STABILIZATION OF THE TEARING MODE BY TURBULENT DIFFUSION AND RUNAWAY ELECTRONS

by

ERIC HANS ESAREY
B.S.N.E., University of Michigan
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Signature of Author . . . . . .
Department of Nuclear Engineering
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Certified by . . . . . .
Professor Kim Molvig
Thesis Supervisor

Accepted by . . . . . .
Professor Allan F. Henry, Chairman
Departmental Committee on Graduate Students

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ABSTRACT

A fully kinetic analysis of the $m = 2$ tearing mode is performed for a tokamak plasma including the effects of turbulent electron diffusion and runaway electrons. Turbulent diffusion is included in the analysis by applying the normal stochastic approximation (NSA) to the collisionless drift kinetic equation (DKE) for electrons. A kinetic analysis inherently allows for the choice of various equilibrium electron velocity distributions, thus enabling a comparison between a drifted Maxwellian and a runaway-type distribution. This analysis is fully electromagnetic, including the effects a magnetic fluctuation potential $\tilde{A}_{||}$ as well as a finite electrostatic potential $\tilde{\phi}$, and is valid in the low-beta, low-frequency regime. The electron response is obtained by applying the NSA to the DKE, and the ion response is given by the linearized Vlasov equation. Ampere's law and quasineutrality are then used to derive a set of coupled, self-adjoint equations for the fluctuation potentials $\tilde{\phi}$ and $\tilde{A}_{||}$. Solutions to this set of equations describe both unstable finite-β drift waves when analyzed for high m modes and the tearing mode when analyzed for low m modes (where m is the poloidal mode number).

In the NSA, the tearing mode is assumed to exist on a background of drift wave turbulence. The underlying drift waves produce overlapping phase space islands, which lead to stochastic electron orbits. The NSA exploits the properties of stochastic orbits to replace the nonlinear fluctuation terms in the orbit operator of the DKE with a radial diffusion operator, $-D_e \partial^2 / \partial^2 x$, where D_e is the electron diffusion coefficient. Results for the tearing mode indicate that stability is obtained for sufficiently large values of the diffusion coefficient. Provided $D_e \sim 1/n$, this implies that a density threshold must be surpassed before the tearing mode is observed. Physically, turbulent electron diffusion prohibits the formation of a perturbed parallel current within a finite diffusive correlation distance $x_c \sim D_e^{1/3}$ of the rational

surface. This cuts into the available energy driving the tearing mode and reduces it from $\Delta'(0)$ to a value of $\Delta'(x_c)$. Here, $\Delta'(x_c)$ is the difference between the logarithmic derivative of $\tilde{A}_{||}$ evaluated at a distance x_c from the rational surface and that value at a distance of $-x_c$. Since $\Delta'(x)$ is typically a decreasing function of x, then stabilization occurs when $x_c > W$, where W is the nonlinear island saturation width given by $\Delta'(W) = 0$.

When a runaway-type distribution is used for the equilibrium electron distribution in the place of a drifted Maxwellian, the real frequency of the tearing mode is shifted to a value above the electron diamagnetic frequency ω_{*e} by an amount $\delta\omega \sim n_b v_b$. Here, n_b is the density and v_b the velocity of the fast electron beam which is used to model the runaway-type current. In addition, a new stabilizing term appears in the expression for the growth rate proportional to $\delta\omega$. Physically, this stabilizing term represents the additional energy necessary to maintain the particle motion at the frequency $\omega_{*e} + \delta\omega$. Since $\delta\omega = 0$ for a drifted Maxwellian equilibrium, this implies that stability is greater when a fast electron population is present. At higher densities, these fast electrons relax back into the bulk population due to the increased collisionality. Hence, the tearing mode stability is enhanced at low densities due to the presence of runaway electrons. This runaway stabilization is a higher order effect, however, and is only important for a tearing mode near marginal stability.

The inclusion of a finite electrostatic potential gives an additional stabilizing term to the dispersion relation, which physically represents ion inertial effects. This ion inertial effect implies that, in the absence of both turbulent diffusion and runaway electrons, the tearing mode is stabilized for ion betas β_i above some critical value, β_c, where $\beta_c \sim \Delta'(0)$. Hence, this ion inertial stabilization at high density, combined with the stabilization by turbulent diffusion and runaway electrons at low density, implies that it may be possible to operate a tokamak in a plasma regime which is stable to the $m = 2$ tearing mode at all densities. For typical Alcator C parameters, a tearing mode island of width $W \sim .1$ cm is suppressed for $D_e > 10^4$ cm^2/sec and highly MHD unstable profiles providing large $\Delta'(0)$ are stabilized for $\beta_i > 10^{-3}$.

Thesis Supervisor: Dr. Kim Molvig
Title: Professor of Nuclear Engineering

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CONTENTS

LIST OF FIGURES

Chapter 1
INTRODUCTION

Tearing modes are a subject of current interest in plasma physics due to their role in both space and laboratory plasma behavior. In space plasmas, tearing modes are an important destabilization mechanism in spontaneous magnetic reconnection phenomena, such as in the onset of substorms in the Earth's magnetotail [1,2]. In laboratory plasmas, such as in a tokamak, tearing modes play an important role in the onset of major disruptions. It is generally agreed that major disruptions must be totally suppressed in an actual fusion reactor in order to prohibit excessive damage to the first wall. Currently, the most widely accepted theoretical model of major disruptions features low poloidal number (low m) tearing modes which saturate to produce magnetic islands. It is possible for such magnetic islands to overlap and thus form large stochastic magnetic regions which enhance particle diffusion, and, in the case of major disruptions, lead to catastrophic plasma confinement loss [3–6]. Hence, control of such disruptions requires the elimination or, at least, suppression of these tearing mode islands.

Recent experimental results on Alcator C indicate the existence of a density threshold which must be surpassed before the $m = 2$ tearing mode is observed [7]. Such an observation is inconsistent with the previous kinetic and resistive magnetohydrodynamic (MHD) theories of the tearing mode. Traditionally, the tearing mode has been analyzed using resistive MHD theory which predicts instability for $\Delta' > 0$, independent of plasma density [3]. Here Δ' is the jump in the logarithmic radial derivative of the perturbed magnetic potential, $\tilde{A}_{\|}$, across the rational surface. This is determined by integrating the ideal MHD equation for the vector potential from the external region inward towards the rational surface. Typically, experimental profiles indicate $\Delta' > 0$, and hence the $m = 2$ tearing mode should be observed at all densities according to resistive MHD. Besides resistive MHD, the tearing mode has also been studied using a kinetic theory based on the collisionless

drift kinetic equation (DKE)[8,9]. These studies [10–12] have shown that the "collisionless" tearing mode can be stabilized by sufficiently large plasma beta β even though $\Delta' > 0$. Here the plasma beta β is defined to be the ratio of the plasma kinetic pressure to the magnetic pressure. This stabilization at high β is the result of the finite inertia of the ions combined with the fact that the collisionless tearing mode has a real frequency given by ω_{*e}, where ω_{*e} is the electron diamagnetic drift frequency. This high beta stabilization, however, indicates that the tearing mode should only be observed at low densities which is contrary to observations on Alcator C. It is the existence of a density threshold before the onset of instability, which is in qualitative disagreement with the previous theories of the tearing mode, that has motivated the present work.

In this study, two mechanisms are proposed as possible explanations of the observed stabilization of the $m = 2$ tearing mode at low densities. The first is stabilization by turbulent electron diffusion and the second is stabilization by runaway electrons. In the calculation of the energy drive (represented by Δ') used in the previous theories of the tearing mode, the positivity of Δ' is dependent on the perturbed parallel current $\tilde{J}_{||}$ becoming singular with odd parity about the rational surface. It is the existence of a very large $\tilde{J}_{||}$ near the rational surface which gives rise to a positive value of Δ'. Turbulent electron diffusion, however, prohibits the formation of a large $\tilde{J}_{||}$ within a finite diffusive correlation distance x_c of the rational surface. As will be shown below, $x_c \sim {D_e}^{1/3}$, where D_e is the turbulent electron diffusion coefficient. Hence, if turbulent electron diffusion is included in the analysis of the tearing mode, then stability may be obtained for sufficiently large D_e. Provided $D_e \sim 1/n$, this implies that the tearing mode may be stabilized below some critical density.

Another important observation in tokamaks at low densities is the existence of runaway electrons. At high densities, the equilibrium plasma current is described by an equilibrium electron velocity distribution in the form of a drifted Maxwellian with a drift speed much smaller than the thermal speed. In other words, the equilibrium current is composed of a small drift of the bulk electron population. At low

densities, however, the equilibrium current is carried by a small fraction of the electron population ("runaway" electrons) possessing speeds greater than the thermal velocity. As the plasma density increases, so does the plasma collisionality, and the runaway electrons relax back into the bulk distribution. This raises the question of whether the MHD stability can be enhanced by the presence of a high energy subpopulation carrying the current. The inclusion of a fast population of electrons into the tearing mode analysis may change the electron response in the region about the rational surface and provide additional stability. Hence, the shift in the equilibrium electron distribution from a runaway–type distribution to a drifted Maxwellian as the density is increased may correspond to a shift from stability to instability for the $m = 2$ tearing mode.

This study uses a fully kinetic approach to the tearing mode which includes the effects of turbulent electron diffusion and treats the tearing mode as an electromagnetic fluctuation. A kinetic theory also allows the choice of various equilibrium velocity distributions which give rise to the same equilibrium spatial current distributions. In such a way the effects of a drifted Maxwellian to those of a runaway distribution can be compared while fixing the equilibrium current distribution.

The effects of turbulent electron diffusion enter the analysis by applying the normal stochastic approximation (NSA) to the collisionless drift kinetic equation (DKE) for electrons, as is described below. The end effect of the turbulent electron diffusion is to stabilize the tearing mode at sufficiently large values of the diffusion coefficient [13,14]. This implies the existence of a density threshold provided the electron diffusion coefficient scales inversely with density, $D_e \sim 1/n$. Physically, electron diffusion prohibits the tearing region from becoming too small, whereas in resistive MHD the layer thickness is limited only by dissipation, which alters the growth rate without affecting stability. Electron diffusion prohibits the formation of a perturbed parallel current channel narrower than the correlation distance, $x_c = (D_e/k'_{\|}v_e)^{1/3}$. This flattening of the perturbed current thus cuts into the available free energy driving the tearing mode (represented by Δ') hence reducing

the available energy to $\Delta'(x_c)$ (as opposed to $\Delta'(0)$ in resistive MHD). Here, $\Delta'(x_c)$ is the difference between the logarithmic derivative of $\tilde{A}_{||}$ evaluated at a distance x_c from the rational surface and that value at a distance of $-x_c$. For typical experimental profiles, $\Delta'(x)$ is a decreasing function of the distance from the rational surface, x. The below results for the "magnetic" tearing mode (which neglects the effects of the electrostatic potential $\tilde{\phi}$) give a growth rate scaling as $\gamma \sim \Delta'(x_c)$[13]. Hence, stability occurs when $x_c > W$, W being the distance where $\Delta'(W) = 0$. Using the Alcator C parameters of $L_s/L_n \simeq 20$, $T \sim 1$ keV, $a \sim 20$ cm and $W/a \sim .05$ indicates stability is obtained for $D_e > 10^4$ cm^2/sec. This gives a value for the density threshold on the order of that observed in Alcator C.

By treating the tearing mode as an electromagnetic fluctuation, the effects of a finite electrostatic potential, $\tilde{\phi}$, are investigated and found to be stabilizing. Physically, the relevant stabilizing terms involving the electrostatic potential represent ion inertia. This mechanism is completely independent of electron diffusion. In fact, when the diffusion is neglected, the tearing mode is stabilized when $\beta_i(\tau L_s/L_n)^{1/2} \gtrsim \Delta'(0)$, indicating stability at sufficiently high plasma beta. Here, τ is the temperature ratio, L_s is the shear length, L_n the density scale length, and β_i is the ratio of ion pressure to magnetic pressure. Using the Alcator C parameters $L_s/L_n \sim 20$, $\rho_i/a \sim .002$ and $\Delta' a \sim 10$ indicates stability for $\beta_i > 5 \times 10^{-3}$. A result identical to this was obtained previously by Basu and Coppi [11]. Physically, this stabilizing effect represents that portion of the available magnetic energy driving the tearing mode which must be used to maintain the ion motion. In the kinetic theory description the tearing mode has a real frequency ($\omega = \omega_{*e}$) and large ion inertia becomes a stabilizing effect. In contrast, for the purely growing modes of the resistive MHD description, no ionic energy is required by the perturbation itself and ion inertia does not influence stability but only the magnitude of the growth rate. In this picture, the tearing mode is actually interpreted as an electron drift wave under modification of the of the equilibrium, current gradient. This current gradient introduces an additional energy source which drives the low m "drift–tearing" modes unstable. Hence, the inclusion of a finite electrostatic potential indicates

that the tearing mode can be stabilized at sufficiently high densities in addition to the stability at low densities discussed in the previous paragraph.

The analysis of the magnetic tearing mode using a drifted Maxwellian for the equilibrium electron distribution gives a real frequency equal to ω_{*e} and a growth rate $\gamma \sim \Delta'(x_c)$. When the analysis is repeated using a runaway–type distribution, the real frequency of the tearing mode is shifted to a value slightly above ω_{*e}. The expression for the growth rate then contains a new stabilizing term proportional to this frequency shift $\delta\omega = \omega - \omega_{*e}$. Physically, this additional stabilization represents the additional energy required to sustain the particle oscillation at the frequency $\omega_{*e} + \delta\omega$. This runaway stabilization is a higher order effect and is only important for a tearing mode near marginal stability; i.e., for a low density case where the MHD energy drive is nearly cancelled by the ion inertial stabilization.

Chapter 2 discusses the mathematical model used in this treatment of the tearing mode and presents the derivation of a coupled, self–adjoint system of equations for the fluctuation potentials $\tilde{A}_{||}$ and $\tilde{\phi}$. Section 2.1 deals with the electron response and the development of the normal stochastic approximation (NSA), which enables the effects of turbulent electron diffusion to be treated in a self–consistent manner [15–18]. The electron response is obtained by applying the NSA to the drift kinetic equation. In effect, the NSA amounts to a coarse grain averaging over the micro–scale structure characterizing the stochastic electron orbits. These stochastic electron orbits are the direct result of the presence of drift wave microturbulence. The end result is that the nonlinear effects of turbulence are represented by the appearance of a turbulent diffusion coefficient. In the limit where the electrons experience stochastic orbits, the NSA is essentially equivalent to the direct-interaction approximation (DIA) [17]. Section 2.4 discusses the validity of treating the full spatial diffusion operator with an approximate Krook–type diffusion frequency. This Krook approximation amounts to replacing the diffusion operator, $-D\partial^2/\partial x^2$, with a diffusion frequency, $\omega_c \equiv (\frac{1}{3}(k'_{||}v_{||})^2 D)^{1/3}$, in the equation for the nonlinear electron response [18]. This leads to a much simpler mathematical treatment of the

diffusion (both analytically and numerically). In physical terms, this model assumes that the tearing mode exists on a background of drift wave turbulence. The presence of drift waves lead to stochastic electron orbits and hence turbulent diffusion, which is treated in the electron response via the NSA.

The ion response is virtually unaffected by turbulent diffusion, and thus the ions are treated with the linearized Vlasov equation in Sec. 2.2. In Section 2.3, Ampere's law and quasineutrality are invoked to give a set of coupled, self–adjoint equations for the fluctuation potentials, $\tilde{A}_{||}$ and $\tilde{\phi}$. This coupled system is fully kinetic, globally valid over the entire plasma, and reduces to ideal MHD at large distances from the rational surface. Since this system of equations for $\tilde{\phi}$ and $\tilde{A}_{||}$ is self–adjoint, a variational principle can be formed. This variational principle is later used in the calculation of the dispersion relation for the tearing mode.

Chapter 3 examines the stabilization of the tearing mode by turbulent electron diffusion. This is done by analyzing the full set of kinetic equations for $\tilde{\phi}$ and $\tilde{A}_{||}$ using a drifted Maxwellian for the equilibrium electron distribution. The goal of Sec. 3.1 is to show what approximations are necessary to obtain the resistive MHD results for the tearing mode from the above set of coupled equations for $\tilde{A}_{||}$ and $\tilde{\phi}$. Resistivity is introduced in the Krook approximate model by replacing ω_c with a velocity collision frequency, ν. After further reducing the coupled set of equations for $\tilde{A}_{||}$ and $\tilde{\phi}$ in an inner and outer region, a simple variational calculation is performed which yields the resistive growth rate [3] for the tearing mode. The important limits taken (unjustifiably from the kinetic point of view) in order to obtain the resistive MHD results were (a) the asymptotic limit of taking the tearing layer to be zero and thus obtaining the energy drive term as $\Delta'(0)$ and (b) the limit of setting the electron diamagnetic frequency to be zero.

Section 3.2 presents a formal analytic proof demonstrating that the tearing mode is stabilized for sufficiently large values of the diffusion coefficient, D_e, for a system which is resistive MHD unstable, $\Delta'(0) > 0$. This is done by identifying the appropriate terms in the variational integral as the energy drive for the tearing

mode. In the ideal MHD limit, this drive yields the quantity $\Delta'(0)$. Using the full kinetic operators, however, the energy drive goes to zero for large values of the diffusion coefficient and stabilization is achieved. Section 3.3 derives a simple dispersion relation for the magnetic tearing mode in which the effects of the electrostatic potential, $\tilde{\phi}$, are neglected. The equation governing $\tilde{A}_{||}$ is solved to leading order in an inner and outer region, and the dispersion relation is determined by applying matching conditions to the boundaries at $\pm x_c$. The resulting dispersion relation indicates the magnetic tearing mode to have a real frequency equal to the electron diamagnetic frequency, ω_{*e}, and a linear growth rate proportional to $\Delta'(x_c)$ [13]. This result is similar to the collisionless tearing mode as calculated by Laval et al. [8] where they find $\gamma \sim \Delta'(0)$. Numerical calculations indicate $\Delta'(x)$ to be a decreasing function of x; hence, stability is given by $x_c > W$, where $\Delta'(W) = 0$. This condition for stability, $\Delta'(x_c) = 0$, indicates that the magnetic energy involved in $\tilde{A}_{||}$ and $\tilde{J}_{||}$ outside the tearing region, $|x| > x_c$, is zero and thus no free energy is available to drive the tearing mode. This is analogous to nonlinear estimates of the magnetic island saturation width, W, given by the value where $\Delta'(W) = 0$ [4,20].

The goal of Section 3.4 is to determine the dispersion relation for the tearing mode including the effects of a finite electrostatic potential. This is done by a variational calculation. The end result is that the effect of a finite electrostatic potential, $\tilde{\phi}$, produces additional stabilizing terms in the dispersion relation which are independent of electron turbulent diffusion. This additional stabilization is due to the inclusion of terms involving the electrostatic potential in the variational integral which represent parallel and poloidal ion inertia. In the absence of electron diffusion, the dispersion relation indicates that the tearing mode is stabilized by ion inertial effects when $\beta_i(\tau L_s/L_n)^{1/2} \gtrsim \Delta'(0)$, a result identical to that obtained by Basu and Coppi [11]. The appearance of factor β_i is due to the available free energy in the outer region scaling as B_0^2 whereas the ion inertial terms scale as nT_i. Thus, in addition to the stabilization at low densities due to turbulent electron diffusion, the tearing mode is also stabilized at high density due to ion inertial effects.

Chapter 4 examines the stabilization of the tearing mode by runaway electrons. Section 4.1 derives the modified set of coupled equations for $\tilde{\phi}$ and $\tilde{A}_{||}$ in the case where the equilibrium electron distribution is a runaway–type distribution. For simplicity, this distribution is modeled by a low density monoenergetic electron beam superimposed of the tail of a Maxwellian. To see the effects of the fast beam electrons, corrections of order ω_{*e}/ω_c must be calculated to the dispersion relation presented in Chapter 3. Section 4.2 calculates these corrections of order ω_{*e}/ω_c for the "regular" tearing mode of Chapter 3, where the equilibrium distribution is a drifted Maxwellian. Section 4.3 then calculates the dispersion relation for the runaway tearing mode including terms of order ω_{*e}/ω_c. These results indicate that the real frequency of the runaway tearing mode is shifted above ω_{*e} by $\delta\omega$, where $\delta\omega/\omega_{*e} \sim n_b v_b/(n_0 v_e)$. Here n_b and v_b are the beam density and beam speed; and n_0 and v_e are the bulk density and thermal speed. Consequently, an additional stabilizing term appears in the expression for the growth rate of the form $\delta\omega/\omega_c$. This additional stabilization represents the additional energy required to maintain the particle oscillation at the beam shifted frequency. Hence, enhanced stability results when the plasma current is carried by a beam of fast electrons as opposed to a slow drift of the bulk electrons.

The goal of Chapter 5 is to develop a fluid model which is equivalent to the kinetic theory in the NSA model. By using such an equivalent fluid model, the physical processes in the tearing mode dynamics can be illuminated more simply than in the NSA kinetic theory. Section 5.1 presents the derivation of the fluid model by performing velocity moments of the NSA drift kinetic equation. In the Krook approximation, the fluid model demonstrates that the NSA diffuses both the perturbed density and the perturbed momentum at equal rates. The effective rate at which this diffusion occurs is ω_c. Section 5.2 uses this fluid model to examine the regular tearing mode and shows that the results are equivalent to those obtained in Chapter 3 using the kinetic theory. Section 5.3 uses the fluid model to examine the runaway tearing mode and shows that the results are equivalent to those obtained in Chapter 3 using the kinetic theory. Section 5.4 presents a simplified fluid theory,

in which the fluid model is reduced to the "bare basics" necessary to describe the tearing mode including the stabilization mechanisms of turbulent diffusion and runaway electrons. This simplified model reduces the mathematical analysis to a minimum and, thus, allows the physics to be seen more easily.

Numerical results to the full kinetic description are presented in Chapter 6. These kinetic equations are solved using two different numerical methods. The first, and the more exact method, utilizes the initial value code TEDIT [19,21,22] and solves this problem including the full diffusion operator, $-D\partial^2/\partial x^2$, in the equation for the electron response. The second method uses a shooting code in solving this problem in the Krook approximation, $-D\partial^2/\partial x^2 \rightarrow \omega_c$. Both codes are observed to give the same qualitative results, as predicted by the analytical discussion in Sec. 2.4. The numerical calculations for the regular tearing mode agree very well with the analytical expression for the growth rate as given by the variational calculation in Chapter 3. The numerical results for the runaway tearing mode are not quite as extensive as for the regular tearing mode. However, the basic qualitative picture described by the analytic theory is supported. In particular, the real frequency shift $\delta\omega$ is observed to scale linearly with $n_b v_b$, and the stability is observed to be enhanced over that of the regular tearing mode for a fixed value of the equilibrium current.

A summary of this study is presented in Chapter 7 and suggestions for future study are presented in Chapter 8. The remainder of this introductory chapter presents the background theory of the tearing mode. Section 1.1 introduces the concept of spontaneous magnetic reconnection. Section 1.2 and Section 1.3 present the basic theory of the resistive and collisionless tearing modes.

1.1 Spontaneous Magnetic Reconnection

Spontaneous magnetic reconnection [2,9] can be viewed as a process in which a specific magnetic field topology relaxes to a structure of lower potential energy. The tearing mode instability is a mechanism through which this potential energy is released during relaxation of the magnetic topology [1]. Such a relaxation, however, is prohibited in the ideal MHD model due to the "frozen-in law", which states that the plasma and magnetic field lines must move together [23]. Thus, some "non-ideal" form of dissipation is needed in order for magnetic tearing to occur, such as particle collisions (resistivity), turbulence, or wave-particle resonant damping. Despite this, ideal MHD is useful in describing the energy drive for the tearing mode [8,24,25]. This is done by determining if a particular magnetic topology has sufficient free energy available to render a tearing mode unstable and, hence, allow magnetic reconnection to occur. Non–ideal effects are only important in a narrow region in the plasma in which magnetic reconnection first occurs. Although such dissipation is necessary for the existence of the tearing mode, the stability of the tearing mode is determined by the global magnetic structure of the plasma outside of this narrow region. For example, ideal MHD can be used to determine the available free energy for various global magnetic structures such as magnetically confined laboratory plasmas or the plasma in the Earth's magnetotail [26]. This paper will focus on magnetic reconnection in a tokamak plasma.

A tokamak plasma can be idealized as a straight cylinder with an axial, or toroidal, plasma current and a magnetic field primarily in the toroidal direction with a somewhat smaller poloidal (azimuthal) component. Such a system exhibits magnetic shear, that is, the strength of the poloidal magnetic field varies as a function of plasma radius, whereas the toroidal magnetic field remains constant. In the ideal MHD description, these magnetic lines trace out concentric cylindrical surfaces. Magnetic shear prevents one magnetic surface at a given radius from slipping pass another surface, hence, dissipation is necessary for these magnetic surfaces to rearrange themselves. If the current profile within the plasma becomes

too peaked, thus developing steep gradients, it is possible for there to exist sufficient free magnetic energy which will, in turn, drive a tearing mode unstable [3]. When this occurs, it is possible for the magnetic surfaces to rearrange themselves through magnetic reconnection. This relaxation leads to the formation of magnetic islands about the mode rational surfaces [27], as illustrated in Fig. 1.1.

The fundamental physical processes involved in magnetic island formation [27] are illustrated in Fig. 1.2. Consider introducing a radial magnetic field perturbation of the form

$$\tilde{B}_r(r,t) = \tilde{B}_r(r)\exp[i(m\theta - nz/R - \omega t)] \tag{1.1}$$

near the mode rational surface, r_s, where $q(r_s) = m/n$. Here $q(r) = rB_T/RB_\theta$, where B_θ is the poloidal magnetic field, B_T is the toroidal magnetic field, and R is the major radius of the tokamak. In resistive MHD, $\omega = i\gamma$, as will be shown below. The poloidal field lines now experience a radial undulation. Those lines close enough to r_s are pulled completely through the rational surface to form magnetic island structures. If $\tilde{B}_r$ is allowed to grow in time, then Faraday's law requires there to exist an electric field in the axial direction (according to $\gamma\tilde{B}_r = -im\tilde{E}_z/r$). In ideal MHD this is not allowed, hence, such a perturbation would remain infinitesimal. However, if a finite resistivity is permitted, then there will exist an axial electric field which thus enables magnetic reconnection to occur. As the magnetic islands expand radially, the plasma fluid flows with it and forms vortex–type flow patterns. (See Fig. 1.2.) These islands would continue to grow until they saturated nonlinearly by exhausting all the available free magnetic energy.

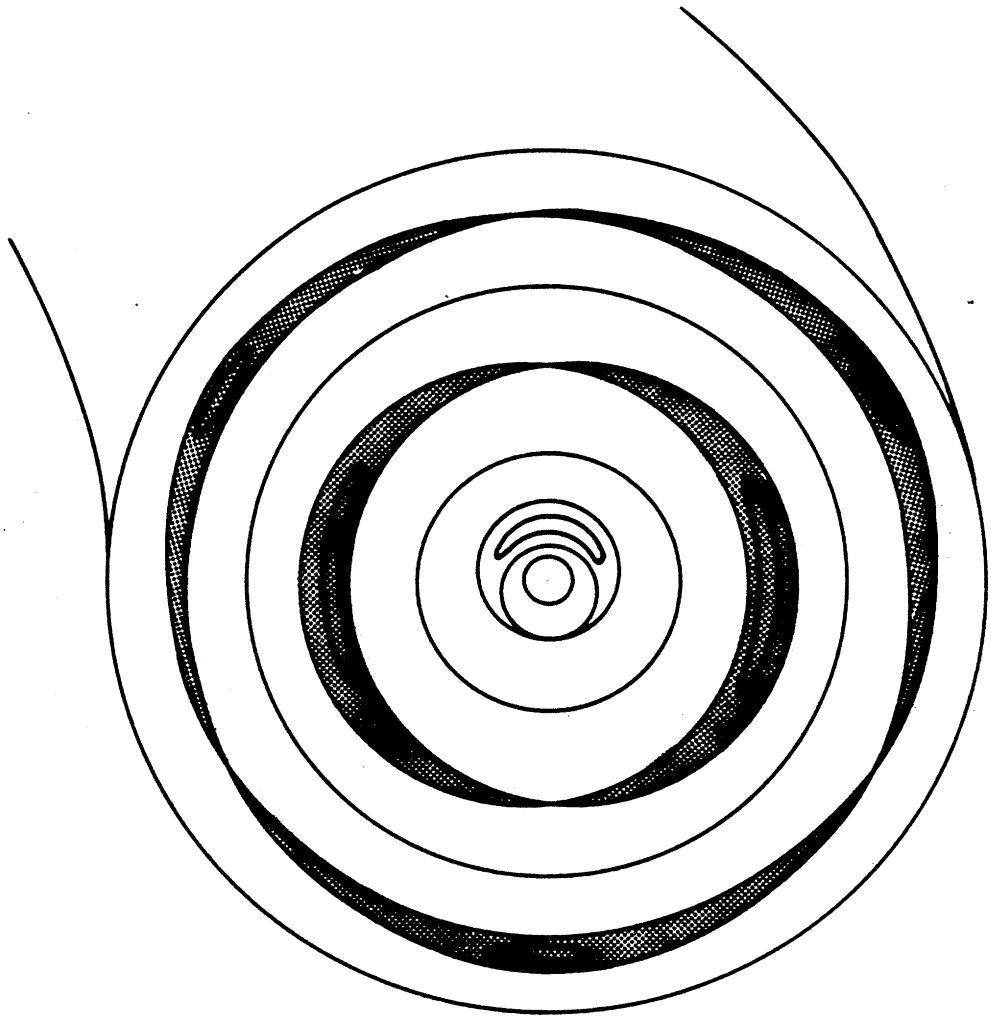

Figure 1.1

Magnetic flux surface contours showing $m = 1$, 2 and 3 magnetic islands.

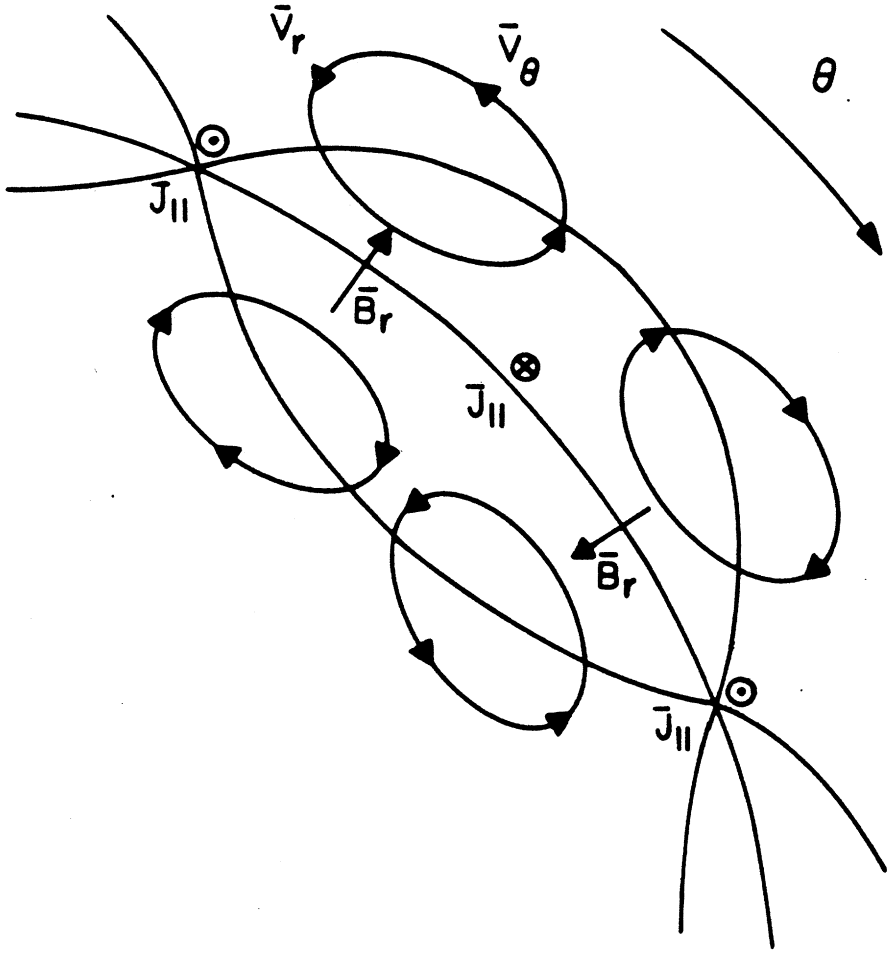

Figure 1.2

Schematic diagram showing the perturbations produced by the tearing mode, and the process leading to magnetic island formation.

It is important to note that a nonzero $\tilde{E}_{\|}$ is only needed in a thin layer about the rational surface in order for the magnetic reconnection process to occur. The existence of this $\tilde{E}_{\|}$ requires the presence of dissipation or some other non–ideal process (such as finite electron inertia). Outside of this thin layer, the plasma is adequately described by ideal MHD. As mentioned previously, the structure of the outer ideal MHD region determines the available free energy. This free energy is then dissipated in the thin non–ideal region about the rational surface where reconnection occurs.

The available free energy for the tearing mode can be calculated using Ampere's law,

$$\nabla^2 \tilde{\mathbf{A}} + \frac{4\pi}{c}\tilde{\mathbf{J}} = 0, \tag{1.2}$$

where the displacement current has been neglected due to the low frequency of the tearing mode. For this problem a sheared, slab geometry is chosen with the equilibrium magnetic field given by $\mathbf{B} = B(\mathbf{e}_z + x/L_s\mathbf{e}_y)$ with $\mathbf{b} = \mathbf{B}/B$. Here, $L_s = -Rq^2/rq'$ is the shear length, and $x = 0$ is chosen to be the position of the rational surface. Throughout the following, a tilde is used to denote fluctuation quantities. Here, $\tilde{\phi}$ and $\tilde{\mathbf{A}}$ represent the perturbed electrostatic and vector potentials respectively, $\tilde{\mathbf{E}}$ represents the perturbed electric field, and $\tilde{\mathbf{B}}$ represents the perturbed magnetic field, such that $\tilde{\mathbf{B}} = \nabla \times \tilde{\mathbf{A}}$. Also, the spatial coordinate x will be normalized in units of the ion gyro radius, ρ_i.

The tearing mode is defined to be a magnetic perturbation arising from a perturbed parallel vector potential. That is, $\tilde{\mathbf{A}} = \tilde{A}_{\|}\mathbf{b}$ and $\tilde{\mathbf{B}} = \nabla \times \tilde{A}_{\|}\mathbf{b}$. Note that the perpendicular components of $\tilde{\mathbf{A}}$ give rise to parallel components of $\tilde{\mathbf{B}}$. These parallel components of $\tilde{\mathbf{B}}$ compress the equilibrium magnetic field, and this compression requires much more energy than that required to bend the field lines perpendicularly. Since such compression of the magnetic field lines is stabilizing, an unstable magnetic perturbation tends to minimize its parallel component.

The parallel component of the low-frequency version of Ampere's law is given by

$$\mathbf{b}\cdot\nabla^2\tilde{A}_{\|}\mathbf{b}+\frac{4\pi}{c}\tilde{J}_{\|}=0. \tag{1.3}$$

Note that $\nabla^2(\mathbf{b}\cdot\tilde{\mathbf{A}})=\mathbf{b}\cdot\nabla^2\tilde{\mathbf{A}}+\tilde{\mathbf{A}}\cdot\nabla^2\mathbf{b}+2(\nabla\tilde{\mathbf{A}}):(\nabla\mathbf{b})$. Recall that this problem assumes slab geometry with $\mid x \mid < a$ and $a/L_s \sim 10^{-1}$. For this geometry, the above equation becomes

$$\mathbf{b}\cdot\nabla^2\tilde{A}_{\|}\mathbf{b}=\nabla^2\tilde{A}_{\|}+L_s^{-2}\tilde{A}_z+L_s^{-1}(\tilde{A}_z'x/L_s+\tilde{A}_y'),$$

where higher order terms in a/L_s have been neglected. Notice that $\tilde{A}_z \simeq \tilde{A}_{\|}$ and $\tilde{A}_y \simeq \tilde{A}_{\|}x/L_s$. Assuming that $\tilde{A}_{\|}'/\tilde{A}_{\|} \sim 1/a$, then this implies that the last term in the above equation scales as $L_s^{-2}\tilde{A}_{\|}$. Hence,

$$\mathbf{b}\cdot\nabla^2\tilde{A}_{\|}\mathbf{b}\simeq\nabla^2\tilde{A}_{\|},$$

where terms of order a^2/L_s^2 have been neglected.

Thus, in the low-frequency, low-shear limit, the parallel component of Ampere's law is given by

$$\nabla^2\tilde{A}_{\|}+\frac{4\pi}{c}\tilde{J}_{\|}=0. \tag{1.4}$$

Multiplying this equation by $\tilde{A}_{\|}$, integrating over the plasma and performing an integration by parts gives the integral

$$S=\int_{-a}^{a}dx\left[\tilde{A}_{\|}'^2+\rho_i^2k_y^2\tilde{A}_{\|}^2-\frac{4\pi}{c}\rho_i^2\tilde{J}_{\|e}[\tilde{A}_{\|}]\cdot\tilde{A}_{\|}\right], \tag{1.5}$$

where, by construction, $S=0$. Here, $\tilde{J}_{\|e}[\tilde{A}_{\|}]$ is a general operator representing the perturbed parallel current, and $\pm a$ represents the edge of the plasma. Physically,

$S = 0$ represents the conservation of energy across the plasma for the perturbation $\tilde{A}_{||}$. By excluding the narrow region about the singular surface, however, the resulting expression for the integral S can be interpreted as the energy drive for the tearing mode[8,13,24]. Physically, the integral over the outer region represents the free energy released by the perturbation $\tilde{A}_{||}$, while the integral over the narrow region about $x = 0$ represents the energy absorbed by the plasma during the tearing process.

To demonstrate that the integral S represents the energy associated with the tearing perturbation $\tilde{A}_{||}$, it is helpful to consider the following heuristic arguments based on Poynting's theorem,

$$\frac{\partial}{\partial t}\tilde{u} + \nabla \cdot \tilde{\mathbf{s}} = -\tilde{w}$$

where $\tilde{u} = (\tilde{B}^2 + \tilde{E}^2)/8\pi$ is the energy associated with the fluctuations $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{E}}$; $\tilde{\mathbf{s}} = (\tilde{\mathbf{E}} \times \tilde{\mathbf{B}})c/4\pi$ is the Poynting vector for the fluctuating fields; and $\tilde{w} = (\tilde{\mathbf{J}} \cdot \tilde{\mathbf{E}})$ is the power due to interaction of $\tilde{\mathbf{J}}$ and $\tilde{\mathbf{E}}$. Integrating the above expression over the plasma volume gives

$$\delta P = \int d^3x \left[\frac{\partial}{\partial t}\left(\tilde{B}^2 + \tilde{E}^2\right)\frac{1}{8\pi} + \left(\tilde{\mathbf{J}} \cdot \tilde{\mathbf{E}}\right)\right] = 0$$

where it was assumed that $\tilde{\mathbf{s}} = 0$ on the plasma surface. Physically, δP represents conservation of power for the fluctuating fields.

