

ASYMPTOTIC ORDERS OF REACHABILITY
IN LINEAR DYNAMIC SYSTEMS

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ABSTRACT

An algebraic approach to high gain controls for linear dynamic systems with varying orders of reachability is developed. Based on this approach, the issues of high gain inputs for reaching target states, high gain feedback for pole placement and high gain inputs for steering trajectories arbitrarily close to almost (A,B)-invariant subspaces and almost (A,B)-controllability subspaces are addressed.

Systems that are parametrized by a small parameter ϵ are considered. Orders of reachability and feedback results are developed first for discrete time systems and then shown to hold for continuous time. Also, for continuous time systems the notions of almost (A,B)-invariance and almost (A,B)-controllability are analyzed. It is illustrated that the approach of this thesis provides a quantitative insight into these notions.

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I INTRODUCTION

I.1 PROBLEM STATEMENT

In this thesis, we develop a theory of asymptotic orders of reachability in linear dynamic systems. We use this theory in analyzing and controlling systems with couplings and control effectiveness of different orders. To provide a motivation for the key issues in our approach, let us consider the following discrete time system as an example:

Example 1.1.1

$$x[k+1] = \begin{bmatrix} 1 & 1 \\ .01 & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ .01 \end{bmatrix} u[k] \quad (1.1.1)$$

This system is reachable but the reachability matrix $[b|Ab] = \begin{bmatrix} 1 & 1.01 \\ .01 & .03 \end{bmatrix}$ is not very far from a singular matrix, in that its condition number is approximately 10^4 . This leads to numerical difficulties in determining reachability as shown in [14]. Also, consider the minimum energy control problem for this system. For example, the minimum energy control to reach $x[2] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ from $x[0] = 0$ is $u_1[1] = -.5$ and $u_1[2] = 1.5$, while the minimum energy control for $x[2] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is $u_2[1] = 49.7$ and $u_2[2] = -49.3$. This order of magnitude difference between u_1 and u_2 is another indication of near unreachability.

To model problems such as the above, this thesis focuses on models in which a small parameter ϵ implicitly indicates the presence of different orders of couplings, scalings, reachability, etc. Specifically, we consider continuous time and discrete time systems of the form

$$\dot{x}(t) = A(\epsilon)x(t) + B(\epsilon)u(t) \quad (1.1.2)$$

$$x[k+1] = A(\epsilon)x[k] + B(\epsilon)u[k] \quad (1.1.3)$$

where ϵ is a small parameter.

In the rest of this chapter, we review the work related to the problems of interest to us, illustrate our motivations via some examples, and describe the above models in more detail. In Chapter II, we develop a theory of orders of reachability for discrete time systems. In Chapter III, we extend this theory to pole placement with full state feedback. In Chapter IV, we show that equivalent results are obtained for continuous time systems, and also provide connections with Willems' work on almost invariance [3]. In Chapter V, we summarize our results, and suggest problems for further research.

I.2 REVIEW OF RELATED WORK

This work was particularly motivated by the pole placement theory for linear time-invariant systems. Specifically, the numerical problems encountered in various pole placement methods and in evaluating system reachability were important elements in our motivation.

Pole placement and related numerical issues are addressed using various approaches in the current literature [4-7]. Patel [6] discusses numerical algorithms for pole placement using state feedback for single input systems. Miminis [5] discusses pole assignment in multi-input systems using the inputs one by one. In multi-input systems, unlike single-input systems, the feedback matrix that produces a given set of poles is not unique, and the additional degrees of freedom may be used to attain other control objectives (see [7]). One may, for example, attempt to minimize the maximum feedback gain; Patel addresses this problem via numerical examples on redistribution of the feedback task among the inputs and balancing the A and B matrices. These examples contain some intuitive ideas, but have not led to systematic procedures that work well for well-defined and substantial classes of systems. Petkov et al. [4] use the freedom in the feedback matrix to analyze the flexibility of assigning

eigenvectors in addition to assigning a given set of eigenvalues, thereby implementing numerically the ideas in [7].

Another area of numerical work involves criteria to measure controllability. Boley et al. [9,10] use the "distance to the nearest uncontrollable system" as a criterion. They define this by the minimum norm perturbation that would make a system uncontrollable. They also relate this concept to state feedback by measuring the amount that the eigenvalues move due to state feedback of bounded magnitude.

Our treatment of problems of this type is qualitative rather than numerical in nature: we assume that small values in the system are modelled by functions of a small parameter ϵ , and we look at how unreachable this system is in terms of "orders of ϵ ". Specifically, the starting point for our work is a regularly perturbed system of the form (1.1.2) and (1.1.3). The issue of controllability in perturbed systems of the form (1.1.1) has been examined by Chow [8]. He defines a system to be strongly controllable if the system is controllable at $\epsilon = 0$. Otherwise, he calls it weakly controllable and concludes that pole placement of such systems will require controls with large gains. Chow looks at systems with two time scales (slow and fast), and he proves that a necessary and sufficient condition for such a "singularly perturbed" system to be strongly controllable is the

controllability of its slow and fast subsystems.

Our analysis goes further than Chow's in that we examine the relative orders of reachability of different parts of the state space. The methods we use have some similarity to those used by Lou et al. [1,2] who analyze the multiple time scale structure of the systems

$$\dot{x}(t) = A(\epsilon)x(t) \tag{1.2.1}$$

$$\dot{x}(t) = A(\epsilon)x(t) + B(\epsilon)u(t) \tag{1.2.2}$$

They relate the time scales of (1.2.1) to the invariant factors of $A(\epsilon)$ viewed as a matrix over the ring of functions analytic at $\epsilon = 0$. Consequently, the Smith Decomposition (Appendix A.1) plays a key role in their analysis as it does in ours (although our decompositions do not involve time scales; see Chapter II). While the primary focus of the work in [1,2] is on time scale structure, some attention is paid to control. In particular, results are given on the use of feedback in (1.2.2) to change the time scale structure of the system. This thesis may be seen as a continuation of the work in [1,2] in that it analyzes the effect of control and feedback on the system of (1.2.2).

I.3 MOTIVATING EXAMPLES

A problem that frequently arises in numerical computations is the determination of these terms that may be neglected what to neglect so that computations are simplified without significantly altering results. This is essentially equivalent to determining the accuracy to which numerical computations must be carried out. To illustrate this problem, consider the following simple example which is a sampled-data model for an undamped rotor of inertia $J=1$, acted on by a piecewise constant torque $u[k]$ in the k^{th} sampling interval, where T is the length of the sampling interval:

Example 1.3.1:

$$x[k+1] = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x[k] + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} u[k] \quad (1.3.1)$$

Suppose that $T \ll 1$, and that we decided to neglect $T^2/2$ and therefore to work with the simplified model:

$$x[k+1] = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ T \end{bmatrix} u[k] \quad (1.3.2)$$

A little calculation shows that a feedback $[-1/T^2 \quad -2/T]$ shifts both of the poles of (1.3.2) to 0. On the other hand, using this feedback in (1.3.1) produces the poles $1/2$ and -1 , which do not stabilize the system. The reason for this is that T^2 terms are not negligible. One way to see this is to examine the

reachability matrices. Specifically if we consider the reachability matrix, $[b|Ab] = \begin{bmatrix} T^2/2 & 3T^2/2 \\ T & T \end{bmatrix}$ of (1.3.1) or $\begin{bmatrix} 0 & T^2 \\ T & T \end{bmatrix}$ of (1.3.2) we realize that these models are no longer reachable when we neglect the T^2 terms in their reachability matrices. In fact, the coupling from the input to the first state is zero, to the second state is T , and the coupling between the two states is T . Thus, the coupling from the input to the first state, through the second state is T^2 . As a result, T^2 plays an important role in reachability, and we should not neglect it. The theory developed in Chapters II and III addresses this problem, and it is illustrated for this example in Section 3.3.

In the same fashion, small values that could play a significant role in reachability might be neglected during computation, due to insufficient numerical accuracy of the computation device used. Consider the following simple example of a discrete time system:

Example 1.3.2:

$$x[k] = \begin{bmatrix} 0 & 1 \\ -1.999 & 3 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1.001 \end{bmatrix} u[k] \quad (1.3.3)$$

If the numerical accuracy of the computation device used is 3 digits, then the entries -1.999 and 1.001 would be rounded to -2 and 1 , respectively, making this reachable system appear unreachable. On the other hand, consider the following

counterpart of (1.3.3)

$$\mathbf{x}[k] = \begin{bmatrix} 0 & 1 \\ -2+\epsilon & 3 \end{bmatrix} \mathbf{x}[k] + \begin{bmatrix} 1 \\ 1+\epsilon \end{bmatrix} u[k] \quad (1.3.4)$$

which is equivalent to (1.3.3) for $\epsilon = 0.001$. By a similarity transformation, we get the following system

$$\mathbf{y}[k] = \begin{bmatrix} 1 & 1 \\ \epsilon & 2 \end{bmatrix} \mathbf{y}[k] + \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} u[k] \quad (1.3.5)$$

If this model is evaluated at $\epsilon = 0.001$, then, assuming that floating point calculations are used, rounding errors will be prevented and it would be possible to correctly detect reachability. Chapter II provides an algorithm to compute the transformation necessary to derive the system of (1.3.5) from (1.3.4). Chapter III illustrates the above numerical problem and compares the result attained by the approach of this thesis to conventional methods for pole placement on a relatively complicated example.

I.4 MODEL DESCRIPTION

In this section, we first provide some algebraic background and then define the models that we use, in detail.

I.4.a ALGEBRAIC BACKGROUND

Let $\mathbb{R}[[\epsilon]]$ be the ring of real valued functions that have Taylor expansions in ϵ . The units in $\mathbb{R}[[\epsilon]]$ are precisely those elements with nonzero constant terms. Let \mathfrak{r} be the set of elements of $\mathbb{R}[[\epsilon]]$ with zero constant terms, then \mathfrak{r} is an ideal of $\mathbb{R}[[\epsilon]]$. Thus, $\mathbb{R}[[\epsilon]]$ is a local ring and \mathfrak{r} its maximal ideal.

$\mathbb{R}^n[[\epsilon]] \equiv \mathbb{R}[[\epsilon]] \oplus \dots \oplus \mathbb{R}[[\epsilon]]$ (n summands) is a finitely generated free $\mathbb{R}[[\epsilon]]$ -module. Let x_i ($1 \leq i \leq n$) form a basis of \mathbb{R}^n . All x_i are in $\mathbb{R}^n[[\epsilon]]$ and form a basis of $\mathbb{R}^n[[\epsilon]]/\mathfrak{r}\mathbb{R}^n[[\epsilon]]$ which is isomorphic to $\mathbb{R}^n + \mathfrak{r}\mathbb{R}^n[[\epsilon]]$. Thus, any basis of \mathbb{R}^n generates $\mathbb{R}^n[[\epsilon]]$. We can also represent $\mathbb{R}^n[[\epsilon]]$ as $\mathbb{R}^n[[\epsilon]] = \mathbb{R}^n + \epsilon\mathbb{R}^n + \epsilon^2\mathbb{R}^n + \epsilon^3\mathbb{R}^n + \dots$

Let $\mathbb{R}((\epsilon))$ be the field of real valued functions which have Laurent expansions in ϵ . $\mathbb{R}^n((\epsilon)) \equiv \mathbb{R}((\epsilon)) \oplus \dots \oplus \mathbb{R}((\epsilon))$ (n summands) is then a vector space.

I.4.b DESCRIPTION OF THE SYSTEM

We use continuous time and discrete time models of the form

$$\dot{x}(t) = A(\epsilon)x(t) + B(\epsilon)u(t) \quad (1.4.1)$$

$$x[k+1] = A(\epsilon)x[k] + B(\epsilon)u[k] \quad (1.4.2)$$

where $A(\epsilon) = [a_{ij}(\epsilon)]_{n \times n}$, $a_{ij}(\epsilon) \in \mathbb{R}((\epsilon))$ (1.4.3)

$$B(\epsilon) = [b_{ij}(\epsilon)]_{n \times m}$$
, $b_{ij}(\epsilon) \in \mathbb{R}((\epsilon))$ (1.4.4)

such that $A(\epsilon) : \mathbb{R}^n((\epsilon)) \rightarrow \mathbb{R}^n((\epsilon))$ (1.4.5)

$$B(\epsilon) : \mathbb{R}^m((\epsilon)) \rightarrow \mathbb{R}^n((\epsilon))$$
 (1.4.6)

and the reachability matrix

$$\mathcal{Q}(\epsilon) = [B(\epsilon) \mid A(\epsilon)B(\epsilon) \mid \dots \mid A^{n-1}(\epsilon)B(\epsilon)] : \mathbb{R}^{mn}((\epsilon)) \rightarrow \mathbb{R}^n((\epsilon))$$

We consider $\epsilon \in (0, a)$, where $a \in \mathbb{R}^+$. The state and input vectors are in fact functions of both t and ϵ . For example, $u(t, \epsilon)$ (denoted above as just $u(t)$), for fixed t , is a family of input values as a function of ϵ . On the other hand, for fixed ϵ , $u(t)$ is some input function. For fixed t , $u(\epsilon) \in \mathbb{R}^m((\epsilon))$ and $x(\epsilon) \in \mathbb{R}^n((\epsilon))$. These systems are defined over $\mathbb{R}((\epsilon))$ primarily because the objective of this thesis is to examine the effect or necessity of high gain feedback. For most of this thesis, $A(\epsilon)$ is restricted to have eigenvalues in $\mathbb{R}[[\epsilon]]$. Also note that $\mathcal{Q}(\epsilon)$ can be made analytic at $\epsilon = 0$ (equivalently over $\mathbb{R}[[\epsilon]]$) by a simple

input scaling. This fact is used at points which make it more convenient to represent the results. In addition, we assume that the reachability matrix is full row rank for all $\epsilon \in (0,a)$, i.e. either the system is reachable for all $\epsilon \in (0,a)$ or the reachable part of the system is being used. In the cases of most interest to us, the reachability matrix will lose rank for $\epsilon = 0$, and a will be the smallest positive value of ϵ for which the reachability matrix loses rank. Under these conditions, we analyze the asymptotic behavior of the system as $\epsilon \downarrow 0$.

II REACHABILITY OF DISCRETE TIME SYSTEMS

In this chapter, we examine the orders of reachability of discrete time linear dynamic systems of the form (1.4.2), satisfying (1.4.3) - (1.4.6).

II.1 ϵ^j -REACHABILITY

We start by developing our theory of asymptotic orders of reachability in an analogous way to existing linear control theory. In order to provide a motivation for our approach, let us start with the following counterpart of Example 1.1.1 in the framework defined in I.4:

Example 2.1.1:

$$x[k+1] = \begin{bmatrix} 1 & 1 \\ \epsilon & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} u[k]$$

so

$$\mathcal{Q}(\epsilon) = \begin{bmatrix} 1 & 1+\epsilon \\ \epsilon & 3\epsilon \end{bmatrix}$$

This system is reachable for all $\epsilon \in (0,2)$. The minimum energy control sequence needed to go from the origin to

$x_1[2] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is $u_1[1] = -1/(2-\epsilon)$ and $u_1[2] = 3/(2-\epsilon)$, which is $O(1)$.⁽¹⁾ The minimum energy control sequence for $x_2[2] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is $u_2[1] = (-\epsilon+1)/\epsilon(2-\epsilon)$ and $u_2[2] = (2\epsilon-1)/\epsilon(2-\epsilon)$, which is $O(1/\epsilon)$.

Let us start by characterizing target states by the order of control sufficient to reach them.

Definition 2.1.2: $x(\epsilon) \in \mathbb{R}^n[[\epsilon]]$ is ϵ^j -reachable if there exists an $O(1/\epsilon^j)$ input sequence $\mathcal{U}(\epsilon) \equiv \begin{bmatrix} u[n-1] \\ \vdots \\ u[0] \end{bmatrix}$ such that $x(\epsilon)$ is reached from zero in n steps using $\mathcal{U}(\epsilon)$ (i.e. $x(\epsilon) = \mathcal{C}(\epsilon)\mathcal{U}(\epsilon)$).

Let \mathcal{X}^j be the set of all ϵ^j -reachable states.

Proposition 2.1.3: $\mathcal{X}^0 \subset \mathcal{X}^1 \subset \mathcal{X}^2 \subset \dots$ and \mathcal{X}^j is an $\mathbb{R}[[\epsilon]]$ -submodule of $\mathbb{R}^n[[\epsilon]]$.

Proof: The first statement follows from the fact that an $O(1/\epsilon^j)$ input sequence is also $O(1/\epsilon^{j+1})$.

To reach $x[n] = c(\epsilon)x(\epsilon)$, where $c(\epsilon) \in \mathbb{R}[[\epsilon]]$ we can use $c(\epsilon)\mathcal{U}(\epsilon)$ which is $O(1/\epsilon^j)$. Thus, $c(\epsilon)x(\epsilon) \in \mathcal{X}^j$.

¹ $f(\epsilon)$ is $O(\epsilon^k)$ if

$$\lim_{\epsilon \downarrow 0} \frac{\|f(\epsilon)\|}{\epsilon^k}$$

exists, where k is an integer, $f(\epsilon)$ is a scalar, vector or matrix, and $\|\ast\|$ denotes the appropriate norm. Note that if $f(\epsilon)$ is $O(\epsilon^k)$ then it is also $O(\epsilon^{k-1})$, $O(\epsilon^{k-2})$ etc.

Let $x_1(\epsilon), x_2(\epsilon) \in \mathcal{X}^j$, and let $u_1(\epsilon), u_2(\epsilon)$ be $O(1/\epsilon^j)$ input sequences that cause these respective states to be reached. To reach $x[n] = x_1(\epsilon) + x_2(\epsilon)$, we can use $u_1(\epsilon) + u_2(\epsilon)$ which is an $O(1/\epsilon^j)$ input sequence. Thus, $x_1(\epsilon) + x_2(\epsilon) \in \mathcal{X}^j$. Therefore, \mathcal{X}^j is a submodule of $\mathbb{R}^n[[\epsilon]]$.

We term \mathcal{X}^j the ϵ^j -reachable submodule. (Note that if $x(\epsilon)$ is ϵ^j reachable, then $(1/\epsilon)x(\epsilon)$ is not necessarily ϵ^j reachable. Thus if we had considered target states in $\mathbb{R}^n((\epsilon))$ in Definition 2.1.2, then the set of ϵ^j reachable states would not be $\mathbb{R}((\epsilon))$ -subspaces).

In Example 2.1.1, $\mathcal{X}^0 = \text{Im} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \epsilon \mathbb{R}^2[[\epsilon]]$, $\mathcal{X}^1 = \mathcal{X}^2 = \dots = \mathbb{R}^2[[\epsilon]]$.

An interesting property of the set of ϵ^j -reachability submodules is that all the structure is embedded in the ϵ^0 -reachable submodule. First of all, note that \mathcal{X}^0 is the image of the reachability matrix under the set of all control sequences in $\mathbb{R}^{mn}[[\epsilon]]$. Also, the ϵ^j -reachable submodule is simply obtained by scaling the ϵ^{j-1} -reachable submodule by $1/\epsilon$. To state this formally:

Proposition 2.1.4: $\mathcal{X}^0 = \{\mathcal{C}(\epsilon)\mathbb{R}^{mn}[[\epsilon]]\} \cap \mathbb{R}^n[[\epsilon]]$ and

$$\mathcal{X}^j = \frac{1}{\epsilon} \{\mathcal{X}^{j-1} \cap \epsilon \mathbb{R}^n[[\epsilon]]\}, \text{ or in general,}$$

$$\mathcal{X}^i = \frac{1}{\epsilon^{i-j}} \{\mathcal{X}^j \cap \epsilon^{i-j} \mathbb{R}^n[[\epsilon]]\}, \text{ for nonnegative integers } i, j \text{ and } i \geq j.$$

Proof: By Definition 2.1.2, $\mathcal{X}^0 = \{\mathcal{C}(\epsilon)\mathbb{R}^{mn}[[\epsilon]]\} \cap \mathbb{R}^n[[\epsilon]]$, or in general $\mathcal{X}^j = \{\mathcal{C}(\epsilon)1/\epsilon^j \mathbb{R}^{mn}[[\epsilon]]\} \cap \mathbb{R}^n[[\epsilon]]$. Then,

$$\begin{aligned} \frac{1}{\epsilon^{i-j}} \{\mathcal{X}^j \cap \epsilon^{i-j} \mathbb{R}^n[[\epsilon]]\} &= \frac{1}{\epsilon^{i-j}} \left\{ \frac{1}{\epsilon^j} \mathcal{C}(\epsilon) \mathbb{R}^{mn}[[\epsilon]] \cap \epsilon^{i-j} \mathbb{R}^n[[\epsilon]] \right\} \\ &= \left\{ \frac{1}{\epsilon^i} \mathcal{C}(\epsilon) \mathbb{R}^{mn}[[\epsilon]] \right\} \cap \mathbb{R}^n[[\epsilon]] = \mathcal{X}^i \end{aligned}$$

Using Proposition 2.1.4, we can find \mathcal{X}^0 from the reachability matrix, and recover \mathcal{X}^j , $\forall j > 0$, from \mathcal{X}^0 .

