# RESULTS ON BROWN-GITLER TYPE SPECTRA

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## ABSTRACT

Specialized Brown-Gitler spectra are constructed. The first collection, $B_1(k)$, is suitable for studying orientable manifolds; the second collection, $B_2(4k+3)$ is suitable for studying Spin manifolds. For any CW complex Z, $B_1(k)_*Z$ and $B_2(4k+3)_*Z$ are computed in a range; the images of $B_1(k)_*Z \to H_*(Z; \mathbb{Z}_{\hat{2}})$ and $B_2(4k+3)_*Z \to bo_*Z$ is discussed. In particular, $\pi_*B_1(k)$ and $\pi_*B_2(4k+3)$ are computed in a range. Odd primary analogues are produced.

Then, we turn to the study of the structure of ordinary Brown-Gitler spectra. Applying these results and work of R. L. Cohen [14], we show that for primes $p \geq 3$ $h_0h_j \in Ext_A(\mathbb{Z}_p, \mathbb{Z}_p)$ is an infinite cycle in the Adams Spectral Sequence and represents an element $\eta_j \in \pi_*S^0$. If $p \geq 5$, $\eta_j$ can be chosen to be of order p. Finally, we use the Adams-Novikov Spectral Sequence to produce more elements in $\pi_*S^0$.

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## Introduction

Brown-Gitler spectra have had two applications. The first is to the study of manifolds and the second is to stable homotopy theory, particularly to stable homotopy groups of spheres. This thesis is divided into two parts. In the first we produce Brown-Gitler type spectra suited for the study of oriented and Spin manifolds. This is the bulk of the work. In the second part, which is joint work with Ralph Cohen, we finish the program begun by him and completely determine, for primes greater than three, all secondary cohomology operations that can detect stable maps of spheres. A central element in our discussion will be sty of the structure of odd primary Brown-Gitler spectra.

Let us examine these results a bit more closely. Brown and Gitler's original result was this: There exists a collection of spectra $\{B(k)\}_{k\geq 0}$ that filters the Eilenberg-MacLane spectrum $K\mathbb{Z}_2$ and so that the associated generalized homology theories $B(k)_*$ filter ordinary homology $H_*( \ ; \mathbb{Z}_2)$ in a particularly nice way. To say that we have Brown-Gitler type spectra suitable for studying orientable manifolds is to say this: There exists a collection of spectra $\{B_1(k)\}_{k>0}$ that filters the Eilenberg-MacLane spectrum $K\hat{\mathbb{Z}}_2$ and so that $B_1(k)_*$, $k \geq 1$, filters $H_*( \ ; \hat{\mathbb{Z}}_2)$ in a nice way. $\hat{\mathbb{Z}}_2$ is the completion of the integers at 2 . Similarly, to say that we have Brown-Gitler type spectra suitable for studying Spin manifolds is to say that we have a collection of spectra $\{B_2(4k+3)\}_{k\geq 0}$ so that $B_2(4k+3)_*$ filters real connective K-theory. This last homology theory is called bo ; it is a direct summand of Spin bordism completed at 2 .

The exact statements of these theorems and their connection to the study of manifolds will be found in the next section.

In the final chapter we - the author and Ralph Cohen - produce for each odd prime an infinite family of elements in the stable homotopy groups of spheres detected by secondary cohomology operations. The key to our construction is a lemma which studies the homotopy theory of Brown-Gitler spectra. Here is the 2-primary result.

<u>Lemma</u>: Let $k > 1$ be an integer and $k \neq 2^j$ some $j$ . Let

$$k = 2^{j_1} + 2^{j_2} + \dots + 2^{j_n}$$

be the diadic expansion of $k$ . $j_1 > j_2 > \dots > j_n$ . Then $B(k)$ is a wedge summand of

$$B(2^{j_1}) \wedge B(2^{j_2}) \wedge \dots \wedge B(2^{j_n}) \quad .$$

In effect this lemma says that the only "indecomposable" Brown-Gitler spectra are the $B(2^j)$ . There is an odd primary analogue.

Apply this lemma and results of R. Cohen we then show that for $p \geq 3$ , $j \geq 2$

$$h_o h_j \in \mathrm{Ext}_A(\mathbb{Z}_p, \mathbb{Z}_p)$$

is an infinite cycle in the Adams Spectral Sequence and represents an element in the stable homotopy groups of spheres. Here $A$ is the mod p Steenrod algebra.

A more detailed outline is given in the introduction to Chapter III.

The author wishes to thank Frank Peterson for many helpful conversations; in particular he heard far more convoluted attempts to prove the theorems of Chapter III. Ralph Cohen, too, is due thanks for his suggestions and insights; a chance remark of his is responsible for the first two chapters of this work. Mark Mahowald, Haynes Miller, Doug Ravenel, and Don Shimamoto all spent some time with the authors over these problems and deserve thanks. Finally, many thanks to Anne Clee and Maggie Sullivan for typing this lengthy manuscript.

## INTRODUCTION: CHAPTERS I AND II

When Brown-Gitler spectra first appeared in 1973 [6], surely they were an answer to this question: Can we define, for all smooth manifolds, higher order characteristic classes? Given the amount of geometric information inherent in primary characteristic classes - Stiefel-Whitney classes - the existence of even a secondary characteristic class would presumably supply topological data.

The existence of Brown-Gitler spectra implies, in effect, that no such higher characteristic classes exist. The negative answer has proved to be of interest, however; Brown-Gitler spectra have since found application in the study of manifolds ([10], [11], [16]), loop space theory ([9], [14], [15]) and stable homotopy theory ([14], [12], [20]). The question we address here is this: Do there exist Brown-Gitler type spectra suited for studying oriented manifolds? or Spin manifolds? or complex manifolds?

The purpose of the first two chapters of this work is to produce Brown-Gitler spectra for oriented and Spin manifolds. The complex case is beyond our methods. In the oriented case we give a complete discussion; the Spin case is less amenable to total analysis.

To begin, let us point out what is remarkable about the original Brown-Gitler spectra. To this end, we discuss manifolds for a moment. Fix an integer n and let M be a closed, differentiable n-manifold. Then, let $T(\nu)$ denote the Thom spectrum of the stable normal bundle of M. We will make it a convention that the Thom class $\mu \in H^0 T(\nu)$

and that cohomology has $\mathbb{Z}_2$ coefficients. Then, $\mu$ may be thought of as a map

$$\mu : T(\nu) \longrightarrow K\mathbb{Z}_2$$

where $K\mathbb{Z}_2$ is the Eilenberg-MacLane spectrum. Then, in cohomology, we have a map

$$\mu^* : A = H^*K\mathbb{Z}_2 \longrightarrow H^*T(\nu)$$

where A is the mod-2 Steenrod Algebra. Let ker(M) = kernel of $\mu^*$ and set

$$I_n = \cap \ker(M)$$

where the intersection is taken over all n-manifolds. Brown and Peterson [7] computed $I_n$:

$$I_n = A\{\chi(Sq^i) : i > [\tfrac{n}{2}]\}$$

That is, $I_n$ is the left ideal in A generated by $\chi(Sq^i)$ for i greater than the greatest integer less than or equal to n/2. $\chi$ is the canonical anti-automorphism of the Steenrod Algebra.

This, then, is Brown and Gitler's original construction:

<u>Theorem</u>: There exist spectra B(k), $k \geq 0$, so that

1) $H^*B(k) = A/I_{2k} = A/A\{\chi(Sq^i) : i > k\}$

2) Let B(k) be the homology theory based on the spectrum B(k), and let $1 : B(k) \longrightarrow K\mathbb{Z}_2$ be the generator of $H^*B(k)$. Then, for any CW

$$1_* : B(k)_q Z \rightarrow H_q Z$$

is surjective for $q \leq 2k+1$.

Let us examine 2) where $Z = M$ is a manifold of dimension $2k$ or $2k+1$. Then the fundamental class $[M] \in H_*M$ is in the image of $1_*$. Since $T(\nu)$ is the Spanier-Whitehead dual of $M \cup \{\text{disjoint basepoint}\}$, we can conclude that there is a lifting

$$\begin{array}{ccc} & \nearrow & B(k) \\ & & \downarrow 1 \\ T(\nu) & \xrightarrow{\mu} & K\mathbb{Z}_2 \end{array}$$

1) implies that this could not occur if $H^*B(k)$ were a smaller quotient of $A$.

Let us now state the results of the first two chapters of this work. Let

$$I_n^{so} = \cap \ker(M)$$

where the intersection is taken over all <u>orientable</u> n-manifolds $M$. Then by [7]

$$I_n^{so} = A\{Sq^1, \chi(Sq^i) : i > [\tfrac{n}{2}]\}$$

if $n = 4k+1$, $4k+2$, or $4k+3$. Note that $I_{4k+1}^{so} = I_{4k+2}^{so}$. This is because $\chi(Sq^{2k+1}) = \chi(Sq^{2k})Sq^1$.

<u>Theorem I</u>: There exist spectra $B_1(2k+1)$ so that

(1) $H^*B_1(2k+1) = A/A\{Sq^1, \chi(Sq^i) : i > 2k\} = A/I^{so}_{4k+1}$

(2) Let $1 : B_1(2k+1) \longrightarrow K\hat{\mathbb{Z}_2}$ classify a generator of $H^0(B_1(2k+1); \hat{\mathbb{Z}_2})$. Then, for any CW complex

$$1_* : B_1(2k+1)_q Z \longrightarrow H_q(Z; \hat{\mathbb{Z}_2})$$

is surjective for any CW complex Z and $q \leq 4k+3$.

$\hat{\mathbb{Z}_2} = \lim \mathbb{Z}/2n\mathbb{Z}$ is the integers completed at 2.

The existence of spectra $\overline{B}_1(2k+1)$ satisfying property 1) of Theorem I was first noted by Mahowald [19] and elaborated by Shimamoto [31]. The method was this: Let $S^3\langle 3\rangle$ be the three-connected cover of the three sphere, $S^3$. Then, if $\eta$ is the generator of $\pi_1 BO$, there is an induced bundle $\gamma$

$$\gamma : \Omega^2 S^3\langle 3\rangle \longrightarrow \Omega^2 S^3 \xrightarrow{\Omega^2\Sigma^2\eta} \Omega^2\Sigma^2 BO \longrightarrow BO .$$

The last composition exists because BO is an infinite loop space. Mahowald asserts the existence of a filtration

$$\{F_m\Omega^2 S^3\langle 3\rangle\}_{m\geq 0}$$

of $\Omega^2 S^3\langle 3\rangle$. Then there is a bundle

$$\gamma_{2k} : F_{2k}\Omega^2 S^3\langle 3\rangle \subseteq \Omega^2 S^3\langle 3\rangle \xrightarrow{\gamma} BO .$$

The 2-completion of the Thom spectrum of $\gamma_{2k}$ is $\overline{B}_1(2k+1)$. Concurrently with the work described here, D. Shimamoto [31] demonstrated that $\overline{B}_1(2k+1)$

also satisfy property 2) of Theorem I. This is enough to produce a homotopy equivalence

$$\overline{B}_1(2k+1) \xrightarrow{\cong} B_1(2k+1) \ .$$

This is proved in Section 2 of Chapter 2.

Referring to Theorem I, note that if $T(\nu)$ is the Thom spectrum of the normal bundle of an orientable n-manifold, $n \leq 4k+3$, there is a factoring

$$\begin{array}{ccc} & & B_1(2k+1) \\ & \nearrow & \downarrow \\ T(\nu) & \xrightarrow{\tilde{\mu}} & K\mathbb{Z}_2^{\wedge} \end{array}$$

where $\tilde{\mu}$ is any generator of $H^0(T(\nu); \mathbb{Z}_2^{\wedge})$. $\tilde{\mu}$ is a choice for the $\mathbb{Z}_2^{\wedge}$ oriented Thom class of $T(\nu)$.

Let us now describe $I^{so}_{4k}$. Let $2k = 2^S(2t+1)$. Then, again by [7],

$$I^{so}_{4k} = A\{Sq^1, \chi(Sq^{2^S(2t+1)}\, Sq^{2^{S-1}(2t-1)} \ldots Sq^{2^{S-1}(2t+1)} Sq^1), \chi(Sq^i) : 1 \leq j \leq S,\ i > 2k\}.$$

We have the following result.

<u>Theorem</u> <u>II</u>: Let $2k = 2^S(2t+1)$. Then there exist spectra $B_1(2k)$ so that

1) $H^*B_1(2k) \simeq A/I^{so}_{4k}$

2) Let $1 : B_1(2k+1) \longrightarrow K\mathbb{Z}_2^{\wedge}$ classify a generator of $H^0(B_1(2k); \mathbb{Z}_2^{\wedge})$ then

is surjective for any CW complex Z and $q \leq 4k-1$.

3) Let $z \in H_{4k}(X; \mathbb{Z}_2^\wedge)$. There are s primary obstructions, s-1 secondary obstructions, and possibly higher obstructions to z being in the image of

$$1_* : B_1(2k)_{4k}Z \longrightarrow H_{4k}(Z; \mathbb{Z}_2^\wedge)$$

Let us examine 3). The primary obstructions will vanish for

$$z = [M] \in H_{4k}(M; \mathbb{Z}_2^\wedge)$$

where [M] is the fundamental class of a $\mathbb{Z}_2^\wedge$ oriented 4k manifold. If $s = 1$, that is, $2k = 4t+2$, then there are no higher obstructions and we have a factoring

$$\begin{array}{ccc} & & B_1(2k) \\ & \nearrow & \downarrow \\ T(\nu) & \xrightarrow{\tilde{\mu}} & K\mathbb{Z}_2 \end{array}$$

If $s > 1$, and $M = \mathbb{C}P^{2k}$, all the secondary obstructions vanish modulo indeterminary on

$$z = [\mathbb{C}P^{2k}] \in H_{4k}(\mathbb{C}P^{2k}; \mathbb{Z}_2)$$

We say nothing, at this time, about other manifolds, or higher obstructions.

Let us now turn to Spin manifolds. If the first and second Stiefel-Whitney classes of the normal bundle of a manifold M vanish, then M has a Spin structure; that is, we have a factoring

$$\begin{array}{ccc} & & BSpin \\ & \nearrow & \downarrow \\ M & \xrightarrow{\nu} & BO \end{array}$$

$\nu$ classifies the normal bundle and $\tilde{\nu}$ induces a map

$$g : T(\nu) \longrightarrow MSpin$$

where MSpin is the Thom spectrum of the universal Spin bundle. Completed at 2, MSpin splits as a wedge of indecomposable spectra; in particular, after completion, there is a projection onto a wedge summand (see [34])

$$MSpin \xrightarrow{j} bo.$$

which is an isomorphism on $H^0$. bo is the 2-completion of the representing spectrum for connective real K-theory. bo has been extensively studied ([17], [19]). We are interested in the composite $j \circ g : T(\nu) \longrightarrow bo$.

We know that $H^*bo = A/A\{Sq^1, Sq^2\}$ and that

$$\pi_n bo \cong \begin{cases} \mathbb{Z}_2^{\wedge} & n = 0,4 \quad \text{mod } 8, \quad n \geq 0 \\ \mathbb{Z}_2 & n = 1,2 \quad \text{mod } 8, \quad n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $1 \quad H^*bo$ be the Steenrod Algebra generator. Then, there are maps $i_{4_j} : bo \longrightarrow K\mathbb{Z}_2^{\wedge}$, unique up to a homotopy equivalence of $K\mathbb{Z}_2^{\wedge}$, so that we have a commutative diagram

$$\begin{array}{ccc} bo & \xrightarrow{i_{4j}} & \Sigma^{4j} K\mathbb{Z}_2^{\wedge} \\ \downarrow 1 & & \downarrow 1 \\ K\mathbb{Z}_2 & \xrightarrow{\chi(Sq^{4j})} & \Sigma^{4j} K\mathbb{Z}_2 \end{array}$$

$i_{4j}$ induces, for every CW complex, a map

$$(i_{4j})_* : bo_q Z \longrightarrow H_{q-4j}(Z; \mathbb{Z}_2^{\wedge})$$

<u>Theorem</u> <u>III</u>: There exist spectra $B_2(4k+3)$ and maps $1 : B_2(4k+3) \longrightarrow bo$ so that

1) $H^*B_2(4k+3) = A/A\{Sq^1, Sq^2, \chi(Sq^i)\ i > 4k\}$

2) For any CW complex Z, $1_* : B_2(4k+3)_q Z \longrightarrow bo_q Z$ is onto

$$\bigcap_{j>k} \ker(i_{4j})_*$$

for $q \leq 8k+7$.

<u>Remark</u>: Given that $H^*B_2(4k+3)$ is as in Theorem III, part 1), the image of $1_*$ in III, part 2) can be no larger than $\bigcap_{j>k} \ker(i_{4j})_*$. The result asserts that the image of $1_*$ is as large as possible.

John Jones and Mark Mahowald have produced spectra $\overline{B}_2(4k+3)$ so that $H^*\overline{B}_2(4k+3) = A/A\{Sq^1, Sq^2, \chi(Sq^i) : i > 4k\}$. At this point, I have no knowledge of whether or not they have an analogue of Theorem III, part 2).

There are odd primary analogues of our theorems, just as there are odd primary analogues of the original Brown-Gitler Spectra ([14]).

For instance, the methods used to prove Theorems I and III give the following result.

<u>Theorem</u> <u>IV</u>: Let $p \geq 3$ be any odd prime and let $A_p$ be the mod-p Steenrod Algebra. Then there exist p-complete spectra $B_1(pk+1)$ so that

1) $H^*(B_1(pk+1), \mathbb{Z}_p) \simeq A_p/A_p\{\beta, \chi(p^i), i > k\}$.

2) Let $1 : B_1(pk+1) \longrightarrow K\mathbb{Z}_p^\wedge$ classify a generator of $H^0(B_1(pk+1); \mathbb{Z}_p^\wedge)$. Then

$$1_* : B_1(pk+1)_q Z \longrightarrow H_q(Z; \mathbb{Z}_p^\wedge)$$

is surjective for any CW complex and $n \leq 2p(k+1) - 1$.

Presumably, there is an odd primary analogue of Theorem II; however, the proof of Theorem II is grueling. One should not undertake an odd primes version unless one sees a need.

One remark should be made. In each of the theorems above, a spectrum B is constructed along with a map $1 : B \longrightarrow \mathcal{H}$ where $\mathcal{H}$ is the representing spectrum of a homology theory derived from a cohordism theory. Then we asserted that

$$1_* : B_*Z \longrightarrow \mathcal{H}_*Z$$

was surjective in a range. More is true: we will actually compute $B_*Z$ in a range and give a good characterization of the kernel of $1_*$. In particular, we may compute $\pi_*B$ in a range. (The computation of $\pi_*B_1(2k+1)$ is accomplished also by Shimamoto [31].) The statement of these results is complicated and we leave it until we have proofs in hand.

In Chapter I, we prove Theorems I, III, and IV. In Chapter II, we prove Theorem II, characterize the homotopy type of the spectra $B_1(2k+1)$, and discuss some applications to manifolds. Each chapter begins with a detailed outline of the proofs to be presented. Hopefully, these outlines will provide some form to the intricacies of our techniques.

All spectra will be completed at a prime p - usually p = 2. See Bousfield [3] and note that the 2-completion of $K\mathbb{Z}$ is $K\hat{\mathbb{Z}}_2$. This is essential; a cornerstone of our proofs are the results of Section I.3 and these theorems do not apply to spectra such as $K\mathbb{Z}$ or even $K\mathbb{Z}_{(2)}$ - $K\mathbb{Z}$ localized at two.

Finally, we are working the stable category; the framework for this setting is provided by Adams [2]. Often, however, we will wish to restrict ourselves to $\Omega$ spectra; that is collections of spaces $X = \{X_i\}$ with homotopy equivalences

$$f_i : X_i \xrightarrow{\cong} \Omega X_{i+1} .$$

For a discussion of these spectra, one should turn to May [22]. The reader should note that if we refer to a spectrum X and then use the notation $X_i$, we are referring to the $i^{th}$ space in the $\Omega$-spectrum X.

## Chapter I

## CONSTRUCTION OF $B_1(2k+1)$, $B_2(4k+3)$, AND $B_1(pk+1)$

In this first chapter, we use the techniques of Brown and Gitler to produce the spectra we want. We concentrate our attention on $B_1(2k+1)$ and $B_2(4k+3)$. The modifications needed to produce $B_1(pk+1)$, $p \geq 3$, are slight and I outline them in a final section.

### 1. Outline of the Proof

The proof of the existence of the spaces $B_1(2k+1)$ and $B_2(4k+3)$ is technical. For that reason, we supply, in this section, a careful outline, providing definitions and statements of lemmas, but postpone the lengthy proofs. This is done for two other reasons: to provide a roadmap through the subsequent sections and to furnish, if not motivation, at least a framework for the technical arguments that follow.

Our argument parallels that of Brown and Gitler [6], and is obviously indebted to their paper. We refer freely to it.

The thrust of the argument is to produce a tower of spectra

$$\begin{array}{ccccccccccccc} \cdots \longrightarrow & X_q & \longrightarrow & X_{q-1} & \longrightarrow \cdots \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_o \\ & {\scriptstyle i_q}\uparrow & & & & {\scriptstyle i_2}\uparrow & & {\scriptstyle i_1}\uparrow & & \\ & F_q & & & & F_2 & & F_1 & & \end{array} \qquad (1.1)$$

so that $p_q : X_q \longrightarrow X_{q-1}$ is a fibration with fiber $F_q$. Additionally, we want a good hold on the homology theory induced by the spectrum $X_q$. $X_o$ will be $K\hat{\mathbb{Z}}_2$ or bo depending on whether we are constructing $B_1(2k+1)$ or $B_2(4k+3)$. Then we will set $B_1(2k+1)$ (or $B_2(4k+3)$) equal to $\varprojlim X_q$.

$F_q$ will be a product (that is, wedge) of Eilenberg-MacLane spectra.

As usual in such constructions, the first step is to determine what $F_q$ should be. As a start, note that if $X_q$ existed, then the fact that

$$F_q \xrightarrow{i_q} X_q \longrightarrow X_{q-1}$$

is a fibration allows us to produce a map $\varepsilon_q : X_{q-1} \longrightarrow F_q$ of degree one. Here and later we will refer to a map of spectra $f : X \longrightarrow \Sigma^n Y$ as being of degree n. When dealing with $\Omega$-spectra this is often more convenient. Let $\delta_q$ be the composition

$$\delta_q = \varepsilon_q \circ i_{q-1} : F_{q-1} \longrightarrow F_q \ .$$

Then we would have a sequence

$$F_o \xrightarrow{\delta_1} F_1 \longrightarrow \cdots \longrightarrow F_{q-1} \xrightarrow{\delta_q} F_q \longrightarrow \cdots \qquad (1.2)$$

Each $\delta_q$ is of degree one. Our first step is to produce (1.2). Then, inductively, we produce $\varepsilon_q$ and, hence, (1.1).

We are trying to realize, as the cohomology of a spectrum, certain modules over the Steenrod Algebra. Set

$$M_1(2k+1) = A/A\{Sq^1, \chi(Sq^i) : i > 2k\}$$

and

$$M_2(4k+3) = A/A\{Sq^1, Sq^2, \chi(Sq^i) : i > 4k\}.$$

We require that the cohomology of (1.2) be a resolution of sorts of $M_1(2k+1)$ or $M_2(4k+3)$, as required. That is, apply cohomology to (1.2) and obtain

$$\cdots \longrightarrow H^*F_q \xrightarrow{\delta_q^*} H^*F_{q-1} \longrightarrow \cdots \longrightarrow H^*F_1 \xrightarrow{\delta_1^*} H^*F_0 \xrightarrow{\pi} M \longrightarrow 0 \tag{1.3}$$

where $M = M_1(2k+1)$ or $M_2(4k+3)$. $\pi$, at this point, is only algebraically defined; it is the quotient map from $H^*K\mathbb{Z}_2$ or $H^*bo$ to M. We ask that (1.3) be a long exact sequence; that is an acyclic resolution of M by modules over the Steenrod Algebra.

We first produce an algebraic resolution of M then produce spectra and maps which give in cohomology (that is, realize) this algebraic resolution.

Such resolutions require a bookkeeping device; here that device will be the $\Lambda$ algebra of Bousfield, Curtis, et.al. [4]. Let us recapitulate their results.

Let $\Lambda$ be the $\mathbb{Z}_2$ bi-graded algebra generated by elements $\lambda_i$ ($i \geq 0$) of grading $(1,i)$. The unit has grading $(0,0)$. We have relations

$$\lambda_i\lambda_j = \sum^{S-1} \binom{}{2s-(j-2i)} \lambda_{i+s}\lambda_{j-s} \tag{1.4}$$

In dealing with this algebra, one has certain conventions. First, if $I = (i_1, \ldots, i_n)$ is an n-tuple of non-negative integers, we write $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_n}$. It is then natural to say that $\lambda_I$ has length n. Secondly, we call $\lambda_I$ admissible if $2i_j \geq i_{j+1}$. It is a result of [4] that admissible monomials in the $\lambda_i$ form a $\mathbb{Z}_2$ vector space basis for $\Lambda$. Finally, we say $\dim\lambda_I = i_1 + i_2 + \ldots + i_n$.

$\Lambda$ has a very specific and rigid structure, and is replete with interesting ideals and sub-algebras. We detail a few in the following result.

<u>Lemma 1.5</u>: 1) Let $\Lambda_k$ be the left ideal generated by $\lambda_i$, $0 \leq i < k$. Then an additive basis for $\Lambda_k$ is given by all admissible monomials $\lambda_{i_1} \ldots \lambda_{i_n}$ with $i_n < k$.

2) Let $\hat{\Lambda}$ be the left ideal generated by $\lambda_{2i-1}$, $i \geq 1$. Then an additive basis for $\hat{\Lambda}$ is all admissible monomials $\lambda_{i_1} \ldots \lambda_{i_n}$ with $i_n$ odd.

3) Let $\overline{\Lambda}$ be the sub $\mathbb{Z}_2$ vector space of $\Lambda$ generated by elements $\lambda_{i_1} \ldots \lambda_{i_{n-1}}\lambda_{i_n}$ with $i_{n-1}$ odd and $i_n \equiv -1 \bmod 4$. If $\lambda_I$ is an element of $\overline{\Lambda}$ and of length greater than one and $\lambda_J \in \Lambda$, then $\lambda_J\lambda_I \in \overline{\Lambda}$.

The proof of 1.5 is in Section 2. It is an easy consequence of 1.4. (See Lemma 2.10.)

Let us see how this algebra was important in Brown and Gitler's work. Define $\Lambda(q,k)$ to be monomials of length q in $\Lambda/\Lambda_k$. And let us now, for strict accuracy, consider the $\mathbb{Z}_2$ dual $\Lambda^*(q,k)$. Then there exist maps $\delta_q^*$ so that the following is an acyclic resolution.

$$\ldots \longrightarrow A \otimes {}^*(q,k) \longrightarrow A \otimes {}^*(q-1,k) \longrightarrow \ldots$$

$$\ldots \longrightarrow A \otimes \Lambda^*(1,k) \xrightarrow{\delta_1^*} A \xrightarrow{\pi} A/A\{\chi(Sq^i) : i > k\} \longrightarrow 0 \qquad (1.6)$$

$\delta_q^*$ is complicated; see Lemma 2.13. If $\overline{\lambda}_j$ is one of the obvious generators of the dual vector space $\Lambda^*(1,k)$, then $\delta_1^*\overline{\lambda}_j = \chi(Sq^{j+1})$.

$$H^*B(k) \cong A/A\{\chi(Sq^i) : i > k\} = M(k) .$$

Notice that since $A \otimes \Lambda^*(q,k)$ is free over the Steenrod Algebra, we may find spectra $I_q$ so that $I_q$ is a product of Eilenberg MacLane spectra, $\pi_*I_q \simeq \Lambda(q,k)$ as a $\mathbb{Z}_2$ vector space, and $H^*I_q = A \otimes \Lambda^*(q,k)$. Further, we can find maps $\delta_q : I_{q-1} \longrightarrow I_q$ which, in cohomology, give $\delta_q^*$. Thus, we have a sequence of spectra

$$\cdots \longleftarrow I_q \xleftarrow{\delta_1} I_{q-1} \longleftarrow \cdots \longleftarrow I_1 \xleftarrow{\delta_1} I_o = K\mathbb{Z}_2 \qquad (1.7)$$

(1.7) realizes (1.6). Notice that, because of our remarks around (1.2), we should ask that $\delta_q$ be of degree one. This is Brown and Gitler's first step.

We turn to the problem of resolving $M_1(2k+1)$. Let $\hat{\Lambda}(q,2k+1)$ be the monomials of length q in $\hat{\Lambda}/\hat{\Lambda} \cap \Lambda_{2k+1}$.

<u>Lemma 1.8</u>: There exist maps $\delta_q^*$ so that

$$\cdots \longrightarrow A \otimes \hat{\Lambda}^*(q,2k+1) \longrightarrow A \otimes \hat{\Lambda}^*(q-1,\ 2k+1) \longrightarrow \cdots$$

$$\cdots \longrightarrow A \otimes \Lambda^*(1,\ 2k+1) \xrightarrow{\delta_1^*} A/ASq^1 \xrightarrow{\pi} M_1(2k+1) \longrightarrow 0$$

is an acyclic resolution of $M_1(2k+1)$ by modules over the Steenrod Algebra.

This is Theorem 2.11 of Section 2. From 2.13 we see that $\delta_1^*\overline{\lambda}_{2j-1} = \chi(Sq^{2j})$. We may now produce spectra $F_q$ so that $F_o = K\hat{\mathbb{Z}}_2$

and $F_q$ $(q \geq 1)$ is a produce of Eilenberg-MacLane spectra with $\pi_* F_q = \hat{\Lambda}(q, 2k+1)$ and $H^* F_q = A \otimes \hat{\Lambda}^*(q, 2k+1)$. Further, we may produce maps $\delta_q : F_{q-1} \longrightarrow F_q$ (of degree one) so that

$$\cdots \longleftarrow F_q \xleftarrow{\delta_q} F_{q-1} \longleftarrow \cdots \longleftarrow F_1 \xleftarrow{\delta_1} F_0 = K\mathbb{Z}_2^{\wedge} \qquad (1.9)$$

realizes the resolution of (1.8).

Now consider $M_2(4k+3) = A/A\{Sq^1, Sq^2, \chi(Sq^i) : i > 4k\}$. Let $\overline{\Lambda}(q, 4k+3)$ be all monomials of length q in $\overline{\Lambda}/\overline{\Lambda} \cap \Lambda_{4k+3}$. Refer to 1.5.

<u>Lemma 1.10</u>: There exist maps $\delta_q^*$ so that

$$\cdots \longrightarrow A \otimes \overline{\Lambda}^*(q, 4k+3) \xrightarrow{\delta_q^*} A \otimes \overline{\Lambda}^*(q-1, 4k+3) \longrightarrow \cdots$$
$$\cdots \longrightarrow A \otimes \overline{\Lambda}^*(2,4k+3) \xrightarrow{\delta} A/ASq^1 \otimes \overline{\Lambda}^*(1,4k+3) \xrightarrow{\delta} A/A\ Sq^1, Sq^2\} \xrightarrow{\pi}$$
$$M_2(4k+3) \longrightarrow 0$$

is an acyclic resolution of $M_2(4k+3)$ by modules over the Steenrod Algebra.

This, too, is proved in Section 2. We may produce spectra so that $F_q$ $(q \geq 2)$ is a product of Eilenberg-MacLane spectra with $\pi_* F_q = \overline{\Lambda}(q, 4k+3)$ and $H^* F_q = A \otimes \overline{\Lambda}^*(q, 4k+3)$. $F_1$ should be a product of Eilenberg-MacLane spaces of type $K\mathbb{Z}_2^{\wedge}$ and

$$\pi_n F_1 = \begin{cases} \mathbb{Z}_2^{\wedge} & n = 4j-1,\ j \geq k+1 \\ 0 & \text{otherwise} \end{cases}$$

$F_o$ = bo. We will be able to produce maps $\delta_q : F_{q-1} \longrightarrow F_q$ of degree one so that

$$\cdots \longleftarrow F_q \xleftarrow{\delta_q} F_{q-1} \longleftarrow \cdots \longleftarrow F_1 \xleftarrow{\delta_1} F_o = bo \qquad (1.11)$$

realizes 1.10. $\delta_1$ is a product of maps $i_{4j} : bo \longrightarrow K\mathbb{Z}_2^{\wedge}$ of degree 4j so that

$$\begin{array}{ccc} bo & \xrightarrow{i_{4j}} & K\mathbb{Z}_2^{\wedge} \\ 1\downarrow & & \downarrow 1 \\ K\mathbb{Z}_2 & \xrightarrow{\chi(Sq^{4j})} & K\mathbb{Z}_2 \end{array}$$

commutes. 1 is the Steenrod Algebra generator of H*bo or $H^*K\mathbb{Z}_2$. Thus $i_{4j}$ is only determined up to a homotopy equivalence of $F_1$. We make a more careful choice in Section 3.

The fact that we have used the symbol $\mu_q$ in 1.6 through 1.11 is meant to be suggestive. There are obvious projection maps

$$\mu_q^* : A \otimes \Lambda^*(q,\ 2k+1) \longrightarrow A \otimes \hat{\Lambda}^*(q,\ 2k+1) \qquad q \geq 1$$

$$\mu_o^* : A \longrightarrow A/ASq^1 = H^*K\mathbb{Z}_2^{\wedge}$$

$$\mu_q^* : A \otimes \Lambda^*(q,\ 4k+3) \longrightarrow A \otimes \bar{\Lambda}^*(q,\ 4k+3) \qquad q \geq 2$$

$$\mu_1^* : A \otimes \Lambda^*(1,\ 4k+3) \longrightarrow A/ASq^1 \otimes \bar{\Lambda}^*(1,\ 4k+3)$$

$$\mu_o^* : A \longrightarrow A/A\{Sq^1,\ Sq^2\} = H^*bo$$

These maps have the property that $\delta_q^*\mu_q^* = \mu_{q-1}^*\delta_q^*$. Since we are producing maps into Eilenberg-MacLane spectra, we have maps $\mu_q : F_q \longrightarrow I_q$ (see 1.7) and a diagram:

$$\begin{array}{ccccccccccc} F_0 & \longrightarrow & F_1 & \longrightarrow & \cdots & \longrightarrow & F_{q-1} & \longrightarrow & F_q & \longrightarrow & \cdots \\ \mu_0\downarrow & & \mu_1\downarrow & & & & \downarrow & & \mu_q\downarrow & & \\ I_0 & \longrightarrow & I_1 & \longrightarrow & \cdots & \longrightarrow & I_{q-1} & \longrightarrow & I_q & \longrightarrow & \cdots \end{array} \qquad (1.12)$$

Thus we get maps of resolutions 1.9 or 1.11 to the resolution 1.7. The existence of 1.12 is important: we wish to use it to play off the results of Brown and Gitler.

Now we return to the original problem: that of constructing the tower (1.1). To do this, we wish to inductively produce the maps $\varepsilon_q : X_{q-1} \longrightarrow F_q$ defined prior to (1.2). This is in general quite difficult, if not impossible. The approach that works for these Brown-Gitler type spectra is to dualize the problem. That is, we apply a functor $\chi$ to 1.9 or 1.11 and obtain a sequence of spectra

$$\cdots \longrightarrow K_q \xrightarrow{d_q} K_{q-1} \longrightarrow \cdots \longrightarrow K_1 \xrightarrow{d_1} K_0 .$$

Here $K_q = \chi(F_q)$ is a product of Eilenberg-MacLane spectra if $q \geq 1$, and $K_0 = \chi(K\mathbb{Z}_2)$ or $\chi(bo)$.

Then, inductively, we produce spaces $G_{q-1}$ and maps $e_q : K_q \longrightarrow G_{q-1}$. Finally, $X_{q-1} = \chi(G_{q-1})$ and $\varepsilon_q = \chi(e_q)$. So we must describe $\chi$.

This functor $\chi$ is essentially the Pontrjagin duality of Brown and Gomenetz [5], and has the property that, for a spectrum X and a CW complex Z one can compute $X_*Z$ from $\chi(X)^*Z$. The functor of [5], however, is only defined for spectra X so that $\pi_n X$ is finite for each n.

Since $\pi_0 K\mathbb{Z}_2^\wedge \cong \pi_0 bo \cong \mathbb{Z}_2^\wedge$ is uncountable, we need to extend their results slightly.

So saying, let $s^\wedge$ be the category of two complete spectra; see Bousfield [3]. The $\chi$-dual category of $s^\wedge$ is a bit more difficult to describe. Let $M(2^n)$ be the $\mathbb{Z}/2^n\mathbb{Z}$ Moore Spectrum, that is $H(M(2^n); \mathbb{Z}) = \mathbb{Z}/2^n\mathbb{Z}$ concentrated in grading zero. Then we have canonical maps $M(2^n) \longrightarrow S^1$ and $M(2^n) \longrightarrow M(2^{n+1})$; the latter induces $\mathbb{Z}/2^n\mathbb{Z} \longrightarrow \mathbb{Z}/2^{n+1}\mathbb{Z}$ in homology. Then we have a commutative diagram

$$\begin{array}{ccc} M(2^n) & \longrightarrow & M(2^{n+1}) \\ \downarrow & & \downarrow \\ S^1 & = & S^1 \end{array}$$

Thus, for any spectrum Y we have a map of degree one

$$\varinjlim Y \wedge M(2^n) \longrightarrow Y$$

We define a category $s^\infty$ by saying that Y is an object in $s^\infty$ if and only if this map is a homotopy equivalence.

Some examples: if for each n $\pi_n Y$ is a finite group consisting only of two torsion, then Y is an object in both $s^\wedge$ and $s^\infty$. $K\mathbb{Z}_2^\wedge$ and bo are objects in $s^\wedge$, and $K\mathbb{Z}_{2^\infty} = \varinjlim K\mathbb{Z}/2^n\mathbb{Z}$ is an object in $s^\infty$. $\mathbb{Z}_{2^\infty}$ is the Prouffer group and may be identified with the two torsion in $\mathbb{R}/\mathbb{Z}$, the real numbers modulo the integers.

If G is an abelian group, Let $G^t$ denote the character group:

$$G^t = \mathrm{Hom}(G; \mathbb{R}/\mathbb{Z}).$$

$(\mathbb{Z}_2^\wedge)^t = \mathbb{Z}_{2^\infty}$.

In Section 3 we prove the following result.

Theorem 1.13: There exist contravariant functors $\chi : s^{\wedge} \longrightarrow s^{\infty}$ and $\chi : s^{\infty} \longrightarrow s^{\wedge}$ satisfying:

1) $\chi(K\mathbb{Z}_2) \simeq K\mathbb{Z}_2$ and $\chi : [K\mathbb{Z}_2, K\mathbb{Z}_2] \longrightarrow [K\mathbb{Z}_2, K\mathbb{Z}_2]$ is the canonical anti-automorphism of the Steenrod Algebra.

2) For a spectrum Y, let $Y^q$ be the cohomology theory based on Y. Then, for each $Y \in s^{\wedge}$ (or $s^{\infty}$) and any finite CW complex Z there is a natural isomorphism

$$S_y : (\chi(Y)_q\ Z)^t \longrightarrow Y^q Z.$$

$S_y$ is natural in Y.

3) There is a natural equivalence between $\chi\chi$ and the identity functor of $s^{\wedge}$ (or $s^{\infty}$) to itself.

4) $\chi$ induces a group isomorphism: $\chi : [X,Y]_q \longrightarrow [\chi(X),\chi(Y)]_q$. $[X,Y]_q$ is the maps of degree q.

5) $\pi_q\chi(Y) \cong (\pi_{-q}Y)^t$ and the isomorphism is natural in Y.

Note: 1.13.2 is the reason this theorem applies to the problem at hand.

We now restate the main theorem of Brown and Gitler's work. Let the following be the $\chi$-dual of the resolution 1.7, which realized 1.6:

$$\longrightarrow L_q \xrightarrow{d_q} L_{q-1} \longrightarrow \cdots \longrightarrow L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_o = K\mathbb{Z}_2$$

Each $L_q$ is a product of Eilenberg-MacLane spectra of type $K\mathbb{Z}_2$.

Theorem 1.14: There exist spectra $E_q$ and maps (of degree one) $e_q : L_q \longrightarrow E_{q-1}$ so that:

1) $E_o = L_o = K\mathbb{Z}_2$ and $e_1 = d_1$.

2) $E_{q-1} \xrightarrow{i_q} E_q \xrightarrow{p_q} L_q$ is the fibration induced from the contractible path fibration over $E_{q-1}$ by $e_q$.

3) $L_q \xrightarrow{e_q} E_{q-1} \xrightarrow{p_{q-1}} L_{q-1}$ is $d_q$.