In applying the above expression to the tearing mode perturbation, $\tilde{\mathbf{B}} = \nabla \times \tilde{A}_{||}\mathbf{b}$. Recall that $\tilde{\mathbf{A}} = \tilde{A}_{||}\mathbf{b}$, since perpendicular components of $\tilde{\mathbf{A}}$ lead to parallel components of $\tilde{\mathbf{B}}$ which provide strong stabilization from magnetic line compression. An unstable tearing mode inherently minimizes $\tilde{B}_{||}$ and hence the compressional stabilization.

To analyze the energy drive for the tearing mode, the above expression for δP is analyzed in the region outside the thin tearing layer at $x = 0$. In this region,

ideal MHD is valid which states $\tilde{E}_{\|} = 0$. Hence, $\tilde{\phi} = \omega \tilde{A}_{\|} / k_{\|} c$. The perpendicular electric field is given by $\tilde{\mathbf{E}}_{\perp} = -\nabla_{\perp}\tilde{\phi}$. Notice

$$\int \mathrm{d}^3x \left(-\nabla_{\perp}\tilde{\phi}\right) \cdot \tilde{\mathbf{J}}_{\perp} = \int \mathrm{d}^3x \tilde{\phi} \left(\nabla_{\perp} \cdot \tilde{\mathbf{J}}_{\perp}\right)$$
$$= -\int \mathrm{d}^3x i k_{\|} \tilde{\phi} \tilde{J}_{\|}$$

where $\nabla \cdot \tilde{\mathbf{J}} = 0$ was used. Hence,

$$\int \mathrm{d}^3x \tilde{\mathbf{E}}_{\perp} \cdot \tilde{\mathbf{J}}_{\perp} = \int \mathrm{d}^3x \left(-\frac{i\omega}{c} \tilde{A}_{\|} \tilde{J}_{\|}\right),$$

and the expression for δP becomes

$$\delta P = \int \mathrm{d}^3x \left\{ \frac{1}{8\pi} \frac{\partial}{\partial t} \left[\mid \tilde{\mathbf{B}}_{\perp} \mid^2 + \mid \nabla_{\perp} \left(\omega \tilde{A}_{\|} / k_{\|} c\right) \mid^2 \right] - \left(\frac{i\omega}{c} \tilde{A}_{\|} \tilde{J}_{\|}\right) \right\}$$

where $\mid \tilde{B}_{\perp} \mid^2 \simeq \mid \nabla_{\perp} \tilde{A}_{\|} \mid^2$. Notice that in the outer region the second term in the above equation can be neglected due to the smallness of $\omega / k_{\|} c$.

To find an expression for the energy, the power δP is integrated over time. This gives

$$\delta W = \frac{1}{8\pi} \int \mathrm{d}^3x \left[\mid \nabla_{\perp} \tilde{A}_{\|} \mid^2 - \frac{4\pi}{c} \tilde{A}_{\|} \tilde{J}_{\|} \right].$$

Hence, the identification can be made that

$$S = 8\pi \frac{\rho_i^2}{L_y L_z} \delta W$$

and, thus, S represents the energy associated with the perturbation $\tilde{A}_{\|}$. Here, L_y and L_z represent the length of the slab of plasma in the y and z directions and the factor ρ_i^2 has appeared since x is normalized in units of ρ_i in the expression for

S. Note, however, that the above heuristic arguments do not correctly describe the plasma at $x = 0$. In order to do so, the effects of finite $\tilde{E}_{||}$ must be considered.

In order to derive an expression for the energy drive of the tearing mode, it is necessary to calculate the ideal MHD form for the perturbed parallel current. To do this, the linear form of the ideal MHD momentum equation is used while neglecting the plasma pressure (low-β) and inertia (marginal stability). Hence, in the low-beta limit at marginal stability ($\alpha = 0$),

$$\tilde{\mathbf{J}}_{\perp} = J_{0||} \frac{\tilde{\mathbf{B}}_{\perp}}{B_0}.$$

The above expression indicates that the perturbed perpendicular current is due to the particles free streaming along the undulating magnetic field lines. Using $\nabla \cdot \tilde{\mathbf{J}} = 0$ then gives

$$\tilde{J}_{||} = -\frac{k_y J'_{||0}}{k_{||} B_0} \tilde{A}_{||}, \tag{1.6}$$

which is the ideal MHD form for the perturbed parallel current. Here, $k_{||} = k'_{||} x$, where $k'_{||} = k_y / L_s$. Notice that in the ideal MHD limit $\tilde{J}_{||}$ is singular at the rational surface, $x = 0$.

The energy drive integral, S, is then evaluated by letting $\tilde{A}_{||}$ satisfy the ideal MHD form of Ampere's law,

$$\left[\frac{d^2}{dx^2} - k_y^2 \rho_i^2 - \frac{4\pi}{c} \rho_i^2 \frac{k_y J'_{||}}{k_{||} B_0} \right] A_{||MHD} = 0. \tag{1.7}$$

This then gives

$$S_{MHD} = -A^2_{||_0} \Delta'_0, \tag{1.8}$$

where $\lim_{x_0 \to 0} [A_{||} A'_{||}(x_0) - A_{||} A'_{||}(-x_0)]_{MHD} \equiv A^2_{||_0} \Delta'_0$. Here, Δ'_0 represents the jump in $\tilde{A}'_{||}$ across the singular surface. If $\Delta'_0 > 0$, then there exists sufficient free

energy in the outer MHD region to drive the tearing mode [28]. Notice that in "pure" ideal MHD alone, such singular behavior of $\tilde{A}_{||}$ and jump conditions in $\tilde{A}'_{||}$ are not allowed to occur in the plasma. Hence, a nonzero Δ'_0 is only allowed by assuming the existence as a non-ideal MHD region in a thin layer about the rational surface [29]. The above development assumed low shear in the outer region. However, the above definition of the energy drive Δ'_0 as being the jump in $A'_{||}$ across the rational surface holds even in systems with high shear in the outer region.

In the above calculation of the energy drive, Δ'_0, the tearing region in which dissipative effects are important was assumed to be infinitesimally thin. In a kinetic theory treatment, as will be outlined below, the effects of spatial turbulent diffusion broaden this dissipative layer to a significant width [13,14]. This has the effect of cutting into the available energy in the outer MHD region and thus reducing the energy drive to a value smaller the Δ'_0. In general, it is possible to define the function

$$A^2_{||_0}\Delta'(x) \equiv [A_{||}A'_{||}(x) - A_{||}A'_{||}(-x)]_{MHD}. \tag{1.9}$$

For typical tokamak profiles, $\Delta'(x)$ is a monotonically decreasing function of x [30–33]. For simplicity, a linear form for Δ' will be used

$$\Delta'(x) \simeq \Delta_0(1 - x/W). \tag{1.10}$$

Physically, Δ_0 represents the energy drive in the limit of an infinitesimally thin dissipative layer and W represents the width at which a magnetic island will nonlinearly saturate [20].

The expression for the ideal MHD energy drive S_{MHD}, given by Eq. (1.8), implicitly assumes that the non-ideal dissipative layer is infinitesimally thin. However, if there exists a dissipation mechanism which produces a layer of finite width, then this inner non-ideal region must be excluded from the ideal MHD energy in-

tegral. If this is done, then the expression for the ideal MHD energy drive becomes $S_{MHD} \sim \Delta'(x_0)$, where x_0 is the width of the dissipation region. For example, as will be discussed below, the effect of radial turbulent diffusion on the tearing mode will be to reduce the available energy from Δ'_0 to $\Delta'(x_c)$. Here, x_c is the diffusion correlation distance (and the width of the dissipative layer) whose value increases with increasing value of the particle diffusion coefficient D. In general, if $\Delta'(x_0) > 0$, then there exists sufficient free energy, thus enabling spontaneous magnetic reconnection to occur.

1.2 Resistive Tearing

As mentioned above, ideal MHD is inadequate to describe magnetic reconnection. The "frozen-in" law of ideal MHD prohibits magnetic lines and surfaces from slipping past one another, hence, prohibiting magnetic reconnection [23]. However, if a form of "non-ideal" dissipation is introduced in a small region about the rational surface, then it becomes possible for magnetic reconnection to occur. In the resistive tearing mode, this non-ideal dissipation appears in the form of collisional resistivity. Traditionally, the resistive tearing mode is described using resistive MHD [3,34]. In resistive MHD, the ideal Ohm's law $\mathbf{E} + (\mathbf{V} \times \mathbf{B})/c = 0$, which describes frozen-in field lines, is modified to include resistivity, $\mathbf{E} + (\mathbf{V} \times \mathbf{B})/c = \eta \mathbf{J}$. This then allows the formation of parallel perturbed electric fields and, hence, allows magnetic reconnection to occur. Here the resistivity η is a Spitzer-type collisional resistivity [35], $\eta = (m_e \nu_e / 2ne^2)$, where ν_e is the Braginskii electron collision frequency [36]. In practice, this resistivity is neglected everywhere except in a small resistive layer within $\pm x_0$ of the rational surface. Hence, the resistive MHD model reduces to ideal MHD outside this resistive layer.

As discussed in Sec. 1.1, ideal MHD is used to describe the energy drive for the tearing mode in the region outside of the dissipation layer. The ideal MHD energy drive in the outer region is defined as the negative energy resulting from the integral

$$S_{OUT} = \left(\int_{-a}^{-x_0} dx + \int_{x_0}^{a} dx \right) \left[\left(\frac{d\tilde{A}_{||}}{dx} \right)^2 + \rho_i^2 k_y^2 \tilde{A}_{||}^2 - \frac{4\pi}{c} \rho_i^2 \tilde{J}_{||} \tilde{A}_{||} \right]. \qquad (1.11)$$

Here, $\pm a$ represents the plasma edge, $\pm x_0$ represents the edge of the resistive layer, and x has been normalized in units of the ion gyroradius ρ_i. The first two terms in S_{OUT} represent stabilizing line bending energy, whereas the last term represents the energy resulting from the interaction of the perturbed parallel current with the magnetic potential. In ideal MHD, $\tilde{J}_{||}$ is given by Eq. (1.6), as discussed in Sec. 1.1.

Equation (1.11) can be evaluated by requiring $\tilde{A}_{||}$ to satisfy the ideal form of Ampere's law, given by Eq. (1.7). After integrating by parts, S_{OUT} becomes

$$\begin{aligned} S_{OUT} &= \lim_{x_0 \to 0} \left[\tilde{A}_{||} \tilde{A}'_{||}(x_0) - \tilde{A}_{||} \tilde{A}'_{||}(-x_0) \right]_{MHD} \\ &\equiv -A^2_{||0} \Delta'_0. \end{aligned} \tag{1.12}$$

Here, Δ'_0 represents the jump in $\tilde{A}'_{||}$ across the singular surface in the limit of an infinitesimally small resistive layer. Physically, Δ'_0 represents the free magnetic energy of the outer region which is available to drive the tearing mode [28].

To determine the dispersion relation and, hence, the growth rate of the resistive tearing mode, it is necessary to consider the dynamics within the dissipative layer. Specifically, the energy source in the outer region S_{OUT} must be balanced against the energy sinks in the inner region, S_{IN}. In the resistive MHD model, the energy integral in the inner region, S_{IN}, has two significant contributions. The first is the kinetic energy associated with the fluid motion of the plasma and the second is the energy dissipated through existence of a finite resistivity. Hence, a heuristic energy integral in the inner region can be written as [37]

$$\frac{S_{IN}}{8\pi} = \int_{-x_0}^{x_0} dx \left[\frac{1}{2} m_i n |\tilde{V}|^2 - \frac{\rho_i^2}{2c\eta} \tilde{E}_{||} \tilde{A}_{||} \right]. \tag{1.13}$$

The first term in Eq. (1.13) represents the kinetic energy of the fluid motion of the plasma (finite ion inertia) and the second term represents the energy dissipated into thermal motion of the plasma through collisional resistivity. The second term results form the interaction term $\rho_i^2 \tilde{J}_{||} \tilde{A}_{||}/2c$ [see Eq. (1.11)], and the relation $\tilde{E}_{||} = \eta \tilde{J}_{||}$.

In the resistive MHD model, the primary fluid motion of the plasma is the $\tilde{\mathbf{E}} \times \mathbf{B}$ motion, $\tilde{\mathbf{V}} = c(\tilde{\mathbf{E}} \times \mathbf{B})/B^2$. Using the fact that $\tilde{\mathbf{E}} = -i\mathbf{k}\tilde{\phi} + i\omega \tilde{A}_{||} \mathbf{b}/c$, then Eq. (1.13) can be rewritten as

$$\frac{S_{IN}}{8\pi} = \int_{-x_0}^{x_0} dx \left[\frac{c^2 m_i n}{2B^2} \left(\left| \frac{d\tilde{\phi}}{dx} \right|^2 + \rho_i^2 k_y^2 |\tilde{\phi}|^2 \right) + \frac{i k_\| \rho_i^2}{2c\eta} \left(\tilde{\phi} - \frac{\omega}{k_\| c} \tilde{A}_\| \right) \tilde{A}_\| \right]. \quad (1.14)$$

The integral S_{IN} is evaluated using a variational technique [37]. To perform a variational calculation, suitable trial functions for $\tilde{\phi}$ and $\tilde{A}_\|$ must be chosen. This is done by noting that at the boundary $\pm x_0$ the trial functions for $\tilde{\phi}$ and $\tilde{A}_\|$ must match onto the ideal MHD solutions. Numerical solutions of the ideal MHD Ampere's law, Eq. (1.7), indicate that for realistic geometries and current profiles $\tilde{A}_{\|MHD}$ is a positive, slowly varying function near the rational surface. Hence, as is done in resistive MHD calculations, $\tilde{A}_\|$ is chosen to be a constant in the inner region. (This corresponds to the "constant ψ" approximation of resistive MHD [3].) Using the fact that in the outer region $\tilde{E}_\| = 0$ implies $\tilde{\phi} = (\omega \tilde{A}_\| / k'_\| x c)$. This fact along with the observation $\tilde{A}_\| = A_{\|0} =$ constant in the inner region implies that $\tilde{\phi}$ must be an odd function about the rational surface. Hence, a suitable trial function for $\tilde{\phi}$ is a linear function which satisfies the condition $\tilde{\phi} = (\omega A_{\|0} / k'_\| x_0 c)$ at $x = \pm x_0$. Thus, S_{IN} is evaluated using the following trial functions:

$$\tilde{A}_\| = A_{\|0} = \text{constant}$$

and

$$\tilde{\phi} = \frac{x}{x_0} \phi_0, \quad \text{where} \quad \phi_0 = \frac{\omega A_{\|0}}{k'_\| \rho_i c} \frac{1}{x_0}. \quad (1.15)$$

In the evaluation of S_{IN}, the boundary x_0 serves as the variational parameter.

Since the trial function for $\tilde{\phi}$ is linear in x in the inner region implies that the second and third terms in Eq. (1.14) can be neglected. Inserting the above trial functions into Eq. (1.14) and performing the integrals gives

$$\frac{S_{IN}}{8\pi} = \frac{c^2 m_i n}{B^2} \left| \frac{\omega A_{\|0}}{k'_{\|} \rho_i c x_0} \right|^2 \frac{1}{x_0} - \frac{i\omega \rho_i^2 A_{\|0}^2}{c^2 \eta} x_0$$

$$= A_{\|0}^2 \left[\frac{1}{4\pi} \left(\frac{\gamma}{v_A k'_{\|} \rho_i} \right)^2 \frac{1}{x_0^3} + \frac{\gamma \rho_i^2}{c^2 \eta} x_0 \right] \tag{1.16}$$

where $v_A^2 = (B^2/4\pi n_0 m_i)$ and $\omega = i\gamma$. Notice that in the resistive MHD picture, the tearing mode has zero real frequency [3].

To find the resistive layer width x_0, the variation of S_{IN} with respect to x_0 is set equal to zero, $\delta S_{IN}/\delta x_0 = 0$. This gives

$$x_0 = \left[\frac{3}{4\pi} \frac{\gamma c^2 \eta}{v_A^2 k'^2_{\|} \rho_i^4} \right]^{1/4} . \tag{1.17}$$

Inserting this into Eq. (1.16) then gives

$$\frac{S_{IN}}{8\pi} = A_{\|0}^2 \frac{4}{3} \rho_i \left[\frac{3}{4\pi} \frac{\gamma^5}{v_A^2 k'^2_{\|} c^6 \eta^3} \right]^{1/4} . \tag{1.18}$$

The growth rate for the resistive tearing mode is found by setting the total energy equal to zero. Hence, setting $S = S_{OUT} + S_{IN} = 0$, using Eq. (1.12) and Eq. (1.18), gives

$$\frac{1}{8\pi} \Delta'_0 = \frac{4}{3} \rho_i \left[\frac{3}{4\pi} \frac{\gamma^5}{v_A^2 k'^2_{\|} c^6 \eta^3} \right]^{1/4} . \tag{1.19}$$

The left hand side represents the free magnetic energy from the outer region and the right hand side represents the energy absorbed in the inner resistive layer. The energy absorbed in the inner region has two contributions. The first is the energy required to sustain the perturbed fluid motion of the ions and the second is the energy dissipated into the plasma through finite resistivity.

Solving Eq. (1.19) for the growth rate gives

$$\gamma = \left(\frac{3}{32\pi}\frac{\Delta_0'}{\rho_i}\right)^{4/5}\left(\frac{4\pi}{3}v_A^2 k_\parallel'^2 c^6 \eta^3\right)^{1/5}. \tag{1.20}$$

Rewriting the above expression in standard notation [27] gives

$$\gamma \simeq .37\tau_R^{-3/5}\tau_A^{-2/5}(\Delta' a)^{4/5}\left(\frac{a}{R}m\frac{aq'}{q^2}\right)^{2/5} \tag{1.21}$$

where $\tau_R = (4\pi a^2/c^2\eta)$, $\tau_A = a/v_A$, a = plasma minor radius, R = plasma major radius, m = poloidal mode number and q = plasma safety factor. More accurate resistive MHD calculations indicate the constant .37 should be replaced by .55 [3,38]. The analytical calculations necessary to obtain the factor .55 are very tedious and they yield the same parameter scaling as indicated by the above expression.

To summarize, the resistive growth rate is determined by balancing the ideal MHD energy drive from the outer region, represented by Δ_0', against the energy required by finite ion inertia and finite resistivity in the inner region. In the resistive MHD model, the tearing mode has zero real frequency and a growth rate given by Eq. (1.21). Instability occurs provided $\Delta_0' > 0$. Recall that in the evaluation of the outer energy drive Δ_0', the limit of an infinitesimally thin layer width is assumed, $x_0 \to 0$. Notice, from Eq. (1.17), that $x_0 \sim \gamma^{1/4}$ and, hence, $x_0 \to 0$ as $\gamma \to 0$. Likewise, the energy absorbed in the inner region by both finite ion inertia and resistivity goes to zero with the growth rate, as indicated by Eq. (1.18). The key dissipation mechanism in this model is, of course, collisional resistivity.

Resistive MHD calculations of the tearing mode have also been performed for the general toroidal case, including finte-beta effects and incompressibility [32,39]. These results indicate that stability is obtained when $\Delta_0' < \Delta_c(\beta)$. Here $\Delta_c(\beta)$ increases with β. However, Δ_c is close to zero except for high values of β [32,40].

1.3 Collisionless Tearing

In Sec. 1.2, the dissipation mechanism which allowed field line reconnection to occur was collisional resistivity. It is often the case, however, that the plasma collision rate is too low to yield large enough linear growth rates corresponding to the observed time scales on which magnetic reconnection occurs. Such is the case in the Earth's magnetotail [41,42]. In such collisionless plasmas, the dominant dissipation mechanism which allows magnetic reconnection to occur is either that of wave-particle resonant interaction [43] (Landau damping) or that of some turbulent (or anomalous) dissipation [44,45]. In this section, a non-collisional and non-turbulent plasma is considered, where the dominant dissipation mechanism is that of wave-particle resonant damping.

To study the basic collisionless tearing mode [1,8,9], a sheared-slab geometry identical to that in Sec. 1.2 is used. Instead of an MHD fluid treatment as used in Sec. 1.2, however, a kinetic approach is used here based on the drift kinetic equation (DKE) [46,47]. To simplify the analysis, only the electron dynamics need be considered, whereas the ions are assumed to form a stationary background. Also, only the effects of a perturbed perpendicular magnetic field $\tilde{\mathbf{B}}_\perp = \nabla \tilde{A}_\parallel \times \mathbf{b}$ need be considered, whereas the effects of a finite electrostatic potential can be neglected, $\tilde{\phi} = 0$. To calculate the dispersion relation for the collision tearing mode, the energy integral obtained from Ampere's law, Eq. (1.5), is again utilized. Hence, only the perturbed parallel electron current need be calculated. This is done by solving the linearized DKE for the perturbed electron distribution $\tilde{f}_e$ and then performing the appropriate velocity moment.

To begin with, the electron dynamics in the presence of a magnetic field perturbation $\tilde{\mathbf{B}}_\perp$ are described by a DKE of the form

$$\left(\frac{\partial}{\partial t} + v_\parallel \mathbf{b}\cdot\nabla + \tilde{\mathbf{v}}_{d_\perp}\cdot\nabla_\perp - \frac{e}{m}E_\parallel\frac{\partial}{\partial v_\parallel}\right) f_e = 0. \qquad (1.22)$$

The perpendicular motion of the guiding centers is given by

$$\tilde{\mathbf{v}}_{d_\perp} = v_{\|} \frac{\tilde{\mathbf{B}}_\perp}{B_0} \tag{1.23}$$

which states that the perpendicular motion arises from the guiding centers free streaming along the undulating field lines $\mathbf{B} = \mathbf{B}_0 + \tilde{\mathbf{B}}_\perp$.

Letting $f_e = \tilde{f}_e + \overline{f}_e$, where $\overline{f}_e$ is the equilibrium electron distribution, then the linearized equation describing $\tilde{f}_e$ is

$$\left(\frac{\partial}{\partial t} + v_{\|}\mathbf{b}\cdot\nabla\right)\tilde{f}_e = -\left(v_{\|}\frac{\tilde{\mathbf{B}}_\perp}{B_0}\cdot\nabla_\perp - \frac{e}{m}\tilde{E}_{\|}\frac{\partial}{\partial v_{\|}}\right)\overline{f}_e. \tag{1.24}$$

The effects of the equilibrium electric field enter the above equation only through the current-carrying piece of $\overline{f}_e$. The equilibrium distribution is chosen to be a drifted Maxwellian with drift speed v_D and thermal speed $v_e = (2T_e/m_e)^{1/2}$ satisfying $v_D/v_e \ll 1$. Hence, $\overline{f}_e$ is given by

$$\overline{f}_e = f_0 + f_1 = \frac{n_0(x)}{\sqrt{\pi}v_e}\exp\left(-\frac{v_{\|}^2}{v_e^2}\right) + \frac{2v_D v_{\|}}{v_e^2}f_0. \tag{1.25}$$

As it turns out, the term proportional to $\tilde{E}_{\|}(\partial f_1/\partial v_{\|})$ on the right hand side of Eq. (1.24) can be neglected. This is because at large x (outside the dissipation layer) the plasma is described adequately by ideal MHD which states $\tilde{E}_{\|} = 0$; while at small x (inside the dissipation layer) the plasma dissipative process of Landau damping is independent of the plasma equilibrium current. Assuming the Fourier dependence $\tilde{f}_e \sim \tilde{f}_e(x)\exp[i(k_y y + k_z z - \omega t)]$, then Eq. (1.24) becomes

$$-i\left(\omega - k_{\|}v_{\|}\right)\tilde{f}_e = -\frac{ie\tilde{A}_{\|}}{cT}v_{\|}\left[\omega f_0 + \omega_{*e}L_n\frac{\partial}{\partial x}(f_0 + f_1)\right] \tag{1.26}$$

where $\omega_{*e} = (cTk_y/eBL_n)$ and $L_n^{-1} = -(d\ln n/dx)$.

The perturbed parallel current is given by taking the first velocity moment of $\tilde{f}_e$. Using Eq. (1.26) gives

$$\tilde{J}_{\|e} = \frac{e^2 n}{cT}\frac{\tilde{A}_\|}{k_\|}\int_{-\infty}^{\infty} dv_\| \left[(\omega - \omega_{*e})\, v_\|^2 - \eta_J \omega_{*e} v_\|^3\right]\left(v_\| - \frac{\omega}{k_\|}\right)^{-1} f_0 \qquad (1.27)$$

where $\eta = (2J'_{\|0}L_n/envv_e)$.

To determine a dispersion relation, the energy integral given by Eq. (1.5) is used. Again, as in Sec. 1.2, this integral is divided into an inner (dissipative) and outer (ideal MHD) region. For the collisionless tearing mode, the boundary of the dissipative layer occurs at $|x| = x_0$, where x_0 is a few times $x_e \equiv (\omega/k'_\| v_e)$. This is chosen since Landau damping becomes significant at $|x| = x_e$. To find $\tilde{J}_\|$ at large x, $|x| > x_0$, the resonance function can be approximated as $(v_\| - \omega/k_\|)^{-1} \simeq v_\|^{-1}$. In this case, $\tilde{J}_\|$ becomes in the outer region

$$\tilde{J}_{\|OUT} \simeq -\frac{k_y J'_{\|0}}{k_\| B_0}\tilde{A}_\|, \quad |x| > x_0 \qquad (1.28)$$

which, as discussed previously, is the ideal MHD response.

In the inner region, $|x| < x_0$, the leading order contribution is from the imaginary piece of the resonance function. Hence, approximating $(v_\| - \omega/k_\|)^{-1} \simeq i\pi\delta(v_\| - \omega/|k_\|)$ gives

$$\tilde{J}_{\|IN} \simeq i\sqrt{\pi}\frac{e^2 n}{cT}v_e^2\tilde{A}_\|\left(1 - \frac{\omega_{*e}}{\omega}\right)\left|\frac{x_e}{x}\right|^3 \exp\left(-\left|\frac{x_e}{x}\right|\right)^2, \quad |x| < x_0. \qquad (1.29)$$

The dispersion relation is found by setting the total energy equal to zero, $S = S_{OUT} + S_{IN} = 0$, where

$$S_{OUT} = \left(\int_{-a}^{x_0} dx + \int_{x_0}^{a} dx\right)\left[\tilde{A}'^2_\| + k_y^2\tilde{A}^2_\| - \frac{4\pi}{c}\tilde{J}_{\|OUT}\tilde{A}_\|\right] \qquad (1.30a)$$

$$S_{IN} = \int_{-x_0}^{x_0} dx \left[\tilde{A}_{||}'^2 + k_y^2 \tilde{A}_{||}^2 - \frac{4\pi}{c} \tilde{J}_{||IN} \tilde{A}_{||} \right]. \tag{1.30b}$$

In the outer ideal MHD region, S_{OUT} can be evaluated by requiring $\tilde{A}_{||}$ to satisfy the ideal MHD version of Ampere's law given by Eq. (1.7). Hence, S_{OUT} can be evaluated approximately to give

$$\begin{aligned} S_{OUT} &\simeq - \lim_{x_0 \to 0} \left[\tilde{A}_{||} \tilde{A}_{||}'(x_0) - \tilde{A}_{||} \tilde{A}_{||}'(-x_0) \right]_{MHD} \\ &= -A_{||0}^2 \Delta_0'. \end{aligned} \tag{1.31}$$

Again, the thickness of the dissipation layer has been neglected to yield the ideal MHD energy Δ_0'.

In the inner dissipative region, S_{IN} is evaluated in the "constant ψ" approximation where $\tilde{A}_{||}$ is taken as constant. Using the approximation

$$\int_0^{x_0} dx \left| \frac{x_e}{x} \right|^3 \exp\left(- \left| \frac{x_e}{x} \right|^2 \right) \simeq x_e \int_0^{\infty} dy y e^{-y^2} = \frac{x_e}{2},$$

then S_{IN} becomes

$$S_{IN} \simeq -2i\sqrt{\pi} \frac{\omega_{pe}^2}{c^2} \frac{(\omega - \omega_{*e})}{k_{||}' v_e} A_{||0}^2, \tag{1.32}$$

where the term proportional to k_y^2 in Eq. (1.30b) has been neglected.

Setting $S = S_{OUT} + S_{IN} = 0$ gives the dispersion relation

$$\Delta_0' = -2i\sqrt{\pi} \frac{\omega_{pe}^2}{c^2} \frac{(\omega - \omega_{*e})}{k_{||}' v_e}. \tag{1.33}$$

Hence, the collisionless tearing mode has a real frequency $\omega_r = \omega_{*e}$ and a growth rate given by

$$\gamma = \frac{k_{\parallel}' v_e}{2\sqrt{\pi}} \frac{c^2}{\omega_{pe}^2} \Delta_0' \tag{1.34}$$

which is smaller by a factor of $1/2$ than the value of γ given by more accurate calculations [1,8,9].

The dispersion relation, Eq. (1.33), can be viewed as a balance between the ideal MHD energy produced in the outer region and the energy dissipative near the rational surface due to resonant wave-particle interactions. The drift frequency ω_{*e} appearing in the expression for the dissipation can be viewed as a Doppler frequency shift resulting from the perturbed motion of the electron guiding centers relative to the laboratory frame. Also, notice that in the calculation of the outer ideal MHD energy, the thickness of the dissipation layer was ignored, as was done in the resistive MHD calculation. An important difference between the description of collisionless tearing and resistive tearing is that the collisionless mode has a real frequency ω_{*e}, whereas the resistive mode was purely growing, $\omega = i\gamma$.

Recall from Sec. 1.2 that the resistive tearing mode can be viewed as a balance between the ideal MHD energy of the outer region and the energy absorbed in the inner region. The energy absorbed in the inner region went into (1) resistive dissipative heating and (2) sustaining the ion motion due to their finite inertia. For the basic collisionless tearing mode represented by Eq. (1.33), however, the ions were assumed to be stationary and, hence, ion inertial effects were neglected. If the ion motion was included in the collisionless tearing mode energetics, then an additional stabilizing term would appear in the dispersion relation representing the energy necessary to sustain the ion oscillation at frequency ω_{*e}. More accurate calculations [9,10] indicate that the collisionless tearing mode can then be stabilized provided

$$\beta_i I_0 \left(\frac{L_S}{2L_n}\right)^{1/2} > \left(\frac{2T_e}{T_i}\right)^{1/2} \rho_i \Delta_0', \tag{1.35}$$

where I_0 is a constant of order unity. Hence, by including the ion dynamics, the collisionless tearing mode can be stabilized at sufficiently high β_i.

Chapter 2
MATHEMATICAL MODEL

In a tokamak plasma, the effects of plasma turbulence play a major role in determining the overall properties of plasma transport. Thus, it is clear that any correct theory of plasma transport must include turbulence. Likewise, in the study of plasma instabilities, it is apparent that the effects of plasma turbulence on the instability itself may play an important role. In particular, in this study the major turbulent effect to be considered is that of electron spatial diffusion.

A mathematical model is developed, based on the normal stochastic approximation (NSA), which includes self-consistently the effects of turbulent electron diffusion [15–18]. The NSA is a theory for the nonlinear, turbulent response in a system with intrinsic stochasticity and long-lived fluctuations. Intrinsic stochasticity [48–50] implies that a diffusive process arises from the chaotic structure of the particle orbits resulting from overlapping phase space islands. Hence, it is necessary for the underlying turbulent fluctuations to be sufficiently large as to satisfy the island overlap condition [51]. The assumption of long-lived fluctuations implies that a hierarchy of statistical averages can be performed. That is, a microscale average over the fine-scale structure of the particle orbits can be performed while holding the statistics of the wave fluctuations fixed. This disparity in scales between the orbits and the fluctuations arises when the lifetime of the long-scale fluctuations associated with the waves, τ_{ac}, is long compared to the Kolmogorov time [52], τ_k, characterizing the rate of the randomization in the particle orbits. Hence, the NSA assumes $\tau_{ac} \gg \tau_k$. In contrast, theories such as the direction interaction approximation (DIA) [53–55] are capable of treating a system which is described by only a single scale length and, hence, are more likely to be valid in the regime $\tau_k > \tau_{ac}$. In such a system, the important turbulent phenomenon is the nonlinear interaction of the waves themselves as opposed to the stochastic behavior of the particle orbits, as is treated in the NSA.

In Sec. 2.1, the electron response is derived by applying the NSA to the drift kinetic equation. In the NSA formalism developed here, the tearing mode is assumed to exist on a background of drift wave turbulence. These turbulent drift waves produce overlapping islands which in turn lead to stochastic electron orbits. The NSA makes use of the observation that the turbulent perturbations to the electron orbits exhibit pathologically complex, fine-scale structures which are produced from relatively simple, long-scale wave fluctuations. Since the spatial structure characterizing the electron orbits is much finer than that characterizing the wave potentials, it is possible to separate the statistics of the particle orbits from the statistics of the wave potentials by a coarse-graining procedure. This procedure is a microscale averaging over the orbit perturbations which is performed while holding the statistics of the wave potentials fixed. The NSA assumes that the orbit perturbations can be treated as a normally distributed random variable with variance $\langle \delta x^2 \rangle = 2Dt$. The end result of the NSA is that the nonlinear effects of turbulence are represented by the appearance of a radial diffusion operator, $-D\partial^2/\partial^2 x$, in the drift kinetic equation.

The ion response is presented in Sec. 2.2 according to the linearized Vlasov equation. Note that the effect of turbulent diffusion is to smooth out the structure of the response functions over the length scale x_c. Since the ion response functions are characterized by a scale length x_i satisfying $x_i > x_c$, the turbulent diffusion has no appreciable effect on the ion response. In Sec. 2.3, the electron and ion responses are combined through Ampere's law and quasineutrality to yield a coupled, self-adjoint system for the electric and magnetic potentials $\tilde{\phi}$ and $\tilde{A}_{||}$. Since this system is self-adjoint, a variational principle is then formulated. Section 2.4 introduces approximate forms for the resonance operators which appear in the NSA treatment of the electron response. Also presented is the Krook approximation to the NSA, in which the operator $-D\partial^2/\partial x^2$ is replaced by an effective diffusion frequency ω_c.

2.1. Electron Response: The Normal Stochastic Approximation

The major goal here is to develop a model which includes the effects of turbulent electron diffusion in a self–consistent manner. In the NSA formalism, the lifetime of the long–scale potential fluctuations of the waves, τ_{ac}, is assumed to be long compared to the Kolmogorov time, τ_k, characterizing the rate of the randomization in the particle orbits [52]. The amplitude of the wave fluctuations must be sufficiently large such that the phase–space islands overlap, thus, producing stochastic particle orbits. For example, in the case of drift waves, the island overlap condition is satisfied for very small fluctuation amplitudes [16,17]. Hence, it is the drift waves themselves that produce the stochastic electron behavior and thus lead to turbulent diffusion. Using the NSA, this phenomenon of stochastic electron behavior is implemented in the stability analysis for the drift wave via a turbulent diffusion coefficient. In particular, one can show that the drift wave is unstable for very small values of the diffusion coefficient and then saturates at some finite value. In fact, it is possible to calculate the saturation value of the diffusion coefficient at which the drift wave stabilizes [16,19].

In the case of the tearing mode, the tearing mode itself will not (in cases of interest) produce island overlap and lead to stochastic electron behavior. To correctly account for turbulent electron behavior in this case, one must consider the tearing mode as existing among a background of turbulence such as that produced by finite–beta drift waves. Due to the large discrepancy in the poloidal wave numbers, m, of the tearing mode and the drift waves, their respective stability analyses can be performed largely independently. This discrepancy in the wave numbers also allows a spatial averaging over the scale length of the drift waves while keeping the tearing mode potentials fixed. (This averaging is unnecessary if the coarse–grain averaging of the stochastic particle orbits has been performed.) The major effect of the drift waves on the tearing mode is that the electrons behave stochastically due to the presence of drift wave turbulence. This stochastic behavior manifests itself as a turbulent diffusion coefficient whose value is independent of the presence

of the tearing mode. Hence, in the tearing mode analysis, the electron diffusion coefficient is treated as an independent external parameter, whose value is to be specified either by experimental observation or calculated through the use of an appropriate microturbulence theory for drift waves.

For this problem a sheared, slab geometry is chosen with the equilibrium magnetic field given by

$$\mathbf{B} = B\left(\mathbf{e}_z + \frac{x}{L_S}\mathbf{e}_y\right)$$

where L_S is the shear length and $x = 0$ is chosen to be the position of the rational surface. Throughout the following, the parallel direction is defined according to the unit vector $\mathbf{b} = \mathbf{B}/B$. The slab coordinates (x, y, z) can be related to the straight tokamak coordinates (r, θ, ϕ) by the following

$$x = r - r_s, \quad y = r\theta \text{ and } z = R\phi.$$

Here, r is the plasma minor radius, R is the plasma major radius and r_s is the location of the plasma rational surface which occurs at $q(r_s) = m/n$. The safety factor is given by $q(r) = rB_\phi/RB_\theta$ (where B_θ and B_ϕ are the poloidal and toroidal magnetic fields). The poloidal mode number m and the toroidal mode number n are related to the wavenumbers k_y and k_z according to $k_y = m/r$ and $k_z = -n/R$. Hence,

$$\mathbf{k} \cdot \mathbf{r} = k_y y + k_z z = m\theta - n\phi$$

and

$$k_\parallel = \mathbf{k} \cdot \mathbf{b} = \frac{nB_\theta}{rB}\left(\frac{m}{n} - q(r)\right) \equiv k'_\parallel x$$

where $k'_{||} = k_y/L_S$ with $L_S \equiv -Rq^2/rq'$. It will be shown that this model reduces to ideal MHD away from the rational surface. Because of this reduction to ideal MHD at large x, the following results for the tearing mode can be generalized quite easily to more complicated geometries simply by calculating the function Δ' for those geometries [30].