Another property of the ϵ^j -reachability submodules is the invariance under $A(\epsilon)$, if the coefficients of the characteristic polynomial of $A(\epsilon)$ are analytic at $\epsilon = 0$:

Proposition 2.1.5: If the coefficients of the characteristic polynomial of $A(\epsilon)$ are analytic at $\epsilon = 0$, then $\epsilon^j A(\epsilon) \mathcal{X}^j \subset \mathcal{X}^0$.

Proof: $A(\epsilon) \mathcal{X}^0 = A(\epsilon) \mathcal{C}(\epsilon) \mathbb{R}^{mn}[[\epsilon]] \subset \mathcal{X}^0$ by the Cayley-Hamilton theorem (note that we need the restriction on the coefficients of the characteristic polynomial here). $\epsilon^j A(\epsilon) \mathcal{X}^j = \epsilon^j \frac{1}{\epsilon^j} A(\epsilon) \{\mathcal{X}^0 \cap \epsilon^j \mathbb{R}^n[[\epsilon]]\} \subset \{A(\epsilon) \mathcal{X}^0 \cap \epsilon^j A(\epsilon) \mathbb{R}^n[[\epsilon]]\} \subset \mathcal{X}^0$

(Note that this result always holds if $A(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$.)

An immediate consequence of the above proposition is the following:

Corollary 2.1.6: $\epsilon^j A(\epsilon) \mathfrak{X}^j \subset \mathfrak{X}^0$ iff $A(\epsilon) \mathfrak{X}^0 \subset \mathfrak{X}^0$.

The reason for scaling $A(\epsilon) \mathfrak{X}^j$ by ϵ^j is that Definition 2.1.2 is stated in terms of target states in $\mathbb{R}^n[[\epsilon]]$, not $\mathbb{R}^n((\epsilon))$. The issue is made clear in the following example:

Example 2.1.7: Let $A(\epsilon) = \begin{bmatrix} 0 & 1/\epsilon \\ \epsilon & 0 \end{bmatrix}$ and $B(\epsilon) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is ϵ -reachable, but $\begin{bmatrix} 1/\epsilon \\ 0 \end{bmatrix}$ is not even valid as a target state. On the other hand, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is reachable by $O(1)$ control, so $\begin{bmatrix} 1/\epsilon \\ 0 \end{bmatrix}$ is reachable by $O(1/\epsilon)$ control. Proposition 2.1.5 aims to capture this fact. By this proposition, any trajectory that starts in the ϵ^j -reachable submodule stays in this submodule if no input is applied.

The following example illustrates the case where the coefficients of the characteristic polynomial of $A(\epsilon)$ are not analytic at $\epsilon=0$:

Example 2.1.8: Let $A(\epsilon) = \begin{bmatrix} 0 & 1/\epsilon^2 \\ \epsilon & 0 \end{bmatrix}$ and $B(\epsilon) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\begin{bmatrix} 0 \\ \epsilon \end{bmatrix}$ is 1-reachable and $\begin{bmatrix} 1/\epsilon \\ 0 \end{bmatrix}$ is not a valid target state, but again intuitively, it is reachable by $O(1/\epsilon)$ control but not by $O(1)$ control. Also, consider $[B(\epsilon) \mid \dots \mid A^3(\epsilon)B(\epsilon)] = \begin{bmatrix} 1 & 0 & 1/\epsilon & 0 \\ 0 & \epsilon & 0 & 1 \end{bmatrix}$, then $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is ϵ -reachable but it is also reachable by $O(1)$ control in 4 steps. This case is examined in more detail in Section II.4

where we introduce the notion of proper systems.

The structure of the ϵ^j -reachability submodules is not always easily obtained by inspection from the pair $(A(\epsilon), B(\epsilon))$, as it was the case in Example 2.1.1. To illustrate this, consider an ϵ perturbation of Example 2.1.1:

Example 2.1.9:

$$x[k+1] = \begin{bmatrix} 1 & 1 \\ -\epsilon & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} u[k]$$

where $\varphi(\epsilon) = \begin{bmatrix} 1 & 1+\epsilon \\ \epsilon & \epsilon \end{bmatrix}$

This is a system reachable for all $\epsilon \in (0, \infty]$. In this case, we find that $x_1[2] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is ϵ -reachable, and $x_2[2] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is ϵ^2 -reachable. Therefore, even an ϵ perturbation may cause drastic changes in our submodules. We have discovered that the invariant factors and the Smith decomposition of the reachability matrix expose the structure of the ϵ^j -reachability submodules, as developed in the following section.

II.2 SMITH DECOMPOSITION

In this section, we examine the role of the Smith decomposition of the reachability matrix in exposing the ϵ^j -reachability structure of our system. To ensure that the minimum possible order of control is used, consider the minimum energy control to reach a target state $x[n]$ from the origin in n

steps in a reachable discrete time system. This is given by:

$$\mathcal{U}(\epsilon) = \mathcal{Q}^T(\epsilon)(\mathcal{Q}(\epsilon)\mathcal{Q}^T(\epsilon))^{-1}\mathbf{x}[n] \quad (2.2.1)$$

where

$$\mathcal{U}(\epsilon) = \begin{bmatrix} u[n-1] \\ \vdots \\ u[0] \end{bmatrix}$$

Consider the Smith decomposition [Appendix A.1] of $\mathcal{Q}(\epsilon)$:

$$\mathcal{Q}(\epsilon) = P(\epsilon)D(\epsilon)Q(\epsilon) \quad (2.2.2)$$

where $P(\epsilon)$ is unimodular, $Q(\epsilon)$ is full row rank at $\epsilon=0$ and $D(\epsilon) = \text{diag}\{I, \epsilon I, \dots, \epsilon^k I\}$ with indices p_0, p_1, \dots, p_k [as defined in Appendix A.1]. Here it is assumed that $D(\epsilon)$ has no $1/\epsilon$ terms on its diagonal or equivalently that $\mathcal{Q}(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$.

We can achieve this by scaling the input vector (i.e. if the first element on the diagonal of $D(\epsilon)$ is $1/\epsilon^i$ then let $u[k] = \epsilon^i v[k]$).

The reason for this assumption is to avoid $1/\epsilon^j$ -reachable submodules so that the representations for ϵ^j -reachable submodules are "cleaner" and easier to interpret. This assumption is elaborated on in Section 2.3, specifically see Proposition 2.3.5. Also, we assume that $D(\epsilon)$ has no zeroes on its diagonal, meaning that the system is reachable for all $\epsilon \in (0, a)$, where $a \in \mathbb{R}^+$. We assume these in the remainder of this chapter unless stated otherwise.

Now, the minimum energy control becomes

$$\begin{aligned} \mathcal{U}(\epsilon) &= Q^T(\epsilon)(Q(\epsilon)Q^T(\epsilon))^{-1}D^{-1}(\epsilon)P^{-1}(\epsilon)\mathbf{x}[n] \\ &= S(\epsilon)D^{-1}(\epsilon)\mathbf{y}[n] \end{aligned} \quad (2.2.3)$$

where $S(\epsilon) = Q^T(\epsilon)(Q(\epsilon)Q^T(\epsilon))^{-1}$, and $\mathbf{y}[n] = P^{-1}(\epsilon)\mathbf{x}[n]$

Since $S(\epsilon)$ is full column rank for $\epsilon = 0$ and $P^{-1}(\epsilon)$ is unimodular, the order of control will be determined by $D^{-1}(\epsilon)$. In Example 2.1.1,

$$\mathcal{Q}(\epsilon) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} \begin{bmatrix} 1 & 1+\epsilon \\ 1 & 3 \end{bmatrix}$$

and Example 2.1.9,

$$\mathcal{Q}(\epsilon) = \begin{bmatrix} 1 & 0 \\ \epsilon & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \epsilon^2 \end{bmatrix} \begin{bmatrix} 1 & 1+\epsilon \\ 0 & -1 \end{bmatrix}$$

and the structure of $D(\epsilon)$ in each case is seen to correlate precisely with the observations made on the examples.

Let $n_i = \sum_{j=0}^i p_j$, $\mathcal{E}_{n_i} = \text{Im} \begin{bmatrix} I \\ 0 \end{bmatrix}$, where I is $n_i \times n_i$ and 0 is $(n-n_i) \times n_i$. If we transform (1.4.2) via $y[k] = P^{-1}(\epsilon)x[k]$, we get

$$y[k+1] = \bar{A}(\epsilon)y[k] + \bar{B}(\epsilon)u[k] \quad (2.2.4)$$

where $\bar{A}(\epsilon) = P^{-1}(\epsilon)A(\epsilon)P(\epsilon)$ and $\bar{B}(\epsilon) = P^{-1}(\epsilon)B(\epsilon)$.

Equation (2.2.3) shows that the ϵ^j -reachable submodules of (2.2.4), \mathcal{Y}^j , are as follows ⁽²⁾

$$\mathcal{Y}^j = \mathcal{E}_{n_j} + \epsilon \mathcal{E}_{n_{j+1}} + \dots + \epsilon^{k-1-j} \mathcal{E}_{n_{k-1}} + \epsilon^{k-j} \mathbb{R}^n[[\epsilon]] \quad (2.2.5)$$

Therefore, (2.2.4) is in such a form that the ϵ^j -reachable submodules can be found immediately from the indices. This form will be discussed in more detail in Section 2.3. Furthermore, we can find the ϵ^j -reachable submodules of the original system (1.4.2) via transformation by $P(\epsilon)$, as shown in the following proposition.

²Note that $\mathcal{E}_{n_k} = \mathbb{R}^n$, and $\mathcal{Y}^k = \mathcal{Y}^{k+1} = \mathcal{Y}^{k+2} = \dots = \mathbb{R}^n[[\epsilon]]$.

Proposition 2.2.1:

$$\alpha^j = P(\epsilon)D_j(\epsilon)\mathbb{R}^n[[\epsilon]], \quad (2.2.6)$$

where $D_j(\epsilon) = \text{diag}\{I_{n_j}, \epsilon I_{p_{j+1}}, \dots, \epsilon^{k-j} I_{p_k}\}$, and $P(\epsilon)$, n_i , p_i are given by the Smith decomposition of the reachability matrix.

Equivalently,

$$\alpha^j = P(\epsilon)\{\epsilon_{n_j} + \epsilon\epsilon_{n_{j+1}} + \dots + \epsilon^{k-1-j}\epsilon_{n_{k-1}} + \epsilon^{k-j}\mathbb{R}^n[[\epsilon]]\} \quad (2.2.7)$$

or,

$$\alpha^j = P(\epsilon)\{\mathbb{R}^{n_j}[[\epsilon]] \oplus \epsilon\mathbb{R}^{p_{j+1}}[[\epsilon]] \oplus \dots \oplus \mathbb{R}^{p_k}[[\epsilon]]\} \quad (2.2.8)$$

Proof: $y^0 = D(\epsilon)Q(\epsilon)\mathbb{R}^{mn}[[\epsilon]] = D(\epsilon)\mathbb{R}^n[[\epsilon]]$, since $Q(\epsilon)$ is unimodular (note that $D(\epsilon) = D_0(\epsilon)$). Thus,

$$y^0 = \epsilon_{n_0} + \dots + \epsilon^{k-1}\epsilon_{n_{k-1}} + \epsilon^k\mathbb{R}^n[[\epsilon]]$$

Assume $y^{j-1} = \epsilon_{n_{j-1}} + \epsilon\epsilon_{n_j} + \dots + \epsilon^{k-j}\epsilon_{n_{k-1}} + \epsilon^{k-j+1}\mathbb{R}^n[[\epsilon]]$

then

$$\begin{aligned} y^j &= \frac{1}{\epsilon}\{y^{j-1} \cap \epsilon\mathbb{R}^n[[\epsilon]]\} \\ &= \epsilon_{n_j} + \epsilon\epsilon_{n_{j+1}} + \dots + \epsilon^{k-1-j}\epsilon_{n_{k-1}} + \epsilon^{k-j}\mathbb{R}^n[[\epsilon]] \\ &= D_j(\epsilon)\mathbb{R}^n[[\epsilon]] \end{aligned}$$

Also,

$$\begin{aligned} &\epsilon_{n_j} + \epsilon\epsilon_{n_{j+1}} + \dots + \epsilon^{k-1-j}\epsilon_{n_{k-1}} + \epsilon^{k-j}\mathbb{R}^n[[\epsilon]] \\ &= \mathbb{R}^{n_j}[[\epsilon]] \oplus \epsilon\mathbb{R}^{p_{j+1}}[[\epsilon]] \oplus \dots \oplus \epsilon^{k-j}\mathbb{R}^{p_k}[[\epsilon]] \end{aligned}$$

Finally, noting that $\alpha^j = P(\epsilon)y^j$ completes the proof.

In Example 2.1.1,

$$\mathcal{X}^0 = \text{Im} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \epsilon \mathbb{R}^2[[\epsilon]], \quad \mathcal{X}^1 = \mathbb{R}^n[[\epsilon]]$$

and in Example 2.1.9,

$$\mathcal{X}^0 = \text{Im} \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} + \epsilon \text{Im} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \epsilon^2 \mathbb{R}^2[[\epsilon]]$$

$$\mathcal{X}^1 = \text{Im} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \epsilon \mathbb{R}^2[[\epsilon]], \quad \mathcal{X}^2 = \mathbb{R}^2[[\epsilon]]$$

In a Smith decomposition, $D(\epsilon)$ is unique, but $P(\epsilon)$ and $Q(\epsilon)$ are not. Thus, the part of (2.2.7) in brackets is unique. Different $P(\epsilon)$ yield different bases for \mathcal{X}^j , but they will all span the same submodule since the \mathcal{X}^j are unique.

We can now characterize systems by the order of control sufficient to reach all target states in $\mathbb{R}^n[[\epsilon]]$.

Definition 2.3.1: A system of the form (1.4.2) is an ϵ^k -reachable system with indices n_0, n_1, \dots, n_k if the reachability matrix has a Smith decomposition such that

$$D(\epsilon) = \text{diag}\{I_{p_0}, \epsilon I_{p_1}, \dots, \epsilon^k I_{p_k}\} \quad \text{where } n_i = \sum_{j=0}^i p_j .$$

Thus, Example 2.1.1 is an ϵ -reachable system with indices $n_0 = 1, n_1 = 2$. Example 2.1.9 is an ϵ^2 -reachable system with indices $n_0 = 1, n_1 = 1, n_2 = 2$.

As seen from (2.2.5), the ϵ^j -reachable submodules of (2.2.4) have a unique structure which can be constructed using the ϵ^j -reachability indices. We wish to examine this and other properties of (2.2.4) in the next section.

II.3 STANDARD FORM

In this section, we define and analyze a form that captures the structure in (2.2.4), (2.2.5):

Definition 2.3.1: Let $(A(\epsilon), B(\epsilon))$ be an ϵ^k -reachable system with indices n_0, \dots, n_k . Let $\mathcal{Q}(\epsilon) = P(\epsilon)D(\epsilon)Q(\epsilon)$ be the Smith decomposition of its reachability matrix. Let $\bar{A}(\epsilon) = P^{-1}(\epsilon)A(\epsilon)P(\epsilon)$ and $\bar{B}(\epsilon) = P^{-1}(\epsilon)B(\epsilon)$. Then, the pair $(\bar{A}(\epsilon), \bar{B}(\epsilon))$ is a standard form for $(A(\epsilon), B(\epsilon))$.

The system in Example 2.1.1 is already in standard form, whereas transforming the system in Example 2.1.9 by $P(\epsilon) = \begin{bmatrix} 1 & 0 \\ \epsilon & 1 \end{bmatrix}$ yields

$$y[k+1] = \begin{bmatrix} 1+\epsilon & 1 \\ -\epsilon^2 & 2-\epsilon \end{bmatrix} y[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which uncovers the previously hidden ϵ^2 structure.

Definition 2.3.2: A standard form of a system is a proper standard form if it has the following structure:

$$A(\epsilon) = \begin{bmatrix} A_{0,0}(\epsilon) & 1/\epsilon A_{0,1}(\epsilon) & \dots & 1/\epsilon^k A_{0,k}(\epsilon) \\ \epsilon A_{1,0}(\epsilon) & A_{1,1}(\epsilon) & \dots & 1/\epsilon^{k-1} A_{1,k}(\epsilon) \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon^k A_{k,0}(\epsilon) & \epsilon^{k-1} A_{k,1}(\epsilon) & \dots & A_{k,k}(\epsilon) \end{bmatrix} \begin{matrix} \} p_0 \\ \} p_1 \\ \vdots \\ \} p_k \end{matrix} \quad (2.3.1a)$$

$$B(\epsilon) = \begin{bmatrix} B_0(\epsilon) \\ \epsilon B_1(\epsilon) \\ \vdots \\ \epsilon^k B_k(\epsilon) \end{bmatrix} \begin{matrix} \} p_0 \\ \} p_1 \\ \vdots \\ \} p_k \end{matrix} \quad (2.3.1b)$$

where $A_{i,j}(\epsilon) : \mathbb{R}^{p_j}(\epsilon) \rightarrow \mathbb{R}^{p_i}(\epsilon)$ and $B_i(\epsilon) : \mathbb{R}^m(\epsilon) \rightarrow \mathbb{R}^{p_i}(\epsilon)$, are analytic at $\epsilon=0$, and $n_i = \sum_{j=0}^i p_j$.

Example 2.1.1 and the above example are both in proper standard form. In fact, finding one proper standard form is enough to conclude that all standard forms of a pair are proper:

Proposition 2.3.3: If a pair $(A(\epsilon), B(\epsilon))$ has a proper standard form, then all standard forms of $(A(\epsilon), B(\epsilon))$ are proper.

Proof: Let $\mathcal{Q}(\epsilon) = P_1(\epsilon)D(\epsilon)Q_1(\epsilon) = P_2(\epsilon)D(\epsilon)Q_2(\epsilon)$, then $A_i(\epsilon) = P_i^{-1}(\epsilon)A(\epsilon)P_i(\epsilon)$, $B_i(\epsilon) = P_i^{-1}(\epsilon)B(\epsilon)$ for $i=1,2$ are two standard forms. Suppose that the pair $(A_1(\epsilon), B_1(\epsilon))$ is a proper standard form. Let $\bar{A}_i(\epsilon) = D^{-1}(\epsilon)A_i(\epsilon)D(\epsilon)$, $\bar{B}_i(\epsilon) = D^{-1}(\epsilon)B_i(\epsilon)$ for $i=1,2$. Note $\bar{A}_1(\epsilon)$ and $\bar{B}_1(\epsilon)$ are both over $\mathbb{R}[[\epsilon]]$. We wish to

show that the same is true for $\bar{A}_2(\epsilon)$ and $\bar{B}_2(\epsilon)$. Let $R(\epsilon) = D^{-1}(\epsilon)P_2^{-1}(\epsilon)P_1(\epsilon)D(\epsilon)$ then $R(\epsilon)$ is invertible, and $Q_2(\epsilon) = R(\epsilon)Q_1(\epsilon)$. But, then $R(\epsilon) = Q_2(\epsilon)Q_1^+(\epsilon)$ and $R^{-1}(\epsilon) = Q_1(\epsilon)Q_2^+(\epsilon)$, where $Q_i^+(\epsilon)$ denotes the right inverse of $Q_i(\epsilon)$, which exists over $\mathbb{R}[[\epsilon]]$. Thus, $R(\epsilon)$ is unimodular. Since $(\bar{A}_1(\epsilon), \bar{B}_1(\epsilon))$ is over $\mathbb{R}[[\epsilon]]$ and $\bar{A}_2(\epsilon) = R(\epsilon)\bar{A}_1(\epsilon)R^{-1}(\epsilon)$, $\bar{B}_2(\epsilon) = R(\epsilon)\bar{B}_1(\epsilon)$, the pair $(\bar{A}_2(\epsilon), \bar{B}_2(\epsilon))$ is also over $\mathbb{R}[[\epsilon]]$. Therefore, $(A_2(\epsilon), B_2(\epsilon))$ is a proper standard form.

A pair $(A(\epsilon), B(\epsilon))$ is termed proper if it has a proper standard form. Thus, both of the systems in Examples 2.1.1 and 2.1.9 are proper. On the other hand, consider the following example:

Example 2.3.4: Let

$$A(\epsilon) = \begin{bmatrix} 1 & 0 \\ 1 & 1/\epsilon \end{bmatrix}, \quad B(\epsilon) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (2.3.2)$$

which is ϵ^0 -reachable and in standard form, since $\mathcal{C}(\epsilon) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

However, this system is not in proper standard form.

For notational convenience, let

$$\mathcal{C}_r(\epsilon) = [B(\epsilon) \mid A(\epsilon)B(\epsilon) \mid \dots \mid A^{r-1}(\epsilon)B(\epsilon)] \quad (2.3.3)$$

For the above example, observe that $\mathcal{C}_3(\epsilon) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1+1/\epsilon \end{bmatrix}$ is not over $\mathbb{R}[[\epsilon]]$. Also, the characteristic polynomial of $A(\epsilon)$,

$\sigma(A) = \lambda^2 - (1+1/\epsilon)\lambda + 1/\epsilon$, has a coefficient which is not in $\mathbb{R}[[\epsilon]]$.

