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AN EFFECTIVE REPRESENTATION OF THE REACHABILITY SET OF PERSISTENT PETRI NETS

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Abstract:

In a persistent Petri net, an enabled transition can become disabled only by firing itself. Here, an algorithm is presented which constructs a semilinear representation of the set of states reachable in an arbitrary persistent Petri net.

Key Words and Phrases: Vector Replacement System, Petri Net, Persistence, Representation of Reachability Set

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1. Introduction

In this report, an effective algorithm is presented which, for any given persistent Petri net, constructs a semilinear representation of its reachability set. The notion of persistence appears in connection with Parallel Program Schemata [7], where persistent operators, once they are enabled, stay so until they are fired, or in connection with the "Church-Rosser-Property" [16]. Also, Lipton et al. [11] use a similar property when studying linear asynchronous systems, as do Muller and Bantky [13] for switching circuits. In [6] it is shown that the decision problem of the persistence of one transition is recursively equivalent to the decidability of the reachability problem for Petri nets, but it is also conjectured that the decision problem of the persistence of Petri nets is totally independent of the reachability problem (and closer to be solved). In the algorithm discussed in this report for the effective construction of (semilinear) representations for the reachability sets of persistent Petri nets, persistence of the given (P, m) is presumed. H. Müller [12] has, independently, obtained a result equivalent to the one presented here.*

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*His algorithm is somewhat more difficult because it does not make use of the properties stated in Definition 5 and Lemma 4 allowing us a recursive approach.

2. Basic Concepts

A *Petri net* P is a triple (S, T, K) with

(i) $S = \{s_1, \ldots, s_v\}$ a finite set of *places*,

(ii) $T = \{t_1, \ldots, t_w\}$ a finite set of *transitions*, $S \cap T = \emptyset$,

(iii) K: $S \times T \cup T \times S \to \mathbf{N}$ a mapping giving the *multiplicity* of *edges* between places and transitions.*

A *marking* of P is a mapping m: $S \to \mathbf{N}$ which usually gets represented as a vector $m \in \mathbf{N}^v$. A *pseudomarking* of P is a mapping $\overline{m}$:$S \to \overline{\mathbf{N}}$ (or a vector $\overline{m} \in \overline{\mathbf{N}}^v$), where $\overline{\mathbf{N}}: = \mathbf{N} \cup \{\omega\}$ is $\mathbf{N}$ augmented by the infinite number ω with $\pm n + \omega = \omega \pm n = \omega$ and $n < \omega$ for all $n \in \mathbf{N}$.

The *marking difference* $\delta t \in \mathbf{Z}^v$ effected by $t \in T$ is given by $(\delta t)_i = K(t, s_i) - K(s_i, t)$ for $i \in I_v$. $t \in T$ is *firable* at pseudomarking $\overline{m}$ (written a $(t, \overline{m})$) iff $(\forall i \in I_v)[\overline{m}_i \geq K(s_i, t)]$. If t is firable at $\overline{m}$, the firing of t takes $\overline{m}$ to $\overline{m} + \delta t$: $\overline{m} \xrightarrow{t} \overline{m} + \delta t$.

For sequences $\varphi = t_{i_1} \ldots t_{i_r} \in T^*$, $\delta\varphi, a(\varphi, \overline{m})$, and $\overline{m} \xrightarrow{\varphi} \overline{m}'$ are defined inductively:

(i) $a(\varphi, \overline{m}): = r = 0 \vee a(t_{i_1}, \overline{m}) \wedge a(t_{i_2} \ldots t_{i_r}, \overline{m} + \delta t_{i_1})$;

(ii) $\delta\varphi: = \sum_{j=1}^{r} \delta t_{i_j}$;

(iii) $\overline{m} \xrightarrow{\varphi} \overline{m}': = a(\varphi, \overline{m}) \wedge \overline{m}' = \overline{m} + \delta\varphi$.

The *reachability set* $R(P, \overline{m})$ of the (pseudo)marked Petri net $(P, \overline{m})$ is given by

$$R(P, \overline{m}): = \{\overline{m}'; (\exists \varphi \in T^*)[\overline{m} \xrightarrow{\varphi} \overline{m}']\}.$$

Let Φ:$T^* \to \mathbf{N}^w$ denote the Parikh mapping indicating, for each $i \in I_w$ and $\varphi \in T^*$, the number of occurrences of t_i in φ. For $\varphi \in T^*$, $\Phi(\varphi)$ is called the *counter* of φ.

