ON ROBUST CONTROL DESIGNS FOR INFINITE DIMENSIONAL SYSTEMS

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Abstract

This thesis deals with the robustness of stability of distributed, linear-time-invariant (DLTI) feedback control systems. The main goal is to formulate a practical method for evaluating feedback designs based on the actual DLTI system characteristics. As a result, a design procedure can be developed for DLTI systems to synthesize feedback controllers that are guaranteed to be closed-loop stable. We have developed a robustness characterization for DLTI systems, and have shown that linear quadratic (LQ) optimal control systems have nice robustness properties and can serve as good reference designs for the actual implementation of the feedback controller. We have studied in detail linear hereditary differential systems and a vibration suppression problem for a flexible beam. We stress the study of implementable controllers, which are finite dimensional, in contrast to optimal controllers that are typically infinite-dimensional. However, one can integrate our multivariable robustness results with the LQ optimal control to derive a finite-dimensional suboptimal control law which is closed-loop stable. We show how this can be done by using spatially-sampled measurements along the flexible beam. Also we have demonstrated by using our robustness results that the inherent damping in the flexible structure plays a vital role in determining whether a physically realizable closed-loop stabilizing controller, based on spatially-sampled measurements, can be achieved.

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INTRODUCTION

1.1 An Overview and Motivation

This thesis deals with the robustness of stability of distributed, linear-time-invariant (DLTI) feedback control systems. The main goal is to formulate a practical method for evaluating feedback designs based on the actual DLTI system characteristics. As a result, a design procedure can be developed for DLTI systems to synthesize feedback controllers that are guaranteed to be closed-loop stable.

This research is motivated by the fact that although infinite dimensional systems are commonly found in practice and it is widely acknowledged [60] that robustness considerations are of utmost importance in the implementation of feedback compensators for infinite dimensional systems, a formal treatment of the robustness issue has not been found in literature. In our research, we have developed a robustness characterization for DLTI systems, and have shown that linear quadratic (LQ) optimal control systems have nice robustness properties and can serve as good reference designs for the actual implementation of the feedback controller. The "goodness" of this reference is measured by the robustness of the optimal system. We stress in this thesis that the optimal controllers are in general infinite dimensional, and the control designer must achieve a design that is finite dimensional and physically realizable [1]. We use a flexible beam vibration suppression example to highlight this discussion.

Often in current control design practice for infinite dimensional systems, a reduced-order model (e.g. [57], [58], [59]) is first generated to approximate the system, and the control design is synthesized using this reduced-order model. These reduced-order design methods, however, do not always produce feedback control designs that are robust, i.e., remain stable in the presence of uncertainties. In fact, the resulting controllers are often unstable when evaluated with the true infinite dimensional plant.

Another shortcoming of reduced-order design methods is that they are all problem-dependent. There is no simple a priori rule to determine which model reduction method will produce the best result, or what degree of approximation is adequate. In particular, one is required to perform the model reduction on the open-loop system (such as selecting open-loop poles), but the validation of the procedure has to be checked on the resulting closed-loop design (where the poles have changed). Modeling errors resulting from the approximation, such as parameter variations, nonlinearities, neglected disturbances, and control and observation spillovers [2], [56] can all cause the system to become unstable [3], [4]. If they do, it becomes necessary to iterate the entire approach. A fresh approximate model is selected and the complete design and analysis process is repeated. In addition, the computational requirements may become stringent when a larger-order model is used.

It is hoped that by using the optimal control as a reference design, the control designer can avoid the aforementioned limitations of the reduced-order design method. That is, they can start with an infinite-dimensional structural model, develop an exact optimal solution, which is robust, and use the robustness properties of the optimal solution to facilitate the actual (finite dimensional) implementation.

We take a control design approach that has found a great deal of success in dealing with finite dimensional systems [5], [6]. This approach integrates the multivariable robustness theory with the LQ control theory. The crucial difference between the finite dimensional study and ours is that in general one cannot use the optimal design directly in the infinite dimensional system. Usually a suboptimal scheme is required for the implementation. This additional design consideration underscores the importance of a robustness study for infinite dimensional systems. It motivated us to investigate the significance of the LQ robustness results in the synthesis of a closed-loop stable and implementable suboptimal control design.

LQ optimal control [54], [55] has long been a popular method with feedback control design engineers due to its conceptual simplicity and ease of computation. However it was recognized just over ten years ago that optimal control is merely a convenient tool to synthesize a design. In order for a control design to be called truly successful, one must do a commendable job in trading off the system design performance, robustness, sensitivity and other system characteristics. These trade-offs are easier to accomplish in the frequency domain. In recent years we have witnessed the development of many practical robustness results for multivariable finite dimensional systems, including the guaranteed stability margins of LQ systems [7], [8], [9], the introduction of the singular values of the return difference of a feedback system as a robustness measure [10], [5], and the loop transfer recovery technique [3], [6]. Consequently, a much more unified treatment for multi-input-multi-output control design methods has been made possible. These methods generally fall into two categories: (1) the full-state feedback type [11], [12], [13]; and (2) the observer type, i.e. the linear-quadratic-guassian/loop-transfer-recovery approach [6], [7], [14]. Although this theory is by no means complete, much success in actual applications and feasibility studies have been reported [15], [16], [17], [18].

We believe that by following a similar development, we have obtained a framework for dealing with control designs for infinite dimensional systems that is equally effective as that for finite dimensional systems.

1.2 A Summary of the Results

Two important examples of infinite dimensional systems are systems with time delays and large flexible structures. They are used in this thesis to develop the concepts behind our research. In particular, instead of beginning with an abstract framework of describing infinite dimensional systems using semigroups, we start by considering a general class of time-delay systems called the linear hereditary differential system [19]. We show that by using only elementary complex function theory, a very general multivariable robustness characterization, using singular values of the return difference, of this class of time-delay systems can be obtained. Based on this characterization, we present an extension of the well known [7] [8] robustness results of finite dimensional LQ control systems.

It has been shown in one form or another [20], [21], [7] that the LQ approach, when applied to finite dimensional systems:

(1) produces a closed-loop system that satisfies an optimal frequency domain condition which is commonly referred to as the Kalman frequency domain inequality;

(2) guarantees that the closed-loop control system has at least 60° phase margins and infinite gain margins simultaneously in all control input channels when a diagonal control weighting matrix is used.

The robustness properties in (2) are direct consequences of property (1). In this thesis, both of these results (1) and (2) are extended to include all linear hereditary differential systems. The results are derived directly in the frequency domain.

A natural question to raise here is "how generally can one extend the results (1) and (2) above?" We provide an answer to this question by combining the Yakubovich frequency domain theorem [22], [23], [26], which is the most general (infinite dimensional) version of the Kalman frequency domain equality known to date, and Desoer and Wang's generalized Nyquist thoerem [24] to establish that for linear systems described as a semigroup over a Hilbert space, (1) and (2) above are true if

(a) the control space is finite dimensional; and

(b) the loop transfer function belongs to a class of transfer functions which have finitely many poles in the closed right-half plane (CRHP).

Their result allows us to generalize the singular value robustness characterization to DLTI systems.

In this thesis we also study the infinite dimensional Lyapunov control approach. The finite dimensional Lyapunov control was investigated by Wong [12], and later by Lehtomaki [5]. This kind of design is a subclass of the LQ systems. It has the additional property that the optimal control system is guaranteed to have at least 90° phase margins and 100% gain reduction margins. These superior stability margins are especially useful because they provide protection against instability due to actuator failures.

We employ a flexible beam, simply supported at both ends, to illustrate the use of the Lyapunov control methodology to synthesize a closed-loop stable control design for the vibration suppression problem. We demonstrate by using our robustness results that the inherent damping in the large flexible structure plays a vital role in determining whether or not a physically realizable closed-loop stable suboptimal (finite-dimensional) controller can be achieved by spatial sampling. Specifically we show that we can design such a controller if the modes of the structure has a constant damping ratio. The number of spatial samples required to stabilize the system increases as this damping ratio decreases. If instead, the damping coefficient is constant, then a dynamical compensator will be required in addition to increasing the number of spatial samples to achieve stability.

Our approach focuses on the loop characteristics. Quite often researchers (e.g. [27]) in the infinite dimensional system control area equate the implementability of the controller with whether the controller can be represented as a compact operator. This is not adequate because in order to realize the basis for that operator, a certain basis-building device must be implemented. This device may be unrealizable. Indeed it may actually be noncausal. For example, in large flexible structures that use modal feedback control [28] even though it can be shown that a compact optimal solution (for the modal basis) exists and the optimal system is closed-loop robust, a closed-loop stable suboptimal implementation might not be possible because ideal band-pass filters are not available in practice.

1.3 Organization of this Thesis

The rest of this thesis is organized as follows.

Chapter 2 contains all results on the linear hereditary differential systems. We present a general robustness characterization, using singular values for this whole class of systems. We show that these systems have a finite number of open-loop poles in the CRHP. When restricted to the LQ control, the optimal system satisfies a Kalman frequency domain inequality. Hence a set of guaranteed stability margins can be readily derived for the feedback system. All results in this chapter contains the known [5] finite dimensional system results as a special case. Unlike the usual abstract treatment for linear hereditary differential systems (e.g. M^2 space and semigroup treatment in [29], [30]), we are able to obtain all results in this chapter using only elementary complex function theory.

Chapter 3 contains a generalization of the result in Chapter 2 to include a much wider class of DLTI systems, which have a finite-dimensional control space and a finite number of closed-loop poles in the CRHP. These results are obtained by formalizing the concepts behind the Yakubovich frequency domain theorem and the Desoer and Wang generalized Nyquist theorem. The Lyapunov control problem along with the associated robustness properties are introduced. As an illustration we study the Lyapunov control problem for the linear hereditary differential system. Also we show that one can use the Yakubovich theorem to rederive the Kalman inequality in Chapter 2.

In Chapter 4 we present an extensive discussion on the use of the Lyapunov control to suppress the vibration of a simply supported, uniform, flexible beam. Since the optimal solution requires the use of perfect measurements, all along the beam, suboptimal controls using spatial sampling are analyzed. Also we use this opportunity to study, using classical control arguments, the importance of inherent damping in a flexible structure. It is shown that if the inherent damping is sufficiently large, then the guaranteed stability margins of the optimal system allow us to synthesize a closed-loop stable suboptimal control.

Chapter 5 contains the conclusions and some suggestions for future research.

CHAPTER 2

ROBUSTNESS PROPERTIES OF LINEAR HEREDITARY DIFFERENTIAL SYSTEMS

2.1 Introduction

In this chapter we study the robustness properties of a class of time delay systems whose dynamics are described by a linear hereditary differential equation [19]. We present a characterization of robustness for this class of systems under feedback control. Also we demonstrate that an optimal frequency domain condition of the Kalman inequality type [20], [7] can be derived for linear quadratic control with infinite horizon [31]. This result provides us with a uniform lower bound on the minimum singular value of the optimal return difference matrix. This bound allows us to establish some guaranteed robustness properties for the optimal system in an efficient manner.

In their paper, Lehtomaki, Sandell and Athans [7] considered the LQ controller formulation for finite dimensional linear time invariant systems, and used the finite dimensional version of the multivariable Kalman inequality to study the robustness of the LQ feedback system. All results presented in this chapter contain the finite dimensional system as a special case.

This chapter is organized as follows. In Section 2.2 we derive the optimal frequency domain condition for the linear quadratic hereditary differential (LQHD) system. The optimal condition yields an inequality which we refer to as the Kalman frequency domain inequality (KFDI). In Section 2.3 we characterize the robustness of linear hereditary differential feedback (LHDF) systems. This characterization is based on a generalized Nyquist theorem for LHDF systems. In Section 2.4 we combine the results in Sections 2.2 and 2.3 to show that LQHD systems have good robustness properties. In particular, these systems have guaranteed 60° phase margins, infinite gain margins and 50% reduction gain margins simultaneously in all control channels. Section 2.5 contains a discussion about the robustness results, highlighted by two numerical examples.

2.2 Linear Quadratic Hereditary Differential Systems and the Multivariable Kalman Frequency Domain Inequality

We consider a general class of linear hereditary differential systems. This class of systems includes systems with time delays, of the retarded type [32]. It is well known that these systems require an infinite dimensional state space realization, i.e. they have infinitely many modes. Multiple discrete time delays as well as distributed time delays are allowed in this formulation.

A number of researchers have investigated the LQ problem for the linear hereditary differential system. They include Gibson [30], Delfour and Mitter [19], Delfour, McCalla and Mitter [31], Kwong and Willsky [33], [39] and Kwong [34], Vinter [35]. In their research, the optimal control solution as well as its closed-loop stability were established. However, none of these researchers have addressed the robustness problem.

We organize this section as follows. In subsection 2.2.1, we review the hereditary differential system and the solution to its LQ problem with infinite horizon. In subsection 2.2.2 we state the Kalman frequency domain inequality for this class of linear quadratic hereditary differential systems. We present the proof of this result in subsection 2.2.3.

2.2.1 Linear Quadratic Hereditary Differential System

We define the LQHD system by following [31]. Let $N \geq 1$ be an integer and $\tau > 0$, and also

$$-\tau = \theta_N < \ldots\ldots < \theta_1 < 0$$

Then we consider the following autonomous hereditary differential system

$$\frac{dx}{dt}(t) = Ax(t) + \sum_{i=1}^{N} A_i \begin{Bmatrix} x(t+\theta_i) & , \ t+\theta_i \geq 0 \\ 0 & , \ \text{otherwise} \end{Bmatrix}$$

$$+ \int_{-\tau}^{0} A_0(\theta) \begin{Bmatrix} x(t+\theta) & , \ t+\theta > 0 \\ 0 & , \ \text{otherwise} \end{Bmatrix} d\theta \qquad (2.1)$$

$$+ Bu(t)$$

and

$$x(\sigma) = h(\sigma),\ \sigma \in [-\tau,0],\ \text{is known,} \tag{2.2}$$

where $x(t)\in\mathbf{R}^n$, and $u(t)\in\mathbf{R}^m$, and A, A_i, $A_0(\theta)$ are matrices of dimension nxn, and B is an nxm matrix; $A_0(\theta)$ is bounded for $\theta\in[-\tau,0]$. Also we assume that the system is stabilizable, i.e. there exist matrices C_0 and $C_1(\theta)$, $\theta \in [-\tau, 0]$ such that the control law

$$u(t) = C_0x(t) + \int_{-\tau}^{0} C_1(\theta)x(t+\theta)d\theta$$

is closed-loop stable.

The quadratic cost criterion is given by

$$J(u) = \int_0^{\infty} [x'(t)Qx(t)+u'(t)Ru(t)]dt \tag{2.3}$$

where the weighting matrices Q and R are symmetric and positive definite. The system of equations from Eq. (2.1) to (2.3) is called the LQHD system.

The optimal control for the LQHD system is given in the feedback compensation form [31] by

$$u(t) = -R^{-1}B'K_0x(t) - \int_{-\tau}^{0} R^{-1}B'K_1(\theta)x(t+\theta)d\theta \tag{2.4}$$

where K_0, $K_1(\theta)$ satisfy the following set of equations,

$$A'K_0 + K_0BR^{-1}B'K_0 + Q + K_1'(0) + K_1(0) = 0 \tag{2.5}$$

$$K_0' = K_0 > 0 \tag{2.6}$$

$$\frac{d}{d\theta} K_1(\theta) = [A'-K_0BR^{-1}B']K_1(\theta) + \sum_{i=1}^{N-1} K_0A_i\delta(\theta-\theta_i) + K_0A_0(\theta) + K_2(0,\theta); \tag{2.7}$$

$$K_1(-\tau) = K_0A_N \tag{2.8}$$

where $\delta(\theta-\theta_i)$ is the Dirac delta function at $\theta=\theta_i$, and

$$K_2(0,\theta) = K_1'(-\theta-\tau)A_N - \int_{-\tau}^{\theta} K_1'(-\theta+\sigma)BR^{-1}B'K_1(\sigma)d\sigma$$

$$+ \sum_{i=1}^{N-1} \left\{ \begin{array}{ll} A_i'K_1(\theta+\theta_i) & , \ -\tau\leq\theta+\theta_i \\ 0 & , \ \text{otherwise} \end{array} \right\}$$

$$+ \sum_{i=1}^{N-1} \left\{ \begin{array}{ll} K_1'(-\theta+\theta_i)A_i & , \ \theta<\theta_i \\ 0 & , \ \text{otherwise} \end{array} \right\}$$

$$+ \int_{-\tau}^{\theta} \left\{ \begin{array}{ll} A_0'(\xi)K_1(\xi+\theta) & , \ \xi\geq-\theta-\tau \\ 0 & , \ \text{otherwise} \end{array} \right\} d\xi$$

$$+ \int_{-\tau}^{\theta} \begin{cases} K_1'(\xi-\theta)A_0(\xi) & , \quad \xi \geq \theta-\tau \\ 0 & , \quad \text{otherwise} \end{cases} d\xi \; . \tag{2.9}$$

The optimal LQHD system is closed-loop stable [31]. Thus it makes sense to talk about the robustness of LQHD systems.

Before we turn to give a statement of the optimality condition in the frequency domain, we remark that by following [31] we have assumed $Q>0$ in the system. Partial work in relaxing this assumption for the LQ problem to the observability condition is given in [34]. The results in this thesis can be adapted to such cases with only minimal changes.

2.2.2 Kalman Inequality for LQHD Systems - Statement

The LQHD system is depicted in its feedback form in Figure 2.1. In the figure, $F(s)$ and $H(s)$ denote the open-loop and feedback compensator transfer function matrices, respectively. From Eq. (2.1), one obtains

$$F(s) = [sI - A - \sum_{i=1}^{N} e^{s\theta_i} A_i - A_0(s)]^{-1} B \tag{2.10}$$

where

$$A_0(s) = \int_{-\tau}^{0} A_0(\sigma) e^{s\sigma} d\sigma \; . \tag{2.11}$$

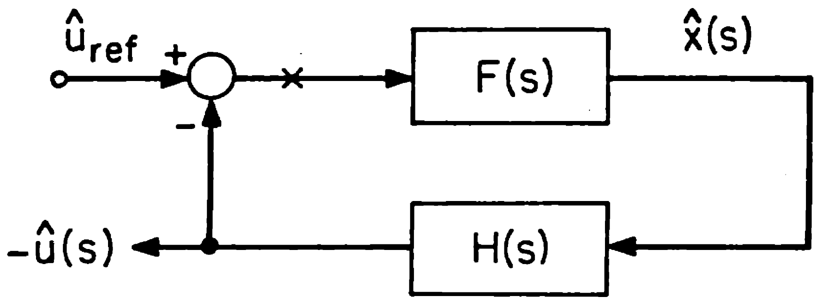


Figure 2.1 LQHD Feedback System

Similarly, from Eq. (2.4),

$$H(s) = R^{-1}B'K_0 + R^{-1}B'\mathcal{K}_1(s) \tag{2.12}$$

where

$$\mathcal{K}_1(s) = \int_{-\tau}^{0} K_1(\theta)e^{s\theta}d\theta \ . \tag{2.13}$$

Therefore, the loop transfer function G(s), with the loop broken at the control channel, i.e., the point X in Figure 2.1 is

$$G(s) = H(s)F(s) = R^{-1}B'[K_0 + \mathcal{K}_1(s)][sI - A - \sum_{i=1}^{N} e^{s\theta_i}A_i - \mathcal{A}_0(s)]^{-1}B. \tag{2.14}$$

Figure 2.2 illustrates the Nyquist Contour D_r of radius r. It is a closed contour constructed such that it encloses all the open-loop poles of F(s) in the closed right-half plane. Indentation of radius 1/r is used to avoid any such pole on the imaginary axis. We shall see in the next section that F(s) has only finitely many poles within the Nyquist contour.

Now we can state the Kalman frequency domain inequality (KFDI)

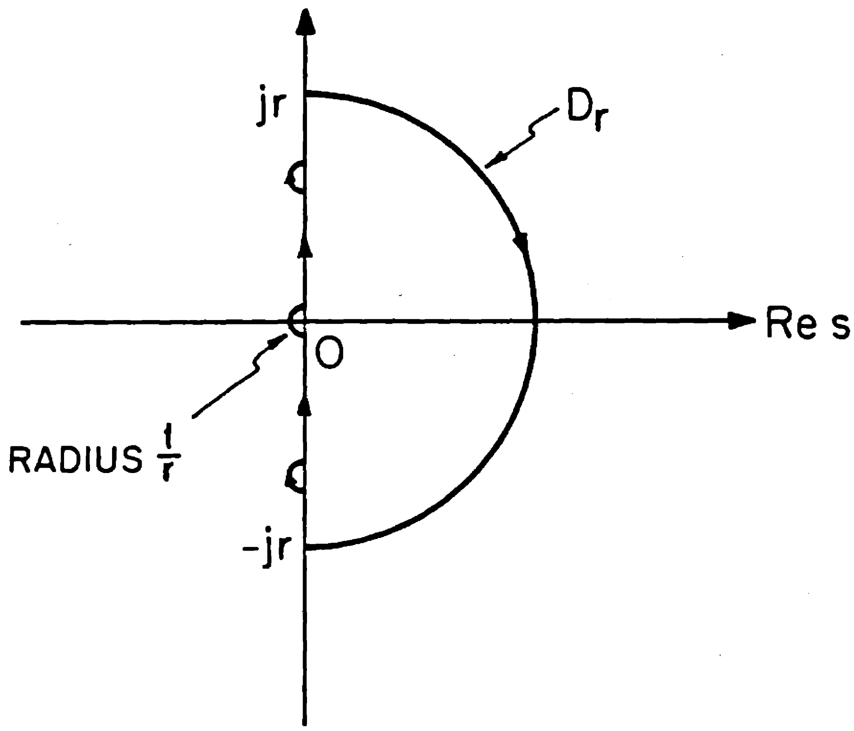


Figure 2.2 Nyquist D_r Contour

as the following theorem.

Theorem 2.1

Let

$$\Omega = \{s \in \lim_{r\to\infty} D_r \ : \ Re(s) \leq 0\} \tag{2.15}$$

Then

$$[I+G(s)]^* R[I+G(s)] > R, \quad \text{for all } s\epsilon\Omega \tag{2.16}$$

where I+G(s) is the return difference transfer function matrix, and $[I+G(s)]^*$ denotes its complex conjugate transpose, and R is the control weighting matrix.

Proof of Theorem 2.1 is presented in the next subsection. In Section 2.4 we shall discuss some consequences of this theorem.

2.2.3 Kalman Inequality for LQHD Systems - Derivation

First let us compute the left-hand-side of Eq. (2.16) using Eq. (2.14).

$$\begin{aligned} [I+G(s)]^* R[I+G(s)] = & R+B'[K_0+K_1(s)]A^{-1}(s)B \\ & + B'A^{-1^*}(s)[K_0+K_1^*(s)]B \\ & + B'A^{-1^*}(s)[K_0+K_1^*(s)]BR^{-1}B'[K_0+K_1(s)]A^{-1}(s)B \end{aligned} \tag{2.17}$$

where we have let

$$A(s) = sI-A-\sum_{i=1}^{N} e^{s\theta_i}A_i-A_0(s) \tag{2.18}$$

Comparing this with Eq. (2.16), we see that it suffices to show

$$A^*(s)[K_0+K_1(s)]+[K_0+K_1^*(s)]A(s)$$

$$+ [K_0+K_1^*(s)]BR^{-1}B'[K_0+K_1(s)] > 0 \tag{2.19}$$

For the case in which the Nyquist contour does not coincide with the imaginary axis, it is somewhat tedious to carry out all the algebra for the rest of this proof here. Since there is no loss in the central idea of the proof, for the sake of simplicity, from hereon, we shall assume that F(s) has no open-loop poles on the imaginary axis, i.e. Ω = imaginary axis. We remark, however, that the derivation presented here can be adapted to the general case in a straightforward fashion.

We establish Eq. (2.19) by deriving the following identities:

<u>Identity 1</u>

$$sK_1(s) = K_1(0)-K_0A_Ne^{s\tau} - \int_{-\tau}^{0} e^{s\theta} \frac{d}{d\theta} K_1(\theta)d\theta \tag{2.20}$$

Identity 2

$$\int_{-\tau}^{0} e^{s\theta} \frac{d}{d\theta} K_1(\theta)d\theta = [A'-K_0BR^{-1}B']K_1(s)$$

$$+ \sum_{i=1}^{N-1} K_0A_ie^{s\theta_i}+K_0A_0(s)$$

$$+ e^{-s\tau}K_1'(-s)A_N \tag{2.21}$$

$$- \Psi(s)$$

$$+ \Gamma_1(s)+\Gamma_2(s)$$

$$+ \Xi_1(s)+\Xi_2(s)$$

where

$$\Psi(s) = \int_{-\tau}^{0} e^{s\theta} \int_{-\tau}^{\theta} K_1'(-\theta+\sigma)BR^{-1}B'K_1(\sigma)d\sigma d\theta \tag{2.22}$$

$$\Gamma_1(s) = \sum_{i=1}^{N-1} \int_{-\tau-\theta_i}^{0} A_i'K_1(\theta+\theta_i)e^{s\theta}d\theta \tag{2.23}$$

$$\Gamma_2(s) = \sum_{i=1}^{N-1} \int_{\theta_i}^{0} K_i'(-\theta+\theta_i)A_ie^{s\theta}d\theta \tag{2.24}$$

$$\Xi_1(s) = \int_{-\tau}^{0}\int_{-\tau-\theta}^{0} A_0'(\xi)K_1'(\xi+\theta)d\xi e^{s\theta}d\theta \tag{2.25}$$

$$\Xi_2(s) = \int_{-\tau}^{0}\int_{-\tau}^{0} K_0'(\xi-\theta)A_0(\zeta)d\zeta e^{s\theta}d\theta \tag{2.26}$$

Identity 3

$$\Psi(s) + \Psi'(-s) = K_1'(-s)BR^{-1}K_1(s) \tag{2.27}$$

Identity 4

$$\Gamma_1(s)+\Gamma_2'(-s) = \sum_{i=1}^{N-1} A_i'K_1(s)e^{-s\theta_i} \tag{2.28}$$

Identity 5

$$\Xi_1(s) + \Xi_2'(-s) = A_0'(-s)K_1(s) \tag{2.29}$$

We refer the readers to Appendix A for the proofs of Identities 1-5. Combining Identities 1-5, we see that

$$sK_1(s)+(-s)K_1'(-s) = K_1(0)+K_1'(0)A_Ne^{s\tau}-A_N'K_0e^{-s\tau}$$

$$- \{[A'-K_0BR^{-1}B']K_1(s)+K_1'(-s)[A-BR^{-1}B'K_0]\}$$

$$- \left(\sum_{i=1}^{N-1} A_i e^{s\theta_i} + \sum_{i=1}^{N-1} A_i' K_0 e^{-s\theta_i} \right)$$

$$- [K_0 A_0(s) + A_0'(-s) K_0]$$

$$+ K_1'(-s) BR^{-1} B' K_1(s)$$

$$- \left[\sum_{i=1}^{N-1} A_i' K_1(s) e^{-s\theta_i} + \sum_{i=1}^{N-1} K_1'(-s) A_i e^{s\theta_i} \right]$$

$$- [A_0'(-s) K_1(s) + K_1'(-s) A_0(s)] \qquad (2.30)$$

After some further manipulations, Eq. (2.30) can be rewritten as

$$A'(-s)[K_0+K_1(s)]+[K_0+K_1'(-s)]A(s)+[K_0+K_1'(-s)]BR^{-1}B'[K_0+K_1(s)]$$

$$= -A'K_0-K_0A+K_0BR^{-1}B'K_0-K_1(0)-K_1'(0) \qquad (2.31a)$$

But by Eq. (2.5),

$$-A'K_0-K_0A+K_0BR^{-1}B'K_0 - K_1(0)-K'(0) = Q > 0 \qquad (2.31b)$$

The inequality of Eq. (2.19) is established by combining Eqs. (2.31a) and (2.31b). Hence Theorem 2.1 is proven.

By following the above proof closely, one can actually give a stronger statement than the Kalman Inequality. Specifically, by combining Eqs. (2.10), (2.14), (2.17), (2.31a) and (2.31b), one obtains

$$(I+G(s))^{*}R(I+G(s)) = F^{*}(s)QF(s) + R \tag{2.32}$$

This identity is called the Kalman equality. By letting $R = \rho I$, where ρ is a positive number, the singular values of the return difference can be readily determined without solving the Riccati equation. This is a very useful fact. In Section 2.5, we use this result to study two design examples.

We remark that by considering degenerate operators and M^2 spaces [29], [35], the LQHD system can be formulated as a semigroup [30]. This leads to an alternate proof for Theorem 2.1, which we present in Chapter 3 as a special case of the more general semigroup formulation for the LQ robustness problem.

2.3 Robustness Characterization of Linear Hereditary Differential Feedback Systems

In this section we develop a robustness characterization of LHDF systems by:

(1) showing that the open-loop dynamics of the linear hereditary differential system have finitely many poles in the closed right-half complex plane (CRHP), Re $s \geq 0$;

(2) stating a version of the multivariable Nyquist theorem for LHDF systems;

(3) characterizing the robustness measure by using the singular values of the return difference and error matrices.

Steps (2) and (3) are a generalization of the robustness characterization presented by Lehtomaki [5] for finite dimensional systems.

The robustness characterization is stated in terms of the nominal and perturbed LHDF systems. The formulation we used is very general, and includes all the LQHD system that we consider in Section 2.2.

2.3.1 Nominal and Perturbed LHDF Systems

The nominal LHDF system is depicted in Figure 2.3 where the loop transfer function matrix is assumed to incorporate both the open-loop plant dynamics and compensation employed.

Specifically, we assume

$$G(s) = H(s)F(s) \tag{2.33}$$

where F(s) is the combined pre-compensator and open-loop plant transfer function and H(s) represents the post-compensator. It is further assumed that H(s) is analytic in the CRHP except at a finite number of isolated points, and F(s) has the following form

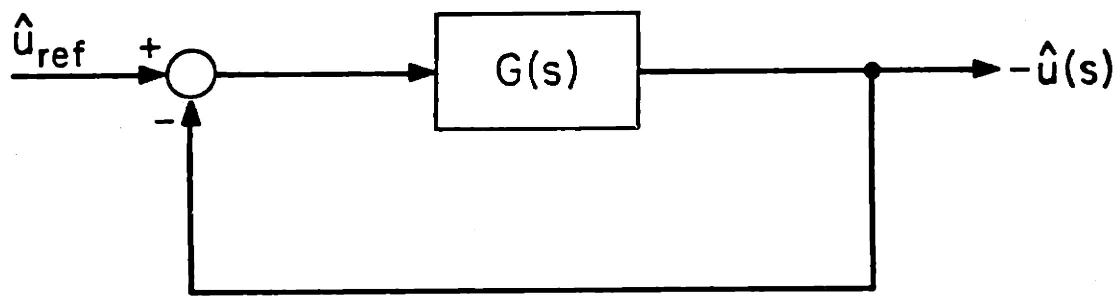


Figure 2.3 Nominal LHDF System

$$F(s) = [sI - A - \sum_{i=1}^{N} e^{s\theta_i} A_i - \int_{-\tau}^{0} A_0(\sigma)e^{s\sigma}d\sigma]^{-1}B \qquad (2.34)$$

where A, A_i, $A_o(\sigma)$ are matrices of dimension nxn, and B is an nxm matrix; $A_0(\sigma)$ is bounded for $\sigma \in [-\tau, 0]$; N is a positive integer and $-\tau = \theta_N < \ldots < \theta_1 < 0$.