In general, we have the following (3):

Proposition 2.3.5: The following statements are equivalent for any pair $(A(\epsilon), B(\epsilon))$ such that $\mathcal{C}_n(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$:

1. $(A(\epsilon), B(\epsilon))$ is proper.
2. $\mathcal{C}_\infty(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$.
3. The coefficients of the characteristic equation of $A(\epsilon)$, $\sigma(A(\epsilon))$ are in $\mathbb{R}[[\epsilon]]$.

To prove this result, let us first consider the following two lemmas:

Lemma 2.3.6: For a pair $(A(\epsilon), B(\epsilon))$ with $\mathcal{C}_n(\epsilon)$ over $\mathbb{R}[[\epsilon]]$, $\mathcal{C}_\infty(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$ iff the coefficients of $\sigma(A(\epsilon))$ are in $\mathbb{R}[[\epsilon]]$.

Proof: (\rightarrow) Follows using the Cayley-Hamilton theorem.

(\leftarrow) If $\mathcal{C}_\infty(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$ then all eigenvalues of $A(\epsilon)$ are $O(1)$, since otherwise $\lim_{\epsilon \downarrow 0} A^j(\epsilon)$ does not exist and thanks to the assumption that $\mathcal{C}(\epsilon)$ is of full row rank, neither does $\lim_{\epsilon \downarrow 0} A^j(\epsilon)B(\epsilon)$ (Note that the eigenvalues of $A(\epsilon)$ are not necessarily over $\mathbb{R}[[\epsilon]]$ since they may include rational but noninteger powers of ϵ). But, the coefficients of $\sigma(A(\epsilon))$ are in $\mathbb{R}((\epsilon))$. Thus, these coefficients are also in $\mathbb{R}[[\epsilon]]$.

³For convenience we use the shorthand $\mathcal{C}_\infty(\epsilon)$ to denote $\mathcal{C}_r(\epsilon)$ for all integers $r \geq n$, i.e. $\mathcal{C}_\infty(\epsilon) = [B(\epsilon) | A(\epsilon)B(\epsilon) | \dots]$.

Lemma 2.3.7: Let $\bar{A}(\epsilon) = D^{-1}(\epsilon)P^{-1}(\epsilon)A(\epsilon)P(\epsilon)D(\epsilon)$,
 $\bar{B}(\epsilon) = D^{-1}(\epsilon)P^{-1}(\epsilon)B(\epsilon)$, then $\mathcal{C}_\omega(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$ iff $\bar{\mathcal{C}}_\omega(\epsilon)$ is
over $\mathbb{R}[[\epsilon]]$.

Proof: (\rightarrow) Follows from the transformation.

(\leftarrow) Clearly $\bar{\mathcal{C}}_n(\epsilon) = Q(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$, and the rest follows using
Lemma 2.3.6 and the Cayley-Hamilton theorem.

We can now prove Proposition 2.3.5:

Proof (of Proposition 2.3.5): (1 \rightarrow 2) Follows from Definition 2.3.2.

(2 \rightarrow 1) By Lemma 2.3.7, $\bar{\mathcal{C}}_\omega(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$. Consider
 $\bar{\mathcal{C}}_{n+1}(\epsilon) = [\bar{B}(\epsilon) \mid \bar{A}(\epsilon)\bar{\mathcal{C}}_n(\epsilon)]$, which is also over $\mathbb{R}[[\epsilon]]$. Then,
 $\bar{B}(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$. Also, $\bar{A}(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$ since $\bar{\mathcal{C}}_n(\epsilon)$ is
full row rank at $\epsilon=0$ and therefore has a right inverse over
 $\mathbb{R}[[\epsilon]]$. Thus, $(D(\epsilon)\bar{A}(\epsilon)D^{-1}(\epsilon), D(\epsilon)\bar{B}(\epsilon))$ is a proper standard
form.

(2 \leftrightarrow 3) Lemma 2.3.6

As an immediate consequence of statement 2 of Proposition
2.3.5 we have the following important property of proper systems:

Corollary 2.3.8: Given a proper pair $(A(\epsilon), B(\epsilon))$, $x \in \mathcal{X}^j$ iff x is
reachable in $p \geq n$ steps with $O(1/\epsilon^j)$ control for all such p .

Note that p is assumed to be independent of ϵ since otherwise
the above corollary may prove to be wrong in some cases, like the
following example:

Example 2.3.9: Let $A(\epsilon) = \begin{bmatrix} 0 & 0 \\ \epsilon & 1 \end{bmatrix}$ and $B(\epsilon) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which is a proper pair and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is ϵ -reachable. But, it is also reachable in p steps where $p-1 < \frac{1}{\epsilon} \leq p$ using the $O(1)$ input sequence $u[k] = 1$ for $0 \leq k \leq p-2$, $u[p-1] = \frac{1}{\epsilon} - p$ and $u[p] = \frac{-1}{\epsilon} + p - 1$. We do not consider such cases in this thesis since p is unbounded as $\epsilon \downarrow 0$ and thus not very useful in practice.

Note that the system in Example 2.1.8 is not proper.

Let us supplement Proposition 2.3.5 with the following:

Corollary 2.3.10: $\bar{\mathcal{C}}_{\infty}(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$ iff $\bar{\mathcal{C}}_{n+1}(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$.

Proof: (\rightarrow) Since $\bar{\mathcal{C}}_{n+1}(\epsilon) = [\bar{B}(\epsilon) \mid \bar{A}(\epsilon)\bar{\mathcal{C}}_n(\epsilon)]$, and $\bar{\mathcal{C}}_n(\epsilon)$ is full row rank at $\epsilon=0$, $\bar{A}(\epsilon)$ are $\bar{B}(\epsilon)$ are over $\mathbb{R}^n[[\epsilon]]$. Thus, $\bar{\mathcal{C}}_{\infty}(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$.

(\leftarrow) Trivial.

For computational efficiency in testing if $\mathcal{C}_{\infty}(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$ or not, one would be interested in finding the minimum index, say r , such that $\mathcal{C}_r(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$ iff $\mathcal{C}_{\infty}(\epsilon)$ is. The above proposition shows that for ϵ^0 -reachable systems we only need to look at one more term. Let us consider the following examples to illustrate the general case:

Example 2.3.11: Let $A(\epsilon) = \begin{bmatrix} 1 & 0 \\ 0 & 1/\epsilon \end{bmatrix}$, $B(\epsilon) = \begin{bmatrix} 1 \\ \epsilon^p \end{bmatrix}$, $p > 0$. Then,

$\mathcal{C}_{\infty}(\epsilon) = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 & 1 \\ \epsilon^p & \epsilon^{p-1} & \dots & \epsilon & 1 & 1/\epsilon & \dots \end{bmatrix}$. The minimum index here is $p+2$.

Example 2.3.12: Let $A(\epsilon) = \begin{bmatrix} 0 & 0 & 0 \\ \epsilon^p & 0 & 1/\epsilon \\ 0 & 1 & 0 \end{bmatrix}$, $B(\epsilon) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $p > 0$. Then,

$\mathcal{C}_\infty(\epsilon) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon^p & 0 & \epsilon^{p-1} & 0 & \dots & 1 & 0 & 1/\epsilon & 0 & \dots \\ 0 & 0 & \epsilon^p & 0 & \epsilon^{p-1} & \dots & 0 & 1 & 0 & 1/\epsilon & \dots \end{bmatrix}$. The minimum index here is $2p+4$.

Thus, it is not an easy task to determine r for a given system, and r may typically be too large to provide any significant computational advantage over, for example, transforming the system into an ϵ^0 -reachable form.

The standard form will prove to be very useful to us, especially for finding feedback to place eigenvalues (Chapter 3). Unfortunately, at this point we first need to compute the Smith decomposition of the reachability matrix to get to a standard form. This clearly is a very expensive operation in terms of both the number of computations and the numerical accuracy. In the remainder of this section, we develop an algorithm to get to a standard form without using the Smith decomposition of the reachability matrix. Our algorithm can only deal with a pair $(A(\epsilon), B(\epsilon))$ over $\mathbb{R}[[\epsilon]]$. Thus, in the remainder of this section, we only consider such pairs. Then, the structure of a pair $(A(\epsilon), B(\epsilon))$ in standard form is as follows:

$$A(\epsilon) = \begin{bmatrix} A_{0,0}(\epsilon) & A_{0,1}(\epsilon) & \dots & A_{0,k}(\epsilon) \\ \epsilon A_{1,0}(\epsilon) & A_{1,1}(\epsilon) & \dots & A_{1,k}(\epsilon) \\ \vdots & & \ddots & \vdots \\ \epsilon^k A_{k,0}(\epsilon) & \epsilon^{k-1} A_{k,1}(\epsilon) & \dots & A_{k,k}(\epsilon) \end{bmatrix} \begin{matrix} \} p_0 \\ \} p_1 \\ \vdots \\ \} p_k \end{matrix} \quad (2.3.4a)$$

$$B(\epsilon) = \begin{bmatrix} B_0(\epsilon) \\ \epsilon B_1(\epsilon) \\ \vdots \\ \epsilon^k B_k(\epsilon) \end{bmatrix} \begin{matrix} \} p_0 \\ \} p_1 \\ \vdots \\ \} p_k \end{matrix} \quad (2.3.4b)$$

Hence all pairs $(A(\epsilon), B(\epsilon))$ over $\mathbb{R}[[\epsilon]]$ are proper.

Proposition 2.3.13 : An ϵ^k -reachable pair $(A(\epsilon), B(\epsilon))$ over $\mathbb{R}[[\epsilon]]$ is in proper standard form with indices p_0, \dots, p_k iff $A(\epsilon)$ and $B(\epsilon)$ satisfy the following condition: Let $F_i(\epsilon) = D_i^{-1}(\epsilon)A(\epsilon)D_i(\epsilon)$, $G_i(\epsilon) = D_i^{-1}(\epsilon)B(\epsilon)$ where $D_i(\epsilon) = \text{diag}\{I_{p_0}, \dots, \epsilon^i I_{p_i + \dots + p_k}\}$ then the reachable subspace of $(F_i(0), G_i(0))$ is $\mathcal{E}_i = \text{Im} \begin{bmatrix} I_{n_i} \\ 0 \end{bmatrix}$, for $\forall i \in [0..k]$.

Proof: (\rightarrow) $(F_k(\epsilon), G_k(\epsilon))$ is over $\mathbb{R}[[\epsilon]]$ and ϵ^0 -reachable. But, $A(\epsilon) = D(\epsilon)F_k(\epsilon)D^{-1}(\epsilon)$ and $B(\epsilon) = D(\epsilon)G_k(\epsilon)$. Thus, $(A(\epsilon), B(\epsilon))$ is in standard form and has the structure of (2.3.1). But, since $(A(\epsilon), B(\epsilon))$ is over $\mathbb{R}[[\epsilon]]$, it has the structure of (2.3.4).

Therefore, $(A(\epsilon), B(\epsilon))$ is in proper standard form.

(\leftarrow) Follows from (2.3.5) and the form of the reachability matrix of $(F_k(\epsilon), G_k(\epsilon))$.

Definition 2.3.14: Let

$$\bar{A}_i(\epsilon) = \begin{bmatrix} A_{0,0}(\epsilon) & A_{0,1}(\epsilon) & \dots & A_{0,i}(\epsilon) \\ \epsilon A_{1,0}(\epsilon) & A_{1,1}(\epsilon) & \dots & A_{1,i}(\epsilon) \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon^i A_{i,0}(\epsilon) & \epsilon^{i-1} A_{i,1}(\epsilon) & \dots & A_{i,i}(\epsilon) \end{bmatrix} \begin{matrix} \} p_0 \\ \} p_1 \\ \vdots \\ \} p_i \end{matrix} \quad (2.3.5a)$$

$$\bar{B}_i(\epsilon) = \begin{bmatrix} B_0(\epsilon) \\ \epsilon B_1(\epsilon) \\ \vdots \\ \epsilon^i B_i(\epsilon) \end{bmatrix} \begin{matrix} \} p_0 \\ \} p_1 \\ \vdots \\ \} p_i \end{matrix} \quad (2.3.5b)$$

then $(\bar{A}_i(\epsilon), \bar{B}_i(\epsilon))$ is the ϵ^i -reachable subsystem of $(A(\epsilon), B(\epsilon))$ with indices n_0, \dots, n_i .

Similar to the submodule structure, the ϵ^i -reachable subsystem contains all ϵ^j -reachable subsystems for $j = 0, \dots, i-1$. The subsystems are layered with weak couplings of orders of ϵ between each component, as shown in Figure 2.3.1. Also,

$$\mathcal{C}_i(\epsilon) \mathbb{R}^{mn_i} [[\epsilon]] \oplus \epsilon^{i+1} \mathbb{R}^{n-n_i} [[\epsilon]] \supset \mathcal{Y}^0 \quad (2.3.6)$$

and the sequence $\{\mathcal{C}_i(\epsilon) \mathbb{R}^{mn_i} [[\epsilon]] \oplus \epsilon^{i+1} \mathbb{R}^{n-n_i} [[\epsilon]]\}$ converges to \mathcal{Y}^0 in k steps. In other words, ϵ^0 -reachable submodules of the ϵ^i -reachable subsystems approximate the ϵ^0 -reachable submodule of the system in standard form upto ϵ^{i+1} accuracy. We use this in Algorithm 2.3.15 below.

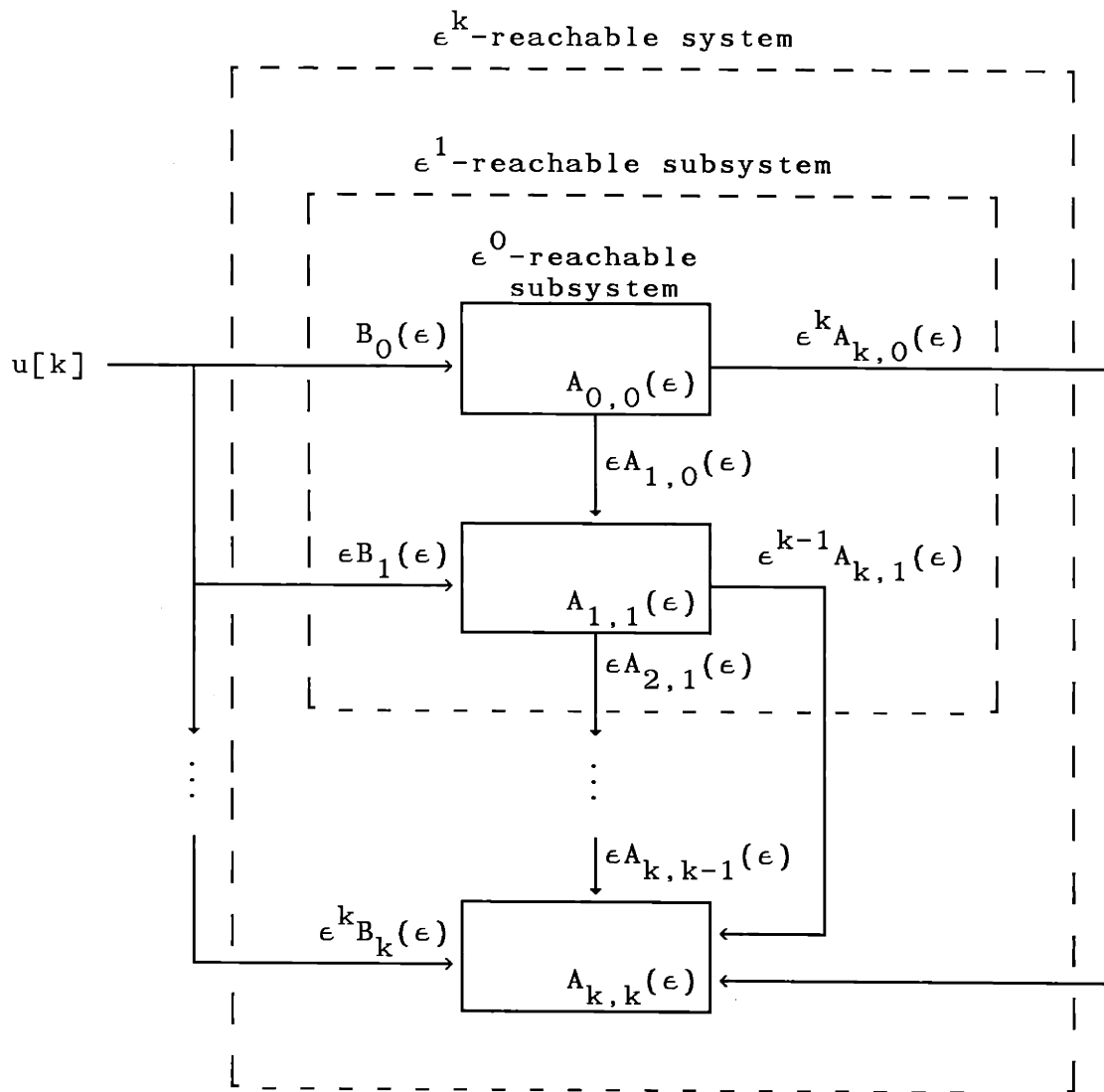


FIGURE 2.3.1 Block diagram showing the structure of an ϵ^k -reachable system in standard form (upper off-diagonal blocks of $A_s(\epsilon)$ have been omitted for clarity).

Computation of the reachability matrix is very expensive. One has to calculate $A^i(\epsilon)B(\epsilon)$ for all the terms in the expansions of $A(\epsilon)$ and $B(\epsilon)$. Thus, it is desirable to work directly with the pair $(A(\epsilon), B(\epsilon))$. The following algorithm takes advantage of Proposition 2.3.13 to recover the ϵ^j -reachability indices. At every step, the reachable subspace of a pair, evaluated at $\epsilon=0$, is computed. Then, the pair is updated by an appropriate scaling of the unreachable part by $1/\epsilon$. The algorithm uses the coefficients of the Taylor expansions of the higher order terms only when necessary. Also, it is possible to recover the actual Smith decomposition of the reachability matrix from the algorithm, if the transformations used in the algorithm are restricted to be permutation matrices and lower triangular matrices (Proposition 2.3.18), though this restriction compromises numerical stability.

Algorithm 2.3.15:

Initialize: $A_0(\epsilon) = A(\epsilon)$, $B_0(\epsilon) = B(\epsilon)$, $i = 0$

Step i:

1. Find T_i such that

$$T_i^{-1}A_i(0)T_i = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \}^{n_i}, \quad T_i^{-1}B_i(0) = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \}^{n_i}$$

with (A_1, B_1) reachable. This determines n_i .

2. If $n_i = n$ then go to End, else continue.

3. Let $A_{i+1}(\epsilon) = D_i^{-1}(\epsilon)T_i^{-1}A_i(\epsilon)T_iD_i(\epsilon)$, $B_{i+1} = D_i^{-1}(\epsilon)T_i^{-1}B_i(\epsilon)$ where $D_i(\epsilon) = \text{diag}\{I_{n_i}, \epsilon I_{n-n_i}\}$.

(It is not necessary to carry out the computation for all the coefficients of $A_i(\epsilon)$ and $B_i(\epsilon)$; see Note 1 below.)

4. Increment i , go to Step i .

End: $k = i$, the system is ϵ^k -reachable with indices n_0, \dots, n_k .

Note 1: Step i of the algorithm recovers the ϵ^i -reachability index, n_i . The $O(\epsilon^{i+1})$ parts of $A(\epsilon)$ and $B(\epsilon)$ have no impact on n_i . Thus, we could perform the transformations up to step i on only the lower order terms. Since the system is ϵ^k -reachable, we do not need to consider $O(\epsilon^{k+1})$ terms at all.

Note 2: The reachability matrix of $(A(\epsilon), B(\epsilon))$ is:

$$\mathcal{C}(\epsilon) = T_0 D_0(\epsilon) T_1 D_1(\epsilon) \dots T_{k-1} D_{k-1}(\epsilon) \mathcal{C}_k(\epsilon) = S(\epsilon) \mathcal{C}_k(\epsilon) \quad (2.3.7)$$

where $\mathcal{C}_k(\epsilon)$ is the reachability matrix of $(A_k(\epsilon), B_k(\epsilon))$ and it is full row rank at $\epsilon=0$. (Note that T_k may be chosen to be the identity.) Thus, by Algorithm 4.1 and Theorem 4.2 of Van Dooren et al. [12] the above algorithm recovers the indices of $\mathcal{C}(\epsilon)$, or the ϵ^j -reachability indices. Also, $D(\epsilon) = \prod_{j=0}^{k-1} D_j(\epsilon)$ where $\mathcal{C}(\epsilon) = P(\epsilon)D(\epsilon)Q(\epsilon)$ is the Smith decomposition of $\mathcal{C}(\epsilon)$, for some unimodular $P(\epsilon)$ and $Q(\epsilon)$.

As an illustration for Algorithm 2.3.15, let us run Example 2.1.9 through this algorithm:

Example 2.3.16:

Step 0: 1. Let $T_0 = I$.

2. $n_0 = 1$.

3. $D_0(\epsilon) = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}$ then $A_1(0) = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, $B_1(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(Assuming that we only need to find the indices, this is the first time we had to use the coefficients of ϵ , and also we do not actually need to calculate $A_1(\epsilon)$ and $B_1(\epsilon)$. $A_1(0)$ and $B_1(0)$ are sufficient for the next step)

4. $i = 1$.

