepsilon^{-2})$ that the higher harmonies have already been generated and in some way would have affected the system at this stage. In order to properly describe the motion, one would require a solution to the fully nonlinear initial value problem. In the next section permanent envelope solutions are found which would include all harmonics to potentially the same order. It is possible that the above solution, when one includes the effects of all the harmonics could indeed tend to the permanent envelope solution if this were compatible with the initial conditions.

Case (2). $\ell < 0$. In this case, the energy in the sidebands stays the same order as it was initially. If the energy in the sidebands was initially very small, then the system would essentially be governed by the equation

$\frac{da_0}{dt} = i\beta a_0^2 a_0^*$, which simply shows that the primary wave undergoes a Stokes frequency modulation. However if the energy in the sidebands was initially of the same order as the energy in the primary, then there still is an energy transfer mechanism, which also might lead to a permanent envelope solution.

Permanent Envelope Solution.

For a certain class of initial conditions, a permanent wave solution of equation (42),

$$\frac{\partial v}{\partial t} = i\gamma \frac{\partial^2 v}{\partial x^2} + i\beta v^2 v^*, \qquad \text{---- (42)}$$

rememher is possible.

Let $\xi = x - vt$, v constant, and let $v = v(\xi)$. Equation (42) becomes,

$$-V \frac{\partial v}{\partial \xi} = i\gamma \frac{\partial^2 v}{\partial \xi^2} + i\beta v^2 v^*. \qquad \text{---- (43)}$$

Let $v = u\, e^{i\frac{V}{2\gamma}\xi}$ in equation (43), which then becomes

$$u'' + \alpha^2 u + \beta/_\gamma\, u^2 u^* = 0, \qquad \text{--- (44)}$$

where $\alpha^2 = \frac{V^2}{4\gamma^2}$. Setting $u = r(\xi)\, e^{i\theta(\xi)}$ in equation (44) and equating real and imaginary parts, one obtains

$$2r'\theta' + r\theta'' = 0, \qquad \text{--- (45)}$$

and

$$r'' - r\theta'^2 + \alpha^2 r + \beta/_\gamma\, r^3 = 0. \qquad \text{--- (46)}$$

Equation (45) implies that

$$r^2\theta' = h, \text{ a constant}, \qquad \text{--- (47)}.$$

Substituting equation (47) in (46), one obtains the differential equation

$$r'' - \frac{h^2}{r^3} + \alpha^2 r + \frac{\beta}{\gamma} r^3 = 0. \quad \cdots (48)$$

Multiplying (48) by r' and integrating, one obtains

$$r'^2 = -\frac{h^2}{r^2} - \alpha^2 r^2 - \frac{\beta}{2\gamma} r^4 + 2E. \quad \cdots (49)$$

Setting $r'^2 = y_1$ and $r^2 = x$, one can draw the phase plane diagram, given by the equation,

$$y = 2E - \alpha^2 x - \frac{\beta}{2\gamma} x^2 - \frac{h^2}{x} \quad \cdots (50)$$

Let $y_1 = -\frac{h^2}{x}$ and $y_2 = \frac{\beta}{2\gamma} x^2 + \alpha^2 x - 2E$. It is possible for the solution to exist for regions in the (x,y) plane where $x > 0$ and $y_1 > y_2$. The different possibilities are shown in Figs. 1, 2, and 3. In Fig. 3, one has perhaps the case of most interest, for it is when $\frac{\beta}{2\gamma} > 0$ that the exponential growth arises in the previous harmonic analysis. One can see that there are permanent envelope solutions, (which in general are elliptic functions) which are periodic. In Fig. 1, there is the possibility of a solitary wave, when $\alpha = \frac{V}{2\gamma}$ is chosen such that the curves y_1 and y_2 share a common tangent. This occurs when the right hand side of equation (50) has a double root, which implies (after some manipulation) that

$$\alpha^6 - \frac{E^2}{h^2}\alpha^4 + \frac{9}{2}\frac{\beta E}{\gamma}\alpha^2 + \frac{27}{16}\frac{\beta^2}{\gamma^2 h^2}\left(h^4 - \frac{64}{27}\frac{\gamma E}{\beta}E^2\right) = 0,$$

