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$\Omega(n \log n)$  LOWER BOUNDS ON LENGTH OF BOOLEAN FORMULAS

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*Abstract. A property of Boolean functions of  $n$  variables is described and shown to imply lower bounds as large as  $\Omega(n \log n)$  on the number of literals in any Boolean formula for any function with the property. Formulas over the full basis of binary operations ( $\wedge$ ,  $\oplus$ , etc.) are considered. The lower bounds apply to all but a vanishing fraction of symmetric functions, in particular to all threshold functions with sufficiently large threshold and to the "congruent to zero modulo  $k$ " function for  $k > 2$ . In the case  $k = 4$  the bound is optimal.*

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**1. Introduction.** We describe a property of Boolean functions of  $n$  variables which implies lower bounds on the size of all Boolean formulas for functions with the property. Let  $C_k^n$  be the Boolean function "congruent to zero modulo  $k$ " of  $n$  arguments, that is,  $C_k^n(x_1, \dots, x_n)$  iff  $\sum_{i=1}^n x_i \equiv 0 \pmod{k}$ . We show that  $C_k^n$  has the property and conclude that there is a constant  $\epsilon > 0$  such that any Boolean formula for  $C_k^n$  over the full basis of binary and unary Boolean operations ( $\wedge$ ,  $\vee$ ,  $\neg$ ,  $\oplus$ , *NAND*, etc.) is of length exceeding  $\epsilon n \log(n/k)$  for all  $k \geq 3$  and all  $n$ . There are formulas for  $C_4^n$  of length asymptotic to  $n \log_2 n$ , so our bound is achieved to within a constant multiple in this case.

The logarithm of the minimum length of a formula for a Boolean function gives the minimum *time*, i.e., depth, of a combinational circuit computing the function. This remark provides some technological motivation for our results. The depth of formulas is also related to the space and parallel time of computations and so is of basic concern in the theory of computational complexity; see [Pat 76, McC 78a, 78b, Bor 77] for further discussion.

General counting arguments allow one to conclude that *most* Boolean functions of  $n$  variables require formulas of size asymptotic to  $2^n / \log_2 n$  [RiS 42, Lup 60, Kri 61]. The largest lower bound provable for *explicit* examples however is proportional to  $n^2 / \log n$  by Neciporuk [Nec 66].<sup>1</sup> Although Neciporuk's method yields lower bounds for many explicit examples [cf. Pat 76, 77], no symmetric function possesses the property which implies Neciporuk's lower bounds. Hodes and Specker [HoS 68] provide another general property of functions which implies nonlinear lower bounds on the length of formulas, and Hodes [Hod 70] demonstrates that it is widely applicable.<sup>2</sup> For example, Hodes' and Specker's results imply that formulas for all but sixteen of the  $2^{n+1}$  symmetric Boolean functions of  $n$  variables grow nonlinearly in  $n$  [Khr 76, Pat 76,77].

Our main theorem resembles that of Hodes and Specker. We essentially show that any function which can be defined by a "small" formula can be restricted to a "large" subset of its variables so that the resulting restricted formula is equivalent to the sum modulo two of a subset of its variables. Since  $C_k^n$  and indeed almost all symmetric functions do not have such large simple restrictions, they cannot have small formulas. Comparing our results to Hodes' and Specker's in the most interesting case of symmetric functions, we note that their theorem yields nonlinear lower bounds whenever ours does, but their bounds are much smaller.<sup>3</sup> Indeed, our bounds of  $\Omega(n \log n)$  are the largest lower bounds on formula length known for *any* symmetric Boolean function. (We remind the reader that  $\alpha(n) = \Omega(\beta(n))$  iff  $\beta(n) = O(\alpha(n))$  iff  $\liminf \alpha(n)/\beta(n) > 0$ .)

In the next section we state the main theorem giving lower bounds and apply it to  $C_k^n$  and a related example. In Section 3 we derive a corollary which is easily applicable to arbitrary symmetric functions and then prove that all but a vanishing fraction of symmetric functions require formulas of length  $\Omega(n \log n)$ . Section 4 contains the proof of the main theorem. In the final Section 5, we compare known upper and lower bounds on formula length and mention some open problems.

**2. The Lower Bound.** Boolean formulas over the *full unary-binary basis* are constructed from variables and constants (0 and 1) possibly using any of the unary and binary Boolean connectives ( $\wedge$ ,  $\vee$ ,  $\neg$ ,  $\oplus$ , *NAND*, etc.). Let  $L(f)$ , the *length* of the formula  $f$ , be the number of occurrences of variables (not constants) in  $f$ . Let  $\text{var}(f)$  be the set of variables that appear in  $f$ . Formulas  $f$  and  $g$  are *equivalent*, denoted  $f \equiv g$ , iff  $f$  and  $g$  define the same function on  $\text{var}(f) \cup \text{var}(g)$ . Every Boolean formula  $g$  is easily shown to be equivalent to a Boolean formula  $f$  constructed from variables and constants using *only* the two connectives  $\oplus$  ("exclusive or") and  $\wedge$  ("and") such that  $f$  contains *exactly* the same number of occurrences of each variable as  $g$ . In particular  $L(g) = L(f)$ , so without loss of generality we henceforth consider Boolean formulas constructed from variables and constants using only  $\oplus$  and  $\wedge$ .

An *assignment*,  $A$ , over a set of variables,  $V$ , is a partial map from  $V$  into  $\{0,1\}$ ;  $\text{dom}(A) \subseteq V$  is the set of variables on which  $A$  is defined, i.e., the variables which  $A$  *fixes*. The *eccentricity*,  $\text{ecc}(A)$ , of an assignment  $A$  is the excess of 1's over 0's in the assignment, that is,  $\text{ecc}(A) = |A^{-1}(1)| - |A^{-1}(0)|$ .  $A$  is *central* if  $\text{ecc}(A)$  is zero or one. Given a formula  $f$  and an assignment  $A$ , the *restriction*,  $f|_A$ , is the formula obtained by substituting  $A(x)$  for each occurrence of  $x$  in  $f$ , where  $x$  ranges over  $\text{dom}(A)$ . If  $A$  is central and  $\text{dom}(A) \subseteq \text{var}(f)$ , then  $f|_A$  is called a *central restriction* of  $f$ .