Step 1: 1. Let $T_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ then $T_1^{-1}A_1(0)T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $T_1^{-1}B_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

2. $n_1 = 1$

3. $D_1(\epsilon) = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}$ then $A_2(0) = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, $B_2(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

4 $i = 2$.

Step 2: 1. Let $T_2 = I$

2. $n_2 = 2$, go to end.

End: ϵ^2 -reachable system with indices $n_0 = n_1 = 1$, $n_2 = 2$.

It is also possible to compute a proper standard form and a Smith decomposition of $\mathcal{C}(\epsilon)$ if T_i are chosen to consist of permutation matrices and elementary row operations, i.e. $T_i = \Pi_i L_i$ where Π_i is a permutation matrix and L_i is a lower triangular matrix. Then, $S(\epsilon)$ defined in equation (2.3.7) has the following form:

$$S(\epsilon) = \prod_{j=0}^{k-1} \Pi_j L_j D_j(\epsilon) = \Pi_0 L_0 D_0(\epsilon) \dots \Pi_{k-1} L_{k-1} D_{k-1}(\epsilon) \quad (2.3.8)$$

In fact, it is not necessary to permute the first n_{i-1} rows of $A_i(\epsilon)$ and $B_i(\epsilon)$ at step i . Thus, Π_i may be chosen to have the special form $\Pi_i = \begin{bmatrix} I_{n_{i-1}} & 0 \\ 0 & R_i \end{bmatrix}$, where R_i is an $(n-n_{i-1}) \times (n-n_{i-1})$ permutation matrix. Therefore,

$$D_j(\epsilon)\Pi_i = \Pi_i D_j(\epsilon) \quad \forall j < i \text{ and } i, j \in [1..k-1] \quad (2.3.9)$$

Lemma 2.3.17: $S(\epsilon) = V(\epsilon)D(\epsilon)$, where $V(\epsilon)$ is some unimodular matrix and $D(\epsilon) = \prod_{j=0}^{k-1} D_j(\epsilon)$.

Proof: We prove this inductively, as follows:

Let $S_i(\epsilon) = \prod_{j=0}^i \Pi_j L_j D_j(\epsilon)$, then we claim that $S_i(\epsilon) = V_i(\epsilon)E_i(\epsilon)$

for all $i \in [1..k-1]$, where $E_i(\epsilon) = \prod_{j=0}^i D_j(\epsilon)$ and $V_i(\epsilon)$ is some unimodular matrix. Clearly, $S_0(\epsilon)$ satisfies this. Assume that $S_i(\epsilon)$ also satisfies this. Then,

$$S_{i+1}(\epsilon) = V_i(\epsilon)E_i(\epsilon)\Pi_{i+1}L_{i+1}D_{i+1}(\epsilon) \quad (2.3.10)$$

By equation (2.3.9) above, $E_i(\epsilon)\Pi_{i+1} = \Pi_{i+1}E_i(\epsilon)$. Moreover, since L_{i+1} is lower triangular and $E_i(\epsilon)$ is diagonal with increasing powers of ϵ on its diagonal, $\bar{L}_{i+1}(\epsilon) = E_i(\epsilon)L_{i+1}E_i^{-1}(\epsilon)$ is unimodular. Thus:

$$S_{i+1}(\epsilon) = V_i(\epsilon)\Pi_{i+1}\bar{L}_{i+1}(\epsilon)E_i(\epsilon)D_{i+1}(\epsilon) = V_{i+1}(\epsilon)E_{i+1}(\epsilon) \quad (2.3.11)$$

where $V_{i+1}(\epsilon)$ is unimodular, and the proof is complete since

$$S(\epsilon) = S_{k-1}(\epsilon) \text{ and } D(\epsilon) = E_{k-1}(\epsilon).$$

Proposition 2.3.18: If permutations and elementary row operations are used, a proper standard form and a Smith decomposition of the reachability matrix for a pair $(A(\epsilon), B(\epsilon))$, over $\mathbb{R}[[\epsilon]]$, can be constructed via algorithm 2.3.15 as follows:

1. Let $P(\epsilon) = S(\epsilon)D^{-1}(\epsilon)$, and $Q(\epsilon) = \mathcal{C}_k(\epsilon)$, where

$$D(\epsilon) = \prod_{j=0}^{k-1} D_j(\epsilon), \text{ then a Smith decomposition of } \mathcal{C}(\epsilon) \text{ is:}$$

$$\mathcal{C}(\epsilon) = P(\epsilon)D(\epsilon)Q(\epsilon).$$

2. Let $A_s(\epsilon) = D(\epsilon)A_k(\epsilon)D^{-1}(\epsilon)$, $B_s(\epsilon) = D(\epsilon)B_k(\epsilon)$ then $(A_s(\epsilon), B_s(\epsilon))$ is a proper standard form for $(A(\epsilon), B(\epsilon))$.

Proof:

1. By Lemma 2.3.17, $P(\epsilon)$ is unimodular. Since $(A_k(\epsilon), B_k(\epsilon))$ is ϵ^0 -reachable, $\mathcal{C}_k(\epsilon)$ is full row rank at $\epsilon = 0$. Also,

$\mathcal{C}(\epsilon) = S(\epsilon)\mathcal{C}_k(\epsilon) = P(\epsilon)D(\epsilon)\mathcal{C}_k(\epsilon)$. Thus, we have the desired Smith decomposition.

2. By part 1 above, $A_s(\epsilon) = P^{-1}(\epsilon)A(\epsilon)P(\epsilon)$, $B_s(\epsilon) = P^{-1}(\epsilon)B(\epsilon)$ and it is a standard form by definition. Since, $(A(\epsilon), B(\epsilon))$ is over $\mathbb{R}[[\epsilon]]$, $(A_s(\epsilon), B_s(\epsilon))$ is also a proper standard form.

Let us illustrate this for Example 2.3.16:

Example 2.3.19: First of all $S(\epsilon) = T_0 D_0(\epsilon) T_1 D_1(\epsilon) = \begin{bmatrix} 1 & 0 \\ \epsilon & \epsilon^2 \end{bmatrix}$ and

$$D(\epsilon) = D_0(\epsilon)D_1(\epsilon) = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon^2 \end{bmatrix}. \text{ Thus, } P(\epsilon) = S(\epsilon)D^{-1}(\epsilon) = \begin{bmatrix} 1 & 0 \\ \epsilon & 1 \end{bmatrix}.$$

Recall that $A_2(\epsilon) = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, and $B_2(\epsilon) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, $\mathcal{C}_2(\epsilon) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

and therefore a Smith decomposition for $\mathcal{C}(\epsilon)$ is

$$\mathcal{C}(\epsilon) = \begin{bmatrix} 1 & 0 \\ \epsilon & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \epsilon^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}. \text{ Also, a proper standard form is given}$$

by $A_s(\epsilon) = D(\epsilon)A_k(\epsilon)D^{-1}(\epsilon) = \begin{bmatrix} 1 & 0 \\ -\epsilon^2 & 2 \end{bmatrix}$ and $B_s(\epsilon) = D(\epsilon)B_k(\epsilon) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

In this section, we developed and analyzed the standard form of a pair $(A(\epsilon), B(\epsilon))$. When a system is in this form, its ϵ^j -reachability subspaces can be expressed as $\mathcal{E}_{n_i} = \text{Im} \begin{bmatrix} I_{n_i} \\ 0 \end{bmatrix}$ (equation (2.2.5)). We have shown that, if a pair is proper, the indices n_i can be found from the structure of any of its standard forms. For the case in which $A(\epsilon)$ and $B(\epsilon)$ are over $\mathbb{R}[[\epsilon]]$, we have developed an algorithm which recovers these reachability indices. If orthogonal transformations are used, this is a stable algorithm. On the other hand, if these transformations are restricted to permutations and elementary row operations, one can in addition easily recover the Smith decomposition of the reachability matrix, and a proper standard form using this algorithm. It is also possible to recover a proper standard form and a Smith decomposition of the reachability matrix even if Algorithm 2.3.15 is run with orthogonal transformations. Unfortunately, this additional operation is not necessarily stable since it involves LU decompositions without allowing for permutations. However, a byproduct of our research in this issue was recovering a Smith decomposition for the product of two matrices from Smith decompositions of each matrix under a weak condition. This is shown in Appendix A.2 for the interested reader, but it is not a part of the main body of this thesis.

These concepts are heavily utilized in Chapter 3, where we deal with feedback issues related to orders of reachability. We conclude this chapter by introducing a concept, in the next section, which gives us the flexibility of nearly reaching target states, say within $O(\epsilon^p)$, for some positive p . This allows us to decrease the orders of control that are used, as compared to those required to reach target states exactly.

II.4 NEAR REACHABILITY

In some applications, one might be satisfied with reaching a target state within some error margin instead of reaching it exactly, in order to get away with using less control energy. In Example 2.1.9, $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is ϵ^1 -reachable, whereas $x = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$ is ϵ^0 -reachable. In other words, we could come $O(\epsilon)$ close to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ using $O(1)$ control instead of using $O(1/\epsilon)$ control to reach it exactly. Let us define this formally as follows:

Definition 2.4.1: $x \in \mathbb{R}^n$ is nearly ϵ^j -reachable if $\exists x(\epsilon) \in \mathcal{X}^j$ such that $x(\epsilon) - x$ is $O(\epsilon)$.

Let \mathcal{Y}^j be the set of all nearly ϵ^j -reachable states.

As an immediate consequence of this definition we have that

$\mathcal{Y}^j = \mathcal{X}^j|_{\epsilon=0} \equiv \{x: x(\epsilon) \in \mathcal{X}^j \text{ and } x = x(0)\}$, also $\mathcal{Y}^0 \subset \mathcal{Y}^1 \subset \dots$ and \mathcal{Y}^j is a subspace of \mathbb{R}^n .

We term \mathcal{V}^j the nearly ϵ^j -reachable subspace.

The computation of a standard form (using Algorithm 2.3.15 and Proposition 2.3.16 of the previous section) is an expensive operation since it involves polynomial matrix multiplications. Specifically, computing the pair $(A_k(\epsilon), B_k(\epsilon))$ is an expensive operation. On the other hand, $P(0)$ can be computed in a much cheaper way, for example via Algorithm 2.3.15. Let us transform the pair $(A(\epsilon), B(\epsilon))$ by $P(0)$ instead of $P(\epsilon)$ and term the new pair a near standard form. In the remainder of this section, it is shown that the nearly ϵ^j -reachable subspaces of this form are equivalent to those of the standard form.

Proposition 2.4.2: $\mathcal{Q}^j = \mathcal{W}^j + \epsilon\mathcal{W}^{j+1} + \dots$ where \mathcal{Q}^j is the ϵ^j -reachable submodule of the standard form, and \mathcal{W}^j is the nearly ϵ^j -reachable subspace of the standard form. Also, $\mathcal{W}^j = \xi_{n_j}$.

Proof: $\mathcal{W}^j = \mathcal{Q}^j|_{\epsilon=0} = \xi_{n_j}$, and also

$$\mathcal{Q}^j = \xi_{n_j} + \epsilon\xi_{n_{j+1}} + \dots = \mathcal{W}^j + \epsilon\mathcal{W}^{j+1} + \dots$$

Proposition 2.4.3: $v \in \mathbb{R}^n$ is nearly ϵ^j -reachable iff

$\exists V(\epsilon): \mathbb{R}^n[[\epsilon]] \rightarrow \mathbb{R}^n[[\epsilon]]$ such that $V(0) = I$ and $V(\epsilon)v \in \mathcal{X}^j$.

Proof: If there exists such a $V(\epsilon)$, then $V(\epsilon)v - v$ is $O(\epsilon)$ since $V(0) = I$. Thus, v is nearly ϵ^j -reachable.

If v is nearly ϵ^j -reachable then $\exists x(\epsilon) \in \mathcal{X}^j$ such that $x(\epsilon) - v$ is $O(\epsilon)$, or $x(0) = v$. If $v = 0$, then $V(\epsilon) = I$ and the proof is

complete. Otherwise, one can always construct a $V(\epsilon)$ such that $V(0) = I$ and $x(\epsilon) = V(\epsilon)v$ (For example let $V(\epsilon) = I + \epsilon V_1 + \dots$ and $x(\epsilon) = v + \epsilon x_1 + \dots$, then find V_i such that $x_i = V_i v$ for all V_i , $i \in I^+$).

Since a near standard form is a transformation of a standard form by $V(\epsilon) = P(0)P^{-1}(\epsilon)$ and $V(0) = I$, it follows from Proposition 2.4.2 and Proposition 2.4.3 that the nearly ϵ^j -reachable subspaces of the near standard form are equivalent to those of the standard form.

The nearly ϵ^j -reachable subspaces could be further distinguished according to the amount of error one could tolerate, in powers of ϵ . This section has treated the case of $O(\epsilon)$ error. We develop the refinement in Appendix A.3.

III FEEDBACK

In this chapter we relate the notion of ϵ^j -reachability to the order of state feedback gains used to shift the eigenvalues by $O(1)$.

III.1 SHIFTING EIGENVALUES BY $O(1)$

In this chapter, we restrict our attention to reachable systems over $\mathbb{R}[[\epsilon]]$. Recall that these systems are proper and all eigenvalues of $A(\epsilon)$ are $O(1)$. We address the problem of arbitrarily shifting these eigenvalues by $O(1)$, using full state feedback. In other words, we wish to find $F(\epsilon)$ over $\mathbb{R}((\epsilon))$ such that $A_F(\epsilon) = A(\epsilon) + B(\epsilon)F(\epsilon)$ has the desired eigenvalues at $\epsilon=0$.

Example 3.1.1: Recall Example 2.1.1:

$$A(\epsilon) = \begin{bmatrix} 1 & 1 \\ \epsilon & 2 \end{bmatrix}, \quad B(\epsilon) = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} \quad (3.1.1)$$

The eigenvalues of $A(\epsilon)$ are at $1+O(\epsilon)$ and $2+O(\epsilon)$. State feedback of $[2 \ 4]$ shifts these eigenvalues to $3+O(\epsilon)$ and $2+O(\epsilon)$. However, there is no $O(1)$ state feedback that would move the eigenvalue at $2+O(\epsilon)$ by $O(1)$, but state feedback of $[5 \ -1/\epsilon]$ shifts the eigenvalues to $3+O(\epsilon)$ and $4+O(\epsilon)$. Here, both eigenvalues are moved by $O(1)$, and an $O(1/\epsilon)$ feedback gain has to be used. Note

that the closed loop system

$$A_F(\epsilon) = \begin{bmatrix} 6 & 1-1/\epsilon \\ 6\epsilon & 1 \end{bmatrix}, \quad B(\epsilon) = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} \quad (3.1.2)$$

is not over $\mathbb{R}[[\epsilon]]$ but it is ϵ -reachable with the same indices, $n_0=1$ and $n_1=1$, as the original system, and is in proper standard form.

We now proceed with showing that, for systems over $\mathbb{R}[[\epsilon]]$, the magnitude of feedback gains necessary and sufficient to move all eigenvalues by $O(1)$ is directly given by the order of reachability of the system. Let us start by looking at ϵ^0 -reachable systems:

Proposition 3.1.2: The pair $(A(\epsilon), B(\epsilon))$, over $\mathbb{R}[[\epsilon]]$, is ϵ^0 -reachable iff

$$\min_{r \in \mathbb{Z}} \{r \mid \forall \Lambda, \exists F(\epsilon) \text{ where } \epsilon^r F(\epsilon) \text{ is over } \mathbb{R}^m[[\epsilon]]\},$$

$$\text{s.t. } \lambda(A(\epsilon) + B(\epsilon)F(\epsilon)) \big|_{\epsilon=0} = \Lambda \} = 0 \quad (3.1.3)$$

where Λ is any self conjugate set of n eigenvalues, $\lambda(A)$ denotes the eigenvalues of A , and \mathbb{Z} denotes the set of all integers.

Proof: (\rightarrow) If the pair $(A(\epsilon), B(\epsilon))$ is ϵ^0 -reachable, then, $\mathcal{Q}(\epsilon) \big|_{\epsilon=0}$ is full row rank. Thus, the pair $(A(0), B(0))$ is reachable, and $\forall \Lambda, \exists F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ s.t. $\lambda(A(\epsilon) + B(\epsilon)F) \big|_{\epsilon=0} = \lambda(A(0) + B(0)F) = \Lambda$. To prove that the minimum such r is 0, assume that equation (3.1.3) holds for some negative r . Then, $\lim_{\epsilon \downarrow 0} F(\epsilon) = 0$, and $\lim_{\epsilon \downarrow 0} (A(\epsilon) + B(\epsilon)F(\epsilon)) = A(0)$, so no $O(1)$ eigenvalues are moved, and

therefore (3.1.3) holds.

(\leftarrow) Conversely, assume that (3.1.3) holds, then $\forall \Lambda, \exists F = F(\epsilon) \big|_{\epsilon=0}$ s.t. $\lambda(A(0)+B(0)F) = \Lambda$. Thus, the pair $(A(0), B(0))$ is reachable, and $\mathcal{Q}(\epsilon) \big|_{\epsilon=0}$ is full row rank. Therefore, the pair $(A(\epsilon), B(\epsilon))$ is ϵ^0 -reachable.

Proposition 3.1.3: The pair $(A(\epsilon), B(\epsilon))$, over $\mathbb{R}[[\epsilon]]$, is ϵ^k -reachable iff

$$\min_{r \in \mathbb{Z}} \{r \mid \forall \Lambda, \exists F(\epsilon) \text{ where } \epsilon^r F(\epsilon) \text{ is over } \mathbb{R}^m[[\epsilon]],$$

$$\text{s.t. } \lambda(A(\epsilon)+B(\epsilon)F(\epsilon)) \big|_{\epsilon=0} = \Lambda\} = k \quad (3.1.4)$$

where Λ is any self conjugate set of n eigenvalues and \mathbb{Z} denotes the set of integers.

Proof: If the pair $(A(\epsilon), B(\epsilon))$ is ϵ^k -reachable, then the pair $(\bar{A}(\epsilon), \bar{B}(\epsilon))$ is ϵ^0 -reachable and by Lemma 2.3.7 over $\mathbb{R}[[\epsilon]]$, where $\bar{A}(\epsilon) = D^{-1}(\epsilon)P^{-1}(\epsilon)A(\epsilon)P(\epsilon)D(\epsilon)$ $\bar{B}(\epsilon) = D^{-1}(\epsilon)P^{-1}(\epsilon)B(\epsilon)P(\epsilon)D(\epsilon)Q(\epsilon)$. Thus, by Proposition 3.1.2, $\forall \Lambda, \exists$ an $O(1)$ $\bar{F}(\epsilon)$ s.t. $\lambda(\bar{A}(\epsilon)+\bar{B}(\epsilon)\bar{F}(\epsilon)) \big|_{\epsilon=0} = \Lambda$. Let $F(\epsilon) = \bar{F}(\epsilon)D^{-1}(\epsilon)P^{-1}(\epsilon)$, then $F(\epsilon)$ is $O(1/\epsilon^k)$. By Lemma 2.3.7, the closed loop pair is proper and so the coefficients of its characteristic equation are over $\mathbb{R}[[\epsilon]]$.

Thus,

$$\lim_{\epsilon \downarrow 0} \lambda(\bar{A}(\epsilon)+\bar{B}(\epsilon)\bar{F}(\epsilon)) = \lim_{\epsilon \downarrow 0} \lambda(A(\epsilon)+B(\epsilon)F(\epsilon)) \quad (3.1.5)$$

On the other hand (to show that the minimum index r is k),

$$\bar{A}(\epsilon) = \begin{bmatrix} A_{0,0}(\epsilon) & \epsilon A_{0,1}(\epsilon) & \dots & \epsilon^i A_{0,k}(\epsilon) \\ A_{1,0}(\epsilon) & A_{1,1}(\epsilon) & \dots & \epsilon^{i-1} A_{1,k}(\epsilon) \\ \vdots & & \ddots & \vdots \\ A_{k,0}(\epsilon) & A_{k,1}(\epsilon) & \dots & A_{k,k}(\epsilon) \end{bmatrix} \quad (3.1.6)$$

$n-n_{k-1}$ columns

Then, $\exists \Lambda$ s.t. the last $n-n_{k-1}$ columns of $\bar{F}(\epsilon)$ are $O(1)$ since otherwise

$$\lim_{\epsilon \downarrow 0} (\bar{A}(\epsilon) + \bar{B}(\epsilon) \bar{F}(\epsilon)) = \begin{bmatrix} * & 0 \\ * & A_{k,k}(0) \end{bmatrix} \quad (3.1.7)$$

where $*$ denotes some constant entries, and the eigenvalues corresponding to $A_{k,k}(\epsilon)$ are not moved by $O(1)$. Since $P^{-1}(\epsilon)$ is unimodular, $F(\epsilon) = \bar{F}(\epsilon) D^{-1}(\epsilon) P^{-1}(\epsilon)$ is $O(1/\epsilon^k)$. Therefore, the minimum index r equals k and equation (3.1.4) holds.