The starting point in this model for determining the electron response shall be the drift kinetic equation [18,46,47] of the following form:

$$\left[\frac{\partial}{\partial t} + v_{||}\mathbf{b}\cdot\nabla - \frac{c}{B}\nabla\left(\tilde{\phi} - \frac{v_{||}}{c}\tilde{A}_{||}\right)\times\mathbf{b}\cdot\nabla - \frac{e}{m}(\tilde{\mathbf{E}} + \mathbf{E}_0)\cdot\mathbf{b}\frac{\partial}{\partial v_{||}}\right] f_e = C(f_e, f_e). \quad (2.1)$$

The third term on the left side of the above equation represents the perpendicular motion of the guiding centers resulting from the $\tilde{\mathbf{E}}\times\mathbf{B}$ drift and the $v_{||}\tilde{\mathbf{B}}_\perp/B$ free streaming motion. The right side of the above equation represents an appropriate velocity space collision operator and E_0 denotes the applied electric field. Throughout the following, a tilde is used to denote fluctuation quantities.

Statistically averaging Eq. (2.1) gives

$$\frac{\partial}{\partial t}\overline{f} - \nabla\cdot\langle\frac{c}{B}\left(\tilde{\phi} - \frac{v_{||}}{c}\tilde{A}_{||}\right)\times\mathbf{b}\tilde{f}\rangle - \frac{\partial}{\partial v_{||}}\langle\frac{e}{m}\tilde{\mathbf{E}}\cdot\mathbf{b}\tilde{f}\rangle$$
$$- \frac{e}{m}\mathbf{E}_0\cdot\mathbf{b}\frac{\partial}{\partial v_{||}}\overline{f} = C(\overline{f},\overline{f}) + \langle C(\tilde{f},\tilde{f})\rangle, \quad (2.2)$$

where $f_e = \tilde{f} + \overline{f}$, with $\overline{f} = \langle f_e\rangle$ = equilibrium electron distribution. The second and third terms on the left represent a turbulent collision operator acting on $\overline{f}$ denoted by $C_T(\overline{f})$, provided $\tilde{f}$ is a known functional of $\overline{f}$.

Setting the fluctuation quantities to zero and assuming $\partial\overline{f}/\partial t = 0$, then the above equation reduces the the familiar Spitzer problem [35] of solving for $\overline{f}$ in the presence of an applied electric field. The effect of $\mathbf{E}_0$ on the equilibrium will be the

production of an equilibrium plasma current. For the case of the tearing mode, the equilibrium plasma current is included in the equilibrium electron distribution by writing $\overline{f} = f_0 + f_1$, where

$$f_0 = \frac{n_o}{\sqrt{\pi} v_e} \exp\left(-\frac{v_\parallel^2}{v_e^2}\right) \quad \text{and} \quad f_1 = \frac{2 v_D v_\parallel}{v_e^2} f_0.$$

Here, $\overline{f}$ is a drifted Maxwellian and it is assumed $v_D/v_e \ll 1$ where $v_e^2 = 2T_e/m_e$. In the above expression, f_1 denotes the current-carrying piece of the equilibrium distribution. The equilibrium current is given by $J_{\parallel 0} = -e v_D n_0$.

Subtracting the equilibrium equation, Eq. (2.2), from Eq. (2.1) yields the fluctuation equation

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} + v_\parallel \mathbf{b}\cdot\nabla - \frac{c}{B}\nabla\left(\tilde{\phi} - \frac{v_\parallel}{c}\tilde{A}_\parallel\right)\times\mathbf{b}\cdot\nabla - \frac{e}{m}(\mathbf{E}_0 + \tilde{\mathbf{E}})\cdot\mathbf{b}\frac{\partial}{\partial v_\parallel}\right]\tilde{f} \\
&\qquad - \frac{c}{B}\nabla\left(\tilde{\phi} - \frac{v_\parallel}{c}\tilde{A}_\parallel\right)\times\mathbf{b}\cdot\nabla\overline{f} - \frac{e}{m}\tilde{\mathbf{E}}\cdot\mathbf{b}\frac{\partial}{\partial v_\parallel}\overline{f} \\
&\qquad = C_T(\overline{f}) + C_L(\tilde{f}) + C(\tilde{f},\tilde{f}) - \langle C(\tilde{f},\tilde{f})\rangle \qquad (2.3)
\end{aligned}$$

where $C_L(\tilde{f})$ represents the linearized collision operator and where the assumption has been made that $\partial\overline{f}/\partial t = 0$.

Since $E_0/E_R \ll 1$, where E_R is the "runaway" value of the electric field [56], and since the equilibrium current is already contained in the f_1 part of $\overline{f}$, the applied electric field term on the left of Eq. (2.3) is ignored. For simplicity the collision operators of the right side of Eq. (2.3) are neglected; however, the question of velocity space collisions will be returned to later.

Using the above approximations, the fluctuation drift kinetic equation is reduced to the actual starting equation for this analysis, which is the collisionless DKE for straight magnetic fields:

$$\left[\frac{\partial}{\partial t} + v_{\|}\mathbf{b}\cdot\nabla - \frac{c}{B}\nabla\left(\tilde{\phi} - \frac{v_{\|}}{c}\tilde{A}_{\|}\right)\times\mathbf{b}\cdot\nabla - \frac{e}{m}\tilde{\mathbf{E}}\cdot\mathbf{b}\frac{\partial}{\partial v_{\|}}\right]\tilde{f}$$

$$= \left[\frac{c}{B}\nabla\left(\tilde{\phi} - \frac{v_{\|}}{c}\tilde{A}_{\|}\right)\times\mathbf{b}\cdot\nabla + \frac{e}{m}\tilde{\mathbf{E}}\cdot\mathbf{b}\frac{\partial}{\partial v_{\|}}\right]\bar{f}. \qquad (2.4)$$

This equation is denoted as

$$\frac{D}{Dt}\tilde{f} = S_0(x).$$

In light of the above discussion, the leading order effect of the third and fourth terms involving the fluctuation potentials in the orbit operator defined by D/Dt is the production of stochastic electron orbits resulting from the underlying drift wave turbulence.

To solve for the electron response, $\tilde{f}$, one integrates over the exact perturbed orbits represented by D/Dt. Thus,

$$\tilde{f}(\mathbf{x},t) = \int_{-\infty}^{t} d\tau' \sum_{k,\omega} \exp\{i\,(m\theta_0(\mathbf{x},\tau'-t) - n\phi_0(\mathbf{x},\tau'-t) - \omega\tau')\}\, S_{0_{k\omega}}\,(r_0(\mathbf{x},\tau'-t)) \qquad (2.5)$$

where $S_{0_{k\omega}}$ is the Fourier decomposition of $S_0(\mathbf{x},t)$:

$$S_0(\mathbf{x},t) = \sum_{k,\omega} \exp\{i(m\theta - n\phi - \omega t)\}\, S_{0_{k,\omega}}(r)$$

where $k = (m,n)$ and $\mathbf{x} = (r,\theta,\phi)$.

Strictly speaking, the wave potentials can only be Fourier decomposed into modes with a well-defined frequency, ω, for times $t < \tau_{ac}$, where τ_{ac} = the autocorrelation time [57] for the fluctuation potentials. In general, $S_{0_{k\omega}}(\underline{r}) \rightarrow |S_{0_{k\omega}}(r)|\exp(-i\Phi_{k\omega}(\theta,\phi,t))$, where $\Phi_k(\theta,\phi,t)$ is the slow variation of the Fourier

coefficient $S_{0_{k_\omega}}(r,\theta,\phi,t)$. However, for $t < \tau_{ac}$, the variation of $\exp(-i\Phi_k(\theta,\phi,t))$ is much smaller than the variation of $\exp(i(m\theta - n\phi - \omega t))$ and, hence, will be neglected. In the following discussion, the times of interest, t, occur on a scale such that $\tau_k < t < \tau_{ac}$, where τ_k = the Kolmogorov [52] time for orbit exponentiation. Hence, a necessary condition [17] for the validity of the following development is $\tau_k \ll \tau_{ac}$.

In Eq. (2.5) the subscript zero signifies the exact electron orbits. These orbits obey the equations

$$\begin{aligned} \mathbf{v}_{\perp gc} &= -\frac{c}{B}\nabla\left(\tilde{\phi} - \frac{v_\parallel}{c}\tilde{A}_\parallel\right) \times \mathbf{b} \\ \dot{\mathbf{v}}_\parallel &= i\frac{e}{m}k_\parallel\left(\tilde{\phi} - \frac{\omega}{k_\parallel c}\tilde{A}_\parallel\right) \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} \mathbf{v}_{\perp gc} &= \dot{r}_0(\mathbf{x},t)\mathbf{e}_x + r\dot{\theta}_0(\mathbf{x},t)\mathbf{e}_y \\ \mathbf{v}_\parallel &= \left(r\dot{\theta}_0(\mathbf{x},t)\mathbf{e}_y + R\dot{\phi}_0(\mathbf{x},t)\mathbf{e}_z\right)\cdot\mathbf{b}. \end{aligned}$$

Here, $\mathbf{v}_{\perp gc}$ and $\mathbf{v}_\parallel$ refer to the velocity of the guiding centers.

The above equations are solved giving the orbits

$$\begin{aligned} r_0(\mathbf{x},\tau-t) &= r + \delta r_0(\mathbf{x},\tau-t) \\ \theta_0(\mathbf{x},\tau-t) &= \theta + \frac{1}{q}\frac{v_\parallel}{R}(\tau-t) + \delta\theta_0(\mathbf{x},\tau-t) + \delta\theta_1(\mathbf{x},\tau-t) + \delta\theta_2(\mathbf{x},\tau-t) \\ \phi_0(\mathbf{x},\tau-t) &= \phi + \frac{v_\parallel}{R}(\tau-t) + \delta\phi_2(\mathbf{x},\tau-t) \end{aligned} \tag{2.7}$$

such that $X_0(\tau - t) = X$ at $\tau = t$ where $\delta\mathbf{x}(0) = 0$. The turbulent perturbations to the orbits are defined as

$$\delta r_0(\mathbf{x},t) = \frac{-ick_\perp}{B}\int_0^t dt'\left(\tilde{\phi} - \frac{v_\parallel}{c}\tilde{A}_\parallel\right)$$

$$\delta\theta_0(\mathbf{x},t) = \frac{-k'_\parallel v_\parallel}{m}\int_0^t dt'\delta r_0(t')$$

$$\delta\theta_1(\mathbf{x},t) = \frac{c}{B}\int_0^t dt'\frac{1}{r}\frac{\partial}{\partial r}\left(\tilde{\phi} - \frac{v_\parallel}{c}\tilde{A}_\parallel\right) \tag{2.8}$$

$$\delta\theta_2(\mathbf{x},t) = \frac{ie}{m_e}\frac{k_\parallel}{qR}\int_0^t dt'\int_0^{t'} dt''\left(\tilde{\phi} - \frac{\omega}{k_\parallel c}\tilde{A}_\parallel\right)$$

$$\delta\phi_2(\mathbf{x},t) = \frac{ie}{m_e}\frac{k_\parallel}{R}\int_0^t dt'\int_0^{t'} dt''\left(\tilde{\phi} - \frac{\omega}{k_\parallel c}\tilde{A}_\parallel\right).$$

This analysis is concerned with the effect of *spatial* diffusion on the tearing mode. In the above, δr_0 represents spatial diffusion due to random $\tilde{\mathbf{E}}\times\mathbf{B}$ fluctuations of drift waves. (Similarly with the term $\delta\theta_1$). The term $\delta\theta_0$ reflects the presence of magnetic shear; that is, radial diffusion leads to poloidal spatial motion in the presence of shear. The terms $\delta\theta_2$ and $\delta\phi_2$ represent velocity turbulent diffusion. These effects are subdominant by the estimate

$$\frac{\langle(m\delta\theta_2 - n\delta\phi_2)^2\rangle}{m^2\langle\delta\theta_0^2\rangle} \sim \frac{k_\parallel^2 D_{vv}}{(k'_\parallel v_\parallel)^2 D_{xx}} \sim \frac{m_i}{m_e}\left(\frac{\rho_i}{L_s}\right)^2 \ll 1$$

and, hence, these terms are neglected. Keep in mind that the above turbulent orbit perturbations are due to the presence of a drift wave background.

Introducing the transform

$$\tilde{f}_{k'}(r,t) = \int_0^{2\pi}\frac{d\theta}{2\pi}\int_0^{2\pi}\frac{d\phi}{2\pi}\exp\left\{-i(m'\theta - n'\phi)\right\}\tilde{f}(\mathbf{x},t)$$

along with the variable transformation $\tau = t - \tau'$, then Eq. (2.5) can be written as

$$\tilde{f}_{k'}(r,t) = \sum_{k,\omega} \exp(-i\omega t) \int_0^\infty d\tau \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} \exp\{-i(m\theta_0(\mathbf{x},-\tau) - m'\theta) \\ -i(n\phi_0(\mathbf{x},-\tau) - n'\phi) + i\omega\tau\} S_{0_{k\omega}}(r_0(\mathbf{x},-\tau)).$$

Substituting the characteristic orbits defined by Eqs. (2.7) into the above expression gives

$$\tilde{f}_{k'}(r,t) = \sum_{k,\omega} \exp(-i\omega t) \int_0^\infty d\tau \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} \exp\{i(m-m')\theta \\ -i(n-n')\theta + i(\omega - k_\| v_\|)\tau + im\delta\theta(\mathbf{x},-\tau)\} \\ \times S_{0_{k\omega}}(r + \delta r(\mathbf{x},-\tau)). \tag{2.9}$$

The source term is given by

$$S_{0_k}(r) = \left[\frac{ie}{T_e} k_\| v_\| \left(\tilde{\phi} - \frac{\omega}{k_\| c}\tilde{A}_\|\right) f_0 + i\frac{c}{B}\mathbf{k}\left(\tilde{\phi} - \frac{v_\|}{c}\tilde{A}_\|\right) \times \mathbf{b}\cdot\nabla\overline{f} + \frac{e}{m}\tilde{\mathbf{E}}_k \cdot \mathbf{b}\frac{\partial}{\partial v_\|} f_1\right]$$

where $k_\| = m/Rq - n/R \equiv k'_\| x$, $k'_\| = k_y/L_s$ and $k_y = m/r$.

Physically, the applicability of the NSA makes use of the observation that for sufficiently long times, $t > \tau_k$ (τ_k = Kolmogorov time for entropy production), [52] the exact particle orbit perturbations, $\delta\theta(t)$, exhibit stochastic behavior. That is, for $t > \tau_k$ the orbits $\delta\theta(\mathbf{x},t)$ develop pathological spatial structure leading to sharp spatial gradients. In particular, orbits with neighboring initial data separate exponentially with time. The NSA makes use of the property that such stochastic orbits are characterized by an extremely fine spatial structure, much finer than the spatial structure of the corresponding fluctuation potentials, $\tilde{\phi}$ and $\tilde{A}_\|$. The wave lengths characterizing the fluctuations in the orbits δr and $\delta\theta$ are much shorter than those characterizing the potentials $\tilde{\phi}$ and $\tilde{A}_\|$. This allows a hierarchical averaging

procedure in which $\tilde{A}_{||}$ and $\tilde{\phi}$ are approximated as constants in the microscale statistical average over the fluctuations in $\delta\theta$ and δr. The statistics of the orbits and the waves then become essentially independent, thus enabling a spatial averaging to be performed on the orbits over a distance in which the wave potentials do not exhibit random behavior.

The above argument implicitly assumes that $\tau_k \ll \tau_{ac}$, where τ_{ac} = the autocorrelation time [57] for the fluctuation potential. When this is satisfied, the orbit functions exhibit random behavior on a time scale (represented by τ_k) much shorter than the time scale on which the fluctuation potentials randomize (represented by τ_{ac}). A typical estimate for the autocorrelation time gives $\tau_{ac} \sim \omega_{*e}{}^{-1}$, where ω_{*e} is on the order of the real frequency. This is implied by experimental observations [58] in which the frequency spectrum $\Delta\omega(k)$ is observed to be of order $\Delta\omega(k)/\omega \sim 1$. In the NSA model [17], the electron orbits exhibit diffusive behavior on the time scale $\tau_k \sim \omega_c{}^{-1}$,where $\omega_c{}^3 = (k'_{||}v_e)^2 D/3$. For the $m = 2$ tearing mode, $\omega_c/\omega_{*e} \sim 10$, and hence $\tau_k < \tau_{ac}$. This procedure of performing a statistical average over the fine microscale of the orbits while holding the statistics of the waves fixed is known as "coarse graining" [17]. (Note, that since $\omega_{*e}/\omega_c \sim m^{1/3}$, then the criterion $\tau_k < \tau_{ac}$ is satisfied more strongly for the $m = 2$ tearing mode than it is for drift waves.)

For short times, $t < \tau_k$, this procedure breaks down. Since τ_k is typically small when compared to the time scales of interest, the small time contribution, $t < \tau_k$, to the integral in Eq. (2.9) can be neglected for most of the terms in the source function, S_{0_k}. It is only for the dominant term in S_{0_k}, namely $(ie/T_e)k_{||}v_{||}\tilde{\phi}$, for which the small time contribution to the overall time integral in Eq. (2.9) is significant and, hence, must be treated more carefully. Here, $k_{||}v_{||}$ scales as the transit frequency, ω_t, and it is the largest order term in S_{0_k}. Physically, for small times $t < \tau_k$, the electron orbits are characterized primarily by free streaming, hence, it is only those terms proportional to $k_{||}v_{||}$ in S_{0_k} which are significant for $t < \tau_k$.

To account for the short time behavior, $t < \tau_k$, the dominant drive term $(ie/T_e)k_{||}v_{||}\tilde{\phi}_{k\omega}$ when combined with the free streaming orbit operator gives the adiabatic response. That is,

$$ik_{\|}v_{\|}\tilde{f}^{ad}_{k\omega} \simeq \frac{ie}{T_e}k_{\|}v_{\|}\tilde{\phi}_{k\omega}f_0$$

gives

$$\tilde{f}^{ad}_{k\omega} \simeq \frac{e\tilde{\phi}_{k\omega}}{T_e}f_0,$$

which is the adiabatic response. In order to preserve this short time behavior in the NSA, the adiabatic response is extracted from the onset. Inserting

$$\tilde{f}_e = \frac{e\tilde{\phi}}{T_e}f_0 + \tilde{h}_e \tag{2.10}$$

into the fluctuation DKE, Eq. (2.4), then the nonlinear electron response is given according to

$$\begin{aligned}\left[\frac{\partial}{\partial t} + v_{\|}\mathbf{b}\cdot\nabla - \frac{c}{B}\nabla\left(\tilde{\phi} - \frac{v_{\|}}{c}\tilde{A}_{\|}\right)\times\mathbf{b}\cdot\nabla - \frac{e}{m}\tilde{\mathbf{E}}\cdot\mathbf{b}\frac{\partial}{\partial v_{\|}}\right]\tilde{h}_e &= S(x)\\ \equiv \frac{c}{B}\nabla\left(\tilde{\phi} - \frac{v_{\|}}{c}\tilde{A}_{\|}\right)\times\mathbf{b}\cdot\nabla(f_0+f_1)&\\ - \frac{e}{T_e}\frac{\partial}{\partial t}\left(\tilde{\phi} - \frac{v_{\|}}{c}\tilde{A}_{\|}\right)f_0 + \frac{e}{m}\tilde{\mathbf{E}}\cdot\mathbf{b}\frac{\partial}{\partial v_{\|}}f_1 &\end{aligned} \tag{2.11}$$

where the inherently nonlinear terms in the fluctuation potentials have been neglected in the expression for $S(x)$. Only the nonlinear terms involving the gradients of $\tilde{h}_e$ are significant, since orbital stochasticity leads to very large gradients in $\tilde{h}_e$ compared to the relatively long wavelengths characterizing the gradients of $\tilde{\phi}$ and $\tilde{A}_{\|}$.

Integrating the above equation over the exact orbits gives the following integral expression for $\tilde{h}_e$, which is the equivalent of Eq. (2.9):

$$\tilde{h}_{k'\omega}(r) = \sum_k \int_0^\infty d\tau \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} \exp\{i(m-m')\theta - i(n-n')\phi + i(\omega - k_\parallel v_\parallel)\tau + im\delta\theta(\mathbf{x}, -\tau)\} \times S_{k\omega}(r + \delta r(\mathbf{x}, -\tau)) \tag{2.12}$$

with the source term

$$S_{k\omega}(r) = \frac{ic}{B}\left(\tilde{\phi} - \frac{v_\parallel}{c}\tilde{A}_\parallel\right)\mathbf{k}\times\mathbf{b}\cdot\nabla(f_0 + f_1) + \frac{ie}{T_e}\omega\left(\tilde{\phi} - \frac{v_\parallel}{c}\tilde{A}_\parallel\right)f_0 - \frac{ie}{m}k_\parallel\left(\tilde{\phi} - \frac{\omega}{k_\parallel c}\tilde{A}_\parallel\right)f_1$$

where $\tilde{h}_{k'}(r,t) = \sum_\omega \exp(-i\omega t)\tilde{h}_{k'\omega}(r)$.

By explicitly separating the adiabatic response from the nonlinear response at the start, then the largest term in the expression S_{0_k}, proportional to $k_\parallel v_\parallel \sim \omega_t =$ the transit frequency, has been replaced by a smaller term in S_k proportional to $\omega \sim \omega_{*e}$, where $\omega_{*e} \ll \omega_t$. The separation $\tilde{f}_e = (e\tilde{\phi}/T_e)f_0 + \tilde{h}_e$ is valid since the adiabatic response remains virtually unaffected by the nonlinear process of $\tilde{\mathbf{E}}\times\mathbf{B}$ turbulent diffusion [15,18].

To solve to $\tilde{h}_{k'\omega}$, the transform

$$S_{k\omega}(r+\delta r) = \int_{-\infty}^{\infty} dk_r \exp\{ik_r(r+\delta r)\} S_{k\omega k_r}$$

is introduced such that

$$\tilde{h}_{k'\omega}(r) = \int_{-\infty}^{\infty} dk_r \exp(ik_r r) \int_0^\infty d\tau \sum_k \exp\{i(\omega - k_\parallel v_\parallel)\tau\} \times \langle \exp\{i(m-m')\theta - i(n-n')\phi + ik_r\delta r(\mathbf{x}, -\tau) + im\delta\theta(\mathbf{x}, -\tau)\}\rangle \times S_{k\omega k_r} \tag{2.13}$$

where $\langle Q \rangle = (2\pi)^{-2} \int d\theta \int d\phi Q$ denotes the angular average.

Consider the quantity

$$A_{kk'} = \langle \exp\{i(m-m')\theta - i(n-n')\phi + ik_r \delta r(\mathbf{x}, -\tau) + im\delta\theta(\mathbf{x}, -\tau)\}\rangle \qquad (2.14)$$

For small τ, $\delta r/r$ and $\delta\theta \ll 1$, hence $A_{kk'} \simeq \delta_{kk'}$. Here, $\delta_{kk'} = 1$ for $k = k'$ and zero otherwise. For large τ/τ_k, the angular dependence of δr and $\delta\theta$ is the dominant variation in the integrand, and $A_{kk'}$ can be computed as a series of integrals over small angular intervals. Letting $\theta_j = 2\pi j/M$ and $\phi_\ell = 2\pi\ell/N$, then

$$\begin{aligned} A_{kk'} = \sum_{j=1}^{M} \int_0^{2\pi/M} \frac{d\theta}{2\pi} \sum_{\ell=1}^{N} \int_0^{2\pi/N} \frac{d\phi}{2\pi} \exp\{&i(m-m')(\theta_j + \theta) \\ &- i(n-n')(\phi_\ell + \phi) + im\delta\theta(r, \theta_j + \theta, \phi_\ell + \phi, -\tau) \\ &- ik_r \delta r(r, \theta_j + \theta, \phi_\ell + \phi, -\tau)\} \end{aligned}$$

or

$$\begin{aligned} A_{kk'} = \sum_{j=1}^{M} \exp\{i(m-m')\theta_j\} \sum_{\ell=1}^{N} \exp\{-i(n-n')\phi_\ell\} \\ \times \int_0^{2\pi/M} \frac{d\theta}{2\pi} \int_0^{2\pi/N} \frac{d\phi}{2\pi} \exp\{im\delta\theta(r, \theta_j + \theta, \phi_\ell + \phi, -\tau) \\ + ik_r r(r, \theta_j + \theta, \phi_\ell + \phi, -\tau)\} [1 + i(m-m')\theta + \cdots][1 - i(n-n')\phi + \cdots] \end{aligned}$$

The essential feature of stochasticity is that an M and N can be chosen such that $\delta\theta$ and $\delta\phi$ undergo many oscillations with respect to θ and ϕ in the period $2\pi/M$ and $2\pi/N$, but $(m-m')/M$ and $(n-n')/N$ remain small. Thus, the above angular integrals approach an average, independent of θ_j and ϕ_ℓ, and the sums give the Kronecker delta, $\delta_{kk'}$. Hence,

$$A_{kk'} \simeq \delta_{kk'} \left\langle \exp\left\{ im\delta\theta(\mathbf{x}, -\tau) + ik_r\delta r(\mathbf{x}, -\tau) \right\} \right\rangle . \tag{2.15}$$

For stochastic orbits, M and N increase with τ such that $M, N \to \infty$ for large τ/τ_k, making the above expression for $A_{kk'}$ asymptotically exact. The important element allowing the above evaluation of $A_{kk'}$ is the disparity between the wavelengths of the potentials and the spatial scale of the orbit perturbations.

The basic assertion of the NSA states that the microscale ensemble average $A_{kk'} \to [A_{kk'}]_0$ is equivalent to an average with $\delta\theta$ and δr taken as normally distributed random variables. For a normally distributed random variable, σ,

$$\langle \exp(ic\sigma) \rangle = \exp\left(-\frac{1}{2} c^2 \langle \sigma^2 \rangle \right)$$

where c = constant. Hence,

$$\begin{aligned} [A_{kk'}]_0 &= \delta_{kk'} \left[\langle \exp(im\delta\theta + ik_r\delta r) \rangle \right]_0 \\ &= \delta_{kk'} \exp\left\{ -\frac{1}{2} \left(m^2 \left[\delta\theta^2\right]_0 + 2mk_r \left[\delta\theta\delta r\right]_0 + k_r^2 \left[\delta r^2\right]_0 \right) \right\} . \end{aligned} \tag{2.16}$$

Assuming δr to be a Wiener process [59] with

$$\left[\delta r^2\right]_0 = 2D\tau,$$

then

$$\left[\delta r \delta\theta\right]_0 = \frac{-k'_{\|} v_{\|}}{m} D\tau^2$$

and

$$\left[\delta\theta^2\right]_0 = \frac{2}{3} \left(\frac{k'_{\|} v_{\|}}{m} \right)^2 D\tau^3$$

where the term $\delta\theta_1$ in Eq. (2.7) has been neglected. Note that since $S_{k\omega k_r}$ belongs to the long-scale statistical ensemble, the microscale average acts only on $A_{kk'}$.

Inserting the above expression for $A_{kk'}$ back into the integrand for $\tilde{h}_{k'\omega}$, gives the following "coarse-grained" value of $\tilde{h}_{k'\omega}$:

$$\left[\tilde{h}_{k\omega}\right]_0 = \int_0^\infty d\tau \exp\left\{i(\omega - k_\parallel v_\parallel)\tau - \frac{1}{3}(k'_\parallel v_\parallel)D\tau^3\right\} \int_{-\infty}^\infty \frac{dr'}{2\pi} S_{k\omega}(r') \times \int_{-\infty}^\infty dk_r \exp\left\{ik_r(r-r') + k_r k'_\parallel v_\parallel D\tau^2 - k_r^2 D\tau\right\} \tag{2.17}$$

where the transform

$$S_{k\omega k_r} = \int_{-\infty}^\infty \frac{dr'}{2\pi} \exp(-ik_r r') S_{k\omega}(r')$$

was used. In the expressions which follow, $\tilde{h}_{k\omega}$ implicitly stands for the coarse-grained value $[\tilde{h}_{k\omega}]_0$. Recognizing the last integral in the above expression for $\tilde{h}_{e_k}$ as a transform of a Gaussian finally yields:

$$\tilde{h}_{e_k} = \int_0^\infty d\tau \int_{-\infty}^\infty dx' G(x, x'; v_\parallel, \tau) S_k(x') \tag{2.18}$$

where the kernel $G(x, x'; v_\parallel, \tau)$ is given by

$$G(x, x'; v_\parallel, \tau) = \frac{1}{\sqrt{4\pi D\tau}} \exp\left[i(\omega - k'_\parallel v_\parallel x)\tau - \frac{1}{3}(k'_\parallel v_\parallel)^2 D\tau^3 - \frac{1}{4D\tau}(x - x' - iDk'_\parallel v_\parallel \tau^2)^2\right]. \tag{2.19}$$

Note that $G(x, x'; v_\parallel, \tau)$ decays with a characteristic time $\tau_c \equiv \left[1/3(k'_\parallel v_e)^2 D\right]^{-1/3} \equiv \omega_c^{-1}$ and represents a peaked function of $x - x'$ with a characteristic width $x_c \equiv \omega_c / k'_\parallel v_e$.

It also can be shown that the solution for the nonlinear electron response, $\tilde{h}_{e_k}$, Eq. (2.18), is equivalent to the solution of the following equation:

$$\left(\frac{\partial}{\partial t} + v_{||}\mathbf{b}\cdot\nabla - D\frac{\partial^2}{\partial x^2}\right)\tilde{h}_e = S(x). \tag{2.20}$$

where $S(x)$ is defined in Eq. (2.11). By comparing the above equation to Eq. (2.11), it is clear that the effect of applying the NSA to the drift kinetic equation is to convert the terms in the orbit operator [the left hand side of Eq. (2.11)] involving the fluctuation potentials into a spatial diffusion operator. This is a reflection of the physical assertion that the major contribution of these terms in the orbit operator is to produce stochastic orbits leading to turbulent spatial diffusion.

In the case of the tearing mode, the drive term $S(x)$ is given by the right hand side of Eq. (2.11). When the source terms, $S(x)$, appear in the combination $(\tilde{\phi} - (v_{||}/c)\tilde{A}_{||})$, then the coupled system obtained from quasineutrality and Ampere's law is self–adjoint. The last source term in the above equation destroys this property. However, for the case of the tearing mode, this term can be dropped to give a self–adjoint system. This is due to the observation that the coupled equations obtained from Ampere's law and quasineutrality reduce to ideal MHD at large x (at large distances from the rational surface, $x = 0$), which implies $\tilde{E}_{||} = 0$ at large x. At small x (inside the dissipative layer), the dissipative process of electron diffusion is insensitive to an equilibrium current.

Hence, we have for the nonlinear electron response

$$\tilde{h}_{e_k} = \int_0^\infty d\tau \int_{-\infty}^\infty dx' G(x,x';v_{||},\tau)\left\{i\left[(\omega-\omega_{*e}) - \omega_{*e}\eta_J\frac{v_{||}}{v_e}\right]\left(\tilde{\phi} - \frac{v_{||}}{c}\tilde{A}_{||}\right)\frac{e\overline{f_0}}{T_e}\right\} \tag{2.21}$$

where

$$\omega_{*e} \equiv \frac{ek_y T_e}{cBL_n}, \quad L_n \equiv -\frac{d\ln f_0}{dx} \quad \text{and} \quad \eta_J = \frac{v_e}{v_\parallel}\frac{\partial f_1/\partial x}{\partial f_0/\partial x} = \frac{-2J'_\parallel}{env_e(\ln f_0)'}.$$

Defining the resonant operators

$$R_n[\psi] = \int_{-\infty}^{\infty} dv_\parallel \int_{-\infty}^{\infty} dx' \int_0^{\infty} d\tau G(x, x'; v_\parallel, \tau)\left(\frac{v_\parallel}{v_e}\right)^n f_0(v_\parallel)\psi(x') \tag{2.22}$$

then the perturbed electron density is given by

$$\frac{\tilde{n}_e}{n_0} = \frac{e}{T_e}\left\{\tilde{\phi} + i[(\omega-\omega_{*e})R_0 - \omega_{*e}\eta_J R_1]\tilde{\phi} - i[(\omega-\omega_{*e})R_1 - \omega_{*e}\eta_J R_2]\frac{v_e}{c}\tilde{A}_\parallel\right\} \tag{2.23}$$

and the perturbed parallel electron current is given by

$$\begin{aligned}\tilde{J}_{\parallel_e} = \frac{-ie^2 n_0}{T_e} v_e \Big\{&[(\omega-\omega_{*e})R_1 - \omega_{*e}\eta_J R_2]\tilde{\phi} \\ &-[(\omega-\omega_{*e})R_2 - \omega_{*e}\eta_J R_3]\frac{v_e}{c}\tilde{A}_\parallel\Big\}\end{aligned} \tag{2.24}$$

where the appropriate velocity moments of $\tilde{h}_e$ have been performed. Utilizing an appropriate ion response and applying Ampere's law and quasineutrality yields a coupled system for the potentials $\tilde{\phi}$ and $\tilde{A}_\parallel$.

2.2. Ion Response

The ion response is assumed to be "classical"; that is, the response is described by the linearized Vlasov equation. Although the island overlap condition is satisfied for the ions, the resulting turbulent diffusion has no appreciable effect on the ion response. The end result of turbulent diffusion is a smoothing of the structure of the response functions over a scale length x_c. Since the ion response is characterized by a scale length x_i satisfying $x_i > x_c$, the ion response virtually reduces to the linearized Vlasov response. For slab geometry with a density gradient in the x–direction, the linearized Vlasov equation is solved by using the standard techniques [60] to yield, after integrating over perpendicular velocities,

$$\tilde{f}_i(v_\parallel) = -\frac{e\tilde{\phi}}{T_i}\overline{f}_{i_0}(v_\parallel) + \tilde{g}_i(v_\parallel), \tag{2.25}$$

where

$$\tilde{g}_i(v_\parallel) = -\frac{ie\overline{f}_{0_i}(v_\parallel)}{T_i}(\omega - \omega_{*i})\int dk_x e^{ik_x}\hat{U}\left[\frac{i\Gamma_0}{\omega - k_\parallel v_\parallel} + k'_\parallel k_x \rho_i^2 \frac{\Gamma'_0 v_\parallel}{(\omega - k_\parallel v_\parallel)^2 c}\right]$$

$$\hat{U} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx' e^{-ik_x x'}\left[\tilde{\phi}(x') - \frac{v_\parallel}{c}\tilde{A}_\parallel(x')\right].$$

From this the perturbed density and current can be calculated by taking the appropriate parallel velocity moments. The perturbed ion density is given by

$$\frac{\tilde{n}_i}{n_0} = -\frac{e}{T_i}\left\{\tilde{\phi} + \frac{\omega - \omega_{*i}}{\omega}\left[\Gamma_0 \varsigma Z(\varsigma)\tilde{\phi} + (\Gamma_0 - \Gamma_1)\rho_i^2 \frac{d}{dx}\left(\varsigma Z \frac{d\tilde{\phi}}{dx}\right)\right]\right.$$

$$\left. - \frac{\omega - \omega_{*i}}{\omega}\Gamma_0 \varsigma(1 + \varsigma Z)\frac{v_i}{c}\tilde{A}_\parallel\right\} \tag{2.26}$$

and the perturbed parallel ion current is given by

$$\tilde{J}_{\|_i} = -\frac{e^2 n_0 v_i}{\Gamma_i}\left\{\frac{\omega-\omega_{*i}}{\omega}\varsigma(1+\varsigma Z)\Gamma_0\left(\tilde{\phi}-\frac{v_i}{c}\varsigma\tilde{A}_{\|}\right)\right\}. \tag{2.27}$$

In the above expressions, $Z(\varsigma)$ is the plasma dispersion [61] function with $\varsigma = \omega/(|k_{\|}|v_i)$ and $\Gamma_n = \Gamma_n(b) = I_n(b)\exp(-b)$ where $I_n(b)$ is the modified Bessel function with $b = k_y^2\rho_i^2$.

In deriving the above expression for the perturbed ion density, Eq. (2.26), the electric potential, $\tilde{\phi}(x')$, was expanded about $x' = x$ to second order thus yielding the second derivative term in Eq. (2.26). This expansion is only valid when $x/x_i < 1$, $x_i \equiv \omega/k'_{\|}v_i$. Hence, in evaluating the above ion Z–functions, the large argument expansions (for $\varsigma > 1$) should be used.

2.3. Coupled Equations

Knowing the perturbed density and current for both the electrons and ions, a coupled system of equations for $\tilde{\phi}$ and $\tilde{A}_{||}$ can be formed by utilizing quasineutrality and Ampere's law.

Quasineutrality states that $\tilde{n}_i = \tilde{n}_e$. Using Eqs. (2.23) and (2.26), then this implies

$$\left[\frac{d}{dx}\varsigma Z(\varsigma)\frac{d}{dx} + (\Lambda + \chi^2) + \frac{i}{d}(\omega - \omega_{*e})R_0 - \frac{i}{d}\omega_{*e}\eta_J R_1\right]\tilde{\phi}$$
$$- \left[\frac{i}{d}(\omega - \omega_{*e})R_1 - \frac{i}{d}\omega_{*e}\eta_J R_2 + x_e\chi^2\right]\frac{v_e}{c}\tilde{A}_{||} = 0. \quad (2.28)$$

The parallel component of Ampere's law states $\nabla^2\tilde{A}_{||} + (4\pi/c)(\tilde{J}_{||e} + \tilde{J}_{||i}) = 0$. Utilizing Eqs. (2.24) and (2.27) gives

$$\frac{\tau v_A^2}{dc^2}\left[\frac{d^2}{dx^2} - b + \alpha_i^2 + \frac{v_e^2}{\tau v_A^2}i(\omega - \omega_{*e})R_2 - \frac{v_e^2}{\tau v_A^2}i\omega_{*e}\eta_J R_3\right]\tilde{A}_{||}$$
$$- \left[\frac{i}{d}(\omega - \omega_{*e})R_1 - \frac{i}{d}\omega_{*e}\eta_J R_2 + x_e\chi^2\right]\frac{v_e}{c}\tilde{\phi} = 0. \quad (2.29)$$

In the above expressions, the following definitions were used:

$$d = (\Gamma_0 - \Gamma_1)\left(\tau + \frac{\omega_{*e}}{\omega}\right), \quad \tau = T_e/T_i$$
$$\Lambda = \frac{1}{d}\left(1 + \tau - \Gamma_0\left(\tau + \frac{\omega_{*e}}{\omega}\right)\right),$$
$$\chi^2 = \frac{\Gamma_0}{d}\left(\tau + \frac{\omega_{*e}}{\omega}\right)(1 + \varsigma Z)$$
$$\alpha_i^2 = \frac{v_i^2}{\tau v_A^2}\left(\tau + \frac{\omega_{*e}}{\omega}\right)\varsigma^2(1 + \varsigma Z)\Gamma_0, \quad b = \rho_i^2 k_y^2$$

where $\zeta = \omega/(|k_{||}|v_i)$, $x_e = \omega/(|k_{||}|v_e)$, and Z = plasma dispersion function. In the above expressions, R_n denote the electron resonance operators and are defined in Eq. (2.22), and x is normalized in units of the ion gyroradius ρ_i.