The *dimension*,  $\text{dim}(f)$ , of  $f$  is the cardinality,  $|\text{var}(f)|$ , of  $\text{var}(f)$ . The formula  $f$  is *affine* iff  $f$  is equivalent to some formula of the form  $\oplus W \oplus c$  where  $c \in \{0,1\}$  and  $W \subseteq \text{var}(f)$ . The theorem below shows that any Boolean formula of  $n$  variables, all of whose affine central restrictions have small dimension, has length  $\Omega(n \log n)$ . More precisely, let the *affine diameter*,  $\text{diam}(f)$ , of  $f$  be the largest dimension of any affine central restriction of  $f$ .

**Lower Bound Theorem.** There is an  $\epsilon > 0$  such that for any Boolean formula  $f$  with  $n$  variables

$$L(f) \geq \epsilon n \log(n/\text{diam}(f)).$$

The theorem immediately applies to formulas for  $C_k^n$ . To see this, note that the only affine restrictions of  $C_k^n$  either are of dimension one or are equivalent to constant functions of dimension less than  $k$ , so  $\text{diam}(C_k^n) < k$ . Therefore,

*Example 1.*  $L(C_k^n) > \epsilon n \log(n/k)$ .

As another example, consider  $n = km$  variables  $x_{ij}$  for  $1 \leq i \leq k$ ,  $1 \leq j \leq m$  and refer to the variables with second index  $j$  as the  $j^{\text{th}}$  block of variables. Let  $p_j$  denote the mod 2 sum of the  $j^{\text{th}}$  block, namely,  $p_j = \bigoplus_{i=1}^k x_{ij}$ , and let  $f^{k,m}$  be the function  $C_4^m(p_1, \dots, p_m)$  of  $n$  variables. It is not hard to see that no restriction of a formula for  $f^{k,m}$  which contains variables from three or more blocks is affine. Hence  $\text{diam}(f^{k,m}) \leq 2k$ , so

*Example 2.*  $L(f^{k,m}) \geq \epsilon km \log(m/2)$  for  $\epsilon$  as in the Lower Bound Theorem.

We remark that choosing  $k = n^{1-\delta}$  still yields  $\Omega(n \log n)$  lower bounds on  $L(f^{k,m})$  even though  $f^{k,m}$  has "large" affine diameter  $n^{1-\delta}$ . This is an example where Hodes' and Specker's results do not apply.

To establish an upper bound on  $L(C_4^n)$ , let  $x$  denote Boolean variables  $x_1, \dots, x_n$ . Construct formulas  $D_0^n(x)$  and  $D_1^n(x)$  for the low order and second lowest order digits of the binary representation of  $\sum_{i=1}^n x_i$  as follows.  $D_0^1(x_1) = x_1$  and  $D_1^1(x_1) = 0$ . Let  $y$  denote  $x_{n+1}, \dots, x_{2n}$ . Then

$$D_0^{2n}(x,y) = \bigoplus_{i=1}^{2n} x_i, \text{ and}$$

$$D_1^{2n}(x,y) = D_1^n(x) \oplus D_1^n(y) \oplus (D_0^n(x) \oplus D_0^n(y)).$$

Hence  $L(D_0^n) = n$ , and  $L(D_1^{2n}) = 2(L(D_1^n) + L(D_0^n))$ . This recurrence implies that  $L(D_1^n) \leq n \log_2 n$  when  $n$  is a power of two. Now a formula for  $C_4^n$  is  $\text{NOR}(D_0^n, D_1^n)$ , so  $L(C_4^n) \leq n(1 + \log_2 n)$  when  $n$  is a power of two. For arbitrary  $n$ , one can obtain a formula for  $C_4^n$  of length  $n \lceil \log_2 n \rceil + 2n - 2^{\lceil \log_2 n \rceil}$ , so

*Proposition 1.*  $L(C_4^n) < n \lceil 1 + \log_2 n \rceil$  for all  $n$ .

Since  $L(f^{k,m}) \leq L(C_4^m) \cdot L(p_j)$  and  $L(p_j) = k$ , we also have

**Proposition 2.**  $L(f^{k,m})$  is asymptotically at most  $km \log_2 m$ .

So the lower bounds on  $L$  in Example 1 for  $k = 4$  and in Example 2 are achievable to within a multiplicative factor.

**3. Lower Bounds For Symmetric Functions.** For any Boolean formula  $f$  of dimension  $n$  which defines a symmetric function, there is by definition a *characteristic function*,

$$\chi_f: \{0, \dots, n\} \rightarrow \{0, 1\}, \text{ such that } f(x_1, \dots, x_n) = \chi_f(\sum_{i=1}^n x_i).$$

**Lemma.** If  $\chi_f(\lfloor n/2 \rfloor) \neq \chi_f(\lfloor n/2 \rfloor + 2)$ , then  $L(f) \geq \epsilon n \log(n/2)$ .

*Proof.* The reader can easily verify that no *central* restriction of  $f$  which has three or more variables can be affine, viz.,  $\text{diam}(f) \leq 2$ . The bound on  $L(f)$  now follows immediately from the Lower Bound Theorem.  $\square$

**Symmetric Function Lower Bound Theorem.** There is an  $\epsilon > 0$  such that for every formula  $f$  of dimension  $n$  which defines a symmetric function, if  $\chi_f(k) \neq \chi_f(k+2)$  for some  $k$ ,  $0 \leq k \leq n - 2$ , then

$$L(f) \geq \epsilon n \log \min(k, n - k).$$

*Proof.* Assume without loss of generality that  $k \leq n/2$ . Let  $A$  be any assignment such that  $|\text{dom}(A) \cap \text{var}(f)| = n - 2k$  and  $A(x) = 0$  for all  $x \in \text{dom}(A)$ . Now  $\chi_{f|_A}(j) = \chi_f(j)$  for  $0 \leq j \leq 2k = \text{dim}(f|_A)$ , so applying the Lemma above to  $f|_A$  yields

$$L(f|_A) \geq \epsilon 2k \log(2k/2).$$

Therefore, at least one of the  $2k$  variables of  $f|_A$  occurs  $\epsilon \log k$  or more times in  $f|_A$ , and *a fortiori* also occurs that often in  $f$ .