(\leftarrow) Clearly, the pair $(A(\epsilon), B(\epsilon))$ is reachable. Assume that $(A(\epsilon), B(\epsilon))$ is ϵ^j -reachable for some $j \neq k$. Then, by the first part of this proof,

$$\min_{r \in \mathbb{I}} \{r \mid \forall \Lambda, \exists F(\epsilon) \text{ where } \epsilon^r F(\epsilon) \text{ is over } \mathbb{R}^m[[\epsilon]]\},$$

$$\text{s.t. } \sigma(A(\epsilon) + B(\epsilon)F(\epsilon))|_{\epsilon=0} = \Lambda \} = j \quad (3.1.8)$$

but this contradicts (3.1.4). Thus, $j=k$ and therefore the pair $(A(\epsilon), B(\epsilon))$ is ϵ^k -reachable.

Note that if $(A(\epsilon), B(\epsilon))$ is ϵ^0 -reachable then the closed loop pair $(A_F(\epsilon), B(\epsilon))$, where $A_F(\epsilon) = A(\epsilon) + B(\epsilon)F(\epsilon)$, is ϵ^0 -reachable for all $F(\epsilon)$ of $O(1)$. Thus we have the following result:

Corollary 3.1.4: Given a pair $(A(\epsilon), B(\epsilon))$ over $\mathbb{R}[[\epsilon]]$, the reachability indices are invariant under any feedback of the form $F(\epsilon) = \bar{F}(\epsilon)D^{-1}(\epsilon)P^{-1}(\epsilon)$ where $\bar{F}(\epsilon)$ is $O(1)$. Also, the closed loop pair is proper.

On the other hand, the ϵ^j -reachability submodules of the standard form are uniquely determined by the indices, and the ϵ^j -reachability submodules of the original system are uniquely determined by the ϵ^j -reachability submodules of the standard form, via $P(\epsilon)$. Thus:

Corollary 3.1.5: Given a pair $(A(\epsilon), B(\epsilon))$ over $\mathbb{R}[[\epsilon]]$, the ϵ^j -reachability submodules are invariant under any feedback of the form $F(\epsilon) = \bar{F}(\epsilon)D^{-1}(\epsilon)P^{-1}(\epsilon)$, where $\bar{F}(\epsilon)$ is $O(1)$.

For the more general class of proper systems over $\mathbb{R}((\epsilon))$, the orders of feedback gains do not necessarily match the orders of reachability. Let us consider the following example:

Example 3.1.6: Consider the following pair

$$A(\epsilon) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\epsilon \\ 0 & 2\epsilon & 0 \end{bmatrix}, \quad B(\epsilon) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (3.1.9)$$

which is an ϵ -reachable system in proper standard form. Let

$$F(\epsilon) = \begin{bmatrix} f_1 & f_2 & 0 \\ f_3 & f_4 & 0 \end{bmatrix}, \text{ where } f_i \text{ are all scalar constants, then}$$

$$\det(\lambda I - A_F(\epsilon)) = \lambda^3 - (f_1 + f_4)\lambda^2 + (f_1 f_4 - f_2 f_3 - 2)\lambda + 2f_1. \text{ Clearly, } f_i \in \mathbb{R}$$

can be chosen appropriately to match any third degree polynomial with real coefficients. Therefore, all eigenvalues of $A(\epsilon)$ can be arbitrarily moved by $O(1)$ using only $O(1)$ feedback gains. What happens in this example is that an $O(1)$ magnitude for the third state component produces an $O(1/\epsilon)$ magnitude for the second component. Therefore, even with $O(1)$ gains, the input values themselves will be $O(1/\epsilon)$ as would be expected given that this is an ϵ -reachable system.

The overall effect of $O(1)$ feedback on the eigenvalues, even for systems over $\mathbb{R}[[\epsilon]]$, is a more subtle issue than the order of feedback necessary to shift the eigenvalues by $O(1)$. Consider the following example:

Example 3.1.7: Let

$$A(\epsilon) = \begin{bmatrix} 0 & 1 \\ \epsilon & 0 \end{bmatrix}, \quad B(\epsilon) = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} \quad (3.1.10)$$

The reachability indices are $n_0=1$ and $n_1=2$. The eigenvalues of $A(\epsilon)$ are at $\pm\sqrt{\epsilon}$. Feedback of $[1 \ 1]$ moves the eigenvalues to $1+O(\epsilon)$ and $\epsilon+O(\epsilon^2)$. Thus, the effect of feedback on one of the eigenvalues is $O(1)$, but on the other is larger than $O(\epsilon)$ (in fact $O(\sqrt{\epsilon})$). As it turns out, this system does not have well-behaved time scale structure since the eigenvalues do not have Taylor expansions in ϵ , and the feedback also alters its time scale structure [1,2].

We leave these problems for further research. Chapter V

suggests some potential extensions.

In the next section, we present an extension of Algorithm 2.3.15 to compute the feedback gains for a given set of eigenvalues for systems over $\mathbb{R}[[\epsilon]]$.

III.2 ALGORITHM

In this section, we present an algorithm to compute the feedback matrix necessary to shift eigenvalues by some desired amount. Recall that at the end of Algorithm 2.3.15, we have a pair $(A_k(\epsilon), B_k(\epsilon))$, where $A_k(\epsilon) = R^{-1}(\epsilon)A(\epsilon)R(\epsilon)$, $B_k(\epsilon) = R^{-1}(\epsilon)B(\epsilon)$, with $(A_k(0), B_k(0))$ reachable, and $R(\epsilon)$ is the product of all the similarity transformations used to achieve the final pair. From the pair $(A_k(0), B_k(0))$, we can compute a feedback matrix F such that the eigenvalues of $A_{kF}(0) = A_k(0) + B_k(0)F$ are as desired. We have that $\sigma(A_{kF}(\epsilon))|_{\epsilon=0} = \sigma(A_{kF}(0))$ since $(A_{kF}(\epsilon), B(\epsilon))$ is proper. Let $F(\epsilon) = FR^{-1}(\epsilon)$ and $A_F(\epsilon) = A(\epsilon) + B(\epsilon)F(\epsilon)$. Since $R(\epsilon)$ is invertible for $\epsilon \in (0, a)$ for some $a \in \mathbb{R}$, $(A_F(\epsilon), B(\epsilon))$ is also proper. Therefore, as in the proof of Proposition 3.1.3 the eigenvalues of $A_F(\epsilon)$ are as desired. Thus, we have the following algorithm:

Algorithm 3.2.1:

1. Run Algorithm 2.3.13.
2. Pick the desired self conjugate set of n eigenvalues Λ .
3. Find F s.t. $\sigma(A_k(0)+B_k(0)F) = \Lambda$.
4. $F(\epsilon) = FR^{-1}(\epsilon)$ is the desired feedback matrix s.t.
 $\sigma(A(\epsilon)+B(\epsilon)F(\epsilon))|_{\epsilon=0} = \Lambda$.

Note: For a stable algorithm, we can choose orthogonal transformations for Algorithm 2.3.13. Even though $(A_k(\epsilon), B_k(\epsilon))$ is not a proper standard form, we still have a valid $F(\epsilon)$ for the original pair. Also, $(A_k(0), B_k(0))$ is sufficient to compute $F(\epsilon)$, i.e. we can still take advantage of Note 1 of Algorithm 2.3.13.

Finally, to illustrate the numerical impacts of our approach, let us consider the examples in the next section.

III.3 NUMERICAL ISSUES

In this section, we first analyze the numerical problems in Example 1.3.1 and then present a more complicated version of Example 1.3.2 which is worked out using a computer package program for pole placement. Recall the system of Example 1.3.1:

Example 3.3.1:

$$x[k+1] = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x[k] + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} u[k] \quad (3.3.1)$$

We have seen in Chapter 1 that simplifying this system by

neglecting the $T^2/2$ term leads to numerical problems. The reason is that the system is T^2 -reachable, and only $O(T^3)$ terms could be neglected. Let us formalize this with the following result:

Proposition 3.3.2: Given an ϵ^k -reachable pair $(A(\epsilon), B(\epsilon))$, let $\bar{A}(\epsilon) = A(\epsilon) + \epsilon^{k+1}\tilde{A}(\epsilon)$ and $\bar{B}(\epsilon) = B(\epsilon) + \epsilon^{k+1}\tilde{B}(\epsilon)$, where $\tilde{A}(\epsilon)$ and $\tilde{B}(\epsilon)$ are $O(1)$, then $(\bar{A}(\epsilon), \bar{B}(\epsilon))$ is ϵ^k -reachable with the same indices.

Proof: Let $\mathcal{Q}(\epsilon) = P(\epsilon)D(\epsilon)Q(\epsilon)$, then $\bar{\mathcal{Q}}(\epsilon) = \mathcal{Q}(\epsilon) + \epsilon^{k+1}\tilde{\mathcal{Q}}(\epsilon) = P(\epsilon)D(\epsilon)\bar{Q}(\epsilon)$, where $\tilde{\mathcal{Q}}(\epsilon)$ is some matrix over $\mathbb{R}[[\epsilon]]$, $\bar{Q}(\epsilon) = Q(\epsilon) + \tilde{Q}(\epsilon)$ and $\tilde{Q}(\epsilon) = \epsilon^{k+1}P^{-1}(\epsilon)D^{-1}(\epsilon)\tilde{\mathcal{Q}}(\epsilon)$. Thus, $\tilde{Q}(0) = 0$, and $\bar{Q}(0)$ is full row rank since $Q(0)$ is.

In the same fashion, small values, which could play a significant role in reachability, might be neglected during computation due to the numerical accuracy of the computation device used. In general, the above proposition states that this numerical accuracy should at least be $O(\epsilon^{k+1})$ for working out an ϵ^k -reachable system. Consider the following example:

Example 3.3.3: Suppose that we wish to do pole placement for the following pair:

$$A = \begin{bmatrix} 11 & -2 & -11 & 5 & -4 \\ 5 & -11 & 16 & -13 & 13 \\ 3.00004 & 6.00001 & -12 & 15 & -11 \\ 2.99995 & -27 & 25.00001 & -13 & 19 \\ 0.99986 & -49.00006 & 71.00006 & -52.99995 & 54 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$$

This system turns out to be ill-conditioned for pole placement purposes. To apply our methods to this problem, we first have to parametrize this pair. The following parametrization seems to be reasonable:

$$A(\epsilon) = \begin{bmatrix} 11 & -2 & -11 & 5 & -4 \\ 5 & -11 & 16 & -13 & 13 \\ 3+4\epsilon & 6+\epsilon & -12 & 15 & -11 \\ 3-5\epsilon & -27 & 25+\epsilon & -13 & 19 \\ 1-14\epsilon & -49-6\epsilon & 71+6\epsilon & -53+5\epsilon & 54 \end{bmatrix} \text{ and } B(\epsilon) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$$

where $A = A(\epsilon=10^{-5})$ and $B = B(\epsilon=10^{-5})$. Consider the following proper standard form of $(A(\epsilon), B(\epsilon))$:

$$A_s(\epsilon) = \begin{bmatrix} 5 & 3 & -8 & 1 & -4 \\ 1 & 2 & -5 & 1 & 9 \\ 3\epsilon & \epsilon & 6 & 3 & 6 \\ 0 & 0 & \epsilon & 9 & -1 \\ 0 & 0 & 0 & 5\epsilon & 7 \end{bmatrix} \text{ and } B_s(\epsilon) = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $A_s(\epsilon) = WA(\epsilon)W^{-1}$, $B_s(\epsilon) = WB(\epsilon)$ and $W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}$.

Thus, our system is ϵ^3 -reachable. An ϵ^0 -reachable form is:

$$A_k(\epsilon) = \begin{bmatrix} 5 & 3 & -8\epsilon & \epsilon^2 & -4\epsilon^3 \\ 1 & 2 & -5\epsilon & \epsilon^2 & 9\epsilon^3 \\ 3 & 1 & 6 & 3\epsilon & 6\epsilon^2 \\ 0 & 0 & 1 & 9 & -\epsilon \\ 0 & 0 & 0 & 5 & 7 \end{bmatrix} \quad \text{and} \quad B_k(\epsilon) = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $A_k(\epsilon) = D^{-1}(\epsilon)A_s(\epsilon)D(\epsilon)$, $B_k(\epsilon) = D^{-1}(\epsilon)B_s(\epsilon)$ and

$$D(\epsilon) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & \epsilon^2 & 0 \\ 0 & 0 & 0 & 0 & \epsilon^3 \end{bmatrix}. \quad \text{Thus, we could do pole placement for the}$$

$$\text{pair } A_k(0) = \begin{bmatrix} 5 & 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 3 & 1 & 6 & 0 & 0 \\ 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 5 & 7 \end{bmatrix}, \quad B_k(0) = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and transform the}$$

achieved feedback matrix to apply to the original system (see Method 2 below). This way we place the $O(1)$ eigenvalues of the original system. ⁽⁴⁾

We performed our experiment using the PC package Control Systems Toolbox [15] with Matlab [16] on an IBM PC-AT. The pole placement routine in [15] uses Ackermann's formula [17], which is really not numerically stable. To improve its numerical performance, the pair is first put into staircase form, which can be done by triangularizing the matrix $[B|A]$, before the use of the pole placement algorithm.

⁴ $A_k(0)$ and $B_k(0)$ can be found via Algorithm 2.3.15, and we really do not need to know the pairs $(A_s(\epsilon), B_s(\epsilon))$ and $(A_k(\epsilon), B_k(\epsilon))$ for pole placement, but we use them for illustration purposes here.

The goal is now to find a feedback for $(A(\epsilon), B(\epsilon))$, a family of systems indexed by ϵ , that places the closed loop eigenvalues at $\lambda_d = [1 \ 2 \ 3 \ 4 \ 5]$ (picked arbitrarily). The reader should keep in mind that at $\epsilon=10^{-5}$, we have the original system (A,B) . In this set-up, we compare the following two methods:

Method 1: Given $\epsilon=\epsilon_0$ for some "small" $\epsilon_0 > 0$, find a feedback $f_1(\epsilon_0)$ for the pair $(A(\epsilon_0), B(\epsilon_0))$ such that the closed loop eigenvalues are as desired using a standard pole placement algorithm.

Method 2 (our approach): Find a feedback f_0 for the pair $(A_k(0), B_k(0))$ such that the closed loop eigenvalues are as desired. Let $f_2(\epsilon) = f_0 D^{-1}(\epsilon)W$, then for any given ϵ_0 use $f_2(\epsilon_0)$ as a feedback for the pair $(A(\epsilon_0), B(\epsilon_0))$ to achieve the desired closed loop eigenvalues.

Both of the above methods have been tested for six different values of ϵ . The results of this experiment are summarized in Tables 3.3.1.a and 3.3.1.b. Here f_1 represents the feedback computed by the first method, f_2 represents the feedback computed using the second method, and λ_1, λ_2 represent the computed eigenvalues of the closed loop matrix, after feedback, for each method. The feedback f_0 for the second method was computed as:

$$f_0 = [7.134252744159868 \ 3.432873627920071 \ 15.07177033492827 \\ 60.52631578947373 \ 3.789473684210530]$$

The following could be observed from Table 3.3.1:

1. For the first four values of ϵ , the mantissa of the entries of f_1 seem to converge, meanwhile the orders of magnitude increase in multiples of 1000 as ϵ is decreased in multiples of 10. On the other hand, the signs flip for $f_1(10^{-4})$, and there is a drop in magnitude for $f_1(10^{-5})$.
2. The computed eigenvalues, λ_1 , are reasonably good for $\epsilon=10^{-1}$, reasonable for $\epsilon=10^{-2}$, and useless thereafter.
3. For all values of ϵ , the mantissa of the entries of f_2 seem to converge, meanwhile the orders of magnitude increase in multiples of 1000 as ϵ is decreased in multiples of 10.
4. The computed eigenvalues, λ_2 , have the correct orders of magnitude for $\epsilon=10^{-1}$, 10^{-2} , and $\lambda_2(10^{-2})$ seem to be better than $\lambda_2(10^{-1})$. The rest deviate significantly from the desired values.
5. For the first four values of ϵ , the mantissa of the entries of f_1 and f_2 seem to converge, meanwhile the orders of magnitude are the same for both. For the other two values of ϵ they look totally different.

Let us analyze these observations:

1. Recall that the pair $(A(\epsilon), B(\epsilon))$ is ϵ^3 -reachable. Therefore, we expect the orders of magnitude of feedback gains to increase in multiples of 1000 as ϵ is decreased in multiples of 10.
2. Observation 1 suggests that f_1 can not be trusted for $\epsilon \leq 10^{-5}$. This could be explained by the insufficiency of machine accuracy.

We estimated that the machine accuracy is about 14 digits after the decimal point (note that floating point computations are used). Since the system is ϵ^3 -reachable, we expect f_1 to be erroneous for $\epsilon \leq 10^{-5}$. This can be seen in the lack of a consistent trend in the values of f_1 as ϵ is decreased from 10^{-3} to 10^{-4} , 10^{-5} and 10^{-6} .

3. Observations 2 and 4 suggest that the computed closed loop eigenvalues are erroneous for small values of ϵ . We also expect this since the closed loop matrix for small ϵ has very large entries due to feedback and thus the computation of eigenvalues is ill-conditioned.

4. The second method gets around the problem of machine accuracy by solving the pole placement problem for a well-behaved system. The feedback computed for this system is then transformed to get the desired feedback for the original system. Consequently the consistent trend in the values of f_2 are maintained for all values of ϵ . Since the goal of the second method is to place $O(1)$ eigenvalues, we expect some error in the actual eigenvalues, but this error is strictly less than $O(1)$ since the basic computations are on a well-conditioned system.

Therefore, this experiment illustrates that the second method achieves asymptotically better results in pole placement compared to the first method assuming that the system is appropriately parametrized by ϵ .

	$\epsilon = 10^{-1}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$
λ_1	1.000000649719127E00	1.2187E00 +j6.907E-1	-2.1776E01 +j2.470E01
	1.999998276328651E00	1.2187E00 -j6.907E-1	-2.1776E01 -j2.470E01
	3.000001547698876E00	3.7549E00 +j6.211E-1	4.349131130951719E01
	5.000000044611793E00	3.7549E00 -j6.211E-1	5.998837078155298E00
	3.999999481458782E00	5.052495954339675E00	9.062308203344981E00
λ_2	1.6600E00 +j3.137E00	5.628343007012566E-1	-1.962096308081683E02
	1.6600E00 -j3.137E00	2.1695E00 +j1.553E00	1.755296146977247E02
	5.8303E00 +j1.583E00	2.1695E00 -j1.553E00	2.066625397015819E01
	5.8303E00 -j1.583E00	5.0489E00 +j8.608E-1	6.002204129392923E00
	1.911685999697308E-2	5.0489E00 -j8.608E-1	9.011557492922475E00
f_1	-3.943896043759587E03	3.007362825791221E06	3.711181392087139E09
	-8.017128802560767E03	-4.216452071434855E06	-3.832219666858551E09
	8.309758387438543E03	4.219457621975910E06	3.832249801604952E09
	3.809863645633179E03	-3.008853317618679E06	-3.711196447176790E09
	2.177648777527002E03	3.613404940440496E06	3.771715591562496E09
f_2	-2.403308471714049E03	3.182713916408672E06	3.728932307217847E09
	-9.687954686180697E03	-4.393226232198147E06	-3.849984924796795E09
	9.992822966507180E03	4.396244019138761E06	3.850015071770338E09
	2.263157894736846E03	-3.184210526315793E06	-3.728947368421055E09
	3.789473684210529E03	3.789473684210530E06	3.789473684210529E09

TABLE 3.3.1.a RESULTS FOR EXAMPLE 3.3.2 FOR $\epsilon = 10^{-1}$, 10^{-2} , 10^{-3}

	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$	$\epsilon = 10^{-6}$
λ_1	-1.3322E00 +j9.638E04	1.067169897228551E07	-2.9156E01 -j7.691E06
	-1.3322E00 -j9.638E04	-1.067168281428935E07	-2.9156E01 -j7.691E06
	6.001775180382936E00	6.305048283046647E00	5.699866527638253E01
	2.337672181170603E00	5.846594618545338E00	8.995957587547895E00
	9.007949867449816E00	9.199613481943446E00	5.999828196486538E00
λ_2	-1.3553E01 +j1.044E05	-4.217955904296881E07	-1.0240E03 +j6.242E10
	-1.3553E01 -j1.044E05	4.217960137109382E07	-1.0240E03 -j6.242E10
	2.710678276143728E01	-4.191570582143022E01	1.000212207642650E03
	5.997725160398033E00	9.898241493445850E00	-6.732027920523609E02
	9.004296958906786E00	5.028794946790915E00	4.891644677416111E01
f_1	3.784803587906603E12	-2.423067655721092E16	5.181756878352025E15
	-3.796907131758365E12	2.422806509616012E16	-5.321413489700029E15
	3.796907433027197E12	-2.422806507913671E16	5.321413538658905E15
	-3.784803738528736E12	2.423067654869921E16	-5.181756902831453E15
	3.790855510461616E12	-2.422937081817381E16	5.251585208505463E15
f_2	3.783420901924446E12	3.788868419545468E15	3.789413157879669E18
	-3.795526165068340E12	-3.790078945861243E15	-3.789534210511248E18
	3.795526466507181E12	3.790078948875600E15	3.789534210541392E18
	-3.783421052631582E12	-3.788868421052635E15	-3.789413157894741E18
	3.789473684210529E12	3.789473684210529E15	3.789473684210530E18

TABLE 3.3.1.b RESULTS FOR EXAMPLE 3.3.2 FOR $\epsilon = 10^{-4}$, 10^{-5} , 10^{-6}

Example 3.3.4: Consider the system in the previous example. Suppose that we wish to place the eigenvalues at $\lambda_d=[1 \ 2 \ 6 \ 7 \ 9]$. It is clear from $A_k(0)$ that three of the eigenvalues of $A_k(\epsilon)$, and thus $A(\epsilon)$, for $\epsilon=0$ are at 6, 9, and 7. So, as far as the second method is concerned, we do not need to move these eigenvalues. It turns out that the other two eigenvalues correspond to the ϵ^0 -reachable subsystem. Thus, the feedback computed via the second method will be $O(1)$.