The *counter set* $C(P, \overline{m})$ of $(P, \overline{m})$ is

$$C(P, \overline{m}): = \{\Phi(\varphi); \varphi \in T^* \wedge a(\varphi, \overline{m})\}.$$

Let $V \in \mathbf{Z}^{v,w}$ be the integer matrix whose i-th column is given by δt_i, for all $i \in I_w$. We immediately have the following

*$\mathbf{N} = \{0, 1, \ldots\}$ denotes the set of nonnegative integers, $\mathbf{Z}$ the set of integers, I_n, for $n \in \mathbf{N}$, the set $\{1, 2, \ldots, n\}$.

Corollary:

(a) $(\forall\varphi \in T^*)[\delta\varphi = V\Phi(\varphi)]$;

(b) $R(P, \overline{m}) = \{\overline{m} + Vc; c \in C(P, \overline{m})\}$.

A *linear set* $L \subseteq \mathrm{N}^w$ is a set of the form $L = \{a + \sum_{i=1}^{r} n_i b^i; (n_1, \ldots, n_r) \in \mathrm{N}^r\}$ for some $r \in \mathrm{N}, a, b^1, \ldots, b^r \in \mathrm{N}^w$. A *semilinear set* is a finite union of linear sets.

Semilinear sets are exactly those sets definable by expressions in Presburger Arithmetic, i.e. the first order theory of the nonnegative integers with addition [15]. Semilinear sets are closed under Boolean operations [4], and there is an effective procedure to construct semilinear representations of the sets defined by Presburger expressions [4,14].

Definition 1:

A Petri net P with initial (pseudo)marking $\overline{m}$ is called *persistent* iff

$$(\forall \overline{m}' \in R(P, \overline{m}), \forall i, j \in I_w)[(i \neq j \wedge a(t_i, \overline{m}') \wedge a(t_j, \overline{m}')) \Rightarrow a(t_i t_j, \overline{m}')]$$

(i.e. once a transition is firable in a persistent Petri net it can lose this property only by firing itself).

3. Basic facts about persistent Petri nets

Given some Petri net $P = (S, T, K)$ with initial marking m we may assume w.l.g. that each transition $t \in T$ has attached to it a so-called *indicator place*, i.e. a place s with no edge incident on it except (t, s) with $K(t, s) = 1$, and $m(s) = 0$, because adding such a place counting the firings of t does not affect the persistence of the net [8].

Let, in the sequel, (P, m) be a persistent marked Petri net with such indicator places.

Lemma 1:

(a) $R(P, m)$ is semilinear.
(b) $C(P, m)$ is semilinear.

Proof:

A nonconstructive proof of (a) is given in [10]. (b) follows from (a) using the projection of $R(P, m)$ on the indicator places and the fact that semilinear sets are closed under projection. ∎

Definition 2: (cf. [10])

Let $\varphi, \rho \in T^*$. Then $\varphi \dot{-} \rho$ is obtained by deleting from φ, for $i = 1, \ldots, w$, the $\min\{(\Phi(\varphi))_i, (\Phi(\rho))_i\}$ leftmost occurrences of t_i.
$\varphi\rho$ denotes the concatenation of φ and ρ.

Lemma 2:

(a) $(\forall \varphi, \rho \in T^*, \forall m' \in R(P, m))[(a(\varphi, m') \wedge a(\rho, m')) \Rightarrow a(\rho(\varphi \dot{-} \rho), m')]$;
(b) $c, c' \in C(P, m) \Rightarrow \max\{c, c'\} \in C(P, m)$ (where max is taken coordinatewise);
(c) $c, c' \in C(P, m), c' \geq c \Rightarrow c' - c \in C(P, m + Vc)$;
(d) $c, c' \in C(P, m), c' \geq c, \varphi \in T^*, a(\varphi, m + Vc), \delta\varphi \geq 0 \Rightarrow \Phi(\varphi) \in C(P, m + Vc')$.

Proof:

a) A proof can be found in [9]. For completeness, another one is given in the following. This proof proceeds by induction on the length $|\varphi \dot{-} \rho|$ of $\varphi \dot{-} \rho$.

Let $\varphi = \varphi_1\varphi_2...\varphi_s, \varphi_i \in T$, and $\rho = \rho_1...\rho_r, \rho_i \in T$.