4) $e_q : L_{q,2k} \longrightarrow E_{q,2k+1}$ is zero.

This is in [6]. We should remark that 3) and 4) together imply that $d_q : L_{q,2k} \longrightarrow L_{q-1,2k+1}$ is zero. This can be proved directly and is a key point in the proof of 1.14.

Theorem 1.14 gives a diagram

$$\begin{array}{ccccccccccc} E_o & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_q & \longrightarrow & E_{q+1} \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ & & L_1 & & L_2 & & & & L_q & & \end{array}$$

Apply the functor $\chi$ to obtain a diagram

$$\begin{array}{ccccccccccc} Y_o & \longleftarrow & Y_1 & \longleftarrow & Y_2 & \longleftarrow & \cdots\cdots & \longleftarrow & Y_q & \longleftarrow & Y_{q+1} \\ & & \uparrow & & \uparrow & & & & & & \uparrow \\ & & I_1 & & I_2 & & & & & & I_{q+1} \end{array} \qquad (1.15)$$

Assume each $p_q$ is a fibration. Then $B(k) = \varprojlim Y_q$ .

We then wish to show that, for any CW complex Z

$$B(k)_n\, Z \longrightarrow H_n Z$$

is surjective for $n \leq 2k+1$. First suppose Z is a finite complex. Then, since $Y_o = K\mathbb{Z}_2$, it is sufficient to show that $(Y_q)_n Z \longrightarrow (Y_{q-1})_n Z$ is surjective for all q and $n \leq 2k+1$. By 1.13.2 it is equivalent to show that

$$E^n_{q-1}Z \longrightarrow E^n_q Z$$

is injective. But this follows from 1.14.2 and 1.14.4. To pass to general CW complexes, we take a direct limit over finite subcomplexes.

Now, we wish to apply 1.13 to prove our theorems. In what follows, let $E_q$ and $L_q$ be those spaces used to construct B(2k+1) (if we are working to $B_1(2k+1)$) or B(4k+3) (for $B_2(4k+3)$).

Apply $\chi$ to 1.12 to obtain a new diagram:

$$\begin{array}{ccccccccc}
\longrightarrow & L_q & \xrightarrow{d_q} & L_{q-1} & \longrightarrow & \cdots & \xrightarrow{d_1} & L_o \\
 & \downarrow{u_q} & & \downarrow & & & & \downarrow{u_o} \\
\longrightarrow & K_q & \xrightarrow{d_q} & K_{q-1} & \longrightarrow & \cdots & \xrightarrow{d_1} & K_o
\end{array}$$

In Section 4 we will prove the following result.

<u>Theorem 1.16</u>: There exist spectra $G_q$ and maps $e'_q : K_q \longrightarrow G_{q-1}$ of degree one and $\ell_q : E_q \longrightarrow G_q$ of degree zero so that

1) $G_o = K_o$, $e'_1 = d_1$ and $\ell_o = u_o$ .

2) $G_{q-1} \xrightarrow{i_q} G_q \xrightarrow{p_q} K_q$ is the fibration induced by $e'_q$ from the contractible path fibration over $G_{q-1}$.

3) The following diagram commutes:

$$\begin{array}{ccccccc}
E_{q-1} & \xrightarrow{i_q} & E_q & \xrightarrow{p_q} & L_q & \xrightarrow{e_q} & E_{q-1} \\
\downarrow{\ell_{q-1}} & & \downarrow{\ell_q} & & \downarrow{u_q} & & \downarrow{\ell_{q-1}} \\
G_{q-1} & \xrightarrow{i_q} & G_q & \xrightarrow{p_q} & K_q & \xrightarrow{e'_q} & G_{q-1}
\end{array}$$

4) $K_q \xrightarrow{e_q} G_{q-1} \xrightarrow{p_{q-1}} K_{q-1}$ is $d_q$.

5) If we are working toward $B_1(2k+1)$

$$e'_q : K_{q,4k+2} \longrightarrow G_{q-1,4k+3} \text{ is null-homotopic for } q \geq 1.$$

If we are working toward $B_2(4k+3)$

$$e'_q : K_{q,8k+6} \longrightarrow G_{q-1,8k+7} \text{ is null-homotopic for } q \geq 2.$$

Just as before, we can now construct $B_1(2k+1)$ or $B_2(4k+3)$. Let $X_q = \chi(G_q)$. Then there is a tower

$$\begin{array}{ccccccc} \longrightarrow X_q & \longrightarrow & X_{q-1} & \longrightarrow & \cdots & \longrightarrow & X_o \\ & & \varepsilon_q \big\downarrow & & & & \big\downarrow \\ & & F_q & & & & F_1 \end{array} \qquad (1.17)$$

We assume each $p_q$ is a fibration and set $B_1(2k+1)$ or $B_2(4k+3)$ equal to $\varprojlim X_q$. Since $X_o = K\hat{\mathbb{Z}}_2$ or bo, as desired, we have canonical maps

$$B_1(2k+1) \longrightarrow K\hat{\mathbb{Z}}_2 \quad \text{and} \quad B_2(4k+3) \longrightarrow bo .$$

Notice, too that 1.16.3 implies that there is a map from 1.17 to 1.15 for $B(2k+1)$ (or $B(4k+3)$) and, thus, we have canonical maps

$$B_1(2k+1) \longrightarrow B(2k+1) \quad \text{and} \quad B_2(4k+3) \longrightarrow B(4k+3) .$$

To compute $H^*B_1(2k+1)$ or $H^*B_2(4k+3)$, we first let

$$i_q : F_q \longrightarrow X_q$$

be the exclusion of the fiber. Then 1.16.4 implies that $\varepsilon_q i_{q-1} = \delta_q$ of 1.9 or 1.11. Then standard techniques (see Proposition 5.1) say that

$$H^*B_1(2k+1) = M_1(2k+1) \quad \text{and} \quad H^*B_2(4k+3) = M_2(4k+3) .$$

Finally, to discuss the homology theories $B_1(2k+1)_*$ and $B_2(4k+3)_*$, we argue exactly as we did between 1.14 and 1.16.

We should note that for $n \leq 4k+1$ (say we are discussing $B_1(2k+1)$)

$$p_{q_*} : (X_q)_n Z \longrightarrow (X_{q-1})_n Z$$

is not just surjective, but split surjective and the splitting is natural in Z This is a consequence of 1.16.5. A detailed discussion is given in Section 5.

Section 2 discusses resolutions, Section 3 discusses Pontrjagin duality and some lemmas. Sections 4 and 5 are devoted to the proof of 1.16 and Theorems I and III of the Introduction. In Section 6, we compute $\pi_* B_1(2k+1)$ and $\pi_* B_2(4k+3)$ in a range.

In Section 7, we discuss changes need for odd primes.

Naturally, the notation Lemma 2.14.3 refers to statement 3 of Lemma 14 found in Section 2.

## 2. Resolution of $M_1(2k+1)$ and $M_2(4k+3)$

In this section we provide the proofs of Lemmas 1.8 and 1.10. We could mimic the techniques of Brown and Gitler, but another technique actually seems to be more illuminating. This is true for two reasons. First, it makes explicit the relationship between a certain weight function (2.2) on the dual Steenrod Algebra and the homology of Brown-Gitler spectra. This relationship has been noted empirically before -

see [14]. Secondly, these new techniques give an elegant description of the $\Lambda$ algebra, a description that seems worth recording.

For some pages, then, we will work with comodules over the dual Steenrod Algebra rather than with modules over the Steenrod Algebra. Recall the dual Steenrod Algebra is a polynomial algebra

$$A^* = \mathbb{Z}_2[\zeta_1, \zeta_2, \ldots]$$

where $\zeta_i$ has grading $2^i-1$. The $\zeta_i$ are the Hopf algebra conjugate of Milnor's $\xi_i$ [24]. There is a coproduct map $\psi : A^* \longrightarrow A^* \otimes A^*$ which is a map of algebras, defined on generators as

$$\psi(\zeta_i) = \sum \zeta_j \otimes \zeta_{i-j}^{2^j} \qquad (2.1)$$

Let M be a left module over the Steenrod Algebra and $M^* = \mathrm{Hom}(M; \mathbb{Z}_2)$ be its dual. The left action of A on M can be transformed into a right action via $\chi$: for $m \in M$ and $a \in A$, set $ma = \chi(a)m$. This gives a map

$$M \otimes A \longrightarrow M$$

The $\mathbb{Z}_2$ dual of this map gives the (conjugate) comodule map on $M^*$:

$$\psi_M : M^* \text{ —— } M^* \otimes A^*$$

$\psi : A^* \longrightarrow A^* \otimes A^*$ is such a map.

$A^*$ has a weight function w defined on it as follows. Set $w(\zeta_i) = 2^{i-1}$ and $w(xy) = w(x) + w(y)$.

Now $H_*K\mathbb{Z}_2^\wedge = \mathbb{Z}_2[\zeta_1^2, \zeta_2, \ldots] \subseteq A^*$ as comodules; thus $H_*K\mathbb{Z}_2^\wedge$ has, as a $\mathbb{Z}_2$-basis, all monomials of $A^*$ of even weight. $H_*bo = \mathbb{Z}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \ldots] \subseteq A^*$ as comodules; therefore, $H_*bo$ has, as a $\mathbb{Z}_2$-basis, all monomials $x \in A$ so that $w(x) \equiv 0 \bmod 4$.

Let $M(k) = A/A\{\chi(Sq^i), i > k\}$, $M_1(2k+1) = A/A\{Sq^1)\ \ i > 2k\}$, and $M_2(4k+3) = A/A\{Sq^1, sq^2, \chi(Sq^i)\ i > 4k\}$. We characterize the duals of these modules.

<u>Lemma 2.2</u>: 1) $M(k)^* \subseteq A^*$ as comodules and has a $\mathbb{Z}_2$-basis consisting of all monomials of weight $\leq k$.

2) $M_1(2k+1)^* \subseteq H_*K\mathbb{Z}_2^\wedge$ as comodules and has a $\mathbb{Z}_2$ basis consisting of all monomials of weight $\leq 2k$ in $H_*K\mathbb{Z}_2^\wedge$.

3) $M_2(4k+3)^* \subseteq H_*bo$ as comodules and has a $\mathbb{Z}_2$ basis consisting of all monomials of weight $\leq 4k$ in $H_*bo$.

<u>Proof</u>: I will do 1). The technique is suggestive of things to come. Let $\Lambda(1,k)$ be monomials of length one in $\Lambda/\Lambda_k$. See Section 1. Define a vector space map $\phi_1 : A^* \longrightarrow \Lambda(1,k)$ by $\phi_1(\zeta_1^j) = \lambda_{j-1}$ $j \geq k+1$ and $\phi_1$ of all other monomials is zero. Then define a map $\partial_1 : A^* \longrightarrow A^* \otimes \Lambda(1,k)$ by the composition

$$\partial_1 : A^* \xrightarrow{\psi} A^* \otimes A^* \xrightarrow{id \otimes \phi_1} A^* \otimes \Lambda(1,k).$$

Using 2.1, we compute that $\ker(\partial_1)$ = all monomials of weight $\leq k$. To finish, we show that $A \otimes \Lambda^*(1,k) \longrightarrow A \xrightarrow{\partial_1} M(k) \longrightarrow 0$ is exact. To this end, let $Sq_i \in A$ be the dual of $\xi_1^i \in A^*$ where $\xi_1$ is the Milnor generator. $\chi(\zeta_1^i) = \xi_1^i$. We compute that

$$\partial_1^* \lambda_{j-1} = \chi(Sq_j) .$$

But it is known [25] that $Sq_j = Sq^j$.

$\partial_1$ is the idea that makes all that follows work; perhaps the reader should take some care over the proof of this lemma.

Let us return to the $\Lambda$ algebra of Section 1. The relations given there (1.4) can be restated as follows: See [4]

$$\sum \binom{n}{k} \lambda_{m+n-k-1} \lambda_{2m+k-1} = 0 \quad (m \geq 1, n \geq 0) \tag{2.3}$$

$\Lambda$ can be given the structure of a differential (bi-) graded algebra by requiring

$$\partial \lambda_{n-1} = \sum \binom{n}{k} \lambda_{n-k-1} \lambda_{k-1} \tag{2.4}$$

Recall that if $I = (i_1, \ldots, i_q)$ is a q-tuple of non-negative integers, we write $\lambda_{i_1} \ldots \lambda_{i_q} = \lambda_I$. The symbols $I = (i_1, \ldots, i_q)$ of length q can be ordered lexigraphically from the right; that is, $(1,1,2) > (1,1,1)$ and $(0,2,1) > (1,1,1)$, for example. The ordering on the I's gives an ordering on the $\lambda_I$'s: $\lambda_I > \lambda_J$ if and only if $I > J$.

Recall that $\lambda_I = \lambda_{i}, \ldots, \lambda_{i_q}$ is admissible if $2i_j \geq i_{j+1}$ for every j.

<u>Lemma</u> <u>2.5</u>: 1) For any $\lambda_I$ not admissible, $\lambda_I = \sum \lambda_J$ with J admissible and $\lambda_J < \lambda_I$.

2) If $I = (i_1, \ldots, i_q)$ then $\partial \lambda_I = \sum \lambda_J$ with $J = (j_1, \ldots, j_{q+1})$, J admissible and $j_{q+1} < i_q$.

<u>Proof</u>: See [6]. This is an easy application of the relation (1.4)

and the differential $\partial$ in $\Lambda$ which reads, in admissible form

$$\partial\lambda_n = \sum \binom{n-j}{j} \lambda_{n-j}\lambda_{j-1}$$

We now turn to the resolutions of Section 1. In particular, we now define the $\mathbb{Z}_2$-duals of the map $\delta_q^*$.

Denote by $\Lambda(q) = \Lambda(q,o)$ all monomials of length q in $\Lambda$. In 2.2 we defined $\phi_1$ and $\partial_1$. Let us make some further definitions. Let $\phi_2 : A^* \otimes A^* \longrightarrow \Lambda(2)$ be defined by the requirements that

$$\phi_2(\zeta_1^i \otimes \zeta_1^j) = \lambda_{i-1}\lambda_{j-1}$$

and $\phi_2$ applied to all other monomials is zero.

Define $\partial_q : A^* \otimes \Lambda(q-1) \longrightarrow A^* \otimes \Lambda(q)$ by the formula

$$\partial_q(a \otimes \lambda_I) = \partial_1(a)\ \lambda_I + a \otimes \partial\lambda_I\ .$$

<u>Lemma</u> <u>2.6</u>: 1) Give $A^* \otimes \Lambda(q)$ the comodule structure $A^* \otimes \Lambda(q) \xrightarrow{\psi\otimes id} A^* \otimes A^* \otimes \Lambda(q)$. Then $\partial_q$, $q \geq 1$, is a comodule map.

2) $\partial_q \circ \partial_{q-1} = 0$.

Proof: 1) follows from the definitions. For 2), the following diagram implies that $\partial_2 \circ \partial_1 = 0$.

$$\begin{array}{ccccc} A^* & \xrightarrow{\psi} & A^* \otimes A^* & \xrightarrow{\Delta_1} & A^* \otimes A^* \otimes A^* \\ \| & & \downarrow{\scriptstyle id\otimes\phi_1} & & \downarrow{\scriptstyle id\otimes\phi_2} \\ A^* & \xrightarrow{\partial_1} & A^* \otimes \Lambda(1) & \xrightarrow{\partial_2} & A^* \otimes \Lambda(2) \end{array} \qquad (2.7)$$

$\Delta_1(a \otimes b) = \psi(a) \otimes b + a \otimes \psi(b)$. Then

$$\partial_q\partial_{q-1}(a \otimes \lambda_I) = \partial_q(\partial_1(a)\lambda_I + a \otimes \partial\lambda_I)$$

$$= \partial_2\partial_1(a)\lambda_I + \partial_1(a)\partial\lambda_I + \partial_1(a)\partial\lambda_I + a \otimes \partial^2\lambda_I = 0 .$$

Note: The relations (2.3) and differential (2.4) in $\Lambda$ are forced by the requirement that 2.7 commute. For instance:

$$\partial_2 \circ (\mathrm{id} \otimes \phi_1)(1 \otimes \zeta_1^n\zeta_2^n) = 0$$

and

$$(\mathrm{id} \otimes \phi_2) \circ \Lambda_1(1 \otimes \zeta_1^n\zeta_2^m) = \sum\binom{n}{k} \lambda_{n+m-k-1}\lambda_{2m+k-1}$$

Thus we could deduce (2.3).

Let $\Lambda_k \subseteq \Lambda$ be the left ideal generated $\lambda_0, \ldots, \lambda_{k-1}$.

Lemma 2.8: 1) A $\mathbb{Z}_2$-basis for $\Lambda_k$ is given by all admissible monomials in $\Lambda$, $\lambda_i, \ldots, \lambda_{i_q}$ so that $i_q < k$.

2) $\Lambda_k$ is closed under $\partial$.

Proof: In [6], but easily follows from 2.5.

From 2.8 it follows that the maps $\partial_q : A^* \otimes \Lambda(q-1) \longrightarrow A^* \otimes \Lambda(q)$ restrict to maps $\overline{\partial}_q : A^* \otimes \Lambda_k(q-1) \longrightarrow A^* \otimes \Lambda_k(q)$. $\Lambda_k(q)$ is monomials of length q in $\Lambda_k$. Thus we get new maps

$$\partial_q : A^* \otimes \Lambda(q-1,k) \longrightarrow A^* \otimes \Lambda(q,k) .$$

$\partial_q$ is the $\mathbb{Z}_2$-dual of $\delta_q^*$ in 1.6. For all k, $\partial_q\partial_{q-1} = 0$.

The main idea of this section is contained in the following result.

Theorem 2.9: $0 \longrightarrow M(k)^* \xrightarrow{i} A^* \xrightarrow{\partial_1} A^* \otimes \Lambda(1,k) \longrightarrow \ldots$

$$\ldots \longrightarrow A^* \otimes \Lambda(q-1,k) \xrightarrow{\partial_q} A^* \otimes \Lambda(q,k) \longrightarrow \ldots$$

is an acyclic resolution of $M(k)^*$ by comodules over the dual Steenrod Algebra. $i$ is the inclusion of 2.2.

Proof: We need only show that it is acyclic; that is 1) her$\partial_1$ = $M(k)^*$ and 2) the complex is exact at $A^* \otimes \Lambda(k,q)$. In $A^*$, if $J = (j_1, \ldots, j_r)$, let $\zeta^J = \zeta_1^{j_1} \ldots \zeta_r^{j_r}$.

1) This is the content of the proof of Lemma 2.2. Note that we have

$$\partial_1(\zeta^J) = \zeta_1^{j_2} \ldots \zeta_{r-1}^{j_r} \otimes \lambda_{w(J)-1} + \sum a_i \otimes \lambda_{i-1} \qquad a_i \in A^*$$

with $i < w(J)$. $w(J) = w(\zeta^J) = j_1 + 2j_2 + \ldots + 2^{r-1}j_r$.

(2) Let $z = \sum a_I \otimes \lambda_I$ be an element of $A^* \otimes \Lambda(q,k)$ and suppose $\partial_{q+1}z = 0$. Assume $\lambda_I$ is admissible for each I. We first write z in a form amenable to induction. Set

$$z = \sum \zeta^{J_s} \otimes \lambda_I + \sum b_L \otimes \lambda_L \qquad b \in A^*$$

where i) $L \leq I$ and ii) if $L = I$, then the monomial of highest weight in $b_L$ has weight less the $w(J_s)$ for all s. Then I is maximal and $w(J_1) = w(J_2) = \ldots = w(J_s)$ is maximal among the monomial coefficients of $\lambda_I$. Set $w(J_s) = w$.

If $J_s = (j_1, \ldots, j_r)$ then set $f(J_s) = (j_2, \ldots, j_r)$. We have

$$0 = \partial_{q+1}z = \sum \zeta^{f(J_s)} \otimes \lambda_{w-1}\lambda_I + \sum C_N \otimes \lambda_N \qquad C_N \in A^*$$

with $N < (w-1, I)$. This follows from 2.5. $(w-1, I)$ cannot be admissible or $\partial_n(z) \neq 0$. If $I = (i_1, \ldots, i_q)$ set $t = i_1 - 2w+1 \geq 0$ and, for each $s$, set $g(J_s) = (t, J_s)$. Then

$$\partial_q(\zeta^{q(J_s)} \otimes \lambda_{f(I)}) = \zeta^{J_s} \otimes \lambda_I + \sum d_L \otimes \lambda_L \qquad d_L \in A^*$$

and $L < I$. Setting $y = \sum \zeta^{g(J_s)} \otimes \lambda_{f(I)}$, we have

$$\partial_{q+1}(z + \partial_q y) = 0$$

and $z + \partial_q y = \sum a'_L \otimes \lambda_L$ with $L \leq I$ and if $L = I$ then the monomial of highest in $a'_L$ is less than $w$. Now one works by induction to show that $z$ is in the image of $\partial_q$. This completes the proof of 2.9.

The resolution of 2.9 is the $\mathbb{Z}_2$-dual of 1.6. We give an explicit formula for $\delta_q^*$ in Lemma 2.13.

We turn to the proofs of Lemmas 1.5, 1.8, and 1.10.

<u>Lemma</u> <u>2.10</u>: 1) Let $\hat{\Lambda} \subseteq \Lambda$ be the left ideal generated by $\lambda_{2i-1}$, $i \geq 1$. Then $\hat{\Lambda}$ is closed under $\partial$ and an additive basis for $\hat{\Lambda}$ is all admissible monomials $\lambda_I = \lambda_{i_1}, \ldots, \lambda_{i_q}$ with $i_q$ odd.

2) $\hat{\Lambda} \cap \Lambda_k$ is closed under $\partial$.

3) Let $\bar{\Lambda}$ be the sub $\mathbb{Z}_2$-vector space of $\Lambda$ generated by elements $\lambda_{i_1}, \ldots, \lambda_{i_q}$ with $i_{q-1}$ odd and $i_q \equiv -1 \bmod 4$. The monomial $\lambda_{2j-1}\lambda_{4i-1} \in \bar{\Lambda}$ whether or not it is admissible. If $\lambda_I \in \bar{\Lambda}$ and $\lambda_I$ is of length greater than one, then $\lambda_i\lambda_I \in \bar{\Lambda}$ for all $i$.

4) $\bar{\Lambda}$ and $\bar{\Lambda} \cap \Lambda_k$ are closed under $\partial$.

Proof: We work with the relations 1.4 and differential 2.4.

1) Let $\lambda_i \in \Lambda$. Then, if $\lambda_i\lambda_{2i-1}$ is not admissible

$$\lambda_i\lambda_{2j-1} = \sum \binom{s-1}{2s-2j+2i+1} \lambda_{i+s}\lambda_{2j-1-s}$$

When s is odd, s-1 is even and, thus, the coefficient of $\lambda_{i+s}\lambda_{2j-1-s}$ is zero. Similarly,

$$\partial\lambda_{2j-1} = \sum \binom{2j}{k} \lambda_{2j-k-1}\lambda_{k-1}$$

and $\binom{2j}{k} = 0$ when k is odd.

2) Follows from 1) and 2.8.2.

3) $\lambda_{2j-1}\lambda_{4i-1} = \sum \binom{s-1}{2s-4i+4j-1} \lambda_{2j-1+s}\lambda_{4i-1-s}$.

If $s \equiv 1,3 \bmod 4$, s-1 is even and the coefficient of $\lambda_{2j-1+s}\lambda_{4i-1-s}$ is zero. If $s \equiv 2 \bmod 4$, the coefficient of $\lambda_{2j-1+s}\lambda_{4i-1-s}$ is of the form $\binom{4t+1}{4r-1} = 0$. The second claim follows from this and 1).

4) $\partial\lambda_{4j=1} = \sum \binom{4j}{k} \lambda_{4j-k-1}\lambda_{k-1}$. $\binom{4j}{k} = 0$ unless $k \equiv 0 \bmod 4$. The result follows from 1), 3), and 2.8.2.

Recall that $\hat{\Lambda}(q, 2k+1)$ is the vector space of monomials of length q in $\hat{\Lambda}/\hat{\Lambda} \cap \Lambda_{2k+1}$. 2.10.1 and 2 allows us to construct comodule maps

$$\partial_q : A^* \otimes \Lambda\hat{}(q-1,2k+1) \longrightarrow A^* \otimes \Lambda\hat{}(q,2k+1)$$

so that the following diagram commutes.

$$\begin{array}{ccc} A^* \otimes \hat{\Lambda}(q-1,2k+1) & \xrightarrow{\partial_q} & A^* \otimes \hat{\Lambda}(q,2k+1) \\ \downarrow & & \downarrow \\ A^* \otimes \Lambda(q-1,2k+1) & \xrightarrow{\partial_q} & A^* \otimes \hat{\Lambda}(q,2k+1) \end{array}$$

The verticle maps are inclusions. The techniques of Theorem 2.9 allow us to prove the following result.

Theorem 2.11: $0 \longrightarrow M(2k+1)^* \xrightarrow{\partial_i} \mathbb{Z}_2[\zeta_1^2, \zeta_2, \ldots] \longrightarrow A^* \otimes \hat{\Lambda}(1,2k+1)$

$$\ldots \longrightarrow A^* \otimes \hat{\Lambda}(q-1, 2k+1) \xrightarrow{\partial_q} A^* \otimes \hat{\Lambda}(q,2k+1) \longrightarrow \ldots$$

is an acyclic resolution of $M(2k+1)^*$ by comodules over the dual Steenrod Algebra. i is the inclusion map.

This is the $\mathbb{Z}_2$ dual of 1.8.

Similarly, recall that $\overline{\Lambda}(q,4k+3)$ is the vector space of monomials of length q in $\overline{\Lambda}/\overline{\Lambda} \cap \Lambda_{4k+3}$. 2.10.3 and 4 allow us to construct comodule maps

$$\partial_q : A^* \otimes \overline{\Lambda}(q-1,k) \longrightarrow A^* \otimes \overline{\Lambda}(q,k) \qquad q \geq 3$$

$$\partial_2 : \mathbb{Z}_2[\zeta_1^2, \zeta_2, \ldots] \otimes \Lambda(1,k) \longrightarrow A^* \otimes \overline{\Lambda}(2,k)$$

$$\partial_1 : \mathbb{Z}_2[\zeta_1^4, \zeta_1^2, \zeta_2 \ldots ] \longrightarrow \mathbb{Z}_2[\zeta_1^2, \zeta_2 ,,,] \otimes \overline{\Lambda}(1,k)$$

so that all the obvious diagrams commute; for instance, the following commutes.

$$\begin{array}{ccc} \mathbb{Z}_2[\zeta_1^2, \zeta_2 \ldots] \otimes \overline{\Lambda}(1,4k+3) & \xrightarrow{\partial_2} & A^* \otimes \overline{\Lambda}(2,4k+3) \\ \downarrow & & \downarrow \\ A^* \otimes \Lambda(1,4k+3) & \xrightarrow{\partial_2} & A^* \otimes \Lambda(2,4k+3) \end{array}$$

The verticle maps are inclusions. The techniques of 2.9 allow us to prove the following result.

Theorem 2.12: $0 \longrightarrow M_2(4k+3)^* \xrightarrow{i} \mathbb{Z}_2[\zeta_1^4, \zeta_2^2, \zeta_3 \ldots]$

$$\xrightarrow{\partial_1} \mathbb{Z}_2[\zeta_1^2, \zeta_2 \ldots] \otimes \overline{\Lambda}(1,4k+3) \xrightarrow{\partial_2} A^* \otimes \overline{\Lambda}(2,4k\pm3) \longrightarrow \ldots$$

$$\ldots \longrightarrow A^* \otimes \overline{\Lambda}(q-1,4k+3) \xrightarrow{\partial_q} A^* \otimes \overline{\Lambda}(q,4k+3) \longrightarrow \ldots$$

is an acyclic resolution of $M_2(4k+3)^*$ by comodules over the dual Steenrod Algebra.

This is the $\mathbb{Z}_2$-dual of 1.10.

Let us examine the $\mathbb{Z}_2$-duals of the maps $\partial_q$. These are A-module maps

$$\delta_q^* : A \otimes \Lambda^*(q,k) \longrightarrow A \otimes \Lambda^*(q-1, k)$$

$$\delta_q^* : A \otimes \hat{\Lambda}^*(q, 2k+1) \longrightarrow A \otimes \hat{\Lambda}^*(q-1, 2k+1)$$

and so on. Let $\overline{\lambda}_I$ the $\mathbb{Z}_2$- dual of $\lambda_I$.

Lemma 2.13: $\delta_q^* : A \otimes \Lambda^*(q,k) \longrightarrow A \otimes \Lambda^*(q-1,k)$ is the A-module map determined by

$$\delta_q^* \overline{\lambda}_I = \sum \overline{\lambda}_I(\lambda_i\lambda_J)\ \chi(Sq^{i+1})\ \overline{\lambda}_J .$$

The sum is over $i \geq -1$ where $\lambda_{-1}\lambda_J$ is interpreted as $\partial\lambda_J$.

Proof: $\partial_q(\zeta_1^{i_1}, \ldots, \zeta_n^{i_n} \otimes \lambda_J) = 1 \otimes \lambda_I + \sum a_L \otimes \lambda_L$ with $L \neq I$ if and only if $i_2 = i_3 = \ldots = i_n = 0$. If $i_1 = 0$ also, then $\lambda_I$ is a summand of $\partial\lambda_J$. Thus, we have

$$\delta_q^*(\lambda_I) = \sum_{i,J} \overline{\lambda}_I(\lambda_i\lambda_J)(\zeta_1^{i+1})^* \overline{\lambda}_J .$$

Here the sum is over $i \geq -1$ (with the convention above) and over J so that the grading of $\lambda_i\lambda_J$ equals the grading of $\lambda_I$. As in 2.2, we may identify $(\zeta_1^{i+1})^*$ as $\chi(Sq^{i+1})$.

2.13 follows.

This lemma determines all maps labelled $\delta_q^*$. For instance

$$\delta_2^* : A \otimes \overline{\Lambda}(2,k)^* \longrightarrow A/ASq^1 \otimes \overline{\Lambda}^*(1,k)$$

is given by

$$\delta_2^*(\overline{\lambda}_I) = \sum_{i,j} \overline{\lambda}_I(\lambda_{2i-1}\lambda_{4j-1})\ \chi(Sq^{2i})\ \overline{\lambda}_{4j-1}$$

and

$$\delta_1^* : A/ASq^1 \otimes \overline{\Lambda}(1,k) \longrightarrow A/A\{Sq^1,\ Sq^2\}$$

is given by

$$\delta_1^*(\lambda_{4i-1}) = \chi(Sq^{4i}).$$

It is instructive to compute that this is well-defined:

$$\begin{aligned} Sq^1\chi(Sq^{4i}) &= \chi(Sq^{4i}Sq^1) = \chi(Sq^2Sq^{4i-1}) + \chi(Sq^1Sq^{4i}) \\ &= \chi(Sq^{4i-1})\ Sq^2 + \chi(Sq^{4i})\ Sq^1 = 0 \quad \text{in } A/A\{Sq^1,\ Sq^2\}. \end{aligned}$$

## 3. Pontrjagin Duality

In this section, we extend the Pontrjagin Duality functor of Brown and Comenetz [5] so that we may apply it to the spectra that we are considering. After this theoretical discussion, we make our first application: we apply Pontrjagin duality to sequences of spectra realizing

2.9, 2.11 and 2.12. Then we discuss the unstable properties of these dualized sequences. The primary results are 3.11 and 3.12.

Let s be the full subcategory of the homotopy category of CW spectra so that Y is an object in s if and only if $\pi_n Y$ is finite for each integer n.

If G is an abelian group let $G^t$ be its character group; that is, its Pontrjagin dual. $G^t = \mathrm{Hom}(G; \mathbb{R}/\mathbb{Z})$ where $\mathbb{R}/\mathbb{Z}$ is the real numbers modulo the integers.

In [5] Brown and Comenetz defined a functor $\chi$ whose properties we delineate here. If Y is a spectrum let $Y_*$ and $Y^*$ denote the (reduced) homology and cohomology theories based on Y.

Theorem 3.1: There is a contravariant functor $\chi : s \longrightarrow s$ satisfying

1) $\chi(K\mathbb{Z}_p) \cong K\mathbb{Z}_p$ for all primes p.

2) For each $Y \in s$ there is a natural equivalence

$$S_Y : \chi(Y_q)^t \longrightarrow Y^q$$

$S_Y$ is natural in Y.

3) There is a natural equivalence between $\chi\,\chi$ and the identity functor.

4) $\chi : [X,Y]_q \longrightarrow [\chi(Y), \chi(X)]$ is a group isomorphism.

5) $\chi : [K\mathbb{Z}_2, K\mathbb{Z}_2] \longrightarrow [K\mathbb{Z}_2, K\mathbb{Z}_2]$ is the canonical anti-automorphism of the Steenrod Algebra.

6) $\pi_q\chi(Y) = (\pi_{-q}Y)^t$

We record the following result, also from [5].

<u>Lemma 3.2</u>: If $F \longrightarrow E \longrightarrow B$ is a fibration, then $\chi(B) \longrightarrow \chi(E) \longrightarrow \chi(F)$ is a fibration.

The restriction of $\chi$ to s arises in this manner: Hom( ; $\mathbb{R}/\mathbb{Z}$) is a functor which maps compact abelian groups to discrete groups and visa versa. This is done by keeping track of the topology on $\mathbb{R}/\mathbb{Z}$. Given an arbitrary spectrum E, there is no canonical choice for a topology on E*. If $\pi_q Y$ is finite for each q, however, there is such a choice of a topology. Let X be an arbitrary CW complex and let $\{X_\alpha\}$ be the directed set of finite subcomplexes of X. Then $Y^q(X_\alpha)$ is finite for each $\alpha$ and, thus, compact in the discrete topology. Then, because $\varprojlim$ is exact on compact groups

$$Y^q X \cong \varprojlim Y^q X_\alpha \ .$$

The requirement that this be a homeomorphism gives $Y^q X$ a unique topology. In addition if we give $\chi(Y)_q X$ the discrete topology, then the equivalence of 3.1.2 is a homeomorphism.

Clearly, this would not work, say, for $E = K\mathbb{Z}$. It can be extended to spectra such as $K\mathbb{Z}_2^\wedge$ and bo; however. For instance, since $K\mathbb{Z}_2^\wedge = \varprojlim K\mathbb{Z}_{2^n}$ and since $H^*(X; \mathbb{Z}_{2^n})$ is compact

$$H^*(X; \mathbb{Z}_2^\wedge) = \varprojlim H^*(X; \mathbb{Z}_{2^n})$$

and, by the above argument, this gives $H^*(X; \mathbb{Z}_2^\wedge)$ a unique compact topology. Note that $\pi_q K\mathbb{Z}_2$ is compact for each q.

Let us extend $\chi$. $s^\wedge$ will be the full subcategory of the homotopy category of CW spectra so that Y is in $s^\wedge$ if and only if

1) $\pi_q Y$ is a finitely generated $\mathbb{Z}_2^{\wedge}$ module for each q

2) Y is 2-complete.

By the last, I mean this: Let $M(2^n)$ is the $\mathbb{Z}/2^n\mathbb{Z}$ Moore space and define inclusions $Y = Y \wedge S^0 \xrightarrow{id \wedge \phi} Y \wedge M(2^n)$ where $\phi$ is the inclusion of the bottom cell. 2) is the requirement that the induced map $Y \longrightarrow \varprojlim Y \wedge M(2^n)$ be a homotopy equivalence. For Y an object in $s^{\wedge}$, $\pi_q\ Y \wedge M(2^n)$ is finite for each q; therefore $(Y \wedge M(2^n))^*X$ acquires a unique compact topology and

$$Y^*X \cong \varprojlim (Y \wedge M(2^n))^*X .$$

We require that this equivalence be a homeomorphism.

$K\mathbb{Z}_2^{\wedge}$ and bo are objects in $s^{\wedge}$.

$s^{\wedge}$ is a category of compact cohomology theories. We define a dual category of discrete cohomology theories. Let $\mathbb{Z}_{2^\infty} = \varinjlim \mathbb{Z}/2^n\mathbb{Z}$. Then $(\mathbb{Z}_2^{\wedge})^t = \mathbb{Z}_{2^\infty}$. $\mathbb{Z}_{2^\infty}$ can be thought of as 2-torsion in $\mathbb{R}/\mathbb{Z}$. Let $s^\infty$ be the full subcategory of the homotopy category of CW spectra so that Y is an object in $s^\infty$ if and only if

1) $\pi_q Y$ is a finite direct sum of quotients of $\mathbb{Z}_{2^\infty}$

2) Y is 2-pro-complete.

By 2) we mean this: define maps of degree 1, $Y \wedge M(2^n) \xrightarrow{id \wedge \psi} Y \wedge S^0 = Y$ where $\psi$ is projection onto the top cell. By 2-pro-complete we mean that the induced map

$$\varinjlim Y \wedge M(2^n) \longrightarrow Y$$

is an equivalence. For each X we give $(Y \wedge M(2^n))^*X$ the discrete topology.

Thus Y*X has the discrete topology.

$K\mathbb{Z}_{2^\infty} = \varinjlim K\mathbb{Z}_{2^n}$ is an object in $s^\infty$ and should be the $\chi$ dual of $K\mathbb{Z}_2^\wedge$.

Definition 3.3.1) For Y in $s^\wedge$ define $\chi(Y) = \varinjlim \chi(Y_\wedge M(2^n))$. We assume the induced maps $\chi(Y_\wedge M(2^n)) \longrightarrow \chi(Y_\wedge M(2^{n+1}))$ are inclusions. We make $\chi$ a functor as follows. If X, Y are in $s^\wedge$ and $f : X \longrightarrow Y$ then there are commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi\downarrow & & \downarrow\phi \\ X_\wedge M(2^n) & \xrightarrow{f_\wedge id} & Y_\wedge M(2^n) \end{array} \tag{3.4}$$

and 3.4 for n projects commutatively to 3.4 for n-1. Let $f_n = f_\wedge id$ : $X_\wedge M(2^n) \longrightarrow Y_\wedge M(2^n)$. Then $\varprojlim f_n = f$, by our assumptions. Set $\chi(f) = \varinjlim \chi(f_n)$.

3.3.2) For Y in $s^\infty$ define $\chi(Y) = \varprojlim \chi(Y_\wedge M(2^n))$. We assume the induced maps $\chi(Y_\wedge M(2^{n+1})) \longrightarrow \chi(Y_\wedge M(2^n))$ are fibrations. $\chi$ may be extended, as in 1), to a functor.

Let us remark immediately that it follows from 3.1.6 and the properties of $(\ \ )^t$ that

$$\pi_q\chi(Y) = (\pi_{-q}Y)^t$$

in both cases. For instance

$$\pi_q\chi(bo) = \begin{cases} \mathbb{Z}_{2^\infty} & q \le o,\ q \equiv 0,4 \bmod 8 \\ \mathbb{Z}_2 & q < o,\ q \equiv 6,7 \bmod 8 \\ 0 & \text{otherwise} \end{cases}$$

The following result implies that $\chi$ carries objects of $s^\wedge$ to object of $s^\infty$ and vice versa.

Lemma 3.5: 1) For Y in $s^\wedge$, there are degree one equivalences and a commutative diagram

$$\begin{array}{ccc} \chi(Y)\wedge M(2^n) & \xrightarrow{\cong} & \chi(Y\wedge M(2^n)) \\ \downarrow & & \downarrow \\ \chi(Y)\wedge M(2^{n+1}) & \xrightarrow{\cong} & \chi(Y\wedge M(2^{n+1})) \end{array} \qquad (3.6)$$

These equivalences are natural in Y.

2) For Y in $s^\infty$ there are equivalences and a commutative diagram

$$\begin{array}{ccc} \chi(Y)\wedge M(2^{n+1}) & \xrightarrow{\cong} & \chi(Y\wedge M(2^{n+1})) \\ \downarrow & & \downarrow \\ \chi(Y)\wedge M(2^n) & \xrightarrow{\cong} & \chi(Y\ M(2^n)) \end{array}$$

These equivalences are natural in Y.

Proof: I will do 1). First suppose that $\pi_q Y$ is a finite for each q. Then, notice that the two definitions of $\chi(Y)$ (3.1 and 3.3.1) coincide. Since $[Y,Y] \xrightarrow{\cong} [\chi(Y), \chi(Y)]$ is a group map and $\chi(id) = id$, $\chi(2^n \cdot id) = 2^n \cdot id$. Thus we have a degree one equivalence, natural in Y.

$$\chi(Y)\wedge M(2^n) \longrightarrow \chi(Y\wedge M(2^n))$$

and the appropriate diagram commutes (3.6).