The perturbed feedback system is depicted in Figure 2-4 where the loop transfer matrix is assumed to incorporate both the open-loop plant dynamics and compensation employed. In this case, however, either or both of the plant and compensator dynamics may be perturbed from their nominal values.

Specifically, we assume

$$\tilde{G}(s) = \tilde{H}(s)\tilde{F}(s) \qquad (2.35)$$

where $\tilde{F}(s)$ is the perturbed pre-compensator and open-loop plant transfer function and $\tilde{H}(s)$ represents the post-compensator. It is further assumed that $\tilde{H}(s)$ is analytic in the CRHP except at a finite number of isolated points, and $\tilde{F}(s)$ has the form

$$\tilde{F}(s) = [sI - \tilde{A} - \sum_{i=1}^{\tilde{N}} e^{s\tilde{\theta}_i}\tilde{A}_i - \int_{-\tilde{\tau}}^{0} \tilde{A}_0(\sigma)e^{s\sigma}d\sigma]^{-1}\tilde{B} \qquad (2.36)$$

where $\tilde{A}$, $\tilde{A}_i$, $\tilde{A}_0(\sigma)$ are matrices of dimension $\tilde{n}\tilde{x}n$, and $\tilde{B}$ is an $\tilde{n}$xm matrix; $\tilde{A}_0(\sigma)$ is bounded for $\sigma \in [-\tilde{\tau}, 0]$; $\tilde{N}$ is a positive integer

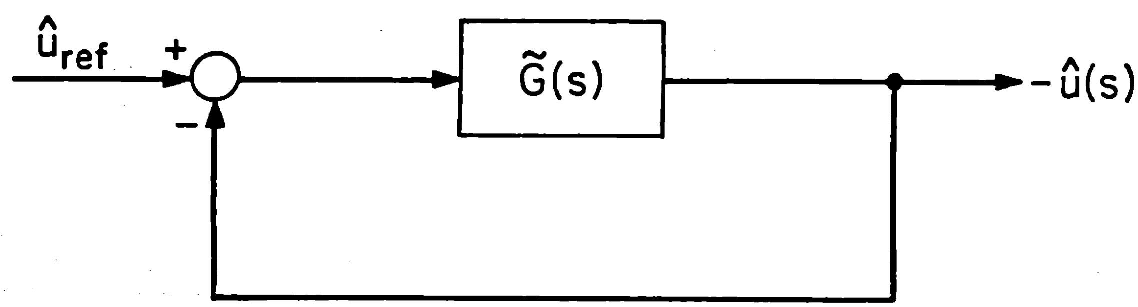


Figure 2.4 Perturbed LHDF System

and $-\tilde{\tau} = \theta_{\tilde{N}} < \ldots\ldots < \tilde{\theta}_1 < 0.$

Note that both G(s) and $\tilde{G}(s)$ are mxm matrices.

2.3.2 Open-Loop Unstable Poles of LHDF Systems

In this subsection we present a study on the open-loop poles of the LHDF system in the CRHP. It is shown that for the nomial system described by Eqs. (2.33) and (2.34), the system has at most a finite number of open-loop poles in the CRHP.

Various stability tests for time delay systems have been published in literature. For examples, Lee and Dianat [36] used the Lyapunov approach. Thowsen [37] presented a Routh-Hurwitz type of criterion. Tsypkin [38] discussed a graphical method that determines the critical frequency and critical delay time. None of these methods, however, deals with distributed time delays. More importantly, they do not seem to provide a convenient way to count the number of unstable poles. These two problems are resolved by using an algebra $\mathcal{A}$ studied by a number of researchers, e.g. Desoer and Vidyasagar [40], Callier and Desoer [41].

Let $\mathbf{R}$ and $\mathbf{R}_+$ denote the real line and the nonnegative real line, respectively, and $\mathbf{L}_1$ denotes the set

$$\mathbf{L}_1 = \{f(\bullet)\,|\,f(\bullet):\mathbf{R}_+ \rightarrow \mathbf{R},\ \int_0^{\infty} |f(t)|dt < \infty\} \tag{2.38}$$

The convolution algebra $\mathcal{A}$ consists of the elements of the form

$$f(t) = \begin{cases} 0 & t < 0 \\ f_a(t) + \sum_{i=0}^{\infty} f_i \delta(t-t_i), & t \geq 0 \end{cases} \tag{2.39}$$

where

(1) $f_a(\bullet) \in L_1$

(2) $t_0 = 0$ and $t_i > 0$ for $i = 1, 2, \ldots.$

(3) $f_i \in \mathbf{R}$; and

(4) $\sum_{i=0}^{\infty} |f_i| < \infty$

Let $\hat{f}$ denote the Laplace transform of $f(\bullet)$. Let $\hat{\mathcal{A}}$ be the set

$$\hat{\mathcal{A}} = \{\hat{f} \,|\, f \in \mathcal{A}\} \tag{2.40}$$

Associate with $\hat{\mathcal{A}}$ the pointwise product. Then $\hat{\mathcal{A}}$ is a commutative algebra with the following property (Callier and Desoer [41]): If $\hat{f} \in \hat{\mathcal{A}}$ and $\hat{f}$ is bounded away from zero at infinity in the CRHP, then $\hat{f}$ has a finite number of zeros in the CRHP. The function $\hat{f}$ is said to be bounded away from zero at infinity in the CRHP if and only if there exist $\eta > 0$, $\rho > 0$ such that for all $|s| \geq \rho$ in the CRHP, $|\hat{f}(s)| > \eta$.

Now consider G(s) described by Eqs (2.33) and (2.34). Since

$$\det(I + H(s)F(s)) = \det(I + F(s)H(s)) \tag{2.41}$$

it is easy to see that

$$\det(I + G(s)) = \frac{\phi_{c\ell}(s)}{\phi_{o\ell}(s)} \tag{2.42}$$

where

$$\phi_{o\ell}(s) = \det(sI - A - \sum_{i=1}^{N} e^{s\theta_i} A_i - \int_{-\tau}^{0} A_0(\sigma)e^{s\sigma}d\sigma) \tag{2.43}$$

and

$$\phi_{c\ell}(s) = \det(sI - A - \sum_{i=1}^{N} e^{s\theta_i} A_i - \int_{-\tau}^{0} A_0(\sigma)e^{s\sigma}d\sigma + BH(s)) \tag{2.44}$$

The following theorem is fundamental to the derivation of a Nyquist theorem for LHDF systems, because it allows us to count the number of encirclements.

<u>Theorem 2.2</u>

$\phi_{o\ell}(s)$ as defined by Eq. (2.43) has finitely many zeros in the CRHP.

Proof of Theorem 2.2

We consider two cases.

Case 1: $A_0(\sigma) = 0$ for all $\sigma \in [-\tau, 0]$. The proof of this case has been briefly described by Callier and Desoer [41]. It is trivial to see that $\frac{1}{(s+1)}$, $\frac{s}{(s+1)}$ and $\frac{e^{s\theta_i}}{(s+1)}$ all belong to $\hat{\mathcal{A}}$, as are their products. Applying Cramer's rule shows that $\phi_{o\ell}(s)$ is a linear combination of 1, s and $e^{s\theta_i}$ and their products. Therefore $\frac{\phi_{o\ell}(s)}{(s+1)^n} \in \hat{\mathcal{A}}$.

Case 2: $A_0(\sigma) \neq 0$. Note that $\int_{-\tau}^{0} a_0(\sigma)e^{s\sigma}d\sigma$, $a_0(\sigma)$ bounded, is equivalent to the Laplace transform of a bounded function with compact support. Hence it is an element of $\hat{\mathcal{A}}$. The product $\int_{-\tau}^{0} a_0(\sigma)e^{s\sigma}d\sigma \int_{-\tau}^{0} b_0(\sigma)e^{s\sigma}d\sigma$ is also in $\hat{\mathcal{A}}$ because it corresponds to the Laplace transform of the convolution of two functions which both have a compact support. Finally, multiplication of $\int_{-\tau}^{0} a_0(\sigma)e^{s\sigma}d\sigma$ with $\frac{1}{1+s}$, $\frac{s}{s+1}$, or $\frac{e^{s\theta_i}}{s+1}$ also yields a product that is in $\hat{\mathcal{A}}$. Extending this argument as in the previous case, we see that $\frac{\phi_{o\ell}(s)}{(s+1)^n} \in \hat{\mathcal{A}}$.

Now observe that $e^{s\theta_i}$ and $\int_{-\tau}^{0} a_0(\sigma)e^{s\sigma}d\sigma$ are uniformly bounded for all $s \in$ CRHP. Therefore as s tends to ∞, $\phi_{o\ell}(s)$ is of the order s^n. Consequently $\frac{\phi_{o\ell}(s)}{(s+1)^n}$ tends to 1 as $|s|$ tends to ∞ in the CRHP. Thus according to the property of $\hat{\mathcal{A}}$, $\frac{\phi_{o\ell}(s)}{(s+1)^n}$, hence $\phi_{o\ell}(s)$, has only a finite number of zeros in the CRHP. (End of Proof).

2.3.3 Multivariable Nyquist Theorem

Because of Theorem 2.2, a version of the multivariable Nyquist theorem can now be stated in the following form. This version follows easily from the standard application of the principle of arguments of complex variable theory, e.g. [42] [43].

Theorem 2.3 (Nyquist Theorem for LHDF Systems)

Suppose that G(s) is defined by Eqs. (2.33) and (2.34). Then the system of Figure 2.3 is closed-loop stable, (in the sense that $\phi_{c\ell}$ (s) Eq.(2.44) has no zero in the CRHP,) if and only if both the following conditions are satisfied:

(1) $\det [I + G(s)] \neq 0$, for all $s \in \Omega$, where Ω is defined by Eq. (2.15);

(2) $\det[I + G(s)]|_{s\epsilon\Omega}$ encircles the origin p times in the counterclockwise sense, where p denotes the number of zeros of $\phi_{o\ell}(s)$ (Eq. 2.43) in the CRHP, counting multiplicities.

2.3.4. Robustness Theorems for Hereditary Differential Systems

In this subsection, we develop theorems that guarantee the stability of the perturbed closed-loop system of Figure 2.4. The development here closely parallels that in [5]. Theorem 2.3 allows us to derive a simple test of robustness for linear hereditary differential systems. As in the well known robustness theorems for finite dimensional systems by Doyle [10], Lehtomaki, et al. [7], Doyle and Stein [3], and Lehtomaki [5], the notion of singular values of matrices are used to establish these robustness tests.

Let G(s) satisfy Eqs. (2.33) and (2.34), and let $\tilde{G}(s)$ satisfy Eqs. (2.35) and (2.36). Then define the error matrix of the perturbed system by

$$E(s) = \tilde{G}(s) - G(s) \tag{2.45}$$

Also we know that

$$\det (I + \tilde{G}(s)) = \frac{\tilde{\phi}_{c\ell}(s)}{\tilde{\phi}_{o\ell}(s)} \tag{2.46}$$

where

$$\tilde{\phi}_{o\ell}(s) = \det (sI - \tilde{A} - \sum_{i=1}^{\tilde{N}} e^{s\tilde{\theta}_i}\tilde{A}_i - \int_{-\tilde{\tau}}^{0} \tilde{A}_0(\sigma)e^{s\sigma}d\sigma) \tag{2.47}$$

$$\tilde{\phi}_{c\ell}(s) = \det (sI - \tilde{A} - \sum_{i=1}^{\tilde{N}} e^{s\tilde{\theta}_i}\tilde{A}_i - \int_{-\tilde{\tau}}^{0} \tilde{A}_0(\sigma)e^{s\sigma}d\sigma + \widetilde{BH}(s)) \tag{2.48}$$

We can now state the robustness theorem for additive modeling error as follows.

Theorem 2.4 (Robustness Theorem for Additive Error):

The perturbed system of Figure 2.4 is closed-loop stable (in the sense that $\tilde{\phi}_{c\ell}(s)$ has no zeros in the CRHP) if the following conditions hold:

(1) a. $\phi_{o\ell}(s)$ and $\tilde{\phi}_{o\ell}(s)$ have the same number of zeros in the CRHP;

b. if $\tilde{\phi}_{o\ell}(j\omega_0) = 0$, then $\phi_{o\ell}(j\omega_0) = 0$;

c. $\phi_{c\ell}(s)$ has no zeros in the CRHP.

(2) $\sigma_{min}[I + G(s)] > \sigma_{max}[E(s)]$ for all $s \in \Omega$.

In the above, $\phi_{o\ell}(s)$, $\phi_{c\ell}(s)$, $\tilde{\phi}_{o\ell}(s)$, $\tilde{\phi}_{c\ell}(s)$ are defined by Eqs. (2.43), (2.44), (2.47), (2.48), respectively. The additive error matrix E(s) is defined by Eq. (2.45). The path Ω is defined by Eq. (2.15). The notations $\sigma_{min}[\bullet]$ and $\sigma_{max}[\bullet]$ denote the minimum and maximum singular values of the matrix.

This theorem says that the size of the modeling and implementation error that a feedback system can tolerate without becoming unstable is given by the quantity σ_{min} [I+G(s)], when no structure of the error is assumed. The proof of this theorem is of the imbedding type [45].

Proof of Theorem 2.4

Let $\lambda \in [0,1]$. It is well known (e.g. [5] [44]) in singular value theory that

$$\sigma_{min}[I + G(s) + \lambda E(s)] > \sigma_{min}[I + G(s)] - \lambda\sigma_{max}[E(s)] \quad (2.49)$$

Therefore condition (2) guarantees that $I + G(s) + \lambda E(s)$ is nonsingular for all $s \in \Omega$.

Now suppose that as λ is varied continuously from zero to unity, the number of encirclements of

$$f(\lambda, s) = \det (I + G(s) + \lambda E(s)) \tag{2.50}$$

around the origin changes. Since $f(\lambda, s)$ is continuous in $(\lambda, s) \in [0, 1] \times \Omega$, its locus on the path Ω forms a closed bounded contour in the complex plane for any $\lambda \in [0, 1]$. Therefore the only way for a change in the number of encirclements to occur is for the locus of $f(\lambda_0, s)$ to pass through the origin for some $\lambda_0 \in [0, 1]$. This is equivalent to requiring

$$\det(I + G(s_0) + \lambda_0 E(s_0)) = 0 \quad \text{for some } (\lambda_0, s_0) \in [0, 1] \times \Omega. \tag{2.51}$$

However we have shown that $I + G(s) + \lambda E(s)$ cannot be singular by assuming condition (2). Hence we conclude that $\det(I + \tilde{G}(s))$, corresponding to $\lambda = 1$, encircles the origin the same number of times as $\det(I + G(s))$, corresponding to $\lambda = 0$, along the path $s \in \Omega$. By applying Theorem 2.3 (Nyquist Theorem), we see that the perturbed system is closed-loop stable. (End of Proof)

In his thesis, Lehtomaki [5] presented six different error models for describing the perturbed system. For each model, a corresponding robustness theorem similar to Theorem 2.4 was stated and proved. Due to our results in subsection 2.3.2, Lehtomaki's robustness theorems of the perturbed system can be carried over to the LQHD case without any modification. Theorem 2.4 is an example of this generalization. Of the other five models, we shall also describe the multiplicative error model below. Readers who are interested to see details on the rest of the error model types are referred to [5, Ch. 3] for details.

A useful way to describe the perturbed system with respect to the nominal system is to define a multiplicative factor matrix L(s) by

$$\tilde{G}(s) = G(s)\, L(s) \tag{2.52}$$

This representation generalizes the gain and phase margin description of the single-input-single-output case, and is particularly useful in describing the robustness properties of the LQHD system that we discuss in the next section. The corresponding robustness theorem for multiplicative modeling error is the following.

<u>Theorem 2.5 (Robustness Theorem for Multiplicative Error):</u>

The perturbed system of Figure 2.4 is closed-loop stable if the following conditions hold:

(1) a. $\phi_{o\ell}(s)$ and $\tilde{\phi}_{o\ell}(s)$ have the same number of zeros in the CRHP;

b. if $\tilde{\phi}_{o\ell}(j\omega_0) = 0$, then $\phi_{o\ell}(j\omega_0) = 0$;

c. $\phi_{c\ell}(s)$ has no zeros in the CRHP.

(2) L(s) has no eigenvalue in $\mathbf{R}_+$, for all $s \in \Omega$.

(3) $$\sigma_{min}[I + G(s)] > \sigma_{max}[L^{-1}(s) - I] \quad \text{for all } s \in \Omega. \qquad (2.53)$$

In the above $\phi_{o\ell}(s)$, $\phi_{c\ell}(s)$, $\tilde{\phi}_{o\ell}(s)$, $\tilde{\phi}_{c\ell}(s)$ are defined by Eqs. (2.43), (2.44), (2.47), (2.48), respectively. The multiplicative error matrix L(s) is defined by Eq. (2.52). The path Ω is defined by Eq. (2.15).

Proof of Theorem 2.5

Let $f(\lambda,s)$: $[0,1] \times \Omega \rightarrow \mathbf{C}$ be the complex-valued function defined by

$$f(\lambda,s) = \det(I + G(s)[I + \lambda(L^{-1}(s) - I)]^{-1}) \qquad (2.54)$$

The inverses exist because of condition (2). It is easy to see that

(i) $$f(0,s) = \det(I + G(s)); \qquad (2.55)$$

(ii) $$f(1,s) = \det(I + \tilde{G}(s)); \qquad (2.56)$$

(iii) $f(\lambda,s)$ is continuous in $[0, 1] \times \Omega$.

Now suppose that as λ is varied continuously from zero to unity, the number of encirclements of $f(\lambda, s)$ around the origin changes. Since $f(\lambda, s)$ is continuous in $(\lambda, s) \in [0, 1] \times \Omega$, its locus on the path Ω forms a closed bounded contour in the complex plane for any $\lambda \in [0, 1]$. Therefore the only way for a change in the number of encirclements to occur is for the locus of $f(\lambda_0, s)$ to pass through the origin for some $\lambda_0 \in [0, 1]$. This is equivalent to requiring

$$f(\lambda_0, s_0) = 0 \quad \text{for some } (\lambda_0, s_0) \in [0, 1] \times \Omega. \tag{2.57}$$

Since

$$I+G(s)[I+\lambda(L^{-1}(s)-I)]^{-1} = [I+G(s)+\lambda(L^{-1}(s)-I)][I+\lambda(L^{-1}(s)-I)]^{-1} \tag{2.58}$$

Eq. (2.57) is the same as requiring $I + G(s_0) + \lambda_0(L^{-1}(s_0) - I)$ be singular for some (λ_0, s_0) in $[0, 1] \times \Omega$. However, condition (3) guarantees that for all $|\lambda| \leq 1$, and all $s \in \Omega$

$$\sigma_{min}[I + G(s) + \lambda(L^{-1}(s) - I)] > 0 \tag{2.59}$$

contradicting Eq. (2.57). Therefore we conclude that $\det(I + \tilde{G}(s))$, corresponding to $\lambda = 1$, encirles the origin the same number of times as $\det(I + G(s))$, corresponding to $\lambda = 0$, along the path $s \in \Omega$. By applying Theorem 2.3 (Nyquist theorem), we see that the perturbed system is closed-loop stable. (End of Proof)

In the next section we show that LQHD systems have many of the robustness properties which have been known to be enjoyed by finite dimensional LQ system. These robustness properties are consequences of Theorem 2.5 and the KFDI derived in Section 2.2.

2.4 Robustness Properties of Linear Quadratic Hereditary Differential Systems

The KFDI proven in Section 2.2 for LQHD systems is the same as the KDFI for LQ systems with finite dimensional state representations [7], [21]. As in the finite dimensional case, the KFDI has some very important consequences in terms of robustness for LQHD systems. The purpose of this section is to discuss these robustness properties of the LQHD system.

For the purposes of investigating the robustness properties of the LQHD system, depicted in Figure 2.1, it is convenient to represent the model uncertainties as a multiplicative perturbation factor, as shown in Figure 2.5. In this configuration, the matrix L(s) represents the model uncertainties of the system reflected to the point X in Figure 2.1 where the loop is broken, as a multiplicative perturbation in the control channel. The matrix G(s) is the nominal system loop transfer function matrix, given by Eqs. (2.33), (2.34). Nominally L(s) is the identity matrix and the LQHD

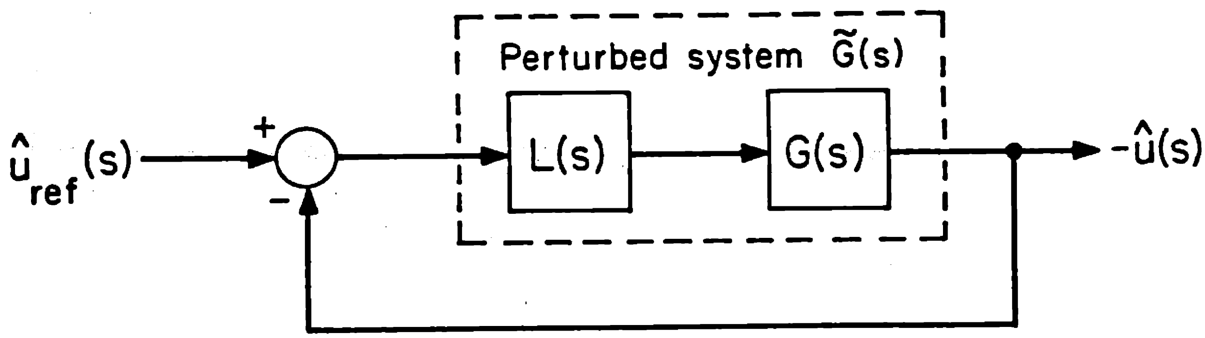


Figure 2.5 Multiplicatively Perturbed LQHD System

system is closed-loop stable. However because of modeling errors and implementation constraints, the matrix L(s) is subject to changes from its nominal value. Consequently, the product G(s)L(s) can be viewed as the actual system loop transfer function matrix. This closed-loop system is stable if and only if the closed-loop system of Figure 2.6 is stable. In the figure, $R^{1/2}$ is a square root of the control weighting matrix, in the sense that

$$R = R^{1/2^*} R^{1/2} \tag{2.60}$$

With this formulation, we can state some of the robustness properties of the LQHD system as the following theorem.

<u>Theorem 2.6 (Stability Margin of LQHD Systems)</u>

The multiplicatively perturbed LQHD system in Figure 2.5 is closed-loop stable, i.e. the polynomial $\tilde{\phi}_{c\ell}(s)$ (Eq. (2.48)) has no CRHP zeros, provided that the following conditions are satisfied:

(1) G(s) is specified by Eqs. (2.10) - (2.14), as the closed-loop solution to the LQHD control problem.

(2) (a) $\phi_{o\ell}(s)$ and $\tilde{\phi}_{o\ell}(s)$ have the same number of CRHP zeros

(b) if $\tilde{\phi}_{o\ell}(j\omega_0) = 0$ then $\phi_{o\ell}(j\omega_0) = 0$

where $\phi_{o\ell}(s)$, $\tilde{\phi}_{o\ell}(s)$, are defined in Eqs. (2.43), (2.47), respectively.

(3) L(s) has no eigenvalue in $\mathbf{R}_+$.

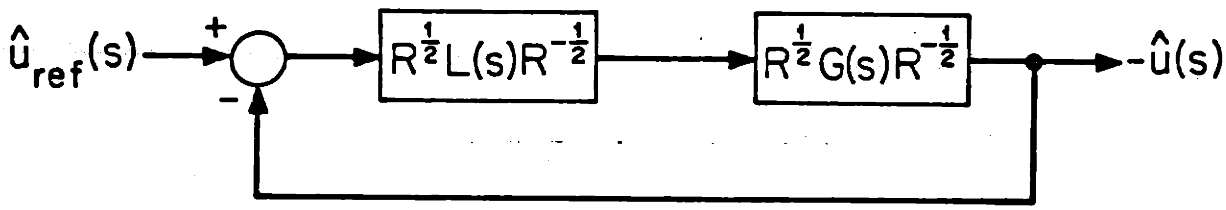


Figure 2.6 Perturbed Feedback System Representation for the Derivation of LQHD Robustness Properties

(4) $\sigma_{max}(R^{1/2}L^{-1}(s)R^{-1/2} - I) \leq 1$ for all $s \in \Omega$.

Proof of Theorem 2.6

It is well known that $\phi_{c\ell}(s)$ (Eq. (2.44)) has no zero in the CRHP [31]. Note that

$$\det[I + (R^{1/2}G(s)R^{-1/2})(R^{1/2}L(s)R^{-1/2})] = \det[I + G(s)L(s)] \tag{2.61}$$

Hence

$$\det[I + (R^{1/2}G(s)R^{-1/2})(R^{1/2}L(s)R^{-1/2})] = \frac{\tilde{\phi}_{o\ell}(s)}{\tilde{\phi}_{c\ell}(s)} \tag{2.62}$$

Similarly,

$$\det[I + R^{1/2}G(s)R^{-1/2}] = \frac{\phi_{o\ell}(s)}{\phi_{c\ell}(s)} \tag{2.63}$$

Theorem 2.1 (KFDI) impiles that

$$\sigma_{max}[I + R^{1/2}G(s)R^{-1/2}] > 1 \quad \text{for all } s \in \Omega \tag{2.64}$$

Therefore in conjunction with Theorem 2.5, condition (3) guarantees that $\tilde{\phi}_{c\ell}(s)$ has no zeros in the CRHP. (End of Proof)

The following are corollaries to Theorem 2.6. Their proofs are omitted because they are completely identical to the proofs of their counterparts for finite dimensional systems. For examples interested readers are referred to [5, Ch. 5].

Corollary 2.7

The multiplicatively perturbed LQHD feedback system is stable if

$$\sigma_{max}(L^{-1}(s)-I) < [\text{cond}(R)]^{-1/2} \quad , \; s \in \Omega \tag{2.65}$$

where cond(R) denotes the condition number of the matrix R,

$$\text{cond}(R) = \frac{\sigma_{max}(R)}{\sigma_{min}(R)} \tag{2.66}$$

defined as the ratio of its maximum singular value to its minimum singular value.

Corollary 2.8

If $R > 0$ is diagonal, then simultaneously in each feedback loop, the LQHD system has the following guaranteed stability margins,

1. $[1/2, \infty]$ gain margin
2. $\pm 60^{\circ}$ phase margin

In other words, let L(s) be diagonal and let each of these diagonal elements be either a positive number larger than $\frac{1}{2}$, or complex number of the form $e^{j\phi}$, $-60^\circ \leq \phi \leq 60^\circ$. Then the perturbed system is closed-loop stable.

Corollary 2.9

If R is block diagonal of the form

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \tag{2.67}$$

and L(s) consists of crossfeed perturbation of the form

$$L(s) = \begin{bmatrix} I & X(s) \\ 0 & I \end{bmatrix} , \tag{2.68}$$

Then the LQHD system is stable if

$$\sigma_{max}(X(s)) < \left[\frac{\sigma_{min}(R_2)}{\sigma_{max}(R_1)}\right]^{1/2} , \quad s \in \Omega \tag{2.69}$$

Similarly, if L(s) is of the form

$$L(s) = \begin{bmatrix} I & 0 \\ X(s) & I \end{bmatrix} , \tag{2.70}$$

then the LQHD system is stable if

$$\sigma_{max}(X(s)) < [\frac{\sigma_{min}(R_1)}{\sigma_{max}(R_2)}]^{1/2} \quad , \; s \in \Omega \tag{2.71}$$

In particular, if $R = \rho I$, where ρ is a positive scalar, then the LQHD system is stable if

$$\sigma_{max}(X(s)) < 1 \; , \; s \in \Omega \tag{2.72}$$

for both types of crossfeed perturbations (Eqs. (2.68) and (2.70)).

2.5 Discussion on LQHD Control Designs

In this section, we look at two scalar control examples to highlight the previous results for LQHD systems, and discuss the applicability of some of the well-known design techniques for finite-dimensional systems to the LQHD case.

2.5.1 Examples of LQHD Systems

The first example is a special case which shows that one can often obtain the same robustness measure as the no time delay case as long as one has perfect knowledge of the time lag. The second example compares a suboptimal design with the optimal one.

Example 1:

Consider the following single-input/single-output system in which there is a time delay τ in its actuator dynamics.

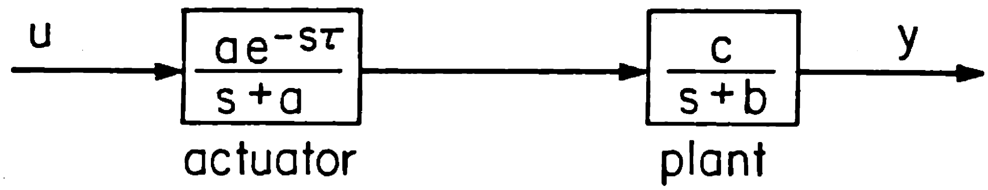


Figure 2.7 Single-Input/Single-Output LQHD Example

Let the quadratic criterion[1] be

$$J(u) = \int_0^\infty [y^2(t) + u^2(t)]dt \tag{2.73}$$

Then the Kalman equality (Eq. (2.32)) can be used to compute the magnitude of the optimal return difference, which is given by

$$|1 + g(j\omega)|^2 = 1 + \left|\frac{ac}{(j\omega+a)(j\omega+b)}\right|^2 \tag{2.74}$$

It is interesting to note in this equation that the quantity $|1+g(j\omega)|$, which is the robustness measure formulated in Section 2.3, is independent of τ. The reason for this is that if the LQ system has full knowledge of the time delay element, then as a consequence of the full state feedback, the LQ system will compensate so that the same stability margin is obtained as in the $\tau = 0$ case.

[1]Strictly speaking, this corresponds to a positive semi-definite Q matrix which is not considered in this chapter. However, this limitation is removed in Chapter 3.

Example 2

This example is drawn from Gibson [30], who has computed the control gains for the example numerically. We consider the suboptimal design using those control gains here.

Let the dynamics of the system be

$$\dot{x}(t) = x(t) + x(t-1) + u(t) \tag{2.75}$$

where $x(\bullet)$ and $u(\bullet)$ are scalar functions. Gibson divides the unit interval into N subintervals in order to approximate the state $x(t+\bullet)$ over the unit interval as a piecewise constant function. Gibson then derived an N-th order approximate model for the system. He solved this system numerically for the control gains k_0, $k_1(\bullet)$ in

$$u(t) = -k_0x(t) - \int_{-1}^{0} k_1(\theta)x(t+\theta)d\theta$$

The resulting values are given for $\theta = 0.0, -0.1, -0.2, \ldots, -1.0$, and $N = 17, 29, 50, 74$, by the following table.