If $|\varphi \dot{-} \rho| = 0$ the conclusion is obvious. Now assume that $a(\varphi, m'), a(\rho, m'), |\varphi \dot{-} \rho| = 1$, and $\varphi \dot{-} \rho = \varphi_i$. Then, $a(\rho\varphi_i, m')$ is shown by induction on i. If $i = 1$, the transition φ_i does not appear in ρ, and from $a(\rho, m')$ and $a(\varphi_1, m')$ one can conclude $a(\rho\varphi_1, m')$ by iterated application of Definition 1. If $i > 1$, let j be minimal s.t. $\rho_j = \varphi_1$. Then, we have by hypothesis $a(\rho_1 \ldots \rho_j, m')$, and by the definition of persistence $a(\varphi_1\rho_1 \ldots \rho_{j-1}, m')$. As $\delta(\rho_1 \ldots \rho_j) = \delta(\varphi_1\rho_1 \ldots \rho_{j-1})$ we obtain from the hypothesis and $\delta\varphi_1 = \delta\rho_j$ that $a(\rho_1 \ldots \rho_{j-1}\rho_{j+1} \ldots \rho_r, m' + \delta\rho_j)$ and $a(\varphi_2 \ldots \varphi_s, m' + \delta\rho_j)$, hence by induction (on i)

$$a(\rho_1 \ldots \rho_{j-1}\rho_{j+1} \ldots \rho_r\varphi_i, m' + \delta\rho_j) \Rightarrow a(\varphi_1\rho_1 \ldots \rho_{j-1}\rho_{j+1} \ldots \rho_r\varphi_i, m') \Rightarrow$$
$$a(\rho_1 \ldots \rho_{j-1}\varphi_1\rho_{j+1} \ldots \rho_r\varphi_i, m') = a(\rho\varphi_i, m').$$

Now, if $|\varphi \dot{-} \rho| > 1$, let φ_i be the first element in φ not eliminated when forming $\varphi \dot{-} \rho$, and let $\varphi' := \varphi_1 \ldots \varphi_i$, $\varphi'' := \varphi_{i+1} \ldots \varphi_s$. Then, by induction hypothesis,
$(a(\rho, m') \wedge a(\varphi', m')) \Rightarrow a(\rho(\varphi' \dot{-} \rho), m')$, and
$(a(\rho(\varphi' \dot{-} \rho), m') \wedge a(\varphi'\varphi'', m')) \Rightarrow a(\rho(\varphi' \dot{-} \rho)(\varphi'\varphi'' \dot{-} \rho(\varphi' \dot{-} \rho)), m') = a(\rho(\varphi \dot{-} \rho), m')$,
as $|\varphi'\varphi'' \dot{-} \rho(\varphi' \dot{-} \rho)| < |\varphi \dot{-} \rho|$.
This concludes the proof of a).
b) follows from a) as $\Phi(\rho(\varphi \dot{-} \rho)) = \max\{\Phi(\varphi), \Phi(\rho)\}$, $\varphi, \rho \in T^*$,
c) follows from a) if one considers the case $\Phi(\varphi) \geq \Phi(\rho)$.
d) As $c' - c \in C(P, m + Vc)$ because of c), and as $\delta\varphi \geq 0$ we have $c' - c \in C(P, m + Vc + \delta\varphi)$, i.e. $\Phi(\varphi) + c' \in C(P, m)$. Hence, by c), $\Phi(\varphi) = \Phi(\varphi) + c' - c' \in C(P, m + Vc')$. ∎

4. The Reachability Graph $RG(P, \overline{m})$

The following algorithm for the construction of the *reachability graph* $RG(P, \overline{m})$ is a slight modification of an algorithm given in [7]. In the algorithm, a digraph is constructed whose nodes and edges are labelled: each edge e carries a label $t(e) \in T$, and each node k is labelled with a pseudomarking $\overline{m}(k) \in \overline{\mathbf{N}}^v$.