Therefore, we have a diagram, for each q

$$\begin{array}{ccc} \chi(Y_\wedge M(2^q)_\wedge M(2^n)) & \longrightarrow & \chi(Y_\wedge M(2^{q+1})_\wedge M(2^n)) \\ \uparrow & & \uparrow \\ \chi(Y_\wedge M(2^q))_\wedge M(2^n) & \longrightarrow & \chi(Y_\wedge M(2^{q+1}))_\wedge M(2^n) \end{array} \qquad (3.7)$$

assuming all horizontal maps are inclusions, we can conclude the existence of an equivalence $\chi(Y_\wedge M(2^n)) \longleftarrow \chi(Y)_\wedge M(2^n)$. One need now check that 3.6 commutes. But 3.7 for n maps commutatively to 3.7 for n+1. To see that the induced equivalences are natural in Y, notice that 3.7 is natural in Y.

The following is implied immediately by the definitions.

Lemma 3.8. 1) There is a natural equivalence between $\chi\cdot\chi$ and the identity functor of $s^\wedge$ (or $s^\infty$) and itself.

2) There is a group isomorphism $[X,Y] \xrightarrow{\cong} [\chi(Y), \chi(X)]$.

Proof: 2) follows from 1) and the fact that $\chi$ is additive; that is if f and g : $X \longrightarrow Y$ are maps, then $\chi(f+g) = \chi(f) + \chi(g)$.

Finally, we wish to prove the following result.

Lemma 3.9. 1) For $Y \in s^\wedge$ and for any CW complex Z there is an equivalence, natural in Y and Z

$$S_Y : \chi(Y)^t_q Z \xrightarrow{\cong} Y^q Z .$$

2) For $Y \in s^\infty$ and any finite CW complex Z there is an equivalence, natural in Y and Z

$$S_Y : \chi(Y)^t_q Z \xrightarrow{\cong} Y^q Z .$$

Proof: 1) $S_Y$ is the isomorphism

$$\chi(Y)_q^t Z = (\varinjlim \chi(Y\wedge M(2^n))_q Z)^t \cong \varprojlim \chi(Y\wedge M(2^n))_q^t Z$$
$$\xrightarrow{\cong} \varprojlim (Y\wedge M(2^n))^q Z \cong Y^q Z .$$

The second isomorphism exists because $(\ )^t$ carries direct limits of discrete groups to inverse limits of compact groups.

2) Because Z is a finite CW complex, $\chi(Y\wedge M(2^n)_q Z$ is finite; therefore, we have an isomorphism

$$\chi(Y)_q Z \cong \varprojlim \chi(Y\wedge M(2^n))_q Z$$

$\chi(Y)_q Z$ has a unique compact topology determined by this equivalence. Then $S_y$ is the isomorphism

$$\chi(Y)_q^t Z \cong (\varprojlim \chi(Y\wedge M(2^n))_q Z)^t = \varinjlim \chi(Y\wedge M(2^n))_q^t Z$$
$$\cong \varinjlim (Y\wedge M(2^n))^q Z = Y^q Z.$$

This completes the proof of Theorem 1.13 and of our abstract discussion of Pontrjagin duality. We now make our first application.

Let the following realize the resolution given in 2.9 for M(2k+1):

$$\cdots \longleftarrow I_q \xleftarrow{\delta_q} I_{q-1} \longleftarrow \cdots \longleftarrow I_1 \xleftarrow{\delta_1} I_o = K\hat{\mathbb{Z}}_2$$

and let the following realize the resolution given in 2.11 for $M_1(2k+1)$:

$$\cdots \longleftarrow F_q \xleftarrow{\delta_q} F_{q-1} \longleftarrow \cdots \longleftarrow F_1 \xleftarrow{\delta_1} F_o = K\hat{\mathbb{Z}}_2$$

Assume that $\delta_q$, in both cases, is of degree one.

Note that we have maps $\mu_q : F_q \longrightarrow I_q$ so that we have a commutative diagram:

$$\begin{array}{ccc} F_q & \xleftarrow{\delta_q} & F_{q-1} \\ \mu_q \downarrow & & \downarrow \mu_{q-1} \\ I_q & \xleftarrow{\delta_q} & I_{q-1} \end{array}$$

Apply $\chi$ to obtain a diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & L_q & \xrightarrow{d_a} & L_{q-1} & \longrightarrow & \cdots & \longrightarrow & L_1 & \xrightarrow{d_1} & L_o = K\mathbb{Z}_2 \\ & & u_q \downarrow & & \downarrow u_{q-1} & & & & \downarrow & & u_o \downarrow \\ \cdots & \longrightarrow & K_q & \xrightarrow{d'_q} & K_{q-1} & \longrightarrow & \cdots & \longrightarrow & K_1 & \xrightarrow{d'_1} & K_o = K\mathbb{Z}_{2^\infty} . \end{array}$$

Now, we want to record that $d_q : L_{q,4k+2} \longrightarrow L_{q-1,4k+3}$ and $d'_q : K_{q,4k+2} \longrightarrow K_{q-1,4k+3}$ are null-homotopic, but we need something stronger. So we make a definition. For the moment, let us work in the topological category, instead of the homotopy category. Suppose (in the topological category) we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y' \\ \mu \downarrow & & \downarrow \mu' \\ Y & \xrightarrow{g} & y' \end{array}$$

Suppose further that we have null-homotopies F of f and G of g so that we have a diagram:

$$\begin{array}{ccc} X \times I & \xrightarrow{F} & X' \\ u \times id \downarrow & & \downarrow u' \\ Y \times I & \xrightarrow{F'} & Y' \end{array}$$

Then we say that f and g are concurrently null-homotopic. The use of such things will become clear in the proof of Theorems 4.8 and 4.9.

Lemma 3.11: For $q \geq 1$, $d_q$ and $d'_q$ are concurrently null-homotopic:

$$\begin{array}{ccc} L_{q,4k+2} & \xrightarrow{d_q} & L_{q,4k+3} \\ {\scriptstyle u_q}\downarrow & & \downarrow{\scriptstyle u_{q-1}} \\ K_{q,4k+2} & \xrightarrow{d'_q} & K_{q-1,4k+3} \end{array}$$

Because all the spaces concerned are Eilenberg-MacLane spaces, this is essentially the homology calculation done by Brown and Gitler [6]. Before filling in the proof, though, let us record a similar lemma for the bo case.

Let the following realize the resolution given in 2.9 for $M(4k+3)$

$$\cdots \longleftarrow I_q \xleftarrow{\delta_q} I_{q-1} \longleftarrow \cdots \longleftarrow I_1 \xleftarrow{\delta_1} I_o = K\mathbb{Z}_2$$

And let the following realize the resolution givenin 2.12 for $M_2(4k+3)$.

$$\cdots \longleftarrow F_q \xleftarrow{\delta_q} F_{q-1} \longleftarrow \cdots \longleftarrow F_1 \xleftarrow{\delta_1} F_o = bo\ .$$

Actually, let us be a little careful about the construction of $\delta_1 : bo \longrightarrow F_1$. This is a map

$$\underset{j>k}{X}\ i_{4j} : bo \longrightarrow X\ K\hat{\mathbb{Z}}_2$$

where $i_{4j}$ makes the following diagram commute.

$$\begin{array}{ccc} bo & \xrightarrow{i_{4j}} & K\mathbb{Z}_2^{\wedge} \\ \downarrow 1 & & \downarrow 1 \\ K\mathbb{Z}_2 & \xrightarrow{\chi(Sq^{4j})} & K\mathbb{Z}_2 \end{array} \qquad (1 = \text{generator of } H^*)$$

$i_{4j}$ is unique only up to homotopy equivalence of $K\mathbb{Z}_2^{\wedge}$ . We want to fix a particular $i_{4j}$. To that end, pick $i_4 : bo \longrightarrow K\mathbb{Z}_2^{\wedge}$. Then $i_{4j}$ will be the unique map that makes the following diagram commute.

$$\begin{array}{ccc} \Lambda_j bo & \xrightarrow{*} & bo \\ \Lambda_j i_4 \downarrow & & \downarrow i_{4j} \\ \Lambda_j K\mathbb{Z}_2^{\wedge} & \xrightarrow{*} & K\mathbb{Z}_2^{\wedge} \end{array}$$

Here the horizontal maps are multiplications of ring spectra. Then form $\delta_1 : bo \longrightarrow F_1$, now assumed to be of degree one.

Now, we have maps $\mu_q : F_q \longrightarrow I_q$ so that we have commutative diagrams

$$\begin{array}{ccc} F_q & \xleftarrow{\delta_q} & F_{q-1} \\ \mu_q \downarrow & & \downarrow \mu_{q-1} \\ I_q & \xleftarrow{\delta_q} & I_{q-1} \end{array}$$

Apply $\chi$ to obtain

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & L_q & \xrightarrow{d_q} & L_{q-1} & \longrightarrow & \cdots & \longrightarrow & L_1 & \xrightarrow{d_1} & L_o = K\mathbb{Z}_2 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & K_q & \xrightarrow{d'_q} & K_{q-1} & \longrightarrow & \cdots & \longrightarrow & K_1 & \xrightarrow{d'_1} & K_o = \chi(bo) \end{array}$$

Then we have

Lemma 3.12: For $q \geq 2$, $d_q$ and $d'_q$ are concurrently null-homotopic

$$\begin{array}{ccc} L_{q,8k+6} & \xrightarrow{d_q} & L_{q-1,8k+7} \\ {\scriptstyle u_q}\downarrow & & \downarrow {\scriptstyle u_{q-1}} \\ K_{q,8k+6} & \xrightarrow{d'_q} & K_{q-1,8k+1} \end{array}$$

Note: $K_{1,n} \longrightarrow K_{o,n+1} = \chi(bo)_{n+1}$ is not zero unless $n < 4k+3$. And, if $n < 4k+3$, $K_{1,n}$ is contractible.

I will prove 3.11. 3.12 is exactly the same.

Proof of Lemma 3.11: To begin, we note three facts. First, for all q, $K_q$ and $L_q$ are products of Eilenberg-MacLane Spectra; second, for $q \geq 1$, $u_q : L_q \longrightarrow K_q$ has a section $s : K_q \longrightarrow L_q$ so that $\mu_q \circ s \cong id$; and third, $d_q : L_{q,4k+2} \longrightarrow L_{q-1,4k+3}$ is null-homotopic (see Theorem 1.14).

These facts imply that $d'_q : K_{q,4k+2} \longrightarrow K_{q-1,4k+3}$ is null-homotopic. We wish to show that there exists a null-homotopy of $d_q$

$$D : L_{q,4k+2} \times I \longrightarrow L_{q-1,4k+3}$$

with the following property: if we define D' to be the composition

$$D' : K_{q,4k+2} \times I \xrightarrow{s \times id} L_{q,4k+2} \times I \xrightarrow{D'} L_{q-1,4k+3} \xrightarrow{u_{q-1}} K_{q-1,4k+2}$$

then

$$D' \circ (u_q \times id) = u_{q-1} \circ D.$$

It is clear that we may do this. For the author's sake, if for no other reason, let us belabor the point.

The following remark should clear up the issue. If $g : K\mathbb{Z}_2 \longrightarrow L_q$ is a map (of any degree) so that $u_q \circ g$ is null-homotopic, we may assume the following composition is the one-point map ($\mu_q \circ g(x) =$ basepoint for each x)

$$K(\mathbb{Z}_2;\ n) \xrightarrow{g} L_{q,4k+3} \xrightarrow{u_q} K_{q,4k+3}$$

If, in addition, $g : K(\mathbb{Z}_2;\ n) \longrightarrow L_{q,4k+3}$ is null-homotopic, we wish to find a null-homotopy G so that $u_q \circ G$ is also the one point map. Let $J_q$ be the fiber of $u_q : L_q \longrightarrow K_q$. Then there is a lifting

$$\begin{array}{ccc} & \overset{g}{\nearrow} & J_q \\ & & \big\downarrow \\ K\mathbb{Z}_2 & \xrightarrow{g} & L_q \end{array}$$

Now one checks that there exists such a lifting so that $g : K(\mathbb{Z}_2\ ;\ n) \longrightarrow L_{q,4k+2}$ being null-homotopic implies that $\hat{g} : K(\mathbb{Z}_2\ ;\ n) \longrightarrow J_{q,4k+3}$ is null-homotopic. The remark, and the lemma, follows.

<u>Note</u>: $J_{o,4k+3} = K(\mathbb{Z}_{2^\infty}\ ;\ 4k+2)$ and if

$$Sq^I = Sq^{i_1} \ldots Sq^{i_j} : K\mathbb{Z}_2 \longrightarrow K\mathbb{Z}_2$$

is a Steenrod Square with $i_1$ odd, then $u_o \circ Sq^I$ is null-homotopic and the lifting of $Sq^I$ to $K\mathbb{Z}_{2^\infty}$ is the composition

$$Sq^{I'} = Sq^{i_1 - 1} \ldots Sq^{i_j} : K\mathbb{Z}_2 \longrightarrow K\mathbb{Z}_2 \longrightarrow K\mathbb{Z}_{2^\infty}$$

Note that excess (I') = excess (I) -1.

## 4. A Sequence of Spectra

In this section we prove Theorem 1.16. As a preliminary we need some information on infinite loop spaces.

Let $C_\infty$ be May's "little $\infty$-cubes" operad. For a CW complex X let $C_\infty X = \coprod_{k \geq 0} C_\infty(K) \times_{\Sigma_K} X^K/\sim$ as described by May [21]. Let $\eta : X \longrightarrow C_\infty X$ be the natural inclusion and $\mu : C_\infty(C_\infty X) \longrightarrow C_\infty X$ be the map given by the operad action. For details see [21]. Recall that a $C_\infty$ space is a pair $(X,\xi)$ where $\xi : C_\infty X \longrightarrow X$ is a map such that the following diagrams commute.

$$\begin{array}{ccc} X & \xrightarrow{\eta} & C_\infty X \\ & {}_{id}\searrow & \downarrow \xi \\ & & X \end{array} \qquad\qquad \begin{array}{ccc} C_\infty C_\infty X & \xrightarrow{\mu} & C_\infty X \\ C_\infty \xi \downarrow & & \downarrow \xi \\ C_\infty X & \xrightarrow{\xi} & X \end{array}$$

By commute, we mean "commute in the topological category."

Let Y be an $\Omega$-spectrum. Each $Y_n$ is an infinite loop space and therefore has a canonical $C_\infty$ structure.

Let X be another $\Omega$-spectrum. A map $f : X_k \longrightarrow Y_n$ is a $C_\infty$ map if the diagram

$$\begin{array}{ccc} C_\infty X_k & \xrightarrow{\xi} & X_k \\ C_\infty f \downarrow & & \downarrow f \\ C_\infty Y_n & \xrightarrow{\xi} & Y_n \end{array} \qquad (4.1)$$

commutes. If $f' : X'_k \longrightarrow Y'_n$ is another $C_\infty$ map then the commutative diagram

$$\begin{array}{ccc} X_k & \xrightarrow{f} & Y_n \\ i \downarrow & & \downarrow j \\ X'_k & \xrightarrow{f'} & Y'_n \end{array} \qquad (4.2)$$

is a commutative diagram of $C_\infty$ maps if 4.1 for f maps to 4.1 for f'. This will happen if and only if the vertical maps i and j are $C_\infty$ maps. The following is our principal result on the structure of $C_\infty$ maps.

<u>Lemma 4.3</u>: 1) Let X and Y be $\Omega$-spectra so that X is a product of Eilenberg-MacLane spectra. If $f : X_k \longrightarrow Y_n$ is a $C_\infty$ map, then there exists a map of spectra $F : X \longrightarrow Y$ that induces f.

2) If $i : X \longrightarrow X'$ and $f : Y \longrightarrow Y'$ are maps of $\Omega$-spectra and f and f' are $C_\infty$ maps so that

$$\begin{array}{ccc} X_k & \longrightarrow & Y_\mu \\ \downarrow & & \downarrow j \\ X'_k & \longrightarrow & Y'_n \end{array}$$

commutes, then this diagram is induced by a diagram of spectra.

<u>Proof</u>: May's work [21] or Lada's work [13] gives a spectrum $BY_n$ and a map $BY_n \longrightarrow Y$ so that $\pi_q BY_n \xrightarrow{\cong} \pi_q Y$ for $q \geq -n$. Note that $(BY_n)_n = Y_n$. If $j : Y \longrightarrow Y'$ is a map of spectra, then there exists a map $Bj : BY_n \longrightarrow BY'_n$ so that

$$\begin{array}{ccc} BY_n & \longrightarrow & Y \\ Bj \downarrow & & \downarrow j \\ BY_n & \longrightarrow & Y \end{array}$$

commutes. Similarly, since X is a product of Eilenberg-MacLane spaces, we may use this construction to find a space $BX_k$ and a map

$$X \longrightarrow BX_k$$

so that $\pi_q X \xrightarrow{\cong} \pi_q BX_k$ for $q \geq -k$. This, too, is natural. If $f : X_k \longrightarrow Y_n$ is a $C_\infty$ map then, there exists a map

$$Bf : BX_k \longrightarrow BY_n$$

inducing f. This, in turn, is natural. F, then is the composition

$$F : X \longrightarrow BX_k \xrightarrow{Bf} BY_n \longrightarrow Y.$$

2) follows because the appropriate diagrams commute.

We also need the following result, culled from R. Cohen [14].

<u>Lemma 4.4</u>: Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration of spectra and suppose the induced map $B_{n-1} \longrightarrow F_n$ is zero. If $f : X_k \longrightarrow E_n$ is a $C_\infty$ map so that $pf = 0$, then there exists a $C_\infty$ map $p' : Y_k \longrightarrow F_n$ so that $ip' = p$.

We now recall in detail the results of Brown and Gitler. Let the following sequence be the $\chi$-dual of the realization of 2.9 for M(K) (refer to 3.11):

$$\cdots \longrightarrow L_q \xrightarrow{d_q} L_{q-1} \longrightarrow \cdots \longrightarrow L_1 \xrightarrow{d_1} L_o = K\mathbb{Z}_2$$

Then we have from Brown and Gitler [6]

Theorem 4.5: There is a sequence of spectra $E_q$, $q \geq 0$ and maps $e_q : L_q \longrightarrow E_{q-1}$ of degree 1 satisfying

1) $E_o = L_o = K\mathbb{Z}_2$ and $e_1 = d_1$

2) $E_{q-1} \xrightarrow{i_q} E_q \xrightarrow{p_q} L_q$ is the fibration induced by $e_q$ from the contractible path fibration over $E_{q-1}$.

3) $L_q \xrightarrow{e_q} E_{q-1} \xrightarrow{p_{q-1}} L_{q-1}$ is $d_q$.

4) $e_q : L_{q,2k} \longrightarrow E_{q-1,2k+1}$ is zero.

5) Let $j_q : E_o \longrightarrow E_q$ be the composition of the maps $i_p : E_{p-1} \longrightarrow E_p$.

Then the sequence

$$[K\mathbb{Z}_2 , E_o] \xrightarrow{j_{q_*}} [K\mathbb{Z}_2 , E_q] \xrightarrow{p_{q_*}} [K\mathbb{Z}_2, L_q]$$

is exact.

Theorem 4.6: $e_q$ is uniquely determined (up to homotopy) by 4.5.3 and 4.5.4.

Proof: This is from Brown and Peterson [9].

Lemma 4.7: Let $g \in [K\mathbb{Z}_2 , L_o] = [K\mathbb{Z}_2 , K\mathbb{Z}_2] = A$. Then g is in the image of $d_1 : [K\mathbb{Z}_2, L_1] \longrightarrow [K\mathbb{Z}_2, L_o]$ if and only, for every CW complex X, the induced map $g : H^qX \longrightarrow H^{2k+1}X$ is zero.

Proof: This is from Brown and Gitler [6].

Consider the diagram of spectra from 3.11

$$\begin{array}{ccccccccc} \longrightarrow L_q & \xrightarrow{d_q} & L_{q-1} & \longrightarrow & \cdots & \longrightarrow L_1 & \xrightarrow{d_1} & L_o = K\mathbb{Z}_2 \\ u_q \downarrow & & \downarrow u_{q-1} & & & u_1 \downarrow & & \downarrow u_o \\ \longrightarrow K_q & \xrightarrow{d'_q} & K_{q-1} & \longrightarrow & \cdots & \longrightarrow K_1 & \xrightarrow{d'_1} & K_o = K\mathbb{Z}_{2^\infty} \end{array}$$

The top is the $\chi$-dual of the resolution realizing 2.9 for $M(2k+1)$, the bottom is the $\chi$-dual of the resolution realizing 2.11 for $M_1(2k+1)$. Our main result is this:

Theorem 4.8: There are spectra $G_q$ and maps $e'_q : K_q \longrightarrow G_{q-1}$ of degree 1 and $\ell_q : E_q \longrightarrow G_q$ of degree zero so that

1) $G_o = K_o = K\mathbb{Z}_{2^\infty}$ and $e'_1 = d'_1$, $\ell_o = u_o$.

2) $G_{q-1} \xrightarrow{i_q} G_q \longrightarrow K_q$ is the fibration induced by $e'_q$ from the contractible path fibration over $G_{q-1}$.

3) Both these diagrams commute:

$$\begin{array}{ccccc} E_{q-1} & \xrightarrow{i_q} & E_q & \xrightarrow{p_q} & L_q \\ \downarrow{\scriptstyle \ell_{q-1}} & & \downarrow{\scriptstyle \ell_q} & & \downarrow{\scriptstyle u_q} \\ G_{q-1} & \xrightarrow{i_q} & G_q & \xrightarrow{p_q} & K_q \end{array} \qquad \begin{array}{ccc} L_q & \xrightarrow{e_q} & E_{q-1} \\ \downarrow{\scriptstyle u_q} & & \downarrow{\scriptstyle \ell_{q-1}} \\ K_q & \xrightarrow{e'_q} & G_{q-1} \end{array}$$

4) $K_q \xrightarrow{e'_q} G_{q-1} \xrightarrow{p_{q-1}} K_{q-1}$ is $d'_q$.

5) For $q \geq 1$, $e'_q : K_{q,4k+2} \longrightarrow G_{q-1,4k+3}$ is zero, and $e_q$ and $e'_q$ are concurrently null-homotopic.

We postpone the proof, for a moment, so that we may state a similar theorem for the bo case. Consider the diagram of spectra from 3.12.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & L_q & \xrightarrow{d_q} & L_{q-1} & \longrightarrow & \cdots & \longrightarrow & L_1 & \xrightarrow{d_1} & L_o = K\mathbb{Z}_2 \\ & & \downarrow{\scriptstyle u_q} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle u_o} \\ \cdots & \longrightarrow & K_q & \xrightarrow{d'_q} & K_{q-1} & \longrightarrow & \cdots & \longrightarrow & K_1 & \xrightarrow{d'_1} & K_o = \chi(bo) \end{array}$$

The top is the $\chi$-dual of the resolution realizing 2.9 for $M(4k+3)$, the bottom is the $\chi$-dual of the resolution realizing 2.12 for $M_2(4k+3)$. The main result in this direction is this.

Theorem 4.9: There are spectra $G_q$ and maps $e'_q : K_q \longrightarrow G_{q-1}$ of degree 1 and $\ell_q : E_q \longrightarrow G_q$ of degree o so that

1) $G_o = K_o = \chi(bo)$ and $e'_1 = d'_1$, $\ell_o = u_o$.

2) $G_{q-1} \xrightarrow{i_q} G_q \xrightarrow{q-1} K_q$ is the fibration induced by $e'_q$ from the contractible path fibration over $G_{q-1}$.

3) Both these diagrams commute:

$$\begin{array}{ccccc} E_{q-1} & \xrightarrow{i_q} & E_q & \xrightarrow{p_q} & L_q \\ \downarrow{\scriptstyle \ell_{q-1}} & & \downarrow{\scriptstyle \ell_q} & & \downarrow{\scriptstyle u_q} \\ G_{q-1} & \xrightarrow{i_q} & G_q & \xrightarrow{p_q} & K_q \end{array} \qquad \begin{array}{ccc} L_q & \xrightarrow{e_q} & E_{q-1} \\ \downarrow{\scriptstyle u_q} & & \downarrow{\scriptstyle \ell_{q-1}} \\ K_q & \xrightarrow{e'_q} & G_{q-1} \end{array}$$

4) $K_q \xrightarrow{e'_q} G_{q-1} \xrightarrow{p_{q-1}} K_{q-1}$ is $d'_q$

5) For $q \geq 2$, $e'_q : K_{q,8k+6} \longrightarrow G_{q-1,8k+7}$ is zero, and $e_q$ and $e'_q$ are concurrently null-homotopic.

I will prove 4.8. The proof of 4.9 proceeds verbatim.

Proof of 4.8: We proceed by induction on q.

$q = 1$: $e'_1$ and $\ell_o$ are determined by 1); $G_1$ and $\ell_1$ by 2) and 3). 3) follows and 4) is a tautology. 5) follows from 3.11.

$q > 1$ : Now suppose that, for all $p \leq q$, we have constructed $G_q$, $e'_q$ and $\ell_q$ satisfying 1) - 5). Then we have a diagram

$$\begin{array}{ccccc} L_{q+1} & \xrightarrow{d_{q+1}} & L_q & \xrightarrow{e_q} & E_{q-1} \\ \downarrow & & \downarrow u_q & & \downarrow q-1 \\ K_{q+1} & \xrightarrow{d'_{q+1}} & K_q & \xrightarrow{e'_q} & G_{q-1} \end{array}$$

Let $j_q = i_1 \circ i_2 \circ \ldots\ i_{q-1} : E_o \longrightarrow E_{q-1}$ or $G_o \longrightarrow G_{q-1}$. Then we have a diagram

$$\begin{array}{ccccc} L_1 & \longrightarrow & E_o & \longrightarrow & E_{q-1} \\ \downarrow & & \downarrow & & \downarrow \\ K_1 & \longrightarrow & G_o & \longrightarrow & G_{q-1} \end{array}$$

As in [6], we may use the facts that $p_{q-1}e_q d_{q+1} = d_q d_{q+1} = 0$, 4.5.5, 4.5.4 and 4.7 to conclude that $e_q d_{q+1} : L_{q+1} \longrightarrow E_{q-1}$ lifts to a map $\phi : L_{q+1} \longrightarrow L_1$ so that $j_q d_q \phi = e_q d_{q+1}$.

Let $s : K_{q+1} \longrightarrow L_{q+1}$ be any section of $u_{q+1} : u_{q+1}s =$ identity. Then I claim that $\phi' = u_1 \phi s$ is a lift of $e'_q d_{q+1}$ to $K_1$. ($\phi' : K_{q+1} \longrightarrow K_1$).

First, $e'_q d'_{q+1} = e_q d_{q+1}\ u_{q+1} s$

$= \ell_{q-1}\ e_q d_{q+1} s$

Second, $j_q d_1 u_1 \phi s = \ell_{q-1} j_q d_1 \phi s$

$= \ell_{q-1} e_q d_{q+1} s$

Now note that the following diagram need <u>not</u> commute:

$$\begin{array}{ccc} L_{q+1} & \xrightarrow{\phi} & L_1 \\ u_{q+1}\downarrow & & \downarrow u_1 \\ K_{q+1} & \xrightarrow{\phi'} & K_1 \end{array}$$

Consider $\phi' u_{q+1} + u_1\phi : L_{q+1} \longrightarrow K_1$. Since $(u_1)_* : [K\mathbb{Z}_2 , L_1]_* \longrightarrow [K\mathbb{Z}_2 , K_1]_*$ is surjective, there exists a map $\alpha : L_{q+1} \longrightarrow L_1$ so that $u_1\alpha = \phi' u_{q+1} + u_1\phi$. Set $\phi'' = \phi + \alpha$. Then $\phi' u_{q+1} = u_1\phi'$.

And, $\phi''$ is still a lifting of $e_q d_{q+1}$ to $L_1$. This is because $j_{q-1}d_1 = 0$. Which is exactly the point: apply 3) in 4.8 for $p \leq q$ to conclude that $e_q d_{q+1}$ and $e'_q d'_{q+1}$ are concurrently null-homotopic. So, we may conclude that there exist maps f and f' so that the following diagrams commute:

$$\begin{array}{ccc} & \overset{f}{\nearrow} & E_q \\ & & \downarrow p_q \\ L_{q+1} & \xrightarrow{d_{q+1}} & L_q \end{array} \qquad \begin{array}{ccc} & \overset{f'}{\nearrow} & G_q \\ & & \downarrow p_q \\ K_{q+1} & \xrightarrow[d_{q+1}]{} & K_q \end{array} \qquad \begin{array}{ccc} L_{q+1} & \xrightarrow{f} & E_q \\ {\scriptstyle u_{q+1}}\downarrow & & \downarrow{\scriptstyle \ell_q} \\ K_{q+1} & \xrightarrow{f'} & G_q \end{array}$$

Now, because $d_{q+1} : L_{q+1,4k+2} \longrightarrow L_{q,4k+3}$ and $d'_q : K_{q+1,4k+2} \longrightarrow K_{q,4k+3}$ are concurrently null-homotopic (3.11), there exist maps $\gamma$ and $\gamma'$ so that the following diagrams commute

$$\begin{array}{ccc} & \overset{\gamma}{\nearrow} & E_{q-1,4k+3} \\ & & \downarrow \\ L_{q+1,4k+2} & \longrightarrow & E_{q,4k+3} \end{array} \qquad \begin{array}{ccc} & \overset{\gamma}{\nearrow} & G_{q-1,4k+3} \\ & & \downarrow \\ K_{q+1,4k+2} & \longrightarrow & G_{q,4k+3} \end{array}$$

$$\begin{array}{ccc} L_{q+1,4k+2} & \xrightarrow{\gamma} & E_{q-1,4k+3} \\ \downarrow{\scriptstyle u_{q+1}} & & \downarrow{\scriptstyle \ell_{q-1}} \\ K_{q+1,4k+2} & \xrightarrow{\gamma'} & G_{q-1,4k+3} \end{array} \qquad (4.10)$$

I claim that this last diagram (4.10) is a diagram of $C_\infty$ maps. Again, let $s : K_{q+1} \longrightarrow L_{q+1}$ be any section of $u_{q+1}$. Then s induces a $C_\infty$ map $s : K_{q+1,4k+2} \longrightarrow L_{q+1,4k+2}$ which is a section of (the unstable) $u_{q+1}$.

$$\gamma' = \gamma u_{q+1} s = \ell_{q-1} \gamma s$$

$\gamma$ is a $C_\infty$ map by Lemma 4.4; therefore $\gamma'$ is a $C_\infty$ map.

Now, apply Lemma 4.3 to conclude that there exist maps g,g' so that the following diagram commutes.

$$\begin{array}{ccc} L_{q+1} & \xrightarrow{g} & E_{q-1} \\ {\scriptstyle u_{q+1}}\downarrow & & \downarrow{\scriptstyle \ell_q} \\ K_{q+1} & \xrightarrow{g_1} & G_{q-1} \end{array}$$

First note that the map

$$f - i_q g : L_{q+1} \longrightarrow E_q$$

has the properties that $d_{q+1} = p_q(f - i_q g) = p_q f$ and that

$$f - i_q g : L_{q+1,4k+2} \longrightarrow E_{q,4k+3}$$

is zero. Thus, by 4.6, $f - i_q g = e_{q+1}$. Set $e'_{q+1} = f' - i_q g' : K_{q+1} \longrightarrow G_q$. Property 2) of Theorem 4.8 defines $G_{q+1}$ and 3) defines $\ell_{q+1}$. 3) is satisfied. 4) follows because

$$p_q e_q = p_q(f' - i_q g') = p_q f' = d'_{q+1}$$

and 5) follows from the definition of $e'_q$.

This completes the induction step. The theorem follows.

Remark: The two principle ingredients in this proof are the facts that

1) for $q \geq 2$, there is a section of $u_q : L_{q+1} \xrightarrow{u_q} K_q$

and

2) for $q \geq 1$, that $d_{q+1}$ and $d'_{q+1}$ are concurrently null-homotopic

$$\begin{array}{ccc} L_{q+1,4k+2} & \xrightarrow{d_{q+1}} & L_{q,4k+3} \\ \downarrow & & \downarrow \\ K_{q+1,4k+2} & \xrightarrow{d'_{q+1}} & K_{q,4k+3} \end{array}$$

So, there is no obstacle to applying these techniques to proving Theorem 4.9; therefore we consider 4.9 to be proved.

## 5. Construction of $B_1(2k+1)$ and $B_2(4k+3)$

In this section, we prove our main theorems.

Let me first describe how to produce a spectrum $B_1(2k+1)$ so that

1) $H^*B_1(2k+1) \simeq A/A\{Sq^1, \chi(Sq^i), i > 2k\}$.

2) If $1 : B_1(2k+1) \longrightarrow K\mathbb{Z}_2^{\wedge}$ classifies a generator of $H^0(B_1(2k+1), \mathbb{Z}_2^{\wedge})$ then $1_* : B_1(2k+1) \longrightarrow H_n(X; \mathbb{Z}_2^{\wedge})$ is onto for $n \leq 4k+3$.

3) If $B(2k+1)$ is the $\mathbb{Z}_2$-Brown-Gitler Spectrum, then there is a map $B_1(2k+1) \longrightarrow B(2k+1)$ inducing the obvious quotient in cohomology.

Reaching from Theorem 4.8, we have constructed, for every q, a diagram

$$\begin{array}{ccccccc} E_{q-1} & \longrightarrow & E_q & \longrightarrow & L_q & \longrightarrow & E_{q-1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G_{q-1} & \longrightarrow & G_q & \longrightarrow & K_q & \longrightarrow & G_{q-1} \end{array}$$

We take the $\chi$-dual of this diagram. $Y_q = \chi(E_q)$, $X_q = (G_q)$,

$I_q = \chi(L_q)$ and $F_q = \chi(K_q)$. Then we have a diagram

$$\begin{array}{ccccccc} X_{q-1} & \xleftarrow{p_q} & X_q & \xleftarrow{i_q} & F_q & \xleftarrow{\varepsilon_q} & X_{q-1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y_{q-1} & \xleftarrow{p_q} & Y_q & \xleftarrow{i_q} & I_q & \xleftarrow{\varepsilon_q} & Y_{q-1} \end{array}$$

Notice that, for every $q \geq 0$, $I_q$ is a product of Eilenberg-MacLane spaces of type $K\mathbb{Z}_2$, for every $q \geq 1$, $F_q$ is a produce of Eilenberg-MacLane spaces of type $K\mathbb{Z}_2$, and $F_o = K\mathbb{Z}_2^{\wedge}$. In fact, if

$$\delta_q = \varepsilon_q \circ i_{q-1} : F_{q-1} \longrightarrow F_q ,$$

then, in cohomology, $\delta_q^* : A \otimes \Lambda^{\wedge}(q,k) \longrightarrow A \otimes \Lambda^{\wedge}(q-1,k)$ ($\delta_1^* : A \otimes \Lambda^{\wedge}(1,k) \longrightarrow A/ASq^1$) is the map of 2.11. Assume $p_q$ is a fibration for each q.

Define $B_1(2k+1) = \varprojlim X_q$.

Then, of course, we have a map

$$B_1(2k+1) = \varprojlim X_q \longrightarrow \varprojlim Y_q = B(2k+1)$$

<u>Proposition</u> <u>5.1</u>: $H^*B_1(2k+1) \cong A/A\{Sq^1, \chi(Sq^i), i > 2k\}$.

Proof: Since we are working with mod 2 cohomology

$$H^*B_1(2k+1) \cong \varinjlim H^*X_q .$$

We will show that

$$H^*F_q \xleftarrow{i_q^*} H^*X_q \xleftarrow{j_q^*} H^*K\mathbb{Z}_2^{\wedge} \xleftarrow{\delta_1^*} H^*F_1$$

is exact where $j_q^* = (p_1 \circ \dots p_{q-1} \circ p_q)^*$. Thus we have a sequence

$$0 \longrightarrow A/A\{Sq^1, \chi(Sq^i)\ i > 2k\} \longrightarrow H^*X_q \longrightarrow H^*F_q$$

for every q. Thus $\varinjlim H^*X_q = A/A\{Sq^1, \chi(Sq^i), i > 2k\}$. The sequence is exact at $H^*X_q$. (The reader is advised to draw diagrams.)

We prove this by induction. This is true for q = 1. Suppose it is true for q - 1. If $i_q^* v = 0$, then there exists a w so that $p_q^* w = v$. Then $\varepsilon_{q-1}^* i_{q-1}^* w = 0$ (because $\varepsilon_{q-1}^* i_{q-1}^* = 0$); therefore

$$\delta_{q-1}^* i_{q-1}^* w = i_{q-2}^* \varepsilon_{q-1}^* i_{q-1}^* w = 0 .$$

So there exists $x \in H^*F_q$ so that $\delta_q^* x = i_{q-1} w$. Then

$$p_q^*(w - \varepsilon_q^* x) = p_q^* w = v \quad \text{and} \quad i_{q-1}(w - \varepsilon_q^* x) = 0 .$$

So exactness at $H^*X_q$ follows from exactness at $H^*X_{q-1}$. The sequence is exact at $H^*K\hat{\mathbb{Z}}_2$.

Let $v \in H^*K\mathbb{Z}_2$ and suppose $j_q^* v = 0$. Let s be the least integer so that $j_s^* v = 0$. If s = 1, v is in the image of $\delta_1^*$. If $s \neq 1$, then $j_{s-1}^* v = \varepsilon_s^* w$ for some w. Since $i_{s-1}^* j_{s-1}^* = 0$, $i_{s-1}^* j_{s-1}^* v = \delta_s^* w = 0$. Thus there exists x, so that $\delta_{s+1}^* x = w$. Then, $0 = \varepsilon_s^* \delta_{s+1}^* x = j_{x-1} v$. This contradicts the choice of s, and completes the proof.

<u>Theorem 5.2</u>: Let Z be any CW complex. Then consider the following sequence

$$0 \longrightarrow (F_q)_n Z \longrightarrow (X_q)_n Z \xrightarrow{(p_q)_*} (X_{q-1})_n Z \longrightarrow 0$$

This sequence is split short exact for $n \leq 4k+1$ and short exact for $n = 4k+2$. If $n \leq 4k+1$, the splitting is natural in Z. If $n = 4k+3$, $(p_q)_*$ is surjective.

Proof: We first prove the result for finite CW complexes Z. By the properties of Pontrjagin duality (3.9) and of cohomology, we need only examine the sequence

$$\cdots \longrightarrow [Z,G_{q-1},n] \longrightarrow [Z,G_{q,n}] \longrightarrow [Z,K_{q,n}] \xrightarrow{e_q^*} [Z,G_{q-1,n+1}]$$

Since $e_q : K_{q,n} \longrightarrow G_{q-1,n+1}$ is zero for $n \leq 4k+2$, Theorem 4.8.2 implies that for every connected finite CW complex Z

$$0 \longrightarrow [Z,G_{q-1,n}] \longrightarrow [Z,G_{q,n}] \longrightarrow [Z,K_{q,n}] \longrightarrow 0$$

is naturally split short exact for $n \leq 4k+2$. Thus, using the suspension isomorphism

$$[Z,\ G_{q,n}] \simeq [\Sigma Z,\ G_{q,n+1}]$$

(for example) we have proved the theorem for finite CW complexes.

For arbitrary CW complexes Z, let $\{Z_\alpha\}$ be the directed system of finite subcomplexes of Z. Then, for any homology theory $K_*$, $K_*Z = \varinjlim K_*Z_\alpha$. Since $\varinjlim$ is an exact functor and because the splittings were natural, the result follows for any CW complex.

Remark: We can only say that

$$0 \longrightarrow [Z,\ G_{q-1,4k+2}] \longrightarrow [Z,\ G_{q,4k+2}] \longrightarrow [Z,\ K_{q,4k+2}] \longrightarrow 0$$

is <u>split</u> for connected CW complexes because $G_{q,4k+2}$ is not a connected space. $\pi_0 G_{q,4k+2}$ is a group but it does <u>not</u> follow that

$$\pi_0 G_{q,4k+2} = \pi_0 G_{q-1,4k+2} \times \pi_0 K_{q,4k+2}$$

Let me now describe how to produce the spectrum $B_2(4k+3)$.

Recall that for every integer j we chose (in Section 3) map $i_{4j}$ so that the following diagram commutes

$$\begin{array}{ccc} bo & \xrightarrow{i_{4j}} & K\hat{\mathbb{Z}}_2 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ K\mathbb{Z}_2 & \xrightarrow{\chi(Sq^{4j})} & K\mathbb{Z}_2 \end{array}$$

$i_{4j}$ induces, for every CW complex Z, a map $(i_{4j})_* : bo_n Z \longrightarrow H_{n-4j}(Z; \hat{\mathbb{Z}}_2)$

Then $B_2(4k+3)$ has the properties that

1) $H^*B_2(4k+3) = A/A\{Sq^1, Sq^2, \chi(Sq^i), i > 4k\}$.