By choosing  $\text{dom}(A)$  to be the  $n - 2k$  most frequently occurring variables in  $f$ , we conclude that each variable in  $\text{dom}(A)$  occurs at least  $\epsilon \log k$  times in  $f$ , so

$$L(f) \geq (n - 2k)\epsilon \log k + L(f|_A) \geq (n - 2k)\epsilon \log k + 2k\epsilon \log k = \epsilon n \log k. \quad \square$$

Let  $T_k^n$  be the threshold  $k$  function of  $n$  variables, that is,

$$T_k^n(x_1, \dots, x_n) = 1 \text{ iff } \sum_{i=1}^n x_i \geq k.$$

Since  $\chi_{T_k^n}(k) = 0$  and  $\chi_{T_k^n}(k + 2) = 1$ , we have

**Example 3.**  $L(T_k^n) \geq \epsilon n \log \min(k, n - k)$ .

More generally, there are exactly  $4 \cdot 2^{2b}$  symmetric functions  $f$  of  $n$  variables such that  $\chi_f(k) = \chi_f(k+2)$  for all  $k$ ,  $b \leq k \leq n-b$ . The preceding Theorem implies a bound of  $\epsilon n \log b$  on length of formulas for the remaining  $2^{n+1} - 4 \cdot 2^{2b}$  symmetric functions. Choosing any  $\delta$ ,  $0 < \delta < 1$ , and  $b = \delta n$ , we have:

*Corollary.* The minimum formula length for all but  $o(2^{n+1})$  of the  $2^{n+1}$  symmetric functions of  $n$  variables is  $\Omega(n \log n)$ .

Finally, we note that the Symmetric Function Lower Bound Theorem also applies to nonsymmetric functions  $f$  as long as  $\chi_f(k)$  and  $\chi_f(k+2)$  are well defined, i.e., as long as

$$\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = m \text{ implies } f(x_1, \dots, x_n) = f(y_1, \dots, y_n)$$

for  $m = k, k+2$ . For example, the length of formulas for *any* function which agrees with  $T_{\lfloor n/2 \rfloor}^n$  on arguments of weight  $\lfloor n/2 \rfloor$  and  $\lfloor n/2 \rfloor + 2$  is  $\Omega(n \log n)$ .

**4. Proof of the Lower Bound.** The Lower Bound Theorem follows directly from the Main Lemma below. The proof of the Main Lemma requires four elementary lemmas which are presented first.

Let  $f, g$  be formulas. We call  $g$  an *affine variant* of  $f$  iff  $f \oplus g$  is affine. A formula  $f$  is an *r-formula* if no variable in  $f$  occurs more than  $r$  times;  $f$  is *r-minimal* with respect to some property of formulas if  $f$  is an  $r$ -formula and  $L(f)$  is minimal among the  $r$ -formulas with the property. (Note that  $L(f) \leq r \cdot \dim(f)$  for any  $r$ -formula  $f$ , but this condition does not imply that  $f$  is necessarily an  $r$ -formula.)

*Affine Variant Lemma:* Let  $g$  be an affine variant of  $f$ . For all assignments  $A$ ,

- (i)  $f|_A$  is affine if  $g|_A$  is affine, and
- (ii) if for some  $r \geq 1$ ,  $g$  is an  $r$ -minimal affine variant of  $f$  and

$$\text{dom}(A) \subseteq \text{var}(g), \text{ then } \dim(f|_A) - \dim(g|_A) = \dim(f) - \dim(g).$$

*Proof:*(i) The formula  $f \oplus g$  is affine by hypothesis, hence  $f|_A \oplus g|_A$  is affine. If also  $g|_A$  is affine, then adding the two together gives the affine function  $f|_A$ .

(ii)  $\text{var}(g) \subseteq \text{var}(f)$ , for if not, substituting constants for the variables in  $g$  which do not appear in  $f$  yields a shorter affine variant which is an  $r$ -formula, contradicting the  $r$ -minimality of  $g$ . The result follows easily.  $\square$

An assignment  $B$  is an *extension* of  $A$  if  $B$  extends the partial function  $A$ . Let  $\text{dom}(B,A)$  denote  $\text{dom}(B) - \text{dom}(A)$ , the set of new variables fixed by  $B$ .

*Conjunction Lemma:* Given a central assignment  $A$  and a formula  $f$  such that  $f|_A \equiv g \wedge h$ , where  $g$  and  $h$  are affine, there is a central extension  $B$  of  $A$  such that  $\text{dom}(B,A) \subseteq \text{var}(f|_A)$ ,  $f|_B$  is affine, and  $\dim(f|_B) \geq \dim(f|_A)/3$ .

*Proof:* We have

$$G \equiv \oplus P \oplus \oplus R \oplus c \quad \text{and} \quad H \equiv \oplus Q \oplus \oplus R \oplus d$$

where  $P, Q, R$  are disjoint subsets of  $\text{var}(f|_A)$  and  $c, d \in \{0,1\}$ . Let  $B_1, B_2, B_3$  be central extensions of  $A$  fixing additionally the variables of  $QUR, PUR, PUQ$  respectively. Each of  $f|_{B_i}$  for  $i=1,2,3$  is affine and

$$\text{var}(f|_{B_1}) \cup \text{var}(f|_{B_2}) \cup \text{var}(f|_{B_3}) = PUQUR = \text{var}(f|_A).$$

Hence, for some  $i$ ,  $\dim(f|_{B_i}) \geq \dim(f|_A)/3$ .  $\square$

*Partition Lemma:* Given sets  $S_1, S_2, \dots, S_t$ , let  $T$  be the elements which occur in two or more of the  $S_i$ . That is,

$$T = \cup_{i < j \leq t} (S_i \cap S_j).$$

Then there exists a partition  $\{\lambda, \mu\}$  of  $\{1, \dots, t\}$  such that if  $L = \cup_{i \in \lambda} S_i$  and  $M = \cup_{i \in \mu} S_i$ , then  $|L \cap M| \geq |T|/2$ .