Algorithm A:

```
begin
  start with a node r (the "root") with m̄(r): = m̄, which is not marked.
  while there is an unmarked node do
    k: = some unmarked node;
    mark k;
    for all t ∈ T with a(t, m̄(k)) do
      add to the graph constructed so as far a new unmarked node k' and an edge e from k to k' with t(e): = t;
      for i: = 1, . . . , v do
        if there is a node k'' on a simple path from r to k with m̄(k'') ≤ m̄(k) + δt and (m̄(k''))_i < (m̄(k) + δt)_i
        then
          (m̄(k'))_i: = ω
        else
          (m̄(k'))_i: = (m̄(k) + δt)_i
        fi
      od;
      if there is a node k'' ≠ k' in the graph constructed so far with m̄(k'') = m̄(k')
      then
        identify k' with k''
      fi
    od
  od
end Algorithm A.
```

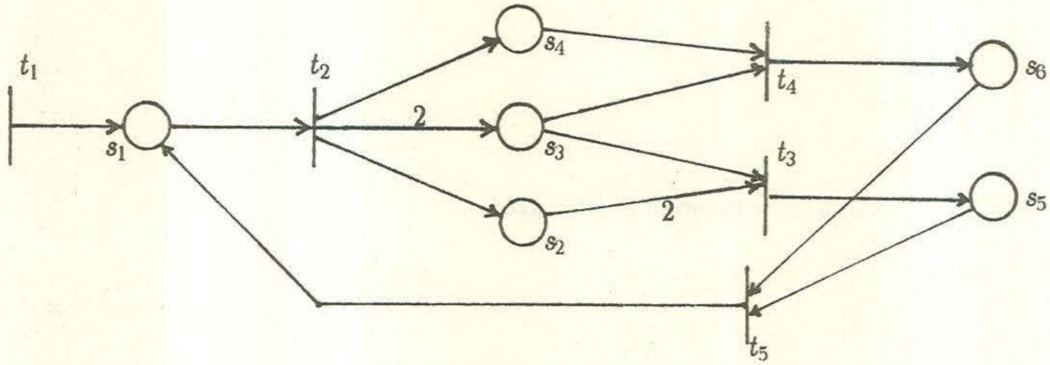
The proof for the termination of Algorithm A is along the same lines as in [7] and will not be given here.

A very important property of $RG(P, m)$ is the following: whenever $m' \in \mathbf{N}^v$ is a marking of P, k is a node in $RG(P, m)$ with $\overline{m}(k) \geq m'$, and $\varphi \in T^*$ is a sequence firable at m', then there is a (unique) path in $RG(P, m)$ starting from k whose edge labelling sequence is φ, and the node marking of the endpoint of this path is $\geq m' + \delta\varphi$.

It is easy to prove this observation by induction on the length of φ, however, no such proof will be given here.

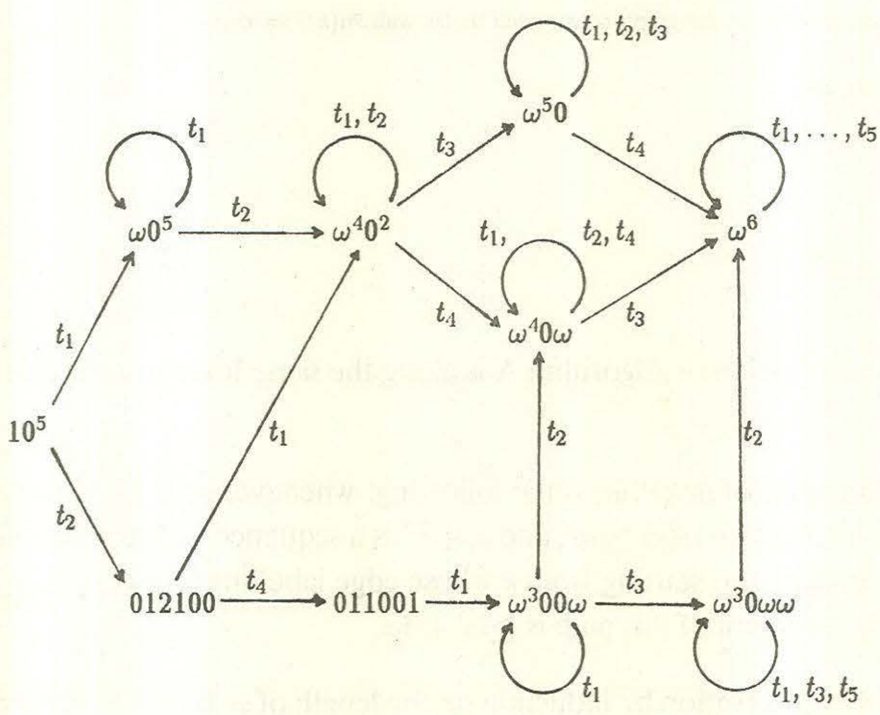
Example (without indicator places):