2) There is a map $B_2(4k+3) \xrightarrow{\omega} bo$ so that $B_2(4k+3) \xrightarrow{\omega} bo \xrightarrow{1} K\mathbb{Z}_2$ classifies the generator of $H^*B_2(4k+3)$ and that the map

$$\omega_* : B_2(4k+3)_n Z \xrightarrow{\omega} bo_n Z$$

is onto the subgroup $\bigcap_{j>k} \ker(i_{4j})_*$ for $n \leq 8k+7$ and any CW complex Z.

3) If $B(4k+3)$ is the $\mathbb{Z}_2$ Brown-Gitler Spectrum, then there is a map $B_2(4k+3) \longrightarrow B(4k+3)$ inducing the obvious quotient in cohomology.

Reading from Theorem 4.9, we have constructed, for every q, a diagram

$$\begin{array}{ccccccc} E_{q-1} & \longrightarrow & E_q & \longrightarrow & L_q & \longrightarrow & E_{q-1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G_{q-1} & \longrightarrow & G_q & \longrightarrow & K_q & \longrightarrow & G_{q-1} \end{array}$$

We take the $\chi$-dual of this diagram. $Y_q = \chi(E_q)$, $X_q = \chi(G_q)$, $I_q = \chi(L_q)$ and $F_q = \chi(K_q)$. Then we have a diagram

$$\begin{array}{ccccccc} X_{q-1} & \xleftarrow{p_q} & X_q & \xleftarrow{i_q} & F_q & \xleftarrow{\varepsilon_q} & X_{q-1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y_{q-1} & \xleftarrow{p_q} & Y_q & \xleftarrow{i_q} & I_q & \xleftarrow{\varepsilon_q} & Y_{q-1} \end{array}$$

For every $q \geq 2$, $F_q$ is a product of Eilenberg-MacLane spaces of type $K\mathbb{Z}_2$, so that $H^*F_q = A \otimes \overline{\Lambda}^*(q,k)$; $F_1$ is a product of Eilenberg-MacLane spaces of type $K\mathbb{Z}_2$ so that $H^*F_1 = A/ASq^1 \otimes \Lambda^*(1,k)$, and $F_o = bo$. If $\delta_q = \varepsilon_q \circ i_{q-1} : F_{q-1} \longrightarrow F_q$, then $\delta_q^*$ is the map of 2.12.

Define $B_2(4k+3) = \varprojlim X_q$ (assuming $p_q$ is a fibration for each q).

Then we have a map

$$B_2(4k+3) = \varprojlim X_q \longrightarrow \varprojlim Y_q = B(4k+3) .$$

We have the following.

<u>Proposition</u> <u>5.3</u>: $H^*B_2(4k+3) = A/A\{Sq^1, Sq^2, \chi(Sq^i), i > 4k\}$

<u>Theorem</u> <u>5.4</u>: 1) $\delta_1 : F_o \longrightarrow F_1$ is the map $\underset{j>k}{\times} i_{4j} : bo \longrightarrow \underset{j>k}{\times} K\hat{\mathbb{Z}}_2$. Thus we have a fibration sequence

$$X_1 \longrightarrow bo \xrightarrow{\times i_{4j}} \underset{j>k}{\times} K\hat{\mathbb{Z}}_2 .$$

2) Let Z be any CW complex. Then consider the following sequence.

$$0 \longrightarrow (F_q)_n Z \longrightarrow (X_q)_n Z \xrightarrow{(p_q)_*} (X_{q-1})_n Z \longrightarrow 0$$

For $q \geq 2$, this sequence is naturally split short exact for $n \leq 8k+5$, short exact for $n = 8k+6$. If $n = 8k+7$, $(p_q)_*$ is surjective.

Lemma 5.3 and Theorem 5.4 are directly comparable to 5.1 and 5.2. Theorem 5.4.2 says that, in particular, for any CW complex Z

$$B_2(4k+3)_n Z \longrightarrow (X_1)_n Z$$

is onto for $n \leq 8k+7$. 5.4.1 then implies that the image of

$$(X_1)_n Z \longrightarrow bo_n Z$$

is $\cap$ her$(i_{4j})_*$. Thus the image of

$$\omega_* : B_2(4k+3)_n Z \to bo_n Z \quad , \qquad n \leq 8k+7$$

is $\bigcap_{i>j}$ her$(i_{4j})_*$.

This completes the proof of Theorems 1 and 2 of the introduction.

6. $\underline{\pi_* B_1(2k+1) \text{ and } \pi_* B_2(4k+3)}$

As the first and simplest application of our structure theorems (5.2 and 5.4), I will show how to compute $\pi_* B_1(2k+1)$ and $\pi_* B_2(4k+3)$ in a range.

The first of these is easy. Recall that $\hat{\Lambda} \subseteq \Lambda$ is the left ideal generated by $\lambda_{2i-1}$ and that $\Lambda_{2k+1} = \Lambda\{\lambda_0, \ldots, \lambda_{2k}\} \subseteq \Lambda$ is the left ideal generated by $\lambda_i$, $i \leq 2k$. Furthermore, $\Lambda$ is bigraded with each $\lambda_i$ having bigrading $(i,1)$. Let $(\hat{\Lambda}/\hat{\Lambda} \cap \Lambda_{2k+1})_n$ be all elements of $\hat{\Lambda}/\hat{\Lambda} \cap \Lambda_{2k+1}$ of bigrading $(n,r)$, $r > 0$.

Corollary 6.1:

$$\pi_0 B_1(2k+1) \cong \hat{\mathbb{Z}}_2$$

$$\pi_n B_1(2k+1) = (\Lambda/\Lambda \cap \Lambda_{2k+1})_n \qquad 1 \leq n \leq 4k+1 \ .$$

Proof: Set $Z = S^0$ in Theorem 5.2.

Remark: Let $B(2k+1)$ be the $\mathbb{Z}_2$ Brown-Gitler Spectrum and $M(2)$ the mod-2 Moore space. Then, we will, in Chapter II, Section 2, see that

$$B_1(2k+1) \wedge M(2) \cong B(2k+1)$$

From this we will be able to conclude that

$$\pi_{4k+2} B_1(2k+1) \cong (\hat{\Lambda}/\hat{\Lambda} \cap \Lambda_{2k+1})_{4k+2} \ .$$

Theorem 5.2 only gives us a set isomorphism between these groups.

Now let us consider $B_2(4k+3)$. Let $\Lambda' \subseteq \Lambda$ be the left ideal generated by admissible monomials $\lambda_{2i-1}\lambda_{4j-1}$. Recall (Lemma 2.10.3) that a $\mathbb{Z}_2$ basis for $\Lambda'$ is given by all admissible monomials $\lambda_{i}, \ldots, \lambda_q$ with $q \geq 2$, and $i_{q-1} \equiv 1 \bmod 2$, and $i_q \equiv -1 \bmod 4$. Then, Theorem 5.5 tells us that

$$\pi_n B_2(4k+3) \cong \pi_n X_1 \times (\Lambda'/\Lambda' \cap \Lambda_{4k+3})_n \qquad n \leq 8k+5$$

$X_1$ is the first stage in the tower whose inverse limit is $B_2(4k+3)$. We have a fibration sequence

$$X_1 \longrightarrow bo \xrightarrow{\times i_{4j}} \underset{j>k}{\times} K\hat{\mathbb{Z}}_2 .$$

Let us compute $\pi_* X_1$ and, thus, $\pi_* B_2(4k+3)$. Recall that

$$\pi_n bo = \begin{cases} \hat{\mathbb{Z}}_2 & n \equiv 0,4 \quad (8) \\ \mathbb{Z}_2 & n \equiv 1,2 \quad (8) \\ 0 & \text{otherwise} \end{cases}$$

Let $\beta_n$ be the generator $\pi_n bo$.

The following is an obvious first step.

<u>Lemma 6.2</u>: 1) $i_{8j} : bo \longrightarrow K\mathbb{Z}_2$ induces multiplication by $2^{4j}$ in homotopy for $j \geq 1$.

2) $i_{8j+4} : bo \longrightarrow K\hat{\mathbb{Z}}_2$ induces multiplication by $2^{4j+3}$ in homotopy for $j \geq 0$.

<u>Proof</u>: We show 1) for j = 1 and 2) for j = 0, then use Bott periodicity to extend the results. In the Adams Spectral Sequence converging to $\pi_* bo$, $\beta_4$ is on the three line and $\beta_8$ is on the four line. See [17].

Therefore, $i_4$ induces multiplication by $2^k$ with $k \geq 3$ and $i_8$ induces multiplication by $2^\ell$ with $\ell \geq 4$. We wish to show these inequalities are strict.

$$\text{Let } i = \underset{j>0}{X}\ i_{4j} : bo \longrightarrow \underset{j}{X}\ K\mathbb{Z}_2^\wedge$$

$$Sq = \underset{j\geq 0}{X}\ \chi(Sq^{2^j}) : K\mathbb{Z}_2 \longrightarrow \underset{j}{X}\ K\mathbb{Z}_2 .$$

Then we have commutative diagram of fibration sequences

$$\begin{array}{ccccc} X_1 & \longrightarrow & bo & \xrightarrow{i} & XK\mathbb{Z}_2^\wedge \\ \downarrow & & 1\downarrow & & \downarrow \\ E_1 & \longrightarrow & K\mathbb{Z}_2 & \xrightarrow{Sq} & XK\mathbb{Z}_2 \end{array}$$

$X_1$ is the first stage of the tower whose inverse limit is $B_2(3) = S^0$. $E_1$ is the first stage in an Adams Resolution for $S^0$. Thus we have a map

$$f : S^0 \longrightarrow X_1 \longrightarrow E_1$$

Let $\nu$ generate $\pi_3 S^0 = \mathbb{Z}_8$. Then $f_*\nu \neq 0$. Thus $\pi_3 X_1 \cong \mathbb{Z}_{2^s}$ with $s \leq 3$. Let $\sigma$ generate $\pi_7 S^0 = \mathbb{Z}_{16}$. Then $f_*\sigma \neq 0$. Thus $\pi_7 X_1 = \mathbb{Z}_{2^t}$ with $t \leq 4$. So, $i_4$ induces multiplication by 8 and $i_8$ induces multiplication by 16. Let $\beta_8 : S^8 \longrightarrow bo$ be a generator of $\pi_8 bo = \mathbb{Z}_2$. Then we have a map

$$b : S^8 \wedge bo \xrightarrow{\beta_8 \wedge id} bo \wedge bo \longrightarrow bo$$

so that $b_* : \pi_n S^8 \wedge bo \longrightarrow bo$ is an isomorphism for $n \geq 8$. This is the Bott periodicity map [31]. Iterates produces maps

$$b^j : S^{8j} \wedge bo \longrightarrow bo$$

and

$$\beta_{8j+4} = b_*^j \beta_4 \text{ and } \beta_{8j+8} = b_*^j \beta_8 \ .$$

(up to units in $\hat{\mathbb{Z}}_2$). Finally, using the ring spectrum properties of bo, and our choices of $i_{4j}$ (see Section 3, before 3.12), we see that the following commute

$$\begin{array}{ccc} \Lambda_j bo \wedge bo & \longrightarrow & bo \\ \Big\downarrow (\Lambda i_8) \wedge i_4 & & \Big\downarrow i_{8j+4} \\ \Lambda_j K\hat{\mathbb{Z}}_2 \wedge K\hat{\mathbb{Z}}_2 & \longrightarrow & K\hat{\mathbb{Z}}_2 \end{array} \qquad \begin{array}{ccc} \Lambda_j bo \wedge bo & \longrightarrow & bo \\ \Big\downarrow (\Lambda i_8) \wedge i_8 & & \Big\downarrow i_{8j+8} \\ \Lambda_j K\hat{\mathbb{Z}}_2 \wedge K\hat{\mathbb{Z}}_2 & \longrightarrow & K\hat{\mathbb{Z}}_2 \end{array}$$

The result follows.

<u>Lemma 6.3</u>: Let $X_1$ be the fiber of $\underset{j>k}{X} i_{4j} : bo \longrightarrow XK\hat{\mathbb{Z}}_2$. Then

$$\pi_n X_1 = \pi_n bo \quad \text{for} \quad n \leq 4k+2$$

and for $n \geq 4k+3$

$$\pi_n X_1 \cong \begin{cases} \mathbb{Z}_{2^{4j}} & n = 8j-1 \\ \mathbb{Z}_2 & n = 8j+1,\ 8j+2 \\ \mathbb{Z}_{2^{4j+3}} & n = 8j+3 \\ 0 & \text{otherwise} \end{cases}$$

Let $\alpha_n$ generate $\pi_n X_1$, $n \geq 4k+3$ and let $\eta$ generate $\pi_1 S^0$. Then $\eta\alpha_{8j+1} = \alpha_{8j+2}$, and $\eta^2\alpha_{j+1} = 2^{4j+2}\alpha_{8j+3}$ (up to a unit in $\hat{\mathbb{Z}}_2$).

<u>Proof</u>: The first statement follows from 6.2. We need to prove the composition results. We consider all k at once, and let $X_1^k = X_1$ for $B_2(4k+3)$. Then

$$\pi_n X_1^k \cong \pi_n X_1^{k+1}$$

for $n \geq 4(k+1) + 3$. The isomorphism is induced by the obvious map. In $\pi_* X_1^0$, $\eta\alpha_1 = \alpha_2$ and $\eta^2\alpha_1 = 4\cdot\alpha_3$. This is because the inclusion

$$S^0 \longrightarrow X_1^0$$

is a homotopy isomorphism for dimensions less than six.

Now for any product of Eilenberg-MacLane spaces, let

$$2^j\mathrm{id} : X K\mathbb{Z}_2^{\wedge} \longrightarrow X K\mathbb{Z}_2^{\wedge}$$

be $2^j$ times the identity. Then we have a commutative diagram

$$\begin{array}{ccccc} X_1^0 & \longrightarrow & S^{8k}\wedge bo & \longrightarrow & X \quad K\mathbb{Z}_2 \\ \phi\downarrow & & \downarrow b^k & & 2^{4k}\,\mathrm{id}\downarrow \\ X_1^{2k} & \longrightarrow & bo & \longrightarrow & X \quad K\mathbb{Z}_2 \end{array}$$

$b^k$ is the $k^{th}$ iterate of the Bott periodicity map.

Let $\phi_* : \pi_{n-8k} X_1^0 \longrightarrow \pi_n X_1^{2k}$ be the induced map. Then $\phi_*\alpha_1 = \alpha_{8k+1}$, $\phi_*\alpha_2 = \alpha_{8k+2}$ and $\phi_*\alpha_3 = 2^{4k}\cdot\alpha_{8k+3}$. Thus $\eta\alpha_{8k+3} = \alpha_{8k+2}$, $\eta^2\alpha_{8k+2} = 4\cdot 2^{4k}\cdot\alpha_{8k+3} = 2^{4k+2}\cdot\alpha_{8k+3}$. The result follows.

Corollary 6.4. For $n \leq 4k+2$

$$\pi_n B_2(4k+3) \cong \pi_n bo\ .$$

For $4k+3 \leq n \leq 8k+5$,

$$\pi_n B_2(4k+3) \cong \begin{cases} \mathbb{Z}_{2^{4j}} \times (\Lambda'/\Lambda' \cap \Lambda_{4k+3})_n & n = 8j-1 \\ \mathbb{Z}_2 \times (\Lambda'/\Lambda \cap \Lambda_{4k+3})_n & n = 8j+1,\ 8j+2 \\ \mathbb{Z}_{2^{4j+3}} \times (\Lambda'/\Lambda \cap \Lambda_{4k+3})_n & n = 8j+3 \\ (\Lambda'/\Lambda \cap \Lambda_{4k+3})_n & \text{otherwise} \end{cases}$$

Let $\alpha_n$ generate the cyclic summand of $\pi_n B_2(4k+3)$ that does not arise from the $\Lambda$-algebra and let $\eta$ generate $\pi_1 S^0$. Then $\eta\alpha_{8j+1} = \alpha_{8j+2}$ and $\eta^2\alpha_{8j+1} = 2^{4j+2}\alpha_{8j+3}$.

Proof: In Theorem 5.4, set $Z = S^0$ and then apply Lemma 6.3.

## 7. The Construction of $B_1(pk+1)$

In this section we outline the changes needed to construct the odd primary analogue of the spectra $B_1(2k+1)$; that is we will prove Theorem IV of the introduction. The techniques are the same and are modelled on Brown and Gitler [6] and we refer often to the R. L. Cohen's construction of odd primary Brown-Gitler Spectra ([14], Chapter I). Fix a prime $p \geq 3$. The following is Cohen's result. Cohomology has $\mathbb{Z}_p$ coefficients.

Theorem 7.1: There exist spectra $B(pk+1)$ so that

1) $H^*B(pk+1) = A/A\{\chi(\beta^{\varepsilon}p^i) : i > k, \varepsilon = 0, 1\}$

2) If $B(pk+1) \longrightarrow K\mathbb{Z}_p$ is the generator of $H^0B(pk+1)$, then, for every CW complex $Z$,

$$B(pk+1)_n Z \longrightarrow H_n Z$$

is surjective for $n \leq 2p(k+1) - 1$.

Note: Cohen calls the spectra $B(pk+1)$ by the name $B(k)$. However, his work implies the existence of spectra $B(pk)$ with

$$H^*B(pk) = A/A\{\chi(\beta^{\varepsilon}p^i) : i + \varepsilon > k\}$$

and

$$B(pk)_n Z \longrightarrow H_n Z$$

onto for $n \leq 2p(k+1) - 3$. Then $B(pk)$ would be analogous to $B(2k)$ and

B(pk+1) analogous to B(2k+1). This is the reason I have chosen the notation in Theorem 7.1. See also 7.4 below.

We wish to prove the following result.

Theorem 7.2: There exist spectra $B_1(pk+1)$ so that

1) $H^*B_1(pk+1) = A/A\{\beta, \chi(P^i) : i > k\}$

2) If $B_1(pk+1) \longrightarrow K\mathbb{Z}_p^\wedge$ is a generator of $H^0(B_1(pk+1); \mathbb{Z}_p^\wedge)$, then, for every CW complex Z,

$$B_1(pk+1)_n Z \longrightarrow H_n(Z; \mathbb{Z}_p^\wedge)$$

is onto for $n \leq 2p(k+1) - 1$.

3) There is a map $B_1(pk+1) \longrightarrow B(pk+1)$ inducing the obvious quotient in homology.

The first step is to provide resolutions of various modules over the Steenrod Algebra. Our tool, as before, in the odd primary $\Lambda$ algebra of [4]. We recall these results.

The p-primary $\Lambda$ algebra is a differential bigraded $\mathbb{Z}_p$ algebra with generators $\lambda_{n-1}$ $(n \geq 1)$ of bigrading $(2n(p-1)-1,1)$ and $\mu_{n-1}$ $(n \geq 0)$ of bigrading $(2n(p+1),1)$. Relations in $\Lambda$ are completely determined by

$$\begin{array}{ll}
\sum\binom{n}{k}\lambda_{n+m-k-1}\lambda_{pm+k-1} = 0 & n \geq 0,\ m \geq 1 \\
\sum\binom{n}{k}\mu_{n+m-k-1}\lambda_{pm+k-1} + \sum\binom{n}{k}\lambda_{n+m-k-1}\mu_{pm+k-1} = 0 & n \geq 0,\ m \geq 1 \\
\sum\binom{n}{k}\mu_{n+m-k-1}\lambda_{p(m+1)+k-1} = 0 & n \geq 0,\ m \geq 0 \\
\sum\binom{n}{k}\mu_{n+m-k-1}\mu_{p(m+1)+k-1} = 0 & n \geq 0,\ m \geq 0
\end{array}$$

and the differential is given by

$$\partial\lambda_{n-1} = \sum\binom{n}{k}\lambda_{n-k-1}\lambda_{k-1}$$

$$\partial\mu_{n-1} = \sum\binom{n}{k}\mu_{n-k-1}\lambda_{k-1} + \sum\binom{n}{k}\lambda_{n-k-1}\mu_{k-1}$$

As with the 2-primary case, one has certain conventions for this algebra. We write $\nu_n$ to mean either $\lambda_n$ or $\mu_n$ and, if $I = (i_1, \ldots, i_q)$ is a sequence of integers with $i_n \geq -1$, then we write

$$\nu_I = \nu_{i_1}, \ldots, \nu_{i_q}.$$

If $i_n = -1$, then $\nu_{i_n} = \mu_{-1}$. $\nu_I$ is admissible if

$$p(i_n+1) \geq \begin{cases} i_{n+1} + 2 & \nu_{i_n} = \lambda_{i_n} \\ i_{n+1} + 1 & \nu_{i_n} = \mu_{i_n} \end{cases}$$

Admissible monomials form an additive basis for $\Lambda$. The relations and differential may be expressed in terms of this basis. See Cohen [14].

We delineate a few ideals in $\Lambda$.

Lemma 7.3.1) Let $\Lambda_{pk} = \Lambda\{\mu_{-1}, \lambda_o, \mu, \ldots, \mu_{k-2}, \lambda_{k-1}\}$; this is the left ideal generated by $\mu_{-1}, \ldots, \lambda_{k-1}$. Then, an additive basis for $\Lambda_{pk}$ is $\nu_{i_1}, \ldots, \nu_{i_q}$ admissible with $i_q \leq k-2$ if $\nu_{i_q} = \mu_{i_q}$ or $i_q \leq k-1$ if $\nu_{i_q} = \lambda_{i_q}$.

2) Let $\Lambda_{pk+1} = \Lambda\{\mu_{-1}, \lambda_o, \mu_o, \ldots, \lambda_{k-1}, \mu_{k-1}\}$. Then, an additive basis for $\Lambda_{pk+1}$ is $\nu_{i_1}, \ldots, \nu_{i_q}$ admissible with $i_q \leq k-1$.

3) Let $\hat{\Lambda} = \Lambda\{\lambda_o, \lambda_1, \ldots, \lambda_b, \ldots\}$. Then an additive basis for $\hat{\Lambda}$ is $\nu_{i_1}, \ldots, \nu_{i_q}$ admissible with $\nu_{i_q} = \lambda_{i_q}$.

4) $\Lambda_{pk}$, $\Lambda_{pk+1}$, and $\hat{\Lambda}$ are closed under the differential.

Proof: 1) and 2) are in R. Cohen [14]. All the results are easy consequences of the relations as given in [14] and the differential as given above.

Recall that the dual dual mod p Steenrod Algebra is as an algebra;

$$A^* \simeq E(e_0, e_1, \ldots, ) \otimes \mathbb{Z}_p [t_1, \ldots] .$$

This is an exterior algebra on symbols $e_i$ tensor a polynomial algebra on generator $t_i$. $e_i$ and $t_i$ are the Hopf algebra conjugates Milnor's $\tau_i$ are $\xi_i$ respectively. See [29]. $e_i$ has grading $2p^i-1$ and $t_i$ has grading $2(p^i-1)$. The (conjugate) coproduct structure of $A^*$ is given by the formulas

$$\psi(t_i) = \sum t_j \otimes t_{i-j}^{p^j}$$

$$\psi(e_i) = \sum e_j \otimes t_{i-j}^{p^j} + 1 \otimes e_i$$

$A^*$ has a weight function w: given by the requirements that $w(t_i) = w(e_i) = p^i$, $w(e_0) = 1$, and $w(xy) = w(x) + w(y)$.

If $K\mathbb{Z}_p^\wedge$ is Eilenberg-MacLane Spectrum of the integers completed at p, then $H_*K\mathbb{Z}_p^\wedge \subseteq A^*$ (as a comodule) and is given by

$$H_*K\mathbb{Z}_p^\wedge = E(e_1, e_2, \ldots) \otimes \mathbb{Z}_p [t_1, \ldots] .$$

That is, an additive basis for $H_*K\mathbb{Z}_p^\wedge$ is all $x \in A^*$ so that $w(x) \equiv 0$ mod p.

We are concerned with three cyclic quotients of the Steenrod Algebra. Set

$$M(pk) = A/A\{\chi(\beta^{\varepsilon}P^{i}) : pi + \varepsilon > pk\}$$

$$M(pk+1) = A/A\{\chi(\beta^{\varepsilon}P^{i}) : i > k, \varepsilon = 0,1\}$$

$$M_1(pk+1) = A/A\{\beta, \chi(P^{i}) : i > k\} .$$

If M is a module over the Steenrod Algebra we give M* - the $\mathbb{Z}_p$ dual - the conjugate comodule structure.

Lemma 7.4: 1) $M(pk)^* \subseteq A^*$ as a comodule and has an additive basis consisting of monomials $x \in A^*$ with $w(x) \leq pk$.

2) $M(pk+1)^* \subseteq A^*$ as a comodule and has an additive basis of consisting monomials $x \in A^*$ with $w(x) \leq pk+1$.

3) $M_1(pk+1)^* \subseteq H_*K\mathbb{Z}_p^{\wedge}$ and has an additive basis consisting of monomials $x \in H_*K\mathbb{Z}_p$ with $w(x) \leq pk$.

Proof: 1) for instance, proceeds as follows. Let $\Lambda(1)$ be monomials of length one in $\Lambda$. Define a $\mathbb{Z}_p$ vector space map $\phi : A^* \longrightarrow \Lambda(1)$ by requiring that $\phi(t_1^n) = \lambda_{n-1}$, $\phi(e_0 t_1^n) = \mu_{n-1}$ and that $\phi$ applied to all other monomials is zero. Then let $\Lambda(1,pk)$ be monomials of length 1 in $\Lambda/\Lambda_{pk}$ and define a map $_1 : A^* —\ A^* \otimes\ (1,pk)$ by the composition

$$\partial_1 : A^* \xrightarrow{\psi} A^* \otimes A^* \longrightarrow A^* \otimes \Lambda(1) \longrightarrow A^* \otimes \Lambda(1,pk) .$$

One now checks that the kernel of $\partial_1$ is all monomials $x \in A^*$ with $w(x) \leq$ pk and that the following sequence is exact.

$$A \otimes \Lambda^*(1,pk) \longrightarrow A \longrightarrow M(pk) \longrightarrow 0 .$$

The others (2 and 3) are proved in a similar manner.

Now, let $\Lambda(q,pk)$ be monomials of length q in $\Lambda/\Lambda_{pk}$, and let $\Lambda(q,pk+1)$ be monomials of length q in $\Lambda/\Lambda_{pk+1}$, and let $\hat{\Lambda}(q,pk+1)$ be monomials of length q in $\hat{\Lambda}/\hat{\Lambda} \cap \Lambda_{pk+1}$.

Theorem 7.5: There exist comodule maps $\partial_q$ so that

$$1) \quad 0 \longrightarrow M(pk)^* \longrightarrow A^* \xrightarrow{\partial_1} A^* \otimes \Lambda(1,pk) \xrightarrow{\partial_2} \cdots$$
$$\longrightarrow A^* \otimes \Lambda(q-1,pk) \xrightarrow{\partial_q} A^* \otimes \Lambda(q,pk) \longrightarrow \cdots$$

is a resolution of $M(pk)^*$ by comodules over the dual Steenrod Algebra.

$$2) \quad 0 \longrightarrow M(pk+1)^* \longrightarrow A^* \xrightarrow{\partial_1} A^* \otimes \Lambda(1,pk+1) \longrightarrow \cdots$$
$$\longrightarrow A^* \otimes \Lambda(q-1,pk+1) \xrightarrow{\partial_q} A^* \otimes \Lambda(q,pk) \longrightarrow \cdots$$

is a resolution of $M(pk+1)^*$ by comodules over the dual Steenrod Algebra.

$$3) \quad 0 \longrightarrow M_1(pk+1)^* \longrightarrow H_*K\mathbb{Z}_p \xrightarrow{\partial_1} A^* \otimes \hat{\Lambda}(1,pk+1) \longrightarrow \cdots$$
$$\longrightarrow A^* \otimes \hat{\Lambda}(q-1,pk+1) \xrightarrow{\partial_q} A^* \otimes \hat{\Lambda}(q,pk) \longrightarrow \cdots$$

is a resolution of $M_1(pk+1)^*$ by comodules over the dual Steerod Algebra.

Note: 1) and 2) are in [14]. The methods of Section 2 suffice. One should refer to [14], Chapter I. Clearly, there is a map from resolution 3) to resolution 2) inducing the inclusion $M_1(pk+1)^* \subseteq M^*(pk+1)$.

Lemma 7.6: Let $\delta_q^*$ be the $\mathbb{Z}_p$ dual of $\partial_q$, then $\delta_q^*$ is given by:

$$\delta_q^* \bar{\nu}_I = \sum \bar{\nu}_I(\lambda_i \nu_J)\ \chi(P^{i+1})\ \bar{\nu}_J + \sum \bar{\nu}_I(\mu_i \nu_J)\ \chi(\beta P^{i+1})\ \bar{\nu}_J \ .$$

Here $\bar{\nu}_I$ is the $\mathbb{Z}_p$ dual of $\nu_I$, both sums are over $i \geq -1$, and $\lambda_{-1}\nu_I = \partial\nu_I$.

After producing these resolutions (7.5 and 7.6) the next step in our program is to extend Pontrjagin duality [5] to p-complete spectra. However, the reader sensitive to generalizations will have realized that there is nothing special about the prime 2 is our discussion is Section 3. We could just have easily used an arbitrary prime p. We need say no more.

Now, let $I_q$ be a spectrum so that $I_q$ is a product of Eilenberg-MacLane spaces of type $K\mathbb{Z}_p$ and $H^*I_q = A \otimes \Lambda^*(q,pk+1)$. $I_o = K\mathbb{Z}_p$. Similarly, let $F_q$ $(q \geq 1)$ be a spectrum so that $F_q$ is a product of Eilenberg-MacLane spaces so that $H^*F_q = A \otimes \hat{\Lambda}^*(q,pk+1)$. $F_o = K\hat{\mathbb{Z}}_p$. Then there exist maps $\delta_q : F_{q-1} \longrightarrow F_q$ and $\delta_q : I_{q-1} \longrightarrow I_q$ $(q \geq 1)$ so that $\delta_q^*$ is the map of 7.6. And, there exist maps $\omega_q : F_q \longrightarrow I_q$ so that $w_q^*$ is the obvious quotient. Then we have a commutative diagram

$$\begin{array}{ccccccc}
F_q & \xleftarrow{\delta_q} & F_{q-1} & \cdots\cdots \longleftarrow & F_1 & \xleftarrow{\delta_1} & F_o \\
\omega_q \downarrow & & \downarrow \omega_{q-1} & & \downarrow \omega_1 & & \downarrow \omega_o \\
I_q & \xleftarrow{\delta_q} & I_{q-1} & \cdots\cdots \longleftarrow & I_1 & \xleftarrow{\delta_1} & I_o
\end{array}$$

Let $\chi$ be the Pontrjagin duality functor. Set $L_q = \chi(I_q)$ and $K_q = \chi(F_q)$. If $d_q = \chi(\delta_q)$ and $w_q = \chi(\omega_q)$, then we have a diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow L_q & \xrightarrow{d_q} & L_{q-1} & \longrightarrow \cdots\cdots & L_1 & \longrightarrow & L_o = K\mathbb{Z} \\
w_q \downarrow & & \downarrow & & \downarrow w_1 & & \downarrow w_o \\
\cdots \longrightarrow K_q & \xrightarrow{d_q} & K_{q-1} & \longrightarrow \cdots\cdots & K_1 & \longrightarrow & K_o = K\mathbb{Z}_{p^\infty}
\end{array}$$

Referring to Section 3 for definitions and techniques of proof, we record the following result.

Lemma 7.7: Let $j = 2p(k+1) - 1$, then we have a diagram $(q \geq 1)$.

$$\begin{array}{ccc} L_{q,j-1} & \xrightarrow{d_1} & L_{q-1,j} \\ {\scriptstyle w_q}\downarrow & & \downarrow{\scriptstyle w_{q-1}} \\ K_{q,j-1} & \xrightarrow{d_1} & K_{q-1,j} \end{array}$$

Then $d_1 : L_{q,j-1} \longrightarrow L_{q-1,j}$ and $K_{q,j-1} \longrightarrow K_{q-1,j}$ are concurrently null-homotopic.

The fact that $d_1 : L_{q,j-1} \longrightarrow L_{q-1,j}$ is null-homotopic is in [14]. See the next result.

From R. Cohen's work we cull the following result.

Theorem 7.8: There is a sequence of spectra $E_q$, $q \geq 0$, and maps (of degree 1) $e_q : L_q \longrightarrow E_{q-1}$ satisfying

1) $E_o = L_o = K\mathbb{Z}_p$ and $e_1 = d_1$

2) $E_{q-1} \xrightarrow{i_q} E_q \xrightarrow{p_q} L_q$ is the fibration induced by $e_q$ from the contractible path fibration over $E_{q-1}$.

3) $L_q \xrightarrow{e_q} E_{q-1} \xrightarrow{p_{q-1}} L_{q-1}$ is $d_q$.

4) $e_q : L_{q,j-1} \longrightarrow E_{q-1,j}$ is null-homotopic $(j = 2p(k+1) - 1)$

5) Let $j_q : E_o = E_q$ be the composition of the map $i_s : E_{s-1} \longrightarrow E_s$. Then the sequence

$$[K\mathbb{Z}_p , E_o] \longrightarrow [K\mathbb{Z}_p , E_q] \longrightarrow [K\mathbb{Z}_p , L_q]$$

is exact.

We also have the following from [14].

<u>Theorem</u> <u>7.9</u>: $e_q$ is uniquely determined by 7.8.3 and 7.8.4.

So, now, referring to [14] for any other necessary lemmas, we can now prove the main result.

<u>Theorem</u> <u>7.10</u>: There are spectra $G_q$ and maps $e'_q : K_q \longrightarrow G_{q-1}$ of degree 1 and $\ell_q : E_q \longrightarrow G_q$ of degree zero so that

1) $G_o = K_o = K\mathbb{Z}_{p\infty}$ and $e'_1 = d_1$, $\ell_o = w_o$.

2) $G_{q-1} \xrightarrow{i_q} G_q \xrightarrow{p_q} K_q$ is the fibration induced by $e'_q$ from the contractible path fibration over $G_{q-1}$.

3) The following diagram commutes:

$$\begin{array}{ccccccc} E_{q-1} & \xrightarrow{i_q} & E_q & \xrightarrow{p_q} & L_q & \xrightarrow{e_q} & E_{q-1} \\ \downarrow{\scriptstyle \ell_{q-1}} & & \downarrow{\scriptstyle \ell_q} & & \downarrow{\scriptstyle w_q} & & \downarrow{\scriptstyle \ell_{q-1}} \\ G_{q-1} & \xrightarrow{i_q} & G_q & \xrightarrow{p_q} & K_q & \xrightarrow{e'_q} & G_{q-1} \end{array}$$

4) $K_q \xrightarrow{e_q} G_{q-1} \xrightarrow{p_{q-1}} K_{q-1}$ is $d_q$.

5) $e'_q : K_{q,j-1} \longrightarrow G_{q-1,j}$ is null-homotopic $(j = 2p(k+1) - 1)$.

We can, with 7.10, construct $B_1(pk+1)$.

Referring to 7.8, let $Y_q = \chi(E_q)$. Recall that $I_q = \chi(L_q)$. Then there is a tower

$$\begin{array}{ccccccccc} \cdots \longrightarrow & Y_q & \xrightarrow{p_q} & Y_{q-1} & \longrightarrow \cdots\cdots \longrightarrow & Y_1 & \xrightarrow{p_1} & Y_o = K\mathbb{Z}_p & \quad (7.11) \\ & \uparrow{\scriptstyle i_q} & & & & \uparrow{\scriptstyle i_1} & & & \\ & I_q & & & & I_1 & & & \end{array}$$

We assume each $p_q$ is a fibration and set $B(pk+1) = \varprojlim Y_q$. This is Cohen's construction. So, referring to 7.10, let $X_q = \chi(G_q)$. Recall that $F_q = \chi(K_q)$. Then there is a tower

$$\begin{array}{ccccccccc} \cdots \longrightarrow & X_q & \xrightarrow{p_q} & X_{q-1} & \longrightarrow \cdots\cdots \longrightarrow & X_1 & \xrightarrow{p_1} & X_o = K\mathbb{Z}_p \\ & {\scriptstyle i_q}\uparrow & & & & {\scriptstyle i_1}\uparrow & & \\ & F_q & & & & F_1 & & \end{array} \qquad (7.12)$$

Assume each $p_q$ is a fibration and that $i_q$ is the inclusion of the fiber. Then, set $B_1(pk+1) = \varprojlim X_q$. Obviously, 7.10.3 insures that there is a map from tower 7.12 to tower 7.11. Thus we have an induced map $B_1(pk+1) \longrightarrow B(pk+1)$.

We now record our last two results. This will complete the proof of 7.2 and, thus, of Theorem IV of the introduction.

Proposition 7.13: $H^*B_1(pk+1) = A/A\{\beta, \chi(P^i) : i > k\}$.

See Proposition 5.1

Theorem 7.14: Let Z be any CW complex. Consider the following sequence of homology theories ($q \geq 1$).

$$(F_q)_n Z \xrightarrow{i_{q*}} (X_q)_n Z \xrightarrow{p_{q*}} (X_{q-1})_n Z$$

Then:

1) This sequence is split short exact for $n \leq 2p(k+1) - 3$. The splittings are natural in Z.

2) The sequence is short exact for $n = 2p(k+1) - 2$.

3) If $n = 2p(k+1) - 1$, $p_{q*}$ is surjective.

This completes the proof of Theorem 7.2.

Clearly one can use 7.14 to compute $\pi_* B_1(pk+1)$ in a range - set $Z = S^0$. Define $(\hat{\Lambda}/\hat{\Lambda} \cap \Lambda_{pk+1})_n$ to be the elements of $\hat{\Lambda}/\hat{\Lambda} \cap \Lambda_{pk+1}$ of bigrading $(s,n)$ where $s \geq 0$.

Corollary 7.15: For $n \leq 2p(k+1) - 3$

$$\pi_n B_1(pk+1) = (\hat{\Lambda}/\hat{\Lambda} \cap \Lambda_{pk+1})_n \cdot$$

# Chapter II

## The Construction of $B_1(2k)$

The next five sections are devoted to the construction and analysis of the spectra $B_1(2k)$ promised by Theorem II of the Introduction. The techniques are substantially different from those of Chapter I. It is true, however, that we depend heavily on the results of Chapter I and an important step in this new chapter is a characterization and exploitation of the homotopy type of $B_1(2k+1)$. We begin - in Section 1 - with an outline of our ideas and proofs, but leave long proofs and a welter of details to the later sections. The final section discusses the application of $B_1(2k)$ to the study of 4k manifolds. We will often refer to results in Chapter I and will do so by number; the symbol I.4.3 refers to Lemma 3 of Section 4 of Chapter I.

### 1. Outline of the Proof

As we mentioned, our approach to the construction of the spectra $B_1(2k)$ differs from that of Brown and Gitler. To explain a bit, let $B(n)$ be the $n^{th}$ $\mathbb{Z}_2$-Brown-Gitler spectrum. Then, the work of Brown and Peterson [9] implies that there is a cofibration sequence

$$B(n-1) \longrightarrow B(n) \xrightarrow{j_n} \Sigma^n B[\tfrac{n}{2}] \qquad (1.1)$$

which induces, in cohomology, the sequence

$$0 \longrightarrow M[\tfrac{n}{2}] \xrightarrow{\chi(Sq^n)} M(n) \longrightarrow M(n-1) \longrightarrow 0.$$

Recall that $H^*B(n) = M(n) = A/A\{\chi(Sq^i) : i > n\}$. The maps $\rho$ and j are in no sense unique; in fact, if $\rho : B(n-1) \longrightarrow B(n)$ is any map that induces, in cohomology, a non-zero map $H^*B(n) \longrightarrow H^*B(n-1)$, then the cofiber of $\rho$ is $\Sigma^n B[\frac{n}{2}]$. See [9]. Similarly, one would hope that, given a suitable choice of $j_n$, there existed an $\rho$ so that (1.1) was a cofibration sequence. And, in fact, this is the case - at least for n even.