*Proof:* The proof is by induction on  $t$ . The case  $t=1$  is trivial. Given  $S_1, \dots, S_t, S_{t+1}$ , we have by induction a partition  $\{\lambda, \mu\}$  of  $\{1, \dots, t\}$  and sets  $L, M$ , and  $T$  satisfying the lemma. We now define a partition  $\{\lambda', \mu'\}$  of  $\{1, \dots, t+1\}$  as follows.

Let  $T_L = (S_{t+1} \cap L) - T$  and  $T_M = (S_{t+1} \cap M) - T$ . Assume without loss of generality that  $|T_M| \geq |T_L|$  and define  $\lambda' = \lambda \cup \{t+1\}$ ,  $\mu' = \mu$ . Now let

$$L' = \cup_{i \in \lambda'} S_i = L \cup S_{t+1}, \quad M' = \cup_{i \in \mu'} S_i = M,$$

and let  $T'$  be the elements which appear two or more times among  $S_1, \dots, S_{t+1}$ . Note that  $T'$  is the disjoint union of  $T, T_L$ , and  $T_M$ .

Since  $|T_M| \geq |T_L|$ , at least half the elements in  $T' - T$  are in  $T_M$ . Also since  $L$

$\cap M \subseteq T$ , the hypothesis implies that at least half the elements in  $T$  are in  $L \cap M$ . But  $T_M \cup (L \cap M) \subseteq L' \cap M'$  by definition. Hence at least half the elements in  $T'$  are in  $L' \cap M'$ .  $\square$ .

*Beta Lemma.* There exist constants  $\alpha > 0$ ,  $a > 1$  such that if we define  $\beta(r) = (\alpha a^r C_r)^{-1}$  where  $C_r = \binom{2r-2}{r-1}/r$  is the Catalan number, then

$$(i) \quad \beta(r) = \alpha / \sum_{s=1}^{r-1} (\beta(r)\beta(r-s))^{-1} \text{ for } r > 1,$$

$$(ii) \quad \beta(r) \leq (1 - 15\alpha)/6 < 1 \text{ for } r \geq 1,$$

$$(iii) \quad \beta(r) \leq (1 - 5\alpha)/(1 - 5\alpha + 4r).$$

*Proof:* (i) The Catalan numbers satisfy the convolution property

$$C_r = \sum_{s=1}^{r-1} C_s C_{r-s}$$

[Knu 73, Section 2.3.4.4] from which the corresponding property (i) of  $\beta$  follows immediately.

(ii), (iii). Moreover,  $C_r$  is asymptotic to  $dr^{-3}/24r$  for some fixed  $d > 0$  [Knu 73, Section 2.3.4.4]. This estimate makes it obvious that for any sufficiently small  $\alpha$  one can choose a value for  $a$  which guarantees (ii) and (iii). Suitable values are  $\alpha = 1/30$  and  $a = 360$ .  $\square$

*Main Lemma.* Let  $f$  be an  $r$ -formula with  $r \geq 1$ , and let  $A_0$  be a central assignment. There exists a central extension  $A$  of  $A_0$  such that  $f|_A$  is affine,  $\text{dom}(A, A_0) \subseteq \text{var}(f)$ , and

$$\dim(f|_A) \geq \beta(r) \cdot \dim(f|_{A_0}).$$

*Proof of Main Lemma.* The proof is by course-of-values induction on  $r$ . Hence we assume  $r \geq 1$  and that the lemma holds for all  $r'$ -formulas with  $r' < r$ . To show the lemma holds for all  $r$ -formulas, we proceed using a course-of-values subinduction on  $L(f)$ . Hence we consider some  $r$ -formula  $f$  and some central assignment  $A_0$ , and further assume that the lemma holds for all  $r$ -formulas of length less than  $L(f)$ .

Suppose that  $g$  is an  $r$ -minimal affine variant of  $f|_{A_0}$  and  $L(g) < L(f)$ . Then by the subinduction hypothesis, there is a central extension  $A$  of  $A_0$  satisfying the lemma for  $g$ .  $f|_A$  is affine by the Affine Variant Lemma (i). Moreover,

$$\begin{aligned}
\dim(f|_A) &= \dim(g|_A) + (\dim(f|_{A_0}) - \dim(g)) && \text{by Affine Variant Lemma (ii)} \\
&\geq \beta(r) \cdot \dim(g) + (\dim(f|_{A_0}) - \dim(g)) && \text{by induction} \\
&\geq \beta(r) \cdot \dim(f|_{A_0}) && \text{since } \beta(r) < 1 \text{ by Beta Lemma (ii).}
\end{aligned}$$

This shows the lemma holds for  $f$  using the same  $A$ .

Hence we can assume that  $f$  is an  $r$ -minimal affine variant of  $f|_{A_0}$ . In particular, we have  $f = f|_{A_0}$  and  $\text{var}(f) \cap \text{dom}(A_0) = \emptyset$ .

Express  $f$  as  $\bigoplus_{i=1}^k F_i$  where no  $F_i$  has  $\oplus$  as its main connective. Clearly no  $F_i$  is affine since otherwise  $(\bigoplus_{j \neq i} F_j)$  is an affine variant of  $f$ , contradicting the minimality of  $f$ . Hence, each  $F_i$  equals  $G_i \wedge H_i$ , and furthermore the minimality of  $f$  ensures that neither of the formulas  $G_i$  nor  $H_i$  are equivalent to constant functions.