Results similar to the previous example are illustrated in Table 3.3.2 for $\epsilon = 10^{-4}, 10^{-5}, 10^{-6}$. Note that f_1 and f_2 are completely different for this case. The feedback f_1 consists of very large gains trying to correct for small (less than $O(1)$) discrepancies in the eigenvalues. On the other hand, these large gains result in an ill-conditioned closed loop system and therefore the resulting eigenvalues could actually be quite different from the desired eigenvalues. On the other hand, f_2 consists of $O(1)$ gains and λ_2 converges to the desired eigenvalues as $\epsilon \downarrow 0$.

This special case illustrates the following phenomenon: The feedback for pole placement of even SISO system is nonunique from a numerical point of view. In the case that the desired eigenvalues match the $O(1)$ parts of the existing eigenvalues, the feedback gains calculated by the second method may turn out to be

significantly smaller than the values obtained by a direct calculation of feedback, for example via the first method. Since the closed loop matrix is better conditioned in the first case, better accuracy of the actual closed loop eigenvalues could be achieved.

	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$
λ_1	9.952854516667541E-1	1.1292E00 +j3.066E00	1.005001193743054E02
	2.005044537392085E00	1.1292E00 -j3.066E00	-2.927266446538739E01
	6.999615179341110E00	7.823895697283298E00	3.406348114334948E01
	6.000001280345070E00	5.999787275229292E00	5.999971130950930E00
	9.000002324854214E00	9.000014238104945E00	9.000104984387471E00
λ_2	9.984044024457135E-1	9.998400442055826E-1	9.999840004421892E-1
	2.003246838429415E00	2.000324968187798E00	2.000032499681668E00
	8.998499391574481E00	8.999849993936724E00	8.999984999939423E00
	7.002496899398323E00	7.000249968774466E00	7.000024999687599E00
	5.997352468152110E00	5.999735024895445E00	5.999973500249168E00
f_1	-7.850156450365858E05	-7.715835668188436E07	2.258752919097173E09
	7.915688000773013E05	7.722284913526829E07	-2.259142267627447E09
	-7.915703294962894E05	-7.722285070843991E07	2.259142217546636E09
	7.850197920401807E05	7.715836083743484E07	-2.258752924580083E09
	-7.882943695349249E05	-7.719060510521657E07	2.258947553199712E09
f_2	2.764705882352945E00	2.764705882352945E00	2.764705882352945E00
	1.235294117647063E00	1.235294117647063E00	1.235294117647063E00
	0	0	0
	0	0	0
	0	0	0

TABLE 3.3.2 RESULTS FOR EXAMPLE 3.3.3 FOR $\epsilon = 10^{-3}$, 10^{-4} , 10^{-5}

Similar concerns have been expressed by authors interested in numerical issues of multivariable pole placement for linear time invariant systems (as explained in I.2). Our approach would address those issues by scaling the pair (A,B) appropriately. Unfortunately, (A,B) has to be parametrized by ϵ first. This problem has been left for future research and some heuristic suggestions for parametrizations are made in Chapter V.

IV CONTINUOUS TIME

In this chapter, we show the counterparts to our discrete time results for continuous time systems. Also, we examine the notion of almost invariance [3].

IV.1 ϵ^j -REACHABILITY

Consider the continuous time system of equation (1.4.1):

$$\dot{x}(t) = A(\epsilon)x(t) + B(\epsilon)u(t) \quad (1.4.1)$$

where the input function $u(t)$ is an m -dimensional vector of real valued functions defined on $\mathbb{R}(\epsilon)$, and the state $x(t)$ is an n -dimensional vector of real valued functions in $\mathbb{R}(\epsilon)$ generated by $u(t)$, the system (1.4.1), and an initial state $x(0)$. To examine reachability, we assume that $x(0) = 0$. Let us state the counterpart to Definition 2.1.2 for continuous time, as follows:

Definition 4.1.1: $x \in \mathbb{R}^n[[\epsilon]]$ is ϵ^j -reachable if $\exists \tau \in \mathbb{R}^+$ and $u(t) \in 1/\epsilon^j \mathbb{R}^m[[\epsilon]] \forall t \in [0, \tau]$ such that $x(\tau) = x$, with $x(0) = 0$.

Let \mathcal{X}^j be the set of all ϵ^j -reachable states, then:

Proposition 4.1.2: $\mathcal{X}^0 \subset \mathcal{X}^1 \subset \mathcal{X}^2 \subset \dots$ and \mathcal{X}^j is a submodule of $\mathbb{R}^n[[\epsilon]]$.

Proof: Same as Proposition 2.1.1.

We term \mathcal{X}^j the ϵ^j -reachable submodule.

To prove that these submodules are equivalent to those of discrete time for proper systems, consider the following proposition and corollary:

Proposition 4.1.3: Given a pair $(A(\epsilon), B(\epsilon))$, if the coefficients of the characteristic polynomial of $A(\epsilon)$ are over $\mathbb{R}[[\epsilon]]$ then

$\mathcal{X}^0 = \langle A(\epsilon) | \mathfrak{B}_\epsilon \rangle \cap \mathbb{R}^n[[\epsilon]]$ where $\langle A(\epsilon) | \mathfrak{B}_\epsilon \rangle \equiv \sum_{i=1}^n A^{i-1}(\epsilon) \mathfrak{B}_\epsilon$ and \mathfrak{B}_ϵ is the

image of $B(\epsilon)$ over $\mathbb{R}[[\epsilon]]$.

Proof: If $x \in \mathcal{X}^0$ then there exists $\tau \in \mathbb{R}^+$ and $u(t)$ of $O(1)$ such that

$$x(\tau) = x = \int_0^\tau e^{A(\epsilon)(\tau-t)} B(\epsilon) u(t) dt = \sum_{i=1}^n A^{i-1}(\epsilon) B(\epsilon) \int_0^\tau \psi_i(\tau-t) u(t) dt =$$

$$\sum_{i=1}^n A^{i-1}(\epsilon) B(\epsilon) \psi_i(\epsilon). \quad \psi_i(\epsilon) \in \mathbb{R}^m[[\epsilon]] \text{ and also } x \in \mathbb{R}^n[[\epsilon]]. \text{ Thus,}$$

$$\mathcal{X}^0 \subset \langle A(\epsilon) | \mathfrak{B}_\epsilon \rangle \cap \mathbb{R}^n[[\epsilon]].$$

For the converse, let $x \in \langle A(\epsilon) | \mathfrak{B}_\epsilon \rangle \cap \mathbb{R}^n[[\epsilon]]$. Then $x = \sum_{i=1}^n A^{i-1}(\epsilon) B(\epsilon) \alpha_i(\epsilon)$

for $\alpha_i(\epsilon)$ over $\mathbb{R}[[\epsilon]]$. To find the control to reach x , let us use the minimum energy control to reach a target state, as before, to

insure that we do our best in finding the minimum order of control required to reach that state. Recall that the minimum energy control is:

$$u(t) = B^T(\epsilon)e^{(\tau-t)A^T(\epsilon)}W_\tau^{-1}(\epsilon)x, \quad 0 < t < \tau \quad (4.1.1)$$

where $W_\tau(\epsilon) = \int_0^\tau e^{tA(\epsilon)}B(\epsilon)B^T(\epsilon)e^{tA^T(\epsilon)}dt$ is positive definite for $\tau > 0$ and also symmetric.

To determine the order of control, consider

$$\int_0^\tau u^T(t)u(t)dt \quad (4.1.2)$$

which we can write as

$$x^T W_\tau^{-1}(\epsilon) \left(\int_0^\tau e^{(\tau-t)A(\epsilon)}B(\epsilon)B^T(\epsilon)e^{(\tau-t)A^T(\epsilon)}dt \right) W_\tau^{-1}(\epsilon)x \quad (4.1.4)$$

The term in the parenthesis of the above equation can be seen to equal $W_\tau(\epsilon)$ by a simple change of variable. Thus, (4.1.1) just equals:

$$x^T W_\tau^{-1}(\epsilon)x \quad (4.1.5)$$

Let $\tilde{A}(\epsilon) = D^{-1}(\epsilon)P^{-1}(\epsilon)A(\epsilon)P(\epsilon)D(\epsilon)$, $\tilde{B}(\epsilon) = D^{-1}(\epsilon)P^{-1}(\epsilon)B(\epsilon)$.

Then, $W_\tau(\epsilon) = P(\epsilon)D(\epsilon)\tilde{W}_\tau(\epsilon)D^T(\epsilon)P^T(\epsilon)$ where

$$\tilde{W}_\tau(\epsilon) = \int_0^\tau e^{t\tilde{A}(\epsilon)}\tilde{B}(\epsilon)\tilde{B}^T(\epsilon)e^{t\tilde{A}^T(\epsilon)}dt. \quad \text{Since } \tilde{Q}(\epsilon) = Q(\epsilon) \text{ is}$$

unimodular, $\tilde{W}_\tau(\epsilon)$ is also unimodular by use of the Cayley Hamilton theorem and the assumption that the coefficients of the characteristic polynomial of $A(\epsilon)$ are over $R[[\epsilon]]$. Let

$$\tilde{x} = D^{-1}(\epsilon)P^{-1}(\epsilon)x = \sum_{i=1}^n \tilde{A}^i(\epsilon)\tilde{B}(\epsilon)\alpha_i(\epsilon). \quad \text{Therefore, } \tilde{x} \text{ is over } R[[\epsilon]]$$

and thus $\int_0^T u^T(t)u(t)dt = \tilde{x}^T \tilde{W}^{-1} \tilde{x}$ is $O(1)$.

Corollary 4.1.4: $\mathcal{X}^0 = P(\epsilon)D(\epsilon)R^n[[\epsilon]]$ where $\mathcal{Q}(\epsilon) = P(\epsilon)D(\epsilon)Q(\epsilon)$ is a Smith decomposition for the reachability matrix.

Using the iterative relation $\mathcal{X}^{j+1} = \frac{1}{\epsilon} \{ \mathcal{X}^j \cap \epsilon R^n[[\epsilon]] \}$, (Proposition 2.1.2), we can recover all the other reachability submodules from the Smith decomposition of the reachability matrix and Corollary 4.1.4. Therefore, all our results for discrete time also hold for continuous time, including the feedback results.

IV.2 (A,B)-INVARIANCE AND ALMOST (A,B)-INVARIANCE

In this section, we use our framework to provide some quantitative insight into the notions of almost (A,B)-invariance and almost (A,B)-controllability introduced by J. C. Willems [3].

Let us begin with a review of some basic geometric notions. An initial condition in an (A,B)-invariant subspace [10] can, by definition, be made to give rise to a trajectory in that subspace by an appropriate choice of input. The stronger notion of an (A,B)-controllability subspace [10] corresponds to a subspace where any initial and final conditions $x(0)$ and $x(T)$ in it can be connected by a trajectory in this subspace. Similarly, a trajectory starting in an almost (A,B)-invariant subspace [3] can,

by definition, be steered arbitrarily close to this subspace. Finally, a trajectory starting in an almost (A,B)-controllability subspace [3] can, by definition, be steered arbitrarily close to this subspace and can be made to pass through any given point in the subspace at some time $T > 0$.

Willems [3] also shows that given an almost (A,B)-invariant subspace, there exists a sequence of (A,B)-invariant subspaces that converges to the almost (A,B)-invariant subspace in the Grassmanian sense (i.e. some sequence of bases for the sequence of (A,B)-invariant subspaces converges to a basis for the almost (A,B)-invariant subspace—this is defined in [3] and also in Definition 4.2.5 of this section). Willems also claims the converse, i.e. that given a sequence of (A,B)-invariant subspaces that converge to a subspace in the Grassmanian sense, that subspace is almost (A,B)-invariant. A counter example for this claim will be presented shortly (see Example 4.2.7).

To give a flavor for our approach, let us consider the following example:

Example 4.2.1: Let

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It is easy to see from the results in [3] that $\mathcal{V}_a = \text{Im} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an almost (A,B)-invariant subspace. Consider the family of

subspaces, $\{\mathcal{V}_\epsilon\}$, generated by $\begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$ for all $\epsilon \in (0, \infty)$. Since

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} (1/\epsilon) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (-1/\epsilon),$$

these subspaces are (A,B)-invariant. As we let $\epsilon \rightarrow 0$,

$\{\mathcal{V}_\epsilon\} \rightarrow \mathcal{V}_a = \text{Im} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which is an almost (A,B)-invariant subspace, so

we have found a sequence satisfying (the correct direction of)

Willems' result. Also, using the relation $(-1/\epsilon) = F(\epsilon) \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$ with

$F(\epsilon) = [-1/\epsilon \ 0]$, these subspaces are $A_F(\epsilon)$ invariant where

$$A_F(\epsilon) = A - BF(\epsilon) = \begin{bmatrix} 1/\epsilon & 0 \\ 1 & 0 \end{bmatrix}.$$

Furthermore, the $\{\mathcal{V}_\epsilon\}$ are coasting subspaces, [3], i.e. they are (A,B)-invariant but they have no (A,B)-controllable part, whereas \mathcal{V}_a is a sliding subspace, [3], i.e. it is almost (A,B)-invariant but it has no (A,B)-invariant part.

Note that an eigenvalue of $A_F(\epsilon) \rightarrow +\infty$ as $\epsilon \rightarrow 0$. On the other hand, consider the family of (A,B)-invariant subspaces, $\{\mathcal{V}'_\epsilon\}$, generated by $\begin{bmatrix} 1 \\ -\epsilon \end{bmatrix}$. As $\epsilon \rightarrow 0$, $\{\mathcal{V}'_\epsilon\} \rightarrow \mathcal{V}_a$ also. By going through the above procedure, we get $F'(\epsilon) = [1/\epsilon \ 0]$ and $A_{F'}(\epsilon) = \begin{bmatrix} -1/\epsilon & 0 \\ 1 & 0 \end{bmatrix}$. Now, the eigenvalue of $A_{F'}(\epsilon)$ that blows up $\rightarrow -\infty$ as $\epsilon \rightarrow 0$.

We proceed with proving some results related to the above observations, but we first state some algebraic properties that we use extensively.

$\mathbb{R}^n((\epsilon))$ is a vector space over the field $\mathbb{R}((\epsilon))$. Let \mathcal{V}_ϵ be a subspace of $\mathbb{R}^n((\epsilon))$ and let $V(\epsilon) = [v_1(\epsilon) \mid \dots \mid v_\mu(\epsilon)]$ such that $\{v_i(\epsilon)\}$ is a linearly independent set that spans \mathcal{V}_ϵ . Since \mathcal{V} is

closed under multiplication by elements in $\mathbb{R}((\epsilon))$, it is possible to pick $v_i(\epsilon)$ such that $v_i(\epsilon) \in \mathbb{R}[[\epsilon]]$ and $V(0)$ is full column rank. The subspaces of Wonham and Willems are naturally defined over \mathbb{R} . Note that the span of the columns of $V(\epsilon=\epsilon_0)$, for some $\epsilon_0 > 0$, is a subspace of \mathbb{R}^n . Thus it is also possible to think of \mathcal{V}_ϵ as a sequence of subspaces of \mathbb{R}^n for different values of ϵ . We use this to connect our results to their counterparts in [3] and [10]. Also, in this section, we assume that the coefficients of the characteristic polynomial of $A(\epsilon)$ are over $\mathbb{R}[[\epsilon]]$.

Definition 4.2.2: $\mathcal{V}_\epsilon \subset \mathbb{R}^n((\epsilon))$ is $(A(\epsilon), B(\epsilon))$ -invariant if $\exists F(\epsilon): \mathbb{R}^n((\epsilon)) \rightarrow \mathbb{R}^m((\epsilon))$ s.t. $A_F(\epsilon)\mathcal{V}_\epsilon \subset \mathcal{V}_\epsilon$, where $A_F(\epsilon) = A(\epsilon) + B(\epsilon)F(\epsilon)$.

We denote the family of $(A(\epsilon), B(\epsilon))$ -invariant $\mathbb{R}((\epsilon))$ -subspaces by $\underline{\mathcal{V}}_\epsilon$. For some cases, we consider $(A(\epsilon), B(\epsilon))$ -invariant $\mathbb{R}((\epsilon))$ -subspaces for $A(\epsilon)=A$ and $B(\epsilon)=B$. We use the same notation, and assume that the reader will infer the relevant underlying system from the context. We denote the family of (A, B) -invariant subspaces by $\underline{\mathcal{V}}$ and the family of almost (A, B) -invariant subspaces by $\underline{\mathcal{V}}_a$, consistent with the notation of Willems.

A straightforward extension of this definition is the following well known result [10]:

Proposition 4.2.3: $\mathcal{V}_\epsilon \in \underline{\mathcal{V}}_\epsilon$ iff $A(\epsilon)\mathcal{V}_\epsilon \subset \mathcal{V}_\epsilon + \mathcal{B}$, where $\mathcal{B} = B(\epsilon)\mathbb{R}^m((\epsilon))$.

Definition 4.2.4: $\mathcal{V}_\epsilon \subset \mathbb{R}^n((\epsilon))$ is ϵ^j - $(A(\epsilon), B(\epsilon))$ -invariant (denoted by $\mathcal{V}_\epsilon \in \underline{V}_\epsilon^j$) if $\exists F(\epsilon): \mathbb{R}^n((\epsilon)) \rightarrow \mathbb{R}^m((\epsilon))$ s.t. $A_F(\epsilon)\mathcal{V}_\epsilon \subset \mathcal{V}_\epsilon$, and $F(\epsilon)$ is $O(1/\epsilon^j)$.

Definition 4.2.5: Given $\mathcal{V}_a \subset \mathbb{R}^n$ and $\mathcal{V}_\epsilon \subset \mathbb{R}^n((\epsilon))$, $\mathcal{V}_\epsilon \xrightarrow{\epsilon \rightarrow 0} \mathcal{V}_a$ if whenever $\{v_1(\epsilon), \dots, v_\mu(\epsilon)\}$, where $v_i(\epsilon) \in \mathbb{R}^n[[\epsilon]]$, is a set of generators for \mathcal{V}_ϵ , the set of vectors $\{v_1, \dots, v_\mu\}$, where $v_i = \lim_{\epsilon \rightarrow 0} v_i(\epsilon)$, forms a basis for \mathcal{V}_a (this is a convergence in the Grassmanian sense).

Note that we can construct a matrix $W(\epsilon)$ over $\mathbb{R}[[\epsilon]]$ such that $W(0) = I$ and $v_i(\epsilon) = W(\epsilon)v_i$. Thus an alternate representation of \mathcal{V}_ϵ would be $W(\epsilon)\mathcal{V}_a$. Similarly, given $\mathcal{V}_a \subset \mathbb{R}^n$ and some $W(\epsilon)$ as above, its counterpart over $\mathbb{R}((\epsilon))$ with respect to $W(\epsilon)$ could be represented as $W(\epsilon)\mathcal{V}_a$.

The following result enables us to establish a connection between our framework and the notion of almost (A, B) -invariance. It provides a method to compute approximations for the distributional inputs required to steer the trajectories of an almost (A, B) -invariant subspace exactly through that subspace. Using these high gain feedback approximations one can steer trajectories arbitrarily close to an almost (A, B) -invariant subspace.

Proposition 4.2.6: For a given pair (A,B) , if $\mathcal{V}_a \in \underline{V}_a$ then \exists

$\mathcal{V}_\epsilon \in \underline{V}_{-\epsilon}$ such that $\mathcal{V}_\epsilon \xrightarrow{\epsilon \rightarrow 0} \mathcal{V}_a$.