This, then, is our method. Fix $k \geq 1$. We produce a spectrum $C(k)$ and a map

$$j_k : B_1(2k+1) \longrightarrow \Sigma^{2k+1} C(k)$$

and <u>define</u> $B_1(2k)$ to be the spectrum so that

$$B_1(2k) \longrightarrow B_1(2k+1) \xrightarrow{j_k} \Sigma^{2k+1} C(k) \qquad (1.2)$$

is a cofibration sequence. Then there will be a short exact sequence in cohomology

$$0 \longrightarrow H^*C(k) \xrightarrow{j_k^*} M_1(2k+1) \longrightarrow H^*B_1(2k) = M_1(2k) \longrightarrow 0 \quad (1.3)$$

$H^*B_1(2k+1) \simeq M_1(2k+1) \simeq A/A\{Sq^1, \chi(Sq^i) : i > 2k\}$. Producing $C(k)$ and a map $j_k$ with the right homological properties is relatively easy, but that is not sufficient. Because we wish to discuss the homology theory $B_1(2k)_*$, we need a good hold on the homotopy theoretic properties of $j_k$. This requires a little more work.

To begin: (1.3) tells us what $H^*C(k)$ should be. Recall that $M_1(2k+1)$ has a $\mathbb{Z}_2$ basis consisting of elements $\chi(Sq^{2j}Sq^I)$ where $(2j,I)$ is admissible and $2j \geq 2k$. Furthermore, if $2k = 2^s(2t+1)$, then

$M_1(2k) = M_1(2^s(2t+1)) =$

$$A/A\{Sq^1,\ \chi(Sq^{2^s(2t+1)} \ldots Sq^{2^i(2t+1)}Sq^1),\ \chi(Sq^j) : j > 2k,\ i \geq 1\} .$$

Thus $M_1(2k)$ has a $\mathbb{Z}_2$ basis of elements

i) $\chi(Sq^{2j}Sq^I)$ with $(2j,I)$ admissible and $2j < 2k$.

ii) $\chi(Sq^{2^s(2t+1)} \ldots Sq^{2^i(2t+1)}Sq^{2j}Sq^I)$ with $(2j,I)$ admissible and $4j \leq 2^i(2t+1)$.

The following is an immediate consequence of this fact.

<u>Lemma 1.4</u>: Let $N(k)$ be the kernel of the projection $M_1(2k+1) \longrightarrow M_1(2k)$. Then, as modules over the Steenrod Algebra

$$N(k) \cong \underset{1<i\leq s}{X} M_1(2^{i-1}(2t+1) - 1) \otimes x_i \ X\, M_1(2t+1) \otimes x_1$$

where $x$ is a class of grading $(2^s + 2^{s-1} + \ldots + 2^i)(2t+1) + 1$. $(Sq^j x_i = 0,\ j > 0)$. The map from $N(k) \longrightarrow M_1(2k+1)$ is given by sending

$$x_i \longrightarrow \chi(Sq^{2s(2t+1)} \ldots Sq^{2^i(2t+1)}Sq^1) .$$

Lemma 1.4 now informs us what $C(k)$ should be; in fact, define $C(k)$ to be

$$C(k) = V\ \textstyle\sum^{a(i)}B_1(2^{i-1}(2t+1) - 1)\ v\ \sum^{a(1)}B_1(2t+1)$$

Here $a(i) = (2^{s-1} + \ldots + 2^i)(2t+1)$. Then $H^*\Sigma^{2k+1}C(k) = N(k)$.

The next step in the proof is to produce the map $j_k$ : $B_1(2k+1) \longrightarrow \sum^{2k+1} C(k)$. Because we are working in the stable category, we need only produce maps

$$f_j : B_1(2k+1) \longrightarrow \Sigma^{b(i)} B_1(2^{i-1}(2t+1) - 1) \qquad s > i > 1$$

and

$$f_1 : B_1(2k+1) \longrightarrow \Sigma^{b(1)} B_1(2t+1)$$

$b(i) = a(i) + 2k+1$.

Producing maps between Brown-Gitler spectra is, in theory, easy. Recall that $B(n)$ is defined to be the inverse limit of a tower of fibrations ($\varepsilon_q$ of degree one; that is $\varepsilon_q$ is a map $Y_{q-1} \longrightarrow \Sigma I_q$)

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & Y_q & \longrightarrow & Y_{q-1} & \longrightarrow \cdots \longrightarrow & Y_1 & \longrightarrow & Y_o = K\mathbb{Z}_2 \\ & & & {\scriptstyle \varepsilon_q}\downarrow & & & & {\scriptstyle \varepsilon_1}\downarrow \\ & & & I_q & & & & I_1 \end{array} \qquad (1.5)$$

$I_q$ is a product of Eilenberg-MacLane spectra of type $K\mathbb{Z}_2$; $H^*I_q = A \otimes \Lambda^*(q,n)$; and, if $i_q : I_q \longrightarrow Y_q$ is the inclusion of the fiber, then $\varepsilon_q \circ i_{q-1} = \delta_q$ where, in cohomology, $\delta_q^* : A \otimes \Lambda^*(q,n) \longrightarrow A \otimes \Lambda^*(q-1,n)$ is the map of I.2.13 (cf. I.2.9).

Similarly $B_1(2n+1)$ is the inverse limit of a tower of fibrations ($\varepsilon_q$ of degree one)

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & X_q & \longrightarrow & X_{q-1} & \longrightarrow \cdots \longrightarrow & X_1 & \longrightarrow & X_o = K\hat{\mathbb{Z}}_2 \\ & & & {\scriptstyle \varepsilon_q}\downarrow & & & & {\scriptstyle \varepsilon_1}\downarrow \\ & & & F_q & & & & F_1 \end{array} \qquad (1.6)$$

$F_q$ is a product of Eilenberg-MacLane spectra of type $K\mathbb{Z}_2$; $H^*F_q = A \otimes \hat{\Lambda}(q,2n+1)$ and, if $i_q : F_q \longrightarrow X_q$ is the inclusion of the fiber, then $\varepsilon_q \circ i_{q-1} = \delta_q$ where, in cohomology, $\delta_q^* : A \otimes \hat{\Lambda}^*(q,2n+1) \longrightarrow A \otimes \hat{\Lambda}^*(q-1,2n+1)$ is the map of I.2.13 (cf. I.2.11).

Suppose we wish to construct a map $g : B_1(2n+1) \longrightarrow \Sigma^p B(m)$ realizing a certain map $g^* : M(m) \longrightarrow M_1(2n+1)$. First, construct a map of resolutions

$$\begin{array}{ccccccccc}
\longrightarrow A \otimes \Lambda^*(q,m) & \xrightarrow{\delta_q^*} & \cdots \longrightarrow A \otimes \Lambda^*(1,m) & \xrightarrow{\delta_1^*} & A & \longrightarrow & M(m) \\
\downarrow g_q^* & & \downarrow g_1^* & & \downarrow g_0^* & & \downarrow g^* \\
\longrightarrow A \otimes \hat{\Lambda}^*(q,2n+1) & \xrightarrow{\delta_q^*} & \cdots \longrightarrow A \otimes \hat{\Lambda}^*(1,2n+1) & \xrightarrow{\delta_1^*} & A/ASq^1 & \longrightarrow & M_1(2n+1)
\end{array} \tag{1.7}$$

Then, because we are constructing maps into Eilenberg-MacLane spectra we may realize this with maps of spectra

$$\begin{array}{ccccccc}
\cdots \longleftarrow F_q & \xleftarrow{\delta_q} & F_{q-1} \longleftarrow \cdots & \longleftarrow & F_1 & \xleftarrow{\delta_1} & K\hat{\mathbb{Z}}_2 \\
\downarrow g_q & & \downarrow g_{q-1} & & \downarrow g_1 & & \downarrow g_0 \\
\cdots \longleftarrow I_q & \xleftarrow{\delta_q} & I_{q-1} \longleftarrow \cdots & \longleftarrow & I_1 & \xleftarrow{\delta_1} & K\mathbb{Z}_2
\end{array}$$

The verticle maps are of degree p, the horizontal maps are of degree one. The results of Section 2 will imply the following result.

<u>Theorem 1.8</u>: If $4n+1-p \leq 2n+1$, then there exist maps $\bar{g}_q : X_q \longrightarrow Y_q$ of degree p so that $\bar{g}_0 = g_0$ and thé following diagram commutes

$$\begin{array}{ccccccc}
F_q & \xrightarrow{i_q} & X_q & \longrightarrow & X_{q-1} & \xrightarrow{\varepsilon_q} & F_q \\
\downarrow g_q & & \downarrow \bar{g}_q & & \downarrow \bar{g}_{q-1} & & \downarrow g_q \\
I_q & \xrightarrow{i_q} & Y_q & \longrightarrow & Y_{q-1} & \xrightarrow{\varepsilon_q} & I_q
\end{array}$$

Using this result, we may define $g = \varprojlim \bar{g}_q : B_1(2n+1) \longrightarrow \Sigma^p B(m)$. The condition that $\bar{g}_0 = g_0$ implies that we have realized

$g^* : M(m) \longrightarrow M_1(2n+1)$.

In a similar way one may produce maps $B_1(2n+1) \longrightarrow \Sigma^p B_1(2m+1)$ for $4n+1-p \leq 4m+3$, and maps $B(n) \longrightarrow \Sigma^p B_1(2m+1)$ for $2n-p \leq 4m+3$, and maps $B(n) \longrightarrow \Sigma^p B(m)$ for $2n-p \leq 2m+1$.

We have taken this approach to constructing maps between Brown-Gitler spectra precisely because we will need to specify the maps

$$g_q^* : A \otimes \Lambda^*(q,m) \text{ — } A \otimes \hat{\Lambda}^*(q,2n+1)$$

(for example). If one is not worried about $g_q^*$, then one can produce many maps $\tilde{g} : B_1(2n+1) \longrightarrow \Sigma^p B(m)$ realizing, in cohomology, a given map $g^* : M(m) \longrightarrow M_1(2n+1)$. The techniques of Brown and Peterson [9] suffice. See Section 2.

Let us use 1.8 to produce quite a number of maps.

There is a projection map

$$M(2n) \simeq A/A\{\chi(Sq^i): \ i > 2n\} \longrightarrow A/A\{Sq^1, \chi(Sq^i): \ i > 2n\} \cong M_1(2n+1)$$

If we set $\delta_0^*$: $A \longrightarrow A/ASq^1$ and $\beta_q^* : A \otimes \Lambda^*(q,2m) \longrightarrow A \otimes \hat{\Lambda}^*(q,2m+1)$ to be the canonical projections, then I.2.13 assures us that (1.7) commutes; that is, $\delta_q^*\beta_q^* = \beta_{q-1}^*\delta_q^*$. Thus 1.8 gives us the following.

<u>Corollary 1.9</u>: There exists a map $i_{2n} : B_1(2n+1) \longrightarrow B(2n)$ inducing the projection $M(2n) \longrightarrow M_1(2n+1)$.

Call $i_{2n}$ the canonical inclusion $B_1(2n+1) \longrightarrow B(2n)$.

Corollary 1.9 has an immediate consequence. There are maps of of degree one, $Sq^1 : M_1(2n+1) \longrightarrow M(2n+1)$ and $Sq^1 : M_1(2n-1) \longrightarrow M(2n)$

both of which, of course, send the Steenrod Algebra generator to $Sq^1$. We will prove the following result in Section 2. It is also consequence of 1.8.

Corollary 1.10: 1) There exists a map $\phi_{2n+1} : B(2n+1) \longrightarrow \Sigma B_1(2n+1)$ inducing, in cohomology, $Sq^1 : M_1(2n+1) \longrightarrow M(2n+1)$.

2) There exists a map $\phi_{2n} : B(2n) \longrightarrow \Sigma B_1(2n-1)$ inducing, in cohomology, $Sq^1 : M_1(2n-1) \longrightarrow M(2n)$.

Now, consider the map $\chi(Sq^{2n}) : M(n) \longrightarrow M(2n)$. The next result is found in Section 3.

Lemma 1.11: There exist maps (of degree 2k)

$$\alpha_q^* : A \otimes \Lambda^*(q,n) \longrightarrow A \otimes \Lambda^*(q,2n)$$

so that $\delta_q^* \alpha_q^* = \alpha_{q-1}^* \delta_q^*$ and

1) $\alpha_0^* = \chi(Sq^{2n}) : A \longrightarrow A$

2) For $q \geq 1$, $\alpha_q^* \overline{\lambda}_I = \sum \chi(Sq^{p(I,J)})\, \overline{\lambda}_J$ with $p(I,J)$ an integer depending on I and J and so that

a) $p(I,J) \geq 2n - \dim\overline{\lambda}_I$

b) if $\lambda_J \in \hat{\Lambda}$, then $p(I,J) > 2n - \dim\overline{\lambda}_I$.

Corollary 1.12: There exists a map $h_{2n} : B(2n) \longrightarrow \Sigma^{2n}B(n)$ inducing, in cohomology, $\chi(Sq^{2n}) : M(n) \longrightarrow M(2n)$.

The strength of Lemma 1.11 lies in the following results.

Theorem 1.13: Let $\overline{B}(2n-1)$ be the spectrum so that

$$\overline{B}(2n-1) \longrightarrow B(2n) \xrightarrow{h_{2n}} \Sigma^{2n}B(n)$$

is a cofibration sequence. Then $\overline{B}(2n-1) \cong B(2n-1)$

Theorem 1.14: Let $i_{2n} : B_1(2n+1) \longrightarrow B(2n)$ be as in 1.9. Then, let $\overline{B}_1(2n+1)$ be the spectrum so that

$$\overline{B}_1(2n-1) \longrightarrow B_1(2n+1) \xrightarrow{h_{2n}\cdot i_{2n}} \Sigma^{n}B(n)$$

is a cofibration sequence. Then $\overline{B}_1(2n-1) \cong B_1(2n-1)$.

We now define the map $j_k : B_1(2k+1) \longrightarrow \Sigma^{2k+1}C(k)$. Define $f_i = \phi_{2^{i-1}(2t+1)} \circ h_{2^i(2t+1)} \circ \ldots \circ h_{2k} \circ i_{2k} : B_1(2k+1) \longrightarrow \Sigma^{b(i)}B_1(2^{j-1}(2t+1) - 1)$ for $s \geq i > 1$. $b(i) = (2^s + \ldots + 2^i)(2t+1) + 1$. Define $f_1 = \phi_{2t+1} \circ h_{4t+2} \circ \ldots \ldots \circ h_{2k} \circ i_{2k} : B_1(2k+1) \longrightarrow \Sigma^{b(1)}B_1(2t+1)$ $b(1) = (2^{s+1} - 2)(2t+1) + 1$. Finally, set

$$j_k = Vf : B_1(2k+1) \longrightarrow \Sigma^{2k+1}C(k)$$

The following is an immediate consequence of our definitions and 1.14.

Theorem 1.15: Let $B_1(2k)$ be the spectrum so that

$$B_1(2k) \longrightarrow B_1(2k+1) \xrightarrow{j_k} \Sigma^{2k+1}C(k)$$

is a cofibration sequence. Then

1) $H^*B_1(2k) = M_1(2k)$

2) There is a map $B_1(2k) \longrightarrow K\hat{\mathbb{Z}}_2$ so that for an CW complex Z

$$B_1(2k)_*Z \longrightarrow H_*(Z;\ \hat{\mathbb{Z}}_2)$$

is onto for $* \leq 4k-1$.

Here is how we will prove 1.13, 1.14, and 1.15. For instance, consider Theorem 1.14. From 1.8 and the discussion before 1.9 and from 1.11 we have maps $\bar{\alpha}_q \circ \bar{\beta}_q : X_q \longrightarrow Y_q^n$ of degree 2n and maps $\alpha_q \circ \beta_q$ : $F_q \longrightarrow I_q^n$ of degree 2n so that the following diagram commutes.

$$\begin{array}{ccccccc} F_q & \longrightarrow & X_q & \longrightarrow & X_{q-1} & \xrightarrow{\varepsilon_q} & F_q \\ \downarrow{\scriptstyle \alpha_q\circ\beta_q} & & \downarrow{\scriptstyle \bar{\alpha}_q\circ\bar{\beta}_q} & & \downarrow{\scriptstyle \bar{\alpha}_{q-1}\circ\bar{\beta}_{q-1}} & & \downarrow{\scriptstyle \alpha_q\circ\beta_q} \\ I_q^n & \longrightarrow & Y_q^n & \longrightarrow & Y_{q-1}^n & \xrightarrow{\varepsilon_q} & I_q^n \end{array}$$

$\varprojlim \bar{\alpha}_q \circ \bar{\beta}_q = h_{2n} \circ i_{2n}$. Let $W_q$ be the spectrum so that

$$W_q \longrightarrow X_q \xrightarrow{\bar{\alpha}_q\circ\bar{\beta}_q} Y_q$$

is a cofibration sequence, and let $J_q$ be the spectrum so that

$$J_q \longrightarrow F_q \xrightarrow{\alpha_q\circ\beta_q} I_q$$

is a cofibration sequence. Then we have a tower

$$\begin{array}{ccccccccccccc} \cdots \longrightarrow W_q & \longrightarrow & W_{q-1} & \longrightarrow & \cdots & \longrightarrow & W_1 & \longrightarrow & W_0 & \longrightarrow & K\hat{\mathbb{Z}}_2 \\ & & \downarrow{\scriptstyle \varepsilon'_q} & & & & & & \downarrow{\scriptstyle \varepsilon'_q} & & \\ & & J_q & & & & & & J_1 & & \end{array}$$

Then using Pontrjagin duality, what we know about $B_1(2k+1)$ and $B(n)$, and the form of $\alpha_q^*$ in 1.11 we will show that

$$(W_{q-1})_*Z \xrightarrow{\varepsilon'_{q*}} (J_q)_*Z$$

is zero for $* \leq 4n-1$. Further, $(\omega_0)_*Z \longrightarrow H_*(Z;\ \mathbb{Z}_2^\wedge)$ will be surjective for $* \leq 4n-1$. Thus, $H^*\overline{B}_1(2n-1) = M_1(2n-1)$ and

$$\overline{B}_1(2n-1)_*Z \longrightarrow H_*(Z;\ \mathbb{Z}_2^\wedge)$$

will be surjective for $* \leq 4n-1$. Then 1.14 follows from results in Section 2.

Of course, 1.15.2 skirts the issue. We are really interested in the image

$$B_1(2k)_{4k}Z \longrightarrow H_{4k}(X;\ \mathbb{Z}_2^\wedge)\ .$$

Section 5 addresses this question. We sketch the results here.

We defined maps $f_i : B_1(2k+1) \longrightarrow \Sigma^{b(i)}B_1(2^{i-1}(2t+1) - 1)$ and $f_i : B_1(2k+1) \longrightarrow \Sigma^{b(1)}B_1(2t+1)$ above. Let $1 : B_1(2n+1) \longrightarrow K\mathbb{Z}_2^\wedge$ classify a generator of $H^0(B_1(2n+1);\ \mathbb{Z}_2^\wedge)$. And let $Sq_i : K\mathbb{Z}_2 \longrightarrow {}^{b(i)}K\mathbb{Z}_2^\wedge$ be the unique map so that $1\circ f_i = Sq_i\circ 1$. Note that the composition $K\mathbb{Z}_2^\wedge \xrightarrow{Sq_i} \Sigma^{b(i)}K\mathbb{Z}_2^\wedge \longrightarrow \Sigma^{b(i)}K\mathbb{Z}_2$ is $\chi(Sq^{2^s(2t+1)} \ldots Sq^{2^i(2t+1)}Sq^2) \in H^*K\mathbb{Z}_2^\wedge$.

$Sq_i$ induces, for every CW complex, a map $Sq_{i*} : H_{4k}(Z;\ \mathbb{Z}_2^\wedge) \longrightarrow H_{4k-b(i)}(Z;\ \mathbb{Z}_2^\wedge)$. The following is fairly clear.

<u>Proposition 1.16</u>: If $x \in H_{4k}(Z;\ \mathbb{Z}_2^\wedge)$ is in the image of

$$B_1(2k)_{4k}Z \longrightarrow H_{4k}(Z;\ \mathbb{Z}_2^\wedge)$$

then $x \in \cap$ her $Sq_{i*}$.

We will show the following result.

Proposition 1.17: Let M be a closed orientable manifold and let $[M] \in H_{4k}(M; \mathbb{Z}_2^\wedge)$ be a choice for the fundamental class of M. Then $[M] \in \cap \ker Sq_{i*}$.

We also consider secondary obstructions. Let $X_0' = V\Sigma^{b(i)}K\mathbb{Z}_2^\wedge$ and let $Sq = VSq_i : K\mathbb{Z}_2^\wedge \longrightarrow X_0'$. Then $W_0$ is the spectrum so that

$$W_0 \xrightarrow{\pi} K\mathbb{Z}_2^\wedge \longrightarrow X_0'$$

is a cofibration sequence. An Adam relation calculation shows that

$$K\mathbb{Z}_2^\wedge \xrightarrow{Sq_i} \Sigma^{b(j)}K\mathbb{Z}_2^\wedge \xrightarrow{\chi(Sq^{2^{i-1}(2t+1)})} \Sigma^{b(j)+2^{i-1}(2t+1)}K\mathbb{Z}_2$$

is zero in $H^*K\mathbb{Z}_2^\wedge$ $(2 \le i \le s)$. Thus there exists a cohomology class

$$\psi_i : W_1 \rightarrow \Sigma^{c(i)}K\mathbb{Z}_2 \tag{1.18}$$

$(c(i) = (2^s + \dots + 2^i + 2^{i-1})(2t+1))$ and a diagram

$$\begin{array}{ccccccc} \Sigma^{b(i)-1}K\mathbb{Z}_2^\wedge & \longrightarrow & \Sigma^{-1}X_0^1 & \longrightarrow & W_1 & \xrightarrow{\psi_i} & \Sigma^{c(i)}K\mathbb{Z}_2 \\ & & & & \downarrow{\scriptstyle\pi} & & \\ & & & & K\mathbb{Z}_2^\wedge & \longrightarrow & X_0' \end{array}$$

The composition across the top is $\chi(Sq^{2^{i-1}(2t+1)})$.

If $x \in \cap \ker Sq_{i*} \subseteq H_*(Z; \mathbb{Z}_2^\wedge)$, then we may define a secondary homology operation $\Psi_i$ on x as follows. Pick $y \in (W_1)_*Z$ so that $\pi_*y = x$. Then set $\Psi_i(x) \in H_*Z$ to be $\psi_{i*}(y)$. $\Psi_i(x)$ has indeterminacy $\chi(Sq^{2^{i-1}(2t+1)})_*H_*(Z; \mathbb{Z}_2^\wedge)$.

Note that 1.17 implies that $\Psi_i[M]$ is defined for all oriented manifolds M and all i, $2 \leq i \leq s$.

Theorem 1.19: There exist a choice for $\psi_i$ (1.18) so that if $x \in \cap$ her $Sq_{i*} \subseteq H_{4k}(Z; \mathbb{Z}_2^{\wedge})$ , then x is in the image of

$$B(2k)_{4k}Z \longrightarrow H_{4k}(X; \mathbb{Z}_2^{\wedge})$$

only if $\Psi_i(x)$ is zero modulo indeterminacy.

Proposition 1.20: Let $[\mathbb{C}P^{2k}] \in H_{4k}(\mathbb{C}P^{2k}; \mathbb{Z}_2^{\wedge})$ be a choice for the fundamental class. Then, for all, $\Psi_i[\mathbb{C}P^{2k}] = 0$ modulo indeterminacy.

We refer the reader to Section 5 for proofs.

The upshot is this: if $2k = 2^s(2t+1)$ and $x \in H_{4k}(Z; \mathbb{Z}_2^{\wedge})$ then there are s primary obstructions and s-1 secondary obstructions and (possibly) higher obstructions to x being in the image of $B_1(2k)_{4k}\, Z \longrightarrow H_{4k}(Z; \mathbb{Z}_2^{\wedge})$ . This can be refined if s = 1.

Theorem 1.21: $B_1(4k+2)_*Z \longrightarrow H_*(Z; \mathbb{Z}_2^{\wedge})$ is onto the kernel of $Sq_{1*}$ for $* \leq 8k+4$. For $* < 8k+4$, $\ker Sq_{1*} = H_*(Z; \mathbb{Z}_2^{\wedge})$ .

## 2. The Homotopy Type of $B_1(2k+1)$

In this section, we give a complete characterization of the homotopy type of $B_1(2k+1)$. In the process, we display a strong connection between the spectra $B_1(2k+1)$ and the stable homotopy theory of manifolds. The concepts here are all in Brown and Peterson [9].

The first idea centers on the $B_1(2k+1)$-cohomology of the Thom spectrum of the stable normal bundle of a closed, differentiable, orientable manifold M. Let $T(\nu)$ denote the Thom spectrum of the normal bundle. We assume that the Thom class $\mu \in H^0T(\nu)$. It is well-known that $T(\nu)$ is the Spanier-Whitehead dual of $M^+$ (+ = disjoint basepoint). Thus, if M is an n-manifold

$$Y^pT(\nu) = [T(\nu), Y]^p \simeq \pi_{n-p}\, Y_\wedge M^+ \simeq Y_{n-p}M^+$$

for any spectrum Y.

Recall that $B_1(2k+1)$ was the inverse limit of a tower of fibrations

$$\begin{array}{ccccccccc} \cdots \longrightarrow & X_q & \xrightarrow{p_q} & X_{q-1} & \longrightarrow \cdots \longrightarrow & X_1 & \longrightarrow & X_0 = K\hat{\mathbb{Z}}_2 & \\ & & & \downarrow{\scriptstyle \varepsilon_q} & & & & \downarrow{\scriptstyle \varepsilon_1} & \qquad (2.1) \\ & & & F_q & & & & F_1 & \end{array}$$

Each $p_q$ is the fibration induced by $\varepsilon_q$ from the contractible path fibration over $F_q$. $\varepsilon_q$ is of degree one, and if $i_q : F_q \longrightarrow X_q$ is the inclusion of the fiber, then $\varepsilon_q \circ i_{q-1}$ is, in cohomology, the map

$$H^*F_q = A \otimes \hat{\Lambda}^*(q,2k+1) \xrightarrow{\delta_q^*} A \otimes \hat{\Lambda}^*(q-1,2k+1) \ .$$

of Section I.2. $F_q$ is a product of Eilenberg-MacLane spaces of type $K\mathbb{Z}_2$. Our first result is this.

Theorem 2.2: Let M be an n-manifold and $v \in H^pT(\nu)$ be a cohomology class so that

1) $n-p \leq 4k+3$

2) v is the reduction of an integral cohomology class.

Then, there is an induced map $v : T(\nu) \longrightarrow \Sigma^p K\mathbb{Z}_2^\wedge = \Sigma^p X_0$ and any lifting of v to $\Sigma^p X_{q-1}$ lifts to $\Sigma^p X_q$.

Proof: By Theorem I.5.2

$$(X_q)_{n-p}M^+ \longrightarrow (X_{q-1})_{n-p}M^+$$

is surjective for $n-p \leq 4k+3$. Thus, by Spanier-Whitehead duality

$$(X_q)^pT(\nu) \longrightarrow (X_{q-1})^pT(\nu)$$

is surjective. The result follows.

We now wish to associate to $B_1(2k+1)$ a particular 4k+1 manifold. Let MSO be the Thom spectrum of the universal bundle over BSO - the classifying space for oriented bundles. Then $H^0(MSO; \mathbb{Z}_2^\wedge) \cong \mathbb{Z}_2^\wedge$. Choose a generator for this group:

$$\mu : MSO \longrightarrow K\mathbb{Z}_2^\wedge$$

Lemma 2.3: For n = 4k+1, 4k+2, or 4k+3 there exists an orientable n-manifold $Q_n$, so that for any orientation of the normal bundle of g

$$g : T(\nu) \longrightarrow MSO$$

the following sequence is exact:

$$0 \longrightarrow A\{Sq^1, \chi(Sq^i): \; i > 2k\} \longrightarrow A/ASq^1 \xrightarrow{(\mu g)^*} H^*T(\nu)$$

<u>Proof</u>: According to Brown and Peterson [7], for each $\alpha \in A/ASq^1$ so that $\alpha$ maps to a non-zero element of $A/A\{Sq^1, \chi(Sq^i): \; i > 2k\}$, there exists an n-manifold M(a) so that for any orientation

$$T(\nu_{M(\alpha)}) \xrightarrow{g} MSO$$

of the normal bundle of $M(\alpha)$, $(ug)^*\alpha \neq 0$. Then

$$Q_n = \underset{\alpha}{\#} M(\alpha)$$

where # denotes the connected sum. The result follows.

So let $\mu_n \in H^*T(\nu)$ be the Thom class of the normal bundle of $Q_n$. Then $\mu_n$ is the reduction of an integral class; in fact $\mu \circ g$ (above) factors $\mu_n : T(\nu) \longrightarrow K\mathbb{Z}_2$ through $K\hat{\mathbb{Z}}_2$. $Q_n$ is of dimension less than or equal to 4k+3; therefore, 2.2 implies that there is a factoring

$$\begin{array}{ccc} & & B_1(2k+1) \\ & \nearrow & \downarrow \\ T(\nu) & \xrightarrow{\mu\circ g} & K\hat{\mathbb{Z}}_2 \end{array}$$

There are many choices for $\tilde{g}$, but, for each choice $\tilde{g}^*$ is an injection.

Theorem 2.4: Let $T(\nu)$ be the Thom spectrum of $Q_{4k+1}$ and let $\mu\circ g : T(\nu) \longrightarrow K\mathbb{Z}_2^{\wedge}$ be an orientation of the Thom class of $T(\nu)$. If $g_{q-1}$ is any lifting of $\mu\circ g$ to $X_{q-1}$, then $\varepsilon_q : X_{q-1} \longrightarrow F_q$ is the unique map so that $(\varepsilon_q \circ i_{q-1})^* = \delta_q^*$ and $\varepsilon_q g_{q-1} = 0$.

Recall that $i_{q-1} : F_{q-1} \longrightarrow X_{q-1}$ is the inclusion of the fiber.

Proof: $\varepsilon_q g_{q-1} = 0$ by Theorem 2.2. In the proof of I.5.1 we demonstrated that the following sequence is exact:

$$0 \longrightarrow M_1(2k+1) \longrightarrow H^*X_{q-1} \xrightarrow{i^*_{q-1}} H^*F_{q-1}$$

The map $g^*_{q-1} : H^*X_{q-1} \longrightarrow H^*T(\nu)$ factors through $M_1(2k+1)$ and, hence, splits the above exact sequence. Therefore

$$\varepsilon_q^* : H^*F_q \longrightarrow H^*X_{q-1}$$

(and, hence, $\varepsilon_q$) is uniquely determined by the conditions that $(\varepsilon_q \circ g_{q-1})^* = 0$ and $(\varepsilon_q i_{q-1})^* = \delta_q^*$.

2.4 completely characterizes the maps $\varepsilon_q$ and, hence, the spectrum $B_1(2k+1)$. We now discuss manifolds adapted to a homology class. The triple $(Q_n, v, B_1(2k+1))$ is a key example; here $Q_n$ is as in 2.3 and $v \in H^*B_1(2k+1)$ is the Steenrod Algebra generator.

Definition 2.5: Suppose W is a spectrum so that there exists a manifold of dimension n and a map $\gamma : T(\nu_M) \longrightarrow W$ (not necessarily of degree zero). Suppose $v \in H^pW$. Then $(M, v, W)$ is adapted to $B_1(2k+1)$ if

1) $\gamma^* : H^*W \longrightarrow H^*T(\nu_M)$ is injective
2) $n-p \leq 4k+3$

and

3) v is the reduction of an integral class.

Theorem 2.6: Suppose (W, v, M) is adapted to $B_1(2k+1)$. There there is an induced map $v : W \longrightarrow \Sigma^p K\mathbb{Z}_2^\wedge$. Any lifting of v to $\Sigma^p X_{q-1}$ lifts to $\Sigma^p X_q$.

Proof: Suppose $\tilde{v}$ is a lifting of v to $\Sigma^p X_{q-1}$. Then we have a diagram

$$\begin{array}{ccccccc} & & & & \Sigma^p X_{q-1} & \xrightarrow{\varepsilon_q} & \Sigma^{q+1} F_q \\ & & & \overset{\tilde{v}}{\nearrow} & \downarrow & & \\ T(\nu) & \xrightarrow{\gamma} & W & \xrightarrow{v} & \Sigma^p K\mathbb{Z}_2^\wedge & & \end{array}$$

We wish to know whether $\varepsilon_q \tilde{v} = 0$. But, by 2.2 $\varepsilon_q \tilde{v}\gamma = 0$. Since $F_q$ is a product of Eilenberg-MacLane spectra of type $K\mathbb{Z}_2$ and because $\gamma^* : H^*W \longrightarrow H^*T(\nu)$ is injective, we conclude that $\varepsilon_q \tilde{v} = 0$.

Corollary 2.7: $B_1(2k+1)$ is the unique two-complete spectrum so that

1) $H^*B_1(2k+1) \simeq A/A\{Sq^1, \chi(Sq^i): \ i > 2k\}$.

2) For any CW complex Z,

$$B_1(2k+1)_n Z \longrightarrow H_n(Z; \mathbb{Z}_2^\wedge)$$

is surjective for $n \leq 4k+3$.

Note: Corollary 2.8 implies that the spectra constructed by Shimamoto [32] are homotopy equivalent to the spectra $B_1(2k+1)$; at least, after they are completed at two.

Corollary 2.8.1) There is a map $f : B_1(2k-1) \longrightarrow B_1(2k+1)$ inducing the quotient $M_1(2k+1) \longrightarrow M_1(2k-1)$.

2) Let $\overline{B}(k)$ be the cofiber of f; that is, there is a cofibration sequence

$$B_1(2k-1) \xrightarrow{f} B_1(2k+1) \longrightarrow \Sigma^{2k}\overline{B}(k)$$

Then $\overline{B}(k)$ is homotopy equivalent to $B(k)$.

Proof: 1) Follows from 2.3 and 2.6. 2) follows from 2.3 and the analogue of 2.6 found in Brown and Peterson [9]. Note that if $T(\nu)$ is the Thom spectrum of $Q_{4k+1}$ then the composition

$$T(\nu) \longrightarrow B_1(2k+1) \longrightarrow \Sigma^{2k}\overline{B}(k)$$

shows that $(Q_{4k+1}, u, \overline{B}(k))$ is adapted to $B(k)$. u is the Steenrod Algebra generator of $H^*\overline{B}(k) = M(k)$. This completes the proof.

Note that one could use 2.6 and its analogue in [9] to produce a plethora of maps between Brown-Gitler spectra, but any map produced by this method is in no sense unique or nicely characterized. We devote the next section to tightening our methods.

3. Maps Between Brown-Gitler Spectra.

The outline of the construction of the spectra $B_1(2k)$ given in section 1 indicated that we would construct some very specific maps between Brown-Gitler type spectra. We do that in this section, producing all the maps claimed and, at the end, prove the algebraic Lemma 1.11.

The situation is this. Let $\tilde{B}(n)$ stand for $B(n)$ or $B_1(n)$, as the case may be, and let $H^*\tilde{B}(n) = \tilde{M}(n)$. Suppose we wish to construct a map $g : \tilde{B}(n) \longrightarrow \tilde{B}(m)$ realizing a certain map $g^* : \tilde{M}(m) \longrightarrow \tilde{M}(n)$. The methods of 2.6 will give us one; however, suppose further that we have a map of resolutions (cf. I.2)

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & A \otimes \tilde{\Lambda}^*(q,m) & \xrightarrow{\delta_q^*} & \cdots \longrightarrow & A \otimes \tilde{\Lambda}^*(1,m) & \longrightarrow & \tilde{A} & \longrightarrow & M(m) \\
& \downarrow g_q^* & & & g_1^* \downarrow & & \downarrow g_0^* & & \downarrow g^* \\
\cdots \longrightarrow & A \otimes \tilde{\Lambda}^*(g,n) & \xrightarrow{\delta_q^*} & \cdots \longrightarrow & A \otimes \tilde{\Lambda}^*(1,m) & \longrightarrow & \tilde{A} & \longrightarrow & M(n)
\end{array}$$

where $\tilde{\Lambda} = \Lambda$ or $\hat{\Lambda}$ and $\tilde{A} = A$ or $A/A\,Sq^1$ depending on whether $\tilde{M}(n) = M(n)$ or $M_1(m)$. Because we are constructing maps into Eilenberg-MacLane spectra we may realize this resolution with maps of spectra

$$\begin{array}{ccccccccc}
\cdots \longleftarrow & J_q^n & \xleftarrow{\delta_q} & J_{q-1}^n & \longleftarrow \cdots \longleftarrow & J_1^n & \xleftarrow{\delta_q} & J_0^n \\
g_p & \downarrow & g_{p-1} & \downarrow & & \downarrow g_1 & & \downarrow g_0 \\
\cdots \longleftarrow & J_q^m & \xleftarrow{\delta_q} & J_{q-1}^m & \longleftarrow \cdots \longleftarrow & J_1^m & \xleftarrow{\sigma_1} & J_0^m
\end{array}$$

Where, of course $J_q^n = I_q$ or $F_q$ depending whether $\tilde{B}(n) = B(n)$ or $B_1(n)$. The verticle maps of degree $p$, the horizonal maps of degree one. The main result is the following. Let $W_q^n = X_q$ or $Y_q$ (for $B_1(n)$ or $B(n)$).

<u>Theorem 3.1</u>. Under conditions stated below, there exist maps $\bar{g}_q : W_q^n \longrightarrow W_q^m$ of degree $p$ so that $\bar{g}_0 = g_0$ and the following diagram commutes

$$\begin{array}{ccccccc} J_q^n & \xrightarrow{i_q} & W_q^n & \longrightarrow & W_{q-1}^n & \xrightarrow{\varepsilon_q} & J_q^n \\ \downarrow{\scriptstyle g_p} & & \downarrow{\scriptstyle \bar{g}_q} & & \downarrow{\scriptstyle \bar{g}_{q-1}} & & \downarrow{\scriptstyle g_q} \\ J_q^m & \xrightarrow{i_q} & W_q & \longrightarrow & W_{q-1} & \xrightarrow{\varepsilon_q} & J_q^m \end{array} \qquad (3.2)$$

The conditions we demand are:

1) $2n-p \leq 2m+1$ if $\tilde{B}(n) = B(n)$ and $\tilde{B}(m) = B(m)$
2) $2n-p \leq 4m'+3$ if $\tilde{B}(n) = B(n)$ and $\tilde{B}(m) = B_1(2m'+1)$
3) $4n'+1-p \leq 2m+1$ if $\tilde{B}(n) = B_1(2n'+1)$ and $\tilde{B}(m) = B(m)$
4) $4n'+1-p \leq 4m'+3$ if $\tilde{B}(n) = B_1(2n'+1)$ and $\tilde{B}(m) = B_1(2m'+1)$

<u>Proof</u>. We will do 3. The rest are similar, citing results from Brown and Peterson [9]. Write $W_q^m = Y_q$, $J_q^m = I_q$, $W_w^{2n'+1} = X_q$ and $J_q^{2n'+1} = F_q$. We have specified $\bar{g}_0 = g_0 : X_0 = K\hat{\mathbb{Z}}_2 \to K\mathbb{Z}_2 = Y_0$. Because we demand that (3.2) commute, we have defined $\bar{g}_1$.

Proceeding inductively, suppose $(q \geq 1)$ $\bar{g}_q : X_q \to Y_q$ is defined. Then we show that the following diagram commutes.

$$\begin{array}{ccc} X_q & \xrightarrow{\varepsilon_{q+1}} & F_{q+1} \\ \bar{g}_q \downarrow & & \downarrow g_{q+1} \\ Y_q & \xrightarrow{\varepsilon_{q+1}} & I_{q+1} \end{array}$$

Then, (3.2) will define $\bar{g}_{q+1}$ .

Let $\Delta = g_{q+1}\, \varepsilon_{q+1} - \varepsilon_{q+1}\, \gamma_q$ . We wish to show that $\Delta = 0$ . Since $I_{q+1}$ is a product of Eilenberg-MacLane spaces of type $K\mathbb{Z}_2$ , it is enough to show that $\Delta^* = 0$ .

Recall (I.5.1) that

$$0 \longrightarrow M_1(2n'+1) \longrightarrow H^*X_q \xrightarrow{i_q^*} H^*F_q \qquad (3.3)$$

is exact. Let $T(v)$ be the Thom spectra of the normal bundle of $Q_{4k+1}$ (2.3) and let $f_q : T(v) \longrightarrow X_q$ be any lifting of any orientation of $T(v)$ : $f : T(v) \longrightarrow K\hat{\mathbb{Z}}_2 = X_0$ . Then since $f_q^* : H^*X_q \longrightarrow H^*T(v)$ factors through $M_1(2n'+1)$ , $f_q^*$ splits (3.3). So to show that $\Delta^* = 0$ we need only show that $i_q^*\Delta^* = 0$ and $f_q^*\Delta^* = 0$ .

But $\Delta i_q = g_{q+1}\, \delta_{q+1} - \delta_{q+1}\, g_q = 0$ .