The above system is a kinetic description globally valid over the entire plasma which reduces to ideal MHD at large distances from the rational surface (large x), and is self–adjoint. When analyzed for high m modes this system yields unstable finite–β drift waves, and when analyzed for low m modes this system yields the tearing mode.

It is convenient to abbreviate the above system in the following shorthand notation:

$$\begin{aligned} L_1\tilde{\phi} + L_x\tilde{A}_{||} &= 0 \\ L_2\tilde{A}_{||} + L_x\tilde{\phi} &= 0 \end{aligned} \tag{2.30}$$

where

$$\begin{aligned} L_1 &= \left[\frac{d}{dx}\zeta Z\frac{d}{dx} + \Lambda + \chi^2 + \frac{i}{d}(\omega - \omega_{*e})R_0 - \frac{i}{d}\omega_{*e}\eta_J R_1\right] \\ L_2 &= \frac{\tau v_A^2}{dc^2}\left[\frac{d^2}{dx^2} - b + \alpha_i^2 + \frac{v_e^2}{\tau v_A^2}i(\omega - \omega_{*e})R_2 - \frac{v_e^2}{\tau v_A^2}i\omega_{*e}\eta_J R_3\right] \\ L_x &= -\left[\frac{i}{d}(\omega - \omega_{*e})R_1 - \frac{i}{d}\omega_{*e}\eta_J R_2 + x_e\chi^2\right]\frac{v_e}{c}. \end{aligned}$$

Since the above system is self-adjoint, a variational principle can be formed. The variational integral, S, is obtained by

$$S = -\int_{-a}^{a} dx\left[\tilde{\phi}L_1\tilde{\phi} + \tilde{A}_{||}L_2\tilde{A}_{||} + 2\tilde{\phi}L_x\tilde{A}_{||}\right], \tag{2.31}$$

such that variation of S with respect to $\tilde{\phi}$ and $\tilde{A}_{\|}$ yields the coupled Eqs. (2.30). The variational parameters characterizing suitable trial functions, $\tilde{\phi}_T$ and $\tilde{A}_{\|T}$, are thus determined by requiring $\delta S = 0$. Once these variational parameters have been determined, the trial functions are again inserted into Eq. (2.31) and the dispersion relation for these trial functions is determined by setting $S = 0$. This variational principle will be utilized later in a calculation of the dispersion relation for the tearing mode.

2.4 Resonance Operators

As a closing comment on the mathematical model, two approximate forms for the resonant operators $R_n[\psi]$, defined by Eq. (2.22), will be presented. Such approximate forms of $R_n[\psi]$ are necessary in order to perform an analytical calculation of the dispersion relation. The first approximation for R_n are interpolation polynomials which preserve the asymptotic behavior of R_n. The second is the so-called "Krook" approximation in which R_n is approximated in terms of the plasma dispersion function.

In the above fully kinetic analysis, calculation of the nonlinear electron response led to the resonance operators, $R_n[\psi]$, as given by Eq. (2.22). In the limit that $\psi(x')$ is a slowly varying function of x', that is, if $x_c/x_T < 1$, where $x_T^{-1} = |d/dx(\ln\psi)|$, then $\psi(x')$ can be expanded about $x' = x$. To leading order in x_c/x_T the above resonance operators can be replaced by multiplicative operators of the form $R_n[\psi] \simeq I_n(x)\psi(x)$, where

$$I_n(x) = \int_{-\infty}^{\infty} dv_{\|} \left(\frac{v_{\|}}{v_e}\right)^n f_0(v_{\|}) \int_0^{\infty} d\tau \exp\left[i(\omega - k'_{\|}v_{\|}x)\tau - \frac{1}{3}(k'_{\|}v_{\|})^2 D\tau^3\right]. \quad (2.32)$$

The above multiplicative resonance functions can be evaluated asymptotically to yield the following interpolation functions [19]:

$$\begin{aligned}
I_0 &\simeq \frac{2.8}{\omega_c}\left[1 + 1.58\left(\frac{|x|}{x_c}\right)\right]^{-1} \\
I_2 &\simeq \frac{.47}{\omega_c}\left[1 + \left(\frac{x}{x_c}\right)^2\right]^{-1} \\
I_1 &\simeq \frac{-.29i}{\omega_c}\frac{x}{x_c}\left[1 + .29\left(\frac{x}{x_c}\right)^2\right]^{-1} \\
I_3 &\simeq \frac{-.24i}{\omega_c}\frac{x}{x_c}\left[1 + .48\left(\frac{x}{x_c}\right)^2\right]^{-1}
\end{aligned} \quad (2.33)$$

This above asymptotic behavior can be reproduced qualitatively [19] if the exponential in Eq. (2.33) involving the factor $-1/3(k'_{\|}v_{\|})^2 D\tau^3 \simeq -\omega_c^3\tau^3$ is replaced by the linear factor $-.36\omega_c\tau$. Under this approximation, the above resonance functions, $I_n(x)$, can be evaluated exactly in terms of the Z–function. This approximation will be referred to as the Krook limit.

$$I_n(x)_{Krook} = \int_{-\infty}^{\infty} dv_{\|} \left(\frac{v_{\|}}{v_e}\right)^n f_0(v_{\|}) \int_0^{\infty} d\tau \exp[i(\omega - k'_{\|}v_{\|}x)\tau - .36\omega_c\tau]$$
$$= \int_{-\infty}^{\infty} dv_{\|} \left(\frac{v_{\|}}{v_e}\right)^n \frac{if_0(v_{\|})}{\omega_0 - k'_{\|}v_{\|}x} \tag{2.34}$$

where $\omega_0 = \omega + i0.36\omega_c$. These integrals can be evaluated in terms of the Z–functions [61].

$$I_0 = \frac{-ix_c}{\omega_c x} Z(\varsigma_e)$$
$$I_1 = \frac{-ix_c}{\omega_c x} (1 + \varsigma_e Z)$$
$$I_2 = \frac{-ix_c}{\omega_c x} \varsigma_e (1 + \varsigma_e Z)$$
$$I_3 = \frac{-ix_c}{\omega_c x} \left[\frac{1}{2} + \varsigma_e^2 (1 + \varsigma_e Z)\right] \tag{2.35}$$

where $\varsigma_e = \omega_0/|k_{\|}|v_e$. Essentially, under the Krook approximation, the turbulent spatial diffusion operator in Eq. (2.20), $-D\partial^2/\partial x^2$, is replaced by a Krook-type diffusion frequency, $0.36\omega_c$.

Chapter 3

STABILIZATION BY TURBULENT ELECTRON DIFFUSION

This chapter analyzes the tearing mode using the kinetic equations containing the effects of turbulent electron diffusion which were derived in Ch. 2. In Sec. 3.1, special limits of the coupled equations governing the potentials $\tilde{\phi}$ and $\tilde{A}_{||}$, Eqs. (2.30), are taken in order to obtain the standard results for the resistive MHD tearing mode. This helps illuminate which approximations and conditions resistive MHD assumes, and points out the physics neglected by such a model. Section 3.2 presents a formal proof demonstrating that the tearing mode is stabilized by sufficiently large turbulent electron diffusion. This is done by showing that the energy drive for the tearing mode goes through zero and becomes stabilizing as the electron diffusion coefficient becomes infinite. An approximate dispersion relation for the tearing mode is calculated in Sec. 3.3. This is done for the "magnetic" tearing mode described by $L_2\tilde{A}_{||} = 0$, where the effects of a finite electrostatic potential $\tilde{\phi}$ are neglected. Section 3.4 presents the calculation of the dispersion relation for the electromagnetic tearing mode including the effects of a finite electrostatic potential. This final dispersion relation is obtained through a variational calculation.

3.1 Resistive Tearing Mode Limit

In this section, a special "resistive limit" of Eqs. (2.30) is taken to show that in such an appropriate limit the basic results of resistive MHD [3] can be reproduced for the tearing mode. As a starting point, the perturbed electron distribution must be calculated in this limit. Recall that in the above fully kinetic analysis, calculation of the nonlinear electron response led to the resonance operators, $R_n[\psi]$, as given by Eq. (2.22). As discussed in Sec. 2.4, in the limit that $\psi(x')$ is a slowly varying function of x', the above resonance operators can be replaced by multiplicative operators of the form $R_n[\psi] \simeq I_n(x)\psi(x)$, where $I_n(x)$ is given by Eq. (2.32). In the Krook limit, the asymptotic behavior of the functions $I_n(x)$ can be reproduced qualitatively [19] if the exponential in Eq. (2.32) involving the factor $-1/3(k'_{\|}v_{\|})^2 D\tau^3 \simeq -\omega_c^3\tau^3$ is replaced by the linear factor $-.36\omega_c\tau$. Under this approximation, the above resonance functions, $I_n(x)$, can be evaluated exactly in terms of the plasma dispersion function as indicated in Eqs. (2.34) and (2.35). Essentially, the Krook approximation replaces the turbulent spatial diffusion operator in Eq. (2.20), $-D\partial^2/\partial x^2$, by a Krook type diffusion frequency, $0.36\omega_c$.

In light of the above discussion, an approximate resistive limit can be obtained by replacing ω_c with a collision frequency, ν. Thus, the nonlinear electron response in this simple collisional limit is given by Eq. (2.20) with $-D\partial^2/\partial x^2 \rightarrow \nu$. The inversion of this operator gives

$$\tilde{h}_{e_k} = \frac{iS_k(x)}{\omega - k'_{\|}v_{\|}x + i\nu}. \tag{3.1}$$

Hence, in this resistive limit the above resonance operators, R_n, are identical to the previous Krook approximate resonance functions, I_n, given by Eq. (2.35), only now ω_c is replaced by ν, that is, $\omega_0 = \omega + i\nu$.

In the resistive MHD problem, it is assumed that a narrow "tearing" region of width $2x_0$ exists about the singular surface. Inside this tearing region the physics

is dominated by "non-ideal" dissipative effects. In the outer region away from the rational surface, the physics reduces to ideal MHD. In order to obtain the resistive MHD results, it is appropriate to consider the following scaling such that $\omega/(k_{\|}v_T) > 1$ in the inner region for both electrons and ions, whereas $\omega/(k_{\|}v_T) < 1$ in the outer region for both electrons and ions.

In the outer region, $x > x_0$, the density fluctuations to leading order in $\omega/(k'_{\|}xv_T)$ are

$$\frac{\tilde{n}_i}{n_0} \simeq -\frac{e}{T_e}\left[\tau\tilde{\phi} - \frac{\tau\omega + \omega_{*e}}{k_{\|}c}\tilde{A}_{\|}\right] \quad \text{and} \quad \frac{\tilde{n}_e}{n_0} \simeq \frac{e}{T_e}\left[\tilde{\phi} - \frac{\omega - \omega_{*e}}{k_{\|}c}\tilde{A}_{\|}\right]. \tag{3.2}$$

Hence, quasineutrality, $\tilde{n}_e = \tilde{n}_i$, implies that in the outer region $\tilde{E}_{\|} = 0$, which is the ideal MHD result.

Similarly, in the outer region, $x > x_0$, one find Ampere's law, reducing in leading order to

$$\frac{\tau v_A^2}{dc^2}\left[\frac{d^2}{dx^2} - k_y^2 - \frac{4\pi}{c}\frac{k_y J'_{\|}}{k_{\|}B_0}\right]\tilde{A}_{\|} = 0, \tag{3.3}$$

which is the equation for marginal stability in ideal MHD. Thus, in this resistive limit, Eqs. (2.30) reduce to the ideal MHD forms at large x, $x > x_0$. (This is also the case for the fully kinetic description).

A dispersion relation for the basic resistive tearing mode can be derived by utilizing the variational integral given above in Eq. (2.31). (This method [37] is similar to that performed in Sec. 1.2.) In the outer region, the resistive limit gives the following energy integral:

$$S_{out} = \int_{|x|>x_0} dx \frac{\tau v_A^2}{dc^2}\left[\tilde{A}'^2_{\|} + \left(b + \frac{4\pi}{c}\rho_i^2\frac{k_y J'_{\|}}{k_{\|}B_0}\right)\tilde{A}^2_{\|}\right], \tag{3.4}$$

where the condition $\tilde{E}_{\|} = 0$ has been imposed as well as requiring $\tilde{A}_{\|}(\pm a) = 0$, $a =$ plasma edge.

Integrating by parts and requiring $\tilde{A}_{||}$ to satisfy the ideal form of Ampere's law, Eq. (3.3), then gives in the asymptotic limit of $x_0 \to 0$

$$S_{out} = -\frac{\tau v_A^2}{dc^2} A_{||_0}^2 \Delta_0' \tag{3.5}$$

where $\lim_{x_0 \to 0}[A_{||}A_{||}'(x_0) - A_{||}A_{||}'(-x_0)] \equiv A_{||_0}^2 \Delta_0'$ and $A_{||}(x_0 \pm) \equiv A_{||_0}$. Here, Δ_0' represents the jump in $\tilde{A}_{||}'$ across the singular surface (for nearly constant $A_{||}$ or the "constant ψ" approximation) [3].

Similarly, in the inner region, $x < x_0$, using the ordering $\omega/k_{||}v_e, \omega/k_{||}v_i > 1$ gives the leading order form of the variation integral, Eq. (2.31), as

$$S_{in} = \int_{-x_0}^{x_0} dx \left[-\tilde{\phi}'^2 + \frac{v_e^2}{dc^2}(\omega - \omega_{*e})\frac{1}{2\omega_0}\tilde{A}_{||}^2 \right] \tag{3.6}$$

where $d \simeq (\Gamma_0 - \Gamma_1)(\tau + \omega_{*e}/\omega)$. Here the first term in S_{in} is the leading order contribution from quasineutrality and the second term the leading order contribution from Ampere's law.

To evaluate the above integral for S_{in}, it is necessary to choose suitable trial functions for $\tilde{\phi}$ and $\tilde{A}_{||}$. As is standard in the basic resistive MHD calculations, $\tilde{A}_{||}$ is chosen to be constant across the inner region (the "constant ψ" approximation). Likewise, $\tilde{\phi}$ is chosen to be a linear function of x with odd parity such that $\tilde{E}_{||} = 0$ at $x = \pm x_0$. Thus, for the inner region, $x < x_0$, $\tilde{A}_{||} = A_{||_0}$ and $\tilde{\phi} = (x/x_0)\phi_0$, where $\phi_0 = \omega/(k_{||}' x_0 c) A_{||_0}$.

Using the above trial functions, the integral over the inner region can be evaluated to give

$$S_{in} = -\left(\frac{\omega A_{||_0}}{k_{||}' \rho_i c}\right)^2 \frac{2}{x_0^3} + \frac{v_e^2}{dc^2}(\omega - \omega_{*e})\frac{x_0}{\omega_0} A_{||_0}^2. \tag{3.7}$$

Combining Eqs. (3.5) and (3.7) through the relation $S_0 = S_{out} + S_{in}$ gives the dispersion relation by setting $S_0 = 0$:

$$\frac{\tau v_A^2}{c^2}\Delta_0' = -\frac{2\tau\omega(\omega-\omega_{*i})}{k_\parallel'^2\rho_i^2c^2x_0^3} + \frac{v_e^2}{c^2}\frac{\omega-\omega_{*e}}{\omega+i\nu}x_0. \tag{3.8}$$

To obtain the classical resistive tearing mode growth rate, one sets $\omega_{*e} = \omega_{*i} = 0$, $\omega = i\gamma$ and assumes $\nu \gg \gamma$. Thus, the above equation becomes

$$\frac{\tau v_A^2}{c^2}\Delta_0' = \frac{2\tau\gamma^2}{k_\parallel'^2\rho_i^2c^2}\frac{1}{x_0^3} + \frac{v_e^2}{c^2}\frac{\gamma}{\nu}x_0 \tag{3.9}$$

which can be interpreted as an energy balance relation. This is the basic dispersion relation which specifies the resistive MHD growth rate.

It can be shown through simple physical arguments that the term on the left of Eq. (3.9) involving Δ_0' represents the available magnetic energy in the outer region which thus provides the drive for the tearing mode. (A detailed study of the magnetic energy drive is given by Adler et al. [28]). The first term on the right represents the energy absorbed in ion inertia and the second term represents resistive electron dissipation. Both terms are zero unless the perturbation is growing, $\gamma > 0$. Hence Eq. (3.9) signifies a balance between the magnetic energy in the outer region, which drives the tearing mode, with the ion inertia and electron dissipation in the inner region.

The variational parameter in Eq. (3.9) is the tearing width, x_0. Varying S_0 with respect to x_0 then gives

$$x_0 = \left[\frac{6\tau\nu\gamma}{\rho_i^2k_\parallel'^2v_e^2}\right]^{1/4}. \tag{3.10}$$

Inserting this into Eq. (3.9) then gives the growth rate [3]

$$\gamma = .56\Delta_0'^{4/5}\nu^{3/5}\tau^{3/5}\left[\frac{\rho_ik_\parallel'v_A^4}{v_e^3}\right]^{2/5} \tag{3.11}$$

or in standard resistive MHD notation [27]

$$\gamma \simeq .37\tau_R^{-3/5}\tau_A^{-2/5}(\Delta_0' a)^{4/5}\left(\frac{a}{R}n\frac{aq'}{q}\right)^{2/5} \tag{3.12}$$

where $\tau_R = 4\pi a^2/(c^2\eta)$, $\tau_A = a/v_A$, and $\eta = 2m_e\nu/(e^2 n_0)$. This expression for the growth rate is identical to that obtained in Sec. 1.2. More accurate calculations [27,38] indicate the factor .37 should be replaced by .55. This numerical difference is a result of the variational approximation method used in the above calculation.

Hence, by implementing the above limit of constant resistivity into the Krook model for the electron response, the coupled Eqs. (2.30) for $\tilde{\phi}$ and $\tilde{A}_{||}$ were shown to contain the fundamental resistive tearing mode in the appropriate limits. It is important to note that to obtain the above resistive limit the following assumptions were made:

(a) In the evaluation of S_{out}, the limit $x_0 \to 0$ was taken yielding $S_{out} \sim \Delta_0' =$ constant, independent of x_0.

(b) Finite Larmor radius effects and other "diamagnetic" effects were neglected; that is, $\omega_* \equiv 0$.

3.2 Proof of Stability for Large Turbulent Diffusion

Recall from the above discussion of the resistive tearing mode that it is the available magnetic energy of the outer region, represented by Δ'_0, which drives the tearing mode unstable. From this, a general energy drive [62] for the tearing mode is then defined to be the negative energy from the integral

$$S = \int_{-a}^{a} dx \left[\tilde{A}_{||}'^2 + \rho_i^2 k_y^2 \tilde{A}_{||}^2 - \frac{4\pi}{c} \rho_i^2 \tilde{J}_{||e}[\tilde{A}_{||}] \cdot \tilde{A}_{||} \right] \tag{3.13}$$

where $\tilde{J}_{||e}$ is the full kinetic operator representing the perturbed parallel current

$$\frac{4\pi}{c} \rho_i^2 \tilde{J}_{||e}[\tilde{A}_{||}] = -\frac{iv_e^2}{\tau v_A^2} \omega_{*e} \eta_J R_3[\tilde{A}_{||}].$$

Hence, $S < 0$ is a necessary condition for instability.

In the resistive MHD limit, $R_3 = -i/2k_{||}v_e$, and the negative energy drive takes on the ideal MHD form as indicated by the expression

$$S_{resistive} = -A_{||0}^2 \Delta'_0. \tag{3.14}$$

Hence, in resistive MHD, Δ'_0 represents the available magnetic energy to drive the tearing mode. Instability is then obtained when $\Delta'_0 > 0$.

Notice that in the energy integral given by Eq. (3.13), the first two terms are stabilizing and it is only the third term involving the perturbed current operator which can be negative and thus drive the tearing mode unstable. In the resistive MHD treatment, the perturbed current operator is singular at the rational surface. In fact, it is this singular behavior which dominates in the integral S and typically causes Δ'_0 to be positive (and hence S to be negative). However, if one retains the full kinetic operator for $\tilde{J}_{||e}[\tilde{A}_{||}]$ involving the resonance operator $R_3[\tilde{A}_{||}]$, then this operator is no longer singular at $x = 0$. In fact, one can easily show that the integral

$$S_J \equiv \int_{-a}^{a} dx \frac{4\pi}{c} \rho_i^2 \tilde{J}_{\|e}[\tilde{A}_{\|}]\tilde{A}_{\|} \tag{3.15}$$

is bounded for $D > 0$. This will be demonstrated with the aid of the inequality [63]

$$\left| \int_a^b dt F(t) \right| \leq \int_a^b dt \, |F(t)|$$

where $a \leq b$. Hence, letting $c_0 = |4\pi\rho_i^2 v_e^2 \omega_{*e} \eta_J / (c\tau v_A^2)|$, then

$$\begin{aligned}
|S_J| &\leq c_0 \int_{-a}^{a} dx \left| R_3 \left[\tilde{A}_{\|}\right] \cdot \tilde{A}_{\|} \right| \\
&\leq c_0 \int_{-a}^{a} dx \, |I_3(x)| \left| \tilde{A}_{\|} \right|_{max}^2 \\
&= \frac{4a}{\sqrt{\pi}} c_0 \left| \tilde{A}_{\|} \right|_{max}^2 \int_0^{\infty} d\tau \int_0^{\infty} dy y^3 \exp\left[-y^2\left(1 + \omega_c^3 \tau^3\right)\right] \\
&= \frac{2a}{\sqrt{\pi}} c_0 \left| \tilde{A}_{\|} \right|_{max}^2 \frac{1}{\omega_c} \int_0^{\infty} dt \left(1 + t^3\right)^{-2}
\end{aligned}$$

where $\left| \tilde{A}_{\|} \right|_{max}^2$ is the maximum value attained with the plasma. Clearly, the last integral is finite and the conclusion is reached that

$$|S_J| \leq c_1 D^{-1/3}$$

where c_1 is the constant indicated by the previous equation. Hence, $|S_J| \to 0$ as $D \to \infty$. Since the energy drive S also contains the line bending stabilization terms, the conclusion is reached that for some critical diffusion coefficient, $D = D_{cr}$, the energy drive S will be zero and the tearing mode is stabilized.

3.3 Approximate Dispersion Relation

By ignoring the effects of the electrostatic potential, a simple expression for the growth rate [13] is obtained which shows that the tearing mode stabilizes when a critical value of the diffusion coefficient is surpassed. This analysis begins by returning to the set of coupled equations for $\tilde{A}_{||}$ and $\tilde{\phi}$, given by Eqs. (2.30).

For typical Alcator C parameters, one finds that the parameter $\epsilon \equiv \omega/\omega_c \sim \omega_{*e}/\omega_c \sim 10^{-1}$ is small, and it is possible to scale the above operators as follows: $L_1 \sim 1$, $L_2 \sim \epsilon$, $L_x \sim \epsilon$. Hence, the contribution of the coupling terms, L_x, to the dispersion relation are subdominant by order ϵ. As a first approximation, it is assumed that Eqs. (2.30) decouple, leaving the equations

$$L_1\tilde{\phi} = 0, \quad L_2\tilde{A}_{||} = 0. \tag{3.16}$$

Hence, defining the tearing mode as a primarily magnetic fluctuation reduces the problem to solving the equation $L_2\tilde{A}_{||} = 0$, which describes the "magnetic" tearing mode [13]. (The equation $L_1\tilde{\phi} = 0$ describes the electrostatic drift wave [15].) The contribution of the coupling terms as well as the electrostatic terms to the dispersion relation will be considered later in a variational calculation.

To leading order, the magnetic tearing mode is given by $L_2\tilde{A}_{||} = 0$ or

$$\left[\frac{d^2}{dx^2} - b + \frac{v_e^2}{\tau v_A^2} i(\omega - \omega_{*e}) R_2 - \frac{v_e^2}{\tau v_A^2} i\omega_{*e}\eta_J R_3\right] \tilde{A}_{||} = 0. \tag{3.17}$$

The approach will be to simplify and solve the above equation in two regions: $|x| > x_c$ and $|x| < x_c$. The dispersion relation will then be determined by matching $\tilde{A}_{||}$ and $\tilde{A}'_{||}$ at $|x| = x_c$.

Provided that $\tilde{A}_{||}$ is a sufficiently slowly varying function of x, then the resonance operators in Eq. (3.17) can be replaced by multiplicative functions, as discussed in Sec. 2.4. That is, if $x_c/x_T < 1$ then $R_n[\tilde{A}_{||}] \simeq I_n(x)\tilde{A}_{||}(x)$ where I_n is given by Eqs. (2.33). Noting this, then Eq. (3.17) can be approximated by its

leading order terms in an outer region, $|x| > x_c$, and likewise in an inner region, $|x| < x_c$. Hence,

$$\text{For} \quad |x| > x_c, \quad \left[\frac{d^2}{dx^2} - b - \frac{4\pi}{c}\rho_i^2 \frac{k_y J_{||}'}{k_{||} B_0}\right] \tilde{A}_{||} = 0 \tag{3.18}$$

$$\text{For} \quad |x| < x_c, \quad \left[\frac{d^2}{dx^2} + \frac{v_e^2}{\tau v_A^2} i(\omega - \omega_{*e}) \frac{.47}{\omega_c}\right] \tilde{A}_{||} = 0. \tag{3.19}$$

Notice that in the outer region, $|x| > x_c$, Ampere's law has reduced to the marginal stability equation of ideal MHD. Formally, the solution to Eq. (3.18) in the outer region for $|x| > x_c$ will be denoted as the ideal MHD solution, A_{MHD}.

Defining

$$\alpha^2 = \frac{-v_e^2}{\tau v_A^2} i(\omega - \omega_{*e}) \frac{.47}{\omega_c} \simeq \frac{v_e^2}{\tau v_A^2} .47 \frac{\gamma}{\omega_c}, \quad \text{for} \quad \omega \simeq \omega_{*e} + i\gamma,$$

then, in the inner region, $|x| < x_c$, one has the solution

$$A_{IN} = C_1 e^{-\alpha x} + C_2 e^{+\alpha x}.$$

To determine the dispersion relation, matching conditions are imposed at $|x| = x_c$ requiring $A_{||}$ and $A_{||}'$ to be continuous. That is,

$$A_{IN}(\pm x_c) = A_{MHD}(\pm x_c) \equiv A_{\pm}(\pm x_c)$$

$$A_{IN}'(\pm x_c) = A_{MHD}'(\pm x_c) \equiv A_{\pm}(\pm x_c).$$

By requiring $A_{\pm}(\pm a) = 0$, Eq. (3.18) can be integrated inwards to determine the ratios

$$\Delta_{\pm} = \Delta_{\pm}(\pm x) = \frac{A_{\pm}'(\pm x)}{A_{\pm}(\pm x)}.$$

Using the above four matching conditions, C_1 and C_2 can be solved for in terms of Δ_+ and Δ_- leading to the dispersion relation

$$\exp(4\alpha x_c) = \frac{(\alpha+\Delta_+)^2-(\alpha-\Delta_-)^2}{(\alpha^2-\Delta_+^2)(\alpha^2-\Delta_-^2)^2}. \tag{3.20}$$

Typical Alcator C parameters indicate $\Delta_\pm \sim 10^{-2}$ and $x_c \sim 10$. Assuming the scaling $\alpha^2 x_c \sim \Delta_\pm$ implies $\Delta_\pm^2/\alpha^2 \sim 10^{-1}$. Using this fact, along with the definition $\Delta_+(x_c) - \Delta_-(x_c) \equiv \Delta'(x_c)$, the above dispersion relation can be expanded to yield

$$\Delta'(x_c) \simeq -i\frac{v_e^2}{\tau v_A^2}\frac{(\omega-\omega_{*e})}{k'_{||}\rho_i v_e}. \tag{3.21}$$

This implies that the magnetic tearing mode has a real frequency equal to the electron diamagnetic frequency, ω_{*e}, with a growth rate [13] given by

$$\gamma \simeq \frac{\tau v_A^2}{v_e^2}k'_{||}\rho_i v_e \Delta'(x_c) = k'_{||}v_e\frac{c^2}{\omega_{pe}^2}\frac{\Delta'(x_c)}{\rho_i}. \tag{3.22}$$

The above expression is essentially the same as that for the collisionless tearing mode [8] with Δ'_0 replaced by $\Delta'(x_c)$. Hence, instability will occur if $\Delta'(x_c) > 0$. Note, however, that $\Delta'(x_c)$ is now evaluated at a finite distance x_c from the rational surface. Numerical calculations of $\Delta'(x_c)$ for cylindrical geometry [30] and typical Alcator C profiles indicate it to be a primarily linear decreasing function of x, $\Delta'(x) \simeq \Delta_0\,(1 - x/W)$. Here, W corresponds to the island saturation width [20] of resistive MHD. Hence, the tearing mode will be stabilized if $\Delta'(x_c) < 0$ or $x_c > W$. Since $x_c \sim D^{1/3}$, this implies that if $D \sim 1/n$, then there exists a density threshold which must be surpassed before instability occurs.

It is also insightful to present a second approximate calculation to obtain the dispersion relation given by Eq. (3.21). This involves the use of the energy integral S given by Eq. (3.13). This integral is divided into an inner region S_{IN} for $|x| < x_c$ and an outer region S_{OUT} for $|x| > x_c$. The perturbed parallel current $\tilde{J}_{||e}$ is approximated in these two regions in a manner similar to that used to obtain

Ampere's law in the inner and outer region as expressed by Eqs. (3.18) and (3.19). This then gives

$$S_{OUT} = \int_{|x|>x_c} dx \left[\tilde{A}_{||}'^2 + b^2 \tilde{A}_{||}^2 + \frac{4\pi}{c} \rho_i^2 \frac{k_y J_{||}'}{k_{||} B_0} \tilde{A}_{||}^2 \right] \tag{3.23}$$

and

$$S_{IN} = \int_{-x_c}^{x_c} dx \left[\tilde{A}_{||}'^2 - \frac{v_e^2}{\tau v_A^2} \frac{.47}{\omega_c} i (\omega - \omega_{*e}) \tilde{A}_{||}^2 \right]. \tag{3.24}$$

The integral S_{OUT} can be evaluated by requiring $\tilde{A}_{||}$ to satisfy the ideal MHD form of Ampere's law, Eq. (3.18). This then gives

$$S_{OUT} \simeq - \left[\tilde{A}_{||} \tilde{A}_{||}'(x_c) - \tilde{A}_{||} \tilde{A}_{||}'(-x_c) \right]_{MHD} \equiv -A_{||0}^2 \Delta'(x_c). \tag{3.25}$$

The integral S_{IN} is evaluated using the "constant ψ" approximation which states that $\tilde{A}_{||}$ is to be taken as a constant in the inner region. This then gives

$$S_{IN} \simeq - \frac{i v_e^2}{\tau v_A^2} \frac{x_c}{\omega_c} (\omega - \omega_{*e}) A_{||0}^2. \tag{3.26}$$

The dispersion relation is then obtained by setting $S = S_{IN} + S_{OUT} = 0$. This gives the dispersion relation obtained above, Eq. (3.21).

The advantage of using the energy integral formulation in the calculation of the dispersion relation is that it provides a physical interpretation of Eq. (3.21). Specifically, the disperison relation given by Eq. (3.21) represents a balance between the ideal MHD energy drive in the outer region [represented by the left side of Eq. (3.21)] with the energy dissipated in the inner region through turbulent diffusion [represented by the right side of Eq. (3.21)]. Notice that the dissipation is proportional to $(\omega - \omega_{*e})$. Hence, ω_{*e} can be viewed as a Doppler frequency shift resulting from the motion of the electrons relative to the laboratory frame. Physically, $\Delta'(x_c)$ is interpreted as the magnetic energy in the outer region available to drive the tearing mode. In essence, the turbulent electron diffusion prohibits the formation of, or flattens, the perturbed parallel current, $\tilde{J}_{||e}$, within a finite correlation distance, x_c, of the rational surface. This has the effect of reducing the available magnetic energy drive from the value $\Delta'(0)$ to $\Delta'(x_c)$.

3.4 Variational Calculation

A more detailed calculation of the dispersion relation [14] is now presented including the effects of the electrostatic coupling terms. This is done through the use of a variational principle. The starting point is again Ampere's law and quasineutrality, given by Eqs. (2.30). To leading order in ω_*/ω_c, the effects of the resonant operators can be neglected in the electrostatic operator, L_1, and the coupling operator, L_x. This gives

$$L_1 \simeq \left[\frac{d}{dx}\varsigma Z(\varsigma)\frac{d}{dx} + \Lambda + \chi^2\right], \quad L_x \simeq -\frac{v_e}{c}x_e\chi^2.$$

The expansion used to yield the differential operator in the expression for L_1 implicitly assumed that $|x/x_i| < 1$, that is, $\varsigma > 1$. Using the above approximations and expanding the ion Z-functions [61] for large arguments, quasineutrality can then be written as

$$\left[-\frac{d^2}{dx^2} + \Lambda - \frac{1}{2}\frac{x^2}{x_i^2}\right]\tilde{\phi} + \frac{1}{2}\frac{x^2}{x_i^2}\frac{\omega}{k_\| c}\tilde{A}_\| \simeq O\left(\frac{\omega_*^2}{\omega_c^2}\right) \tag{3.27}$$

where $x_i = \omega/(k'_\|\rho_i v_i)$. Scaling $\omega_*/\omega_c \sim \epsilon$, $\omega/\omega_* \sim 1$, $x^2/x_i^2 \sim \epsilon$ and $d^2/dx^2 \sim \epsilon^2$; then, for large x, $|x/x_c| > 1$, quasineutrality implies to leading order,

$$\Lambda \simeq 0 \rightarrow \omega \simeq \omega_{*e},$$

and to first order in ϵ quasineutrality implies

$$\tilde{\phi} - \frac{\omega}{k_\| c}\tilde{A}_\| \simeq 0 \rightarrow \tilde{E}_\| = 0.$$

Thus, in the outer region, the equation for quasineutrality requires $\omega \simeq \omega_{*e}$ and reduces to the ideal MHD constraint $\tilde{E}_\| \simeq 0$. Also at large x, $|x/x_c| > 1$, assuming $\tilde{E}_\| = 0$, Ampere's law reduces to the ideal MHD equation of marginal stability,

given by Eq. (3.18). Hence, the coupled Eqs. (2.30) reduce to ideal MHD with the additional constraint that $\omega \simeq \omega_{*e}$ in the outer region where $|x| > x_c$.

Since the equations governing $\tilde{A}_{||}$ and $\tilde{\phi}$ are self-adjoint, a simple variational integral, Eq. (2.31), is obtained. In this calculation, $\pm a$ represents the edges of an intermediate integration region which is assumed to lie in the region where ideal MHD is approximately valid (that is, $a > x_c$).

Integrating by parts, Eq. (2.31) becomes

$$S = \frac{\tau v_A^2}{dc^2}[\mathcal{L}_A - \beta_A] + [\mathcal{L}_\phi - \beta_\phi] \tag{3.28}$$

where

$$\mathcal{L}_\phi = \int_{-a}^{a} dx \left[\varsigma Z \left(\frac{d\tilde{\phi}}{dx} \right)^2 - \Lambda \tilde{\phi}^2 - \frac{\Gamma_0}{d} \left(\tau + \frac{\omega_{*e}}{\omega} \right) (1 + \varsigma Z) \left(\tilde{\phi} - \frac{\omega}{k_{||} c} \tilde{A}_{||} \right)^2 \right] \tag{3.29}$$

$$\mathcal{L}_A = \int_{-a}^{a} dx \left[\left(\frac{d\tilde{A}_{||}}{dx} \right)^2 + \left(b - \frac{v_e^2}{\tau v_A^2} i(\omega - \omega_{*e}) R_2 + \frac{v_e^2}{\tau v_A^2} i \omega_{*e} \eta_J R_3 \right) \tilde{A}_{||}^2 \right] \tag{3.30}$$

$$\beta_\phi = \varsigma Z \tilde{\phi} \tilde{\phi}' \Big|_{-a}^{a} \quad \text{and} \quad \beta_A = \tilde{A}_{||} \tilde{A}_{||}' \Big|_{-a}^{a}.$$

Here, $\mathcal{L}_A$ represents the contribution of the magnetic terms (involving the electron response only) with the corresponding boundary terms β_A. The contributions of the electrostatic, coupling and the ion terms are represented by $\mathcal{L}_\phi$ with the boundary terms β_ϕ. The magnetic tearing mode considered above in Sec. 3.3 is accounted for in the $\mathcal{L}_A$ and β_A terms. That is,

$$\mathcal{L}_A - \beta_A = \int_{-a}^{a} dx \tilde{A}_{||} L_2 \tilde{A}_{||}$$

where it has been shown above that $L_2\tilde{A}_{||} \quad 0$ yields the basic collisionless magnetic tearing mode [13]. Hence, $\mathcal{L}_\phi$ and β_ϕ represent the contributions to the dispersion relation which are due to coupling to the electrostatic branch as well as the ion contributions to the perturbed current.

At the boundary, $\pm a$, $\tilde{A}_{||}$ and $\tilde{\phi}$ are required to match onto the ideal MHD solution. In particular, at the boundary, $\tilde{E}_{||} = 0$. Using this condition, the magnitude of the boundary terms β_ϕ and β_A can be compared. Noting $\beta_\phi \sim (\omega/k'_{||}ca)^2 \tilde{A}_{||}\tilde{A}'_{||}$, then it is easy to show that the contribution of β_ϕ to the dispersion relation is smaller than that of β_A by ω_*^2/ω_c^2. Hence, the boundary term β_ϕ can be neglected.

The above variational integral will be performed using two steps. First, the electrostatic integral will be calculated using a suitable trial function for $\tilde{\phi}$ while assuming $\tilde{A}_{||}$ to be a constant in the $\tilde{\phi} - (\omega/k_{||}c)\ \tilde{A}_{||}$ term (which is justified by the fact that $\tilde{A}_{||}$ is nearly constant while $\tilde{\phi}$ must be linear in x near the rational surface). One can then solve for $\tilde{\phi}$ variationally. After this, the magnetic terms $\mathcal{L}_A - \beta_A$ are calculated using a linear trial function for $\tilde{A}_{||}$ while allowing the limit of integration, a, to be the variational parameter.

The electrostatic integral, $\mathcal{L}_\phi$, is given by Eq. (3.29). In the above expression, one must keep in mind that the ion Z-functions are to be expanded for large arguments. Here, Λ represents the eigenfrequency and is approximately zero when $\omega \simeq \omega_{*e}$ and hence will be neglected. The first term in Eq. (3.29) represents the $\tilde{\mathbf{E}} \times \mathbf{B}$ ion motion in the poloidal direction whereas the last term represents the ion motion along the field lines.