We define a partition of each set  $\text{var}(F_i)$  into four sets as follows:

$$\begin{aligned}
\text{global}(F_i) &= \text{var}(F_i) \cap (\bigcup_{j \neq i} \text{var}(F_j)); \\
\text{joint}(G_i, H_i) &= (\text{var}(G_i) \cap \text{var}(H_i)) - \text{global}(F_i); \\
\text{own}(G_i) &= \text{var}(G_i) - (\text{joint}(G_i, H_i) \cup \text{global}(F_i)); \\
\text{own}(H_i) &= \text{var}(H_i) - (\text{joint}(G_i, H_i) \cup \text{global}(F_i)).
\end{aligned}$$

Let  $\text{global} = \bigcup_i \text{global}(F_i)$ ,  $\text{joint} = \bigcup_i \text{joint}(G_i, H_i)$ , and  $\text{own} = \bigcup_i (\text{own}(G_i) \cup \text{own}(H_i))$ . So  $\text{var}(f)$  is the disjoint union of  $\text{global}$ ,  $\text{joint}$ , and  $\text{own}$ .

The following four cases, defined solely in terms of the cardinalities of  $\text{var}(f)$ ,  $\text{global}$ ,  $\text{joint}$ , and  $\text{own}$ , are obviously exhaustive. Let  $n = \dim(f)$ .

*Case 1:*  $n \leq 1/\beta(r)$ . In this case we can take any central extension  $A$  of  $A_0$  such that  $\text{dom}(A, A_0) \subseteq \text{var}(f)$  and  $\dim(f|_A) = 1$ . Any formula in a single variable is necessarily affine.

*Case 2:*  $|\text{global}| \geq 2\alpha n$ . Noting that  $\text{global}$  equals the set of variables which occur in two or more of the sets  $\text{var}(F_i)$ , we apply the Partition Lemma to the sets  $\text{var}(F_i)$ ,  $i=1, \dots, k$ , and obtain a partition  $\{\lambda, \mu\}$  of  $\{1, \dots, k\}$  such that

$$|\cup_{i \in \lambda} \text{var}(F_i) \cap \cup_{i \in \mu} \text{var}(F_i)| \geq |\text{global}|/2 \geq \alpha n.$$

Now  $f$  is equivalent to  $L \oplus M$  where  $L = \bigoplus_{i \in \lambda} F_i$ ,  $M = \bigoplus_{i \in \mu} F_i$ . We use the fact that all the variables of  $\text{var}(L) \cap \text{var}(M)$  must occur fewer than  $r$  times in each of  $L$ ,  $M$ , to invoke the Main Lemma successively on the two parts. For  $1 \leq s \leq r-1$ , let  $V_s$  be the set of those variables of  $\text{var}(L) \cap \text{var}(M)$  which occur exactly  $s$  times in  $L$  (and hence at most  $r-s$  times in  $M$ ). For some  $t$ ,

$$|V_t| \geq \beta(r)n/(\beta(t)\beta(r-t))$$

since

$$\sum_{s=1}^{r-1} |V_s| = |\text{var}(L) \cap \text{var}(M)| \geq \alpha n = \sum_{s=1}^{r-1} [\beta(r)n/(\beta(s)\beta(r-s))]$$

by Beta Lemma (i).

Let  $B$  be any central extension of  $A_0$  with  $\text{dom}(B, A_0) = \text{var}(f) - V_t$ , and let  $L' = L|_B$ ,  $M' = M|_B$ . Thus  $\text{var}(L') = \text{var}(M') = V_t$ . The Main Lemma applied to the  $t$ -formula  $L'$  yields an extension  $B'$  of  $B$  such that  $L'|_{B'}$  is affine and

$$\dim(M'|_{B'}) = \dim(L'|_{B'}) > \beta(t) \cdot \dim(L') \geq \beta(r)n/\beta(r-t).$$

The Lemma applied to the  $(r-t)$ -formula  $M'|_{B'}$  yields an extension  $A$  of  $B'$  such that  $M'|_A$  is affine and  $\dim(M'|_A) \geq \beta(r-t) \cdot \dim(M'|_{B'}) \geq \beta(r)n$ . Since  $(L \oplus M)|_A = ((L'|_{B'})|_A) \oplus (M'|_A)$  and is clearly affine, and  $\dim((L \oplus M)|_A) = \dim(M'|_A) \geq \beta(r)n$ , we have concluded the proof of Case 2.

*Case 3:*  $|\text{joint}| \geq 3\alpha n$ .

Each variable in  $\text{joint}$  occurs in exactly one  $F_i$ , and at least once but strictly fewer than  $r$  times in  $G_i$  and in  $H_i$ . We will restrict to a subset of these variables and then apply the induction hypothesis for smaller values of  $r$ .

Let  $u_i = |\text{joint}(G_i, H_i)|$ . As in Case 2, there is some  $t_i$ ,  $1 \leq t_i \leq r-1$ , such that  $V_i$  is the set of variables in  $\text{joint}(G_i, H_i)$  that occur exactly  $t_i$  times in  $G_i$  (and hence at most  $r-t_i$  times in  $H_i$ ) then

$$|V_i| \geq u_i \beta(r) / (\alpha \beta(t_i) \beta(r-t_i)).$$

Let  $B_0$  be any central extension of  $A_0$  fixing all the variables in  $\text{var}(f) - \cup_i V_i$ ,

and let  $F_i' = F_i|_{B_0}$ ,  $G_i' = G_i|_{B_0}$ ,  $H_i' = H_i|_{B_0}$ .  $G_i'$  is a  $t_i$ -formula and  $H_i'$  is an  $(r-t_i)$ -formula.

We now proceed in  $k$  stages. At the  $i^{\text{th}}$  stage, we find a central extension  $B_i$  of  $B_{i-1}$  such that  $F_i|_{B_i}$  is affine and  $\dim(F_i|_{B_i}) \geq u_i\beta(r)/(3\alpha)$ .