And, $g_{q+1}\varepsilon_{q+1\ q} = 0$ by 2.2; $\varepsilon_{q+1}\bar{g}_q f_q = 0$ by the analogue of 2.2 for $B(m)$ . Recall that $\bar{g}_q$ is a map of degree $p$ . Thus $\Delta g_q = 0$ . So, we may conclude that $\Delta^* = 0$ and, thus, that $\Delta = 0$ .

This completes the proof of 3.1.

Let $\beta_q^* : A \otimes \Lambda^*(q,2k) \longrightarrow A \otimes \hat{\Lambda}^*(q\ 2k+1)$ and $\beta_0^* : A \longrightarrow A/A\,Sq^1$ be the quotients. Clearly $\delta_q^* \beta_q^* = \beta_{q-1}^* \delta_q^*$ . $\beta_q^*$ can be realized by a map (of degree zero) $\beta_q : F_q \longrightarrow I_q$ . The following is an immediate consequence of 3.1.

Corollary 3.4. There exist maps $\overline{\beta}_0 : X_q \longrightarrow Y_q$

1) $\overline{\beta}_0 = \beta_0 : K\hat{\mathbb{Z}}_2 \longrightarrow K\mathbb{Z}_2$

2) The following diagram commutes

$$\begin{array}{ccccccc}
F_q & \longrightarrow & X_q & \longrightarrow & X_{q-1} & \xrightarrow{\varepsilon_q} & F_q \\
\beta_q \downarrow & & \downarrow \overline{\beta}_q & & \downarrow \overline{\beta}_{q-1} & & \downarrow \beta_q \\
I_q & \longrightarrow & Y_q & \longrightarrow & Y_{q-1} & \xrightarrow{\varepsilon_q} & I_q
\end{array}$$

3) If $i_{2k} = \varprojlim \overline{\beta}_q : B_1(2k+1) \longrightarrow B(2k)$ the $i_{2k}^*$ is the quotient $M(2k) \longrightarrow M_1(2k+1)$

We extract the following from Brown and Peterson [9].

Proposition 3.5. There exists a manifold $P_{2n}$ of dimension $2n$ so that if $T(v)$ is the Thom spectrum of the normal bundle of $P_{2n}$ , then there is a map of degree zero $\gamma_{2n} : T(v) \longrightarrow B(n)$ that is injective in cohomology.

The methods 2.3 would suffice to prove 3.5.

Corollary 3.6. There is a map of degree one $\phi_{2k} : B(2k) \longrightarrow \Sigma B_1(2k-1)$ inducing, in cohomology, $Sq^1 : M_1(2k-1) \longrightarrow M(2k)$ .

Proof. Let CF be the cofiber of $i_{2k} : B_1(2k+1) \longrightarrow B(2k)$ . Then, let $v \in H^1 CF$ be the element which maps, under

$$H^* CF \longrightarrow H^* B(2k)$$

to $Sq^1 \in M(2k)$ . Then, $(P_{4n}, v, CF)$ is adapted to $B_1(2k-1)$ . The result follows from 2.6.

The next result was first noticed by Mahowald [19].

Corollary 3.7. Let $M(2)$ be the $\mathbb{Z}_2$ Moore spectrum. Then there is a homotopy equivalence

$$B_1(2k+1) \wedge M(2) \longrightarrow B(2k+1) \quad .$$

There is a cofiber sequence

$$B_1(2k+1) \longrightarrow B_1(2k+1) \longrightarrow B(2k+1) \xrightarrow{\phi_{2k+1}} \Sigma B_1(2k+1)$$

In cohomology $\phi^*_{2k+1} = Sq^1 : M_1(2k+1) \longrightarrow B(2k+1)$ .

Proof. $B(1) = M(2)$ . A cohomology calculation shows that the following composition is a cohomology isomorphism and, thus, a homotopy equivalence.

$$B_1(2k+1) \wedge M(2) \xrightarrow{i_{2k} \wedge id} B(2k) \wedge B(1) \longrightarrow B(2k+1) \quad .$$

$m$ is the multiplication given by the work of [9]. The results follow.

As an aside, but as a corollary to 3.7, we have the following result. See [31]. Recall that $(\Lambda^{\wedge}/\Lambda^{\wedge} \cap \Lambda_{2k+1})_n$ is the $\mathbb{Z}_2$ vector space of symbols $\lambda_{i_1} \ldots \lambda_{i}$ in $\Lambda^{\wedge}/\Lambda^{\wedge} \cap \Lambda_{2k+1}$ with $i_1 + \ldots + i_q = n$ .

<u>Theorem 3.8</u>. $\pi_n B_1(2k+1) \cong (\Lambda^{\wedge}/\Lambda^{\wedge} \cap \Lambda_{2k+1})_n$ for $1 \leq n \leq 4k+2$ .

<u>Proof</u>. For $n < 4k+2$ this is I.6.1. An argument similar to I.6.1 shows that $\pi_{4k+2}\, B_1(2k+1) \cong (\Lambda^{\wedge}/\Lambda^{\wedge} \cap \Lambda_{2k+1})_{4k+2}$ as sets. Similarly, we can conclude that

$$\pi_{4k+2}B(2k+1) = (\Lambda/\Lambda_{2k+1})_{4k+2} \qquad \text{as sets.}$$

Thus from 3.7 and the choice of the inclusion $B_1(2k+1) \longrightarrow B(2k+1)$ we see that $\pi_{4k+2}B_1(2k+1)$ injects into $\pi_{4k+2}B(2k+1)$ and that the image of $\pi_{4k+2}B_1(2k+1) \longrightarrow \pi_{4k+2}B(2k+1)$ is of order 2. The result follows.

We now need only construct the map $h_{2k} : B(2k) \longrightarrow \Sigma^{2k}B(k)$ inducing $\chi(Sq^{2k}) : M(k) \longrightarrow M(2k)$ in cohomology. Let $\alpha_q^*$ be the maps of Lemma 1.11 or Lemma 3.10 below. Then 3.1 immediately implies the following. $Y_q^k$ is the $k^{\underline{th}}$ stage for $B(k)$ .

Corollary 3.9. There exist maps of degree $2k$ ; $\alpha_q : I_q^{2k} \longrightarrow I_q^k$ and $\overline{\alpha}_q : Y_q^{2k} \longrightarrow Y_q^k$ so that $\alpha_q^*$ is as in 3.10 and

1) $\overline{\alpha}_0 = \alpha_0 = \chi(Sq^{2k}) : K\mathbb{Z}_2 \longrightarrow K\mathbb{Z}_2$

2) The following diagram commutes

$$\begin{array}{ccccccc} I_q^{2k} & \longrightarrow & Y_q^{2k} & \longrightarrow & Y_{q-1}^{2k} & \xrightarrow{\varepsilon_q} & I_q^{2k} \\ \downarrow{\scriptstyle \alpha_q} & & \downarrow{\scriptstyle \overline{\alpha}_q} & & \downarrow{\scriptstyle \overline{\alpha}_{q-1}} & & \downarrow{\scriptstyle \alpha_q} \\ I_q^k & \longrightarrow & Y_q & \longrightarrow & Y_{q-1} & \xrightarrow{\varepsilon_q} & I_q \end{array}$$

3) If $h_k = \varprojlim \overline{\alpha}_q : B(2k) \longrightarrow \Sigma^{2k}B(k)$ , then

$$h_k^* = \chi(Sq^{2k}) : M(k) \longrightarrow M(2k) .$$

This leaves the construction of $\alpha_q$ , accomplished with the following argument. To lighten notation we write $\overline{\lambda}_I = \lambda_I$ and understand that we are working in the dual vector space. Remember that $\lambda_I(\lambda_J) = \overline{\lambda}_I(\lambda_J)$ .

Lemma 3.10. There exist maps (of degree 2k)

$$\alpha_q^* : A \otimes \Lambda^*(q,k) \longrightarrow A \otimes \Lambda^*(q,2k)$$

so that $\alpha_q^*\delta_q^* = \alpha_{q-1}^*\delta_q$ and

1) $\alpha_0^* = \chi(Sq^{2k}) : A \longrightarrow A$

2) For $q \geq 1$ , $\alpha_q^* \lambda_I = \Sigma\chi(Sq^{p(I,J)})\lambda_J$ with $p(I,J)$ an integer so that $p(I,J) \geq 2k\text{-dim } \lambda_I$ and, if $\lambda_J \in \hat{\Lambda}$ , then $p(I,J) > 2k\text{-dim } \lambda_I$ .

We need two lemmas on the $\Lambda$-algebra. Both are in Brown and Gitler [6].

Lemma 3.11. Let $\lambda_I \in \Lambda$ . If $\lambda_j\lambda_I \neq 0$ in $\Lambda/\Lambda_k$ , then $2j + \dim I \geq 2k$ .

Secondly, recall that the elements $\lambda_I \in \Lambda$ are ordered lexigraphically from the right.

Lemma 3.12. For any $\lambda_I$ not admissible, $\lambda_I = \Sigma\lambda_J$ with $J$ admissible and $\lambda_J < \lambda_I$ . If $I = (i_1,\dots,i_q)$ then $\partial\lambda_I = \Sigma\lambda_J$ with $J = (j_1,\dots,j_{q+1})$ , $J$ admissible, and $j_{q+1} < i_q$ .

Proof of 3.10. We work by induction on $q$ . $\alpha_0^*$ is defined by 1). We may define $\alpha_1^*$ by

$$\alpha_1^*\lambda_i = \Sigma\binom{i-j}{2k-2j}\chi(Sq^j)\lambda_{2k+i-j} .$$

Then $2k-2j \leq i-j$ implies $j \geq 2k-i$ . If $2k+i-j$ is odd, then $j > 2k-i$ . Suppose $\alpha_q^*$ has been defined and satisfies 2) .

Then,

$$\begin{aligned}\alpha_q^*\delta_{q+1}^*\lambda_I &= \Sigma\lambda_I(\lambda_i\lambda_L)\chi(Sq^{i+1})\alpha_q^*\lambda_L\\ &= \Sigma\lambda_I(\lambda_i\lambda_L)\chi(Sq^pSq^{i+1})\lambda_J \qquad (p = p(L,J))\\ &= \Sigma\lambda_I(\lambda_i\lambda_L)\chi(Sq^pSq^{i+1})\lambda_J\\ &+ \Sigma\lambda_I(\lambda_i\lambda_L)\binom{i-j}{p-\partial j}\chi(Sq^{p+i+1-j}Sq^j)\lambda_J \quad .\end{aligned}$$

The first sum is over $p$ and $i$ so that $2i+2 \leq p$ . By Lemma 3.11

$$2i + \dim L \geq k \quad \text{so} \quad i+1 > 2k - \dim L - i = 2k - \dim I \ .$$

On the other hand

$$p - 2j \leq i-j \quad \text{implies} \quad j \geq p-i \geq 2k - \dim L - i = 2k - \dim I \quad .$$

If $\lambda_J \in \hat{\Lambda}$ , then $j \geq p-i > 2k - \dim I$ .

Here is the point: write $\alpha_q^*\delta_{q+1}^*\lambda_I$ in the form

$$\alpha_q^*\delta_{q+1}^*\lambda_I = \Sigma\ \chi(Sq^sSq^t)\lambda_M + \Sigma\ \chi(Sq^sSq^t)\lambda_J$$

with $J > M$ . Obviously, $\Sigma\chi(Sq^sSq^t)$ (the coefficient of $\lambda_M$ or $\lambda_J$) should be indexed to $J$ or $M$ . I drop it to ease notation. $Sq^sSq^t$ is admissible. Now, I claim that, using only the fact that $\delta_q^*\alpha_q^*\delta_{q+1}^*\lambda_I = 0$ , I can produce a $w \in A \otimes \Lambda^*(q+1, 2k)$ so that ($t$ as in the coefficent of $\lambda_M$)

$$w = \Sigma\ \chi(Sq^t)\lambda_{s-1}\lambda_M$$

and that i) $\lambda_{s-1}\lambda_M$ is admissible

ii) $\delta_{q+1}^*\ w = \Sigma\ \chi(Sq^sSq^t)\lambda_M + \Sigma\ \chi(Sq^iSq^j)\lambda_J$ with $J > M$ .

iii) If $j$ is as in ii), then $j \geq 2k - \dim I$ or $j > 2k - \dim I$ if $\lambda_J \in \hat{\Lambda}$ .

Since for $t$ (all $J$ and $M$) we have $t \geq 2k - \dim I$ (or $t > 2k - \dim I$ if $\lambda_M$ or $\lambda_J \in \hat{\Lambda}$) and since $\delta^*_q \delta^*_{q+1} v = 0$ we may proceed inductively to produce

$$\tilde{w} \in A \otimes \Lambda^* (q+1, 2k)$$

so that $\tilde{w} = \Sigma \chi(Sq^{p'})\lambda_N$ $p' \geq 2k - \dim I$ or $p' > 2k - \dim I$ if $\lambda_N \in \hat{\Lambda}$ and so that $\delta^*_{q+1} \tilde{w} = \alpha^*_q \delta^*_{q+1} \lambda_I$. Then, we set $\alpha^*_{q+1} \lambda_I = \tilde{w}$.

So, I must produce $w$. Write $M = (m, M')$ and write the coefficient of $\lambda_M$ in $\alpha^*_q \delta^*_{q+1} \lambda_I$ as

$$\chi(Sq^a Sq^b) + \Sigma\, \chi(Sq^s Sq^t)$$

with $s < a$ for all $s$.

Now,

$$\begin{aligned} 0 = \alpha^*_{q-1} \delta^*_q \delta^*_{q+1} \lambda_I &= \delta^*_q \alpha^*_q \delta^*_{q+1} \lambda_I \\ &= \Sigma\, \chi(Sq^{m+1} Sq^s Sq^t)\lambda_{M'} + \Sigma\, a_T \lambda_T \quad \text{with } T > M' \text{ and} \\ &\qquad a_T \in A . \end{aligned}$$

This a consequence of Lemma 3.12. Thus, we are forced to conclude that $Sq^{m+1} Sq^a$ is not admissible; that is $2a > m+1$, or $2a - 2 \geq m$. Thus $\lambda_{a-1}\lambda_M$ is an admissible monomial in $\Lambda$. Let us examine $\delta^*_{q+1}\ \chi(Sq^b)\lambda_{a-1}\lambda_M$.

$$\begin{aligned} \delta^*_{q+1}\ \chi(Sq^b)\lambda_{a-1}\lambda_M &= \chi(Sq^a Sq^b)\lambda_M \\ &\quad + \Sigma\, \lambda_{a-1}\lambda_M(\lambda_i \lambda_J)\chi(Sq^{i+1} Sq^b)\lambda_J . \end{aligned}$$

$\lambda_J > \lambda_M$. Again, see Lemma 3.12. Since $b \geq 2k - \dim I$ (or $b > 2k - \dim I$ if $\lambda_M \in \hat{\Lambda}$) if $Sq^{i+1} Sq^b$ is admissible, condition iii) as above is satisfied. Suppose $Sq^{i+1} Sq^b$ is not admissible. Then

$$Sq^{i+1}Sq^{b} = \Sigma \binom{b-\ell-1}{i+1-2\ell} Sq^{b+i+1-\ell}Sq^{\ell} \quad .$$

First, we have that $i+1-2\ell \leq b-\ell-1$ or

$$\ell \geq i-b+2 \ .$$

Then, because $\lambda_i\lambda_J \neq 0$ in $\Lambda/\Lambda_{2k}$ we must have (Lemma 3.11)

$$2i \geq 4k - \dim J$$

or

$$i \geq 4k-i-\dim J = 4k-\dim M-a+1 \quad .$$

Thus, combining these two inequalities

$$\ell \geq 4k - (a+b+\dim M) + 3 \ .$$

Since $\chi(Sq^a Sq^b)\lambda_M$ is a summand of $\alpha_q^* \ \chi(Sq^{i+1})\lambda_L$ where $\chi(Sq^{i+1})\lambda_L$ is a summand of $\delta_{q+1}^*\lambda_I$ we see that

$$a+b+\dim M = i+1+\dim L + 2k = \dim I + 1 + 2k$$

Thus,

$$\ell \geq 2k - \dim I + 2 \quad .$$

We may now proceed algorithmically to produce $w$ satisfying i) - iii) above and, thus, inductively, to define $\alpha_{q+1}^*$ .

This completes the proof of 3.6. Therefore, we have constructed the maps of section 1.

## 4. The Construction of $B_1(2k)$ .

In this section, we produce the spectra $B_1(2k)$ and prove Theorems 1.13, 1.14, 1.15, and 1.21, and Propositions 1.16.

Recall that we defined a map

$$j_k : B_1(2k+1) \longrightarrow \Sigma^{2k+1}C(k)$$

in section 1. Then we defined $B_1(2k)$ to be the spectrum so that the following is a cofibration sequence.

$$B_1(2k) \longrightarrow B_1(2k+1) \xrightarrow{j_k} \Sigma^{2k+1}C(k) .$$

As noted, the following is immediate.

<u>Proposition 4.1</u>. $H^*B(2k) = M_1(2k)$ .

The bulk of this section will be devoted to the discussion of the homology theory $B_1(2k)_*$ . This is intricate and we take some come care. The method is this: First we produce a tower whose inverse limit is $C(k)$ . Then, second, we produce a map from the tower for $B_1(2k+1)$ to the tower for $C(k)$ ; the inverse limit of this map is $j_k$ . This will allow us to produce a tower whose inverse limit in $B_1(2k)$ . Then, third, we will use the Pontrjagin duality of Section I.3 and what we know about the maps we have constructed to examine each stage of the tower for $B_1(2k)$ .

The first step is to construct the tower whose inverse limit is $C(k)$ . Recall that $2k = 2^s(2t+1)$ and that

$$C(k) = \vee\Sigma^{a(i)} B_1(2^{i-1}(2t+1)-1) \vee \Sigma^{a(1)} B(2t+1) .$$

Here $a(i) = (2^{s-1} + \ldots + 2^1)(2t+1)$ . For even $n$ , we have a tower whose inverse limit is $B_1(2n+1)$ :

$$\begin{array}{ccccccccc} \cdots \to X_q^{2n+1} & \to & X_{q-1}^{2n+1} & \to & \cdots & \to & X_1^{2n+1} & \to & X_0^{2n+1} \\ & & \varepsilon_q \big\downarrow & & & & & & \varepsilon_1 \big\downarrow \\ & & F_q^{2n+1} & & & & & & F_1^{2n+1} \end{array} \qquad (4.2)$$

Then, of course, set

$$X_q' = \underset{j}{\vee}\ \Sigma^{a(i)}\ X_q^{2^{i-1}(2t+1)-1} \vee \Sigma^{a(1)} X_q^{2t+1}$$

and

$$F_q' = \vee\ \Sigma^{a(i)}\ F_q^{2^{-1}(2t+1)-1} \vee \Sigma^{a(1)} F_q^{2t+1}$$

Then, we have a tower whose inverse limit in $C(k)$

$$\begin{array}{ccccccccc} \cdots \to X_q' & \to & X_{q-1}' & \to \cdots \to & X_1' & \to & X_0' \simeq \vee\Sigma^{a(j)}\ K\,\hat{\mathbb{Z}}_j \\ \varepsilon_q' \big\downarrow & & & & \varepsilon_1' \big\downarrow & & \\ F_q' & & & & F_1' & & \end{array} \qquad (4.3)$$

Let $X_q = X_q^{2k+1}$ and $F_q = F_q^{2k+1}$ in (4.2). Then, our next step is to produce maps of degree 2k+1

$$\bar{j}_q : X_q \to X_q' \quad \text{and} \quad j_q : F_q \to F_q'$$

so that 1) is the following diagram commutes

$$\begin{array}{ccccccc} F_q & \longrightarrow & X_q & \longrightarrow & X_{q-1} & \xrightarrow{\varepsilon_q} & F_q \\ j_q \big| & & \big|\bar{j}_q & & \big|\bar{j}_{q-1} & & \big| j_q \\ F_q' & \longrightarrow & X_q' & \longrightarrow & X_{q-1}' & \xrightarrow{\varepsilon_q} & F_q' \end{array} \tag{4.4}$$

and 2) $\varprojlim \bar{j}_q = j_k : B_1(2k+1) \longrightarrow \Sigma^{2k+1}C(k)$ .

Recall that $j_k$ is a wedge of maps

$$f_i : B_1(2k+1) \longrightarrow \Sigma^{b(i)}B_1(2^{i-1}(2t+1)-1) \quad i > 1$$

and

$$f_1 : B_1(2k+1) \longrightarrow \Sigma^{b(1)}B_1(2t+1)$$

where $b(i) = a(i) + 2k + 1$ . For $i > 1$ , $f_i$ is the composition

$$f_i : B_1(2k+1) \xrightarrow{i_{2k}} B(2k) \xrightarrow{h_{2k}} \Sigma^{2k}B(k) \longrightarrow \cdots$$
$$\cdots \longrightarrow \Sigma^{b(i)-1}B(2^{i-1}(2t+1)) \xrightarrow{\phi} \Sigma^{b(i)}B_1(2^{-1}(2t+1))$$

where $\phi = \phi_{2^{i-1}(2t+1)}$ of Corollary 3.6 and the unlabelled maps are of the form $h_{2n}$ of 3.9. $i_{2k}$ is as in 3.4.

Similarly $f_1$ is the composition

$$f_1 : B_1(2k+1) \xrightarrow{i_{2k}} B(2k) \xrightarrow{h_{2k}} \Sigma^{2k}B(k) \longrightarrow \cdots$$
$$\longrightarrow \Sigma^{b(1)-1}B(2t+1) \xrightarrow{\phi} \Sigma^{b(1)}B_1(2t+1) .$$

$\phi = \phi_{2t+1}$ of Corollary 3.7 and the unlabelled maps are of the form $h_{2n}$ .

Let us first examine the maps $\phi_n$ ($n$ either even or odd). It is an easy argument to show that there exists a commutative diagram

$$\begin{array}{ccccccc} I_q & \longrightarrow & Y_q & \longrightarrow & Y_{q-1} & \xrightarrow{\varepsilon_q} & I_q \\ \downarrow{\phi_q} & & \downarrow{\overline{\phi}_q} & & \downarrow{\overline{\phi}_{q-1}} & & \downarrow{\phi_q} \\ F_q & \longrightarrow & X_q & \longrightarrow & X_{q-1} & \xrightarrow{\varepsilon_q} & F_q \end{array} \tag{4.5}$$

so that $\varprojlim \overline{\phi}_q = \phi_n : B(n) \longrightarrow \Sigma B_1(n)$ $(\Sigma B_1(n-1)$ if $n$ is even$)$ .

So now let

$$\overline{\alpha}(q,2n) = \overline{\alpha}_q : Y_q^{2n} \longrightarrow Y_q^n$$

and

$$\alpha(q,2n) = \alpha_q : I_q^{2n} \longrightarrow I_q^n \quad .$$

Then $\varprojlim \overline{\alpha}(q,2n) = h_{2n}$ . See Corollary 3.9. Let

$$\overline{\beta}_q : X_q \longrightarrow Y_q^{2k}$$

and

$$\beta_q : F_q \longrightarrow I_q^{2k}$$ be as in Corollary 3.4.

We now define a map of degree $b(i)$

$$\overline{f}(i,q) : X_q \longrightarrow X_q^{2^{i-1}(2t+1)}$$

to be the composition $\overline{f}(i,q) = \overline{\phi}_q \circ \overline{\alpha}(q,2^i(2t+1)) \circ \ldots \circ \overline{\alpha}(q,2k) \circ \overline{\beta}_q$ .

$\overline{\phi}_q$ is as in (4.5) for $\phi_{2^{i-1}(2t+1)} : B(2^{i-1}(2t+1)) \longrightarrow \Sigma B_1(2^{i-1}(2t+1)-1)$ .

Similarly we can define a map of degree $b(1)$

$$\overline{f}(1,q) : X_q \longrightarrow X_q^{2t+1}$$

to be the composition $\overline{f}(1,q) = \overline{\phi}_q \circ \overline{\alpha}(q, (2t+1)) \circ \ldots \circ \overline{\alpha}(q,2k) \circ \overline{\beta}_q$ .

$\overline{\phi}_q$ is as in (4.5) for $\phi_{2t+1} : B(2t+1) \longrightarrow \Sigma B_1(2t+1)$ .
Then we can define $\overline{f}_q = V\overline{f}(i,q) : X_q \longrightarrow X_q'$ . $\overline{f}_q$ is of degree $2k+1$ and $\varprojlim \overline{f}_q = V \varprojlim \overline{f}(i,q) = j_k : B_1(2k+1) \longrightarrow \Sigma^{2k+1}C(k)$ .

Let us define maps of degree $b(i)$

$$f(i,q) : F_q \longrightarrow F_q^{2^{i-1}(2t+1)}$$

to be the composition $f(i,q) = \phi_q \circ \alpha(q,2^i(2t+1)) \circ \ldots \circ \alpha(q,2k) \circ B_q$ .
$\phi_q$ is as in (4.5) for $\phi_{2^{i-1}(2t+1)}$ .

Then set $j_q = Vf(i,q) : F_q \longrightarrow F_q'$ . The diagram promised (4.4) now commutes.

Now let us define $W_q$ to be the spectrum so that

$$W_q \xrightarrow{\overline{\theta}_q} X_q \xrightarrow{\overline{j}_q} X_q'$$

is a cofibration sequence. Simirlarly, define $J_q$ to be the spectrum so that

$$J_q \xrightarrow{\theta_q} F_q \xrightarrow{j_q} F_q'$$

is a cofibration sequence. Note that $J_q$ is <u>not</u> a product of Eilenberg MacLane spectra. We may conclude that there is a tower

$$\begin{array}{ccccccccc} \cdots \longrightarrow W_q & \xrightarrow{p_q} & W_{q-1} & \longrightarrow & \cdots & \longrightarrow W_1 \longrightarrow & W_0 \\ & & \downarrow{\scriptstyle \varepsilon_q} & & & & \downarrow{\scriptstyle \varepsilon_1} \\ & & J_q & & & & J_1 \end{array} \qquad (4.6)$$

Further, we may assume that $p_q : W_q \longrightarrow W_{q-1}$ is a fibration induced form the contractible path fibration over $J_q$ by $\varepsilon_q$ .

$$\varprojlim W_q \cong B_1(2k) \quad .$$

$\overline{\theta}_0 : W_0 \longrightarrow K\mathbb{Z}_2^{\wedge}$ induces a canonical choice for a generator of $H^0(B_1(2k);\ \mathbb{Z}_2^{\wedge})$ ; namely,

$$B_1(2k) \longrightarrow W_0 \xrightarrow{\overline{\theta}_0} K\mathbb{Z}_2^{\wedge} \quad .$$

We have completed the first two steps in our discussion of $B_1(2k)_*$ ; the third step is to discuss the induced homology theories $(W_q)_*$ .

Recall that $2k = 2^s(2t+1)$ . Let $Z$ be any CW complex. The first result is this.

<u>Theorem 4.7</u>. 1) If $s > 1$ and $q \geq 1$ , the sequence

$$(J_q)_n Z \xrightarrow{i_{q*}} (W_q)_n Z \xrightarrow{p_{q*}} (W_{q-1})_n Z$$

is split short exact for $n \leq 4k-3$ , short exact for $n = 4k-2$ , and if $n = 4k-1$ , $p_{q*}$ is surjective.

2) If $s = 1$ and $q \geq 1$ , the sequence

$$(J_q)_n Z \xrightarrow{i_{q*}} (W_q)_n Z \xrightarrow{p_{q*}} (W_{q-1})_n Z$$

is split short exact for $n \leq 4k-2$ , short exact for $n = 4k-1$ , and if $n = 4k$ , $p_{q*}$ is surjective. The splittings are natural in $Z$ .

3) If $n \leq 4k-1$ , $\theta_{0*} : (W_0)_n Z \longrightarrow H_n(Z; \mathbb{Z}_{\hat{2}})$ is surjective. If $n = 4k$ , then $(\theta_0)_*$ is onto $\bigcap_i Sq_{i*} \subseteq H_{4k}(Z; \mathbb{Z}_{\hat{2}})$ .

4.7 implies 1.15, 1.16, and 1.21 of the outline in section 1. As usual when we wish to prove results such as 4.7 we employ the Pontrjagin duality functor $\chi$ of I.3. The key lemma is the following. Let $\alpha_q$ be as in 3.10 and $\beta_q$ as in 3.4. $\alpha_q = \alpha(2k,q)$ .

<u>Lemma 4.8</u>. For $q \geq 1$ . $\chi(\alpha_q \beta_q) : \chi(I_q^k)_{2k-1} \longrightarrow \chi(F_q)_{4k}$ is null-homotopic.

We postpone the proof, not because it is difficult, but because it involves a sequence of ideas from Brown and Comenetz which have not been employed yet. Lemma 4.8 follows from 3.10 and 4.8 explains the form of 3.10.

<u>Proof of 4.7</u>. Let $K_q = \chi(F_q)$ , $G_q = \chi(X_q)$ , $K_q' = \chi(F_q')$ , and $G_q' = \chi(X_q')$ . And let $e_q = \chi(\varepsilon_q)$ and $e_q' = \chi(\varepsilon_q')$ . We prove 1) and 2) first. Let $n$ be as in the hypotheses of 4.7.1 or 4.7.2. Consider the diagram:

$$\begin{array}{ccccccc}
K_{q,n-1} & \xrightarrow{\chi(\theta_q)} & \chi(J_q)_{n-1} & \longrightarrow & K'_{q,n-2k-1} & \xrightarrow{\chi(j_q)} & K_{q,n} \\
\Big\downarrow e_q & & \Big\downarrow e_q & & \Big\downarrow e_q' & & \Big\downarrow e_q \\
G_{q-1,n} & \xrightarrow{\chi(\bar{\theta}_q)} & \chi(W_q)_n & \longrightarrow & G'_{q,n-2k} & \xrightarrow{\chi(\bar{j}_q)} & G_{q-1,n+1}
\end{array}$$

$e_q : K_{q,n} \longrightarrow G_{q-1,n+1}$ is null-homotopic by Theorem I.4.8, part 5. $\chi(j_q) : K'_{q,n-2k-1} \longrightarrow K_{q,n}$ is null-homotopic by the construction of $j_q$ and lemma 4.8. We claim that $e'_q : K'_{q,n-2k-1} \longrightarrow G'_{q,n-2k}$ is also null homotopic.

Let us first show that the claim implies 4.7.1 and 4.7.2. $\chi(J_q)_{n-1}$ may be considered to be the pull-back by $\chi(j_q)$ of the contractible path fibration over $K_{q,n}$ . Similarly, $(W_q)_n$ may be considered to be the pull back by $\chi(\bar{j}_q)$ of the contractible path fibration over $G_{q-1,n+1}$ . Thus assuming the claim we conclude that

$$e_q : \chi(J_q)_{n-1} \longrightarrow \chi(W_q)_n$$

is null-homotopic. Then the results follow from an argument exactly like that in I.5.2. Let us now demonstrate our claim.

Let $c(i) = 2^{i-1}(2t+1)-1$ if $j > 1$ and let $c(1) = 2t+1$ . Then

$$e'_q : K'_{q,n-2k-1} \longrightarrow G'_{q-1,n-2k}$$

is the map $Xe_q : \underset{i}{X} K^{c(i)}_{q,n-2k-a(i)-1} \longrightarrow \underset{i}{X} G^{c(i)}_{q,n-2k-a(i)}$ .

For $i > 1$ , $n-2k-a(i) \leq 2k-a(i)-1 = 2^i(2t+1)-1$ . Thus

$$e_q : K^{c(i)}_{q,n-2k-a(i)-1} \longrightarrow G^{c(i)}_{q,n-2k-a(i)}$$

is null-homotopic by I.4.8 part 5.

For $i = 1$ , $n-2k-a(i) \leq 2k-a(1) = 2(2t+1)$ . Thus

$$e_q : K^{c(1)}_{q,n-2k-a(1)-1} \longrightarrow G^{c(1)}_{q,n-2k-a(1)}$$

is null-homotopic; again by I.4.8. The claim follows.

We now remark that 4.7.3 follows froma simple excess plus dimension argument. This completes the proof of 4.7.

To prove Lemma 4.8, we need information about the induced map

$$\chi(\alpha_q\beta_q)^* : H^*\chi(F_q) \longrightarrow H^*\chi(I_q^k)$$

$\chi(F_q)$ and $\chi(I_q^k)$ are both products of Eilenberg-MacLane spectra; therefore homological information will suffice. The necessary results are in Brown and Comenetz.

Let $M$ be a (left) module over the Steenrod Algebra. Let $\mathrm{Hom}_A(M,A)$ be the group of Steenrod Algebra maps from $M$ to $A$. Give $\mathrm{Hom}_A(M,A)$ the structure of a left module over the Steenrod Algebra by the formula

$$a\phi(m) = \phi(m)\chi(a) \quad .$$

$m \in M$ , $\phi \in \mathrm{Hom}_A(M,A)$ and $a \in A$ .

Proposition 4.9: 1) If $Y$ is a spectrum so that $\pi_2 Y$ is finite for each $n$ and $\pi_n Y = 0$ for $n < n_0$ is some integer, then

$$H^*\chi(Y) \cong \mathrm{Hom}_A(H^*Y, A)$$

2) If $f : X \longrightarrow Y$ is a map of spectra so that $\pi_n X$ and $\pi_n Y$ are finite for each $n$ and $\pi_n Y = \pi_n X = 0$ for $n < n_0$ where $n_0$ is some integer, then

$$\chi(f)^* = \mathrm{Hom}_A(f^*, A) \quad .$$

This is in Brown and Comenetz [5].

Proposition 4.10: Let $M$ and $N$ be free modules over the Steenrod Algebra $M$ with basis $\{m_i\}$ and $N$ with basis $\{n_i\}$ . Suppose $f : M \to N$ is a Steenrod Algebra map given by $f(m_i) = \Sigma a_{ij} n_j$ with $a_{ij} \in A$ . Then $\mathrm{Hom}_A(M,A)$ and $\mathrm{Hom}_A(N,A)$ are free over the Steenrod Algebra with dual bases $\{\overline{m}_i\}$ and $\{\overline{n}_i\}$ respectively and

$$\mathrm{Hom}_A(f,A)(\overline{n}_j) = \Sigma\chi(a_{ij})\overline{m}_i \quad .$$

This is Brown and Gitler [6].

4.8 follows immediately for 4.9, 4.10, and 3.10.

To close this section we remark that one can easily adapt the arguments of this section to prove the following two theorems.

Theorem 4.11: Let $\overline{B}(2k-1)$ be the spectrum so that

$$\overline{B}(2k-1) \longrightarrow B(2k) \xrightarrow{h_{2k}} \Sigma^{2k}B(k)$$

is a cofibration sequence. Then

1) $H^*\overline{B}(2k-1) = A/A\{\chi(Sq^i) : i > 2k-1\}$ .

2) For any CW complex $Z$ , $\overline{B}(2k-1)_n Z \longrightarrow H_n Z$ is onto for $n < 4k-1$ .

From 4.11 and an anologue of 2.7 for the original Brown-Gitler spectra we conclude that $\overline{B}(2k-1) = B(2k-1)$ . This is Theorem 1.13.

We also have the following.

Theorem 4.12: Let $\overline{B}_1(2k-1)$ be the spectra so that

$$\overline{B}_1(2k-1) \longrightarrow B_1(2k+1) \xrightarrow{h_{2k}i_{2k}} \Sigma^{2k}B(k)$$

is a cofibration sequence. Then

1) $H^*B_1(2k-1) = A/A\{Sq^1, \chi(Sq^i) : i > 2k-2\}$ .

2) For any CW complex $Z$ , the composition $B_1(2k-1)_n Z \longrightarrow H_n(Z; \mathbb{Z}_2^{\wedge})$ is onto for $n \leq 4k-1$ .

From 4.12 and 2.7 we conclude that $\overline{B}_1(2k-1) \simeq B_1(2k-1)$ . This is Theorem 1.14.

## 5. Secondary Obstructions and Applications to Manifolds.

In this section we produce the secondary homology operations of 1.19 and prove the stated theorem. Then, we show that these operations are defined whenever $[M] \in H^{4k}M$ is the fundamental class of a closed orientable 4k manifold. Finally, we discuss the case $M = CP^{2k}$.

Let $Sq_i' = \chi(Sq^{2^S(2t+1)} \ldots Sq^{2^i(2t+1)} Sq^1) \in A$ and let $Sq_i : K\mathbb{Z}_2^\wedge \longrightarrow K\mathbb{Z}_2^\wedge$ be the unique map of degree $b(i) = (2^S + \ldots + 2^i)(2t+1)+1$ so the following commutes:

$$\begin{array}{ccc} K\mathbb{Z}_2^\wedge & \xrightarrow{Sq_i'} & K\mathbb{Z}_2^\wedge \\ \downarrow & & \downarrow \\ K\mathbb{Z}_2 & \xrightarrow{Sq_i'} & K\mathbb{Z}_2 \end{array}$$

The verticle maps are the Steenrod Algebra generator of $H^*K\mathbb{Z}_2^\wedge$. The Adem relation

$$\chi(Sq^{2^{i-1}(2t+1)})Sq_i' = \chi(Sq^{2^S(2t+1)+1}Sq^{2^{S-1}(2t+1)} \ldots Sq^{2^{i-1}(2t+1)}) \quad 2 \le i \le S$$
$$= 0 \quad \text{in } H^*K\mathbb{Z}_2^\wedge$$

guarantees that we have a diagram

$$\begin{array}{c} \Sigma^{b(i)-1}K\mathbb{Z}_2^\wedge \xrightarrow{\sigma} X_0' \xrightarrow{\eta} W_0 \xrightarrow{\psi_i} \Sigma^{d(i)}K\mathbb{Z}_2 \\ K\mathbb{Z}_2^\wedge \xrightarrow{Sq} X_0' \end{array} \tag{5.1}$$

so that the composition across the is $\chi(Sq^{2^{i-1}(2t+1)}) \in H^*K\mathbb{Z}_2^\wedge$. $W_0$, $X_0'$ are as in section 4; $X_0' = \bigvee_i \Sigma^{b(i)}K\mathbb{Z}_2^\wedge$ and $\sigma$ is the inclusion. $Sq = \bigvee_i Sq_i$ , and $\eta$ is the inclusion of the fiber and is of degree 1.

$$d(i) = (2^s + \ldots + 2^i + 2^{i-1})(2t+1) .$$

Let $\varepsilon_1 : W_0 \longrightarrow J_1$ be the map of degree one of section 4. Our first result is this.

<u>Proposition 5.2</u>: There exists a map $y : J_1 \longrightarrow \Sigma^{d(i)-1} K\mathbb{Z}_2$ so that $y\varepsilon_1 : W_0 \longrightarrow \Sigma^{d(i)} K\mathbb{Z}_2$ is a choice for $\psi_i$ .

<u>Proof</u>. We have a diagram in which theorems are cofibration sequences.

$$\begin{array}{ccccc}
W_0 & \longrightarrow & K\hat{\mathbb{Z}}_2 & \xrightarrow{Sq} & X_0' \\
\varepsilon_1 \downarrow & & \varepsilon_1 \downarrow & & \downarrow \varepsilon_1' \\
J_1 & \longrightarrow & F_1 & \xrightarrow{j_1} & F_1'
\end{array}$$

In cohomology this induces a diagram

$$\begin{array}{ccccc}
A \otimes \hat{\Lambda}^*(1,2k+1) & \longleftarrow & A \otimes \hat{\Lambda}^*(1,2^{i-1}(2t+1)-1) & & \\
\downarrow & & \downarrow & & \\
H^*F_1 & \xleftarrow{j_1^*} & H^*F_1' & \longleftarrow & H^*J_1 \\
\downarrow & & \downarrow (\varepsilon_1')^* & & \downarrow \varepsilon_1^* \\
H^*X_0 & \longleftarrow & H^*X_0' & \longleftarrow & H^*W_0
\end{array}$$

If $x_i \in H^*X_0' \cong \oplus H^* \, {}^{b(i)}K\hat{\mathbb{Z}}_2$ , then

$$Sq^{*}\,\chi(Sq^{2^{i-1}(2t+1)})x_i = 0 \quad .$$

We are asking whether or not there is a class $y \in H^{*}J_1$ so that $\varepsilon_1 y$ maps to $\chi(Sq^{2^{i-1}(2t+1)})x_i$ under $H^{*}W_0 \longrightarrow H^{*}X_0'$ . If

$$\lambda = \overline{\lambda}_{2^{i-1}(2t+1)-1} \in A \otimes \hat{\Lambda}^{*}(1,2^{i-1}(2t+1)-1) \subseteq H^{*}F_1'$$

then $(\varepsilon_1')^{*}\lambda = \chi(Sq^{2^{i-1}(2t+1)})x_i$ . We need only show that $f^{*}\lambda = 0$ .

$$f = f(1,i) = \phi_1 \circ \alpha(1,2^{i}(2t+1)) \circ \ldots \circ \alpha(1,2k) \circ \beta_1 \quad .$$

The definition of $f(1,i)$ is in section 4, just after 4.5.