Table 2.1

θ	N	17	29	50	74
	k_0	2.8260	2.8190	2.8148	2.8130
		←	$k_1(\theta)$		→
0.0		0.6684	0.6547	0.6469	0.6435
-0.1		0.7726	0.7169	0.7179	0.7273
-0.2		0.8467	0.8239	0.8209	0.8258
-0.3		0.9961	0.9508	0.9434	0.9607
-0.4		1.0822	1.1020	1.0895	1.1023
-0.5		1.2811	1.2822	1.2633	1.2694
-0.6		1.5200	1.4963	1.4693	1.4965
-0.7		1.6606	1.7501	1.7125	1.7315
-0.8		1.9802	2.0499	1.9987	2.0480
-0.9		2.3648	2.4033	2.3347	2.3748
-1.0		2.5852	2.6730	2.7284	2.7541

We consider the suboptimal control of the following form,

$$u(t) = - k_0x(t) - \sum_{j=1}^{10} \tilde{k}_j \int_{-\frac{j}{10}}^{-\frac{j-1}{10}} x(t+\theta)d\theta$$

where $\tilde{k}_j$ belongs to one of three cases

(1) $\tilde{k}_j = k_1(-\frac{j-1}{10})$;

(2) $\tilde{k}_j = [k_1(-\frac{j-1}{10}) + k_1(-\frac{j}{10})]/2;$

(3) $\tilde{k}_j = k_1(-\frac{j}{10}).$

The return difference for these suboptimal controllers are plotted in Figure 2.8, for the N = 17 approximation. Also shown is the optimal return difference which is computed using the Kalman equality (Eq. (2.32)).

It is clear from these plots that the suboptimal controller (2) approximates the optimal system very closely. Also by looking at the plots for N = 50, 74, 23, we see that only insignificant improvement can be obtained by using those higher order models.

2.5.2 Extensions of LQHD Control Designs

In this thesis we have not considered the effect of observers in the controller designer. However, the most important control design that has evolved from the multivariable robustness study for finite dimensional systems is probably the linear-quadratic-gaussian/loop-transfer-recovery approach [3], [6]. This approach combines an observer (usually the Kalman filter [61]) design for the system with the asymptotic properties of the LQ regulator [62] to synthesize controllers which have desirable loop properties. It

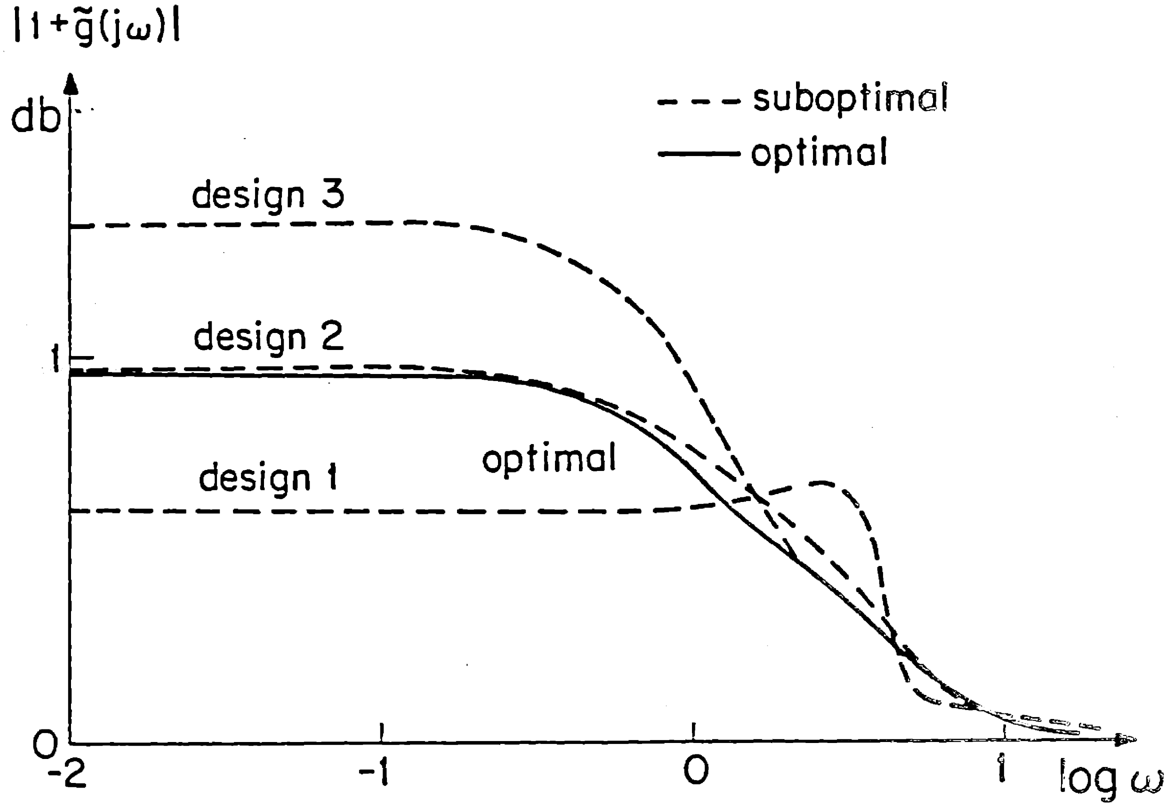


Figure 2.8 Return Difference of Optimal and Suboptimal Controllers for Example 2

seems to us that this approach is also valid for the LQHD system, provided that the system is minimum-phase (a requirement of the finite dimensional case also). A good filter design to use is the one for LQHD systems by Kwong and Willsky [39].

Since our singular-value formulation for robustness is identical to the one for finite dimensional systems, all the finite dimensional results that depend only on the manipulation of the singular value decomposition are also valid for the LQHD system. In particular, the design techniques which exploit the structural information of a system are appropriate for LQHD system designs also. Examples of these techniques are the celebrated design analysis method using the error structure by Lehtomaki, et al. [63], and the design adjustment technique using the control weighting matrix by Lee, et al. [13], and the localizing robustness analysis by Lee, et al. [64]. Without being repetitious here, we refer the interested readers to the last chapter of this thesis for an elaboration of these techniques.

CHAPTER 3

ROBUSTNESS PROPERTIES OF DISTRIBUTED, LINEAR-TIME-INVARIANT SYSTEMS

3.1 Introduction

In Chapter 2 we have developed a multivariable robustness characterization for LHDF systems, using singular values of the return difference. Also we have shown that the LQHD control design

(1) produces a closed-loop system that satisfies an optimal frequency domain condition which is commonly referred to as the Kalman frequency domain inequality (KFDI);

(2) guarantees that the closed-loop control system has at least 60° phase margins and infinite gain margins simultaneously in all control input channels when a diagonal control weighting matrix is used.

For systems described by semigroups, the above results are also valid. Yakubovich [22] [23] proved the result (1) for linear systems in which both the system and input operators are linear bounded, and in a separate paper with Likhtarnikov [26], for systems described by linear unbounded operators.

In order to establish the result (2), we need to extend the singular-value robustness characterization to the present case. It appears that the paper by Desoer and Wang [24] contains the most general Nyquist Theorem available for our purpose. Their result was derived by using a quotient algebra of transfer functions [41]. We use this algebra to describe the nominal and perturbed system. Also we study the infinite dimensional Lyapunov control design approach. The finite dimensional Lyapunov control was investigated by Wong [12]. This kind of design is a subclass of the LQ system. It has the additional property that the optimal control system is guaranteed to have at least 90° phase margins and 100% gain reduction margins.

The rest of this chapter is organized as follows. In Section 3.2, we discuss the Yakubovich Frequency Domain Theorem (YFDT), also we revisit the LQHD system to show that the YFDT can be used to derive the KFDI in Chapter 2. In Section 3.3 we develop the robustness characterization for distributed, linear-time-invariant (DLTI) systems, described by linear operators with a finite dimensional control space. The result is then used to derive the robustness properties of LQ optimal systems. In Section 3.4, we discuss the Lyapunov control system.

In Section 3.3 we emphasize that the robustness characterization is only valid when the control space is finite dimensional, even though the YFDT applies to the infinite dimensional case also. This shortcoming is due to the absence of any Nyquist-type of theorem for infinite dimensional control space. It is, however, possible to formulate a robustness characterization based on the singular values of linear operators, for which the control space is allowed to be infinite dimensional. We present this result in Appendix B.

3.2 Yakubovich Frequency Domain Theorem

Yakubovich [22], [23] proved that under certain rather general assumptions, the KFDI-type of criterion (see Chapter 2) is a necessary and sufficient condition for the existence of LQ optimal control for infinite dimensional systems. The necessary part of his theorem provides a basis for us to study the robustness properties of LQ systems represented as a semigroup.

Let **X** and **U** be two Hilbert spaces. Each of these spaces has an inner product. With an abuse of notation, let each inner product be denoted by $(\bullet,\bullet)$, and the norm associated with the inner product be denoted by $||\bullet||$.

First we present the results for systems described by bounded operators. This covers all finite dimensional systems and is sufficient to treat a number of systems described by linear compact and integral operators. The results for linear unbounded operators are given in subsection 3.2.1.

Let Q be a linear bounded, self-adjoint operator on **X** such that

$$(x, Qx) \geq 0 \quad \text{for all } x \in \mathbf{X} \tag{3.1}$$

Let R be a linear bounded, self-adjoint operator on **U** such that for some $\delta > 0$,

$$(u, Ru) \geq \delta||u||^2 \quad \text{for all } u \in \mathbf{U} \tag{3.2}$$

<u>Definitions:</u> Let A: $\mathbf{X} \to \mathbf{X}$, B: $\mathbf{U} \to \mathbf{X}$ be linear operators and I = identity operator in $\mathbf{X}$ (or $\mathbf{U}$). The spectrum sp(A) of A is the set of all complex numbers λ such that $\lambda I - A$ does not have a linear bounded inverse. A is called Hurwitz if sp(A) is disjoint from the CRHP. The pair A, B is said to be stabilizable if there is a linear bounded operator H such that $A + BH$ is Hurwitz. Let A^*, H^* denote the adjoints [46] of A, H, respectively.

<u>Theorem 3.1 (Yakubovich Frequency Domain Theorem (YFDT))</u>

Let A, B be a stabilizable pair of linear bounded operators and suppose that no point in sp(A) is on the imaginary axis. Then a necessary and sufficient condition for the existence of a linear bounded self-adjoint operator K: $\mathbf{X} \to \mathbf{X}$ and an operator H^*: $\mathbf{X} \to \mathbf{U}$ such that

$$KA + A^*K + Q - HRH^* = 0 \tag{3.3}$$

$$KB - HR = 0 \tag{3.4}$$

and $||\exp[(A - BH^*)t]x_0||$ is squared integrable over t, i.e. $\in L_2((0,\infty), \mathbf{X})$, for all $x_0 \in \mathbf{X}$ is

$$((j\omega I - A)^{-1}Bu, Q(j\omega I - A)^{-1}Bu) + (u, Ru)$$

$$= ((I + G(j\omega))u, R(I + G(j\omega))u), \text{ for all } u \in \mathbf{U} + j\mathbf{U}, \ \omega \geq 0 \tag{3.5}$$

where

$$G(j\omega) = H^{*}(j\omega I - A)^{-1}B \ . \tag{3.6}$$

Eq. (3.5) implies the following domain inequality that corresponds to the KFDI in Chapter 2,

$$((I + G(j\omega))u,\ R(I + G(j\omega))u) \geq (u,\ Ru)$$
$$\text{for all } u \in \mathbf{U} + j\mathbf{U}.$$

In subsection 3.2.1, we state the corresponding theorem for linear unbounded operators.

The relationship of Theorem 3.1 to the LQ optimal control is as follows. Let A be the infinitesimal generator of a strongly continuous semigroup of operators $T(t)$ over $\mathbf{X}$, i.e.

$$\lim_{t\to 0^{+}} ||T(t)x-x|| = 0 \quad \text{for all } x \in \mathbf{X} \tag{3.7}$$

The optimal control problem is defined as [22]:

Find $u(\bullet) \in \mathbf{L}_2([0,\infty);\ \mathbf{U})$ to minimize the quadratic cost functional

$$J(u) = \int_0^{\infty} (x(t),\ Qx(t))dt + \int_0^{\infty} (u(t),\ Ru(t))dt \tag{3.8}$$

where Q and R are defined in Eqs. (3.1) and (3.2), respectively, subject to the dynamics

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \; ; \; x(0) \text{ given} \tag{3.9}$$

Furtherfmore we assume that A^*, $Q^{1/2^*}$ is a stabilizable pair, where $Q^{1/2}$ is the square root of Q:

$$Q = Q^{1/2^*} Q^{1/2}$$

The optimal control solution is characterized by Theorem 3.2.

Theorem 3.2 (Yakubovich [22])

(1) The solutions K, H^* to Eqs. (3.3)-(3.4) with the property of $\exp[(A-BH^*)t]\, x_0 \in L_2((0, \infty), \mathbf{X})$ are unique.

(2) The control given by

$$u^0(t) = - H^* x(t) \tag{3.10}$$

is the optimal control solution to Eqs. (3.8)-(3.9).

(3) The optimal system is closed-loop exponentially stable.

The optimal control can be represented in feedback form in the frequency domain by Figure 3.1, where $\hat{u}(s)$ and $\hat{x}(s)$ are the Laplace transforms of u(t) and x(t), respectively. By breaking the loop at $\hat{u}(s)$, one sees that the loop transfer function is

$$G(s) = H^*(sI - A)^{-1}B \tag{3.11}$$

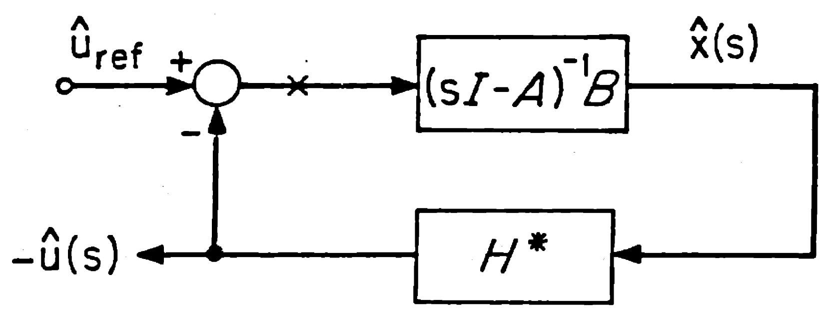


Figure 3.1 LQ Feedback System

Correspondingly the operator-valued function $I + G(s)$ is the return difference.

Theorems 3.1–3.2 allow one to extend the finite dimensional LQ robustness theorems [7] [8] [20] to the infinite dimensional case. This is possible when the dimension of the control space, i.e. **U**, is finite. Section 3.3 deals with this case. A generalized Nyquist theorem by Desoer and Wang [24] is used to extend the finite dimensional results.

In order to characterize the robustness of distributed systems, it is essential that we consider the case in which the operator A is linear unbounded, since the infinitesimal generator of a semigroup is generally unbounded.

3.2.1 Evolution Formulation in which A and B are Unbounded

The case in which A and B are unbounded operators is considered in this subsection. Under a given set of conditions, Likhtarnikov and Yakubovich [26] showed that the YFDT (Theorem 3.1) is also true for this case. We discuss their extended result for the necessary part of Theorem 3.1 here. That result allows us to study the robustness characterization of LQ optimal systems in which A is unbounded.

The problem considered in [26] is an evolution problem of the parabolic type. In subsections 3.2.2 and 3.2.3, we discuss how the

LQHD system and the distributed system described by a second-order partial differential equation can be adapted to use this frequency domain theorem, even though their natural descriptions are not parabolic.

The result in [26] assumes that the parabolic evolution equation of the system satisfies a coercive and a regularity condition. Its assumptions are similar to those in Lions [75, Ch. 3], which contains a detailed study on the optimal control problem. However, [75] did not consider the case in which B is unbounded.

Let **X** and **U** be two Hilbert spaces given as before. Also let **V** be a Hilbert space such that **V** is a dense subspace of **X**. Let **V'** be the dual space of **V**. Then

$$\mathbf{V} \subset \mathbf{X} \subset \mathbf{V}'$$

Let $(\bullet,\bullet)_{\mathbf{V}}$ and $||\bullet||_{\mathbf{V}}$ denote the inner product and its corresponding norm on **V**, and when necessary for the distinction, we will use $(\bullet,\bullet)_{\mathbf{X}}$, $||\bullet||_{\mathbf{X}}$ for **X**.

Let $\mathcal{L}(\mathbf{Y},\mathbf{Z})$ denote the space of linear continuous operators from **Y** to **Z**.

Let A, B be linear unbounded operators:

$$A \in \mathcal{L}(\mathbf{V},\mathbf{V}')$$

$$B \in \mathcal{L}(\mathbf{U},\mathbf{V}')$$

such that the operator A is an infinitesimal generator of a strongly continuous semigroup of linear bounded operators in **X**, which satisfies the following coercive condition: there exists real numbers λ, α such that $\alpha > 0$ and

$$(v,-Av)_{\mathbf{V}} + (-Av,v)_{\mathbf{V}} + \lambda ||v||^2_{\mathbf{X}} \geq \alpha \, ||v||^2_{\mathbf{V}} \quad \text{for all } v \in \mathbf{V} \tag{3.12}$$

Then A and B describe an evolution system as follows,

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \tag{3.13}$$

$$x(0) = x_0, \; x_0 \in \mathbf{X}.$$

where $x(t) \in \mathbf{X}$, $u(t) \in \mathbf{U}$ and $t \geq 0$, and $u(\bullet) \in L_2([0,\infty), \mathbf{U})$.

Other conditions such as the regularity condition for sufficient "smoothness" which is required for the synthesis of the optimal control are described in [26]. As remarked earlier, we do not elaborate on these conditions here since we are only interested in the necessary part of the frequency domain theorem in [26]. Instead we simply assume that the optimal control solution exists and state the frequency domain condition for the case in hand. This is stated as a continuation of theorem 3.1.

Theorem 3.1A (Unbounded Operator Case):

Suppose that no point in sp(A) is on the imaginary axis and there exists a solution $K \in \mathcal{L}(\mathbf{V}',\mathbf{X}) \cap \mathcal{L}(\mathbf{X},\mathbf{V})$, self-adjoint in **X**, for the equations

$$KA + A^{*}K + Q - HRH^{*} = 0$$

$$KB - HR = 0$$

Then Eq. (3.5) holds, i.e.

$$((j\omega I-A)^{-1}Bu,\ Q(j\omega I-A)^{-1}Bu) + (u,Ru)$$
$$= ((I + G(j\omega))u,\ R(I + G(j\omega))u) \quad \text{for all } u \in \mathbf{U} + j\mathbf{U} \qquad (3.14a)$$

where

$$G(j\omega) = H^{*}(j\omega I - A)^{-1}B$$

and $Q \in \mathcal{L}(\mathbf{V},\mathbf{V}')$ and $R \in \mathcal{L}(\mathbf{U},\mathbf{U})$ are self-adjoint.

If in addition,

$$(Qx,x)_{\mathbf{V}} \geq 0 \quad \text{for all } x \in \mathbf{V}$$

and for some $\delta > 0$

$$(Ru,u) > \delta||u||^{2} \text{ for all } u \in \mathbf{U}$$

then

$$((I + G(j\omega))u, R(I + G(j\omega))u) \geq (u, Ru)$$

$$\text{for all } u \in \mathbf{U} + j\mathbf{U} \tag{3.14b}$$

In the next subsection we show that the LQHD system can be represented in the form of Eq. (3.13), and satisfies Eq. (3.12). In subsection 3.2.3, we do the same for hyperbolic systems with sufficient inherent damping.

3.2.2 Semigroup Representation of Linear Hereditary Differential Systems

Consider the linear hereditary differential system that we define in Eqs. (2.1) - (2.2). This can be represented in semigroup form [21] [29] as follows.

Let the space

$$M^2 = \mathbf{R}^n \times L_2((0, -\tau), \mathbf{R}^n)$$

Then Eqs. (2.1) - (2.2) can be described equivalently as

$$\frac{dz}{dt} = Az(t) + Bu(t)$$

$$z(t) = h; \; h \in D(A)$$

where A is a closed linear operator on M^2, with a dense domain in M^2, defined by

$$(Ah)(\theta) = \begin{cases} Ah(\theta) + \sum_{i=1}^{N} A_i h(\theta_i) + \int_{-\tau}^{0} A_0(\theta)h(\theta)d\theta; & \theta = 0 \\ \dfrac{dh(\theta)}{d\theta} & ; \theta \neq 0 \end{cases}$$

and

$$\mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

This linear operator A is generally unbounded. In order to use the result in subsection 3.2.1, we must show that A satisfies Eq. (3.12).

Since sp(A) lies entirely to the left of the line Re s = β for some finite $\beta \geq 0$ in the complex plane [31], A - βI generates a contraction semigroup. Consequently A - βI is dissipative [55], i.e.

$$((A - \beta I)x,x) + (x, (A - \beta I)x) = 0$$

Expanding, this is identical to

$$(Ax,x) + (x,Ax) - 2\beta||x||^2 = 0$$

Therefore Eq. (3.12) is satisfied. Hence Theorem 3.1A is applicable. We remark that in fact, it is possible to draw the same conclusion for a larger class of linear hereditary differential systems (Delfour [76]).

We now show that Eq. (3.14) implies the KFDI in Eq. (2.16) for the LQHD systems. The optimal solution for the semigroup formulation gives

$$G(j\omega) = H^{*}(j\omega I - A)^{-1}B$$

where the operator H^{*} is defined by [31],

$$H^{*}f = K_0 f(t) + \int_{-\tau}^{0} K_1(\theta) f(t+\theta) d\theta$$

where K_0 and $K_1(\theta)$ are matrices given by Eqs. (2.5) - (2.9).

It is a simple algebraic exercise to show that

$$(j\omega I - A)^{-1} B e^{j\omega t} v = F(j\omega) e^{j\omega t} v$$

Let $u = e^{j\omega t} v$, $v \in \mathbf{U}$, in Eq. (3.14b). Then

$$(I + G(j\omega))u = (e^{j\omega t} + G(j\omega) e^{j\omega t})v$$

where F(s) in defined in Eq. (2.10). Also by using the definition of H^{*}, we get

$$G(j\omega) e^{j\omega t} v = [K_0 F(j\omega) e^{j\omega t} + \int_{-\tau}^{0} K_1(\theta) F(j\omega) e^{j\omega(t+\theta)} d\theta] v$$

Hence manipulating the above, we obtain

$$(I + G(j\omega))e^{j\omega t}v = (\mathrm{I} + G(j\omega))e^{j\omega t}v$$

where G(s) is defined in Eq. (2.14). Since ω and v are arbitrary, $I + G(j\omega)$ and $\mathrm{I} + G(j\omega)$ are identical. Therefore the YFDT in this section implies the KFDI in Chapter 2.

3.2.3 Hyperbolic Systems with Inherent Damping

In this subsection, we consider the class of distributed control system described by the second-order differential equation [28]

$$\frac{d^2x(t)}{dt^2} + L\,\frac{dx(t)}{dt} + Px(t) = B_0u(t), \quad t \geq 0 \tag{3.15}$$

where x(t) and u(t) belong to the Hilbert spaces **X** and **U**, respectively.

Let P be a self-adjoint linear operator with domain $\boldsymbol{D}(P)$, and assume that it satisfies the coercive condition: there exists $\alpha > 0$ that

$$(Px,x) \geq \alpha||x||^2 \quad \text{for all } x \in \boldsymbol{D}(P) \tag{3.16a}$$

In addition assume that P^{-1} is compact.

Let L be a nonnegative, symmetric linear operator with domain

$D(L) \supset D(P)$, and assume there exists $\gamma \geq 0$ such that

$$||Lx|| \leq \gamma ||Px|| \qquad \text{for all } x \in D(P) \tag{3.16b}$$

Rewriting Eq. (3.14) in the form of the parabolic description of Eq. (3.13), we get

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \frac{dx(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -P & -L \end{bmatrix}\begin{bmatrix} x(t) \\ \frac{dx(t)}{dt} \end{bmatrix} + \begin{bmatrix} 0 \\ B_0 \end{bmatrix} u(t)$$

Then Theorem 3.1A is applicable if $\tilde{A} = \begin{bmatrix} 0 & I \\ -P & -L \end{bmatrix}$ satisfies Eq. (3.12) when extended to the space $D(P) \times \mathbf{X}$, from $D(P) \times D(P)$.

Gibson [28] proved that $\tilde{A}$ can be extended to $D(P) \times \mathbf{X}$ and the extension generates a contraction semigroup of linear bounded operators on this space; of which the inner product is defined as

$$\left\langle \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right\rangle = (Px_1, x_2)_{\mathbf{X}} + (y_1, y_2)_{\mathbf{X}}$$

and $\tilde{A}$ is dissipative. Therefore [55],

$$\left\langle \tilde{A}\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \tilde{A}\begin{bmatrix} x \\ y \end{bmatrix} \right\rangle \leq 0$$

Consequently for any real $\alpha > 0$, $\lambda = \alpha$, the coercive condition in Eq. (3.12) is satisfied for $\mathbf{V} = D(P) \times D(P)$ dense in $D(P) \times \mathbf{X}$.

3.3 Robustness Characterization for the Case Dim U is Finite

In this section we discuss a version of the multivariable Nyquist theorem due to Desoer and Wang [24] for a distributed system which has a finite dimensional control space. We use this theorem to extend the robustness characterization for LQHD systems in Chapter 2 to more general distributed systems. Then a robustness theorem will be stated for the LQ optimal system.

3.3.1 Preliminaries

Callier and Desoer [41] introduced an algebra $\hat{\mathcal{B}}$ of transfer functions that is suitable for studying distributed, linear time-invariant systems. The algebra $\hat{\mathcal{B}}$ is formed as a quotient algebra [72] of two convolution algebras. It allows us to manipulate distributed systems within the algebra in ways similar to those for the finite dimensional system within the algebra of proper rational transfer functions. Desoer and Wang [24] successfully applied the algebra $\hat{\mathcal{B}}$ to obtain a generalized Nyquist theorem for distributed systems. The definition of $\hat{\mathcal{B}}$ that we adopt in this chapter is similar to that in [24].

Let $\sigma \in \mathbf{R}$, and let $\mathcal{A}(\sigma)$ denote the convolution algebra which consists of elements of the form

$$f(t) = \begin{cases} 0 & , t < 0 \\ f_a(t) + \sum_{i=0}^{\infty} f_i \delta(t-t_i) & , t \geq 0 \end{cases}$$

where

(2) $f_a(t) \in \mathbf{R}$;

(2) $\int_0^{\infty} |f_a(t)| e^{-\sigma t} dt < \infty$;

(3) $t_0 = 0$ and $t_i > 0$ for $i = 1,2,\ldots$;

(4) $f_i \in \mathbf{R}$; and

(5) $\sum_{i=0}^{\infty} |f_i| e^{-\sigma t_i} < \infty$

$f(\bullet)$ is said to belong to $\underline{\mathcal{A}}(\sigma)$ if and only if there exists $\sigma_1 \in \mathbf{R}$, $\sigma_1 < \sigma$ such that $f(\bullet) \in \mathcal{A}(\sigma_1)$.

Let $\hat{f}$ denote the Laplace transform of $f(\bullet)$. Let $\hat{\underline{\mathcal{A}}}(\sigma)$ be the set

$$\hat{\underline{\mathcal{A}}}(\sigma) = \{\hat{f} \mid f \in \underline{\mathcal{A}}(\sigma)\}$$

and $\hat{\underline{\mathcal{A}}}^{\infty}(\sigma)$ be the set

$$\hat{\underline{\mathcal{A}}}^{\infty}(\sigma) = \{\hat{f} \mid \hat{f} \in \hat{\underline{\mathcal{A}}}(\sigma) \cap \hat{\Psi}(\sigma)\}$$

where $\hat{\Psi}(\sigma)$ is the set consisting of all $\hat{f}$ such that $\hat{f}$ is bounded away from zero at infinity in the closed right-half complex plane Re s $\geq \sigma$, i.e. there exist $\eta > 0$, $\rho > 0$ such that for all $s \in \{\text{Re } s \geq \sigma \text{ and } |s - \sigma| \geq \rho\}$, $|\hat{f}(s)| > \eta$.

Finally let $\hat{\mathscr{B}}(\sigma)$ be the algebra with pointwise product,

$$\hat{\mathscr{B}}(\sigma) = \{\hat{f} \,|\, \hat{f} = \hat{n}/\hat{d} \quad \text{with } \hat{n} \in \hat{\underline{\mathscr{A}}}(\sigma) \text{ and } \hat{d} \in \hat{\underline{\mathscr{A}}}^{\infty}(\sigma)\}$$

and $\mathscr{B}(\sigma)$ be the corresponding convolution algebra, i.e.

$$\mathscr{B}(\sigma) = \{f(\bullet) \,|\, \hat{f} \in \hat{\mathscr{B}}(\sigma)\}$$

$\mathscr{A}(0)$, $\underline{\mathscr{A}}(0)$, $\hat{\underline{\mathscr{A}}}(0)$, $\hat{\underline{\mathscr{A}}}^{\infty}(0)$, $\mathscr{B}(0)$ and $\hat{\mathscr{B}}(0)$ are abbreviated as $\mathscr{A}$, $\underline{\mathscr{A}}$, $\hat{\underline{\mathscr{A}}}$, $\hat{\underline{\mathscr{A}}}^{\infty}$, $\mathscr{B}$ and $\hat{\mathscr{B}}$, respectively.

3.3.2 Nyquist Theorem for Distributed Systems

We consider the feedback system in Figure 3.2, where

$$G(s) \in \hat{\mathscr{B}}^{m\times m} \tag{3.17}$$

Let (N,D) be a right coprime representation of G(s) [41], [24].

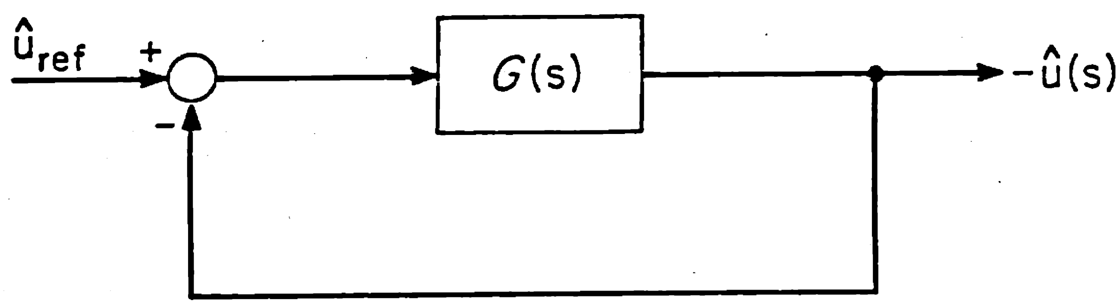


Figure 3.2 Nominal Feedback System

Equivalently,

(1) $G(s) = N(s)D^{-1}(s)$; (3.18)

(2) $N(s) \in \hat{\underline{\mathscr{A}}}^{m\times m}$, $D(s) \in \hat{\underline{\mathscr{A}}}^{\infty m\times m}$;

(3) there exist $U(s) \in \hat{\underline{\mathscr{A}}}^{m\times m}$, $V(s) \in \hat{\underline{\mathscr{A}}}^{m\times m}$ such that

$$U(s)N(s) + V(s)D(s) = I; \tag{3.19}$$

(4) det $D(s)$ is analytic and bounded away from zero at ∞ in the CRHP.

Since $D(s) \in \hat{\underline{\mathscr{A}}}^{\infty m\times m} \subset \hat{\underline{\mathscr{A}}}^{m\times m}$, under condition (4) det $D(s)$ has only a finite number of zeros in the CRHP [41]. Let p denote this number, counting multiplicities. In addition we assume that

(a) $G(s)$ tends to a constant matrix as $|s| \to \infty$ in the CRHP; (3.20)

(b) $G(s)$ has no zero on the imaginary axis. (3.21)

Under this formulation, one can deduce a Nyquist theorem based on Desoer and Wang [24] as follows.