*Stage  $i$ :* Since  $G_i'$  is a  $t_i$ -formula,  $1 \leq t_i < r$ , we apply the induction hypothesis to  $G_i'$  and  $B_{i-1}$  to obtain a central extension  $B_i'$  such that  $G_i|_{B_i'}$  is affine and  $\dim(G_i|_{B_i'}) \geq \beta(t_i)\dim(G_i')$ . Since  $H_i'$  is an  $(r-t_i)$ -formula,  $r-t_i < r$ , we apply the induction hypothesis again to  $H_i|_{B_i'}$  to obtain a central extension  $B_i''$  such that  $H_i|_{B_i''}$  is affine and  $\dim(H_i|_{B_i''}) \geq \beta(r-t_i)\dim(H_i|_{B_i'})$ . The restriction of an affine function is affine, so  $G_i|_{B_i''}$  is affine. Hence, by the Conjunction Lemma there exists a central extension  $B_i$  of  $B_i''$  such that  $F_i|_{B_i}$  is affine and has dimension at least  $\dim(F_i|_{B_i''})/3$ . In calculating the dimension of  $F_i|_{B_i}$ , we make use of the fact that  $\text{var}(F_i') = \text{var}(G_i') = \text{var}(H_i') = V_i$ . We have

$$\begin{aligned} \dim(H_i|_{B_i''}) &\geq \beta(r-t_i)\dim(H_i|_{B_i'}) = \beta(r-t_i)\dim(G_i|_{B_i'}) \\ &\geq \beta(r-t_i)\beta(t_i)|V_i| \geq u_i\beta(r)/\alpha. \end{aligned}$$

Then

$$\dim(F_i|_{B_i}) \geq \dim(H_i|_{B_i''})/3 \geq u_i\beta(r)/(3\alpha).$$

Now let  $A = B_k$ , the central assignment obtained after the final stage. Note that  $f|_A = \bigoplus_{i=1}^k F_i|_{B_i}$ , and so  $f|_A$  is affine. Moreover,

$$\dim(f|_A) = \sum_i \dim(F_i|_{B_i}) \geq \sum_i u_i\beta(r)/(3\alpha) = |\text{joint}| \beta(r)/3\alpha \geq \beta(r)n$$

by the defining condition for this case. This concludes the proof of Case 3.

*Case 4:*  $|\text{own}| \geq (1 - 5\alpha)n$  and  $n > 1/\beta(r)$ .

In this case, we will find a central extension  $B$  of  $A_0$  such that  $\text{dom}(B) \subseteq \text{var}(f)$  and  $f|_B$  is functionally independent of some non-empty subset  $V$  of its variables. Let  $\text{yield} = |V|$  and  $\text{cost} = \text{yield} + |\text{dom}(B, A_0)|$ . If  $\text{yield} \geq \beta(r)\text{cost}$ , then we can find a central extension  $A$  of  $B$  satisfying the Lemma for  $f$ .

To see this, let  $g$  be the restriction of  $f|_B$  obtained from some arbitrary assignment to  $V$ . Note that  $g$  is equivalent to  $f|_B$  since  $f|_B$  does not depend on  $V$ . Also,

$\text{var}(f)$  is the disjoint union of  $\text{dom}(B, A_0)$ ,  $V$ , and  $\text{var}(g)$ ; in particular,  $\text{dim}(f) = \text{cost} + \text{dim}(g)$ . Now  $L(g) < L(f)$  since  $V$  is non-empty, so by the subinduction hypothesis there is a central extension  $A$  of  $B$  such that  $g|_A$  is affine,  $\text{dom}(A, B) \subseteq \text{var}(g)$ , and  $\text{dim}(g|_A) \geq \beta(r) \cdot \text{dim}(g)$ . But  $f|_A$  is equivalent to  $g|_A$ , so  $f|_A$  is also affine. Moreover,  $\text{var}(f|_A)$  is the disjoint union of  $\text{var}(g|_A)$  and  $V$  since  $\text{dom}(A) \cap V = \emptyset$ . Therefore,

$$\text{dim}(f|_A) = \text{dim}(g|_A) + \text{yield} \geq \beta(r) \cdot \text{dim}(g) + \beta(r) \cdot \text{cost} = \beta(r) \cdot \text{dim}(f),$$

as required.

Thus, to complete the proof, we need only describe how to determine  $B$  and  $V$ .

Let  $g_i = |\text{own}(G_i)|$ ,  $h_i = |\text{own}(H_i)|$ . Without loss of generality, we can assume  $g_i \geq h_i$ . We note that  $\sum_i g_i \geq |\text{own}|/2$ .

For each  $i$ , we have two strategies which can remove the dependence of  $f$  on a subset of either  $\text{own}(G_i)$  or  $\text{own}(H_i)$ . We will show below that at least one of these always has an adequate yield/cost ratio for some  $i$ .

*Strategy A:* This strategy is applicable only if there is a central extension of  $A_0$  fixing only  $\text{var}(f)$  and making  $H_i$  equivalent to 0. Find a minimal central extension  $B$  of  $A_0$  for which  $H_i|_B \equiv 0$ ,  $\text{var}(H_i) \subseteq \text{dom}(B) \subseteq \text{var}(f)$ , and  $\text{dom}(B) \cap \text{own}(G_i)$  is as small as possible among such extensions. Since  $F_i = G_i \wedge H_i$ , we have  $F_i|_B \equiv 0$ , and so  $f|_B$  is independent of any remaining variables of  $\text{own}(G_i)$ . Thus  $V$  is  $\text{own}(G_i) - \text{dom}(B)$ .