For dimensional reasons one can conclude that

$$\phi_1^{*}\lambda = \overline{\lambda}_{2^{i-1}(2t+1)} \in A \otimes \Lambda^{*}(1,2^{i-1}(2t+1)) \qquad (i \geq 2)$$

The formula for $\alpha(1,2n)^{*}$ given at the beginning of the proof of 3.10 implies

$$(\alpha(1,2^{i}(2t+1)) \circ \ldots \circ \alpha(1,2k))^{*}\, \overline{\lambda}_{2^{i-1}(2t+1)} = \chi(Sq^{2^{s-1}(2t+1)})\overline{\lambda}_{2k} \quad .$$

Since $\beta_1^{*}\,\overline{\lambda}_{2k} = 0$ , it follows $f(1,i)^{*}\lambda = 0$ . This proves 5.2.

We use this result to observe that there is a diagram

$$\begin{array}{ccccccc}
\Sigma^{b(i)-1}K\mathbb{Z}_2^{\wedge} & \xrightarrow{\sigma} & X_0' & \xrightarrow{\varepsilon_1'} & F_1' & \xrightarrow{\lambda} & K\mathbb{Z}_2 \\
 & & \downarrow \eta & & \downarrow & & \| \\
 & & W_0 & \xrightarrow{\varepsilon_1} & J_1 & \xrightarrow{y} & K\mathbb{Z}_2 \\
 & & \downarrow & & & & \\
 & & K\mathbb{Z}_2^{\wedge} & & & &
\end{array} \qquad (5.3)$$

where $\lambda = \overline{\lambda}_{2^{i-1}(2t+1)-1} \in A \otimes \hat{\Lambda}^*(1, 2^{i-1}(2t+1)-1) \subseteq H^*F_1$ .

$y\varepsilon_1\eta\sigma = \chi(Sq^{2^{i-1}(2t+1)})$ We make a choice for $y$ ; then in 5.1 set $\psi_i = y\varepsilon_1$ . This diagram is the secondary homology operation $\overline{\Psi}_i$ .

<u>Theorem 5.4</u>. Let $Z$ be a CW complex. If $x \in \cap \ker Sq_{i*} \subseteq H_{4k}(Z; \hat{\mathbb{Z}}_2)$ then $x$ is in the image of

$$B_1(2k)_{4k}Z \longrightarrow H_{4k}(Z; \hat{\mathbb{Z}}_2) \quad .$$

only if $\overline{\Psi}_i(x)$ is zero modulo indeterminacy for $i$ , $2 \leq i \leq s$ .

The proof of 5.1 requires the following preliminary. Recall that $K_1^{2n+1} = \chi(F_1^{2n+1})$ and $K\,\mathbb{Z}_{2\infty} = \chi(K\,\hat{\mathbb{Z}}_2)$ .

<u>Lemma 5.5</u>. $X(\varepsilon_1) = e_1 = d_1 : K_{1,4n+3}^{2n+1} \longrightarrow K(\mathbb{Z}_{2\infty};4n+4)$ is completely determined as follows.

$K_{1,4n+3}^{2n+1} = XK(\mathbb{Z}_2;\ 4n+2-2j)$ with $j \geq n$ and the composition

$$K(\mathbb{Z}_2;\ 4n+2-2j) \subseteq K_{1,4n+3}^{2n+1} \xrightarrow{d_1} K(\mathbb{Z}_{2\infty};\ 4n+4)$$

is zero unless $j = n$ . If $j = n$ , this composition is the composite

$$K(\mathbb{Z};\ 2n+2) \xrightarrow{Sq^{2n+2}} K(\mathbb{Z}_2;\ 4n+4) \longrightarrow K(\mathbb{Z}_{2\infty};4n+4) \ .$$

The last map is the canonical inclusion

<u>Proof</u>. From I.2.13 we have a diagram $(\varepsilon_1 = \delta_1)$

$$\begin{array}{ccc} F_1^{2n+1} & \xleftarrow{\delta_1} & K\,\hat{\mathbb{Z}}_2 \\ \| & & \downarrow 1 \\ F_1^{2n+1} & \xleftarrow{\bar{\delta}_1} & K\,\mathbb{Z}_2 \end{array}$$

where $\bar{\delta}_1$ is given in cohomology $\bar{\delta}_1^* 1 = \Sigma\chi(Sq^{2j+2})\bar{\lambda}_{2j+1}$. Then, by Pontrjagin duality, we have a diagram

$$\begin{array}{ccc} K_1^{2n+1} & \xrightarrow{d_1} & K\,\mathbb{Z}_2 \\ \| & & \cap| \\ K_1^{2n+1} & \xrightarrow{d_1} & K\,\mathbb{Z}_{2^\infty} \end{array}$$

where $\bar{d}_1$ is given in cohomology by $\bar{d}_1^*\lambda_{2j+1} = Sq^{2j+2}$. See 4.9 and 4.10. Lemma 5.5 follows.

Proof of 5.4. We take the Ponrjagin dual of diagram 5.3. Set $K_1 = \chi(F_1)$, $G_0 = \chi(K\,\hat{\mathbb{Z}}_2)$, $K_1' = \chi(F_1')$ and $G_0' = \chi(X_0')$. We have $(2n = 2^{i-1}(2t+1))$

$$\begin{array}{ccccccc} & & & K(\mathbb{Z}_2;\ 2k-2n) & & & \\ & & \swarrow{\scriptstyle \chi(\psi_i)} & & \searrow{\scriptstyle \chi(\lambda)} & & \\ & & \chi(J_1)_{4k-1} & \longrightarrow & K'_{1,2k-1} & \xrightarrow{\chi(j_1)} & K_{1,4k} \\ & & e_1\downarrow & & e_1'\downarrow & & \downarrow e_1 \\ G_{0,4k} & \longrightarrow & \chi(W_0)_{4k} & \longrightarrow & G'_{0,2k} & \longrightarrow & G_{0,4k+1} \end{array}$$

$\chi(j_1)$ is null-homotopic by Lemma 4.8; $e_1 : K_{1,4k} \longrightarrow G_{0,4k+1}$ is null-homotopic $e_1' : K_{1,2k-1}' \longrightarrow G_{0,2k}'$ is determined by 5.5.

5.4 can be rephrased in this dualized context to read: Let $Z$ be a finite CW complex and $x \in \chi(W_0)^{4k}Z$ . Then $x$ is not in the kernel of

$$\chi(W_0)^{4k}Z \longrightarrow \chi(W_1)^{4k}Z$$

($W_1$ as in 4.6) if and only if $x$ is not in the union of the images of

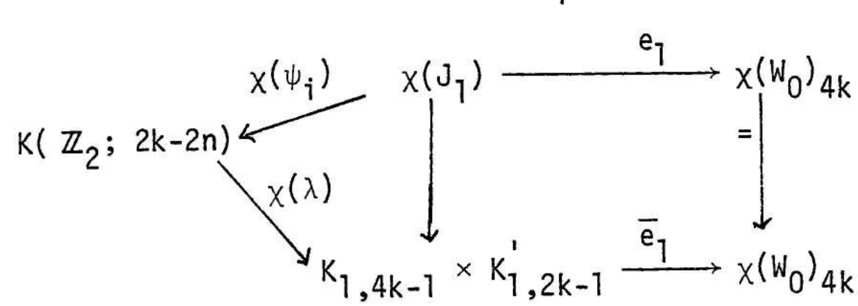
$$(e_1\chi(\psi_i))_* : H^*Z \longrightarrow \chi(W_0)^{4k}Z \quad . \qquad (2 \le i \le s)$$

Thus 5.4 holds if and only if the union of the images of $(e_1\chi(\psi_i))_*$ equals the image of $e_{1*}$ in $\chi(W_0)^{4k}Z$ .

Choose a null-homotopy of $\chi(j_1)$ . Because $K_{1,2k-1}'$ is a product of Eilenberg-MacLane spaces, we may assume that the induced null-homotopy on the composition

$$K(\mathbb{Z}_2;\ 2k-2n) \xrightarrow{\chi(\lambda)} K_{1,2k-1}' \xrightarrow{\chi(j_1)} K_{1,4k}$$

is that induced by the lifting $\chi(\psi_i)$ . Then we have a diagram



The verticle maps are homotopy equivalences. Let

$$r \ : \ K(\mathbb{Z}_2;\ p) \longrightarrow K_{1,4k-1} \times K'_{1_2 2k-1}$$

be the inclusion of any factor. Then, because $e_1 \ : \ K_{1,4k} \longrightarrow G_{0,4k+1}$ is null-homotopic, because of 4.10, and because $\chi(J_1)$ is the pull-back by $\chi(j_1)$ of the contractible path fibration over $K_{1,4k}$, we see that $e_1 r$ is null-homotopic unless $r = \chi(\lambda) = \chi(\overline{\lambda}_{2^{i-1}(2t+1)-1})$, $2 \leq i \leq S$.

Therefore, 5.4 is true when $Z$ is a finite CW complex. For general $Z$, use a limit argument as at the end of I.5.2.

This completes the proof of 5.4.

We now specialize to the case when $Z = M$, an orientable closed manifold of dimension $4k$. The next result implies that if $M$ is an orientable manifold of dimension $8n+4$ and $T(\nu)$ is the Thom spectrum of the normal bundle of $M$, then there is a factoring

$$\begin{array}{ccc} & & B_1(4n+2) \\ & \nearrow & \downarrow \\ T(\nu) & \xrightarrow{\ \mu\ } & K\hat{\mathbb{Z}}_2 \end{array}$$

$\mu$ is a choice for the Thom class $\mu \in H^0(T(\nu);\hat{\mathbb{Z}}_2)$.

<u>Proposition 5.6</u>. Let $M$ be an orientable manifold of dimension $4k$. Then $Sq_{i*} \ : \ H_{4k}M \longrightarrow H_{4k-b(i)}M$ is zero for all $1 \leq i \leq S$.

Proof. By Pontrjagin duality we have a map of degree $b(i)$

$$\chi(Sq_i) : K\mathbb{Z}_{2^\infty} \longrightarrow K\mathbb{Z}_{2^\infty}$$

and we are asking whether or not $\chi(Sq_i)_* : H^{4k-b(i)}(M; \mathbb{Z}_{2^\infty}) \longrightarrow H^{4k}(M; \mathbb{Z}_{2^\infty})$ is zero. Let $Sq = Sq^{2^s(2t+1)} \ldots Sq^{2^i(2t+1)}$. Then we have a diagram

$$\begin{array}{ccccc} H^{4k-b(i)}(M; \mathbb{Z}_{2^\infty}) & \xrightarrow{Sq^1} & H^{4k-b(i)+1}(M; \mathbb{Z}_2) & \xrightarrow{Sq} & H^{4k}(M; \mathbb{Z}_2) \\ & & & \searrow f \quad \swarrow \text{incl} & \\ & & & H^{4k}(M; \mathbb{Z}_{2^\infty}) & \end{array}$$

$f$ is defined by this diagram. Note that $fSq^1 = \chi(Sq_i)$. If $x \in H^{4k-a(i)+1}M$, then $Sq(x) = x^{2^{s-i+1}}$. Now, if $y \in H^{4k-b(i)}(M; Z_{2^\infty})$, then we will show that $Sq^1 y \in H^{4k-b(i)-1}M$ is the reduction of an integral class of finite order; there exists a class $z \in H^{4k-b(i)+1}(M; \mathbb{Z})$ so that the coefficient map

$$H^{4k-b(i)+1}(M; \mathbb{Z}) \longrightarrow H^{4k-b(i)+1}M$$

carries $z$ to $Sq^1 y$ and $z$ is of finite order. Since $H^{4k}(M; \mathbb{Z}) = \mathbb{Z}$, $(z)^{2^{s-i+1}} = 0$; this shows that $(Sq^1 y)^{2^{s-i+1}} = 0$ and, thus that $\chi(Sq_i)_* y = 0$.

Consider the diagram of groups

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{x2^n} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_{2^n} \\ \downarrow & & \downarrow & & \| \\ \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_{2^{n+1}} & \longrightarrow & \mathbb{Z}_{2^n} \\ \| & & \downarrow & & \downarrow \\ \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_{2^\infty} & \xrightarrow{x2} & \mathbb{Z}_{2^\infty} \end{array}$$

This induces a diagram

$$\begin{array}{ccccc} H^p(M;\mathbb{Z}_{2^n}) & \xrightarrow{=} & H^p(M;\mathbb{Z}_{2^n}) & \xrightarrow{\subseteq} & H^p(M;\mathbb{Z}_{2^\infty}) \\ \downarrow\beta & & \downarrow\beta' & & \downarrow Sq^1 \\ H^{p+1}(M;\mathbb{Z}) & \longrightarrow & H^{p+1}(M;\mathbb{Z}_2) & \xrightarrow{=} & H^{p+1}(M;\mathbb{Z}_2) \end{array}$$

$\beta$ and $\beta'$ are the connecting homomorphisms in the appropriate long exact coefficient sequence. $p$ is an integer. Let $y \in H^p(M;\mathbb{Z}_{2^\infty}) = \varinjlim H^p(M;\mathbb{Z}_{2^n})$. Choose $n$ so that there exist $z' \in H^p(M;\mathbb{Z}_{2^n})$ mapping to $y$. Set $z = \beta z'$ in $H^{p+1}(M;\mathbb{Z})$. Then $z$ reduces to $Sq^1 y$ and $z$ is of order less than or equal to $2^{n+1}$.

This completes the proof of 5.6.

Proposition 5.6 implies that the secondary homology operation $\Psi_i$ are defined on the fundamental class $[M] \in H^{4k}(M;\hat{\mathbb{Z}_2})$. We have the following partial result.

<u>Proposition 5.7</u>. If $[CP^{2k}] \in H^{4k}(CP^{2k};\hat{\mathbb{Z}_2})$ is the fundamental class, then $\Psi_i[CP^{2k}]$ is zero modulo indeterminacy for all $i$, $2 \leq i \leq s$.

<u>Proof</u>. We compute the indeterminacy. This is the image of

$$\chi(Sq^{2^{i-1}(2t+1)})_* : H_{4k-b(i)-1}(CP^{2k};\hat{\mathbb{Z}_2}) \longrightarrow H_{4k-d(i)}CP^{2k}$$

where $d(i) = (2^s + \dots + 2^{i-1})(2t+1)$. $(2k = 2^s(2t+1))$. Thus

$4k-b(i)-1 = 2^i(2t+1)$ and let $n = 4k-d(i) = 2^{i-1}(2t+1)$ . There is a factoring

$$\begin{array}{ccc} H_{2n}(CP^{2k};\mathbb{Z}_{\hat{2}}) & \xrightarrow{C_*} & H_{2n}CP^{2k} \\ & {\scriptstyle \chi(Sq^n)_*} \searrow \quad \swarrow {\scriptstyle \chi(Sq^n)_*} & \\ & H_nCP^{2k} & \end{array}$$

$C_*$ is the coefficient map. $C_*$ is surjective. If we show that

$$\chi(Sq^n)_* : H_{2n}CP^{2k} \longrightarrow H_nCP^{2k}$$

is an isomorphism, we will be done. This is equivalent, by Pontrjagin duality to showing that

$$Sq^n : H^nCP^{2k} \longrightarrow H^{2n}CP^{2k}$$

is an isomorphism. This is clearly true.

# Chapter III Secondary Cohomology Operations that Detect Homotopy Classes.

This chapter is joint work with Ralph Cohen.

A basic problem in homotopy theory is to determine which $s^{\underline{th}}$ order stable cohomology operations can act trivially on the cohomology of a two-cell complex, $S^k \cup_\alpha D^{k+m+1}$. The attaching map lies in the $m^{\underline{th}}$ stable stem, $\pi_m S^o$. This problem is equivalent to determining which elements of $\mathrm{Ext}^s(\mathbb{Z}_p, \mathbb{Z}_p)$ survive to $E_\infty$ in the Adams Spectral Sequence. Here $A$ is the mod p Steenrod Algebra.

For $s = 1$ this problem is the Hopf invariant one problem and was solved by Adams for $p = 2$ [1] and by Liulevicius [18] and Shimada and Yamanoshita [30] for $p > 2$. In this chapter we give a complete answer to this question for $s = 2$ and $p \geq 5$.

Our main result is the following

Theorem I: For all primes $p \geq 3$ and integers $j \geq 2$, the element $h_0 h_j \in \mathrm{Ext}_A(\mathbb{Z}_p, \mathbb{Z}_p)$ is an infinite cycle in the Adams Spectral Sequence. Furthermore if $p \geq 5$ or $p = 3$ and $j$ is odd, $h_o h_j$ represents an element $\eta_j \in \pi_n S^o$ of order $p$, where $n = 2(p^j+1)(p-1)-2$.

Said differently, we will prove that there exists a stable map $\eta_j : S^n \longrightarrow S^o$ so that in $H^*(S^o \cup_{\eta_j} D^{n+1}; \mathbb{Z}_p)$ the secondary cohomology operation $\Phi_{o,j}$ determined by the Adem relation involving $P^1 P^{p^j}$ acts nontrivially.

Now for each prime $p \geq 3$ there are four other maps known to be detected by secondary cohomology operations:

$$p^2 : S^o \longrightarrow S^o \qquad \alpha_z : S^{4p-5} \longrightarrow S^o$$
$$\beta_1 : S^{2p(p-1)-2} \longrightarrow S^o \qquad \beta_z : S^{\ell} \longrightarrow S^o .$$

where

$$\ell = 4p(p^2-1) - 2(p-1) - 2 .$$

These classes are detected by $a_o^2$, $g_1$, $b_o$, and $k_1$ respectively in $Ext_A^2(\mathbb{Z}_p, \mathbb{Z}_p)$. (The notation is that of [23, Section 10]). Prior to Theorem I, all other information has been negative. In fact, the results of Miller, Ravenel, and Wilson [23], Toda [33], and Ravenel [28] all tell us that certain classes in $Ext_A^2(\mathbb{Z}_p, \mathbb{Z}_p)$ are not infinite cycles. Combining our results with theirs we obtain the following.

<u>Corollary II</u>: For $p \geq 5$, the only classes in $Ext_A^2(\mathbb{Z}_p, \mathbb{Z}_p)$ that are infinite cycles are $a_o^2$, $g_1$, $b_o$, $k_1$, and $h_o h_j$, $j > 1$.

It is an easy consequence of the results of [23] to produce maps of spheres detected by secondary Brown-Peterson cohomology operations. That is, we shall prove the following.

<u>Theorem III</u>: For $p \geq 5$ or $p = 3$ and $j$ odd, there are non-zero elements

$$\hat{\beta}_{p^{j-1}/k} \in Ext^2_{BP_*BP}(BP_*, BP_*)$$

$1 \leq k \leq p^{j-1} - 1$, that are infinite cycles in the Adams-Novikov Spectral Sequence, and they detect non-trivial homotopy classes of order $p$. As a special case, $\hat{\beta}_{p^{j-1}/p^{j-1}-1}$ detects $\eta_j$.

<u>Remark</u>: In the notation of [23] $\hat{\beta}_{p^{j-1}/k}$ may be written

$$\hat{\beta}_{p^{j-1}/k} = \beta_{p^{j-1}/k} + \Sigma c_i \beta_{a(i)/b(i)} \qquad c_i \in \mathbb{Z}_p$$

where

$$a(i) = [p^{2i+1}+1/p+1]p^{j-2i-1}$$

$$b(i) = k-p^{j-1}+p^{j-2i-1}$$

and

$$1 \leq i \leq \max\{\ell | p^{2\ell} < p^{j-1}/p^{j-1}-k\} .$$

Let us make some remarks on the situation at the primes 2 and 3. At $p = 2$ the $\eta_j$ family is detected by $h_i h_j$ and was shown to exist by Mahowald [19]. $2\eta_j \neq 0$ ; in fact, using the computations of Carlson [12], one can show that one can modify the $\eta_j$ so that $2\eta_j = \eta_{j-1}^2 \neq 0$ . These elements of Mahowald formed the first known infinite family of homotopy classes detected by secondary operations. The work of Ralph Cohen [14] and this work were motivated by the desire to apply Mahowald's techniques to odd primes.

At both the primes 2 and 3 there remains one infinite family of elements in $\mathrm{Ext}_A^2(\mathbb{Z}_p, \mathbb{Z}_p)$ whose Adams Spectral Sequence behavior is not yet understood. At $p = 2$ this is the Arf invariant family $h_j^2$ in bidegree $(2,2^{j+1})$ and at the prime 3 this is the analagous family $b_j = \langle h_j, h_j, h_j \rangle$ in bidegree $(2,2p^{j+1}(p-1))$ . For $p \geq 5$ Ravenel showed that for $j > 0$ , $b_j$ is not an infinite cycle.

The proof of Theorem I breaks into two steps. First, in sections 1, 2, and 3, we provide a proof of the following.

<u>Theorem 1.1</u>: For all primes $p \geq 3$ and integers $j > 1$ , the element $h_o h_j \in \mathrm{Ext}_A^2(\mathbb{Z}_p, \mathbb{Z}_p)$ is an infinite cycle in the Adams Spectral Sequence and represents and element $\hat{\eta}_j \in \pi_* S^o$ .

This result claims nothing about the order of $\hat{\eta}_j$ . The proof of 1.1 is long and is broken into three steps. In section 1 we reduce the proof to a lemma. This lemma is proved in section 3 after we recall some results of Ralph Cohen [14] in section 2.

In section 4 we show that for $p \geq 5$ or $p = 3$ and $j$ odd $\hat{\eta}_j$ can be modified to yield an element $\eta_j$ of order p represented by $h_o h_j$ . Finally, in section 5 we prove Theorem III.

## 1. Reduction to a Lemma.

In this section we will prove, modulo a lemma, the following result.

Theorem 1.1: For all primes $p \geq 3$ and integers $j > 1$, the element $h_o h_j \in \mathrm{Ext}_A^2(\mathbb{Z}_p, \mathbb{Z}_p)$ is an infinite cycle in the Adams spectral sequence. Here $A = A_p$ is the mod $p$ Steenrod algebra.

We begin by fixing some notation that will be used throughout the paper. If $f : X \to Y$ is a map of spectra, let $M(f)$ be the mapping cone

$$M(f) = Y \cup_f c(X) .$$

So in particular $M(p)$ is the mod $p$ Moore spectrum

$$M(p) = S^o \cup_p D^1 .$$

Fix a prime $p \geq 3$ and an integer $j > 1$. Recall that $h_o h_j \in \mathrm{Ext}_A^2(\mathbb{Z}_p, \mathbb{Z}_p)$ corresponds to an Adem relation

$$P^1 P^{p^j} + \sum_{i=0}^{j-1} a_i P^{p^i} = 0$$

with $a_i \in A$. This relation in turn induces a secondary cohomology operation $\Phi_{o,j}$ in the usual way. See, for example [1, 18, or 14.IV.3].

Set $n = 2(p^j+1)(p-1)-2$. In [14.IV] a map

$$\overline{\zeta}_j : \Sigma^{n-1} M(p) \to S^o$$

was constructed and was shown to satisfy the following properties.

Lemma 1.2. a. In $H^*(M(\overline{\zeta}_j))$ $\Phi_{o,j}(\sigma_o) = \sigma_n$ with zero indeterminacy. Here $\sigma_o$ and $\sigma_n$ are the generators of $H^*(M(\overline{\zeta}_j))$ in dimensions 0 and n respectively.

b. There is an element $h_o b_{j-1} \in \mathrm{Ext}_A^3(\mathbb{Z}_p, \mathbb{Z}_p)$ that is an infinite cycle in the Adams spectral sequence and is represented by the composition

$$\zeta_j : S^{n-1} \xrightarrow{i} \Sigma^{n-1}M(p) \xrightarrow[\overline{\zeta}_j]{} S^o$$

where i is the inclusion of the bottom cell.

Lemma 1.2. a. is Theorem IV.c of [14] and 1.2.b is Theorem IV.b of [14].

We will recall the construction of $\overline{\zeta}_j$ in the next section. For now we will use its existence to reduce the proof of Theorem 1.1 to a lemma. In order to state this lemma we need to recall some basic information concerning Adams resolutions. For a thorough treatment see [2].

Let X be a spectrum. Then an Adams resolution for X is a sequence of spectra

$$\begin{array}{ccccccccc} X_s & \xrightarrow{i_s} & X_{s-1} & \longrightarrow \cdots \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_o = X \\ & & \Big\downarrow{\scriptstyle j_{s-1}} & & & & \Big\downarrow{\scriptstyle j_1} & & \Big\downarrow{\scriptstyle j_o} \\ & & K_{s-1} & & & & K_1 & & K_o \end{array}$$

so that 1) $K_s$ is a wedge of suspensions of Eilenberg-MacLane Spectra of type $K\mathbb{Z}_p$,

2) $j_s^* : H^*K_s \longrightarrow H^*X_s$ is surjective, and

3) $X_s \xrightarrow{i_s} X_{s-1} \xrightarrow{j_{s-1}} K_{s-1}$ is a cofibration sequence.

If $Y$ is another spectrum, let $[Y,Z]_q$ be homotopy classes of maps $\Sigma^q Y \longrightarrow Z$. Applying the functor $[Y, ]_*$ we get a spectral sequence, the Adams Spectral Sequence. If $Y$ and $X$ are connected and $X$ is p-complete, then the Adams spectral sequence converges to $[Y,X]_*$ and is well known, $E_2 = \mathrm{Ext}_A^*(H^*X, H^*Y)$.

Suppose $f : Y \longrightarrow X$ is a map and that there is a lifting

$$\begin{array}{ccc} & & X_s \\ & \overset{f'}{\nearrow} & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

and that there is no lifting of $f$ to $X_{s+1}$. Then, $f$ is said to be <u>on the s-line</u> of the Adams Spectral Sequence.

Let
$$\begin{array}{ccccccccc} \cdots \longrightarrow & Y_q & \xrightarrow{i_q} & Y_{q-1} & \longrightarrow \cdots \longrightarrow & Y_1 & \xrightarrow{i_1} & Y_o & = Y \\ & & & \downarrow & & & & \downarrow & \\ & & & L_{q-1} & & & & L_o & \end{array}$$

be an Adams resolution for $Y$. If $f : Y \longrightarrow X$ is on the s-line, there exist maps $f_s : Y_q \longrightarrow X_{s+q}$ so the following diagram commutes:

$$\begin{array}{ccc} Y_q & \longrightarrow & X_{s+q} \\ {\scriptstyle i_q}\downarrow & & \downarrow{\scriptstyle i_{s+q}} \\ Y_{q-1} & \xrightarrow{f_{s-1}} & X_{s+q-1} \end{array} .$$

For our purposes we will need a minimal resolution for $X = S^0$ . This is a resolution

$$\begin{array}{ccccccccc} \longrightarrow E_s & \xrightarrow{i_s} & E_{s-1} & \longrightarrow \cdots \longrightarrow & E_1 & \longrightarrow & E_0 & = & S^0 \\ & & \big\downarrow j_s & & \big\downarrow j_1 & & \big\downarrow j_0 & & \\ & & K_{s-1} & & K_1 & & K_0 & & \end{array}$$

satisfying 1)-3) in the definition of Adams Resolutions but including the additional requirement that

$$j_s^* : H^*K_s \longrightarrow H^*E_s$$

carry the Steenrod Algebra generators of $H^*K_s$ bijectively to a minimal set of Steenrod Algebra generators for $H^*E_s$ . Then, it is standard that there is an isomorphism

$$\pi_*\Sigma^s K_s \cong \mathrm{Ext}_A^s(\mathbb{Z}_p, \mathbb{Z}_p) \ . \qquad (1.3)$$

We list results of A. Liulevicius [18]. The notation is that of [23].

<u>Lemma 1.4</u>. a. $\mathrm{Ext}_A^0(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p$ concentrated in graded zero.

b. $\mathrm{Ext}_A^1(\mathbb{Z}_p, \mathbb{Z}_p)$ has $\mathbb{Z}_p$ basis $a_0$ in grading 1 and $h_i$ of grading $2p^i(p-1)$, $i > 0$ .

c. $\mathrm{Ext}_A^2(\mathbb{Z}_p, \mathbb{Z}_p)$ has $\mathbb{Z}_p$ basis

| | | |
|---|---|---|
| $h_i h_j$ | $i < j-1$ | of grading $2(p-1)(p^i+p^j)$ |
| $g_i$ | $i \geq 0$ | of grading $2(p-1)(p^{i+1}+2p^i)$ |
| $k_i$ | $i \geq 0$ | of grading $2(p-1)(2p^{i+1}+p^i)$ |
| $a_0 h_j$ | $j > 0$ | of grading $2(p-1)p^j+1$ |

$b_i \quad i \geq 0$ of grading $2(p-1)p^{i+1}$

$a_o a_o$ of grading $2$

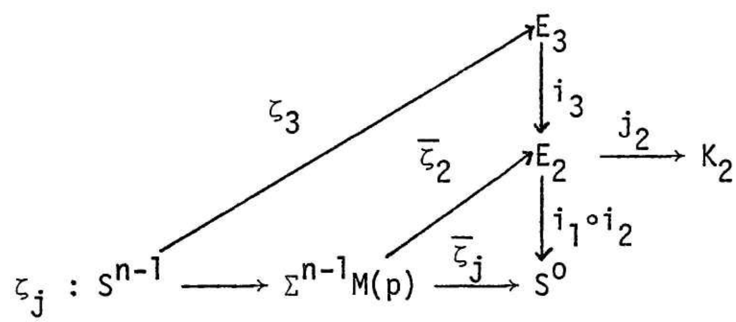
$a_1$ of grading $4(p-1)+1$ .

We can now state the lemma that will allow us to construct an element $\hat{\eta}_j \in \pi_n(S^o)$ represented by $h_o h_j$ and thereby prove Theorem 1.1. Let

$$\bar{\zeta}_j : \Sigma^{n-1}M(p) \longrightarrow S^o$$

$$\zeta_j : S^{n-1} \longrightarrow S^o$$

be the maps of 1.2. Then 1.2 and 1.4a. imply there is a diagram of liftings.




(1.5)

Here $E_3$ , $E_2$ , and $K_2$ are as above, and $\zeta_3$ and $\bar{\zeta}_2$ are liftings of $\zeta_j$ and $\bar{\zeta}_j$ respectively. These exist because $\pi_n K_s = \pi_{n-1} K_s = 0$ for $s = 0,1$ and because $\pi_{n-1} K_2 = 0$ .

<u>Lemma 1.6</u>: Let $h_o h_j \in \pi_* K_2$ under the isomorphism (1.3). Then $h_o h_j$ is a generator for $H^* K_s$ over the Steenrod Algebra and

$(j_2\,\bar{\zeta}_2)^* h_o h_j = \sigma_n \in H^n \Sigma^{n-1} M(p)$ .

Proof: This follows from 1.2.

Here is our main lemma.

Lemma 1.7: In diagram 1.5, there exist liftings $\bar{\zeta}_2$ and $\zeta_3$ so that $\zeta_3$ has order $p$ in $\pi_{n-1} E_3$ .

The proof of this lemma is the content of sections 2 and 3. Assuming 1.7 we can now prove Theorem 1.1.

Proof of 1.1: Let $\delta : \Sigma^{n-1} M(p) \longrightarrow S^n$ be projection onto the top cell. Then, combining 1.4.c and 1.6, we have a commutative diagram in which the rows are cofibration sequences:

$$\begin{array}{ccccccc} S^{n-1} & \longrightarrow & \Sigma^{n-1} M(p) & \longrightarrow & S^n & \xrightarrow{\times p} & S^n \\ \zeta_3 \downarrow & & \bar{\zeta}_2 \downarrow & & h_o h_j \downarrow & & \Sigma\zeta_3 \downarrow \\ E_3 & \longrightarrow & E_2 & \longrightarrow & K_2 & \longrightarrow & \Sigma E_3 \end{array} .$$

Because $\Sigma\zeta_3$ has order $p$ , the composite $S^n \xrightarrow{h_o h_j} K_2 \longrightarrow \Sigma E_3$ is null-homotopic. Therefore there is a lifting

$$\begin{array}{ccc} & & E_2 \\ & \overset{\hat{\eta}_2}{\nearrow} & \downarrow \\ S^n & \xrightarrow{h_o h_j} & K_2 \end{array} .$$

This implies that $h_o h_j$ is an infinite cycle and is represented by the composition $\eta_j : S^n \xrightarrow{\hat{\eta}_2} E_2 \xrightarrow{i_1 \circ i_2} S^o$ .

We are therefore reduced to proving Lemma 1.7. To do this we'll need to recall the construction of the map $\overline{\zeta}_j$ from [14]. We do this in section 2. In section 3 we will prove 1.7 and thereby complete the proof of Theorem 1.1.

## 2. Construction of $\overline{\zeta}_j$ .

In this section we recall some results of the first author [14] and prove a curious lemma about the structure of Brown-Gitler spectra. We begin by recalling an outline of the construction of the map

$$\overline{\zeta}_j : \Sigma^{n-1}M(p) \longrightarrow S^0 .$$

Let $t(V_k) = C_2(k)^+ \wedge_{\Sigma_k} S^{1(k)}$ be the Thom space of the k-plane bundle $V_k$ :

$$C_2(k) \times_{\Sigma_k} \mathbb{R}^k \longrightarrow C_2(k)/\Sigma_k$$

where $C_2(k)$ is the configuration space $C_2(k) = \{(x_1,\dots,x_k) \in (\mathbb{R}^2)^k$ such that $x_i \neq x_j$ if $i \neq j\}$ . $t(V_k)$ is a Snaith stable wedge summand of $\Omega^2S^3$ . That is, there is an equivalence of suspension spectra

$$\Sigma^\infty\Omega^2S^3 \xrightarrow[\simeq]{} \bigvee_{k\geq 1} \Sigma^\infty t(V_k) .$$

See [32] and [9] for a discussion of this splitting.

In [14.II] it was shown that

$$\tilde{H}^* t(V_{pk}) = A/A\{\chi(\beta^\varepsilon P^i): \ pi+\varepsilon > k\} \otimes \mu_k$$

where $\mu_k$ is a class of dimension $2k(p-1)$ and $\chi$ is the canonical antiautomorphism of the Steenrod Algebra. In particular then

$$\tilde{H}^* t(V_{p^{j+1}+p}) = A/A\{\chi(\beta^\varepsilon P^i): \ i > p^{j-1} , \ \varepsilon = 0,1\} \otimes \mu$$

where $\mu$ is a class of dimension $2(p^j+1)(p-1)$ .

One of the thrusts of [14] was to deduce that $t(V_p^{j+1}{}_{+p})$ was homotopy equivalent to $B(p^{j-1})$ -- the $p^{j-1}$ - p-primary Brown-Gitler spectrum. Much follows from this -- 2.2 below for instance.

Set $s = -2p^j-2p+1$ . An easy calculation shows that as a spectrum we have the following:

i) $H^{2p^j(p-1)-1}\Sigma^s t(V_p^{j+1}{}_{+p}) \cong \mathbb{Z}_p$ generated by $a_j = \chi(\beta P^{p^{j-1}} P^{p^{j-2}} \ldots P^p P^1 \beta)(v)$ where $v$ is the Steenrod Algebra generator of $H^*\Sigma^s t(V_p^{j+1}{}_{+p})$ , and

ii) If $\alpha_j = \chi(\beta P^{p^{j-1}} P^{p^{j-2}} \ldots P^1)(v) \in H^*\Sigma^s t(V_p^{j+1}{}_{+p})$ , then $\beta\alpha_j = a_j$ ,

<u>Lemma 2.1</u>. There exists a map $g : \Sigma^s t(V_p^{j+1}{}_{+p}) \longrightarrow S^0$ so that in $H^*M(g)$ :

a) $P^{p^j}(\sigma_0) = \Sigma a_j$ and all other primary operations are zero on $\sigma_0$ . Here $\sigma_0$ is the generator of $H^0M(g)$ .

b) $\Gamma_{b_{j-1}}(\sigma_0) = \Sigma\alpha_j$ modulo indeterminacy, where $\Gamma_{b_{j-1}}$ is the secondary cohomology operation corresponding to $b_{j-1} \in \mathrm{Ext}_A^{2,2p^j(p-1)}(\mathbb{Z}_p, \mathbb{Z}_p)$ .

Lemma 2.2.a. is Theorem IV.2.1 of [14] and 2.1.b. is Theorem IV.3.4. of [14] . $\overline{\zeta}_j$ will factor through $g$ .

<u>Lemma 2.2</u>. There is a map $\overline{\xi} : \Sigma^{n-1}M(p) \longrightarrow \Sigma^s t(V_p^{j+1}{}_{+p})$ so that in $H^*M(\overline{\xi})$

a) $P^1(\alpha_j) = (-1)^j 2\Sigma\sigma_{n-1}$

b) $P^1(a_j) = \Sigma\sigma_n$

where $\sigma_{n-1}$ and $\sigma_n$ generate $H^{n-1}\Sigma^{n-1}M(p)$ and $H^n\Sigma^{n-1}M(p)$ respectively.

Lemma 2.2.a. is Lemma IV.5.2 of [14] and 2.2.b is Theorem IV.1.1 of [14]. The map $\overline{\zeta}_j : \Sigma^{n-1}M(p) \longrightarrow S^0$ was defined to be the composition

$$\overline{\zeta}_j : \Sigma^{n-1}M(p) \xrightarrow[\overline{\xi}]{} \Sigma^s t(V_p{}^{j+1}{}_{+p}) \xrightarrow[g]{} S^0 .$$

Standard composition methods allow us to combine 2.1 with 2.2 to conclude that in $H^*M(\overline{\zeta}_j)$ , $\Phi_{o,j}(\zeta_o) = \Sigma\sigma_n$ modulo indeterminacy. The indeterminacy is then computed to be zero. See section IV.5 of [14].

In order prove lemma 1.7 (and therefore Theorem 1.1) we will need to know that the identity map of $t(V_p{}^{j+1}{}_{+p})$ has stable order $p$ . To do this we will show that $t(V_p{}^{j+1}{}_{+p})$ is a stable wedge summand of $t(V_p{}^{j+1}) \wedge t(V_p)$ . This will be sufficient since $t(V_p) \simeq \Sigma^{2p-2}(M(p))$ .

Recall Snaith's stable equivalence

$$\Sigma^\infty\Omega^2S^3 \simeq \bigvee_{k\geq 1} t(V_k) .$$

Let $m$ be the composition of spectra maps

$$m : t(V_p^{\,j+1}) \wedge t(V_p) \xrightarrow{\subset} \Sigma^\infty(\Omega^2S^2 \times \Omega^2S^3)$$
$$\xrightarrow{*} \Sigma^\infty\Omega^2S^3 \longrightarrow t(V_{p^{j+1}+p}) .$$

Here $*$ is the H-space multiplication of $\Omega^2S^3$ .

Lemma 2.3. There is a map $d : t(V_p^{\,j+1}) \longrightarrow t(V_p^{\,j+1}) \wedge t(V_p)$ so that $m \circ d$ is a self-homotopy equivalence of $t(V_p^{\,j+1})$ . That is, $t(V_{p^{j+1}+p})$ is a wedge summand of $t(V_p^{\,j+1}) \wedge t(V_p)$ .

Proof: Let $\Delta : \Omega^2S^3 \longrightarrow \Omega^2S^3 \times \Omega^2S^3$ be the diagonal map. d is the composition

$$d : t(V_{p^{j+1}+p}) \longrightarrow \Sigma^\infty\Omega^2S^3 \xrightarrow{\Delta} \Sigma^\infty(\Omega^2S^3 \times \Omega^2S^3)$$
$$\longrightarrow t(V_p^{\,j+1}) \wedge t(V_p) .$$

We make a homology calculation. At primes $p \geq 3$ ,

$$H_*\Omega^2S^3 = E(a_0, a_i, \ldots) \otimes \mathbb{Z}_p[b_1, b_2 \ldots]$$

as a Hopf algebra. The multiplication is induced by $m$ and the comultiplication by $\Delta$ . $a_i$ has dimension $2p^i-1$ and $b_i$ has dimension $2p^i-2$ .

As remarked above $H^*t(V_{p^{j+1}+p})$ is a cyclic $A$ module and its generator is the dual of $b_1^{p^{j+1}}$ . Now

$$\Delta_* b_1^{p^j+1} = \sum_k \binom{p^j+1}{k} b_1^{p^j+1-k} \otimes b_1^k \quad .$$

So, $(m \circ d)_* b_1^{p^j+1} = b_1^{p^j+1}$ . Since $H^* t(V_p{}^{j+1}{}_{+p})$ is cyclic, $(m \circ d)^*$ is an isomorphism. Since we are completed at $p$ , this implies that $m \circ d$ is a homotopy equivalence.