For the (persistent) Petri net P



with initial marking $m = 10^5$ (short for $(1,0,0,0,0,0) \in \overline{\mathbb{N}}^6$), Algorithm A produces the graph

$RG(P, m)$:



Definition 3:

For $\overline{m} \in \overline{\mathbf{N}}^v$, $N \in \mathbf{N}$, let $F(\overline{m}, N)$ denote the set

$$F(\overline{m}, N) := \{m \in \mathbf{N}^v; (\forall i \in I_v)[m_i = \overline{m}_i \vee (\overline{m}_i = \omega \wedge m_i \geq N)].$$

Lemma 3:

Let (P, m) be a persistent Petri net, k a node in its reachability graph, P' the Petri net which is obtained from P by eliminating all those places and incident arcs, for which the corresponding component of $\overline{m}(k)$ equals ω, and m' the projection of $\overline{m}(k)$ onto the places of P'.
Then, (P', m') is persistent.

Proof:

Let T' be the set of transitions of P', $t' \in T'$ being obtained by the above construction from $t \in T$, and assume that (P', m') is not persistent. Then there are $\varphi' \in T'^*$ and $t'_i, t'_j \in T'$ with $i \neq j$ s.t. (with $m'' := m' + \delta\varphi'$)

$$a(t'_i, m'') \wedge a(t'_j, m'') \wedge \neg a(t'_i t'_j, m''). \qquad (*)$$

But as has been shown in [5], one can, for given $(P, m), N \in \mathbf{N}$, and node k in $RG(P, m)$, effectively find some $\overline{\varphi} \in T^*$ s.t. $a(\overline{\varphi}, m)$ and $m + \delta\overline{\varphi} \in F(\overline{m}(k), N)$. Choosing N big enough one could obtain $\overline{m} \in R(P, m) \bigcap F(\overline{m}(k), N)$ s.t. $a(\varphi t_i, \overline{m})$ and $a(\varphi t_j, \overline{m})$ (where $\varphi, t_i, t_j \in T^*$ correspond to $\varphi', t'_i, t'_j \in T'^*$). From (*) then follows $\neg a(\varphi t_i t_j, \overline{m})$, contradicting the persistence of (P, m). ∎

5. Strongly connected components of $RG(P, m)$

Let CC be a strongly connected component (SCC) of $RG(T, m)$. Stripping the nodes in CC of their marking $\overline{m}$ and considering CC as the state transition diagram of a finite automaton, one can obtain the regular set of all transition sequences constituting edge labelling sequences of closed paths in CC. The Parikh image CT of this set, then, is a linear set with $0 \in CT$, and a representation of it can effectively be constructed from CC.

Now, let $CT^+ := \{c \in CT; Vc \geq 0\}$. Then, CT^+ also is an effectively constructable linear set. If $c \in CT^+$, then there is a node k in CC and a transition sequence $\varphi \in T^*$ s.t.

$$\Phi(\varphi) = c \text{ and } a(\varphi, \overline{m}(k)).$$

Because of Lemma 3 and Lemma 2d), we have, for any other node k' in CC,

$$c \in C(P, \overline{m}(k')).$$

Therefore, if k is some fixed node in CC, one can effectively find $\varphi^1, \ldots, \varphi^r$ s.t.

(i) $(\forall i \in I_r)[a(\varphi^i, \overline{m}(k))]$;

(ii) CT^+ is generated by $\{\Phi(\varphi^i); i \in I_r\}$, i.e.

$$CT^+ = \{\sum_{i=1}^{r} n_i \Phi(\varphi^i); (n_i, \ldots, n_r) \in \mathbb{N}^r\}.$$

Definition 4:

Let CT^+ and k be as above. A *hurdle* for k is then a number $H_k \in \mathbb{N}$ s.t. there are $\varphi^1, \ldots, \varphi^r$ generating CT^+ and firable at $\overline{m}(k)$ for which

$$m' \in F(\overline{m}(k), H_k) \Rightarrow (\forall i \in I_r)[a(\varphi^i, m')]$$

holds.

Given CC and k, such an H_k can effectively be determined by looking at the marking differences effected by all prefixes of the transition sequences $\varphi^i, i \in I_r$, generating CT^+. Note that it suffices to look at one application of φ^i only because $\delta\varphi^i \geq 0$.