A trial function of the following form is chosen:

$$\tilde{\phi} = \frac{\omega}{k'_{||}c} A_0 \frac{x}{(x^2 \pm i\alpha^2)}, \quad A_0 \equiv \text{constant}. \tag{3.31}$$

Here, α is the variational parameter and it is assumed $(\alpha/a)^2 \ll 1$ such that at the boundary, $x = a$, one has $\tilde{E}_{||} \simeq 0$.

Inserting the above trial functions in Eq. (3.29) gives

$$\mathcal{L}_\phi = \left(\frac{\omega}{k'_\parallel c}\right)^2 A_0^2 \int_{-a}^{a} dx \left[\varsigma Z \frac{(\pm i\alpha^2 - x^2)^2}{(\pm i\alpha^2 + x^2)^4} + \frac{(1+\varsigma Z)}{x^2} \frac{\alpha^2}{(\pm i\alpha^2 + x^2)^2}\right]. \quad (3.32)$$

Using the large argument expansions for the Z–functions along with the variable substitution $y = x/\alpha$ and assuming $a/\alpha > 1$, then the limits of integration can be extended to infinity to yield

$$\mathcal{L}_\phi \simeq \left(\frac{\omega}{k'_\parallel c}\right)^2 A_0^2 \int_{-\infty}^{\infty} dy \left[-\frac{1}{\alpha^3} \frac{(\pm i - y^2)^2}{(\pm i + y^2)^4} - \frac{1}{2x_i^2} \frac{\alpha}{(\pm i + y^2)^2}\right].$$

These integrals can be evaluated explicitly (see Ch. 4 for details) to give the expression

$$\mathcal{L}_\phi \simeq \left(\frac{\omega}{k'_\parallel c}\right)^2 A_0^2 \frac{\sqrt{2}}{8} \pi(1 \pm i) \left[\frac{1}{\alpha^3} + \frac{\alpha}{x_i^2}\right]. \quad (3.33)$$

The variational parameter is then specified according to $\delta\mathcal{L}_\phi/\delta\alpha = 0$. This gives $\alpha^2 = \sqrt{3}x_i$. Inserting this expression for α^2 back into Eq. (3.33) then gives

$$\mathcal{L}_\phi \simeq \frac{\pi}{\sqrt{2}3^{3/4}} A_0^2 \left(\frac{v_i}{c}\right)^{3/2} \left(\frac{\omega}{k'_\parallel \rho_i c}\right)^{1/2} (1 \pm i). \quad (3.34)$$

Note that this expression is independent of a, which serves as the variational parameter in the calculation of the magnetic integrals.

The magnetic contributions to the variational form, S, are now calculated. The magnetic terms are given by Eq. (3.30) with the boundary term β_A. Assuming $x_c < |d \ln A/dx|^{-1}$, then one can approximate the resonance operators, R_n, by the multiplicative operators, I_n, as given in Eq. (2.33). Thus

$$\mathcal{L}_A \quad \beta_A \simeq \int_{-a}^{a} dx \left[A'^2 + \left(b + \frac{\Gamma}{\omega_c} \frac{1}{1+(x/x_c)^2} + \frac{\lambda}{2} \frac{x}{x_c^2} \frac{1}{1+1/2(x/x_c)^2} \right) A^2 \right] - AA' \Big|_{-a}^{a} \tag{3.35}$$

where

$$\Gamma = \frac{-iv_e^2}{\tau v_A^2}(\omega - \omega_{*e}), \quad \lambda = \frac{1}{2}\frac{v_e^2}{\tau v_A^2}\frac{\omega_{*e}}{\omega_c}\eta_J x_c.$$

For the trial functions, the following forms are used. In the outer region, $|x| > a$, $A_{||}$ is required to be the ideal MHD solution. That is, $A_{||} = A_{MHD}$ for $|x| > a$, where A_{MHD} must solve Eq. (3.18). To leading order for small x/x_J, where $x_J^{-1} = |d \ln J_{||}/dx|$, Eq. (3.18) can be solved asymptotically [37] to yield

$$A = \begin{cases} A_+^{MHD} = f + c_+ g & \text{for} \quad x > a \\ A_-^{MHD} = f + c_- g & \text{for} \quad x < -a \end{cases} \tag{3.36}$$

where

$$f \simeq 1 + \frac{\lambda x}{2}\ln x^2 + \frac{\lambda^2 x^2}{4}(\ln x^2 + 2b - 3) + \cdots$$

$$f' \simeq \frac{\lambda}{2}[\ln x^2 + 2] + \frac{\lambda^2 x}{2}[\ln x^2 + 2b - 2] + \cdots$$

$$g \simeq \lambda x + \frac{\lambda^2 x^2}{2} + \cdots, \quad g' \simeq \lambda + \lambda^2 x + \cdots \tag{3.37}$$

and it is assumed $\lambda x \sim \lambda a < 1$.

Consider the following definition: $\Delta'(x) \equiv [A'_+(x) - A'_-(-x)]_{MHD}$. Numerical calculations [30] for Alcator C parameters indicate $\Delta'(x)$ to be primarily a linear function of x of the form $\Delta'(x) = \Delta_0 (1 - x/W)$. Here, W represents the nonlinear island saturation width [20]. Hence, by identifying

$$C_+ - C_- = \Delta_0/\lambda; \quad C_+ + C_- = \frac{-\Delta_0}{\lambda^2 W} \tag{3.38}$$

and by requiring $\lambda^2 a C_\pm > \lambda^2 a$, then to leading order $\Delta'(x) \simeq \Delta_0(1 - x/W)$. For the sake of simplicity in the following calculations, the following approximate forms are used as definitions:

$$f \equiv 1 + \lambda x, \quad g \equiv \lambda x + \frac{\lambda^2 x^2}{2}$$

$$f' \equiv \lambda, \quad g' \equiv \lambda + \lambda^2 x. \tag{3.39}$$

By doing this, then one directly obtains the desired result $\Delta'(x) = \Delta_0\,(1 - x/W)$. Strictly speaking, however, the forms of f and g used in Eq. (3.39) are only approximately valid when $C_\pm > 1$. This constraint can be relaxed, however, if the forms in Eq. (3.39) are taken to define the outer trial functions for $|x| > a$. In any event, requiring $A_{MHD} > 0$ (as is indicated from experimental profiles) implies the restraint $\lambda a C_\pm < 1$. Utilizing the scaling $C_\pm \sim \Delta_0/(\lambda^2 W)$ then requiring $A_{MHD} > 0$ implies $\Delta_0 a/(\lambda W) < 1$.

The trial function for the inner region is taken to be a slowly varying linear function of x.

$$A = \begin{cases} A_+ = 1 + L_+ x & \text{for} \quad 0 < x < a \\ A_- = 1 + L_- x & \text{for} \quad -a < x < 0 \end{cases} \tag{3.40}$$

Requiring this trial function to be continuous at the boundary, $x = \pm a$, with the ideal MHD solution then defines $L_\pm$ to be

$$L_\pm = \lambda + C_\pm \lambda \left(1 \pm \frac{\lambda a}{2}\right). \tag{3.41}$$

Using these trial functions, it is now possible to evaluate the magnetic form $\mathcal{L}_A - \beta_A$. The boundary terms are given by

$$\beta_A = AA' \Big|_{-a}^{a} = A_+ A'_{+_{MHD}} \Big|_a - A'_- A'_{-_{MHD}} \Big|_{-a}$$

$$= \Delta'(a) + a(L_+^2 + L_-^2) + \frac{\lambda^2 a^2}{2}(L_+ C_+ - L_- C_-) \tag{3.42}$$

where $\Delta'(a) = \Delta_0(1 - a/W)$.

In the magnetic variational terms, $S_A = \mathcal{L}_A - \beta_A$, the variational parameter is chosen to be the limit of integration, a, which also characterizes the slope of the trial function. Note that the electrostatic term, $\mathcal{L}_\phi$, was evaluated approximately to be independent of a. Thus, only the magnetic terms determine the variational parameter a.

Variation of the magnetic form, S_A, with respect to a then gives

$$\frac{\delta S_A}{\delta a} = \left[A'^2 + \left(b + \frac{\Gamma}{\omega_c}\frac{1}{1+(x/x_c)^2} + \frac{\lambda}{2}\frac{x}{x_c^2}\frac{1}{1+1/2(x/x_c)^2}\right)A^2\right] \Big|_{+a} + \Big|_{-a} - \frac{\delta \beta_A}{\delta a}. \tag{3.43}$$

Recall that requiring $A_\pm > 0$ leads to the constraint $L_\pm a < 1$. Also, in the evaluation of the boundary term, it suffices to approximate $L_\pm$ as independent of a; that is, $L_\pm \simeq C_\pm \lambda$. Hence, to leading order in $L_\pm a$,

$$\frac{\delta S_A}{\delta a} = \left[(L_+^2 + L_-^2) + 2b + \frac{2\Gamma}{\omega_c}\frac{1}{1+(a/x_c)^2} + \frac{\lambda}{2}\frac{a^2}{x_c^2}\frac{2(L_+ + L_-)}{1+1/2(a/x_c)^2}\right]$$

$$- \left[\frac{-\Delta_0}{W} + (L_+^2 + L_-^2) + \lambda^2 a(L_+ C_+ - L_- C_-)\right].$$

The last term in the above equation can be neglected when compared to the fourth and fifth terms. To leading order it is approximated $L_\pm \simeq C_\pm \lambda$, and, thus, Eq. (3.43) becomes

$$\frac{\delta S_A}{\delta a} = \frac{\Delta_0}{W} + 2b + \frac{2\Gamma}{\omega_c}\frac{1}{1+(a/x_c)^2} - \lambda\frac{a^2}{x_c^2}\frac{1}{1+1/2(a/x_c)^2}\frac{\Delta_0}{\lambda W}. \tag{3.44}$$

Assuming $\omega = \omega_{*e} + i\gamma$, then at marginal stability, $\Gamma \simeq 0$. Setting $\delta S_A/\delta a = 0$ and solving for a yields

$$\frac{a^2}{x_c^2} = \frac{\Delta_0/W + 2b}{1/2(\Delta_0/W) - b}.$$

Typically, $b \sim \lambda^2$ and $\Delta_0 \sim \lambda$. Thus, $bW/\Delta_0 \sim \lambda a \ll 1$, and to leading order $a^2 \simeq 2x_c^2$.

To find the dispersion relation, the integral form S_A is evaluated explicitly. From Eq. (3.35),

$$\begin{aligned} S_A = \Bigg[& a(L_+^2 + L_-^2) + 2ba + 2\frac{\Gamma}{\omega_c}\left(\tan^{-1}\frac{a}{x_c}\right)x_c\frac{a}{\sqrt{2}x_c} \\ & + 2\lambda x_c(L_+ + L_-)\left(\frac{a}{x_c} - \sqrt{2}\tan^{-1}\frac{a}{\sqrt{2}x_c}\right)\Bigg] \\ & - [\Delta'(a) - (L_+^2 + L_-^2)a]. \end{aligned} \tag{3.45}$$

Approximating $L_+ + L_- \simeq -\Delta_0/\lambda W$ and using the result $a \simeq \sqrt{2}x_c$ along with the definition $\Delta'(\sqrt{2}x_c) = \Delta_0(1 - \sqrt{2}x_c/W)$ then gives

$$S_A \simeq -\Delta'\left(\frac{\sqrt{2}}{2}x_c\right) + 2\sqrt{2}bx_c + \frac{2\Gamma}{\omega_c}x_c. \tag{3.46}$$

The overall dispersion relation including the electrostatic terms is given by Eq. (3.28). Using the results of Eqs. (3.34) and (3.46) gives the following dispersion relation:

$$\frac{2\Gamma}{\omega_c}x_c = \Delta'\left(\frac{\sqrt{2}}{2}x_c\right) - 2\sqrt{2}bx_c - \frac{d}{\tau}\left(\frac{v_i}{v_A}\right)^2\left(\frac{\omega}{k_\parallel'\rho_i v_i}\right)^{1/2}(1 \pm i). \tag{3.47}$$

On the right hand side of Eq. (3.47), one can approximate $\omega \simeq \omega_{*e}$, $d \simeq 2$, and $\tau = 1$. Recalling the definition of Γ then the above dispersion relation becomes

$$-i(\omega - \omega_{*e}) = \frac{\omega_c}{2x_c}\frac{v_A^2}{v_e^2}\left[\Delta'\left(\frac{\sqrt{2}}{2}x_c\right) - 2\sqrt{2}bx_c - 2\left(\frac{v_i}{v_A}\right)^2\left(\frac{L_s}{L_n}\right)^{1/2}(1 \pm i)\right] \tag{3.48}$$

where now $x_i = L_s/L_n$. The above expression is valid providing $bW/\Delta_0 < 1$ and $\Delta_0 a/\lambda W < 1$.

Hence, writing $\omega = \omega_R + i\gamma$ with $|\gamma/\omega_R| \ll 1$ then gives the real frequency

$$\omega_R = \omega_{*e} \pm \omega_{*e}\left(\frac{m_e}{m_i}\frac{L_n}{L_s}\right)^{1/2}. \tag{3.49}$$

The growth rate is given by

$$\gamma = \frac{k'_{\|}\rho_i v_e}{2}\left\{\frac{v_A^2}{v_e^2}\left[\Delta'\left(\frac{\sqrt{2}}{2}x_c\right) - 2\sqrt{2}bx_c\right] - 2\left(\frac{L_s}{L_n}\right)^{1/2}\frac{m_e}{m_i}\right\}. \tag{3.50}$$

In the limit $bW/\Delta_0 \ll 1$, then stability is obtained when $\gamma < 0$, or

$$x_c > \sqrt{2}W\left[1 - \frac{2}{\Delta_0}\beta_i\left(\frac{L_S}{L_n}\frac{\tau}{\sqrt{2}}\right)^{1/2}\right]. \tag{3.51}$$

Thus, it is apparent that the coupling to the electrostatic branch, represented by the second term on the right of Eq. (3.51) is a stabilizing effect. Physically, this term represents that fraction of the available energy which is necessary to maintain the ion motion.

The first term on the right in the expression for the growth rate, Eq. (3.50), represents the magnetic energy drive from the ideal MHD region. Recall, $\Delta'(x) \simeq \Delta_0(1 - x/W)$, where W corresponds to the island saturation width of resistive MHD. Hence, the tearing mode will be stabilized [13,14] if $\Delta'(x_c) < 0$ or $x_c > W$. Since

$x_c \sim D^{1/3}$, this implies that if $D \sim 1/n$, then there exists a density threshold which must be surpassed before instability occurs. Physically, $\Delta'(x_c)$ is interpreted as the magnetic energy in the outer region available to drive the tearing mode. In essence, the turbulent electron diffusion prohibits the formation of, or flattens, the perturbed parallel current, $\tilde{J}_{\|e}$, within a finite correlation distance, x_c, of the rational surface. This has the effect of reducing the available magnetic energy drive from the value $\Delta'(0)$ to $\Delta'(x_c)$. The second term on the right of Eq. (3.50) represents line bending in the inner dissipative region and is stabilizing. The last term is ion inertia stabilization [10,14], and represents that portion of energy necessary to sustain the ion oscillation at frequency ω_{*e}.

Chapter 4
STABILIZATION BY RUNAWAY ELECTRONS

In this chapter, the kinetic theory of the tearing mode is expanded to include the effects of runaway electrons [56,64]. The point of performing this calculation with such an electron distribution is to see the effects, if any, which will occur if the equilibrium current is carried by a small fraction of relatively fast electrons as opposed to an equilibrium current carried by a small drift of the bulk electron population. Since runaway electrons are only present at low density, they may provide a low density stabilization mechanism for the tearing mode. The kinetic theory developed here is identical with that of Ch 3, except now the equilibrium current is carried by an electron beam of density $n_b \sim 10^{-3} n_o$ and velocity $v_b \sim 10 v_e$ (as compared to the drifted Maxwellian of the previous chapter). Modeling the equilibrium electron distribution with a realistic distribution observed in a runaway discharge (say, during current drive experiments) becomes too difficult mathematically. Instead, a simple monoenergetic electron beam is used to model this runaway case. If a significant change in the stability of the tearing mode is observed to occur with the equilibrium current modeled crudely by an electron beam as done here, then it may be worthwhile to repeat this calculation using more realistic electron distributions. The results of this chapter, however, indicate the runaway electrons provide a stabilizing correction to the particle inertia which is subdominant by order ω_{*e}/ω_c.

In Sec. 4.1 of this chapter, coupled equations for the fluctuation potentials $\tilde{\phi}$ and $\tilde{A}_{||}$ are again derived only now the the equilibrium current is represented by an electron beam (describing the "runaway" tearing mode) as opposed to a drifted Maxwellian (describing the "regular" tearing mode). To calculate a dispersion relation, this coupled system is solved using a variational calculation similar to that used in Sec. 3.4. In order to see effects of the electron beam, it is necessary to calculate corrections of order ω_*/ω_c to the previously calculated dispersion relation of Sec. 3.4. Section 4.2 involves calculating the corrections of order ω_*/ω_c to the dispersion relation for the "regular" tearing mode (where the current is represented by a drifted Maxwellian). Section 4.3 then calculates the dispersion relation for the "runaway" tearing mode including the effects of the electron beam.

4.1 Coupled Equations

The goal in this section is to derive a coupled set of equations for the fluctuation potentials $\widetilde{\phi}$ and $\widetilde{A}_{\|}$. As discussed in the previous section, the electron response is given by applying the normal stochastic approximation (NSA) [17] to the drift kinetic equation (DKE) [44] written for the fluctuations. The ion response is described by the linearized Vlasov equation. Once the perturbed density and parallel current are calculated by taking the appropriate moments of the perturbed distribution function, quasineutrality and Ampere's law are used to give a closed, coupled system for $\widetilde{\phi}$ and $\widetilde{A}_{\|}$. This system is self-adjoint, hence, a variational principle can be utilized to determine the dispersion relation.

The electron response is given by applying the NSA to the fluctuation DKE. As discussed in the previous section, the perturbed electron response $\widetilde{f}_e$ is then described by

$$\widetilde{f}_e = \frac{e\widetilde{\phi}}{T_e} f_0 + \widetilde{h}_e, \tag{4.1}$$

where the nonlinear electron response $\widetilde{h}_e$ is given by

$$\left(\frac{\partial}{\partial t} + v_{\|}\mathbf{b}\cdot\nabla - D\frac{\partial^2}{\partial x^2}\right)\widetilde{h}_e = S(x) \tag{4.2}$$

with

$$S(x) = \left[\frac{c}{B}\nabla\left(\widetilde{\phi} - \frac{v_{\|}}{c}\widetilde{A}_{\|}\right)\times\mathbf{b}\cdot\nabla(f_0 + f_1) - \frac{e}{T_e}\frac{\partial}{\partial t}\left(\widetilde{\phi} - \frac{v_{\|}}{c}\widetilde{A}_{\|}\right) f_0\right].$$

The end result of the NSA is to convert the terms proportional to the fluctuation potentials in the orbit operator acting on $\widetilde{h}_e$ into the spatial diffusion term appearing on the left hand side of Eq. (4.2). In physical terms, the NSA amounts to introducing a spatial diffusion term in the orbit operator in place of the nonlinear

terms which, through $\widetilde{\mathbf{E}} \cdot \mathbf{B}$ drift wave turbulence, produces such radial diffusion. Equation (4.1) states that the adiabatic piece of the electron response is unaffected by such turbulent diffusion. In the treatment which follows, the diffusion coefficient D appearing in Eq. (4.2) is to be treated as an external constant parameter independent of the tearing mode dynamics, which is to be specified either through experimental observation or through an appropriate theory of the underlying turbulence.

The equilibrium electron distribution is given by $\overline{f}_e = f_0 + f_1$. Here, f_0 is a Maxwellian and f_1 represents the piece of the distribution which caries the equilibrium current. The current carrying part f_1 is given by

$$f_1 = \begin{cases} \dfrac{2v_{\|}v_D}{v_e^2} f_0, & \text{regular tearing} \\ n_b \delta(v_{\|} - v_b), & \text{runaway tearing} \end{cases}$$

where

$$f_0 = \frac{n_o}{\sqrt{\pi} v_e} e^{-v_{\|}^2/v_e^2}$$

and $v_e^2 = 2T_e/m_e$ (with v_e, v_b constant in space). In the regular tearing mode, f_1 is the current carrying part of the drifted Maxwellian (where $v_D/v_e \sim 10^{-2}$). In the runaway tearing mode, f_1 represents a monoenergetic beam of electrons with density n_b and speed v_b. (Typically, $n_b/n_o \sim 10^{-3}$ and $v_b/v_e \sim 10$.)

In comparing the regular tearing mode to the runaway case, the total equilibrium current is to be held fixed:

$$J_{\|_o}(x) = -en_o v_D = -en_b v_b. \tag{4.3}$$

With this constraint, any differences occurring in the dispersion relation between the runaway and the regular tearing mode will be the result of in which way the equilibrium *velocity* distribution represents the equilibrium current (i.e., either by

a current carried by a few fast electrons, or by a current carried by many slowly drifting bulk electrons).

Equation (4.2) will be solved for a sheared, slab geometry with linear gradients assumed for the equilibrium density and current profiles. That is,

$$\mathbf{B}_0 = B_0(\mathbf{e}_z + x/L_s\mathbf{e}_y),$$
$$\frac{d}{dx}n_o = -n_o/L_n \text{ and } \frac{d}{dx}J_{\|_o} = -J_{\|_o}/L_J.$$

The origin in x will be chosen such that

$$k_\| = k_y B_{y_0} + k_z B_{z_0} \equiv k_\|{}' x$$

where $k_\|{}' = k_y/L_s$. These slab quantities can be related to tokamak-type quantities by identifying

$$k_y = m/r \text{ and } L_s = Rq^2/rq'$$

where r and R are the minor and major plasma radius and $q(r)$ is the safety factor profile. Also, throughout the following, a Fourier mode $Q_k(x,\ k_y,\ k_z,\ \omega)$ is related to $Q(x,y,z,t)$ by

$$Q(x,y,z,t) = \sum_{k,\omega} Q_k(x) e^{i(k_y y + k_z z - \omega t)}.$$

The solution to Eq. (4.2) for the nonlinear electron response $\tilde{h}_e$ is given by

$$\tilde{h}_e = \int_0^\infty d\tau \int_{-\infty}^\infty dx' G(x,x';v_\|,\tau) S_k(x') \qquad (4.4)$$

with the propagator

$$G(x,x';v_{\|},\tau) = (1/\sqrt{4\pi D\tau})\exp\Big[i(\omega - k_{\|}{}'v_{\|}x)\tau$$
$$- \frac{1}{3}(k_{\|}{}'v_{\|})^2 D\tau^3 - (1/4D\tau)(x - x' - iDk_{\|}{}'v_{\|}\tau^2)^2\Big]$$

and the source term

$$S_k(x) = \frac{-ie}{T_e}\left[\omega f_0 + \omega_{*e}L_n\frac{\partial}{\partial x}(f_0 + f_1)\right]\left(\widetilde{\phi} - \frac{v_{\|}}{c}\widetilde{A}_{\|}\right)$$

where $\omega_{*e} = (ck_yT_e/eBL_n)$. In Eq. (4.4), the kernal $G(x,x',v_{\|},\tau)$ is a function with a characteristic decay time $\tau_c = \omega_c^{-1}$, where $\omega_c^3 = (k_{\|}{}'v_e)^2D/3$, and with a characteristic width $x_c = \omega_c/k_{\|}{}'v_e$.

Moments of the perturbed electron distribution, as described by Eqs. (4.1) and (4.4), can be taken to determine the perturbed electron density and current. In doing so, it is useful to introduce the following resonance operators, $R_n[\psi]$:

$$R_n[\psi] = \frac{1}{n_o}\int_{-\infty}^{\infty}dv_{\|}\int_0^{\infty}d\tau\int_{-\infty}^{\infty}dx'G(x,x';v_{\|},\tau)\left(\frac{v_{\|}}{v_e}\right)^n f_0(v_{\|})\psi(x') \qquad (4.5)$$

where ψ represents one of the fluctuation potentials.

For $\widetilde{\phi}$ and $\widetilde{A}_{\|}$ sufficiently slowly varying in x, then the resonance operators $R_n[\psi]$ can be replaced by multiplicative functions $I_n(x)\cdot\psi(x)$ by expanding $\psi(x')$ about $x' = x$. That is, for $x_c/x_T \ll 1$, where $x_T = |d\ln\psi/dx|^{-1}$, then to leading order in x_c/x_T

$$R_n[\psi] \simeq I_n(x)\psi(x) \qquad (4.6)$$

where

$$I_n(x) = \frac{1}{n_o}\int_0^{\infty}d\tau\int_{-\infty}^{\infty}dv_{\|}\kappa(v_{\|},\tau)\left(\frac{v_{\|}}{v_e}\right)^n f_0(v_{\|})$$

with the kernal

$$\kappa(v_{\|},\tau) = \exp\left[i(\omega - k_{\|}{}'v_{\|}x)\tau \quad \frac{1}{3}(k_{\|}{}'v_{\|})^2 D\tau^3\right].$$

The perturbed electron density can then be written as

$$\frac{\widetilde{n}_e}{n_o} = \frac{e}{T_e}\left\{\widetilde{\phi} + \left[i(\omega - \omega_{*e})I_0 - i\omega_{*e}(\eta_J I_1)^*\right]\widetilde{\phi} \right. \\ \left. - \left[i(\omega - \omega_{*e})I_1 - i\omega_{*e}(\eta_J I_2)^*\right]\frac{v_e}{c}\widetilde{A}_{\|}\right\}, \tag{4.7}$$

and the perturbed parallel electron current can be written as

$$\frac{\widetilde{J}_{\|}}{-ev_e n_o} = \frac{ie}{T_e}\left\{\left[(\omega - \omega_{*e})I_1 - \omega_{*e}(\eta_J I_2)^*\right]\widetilde{\phi} \right. \\ \left. - \left[(\omega - \omega_{*e})I_2 - \omega_{*e}(\eta_J I_3)^*\right]\frac{v_e}{c}\widetilde{A}_{\|}\right\}. \tag{4.8}$$

In the above equations

$$(\eta_J I_n)^* = \begin{cases} \dfrac{2v_D}{v_e}\dfrac{L_n}{L_J}I_n, & \text{regular tearing} \\ \dfrac{n_b}{n_o}\dfrac{L_n}{L_J}K_n, & \text{runaway tearing} \end{cases}. \tag{4.9}$$

Here, K_n refers to the beam electron resonance functions given by

$$K_n(x) = \int_0^{\infty} d\tau \int_{-\infty}^{\infty} dv_{\|}\kappa(v_{\|},\tau)\left(\frac{v_{\|}}{v_e}\right)^n \delta(v_{\|} - v_b). \tag{4.10}$$

A closed, coupled system for $\widetilde{\phi}$ and $\widetilde{A}_{\|}$ can be obtained using quasineutrality and Ampere's law. The expressions for the perturbed ion density and parallel current are given by the linearized Vlasov equation and can be found in Sec. 2.2

Quasineutrality, $\widetilde{n}_e = \widetilde{n}_i$, can be written as

$$L_1\widetilde{\phi} + L_x\widetilde{A}_{||} = 0. \tag{4.11}$$

Ampere's law, $\nabla^2\widetilde{A}_{||} + (4\pi/c)(\widetilde{J}_{||_e} + \widetilde{J}_{||_i}) = 0$, can be written as

$$L_2\widetilde{A}_{||} + L_x\widetilde{\phi} = 0. \tag{4.12}$$

The operators L_1, L_2 and L_x are given by

$$L_1 = \left[\frac{d^2}{dx^2}\xi Z\frac{d^2}{dx^2} + (\Lambda + \chi^2) + \frac{i}{d}(\omega - \omega_{*e})I_0 - \frac{i}{d}\omega_{*e}(\eta_J I_1)^*\right] \tag{4.13a}$$

$$L_2 = \frac{\tau v_A^2}{dc^2}\left[\frac{d^2}{dx^2} - b + \alpha_i^2 + \frac{v_e^2}{\tau v_A^2}i(\omega - \omega_{*e})I_2 - \frac{v_e^2}{\tau v_A^2}i\omega_{*e}(\eta_J I_3)^*\right] \tag{4.13b}$$

$$L_x = -\left[\frac{i}{d}(\omega - \omega_{*e})I_1 - \frac{i}{d}\omega_{*e}(\eta_J I_2)^* + x_e\chi^2\right]\frac{v_e}{c} \tag{4.13c}$$

with the following definitions:

$$d = (\Gamma_0 - \Gamma_1)(\tau + \omega_{*e}/\omega), \quad \Lambda = \frac{1}{d}\left(1 + \tau - \Gamma_0\left(\tau + \frac{\omega_{*e}}{\omega}\right)\right)$$
$$\chi^2 = \frac{\Gamma_0}{d}\left(\tau + \frac{\omega_{*e}}{\omega}\right)(1 + \xi Z), \quad \tau = T_e/T_i$$
$$\alpha_i^2 = \frac{v_i^2}{\tau v_A^2}\left(\tau\frac{\omega_{*e}}{\omega}\right)\xi^2(1 + \xi Z)\Gamma_0, \quad b = \rho_i^2 k_y^2,$$

where $\xi = \omega/(|k_{||}|v_i)$, $x_e = \omega/(|k_{||}|v_e)$, $v_A^2 = B^2/(4\pi m_i n_o)$, and $Z = Z(\xi)$ = plasma dispersion function. In the above expressions, x is normalized in units of the ion gyroradius ρ_i.

Since the above coupled system given by Eq. (4.11) and Eq. (4.12) is self-adjoint, a variational integral can be formed:

$$S = -\int_{-a}^{a} dx \left[\tilde{\phi} L_1 \tilde{\phi} + \tilde{A}_{\|} L_2 \tilde{A}_{\|} + 2\tilde{\phi} L_x \tilde{A}_{\|}\right], \tag{4.14}$$

where $\pm a$ represents the plasma edge. The above variational form will be used to calculate the dispersion relation.

At this point it is helpful to present a more useful form for the resonance functions $I_n(x)$ and $K_n(x)$. This the so-called "Krook" approximation (see Sec. 2.4) in which the term $-\frac{1}{3}(k_{\|}{}' v_{\|})^2 D\tau^3$ in the expression for $\kappa(v_{\|}, \tau)$, Eq. (4.6), is replaced by the linear factor $-\omega_c \tau$, where $\omega_c^3 = (k_{\|} v_e)^2 D/3$. This amounts to replacing the term $-D\partial^2/\partial x^2$ in Eq. (4.2) with the equivalent diffusive frequency ω_c. Under the Krook approximation, the functions $I_n(x)$ can be evaluated exactly in terms of the plasma dispersion function, which can then be approximated by interpolation polynomials preserving the asymptotic behavior at large and small x.

$$I_0 = \frac{-ix_c}{\omega_c x} Z(\xi_e) \simeq \frac{1}{\omega_c} \left(\frac{1 + ix_e/x_c}{1 + \frac{1}{\sqrt{\pi}} \frac{|x|}{x_c} \left(1 + \frac{ix_e}{x_c}\right)} \right) \tag{4.15a}$$

$$I_1 = \frac{-ix_c}{\omega_c x}(1 + \xi_e Z) \simeq \frac{-i}{\omega_c} \left(\frac{\frac{1}{2}\frac{x}{x_c}\left(1 + \frac{2ix_e}{x_c}\right)}{1 + \frac{1}{2}\frac{x^2}{x_c^2}\left(1 + \frac{2ix_e}{x_c}\right)} \right) \tag{4.15b}$$

$$I_2 = \frac{-ix_c}{\omega_c x} \xi_e (1 + \xi_e Z) \simeq \frac{1}{\omega_c} \left(\frac{\frac{1}{2}\left(1 + \frac{ix_e}{x_c}\right)}{1 + \frac{1}{2}\frac{x^2}{x_c^2}\left(1 + \frac{2ix_e}{x_c}\right)} \right) \tag{4.15c}$$

$$I_3 = \frac{-ix_c}{\omega_c x} \left[\frac{1}{2} + \xi_e^2 (1 + \xi_e Z)\right] \simeq -\frac{i}{\omega_c} \left(\frac{\frac{3}{4}\frac{x}{x_c}\left(1 + \frac{2ix_e}{x_c}\right)}{1 + \frac{3}{2}\frac{x^2}{x_c^2}\left(1 + \frac{2ix_e}{x_c}\right)} \right) \tag{4.15d}$$

where $\xi_e = (x_e + ix_c)/x$, $x_e = \omega/(k_{\|}{}' v_e)$ and $x_c = \omega_c/(k_{\|}{}' v_e)$.

Similarly, under the Krook approximation, the beam electron resonance functions are given by

$$
\begin{aligned}
K_n(x) &= \left(\frac{v_b}{v_e}\right)^n \frac{1}{\omega_b}\left[\left(1 - i\frac{\omega}{\omega_b}\right) + i\frac{x}{x_b}\right]^{-1} \\
&\simeq \left(\frac{v_b}{v_e}\right)^n \frac{1}{\omega_b}\left[1 + \frac{ix}{x_b}\right]^{-1} \\
&= \left(\frac{v_b}{v_e}\right)^n \frac{1}{\omega_b}\left(\frac{1 - \dfrac{ix}{x_b}}{1 + \dfrac{x^2}{x_b^2}}\right)
\end{aligned}
\tag{4.16}
$$

where the term of order $\omega/\omega_b \sim (\omega_*/\omega_c)(v_e/v_b)^{2/3}$ has been neglected.

In Secs. 4.2 and 4.3, the Krook approximate resonance functions, Eqs. (4.15)–(4.16), will be used along with the variational form S, Eq. (4.14), to determine the dispersion relation for the regular and runaway tearing modes. This is done by inserting suitable trial functions for $\widetilde{\phi}$ and $\widetilde{A}_{\|}$ into the integral S. The variational parameters α_m characterizing the trial functions are then determined by setting $\delta S/\delta\alpha_m = 0$. The dispersion relation is then given by setting $S = 0$.

4.2 Regular Tearing

The goal of this section is to calculate the dispersion relation for the regular tearing mode. This will be done using a variational calculation similar to that used in Sec. 3.4, only now the calculation will be presented in greater detail including the corrections of order ω/ω_c necessary to see the runaway stabilization. The coupled equations representing quasineutrality and Ampere's law, Eq. (4.11) and Eq. (4.12), will be solved variationally using the variational form S, Eq. (4.14).

To begin with, it is necessary to find suitable trial functions for $\tilde{\phi}$ and $\tilde{A}_{||}$ which can be inserted into the variational integral S. The asymptotic behavior of these trial functions for large x, $|x| > x_c$, must be consistent with the asymptotic form of quasineutrality and Ampere's law. Recalling that the ion Z-functions which appear in Eq. (4.11) and Eq. (4.12) are only valid for $|x| < x_i$, then to leading order in ω/ω_c, at large x, $x_c < |x| < x_i$, quasineutrality, Eq. (4.11), becomes

$$\left[-\frac{d^2}{dx^2} + \Lambda - \frac{1}{2}\frac{x^2}{x_i^2}\frac{\Gamma_0}{\Gamma_0 - \Gamma_1}\right]\tilde{\phi} + \left(\frac{1}{2}\frac{x^2}{x_i^2}\frac{\Gamma_0}{\Gamma_0 - \Gamma_1}\frac{\omega}{k_{||}c}\right)\tilde{A}_{||} = 0 \tag{4.17}$$

where terms of order $x_e/x < x_e/x_c \sim \omega_*/\omega_c$ have been neglected.

Assuming, in the region $x_c < |x| < x_i$, that $x^2/x_i^2 \sim \epsilon$, $|\tilde{\phi}''/\tilde{\phi}| \sim \epsilon^2$ and $\omega/\omega_{*e} \sim 1$, then to leading order ϵ°, quasineutrality implies

$$\Lambda = 0 \quad \text{or} \quad \omega \simeq \omega_{*e}, \tag{4.18}$$

where the approximation $\Gamma_0 \simeq 1$ has been used. To first order in ϵ, quasineutrality implies

$$\tilde{\phi} - \frac{\omega}{k_{||}c}\tilde{A}_{||} = 0 \quad \text{or} \quad \tilde{E}_{||} = 0. \tag{4.19}$$

Hence, at large x, $|x| > x_c$, quasineutrality reduces to the ideal MHD equation $\tilde{E}_{||} \simeq 0$ with the constraint $\omega \simeq \omega_{*e}$.

Similarly, in the region $x_c < |x| < x_i$. Ampere's law can be written to leading order in ω/ω_c as

$$\frac{\tau v_A^2}{dc^2}\left[\frac{d^2}{dx^2} - b - \frac{\Gamma_0}{2}\frac{v_i^2}{\tau v_A^2}\left(\tau + \frac{\omega_{*e}}{\omega}\right) - \frac{4\pi}{c}\rho_i^2\frac{k_y J'_{\parallel_o}}{k_\parallel B_0}\right]\widetilde{A}_\parallel + \left(\frac{1}{2}\frac{x^2}{x_i^2}\frac{\Gamma_0}{\Gamma_0 - \Gamma_1}\right)\frac{\omega}{k_\parallel c}\widetilde{\phi} = 0, \tag{4.20}$$

where terms of order $x_e/x < x_e/x_c \sim \omega_*/\omega_c$ have been neglected.

Using the fact that for $|x| > x_c$ quasineutrality states $\widetilde{E}_\parallel = 0$, then Ampere's law reduces at large x to

$$\left[\frac{d^2}{dx^2} - b - \frac{4\pi}{c}\rho_i^2\frac{k_y J'_{\parallel_o}}{k_\parallel B_0}\right]\widetilde{A}_\parallel = 0. \tag{4.21}$$

The above equation is the ideal MHD equation for $\widetilde{A}_\parallel$ at marginal stability. (This will be discussed further in Sec. 5.3.) Hence, the conclusion is reached that at large x, $|x| > x_c$, quasineutrality and Ampere's law reduce to ideal MHD, Eq. (4.19) and Eq. (4.21), with the additional constraint $\omega \simeq \omega_{*e}$.