*Strategy B:* This strategy is applicable only if there is a central extension of  $A_0$  fixing only  $\text{var}(H_i) \cup \text{dom}(A_0)$  and making  $H_i$  equivalent to 1. Find a maximal set  $V \subseteq \text{own}(H_i)$  for which there is a central extension  $B$  of  $A_0$  satisfying  $H_i|_B \equiv 1$  and  $\text{dom}(B, A_0) = \text{var}(H_i) - V$ . Since  $f|_B$  is independent of  $V$ , the yield is  $|V|$  and the cost is  $\text{dim}(H_i)$ .

We begin our analysis by noting that since  $f$  is  $r$ -minimal, no subformula of  $f$  is equivalent to a constant. Hence there is an extension  $B'$  of  $A_0$  such that  $H_i|_{B'} \equiv 0$ . Let  $d(H_i)$  be the least integer for which there is an extension  $B'$  of  $A_0$  such that  $\text{dom}(B', A_0) = \text{var}(H_i)$ ,  $H_i|_{B'} \equiv 0$ , and

$$-d(H_i) \leq \text{ecc}(B') \leq d(H_i) + 1.$$

Clearly,  $d(H_i) \leq \text{dim}(H_i)$ .

Suppose Strategy A is applicable and B is the assignment required in the strategy. Let B' be the restriction of the partial function B to  $\text{dom}(A_0) \cup \text{var}(H_i)$ . Since B is minimal central such that  $H_i|_B \equiv 0$ , it must be that  $\text{ecc}(B')$  equals either  $-d(H_i)$  or  $d(H_i) + 1$ , and the variables in  $\text{dom}(B, B')$  are the minimal number which serve to extend B' to a central assignment. Hence,

$$|\text{dom}(B, A_0)| = \text{dim}(H_i) + d(H_i).$$

Since B is defined to fix as few variables from  $\text{own}(G_i)$  as possible, either  $\text{dom}(B) \cap \text{own}(G_i) = \emptyset$  or  $\text{own}(G_i) - \text{dom}(B) = \text{var}(f) - \text{dom}(B)$ . Therefore,

$$\text{yield}_A = |\text{own}(G_i) - \text{dom}(B)| = \min(g_i, n - \text{dim}(H_i) - d(H_i)), \text{ and}$$

$$\text{cost}_A = \text{dim}(H_i) + d(H_i) + \text{yield}_A = \min(g_i + \text{dim}(H_i) + d(H_i), n).$$

If Strategy A is not applicable, let  $\text{yield}_A = 0$  and  $\text{cost}_A = n$ , so the preceding formulas for  $\text{cost}_A$  and  $\text{yield}_A$  always hold.

If  $\text{yield}_A / \text{cost}_A \geq \beta(r)$  for some value of  $i$ , then Strategy A succeeds.

In any application of Strategy B,  $|V| \geq \min(h_i, d(H_i) - 1)$ . To see this, let V' be any subset of  $\text{own}(H_i)$  such that  $|V'| = \min(h_i, d(H_i) - 1)$ , and let B' be any central extension of  $A_0$  with  $\text{dom}(B') = \text{dom}(A_0) \cup (\text{var}(H_i) - V')$ . Let C be an arbitrary assignment with  $\text{dom}(C) = V'$ . Then

$$\begin{aligned} -d(H_i) &< -\min(h_i, d(H_i) - 1) && \text{trivially} \\ &= -|\text{dom}(C)| \\ &\leq \text{ecc}(B' \cup C) && \text{since } B' \text{ is central} \\ &\leq |\text{dom}(C)| + 1 && \text{since } B' \text{ is central} \\ &= \min(h_i, d(H_i) - 1) + 1 \\ &< d(H_i) + 1. \end{aligned}$$

By the minimality condition in the definition of  $d$ ,  $H_i|_{(B' \cup C)} \equiv 1$ . This holds for any such C, so  $H_i|_{B'} \equiv 1$ , and  $H_i|_{B'}$  does not depend on the variables in V'. Since Strategy B chooses V as large as possible, we have  $|V| \geq |V'| \geq \min(h_i, d(H_i) - 1)$  as desired.

Eliminating  $V$  from the expression of cost for Strategy B, we get

$$\text{yield}_B \geq \min(h_i, d(H_i) - 1)$$

and

$$\text{cost}_B = \dim(H_i).$$

If Strategy B is inapplicable, let  $\text{yield}_B = 0$  and  $\text{cost}_B = \dim(H_i)$ . Note that in this case  $d(H_i) = 0$ , so the preceding formulas for  $\text{yield}_B$  and  $\text{cost}_B$  always hold.

If  $\text{yield}_B/\text{cost}_B \geq \beta(r)$  for some value of  $i$ , then Strategy B succeeds.

We prove by contradiction that there exists an  $i$  for which either Strategy A or Strategy B succeeds. Assume neither Strategy succeeds for any  $i$ . Since Strategy A fails,  $\text{yield}_A/\text{cost}_A < \beta$ . (We omit the argument  $r$  from  $\beta$  in the remainder of this analysis.) So for all  $i$

$$(1) (1 - \beta)\min(g_i + \dim(H_i) + d(H_i), n) < \dim(H_i) + d(H_i).$$

Since Strategy B fails,  $\text{yield}_B/\text{cost}_B < \beta$ , so for all  $i$

$$(2) \min(h_i, d(H_i) - 1) < \beta \cdot \dim(H_i).$$

Let  $m = |\text{global} \cup \text{joint}| \leq 5\alpha n$ . Counting up the sizes of the various sets and using the conditions for this case, we get

$$(3) d(H_i) \leq \dim(H_i) \leq m + h_i \leq n - \sum_j g_j \leq n - |\text{own}|/2 \leq (1 + 5\alpha)n/2.$$

From (3) and (2), we get

$$(4) d(H_i) - m - 1 \leq \min(h_i, d(H_i) - 1) < \beta \cdot \dim(H_i) \leq \beta n.$$

Using (3), (4), the fact that  $\beta n > 1$ , and Beta Lemma (ii), we get

$$(5) \dim(H_i) + d(H_i) \leq (1 + 5\alpha)n/2 + m + 1 + \beta n$$

$$< (1 + 15\alpha)n/2 + 2\beta n \leq (1 - \beta)n.$$

Assuming the "min" in (1) equals its second argument contradicts (5). Hence, the first argument is always the smaller, and (1) gives

$$(6) \quad (1 - \beta)g_i < \beta \cdot (\dim(H_i) + d(H_i)) \leq 2\beta \cdot \dim(H_i) \text{ for all } i.$$

Therefore,

$$(7) \quad (1 - \beta)(1 - 5\alpha)n/2 \leq (1 - \beta)|\text{own}|/2 \leq (1 - \beta)\sum_i g_i < 2\beta\sum_i \dim(H_i) < 2\beta rn$$

since no variable occurs more than  $r$  times in all. Now (7) yields an immediate contradiction with Beta Lemma (iii).