Further, as mentioned in the proof of Lemma 3, a $c \in C(P,m)$ can effectively be found s.t. $m + Vc \in F(\overline{m}(k), H_k)$; c is called *appropriate* for k.

Definition 5:

Let (P,m) be an arbitrary marked Petri net, $P = (S,T,K)$.

(a) A transition $t \in T$ is *bounded* in (P,m) iff

$$(\exists N \in \mathbf{N},\ \nexists c \in C(P,m))[N\Phi(t) \leq c].$$

The set of bounded transitions in (P,m) is denoted by $BT(P,m)$.

(b) An SCC CC in $RG(P,m)$ is called *distinguished* iff the labels of all edges leaving CC are in $BT(P,m)$.

It has been shown in [6] that it is decidable whether $t \in T$ is bounded. As a matter of fact, $t \in T$ is not bounded iff it is the label of some edge within some SCC of $RG(P,m)$.

Lemma 4:

Let (P,m) be a persistent Petri net, $P = (S,T,K), t_i \in T$, and $N \in \mathbf{N}$. Let P' be the modification of P obtained by adding a new place s such that only $K(s,t) = 1$ and there is no other edge incident on s. Let, further, m' be the extension of m s.t. $m'(s) = N$. Then

(a) (P',m') is persistent;

(b) $C(P',m') = \{c \in C(P,m); c_i \leq N\}$;

(c) $t_i \in BT(P',m')$.

Proof:

a) follows easily from the definition of persistence, b) can be seen by induction on the length of firing sequences containing at most N occurrences of t_i, and c) is a corollary of b). ∎

6. Construction of a Semilinear Representation of $C(P, m)$

Let (P, m) be a persistent Petri net.

Algorithm B:

begin
  var GSL:**repr of semilinear set;**
  co GSL refers to representations of semilinear sets **oc;**
  procedure slset $((P, m)$:**persistent Petri net);**
  begin
    var HK:**integer;** CTK:**repr of semilinear set;** $c, maxc$:**counter;**
    var Pm:**persistent Petri net;** k:**node;**
    $maxc := 0$ **co** in $\mathbf{N}^{n}$ **oc;**
    construct $RG(P, m)$ using Algorithm A;
    for all CC s.t. CC is a distinguished SCC in $RG(P, m)$ **do**
      $CTK :=$ a representation of CT^{+} of CC;
      $k :=$ some node in CC;
      $c :=$ some appropriate counter for k;
      attach to k a representation of $c + CTK$, to all other nodes in CC a representation of the empty set $\emptyset$;
      while there is some edge in CC labelled t from node k' to k'' (possibly $k' = k''$) with semilinear sets $SL_{k'}$
        and $SL_{k''}$ attached to k' and k'', resp., s.t.
$$(\exists c' \in SL_{k'})[a(t, m + Vc') \wedge c' + \Phi(t) \notin SL_{k''}]$$
        co this can be written as a Presburger expression **oc**
      do
        replace $SL_{k''}$ by a representation for
$$\{c' + \Phi(t); c' \in SL_{k'} \wedge a(t, m + Vc')\}$$
        co this again is a semilinear set **oc**
      od;
      $maxc := \max\{c, maxc\}$ **co** maximum taken coordinatewise **oc**
    od;
    for all k' in distinguished SCC's of $RG(P, m)$ **do**
      $GSL := GSL \bigcup SL_{k'}$;
    for all $t_i \in T - BT(P, m)$ **do**
      $Pm :=$ the persistent Petri net obtained from (P, m) by bounding t_i by $maxc_i$ as in Lemma 4;
      slset(Pm)
    od
  end slset;
  $GSL := \emptyset$;
  slset$((P, m))$;
  print(GSL)
end Algorithm B.

Theorem 1:

Algorithm B terminates for persistent (P, m).

Proof:

Because of Lemma 4, the marked Petri nets in all recursive calls of the procedure *slset* are persistent. As in each recursive step the number of unbounded transitions of the net strictly decreases it suffices to prove termination of the *while*-loop in *slset*.