Theorem 3.3 (Generalized Nyquist Theorem):

The system of Figure 3.3 is closed-loop stable if and only if

(1) $\det[I + G(j\omega)] \neq 0$ for all $\omega \in \mathbf{R}_+$ (i.e. $\omega \geq 0$); and

(2) $\det[I + G(j\omega)]|_{\omega\in\mathbf{R}_+}$ encircles the origin p times in the counterclockwise sense.

3.3.3 Robustness Theorems

We call the feedback system in Figure 3.2 the nominal feedback system. We define the perturbed feedback system to be the system of Figure 3.3, obtained by replacing G(s) with $\tilde{G}(s)$ in Figure 3.2.

Similarly to G(s), we assume

(1) $\tilde{G}(s) \in \hat{\mathcal{B}}^{m \times m}$; (3.22)

(2) $\tilde{G}(s)$ has a right coprime representation $(\tilde{N}, \tilde{D})$. (3.23)

As in Chapter 2, we define the additive error matrix by

$$E(s) = \tilde{G}(s) - G(s) \tag{3.24}$$

For the study of robustness, it is useful to write down the open- and closed-loop characteristic functions.

$$\Phi_{o\ell}(s) = \det(D(s)) \tag{3.25}$$

$$\Phi_{c\ell}(s) = \det(N(s) + D(s)) \tag{3.26}$$

$$\tilde{\phi}_{o\ell}(s) = \det(\tilde{D}(s)) \tag{3.27}$$

$$\tilde{\phi}_{c\ell}(s) = \det(\tilde{N}(s) + \tilde{D}(s)) \tag{3.28}$$

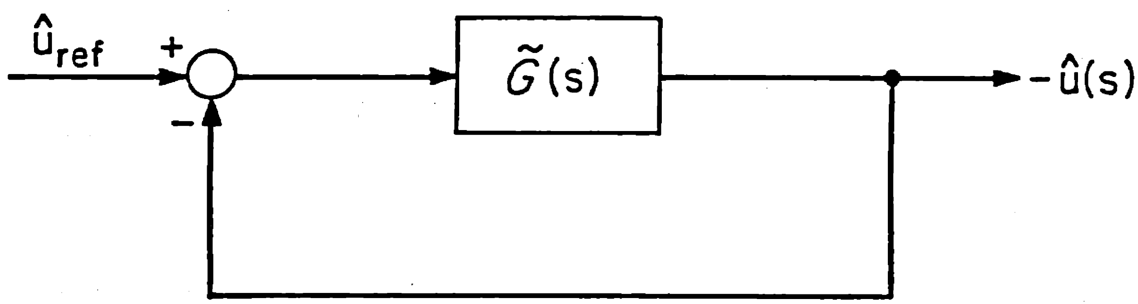


Figure 3.3 Perturbed Feedback System

They are related to the return difference by

$$\det(I + G(s)) = \frac{\phi_{c\ell}(s)}{\phi_{o\ell}(s)} \tag{3.29}$$

$$\det(I + \tilde{G}(s)) = \frac{\tilde{\phi}_{c\ell}(s)}{\tilde{\phi}_{o\ell}(s)} \tag{3.30}$$

By the coprimeness of N, D ($\tilde{N}$, $\tilde{D}$), s is an m-th zero of $\det(I + G(s))$ ($\det(I + \tilde{G}(s))$) if and only if s is an m-th order zero of $\phi_{c\ell}(s)$ ($\tilde{\phi}_{c\ell}(s)$). Also s is an m-th order pole of $\det(I + G(s))$ ($\det(I + \tilde{G}(s))$) if and only if s is an m-th order zero of $\phi_{o\ell}(s)$). Therefore the closed-loop system is stable if and only if $\phi_{c\ell}(s)$ ($\tilde{\phi}_{c\ell}(s)$) has no zero in the CRHP.

We can now state the robustness theorem for additive modeling errors as follows.

<u>Theorem 3.4 (Robustness Theorem for Additive Error):</u>

The perturbed system of Figure 3.3 is closed-loop stable (in the sense that $\tilde{\phi}_{c\ell}(s)$ has no zeros in the CRHP) if the following conditions hold:

(1) a. $\phi_{o\ell}(s)$ and $\tilde{\phi}_{o\ell}(s)$ have the same number of zeros in the CRHP;

b. $\phi_{c\ell}(s)$ has no zeros in the CRHP.

(2) $\sigma_{min}[I + G(j\omega)] > \sigma_{max}[E(j\omega)]$ for all $\omega \in \mathbf{R}_+$.

In the above $\phi_{o\ell}(s)$, $\phi_{c\ell}(s)$, $\tilde{\phi}_{o\ell}(s)$, $\tilde{\phi}_{c\ell}(s)$ are defined by Eqs. (3.45)-(3.28). The additive error matrix $E(s)$ is defined by Eq. (3.24). The notations $\sigma_{min}[\bullet]$ and $\sigma_{max}[\bullet]$ denote the minimum and maximum singular values of the matrix.

Proof of Theorem 3.4

This proof is very similar to that of Theorem 2.4. Let $\lambda \in [0,1]$. It is well known (e.g. [5] [44]) in singular value theory that

$$\sigma_{min}[I + G(j\omega) + \lambda E(j\omega)] > \sigma_{min}[I + G(j\omega)] - \lambda\sigma_{max}[E(j\omega)] \tag{3.31}$$

Therefore condition (2) guarantees that $I + G(j\omega) + \lambda E(j\omega)$ is nonsingular for all $\omega \in \mathbf{R}_+$.

Now suppose that as λ is varied continuously from zero to unity, the number of encirclements of

$$f(\lambda,\omega) = \det\ (I + G(j\omega) + \lambda E(j\omega)) \tag{3.32}$$

around the origin changes. Since $f(\lambda,\omega)$ is continuous in $(\lambda,\omega) \in [0,\ 1] \times \mathbf{R}_+$, its locus on the $\mathbf{R}_+$ forms a closed bounded contour in the complex plane for any $\lambda \in [0,\ 1]$. Therefore the only way for a change in the number of encirclements to occur is for the locus of $f(\lambda_0,\omega)$ to pass through the origin for some $\lambda_0 \in [0,\ 1]$. This is equivalent to requiring

$$\det(I + G(\omega_0) + \lambda_0 E(\omega_0)) = 0 \quad \text{for some } (\lambda_0, \omega_0) \in [0,\ 1] \times \mathbf{R}_+ \tag{3.33}$$

However we have shown that $I + G(j\omega) + \lambda E(j\omega)$ cannot be singular by assuming condition (2). Hence we conclude that $\det(I + \tilde{G}(j\omega))$, corresponding to $\lambda = 1$, encircles the origin the same number of times as $\det(I + G(j\omega))$, corresponding to $\lambda = 0$, along the path $\omega \in \mathbf{R}_+$. By applying Theorem 3.3 (Nyquist Theorem), we see that the perturbed system is closed-loop stable. (End of Proof)

Another useful way to describe the perturbed system with respect to the nominal system is to define a multiplicative factor matrix $L(s)$ by

$$\tilde{G}(s) = G(s)\ L(s) \tag{3.34}$$

This representation generalizes the gain and phase margin description of the single-input-single-output case, and is particularly useful to describe the robustness properties of the LQ system. The corresponding robustness theorem for multiplicative modeling error is the following.

Theorem 3.5 (Robustness Theorem for Multiplicative Error):

The perturbed system of Figure 3.3 is closed-loop stable (in the sense that $\tilde{\phi}_{c\ell}(s)$ (Eq. (3.28)) has no zero in the CRHP) if the following conditions hold:

(1) a. $\phi_{o\ell}(s)$ and $\tilde{\phi}_{o\ell}(s)$ have the same number of zeros in the CRHP;

b. $\phi_{c\ell}(s)$ has no zeros in the CRHP;

where $\phi_{o\ell}$, $\tilde{\phi}_{o\ell}$ and $\phi_{c\ell}$ are defined in Eqs. (3.25) - (3.27).

(2) $L(j\omega)$ has no eigenvalue in $\mathbf{R}_{+}$, for all $\omega \in \mathbf{R}_{+}$.

(3) $\sigma_{min}[I + G(j\omega)] > \sigma_{max}[L^{-1}(j\omega) - I]$ for all $\omega \in \mathbf{R}_{+}$. (3.35)

The proof of this theorem is essentially the same as that of Theorem 2.5, and is therefore omitted.

3.3.4 Robustness Properties of Distributed LQ Systems

The YFDT presented in Section 3.2 for infinite dimensional LQ systems contains the KFDI for LQHD systems as a special case when the control space is finite dimensional. Since by using the algebra $\hat{\mathcal{B}}$, we were able to characterize the robustness of distributed linear time-invariant systems in the same way that we characterize the robustness of LHDF systems, the robustness results for the LQHD optimal system in Chapter 2 can be extended to include a much wider class of distributed systems.

For the purposes of investigating the robustness properties of the LQ system, depicted in Figure 3.1, it is convenient to represent the model uncertainties as a multiplicative perturbation factor, as shown in Figure 3.4. In this configuration, the matrix $L(s)$ represents the model uncertainties of the system reflected to the point X in Figure 3.1 where the loop is broken, as a multiplicative perturbation in the control channel. The matrix $G(s)$ is the nominal system loop transfer function matrix, given by Eq. (3.11). Nominally $L(s)$ is the identity matrix and the LQ system is closed loop stable. However because of modeling errors and implementation constraints, the matrix $L(s)$ is subject to changes from its nominal value. Consequently, the product $G(s)L(s)$ can be viewed as the actual system loop transfer function matrix. This closed-loop system is stable if and only if the closed-loop system of Figure 3.5 is stable. In the figure, $R^{1/2}$ is a square root of the control weighting matrix, in the sense that

$$R = R^{1/2^*} R^{1/2} \tag{3.36}$$

With this formulation, we can state some of the robustness properties of the LQ system as the following theorem.

<u>Theorem 3.6 (Stability Margin of Distributed LQ Systems)</u>

The multiplicatively perturbed LQ system in Figure 3.4 is closed-loop stable, i.e. the characteristic function $\tilde{\phi}_{c\ell}(s)$ (Eq. (3.28)) has no CRHP zeros, provided that the following conditions are satisfied:

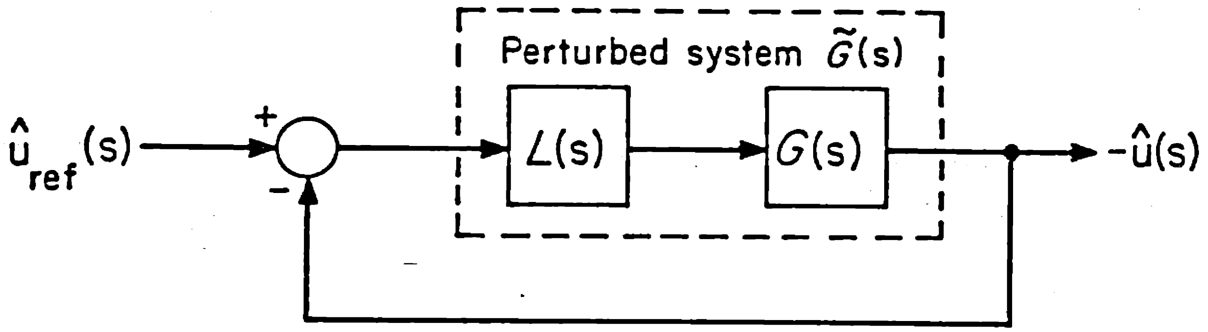


Figure 3.4 Multiplicatively Perturbed Feedback System

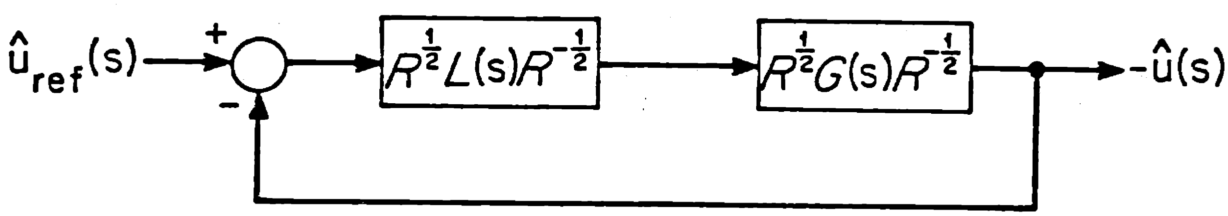


Figure 3.5 Perturbed Feedback System Representation for the Derivation of LQ Robustness Properties

(1) $G(s)$ is specified by Eqs. (3.3), (3.4), (3.11), as the closed-loop solution to the LQ control problem.

(2) $\phi_{o\ell}(s)$ and $\tilde{\phi}_{o\ell}(s)$ have the same number of CRHP zeros, where $\phi_{o\ell}(s)$, $\tilde{\phi}_{o\ell}(s)$, are defined in Eqs. (3.25), (3.27), respectively, and both $\phi_{o\ell}(s)$ and $\tilde{\phi}_{o\ell}(s)$ have no zeros on the imaginary axis.

(3) $L(j\omega)$ has no eigenvalue in $\mathbf{R}_+$.

(4) $\sigma_{max}(R^{1/2}L^{-1}(j\omega)R^{-1/2} - I) \leq 1$ for all $\omega \in \mathbf{R}_+$.

<u>Proof of Theorem 3.6</u>

By theorem 3.2, $\phi_{c\ell}(s)$ (Eq. (3.26)) has no zero in the CRHP. Note that

$$\det[I + (R^{1/2}G(s)R^{-1/2})(R^{1/2}L(s)R^{-1/2})] = \det[I + G(s)L(s)] \tag{3.37}$$

Hence

$$\det[I + (R^{1/2}G(s)R^{-1/2})(R^{1/2}L(s)R^{-1/2})] = \frac{\tilde{\phi}_{o\ell}(s)}{\tilde{\phi}_{c\ell}(s)} \tag{3.38}$$

Similarly,

$$\det[I + R^{1/2}G(s)R^{-1/2}] = \frac{\phi_{o\ell}(s)}{\phi_{c\ell}(s)} \tag{3.39}$$

Theorem (3.1) YFDT impiles that

$$\sigma_{max}[I + R^{1/2}G(j\omega)R^{-1/2}] > 1 \quad \text{for all } \omega \in \mathbf{R}_{+} \tag{3.40}$$

Therefore in conjunction with Theorem 3.5, condition (4) guarantees that $\tilde{\phi}_{c\ell}(s)$ has no zeros in the CRHP. (End of Proof)

The following are corollaries of Theorem 3.6. Their proofs are omitted since they are the same as the standard proofs (e.g. [5]) in the finite dimensional case.

<u>Corollary 3.7</u>

The multiplicatively perturbed LQ feedback system is stable if

$$\sigma_{max}(L^{-1}(j\omega)-I) < [\text{cond}(R)]^{-1/2} \;, \quad \text{for all } \omega \in \mathbf{R}_{+} \tag{3.41}$$

where cond(R) denotes the condition number of R

$$\text{cond}(R) = \frac{\sigma_{max}(R)}{\sigma_{min}(R)} \tag{3.42}$$

defined as the ratio of its maximum singular value to its minimum singular value.

<u>Corollary 3.8</u>

If $R > 0$ is diagonal, then simultaneously in each feedback loop, the LQHD system has the following guaranteed stability margins,

1. $[1/2, \infty]$ gain margin
2. $\pm 60^{\circ}$ phase margin

<u>Remark:</u> Tsitsiklis and Athans [25] have shown that the robustness properties in this corollary actually hold for nonlinear systems, under certain assumptions. Their proof is basically an extension of the one found in Yakubovich [22]. Our formulation in this thesis, however, provides a more general framework for characterizing robustness.

<u>Corollary 3.9</u>

If R is block diagonal of the form

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \tag{3.43}$$

and $L(s)$ consists of crossfeed perturbation of the form

$$L(s) = \begin{bmatrix} I & X(s) \\ 0 & I \end{bmatrix} , \tag{3.44}$$

Then the LQ system is stable if

$$\sigma_{max}(X(j\omega)) < [\frac{\sigma_{min}(R_2)}{\sigma_{max}(R_1)}]^{1/2} \quad , \text{ for all } \omega \in \mathbf{R}_+ \tag{3.45}$$

Similarly, if $L(s)$ is of the form

$$L(s) = \begin{bmatrix} I & 0 \\ X(s) & I \end{bmatrix} \quad , \tag{3.46}$$

then the LQ system is stable if

$$\sigma_{max}(X(j\omega)) < [\frac{\sigma_{min}(R_1)}{\sigma_{max}(R_2)}]^{1/2} \quad , \text{ for all } \omega \in \mathbf{R}_+ \tag{3.47}$$

In particular, if $R = \rho I$, where ρ is a positive scalar, then the LQ system is stable if

$$\sigma_{max}(X(j\omega)) < 1 \quad , \text{ for all } \omega \in \mathbf{R}_+ \tag{3.48}$$

for both types of crossfeed perturbations (Eqs. (3.44) and (3.46)).

3.4 Lyapunov Control Systems

Lyapunov control systems form a subclass of LQ control systems, in which the control gains is computed via the solution of a Lyapunov equation instead of the Riccati equation. A nice introduction to the finite dimensional Lyapunov control system is presented in Wong [12]. In this section, we discuss the semigroup description of Lyapunov control systems, and their robustness properties.

Let A be the infinitesimal generator of a strongly continuous semigroup of operators over $\mathbf{X}$, and let B be a linear operator. Suppose A is Hurwitz. Then the Lyapunov controller for the dynamics

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t); \quad x(0) \text{ given} \tag{3.49}$$

is given in feedback form by

$$u(t) = -H^{*}x(t) \tag{3.50}$$

where H is the solution to the following set of equations,

$$KA + A^{*}K + Q = 0 \tag{3.51}$$

$$KB - HR = 0 \tag{3.52}$$

The self-adjoint operators Q and R are defined in Eqs. (3.1) and (3.2) as in the LQ problem (Eqs. (3.8) - (3.10)). Also assume that A^{*} and $Q^{1/2^{*}}$ are stablizable.

Since A is assumed to be Hurwitz, the solution K to Eq. (3.51) is at least positive semi-definite. By direct manipulation of Eqs. (3.51) - (3.52), we get

$$K(A - BH^{*}) + (A - BH^{*})^{*}K = -Q - 2HRH^{*} \tag{3.53}$$

Thus $A - BH^{*}$ is Hurwitz. In addition, Eq. (3.51) can be written as

$$KA + A^{*}K + (Q + HRH^{*}) - HRH^{*} = 0 \tag{3.54}$$

This shows that the Lyapunov control system belongs to the class of LQ systems.

Yakubovich [23] has a frequency domain theorem which is useful for studying the robustness of Lyapunov control systems.

Theorem 3.10:

For the existence of K and H satisfying Eq. (3.51) - (3.52), it is necessary and sufficient that

$$RG(j\omega) + G^{*}(j\omega)R \geq 0 \quad \text{for all } \omega \geq 0 \tag{3.55}$$

where $G(j\omega)$ is the loop function defined by

$$G(s) = H^{*}(sI - A)^{-1}B \tag{3.56}$$

Recall that a perturbed system can be represented multiplicatively as

$$\tilde{G}(j\omega) = G(j\omega)L(j\omega) \tag{3.57}$$

The Lyapunov control system will remain stable as long as the multiplicative perturbation $L(j\omega)$ is "passive" [40].

For Theorem 3.11 and its Corollary 3.12, we assume that the control space **U** is finite dimensional.

Theorem 3.11:

The perturbed system will remain closed-loop stable as long as

$$RL(j\omega) + L^{*}(j\omega)R \geq 0 \quad \text{for all } \omega \geq 0 \tag{3.58}$$

Proof of Theorem 3.11:

This proof uses an imbedding argument similar to that for Theorem 3.4. In this case, we consider the continuous function

$$f(\lambda,\omega) = \det(I + G(j\omega)(\lambda L(j\omega) + (1 - \lambda)I)) \tag{3.59}$$

where $\lambda \in [0, 1]$, and $\omega \geq 0$. We want to show that $f(\lambda,\omega) \neq 0$ for all λ and ω. Hence there is no change in encirclements when the graph $f(0,\omega)$ is continuously collapsed to that of $f(1,\omega)$.

We prove this by contradiction. Let

$$P(\lambda,\omega) = \lambda L(j\omega) + (1 - \lambda)I \tag{3.60}$$

and suppose that $f(\lambda_0,\omega_0) = 0$ for some λ_0, ω_0. Then there exists a nonzero vector $u \in \mathbf{U}$ such that

$$[I + G(j\omega_0)P(\lambda_0,\omega_0)]u = 0 \tag{3.61}$$

i.e.

$$u = - G(j\omega_0)P(\lambda_0,\omega_0)u \tag{3.62}$$

Define

$$z = P(\lambda_0,\omega_0)u \tag{3.63}$$

Then Eq. (3.62) becomes

$$z = - P(\lambda_0,\omega_0)G(j\omega_0)z \tag{3.64}$$

Thus

$$z^* R G(j\omega_0)z + z^* G^*(j\omega_0)Rz$$

$$= - z^* G^*(j\omega_0)[RP(\lambda_0,\omega_0) + P^*(\lambda_0,\omega_0)R]G(j\omega_0)z \geq 0 \tag{3.65}$$

Since $G(j\omega_0)z = u$ is nonzero,

$$RP(\lambda_0,\omega_0) + P^*(\lambda_0,\omega_0)R \le 0 \qquad (3.67)$$

but this contradicts Eq. (3.58) because of Eq. (3.60).

(End of Proof)

As in the LQ case (see Theorem 3.6), by considering $R^{-\frac{1}{2}*}G(s)R^{-\frac{1}{2}}$ and $R^{-\frac{1}{2}*}L(s)R^{-\frac{1}{2}}$ in place of $G(s)$ and $L(s)$, one can easily obtain the following Corollary.

Corollary 3.12

If R is diagonal, then the Lyapunov control system has, simultaneously in all control channels, at least

(1) $\pm 90^\circ$ phase margins,

(2) 100% reduction gain margins, and

(3) ∞ gain margins.

In Chapter 4, we study a Lyapunov control design for the vibration suppression of a flexible beam.

In case **U** is infinite dimensional, we can state the following theorem. Part (1) of this theorem is an extension of the necessary part of Theorem 3.10. Part (2) provides a test for stability, similar to the condition in Eq. (3.58), except here one must go through the test for the whole CRHP.

Theorem 3.13:

For Lyapunov control systems,

(1) $RG(s) + G^*(s)R \geq 0$ for all $s \in$ CRHP

(2) The perturbed system will remain stable as long as

$RL(s) + L(s)R \geq 0$ for all $s \in$ CRHP

Outline of Proof:

The result (1) can be easily obtained by manipulating Eqs. (3.51) - (3.52). For (2), we want to show that $I + G(s)L(s)$ is invertible for all $s \in$ CRHP. This can be done by using a contradiction argument similar to that in Eqs. (3.60) - (3.67), i.e. by supposing $I + G(s_0)L(s_0)$ is singular for some $s_0 \in$ CRHP.

(End of Proof)

CHAPTER 4

SYNTHESIS OF A ROBUST CONTROLLER FOR THE SUPPRESSION OF VIBRATION OF A FLEXIBLE BEAM

4.1 Introduction

The robustness characterization that we discussed in Chapters 2 and 3 are useful for the robustness analysis of distributed feedback systems, regardless of what the control design method is. However, the guaranteed stability margins for LQ optimal systems that we derived has one additional important impact. They allow the control system designer to synthesize closed-loop stable, realizable (finite dimensional) feedback controllers for infinite dimensional systems without going through any trial and error iteration in design. As discussed in Chapter 1, the trial and error iterative process is a notable shortcoming of design methods which depend on the use of reduced-order models. In this chapter, we use a flexible beam, simply supported at both ends, to illustrate a design synthesis method which leads to a stable closed-loop system when the inherent damping is sufficiently large.

Also we use this opportunity to study the importance of inherent damping in large flexible structure. Stein and Greene [1] showed, by example, that sufficient damping must be present in order to design stable finite dimensional compensators. Our results support their findings. We show that if either all the temporal modes of the flexible structure have the same damping ratio or the same damping coefficient, then a stable finite-dimensional compensator can be implemented by using spatial sampling. In the latter case, a dynamical compensator is required to attenuate the high frequency components in addition to the sampled measurements.

We organize this chapter as follows. In Section 4.2 we describe a uniform slender beam, simply supported at both ends. In Section 4.3 we describe the system we want to control. A point actuator is used to control the vibration of the beam and the damping ratio is assumed to be a constant for all modes. Also we present an open-loop analysis of the beam. In Section 4.4 we formulate a Lyapunov control problem for the flexible beam using the semigroup framework. The closed-loop behavior of the resulting optimal feedback system is discussed. In Section 4.5 we show that the continuous distributed measurements required by the Lyapunov controller can be adequately approximated with finitely many samples. In Section 4.6 we take a closer look at the relationship between the size of the structure's inherent damping and the density of spatial sampling required to maintain a stable approximation.

In Sections 4.3–4.6, a constant damping ratio is assumed for all modal deflections. In Section 4.7, we consider instead the case in which the inherent damping corresponds to a constant damping coefficient for all modes. In this case, since the open-loop system frequency response has an envelope of peaks that does not roll off, a high frequency attenuator is used, in addition to increasing the number of spatial samples.

4.2 A Simply Supported Flexible Beam

The dynamics of a flexible beam freely hinged at both ends with constant density per unit length, denoted by ρ, and constant flexural rigidity EI, can be represented by the following system of partial differential equations [47]

$$\rho \frac{\partial^2}{\partial t^2} y(x,t) + L \frac{\partial}{\partial t} y(x,t) + EI \frac{\partial^4}{\partial x^4} y(x,t) = w(x)u(x,t) \tag{4.1}$$

along with the boundary conditions

$$y(0,t)\ ,\ y(a,t) = 0;\ \frac{\partial^2}{\partial x^2} y(0,t)\ ,\ \frac{\partial^2}{\partial x^2} y(a,t) = 0\ . \tag{4.2}$$

The independent variable x is the longitudinal coordinate ranging from 0 to a, where a is the length of the beam, and the variable $y(\bullet,\bullet)$, which depends on x and time t, denotes the vertical displacement of the beam from its equilibrium position. The function $u(x,t)$ and the product $w(x)u(x,t)$ represent the control input and the force, respectively, acting on the beam at x and t. Figure 4.1 illustrates these arrangements. The control input function $u(\bullet,\bullet)$ is an element of $L_2([0,a]\times[0,\infty); \mathbf{R})$. The function

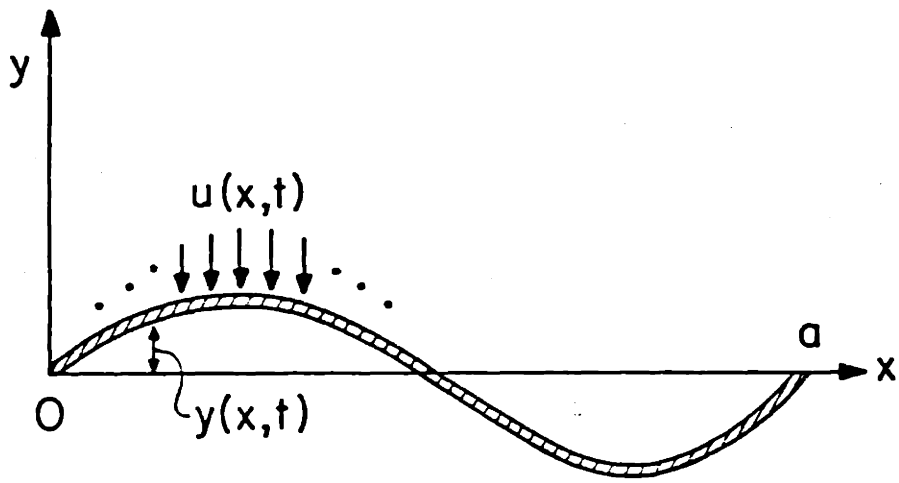


Figure 4.1 Flexible Beam Example

$w(x)$ is real and may vanish for some x and is integrable i.e. $w(\bullet) \in \mathbf{L}_1$. The damping term $L \frac{\partial}{\partial t} y(x,t)$ represents the inherent energy dissipation [47] [48].

The assumption of an inherent damping term is extremely crucial to vibration control problems, as evident from the following in literature.

(1) Gibson [28] showed that positive definiteness of L is a sufficient condition for his approximate method, based on optimal control of a sequence of reduced-order modal models, to converge, and for the limiting control law to be closed-loop stable.

(2) Gibson [27] showed that open-loop uniformly exponential stability is a necessary condition for the existence of a compact feedback compensator that is closed-loop stable.

(3) Stein and Greene [1] used classical control arguments to demonstrate that a practical implementation of a closed-loop stable finite dimensional feedback compensator is impossible unless sufficiently large inherent damping is assumed.

In addition, we have discussed in Chapter 3 that it is possible to use the infinite dimensional robustness theorems when a sufficient amount of inherent damping is present.

4.3 Open-Loop Analysis

Beginning in this section we study a simplified model of the flexible beam. We consider the single actuator case in which a point actuator is assumed to be situated at the midpoint of the beam, exerting a force on the beam for all time $t \geq 0$. Also we assume that the operator L which accounts for the inherent damping in the beam dynamics is given by

$$L = - \zeta \frac{\partial^2}{\partial x^2} \quad , \zeta > 0 \tag{4.3}$$

where ζ is a small positive number representing a small constant damping ratio for the beam's modal responses. In a moment, we will clarify this in terms of a series of second-order transfer functions.

In addition, referring to Eqs. (4.1) - (4.2), we let, without loss of generality,

$$a = 1 \tag{4.4}$$

$$\rho = 1 \tag{4.5}$$

$$EI = 1 \tag{4.6}$$

These assumptions simplify the ensuing analysis considerably.

Rewriting Eqs. (4.1) and (4.2) under the assumptions of Eqs. (4.3) - (4.6), we obtain the following set of equations for the dynamics of the uniform flexible beam simply supported at both ends.

$$\frac{\partial^4}{\partial x^4} y(x,t) - \zeta \frac{\partial^2}{\partial x^2} \frac{\partial}{\partial t} y(x,t) + \frac{\partial^2}{\partial t^2} y(x,t) = u(t) \bullet \delta(x - \tfrac{1}{2}) \quad (4.7)$$

where $\delta(x - \frac{1}{2})$ is the spatial Dirac delta function at $x = \frac{1}{2}$. The boundary conditions that govern the dynamics are

$$y(0,t) = y(1,t) = 0 \quad (4.8)$$

$$\frac{\partial^2}{\partial x^2} y(0,t) = \frac{\partial^2}{\partial x^2} y(1,t) = 0 \quad (4.9)$$

It is easy to show (Appendix C) that the solution to Eqs. (4.7) - (4.9) is representable in modal form,

$$y(x,t) = \sum_{i=1}^{\infty} \sin(i\pi x) T_i(t) \quad (4.10)$$

where the functions $\sin(i\pi x)$ and $T_i(t)$ are the so-called mode shapes and modal deflection of the beam. The modal deflection $T_i(t)$ is given by the equation

$$\frac{d^2}{dt^2} T_i(t) + \zeta i^2\pi^2 \frac{d}{dt} T_i(t) + i^4\pi^4 T_i(t) = 2u(t)\sin\left(\frac{i\pi}{2}\right) \ ; \quad (4.11)$$

$$i = 0,\ 1, \ldots$$

Observe that because of the boundary conditions in Eqs. (4.8) - (4.9), the mode shape functions $\sqrt{2}\sin(i\pi x)$, $i = 1,2,3,\ldots$, form a complete orthonormal basis for the beam shape between x=0 and x=1.