By previous remarks and results of [14,II], $m^*$ is induced by the Cartan diagonal in the Steenrod Algebra:

$$\begin{aligned} m^* : H^* t(V_p{}^{j+1}{}_{+p}) &\cong A/A\{\chi(\beta^\varepsilon P^i) : i > p^{j-1}\} \\ &\to A/A\{\chi(\beta^\varepsilon P^i) : i+\varepsilon > p^{j-1}\} \otimes A/A\{\chi(\beta^\varepsilon P^i) : i \geq 1\} \\ &\cong H^* t(V_p{}^{j+1}) \wedge t(V_p) . \end{aligned}$$

Thus, $m^* : H^k \Sigma^s t(V_p{}^{j+1}{}_{+p}) \longrightarrow H(\Sigma^s t(V_p{}^{j+1}) \wedge t(V_p))$ is an isomorphism for $k \geq 2p^j(p-1)-4$ . Let $G$ be the composition

$$G : \Sigma^s t(V_p{}^{j+1}) \wedge t(V_p) \xrightarrow{m} {}^s t(V_p{}^{j+1}{}_{+p}) \xrightarrow{g} S^0$$

and let $\bar{\mu}$ be the composition

$$\bar{\mu} : \Sigma^{n-1} M(p) \xrightarrow{\bar{\xi}} \Sigma^s t(V_p{}^{j+1}{}_{+p}) \xrightarrow{d} \Sigma^s t(V_p{}^{j+1}{}_{+p}) \wedge t(V_p) .$$

Then, the above remarks, 2.1 and 2.2, imply the following result.

<u>Lemma 2.4.a)</u> In $H^* M(G)$ , $P^{p^j}(\sigma_o) = \Sigma m^* a_j$ and $\Gamma_{b_j}(\sigma_o) = m^* \alpha_j$ modulo indeterminacy. All other primary operations are zero on $\sigma_o$ .

b) In $H^* M(\bar{\mu})$ , $P^1 m^* \alpha_j = (-1)^j 2\, \Sigma\sigma_{n-1}$ and

$P^1 m^* a_j = \Sigma\sigma_n$ .

c) $\overline{\zeta}_{j-1} = G\circ\overline{\mu}$ .

This completes our recapitulation of results for [14]. We might remark, in closing, that 2.3 is true in much greater generality.

If $k \neq p^i$ , then write the p-adic expansion of pk

$$pk = c_1 p^{i_1} + c_2 p^{i_2} + \ldots + c_m p^{i_m} \qquad m \geq 2$$

with $i_1 > i_2 > \ldots > i_m$ and $1 \geq c_j \geq p-1$ for each $j$ .

Then $t(V_{pk})$ is a wedge summand of

$$\underbrace{t(V_{p^{i_1}})\wedge\ldots\wedge t(V_{p^{i_1}})}_{c_1}\wedge\ldots\wedge(\underbrace{t(V_{p^{i_n}})\wedge\ldots\wedge t(V_{p^{i_n}})}_{c_n}) .$$

The methods of 2.3 suffice. One can observe also that similar splittings at the prime $p = 2$ exist.

## 3. The Proof of Lemma 1.7.

This section is devoted to showing that there exist maps $\zeta_3$ and $\overline{\zeta}_2$ so that $\zeta_3$ is of order $p$ and so that we have a commutative diagram

$$\begin{array}{ccccc} S^{n-1} & \longrightarrow & \Sigma^{n-1}m(p) & & \\ \zeta_3 \downarrow & & \downarrow \overline{\zeta}_2 & \searrow \overline{\zeta}_j & \\ E_3 & \longrightarrow & E_2 & \longrightarrow & S^0 \end{array} \quad .$$

As shown in section 1, this will complete a proof of Theorem 1.1.

Our goal is to produce spaces $Y_1$ and $T_1$ , and $v$ , $\overline{v}$ , and $G_1$ , so that we have a commutative diagram

$$\begin{array}{ccccc} S^{n-1} & \xrightarrow{v} & Y_1 & \xrightarrow{f_1} & E_3 \\ \downarrow i & & \downarrow & & \downarrow \\ \Sigma^{n-1}M(p) & \xrightarrow{\overline{v}} & T_1 \wedge t(V_p) & \xrightarrow{G_1} & E_2 \end{array} \quad . \tag{3.1}$$

Then $\zeta_3$ will be $f_1 \circ v$ and $\overline{\zeta}_2$ will be $G_1 \circ v$ . As the notation suggests, $Y_1$ and $T_1$ will be first stages in Adams resolutions for certain spaces $Y$ and $T$ .

For any spectrum $Z$ , let $p : Z \longrightarrow Z$ be p-times the identity. In order to show $\zeta_3$ has order $p$ we will use a null-homotopy of $G_1 \circ p : T_1\ t(V_p) \longrightarrow E_2$ (recall $t(V_p) \simeq \Sigma^{2p-2}M(p)$) to build a lifting $\hat{f}_1$ and a diagram

$$\begin{array}{ccccccc} & & & & & & \Sigma^{-1}K_2 \\ & & & \overset{\hat{f}_1}{\nearrow} & & & \downarrow \\ S^{n-1} & \xrightarrow{v} & Y_1 & \xrightarrow{p} & Y_1 & \xrightarrow{f_1} & E_3 \end{array} \quad .$$

(Recall that $E_3 \xrightarrow{i_3} E_2 \xrightarrow{i_2} K_2$ is a cofibration sequence). This lifting $\hat{f}$ , will be sufficiently explicit so that we will be able to conclude that $\hat{f}_1 \circ v$ is zero in cohomology. Thus, since $\Sigma^{-1}K_2$ is a wedge of Eilenberg-MacLane spaces, we will have that $\hat{f}_1 \circ v$ is null-homotopic. It then follows that $\zeta_3 = \hat{f}_1 \circ v$ is of order $p$ .

The task is now three-fold.

(1) Define $T_1$ , $Y_1$ , and the maps $f_1$ and $G_1$ .

(2) Construct maps $v$ and $\overline{v}$ .

(3) Construct and analyze $\hat{f}_1$ .

We begin with Step (1). Let $T = \Sigma^s t(V_p^{j+1})$ . Then, since $H^*T$ is cyclic over the Steenrod Algebra, let $T \xrightarrow{w} \Sigma^t k\,\mathbb{Z}_p$ classify the Steenrod Algebra generator of $H^*T$ . Let the following be a cofibration sequence

$$T_1 \longrightarrow T \xrightarrow{w} \Sigma^t K\,\mathbb{Z}_p \quad .$$

$T_1$ is the first stage of an Adams resolution for $T$ . Since, for any spectrum $Z$ , $K\,\mathbb{Z}_p \wedge Z$ is a wedge of Eilenberg-MacLane spectra of type $K\,\mathbb{Z}_p$ , we have a cofibration sequence

$$T_1 \wedge t(V_p) \longrightarrow T \wedge t(V_p) \xrightarrow{w \wedge 1} \Sigma^t K\mathbb{Z}_p \wedge t(V_p)$$

and $T_1 \wedge t(V_p)$ is the first stage of an Adams resolution for $T \wedge t(V_p)$ .

Lemma 2.4.a gives us a map $G : T \wedge t(V_p) \longrightarrow S^0$ on the 1-line of the Adams Spectral Sequence converging to $[T \wedge t(V_p), S^0]$ . So, by our remarks in section 1, we have a map $G_1 : T_1 \wedge t(V_p) \longrightarrow E_2$ so that the following diagram commutes

$$\begin{array}{ccc} T_1 \wedge t(V_p) & \xrightarrow{G_1} & E_2 \\ \downarrow & & \downarrow i_1 \circ i_2 \\ T \wedge t(V_p) & \xrightarrow{G} & S^0 \end{array} \qquad (3.2)$$

This is $T_1$ and $G_1$ . Let us now define $Y$ .

Observe that $H^q T \wedge t(V_p) = 0$ for $q \geq 2p^j(p-1)$ and, if $q = 2p^j(p-1)-1$ , $H^q T \wedge t(V_p) \cong \mathbb{Z}_p$ . Let $Y$ be the $2p^j(p-1)-2$ skeleton of $T \wedge t(V_p)$ . Then, we may assume that there is a cofibration sequence

$$Y \xrightarrow[i]{\subseteq} T\ t(V_p) \xrightarrow{\psi} S^{k+1} \quad .$$

where $k = 2p^j(p-1)-2$ .

Since $i^*$ is surjective in cohomology, we can form a commutative diagram of cofibration sequences

$$\begin{array}{ccccc} Y_1 & \longrightarrow & Y & \longrightarrow & \Sigma^t K\,\mathbb{Z}_p \wedge t(V_p) \\ \downarrow & & \downarrow & & \downarrow \\ T_1 \wedge t(V_p) & \longrightarrow & T \wedge t(V_p) & \xrightarrow{w \wedge id} & \Sigma^t K\,\mathbb{Z}_p \wedge t(V_p) \end{array}$$

and $Y_1$ will be the first stage of an Adams resolution for $Y$ . From (3.2) we get a diagram

$$\begin{array}{ccccc} Y_1 & \longrightarrow & T_1 \wedge t(V_p) & \xrightarrow{G_1} & E_2 \\ \downarrow & & \downarrow & & \downarrow \\ Y & \xrightarrow{i} & T \wedge t(V_p) & \xrightarrow{G} & S^o \end{array}$$

Lemma 2.4.a implies that any lifting of $G \circ i : Y \longrightarrow S^o$ to $E_1$ lifts to $E_2$ . Therefore, by our remarks on Adams resolutions, we may find a map $f_1 : Y_1 \longrightarrow E_3$ so that the following diagram commutes

$$\begin{array}{ccc} Y_1 & \xrightarrow{f_1} & E_3 \\ \Big| & & \Big| i_3 \\ T_1 \wedge t(V_p) & \xrightarrow{G_1} & E_2 \end{array} .$$

This completes Step 1 of our proof. We begin Step 2, the construction of $v$ and $\overline{v}$ . Let $\overline{\mu} : \Sigma^{n-1} M(p) \longrightarrow T \wedge t(V_p)$ be the map of 2.4.b and let $\mu$ be the composition
$S^{n-1} \xrightarrow{\subseteq} \Sigma^{n-1} M(p) \xrightarrow{\overline{\mu}} T\ t(V_p)$ .

Lemma 3.4. There is a map $\tilde{\mu} : S^{n-1} \longrightarrow Y$ making the following diagram commute:

$$\begin{array}{ccc} & & Y \\ & \overset{\tilde{\mu}}{\nearrow} & \downarrow i \\ S^{n-1} & \xrightarrow{\mu} & T_\wedge t(V_p) \end{array} .$$

Proof: Referring to the cofibration sequence (3.3) we have a composition

$$S^{n-1} \xrightarrow{\mu} T_\wedge t(V_p) \xrightarrow{\psi} S^{k+1} .$$

Since $n = 2p^j(p-1)+2(p-1)-2$ and $k = 2p^j(p-1)-2$, $\psi \circ \mu \in \pi^s_{2p-4}(S^0) = 0$. The result now follows.

Lemma 3.4 implies that we have a diagram

$$\begin{array}{ccccc} S^{n-1} & \xrightarrow{\tilde{\mu}} & Y & \longrightarrow & \Sigma^t K\mathbb{Z}_{p\wedge} t(V_p) \\ | & & \searrow & & | \\ \Sigma^{n-1}M(p) & \xrightarrow{\bar{\mu}} & T_\wedge t(V_p) & \xrightarrow{w_\wedge id} & \Sigma^t K\mathbb{Z}_{p\wedge} t(V_p) \end{array} .$$

For dimensional reasons $(w_\wedge id) \circ \bar{\mu}$ is null-homotopic, and, thus, there exists liftings $v$ of $\tilde{\mu}$ and $\bar{v}$ of $\bar{\mu}$ so that the following diagram commutes:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{v} & Y_1 \\ \downarrow & & \downarrow \\ \Sigma^{n-1}M(p) & \xrightarrow{\bar{v}} & T_1 \wedge t(V_p) \end{array} .$$

If $\bar{\zeta}_j : \Sigma^{n-1}M(p) \longrightarrow S^o$ and $\zeta_j : S^{n-1} \longrightarrow S^o$ are the maps of Lemma 1.2, then our definitions and construction imply that the following result holds.

<u>Lemma 3.5.</u> $\zeta_3 = f_1 \circ v : S^{n-1} \longrightarrow E_3$ is a lifting of $\zeta_j$ , $\bar{\zeta}_2 = G_1 \circ \bar{v} : \Sigma^{n-1}M(p) \longrightarrow E_2$ is a lifting of $\bar{\zeta}_j$ , and we have a commutative diagram

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\zeta_3} & E_3 \\ \downarrow & & \downarrow \\ \Sigma^{n-1}M(p) & \xrightarrow{\bar{\zeta}_2} & E_2 \end{array} .$$

This completes Step 2 of our proof. We begin Step 3, the construction and study of $\hat{f}_1$ . To this end we study $Y_1$ more closely, produce a space $X_1$ , and a cofibration sequence

$$X_1 \wedge t(V_p) \longrightarrow Y_1 \longrightarrow S^k \qquad k = 2p^j(p-1)-2 .$$

Let us define $X$ and $x_1$ . Recall that $T = \Sigma^s t(V_p{}^{j+1})$ . Then $H^qT = 0$ for $q > 2(p^j-2)(p-1)-2$ and, if $q = 2(p^j-2)(p-1)-2$ , then $H^qT = \mathbb{Z}_p$ , generated by $\chi(\beta P^{p^{j-1}} P^{p^{j-2}} \dots P^p P^1)w$ .

Here, $w$ is the generator of $H^*T$. Let $X$ be the $2(p^j-2)(p-1)-3$ skeleton of $T$. Then we may assume we have a cofibration sequence

$$X \longrightarrow T \longrightarrow S^{2(p^j-2)(p-1)-2} .$$

In fact, we have a diagram in which both rows and columns are cofibration sequences

$$\begin{array}{ccccc} X\wedge t(V_p) & \longrightarrow & Y & \longrightarrow & S^k \\ \downarrow & & \downarrow i & & \downarrow \\ T\wedge t(V_p) & \longrightarrow & T\wedge t(V_p) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^k M(p) & \longrightarrow & S^{k+1} & \xrightarrow{xp} & S^{k+1} \end{array} \qquad (3.6)$$

Notice that $X \longrightarrow T \xrightarrow{w} \Sigma^t K\mathbb{Z}_p$ is surjective in cohomology. ($w$ is the Steenrod Algebra generator of $H^*T$.) Thus, we have a diagram of cofibration sequences

$$\begin{array}{ccccc} X_1 & \longrightarrow & X & \longrightarrow & \Sigma^t K\mathbb{Z}_p \\ \downarrow & & \downarrow & & \downarrow \\ T_1 & \longrightarrow & T & \xrightarrow{w} & \Sigma^t K\mathbb{Z}_p \end{array} .$$

$X_1$ is the first stage of an Adams Resolution for $X$. Referring to (3.6) we have a commutative diagram

$$\begin{array}{ccccc} X_1\wedge t(V_p) & \longrightarrow & Y_1 & \longrightarrow & T_1\wedge t(V_p) \\ \downarrow & & \downarrow & & \downarrow \\ X\wedge t(V_p) & \longrightarrow & Y & \longrightarrow & T\wedge t(V_p) \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^t K\mathbb{Z}_p\wedge t(V_p) & \xrightarrow{=} & \Sigma^t K\mathbb{Z}_p\wedge t(V_p) & = & \Sigma^t K\mathbb{Z}_p\wedge t(V_p) \end{array} . \qquad (3.7)$$

Combining these remarks, we note that the following lemma holds.

Lemma 3.8. There is a commutative diagram in which both rows and columns are cofibrations:

$$\begin{array}{ccccc}
X_1 \wedge t(V_p) & \longrightarrow & Y_1 & \longrightarrow & S^k \\
\downarrow & & \downarrow & & \| \\
X \wedge t(V_p) & \longrightarrow & Y & \longrightarrow & S^k \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^t K\, \mathbb{Z}_p \wedge t(V_p) & \xrightarrow{=} & \Sigma^t K\, \mathbb{Z}_p \wedge t(V_p) & \longrightarrow & *
\end{array}$$

$k = 2p^j(p-1)-2$ and $*$ = point.

This brings us to the construction of $\hat{f}_1$ . We refer to the diagram:

$$\begin{array}{ccccc}
 & & & \overset{\hat{f}_1}{\dashrightarrow} & \Sigma^{-1}K_2 \\
 & & & & \downarrow \\
Y_1 & \xrightarrow{p} & Y_1 & \xrightarrow{f_1} & E_2 \\
\downarrow & & \downarrow & & \downarrow i_3 \\
T_1 \wedge t(V_p) & \xrightarrow{p} & T_1 \wedge t(V_p) & \xrightarrow{G_1} & E_2
\end{array} \quad .$$

Lemma 3.9. There exists an element $\phi \in \pi_k \Sigma^{-1} K_2$ $(k = 2p^j(p-1)-2)$ so that $f_1 \circ p : Y_1 \longrightarrow E_3$ factors as a composition

$$f_1 \circ p : \; Y_1 \xrightarrow{\rho} S^k \xrightarrow{\phi} \Sigma^{-1} K_2 \longrightarrow E \quad .$$

Remark: $\rho$ is as in (3.8). We will set $f_1 = \phi \circ \rho$ .
Before proving (3.9), let us show that (3.9) implies that Lemma 1.7 holds.

<u>Corollary 3.10</u>. Let $\zeta_3 = f_1 \circ v : S^{n-1} \longrightarrow E_3$ . Then $\zeta_3$ has order $p$ .

<u>Proof</u>: We ask whether $S^{n-1} \xrightarrow{p} S^{n-1} \xrightarrow{\zeta_3} E_3$ is null-homotopic. This composition may be written

$$S^{n-1} \xrightarrow{v} Y_1 \xrightarrow{p} Y_1 \xrightarrow{f_1} E_3$$

which, in turn, by (3.9), may be written

$$S^{n-1} \xrightarrow{v} Y_1 \xrightarrow{\rho} S^k \xrightarrow{\phi} \Sigma^{-1}K_2 \longrightarrow E^3 .$$

Since $n-1 = 2p^j(p-1)+2(p-1)-3$ and $k = 2p^j(p-1)-2$ , $\rho \circ v$ is zero in cohomology. Thus, $\phi \circ \rho \circ v$ is null-homotopic.

This completes the proof of (3.10).

Thus proving Lemma 1.7 and therefore Theorem 1.1 has been reduced to proving (3.9).

<u>Proof of (3.9)</u>: Let $p : t(V_p) \longrightarrow t(V_p)$ be $p$ times the identity. Since $t(V_p) \simeq \Sigma^{2p-2}M(p)$ this map is null-homotopic (remember $p \geq 3$ ). Now if $Z$ is any spectrum let $c(Z)$ be the cone on $Z$ . Let

$$h : c(t(V_p)) \longrightarrow t(V_p)$$

extend $p$ . The composition

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$$H_1 \ : \ c(Y_1) \longrightarrow c(T_1 \wedge t(V_p)) = T_1 \wedge c(t(V_p))$$

$$\xrightarrow{1 \wedge h} T_1 \wedge t(V_p) \xrightarrow{G_1} E_2$$

is a null homotopy of $i_3 \circ f_1 \circ p$ . $(i_3 : E_3 \longrightarrow E_2)$ .
When restricted to

$$c(X_1 \wedge t(V_p)) \subseteq c(Y_1) \ .$$

$H_1$ factors through the map (refer to (3.7))

$$H_2 \ : \ c(X_1 \wedge t(V_p)) = X_1 \wedge c(t(V_p)) \xrightarrow{i \wedge h} X_1 \wedge t(V_p) \longrightarrow Y_1 \xrightarrow{f_1} E_3 \ .$$

Define $\overline{\phi} : Y_1 \cup c(X_1 \ t(V_p)) \longrightarrow E_3$ to be the union of $f_1 \circ p$ on $Y_1$ and $H_2$ on $c(X_1 \wedge t(V_p))$ . Observe that

$$Y_1 \cup c(X_1 \wedge t(V_p)) \simeq S^k \ . \qquad \text{(Refer to (3.8).)}$$

We then have a commutative diagram with $\overline{\phi} \circ \rho = f_1 \circ p$

$$\begin{array}{ccccc} Y_1 \longrightarrow S^k \simeq Y_1 \cup & c(X_1 \wedge t(V_p)) & \xrightarrow{\overline{\phi}} & E_3 & \\ & \downarrow & & \downarrow i_3 & \\ & c(Y_1) & \xrightarrow{H_1} & E_2 & . \end{array}$$

Thus $i_3 \circ \overline{\phi}$ is null-homotopic since it factors through $c(Y_1)$ .
Hence there is a lifting $\phi$ :

$$\begin{array}{ccccc} & & & & \Sigma^{-1}K_2 \\ & & & \overset{\phi}{\nearrow} & \downarrow \\ Y_k & \xrightarrow{\rho} & S^k & \xrightarrow{\overline{\phi}} & E_3 \end{array} \quad .$$

This completes the proof of (3.9) and, therefore, Step 3 of our proof of Lemma 1.7.

## 4. $h_o h_j$ detects a map of order $p$ .

The goal of this section is to show that, not only is $h_o h_j$ an infinite cycle in the Adams Spectral Sequence (Theorem 1.1), but it represents an element $\eta_j \in \pi_n S^o$ of order $p$ . This follows from the following result.

Theorem 4.1. Let $p$ be a prime greater than 3 or let $p = 3$ and $j$ be odd. Then there exists a map

$$\bar{\eta}_j : S^n \longrightarrow \Sigma^{-1}M(p) \qquad (n = 2(p^j+1)(p-1)-2)$$

so that in $H^*M(\bar{\sigma}_j)$ , $\phi_{o,j}(\sigma_o) = \Sigma\sigma_n$ , where $\sigma_o \in H^o\Sigma^{-1}M(p)$ and $\sigma_n \in H^n S^n$ are the generators.

Observe that Theorem 4.1 and Theorem 1.1 imply Theorem 1 of the introduction by letting $\eta_j$ be the composition

$$\eta_j : S^n \xrightarrow{\bar{\eta}_j} \Sigma^{-1}M(p) \xrightarrow{\delta} S^o$$

where $\delta$ pinches to the top cell of $\Sigma^{-1}M(p)$ .

Fix a prime $p$ and integer $j$ conforming to the hypotheses of Theorem 4.1.

The proof of 4.1 proceeds in three steps:

1) We produce a map $\tilde{\eta}_j : \Sigma^n M(p) \longrightarrow M(p)$ so that in $H^*M(\tilde{\eta}_j)$ , $\phi_{o,j}(\sigma_o) = \Sigma\sigma_n$ and $\phi_{o,j}(\sigma_1) = \Sigma\sigma_{n+1}$ with zero indeterminacy. Here $\sigma_j$ generates the appropriate cohomology group.

2) We produce a map $\varepsilon_j : \Sigma^n M(p) \longrightarrow M(p)$ so that in $H^* M(\varepsilon_j)$ $\Phi_{o,j}(\sigma_o) = (-1)^{j-1} 2\Sigma\sigma_n$ and $\Phi_{o,j}(\sigma_1) = \Sigma\sigma_{n+1}$ with zero indeterminacy.

3) Then we set $\gamma_j = \varepsilon_j + (-1)^j 2\tilde{\eta}_j : \Sigma^n M(p) \longrightarrow M(p)$ . Then in $H^* M(\gamma_j)$ , $\Phi_{o,j}(\sigma_o) = 0$ and $\Phi_{o,j}(\sigma_1) = (1+(-1)^j 2)\Sigma\sigma_{n+1}$ with zero indeterminacy.

From this we will argue, in an analogous manner to our proof of Theorem 1.1, that there exists a map $\bar{\eta}_j : S^{n+1} \longrightarrow M(p)$ satisfying Theorem 4.1.

<u>Remark</u>: Consider the map $\gamma_j$ of Step 3. If $j$ is even and $p = 3$ , then, in $H^* M(\gamma_j)$ , $\Phi_{o,j}(\sigma_1) = (1+(-1)^j 2)\Sigma\sigma_{n+1} = 0$ . Thus our proof breaks down. This explains the hypotheses of Theorem 4.1 and of Theorem 1 of the introduction.

Step 1 of our program is easy. Let $\hat{\eta}_j : S^n \longrightarrow S^o$ be the map given by Theorem 1.1. Set $\tilde{\eta}_j = \hat{\eta}_j \wedge id : S^n \wedge M(p) \longrightarrow S^o \wedge M(p)$ .

<u>Lemma 4.2</u>. In $H^* M(\tilde{\eta}_j)$ , $\Phi_{o,j}(\sigma_o) = \Sigma\sigma_n$ and $\Phi_{o,j}(\sigma_1) = \Sigma\phi_{n+1}$ with zero indeterminacy.

<u>Proof</u>: $\hat{\eta}_j$ is represented by $h_o h_j$ in the $E_2$ term of the Adams Spectral Sequence. The result now follows.

Step 2 is only a bit harder. For this we must combine and interpret some of the results of Section 2.

Let $U = \Sigma^s t(V_p^{j+1}{}_{+p})$ be the spectrum of Section 2. Then Lemma 2.3 implies that the identity map of $U$ has order $p$. Thus the map

$$1 \wedge \delta : U \wedge \Sigma^{-1} M(p) \longrightarrow U \wedge S^0$$

has a section $\Psi : U \longrightarrow U \wedge \Sigma^{-1} M(p)$ so that $(1 \wedge \delta) \circ \Psi$ is the identity of $U$.

Recall that $H^* U$ is cyclic over the Steenrod Algebra with generator $v$, so we may consider elements $a_j = \chi(\beta P^{p^{j-1}} P^{p^{j-2}} \ldots P^1 \beta)(v)$ and $\alpha_j = \chi(\beta P^{p^{j-1}} \ldots P^1)(v)$ in $H^* U$.

Now, let $g : U \longrightarrow S^0$ be the map of 2.1. Set $\overline{g}$ to be the composition

$$\overline{g} : \Sigma U \xrightarrow{\Psi} U\, M(p) \xrightarrow{g \wedge id} M(p) .$$

Then Lemma 2.1 and a calculation of the cohomological properties of $\Psi$ imply that the following result holds.

<u>Lemma 4.3</u>. In $H^* M(\overline{g})$, $P^{p^j}(\sigma_0) = -\Sigma \alpha_j$ and $P^{p^j}(\sigma_1) = \Sigma a_j$.

We can now define $\varepsilon_j$. Let $\overline{\xi} : \Sigma^{n-1} M(p) \longrightarrow U$ be the map of 2.2. Then in $H^* M(\overline{\xi})$ we saw that $P^1(\alpha_j) = (-1)^j 2 \Sigma \sigma_{n-1}$ and $P^1(a_j) = \Sigma \sigma_n$. We define $\varepsilon_j$ to be the self map of the Moore spectrum given by the composition

$$\varepsilon_j : \Sigma^n M(p) \xrightarrow{\bar{\xi}} \Sigma U \xrightarrow{\bar{g}} M(p) .$$

Lemma 4.4. In $H^*M(\xi_j)$ , $\Phi_{o,j}(\sigma_o) = (-1)^{j-1} 2\, \Sigma\sigma_n$ and $\Phi_{o,j}(\sigma_1) = \Sigma\sigma_n$ both with zero indeterminacy.

Proof: This follows from 4.3 and 2.2.

We now begin Step 3.

Define $\gamma_j = \varepsilon_j + (-1)^j 2\hat{n}j : \Sigma^n M(p) \longrightarrow M(p)$ .

Corollary 4.5. In $H^*(\gamma_j)$

$\Phi_{o,j}(\sigma_o) = 0$ and $\Phi_{o,j}(\sigma_1) = (1 + (-1)^j 2))(\Sigma\sigma_{n+1})$ both with zero indeterminacy.

Proof: This follows from 4.2 and 4.4.

We are now ready to complete the proof of Theorem 4.1.

Proof of Theorem 4.1: Let

$$\begin{array}{ccccccc} \cdots \to E_3 \to & E_2 & \to E_1 \to S^o \\ & \downarrow & \\ & K_2 & \end{array}$$

be the minimal Adams resolution for $S^o$ given in Section 1. Then

$$\begin{array}{ccccccc} \cdots E_3 \wedge M(p) & \to & E_2 \wedge M(p) & \to & E_1 \wedge M(p) & \to & M(p) \\ & & \downarrow & & & & \\ & & K_2 \wedge M(p) & & & & \end{array}$$

is an Adams resolution of $M(p)$ . $\pi_{n+1}(K_2 \wedge M(p)) \cong \mathbb{Z}_p$ generated by an element that represents $\overline{h_o h_j} \in \mathrm{Ext}^2_{A_p}(H^*M(p), \mathbb{Z}_2)$ where under the homomorphism

$$\delta_* : \mathrm{Ext}_{A_p}(H^*M(p), \mathbb{Z}_p) \to \mathrm{Ext}_{A_p}(H^*(S^1), \mathbb{Z}_p) \cong \mathrm{Ext}_A(\mathbb{Z}_p, \mathbb{Z}_p)$$

$\delta_*(\overline{h_o h_j}) = h_o h_j$ . (Recall that $\delta : M(p) \to S^1$ is the pinch map.)

We now essentially repeat the argument given to prove Theorem 1.1 in order to prove Theorem 4.1. Namely, by Corollary 4.5 there exists a diagram of liftings

$$\begin{array}{ccccc} & & & & E_3 \wedge M(p) \\ & \overline{\gamma}_{j,3} \nearrow & & & \downarrow \\ & & & \gamma_{j,2} \nearrow & E_2 \wedge M(p) \\ & & & & \downarrow \\ S^n & \subset\!\!\longrightarrow & \Sigma^n M(p) & \xrightarrow[\gamma_j]{} & M(p) \end{array}$$

Furthermore, by 4.5 there exists a commutative diagram of cofibration sequences

$$\begin{array}{ccccccc}
S^n & \hookrightarrow & \Sigma^n M(p) & \longrightarrow & S^{n+1} & \xrightarrow{\ p\ } & S^{n+1} \\
\downarrow{\scriptstyle \bar{\gamma}_{j,3}} & & \downarrow{\scriptstyle \gamma_{j,2}} & & \downarrow{\scriptstyle (1+(-1)^j 2)\overline{h_o h_j}} & & \downarrow{\scriptstyle \Sigma\bar{\gamma}_{j,3}} \\
E_3 \wedge M(p) & \to & E_2 \wedge M(p) \to K_2 \wedge M(p) & & & \longrightarrow & \Sigma E_3 \wedge M(p)
\end{array}$$

Because p _ 3 , the identity map of $M(p)$ is of order $p$ . Therefore the composition $S^{n+1} \xrightarrow[p]{} S^{n+1} \xrightarrow[\Sigma\bar{\gamma}_{j,3}]{} \Sigma E_3 \wedge M(p)$

Thus there exists a lifting

$$\begin{array}{ccc}
 & \overset{\eta_2'}{\nearrow} & E_2 \quad M(p) \\
 & & \downarrow \\
S^{n+1} & \xrightarrow[(1+(-1)^j 2)\overline{h_o h_j}]{} & K_2 \quad M(p)
\end{array}$$

Define $\eta_j$ to be the composition $S^{n+1} \xrightarrow[\eta_2']{} E_2 \wedge M(p) \to M(p)$ .

$\eta_j$ is represented by $(1 + (-1)^j 2)\overline{h_o h_j}$ in $\mathrm{Ext}_A\,(H^* M(p), Z_p)$ .

Now set $\bar{\eta}_j = \Sigma^{-1} q_{\ j}$ , where $q \in \mathbb{Z}_p$ is the multiplication inverse of $1 + (-1)^j 2$ in $\mathbb{Z}_p$ .

This completes the proof of Theorem 4.1 and therefore of Theorem 1 in the introduction.

## 5. More infinite families in $\pi_* S^0$ .

We now turn to Theorem 3 of the introduction. Let $p$ be an odd prime and $j$ be any positive integer if $p \geq 5$ or if $p = 3$ let $j$ be odd. Theorem 1 yields a map

$$\eta_j : S^n \to S^o , n = 2(p^j + 1)(p-1)-2$$

of order $p$ represented by $h_o h_j$ in the Adams spectral sequence. Consider the following diagram

$$\begin{array}{ccccc} & \overset{\bar{\eta}_j}{\nearrow} & \Sigma^{2i(p-1)-1}M(p) & \xrightarrow{\alpha^i} & \Sigma^{-1}M(p) \to S^o \\ & & \downarrow & & \\ S^{n+2i(p-1)} & \xrightarrow[\eta_j]{} & S^{2i(p-1)} & & \end{array}$$

Here $\alpha : \Sigma^{2(p-1)}M(p) \to M(p)$ is a map of Moore spectra which makes the diagram commute

$$\begin{array}{ccc} \Sigma^{2(p-1)}M(p) & \xrightarrow{\alpha} & M(p) \\ \uparrow & & \downarrow \delta \\ S^{2(p-1)} & \xrightarrow[\alpha_1]{} & S^1 \end{array}$$

where $\alpha_1 \in \pi_{2p-3}(S^o)$ is the generator. $\alpha^i$ is the composition of $\alpha$ with itself $i$ times. $\bar{\eta}_j$ is a lifting of $\eta_j$ which exists because $\eta_j$ has order $p$ (see 4.1).

Define $\eta_{j,i} \in \pi_{n+2i(p-1)}(S^o)$ by $\eta_{j,i} = \pi \circ \alpha^{i-1} \circ \eta_j$. Observe that $\eta_j = \eta_{j,1}$.

By postpoining the definition of $\hat{\beta}_{p^j/k}$ to 5.3, we can state

Theorem 5.2: $\eta_{j,1}$ is essential for $1 \leq i \leq p^j-1$ and is detected by a non-zero element $\hat{\beta}_{p^j/p^{j-1}-1}$ in the Adams Novikov spectral sequence.

We begin the proof by recalling the results of Miller, Ravenel, and Wilson [23] concerning the Adams-Novikov spectral sequence based on the Brown-Peterson spectrum BP.

Theorem 5.3: For $p \geq 3$

(1) $\mathrm{Ext}^o_{BP_*BP}(BP_*,BP_*) = \mathbb{Z}_p$, concentrated in degree zero.

(2) [26] $\mathrm{Ext}^1_{BP_*BP}(BP_*,BP_*)$ is generated by classses $\alpha_{sp^m/n+1}$ (for $m \geq 0$, $p \nmid s \geq 1$) of order $p^m$ and degree $2sp^m(p-1)$.

(3) [23] $\mathrm{Ext}^2_{BP_*BP}(BP_*,BP_*)$ is a direct sum of cyclic subgroups generated by classes $\beta_{sp^m/(j,i+1)}$ for $m \geq 0$, $p \nmid s \geq 1$, $j \geq 1$, $i \geq 0$, and subject to

(a) $j \leq p^m$ if $s = 1$

(b) $p^i | j \leq a_{m-i}$, where $a_o = 1$ and $a_k = p^k + p^{k-1} - 1$, $k \geq j$

and

(c) $a_{m-i-1} < j$ if $p^{i+1} | j$.

$\beta_{sp^m/(j,i+1)}$ has order $p^{i+1}$ and degree $2(p^2-1)sp^m - 2(p-1)j$ .

Note: (1) We write $p^n\alpha_{sp^m/n+1} = \alpha_{sp^m}$ and $p^i\beta_{sp^m/(j,i+1)} = \beta_{sp^m/j}$ .

(2) Novikov showed in [26] that $\alpha_{sp^m/n+1}$ detects an element of rder $p^{n+1}$ in $\pi_{2sp^m(p-1)-1}(S^o)$ contained in the image of $J$ .

Lemma 5.4: Let $\eta_j : S^n \longrightarrow S^o$ be a map of order $p$ detected by $h_oh_j$ . Then $\eta_j$ is represented, in the Adams Novikov Spectral Sequence, by an element

$$\hat{\beta}_{p^{j-1}/p^{j-1}-1} = \beta_{p^{j-1}/p^{j-1}-1} + \Sigma\, c_i\, \beta_{a(i)/b(i)} \qquad c_i \in \mathbb{Z}_p$$

where $a(i) = (p^{2i+1}/p+1)p^{j-2i-1}$

$b(i) = p^{j-2i-1}-1$

and $1 \leq i \leq [j/2] - 1$

Proof: There is a map of ring spectra $\Phi : BP \longrightarrow K\mathbb{Z}_p$ . This gives a map of Adams-type spectral sequences, and thus a map

$$\Phi_* : Ext_{BP_*BP}(BP_*,BP) \longrightarrow Ext_A(\mathbb{Z}_p, \mathbb{Z}_p)$$

In [23] it was shown that $\Phi_*(\beta_{p^{j-1}/p^{j-1}-1}) = h_oh_j$ . Since $Ext^{o,n}_{BP_*BP}(BP_*,BP) = Ext^{1,n+1}_{BP_*BP}(BP_*,BP_*) = 0$ and

$$\mathrm{Ext}^{2,n+2}_{BP_*BP}(BP_*,BP_*) \cong (\mathbb{Z}_p)^{[\frac{j}{2}]}$$

generated by $\beta_{p^{j-1}/p^{j-1}-1}$ and $\beta_{a(i)/b(i)}$, the result follows.

Definition 5.5: Define

$$\hat{\beta}_{p^j/k} = \beta_{p^{j-1}/k} + \Sigma\, c_i\, \beta_{a(i)/b(i,k)}$$

where $c_i$ is an in 5.4 and

$$a(i) = (p^{2i+1}/p+1)p^{j-2i-1}$$

$$b(i,k) = k - p^{j-1} + p^{j-2i-1}$$

and $1 \le i \le \max\{\ell | p^{2\ell} < p^{j-1}/p^{j-1} - k\}$ .

Proof of Theorem 5.2: The cofibration sequence $S^0 \underset{xp}{\to} S^0 \underset{i}{\to} M(p) \overset{\delta}{\to} S^1$ induces a short exact sequence in Brown-Peterson homology:

$$0 \to BP_* \overset{xp}{\to} BP_*/(p) \to 0$$

where $BP_* = BP_*(S^0) = \pi_*BP = \mathbb{Z}_{(p)}[v_i]$ where $\dim v_i = 2(p^i-1)$ . This is induces a long exact sequence in Ext :

$$\cdots \to \mathrm{Ext}^{s}_{BP_*BP}(BP_*,BP_*) \overset{xp}{\to} \mathrm{Ext}^{s}_{BP_*BP}(BP_*,BP_*) \overset{i_*}{\to}$$

$$\mathrm{Ext}^{s}_{BP_*BP}(BP_*,BP_*/(p)) \overset{\delta_*}{\to} \mathrm{Ext}^{s+1}_{BP_*BP}(BP_*BP_*) \to \cdots$$

and $\delta_*$ is induced by a map of spectral sequences [35]. Theorem 5.3 and this long exact sequence imply that $\mathrm{Ext}^1_{BP_*BP}(BP_*BP_*/p))$ is a $\mathbb{Z}_p$-vector spaces generated by elements $\overline{\alpha}_j$ and $\overline{\beta}_{sp^m/j}$ where $i_*\alpha_{sp^m/n+1} = \overline{\alpha}_{sp^m}$ and $\delta_*\overline{\beta}_{sp^m/j} = \beta_{sp^m/j}$ .

Define $\beta'_{p^{j-1}/k} = \overline{\beta}_{p^{j-1}/k} + \Sigma\, c_i\, \overline{\beta}_{a(i)/b(i,k)}$ .

Lemma 5.4 then implies that $\overline{\eta}_j$ (of 5.1) is represented by a class of the form

$$\beta'_{p^j/p^j-1} + c\overline{\alpha}_{p^j+1} \quad , \text{ where } c \in \mathbb{Z}_p \, .$$

(Note: Since $\overline{\alpha}_{p^j+1}$ represents an element in $\pi_*M(p)$ in the image of $i : S^o \to M(p)$ , $\overline{\eta}_j$ can be chosen so that $c = 0$ .)

Now $\alpha^{i-1} : \Sigma^{2(i-1)(p-1)}M(p) \to M(p)$ induces multiplication by $v_1^{i-1}$ in $BP_*(M(p)) = BP_*/(p)$ . Therefore the element $\overline{\eta}_{j,1} = \alpha^{i-1}\circ\overline{\eta}_j : S^{\ \ +2(i-1)(p-1)} \to \Sigma^{-1}M(p)$ is represented by

$$\beta'_{p^j/p^j-i} + c\overline{\alpha}_{p^j+i}$$

by the construction of the $\beta$'s . (See [23].) Thus $\eta_{j,i}$ is represented by $\hat{\beta}_{p^j/p^{j-i}}$ .

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