It is further helpful to rewrite the variational form S, Eq. (4.14), in the following way. After integrating by parts, x can be written

$$S = \frac{\tau v_A^2}{dc^2}[\mathcal{L}_A - \beta_A] + [\mathcal{L}_\phi - \beta_\phi] \tag{4.22}$$

where

$$\mathcal{L}_A = \int_{-a}^{a} dx \left\{\left(\frac{d\widetilde{A}_\parallel}{dx}\right)^2 + \left[b - \frac{v_e^2}{\tau v_A^2} i(\omega - \omega_{*e})I_2 + \frac{v_e^2}{\tau v_A^2} i\omega_{*e}(\eta_J I_3)^*\right]\widetilde{A}_\parallel^2\right\} \tag{4.23}$$

$$\mathcal{L}_\phi = \int_{-a}^{a} dx \Bigg\{ \left[\xi Z \left(\frac{d\widetilde{\phi}}{dx} \right)^2 - \Lambda \widetilde{\phi}^2 - (1+\xi Z) \left(\widetilde{\phi} - \frac{\omega}{k_\| c} \widetilde{A}_\| \right)^2 \right] + \left[-\frac{i}{d}(\omega - \omega_{*e}) I_0 \widetilde{\phi}^2 + 2\frac{i}{d}(\omega - \omega_{*e}) I_1 \widetilde{\phi} \widetilde{A}_\| \frac{v_e}{c} + \frac{i}{d}\omega_{*e}(\eta_J I_1)^* \widetilde{\phi}^2 - 2\frac{i}{d}\omega_{*e}(\eta_J I_2)^* \widetilde{\phi} \widetilde{A}_\| \frac{v_e}{c} \right] \Bigg\} \tag{4.24}$$

with the boundary terms

$$\beta_A = \widetilde{A}_\| \widetilde{A}'_\| |_{-a}^{a} \quad \text{and} \quad \beta_\phi = \xi Z \widetilde{\phi} \widetilde{\phi}' |_{-a}^{a}.$$

In the above expressions, $\pm a$ represents an intermediate position in the plasma (whose value is to be determined variationally) beyond which ideal MHD is approximately valid. That is, for $|x| \geq a > x_c$, the plasma is sufficiently described by ideal MHD.

The tearing mode is typically described by a slowly varying $\widetilde{A}_\|$ which is even (nearly constant) about the rational surface, as indicated by numerical solutions of the ideal MHD equation for marginal stability, Eq. (4.21). Also, since $\widetilde{E}_\| = 0$, this implies that $\widetilde{\phi}$ is primarily odd about the rational surface. A suitable trial function for the electrostatic potential consistent with these observations is

$$\widetilde{\phi}_{Trial} = \phi_0 \frac{x^3 - i\alpha^2 x}{x^4 + \alpha^4}, \; \phi_0 = x_e \frac{v_e}{c} A_0, \tag{4.25}$$

where α^2 is the variational parameter.

For small x, then

$$\widetilde{\phi}_{Trial} \to \phi_0 \frac{-ix}{\alpha^2} \quad \text{as} \quad x \to 0,$$

and for large x, $x^2 > \alpha^2$, then

$$\widetilde{\phi}_{Trial} \simeq \phi_0 \frac{1}{x} = \frac{\omega}{k_{\|} c} A_0$$

which implies $\widetilde{E}_{\|} = 0$ for $x^2 > \alpha^2$ and nearly constant $\widetilde{A}_{\|}, \widetilde{A}_{\|} \simeq A_0$. Hence, the above trial function is consistent with ideal MHD boundary conditions at $\pm a$ provided $\alpha^2 < a^2 \sim x_c^2$, (since ideal MHD is valid for $|x| > x_c$).

In the limit that $\alpha^2 \ll x_c^2$, and since $\widetilde{A}_{\|}$ is slowly varying about $x = 0$, the above trial function for $\widetilde{\phi}$ implies that the limits of integration $\pm a$ can be extended to infinity in the evaluation of the integral $\mathcal{L}_\phi$. By doing so, the integral $\mathcal{L}_\phi$ will be independent of the value of a. Provided $\alpha^2 < x_c^2$, the error in evaluating $\mathcal{L}_\phi$ as independent of a should be at most of order α^2/x_c^2.

The trial function for $\widetilde{A}_{\|}$ is chosen to be of the following form:

$$\widetilde{A}_{\|_{Trial}} = \begin{cases} 1 + L_+ x & \text{, for } 0 < x < a \\ 1 + L_- x & \text{, for } -a < x < 0 \end{cases}. \tag{4.26}$$

The value of the trial function for $\widetilde{A}_{\|}$ is required to match on to the ideal MHD solution at $\pm a$.

$$\widetilde{A}_{\|_{Trial}} = \widetilde{A}_{\|_{MHD}} \text{ at } x = \pm a,$$

where $\widetilde{A}_{\|_{MHD}}$ is the solution to Eq. (4.21).

The ideal MHD equation for marginal stability, Eq. (4.21), can be solved asymptotically [37] at small x to give the following expression for $\widetilde{A}_{\|_{MHD}}$:

$$\widetilde{A}_{\|_{MHD}} = \begin{cases} f + C_+ g & \text{for } x > 0 \\ f + C_- g & \text{for } x < 0 \end{cases}, \tag{4.27}$$

where

$$f \simeq 1 + \frac{\lambda x}{2} \ln x^2 + \frac{\lambda^2 x^2}{4}(\ln x^2 + 2b - 3) + ...$$
$$f' \simeq \frac{\lambda}{2}(\ln x^2 + 2) + \frac{\lambda^2 x}{2}(\ln x^2 + 2b - 2) + ...$$
$$g \simeq \lambda x + \frac{\lambda^2 x^2}{2} + ..., g' \simeq \lambda + \lambda^2 x + ... \tag{4.28}$$

which is valid for $\lambda x < 1$, where $\lambda = (v_e^2/\tau v_A^2)(v_D L_n / v_e L_J) x_e$.

For the sake of choosing a mathematically simple trial function, the following definitions for the functions f and g will be used at $x = \pm a$:

$$f \equiv 1 + \lambda x, \; g \equiv \lambda x + \frac{\lambda^2 x^2}{2}$$
$$f' \equiv \lambda, \; g' \equiv \lambda + \lambda x^2. \tag{4.29}$$

The error made in using the above expressions for f and g, Eq. (4.29), instead of using the more accurate expressions, Eq. (4.28), manifests itself in so far as it being equivalent to using a more approximate form for the trial function used for $\widetilde{A}_{||}$.

Setting $\widetilde{A}_{||_{Trial}} = \widetilde{A}_{||_{MHD}}$ at $\pm a$ gives the following values for the slope of the trial function for $\widetilde{A}_{||}$:

$$L_{\pm} = \lambda + C_{\pm}\lambda \left(1 \pm \frac{\lambda a}{2}\right), \tag{4.30}$$

where $\widetilde{A}_{||_{Trial}}$ has the form given in Eq. (4.26).

To find the value of the constants $C_{\pm}$ in terms of physical parameters, consider the following definition:

$$\Delta'(x) \equiv \widetilde{A}'_{||_{MHD}}(+x) - \widetilde{A}'_{||_{MHD}}(-x). \tag{4.31}$$

Numerical calculations for Alcator C profiles indicates $\Delta'(x)$ to be primarily linear function of x of the form

$$\Delta'(x) = \Delta_0(1 - x/W). \tag{4.32}$$

Here Δ_0 represents the ideal MHD energy available to drive the tearing mode for an infinitesimally thin dissipative layer and W represents the nonlinear island saturation width. Using the approximate form for the ideal MHD solution, given by Eq. (4.27) and Eq. (4.29), specifies $C_\pm$ to be

$$C_+ - C_- = \Delta_0/\lambda;\ C_+ + C_- = \frac{-\Delta_0}{\lambda^2 W}. \tag{4.33}$$

The expressions for f and g assumed that $\lambda a < 1$ and requiring $\widetilde{A}_{||}$ to be positive (as observed numerically) requires $C_\pm \lambda a < 1$. (Typical experimental profiles indicate $\lambda \sim \Delta_0 \sim 10^{-2}$ and $W \sim a \sim 10$.)

Hence, the trial function for $\widetilde{A}_{||}$ is given by Eq. (4.26) with a linear slope given by Eq. (4.30) and (4.33). The variational parameter for this trial function is the limit of integration $\pm a$ which appears in the integral $\mathcal{L}_A$ as well as the boundary term β_A. Since $\mathcal{L}_A$ and β_A are independent of $\widetilde{\phi}$, the variation with respect to the variational parameters α and a can be performed independently of one another. (Recall that in the integral $\mathcal{L}_\phi$, $\widetilde{A}_{||}$ was taken as constant and the limits of integration were extended to infinity thus making $\mathcal{L}_\phi$ independent of the parameter a.)

Consider the evaluation of the electrostatic and coupling terms represent by the integral $\mathcal{L}_\phi$, Eq. (4.24), with the boundary term β_ϕ. Near the boundary $x = \pm a$ the ideal constraint $\widetilde{E}_{||} \simeq 0$ holds. Hence,

$$\widetilde{\phi} \simeq \frac{\omega}{k_{||}' x c}\widetilde{A}_{||},\ \widetilde{A}_{||} \simeq \text{ constant at } x = \pm a.$$

Thus,

$$\tilde{\phi}\tilde{\phi}' \simeq \left(\frac{\omega A_{\|_0}}{k_\|{}'C}\right)^2 \left(-\frac{1}{x^3}\right) \equiv -\frac{\phi_0^2}{x^3}$$

and the boundary term can be approximately evaluated to give

$$\beta_\phi \simeq -\phi_0^2 \frac{2}{a^3}. \tag{4.34}$$

As will be shown below, the integral $\mathcal{L}_\phi$ is evaluated to give approximately

$$\mathcal{L}_\phi \simeq \phi_0^2 \frac{2}{\alpha^3}(1+i). \tag{4.35}$$

Hence, the contribution of a finite electrostatic potential to the dispersion relation is given by

$$\mathcal{L}_\phi - \beta_\phi \simeq \phi_0^2 \frac{2}{\alpha^3}\left[(1+i) + \frac{\alpha^3}{a^3}\right], \tag{4.36}$$

where the last term represents the contribution from the boundary. However, it has been assumed that $\alpha^2/a^2 \ll 1$, thus enabling the limits of integration $\pm a$ to be extended to infinity. Thus, in so far as the limits can be extended to infinity, the boundary term β_ϕ can be neglected. The error in neglecting the boundary term is of order α^3/a^3. As will be shown below, to leading order $\alpha^2 \simeq x_i$ and $a \simeq x_c$. Neglecting β_ϕ amounts to neglecting a small stabilizing term (of order $x_i^{3/2}/x_c^3$) which slightly enhances the particle inertial stabilization as indicated in Eq. (4.36).

Consider the electrostatic variational integral $\mathcal{L}_\phi$ given by Eq. (4.24). $\mathcal{L}_\phi$ can be rewritten as follows:

$$\mathcal{L}_\phi = \mathcal{L}_{\phi_0} + \mathcal{L}_{\phi_1} \tag{4.37}$$

where

$$\mathcal{L}_{\phi_0} = \int_{-a}^{a} dx \left[\xi Z \left(\frac{d\widetilde{\phi}}{dx} \right)^2 - \Lambda \widetilde{\phi}^2 - (1 + \xi Z) \left(\widetilde{\phi} - \frac{\omega}{k_{\|} c} \widetilde{A}_{\|} \right)^2 \right] \tag{4.38}$$

and

$$\mathcal{L}_{\phi_1} = \int_{-a}^{a} dx \left[-\frac{i}{d}(\omega - \omega_{*e}) I_0 \widetilde{\phi}^2 + 2\frac{i}{d}(\omega - \omega_{*e}) I_1 \widetilde{\phi} \widetilde{A}_{\|} \frac{v_e}{c} \right.$$
$$\left. + \frac{i}{d} \omega_{*e} (\eta_J I_1)^* \widetilde{\phi}^2 - 2\frac{i}{d} \omega_{*e} (\eta_J I_2)^* \widetilde{\phi} \widetilde{A}_{\|} \frac{v_e}{c} \right]. \tag{4.39}$$

Recall that the trial function to be used for $\widetilde{\phi}$ in the above integral is given by Eq. (4.25) and that $\widetilde{A}_{\|}$ is to be taken as a constant (since it always appears in combination with $\widetilde{\phi}$).

In this section, the correction to the leading order result for $\mathcal{L}_\phi$, which is given in the previous chapter, is to be calculated. Recall that the leading order value for $\mathcal{L}_\phi$ is determined by the first and third (last) term in the expression for $\mathcal{L}_{\phi_0}$. As will be shown below, the next order corrections to $\mathcal{L}_\phi$ are due to the second (middle) term in $\mathcal{L}_{\phi_0}$, which is proportional to Λ, and the second term in $\mathcal{L}_{\phi_1}$. Notice that the first term in $\mathcal{L}_{\phi_1}$ is smaller than the Λ term in $\mathcal{L}_{\phi_0}$ by ω/ω_c, and thus can be neglected. Similarly, the third term in $\mathcal{L}_{\phi_1}$ is smaller than the Λ term in $\mathcal{L}_{\phi_0}$ by ω/ω_c and will also be neglected. As will be shown below, $\Lambda \sim (\omega - \omega_{*e})/\omega_{*e} \sim (n_b v_b / n_o v_e)$ and recall $\eta_J \sim (n_b v_b / n_o v_e)$. The fourth term in $\mathcal{L}_{\phi_1}$, when written for the regular tearing mode, is odd in x and thus will not contribute to the integral. When written for the runaway tearing mode, the fourth term in $\mathcal{L}_{\phi_1}$ when compared to the second term in $\mathcal{L}_{\phi_1}$ is smaller by the factor ω_c/ω_b, and thus will be neglected. (As is discussed below, this calculation is performed for $2x_c^2 = x_b^2$, hence, $\omega_c/\omega_b \sim v_e/v_b$.)

Neglecting the above mentioned terms, then the electrostatic integral is reduced to $\mathcal{L}_\phi = \mathcal{L}_{\phi_0} + \mathcal{L}_{\phi_1}$ where

$$\mathcal{L}_{\phi_1} \simeq \int_{-a}^{a} dx \left[2\frac{i}{d}(\omega - \omega_{*e}) I_1 \tilde{\phi}\tilde{A}_{\|} \frac{v_e}{c} \right] \tag{4.40}$$

and where $\mathcal{L}_{\phi_0}$ is given by Eq. (4.38).

Recalling that the ion Z-functions must be expanded for $x/x_i < 1$, then $\mathcal{L}_{\phi_0}$ can be written as

$$\mathcal{L}_{\phi_0} = \int_{-a}^{a} dx \left[-\left(\frac{d\tilde{\phi}}{dx}\right)^2 - \Lambda\tilde{\phi}^2 + \frac{x^2}{2x_i^2}\left(\tilde{\phi} - \frac{\omega}{k_{\|}c}\tilde{A}_{\|}\right)^2 \right]. \tag{4.41}$$

Using the trial function for $\tilde{\phi}$ given by Eq. (4.25) with $\tilde{A}_{\|} = A_0 =$ constant, and introducing the normalized variable $s^2 = x^2/\alpha^2$, then the terms appearing in $\mathcal{L}_\phi$ can be written as

$$\left(\frac{d\tilde{\phi}}{dx}\right)^2 = \frac{\phi_0^2}{\alpha^2}(s^4+1)^{-4}\left[s^{12} - 6is^{10} - 15s^8 + 20is^6 + 15s^4 - 6is^2 - 1\right], \tag{4.42a}$$

and

$$\tilde{\phi}^2 = \frac{\phi_0^2}{\alpha^2}(s^4+1)^{-2}[s^6 - 2is^4 - s^2], \tag{4.42b}$$

and

$$\left(\tilde{\phi} - \frac{\omega}{k_{\|}c}\tilde{A}_{\|}\right)^2 = \frac{\phi_0^2}{\alpha^2}s^{-2}(s^4+1)^{-2}[-s^4 + 2is^2 + 1]. \tag{4.42c}$$

Inserting the above expressions in $\mathcal{L}_{\phi_0}$, Eq. (4.41), and expanding the limits of integration to infinity, which assumes $\alpha^2/a^2 \ll 1$, then $\mathcal{L}_{\phi_0}$ becomes

$$\mathcal{L}_{\phi_0} = \alpha\phi_0^2\left\{ -\frac{1}{\alpha^4}\left[I_{12}^b - 6iI_{10}^b - 15I_8^b + 20iI_6^b + 15I_4^b - 6iI_2^b - I_0^b\right] - \frac{\Lambda}{\alpha^2}\left[I_6^a - 2iI_4^a - I_2^a\right] + \frac{1}{2x_i^2}\left[-I_4^a + 2iI_2^a + I_0^a\right]\right\} \tag{4.43}$$

where the following notation is used:

$$I_n^a = \int_{-\infty}^{\infty} \frac{dyy^n}{(y^4+1)^2} = \frac{1}{2}\left(\frac{\Gamma\left(\frac{n+1}{4}\right)\Gamma\left(2-\frac{n+1}{4}\right)}{\Gamma(2)}\right) \tag{4.44a}$$

$$I_n^b = \int_{-\infty}^{\infty} \frac{dyy^n}{(y^4+1)^4} = \frac{1}{2}\left(\frac{\Gamma\left(\frac{n+1}{4}\right)\Gamma\left(4-\frac{n+1}{4}\right)}{\Gamma(4)}\right). \tag{4.44b}$$

As indicated above, the integrals I_n^a and I_n^b can be evaluated analytically. Specifically, the integrals appearing in Eq. (4.43) take on the following values:

$$\begin{aligned} &I_0^a = \frac{3}{8}\sqrt{2}\pi,\ I_2^a = \frac{1}{8}\sqrt{2}\pi,\ I_4^a = \frac{1}{8}\sqrt{2}\pi,\ I_6^a = \frac{3}{8}\sqrt{2}\pi; \\ &I_0^b = \frac{77}{4^4}\sqrt{2}\pi,\ I_2^b = \frac{15}{4^4}\sqrt{2}\pi,\ I_4^b = \frac{7}{4^4}\sqrt{2}\pi,\ I_6^b = \frac{5}{4^4}\sqrt{2}\pi, \\ &I_8^b = \frac{5}{4^4}\sqrt{2}\pi,\ I_{10}^b = \frac{7}{4^4}\sqrt{2}\pi,\ I_{12}^b = \frac{15}{4^4}\sqrt{2}\pi. \end{aligned} \tag{4.45}$$

Substituting the above values into Eq. (4.43), then $\mathcal{L}_{\phi_0}$ can be evaluated to give

$$\mathcal{L}_{\phi_0} = \frac{\sqrt{2}\pi}{8}\phi_0^2\alpha(1+i)\left\{\frac{1}{\alpha^4} + \frac{2i\Lambda}{\alpha^2} + \frac{1}{x_i^2}\right\}. \tag{4.46}$$

Consider now the term $\mathcal{L}_{\phi_1}$ given by Eq. (4.40). Using the trial function for $\widetilde{\phi}$ given by Eq. (4.25) and the Krook interpolation polynomial for I_1 given by Eq. (4.15b), then by letting the limits of integration go to infinity, $\mathcal{L}_{\phi_1}$ becomes

$$\mathcal{L}_{\phi_1} = \phi_0^2\alpha\left\{\frac{\Lambda\sigma_2}{x_c^2}(I_4^c - iI_2^c)\right\} \tag{4.47}$$

where $\Lambda \simeq (\omega - \omega_{*e})/2\omega_{*e}$, $\sigma_2 = (1 + 2ix_e/x_c)$ and where the following integrals have been defined:

$$I_n^c = \int_{-\infty}^{\infty} \frac{dy y^n}{(1+c^2y^2)(1+y^4)} \tag{4.48}$$

with $c^2 = \sigma_2\alpha^2/2x_c^2$. (However, since $\mathcal{L}_{\phi_1}$ is itself a small correction to the dispersion relation, the approximation can be made that $\sigma_2 \simeq 1$, thus neglecting a further correction of order ω_*/ω_c to $\mathcal{L}_{\phi_1}$.) Again, the integrals I_n^c can be evaluated explicitly to give

$$I_2^c = \frac{\sqrt{2}\pi}{2}(1+c^4)^{-1}[c^2+1-\sqrt{2}c] \tag{4.49a}$$

$$I_4^c = \frac{\sqrt{2}\pi}{2}(1+c^4)^{-1}[c^2-1+\sqrt{2}/c]. \tag{4.49b}$$

As will be shown below, $a^2 \simeq 2x_c^2$. Hence the requirement $\alpha^2/a^2 \ll 1$ implies $c^2 = \alpha^2/2x_c^2 \ll 1$. In this limit, the leading order contributions to I_2^c and I_4^c along with their first derivatives become

$$I_2^c \simeq \frac{\sqrt{2}\pi}{2}, \; I_4^c \simeq \frac{\pi}{c}$$

and

$$(cI_2^c)' \simeq \frac{\sqrt{2}\pi}{2}, \; (cI_4^c)' \simeq -\frac{\sqrt{2}\pi}{2} \tag{4.50}$$

where the prime indicates differentiation with respect to $c = \alpha/\sqrt{2}x_c$.

Hence, $\mathcal{L}_\phi = \mathcal{L}_{\phi_0} + \mathcal{L}_{\phi_1}$ becomes, using Eqs. (4.46) and (4.47),

$$\mathcal{L}_\phi = \phi_0^2\alpha\left\{\frac{\sqrt{2}\pi}{8}(1+i)\left[\frac{1}{\alpha^4}+\frac{2i\Lambda}{\alpha^2}+\frac{1}{x_i^2}\right]+\frac{\Lambda}{x_c^2}[I_4^c - iI_2^c]\right\}. \tag{4.51}$$

To find the variational parameter α, one sets $\delta\mathcal{L}_\phi/\delta\alpha = 0$. This gives

$$\frac{\sqrt{2}\pi}{8}(1+i)\left[-\frac{3}{\alpha^4}-\frac{2i\Lambda}{\alpha^2}+\frac{1}{x_i^2}\right]-\frac{\sqrt{2}\pi}{2}\frac{\Lambda}{x_c^2}(1+i)=0 \tag{4.52}$$

where the values given in Eq. (4.50) have been used in Eq. (4.51). The above expression can be rearranged to read

$$\left(1-4\Lambda\frac{x_i^2}{x_c^2}\right)\hat{\alpha}^4-\frac{2i}{\sqrt{3}}\Lambda x_i\hat{\alpha}^2-1=0 \tag{4.53}$$

where the normalization $\hat{\alpha}^4=\alpha^4/3x_i^2$ has been made. (To leading order, $\hat{\alpha}^4=1$.) Typically, $\Lambda\sim 10^{-2}$, $x_i\sim 10$ and $x_i/x_c^2\sim 1$. Introducing the small parameter $\epsilon\equiv\Lambda x_i\sim 10^{-1}$, then

$$\left(1-4\frac{x_i}{x_i^2}\epsilon\right)\hat{\alpha}^4-\frac{2i}{\sqrt{3}}\epsilon\hat{\alpha}^2-1=0. \tag{4.54}$$

Letting $\hat{\alpha}^2=\hat{\alpha}_0^2+\hat{\alpha}_1^2$, where $\hat{\alpha}_1^2/\hat{\alpha}_0^2\sim\epsilon$, then Eq. (4.54) can be solved to leading order, ϵ^0, to give $\hat{\alpha}_0^2=1$ and to first order in ϵ to give

$$\hat{\alpha}_1^2=\epsilon\left(2\frac{x_i}{x_c^2}+\frac{i}{\sqrt{3}}\right). \tag{4.55}$$

Hence, $\hat{\alpha}^2$ can be written to first order in ϵ as

$$\hat{\alpha}^2=\hat{\alpha}_0^2+\hat{\alpha}_1^2=1+\Lambda x_i\left(2\frac{x_i}{x_i^2}+\frac{i}{\sqrt{3}}\right). \tag{4.56}$$

Using the values in Eq. (4.50) for $c^2<1$, then $\mathcal{L}_\phi$ can be written as

$$\mathcal{L}_\phi=\phi_0^2\frac{\sqrt{2}\pi}{8}\left\{\left(\frac{1}{\alpha^3}+\frac{2i\Lambda}{\alpha}+\frac{\alpha}{x_i^2}\right)(1+i)+\frac{4\Lambda}{x_c}\right\} \tag{4.57}$$

where only the leading order term in $c=\alpha/\sqrt{2}x_c<1$ was retained in $\mathcal{L}_{\phi_1}$.

In the above expression, Eq. (4.57), the two terms proportional to Λ are the leading order corrections to the basic dispersion relation for the regular tearing

mode discussed in the previous section. In these correction terms it is sufficient to approximate α by its leading order contribution, $\alpha^2 \simeq \sqrt{3}x_i$. For the remaining terms in Eq. (4.57), it is necessary to use the first order correction to α given by Eq. (4.55). Substituting the expressions

$$\alpha^{-3} \simeq (\sqrt{3}x_i)^{-3/2}\left(1 - \frac{3}{2}\hat{\alpha}_1^2\right)$$

and

$$\alpha \simeq (\sqrt{3}x_i)^{1/2}\left(1 + \frac{1}{2}\hat{\alpha}_1^2\right)$$

into Eq. (4.57) for $\mathcal{L}_\phi$ then gives

$$\mathcal{L}_\phi = \phi_0^2 \frac{\sqrt{2}\pi}{8}(\sqrt{3}x_i)^{-3/2}\left\{\left[\left(1 - \frac{3}{2}\hat{\alpha}_1^2\right) + i\Lambda\sqrt{3}x_i + 3\left(1 + \frac{1}{2}\hat{\alpha}_1^2\right)\right](1+i) + 4\Lambda\frac{(\sqrt{3}x_i)^{3/2}}{x_c}\right\}. \tag{4.58}$$

Notice, however, that the $\hat{\alpha}_1^2$ contribution cancels itself out in the above equation to leave

$$\mathcal{L}_\phi = \phi_0^2 \frac{\sqrt{2}\pi}{8}(\sqrt{3}x_i)^{-3/2}\left\{[4 + i\Lambda\sqrt{3}x_i](1+i) + 4\Lambda\sqrt{3}x_i\frac{(\sqrt{3}x_i)^{1/2}}{x_c}\right\}. \tag{4.59}$$

Recall from the previous section that stability is determined from the real part of $\mathcal{L}_\phi$:

$$R_e\{\mathcal{L}_\phi\} \simeq \phi_0^2 \frac{\sqrt{2}\pi}{8}(\sqrt{3}x_i)^{-3/2}\left[4 + \sqrt{3}x_i\Lambda\left(4\frac{(\sqrt{3}x_i)^{1/2}}{x_c} - 1\right)\right]. \tag{4.60}$$

Hence, $\mathcal{L}_\phi$ represents a stabilizing contribution to the dispersion relation for $R_e\{\mathcal{L}_\phi\} > 0$. Recall that the above derivation of Eq. (4.59) assumed $c^2 = \alpha^2/2x_c^2 = \sqrt{3}x_i/2x_c^2 < 1$. Hence, the correction term, proportional to Λ, represents a stabilizing contribution provided

$$\frac{1}{16} < \frac{\sqrt{3}x_i}{x_c^2} < 2. \tag{4.61}$$

Notice that $x_c = (\omega_c/\omega_*)/(m_e/m_i)^{1/2}x_i$ and $x_i = L_s/L_n$, where typically $\omega_c/\omega_* \simeq 10$ and $L_s/L_n \simeq 16$. Hence, $x_i/x_c^2 \sim 1$.

As will be shown below, the runaway electrons produce a frequency shift scaling as $\delta\omega/\omega_* \sim (n_b v_b/n_o v_e) \sim 10^{-2}$. Hence, the correction term proportional to Λ represents an additional stabilization of the order $x_i\Lambda \sim 10^{-1}$.

Consider now the magnetic terms given by the integral $\mathcal{L}_A$ with the boundary term β_A. In this part of Ch. 4, $S_A \equiv \mathcal{L}_A - \beta_A$ is evaluated for the regular tearing mode. The evaluation of S_A for the runaway tearing mode is presented in Sec. 4.3.

For the regular tearing mode, the variational form for the magnetic terms S_A is given by

$$S_A = \int_{-a}^{a} dx \left\{ \left(\frac{d\tilde{A}_\parallel}{dx} \right)^2 + \left[b - \frac{v_e^2}{\tau v_A^2} i(\omega - \omega_{*e}) I_2 + \frac{v_e^2}{\tau v_A^2} i\omega_{*e}\eta_J I_3 \right] \tilde{A}_\parallel^2 \right\} - \tilde{A}_\parallel \tilde{A}_\parallel' \Big|_{-a}^{a} \tag{4.62}$$

where $\eta_J = 2(v_D L_n/v_e L_J)$ and $\pm a$ represents an intermediate position in the plasma beyond which $\tilde{A}_\parallel$ is determined by the ideal MHD equation, Eq. (4.21). In this calculation, $\pm a$ also serves as the variational parameter.

In the Krook approximation, the resonance functions I_2 and I_3 are given by

$$I_2 = \frac{1}{\omega_c} \frac{\frac{1}{2}\sigma_1}{1 + \frac{1}{2}\sigma_2 x^2/x_c^2} \tag{4.63a}$$

and

$$I_3 = \frac{-i}{\omega_c} \frac{\frac{3}{4}\sigma_2 x/x_c}{1 + \frac{3}{2}\sigma_2 x^2/x_c^2} \tag{4.63b}$$

where $\sigma_1 = 1 + ix_e/x_c$ and $\sigma_2 = 1 + 2ix_e/x_c$.

As discussed above, the following trial function is used:

$$\tilde{A}_{Trial} = \begin{cases} 1 + L_+ x, & x > a \\ 1 + L_- x, & x < -a \end{cases} \tag{4.64}$$

where

$$L_\pm = \lambda + C_\pm \lambda (1 \pm \lambda a/2)$$

with

$$C_+ + C_- = -\Delta_0/\lambda^2 W; C_+ - C_- = \Delta_0/\lambda$$

and $\lambda = (v_e^2/\tau v_A^2) x_e (v_D L_n / v_e L_n)$ for the regular tearing mode. Requiring $\tilde{A}_{Trial}$ to be positive implies $C_\pm \lambda a < 1$. Typically, $\Delta_0 \sim 10^{-2}$, $W \sim 10$ and $\lambda \sim 10^{-2}$.

Consider the boundary term β_A given by

$$\beta_A = \tilde{A}\tilde{A}'_{MHD}|_a - \tilde{A}\tilde{A}'_{MHD}|_{-a} \tag{4.65}$$

where

$$\tilde{A}_{Trial}(\pm a) = \tilde{A}_{MHD}(\pm a) = 1 \pm L_\pm a$$

and

$$\widetilde{A}'_{MHD}(\pm a) = f' + C_{\pm} g' \simeq \lambda + C_{\pm}\lambda(1 \pm \lambda a)$$

where the functions f and g are given by Eq. (4.29). Hence, β_A can be evaluated to give

$$\beta_A = \Delta_0\left(1 - \frac{a}{W}\right) + a(L_+^2 + L_-^2) + \frac{\lambda^2 a^2}{2}(L_+C_+ - L_-C_-). \tag{4.66}$$

At this point, it is also helpful to evaluate the variation of β_A with respect to a, since this needed to find the value of a. Hence, doing this gives

$$\frac{\delta\beta}{\delta a} = -\frac{\Delta_0}{W} + (L_+^2 + L_-^2) + 2a\lambda^2(L_+C_+ - L_-C_-) + \frac{a^2\lambda^4}{4}(C_+^2 + C_-^2). \tag{4.67}$$

The integrals appearing in the expression for S_A can be evaluated using the following identities:

$$\int \frac{dx}{a^2 + b^2x^2} = \frac{1}{ab}\tan^{-1}\frac{bx}{a} \tag{4.68a}$$

$$\int \frac{dx x^2}{a^2 + b^2x^2} = \frac{x}{b^2} - \frac{a}{b^3}\tan^{-1}\frac{bx}{a} \tag{4.68b}$$

This then gives

$$S_A = \left\{ a(L_+^2 + L_-^2) + 2ab - \frac{iv_e^2}{\tau v_A^2}\frac{\delta\omega}{\omega_*}\frac{x_e}{x_c}\sigma_1\sqrt{\frac{2}{\sigma_2}}x_c\tan^{-1}\left(\sqrt{\frac{\sigma_2}{2}}\frac{a}{x_c}\right) \right.$$
$$\left. + 2\frac{\lambda}{x_c}(L_+ + L_-)x_c\left[a - \sqrt{\frac{2}{3\sigma_2}}x_c\tan^{-1}\left(\sqrt{\frac{3\sigma_2}{2}}\frac{a}{x_c}\right)\right]\right\} - \beta_A \tag{4.69}$$

where $\delta\omega = \omega - \omega_{*e}$. The correction terms proportional to ω_*/ω_c contained in σ_1 and σ_2 are necessary to see the effects of a finite frequency shift $\delta\omega \neq 0$.

At this point, it is useful to expand the above expression for small x_e/x_c. This is done using the following expansions:

$$\sigma_1\sqrt{\frac{2}{\sigma_2}}\tan^{-1}\left(\sqrt{\frac{\sigma_2}{2}}\frac{a}{x_c}\right) \simeq \sqrt{2}\left[\tan^{-1}\left(\frac{a}{\sqrt{2}x_c}\right) + \frac{ix_e}{x_c}\frac{a/\sqrt{2}x_c}{1+a^2/2x_c^2}\right] \qquad (4.70a)$$

and

$$\sqrt{\frac{2}{3\sigma_2}}\tan^{-1}\left(\sqrt{\frac{3\sigma_2}{2}}\frac{a}{x_c}\right)$$
$$\simeq \sqrt{\frac{2}{3}}\left[\tan^{-1}\left(\sqrt{\frac{3}{2}}\frac{a}{x_c}\right) + \frac{ix_e}{x_c}\left(\frac{\sqrt{3}a/\sqrt{2}x_c}{1+3a^2/2x_c^2} - \tan^{-1}\left(\sqrt{\frac{3}{2}}\frac{a}{x_c}\right)\right)\right]. \qquad (4.70b)$$

Inserting the above expansions into Eq. (4.69) for S_A gives

$$S_A = \left\{2\sqrt{\frac{2}{3}}x_c b\hat{a} - \frac{iv_e^2}{\tau v_A^2}\frac{\delta\omega}{\omega_{*e}}\sqrt{2}x_e\left[\tan^{-1}\left(\frac{\hat{a}}{\sqrt{3}}\right) + \frac{ix_e}{x_c}\frac{\hat{a}/\sqrt{3}}{1+\hat{a}^2/3}\right]\right.$$
$$\left. + 2\lambda(L_+ + L_-)\sqrt{\frac{2}{3}}x_c\left[\hat{a} - \left(\tan^{-1}\hat{a} + \frac{ix_e}{x_c}\left(\frac{\hat{a}}{1+\hat{a}^2} - \tan^{-1}\hat{a}\right)\right)\right]\right\}$$
$$- \left\{\Delta_0\left(1 - \hat{a}\sqrt{\frac{2}{3}}\frac{x_c}{W}\right) + \frac{\hat{a}^2}{3}\lambda^2 x_c^2(L_+C_+ - L_-C)\right\} \qquad (4.71)$$

where the normalized parameter $\hat{a}^2 = 3a^2/2x_c^2$ was introduced.

To find the variational parameter $\hat{a}$, which represents the width of the dissipation layer, S_A must be varied with respect to $\hat{a}$. Using Eq. (4.69) gives

$$\frac{\delta S_A}{\delta a} = \left\{2b - i\frac{v_e^2}{\tau v_A^2}\frac{\delta\omega}{\omega_{*e}}\frac{x_e}{x_c}\frac{\sigma_1}{1+\sigma_2 a^2/2x_c^2} + \frac{\lambda}{x_c}\frac{3\sigma_2}{2x_c}\frac{2a^2(L_+ + L_-)}{1+3\sigma_2 a^2/2x_c^2}\right\}$$
$$- \left\{-\frac{\Delta_0}{W} + 2a\lambda^2(L_+C_+ - L_-C_-) + \frac{a^2\lambda^4}{4}(C_+^2 + C_-^2)\right\}. \qquad (4.72)$$

Notice $a\lambda^2(L_+C_+ - L_-C_-) \sim \Delta_0^2 a/W$ and $a^2\lambda^4(C_+^2 + C_-^2) \sim \Delta_0^2 a^2/W$. Hence, these two terms are smaller than Δ_0/W by at least $\Delta_0 a < 10^{-1}$ and will be neglected. Also the approximation will be made that $L_+ + L_- = 2\lambda - \Delta_0/(\lambda W) + \lambda a\Delta_0/2 \simeq -\Delta_0/(\lambda W)$. The leading order contributions retained in the above approximations are sufficient, since the goal of this calculation is to determine the correction due to finite ω_{*e}/ω_c, and the terms neglected were independent of ω_{*e}/ω_c. Using the normalized parameter $\hat{a}$ and neglecting the term $b \sim 10^{-4}$, then Eq. (4.72) becomes

$$-i\frac{v_e^2}{\tau v_A^2}\frac{\delta\omega}{\omega_{*e}}\frac{x_e}{x_c}\frac{\sigma_1}{1+\sigma_2\hat{a}^2/3} - \frac{\Delta_0}{W}\frac{2\sigma_2\hat{a}^2}{1+\sigma_2\hat{a}^2} + \frac{\Delta_0}{W} = 0 \tag{4.73}$$

where $\delta S_A/\delta a$ has been set to zero to determine $\hat{a}$.

Assuming $(v_e^2/\tau v_A^2)(\delta\omega/\omega_{*e}) \sim \Delta_0/W$, then to zeroth order in x_e/x_c Eq. (4.73) specifies $\hat{a}_0^2 = 1$. Letting $\hat{a}^2 = \hat{a}_0^2 + \hat{a}_1^2$, where $\hat{a}_0^2/\hat{a}_1^2 \sim x_e/x_c$, then to first order in x_e/x_c Eq. (4.73) becomes,

$$-i\frac{v_e^2}{\tau v_A^2}\frac{\delta\omega}{\omega_{*e}}\frac{x_e}{x_c}\frac{3}{4} - i\frac{\Delta_0}{W}\frac{x_e}{x_c} - \frac{\Delta_0}{W}\frac{\hat{a}_1^2}{2} = 0.$$

Hence, $a^2 = (2x_c^2/3)\hat{a}^2$ with $\hat{a}^2 = \hat{a}_0^2 + \hat{a}_1^2$, where $\hat{a}_0^2 = 1$ and

$$\hat{a}_1^2 = -2i\frac{x_e}{x_c}\left(1 + \frac{3}{4}\frac{v_e^2}{\tau v_A^2}\frac{\delta\omega}{\omega_{*e}}\frac{W}{\Delta_0}\right). \tag{4.74}$$

The above equation for $\hat{a}_1^2$ represents the correction due to finite ω_{*e}/ω_c to the leading order result $\hat{a}_0^2$ calculated in the previous section. This result assumes $(v_e^2/\tau v_A^2)(\delta\omega/\omega_{*e}) \sim \Delta_0/W$ and $\lambda a, \Delta_0 a < x_e/x_c$. Typically, $\Delta_0 \sim \lambda$ and $W \sim a$.