We conclude that Strategy A or Strategy B succeeds for some  $i$ , completing this case and the proof of the Main Lemma.  $\square$

*Proof of Lower Bound Theorem:* Let  $f$  be a Boolean formula on  $n$  variables. Let  $r = \lfloor 2L(f)/n \rfloor$ , and let  $A_0$  be a central assignment with

$$\text{dom}(A_0) = \{x \mid x \text{ occurs more than } r \text{ times in } f\}.$$

Since  $f|_{A_0}$  is an  $r$ -formula, by the Main Lemma there is a central extension  $A$  of  $A_0$  such that  $f|_A$  is affine,  $\text{dom}(A) \subseteq \text{var}(f)$ , and

$$(8) \quad \dim(f|_A) \geq \beta(r) \cdot \dim(f|_{A_0}).$$

By the choice of  $A_0$ ,  $(r+1) \cdot |\text{dom}(A_0)| \leq L(f)$ , so

$$(9) \quad \dim(f|_{A_0}) = n - |\text{dom}(A_0)| \geq n - L(f)/(r+1) \geq n/2.$$

Also,

$$(10) \quad \beta(r) \geq 2/K^r$$

for some  $K > 1$  using the asymptotic estimate for  $C_r$  given in the proof of the Beta Lemma. Hence, from (8), (9), (10), we get

$$\dim(f|_A) \geq (2/K^r)(n/2) = n/K^r.$$

Solving for  $r$ , we obtain

$$(11) \quad r \geq \log(n/\dim(f|_A))/\log K.$$

Therefore,

$$\begin{aligned}
 L(f) &\geq rn/2 && \text{by choice of } r \\
 &\geq \epsilon n \log(n/\dim(f|_A)) && \text{by (11) where } \epsilon = 1/(2 \log K) \\
 &\geq \epsilon n \log(n/\text{diam}(f)) && \text{by definition of } \text{diam}(f). \quad \square
 \end{aligned}$$

**5. Conclusions and Open Problems.** The conditions we have developed above for deducing lower bounds on length of formulas apply to many explicit examples but have their most interesting applications in the case of symmetric Boolean functions. Earlier results of Hodes and Specker [HoS 68] imply that except for sixteen functions, the length of formulas for symmetric functions of  $n$  variables grows nonlinearly in  $n$ .<sup>4</sup> The results in this paper show that all but a vanishing fraction of the symmetric functions require formulas of length  $\Omega(n \log n)$ . These are the strongest known lower bounds on length of formulas for any symmetric functions.

Polynomial *upper* bounds on the length of formulas for symmetric functions were first obtained by Khrapchenko [Khr 72] and Meyer and Vilfan [Vil 72]. The smallest currently known upper bound is  $o(n^{3.37})$  by Peterson [Pet 78] following earlier work of Pippenger [Pip 74] and Paterson [Pat 77]. The constructions used to achieve the upper bounds are extensions of the construction given in Section 2 of formulas for  $C_4^n$ .

It remains an open problem to improve these bounds. We note three particularly challenging instances of this general problem.

The construction of formulas for  $C_4^n$  extends in an obvious way to yield formulas of length  $O(n(\log n)^{p-1})$  for  $C_{2^p}^n$  but even for  $C_3^n$  the best upper bound we can obtain is  $\Omega(n^2)$ .

*Problem 1.* Is  $L(C_3^n) = o(n^2)$ ?

The Lower Bound Theorem above does not apply to threshold functions with bounded threshold, although Hodes' and Specker's theorem yields very slowly growing nonlinear bounds (cf. Note 3). For fixed  $k$ , Khasin [Kha 69] and Pippenger [Pip 76, KIP 77] have shown that  $L(T_k^n) = O(n \log n)$ .

*Problem 2.* Is  $L(T_2^n) = o(n \log n)$ ?

The best currently known upper bound on length of formulas for the majority function  $T_{\lfloor n/2 \rfloor}^n$  is the same as for arbitrary symmetric functions.

*Problem 3.* Is  $n \log n = o(L(T_{\lfloor n/2 \rfloor}^n))$ ?

*Notes.*

1. A slightly larger lower bound of  $\Omega(n^2)$  is due to Khrapchenko [Khr 71] for the special basis of operations  $\wedge, \vee, \neg$ , but our results are concerned with formulas in which *all* binary operations may appear.

2. Vilfan [Vil 72, 76] extends Hodes' and Specker's results to multivalued logic with arbitrary (not necessarily binary) operations and concludes for example that formulas for  $C_k^n$  grow nonlinearly in  $n$  using  $d$ -valued logic for  $k > d!$ .

3. Vilfan [Vi 72] notes that the nonlinear lower bounds of Hodes and Specker can be shown to be  $O(n \log^* n)$  where  $\log^* n$  is the least integer  $m$  such that

$$2^{2^{\dots^2}} \text{ (height } m) \geq n.$$

4. The sixteen functions are all of the form

$$a \oplus b \oplus x_i \oplus c \Pi x_i \oplus d \Pi(1 \oplus x_i)$$

for  $a, b, c, d \in \{0, 1\}$ . Each of these obviously has a formula of length at most  $3n$ .

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