To do this we first note that if CC is some distinguished SCC in $RG(P,m)$ and SL_k the semilinear set attached to some node k in CC in the course of Algorithm B, then $SL_k + CTK \subseteq SL_k \subseteq C(P,m)$. This is true in the beginning because of the choice of c and CTK, and it remains obviously true when new values are assigned to SL_k. Hence, if the *while*-loop did not terminate, then, by König's Infinity Lemma, there would be an infinite sequence of executions of the loop in each of which the same edge in CC is chosen. As every infinite sequence of pairwise different vectors of any finite dimension with nonnegative integer components contains an infinite increasing subsequence [3], there must be then two executions of the loop such that in the first some c' is added to $SL_{k'}$ (where k' is the origin of e) and in the second which comes later, some c'' is added to $SL_{k'}$ with $c'' > c'$ and $Vc'' \geq Vc'$. Because of Lemma 2c), $c'' \in C(P, m + Vc')$. Let φ be a firing sequence such that $m + Vc' \xrightarrow{\varphi} m + Vc''$. Observing the property of $RG(P,m)$ noted after Algorithm A and the fact that all transitions in φ are unbounded it follows, however, that there must be a path in CC from k' to k' with edge labelling sequence φ. But as $\delta\varphi = V(c'' - c') \geq 0$, we have $c'' - c' \in CTK$, and $c'' \in SL_{k'}$ as soon as $c' \in SL_{k'}$ as $SL_{k'}$ is closed under addition of CTK.
Thus, there is no infinite execution of the *while*-loop. ∎

Theorem 2:

Algorithm B outputs GSL s.t. $GSL = C(P,m)$.

Proof:

Because of the remark made in the proof of Theorem 1, only $GSL \supseteq C(P,m)$ needs to be shown.

It follows immediately from the properties of $RG(T,m)$ that if φ and φ' are edge labelling sequences of paths in $RG(P,m)$ both starting from the root and ending in the same SCC of

$RG(P, m)$, then for all $t_i \in BT(P, m)$ $(\Phi(\varphi))_i$ equals $(\Phi(\varphi'))_i$.

Now assume w.l.g. that $BT(P, m) = \{t_1, \ldots, t_{w'}\}, w' \leq w$.

Because of Lemma 2b), there is, then, for every $c' \in \{(c_1, \ldots, c_{w'}); (c_1, \ldots, c_w) \in C(P, m)\}$ exactly one maximal SCC $CC_{c'}$ (i.e. no other SCC's with the same property can be reached from it) such that for all edge labelling sequences φ of paths from the root of $RG(P, m)$ to any node in $CC_{c'}$ $(\Phi(\varphi))_i = c'_i$ holds for all $i \in I_{w'}$. As $CC_{c'}$ is maximal it is distinguished.

Now, if c is the counter chosen in Algorithm B for $CC_{c'}$ it is clear from the loop predicate that after termination of the *while*-loop

$$\bigcup_{k' \text{ node in } CC_{c'}} SL_{k'} \supseteq \{\bar{c} \in C(P, m);\ \bar{c} \geq c \wedge (\forall i \in I_{w'})[\bar{c}_i = c'_i]\}$$

(Note that applying Lemma 3 to $(P, m(k))$ where k is the node in $CC_{c'}$ chosen in Algorithm B gives $c_i = c'_i$ for all $i \in I_{w'}$). Hence, from Lemma 4b), and induction on the number $w - w'$ of unbounded transitions in (P, m)—the case $w - w' = 0$ being trivial as $RG(P, m)$ contains no loops and all node markings are finite—one obtains $GSL \supseteq C(P, m)$. ∎

Corollary:

If (P, m) is a persistent Petri net then its reachability set is an effectively constructable semilinear set.

Proof:

Corollary to the definition of $C(P, m)$ and Theorem 2. ∎

Theorem 3:

Each of the following problems for persistent Petri nets is decidable:

(a) the reachability problem;
(b) the reachability set equality problem;

(c) the reachability set inclusion problem;
(d) the reachability set disjointness problem.

Proof:

Theorem 2 and well known properties of semilinear sets. ∎

7. Conclusion

Algorithm B effectively solves a problem for which until now only a nonconstructive solution was known. Also, to our knowledge, persistent Petri nets are—besides m-reversible nets [1]—the only class of Petri nets for which an effective closed representation of infinite reachability sets has been found so far (of course, finite reachability sets can effectively be enumerated). Because of the undecidability of the general reachability set inclusion problem [cf. 2] a corresponding representation is not possible for general Petri nets. The complexity of the algorithm presented here is still open as no upper bounds are known on the length of the longest repetition-free non-increasing firing sequence in persistent Petri nets.

8. References

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