Let $\hat{u}(s)$, $\hat{T}_i(s)$ denote the Laplace transforms of $u(t)$ and $T_i(t)$, respectively. It is useful to write down the transfer function from the control of $\hat{u}(s)$ to each individual modal deflection $\hat{T}_i(s)$. This is obtained in a straightforward fashion from Eq. (4.11).

$$\frac{\hat{T}_i(s)}{\hat{u}(s)} = \frac{2\sin\left(\frac{i\pi}{2}\right)}{s^2+\zeta i^2\pi^2 s + i^4\pi^4} \ ; \ i = 1,\ 2, \ldots \quad (4.12)$$

By examining this modal deflection transfer function, we see that the temporal frequency is $i^2\pi^2$, and the damping ratio is ζ. This explains our earlier claim in Eq. (4.3) that ζ corresponds to a constant damping ratio for all deflection modes.

Let $\dot{T}_i(t)$ denote the rate of change of $T_i(t)$, i.e.

$$\dot{T}_i(t) = \frac{d}{dt} T_i(t) \quad (4.13)$$

Then

$$\frac{\hat{\dot{T}}_i(s)}{\hat{u}(s)} = \frac{2s \sin(\frac{i\pi}{2})}{s^2 + \zeta i^2\pi^2 s + i^4\pi^4} \; ; \; i = 1, 2, \ldots \tag{4.14}$$

where $\hat{\dot{T}}(s)$ is the Laplace transform of $\dot{T}_i(t)$. This last transfer function is important here because:

(1) we assume that the available sensors are of the linear-velocity type;

(2) it introduces a 90° phase lead which simplifies the control design task.

Let $\dot{y}(x,t)$ denote the time-derivative of $y(x,t)$,

$$\dot{y}(x,t) = \frac{\partial}{\partial t} y(x,t) \tag{4.15}$$

Then

$$\dot{y}(x,t) = \sum_{i=1}^{\infty} \sin(i\pi x)\dot{T}_i(t) \tag{4.16}$$

Observe that in Eqs.(4.12) and (4.14), since

$$\sin(\frac{i\pi}{2}) = 0 \quad \text{for } i \text{ even} \tag{4.17}$$

the even modes of the beam are not controllable by $u(t)$. However all modes are stabilizable because of the inherent damping mechanism. Any vibration suppression control problem for the system of Eqs. (4.7) - (4.9) is in effect a control problem for the odd numbered modes only. This is a consequence of our choice of placing the actuator at the mid-point of the beam. By placing the actuator elsewhere, for example at an irrational point, complete controllability, albeit small for some modes, can be obtained.

4.4 Optimal Lyapunov Control

In this section we derive the optimal Lyapunov control for the system in Eqs. (4.7) - (4.9). A semigroup formulation is used.

Recall from Chapter 3 that the Lyapunov control system is a subclass of the LQ control system. The Lyapunov control system has the additional properties that it is in general easier to compute and has superior stability margins.

Schaechter [49] considered the LQ formulation for the control of the flexible beam, and also for the estimation of distributed parameter systems in [50]. In both cases, the optimal solution calls for solving a matrix Riccati equation over a ring of polynomials [72], i.e., the matrices in the equation as well as the solution contain elements which are polynomials in D, where D denotes the operator $\frac{\partial}{\partial x}$. The solution of this Riccati equation, however, is not easy to obtain [51]. In addition, the exact solution is generally irrational, i.e. it is not an element of the quotient field [72] of the ring of polynomials, and hence is often difficult to characterize. Brockett and Willems [52] considered a wider class of LQ problems for partial differential equations, in their generalized path integral method. Their method also required the control system designer to deal with irrational functions of D.

We use instead an approach by solving a Lyapunov equation. The linearity of the Lyapunov equation eliminates the aforementioned pitfalls associated with solving the Riccati equation. Exact solutions [73] can be obtained for this Lyapunov formulation. In addition, the Lyapunov control sets up a framework in which one can study the dependence of stability on the inherent damping.

We rewrite Eq. (4.12) in the following semigroup representation.

$$\begin{bmatrix} \frac{\partial y}{\partial t} \\ \frac{\partial^2 y}{\partial t^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{\partial^4}{\partial x^4} & \zeta \frac{\partial^2}{\partial x^2} \end{bmatrix} \begin{bmatrix} y \\ \frac{\partial y}{\partial t} \end{bmatrix} + \begin{bmatrix} 0 \\ \delta(x - \frac{1}{2}) \end{bmatrix} u(t) \tag{4.18}$$

The spectrum of the infinitesimal generator of the semigroup,

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{\partial^4}{\partial x^4} & \zeta \frac{\partial^2}{\partial x^2} \end{bmatrix} \tag{4.19}$$

lies entirely in the open left-half plane, thus permitting the use of the Lyapunov method.

For the Lyapunov design, we choose

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 2\zeta \end{bmatrix} \tag{4.20}$$

and

$$R = 1 \tag{4.21}$$

This choice of Q will lead to a control gain that is independent of the damping ratio. The operator A maps the space S(0,1) x S(0,1) into itself, where S(0,1) is the set of smooth functions $f: [0,1] \rightarrow \mathbf{R}$

$$S(0,1) = \{f(\bullet) \mid \frac{d^n f}{dx^n} \text{ exists for all } n;\ f(0) = f(1) = 0$$

$$\text{and } \frac{d^2 f}{dx^2}(0) = \frac{d^2 f}{dx^2}(1) = 0\} \tag{4.22}$$

It is trivial to verify that S(0,1) is a separable Hilbert space to which we can associate the inner product $\langle \bullet, \bullet \rangle$

$$\langle f,g \rangle = \int_0^1 f(x)g(x)dx \ ; \quad f,g \in S(0,1) \tag{4.23}$$

and with an abuse of notation, the inner product for the space $S(0,1) \times S(0,1)$ is

$$< \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} , \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} > = \int_0^1 [f_1(x)g_1(x) + f_2(s)g_2(x)]dx;$$

$$f_1, f_2, g_1, g_2 \in S(0,1) \qquad (4.24)$$

Before turning to the derviation of the optimal controller, we would like to show that the flexible beam control system belongs to the class of systems discussed in Section 3.2; hence we can use the robustness results in Chapter 3 for this application.

Referring to Eqs. (3.15), for the flexible beam,

$$P = \frac{\partial^4}{\partial x^4}$$

$$L = - \zeta \frac{\partial^2}{\partial x^2}$$

Clearly for all mode shapes $\sin(i\pi x)$,

$$<\frac{\partial^4}{\partial x^4} \sin(i\pi x), \sin(i\pi x)> \geq \sin^2(i\pi x)$$

and for small ζ,

$$<-\zeta \frac{\partial^2}{\partial x^2} \sin(i\pi x),\ \sin(i\pi x)> \leq <\frac{\partial^4}{\partial x^4} \sin(i\pi x),\ \frac{\partial^4}{\partial x^4} \sin(i\pi x)>$$

Therefore Eqs. (3.16a) and (3.16b) are satisifed, and it is legitimate for us to use the LQ robustness properties to design the vibration suppression controller.

Let A^* be the adjoint operator of A with respect to the inner product defined by Eq. (4.24), i.e. for all $f_1, f_2, g_1, g_2 \in S(0,1)$,

$$< \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} ,\ A \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} > \ = \ < A^* \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} ,\ \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} > \tag{4.25}$$

Then by direct comparison of the two sides in Eq. (4.25),

$$A^* = \begin{bmatrix} 0 & -\frac{\partial^4}{\partial x^4} \\ 1 & \zeta \frac{\partial^2}{\partial x^2} \end{bmatrix} \tag{4.26}$$

We form the following Lyapunov equation.

$$A^*K + KA = -Q \tag{4.27}$$

where K: $S(0,1) \times S(0,1) \rightarrow S(0,1) \times S(0,1)$ is a self-adjoint and positive definite linear operator. The solution K to this equation is unique [23] and is given by Eqs. (4.28) - (4.30) below.

$$K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \tag{4.28}$$

$$K_1 - K_2 \frac{\partial^4}{\partial x^4} = 0 \tag{4.29}$$

$$K_2 \frac{\partial^2}{\partial x^2} = -1 \tag{4.30}$$

where K_1 and K_2 are self-adjoint and positive definite linear operators on $S(0,1)$. Note that conceptually, K_2 can be intepreted as the double integral operator, which satisfies the appropriate boundary values.

The Lyapunov control law constructed by using the weights Q and R specified by Eqs. (4.20) and (4.21) is given in the feedback form as

$$u(t)\delta(x - \frac{1}{2}) = - G \begin{bmatrix} y(x,t) \\ \frac{\partial y}{\partial t}(x,t) \end{bmatrix} \tag{4.31}$$

where G: S(0,1) x S(0,1) → L_1 is the map

$$G = R^{-1} [0 \quad \delta(x - \frac{1}{2})]K$$

i.e.

$$G = [0 \quad \delta(x - \frac{1}{2})K_2] \tag{4.32}$$

Or simply

$$u(t) = K_2 \frac{\partial y}{\partial t}(x,t)\Big|_{x = \frac{1}{2}} \tag{4.33}$$

The operator K_2 can be represented in terms of a Green's function [69] [70] $h(x,\xi)$,

$$K_2 f = \int_0^1 h(x,\xi) f(\xi) d\xi \ ; \quad f \in S(0,1) \tag{4.34}$$

where

$$h(x,\xi) = 0 \qquad \text{at } \xi = 0 \text{ and } \xi = 1 \tag{4.35}$$

and

$$\frac{\partial^2}{\partial \xi^2} h(x,\xi) = 0 \qquad \text{at } \xi = 0 \text{ and } \xi = 1 \tag{4.36}$$

The Green's function is determined in Appendix D. It is given by

$$h(x,\xi) = \sum_{i=1}^{\infty} \frac{2}{i^2 \pi^2} \sin(i\pi x) \sin(i\pi \xi) \tag{4.37}$$

Combining this result with Eqs. (4.31) - (4.34) we can write down the optimal Lyapunov control law as

$$u(t)\delta(x - \tfrac{1}{2}) = \int_0^1 \Big[\sum_{i=1}^{\infty} \frac{2}{i^2 \pi^2} \sin(i\pi x) \sin(i\pi \xi) \Big] \bullet \delta(x - \tfrac{1}{2}) \frac{\partial y}{\partial t} (\xi, t) d\xi \tag{4.38}$$

Integrating both sides with respect to x to get rid of the delta functions, we obtain

$$u(t) = \int_0^1 \Big[\sum_{i=1}^{\infty} \frac{2}{i^2\pi^2} \sin(\frac{i\pi}{2}) \sin(i\pi\xi)\Big] \frac{\partial y}{\partial t}(\xi, t)d\xi \tag{4.39}$$

To derive the loop transfer function for this optimal system, breaking the loop at the control, we note from Eq. (4.16) that

$$\frac{\partial y}{\partial t}(\xi, t) = \sum_{k=1}^{\infty} \sin(k\pi\xi)\ \dot{T}_k(t) \tag{4.40}$$

Substituting this into Eq. (4.39) we get

$$u(t) = \int_0^1 \Big[\sum_{i=1}^{\infty} \frac{2}{i^2\pi^2} \sin(\frac{i\pi}{2}) \sin(i\pi\xi)\Big] \Big[\sum_{k=1}^{\infty} \sin(k\pi\xi)\dot{T}_k(t)\Big]d\xi \tag{4.41}$$

Because of orthogonality, this is equivalent to

$$u(t) = \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} \sin(\frac{k\pi}{2})\ \dot{T}_k(t) \tag{4.42}$$

Combining this with Eq. (4.14) we obtain the loop transfer function for this optimal feedback system.

$$g(s) = \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} \sin^2(\frac{k\pi}{2}) \frac{2s}{s^2 + \zeta k^2\pi^2 s + k^4\pi^4} \tag{4.43}$$

This loop transfer function has several properties which are significant for the control design.

According to Chapter 3, g(s) is passive [40] [53], i.e.

$$g(j\omega) + g^{*}(j\omega) \geq 0 \qquad \text{for all } \omega \in \mathbf{R}_{+} \tag{4.44}$$

This can be readily verified by inspecting Eq. (4.43), since $\sin^2(\frac{k\pi}{2})$ is either 0 or 1. The passivity guarantees that the optimal system is closed-loop stable (Willems [53]). In fact one can characterize the additive stability margin of this Lyapunov control system as follows.

<u>Lemma 4.1 (Guaranteed Additive Stability Margin for Lyapunov Control):</u>

Let $\tilde{g}(s)$ be the loop transfer function of a unity feedback system, and $\tilde{g}(s)$ has no open-loop poles in the CRHP. Then this system is closed-loop stable if

$$|g(j\omega) - \tilde{g}(j\omega)| < (1 + |g(j\omega)|^2)^{1/2} \text{ for all } \omega \in \mathbf{R}_{+} \tag{4.45}$$

where g(jω) is the loop transfer function of the optimal Lyapunov control system.

Proof of Lemma 4.1

Adding $1 + g(j\omega)g^*(j\omega)$ to both sides of Eq. (4.44), we get

$$(1 + g(j\omega))(1 + g^*(j\omega)) \geq 1 + g(j\omega)g^*(j\omega) \qquad \text{for all } \omega \in \mathbf{R}_+ \tag{4.46}$$

or simply,

$$|1 + g(j\omega)|^2 \geq 1 + |g(j\omega)|^2 \qquad \text{for all } \omega \in \mathbf{R}_+ \tag{4.47}$$

Thus if the condition in Eq. (4.45) holds, then by the triangle inequality, i.e.

$$|1 + \tilde{g}| = |1 + g + \tilde{g} - g| \geq |1 + g| - |\tilde{g} - g|$$

we get

$$|1 + \tilde{g}(j\omega)| > 0 \qquad \text{for all } \omega \in \mathbf{R}_+ \tag{4.48}$$

Hence the system is closed loop stable by the Nyquist theorem (Willems [53]). (End of Proof)

Now let us examine the shape of $g(s)$. We show in Appendix E that at low frequencies, $|g(j\omega)|$ consists of a series of peaks at the natural frequencies

$$\omega = k^2\pi^2 \quad ; \quad k \text{ odd} \tag{4.49}$$

with

$$\text{the size of the peak} \approx \frac{2}{\zeta k^4 \pi^4} \tag{4.50}$$

In addition, $|g(j\omega)|$ has a finite cutoff frequency ω_c, i.e. $|g(j\omega)|$ is less than unity for all $\omega > \omega_c$. At high frequencies, $|g(j\omega)|$ rolls off at the rate of 20 db/decade. This behavior is shown in the frequency plot in Figure 4.2.

4.5 Implementation of a Closed-Loop Stable Suboptimal Design

The optimal feedback control law that we derived in Section 4.4 cannot be realized exactly in actual practice. It requires continuous measurements along the entire length of the beam as indicated by Eq. (4.33) or complete knowledge of all modes as indicated by Eq. (4.42). Neither of these is achievable physically.

We consider using a suboptimal control of the form

$$u(t) = \sum_{k=1}^{M} \frac{1}{k^2\pi^2} \sin\left(\frac{k\pi}{2}\right) \dot{T}_k^N(t) \tag{4.51}$$

where $\dot{T}_k^N(t)$ is an estimate of $\dot{T}_k(t)$ from the measurements $\frac{\partial y}{\partial t}(x,t)$ at N discrete locations, described below, along the beam. The estimate $\dot{T}_k^N(t)$ is obtained by correlating the measurements with the

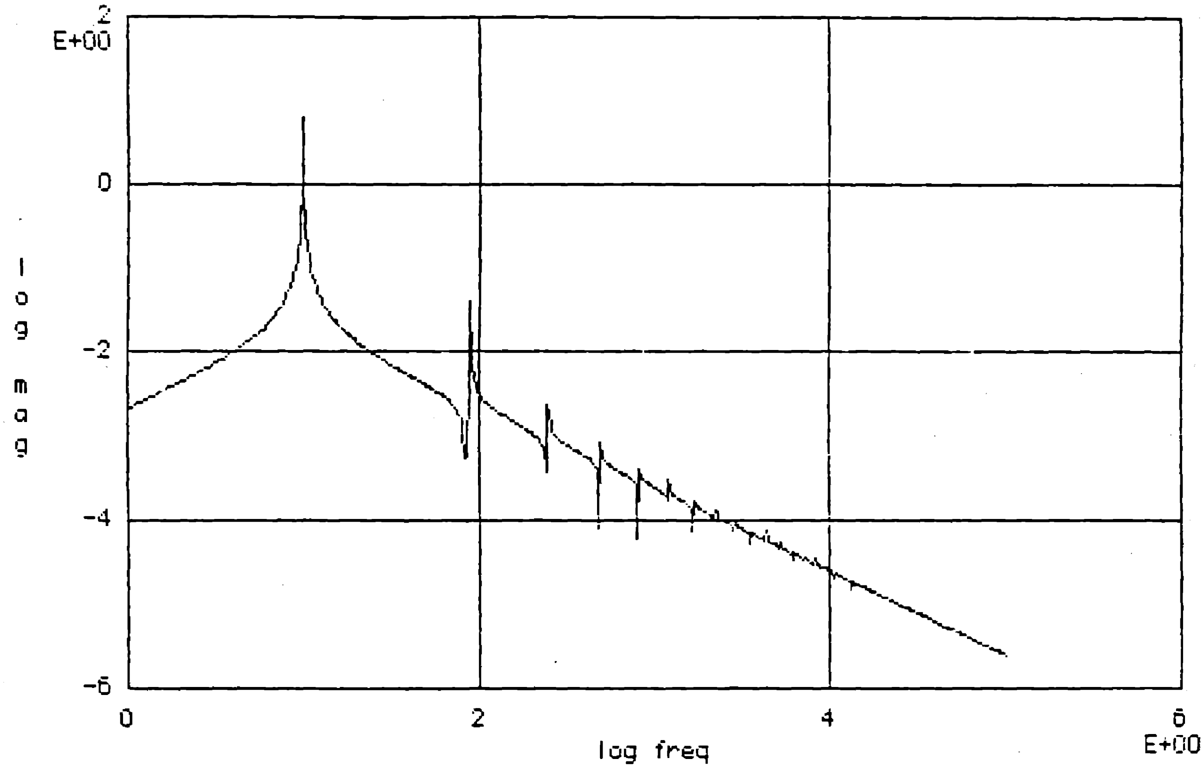
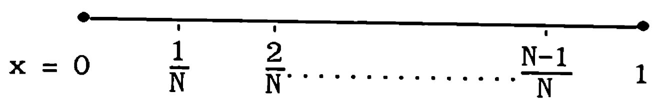


Figure 4.2 Optimal Loop Function

k-th mode shape. In practice, the series summation over k cannot be computed for infinitely many terms. Therefore, we assume that the series summation has been truncated to M terms in actual implementation.

We assume that the beam is divided equally into N parts as follows.



A point sensor that measures the deflection rate is situated at each of the marked point. In other words,

$$z_\ell(t) = \frac{\partial y}{\partial t}\left(\frac{\ell}{N}, t\right) \quad ; \quad \ell = 0,\dots,N \tag{4.52}$$

are known at all times.

We construct the correlation estimate

$$\hat{\dot{T}}_k^N(t) = \frac{2}{N} \sum_{\ell=0}^{N-1} \sin\left(\frac{\ell k\pi}{N}\right) z_\ell(t) \tag{4.53}$$

Note that this estimate uses an approximation of

$$\int_0^1 \sin(k\pi\xi)\, \frac{\partial y}{\partial t}(\xi, t)\, d\xi \simeq \frac{1}{N} \sum_{\ell=0}^{N-1} \sin\left(\frac{\ell k\pi}{N}\right) z_\ell(t) \tag{4.54}$$

in the sense of Riemann integration. A comparison of this with Eq. (4.39) motivates our attempt of using the suboptimal control in Eqs. (4.51) - (4.53).

We summarize the suboptimal control scheme in Figure 4.3.

First we compute the loop transfer function

$$\tilde{g}(s) = \frac{v(s)}{\hat{u}(s)} \tag{4.55}$$

We show in Appendix F that

$$\tilde{g}(s) = \sum_{i=1}^{\infty} \tilde{g}_i(s) \tag{4.56}$$

where

$$\tilde{g}_i(s) = \frac{2s \sin(\frac{i\pi}{2})}{s^2 + \zeta i^2\pi^2 s + i^4\pi^4} c_i^N \tag{4.57}$$

and the quantity c_i^N is evaluated in Appendix F. For M sufficiently large, it is given for i odd by

$$c_i^N = (-1)^{\frac{i-1}{2}} \frac{1}{4N^2} \operatorname{cosec}^2 (\frac{i\pi}{2N}) + O(\frac{1}{N}) \tag{4.58}$$

where $O(\frac{1}{N})$ is a number of the order $\frac{1}{N}$.

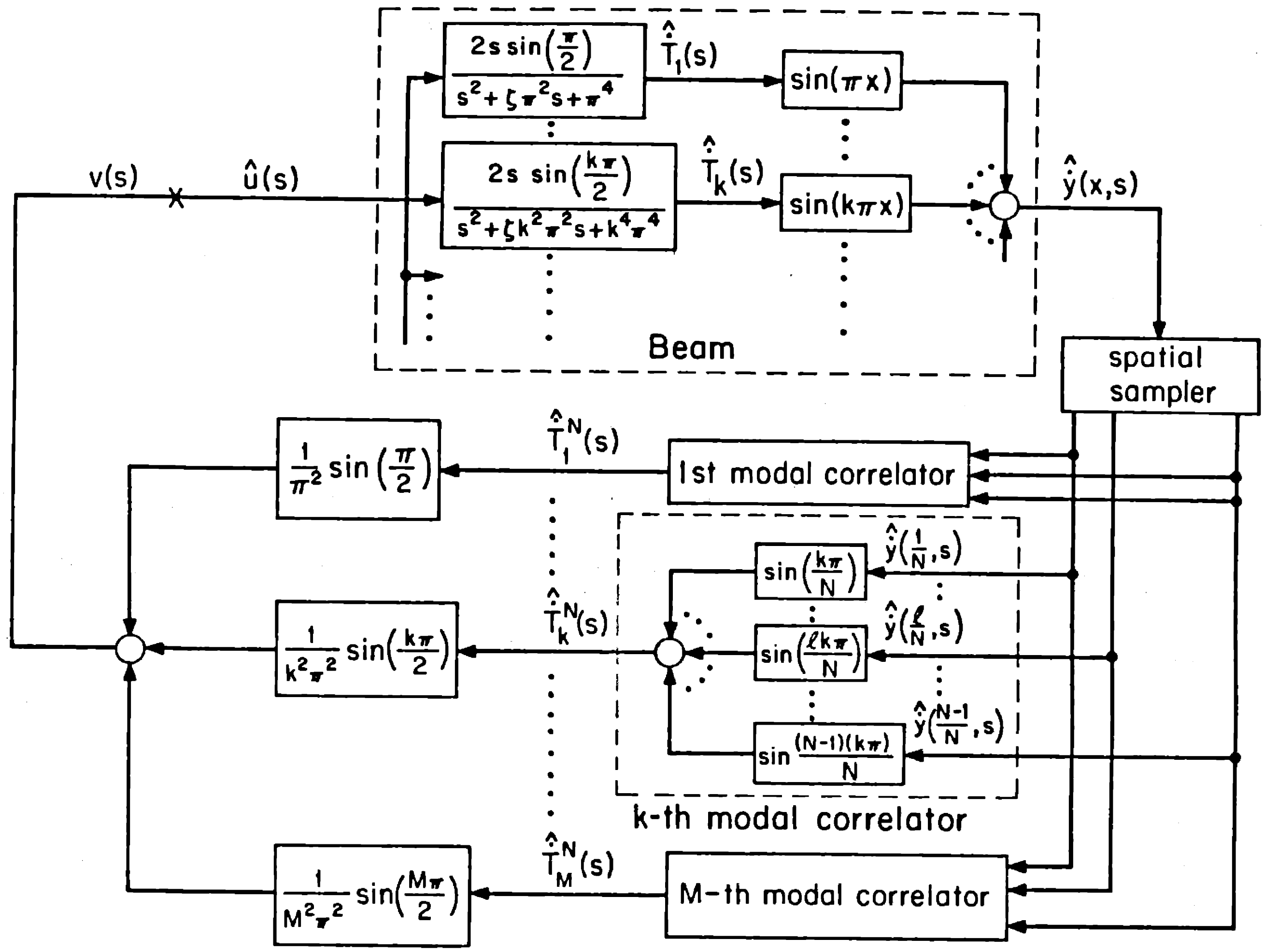


Figure 4.3 Suboptimal Control Implementation,

Constant Damping Ratio Case

Similarly we can decompose the optimal loop transfer function (Eq. (4.43)) as

$$g(s) = \sum_{i=1}^{\infty} g_i(s) \tag{4.59}$$

where

$$g_i(s) = \frac{2s \sin^2(\frac{i\pi}{2})}{s^2 + \zeta i^2\pi^2 s + i^4\pi^4} \frac{1}{i^2\pi^2} \tag{4.60}$$

By Lemma 4.1, the additive stability margin of the optimal system is at least 1. Therefore the suboptimal system is closed-loop stable as long as

$$|\tilde{g}(j\omega) - g(j\omega)| < 1 \qquad \text{for } \omega \in \mathbf{R}_+ \tag{4.61}$$

In Appendix G, we show that there exists an N large enough such that the stability condition in Eq. (4.61) is satisfied. In particular, we show:

(1) The quantity $|\tilde{g}_i(j\omega) - g_i(j\omega)|$ is arbitrarily small for all i and for all $\omega \geq 0$.

(2) Let $\omega = f^2\pi^2$, and k is the nearest odd integer to f. Then the summation

$$\sum_{\substack{i=1 \\ i \neq k}}^{\infty} |\tilde{g}_i(j\omega) - g_i(j\omega)| < 0.05 \tag{4.62}$$

Remark: The bound in (2) is rather conservative. However, it is sufficient for our design. We have not pursued removing this conservatism in this thesis, but we think this bound can be estimated to within the order of $\frac{1}{N}$.

Since

$$|\tilde{g}(j\omega) - g(j\omega)| \leq |\tilde{g}_k(j\omega) - g_k(j\omega)| + \sum_{\substack{i=1 \\ i \neq k}}^{\infty} |\tilde{g}_i(j\omega) - g_i(j\omega)| \tag{4.63}$$

Eq. (4.61) is satisfied. Consequently the suboptimal system is closed-loop stable.

4.6 Relationship between Sample Density and Damping Ratio

Our design procedure in Section 4.5 calls for the determination of N such that for all i,

$$\frac{2}{\zeta i^2 \pi^2} \left| \frac{1}{4N^2} \operatorname{cosec}^2 \left(\frac{i\pi}{2N}\right) - \frac{1}{i^2 \pi^2} + O\left(\frac{1}{N}\right) \right| < 1 \tag{4.64}$$

We discuss in Appendix G how this can be achieved. A finite number of constraints are used to determine N that satisfies Eq. (4.64) which guarantees that the closed-loop system is stable. In particular, we require

$$\frac{2}{\zeta\pi^2}\frac{1}{N} < \frac{\delta}{2} \tag{4.65}$$

where $0.95 > \delta > 0$ is chosen by the control designer for bounding the error term $O(\frac{1}{N})$. This constraint indicates that for our design procedure, N is at least inversely proportional to ζ. Thus when ζ gets smaller, the number N that is computed by the procedure gets larger. Figure 4.4 summarizes our observation, for $\delta = \frac{1}{2}$.

We remark that the sample measurements are meaningful only for $N \geq 2$. In other words, even if the damping ratio increases, there must still be at least one sensor at the mid-point (the other is at the end-point $x = 0$ which does not move) of the beam for feedback measurements. However, when this stage is reached, the size of the damping ratio is probably so large that it would have deviated from the flexible structure assumption, anyway.

Remark: numerical evaluations of the expression inside the absolute value signs in Eq. (4.64) for some i and N indicate that the error term $O_i(\frac{1}{N})$ dominates over the difference $\frac{1}{4N^2}\operatorname{cosec}^2(\frac{i\pi}{4N}) - \frac{1}{i^2\pi^2}$. This has not been verified rigorously, however.

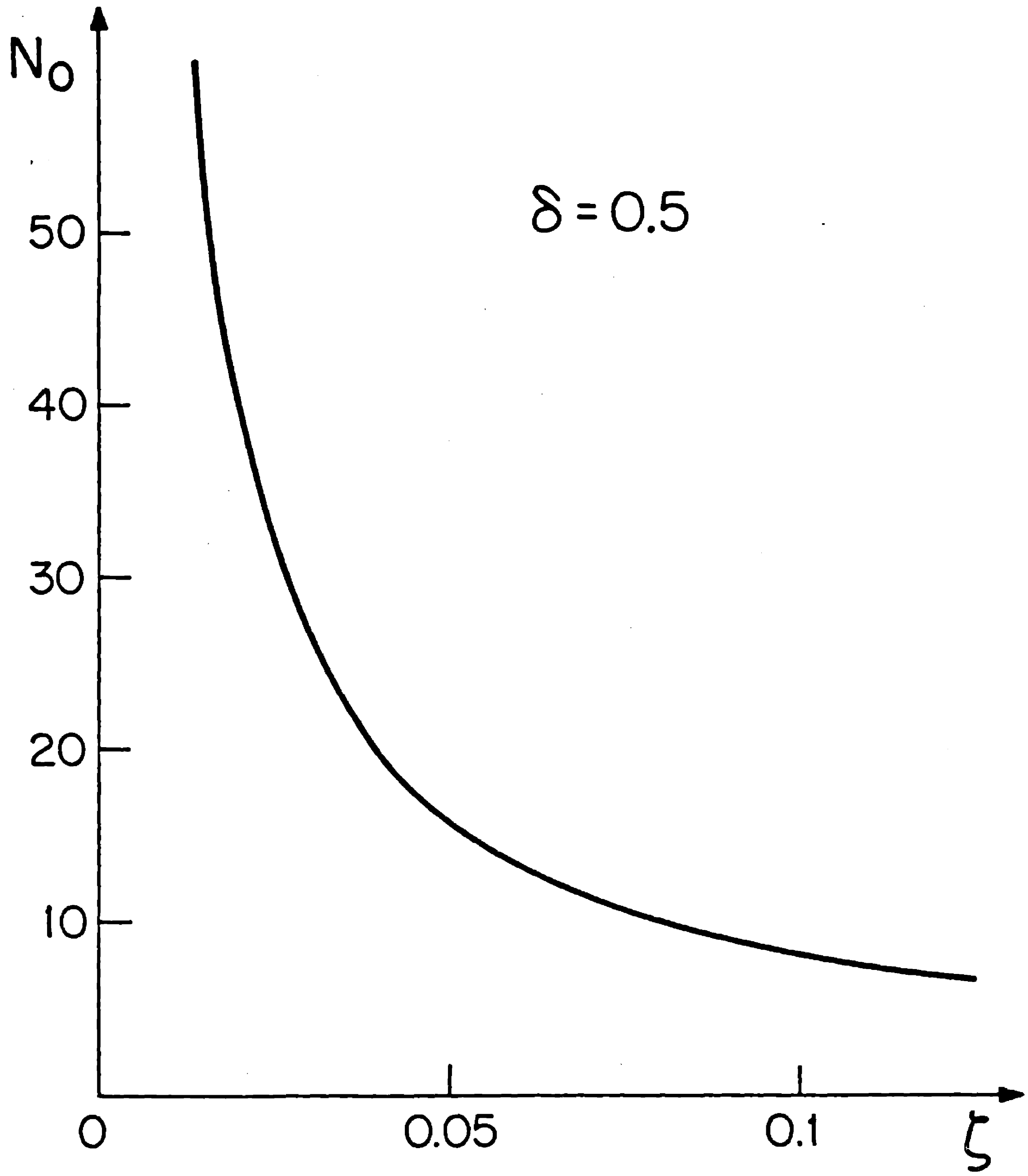


Figure 4.4 Relationship between Damping Ratio and Sample Density

4.7 Closed-Loop Stability of Systems with Constant Inherent Damping Coefficient

In Sections 4.3 - 4.6 we have shown how one can design a practical finite dimensional suboptimal closed-loop stable system for a flexible structure when the inherent damping increases with each higher mode. In this section we discuss whether the same procedure can be applied to cases in which the inherent damping corresponds to a constant damping coefficient for all modal deflection transfer function.