Proof: From the proof of Willems [3], Theorem 6, $\mathcal{V}_a = \mathcal{V} + \mathcal{K}_a$, where \mathcal{V} is (A,B) -invariant, and $\mathcal{K}_a = \mathcal{B}_1 + A_F \mathcal{B}_2 + \dots + A_F^{n-1} \mathcal{B}_{n-1}$, for some F , and a chain $\{\mathcal{B}_i\}$ in \mathcal{B} , $\mathcal{B} \supset \mathcal{B}_1 \supset \dots \supset \mathcal{B}_n$. Thus, it is sufficient to consider $b + A_F b + \dots + A_F^{i-1} b$, $b \in \mathcal{B}$. Furthermore, using the feedback invariance of \underline{V}_ϵ , we consider the following pair with just a bank of integrators

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (4.2.1)$$

Then, $\mathcal{K}_{a,j} = \mathcal{E}_1 + \dots + \mathcal{E}_j$, for $j=1, \dots, n$, where \mathcal{E}_i is the span of the i^{th} unit vector, are the almost (A,B) -controllable subspaces. Let

$$P(\epsilon) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\epsilon & 1 & 0 & 0 \\ \epsilon^2 & -2\epsilon & 1 & 0 \\ \vdots & & & \vdots \\ (-1)^{n-1} \epsilon^{n-1} & (-1)^{n-2} \begin{bmatrix} n-1 \\ 1 \end{bmatrix} \epsilon^{n-2} & (-1)^{n-3} \begin{bmatrix} n-1 \\ 2 \end{bmatrix} \epsilon^{n-3} & \dots & 1 \end{bmatrix} \quad (4.2.2)$$

where the coefficients in each row form the Pascal's triangle with alternating signs. Next, we proceed to show that $P(\epsilon)\mathcal{K}_{a,i}$ is ϵ^i - $(A(\epsilon), B(\epsilon))$ -invariant.

$$A(\epsilon)P(\epsilon)\mathfrak{R}_{a,i} = \begin{bmatrix} 0 \\ 1 \\ -\epsilon \\ \vdots \\ (-1)^{n-2}\epsilon^{n-2} \end{bmatrix} = \begin{bmatrix} 1 \\ -\epsilon \\ \epsilon^2 \\ \vdots \\ (-1)^{n-1}\epsilon^{n-1} \end{bmatrix} (-1/\epsilon) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} (1/\epsilon) \quad (4.2.3)$$

Thus, $\begin{bmatrix} 1 \\ -\epsilon \\ \epsilon^2 \\ \vdots \\ (-1)^{n-1}\epsilon^{n-1} \end{bmatrix}$ is $\epsilon-(A(\epsilon), B(\epsilon))$ -invariant.

Assume that $P(\epsilon)\mathfrak{R}_{a,i}$ is $\epsilon^i-(A(\epsilon), B(\epsilon))$ -invariant. Denote the i^{th}

column of $P(\epsilon)$ by $\begin{bmatrix} 0 \\ \vdots \\ 1 \\ a_1\epsilon \\ \vdots \\ a_{n-i}\epsilon^{n-i} \end{bmatrix}$.

Then, using $O(1/\epsilon^i)$ coefficients we can generate $\begin{bmatrix} 0 \\ \vdots \\ 1 \\ a_1\epsilon \\ \vdots \\ a_{n-i-1}\epsilon^{n-i-1} \end{bmatrix}$.

Consider $P(\epsilon)\mathfrak{R}_{a,i+1}$. Denote the $(i+1)^{\text{th}}$ column of $P(\epsilon)$ by

$\begin{bmatrix} 0 \\ \vdots \\ 1 \\ b_1\epsilon \\ \vdots \\ b_{n-i-1}\epsilon^{n-i-1} \end{bmatrix}$. From Pascal's triangle and alternating

coefficients: $b_1 = a_1 - 1$, $b_2 = a_2 - b_1$, $b_3 = a_3 - b_2$, etc. Thus,

$$\begin{bmatrix} 0 \\ \vdots \\ 1 \\ b_1 \epsilon \\ \vdots \\ b_{n-i-2} \epsilon^{n-i-2} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ a_1 \epsilon \\ \vdots \\ a_{n-i-1} \epsilon^{n-i-1} \end{bmatrix} (1/\epsilon) + \begin{bmatrix} 0 \\ \vdots \\ 1 \\ b_1 \epsilon \\ \vdots \\ b_{n-i-1} \epsilon^{n-i-1} \end{bmatrix} (-1/\epsilon) \quad (4.2.4)$$

Therefore, $P(\epsilon)\mathcal{K}_{a,i+1}$ is ϵ^{i+1} - $(A(\epsilon), B(\epsilon))$ -invariant, and clearly $P(\epsilon)\mathcal{K}_{a,i} \xrightarrow{\epsilon \rightarrow 0} \mathcal{K}_{a,i}$.

$\mathcal{K}_{a,i}$ correspond to sliding subspaces and $P(\epsilon)\mathcal{K}_{a,i}$ correspond to coasting $R((\epsilon))$ -subspaces. $P(\epsilon)$ is chosen such that the (fixed) eigenvalues corresponding to these coasting $R((\epsilon))$ -subspaces approach $-\infty$ as $\epsilon \rightarrow 0$. Let us illustrate this by first considering the feedback $F_i(\epsilon)$ that makes $P(\epsilon)\mathcal{K}_{a,i}$ an $A_F(\epsilon)$ -invariant $R((\epsilon))$ -subspace, where $A_F(\epsilon) = A - BF_i(\epsilon)$. This feedback satisfies

$$\begin{bmatrix} \frac{1}{\epsilon} & \dots & \frac{1}{\epsilon} & 0 & \dots & 0 \end{bmatrix} = F_i(\epsilon)P(\epsilon) \quad (4.2.5)$$

On the other hand,

$$P^{-1}(\epsilon) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \epsilon & 1 & 0 & \dots & 0 \\ \epsilon^2 & 2\epsilon & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \epsilon^{n-1} & \binom{n-1}{1} \epsilon^{n-2} & \binom{n-1}{2} \epsilon^{n-3} & \dots & 1 \end{bmatrix} \quad (4.2.6)$$

(see appendix A.4). Thus, we can pick

$$F_i(\epsilon) = \begin{bmatrix} i \frac{1}{\epsilon} & \dots & \binom{i}{c} \frac{1}{\epsilon^c} & \dots & \frac{1}{\epsilon^i} & 0 & \dots & 0 \end{bmatrix} \quad (4.2.7)$$

and then

$$A_F(\epsilon) = \begin{bmatrix} -i\frac{1}{\epsilon} & \dots & -\begin{bmatrix} i \\ c \end{bmatrix}\frac{1}{\epsilon^c} & \dots & \frac{-1}{\epsilon} & 0 & \dots & 0 & 0 \\ 1 & & & & & 0 & & 0 & \\ \vdots & & & & & \vdots & & \vdots & \\ 0 & & & & & 1 & & 0 & \end{bmatrix} \quad (4.2.8)$$

It follows from the characteristic equation of $A_F(\epsilon)$, which is in a companion form, that it has $n-i$ eigenvalues at 0 and i eigenvalues at $-1/\epsilon$. The eigenvalues at $-1/\epsilon$ approach $-\infty$ as $\epsilon \rightarrow 0$. Since we used similarity transformations and constant feedback to reduce the original pair to the form that we used in the proof, the eigenvalues of the original pair that blow up also approach $-\infty$.

The reverse of the above proposition does not hold. To illustrate this, consider the following example:

Example 4.2.7: Let $A = \begin{bmatrix} 0_3 & 0_3 \\ I_3 & 0_3 \end{bmatrix}$ and $B = \begin{bmatrix} I_3 \\ 0_3 \end{bmatrix}$. Consider

$$\mathcal{V} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \epsilon \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \text{ where } (\cdot) \text{ denotes range over } \mathbb{R}(\epsilon). \quad \mathcal{V} \in \underline{V}_\epsilon \text{ and}$$

$$\mathcal{V} \xrightarrow{\epsilon \rightarrow 0} \mathcal{L} \text{ where } \mathcal{L} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_o + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}_o + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}_o, \text{ where } (\cdot)_o \text{ denotes range over } \mathbb{R}.$$

But \mathcal{L} is not an almost (A,B) -invariant subspace.

Theorem 6 of J.C. Willems [3] is stated as follows:

Theorem 6: $\{V_a \in \underline{V}_a\} \leftrightarrow \{\exists V_\epsilon \in \underline{V} \text{ such that } V_\epsilon \xrightarrow[\epsilon \rightarrow 0]{} V_a \text{ in the Grassmanian sense}\}$.

The above example is also a counterexample to the reverse direction (\leftarrow) of the above result, since \mathcal{V} can also be thought of a sequence of (A,B)-invariant subspaces of \mathbb{R}^n (See Appendix A.5). It is also worth noting that only the forward direction (\rightarrow) of the above theorem has been proved by H. L. Trentelman [18] and no comment on the reverse direction has been made to the best of our knowledge.

In practice, one could calculate the input necessary to steer the trajectories of an almost (A,B)-invariant subspace, arbitrarily close to this subspace, by constructing an $(A(\epsilon), B(\epsilon))$ -invariant $\mathbb{R}((\epsilon))$ -subspace that approaches the almost (A,B)-invariant subspace in the Grassmanian sense, and calculating the feedback that makes the $(A(\epsilon), B(\epsilon))$ -invariant $\mathbb{R}((\epsilon))$ -subspace $A_F(\epsilon)$ -invariant. Naturally, this feedback blows up as one desires to steer these trajectories closer and closer ($\epsilon \rightarrow 0$) to the almost (A,B) invariant subspace since the feedback is meant to approximate distributional inputs. This approximation gets better and better as $\epsilon \rightarrow 0$.

The above procedure specifically addresses an open problem in Willems [3].

Given an almost (A,B)-controllability subspace \mathcal{K}_a , it can be represented as $\mathcal{K}_a = \mathcal{K}_0 \oplus \mathcal{K}_s$ where \mathcal{K}_0 is the supremal

(A,B) -controllability subspace in \mathfrak{R}_a and \mathfrak{R}_s is a sliding subspace. By construction of the proof of Proposition 4.2.6, we can find $\mathcal{V}_c \in \underline{V}_\epsilon$ where $\mathcal{V}_c = Q(\epsilon)\mathfrak{R}_s$, $Q(\epsilon)$ over $\mathbb{R}[[\epsilon]]$ and $Q(0) = I$. \mathcal{V}_c is a coasting $\mathbb{R}((\epsilon))$ -subspace and it can be picked so that eigenvalues corresponding to \mathcal{V}_c approach $-\infty$ as $\epsilon \rightarrow 0$. Thus, via this procedure, the feedback $F(\epsilon)$ that makes \mathcal{V}_c an $A_F(\epsilon)$ -invariant $\mathbb{R}((\epsilon))$ -subspace can be calculated. Let $F(\epsilon_0)$ be the feedback corresponding to $\epsilon = \epsilon_0$ and $A_F(\epsilon_0)$ the closed loop matrix corresponding to $F(\epsilon_0)$. Then, the eigenvalues of $A_F(\epsilon_0)$ corresponding to \mathfrak{R}_s approach $-\infty$ as smaller and smaller values of ϵ_0 are picked. Note that this increases the magnitude of the feedback gains, and they approach impulses in the limit. The eigenvalues corresponding to \mathfrak{R}_0 can be assigned by the usual pole placement methods.

Now, we proceed with the notion of $(A(\epsilon), B(\epsilon))$ -controllability $\mathbb{R}((\epsilon))$ -subspaces adopting Wonham's definition [10] of (A,B) -controllability subspaces:

Definition 4.2.8: $\mathfrak{R} \in \mathbb{R}^n((\epsilon))$ is an $(A(\epsilon), B(\epsilon))$ -controllability subspace if there exists maps $F(\epsilon): \mathbb{R}^n((\epsilon)) \rightarrow \mathbb{R}^m((\epsilon))$ and $G(\epsilon): \mathbb{R}^m((\epsilon)) \rightarrow \mathbb{R}^m((\epsilon))$ such that $\mathfrak{R} = \langle A(\epsilon) + B(\epsilon)F(\epsilon) \mid \text{Im}(B(\epsilon)G(\epsilon)) \rangle$.

We denote the family of $(A(\epsilon), B(\epsilon))$ -controllability $\mathbb{R}((\epsilon))$ -subspaces by \underline{R}_ϵ . For some cases, we consider $(A(\epsilon), B(\epsilon))$ -controllability $\mathbb{R}((\epsilon))$ -subspaces for $A(\epsilon) = A$ and

$B(\epsilon)=B$. Same notation is also used for these cases. We denote the family of (A,B) -controllability subspaces by \underline{R} and the family of almost (A,B) -controllability subspaces by \underline{R}_a , consistent with the notation of Willems.

To put the above definition into a more usable form, consider the following propositions which are the same results as those of Wonham [10] but in the framework that we have developed:

Proposition 4.2.9: $\mathfrak{X} \in \underline{R}_\epsilon$ iff there exists a map $F(\epsilon): \mathbb{R}^n((\epsilon)) \rightarrow \mathbb{R}^m((\epsilon))$ such that $\mathfrak{X} = \langle A(\epsilon)+B(\epsilon)F(\epsilon) | \mathfrak{B} \cap \mathfrak{X} \rangle$ where \mathfrak{B} represents the range of $B(\epsilon)$ over $\mathbb{R}((\epsilon))$.

Proof: (\rightarrow) Just find a $G(\epsilon)$ such that $\text{Im}(B(\epsilon)G(\epsilon)) = \mathfrak{B} \cap \mathfrak{X}$.

(\leftarrow) Let $A_F(\epsilon) = A(\epsilon)+B(\epsilon)F(\epsilon)$ and $\text{Im}(B(\epsilon)G(\epsilon)) = \mathfrak{B}_1$ then $\mathfrak{X} = \langle A_F(\epsilon) | \mathfrak{B}_1 \rangle$. Clearly, $\mathfrak{B}_1 \subset \mathfrak{B} \cap \mathfrak{X}$, thus $\mathfrak{X} \subset \langle A_F(\epsilon) | \mathfrak{B} \cap \mathfrak{X} \rangle$. On the other hand, $A_F(\epsilon)\mathfrak{X} \subset \mathfrak{X}$ thus $\langle A_F(\epsilon) | \mathfrak{B} \cap \mathfrak{X} \rangle = \mathfrak{B} \cap \mathfrak{X} + A_F(\epsilon)\mathfrak{X} \subset \mathfrak{X}$. Therefore, $\mathfrak{X} = \langle A_F(\epsilon) | \mathfrak{B} \cap \mathfrak{X} \rangle$.

Proposition 4.2.10: $\mathfrak{X} = \langle A_F(\epsilon) | \mathfrak{B} \cap \mathfrak{X} \rangle$ for every map $F(\epsilon) \in \mathbf{F}(\mathfrak{X})$. where $\mathbf{F}(\mathfrak{X})$ represents the family of feedback matrices $F(\epsilon)$ such that \mathfrak{X} is $A_F(\epsilon)$ -invariant.

Proof: The proof of Proposition 5.3 of Wonham [10] applies exactly.

Let us examine the properties of the subspaces that

$(A(\epsilon), B(\epsilon))$ -invariant (controllability) $R(\epsilon)$ -subspaces converge to:

Proposition 4.2.11: Given a pair (A, B) let $\mathcal{V}_\epsilon \in \underline{V}_\epsilon$ and $\mathcal{V}_\epsilon \xrightarrow{\epsilon \rightarrow 0} \mathcal{V}_n$, then $\forall x_0 \in \mathcal{V}_n$ and $\delta_0, \delta_1 > 0$, \exists an input function $u(t)$ s.t. $x_u(t)$, the trajectory defined by $u(t)$ and an initial condition $x(0)$ where $\|x(0) - x_0\| < \delta_0$, satisfies $d(x(t), \mathcal{V}_n) < \delta_1 \|x(t)\| \forall t > 0$, where $d(x(t), \mathcal{V}_n) = \inf_{x' \in \mathcal{V}_n} \|x(t) - x'\|$.

Proof: Since $\mathcal{V}_\epsilon \in \underline{V}_\epsilon \forall x_0(\epsilon) \in \mathcal{V}_\epsilon \exists$ an input function $u(t)$ s.t. $x_\epsilon(t)$, the trajectory defined by $u(t)$ and the initial condition $x_0(\epsilon)$, satisfies $x_\epsilon(t) \in \mathcal{V}_\epsilon \forall t > 0$. Since $\mathcal{V}_\epsilon \xrightarrow{\epsilon \rightarrow 0} \mathcal{V}_n$, $\exists Q(\epsilon)$ over $R[[\epsilon]]$ s.t. $Q(0) = I$ and $x_\epsilon(t) = Q(\epsilon)x_a(t) \forall t \geq 0$ where $x_a(t) \in \mathcal{V}_n$. Thus, $\|x_\epsilon(t) - x_a(t)\| \leq \|I - Q^{-1}(\epsilon)\| \|x_\epsilon(t)\| \leq \rho_1 \|x_\epsilon(t)\| \forall t \geq 0$, where $\rho_1 = \|I - Q^{-1}(\epsilon)\|$. Consider $x_0(\epsilon)$ that are also over $R[[\epsilon]]$. Thus, pick ϵ s.t. $\rho_1 < \delta_1$ and $\rho_0 = \rho_1 \|x_0(\epsilon)\| < \delta_0$ and the proof is complete.

Proposition 4.2.12: Given a pair (A, B) let $\mathcal{R}_\epsilon \in \underline{R}_\epsilon$ and $\mathcal{R}_\epsilon \xrightarrow{\epsilon \rightarrow 0} \mathcal{R}_n$, then $\forall x_0, x_\tau \in \mathcal{R}_n \exists \tau > 0$ s.t. $\forall \delta_0, \delta_1, \delta_2 > 0$, \exists an input function $u(t)$ s.t. $x_u(t)$, the trajectory defined by $u(t)$ and an initial condition $x(0)$ where $\|x(0) - x_0\| < \delta_0$, satisfies $d(x(t), \mathcal{R}_n) < \delta_1 \|x(t)\| \forall t > 0$ and $\|x(\tau) - x_\tau\| < \delta_2$.

Proof: Same as the previous proof except that for the target state pick $x(\tau) \in \mathcal{R}_\epsilon$ s.t. $x(\tau)|_{\epsilon=0} = x_\tau$. Also, pick ϵ s.t. $\rho_2 = \rho_1 \|x(\tau)\| < \delta_2$.

In other words, the angle between the trajectories and \mathcal{V}_n or \mathcal{R}_n can be made arbitrarily small. It turns out that \mathcal{R}_n is in fact almost (A,B)-invariant. The following lemma and proposition show this:

Lemma 4.2.13: Given a pair (A,B) let $\mathcal{R}_\epsilon \in \underline{R}_\epsilon$ and $\mathcal{R}_\epsilon \xrightarrow{\epsilon \rightarrow 0} \mathcal{R}_n$, then $\forall x_0$ s.t. $d(x_0, \mathcal{R}_n)$ is $O(\epsilon)$ and $\forall \tau > 0$, \exists an input function $u(t)$ s.t. $d(x_u(t, \epsilon), \mathcal{R}_n)$ is $O(\epsilon)$ for $0 < t \leq \tau$ where $x_u(t, \epsilon)$ is the trajectory defined by $u(t)$ and the initial condition x_0 .

Proof: Here we first need to find a trajectory which is $O(1)$ for $0 < t < \tau$. Find $F(\epsilon)$ s.t. \mathcal{R}_ϵ is $A_F(\epsilon)$ -invariant and the eigenvalues of $A_F(\epsilon)$ corresponding to \mathcal{R}_ϵ are all $O(1)$ and asymptotically stable. Then $\forall x_1 \in \mathcal{R}_\epsilon$, $x_1(t, \epsilon) \in \mathcal{R}_\epsilon \forall t > 0$ where $x_1(t, \epsilon)$ is the trajectory defined by the initial condition x_1 and no input. Since the eigenvalues of $A_F(\epsilon)$ corresponding to \mathcal{R}_ϵ are all $O(1)$ and stable, $x_1(t, \epsilon)$ is also $O(1)$. Therefore, $d(x_1(t, \epsilon), \mathcal{R}_n)$ is $O(\epsilon)$. Consider $x_2(t, \epsilon)$, the trajectory defined by the initial condition $x_2 = x_0 - x_1$. Note that x_2 is $O(\epsilon)$. Since all the eigenvalues of $A_F(\epsilon)$ are $O(1)$, $\forall \tau > 0$ $x_2(t, \epsilon)$ is $O(\epsilon)$ for $0 < t \leq \tau_1$. Therefore, $d(x_u(t, \epsilon), \mathcal{R}_n)$ is $O(\epsilon)$ for $0 < t \leq \tau$.

Proposition 4.2.14: Given a pair (A,B) let $\mathcal{R}_\epsilon \in \underline{R}_\epsilon$ and $\mathcal{R}_\epsilon \xrightarrow{\epsilon \rightarrow 0} \mathcal{R}_n$, then $\forall x_0 \in \mathcal{R}_n$ and $\delta > 0$, \exists an input function $u(t)$ s.t. $d(x_u(t), \mathcal{R}_n) < \delta$ for all t where $x_u(t)$ is the trajectory defined by $u(t)$ and the initial condition x_0 .

Proof: Pick some $\tau > 0$ and apply Lemma 4.2.13. Thus, $\exists u(t, \epsilon)$ s.t. $d(x(t, \epsilon), \mathcal{X}_n)$ is $O(\epsilon)$ for $0 < t \leq \tau$. Then $\exists \epsilon_0 > 0$ s.t. $d(x(t, \epsilon), \mathcal{X}_n) < \delta$ for $0 < t \leq \tau$ and $\forall \epsilon \leq \epsilon_0$. Use $x(\tau, \epsilon)$ as the initial condition to reapply Lemma 4.2.13 for the interval $\tau < t \leq 2\tau$. Find $\epsilon_1 > 0$ s.t. $\epsilon_1 \leq \epsilon_0$ and $d(x(t, \epsilon_1), \mathcal{X}_n) < \delta$ for $\tau < t \leq 2\tau$. Repeated use of Lemma 4.2.13 achieves the desired result.