which can be written as

$$(\alpha^2)^3 - a_1(\alpha^2)^2 - a_2(\alpha^2) + a_3 = 0, \quad a_1, a_2, a_3 > 0,$$

since $\beta/\gamma \, E < 0$ for this case. This equation has two or no real positive roots for α^2 depending on whether

$$a_3 \lessgtr \frac{a_1 a_2}{3} + \frac{2}{27} a_1^3 + \sqrt{\left(\frac{a_1 a_2}{3} + \frac{2}{27} a_1^3\right)^2 + \frac{1}{27}\left(a_1^2 a_2^2 + 4a_2^3\right)}.$$

One can find the form of the solitary wave solution by noting that equation (49) can be written in the form,

$$r'^2 = \frac{\nu^2}{r^2}(r^2 - r_0^2)^2 (r^2 - r_1^2), \quad r_0^2 > r_1^2, \quad \nu^2 = -\frac{\beta}{2\gamma}.$$

The solution of this equation is

$$r^2 = r_0^2 - (r_0^2 - r_1^2)\,\mathrm{Sech}^2 \nu \sqrt{r_0^2 - r_1^2}\; \xi \qquad \ldots [51]$$

Therefore for $\xi = 0$, $r^2 = r_1^2$, and for $\xi = \pm\infty$ $r^2 = r_0^2$.

One notes that these solutions are really permanent envelopes. It is the envelope of the wave train that can move without change of shape.