Let $L = \eta$, then the dynamics of the beam (Eq. (4.1) - (4.6)) are

$$\frac{\partial^4}{\partial x^4} y(x,t) + \eta \frac{\partial}{\partial t} y(x,t) + \frac{\partial^2}{\partial t^2} y(x,t) = u(t)\delta(x - \tfrac{1}{2}) \qquad (4.66)$$

$$y(0,t) = y(1,t) = 0 \qquad (4.67)$$

$$\frac{\partial^2}{\partial x^2} y(0,t) = \frac{\partial^2}{\partial x^2} y(1,t) = 0 \qquad (4.68)$$

The solution to the open-loop system is

$$y(x,t) = \sum_{k=1}^{\infty} \sin(k\pi x) T_k(t) \qquad (4.69)$$

where $T_k(t)$ is related to $u(t)$ by the modal transfer function

$$\frac{\hat{T}_k(s)}{\hat{u}(s)} = \frac{2\sin(\frac{k\pi}{2})}{s^2 + \eta s + k^4\pi^4} \tag{4.70}$$

As can be inferred from this transfer function, the natural frequency is $k^2\pi^2$ for the k-th deflection mode, and the damping coefficient is η for all modes. Except for the difference in the damping coefficient (in the denominator) of the transfer functions, this open-loop solution is the same as the previous case, where the damping coefficients are $\zeta k^2\pi^2$.

In what follows, we describe:

(1) how one can derive an optimal control law which is identical in form to the one used in the previous case;

(2) an attempt to design a suboptimal controller using exactly the same strategy as described in Section 4.5;

(3) why this suboptimal controller cannot be guaranteed closed-loop stability no matter how dense one samples the measurements, unless a dynamical compensator is also used;

(4) an interpretation of the above results.

4.7.1 Optimal and Suboptimal Control Laws

As in the constant damping ratio case, a Lyapunov control problem can be formulated and solved. The details of these derivations, however, are not presented because they are almost identical to those in Sections 4.4 - 4.5.

Referring back to Section 4.4, the infinitesimal generator of the semigroup (Eq. 4.19) is now

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{\partial^4}{\partial x^4} & -\eta \end{bmatrix} \tag{4.71}$$

and we use, in order to achieve rolloff, an integral operator in the control weighting operator

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 2\eta K_2 \end{bmatrix} \tag{4.72}$$

where the operator K_2 is defined in Eqs. (4.34) and (4.37).

After some manipulations, one finds that the optimal Lyapunov feedback control is given in the same manner as the constant damping ratio case, i.e. Eqs. (4.39), (4.42),

$$u(t) = \int_0^1 \sum_{i=1}^{\infty} \frac{2}{i^2\pi^2} \sin(\frac{i\pi}{2}) \sin(i\pi\xi)] \frac{\partial y}{\partial t} (\xi, t) d\xi \tag{4.73}$$

or equivalently,

$$u(t) = \sum_{i=1}^{\infty} \frac{1}{i^2\pi^2} \sin(\frac{i\pi}{2}) \dot{T}_i(t) \tag{4.74}$$

The optimal loop transfer function is

$$g(s) = \sum_{i=1}^{\infty} \frac{1}{i^2\pi^2} \sin^2(\frac{i\pi}{2}) \frac{2s}{s^2+\eta s+i^4\pi^4} \tag{4.75}$$

Eqs. (4.73) - (4.75) motivate us to try to construct the suboptimal control exactly as before (Eqs. (4.51) - (4.54)),

$$u(t) = \sum_{i=1}^{M} \frac{1}{k^2\pi^2} \sin(\frac{k\pi}{2}) \dot{T}_k^N(t) \tag{4.76}$$

$$\dot{T}_k^N(t) = \frac{2}{N} \sum_{\ell=0}^{N-1} \sin(\frac{\ell k\pi}{N}) \frac{\partial y}{\partial t} (\frac{\ell}{N}, t) \tag{4.77}$$

The loop transfer function of this suboptimal system is given by

$$\tilde{g}(s) = \sum_{i=1}^{\infty} \tilde{g}_i(s) \tag{4.78}$$

$$\tilde{g}_i(s) = \frac{2s \sin(\frac{i\pi}{2})}{s^2+\eta s+i^4\pi^4} c_i^N \tag{4.79}$$

$$c_i^N = \frac{1}{N} \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} \sin(\frac{k\pi}{2}) \sum_{\ell=0}^{N-1} (\frac{\ell k\pi}{N}) \sin(\frac{i\ell\pi}{N}) \tag{4.80}$$

This coefficient c_i^N is identical to the one computed for the constant damping ratio case, in Appendix F. Indeed except for the change of the damping coefficient, the representation of $\tilde{g}(s)$ here is the same as the one formulated for the constant damping ratio case.

4.7.2 Suboptimal Implementation

In this subsection we show that when sampled measurements are used without any dynamical filtering, the loop frequency response of the suboptimal system is perturbed by more than the guaranteed additive stability margin from the optimal system for some frequencies. Hence closed-loop stability cannot be guaranteed by this suboptimal implementation when the inherent damping is constant. However, one can remedy this by using a single pole dynamical compensator to attenuate the high frequency componenents.

Let the loop frequency response of the optimal system be represented by

$$g(s) = \sum_{i=1}^{\infty} g_i(s) \tag{4.81}$$

$$g_i(s) = \frac{1}{i^2\pi^2} \sin^2\left(\frac{i\pi}{2}\right) \frac{2s}{s^2+\eta s+i^4\pi^4} \tag{4.82}$$

Using the computed c_i^N from Appendix F, we obtain

$$\tilde{g}_i(ji^2\pi^2) - g_i(ji^2\pi^2) = \begin{cases} \frac{2}{\eta}\left[\frac{1}{4N^2}\operatorname{cosec}^2\left(\frac{i\pi}{2N}\right) - \frac{1}{i^2\pi^2} + O\left(\frac{1}{N}\right)\right], & i \text{ odd} \\ 0 & , i \text{ eve} \end{cases} \tag{4.83}$$

where $O(\frac{1}{N})$ is a number of the order $\frac{1}{N}$. At low $\omega = i^2\pi^2$, this is approximately the peak value of $\tilde{g}(j\omega) - g(j\omega)$, at low frequencies.

Let us assume that N is very large. Then for all positive integers m,

$$\frac{1}{4N^2}\operatorname{cosec}^2\left(\frac{(2mN+1)\pi}{2N}\right) \approx \frac{1}{\pi^2} \tag{4.84}$$

$$O\left(\frac{1}{N}\right) \approx 0 \tag{4.85}$$

$$\frac{1}{(2mN+1)^2\pi^2} \approx 0 \tag{4.86}$$

Thus from Eq. (4.83),

$$\tilde{g}_{2mN+1}(j(2mN+1)^2\pi^2) - g_{2mN+1}(j(2mN+1)^2\pi^2) \approx \frac{2}{\eta\pi^2} \tag{4.87}$$

In general η is an extremely small number, and

$$\frac{2}{\eta\pi^2} > 1 \tag{4.88}$$

unless $\eta > \frac{2}{\pi^2}$, or approximately $\eta > 0.2$. Therefore generally speaking the unity guaranteed stability margin of the LQ system is not sufficient to establish an integer N for sampling such that the suboptimal control (Eq. (4.76)) is closed-loop stable.

The above conclusion is also true when one considers the Lyapunov guaranteed stability margin instead of the unity LQ stability margin, even though the Lyapunov guaranteed stability margin (Eq. (4.45))

$$(1 + \frac{1}{\eta^2 i^4 \pi^4})^{1/2}$$

is larger. This is not enough because when i is sufficiently large, the term $\frac{1}{\eta^2 i^4 \pi^4}$ will no longer be significant. At these high frequencies, the guaranteed stability margin is infinitesimally close to 1.

Figure 4.5 depicts the envelope of the peaks of the suboptimal loop function $\tilde{g}(j\omega)$. As can be seen from the figure, the loop frequency response rolls off initially until the frequency $N^2\pi^2$ is reached, and since the sample density is finite, it rises back to the original peak value at $(2N+1)^2\pi^2$, which is shown in Eqs. (4.83) - (4.87). This phenomenon repeats indefinitely as frequency moves higher.

The above phenomenon can be explained by looking at the open-loop frequency response plot of any actuator-sensor pair. For

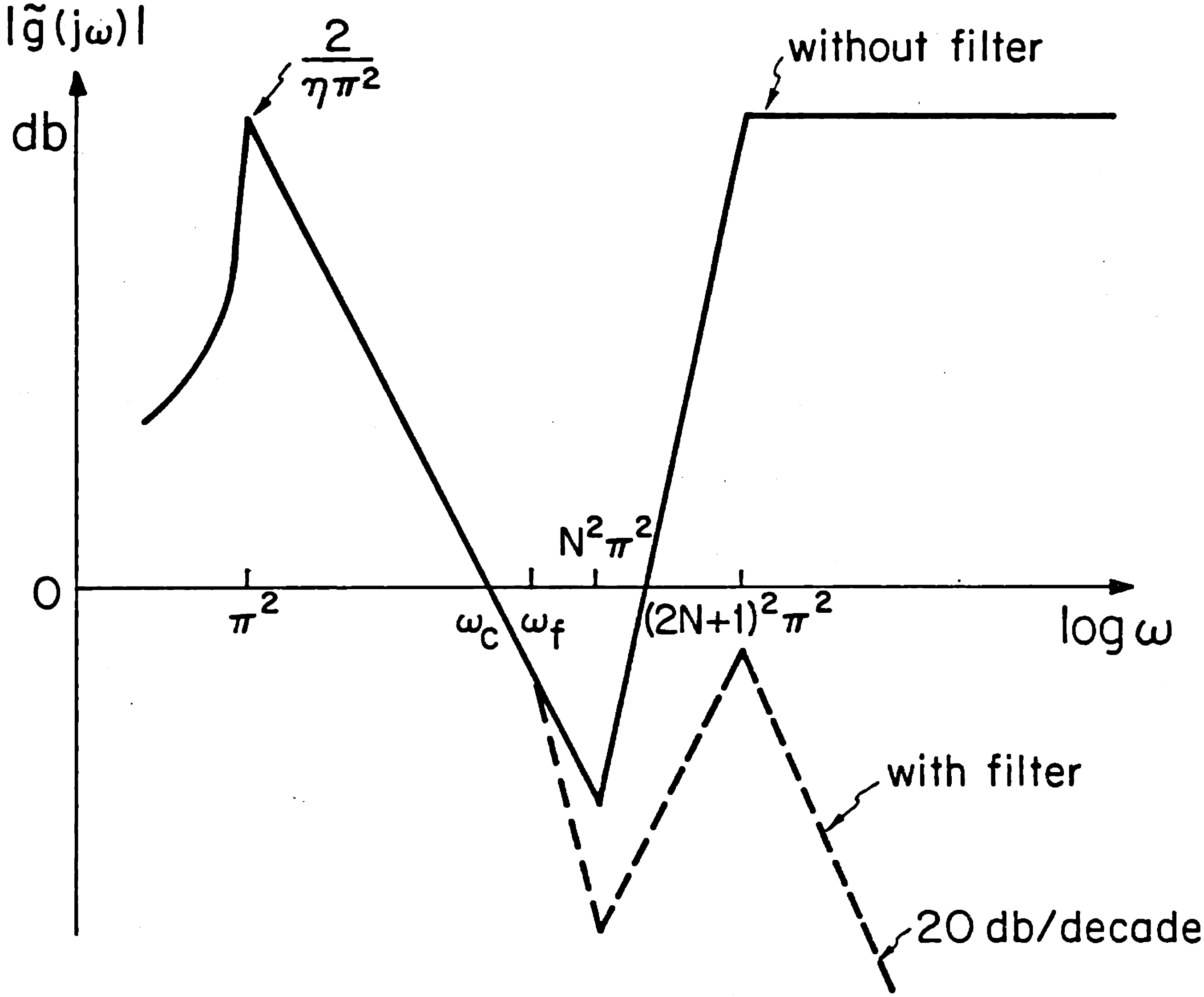


Figure 4.5 Envelope of Peaks for Suboptimal Loop Function, Constant Damping Coefficient Case

example, Figure 4.6 shows the loop function for the sensor colocated with the actuator. This loop function has a constant envelope of the peaks. Thus its high (spatial and temporal) frequency components are aliased to lower spatial frequencies by the sampling-correlation process, and are then multiplied by the high gains appropriate for the lower modes. Consequently we have a problem with the stability of the closed-loop system.

The reason why this does not create a problem for the constant damping ratio case can be seen by observing Figure 4.7. It shows that for the constant damping ratio case, the same actuator-sensor pair gives rise to a loop function whose envelope of peaks rolls off indefinitely. Therefore, spatially aliased components have low amplitudes and do not cause a problem in that case.

The above difficulty with the constant damping coefficient case can be overcome by including a low pass temporal filter to attenuate high frequency components. This corresponds to the suboptimal implementation configured in Figure 4.8.

Recall that a Lyapunov control design guarantees that the system has at least 90° phase margins. Thus by using a single-pole filter, the system is guaranteed to be marginally stable. By placing the pole of the filter ω_f to the right of ω_c, the cutoff frequency of the inital rolloff of $|\tilde{g}(j\omega)|$ in Figure 4.5, additional

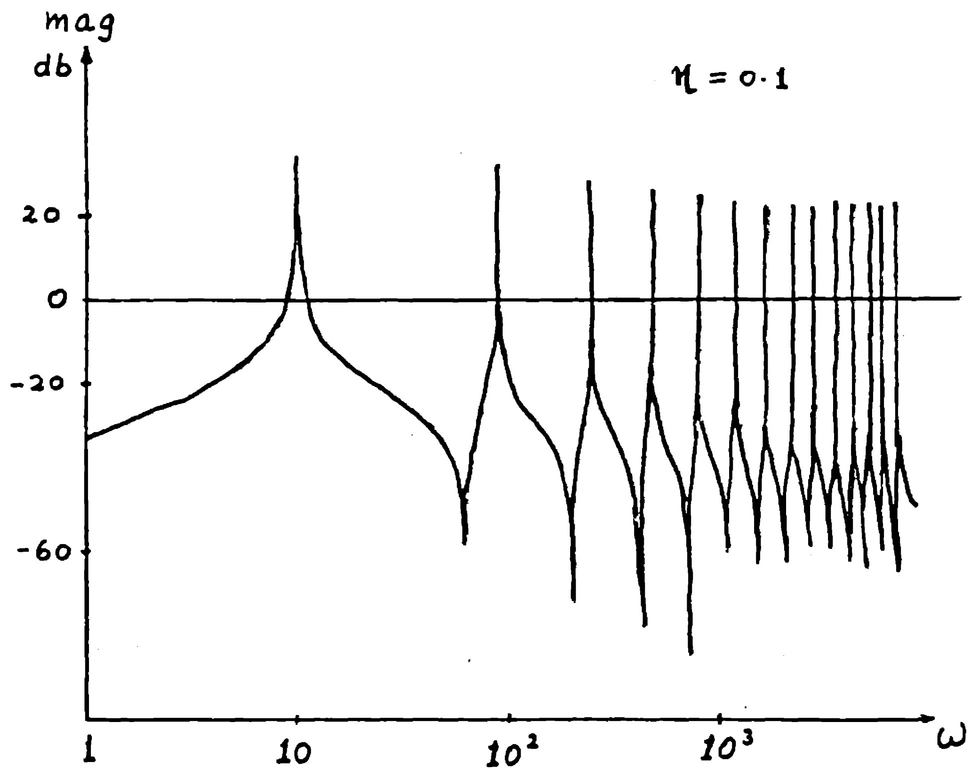


Figure 4.6 Loop Function for Colocated Actuator-Sensor Pair, Constant Damping Coefficient Case

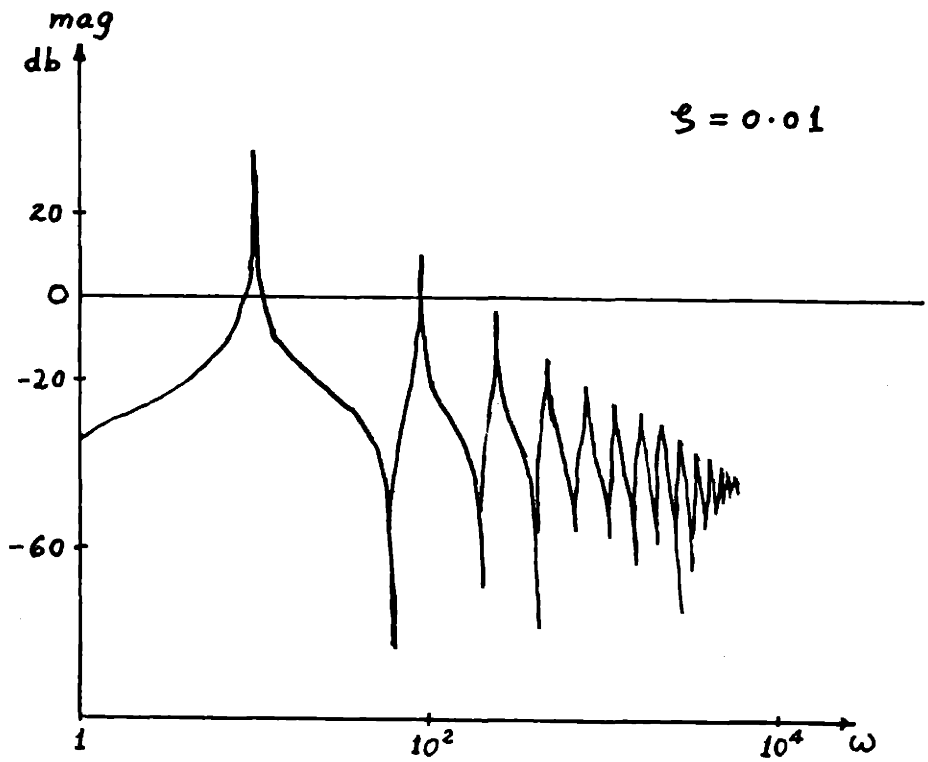


Figure 4.7 Loop Function for Colocated Actuator-Sensor Pair, Constant Damping Ratio Case

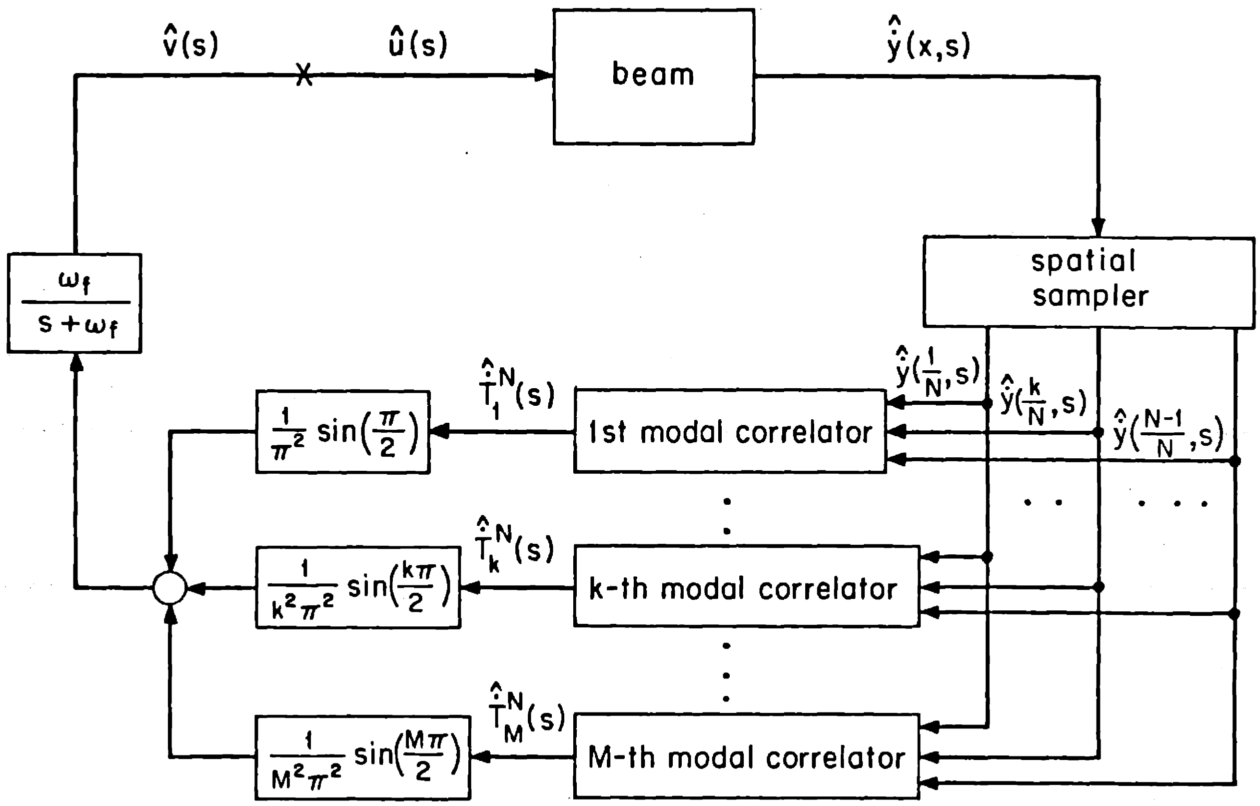


Figure 4.8 Suboptimal Control Implementation, Constant Damping Coefficient Case

stability margin is obtained. Also one should place ω_f sufficiently to the right of $N^2\pi^2$ if one desires to retain ω_c as the cutoff frequency of the overall system. Summarizing, the dynamical filter has a transfer function of $\frac{\omega_f}{s+\omega_f}$, where $\omega_c < \omega_f < N^2\pi^2$. This suboptimal implementation guarantees that the closed-loop system is stable when N is large enough. Also at low frequencies, the system behaves approximately as the optimal system. The loop function of this implementation is shown in Figure 4.5 for a comparison with the one before the filter is inserted in the loop.

CHAPTER 5

CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

5.1 Conclusions

This thesis has addressed the following issue. Given a distributed linear-time-invariant system, there are two sources of errors that can lead to a difference between the actual implemented control system and the computed nominal control system. The sources of errors are:

(a) modeling errors of the plant dynamics; and

(b) inexact implementation of the derived, usually infinite dimensional, controller.

It is imperative that the control system designer knows how much error the system can tolerate without becoming unstable. This knowledge will allow him to synthesize feedback controllers that are implementable and closed-loop stable.

We have studied this problem by looking at the robustness of stability for distributed, linear-time-invariant, feedback systems. We have been successful in developing the following results:

(1) A robustness characterization for linear hereditary differential systems using singular values (Ch. 2). A Nyquist theorem for this class of systems was also established.

(2) The Kalman frequency domain inequality (and equality) for linear quadratic hereditary differential systems (Ch. 2).

(3) A set of guaranteed stability margins for the optimal linear quadratic hereditary differential systems (Ch. 2).

(4) A robustness characterization for distributed, linear-time-invariant systems (Ch. 3). The number of CRHP open-loop poles and the dimension of the control space are both assumed to be finite.

(5) A set of robustness properties for linear quadratic, distributed, linear-time-invariant systems (Ch. 3).

(6) An analysis of a Lyapunov controller for the vibration suppression problem of a flexible beam (Ch. 4).

(7) An analysis of the role played by the inherent damping in the synthesis of an implementable, closed-loop stable controller (Ch. 4). Two cases are studied in detail, the first of which corresponds to a small constant damping ratio for all modes, and the second, a constant damping coefficient.

(8) Synthesis using sampled measurements along the beam of a suboptimal closed-loop stable controller when either a constant inherent damping ratio or damping coefficient is present (Ch. 4). The robustness properties of the optimal system are used in the design procedure.

The finite dimensional version of the results in (1) - (5) have been known for some time. This thesis has extended the finite dimensional results, while at the same time maintaining their practical uses in an infinite dimensional context.

With the advent of techniques for computing the singular value decomposition of a matrix, e.g. Laub [65], Klema and Laub [66], singular values have proven themselves to be an extremely valuable tool for analyzing multi-input-multi-output feedback control systems. By plotting the singular values of the return difference over all frequencies, the class of modeling errors that would not destabilize the control system are exposed.

We exploited the above concepts for distributed, linear-time-invariant systems, and have come up with results that allow the control system designer to use the singular value technique almost in the same way as in the case of finite dimensional systems. The modification here is that the plant dynamics are no longer represented by a rational transfer function, but in most cases, their representations are programmable into the computer. Therefore, there is practically no loss in utility of the singular value analysis in the distributed system case.

Invariably literature on the robustness of finite dimensional systems assumes that since the computed controller is finite dimensional, it can always be implemented exactly. However, this is not true for distributed systems.

Generally the controller design approach for the distributed system fall into one of the following two categories:

(α) Reduced-order model method, and

(β) Infinite dimensional model method, followed by a finite dimensional approximation of the resulting infinite dimensional controller.

Referring back to the two types of errors introduced at the beginning of this section, the approach (α) induces the (a) type error, i.e. the modeling error in the plant dynamics, and the approach (β) induces the (b) type error, i.e. the implementation error. The methods developed in this thesis are suitable for design analysis for either of these approaches. The latter approach has been examined in depth in this thesis using a vibration suppression problem for an idealized model of a flexible beam. Our analysis reveals that the inherent damping of the flexible structure determines the degree of approximation that is required of the suboptimal implementation in order to maintain closed-loop stability.

In what follows, we give an account of the results in Chapters 2-4, and then a few suggestions for future research.

5.2 Account of Results in Chapters 2-4

In Chapter 2, we develop a robustness characterization of linear hereditary differential systems (LHDS), for both additive and multiplicative perturbations. By showing that LHDS have finitely many open-loop poles in the CRHP, we are able to state a Nyquist theorem for LHDS, which provides a basis for robustness characterization. As in the finite dimensional case, the minimum singular value of the return difference is used to characterize the robustness of the system against unstructured error in the system loop. Because of this formulation, the guaranteed stability margins for the linear quadratic optimal system, termed the linear quadratic hereditary differential system, are derived directly from the Kalman frequency domain inequality. We have shown, by directly working with complex-valued matrix functions in the frequency domain, how one can derive the Kalman Inequality for LQHD systems.

In Chapter 3 we generalized the results in Chapter 2 to distributed, linear-time-invariant systems by considering a formulation using linear operators. This is achieved for the case of finite-dimensional control space by combinining the Yakubovich frequency domain theorem and the generalized Nyquist theorem by Desoer and Wang.

The results that we obtain in Chapters 2-3 contain the finite dimensional system case as a special case. In particular, we showed that the guaranteed stability margins of the distributed, LQ optimal systems are, generally speaking, the same as those of the finite dimensional LQ systems.

In Chapter 4 we employ a flexible beam, simply supported at both ends, to illustrate the use of the above robustness results to synthesize a closed-loop stable control design for the vibration suppression problem. Lyapunov control is used, which offers the advantages of being computationally less complex than the general LQ method with better stability margins. By using a suboptimal scheme which calls for discrete spatial sampling of the deflection rate along the beam, we are able to obtain a closed-loop stable finite dimensional controller when there is a sufficient amount of inherent damping. Specifically we show that we can design such a controller if the modes of the structure has a constant, albeit small, damping ratio. The fact that the number of spatial samples required to approximate the controller increases as this damping ratio decreases is exposed. The case of constant damping coefficient bears similar results, except that the suboptimal controller contains spatial sampling as well as a dynamical compensator to attenuate the high frequency components of the loop response.

5.3 Suggestions for Future Research

We believe that the linear quadratic gaussian (LQG) approach is a practical method for designing feedback controllers for distributed systems, provided that the control system designer understands the fundamental characteristics of the compensated system. This thesis has laid down the basic groundwork for the designer to do a credible job. However, there is a whole realm of other design techniques, which have proven to be successful in finite dimensional system design, that this thesis has not explored.

We have studied the implication of full-state LQ feedback for distributed system in detail, and have analyzed the use of spatial sampling to approximate the full-state measurements.

Depending on the application, one or more of the following techniques may be useful for control designs for distributed systems. When combined with the results in this thesis, these techniques are likely to produce control designs which have desirable properties. Theoretical treatments as well as applications of these techniques for distributed systems will allow us to obtain a unified framework for dealing with infinite dimensional system design.

(1) loop-transfer-recovery (LTR) technique

The LQG/LTR technique [3], [6] is one of the most popular methods for designing controllers for multivariable systems. In Chapter 2, we mentioned that this technique seems to be applicable to linear hereditary differential systems also. We think by properly choosing the stochastic filter (observer), the same can be said of distributed systems in general. Although in theory the LQG/LTR method is limited in applicability to minimum phase systems, Athans [6] has reported that desirable results are obtained in actual applications even when the system is non-minimum phase, and has recommended the use of this method under all circumstances. Therefore, we suggest developing this method for infinite dimensional systems, and carrying out a feasiblity study of this method on some concrete applications.

(2) Design Synthesis Technique that Utilizes Crossfeed Structure

In multi-input-multi-output systems, it is possible [13] to adjust the control weighting matrix to trade off the stability margins against the various crossfeed perturbations among the control channels. Lee and Gully [67] have applied this technique to the control of lateral dynamics of a high performance missile. This technique is applicable to all distributed, linear-time-invariant systems without modifications.

(3) Design Analysis technique that Explores Error Structure

Lehtomaki, et al. [63] have shown how one can relax the conservatism when evaluating the feedback design by taking into account the structure of the perturbation error. As a result, the control system designer has more room to trade off the various system characteristics in his design. This technique is applicable to distributed systems without modifications.

A notable limitation of our results for distributed system is that the control space is finite dimensional. This is because all the known Nyquist-type of criteria have the same limitation. However, the Yakubovich theorem is valid for infinite dimensional control spaces also. Perhaps by exploring a function other than det(•), the robustness results in Chapter 3 (which requires verification on the Nyquist path only) can be extended to this case also. Until this suggested research direction is undertaken, one must contend with results such as that in Appendix B which requires the control designer to check the invertibility condition over the whole CRHP, instead of the Nyquist path alone.

There are two important types of modeling errors that we have not considered in this thesis. Quite often the delay time in a time-delay systems is not known exactly. It is useful to know how much error in this delay time can a time-delay system tolerate without becoming unstable. Also we have not considered nonlinearity in the system loop. We think both of these error types are important and more work should be done to expose their impact on the system stability.