In this section, we examined the notions of almost (A,B)-invariant and almost (A,B)-controllability subspaces in the framework that we have developed in this thesis. We outlined a method for calculating inputs that steer trajectories arbitrarily close to almost (A,B)-invariant subspaces or equivalently force the eigenvalues corresponding to sliding parts of almost (A,B)-controllability subspaces to approach $-\infty$. We also analyzed the properties of limits of elements in \underline{V}_ϵ and \underline{R}_ϵ as $\epsilon \downarrow 0$ from a trajectory point of view.

V CONCLUSIONS AND FURTHER RESEARCH

V.1 CONCLUSIONS

In this thesis, we have developed an algebraic approach to high gain controls for linear dynamic systems with varying orders of reachability. Based on this approach, we addressed the issues of high gain inputs for reaching target states, high gain feedback for pole placement and high gain inputs for steering trajectories arbitrarily close to almost (A,B) -invariant subspaces and almost (A,B) -controllability subspaces.

The systems of interest are defined over $\mathbb{R}((\epsilon))$. Since the eigenvalues of $A(\epsilon)$ are not necessarily over $\mathbb{R}((\epsilon))$ it is important to note that eigenvalue dependent results for systems over \mathbb{R} are not applicable in the domain of this thesis. Also, for most of the results, $A(\epsilon)$ is restricted to have $O(1)$ eigenvalues. This restriction is reasonable for continuous time systems since it can be achieved by a simple time scale change forcing the fastest time scale to be $O(1)$.

Chapter II analyzes discrete time systems. Target states are classified by the orders of magnitude of inputs necessary to reach

them. The ϵ^j -reachability submodules formed by these target states are shown to be uniquely determined by the invariant factors of the reachability matrix. A standard form which exposes these submodules is defined via the Smith decomposition of the reachability matrix. A numerically stable algorithm is given to recover the ϵ^j -reachability indices of systems over $\mathbb{R}[[\epsilon]]$. This algorithm also recovers a standard form and a Smith decomposition of the reachability matrix but these operations are not necessarily stable. A concept of near reachability which allows target states to be reached up to $O(\epsilon)$ is also introduced in this chapter.

Chapter III showed that the orders of feedback gains for placement of $O(1)$ eigenvalues are directly related to orders of reachability for systems over $\mathbb{R}[[\epsilon]]$. A stable algorithm to find this feedback is presented. This algorithm was applied to a fifth order system over \mathbb{R} with one input. The system was first parametrized by ϵ . The feedback gain to place $O(1)$ eigenvalues calculated for the parametrized system was evaluated at the specific value of ϵ corresponding to the original system. This approach produced far better numerical results than calculating the feedback directly for the given system.

Chapter IV showed that the counterparts of discrete time results also hold for continuous time in the case of proper systems (i.e. systems where all eigenvalues of $A(\epsilon)$ are $O(1)$).

Finally, $(A(\epsilon), B(\epsilon))$ -invariant $\mathbb{R}((\epsilon))$ -subspaces and $(A(\epsilon), B(\epsilon))$ -controllability $\mathbb{R}((\epsilon))$ -subspaces were analyzed to provide insight into almost (A, B) -invariant subspaces and almost (A, B) -controllability subspaces. It was then easier to explicitly construct sequences of (A, B) -invariant subspaces that converge to almost (A, B) -invariant subspaces or almost (A, B) -controllability subspaces. Using this, a counterexample to a result of Willems was generated. Also, a procedure was presented to construct desired inputs for almost (A, B) -invariant subspaces and almost (A, B) -controllability subspaces using the above sequences.

V.2 SUGGESTIONS FOR FURTHER RESEARCH

1. Algorithm 2.3.15 is only valid for systems over $\mathbb{R}[[\epsilon]]$. A state coupled to the input by $O(\epsilon)$ may be coupled to another state by $O(1/\epsilon)$. Thus, the first state would be a part of the ϵ -reachable subsystem, whereas the second is a part of the ϵ^0 -reachable subsystem as in the following example:

Example 5.2.1: Let $A(\epsilon) = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1/\epsilon \\ \hline \epsilon & 0 & 0 \end{array} \right]$, $B(\epsilon) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. This is a proper system in standard form. Here calculation of the ϵ^0 -reachability index requires the knowledge of ϵ -reachable part of the system.

Further research is needed to extend this algorithm to the more general case of proper systems.

2. With regard to pole placement results, further research could be in three different directions:

i. Analyzing orders of feedback gains for shifting eigenvalues by $O(1)$ for the more general case of proper systems. Intuitively, if a mode is ϵ -reachable but " $1/\epsilon$ -observable", in that it has a $1/\epsilon$ coupling to other states, then it would be possible to shift its eigenvalue by $O(1)$ using $O(1)$ feedback gain. This suggests considering the Smith decomposition of the Hankel matrix

$O(\epsilon)\mathcal{Q}(\epsilon)$ where $O(\epsilon)$ is the observability matrix with observation matrix $C(\epsilon) = I$. Recall that Example 3.1.6 was an example of the above situation. Examination of the Smith decomposition of its

Hankel matrix would yield $D(\epsilon) = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$. The above argument

suggests that it should be possible to shift eigenvalues by $O(1)$ using $O(1)$ feedback gains. This conclusion is also consistent with the feedback calculations on this example in Chapter III.

ii. Changing the dynamics of a given continuous time system with multiple time scales without changing its time scale structure. This would involve arbitrarily shifting $O(\epsilon^j)$ eigenvalues by $O(\epsilon^j)$.

iii. Parametrizing systems over \mathbb{R} . Two heuristic methods could be suggested for this; one is to recognize small entries in the matrix, either isolated or added to another entry, and replace these with powers of ϵ . This could also be done by assuming $\epsilon = 10^{-k}$ for some appropriate positive integer k (This was the

method used in Example 3.3.3). Another method for parametrization could come about by considering the singular value decomposition of the reachability matrix. For weakly reachable systems, the reachability matrix would have small singular values. The choice of ϵ would involve recognizing groupings in singular values and representing them by powers of ϵ .

iv. Computer programs for implementing the algorithms given in this thesis: This would involve writing subroutines for implementing various linear algebra operations on matrices with polynomial entries, using for example Matlab as a basis.

3. The results of Section 4.2 could be extended to the results of Willems on distributionally (A,B) -invariant and (A,B) -controllability subspaces. This seems to be possible by first constructing a sampled version of a given continuous time system with sampling interval ϵ , generating the discrete time model:

$$x[k+1] = A_d(\epsilon)x[k] + B_d(\epsilon)u[k]$$

where $A_d(\epsilon) = e^{\epsilon A}$ and $B_d(\epsilon) = \int_0^\epsilon e^{(\epsilon-s)A} B ds$. This could also relate Willems' results to Schumacher's [19] on discrete time. This discrete time model along with Schumacher's results could be used to approximate distributional inputs in continuous time with piecewise constant inputs. It would then be possible to extend our results to the work on singular optimal control by Willems et. al. [20]. A first order approximation of the above model,

$A_d(\epsilon) = I + \epsilon A$ and $B_d(\epsilon) = \epsilon B$, should also be considered as this may be sufficient for our purposes.

4. Observations

$$y[k] = C(\epsilon)x[k] \text{ or } y(t) = C(\epsilon)x(t)$$

could be included in our model to analyze the input-output behavior of MIMO systems. Orders of reachability ideas could be adapted to observability analysis, and this would then lead to research on connections to optimal control, realization theory and in particular to the work on balanced realizations.

The following example illustrates that optimal control ideas could also be used to find inputs for almost (A,B) -invariant subspaces:

Example 5.2.1: Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Recall that $\text{Im} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an almost (A,B) -invariant subspace. We therefore need a (possibly high gain) feedback matrix that would keep the trajectory arbitrarily close to this subspace. Thus, we could formulate this as an optimal control problem to minimize:

$$\int_0^{\infty} y^T(t)y(t) + u^T(t)R(\epsilon)u(t) dt$$

where $C(\epsilon) = [0 \ 1/\epsilon]$ and $R(\epsilon) = \epsilon^2$. Using the solution to the algebraic Riccati equation we would arrive at the optimal feedback $f^*(\epsilon) = [-\sqrt{2}/\epsilon \ -1/\epsilon^2]$. It turns out that $f^*(\epsilon)$ is a choice of feedback that would make the $(A(\epsilon), B(\epsilon))$ -invariant $R(\epsilon)$ -subspace, $\begin{bmatrix} 1 \\ (-\sqrt{2} - \sqrt{6})\epsilon/2 \end{bmatrix}$, an $A_{f^*(\epsilon)}$ -invariant $R(\epsilon)$ -subspace. Note that this $R(\epsilon)$ -subspace converges to the

almost (A,B) -invariant subspace and both of the eigenvalues approach $-\infty$ as $\epsilon \downarrow 0$. In general, this procedure would pick $C(\epsilon)$ as a basis, for the left nullspace of a given almost (A,B) -invariant subspace, and scale it by ϵ . $R(\epsilon)$ would be picked as $\epsilon^{2k}I$ for some integer k , possibly equal to the dimension of the supremal sliding subspace in that almost (A,B) -invariant subspace (or dimension of the almost (A,B) -invariant subspace minus the dimension of the supremal (A,B) -invariant subspace in that almost (A,B) -invariant subspace).

All of this suggests that the theory developed in this thesis is a good basis for dealing with high gain feedback and/or cheap control problems.

APPENDIX

A.1 SMITH DECOMPOSITION

An $m \times n$ (assume $m \leq n$) matrix $A(\epsilon)$ with a Laurent expansion in ϵ which starts at a finite negative power of ϵ , has a Smith decomposition [11,12]

$$A(\epsilon) = P(\epsilon)D(\epsilon)Q(\epsilon)$$

where $P(\epsilon)$ and $Q(\epsilon)$ are over $\mathbb{R}[[\epsilon]]$ and $P(\epsilon)$ is $m \times m$ and invertible at $\epsilon = 0$, $Q(\epsilon)$ is $m \times n$ and full row rank at $\epsilon = 0$. Also,

$$D(\epsilon) = \text{diag}\{(1/\epsilon^j)I \dots I, \epsilon I, \dots \epsilon^k I, 0\}$$

is $m \times m$. The dimensions of the identity matrices, above, are p_i , where i is the corresponding power of ϵ . We term p_i the indices of $A(\epsilon)$. If the rank of $A(\epsilon)$ is r for $0 < \epsilon < a$ for some $a \in \mathbb{R}^+$, then $D(\epsilon)$ has $m-r$ zeroes on the diagonal. Furthermore, if $A(\epsilon)$ is analytic at $\epsilon=0$, then only nonnegative powers of ϵ appear in $D(\epsilon)$.

A.2 COMPUTATION OF A SMITH DECOMPOSITION OF $X(\epsilon)Y(\epsilon)$
 FROM SMITH DECOMPOSITIONS OF $X(\epsilon)$ AND $Y(\epsilon)$

Let $X(\epsilon) = P_x(\epsilon)D_x(\epsilon)Q_x(\epsilon)$ and $Y(\epsilon) = P_y(\epsilon)D_y(\epsilon)Q_y(\epsilon)$ be Smith decompositions of $X(\epsilon)$ and $Y(\epsilon)$. Assume for simplicity that both are unimodular. Let $Q_x(\epsilon)P_y(\epsilon) = \Pi L(\epsilon)U(\epsilon)$ be a ΠLU decomposition [22] for $Q_x(\epsilon)P_y(\epsilon)$. If $\Pi D_x(\epsilon) = D_x(\epsilon)\Pi$ then $X(\epsilon)Y(\epsilon) = P_x(\epsilon)\Pi\bar{L}(\epsilon)D_x(\epsilon)D_y(\epsilon)\bar{U}(\epsilon)Q_y(\epsilon)$ where $\bar{L}(\epsilon) = D_x(\epsilon)L(\epsilon)D_x^{-1}(\epsilon)$ and $\bar{U}(\epsilon) = D_y^{-1}(\epsilon)U(\epsilon)D_y(\epsilon)$ are both unimodular since $L(\epsilon)$ is lower triangular and $U(\epsilon)$ is upper triangular. Thus, we get a Smith decomposition

$$X(\epsilon)Y(\epsilon) = P_{xy}(\epsilon)D_{xy}(\epsilon)Q_{xy}(\epsilon) \text{ where } P_{xy}(\epsilon) = P_x(\epsilon)\Pi\bar{L}(\epsilon), \\ Q_{xy}(\epsilon) = \bar{U}(\epsilon)Q_y(\epsilon) \text{ and } D_{xy}(\epsilon) = D_x(\epsilon)D_y(\epsilon).$$

Note that the above ΠLU decomposition is computationally expensive since it involves polynomial matrix manipulations. On the other hand if only the indices of $D_{xy}(\epsilon)$ are desired then it is sufficient to only consider ΠLU decomposition of $Q_x(0)P_y(0) = \Pi LU$ and test if $\Pi D_x(\epsilon) = D_x(\epsilon)\Pi$. If so, then $D_{xy}(\epsilon) = D_x(\epsilon)D_y(\epsilon)$.

If $X(\epsilon)$ and $Y(\epsilon)$ are not both square and/or full row/column rank at $\epsilon=0$, then it is possible to derive simple generalizations of the above.

A.3 ϵ^i -NEARLY ϵ^j -REACHABILITY

If it is desired to reach a target state by $O(\epsilon^i)$ accuracy, then the following definition could be used to accomodate for his:

Definition A.3.1: $x \in \mathbb{R}^n$ is ϵ^i -nearly (exactly) ϵ^j -reachable if $\exists x(\epsilon) \in \mathcal{X}^j$ such that $x(\epsilon) - x$ is $O(\epsilon^i)$ ($x(\epsilon) - x = 0$).

Let $\mathcal{V}^{i,j}$ ($\mathcal{V}^{\infty,j}$) be the set of all ϵ^i -nearly (exactly) ϵ^j -reachable states. Then, $\mathcal{V}^{\infty,j} \subset \dots \subset \mathcal{V}^{s_j+1,j} \subset \mathcal{V}^{r_j,j} = \mathcal{V}^j$ for some $r_j \in \mathbb{I}^+$ and $\mathcal{V}^{i,j}$ is a subspace of \mathbb{R}^n . We term $\mathcal{V}^{i,j}$ ($\mathcal{V}^{\infty,j}$) the ϵ^i -nearly (exactly) ϵ^j -reachable subspace.

As an immediate consequence of this definition, $\mathcal{V}^{\infty,j} \subset \mathcal{X}^j$ and $\mathcal{V}^j = \mathcal{V}^{\infty,j}$ iff $\mathcal{V}^j \subset \mathcal{X}^j$. Furthermore, for an ϵ^k -reachable system, using Equation 2.2.5 we get $\mathcal{V}^{k-j,j} = \mathcal{V}^{k-j+1,j} = \dots = \mathcal{V}^{\infty,j}$.

An extension of Proposition 2.4.3 is the following:

Proposition A.3.2: $x \in \mathbb{R}^n$ is ϵ^i -nearly (exactly) ϵ^j -reachable iff $\exists V(\epsilon)$ such that $V(0) = I$, $V(\epsilon) - I$ is $O(\epsilon^i)$ ($V(\epsilon) - I = 0$) and $V(\epsilon)x \in \mathcal{X}^j$.

In addition, we have the following lattice structure relating these subspaces:

$$\begin{array}{l}
 \mathcal{V}^{\infty,0} \subset \dots \subset \mathcal{V}^{1,0} = \mathcal{V}^0 = \mathcal{X}^0|_{\epsilon=0} \\
 \mathcal{V}^{\infty,1} \subset \dots \subset \mathcal{V}^{1,1} = \mathcal{V}^1 = \mathcal{X}^1|_{\epsilon=0} \\
 \mathcal{V}^{\infty,k} \subset \dots \subset \mathcal{V}^{1,k} = \mathcal{V}^k = \mathcal{X}^k|_{\epsilon=0}
 \end{array}$$

A.4 INVERSE OF PASCAL'S TRIANGLE

Recall that

$$P(\epsilon) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\epsilon & 1 & 0 & 0 \\ \epsilon^2 & -2\epsilon & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{n-1} \epsilon^{n-1} & (-1)^{n-2} \binom{n-1}{1} \epsilon^{n-2} & (-1)^{n-3} \binom{n-1}{2} \epsilon^{n-3} & \dots & 1 \end{bmatrix}$$

from Equation 4.2.2 and

$$P^{-1}(\epsilon) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \epsilon & 1 & 0 & 0 \\ \epsilon^2 & 2\epsilon & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \epsilon^{n-1} & \binom{n-1}{1} \epsilon^{n-2} & \binom{n-1}{2} \epsilon^{n-3} & \dots & 1 \end{bmatrix}$$

from Equation 4.2.6. Let us illustrate that $P^{-1}(\epsilon)$ is indeed the inverse of $P(\epsilon)$: Clearly, $P(\epsilon)P^{-1}(\epsilon)$ is lower diagonal with ones on its diagonal. Note that the lower diagonal entry at row r and column c , where $r > c$, is:

$$p(r, c) = \sum_{i=c}^r \binom{r-1}{i-1} \epsilon^{r-i} (-1)^{i-c} \binom{i-1}{c-1} \epsilon^{i-c} = \epsilon^{r-c} \binom{r-1}{c-1} \sum_{i=c}^r (-1)^{i-c} \binom{r-c}{i-c}$$

Let $j=i-c$, then

$$p(r, c) = \epsilon^{r-c} \binom{r-1}{c-1} \sum_{j=0}^{r-c} (-1)^j \binom{r-c}{j} = \epsilon^{r-c} \binom{r-1}{c-1} (1-1)^{r-c} = 0$$

by a summation formula in [21].

An interesting by-product of this is the following summation formula. Using Equations 4.2.5 and 4.2.7 we get:

$$\sum_{j=0}^{r-c} (-1)^{r-c} \binom{i}{r} \binom{r-1}{c-1} = 1, \text{ for all } c \leq r \text{ and } i \geq r.$$

This summation neither existed among nor could be calculated from the summation formulas in [21], to the author's best knowledge.

A.5 COUNTER EXAMPLE TO WILLEMS' CLOSURE THEOREM
FOR ALMOST (A,B) INVARIANT SUBSPACES

Theorem 6 of J.C. Willems [3] is stated as follows:

Theorem 6: $\{V_a \in \underline{V}_a\} \leftrightarrow \{\exists V_\epsilon \in \underline{V}$ such that $V_\epsilon \xrightarrow{\epsilon \rightarrow 0} V_a$ in the Grassmanian sense $\}$.

We believe that the following example is a counter example to the reverse direction (\leftarrow) of the above result:

Example: Let $A = \begin{bmatrix} 0_3 & 0_3 \\ I_3 & 0_3 \end{bmatrix}$ and $B = \begin{bmatrix} I_3 \\ 0_3 \end{bmatrix}$. Consider the family of

(A,B) invariant subspaces $\{V_\epsilon\}$ generated by $\begin{bmatrix} 1 \\ 0 \\ 0 \\ \epsilon \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

$V_\epsilon \xrightarrow{\epsilon \rightarrow 0} L$ where L is spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, but L is not an

almost (A,B) invariant subspace. This can be easily checked via the algorithms ISA and ACSA [1].

In his proof, Willems shows that $B' \cap L(\text{mod } R_{a,L}^*) = \{0\}$, but in order to utilize Lemma 2, he should also show that $L_\epsilon \in \underline{V}^{\tilde{\Sigma}(A', B')}$ (where $L_\epsilon \subset V_\epsilon$ and $L_\epsilon \xrightarrow{\epsilon \rightarrow 0} L$). Since he does not illustrate this Lemma 2 cannot be used as claimed.

On the other hand, it is straightforward to illustrate via

the above example that $L(\text{mod}R_{a,L}^*) \notin \underline{V}^{\Sigma(\text{mod}R_{a,L}^*)}$ (where as Willems erroneously deduces that $L(\text{mod}R_{a,L}^*) \in \underline{V}^{\Sigma(\text{mod}R_{a,L}^*)}$ from the above claim to finally arrive at the result) as follows:

For initial conditons in $\text{Im} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, there will be trajectories in

the form of $\begin{bmatrix} * \\ * \\ * \\ * \\ a \\ * \end{bmatrix}$ where * denotes don't care and a is some nonzero

constant. The component of the trajectory corresponding to a is not factored out by $R_{a,L}^*$. Also, this component is neither in

$L(\text{mod}R_{a,L}^*)$ nor in \mathcal{B} . Therefore, $L(\text{mod}R_{a,L}^*) \notin \underline{V}^{\Sigma(\text{mod}R_{a,L}^*)}$

It is also worth noting that only the forward direction (\rightarrow) of the above theorem has been proved by H. L. Trentelman [18] and no comment on the reverse direction has been made to the best of our knowledge.

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