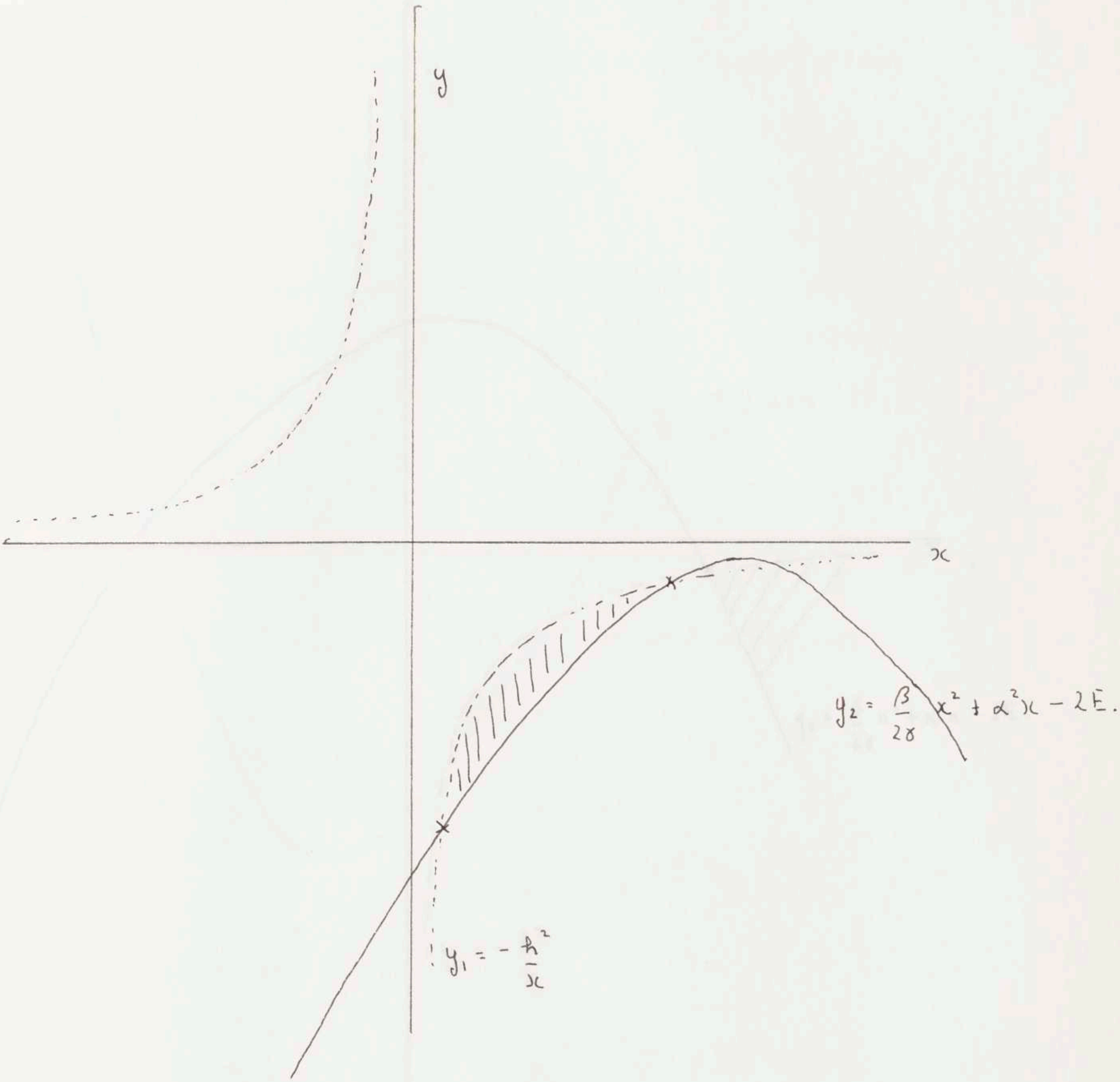

Fig. 1.

$\frac{\beta}{2\gamma} < 0, \quad 2E > 0.$ //////// = region of existence of solⁿ.