In this research, we have not fully analyzed the impact of classifying perturbation errors into modeling errors and implementation errors. For infinite dimensional control designs, it may be advantageous to distinguish between these two classes of errors, and analyze each of them separately, as in the localization theorem in Lee, et al. [64].

Lastly we would like to suggest that the control problem formulation in Chapter 4 for the flexible beam can be extended to control problems for flexible structures with multiple components. The exact Lyapunov solution method by Djaferis and Mitter [73] seems to be suitable for handling these problems.

APPENDIX A

Derivation of Identitites 1 - 5 for LQHD Systems

Derivation of Identity 1:

$$K_1(s) = \int_{-\tau}^{0} K_1(\theta)e^{s\theta}d\theta \tag{A.1}$$

Integrating by parts, we get

$$K_1(s) = \frac{1}{s}(K_1(0)-K_0Be^{s\tau}) - \frac{1}{s}\int_{-\tau}^{0} e^{s\theta}\frac{d}{d\theta}K_1(\theta)d\theta \tag{A.2}$$

Derivation of Identity 2:

Substituting Eq. (2.7) into the integral, we readily see that

$$\int_{-\tau}^{0} e^{s\theta}\frac{d}{d\theta}K_1(\theta)d\theta = [A'-K_0BR^{-1}B']K_1(s) + \sum_{i=1}^{N-1} K_0A_ie^{s\theta_i}+K_0A_0(s)$$

$$+ \int_{-\tau}^{0} e^{s\theta}K_2(0,\theta)d\theta \tag{A.3}$$

where $K_2(0,\theta)$ is given in Eq. (2.9), i.e.

$$\int_{-\tau}^{0} e^{s\theta} K_2(0,\theta)d\theta = \int_{-\tau}^{0} e^{s\theta}K_1^{'}(-\theta-\tau)A_N d\theta$$

$$- \int_{-\tau}^{0} e^{s\theta} \int_{-\tau}^{\theta} K_1^{'}(-\theta+\sigma)BR^{-1}B'K(\sigma)d\sigma d\theta$$

$$+ \int_{-\tau}^{0} e^{s\theta} \sum_{i=1}^{N-1} \begin{cases} A_i^{'}K_1(\theta+\theta_i) & , -\tau \leq \theta+\theta_i \\ 0 & , \text{otherwise} \end{cases}$$

$$+ \int_{-\tau}^{0} e^{s\theta} \sum_{i=1}^{N-1} \begin{cases} K_i^{'}(-\theta+\theta_i)A_i & , \theta_i \leq \theta \\ 0 & , \text{otherwise} \end{cases}$$

$$+ \int_{-\tau}^{0} e^{s\theta} \int_{-\tau}^{0} \left\{ \begin{matrix} A_0^{'}(\zeta)K_1(\zeta+\theta) & , \zeta \geq -\theta-\tau \\ 0 & , \text{otherwise} \end{matrix} \right\} d\zeta \, d\theta$$

$$+ \int_{-\tau}^{0} e^{s\theta} \int_{-\tau}^{0} \left\{ \begin{matrix} K_1^{'}(\zeta-\theta)A_0(\zeta) & , \zeta \geq \theta-\tau \\ 0 & , \text{otherwise} \end{matrix} \right\} d\zeta \, d\theta$$

(A.4)

On the right-hand-side of Eq. (A.4), it is easy to see that the 1st term

$$\int_{-\tau}^{0} e^{s\theta}K_1^{'}(-\theta-\tau)A_N d\theta = e^{-s\tau}K_1^{'}(-s)A_N \qquad (A.5)$$

by a change of variable. The 2nd term is simply $\Psi(s)$ as defined in Eq. (2.21). $\Gamma_1(s)$, $\Gamma_2(s)$, $\Xi_1(s)$, $\Xi_2(s)$ are obtained from the 3rd - 6th terms, respectively, by applying the appropriate upper and lower limits to the integrals.

Derivation of Identity 3:

$$\Psi(s) = \int_{-\tau}^{0} e^{s\theta} \int_{-\tau}^{\theta} \dot{K}_1(-\theta+\sigma)BR^{-1}B'K_1(\sigma)d\sigma d\theta \tag{A.6}$$

Changing the order of integration gives

$$\Psi(s) = \int_{-\tau}^{0} \int_{-\tau}^{\theta} e^{s\theta}\, \dot{K}_1(-\theta+\sigma)\, BR^{-1}B'K_1(\sigma)d\theta d\sigma \tag{A.7}$$

By a change of variable $\gamma = \theta+\sigma$, we get

$$\Psi(s) = \int_{-\tau}^{0} \int_{\sigma}^{0} e^{-s\gamma}\dot{K}_1(\gamma)BR^{-1}B'K_1(\sigma)e^{s\sigma}d\gamma d\sigma$$

$$= \int_{-\tau}^{0} [\int_{\sigma}^{0} e^{-s\gamma}\dot{K}_1(\gamma)d\gamma]BR^{-1}B'K_1(\sigma)e^{s\sigma}d\sigma \tag{A.8}$$

Integrating by parts, using

$$U = \int_{\sigma}^{0} e^{-s\gamma}\dot{K}_1(\gamma)d\gamma \tag{A.9}$$

and

$$dV = BR^{-1}B'K_1(\sigma)e^{s\sigma}d\sigma$$

we obtain

$$\Psi(s) = [\int_{-\tau}^{0} e^{-s\gamma}K_1^{'}(\gamma)d\gamma]\ [\int_{0}^{\sigma} BR^{-1}B'K_1(\zeta)e^{s\zeta}d\zeta]\ \Big|_{\sigma = -\tau}^{\sigma = 0}$$

$$+ \int_{-\tau}^{0} e^{-s\sigma}K_1^{'}(\sigma) \int_{0}^{\sigma} BR^{-1}B'K_1(\zeta)e^{s\zeta}d\zeta d\sigma \qquad (A.10)$$

Evaluating the 1st term on the right-hand-side, we see that it is the same as $K_1^{'}(-s)BR^{-1}B'K_1(s)$. Comparing the 2nd term with Eq. (A.8), we see that it is actually $-\Psi'(-s)$. Therefore,

$$\Psi(s)+\Psi'(-s) = K_1^{'}(-s)BR^{-1}B'K_1(s) \qquad (A.11)$$

as claimed.

<u>Derivation of Identity 4:</u>

From Eqs. (2.22) and (2.23), we know that

$$\Gamma_1(s) = \sum_{i=1}^{N-1} \int_{-\tau-\theta_i}^{0} A_i^{'} K_1(\theta+\theta_i) e^{s\theta} d\theta \tag{A.12}$$

$$\Gamma_2^{'}(-s) = \sum_{i=1}^{N-1} \int_{\theta_i}^{0} A_i^{'} K_1(-\theta+\theta_i) e^{s\theta} d\theta \tag{A.13}$$

By a change of variable, Eq. (A.12) can be written as

$$\Gamma_1(s) = \sum_{i=1}^{N-1} \int_{-\tau}^{\theta_i} A_i^{'} K_1(\zeta) e^{s\zeta} d\zeta e^{-s\theta_i} \tag{A.14}$$

Similarly, Eq. (A.13) can be written as

$$\Gamma_2^{'}(-s) = \sum_{i=1}^{N-1} \int_{\theta_i}^{0} A_i^{'} K_1(\zeta) e^{s\zeta} d\zeta e^{-s\theta_i} \tag{A.15}$$

Adding Eqs. (A.14) and (A.15) produces the desired result.

Derivation of Identity 5:

From Eq. (2.23) we know that

$$\Xi_1(s) = \int_{-\tau}^{0} \int_{-\tau-\theta}^{0} A_0^{'}(\xi) K_1(\xi+\theta) d\xi e^{s\theta} d\theta \tag{A.16}$$

Changing the order of integration, we get

$$\Xi_1(s) = \int_{-\tau}^{0}\int_{-\tau-\xi}^{0} A_0'(\xi)K_1(\xi+\theta)e^{s\theta}d\theta d\xi \tag{A.17}$$

By a change of variable, we obtain

$$\Xi_1(s) = \int_{-\tau}^{0}\int_{-\tau}^{\xi} A_0'(\xi)K_1(\zeta)e^{s\zeta}e^{-s\xi}d\zeta d\xi \tag{A.18}$$

Similarly from Eq. (2.24), we get

$$\Xi_2'(-s) = \int_{-\tau}^{0}\int_{\xi}^{0} A_0'(\xi)K_1(\zeta)e^{s\zeta}e^{-s\xi}d\zeta d\xi \tag{A.19}$$

Adding Eqs. (A.18) and (A.19) gives

$$\Xi_1(s)+\Xi_2'(-s) = \int_{-\tau}^{0}\int_{-\tau}^{0} A_0'(\xi)K_1(\zeta)e^{s\zeta}e^{-s\xi}d\zeta d\xi \tag{A.20}$$

and the claim is immediate.

APPENDIX B

Robustness Theory for Infinite Dimensional Control Space

The purpose of this appendix is two-fold. First we give a tutorial on singular values[2] of linear bounded operators, which are useful for characterizing stability margins of infinite dimensional systems. Second we use this opportunity to show how one can characterize robustness of unity feedback system in general by exploring the invertibility condition within the Banach algebra of linear bounded operators. This result differs from that in Chapter 3 because the control space may be infinite dimensional.

B.1 A Tutorial on Singular Values of Linear Bounded Operators

The definitions and results in this tutorial are largely due to or are easy consequences of Gohberg and Krein [68]. Also we have drawn on Ben-Israel and Greville [44] for a lot of the motivation behind the definitions and interpretations of singular values. We have limited this tutorial to contain only results that are relevant to the study of robustness in this appendix.

For the development of singular values, we list the following definitions and notations.

---

2 Singular values are referred to as s-numbers in [68].

H = a separable Hilbert space with inner product $(\bullet,\bullet)$

$B(H)$ = set of all linear bounded operators on H

$C(H)$ = set of all linear compact operators on H

I = identity operator H

For $A \in B(H)$,

$||A||$ = induced operator norm of A

A^* = adjoint operator of A in the sense that

$(x, Ax) = (A^*x, x)$ for all $x \in H$

$R(A)$ = range of A

$r(A)$ = dimension of $R(A)$

$\lambda_j(A)$ = j-th eigenvalue (in nonincreasing order) of A, where A is compact

(A complex number λ is called an eigenvalue of A if $(\lambda I-A)\,\phi = 0$ has at least one nonzero solution $\phi \in H$, i.e. eigenvector)

Let $A = A^*$ and $(x, Ax) \geq 0$ for all $x \in H$. Then $A^{1/2}$ = square root of A in the sense of

$$A = A^{1/2^*} A^{1/2}$$

B.1.1 Singular Values of Linear Compact Operators

Roughly speaking the set of compact operators is the closest family to matrices. Since a compact operator on a separable Hilbert space admits a Schmidt expansion, defining the singular values of a compact operator is almost as natural as defining its eigenvalues.

Definition (Singular Values of a Compact Operator):

Let $A \in C(H)$. Then the eigenvalues of the operator

$$D = (A^{*}A)^{1/2} \tag{B.1}$$

are called the singular values of the operator A.

Note that $(A^{*}A)^{1/2}$ is linear compact because A is. Hence its eigenvalues are denumerable. We shall enumerate the nonzero singular values, which are all nonnegative, in nonincreasing order, taking their multiplicities into account. Specifically, let

$$\sigma_j(A) = \lambda_j(D) \ ; \quad j=1,2,\ldots,r(A) = r(D) \tag{B.2}$$

$$\sigma_j(A) = 0 \quad ; \ j = r(A) + 1,\ldots \tag{B.3}$$

Much of the significance of singular values is due to the fact that one can decompose any linear compact operator into a series of orthogonal parts. The following theorem is due to the well-known Schmidt expansion of compact operators.

Theorem B.1 (Singular Value Decomposition):

Let $A \in C(H)$. Then A admits a Schmidt expansion, i.e. there exist $\{\phi_j\}$, $\{\psi_j\}$, orthonormal bases for H, such that the operator A can be represented by

$$Ah = \sum_{j=1}^{r(A)} \sigma_j(A)\ (h,\phi_j)\psi_j; \quad h \in H \tag{B.4}$$

This representation of A is called the singular value decomposition (SVD) of A. The series summation converges uniformly in the induced operator norm. Taking multiplicities into account, the basis elements ϕ_j, ψ_j are unique for $j = 1,\ldots,r(A)$, up to a complex scalar multiple.

Theorem B.2 can serve as an alternative definition of singular values. We shall write the SVD of A as

$$A = \sum_{j=1}^{r(A)} \sigma_j(A)\ (\bullet,\phi_j)\ \psi_j \tag{B.5}$$

Because of the orthogonality of the decomposition, one can prove the following theorem, which is the main motivation for formulating stability margin in a singular value context.

Theorem B.2 (Approximation Property of Singular Values):

Let $A \in C(H)$. Then for any $k = 1,2,\ldots,r(A),\ldots$

$$\sigma_k(A) = \min_{K_k \in F_k} ||A-K_k|| \tag{B.6}$$

where

F_k = set of all finite dimensional operators of dimension smaller than k (B.7)

Moreover the solution K_k to Eq. (B.6) is given by a partial sum in the SVD of A,

$$K_k = \sum_{j=1}^{k-1} \sigma_j(A)\ (\bullet,\phi_j)\ \psi_j \tag{B.8}$$

Corollary B.2

(a) $\sigma_1(A) = ||A||$ (B.9)

(b) Let $r(A) = n < \infty$. Then $\sigma_n(A)$ is nonzero, but $\sigma_{n+j}(A)$ is zero, for $j = 1,2,\ldots.$ Suppose $r(A + \Delta A) \leq n-1$, where $\Delta A \in B(H)$. Then

$$||\Delta A|| \geq \sigma_n(A) \tag{B.10}$$

Moveover, equality in Eq. (B.10) occurs if

$$\Delta A = -\ \sigma_n(A)\ (\bullet,\phi_n)\psi_n \tag{B.11}$$

(c) Let A, $B \in C(H)$. Then

$$\sigma_{i+j-1}(A + B) \leq \sigma_i(A) + \sigma_j(B); \quad i,j=1,2,\ldots. \tag{B.12}$$

The proof of Theorem B.2 can be found in [68, pp. 28-29]. Corollary B.2 contains easy consequences of this theorem. Corollary B.2(a) states that the largest singular value of a compact operator is actually equal to the norm of the operator. Since singular values are similar to eigenvalues, they are often computationally more tractable than using notions such as the induced operator norm. It seems, therefore, that singular values are useful as a measure of robustness, provided that we can extend Theorems B.1 - B.2 to include linear bounded operators.

Conceptually the appeal of Theorems B.2 lies in the fact that it relates the singular values of a compact operator, which is an algebraic property, to a geometrical propoerty in the space of linear bounded operators. Loosely speaking, the SVD gives the shortest distance and directions to the "more singular" operators.

If H is finite dimensional, the C(H) = $\mathcal{B}$(H) = space of nxn matrices, where n = dim H. Let A be an nxn matrix. Then A is invertible if and only if $\sigma_n(A) > 0$. By letting i = n, j = 1 in Eq. (B.12) and

$$A = A + \Delta A \tag{B.13}$$

$$B = - \Delta A \tag{B.14}$$

where ΔA is an nxn matrix, one gets

$$\sigma_n(A) \leq \sigma_1(-\Delta A) + \sigma_n(A + \Delta A) \quad (B.15)$$

But by the definition of singular values

$$\sigma_1(-\Delta A) = \sigma_1(\Delta A) \quad (B.16)$$

Hence if

$$\sigma_1(\Delta A) < \sigma_n(A) \quad (B.17)$$

then

$$\sigma_n(A + \Delta A) > 0 \quad (B.18)$$

i.e. $A + \Delta A$ is invertible also.

We shall return to discuss this concept when we discuss the robustness theorems.

B.1.2 Singular Values of Linear Bounded Operators

In general linear compact operators are too restrictive for modeling distributed dynamical systems. In this section we extend the notion of singular values of linear operators to include linear bounded operators. The main difficulty here is that the "eigenvalues" (more precisely, spectrum) of a bounded operator do not form a countable set. However this problem can be resolved by decomposing the linear bounded operator into three parts, the first of which is a linear compact operator.

To give meaning to the singular values of linear bounded operators, we need the following terminology:

Definition (Spectrum of a linear bounded operator):

Let $A \in \mathcal{B}(H)$ and

$$D = (A^{*}A)^{1/2} \tag{B.19}$$

Then

(a) the spectrum sp(A) of A is the set of all complex numbers λ such that $\lambda I - A$ is not invertible in $\mathcal{B}(H)$.

(b) a point λ of sp(D) is called a point of the condensed spectrum of D, denoted by $sp_{\infty}(D)$, if it is either an accumulation point of sp(D) or an eigenvalue of D of infinite multiplicity.

(c) Let

$$\mu = \sup sp(D) \tag{B.20}$$

Case 1: If $\mu \notin sp_{\infty}(D)$, then define

$$\lambda_j(D) = \mu \ ; \ j = 1,2,\ldots \tag{B.21}$$

Case 2: If $\mu \in sp_{\infty}(D)$, then it is be an eigenvalue of finite multiplicity m. Define

$$\lambda_j(D) = \mu \; ; \quad j = 1,2,\ldots,m, \tag{B.22}$$

and the remaining λ_j,

$$\lambda_{m+j}(D) = \lambda_j(D_1) \; ; \; j = 1,2,\ldots \tag{B.23}$$

where

$$D_1 = D - \mu P \tag{B.24}$$

and P is the orthoprojector onto the eigenspace of the operator D corresonding to the eigenvalue μ.

Definition (c) gives a recursive procedure to arrange the largest eigenvalues of D, which are nonnegative, in a nonincreasing order.

By this construction the sequence λ_j has a limit, which we denote by $\lambda_\infty(D)$. Clearly,

$$\lambda_\infty(D) = \sup \, sp_\infty(D) \tag{B.25}$$

Now we can formally define the singular values of a linear bounded operator.

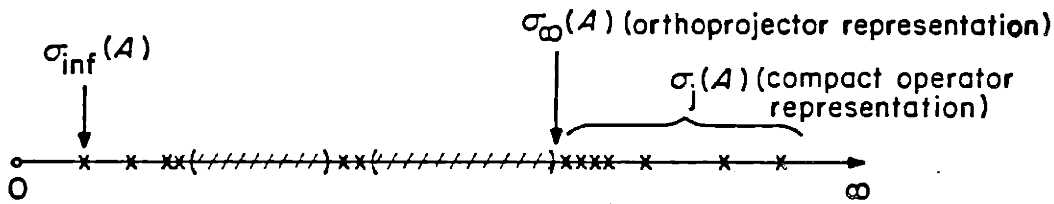
Definition (Singular Value of Linear Bounded Operators):

Let $A \in B(H)$. Then the singular values of A are defined in nonincreasing order as

$$\sigma_j(A) = \lambda_j(D) \; ; \quad j = 1,2,\ldots,\infty \tag{B.26}$$

where $\lambda_j(D)$ is defined in Definition (c) above.

Figure B.1 illustrates the relationship between the spectrum of D and the singular values of A.



The points x and the intervals (/////) are contained in $sp[(A^*A)^{1/2}]$

Figure B.1 Illustration of Singular Values of Linear Bounded Operator A

In the figure, we see that in defining the singular values for the operator A, we have broken A into three parts, the first of which is a compact operator. Hence the singular values can be enumerated. The second part is an orthoprojector on a plane which separates the eigenspace of the compact part from the "non-compact" part. For the readers who desire to know this geometrical interpretaiton more precisely, we refer them to [68, pp. 60-61]. The quantity $\sigma_{inf}(A)$ in the figure will be defined later.

The following theorem provides the motivation behind the construction of the singular values.

Theorem B.3 (Approximation Property of Singular Values):

Let $A \in \mathcal{B}(H)$. Then for any $k = 1,2,\ldots,r(A),\ldots$

$$\sigma_k(A) = \min_{K_k \in F_k} ||A - K_k|| \tag{B.27}$$

where F_k is defined in Eq. (B.7). Moreover

$$\sigma_\infty(A) = \min_{K \in C(H)} ||A - K|| \tag{B.28}$$

The proof of this theorem is given in [68, pp. 61-62]. This theorem has an important corollary.

Corollary B.3 (Fundamental Relationship between Singular Values and Norm):

Let $A \in \mathcal{B}(H)$. Then

$$\sigma_1(A) = ||A|| \tag{B.29}$$

<u>Proof:</u> $F_1 = \{0\} \subset B(H)$. Therefore letting $k = 1$ in Eq. (B.27) gives Eq. (B.29). (End of Proof)

Referring back to Figure B.1, we define the leftmost point of the spectrum as

$$\sigma_{inf}(A) = \inf sp(D) \tag{B.30}$$

We will show that

$$\sigma_1(A^{-1}) = [\sigma_{inf}(A)]^{-1} \tag{B.31}$$

First we establish a lemma.

<u>Lemma B.4:</u>

Let $A \in B(H)$ be invertible. Then

(a) if $\lambda \in sp(A)$, then $\lambda^{-1} \in sp(A^{-1})$; (B.32)

(b) if λ is a nonzero limit point in sp(A), then λ^{-1} is a limit point in $sp(A^{-1})$. (B.33)

<u>Proof of Lemma B.4:</u>

First we note that the invertibility of A implies $0 \notin sp(A)$. Hence $\lambda I - A$ is invertible if and only if $A^{-1} - \lambda^{-1}I$ is invertible. Therefore $\lambda^{-1} \in sp(A^{-1})$. (b) follows from (a) because the function λ^{-1} is analytic in the region $\lambda \neq 0$.

We state Eq. (B.31) formally in a theorem.

Theorem B.5:

Let $A \in B(H)$ be invertible. Then

$$\sigma_{\inf}(A) = [\sigma_1(A^{-1})]^{-1} \tag{B.34}$$

Proof of Theorem B.5:

Since $sp[(A^*A)^{1/2}]$ contains only positive numbers, by Lemma B.4,

$$\{\inf\ sp[(A^*A)^{1/2}]\}^{-1} = \sup\ sp[(A^*A)^{-1/2}] \tag{B.35}$$

But

$$\sigma_1(A^{-1}) = \sup\ sp[(A^*A)^{-1/2}] \tag{B.36}$$

because $A \rightarrow A^*$ is an involution [46]; i.e.

$$(A^*A)^{-1} = A^{-1}(A^{-1})^* \tag{B.37}$$

and

$$\sigma_1(A^{-1}) = \sigma_1[(A^{-1})^*] \tag{B.38}$$

Thus by the definition of $\sigma_{inf}(A)$ in Eq. (B.30)

$$[\sigma_{inf}(A)]^{-1} = \sigma_1(A^{-1}) \tag{B.39}$$

(End of Proof)

This completes our discussion on singular values of linear bounded operators.

B.3 Generalized Robustness Theorem

In this section we present a small gain theorem for additive perturbations, and use Theorems B.3 and B.5 to develop a generalized robustness theorem for feedback systems. We adopt the definition of feedback systems in Willems [53].

B.3.1 Definitions and Notation

We shall use the following notation in this section:

H = a separable Hilbert space with an inner product $(\bullet,\bullet)$

F = a separable Hilbert space of functions

$f: [0,\infty) \rightarrow \mathbf{H}$, with inner product $\langle\bullet,\bullet\rangle$

P_τ = truncation operator on **F** defined by

$$(P_\tau f)(t) = \begin{cases} f(t) \; ; & t \le \tau \\ 0 \; ; & t > \tau \end{cases} \; ; \; f(\bullet) \in \mathbf{F}$$

$\mathbf{F}_e$ = extended space of $\mathbf{F}$

= $\{f(\bullet) | P_\tau f \in \mathbf{F}$ for all $0 \leq \tau < \infty\}$

I = identity operator on $\mathbf{F}_e$(or $\mathbf{F}$)

$B(\mathbf{F}_e)$ = set of all linear bounded operators on $\mathbf{F}_e$

$B(\mathbf{F})$ = set of all linear bounded operators on $\mathbf{F}$

Let $G \in B(\mathbf{F}_e)$ $(B(\mathbf{F}))$. Then

G^* = adjoint of G with respect to $\langle \bullet, \bullet \rangle$

$||G||$ = induced operator norm by $\langle \bullet, \bullet \rangle$

Let G be an operator on $\mathbf{F}_e(\mathbf{F})$. Then G is called causal if $P_\tau G P_\tau = P_\tau G$ for all $\tau \geq 0$, where P_τ has been extended to $\mathbf{F}_e$. Let

$$B^+(\mathbf{F}_e) = \{G \in B(\mathbf{F}_e) | G \text{ is causal}\}$$

and

$$B^+(\mathbf{F}) = \{G \in B(\mathbf{F}) | G \text{ is causal}\}$$

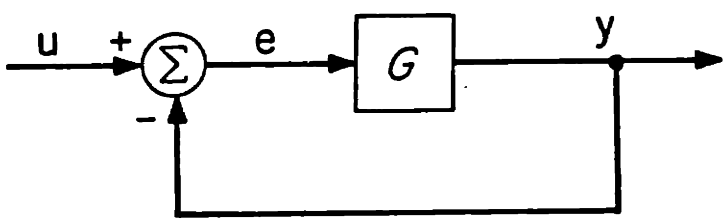
With an abuse of notation, given $G:\mathbf{F}_e \rightarrow \mathbf{F}_e$, the restriction of G to $\mathbf{F}$, $G|\mathbf{F}$, is also denoted by G when no confusion arises.

<u>Definition (Nominal Unity Feedback Control System):</u>

In Figure B.2

$$y = Ge \tag{B.40}$$

$$e = u - y \tag{B.41}$$



B.2 Nominal Unity Feedback System

$$G\colon \mathbf{F}_e \to \mathbf{F}_e \text{ is causal} \tag{B.42}$$

The system in Figure B.2 is called the nominal unity feedback system. It is called linear if G is linear.

<u>Definition (Unity Feedback System subject to Additive Perturbation):</u>

In Figure B.3

$$y = (G + \Delta G)e \tag{B.43}$$

$$e = u - y \tag{B.44}$$

$$G\colon \mathbf{F}_e \to \mathbf{F}_e \text{ is causal} \tag{B.45}$$

$$\Delta G\colon \mathbf{F}_e \to \mathbf{F}_e \text{ is causal} \tag{B.46}$$

The system in Figure B.3 is called a unity feedback system subject to additive perturbation.

<u>Definition (Well-Posedness of Feedback Systems):</u>

Referring to the unity feedback system in Figure B.4, let $F\colon \mathbf{F}_e \to \mathbf{F}_e$ be causal. Then the basic feedback equation is

$$(I + F)e = u \tag{B.47}$$

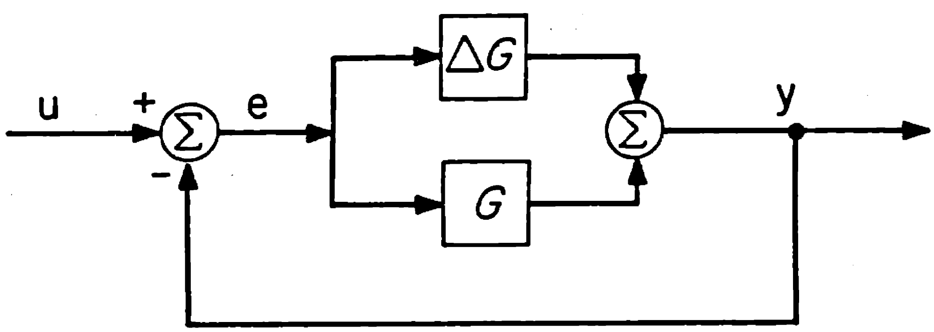


Figure B.3 Unity Feedback System subject to Additive Perturbation

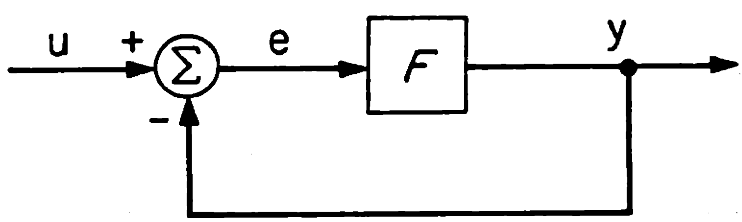


Figure B.4 Unity Feedback System

(a) We say that the unity feedback system is well-posed if

$$(I + F)^{-1}: \mathbf{F}_e \rightarrow \mathbf{F}_e \text{ exists and is causal} \tag{B.48}$$

(b) Given that the unity feedback system is well-posed; we say that the unity feedback system is (finite gain) stable if the inverse in Eq. (B.48) is bounded when restricted to the space **F**, i.e.

$$(I + F)^{-1}|\mathbf{F} \text{ is bounded} \tag{B.49}$$

<u>Remarks:</u>

(1) Referring to the definitions of the nominal unity feedback system, and system subject to additive perturbation, two cases are of interests:

a) F represents the nominal plant operator, i.e. G: $\mathbf{F}_e \rightarrow \mathbf{F}_e$;

b) F represents the nominal plant operator plus the perturbation operator, i.e.

$$F = G + \Delta G: \mathbf{F}_e \rightarrow \mathbf{F}_e \quad . \tag{B.50}$$

(2) A unity feedback system not satisfying the well-posedness is ill conditioned. Therefore in the rest of this appendix, when we speak of a unity feedback system, well-posedness is always assumed implicitly.

(3) Suppose that G, $\Delta G \in B^+(F_e)$, and assuming well-posedness, then the nominal unity feedback system is stable if and only if

$$(I + G)^{-1} \in B^+(F) \tag{B.51}$$

and the unity feedbak system subject to additive perturbation ΔG is stable if and only if

$$(I + G + \Delta G)^{-1} \in B^+(F) \tag{B.52}$$

This is true since a causal operator on **F** can be extended uniquely to a causal operator on $\mathbf{F}_e$.

(4) The notion of stability here says that a bounded input produces bounded output.

Given that the operators G and ΔG are bounded and the nominal unity feedback system is stable, a basic and legitimate question to ask is:

"What is the region of robustness of stability of the unity feedback system?" In other words, "what is the range of ΔG for which the perturbed system is (guaranteed to be) stable?"

One answer to this question is the following theorem.