Introducing the notation

$$\hat{a} = \hat{a}_0(1 + \hat{a}_1^2/2\hat{a}_0^2) = 1 - i\frac{x_e}{x_c}\left(1 + \frac{3}{4}\frac{v_e^2}{\tau v_A^2}\frac{\delta\omega}{\omega_{*e}}\frac{W}{\Delta_0}\right)$$
$$\equiv 1 + \delta\hat{a} \tag{4.75}$$

and using the expansions

$$\tan^{-1}\left(\hat{a}/\sqrt{3}\right) \simeq \frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{4}\delta\hat{a} \quad \text{and} \quad \tan^{-1}\hat{a} \simeq \frac{\pi}{4} + \hat{\delta}\hat{a},$$

then the expression S_A given in Eq. (4.71) can be written

$$\begin{aligned} S_A = \frac{2\sqrt{2}}{3}x_c b - \frac{iv_e^2}{\tau v_A^2}\frac{\delta\omega}{\omega_{*e}}x_e\sqrt{\frac{2}{3}}\left[1 + \frac{3}{4}\left(\delta\hat{a} + i\frac{x_e}{x_c}\right)\right] \\ - 2\frac{\Delta_0}{W}\sqrt{\frac{2}{3}}x_c\left[1 + \delta\hat{a} - \left(\frac{\pi}{4} + \frac{\delta\hat{a}}{2} + \frac{ix_e}{x_c}\left(\frac{1}{2} - \frac{\pi}{4}\right)\right)\right] \\ - \Delta_0\left(1 - \sqrt{\frac{2}{3}}\frac{x_c}{W}(1 + \delta\hat{a})\right) \end{aligned} \tag{4.76}$$

where the approximation $L_+ + L_- \simeq -\Delta_0/(\lambda W)$ has been used and the last term in Eq. (4.71) has been neglected, which is consistent with the above calculation of $\hat{a}^2 = \hat{a}_0^2 + \hat{a}_1^2$.

Using the definition of $\hat{\delta a}$ given in Eq. (4.75) allows the above expression for S_A to be written as

$$\begin{aligned} S_A = \frac{2\sqrt{2}}{3}x_c b - \frac{iv_e^2}{\tau v_A^2}\frac{\delta\omega}{\omega_{*e}}\sqrt{\frac{2}{3}}x_e\left(1 - \frac{9}{16}i\frac{x_e}{x_c}\frac{v_e^2}{\tau v_A^2}\frac{\delta\omega}{\omega_{*e}}\frac{W}{\Delta_o}\right) \\ - \frac{ix_e}{x_c}\frac{1}{2}\sqrt{\frac{2}{3}}x_c\frac{\Delta_o}{W} - \Delta_0\left(1 - \frac{1}{2}\sqrt{\frac{2}{3}}\frac{x_c}{W}\right) \end{aligned} \tag{4.77}$$

The dispersion relation for the regular tearing mode is found using the total variational form $S = \mathcal{L}_\phi + (\tau v_A^2/dc^2)S_A$, where the electrostatic term $\mathcal{L}_\phi$ is given by Eq. (4.59) and where the magnetic term S_A is given by the above expression. This gives

$$S = -i\sqrt{\frac{2}{3}}\frac{x_e}{2}\frac{v_e^2}{c^2}\left\{\frac{\delta\omega}{\omega_{*e}}\left(1 - \frac{9}{16}\frac{iv_e^2}{\tau v_A^2}\frac{\delta\omega}{\omega_{*e}}\frac{W}{\Delta_o}\frac{x_e}{x_c}\right)\right.$$
$$+ i\sqrt{\frac{2}{3}}\frac{\tau v_A^2}{x_e v_e^2}\left[\frac{2\sqrt{2}}{3}x_c b - i\sqrt{\frac{2}{3}}\frac{x_e}{2}\frac{\Delta_o}{W} - \Delta_0\left(1 - \frac{1}{2}\sqrt{\frac{2}{3}}\frac{x_c}{W}\right)\right]$$
$$\left. + i\frac{\pi}{4}\frac{v_i}{v_e}(\sqrt{3}x_i)^{-1/2}\left[\left(4 + i\frac{\sqrt{3}}{2}x_i\frac{\delta\omega}{\omega_{*e}}\right)(1+i) + 2\sqrt{3}x_i\frac{\delta\omega}{\omega_{*e}}\frac{(\sqrt{3}x_i)^{1/2}}{x_c}\right]\right\}. \tag{4.78}$$

Setting the imaginary part of S equal to zero gives the real frequency shift:

$$\frac{\delta\omega_r}{\omega_{*e}} = \pi\left(\frac{m_e}{\sqrt{3}\tau m_i x_i}\right)^{1/2}\left(1 + \frac{\sqrt{3}}{8}\frac{\delta\omega_r}{\omega_{*e}}x_i\right) - \frac{1}{2}\frac{m_e}{m_i}\frac{1}{\beta_i}\frac{\Delta_o}{W} \tag{4.79}$$

where $\delta\omega_r = \omega_{real} - \omega_{*e}$.

Setting the real part of S equal to zero gives the growth rate for the regular tearing mode:

$$\frac{\gamma}{\omega_{*e}} = \left(\frac{3\tau m_e}{2m_i}\right)^{1/2}\frac{1}{x_i\beta_i}\left\{\Delta_0\left(1 - \frac{1}{\sqrt{6}}\frac{x_c}{W}\right) - \frac{2\sqrt{2}}{3}x_c b\right.$$
$$- \frac{\pi}{\sqrt{3}}\left(\frac{2x_i}{\sqrt{3}}\right)^{1/2}\frac{\beta_i}{\tau}\left[1 + \frac{\sqrt{3}}{2}x_i\frac{\delta\omega_r}{\omega_{*e}}\left(\frac{(\sqrt{3}x_i)^{1/2}}{x_c} - \frac{1}{4}\right)\right]$$
$$\left. + \frac{9}{16}\sqrt{\frac{2}{3\tau}}\left(\frac{m_i}{m_e}\right)^{3/2}\beta_i^2 x_i\left(\frac{\delta\omega_r}{\omega_{*e}}\right)^2\frac{W}{\Delta_o}\frac{x_e}{x_c}\right\}. \tag{4.80}$$

In the expression for the real frequency shift $\delta\omega_r/\omega_{*e}$, Eq. (4.79), there are two main contributions. The first, proportional to $(m_e/m_i x_i)^{1/2}$, is the result of a nonzero electrostatic potential $\tilde{\phi}$ and represents a frequency shift due to finite ion inertia. The second, proportional to $(m_e\Delta_0/m_i\beta_i W)$, represents a frequency shift resulting from the correction to the dissipative layer width, $\hat{a} = 1 + \delta\hat{a}$. (The term

proportional to $\delta\omega_r/\omega_{*e}$ on the right of Eq. (4.79) can be neglected, since $\delta\omega/\omega_{*e}$ is small.) Typically, $\beta_i \sim m_e/m_i$, $x_i = L_s/L_n \sim 10$, $\Delta_0 \sim 10^{-2}$ and $W \sim 10$. Hence,

$$\frac{\delta\omega_r}{\omega_{*e}} \simeq \pi \left(\frac{m_e}{\sqrt{3}\tau m_i x_i} \right)^{1/2} - \frac{1}{2}\frac{m_e}{m_i}\frac{1}{\beta_i}\frac{\Delta_0}{W}. \tag{4.81}$$

Typically, the first term is 10^{-2} and the second term is 10^{-3}, which agrees with the numerical results indicating a small positive shift away from ω_{*e} or the order of 10^{-2}. However, at certain parameter regimes (at low β_i and high Δ_0) a small negative frequency shift is observed, which also agrees with the above expression.

In the derivation of the expression for the growth rate, Eq. (4.80), it is important to keep in mind that the following assumptions were made:

$$\frac{\Delta_0 a}{\lambda W} < 1, \frac{(\sqrt{3}x_i)^{1/2}}{x_c} < 1, \text{ and } \frac{v_e^2}{\tau v_A^2}\frac{\delta\omega}{\omega_{*e}}\frac{x_e}{x_c} < \frac{\Delta_o}{W} \tag{4.82}$$

as well as the implicit assumption $\omega_{*e}/\omega_c < 1$.

The first term in the expression for the growth rate, proportional to Δ_0, represents the ideal MHD energy drive for the tearing mode. Included in this term is stabilizing effect resulting from a turbulent broadening of the dissipation layer. Hence, $\Delta_0(1 - x_c/\sqrt{6}W)$ represents the magnetic energy outside of the dissipation layer which is available to drive the tearing mode. Clearly, if $x_c > \sqrt{6}W$, then insufficient free energy exists to drive the mode and stabilization occurs. Here, W represents the width at which the magnetic islands saturate nonlinearly. Since $x_c \sim D^{1/3}$ and if $D \sim 1/n$, then the condition $x_c > \sqrt{6}W$ implies stability at densities below some critical density.

The second term in the expression for the growth rate, proportional to $b = \rho_i^2 k_y^2$, represents the energy required to bend the magnetic field lines within the dissipation layer. This term is stabilizing, but typically small ($x_c b \sim 10^{-3}$).

The third term on the right of Eq. (4.80), which is proportional to $\sqrt{x_i}\beta_i$, represents the energy necessary to sustain the particle oscillation at the real frequency

$\omega = \omega_{*e} + \delta\omega$. This term is the result of including a nonzero electrostatic potential $\widetilde{\phi}$ which couples the wave dynamics of the tearing mode to the drift wave branch. (The electrostatic drift wave [15] is described by $L_1\widetilde{\phi} = 0$, and the magnetic tearing mode [13] is described by $L_2\widetilde{A}_{||} = 0$.) Since this term is proportional to β_i, then this implies that the tearing mode can be stabilized at sufficiently high densities (for β_i above some critical value). Included in this term is a correction proportional to $\delta\omega/\omega_{*e}$. This term represents the additional energy required to maintain the ion oscillation at the shifted frequency $\omega = \omega_{*e} + \delta\omega$.

The fourth term in the growth rate expression, which is proportional to $(\delta\omega_r/\omega_{*e})^2$, is a destabilizing term arising from a nonzero frequency shift $\delta\omega_r$. This term can be viewed as a correction to the energy drive which results when the real frequency of the mode is shifted away from ω_{*e}. Assuming the scaling $(v_e^2/\tau v_A^2)(\delta\omega/\omega_{*e}) \sim \Delta_0/W$ implies that this term scales as and is the order of the correction term to the ion inertial stabilization (that part of the third term proportional to $\delta\omega/\omega_{*e}$). As was assumed in this calculation, these corrections to the growth rate are of order $\omega_{*e}/\omega_c \sim 10^{-1}$.

4.3 Runaway Tearing

The goal of this section of Ch. 4 is to calculate the corrections of order ω_{*e}/ω_c to the dispersion relation for the runaway tearing mode in order to analyze the effects of a finite frequency shift $\delta\omega = \omega - \omega_{*e} \neq 0$. This procedure is identical to that used in Sec. 4.2 where the effects of a finite frequency shift were calculated for the regular tearing mode. As discussed in Sec. 4.1, the equilibrium current for the regular tearing mode is modeled by a drifted Maxwellian; whereas a low density, monoenergetic electron beam on the tail of a Maxwellian is used to model the equilibrium current in the runaway tearing mode.

The coupled equations describing the fluctuation potentials, $L_1\tilde{\phi} + L_x\tilde{A}_{||} = 0$ and $L_2\tilde{A}_{||} + L_x\tilde{\phi} = 0$, are defined in Sec. 4.1 by Eqs. (4.13). As in Sec. 4.2, the variational form S defined by Eq. (4.14) is used to calculate the dispersion relation. Again, the variational form S is separated into magnetic and electrostatic terms, $S = (\tau v_A^2/dc^2)S_A + S_\phi$, where $S_A = \mathcal{L}_A - \beta_A$ and $S_\phi = \mathcal{L}_\phi - \beta_\phi$. The integrals $\mathcal{L}_A$ and $\mathcal{L}_\phi$ are defined by Eq. (4.23) and Eq. (4.24) along with the boundary terms β_A and β_ϕ. As discussed in Sec. 4.2, the coupled equations describing $\tilde{\phi}$ and $\tilde{A}_{||}$ reduced to ideal MHD with the additional constraint $\omega = \omega_{*e}$ at large x, $|x| > x_c$. This holds for both the regular as well as the runaway tearing mode. Hence, the trial functions used to model $\tilde{\phi}$ and $\tilde{A}_{||}$ in Sec. 4.2 for the regular tearing mode, given by Eq. (4.25) and Eq. (4.26), are suitable to use here for the runaway tearing mode.

As discussed in Sec. 4.2, the boundary term β_ϕ can be neglected. The electrostatic integral $\mathcal{L}_\phi$ was calculated in Sec. 4.2 and is given by Eq. (4.59) to be

$$\mathcal{L}_\phi = \phi_0^2 \frac{\sqrt{2}\pi}{8}(\sqrt{3}x_i)^{-3/2}\left\{[4 + i\Lambda\sqrt{3}x_i](1+i) + 4\Lambda\sqrt{3}x_i\frac{(\sqrt{3}x_i)^{1/2}}{x_c}\right\} \qquad (4.83)$$

where it has been assumed that $(\sqrt{3}x_i)^{1/2}/x_c < 1$. This expression for $\mathcal{L}_{\tilde{\phi}}$ was calculated to be independent of the current carrying terms and, hence, it holds for

both the regular and runaway tearing modes. [Notice that neither v_D or v_b appears in Eq. (4.83).]

The boundary term β_A for the runaway tearing mode is again identical to that used in Sec. 4.2 for the regular tearing mode. This is true since β_A depends only on the trial function and the outer ideal MHD solution for $\widetilde{A}_{\|}$. The expression for β_A is given by Eq. (4.66), and the expression for $\delta\beta_A/\delta a$ is given by Eq. (4.67).

The magnetic integral $\mathcal{L}_A$ for the runaway tearing mode is given by

$$\mathcal{L}_A = \int_{-a}^{a} dx \left\{ \left(\frac{d\widetilde{A}_{\|}}{dx} \right)^2 + \left[b - \frac{iv_e^2}{\tau v_A^2} \frac{\delta\omega}{\omega_{*e}} \frac{x_e}{2x_c} \frac{\sigma_1}{1+\sigma_2 x^2/2x_c^2} + i\frac{\lambda}{x_b} \frac{1-ix/x_b}{1+x^2/x_b^2} \right] \widetilde{A}_{\|}^2 \right\} \tag{4.84}$$

where Krook approximate forms for the resonance functions $I_2(x)$ and $K_2(x)$ have been used used. For the runaway case, $\lambda \equiv (v_e^2/\tau v_A^2) x_e (n_b v_b L_n / n_o v_e L_J)$.

The only difference in $\mathcal{L}_A$ between the regular and runaway tearing modes is the last term which is proportional to λ. This is the only term which involves the equilibrium current. As can be seen by comparing Eq. (4.84) with Eq. (4.62), the term proportional to λ for the runaway case has gained a significant imaginary piece whose counterpart is absent for the regular case. The importance of this imaginary term will be the production of a significant frequency shift for the runaway tearing mode which is absent in the regular tearing mode. Combining the imaginary part of the λ term in Eq. (4.84) with the term proportional to $\delta\omega/\omega_{*e}$ implies that a real frequency shift will arise of the form

$$\frac{\delta\omega}{\omega_{*e}} \simeq \frac{\tau v_A^2}{v_e^2} \frac{\lambda}{x_e} \frac{2x_c}{x_b} = \frac{2x_c}{x_b} \frac{n_b}{n_o} \frac{v_b}{v_e} \frac{L_n}{L_J} \tag{4.85}$$

which, as will be shown below, is the correct form for $\delta\omega/\omega_{*e}$ arising from the electron beam.

Using the trial function for $\widetilde{A}_{\|}$ given by Eq. (4.64) along with the integral identities given by Eq. (4.68) allows $\mathcal{L}_A$ to be written as

$$\mathcal{L}_A = \left\{ a(L_+^2 + L_-^2) + 2ab - \frac{iv_e^2}{\tau v_A^2}\frac{\delta\omega}{\omega_{*e}}\frac{x_e}{x_c}\sigma_1\sqrt{\frac{2}{\sigma_2}}x_c \tan^{-1}\left(\sqrt{\frac{\sigma_2}{2}}\frac{\alpha}{x_c}\right) \right.$$
$$\left. + i\frac{\lambda}{x_b}\left[2x_b\tan^{-1}\frac{a}{x_b} - 2\frac{i}{x_b}(L_+ + L_-)\left(x_b^2 a - x_b^3\tan^{-1}\frac{a}{x_b}\right)\right]\right\}. \quad (4.86)$$

The term proportional to $\delta\omega/\omega_{*e}$ can be expanded for small ω_{*e}/ω_c by using Eq. (4.70a). This allows $S_A = \mathcal{L}_A - \beta_A$ to be written as

$$S_A = \left\{ 2ab - \frac{iv_e^2}{\tau v_A^2}\sqrt{2}x_e\left[\frac{\delta\omega}{\omega_{*e}}\tan^{-1}\frac{a}{\sqrt{2}x_c} - \sqrt{2}\frac{n_b}{n_o}\frac{v_b}{v_e}\frac{L_n}{L_J}\tan^{-1}\frac{a}{x_b}\right.\right.$$
$$\left.\left. + i\frac{x_e}{x_c}\frac{\delta\omega}{\omega_{*e}}\frac{a/\sqrt{2}x_c}{1 + a^2/2x_c^2}\right] + 2\lambda(L_+ + L_-)\left(a - x_b\tan^{-1}\frac{a}{x_b}\right)\right\}$$
$$- \{\beta_A - a(L_+^2 + L_-^2)\}. \quad (4.87)$$

The variational parameter a is determined by varying S_A with respect to a. This gives

$$\frac{\delta S_A}{\delta a} = \left\{ 2b - \frac{iv_e^2}{\tau v_A^2}\frac{x_e}{x_c}\left[\frac{\delta\omega}{\omega_{*e}}\frac{\sigma_1}{1 + \sigma_2 a^2/2x_c^2} - 2\frac{x_c}{x_b}\frac{n_b}{n_o}\frac{v_b}{v_e}\frac{L_n}{L_J}\frac{1}{1 + a^2/x_b^2}\right]\right.$$
$$\left. + 2\lambda(L_+ + L_-)\frac{a^2/x_b^2}{1 + a^2/x_b^2}\right\} - \left\{\frac{\delta\beta_A}{\delta a} - (L_+^2 + L_-^2)\right\}. \quad (4.88)$$

The parameter a is then determined by setting $\delta S_A/\delta a = 0$.

Assuming the scaling $\delta\omega/\omega_{*e} \sim (n_b v_b L_n/n_o v_e L_J)$ and $a^2 \sim x_b^2$ implies that the largest terms in Eq. (4.88) are the two terms enclosed by the square brackets. The remaining terms are smaller by at least $\Delta_0 a/\lambda W$. Recall that it is assumed

$\Delta_0 a/\lambda W < 1$ (or $L_\pm a < 1$) which is necessary to insure that $R_e\{\tilde{A}_{||}\}$ remains positive. Letting a_0 signify the leading order contribution to a, then setting the leading order contribution of $\delta S_A/\delta a$ to zero implies

$$\frac{\delta\omega}{\omega_{*e}}\frac{1}{1+a_0^2/2x_c^2} - \frac{2x_c}{x_b}\frac{n_b}{n_o}\frac{v_b}{v_e}\frac{L_n}{L_J}\frac{1}{1+a_0^2/x_b^2} = 0. \tag{4.89}$$

However, the leading order contribution to the real frequency shift, $\delta\omega/\omega_{*e}$, is determined by setting the leading order imaginary part of S_A, as is given by Eq. (4.87), equal to zero. This gives the beam frequency shift to be

$$\frac{\delta\omega}{\omega_{*e}} \simeq \sqrt{2}\frac{n_b}{n_o}\frac{v_b}{v_e}\frac{L_n}{L_J}\frac{\tan^{-1}(a_0/x_b)}{\tan^{-1}(a_0/\sqrt{2}x_c)}. \tag{4.90}$$

Inserting Eq. (4.90) into Eq. (4.91) then gives the following equation to determine a_0:

$$\frac{\tan^{-1}\hat{a}_0}{\tan^{-1}(\hat{a}_0/\hat{\beta})} - \hat{\beta}\frac{1+\hat{a}_0^2/\hat{\beta}^2}{1+\hat{a}_0^2} = 0 \tag{4.91}$$

where the normalized parameters $\hat{a} = a/x_b$ and $\hat{\beta} = \sqrt{2}x_c/x_b$ have been introduced.

Unfortunately, Eq. (4.91) cannot be solved analytically to yield $\hat{a}_0$ for arbitrary $\hat{\beta}$. Actually, $\hat{\beta} = \sqrt{2}(v_b D_0/v_e D_b)^{1/3}$ where D_0 represents the spatial diffusion coefficient for the bulk electrons and D_b represents the spatial diffusion coefficient for the beam electrons. To get around the problem of not being able to solve Eq. (4.91) analytically for arbitrary $\hat{\beta}$, the special case where $D_b/D_0 = 2^{2/3}v_b/v_e$ will be examined. In this special case $\hat{\beta} \equiv 1$ or $x_b = \sqrt{2}x_c$. When this is true, Eq. (4.91) is exactly satisfied for any value of $\hat{a}_0$. Hence, $\hat{a}_0$ must be determined form the higher order terms remaining in Eq. (4.88) for $\delta S_A/\delta a$.

Notice that for $x_b = \sqrt{2}x_c$, then the beam frequency shift becomes, according to Eq. (4.90),

$$\frac{\delta\omega_b}{\delta\omega_{*e}} \equiv \sqrt{2}\frac{n_b}{n_o}\frac{v_b}{v_e}\frac{L_n}{L_J}. \tag{4.92}$$

Letting $\delta\omega_I \equiv \delta\omega - \delta\omega_b$ (as will be shown below, $\delta\omega_I/\omega_{*e} \sim (m_e/m_i x_i)^{1/2}$ and is the frequency shift by finite ion inertia), then for $x_b = \sqrt{2}x_c$ Eq. (4.88) can be written as

$$\frac{\delta S_A}{\delta a} = -\frac{iv_e^2}{\tau v_A^2}\frac{x_e}{x_c}\left(\frac{\delta\omega_I}{\omega_{*e}}\frac{1}{1+\hat{a}^2} + \frac{ix_e}{x_c}\frac{\delta\omega}{\omega_{*e}}\frac{1-\hat{a}^2}{(1+\hat{a}^2)^2}\right) + \frac{\Delta_o}{W}\left(1 - \frac{2\hat{a}^2}{1+\hat{a}^2}\right) \tag{4.93}$$

where $\hat{a}^2 = a^2/x_b^2$. In writing the above expression, the terms $2b, 2a\lambda^2(L_+C_+ - L_-C_-)$ and $a^2\lambda^4(C_+^2 + C_-^2)/4$ were neglected (as they were in Sec. 4.2) and the expansion

$$\frac{\sigma_1}{1+\sigma_2\hat{a}^2} \simeq \frac{1}{1+\hat{a}^2}\left(1 + \frac{ix_e}{x_c}\frac{1-\hat{a}^2}{1+\hat{a}^2}\right)$$

was used.

Assuming $\delta\omega_I/\delta\omega_b \sim x_e/x_c$ and assuming $(v_e^2/\tau v_A^2)x_e/x_c(\delta\omega_I/\omega_{*e}) < \Delta_0/W$, as was done in Sec. 4.2, then to leading order $\delta S_A/\delta a = 0$ becomes

$$\frac{\Delta_o}{W}\left(1 - \frac{2\hat{a}_0^2}{1+\hat{a}_0^2}\right) = 0 \tag{4.94}$$

where $\hat{a}^2 = \hat{a}_0^2 + \hat{a}_1^2$ and $\hat{a}_1^2/\hat{a}_0^2 < 1$. Hence, Eq. (4.94) specifies $\hat{a}_0^2 = 1$.

To calculate the next order correction $\hat{a}_1^2$, $\delta S_A/\delta a = 0$ can be written as

$$-\frac{iv_e^2}{\tau v_A^2}\frac{x_e}{x_c}\left(\frac{\delta\omega_I}{\omega_{*e}}\frac{1}{1+\hat{a}_0^2} + \frac{ix_e}{x_c}\frac{\delta\omega}{\omega_{*e}}\frac{1-\hat{a}_0^2}{(1+\hat{a}^2)^2}\right) + \frac{\Delta_o}{W}\left(\frac{1-\hat{a}_0^2-\hat{a}_1^2}{1+\hat{a}_0^2+\hat{a}_1^2}\right) = 0. \tag{4.95}$$

Setting $\hat{a}_0^2 = 1$ then gives

$$\hat{a}_1^2 \simeq -\frac{iv_e^2}{\tau v_A^2}\frac{x_e}{x_c}\frac{\delta\omega_I}{\omega_{*e}}\frac{W}{\Delta_o}. \tag{4.96}$$

To determine the dispersion relation, the expression for $\hat{a}^2 = \hat{a}_0^2 + \hat{a}_1^2$ is inserted into the integrated expression for S_A. For the case $x_b = \sqrt{2}x_c$, Eq. (4.87) becomes

$$S_A = \left\{ 2\sqrt{2}x_c b\hat{a} - \frac{iv_e^2}{\tau v_A^2}\sqrt{2}x_e\left[\left(\frac{\delta\omega}{\omega_{*e}} - \frac{\delta\omega_b}{\omega_{*e}}\right)\tan^{-1}\hat{a} + \frac{ix_e}{x_c}\frac{\delta\omega}{\omega_{*e}}\frac{\hat{a}}{1+\hat{a}^2}\right]\right.$$
$$\left. - \Delta_0\left[1 - \sqrt{2}\frac{x_c}{W}(2\tan^{-1}\hat{a} - \hat{a})\right]\right\}. \tag{4.97}$$

Notice the following:

i) $$\hat{a} = 1 + \delta\hat{a}, \text{ where } \delta\hat{a} = \hat{a}_1^2/2\hat{a}_0^2$$

ii) $$2\tan^{-1}\hat{a} - \hat{a} \simeq 2(\tan^{-1}1 + \delta\hat{a}/2) - (1 + \delta\hat{a})) = \pi/2 - 1 \simeq 1/2$$

iii) $$\frac{\hat{a}}{1+\hat{a}^2} \simeq \frac{1+\delta\hat{a}}{1+(1+2\delta\hat{a})} = \frac{1}{2}.$$

Hence, to first order in $\delta\hat{a}$, S_A becomes

$$S_A = \left\{ 2\sqrt{2}x_c b - \frac{iv_e^2}{\tau v_A^2}\sqrt{2}x_e\left[\left(\frac{\delta\omega}{\omega_{*e}} - \frac{\delta\omega_b}{\omega_{*e}}\right)\left(\frac{\pi}{4} - \frac{i}{4}\frac{v_e^2}{\tau v_A^2}\frac{\delta\omega_I}{\omega}\frac{W}{\Delta_o}\frac{x_e}{x_c}\right)\right.\right.$$
$$\left.\left. + \frac{ix_e}{2x_c}\frac{\delta\omega}{\omega_{*e}}\right] - \Delta_0\left(1 - \frac{\sqrt{2}}{2}\frac{x_c}{W}\right)\right\}. \tag{4.98}$$

To find the dispersion relation for the runaway tearing mode, the total variational form is used, $S = (\tau v_A^2/dc^2)S_A + S_\phi$. Using Eq. (4.83) and Eq. (4.98), $\underline{S}$ can be written as

$$
\begin{aligned}
S = \; & i\frac{\sqrt{2}}{2}\frac{v_e^2}{c^2}x_e\Bigg\{\Bigg[\left(\frac{\delta\omega}{\omega_{*e}}-\frac{\delta\omega_b}{\omega_{*e}}\right)\left(\frac{\pi}{4}-\frac{i}{4}\frac{x_e}{x_c}\frac{v_e^2}{\tau v_A^2}\frac{\delta\omega_I}{\omega_{*e}}\frac{W}{\Delta_o}\right) \\
& + \frac{i}{2}\frac{x_e}{x_c}\frac{\delta\omega}{\omega_{*e}}\Bigg] + \frac{i}{\sqrt{2}}\frac{\tau v_A^2}{x_e v_e^2}\left[2\sqrt{2}x_c b - \Delta_0\left(1-\frac{\sqrt{2}}{2}\frac{x_c}{W}\right)\right] \\
& + \frac{i\pi}{4\sqrt{3}}\frac{v_i}{v_e}(\sqrt{3}x_i)^{-1/2}\Bigg[\left(4+i\frac{\sqrt{3}}{2}x_i\frac{\delta\omega}{\omega_{*e}}\right)(1+i) \\
& + 2\sqrt{3}x_i\frac{\delta\omega}{\omega_{*e}}\frac{(\sqrt{3}x_i)^{1/2}}{x_c}\Bigg]\Bigg\}. \qquad (4.99)
\end{aligned}
$$

The real frequency shift for the runaway tearing mode is given by setting the imaginary part of S equal to zero. This gives

$$
\frac{\delta\omega_r}{\omega_{*e}} = \sqrt{2}\frac{n_b}{n_o}\frac{v_b}{v_e}\frac{L_n}{L_J} + \frac{4}{\sqrt{3}}\left(\frac{m_e}{\sqrt{3}\tau m_i x_i}\right)^{1/2}\left(1+\frac{\sqrt{3}}{8}\frac{\delta\omega}{\omega_{*e}}x_i\right). \qquad (4.100)
$$

The growth rate of the runaway tearing mode is given by setting the real part of S equal to zero. This gives

$$
\begin{aligned}
\frac{\gamma}{\omega_{*e}} = \; & \frac{4}{\pi}\left(\frac{\tau m_e}{2m_i}\right)^{1/2}\frac{1}{\beta_i x_i}\Bigg\{\Delta_0\left(1-\frac{1}{\sqrt{2}}\frac{x_c}{W}\right) - 2\sqrt{2}x_c b - \left(\frac{2m_i}{\tau m_e}\right)^{1/2}\frac{\beta_i x_i}{2}\frac{x_e}{x_c}\frac{\delta\omega}{\omega_{*e}} \\
& - \pi\sqrt{\frac{2}{3}}\frac{\beta_i}{\tau}\left(\frac{x_i}{\sqrt{3}}\right)^{1/2}\left[1+\frac{\sqrt{3}}{2}x_i\frac{\delta\omega}{\omega_{*e}}\left(\frac{(\sqrt{3}x_i)^{1/2}}{x_c}-\frac{1}{4}\right)\right] \\
& + \frac{1}{4\sqrt{2\tau}}\left(\frac{m_i}{m_e}\right)^{3/2}\beta_i^2 x_i\left(\frac{\delta\omega_I}{\omega_{*e}}\right)^2\frac{W}{\Delta_o}\frac{x_e}{x_c}\Bigg\}, \qquad (4.101)
\end{aligned}
$$

where $\delta\omega_I/\omega_{*e} = \delta\omega_r/\omega_{*e} - \delta\omega_b/\omega_{*e}$ and $\delta\omega_b/\omega_{*e} = \sqrt{2}(n_b v_b L_n/n_o v_e L_J)$.

The expression for the real frequency shift of the runaway tearing mode, Eq. (4.100), has two significant contributions. The first is the beam frequency shift $\delta\omega_b/\omega_{*e} = \sqrt{2}(n_b v_b L_n/n_o v_e L_J)$. This can be viewed as a Doppler frequency shift

by viewing the beam electrons as a fluid of density n_b moving at speed $\delta v_b = \delta\omega_b/k_y$ relative to the rest frame. This point will be discussed further in the last part of Ch. 5. The second contribution to the frequency shift is the ion inertial term, $\delta\omega_I/\omega_{*e} \sim (m_e/\tau m_i x_i)^{1/2}$. This term appears in the frequency shift of the regular tearing mode and is the result of including a non zero electrostatic potential $\tilde{\phi}$. Physically, this shift arises from the finite inertia of the particles (specifically, the electrons). Both terms, $\delta\omega_b$ and $\delta\omega_I$, are positive and typically $\delta\omega_b > \delta\omega_I$.

The expression for the growth rate of the runaway tearing mode, Eq. (4.101), was derived after assuming the following:

$$\frac{\Delta_0 a}{\lambda W} < 1,\ \frac{(\sqrt{3}x_i)^{1/2}}{x_c} < 1,\ \frac{v_e^2}{\tau v_A^2}\frac{\delta\omega_I}{\omega_{*e}}\frac{x_e}{x_c} < \frac{\Delta_o}{W},$$
$$\frac{\delta\omega_I}{\delta\omega_b} < 1 \text{ and } x_b \equiv \sqrt{2}x_c. \tag{4.102}$$

The first three inequalities were also assumed in the analysis of the regular tearing mode whereas the last condition, $x_b = \sqrt{2}x_c$, is necessary in order to solve for $\hat{a}^2$ analytically.

The first term in Eq. (4.101), proportional to Δ_0, represents the MHD energy drive produced in the region outside the dissipative layer. Included in this term is the stabilization which results from turbulent diffusive broadening of the dissipative layer. The second term, proportional to b, represents the energy required to bend the magnetic field lines in the dissipative layer and is stabilizing. The fourth term, proportional to $\beta_i\sqrt{x_i}$, represents the energy required to sustain the particle oscillation at the real frequency $\omega = \omega_{*e} + \delta\omega$. Included in this term is a correction for a finite frequency shift $\delta\omega$. The fifth (last term) in Eq. (4.101), proportional to $(\delta\omega_I/\omega_{*e})^2$, a correction (destabilizing) to the MHD energy drive which results due to the presence of the finite particle inertial frequency shift $\delta\omega_I$. These four terms also appeared in the expression for the growth rate of the regular tearing mode, Eq. (4.80), and were discussed previously at the end of Sec. 4.2.

The new term which appears in the growth rate for the runaway tearing mode is the third term, which is proportional to $(x_e\delta\omega/x_c\omega_{*e})$. This term results from the inclusion of the electron beam as the current carrying part of the electron equilibrium distribution. Physically, this term represents the energy required to maintain the particle oscillation at the beam shifted frequency $\delta\omega_b/\omega_{*e}$. This term scales as and is the same magnitude as the stabilizing correction to the ion inertial term (the fourth term), also resulting from the frequency shift.

In comparing the above results for the runaway tearing mode to the regular tearing mode, there exist several important differences indicating enhanced stability for the runaway tearing mode. First of all, the real frequency shift of the runaway tearing mode is significantly larger than that of the regular tearing mode. The expressions for $\delta\omega/\omega_{*e}$ of both the runaway and regular tearing mode, Eq. (4.100) and Eq. (4.79), contain the particle inertia shift $\delta\omega_I/\omega_{*e} > 0$. The runaway frequency shift has the additional positive term of the beam frequency shift $\delta\omega_b > \delta\omega_I$; whereas the regular tearing mode has the additional negative frequency shift which scales as $(m_e\Delta_0/m_iW\beta_i)$. This negative shift can, in practice, nearly cancel the positive shift $\delta\omega_I$ in the regular tearing mode case.

Comparing the growth rate expressions, Eq. (4.101) and Eq. (4.80), indicates that both runaway and regular tearing modes contain the additional stabilization correction term, proportional to $\delta\omega/\omega_{*e}$, contain in the ion inertial term (proportional to $\beta_i/\sqrt{x_i}$). The growth rate of the runaway tearing mode, however, contains the additional stabilization correction term, proportional to $(x_e\delta\omega/x_c\omega_{*e})$, resulting from the beam electrons. Since $\delta\omega$ is significantly larger (and positive) for the runaway tearing mode, the above two observations indicate greater stability for the runaway case. Notice that the last term in Eq. (4.101) for the runaway case is a destabilizing correction which is proportional to $(\delta\omega_I/\omega_{*e})^2$, independent of the beam frequency shift, and hence, is essentially the same as the corresponding term for the regular tearing mode. In short, enhanced stability for the runaway case results from the appearance of the beam frequency shift $\delta\omega_b > \delta\omega_I$, which then feeds back into the growth rate through stabilizing corrections proportional to $\delta\omega/\omega_{*e}$. These stabilizing corrections, however, are subdominant by order ω_{*e}/ω_c and, hence, are only important when the leading order terms in the expression for the growth rate, Eq. (4.101), nearly cancel.

Chapter 5
FLUID THEORY

In this chapter, a fluid theory is developed which reproduces the basic results of the previous chapters regarding the effects of turbulent electron diffusion and of runaway electrons on the tearing mode. The goal is to determine a set of fluid equations for the study of low-beta, low-frequency electromagnetic fluctuations which is the equivalent of the NSA kinetic theory of Ch. 2. It should be kept in mind that the fluid theory presented in this chapter is implicitly developed for the use of analyzing the stability of fluctuations. This fluid theory is not intended for use in the study of equilibrium transport in a turbulent system, just as the NSA kinetic theory of Ch. 2 is implicitly intended for the use of studying the stability of plasma fluctuations. The fluid model is developed as a set of fluid equations for the full fluid quantities (equilibrium plus perturbations) only in so far as to enable these equations to be linearized about the equilibrium to determine a set of equations for the perturbations. A fluid approach is useful in that it is often an easier model to interpret physically compared to a detailed kinetic mode (in addition to being mathematically simpler). In particular, it is shown that the NSA kinetic theory is equivalent to a fluid model in which both the perturbed density and perturbed momentum are diffused radially at equal rates.

In Sec. 5.1, these fluid equations are derived by taking the appropriate moments of the NSA version of the drift kinetic equation. This set of fluid equations is then used to provide expressions for the perturbed density and the perturbed currents in terms of the potentials $\tilde{\phi}$ and $\tilde{A}_{||}$. These expressions can then be combined through quasineutrality and Ampere's law to give the fluid theory equivalent of the couple equations for $\tilde{\phi}$ and $\tilde{A}_{||}$ given by Eqs. (2.30) in Sec. 2.3. This fluid model is then applied to the tearing mode in the presence of turbulent electron diffusion for two cases: (1) when the equilibrium plasma current is carried by a slow flow of the bulk electron fluid ("regular" tearing), and (2) when the equilibrium plasma current is

carried by a fast flow of a low density electron fluid, which exists in addition to a stationary bulk electron fluid ("runaway" tearing). The regular tearing mode is analyzed in Sec. 5.2 and the runaway tearing mode is analyzed in Sec. 5.3. The results for the perturbed parallel current and perturbed density as well as the growth rates for the regular and runaway tearing modes agree extremely well with the kinetic theory. In Sec. 5.4, the full fluid model is then reduced to the bare basics necessary to describe the tearing mode including the effects of turbulent diffusion and runaway electrons. This is done by keeping only the magnetic potential $\tilde{A}_{\|}$. The physical interpretation of this "bare basics" model is then discussed in detail.