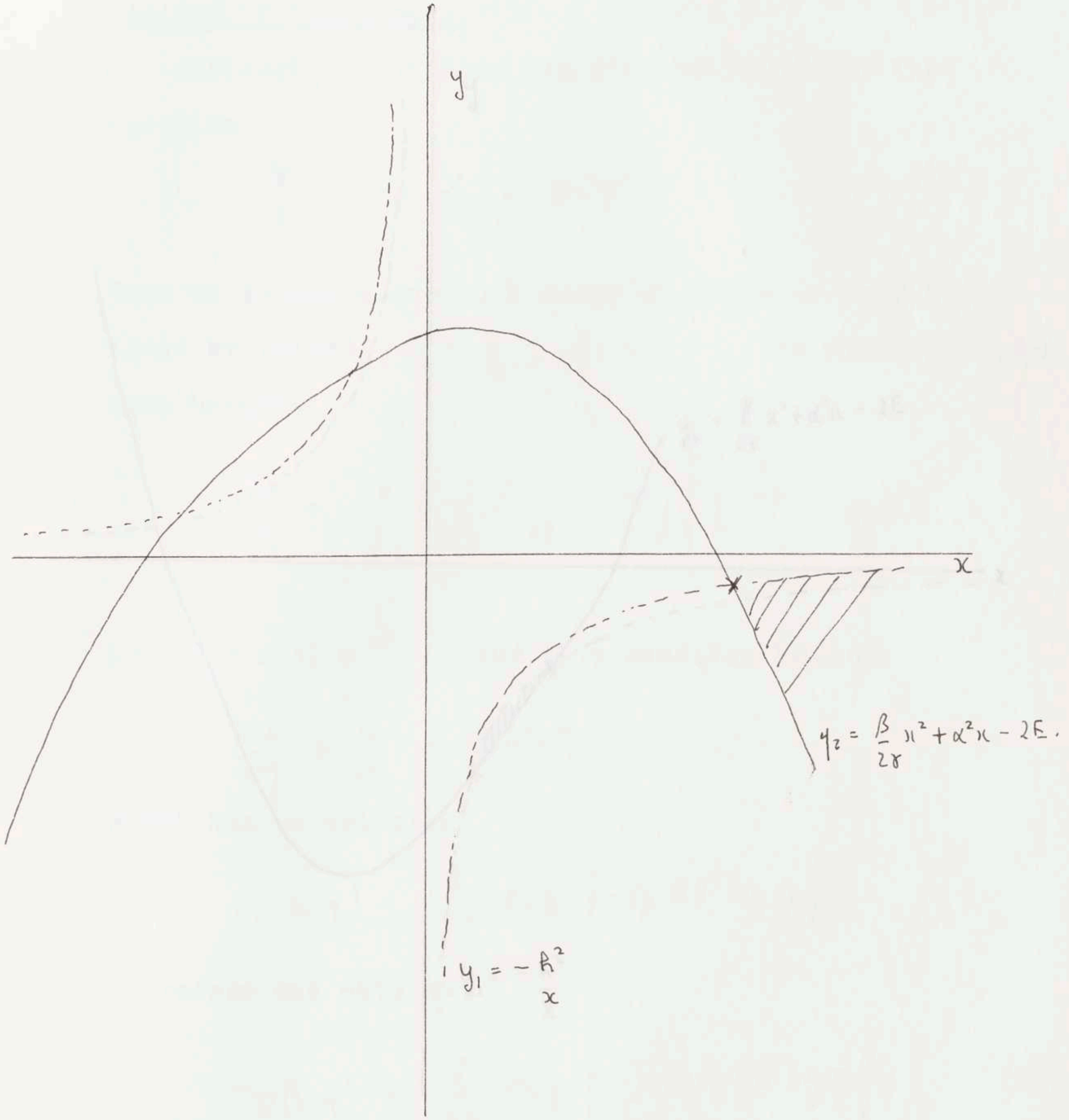

Fig 2.

$\frac{\beta}{2\gamma} < 0, \quad 2E < 0.$ $\qquad$ ////// = region of existence of solⁿ.