<u>Theorem B.6 (Small Gain Theorem for Additive Perturbation):</u>

Consider the feedback system in Figure B.3 (assuming well-posedness). Suppose G: $\mathbf{F}_e \rightarrow \mathbf{F}_e$ and ΔG: $\mathbf{F}_e \rightarrow \mathbf{F}_e$ are both linear and causal and the nominal unity feedback system is stable, i.e. there exists

$$(I + G)^{-1} \in B^{+}(\mathbf{F}) \tag{B.52}$$

Then a sufficient condition for the perturbed unity feedback system in Figure B.3 to be stable, i.e. there exists

$$(I + G + \Delta G)^{-1} \in B^{+}(\mathbf{F}) \tag{B.53}$$

is

$$||(I + G)^{-1}\Delta G|| < 1 \tag{B.54}$$

Corollary B.6

Another sufficient condition is

$$||(I + G)^{-1}|| \; ||\Delta G|| < 1 \tag{B.55}$$

or

$$||\Delta G|| < ||(I + G)^{-1}||^{-1} \tag{B.56}$$

Proof:

Since $\mathbf{F}$ is a Hilbert space, under the induced operator norm

$B(\mathbf{F})$ forms a Banach algebra (see e.g. Rudin [46] or Willems [53]). Moverover $B^{+}(\mathbf{F}) \subset B(\mathbf{F})$ is a subalgebra. Therefore if

$$||(I + G)^{-1}\Delta G|| < 1 \tag{B.57}$$

then $I + (I + G)^{-1}\Delta G$ is invertible in $B^{+}(\mathbf{F})$. Hence $I + G + \Delta G$ is invertible in $B^{+}(\mathbf{F})$, since the invertible elements of $B^{+}(\mathbf{F})$ forms a group. Therefore the unity feedback system in Figure B.3 is stable. The corollary follows since under this norm

$$||(I + G)^{-1}\Delta G|| < ||(I + G)^{-1}|| \; ||\Delta G|| \tag{B.58}$$

(End of Proof)

Remarks:

(1) This theorem shows that the region of robustness of stability can be characterized by the gains, i.e. induced operator norm of the inverse of $I + G$ and ΔG.

$$||\Delta G|| < ||(I + G)^{-1}||^{-1} \tag{B.59}$$

This equation will form the basis for the development of the robustness theorem in terms of singular values below.

(2) The requirement that F be a separable Hilbert space can be relaxed to a Banach space, but the former is needed for the development of singular values.

The following robustness characterization in terms of singular values is a direct consequence of Theorems B.6, B.3 and B.5.

Theorem B.7 (Small Singular Value Theorem for Additive Perturbation):

Under the conditions of Theorem B.6, a sufficient condition for the perturbed unity feedback system in Figure B.3 to be stable, is

$$\sigma_1(\Delta G) < \sigma_{\inf}(I + G) \tag{B.60}$$

By using the Parseval's theorem, one can easily show that Theorem B.7 implies the usual finite dimensional robustness characterization results (Sandell [74], Lehtomaki [5], Doyle [10]).

This completes our discussion on robustness and the singular values of linear bounded operators.

APPENDIX C

Modal Expansion of Flexible Beam

First we use the principle of separation of variables to solve Eqs. (4.7) - (4.9). Let

$$y(x,t) = \sum_{k=0}^{\infty} X_k(x)T_k(t) \tag{C.1}$$

Then Eq. (4.7) can be written as

$$\sum_{k=0}^{\infty} \left[\frac{d^4}{dx^4} X_k(x)T_k(t) - \zeta \frac{d^2}{dx^2} X_k(x) \frac{d}{dt} T_k(t) + X_k(x) \frac{d^2}{dt^2} T_k(t)\right]$$

$$= u(t)\ \delta(x - \tfrac{1}{2}) \tag{C.2}$$

Now try

$$\frac{d^2}{dx^2} X_k(x) + \omega_k^2 X_k(x) = 0 \tag{C.3}$$

Then Eq. (C.2) becomes

$$\sum_{k=0}^{\infty} \left[\omega_k^4 T_k(t) + \zeta\omega_k^2 \frac{d}{dt} T_k(t) + \frac{d^2}{dt^2} T_k(t)\right] X_k(x)$$

$$= u(t)\ \delta(x - \tfrac{1}{2}) \tag{C.4}$$

Using the boundary conditions, Eqs. (4.8) - (4.9), in Eq. (C.3), we get

$$\omega_k = k\pi \qquad ; k = 0,\ 1,\ 2\ldots \tag{C.5}$$

and

$$X_k(x) = \sin(k\pi x) \qquad ; k = 0,\ 1,\ 2,\ldots \tag{C.6}$$

Since $\sqrt{2}\,\sin(k\pi x)$, k = 1, 2,... forms a complete orthonormal basis for the beam shape between x = 0 and x = 1, satisfying the boundary conditions, we conclude that the solution to Eq. (4.7) - (4.9) is given by

$$y(x,t) = \sum_{k=1}^{\infty} \sin(k\pi x)T_k(t) \tag{C.7}$$

where $T_k(t)$ is given by Eq. (C.4). This same result can also be obtained by using the Fourier series expansion.

Now multiplying both sides of Eq. (C.4) by $\sin(k\pi x)$, and integrating over x, from 0 to 1, we obtain

$$\omega_k^4 T_k(t) + \zeta\omega_k^2 \frac{d}{dt} T_k(t) + \frac{d^2}{dt^2} T_k(t) = \frac{1}{2}\, u(t)\, \sin(\frac{k\pi}{2}) \tag{C.8}$$

APPENDIX D

Green's Function Characterization of K_2

The Green's function in Eq. (4.34) is determined as follows. Try

$$h(x,\xi) = \sum_{i=0}^{\infty} \alpha_i \sin(i\pi x)\ \sin(i\pi\xi) \tag{D.1}$$

where α_i's are to be computed. Multiplying both sides by $\sin(n\pi\xi)$ and integrating over the interval between $\xi = 0$ and $\xi = 1$, we get

$$\int_0^1 h(x,\xi)\sin(n\pi\xi)d\xi = \int_0^1 [\sum_{i=0}^{\infty} \alpha_i \sin(i\pi x)\sin(i\pi\xi)]\sin(n\pi\xi)d\xi \tag{D.2}$$

By using Eq. (4.34) and the orthonormality of the sine functions, this is reduced to

$$K_2 \sin(n\pi x) = \frac{\alpha_n}{2} \sin(n\pi x) \tag{D.3}$$

But

$$\frac{\partial^2}{\partial x^2} \sin(n\pi x) = - n^2\pi^2 \sin(n\pi x) \tag{D.4}$$

Therefore,

$$K_2 \frac{\partial^2}{\partial x^2} \sin(n\pi x) = - \frac{\alpha_n n^2 \pi^2}{2} \sin(n\pi x) \tag{D.5}$$

But from Eq. (4.30),

$$K_2 \frac{\partial^2}{\partial x^2} \sin(n\pi x) = - \sin(n\pi x) \tag{D.6}$$

This implies that the value of α_n must be

$$\alpha_n = \frac{2}{n^2 \pi^2}, \qquad n = 1, 2, 3 \tag{D.7}$$

Since $\sin(n\pi x)$ forms a complete basis for $S(0,1)$ and all the boundary conditions are satisfied, we conclude that the operator K_2 can be characterized in the form of Eq. (4.34), by using

$$h(x,\xi) = \sum_{i=1}^{\infty} \frac{2}{i^2 \pi^2} \sin(i\pi x) \sin(i\pi \xi) \tag{D.8}$$

It is easy to verify that K_2 thus defined is positive definite.

APPENDIX E

Optimal Loop Function Analysis

One can write

$$g(s) = \sum_{k=1}^{\infty} g_k(s) \tag{E.1}$$

where

$$g_k(s) = \frac{1}{k^2\pi^2} \sin^2(\frac{k\pi}{2}) \frac{2s}{s^2+\zeta k^2\pi^2 s+k^4\pi^4} \tag{E.2}$$

First we examine $|g(s)|$ at $s = jk^2\pi^2$, $k = 1,2,\ldots$ Separating $g_k(s)$ from the series, we get

$$|g(s)| \leq |g_k(s)| + |h_k(s)| \tag{E.3}$$

where

$$h_k(s) = \sum_{\substack{m=1 \\ m\neq k}}^{\infty} \frac{1}{m^2\pi^2} \sin^2 (\frac{m\pi}{2}) \frac{2s}{s^2+\zeta k^2\pi^2 s+k^4\pi^4} \tag{E.4}$$

Claims:

(1) For k odd,

$$\max_{\omega} |g_k(j\omega)| = |g_k(jk^2\pi^2)| = \frac{2}{\zeta k^4\pi^4} \tag{E.5}$$

(2) For k even, $g_k(j\omega)$ is identically zero,

(3) $$|h_k(jk^2\pi^2)| < \frac{1}{k^2\pi^2}\left(1 + \frac{3}{4k^2\pi^2}\right) \tag{E.6}$$

Proof of Claims:

(1) For k odd, since the function,

$$|g_k(j\omega)|^{-2} = \frac{(k^4\pi^4-\omega^2)^2+\zeta^2k^4\pi^4\omega^2}{4\omega^2}(k^4\pi^4)$$

$$= \frac{k^4\pi^4}{4}\left[\zeta^2k^4\pi^4 + \frac{1}{\omega^2}(k^4\pi^4-\omega^2)^2\right]$$

achieves a minimum when the second term in the sum is zero, i.e. at $\omega = k^2\pi^2$, $|g_k(j\omega)|$ achieves a maximum there also, and

$$|g_k(jk^2\pi^2)| = \frac{2}{\zeta k^4\pi^4}$$

(2) This is trivial since $\sin(\frac{k\pi}{2}) = 0$ for k even.

(3) Let us split $h_k(s)$ into two parts,

$$p_k(s) = \sum_{m=1}^{k-1} \frac{1}{m^2\pi^2} \sin^2(\frac{m\pi}{2}) \frac{2s}{s^2+\zeta m^2\pi^2 s+m^4\pi^4} \tag{E.8}$$

$$q_k(s) = \sum_{m=k+1}^{\infty} \frac{1}{m^2\pi^2} \sin^2(\frac{m\pi}{2}) \frac{2s}{s^2+\zeta m^2\pi^2 s+m^4\pi^4} \tag{E.9}$$

At $s = jk^2\pi^2$, taking the magnitude of each term, and using the fact that $\sin^2(\frac{m\pi}{2}) \leq 1$,

$$|p_k(jk^2\pi^2)| \leq \sum_{m=1}^{k-1} \frac{2}{m^2\pi^4} \frac{k^2}{[(m^4-k^4)^2+\zeta m^4k^4]^{1/2}} \tag{E.10}$$

Droping ζm^4k^4 from the demoninator,

$$|p_k(jk^2\pi^2)| \leq \sum_{m=1}^{k-1} \frac{2}{m^2\pi^4} \frac{k^2}{k^4-m^4} \tag{E.11}$$

Now observe that by partial fraction expansion,

$$\frac{2}{m^2} \frac{k^2}{k^4-m^4} = \frac{1}{k^2} (\frac{2}{m^2} + \frac{1}{k^2-m^2} - \frac{1}{k^2+m^2}) \tag{E.12}$$

Therefore,

$$|p_k(jk^2\pi^2)| \leq \frac{1}{k^2\pi^4} \sum_{m=1}^{k-1} \left(\frac{2}{m^2} + \frac{1}{k^2-m^2}\right) \tag{E.13}$$

By the use of table for series [71],

$$\sum_{m=1}^{k-1} \frac{1}{k^2-m^2} = -\frac{3}{4k^2} + \sum_{m=k+1}^{\infty} \frac{1}{m^2-k^2}$$

$$= -\frac{3}{4k^2} + \sum_{i=1}^{\infty} \frac{1}{2ki+i^2}$$

$$< -\frac{3}{4k^2} + \sum_{i=1}^{\infty} \frac{1}{i^2}$$

$$< \frac{\pi^2}{6} \tag{E.14}$$

Also,

$$\sum_{m=1}^{k-1} \frac{2}{m^2} < \frac{\pi^2}{3} \tag{E.15}$$

Thus

$$|p_k(jk^2\pi^2)| \leq \frac{1}{2k^2\pi^2} \tag{E.16}$$

Similarly,

$$|q_k(jk^2\pi^2)| < \frac{1}{2k^2\pi^2} + \frac{3}{4k^4\pi^4} \tag{E.17}$$

Now since

$$|h_k(s)| \leq |p_k(x)| + |q_k(s)| \tag{E.18}$$

we obtain,

$$|h_k(jk^2\pi^2)| < \frac{1}{k^2\pi^2} + \frac{3}{4k^4\pi^4} \tag{E.19}$$

establishing claim (3).

Using the above claims,

$$|g_k(jk^2\pi^2)| < \frac{1}{k^2\pi^2}\left(1 + \frac{3}{4k^2\pi^2}\right) + \begin{cases} \dfrac{2}{\zeta k^4\pi^4}, & k \text{ odd} \\ 0 & k \text{ even} \end{cases} \tag{E.20}$$

For k relatively small, and odd, since ζ is small,

$$|g(jk^2\pi^2)| \approx \frac{2}{\zeta k^4\pi^4} \tag{E.21}$$

But for k sufficiently large, the term $\frac{2}{\zeta k^4 \pi^4}$ no longer dominates over $\frac{1}{k^2\pi^2}(1 + \frac{3}{4k^2\pi^2})$. Consequently, the values $|g(jk^2\pi^2)|$ are small at high frequencies and rolls off at a rate of no less than 20 dB/decade.

It is not hard to show, by modifying the above analysis, that for $\omega = jf^2\pi^2$, where $f > 0$ is not necessarily an integer,

$$|g(j\omega)| < \frac{1}{f^2\pi^2}(1 + \frac{3}{4\pi^2}) + \frac{2f^2}{k^2\pi^4((k^4-f^4)^2+\zeta k^4 f^4)^{1/2}} \qquad \text{(E.22)}$$

where k is the nearest odd integer to f. Therefore, on the whole, $|g(j\omega)|$ rolls off at 20 dB/decade at high frequencies, and there exists a cutoff frequency ω_c.

APPENDIX F

Derviation of Suboptimal Loop Function

Recall from the open-loop analysis that

$$\frac{\partial y}{\partial t}(x,t) = \sum_{i=1}^{\infty} \sin(i\pi x)\, \dot{T}_i(t) \tag{F.1}$$

Hence $z_\ell(t)$, defined in Eq. (4.52), can be written in terms of $\dot{T}_i(t)$ as

$$z_\ell(t) = \sum_{i=1}^{\infty} \sin\left(\frac{i\ell\pi}{N}\right) \dot{T}_i(t); \qquad \ell = 0,\ 1, \ldots, N \tag{F.2}$$

Substituting this into Eq. (4.53) gives

$$\dot{T}_k^N(t) = \frac{1}{N} \sum_{\ell=0}^{N-1} \sin\left(\frac{\ell k\pi}{N}\right) \sum_{i=1}^{\infty} \sin\left(\frac{i\ell\pi}{N}\right) \dot{T}_i(t) \tag{F.3}$$

Combining this result with Eq. (4.51), and translating the result in the frequency domain, we get the suboptimal feedback

$$v(s) = \sum_{k=1}^{\infty} \frac{2}{k^2\pi^2} \sin\left(\frac{k\pi}{2}\right) \frac{1}{N} \sum_{\ell=0}^{N-1} \sin\left(\frac{\ell k\pi}{N}\right) \sum_{i=1}^{\infty} \sin\left(\frac{i\ell\pi}{N}\right) \hat{\dot{T}}(s) \tag{F.4}$$

where for the moment we assume $M = \infty$ in Eq. (4.51). We shall remove this assumption later.

Using the transfer function $\hat{\dot{T}}_i(s)/\hat{u}(s)$ from Eq. (4.14), we obtain

$$\frac{v(s)}{\hat{u}(s)} = \sum_{k=1}^{\infty} \frac{2}{k^2\pi^2} \sin(\frac{k\pi}{2})$$

$$\times [\frac{1}{N} \sum_{\ell=0}^{N-1} \sum_{i=1}^{\infty} \sin(\frac{ik\pi}{N}) \sin(\frac{i\ell\pi}{N}) \frac{2s \sin(\frac{i\pi}{2})}{s^2 + \zeta i^2\pi^2 s + i^4\pi^4}] \quad (F.5)$$

Rearranging the above, we get (replacing the right-hand-side with $\tilde{g}(s)$),

$$\tilde{g}(s) = \sum_{i=1}^{\infty} \frac{2s \sin(\frac{i\pi}{2})}{s^2 + \zeta i^2\pi^2 s + i^4\pi^4} \{ \sum_{k=1}^{\infty} \frac{2}{k^2\pi^2} \sin(\frac{k\pi}{2})$$

$$\times [\frac{1}{N} \sum_{\ell=0}^{N-1} \sin(\frac{\ell k\pi}{N}) \sin(\frac{i\ell\pi}{N})]\} \quad (F.6)$$

where the order of the summations over i, k and ℓ have been changed. This is possible because the summation over ℓ is finite, and those over i and k are easily seen to be uniformly convergent because of the $i^4\pi^4$ and $k^2\pi^2$ terms in the denominators.

Let c_i^N denote the quantity inside the braces in Eq. (F.6). Then

$$c_i^N = \frac{1}{N} \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} \sin(\frac{k\pi}{2}) \sum_{\ell=0}^{N-1} \sin(\frac{\ell k\pi}{N}) \sin(\frac{i\ell\pi}{N}) \qquad \text{(F.7)}$$

The quantity c_i^N can be evaluated in closed form. Interchanging the summation in Eq. (F.7),

$$c_i^N = \frac{1}{N} \sum_{\ell=0}^{N-1} \sin(\frac{i\ell\pi}{N}) \sum_{k=1}^{\infty} \frac{2}{k^2\pi^2} \sin(\frac{k\pi}{2}) \sin(\frac{\ell k\pi}{N}) \qquad \text{(F.8)}$$

Using a trigonometrical identity,

$$c_i^N = \frac{1}{N} \sum_{\ell=0}^{N-1} \sin(\frac{i\ell\pi}{N}) \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} \{\cos[k\pi(\frac{1}{2} - \frac{\ell}{N})] - \cos[k\pi(\frac{1}{2} + \frac{\ell}{N})]\} \qquad \text{(F.9)}$$

By using the following formula from Gradshteyn and Ryzhik [71],

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4} \ ; \ 0 \leq x \leq 2\pi \qquad \text{(F.10)}$$

we get

$$\sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} \cos[k\pi(\frac{1}{2} + \frac{\ell}{N})] = \frac{1}{6} - \frac{1}{2}(\frac{1}{2} + \frac{\ell}{N}) + \frac{1}{4}(\frac{1}{2} + \frac{\ell}{N})^2 \qquad \text{(F.11)}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} \cos[k\pi(\frac{1}{2} - \frac{\ell}{N})] = \begin{cases} \frac{1}{6} - \frac{1}{2}(\frac{1}{2} - \frac{\ell}{N}) + \frac{1}{4}(\frac{1}{2} - \frac{\ell}{N})^2, & \ell \leq \frac{N}{2} \\ \frac{1}{6} + \frac{1}{2}(\frac{1}{2} - \frac{\ell}{N}) + \frac{1}{4}(\frac{1}{2} - \frac{\ell}{N})^2, & \frac{N}{2} < \ell \end{cases}$$

(F.12)

Therefore,

$$c_i^N = \frac{1}{N} \sum_{\ell=0}^{N-1} \sin(\frac{i\ell\pi}{N}) \begin{cases} \frac{\ell}{2N} & , \ell \leq \frac{N}{2} \\ \frac{1}{2} - \frac{\ell}{2N} & , \frac{N}{2} < \ell \end{cases} \qquad \text{(F.13)}$$

Let N be even and r is the integer N/2. Then

$$c_i^N = -\frac{1}{2N^2} \sum_{\ell=0}^{N-1} \ell \sin(\frac{\ell i\pi}{N}) + \frac{1}{N^2} \sum_{\ell=0}^{r} \ell \sin(\frac{\ell i\pi}{N})$$

(F.14)

$$+ \frac{1}{2N} \sum_{\ell=0}^{N-1} \sin(\frac{\ell i\pi}{N}) - \frac{1}{2N} \sum_{\ell=0}^{r} \sin(\frac{\ell i\pi}{N})$$

Using tables [71]

$$\sum_{\ell=0}^{N-1} \sin\left(\frac{\ell i\pi}{N}\right) = \sin\left(\frac{i\pi}{2}\right) \sin\left(\frac{(N-1)i\pi}{2N}\right) \operatorname{cosec}\left(\frac{i\pi}{2N}\right) \tag{F.15}$$

This can be reduced to

$$\sum_{\ell=0}^{N-1} \sin\left(\frac{\ell i\pi}{N}\right) = \cot\left(\frac{i\pi}{2N}\right) \tag{F.16}$$

Also from tables [71]

$$\sum_{\ell=0}^{N-1} \ell\sin\left(\frac{\ell i\pi}{N}\right) = \frac{\sin(i\pi)}{4\sin^2\left(\frac{i\pi}{2N}\right)} - \frac{N\cos\left(\frac{2N-1}{2N}\, i\pi\right)}{2\sin^2\left(\frac{i\pi}{2N}\right)} \tag{F.17}$$

This can be reduced to

$$\sum_{\ell=0}^{N-1} \ell\sin\left(\frac{\ell i\pi}{N}\right) = \frac{N}{2}\,(-1)^i \cot\left(\frac{i\pi}{2N}\right) \tag{F.18}$$

For the other terms in Eq.(F.14),

$$\sum_{\ell=0}^{r} \sin\left(\frac{\ell i\pi}{N}\right) = \sin\left(\frac{r+1}{2N}\, i\pi\right) \sin\frac{ri\pi}{2N} \operatorname{cosec}\left(\frac{i\pi}{2N}\right) \tag{F.19}$$

$$\sum_{\ell=0}^{r} \ell \sin(\frac{\ell i\pi}{N}) = \frac{\sin \frac{r+1}{2N} i\pi}{4\sin^2(\frac{i\pi}{2N})} - \frac{(r+1)\cos \frac{2r+1}{2N} i\pi}{2\sin^2(\frac{i\pi}{2N})} \tag{F.20}$$

But by construction,

$$r = \frac{N}{2} \tag{F.21}$$

Hence Eqs. (F.19) - (F.20) can be simplified to, for i odd,

$$\sum_{\ell=0}^{r} \sin(\frac{i\ell\pi}{N}) = \frac{1}{2}[\cot(\frac{i\pi}{2N}) + \sin(\frac{i\pi}{2})] \tag{F.22}$$

and for i odd,

$$\sum_{\ell=0}^{r} \ell\sin(\frac{i\ell\pi}{N}) = (-1)^{\frac{i-1}{2}}[\frac{1}{4}\operatorname{cosec}^2(\frac{i\pi}{2N}) + \frac{1}{2}(N+1)] \tag{F.23}$$

Therefore, for i odd

$$c_i^N = (-1)^{\frac{i-1}{2}}[\frac{1}{4N^2}\operatorname{cosec}^2(\frac{i\pi}{2N}) + \frac{N+1}{2N^2} - \frac{1}{4N}] \tag{F.24}$$

Now observe that c_i^N above is obtained by summing over k for infinitely many terms in Eqs. (F.8) - (F.12). In actual implementation, this is not possible, and we can only compute the series from $k = 1$ to $k = M$ for some finite M. However, we can always choose M sufficiently large such that the error introduced by this truncation is less than $\frac{1}{4N^2}$, for each ℓ in Eq. (F.13). Therefore, since ℓ raises from 0 to N-1, the total truncation error between the actual and nominal c_i^N in Eq. (F.24) is less than $\frac{1}{4N}$. This gives us Eq. (4.58), in which the symbol $O(\frac{1}{N})$ is used to denote $\frac{N+1}{2N^2}$.

It is important to note that in the design procedure, M is computed only after N has been chosen (Appendix G).

APPENDIX G

Analysis of Implementation Error

Consider the difference between the corresponding terms $\tilde{g}_i(s) - g_i(s)$.

$$\tilde{g}_i(s) - g_i(s) = \frac{2s}{s^2 + \zeta i^2\pi^2 s + i^4\pi^4}\left[\sin(\frac{i\pi}{2})\ c_i^N - \frac{\sin^2(\frac{i\pi}{2})}{i^2\pi^2}\right] \tag{G.1}$$

Clearly,

$$\max_{\omega\in\mathbf{R}_+} |\tilde{g}_i(j\omega) - g_i(j\omega)| = |\tilde{g}_i(j\omega) - g_i(j\omega)|_{\omega=i^2\pi^2} \tag{G.2}$$

This is identically equal to zero for i even.

By performing an analysis similar to the one in Appendix E, we see that at low frequencies, for ζ very small, $|\tilde{g}_i(j\omega) - g_i(j\omega)|$ is much larger than $\sum_{\substack{n=1 \\ n\neq i}}^{\infty} |\tilde{g}_n(j\omega) - g_n(j\omega)|$ around $\omega = i^2\pi^2$, i.e.

$$|\tilde{g}(j\omega) - g(j\omega)| \simeq |\tilde{g}_i(j\omega) - g_i(j\omega)| \text{ around } \omega = i^2\pi^2 \tag{G.3}$$

but this is not the case at high frequencies.

Let us examine this more closely. For all i odd,

$$|\tilde{g}_i(j\omega) - g_i(j\omega)|_{\omega=i^2\pi^2} = \frac{2}{\zeta i^2\pi^2} \left| c_i^N - \frac{(-1)^{\frac{i-1}{2}}}{i^2\pi^2} \right| \tag{G.4}$$

Let $\delta > 0$ and let N_0 be chosen such that for $N \geq N_0$

$$\frac{2}{\pi^2 N} < \frac{\zeta\delta}{2} \tag{G.5}$$

Then since

$$\frac{N+1}{2N^2} \leq \frac{1}{N} \tag{G.6}$$

we have

$$\frac{2}{i^2\pi^2}\left(\frac{N+1}{2N^2}\right) < \frac{\zeta\delta}{2} \tag{G.7}$$

So a sufficient condition for $|\tilde{g}_i(ji^2\pi^2) - g_i(ji^2\pi^2)| < \delta$ is, by using c_i^N from Eq. (F.24),

$$\frac{1}{i^2\pi^2}\left|\frac{1}{4N^2}\operatorname{cosec}^2\left(\frac{i\pi}{2N}\right) - \frac{1}{i^2\pi^2}\right| < \frac{\zeta\delta}{4} \quad \text{for all i odd} \tag{G.8}$$

In what follows, we show that N can be chosen such that Eq. (G.8) is satisfied.

It is easy to see that as N tends to ∞, for fixed i,

$$\frac{1}{4N^2}\,\mathrm{cosec}^2\left(\frac{i\pi}{2N}\right) \longrightarrow \frac{1}{i^2\pi^2} \tag{G.9}$$

In particular, we can choose a small number $\epsilon > 0$ and N_1 such that for $N \geq N_1$

$$\left|\frac{1}{4N^2}\,\mathrm{cosec}^2\left(\frac{\pi}{2N}\right) - \frac{1}{\pi^2}\right| < \frac{\epsilon\delta}{\pi^2} \tag{G.10}$$

Since the maximum of the discrete sequence $\mathrm{cosec}^2\left(\frac{i\pi}{2N}\right)$ occurs at $i = 1$ and

$$\max_i \frac{1}{4N^2}\,\mathrm{cosec}^2\left(\frac{i\pi}{2N}\right) = \frac{1}{4N^2}\,\mathrm{cosec}^2\left(\frac{\pi}{2N}\right) \tag{G.11}$$

We deduce from Eqs. (G.10) and (G.11) that

$$\frac{1}{4N^2}\,\mathrm{cosec}^2\left(\frac{i\pi}{2N}\right) < \frac{1+\epsilon\delta}{\pi^2} \quad \text{for all } i \tag{G.12}$$

Therefore for all i,

$$\left|\frac{1}{4N^2}\,\mathrm{cosec}^2\left(\frac{i\pi}{2N}\right) - \frac{1}{i^2\pi^2}\right| < \max\left(\frac{1+\epsilon\delta}{\pi^2}, \frac{1}{i^2\pi^2}\right) \tag{G.13}$$

because both terms on the left-hand side are positive. Clearly for all $i \geq 1$, the quantity on the right-hand side is $\frac{1+\epsilon\delta}{\pi^2}$.

If i is large enough such that

$$2\pi^2 \left(\frac{1+\epsilon\delta}{\zeta\delta}\right)^{1/2} < i \tag{G.14}$$

then

$$\frac{1+\epsilon\delta}{i^2\pi^4} < \frac{\zeta\delta}{4} \tag{G.15}$$

Thus according to Eq. (G.13), for these i's,

$$\frac{1}{i^2\pi^2} \left| \frac{1}{4N^2} \operatorname{cosec}^2\left(\frac{i\pi}{2N}\right) - \frac{1}{i^2\pi^2} \right| < \frac{\zeta\delta}{4} \tag{G.16}$$

Let

$$I = \text{the first integer larger than } 2\pi^2 \left(\frac{1+\epsilon\delta}{\zeta\delta}\right)^{1/2} \tag{G.17}$$

Then what remains for us to do is to make sure that our spatial sampling is dense enough that Eq. (G.16) is satisfied for $i < I$ also. Specifically, we want to choose N_2 such that for $N \geq N_2$

$$\frac{1}{i^2\pi^2} \left| \frac{1}{4N^2} \operatorname{cosec}^2\left(\frac{i\pi}{2N}\right) - \frac{1}{i^2\pi^2} \right| < \frac{\zeta\delta}{2} \tag{G.18}$$

for all $i < I$.

Finally let

$$N = \max(N_0, N_1, N_2) \tag{G.19}$$

where N_0, N_1, N_2 are defined along with the inequalities in Eqs. (G.7), (G.16) and (G.18), respectively. This N guarantees that

$$|\tilde{g}(j\omega) - g(j\omega)| < \delta \quad \text{for all } \omega \in \mathbf{R}_+ \tag{G.20}$$

and the suboptimal system is closed-loop stable.

Next we establish that for small ζ, ϵ, and δ

$$\sum_{\substack{i=1 \\ i \neq k}}^{\infty} |\tilde{g}_i(j\omega) - g_i(j\omega)| < 0.05 \tag{G.21}$$

where k is the nearest odd integer to $f > 0$, and f is related to ω by

$$\omega = f^2\pi^2 \tag{G.22}$$

Note that by Eqs. (4.56) - (4.60),

$$|g_i(j\omega) - g_i(j\omega)| = \frac{2f^2}{[(f^4-i^4)^2+\zeta i^4 f^4]^{1/2}\pi^2} e_i \tag{G.23}$$

$$< \frac{2f^2}{|f^4-i^4|\pi^2} e_i$$

where

$$e_i = \left| \frac{1}{4N^2} \operatorname{cosec}^2 \left(\frac{i\pi}{2N}\right) - \frac{1}{i^2\pi^2} + 0_i\left(\frac{1}{N}\right) \right| \tag{G.24}$$

By Eqs. (G.13), (G.5), (G.6),

$$|e_i| < \frac{1+\epsilon\delta}{\pi^2} + \frac{\zeta\delta\pi^2}{4} = \alpha \ , \quad \text{for all } i \tag{G.25}$$

Since ϵ, δ, and ζ are usually small numbers, α is very close to $\frac{1}{\pi^2}$.
Also,

$$\frac{2f^2}{f^4-i^4} = \frac{1}{f^2-i^2} + \frac{1}{f^2+i^2} \tag{G.25}$$

Clearly

$$\sum_{\substack{i=1 \\ i\neq k}}^{\infty} \frac{1}{f^2 + i^2} \leq \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} \tag{G.26}$$

and using similar analysis to that in Appendix E, we obtain

$$\sum_{\substack{i=1 \\ i\neq k}}^{\infty} \frac{1}{|f^2-i^2|} \leq \frac{\pi^2}{6} + \frac{3}{4} \tag{G.27}$$

Therefore, summing Eq. (G.23) over i, and using Eqs. (G.24) - (G.27), we get

$$\sum_{\substack{i=1 \\ i=k}}^{\infty} |\tilde{g}_i(j\omega)| < \left(\frac{1}{3} + \frac{3}{4\pi^2}\right) \alpha \qquad \text{(G.28)}$$

For $\alpha \gtrsim \frac{1}{\pi^2}$, which is usually the case, the last quantity in Eq. (G.28) is less than 0.05.

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