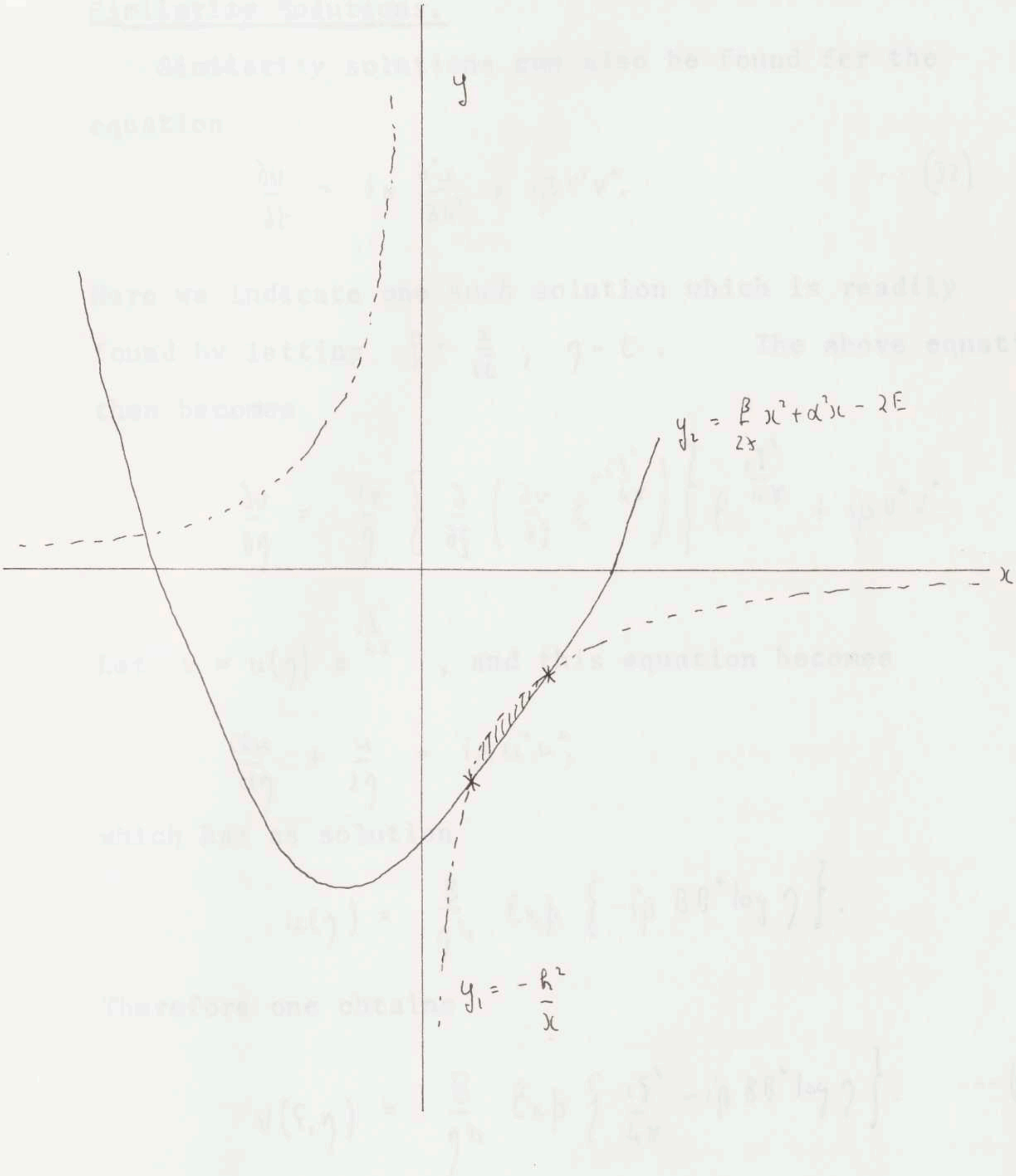

Fig 3.

$\frac{\beta}{2\gamma} > 0, \quad 2E > 0.$ ////// = region of existence of sol?.

Similarity Solutions.

Similarity solutions can also be found for the equation

$$\frac{\partial v}{\partial t} = i\gamma \frac{\partial^2 v}{\partial x^2} + i\beta v^2 v^* . \qquad \cdots (52)$$

Here we indicate one such solution which is readily found by letting $\xi = \frac{x}{\sqrt{t}}$, $\eta = t$. The above equation then becomes

$$\frac{\partial v}{\partial \eta} = \frac{i\gamma}{\eta} \left\{ \frac{\partial}{\partial \xi} \left(\frac{\partial v}{\partial \xi} e^{-\frac{i\xi^2}{4\gamma}} \right) \right\} e^{\frac{i\xi^2}{4\gamma}} + i\beta v^2 v^* .$$

Let $v = u(\eta)\, e^{\frac{i\xi^2}{4\gamma}}$, and this equation becomes

$$\frac{du}{d\eta} + \frac{u}{2\eta} = i\beta u^2 u^* ,$$

which has as solution

$$u(\eta) = \frac{B}{\eta^{1/2}} \exp\left\{ -i\beta B B^* \log \eta \right\} .$$

Therefore one obtains

$$v(\xi, \eta) = \frac{B}{\eta^{1/2}} \exp\left\{ \frac{i\xi^2}{4\gamma} - i\beta B B^* \log \eta \right\} . \qquad \cdots (53).$$

This solution is possibly valid when the energy is initially smeared over a continuous band of frequencies in the local neighborhood of the wave number of the primary travelling wave k . It is certainly true that if no nonlinearity was present ($\beta = 0$), that one would expect a time decay $\frac{1}{\sqrt{t}}$ of the envelope due to dispersion.

Summary and Conclusions.

From the preceding analysis it can be seen that there is a mechanism present which is capable of transferring energy between local sidebands and a primary travelling wave. This is clear from equation (8), which demonstrates that if there is any energy initially in a discrete side-band, that this sideband along with its higher harmonics potentially becomes of the same order as the primary travelling wave, in long time. It is not quite as strong a mechanism as the quartet resonance mechanism in the sense that in the resonance mechanism a discrete wave can be generated from an initially zero state by three other discrete waves.

It is then purely a matter of conjecture as to what the final state of the system is in long time. It has been shown that there is the possibility of permanent envelope solutions and in particular a solution which describes a solitary permanent envelope. As to whether these states are reached depends on the initial conditions.

It seems plausible however that if initially the energy was distributed in a continuous band of wave numbers around a primary travelling wave, that one could expect a solution which behaved in long time like the similarity solution which is given on page (315). For if one had a system with no nonlinearity then the method of stationary phase would certainly indicate that the solution (53), with $\beta = 0$ is valid in long time.

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