5.1 Fluid Model

The basis for this fluid model will be a drift kinetic equation similar to that used in the preceding kinetic analysis. This drift kinetic equation has the following general form.

$$\left[\frac{\partial}{\partial t} + v_{||}\mathbf{b}\cdot\nabla + \tilde{\mathbf{v}}_{D_\perp}\cdot\nabla_\perp + \frac{q}{m}E_{||}\frac{\partial}{\partial v_{||}}\right] f = C(f) \tag{5.1}$$

where $\tilde{\mathbf{v}}_{D_\perp}$ represents the motion of the particle's guiding center due the fluctuating fields $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}_\perp$, and $C(f)$ represents changes in the particle distribution function due to transport processes involving both Coulomb collisions and turbulent effects. The above form of the drift kinetic equation also neglects the slow drift caused by the curvature of the equilibrium field lines.

For the present problem in which the magnetic fields are straight and for fluctuation frequencies $\omega \sim \omega_{*e} \ll \omega_{ce}$, the perpendicular guiding center motion is described by

$$\tilde{\mathbf{v}}_{D_\perp} = \tilde{\mathbf{v}}_{E_\perp} + v_{||}\frac{\tilde{B}_\perp}{B_0} \tag{5.2}$$

where

$$\tilde{\mathbf{v}}_{\perp E} = -\frac{c}{B}\nabla\tilde{\phi}\times\mathbf{b} \qquad \text{and} \qquad \tilde{\mathbf{B}}_\perp = \nabla\times\tilde{A}_{||}\mathbf{b}.$$

The first term in Eq. (5.2) is the $\tilde{\mathbf{E}}\times\mathbf{B}$ drift, and the second term is the perturbed perpendicular velocity resulting from the particles free streaming along the total field lines $\tilde{\mathbf{B}}_\perp + \mathbf{B}_0$.

A detailed form of the collision operator will not be presented; instead, the collision operator will be represented symbolically as follows:

$$C(f) = C^L(f) + C^T(f). \tag{5.3}$$

Here, $C^L(f)$ represents a Landau-type Coulomb collision operator [65] transformed to guiding center coordinates and $C^T(f)$ represents the turbulent collision operator. This fluid theory is concerned only with how the turbulent collision operator enters into the perturbed fluid equations as a spatial diffusion operator as described by the NSA kinetic theory in Ch. 2.

The detailed form of the full collision operator, however, is needed if one wishes to study the equilibrium transport in a turbulent system [66]. This is a very complicated problem in itself and will not be discussed here. For example, Swartz and Molvig [67–69] calculated $C^T(f)$ for electrostatic drift wave turbulence using the NSA to be

$$C^T(f) = \frac{q_e^2}{m_e^2} \sum_{k\omega} \left(2k_{\|}(x) v_{\|} \frac{\partial^2}{\partial v_{\|}^2} + \frac{k_y}{\Omega_{ce}} \frac{\partial}{\partial x} \right) Re I_{k\omega}(x, v_{\|}) \times S_{k\omega}(x) \left(2k_{\|}(x) v_{\|} \frac{\partial^2}{\partial v_{\|}^2} + \frac{k_y}{\Omega_{ce}} \frac{\partial}{\partial x} \right) f_e, \tag{5.4}$$

where Ω_{ce} is the electron cyclotron frequency. $S_{k\omega}(x)$ is the spectrum of the potential fluctuations and the resonance function $I_{k\omega}$ is given by

$$I_{k\omega}(x, v_{\|}) = \int_0^\infty d\tau \exp \left\{ i(\omega - k_{\|}(x) v_{\|})\tau - \frac{1}{3}(k_{\|}' v_{\|})^2 D \tau^3 \right\},$$

where the spatial diffusion coefficient D is a function of the spectrum $S_{k\omega}(x)$ and must be determined self-consistently. In general, the spectrum $S_{k\omega}(x)$ is determined by a nonlinear mode coupling equation [70].

In a fluid model, the precise structure of the collision operator is needed in so far as enabling one to obtain the appropriate velocity moments of such an operator. In this model the zeroth moment of the collision operator will be used as well as the first moment. The zeroth moment and the first moment will be defined as follows:

$$C_0 \equiv \int dv_{||} C(f) = C_0^L + C_0^T$$
$$C_1 \equiv m \int dv_{||} v_{||} C(f) = C_1^L + C_1^T. \tag{5.5}$$

In the above expressions, the moments of the collision operator were again divided into two parts, one representing the effects of Coluomb collisions, C^L, and one representing the effects of turbulence, C^T. The final form in which these collision operators appear in the fluid equations is in the form written for the equilibrium and the perturbations. The equilibrium forms for the collision operators are denoted as

$$\overline{C}_0 = \overline{C}_0^L + \overline{C}_0^T, \qquad \overline{C}_1 = \overline{C}_1^L + \overline{C}_1^T$$

and the perturbed forms are denoted as

$$\tilde{C}_0 = \tilde{C}_0^L + \tilde{C}_0^T, \qquad \tilde{C}_1 = \tilde{C}_1^L + \tilde{C}_1^T.$$

In the NSA drift kinetic equation for the perturbed electron distribtuion $\tilde{f}_e$, Eq. (2.20), the perturbed turbulent collision operator appears in the form

$$\tilde{C}^T = D \frac{\partial^2}{\partial x^2} \tilde{h}_e,$$

where $\tilde{f}_e = (e\tilde{\phi}/T_e) f_0 + \tilde{h}_e$. (Recall that the adiabatic response was found to be unaffected by the presense of $\tilde{\mathbf{E}} \times \mathbf{B}$ turbulent diffusion.) Hence, taking the zeroth and first velocity moments of the above expression indicates $\tilde{C}_o^T$ and $\tilde{C}_1^T$ to be of the form

$$\tilde{C}_0^T = \frac{\partial}{\partial x} D_n^T \frac{\partial}{\partial x} \left(\tilde{n} - \frac{e\tilde{\phi}}{T_e} n_0 \right)$$
$$\tilde{C}_1^T = m \frac{\partial}{\partial x} D_m^T \frac{\partial}{\partial x} \left(n_0 \tilde{V}_{||} + \tilde{n} V_{||0} \right), \tag{5.6}$$

where the fluid density and parallel velocity are defined by

$$n_0 = \int dv_{\|} \overline{f}, \qquad nV_{\|} = \int dv_{\|} v_{\|} \overline{f}$$

and

$$\tilde{n} = \int dv_{\|} \tilde{f}, \qquad \tilde{n} V_{\|0} + n_0 \tilde{V}_{\|} = \int dv_{\|} v_{\|} \tilde{f}.$$

In Eq. (5.6), D_n^T represents the turbulent diffusion coefficient for the diffusion of the perturbed density and D_m^T represents the turbulent diffusion coefficient for the perturbed moment. In general, D_n^T and D_m^T need not be equal since D is in general a function of velocity. However, for the NSA theory of Ch. 2, D was taken as a constant, hence, $D_m^T = D_n^T = D$.

Recall that in the NSA kinetic theory presented in Ch. 2, the Krook approximation was introduced in order to simplify the mathematical treatment of the diffusion operators. This amounted to replacing the diffusion operator $-D\partial^2/\partial x^2$ in Eq. (2.20) with the effective diffusive frequency $\omega_c \equiv [(k'_{\|} v_e)^2 D/3]^{1/3}$. (See Sec. 2.4.) Hence, the Krook approximation to the NSA is equivalent to the following approximate forms for the turbulent collision operators:

$$\begin{aligned} \tilde{C}_0^T &\simeq -\omega_n^T \left(\tilde{n} - \frac{e\tilde{\phi}}{T_e} n_0 \right) \\ \tilde{C}_1^T &\simeq -m\omega_m^T \left(\tilde{n} V_{\|0} + n_0 \tilde{V}_{\|} \right), \end{aligned} \tag{5.7}$$

where

$$\omega_{m,n}^T = \left[\frac{1}{3} (k'_{\|} v_e)^2 D_{m,n}^T \right]^{1/3}.$$

Again, in general, ω_n^T and ω_m^T need not be equal. However, the NSA Krook-approximate kinetic theory specifies $\omega_n^T = \omega_m^T = \omega_c$, which physically states that perturbed density and perturbed momentum are diffused at equal rates.

In order that a collisional limit of the fluid equations may be taken, a Krook Coulomb collision operator will also be introduced. Hence,

$$\begin{aligned} \tilde{C}_0^L &= 0 \\ \tilde{C}_1^T &= -m\nu\left(\tilde{n}V_{\|0} + n_0\tilde{V}_{\|}\right), \end{aligned} \tag{5.8}$$

where ν is the collision frequency. Recall that the Krook collision operator only affects momentum transfer. Although the NSA kinetic theory neglects Coulomb collisions, they will be included in this discussion in order to compare the dissipative effects of turbulent diffusion to the dissipative effects of Coulomb collisions (resistivity).

To obtain a set of fluid equations, the appropriate parallel velocity moments of the drift kinetic equation, Eq. (5.1), are performed. The zeroth moment yields the following form of the continuity equation describing the evolution of the plasma density:

$$\left[\frac{\partial}{\partial t} + \nabla\cdot\left(V_{\|}\mathbf{b} + \tilde{\mathbf{V}}_{E_\perp} + V_{\|}\frac{\tilde{\mathbf{B}}_\perp}{B}\right)\right] n = C_o^L + C_o^T \tag{5.9}$$

where $\tilde{\mathbf{V}}_{B_\perp} = (-c/B)\nabla\tilde{\phi}\times\mathbf{b}$ and the following definitions for the fluid variables have been used:

$$n \equiv \int dv_{\|} f \quad \text{and} \quad nV_{\|} \equiv \int dv_{\|} v_{\|} f.$$

The first moment of Eq. (5.1) gives the following form for the parallel momentum equation:

$$m\Bigg[V_{\|}\left\{\frac{\partial}{\partial t}+\nabla\cdot\left(V_{\|}\mathbf{b}+\widetilde{\mathbf{V}}_{E_{\perp}}+V_{\|}\frac{\widetilde{\mathbf{B}}_{\perp}}{B}\right)\right\}n$$
$$+n\left\{\frac{\partial}{\partial t}+\left(V_{\|}\mathbf{b}+\widetilde{\mathbf{V}}_{E_{\perp}}+V_{\|}\frac{\widetilde{\mathbf{B}}_{\perp}}{B}\right)\cdot\nabla\right\}V_{\|}\Bigg]$$
$$=-\nabla_{\|}P_{\|}-\nabla_{\perp}\cdot\frac{\widetilde{\mathbf{B}}_{\perp}}{B}P_{\|}+qnE_{\|}+C_1^L+C_1^T. \qquad (5.10)$$

In deriving the above equation, the following definition was used:

$$\begin{aligned} m\int dv_{\|}v_{\|}v_{\|}f =& m\int dv_{\|}(V_{\|}+\tilde{v_{\|}})(V_{\|}+\tilde{v_{\|}})f \\ =& mV_{\|}{}^2\int dv_{\|}f+m\int dv_{\|}\tilde{v_{\|}}{}^2f \\ \equiv& mV_{\|}{}^2n+P_{\|}. \end{aligned}$$

The above form of the momentum equation, Eq. (5.10), can be put into the standard form by substituting the continuity equation, Eq. (5.9), into the left hand side of Eq. (5.10). Then, the momentum equation reads

$$mn\left[\frac{\partial}{\partial t}+\left(V_{\|}\mathbf{b}+\widetilde{\mathbf{V}}_{E_{\perp}}+V_{\|}\frac{\widetilde{\mathbf{B}}_{\perp}}{B}\right)\cdot\nabla\right]V_{\|}$$
$$=-\nabla_{\|}P_{\|}-\nabla_{\perp}\cdot\frac{\widetilde{\mathbf{B}}_{\perp}}{B}P_{\|}+qnE_{\|}+\left(C_1^L+C_1^T\right)-mV_{\|}\left(C_o^L+C_o^T\right). \qquad (5.11)$$

This set of fluid equations Eq. (5.9) and Eq. (5.10), is closed by using the isothermal equation of state relating the pressure to the density:

$$P_{\|}=T_{\|}n, \quad \text{where} \quad T_{\|}=\text{ constant}. \qquad (5.12)$$

This isothermal equation of state is a reasonable approximation in a plasma in which the parallel thermal conductivity is very large. Moreover, the isothermal response for the perturbations, $\widetilde{P}_{\|}=\tilde{n}T_{\|}$, is valid in the case when the electrons

experience primarily an adiabatic response, $\widetilde{f}_e \simeq f_0\, e\widetilde{\phi}/T_e$, as is true in the kinetic theory. Such a response is dominant when the following two terms balance in a kinetic equation:

$$v_{||}\mathbf{b}\cdot\nabla\widetilde{f}_e = -\frac{q}{m}\widetilde{E}_{||}\frac{\partial}{\partial v_{||}}f_0. \tag{5.13}$$

The above relation states that the electrons respond instantaneously to electric field perturbations along the field lines (i.e. infinite parallel conductivity).

To study the equilibrium version of these fluid equations, it is assumed that spatial gradients along the field lines are zero. The equilibrium version of the continuity equation, Eq. (5.9), states

$$\frac{\partial}{\partial t}n_0 = \overline{C}_0 \simeq \frac{\partial}{\partial x}D_p\frac{\partial}{\partial x}n_0 \tag{5.14}$$

where a typical form of $\overline{C}_0$ was introduced for the sake of discussion. Here D_p is the particle transport coefficient. In this form the equilibrium continuity equation becomes the familiar diffusion equation for the equilibrium density in the presence of a radial flow given by $\Gamma_x = -D_p(\partial/\partial x)n_0$ due to transport processes (collisions and turbulence). For example, classical transport [36] specifies $D_p \simeq \rho_i^2\nu_{ei}$.

The equilibrium version of the momentum equation, Eq. (5.11), states

$$mn_0\frac{\partial}{\partial t}V_{||_o} = qn_0E_{||o} + \overline{C}_1 - mV_{||_o}\overline{C}_0. \tag{5.15}$$

Note that Eq. (5.14) states $(\partial/\partial t)n_0 = \overline{C}_0$. Substituting this into the above equation gives the following equilibrium relation:

$$m\frac{\partial}{\partial t}(n_0V_{||_o}) = qn_0E_{||_0} + \overline{C}_1. \tag{5.16}$$

The leading order form of $\overline{C}_1$ can be determined in order to allow for a Spitzer-type resistivity [35]. Assuming the time derivative can be neglected to leading order,

then by approximating $\overline{C}_1 \simeq -m\nu n_0 v_{\|0}$, the equilibrium version of the momentum equation states

$$qn_0 E_{\|_o} = m\nu n_o V_{\|_o}. \tag{5.17}$$

Equation (5.17) determines the equilibrium current for a given applied parallel electric field via the plasma resistivity.

To determine the fluctuation equations, the fluid equations are linearized to first order in the perturbed quantities. It is assumed a perturbed quantity, $\widetilde{Q}$, has the following form.

$$\widetilde{Q} = \widetilde{Q}(x) \exp\left[i(k_y y + k_z z - \omega t)\right].$$

Linearizing the continuity equation gives the following equation for the perturbed density in terms of the perturbed parallel velocity:

$$-i(\omega - k_\| V_{\|_o})\widetilde{n} + \nabla \cdot \left(\widetilde{V}_\| \mathbf{b} + \widetilde{\mathbf{V}}_{E_\perp} + V_{\|_o} \frac{\widetilde{\mathbf{B}}_\perp}{B}\right) n_o = \widetilde{C}_0. \tag{5.18}$$

Using the expression for $\widetilde{C}_0$, Eq. (5.7) gives

$$\widetilde{n} = (\omega_{n_0} - k_\| V_{\|_o})^{-1} \left[-i\nabla \cdot \left(\widetilde{V}_\| \mathbf{b} + \widetilde{\mathbf{V}}_{E_\perp} + V_{\|_o} \frac{\widetilde{\mathbf{B}}_\perp}{B}\right) n_o + i\omega_n^T \frac{e\tilde{\phi}}{T} n_0\right] \tag{5.19}$$

where $\omega_{n_0} = \omega + i\omega_n^T$.

The linearized momentum equation is given directly from Eq. (5.11) as

$$\begin{aligned} -imn_o(\omega - V_{\|_o} k_\|)\widetilde{V}_\| + mn_o \left(\widetilde{\mathbf{V}}_\| + \widetilde{\mathbf{V}}_{E_\perp} + V_{\|_o} \frac{\widetilde{\mathbf{B}}_\perp}{B}\right) \cdot \nabla V_{\|_o} \\ + m\widetilde{n} \left(\frac{\partial}{\partial t} + \mathbf{V}_{\|_0} \cdot \nabla\right) V_{\|_o} \\ = -\nabla_\| \widetilde{P}_\| - \nabla_\perp \cdot \frac{\widetilde{\mathbf{B}}_\perp}{B} P_{\|_0} + q\widetilde{n} E_{\|o} + qn_0 \widetilde{E}_\| + \widetilde{C}_1 \\ - m\widetilde{V}_{\|_0} \overline{C}_0 - mV_{\|_o} \widetilde{C}_0 \,. \end{aligned} \tag{5.20}$$

Substituting in the equilibrium relations Eq. (5.14) and Eq. (5.17) into the above equation gives

$$-imn_o(\omega - k_{||}V_{||_o})\tilde{V}_{||} + mn_o\left(\tilde{\mathbf{V}}_{||} + \tilde{\mathbf{V}}_{E_\perp} + V_{||_o}\frac{\tilde{\mathbf{B}}_\perp}{B}\right)\cdot\nabla V_{||_o} \cdot$$
$$= -\nabla_{||}\tilde{P}_{||} - \nabla_\perp\cdot\frac{\tilde{\mathbf{B}}_\perp}{B}P_{||_o} + qn_o\tilde{E}_{||} + m\nu V_{||_o}\tilde{n} + \tilde{C}_1 - mV_{||_o}\tilde{C}_0$$
$$- m\tilde{V}_{||_o}\frac{\partial}{\partial x}D_p\frac{\partial}{\partial x}n_o. \qquad (5.21)$$

Assuming

$$\left|\frac{\partial}{\partial x}D_p\frac{\partial}{\partial x}n_o\right| \ll \omega_n^T n_o$$

implies that the last term in Eq. (5.21) can be neglected. Using the expressions for $\tilde{C}_1$ and $\tilde{C}_0$, Eqs. (5.7) and (5.8), along with the isothermal equation of state, $\tilde{P}_{||} = \tilde{n}T_{||_o}$, then gives the following expression for the perturbed momentum equation:

$$(\omega_{m_0} - k_{||}V_{||_o})\tilde{V}_{||} = -i\left(\tilde{\mathbf{V}}_{E_\perp} + V_{||_o}\frac{\tilde{\mathbf{B}}_\perp}{B}\right)\cdot\nabla V_{||_o} - \frac{iT}{mn}\nabla_\perp\cdot\frac{\tilde{\mathbf{B}}_\perp}{B}n_o + \frac{iq}{m}\tilde{E}_{||}$$
$$- iV_{||_o}\omega_n^T\frac{e\tilde{\phi}}{T} + \frac{\tilde{n}}{nk_{||}}\left(k_{||}^2\frac{T}{m} - iV_{||_o}k_{||}\delta\omega_{mn}\right) \qquad (5.22)$$

where

$$\omega_{m_0} = \omega + i(\omega_m^T + \nu) \text{ and } \delta\omega_{mn} = \omega_m^T - \omega_n^T.$$

The perturbed continuity and momentum equations, Eq. (5.19) and Eq. (5.22), give a set of coupled equations for $\tilde{n}$ and $\tilde{V}_{||}$ in terms of $\tilde{\phi}$ and $\tilde{A}_{||}$. These two equations can be combined to yield an expression for $\tilde{n}$ in terms of $\tilde{\phi}$ and $\tilde{A}_{||}$, along with an equation for $\tilde{V}_{||}$ in terms of $\tilde{\phi}$ and $\tilde{A}_{||}$. Substituting Eq. (5.22) into

the perturbed continuity equation gives the following expression for the perturbed density:

$$\left[(\omega_{m_0}-k_{\|}V_{\|_e})(\omega_{n_e}-k_{\|}V_{\|_e})-\left(\frac{T}{m}k_{\|}^2-iV_{\|_e}k_{\|}\delta\omega_{mn}\right)\right]\tilde{n}$$
$$=-in_ok_{\|}\left[\left(\tilde{\mathbf{V}}_{E_\perp}+V_{\|_e}\frac{\tilde{\mathbf{B}}_\perp}{B}\right)\cdot\nabla V_{\|_e}+\frac{T}{nm}\nabla_\perp\cdot\frac{\tilde{\mathbf{B}}_\perp}{B}n_0-\frac{q}{m}\tilde{E}_{\|}+V_{\|_e}\omega_n^T\frac{e\tilde{\phi}}{T}\right]$$
$$-in_o(\omega_{m_0}-k_{\|}V_{\|_e})\left[\frac{1}{n_0}\nabla_\perp\cdot\left(\tilde{\mathbf{V}}_{E_\perp}+V_{\|_e}\frac{\tilde{\mathbf{B}}_\perp}{B}\right)n_o-\omega_n^T\frac{e\tilde{\phi}}{T}\right]. \qquad (5.23)$$

The perturbed current can be found using Eq. (5.19) and Eq. (5.22) along with the definition $\tilde{J}_{\|}=q(n_o\tilde{V}_{\|}+\tilde{n}V_{\|_e})$. This gives the expression

$$\left[(\omega_{m_0}-k_{\|}V_{\|_e})(\omega_{n_e}-k_{\|}V_{\|_e})-\left(\frac{T}{m}k_{\|}^2-ik_{\|}V_{\|_e}\delta\omega_{mn}\right)\right]\frac{\tilde{J}_{\|}}{q}$$
$$=-in_o\omega_{n_e}\left[\left(\tilde{\mathbf{V}}_{E_\perp}+V_{\|_e}\frac{\tilde{\mathbf{B}}_\perp}{B}\right)\cdot\nabla V_{\|_e}+\frac{T}{m}\nabla_\perp\cdot\frac{\tilde{\mathbf{B}}_\perp}{B}n_o-\frac{q}{m}\tilde{E}_{\|}+V_{\|_e}\omega_n^T\frac{e\tilde{\phi}}{T}\right]$$
$$-\frac{i}{k_{\|}}\left[k_{\|}V_{\|_e}(\omega_{m_0}-k_{\|}V_{\|_e})+\left(\frac{T}{m}k_{\|}^2-ik_{\|}V_{\|_e}\delta\omega_{mn}\right)\right]$$
$$\times\left[\nabla_\perp\cdot\left(\tilde{\mathbf{V}}_{E_\perp}+V_{\|_e}\frac{\tilde{\mathbf{B}}_\perp}{B}\right)n_o-\omega_n^T\frac{e\tilde{\phi}}{T}n_o\right]. \qquad (5.24)$$

When combined with quasineutrality and Ampere's law, Eq. (5.23) and Eq. (5.24) represent a closed system which can be analyzed to determine the dispersion relationship for the tearing mode including the effects of collisions and turbulence. In the following analysis it will be assumed $\delta\omega_{mn}=0$ or, in other words, $\omega_m^T=\omega_n^T$. This implies that the underlying kinetic turbulence diffuses particles at the same rate in which it diffuses momentum. This is what is implied in the kinetic theory of the preceding sections where the turbulence operator appeared as $-D(\partial^2/\partial x^2)\tilde{h}_e$. Hence, $D_n^T=D_m^T=D$ and $\omega_n^T=\omega_m^T=\omega_c$.

To make connection with the kinetic theory of the previous sections, $V_{\|_o}$ will be treated as a constant, as was v_D and v_b in the kinetic theory. (The perpendicular gradient of $V_{\|_o}$ can be retained, however, and no significant difference occurs in the final dispersion relation.) To rewrite Eqs. (5.23) and (5.24) in a more useful form, $\tilde{\mathbf{V}}_{E_\perp}$, $\tilde{\mathbf{B}}_\perp$ and $\tilde{E}_\|$ are rewritten in terms of $\tilde{\phi}$ and $A_\|$ and the substitution $J_{\|_o} = qn_0V_{\|_o}$ is made where appropriate. The expressions for the perturbed current and density become

$$\left[(\omega_{m_0}-k_\|V_{\|_o})(\omega_{n_o}-k_\|V_{\|_o})-\frac{T}{m}k_\|^2\right]\tilde{n}$$
$$=\frac{en_o}{T}\left\{\frac{q}{e}\frac{T}{m}k_\|^2\left(\tilde{\phi}-\frac{\omega}{k_\|c}\left(1+\frac{e}{q}\frac{\omega_*}{\omega}\right)\tilde{A}_\|\right)-ik_\|V_{\|_o}\omega_n^T\tilde{\phi}\right.$$
$$\left.+(\omega_{m_0}-k_\|V_{\|_o})\left[\omega_*\left(\tilde{\phi}-\tilde{A}_\|\frac{J_{\|_o}}{qnc}\frac{L_n}{L_J}\right)+i\omega_n^T\tilde{\phi}\right]\right\} \qquad (5.25)$$

and

$$\left[(\omega_{m_0}-k_\|V_{\|_o})(\omega_{n_o}-k_\|V_{\|_o})-\frac{T}{m}k_\|^2\right]\frac{\tilde{J}_\|}{q}$$
$$=\frac{en_o}{T}\left\{\omega_{n_o}\left[\frac{q}{e}\frac{T}{m}k_\|\left(\tilde{\phi}-\frac{\omega}{k_\|c}\left(1+\frac{e}{q}\frac{\omega_*}{\omega}\right)\tilde{A}_\|\right)-iV_{\|_o}\omega_n^T\tilde{\phi}\right]\right.$$
$$+\left[\frac{T}{m}k_\|^2+k_\|V_{\|_o}(\omega_{m_0}-k_\|V_{\|_o})\right]$$
$$\left.\times\left[\frac{\omega_*}{k_\|}\left(\tilde{\phi}-\frac{J_{\|_0}}{qnc}\frac{L_n}{L_J}\tilde{A}_\|\right)+i\frac{\omega_n^T}{k_\|}\tilde{\phi}\right]\right\}. \qquad (5.26)$$

In the above equations, $\omega_* \equiv ck_yT/(eBL_n)$, $dn_o/dx \equiv -n_o/L_n$ and $dJ_{\|_o}/dx \equiv -J_{\|_o}/L_J$.

Equations (5.25) and (5.26) represent a coupled system for $\tilde{\phi}$ and $\tilde{A}_\|$ which, when combined with quasineutrality and Ampere's law, can be analyzed to yield a dispersion relation for the tearing mode. Equations (5.25) and (5.26) hold for

either electrons or ions, although when written for ions, finite Larmor radius effects should be included. (The drift kinetic equation used assumed zero Larmor radius.) The "regular" tearing mode is analyzed by using a single fluid of "bulk" electrons with $V_{\|_e} = v_D$, where $v_D/v_e \sim 10^{-2}$. The "runaway" tearing mode is analyzed by using a two fluid model for the electrons containing a bulk electron fluid with $V_{\|_e} = 0$, and a beam electron fluid with $V_{\|_e} = v_b$ and $n_o = n_b$, where $v_b/v_e \sim 10$ and $n_b/n_0 \sim 10^{-3}$.

5.2 Regular Tearing

To examine the "regular" tearing mode, Eqs. (5.25) and (5.26) for the perturbed density and perturbed parallel current will be used in the equation for quasineutrality as well as in Ampere's law. This gives a set of coupled equations for the fluctuation potentials $\widetilde{\phi}$ and $\widetilde{A}_{||}$. In the fluid model, the regular tearing mode exists in a two fluid plasma composed of (1) an electron population drifting with an equilibrium parallel velocity $V_{||_e} = v_D$, where $v_D/v_e \sim 10^{-2}$, and (2) an ion population with no parallel equilibrium flow, $V_{||_i} = 0$.

To find the perturbed electron density, Eq. (5.25), which is written for either ions or electrons with an arbitrary equilibrium flow, is written for electrons with $V_{||_e} = v_D$. Using the fact that $v_D/v_e \sim 10^{-2}$ and assuming $k_{||}V_D/\omega < 1$, then to leading order

$$\left[\omega_{m_e}\omega_{n_e} - \frac{1}{2}v_e^2 k_{||}^2\right]\widetilde{n}_e \cong \frac{en_o}{T_e}\left\{-\frac{1}{2}v_e^2 k_{||}^2\left(\widetilde{\phi} - \frac{\omega}{k_{||}c}\left(1 - \frac{\omega_{*e}}{\omega}\right)\widetilde{A}_{||}\right)\right.$$
$$- ik_{||}v_D\omega_n\widetilde{\phi} + (\omega_{m_0} - k_{||}v_D)$$
$$\left.\times\left[\omega_{*e}\left(\widetilde{\phi} + \widetilde{A}_{||}\frac{J_{||_e}}{enc}\frac{L_n}{L_J}\right) + i\omega_n^T\widetilde{\phi}\right]\right\}. \qquad (5.27)$$

Rearranging the above terms gives

$$\frac{\widetilde{n}_e}{n_o} = \frac{e}{T_e}\left\{\widetilde{\phi} + \left(\omega_{m_e}\omega_{n_e} - \frac{1}{2}v_e^2 k_{||}^2\right)^{-1}\left[-\omega_{m_0}(\omega - \omega_{*e})\widetilde{\phi}\right.\right.$$
$$- k_{||}v_D(\omega_{*e} + 2i\omega_n^T)\widetilde{\phi} + \frac{1}{2}v_e^2 k_{||}^2\frac{\omega}{k_{||}c}\left(1 - \frac{\omega_{*e}}{\omega}\right)\widetilde{A}_{||}$$
$$\left.\left.+(\omega_{m_e} - k_{||}v_D)\omega_{*e}\frac{J_{||_e}}{enc}\frac{L_n}{L_J}\widetilde{A}_{||}\right]\right\}. \qquad (5.28)$$

Recalling the variational calculation presented in Ch. 4, it is only necessary to keep the leading order terms in $\widetilde{\phi}$ and $\widetilde{A}_{||}$, the rest can be neglected. Hence,

$$\frac{\tilde{n}_e}{n_o} \simeq \frac{e}{T_e}\left\{\tilde{\phi} + \frac{\frac{1}{2}v_e^2 k_\parallel^2(\omega/k_\parallel c)(1-\omega_{*e}/\omega)}{\left(\omega_{m_\perp}\omega_{n_\perp} - \frac{1}{2}v_e^2 k_\parallel^2\right)}\tilde{A}_\parallel\right\}. \tag{5.29}$$

The remaining terms which have been neglected in Eq. (5.28) are smaller by either ω_{*e}/ω_c or by v_D/v_e.

Neglecting collisions and letting $\omega_m^T = \omega_n^T = \omega_c$, then by defining $x/x_e = v_e k_\parallel/\omega_{*e}$ and $x/x_c = v_e k_\parallel \omega_c$, the second term in Eq. (5.29) can be written (using $\omega_{m_\perp}\omega_{n_\perp} \simeq -\omega_c^2 + 2i\omega\omega_c$)

$$\frac{\frac{1}{2}v_e^2 k_\parallel^2 \frac{\omega}{k_\parallel c}\left(1-\frac{\omega_{*e}}{\omega}\right)\tilde{A}_\parallel}{\omega_{m_\perp}\omega_{n_\perp} - \frac{1}{2}v_e^2 k_\parallel^2} \simeq \frac{-\frac{1}{2}\frac{x^2}{x_c^2}\left(1+\frac{2ix_e}{x_c}\right)\frac{x_e}{x}\left(1-\frac{\omega_{*e}}{\omega}\right)\frac{v_e}{c}\tilde{A}_\parallel}{1+\frac{1}{2}\frac{x^2}{x_c^2}\left(1+\frac{2ix_e}{x_c}\right)}. \tag{5.30}$$

Recall from Sec. 4.1 that the diffusive resonance function, I_1, is given approximately by the following interpolation formula,

$$I_1 = \frac{-ix_c}{\omega_c x}\xi_e(1+\xi_e Z) \simeq \frac{-i}{\omega_c}\left(\frac{\frac{1}{2}\frac{x}{x_c}\left(1+\frac{2ix_e}{x_c}\right)}{1+\frac{1}{2}\frac{x^2}{x_c^2}\left(1+\frac{2ix_e}{x_c}\right)}\right).$$

Hence, using the above approximation, the perturbed electron density becomes

$$\frac{\tilde{n}_e}{n_o} \cong \frac{e}{T_e}\left\{\tilde{\phi} - i(\omega-\omega_{*e})I_1\frac{v_e}{c}\tilde{A}_\parallel\right\}. \tag{5.31}$$

Thus, to leading order, the fluid theory reproduces the result of kinetic theory for the perturbed electron density, as used in Ch. 4.

In a similar way, Eq. (5.26) can be used to determine the perturbed parallel current for the electrons. Using $J_{\parallel_e} = -en_o v_D$, $v_D/v_e \sim 10^{-2}$, gives

$$-\left(\omega_{m_e}\omega_{n_e}-\frac{1}{2}v_e^2k_\parallel^2\right)\frac{\widetilde{J}_{\parallel_e}}{e}\simeq\frac{en_o}{T_e}\left\{\omega_{n_e}\left[-\frac{1}{2}v_e^2k_\parallel^2\left(\widetilde{\phi}-\frac{\omega}{k_\parallel c}\left(1-\frac{\omega_{*e}}{\omega}\right)\widetilde{A}_\parallel\right)\right.\right.$$
$$\left.-iv_D\omega_n^T\widetilde{\phi}\right]+\left(\frac{1}{2}v_e^2k_\parallel^2+k_\parallel v_D\omega_{m_e}\right)$$
$$\left.\times\left[\frac{\omega_{*e}}{k_\parallel}\left(\widetilde{\phi}+\frac{J_{\parallel_e}}{enc}\frac{L_n}{L_J}\widetilde{A}_\parallel\right)+\frac{i\omega_n^T}{k_\parallel}\widetilde{\phi}\right]\right\}.$$

Rearranging the above terms gives

$$\frac{\widetilde{J}_{\parallel_e}}{-e}=\left(\omega_{m_e}\omega_{n_e}-\frac{1}{2}v_e^2k_\parallel^2\right)^{-1}\frac{en_o}{T_e}\left\{\left[-\frac{1}{2}v_e^2k_\parallel(\omega-\omega_{*e})+\omega_{m_e}v_D\omega_{*e}\right]\widetilde{\phi}\right.$$
$$\left.+\left[\omega_{n_e}\frac{1}{2}v_e^2\frac{1}{c}(\omega-\omega_{*e})+\left(\frac{1}{2}v_e^2k_\parallel^2+k_\parallel v_D\omega_{m_e}\right)\frac{\omega_{*e}}{k_\parallel}\frac{J_{\parallel_e}}{enc}\frac{L_n}{L_J}\right]\widetilde{A}_\parallel\right\}. \quad (5.32)$$

As in the kinetic theory of Ch. 4, the following terms can be neglected (due to the smallness of v_D/v_e) and the perturbed current to leading order becomes

$$\frac{\widetilde{J}_{\parallel_e}}{-e}\cong\frac{en_o}{T_e}\left(\omega_{m_e}\omega_{n_e}-\frac{1}{2}v_e^2k_\parallel^2\right)^{-1}\left\{-\frac{1}{2}v_e^2k_\parallel(\omega-\omega_{*e})\widetilde{\phi}\right.$$
$$\left.+\left[\frac{1}{2}\omega_{n_e}\frac{v_e^2}{c}(\omega-\omega_{*e})+\frac{1}{2}v_e^2k_\parallel^2\frac{\omega_{*e}}{k_\parallel}\frac{J_{\parallel_e}}{enc}\frac{L_n}{L_J}\right]\widetilde{A}_\parallel\right\}. \quad (5.33)$$

Using the approximation, $\omega_{m_e}\omega_{n_e}\simeq-\omega_c^2+2i\omega_c\omega$, then the following relations hold:

$$\frac{-\frac{1}{2}v_e^2k_\parallel}{\omega_{m_e}\omega_{n_e}-\frac{1}{2}v_e^2k_\parallel^2}\simeq\frac{v_e}{\omega_c}\left(\frac{\frac{1}{2}\frac{x}{x_c}\left(1+\frac{2ix_e}{x_c}\right)}{1+\frac{1}{2}\frac{x^2}{x_c^2}\left(1+\frac{2ix_e}{x_c}\right)}\right)\simeq iv_eI_1 \quad (5.34a)$$

$$\frac{\frac{1}{2}\omega_{n_e}\frac{v_e^2}{c}}{\omega_{m_e}\omega_{n_e}-\frac{1}{2}v_e^2k_\parallel^2}\simeq\left(\frac{-\frac{v_e^2}{c}\frac{i}{\omega_c}\left(1+\frac{ix_e}{x_c}\right)}{1+\frac{1}{2}\frac{x^2}{x_c^2}\left(1+\frac{2ix_e}{x_c}\right)}\right)\simeq\frac{-iv_e^2}{c}I_2 \quad (5.34b)$$

$$\frac{\frac{1}{2}v_e^2k_\parallel\omega_{*e}}{\omega_{m_e}\omega_{n_e}-\frac{1}{2}v_e^2k_\parallel^2}\simeq\left(\frac{\omega_{*e}\frac{v_e}{\omega_c}\frac{1}{2}\frac{x}{x_c}\left(1+\frac{2ix_e}{x_c}\right)}{1+\frac{1}{2}\frac{x^2}{x_c^2}\left(1+\frac{2ix_e}{x_c}\right)}\right)\simeq iv_e\omega_{*e}I_3 \quad (5.34c)$$

where the interpolation functions were used for $I_n(x)$ in the Krook limit. [See Eq. (4.15).]

If the definition $\eta_J = 2v_D L_n / v_e L_J$ is used ($J_{\|_o} = -en_o v_D$), then Eq. (5.33) becomes

$$\frac{\widetilde{J}_{\|e}}{-e} \simeq \frac{ie}{T_e} n_o v_e \left\{ \left[-(\omega - \omega_{*e}) I_2 + \omega_{*e} \eta_J I_3 \right] \frac{v_e}{c} \widetilde{A}_{\|} + (\omega - \omega_{*e}) I_1 \widetilde{\phi} \right\} \tag{5.35}$$

which are the leading order terms derived in the previous kinetic theory.

To obtain the ion perturbations, Eqs. (5.25) and (5.26) are again used for the case of fluid ions in which no equilibrium flows are present, $V_{\|_o} = 0$. As in the kinetic theory, the ions are assumed to be unaffected by turbulent diffusion. Hence $\omega_{m_o} = \omega_{n_o} = \omega$. Eq. (5.25) then gives

$$\left[\omega^2 - \frac{1}{2} v_e^2 k_{\|}^2 \right] \widetilde{n}_i = \frac{e n_o}{T_i} \left\{ \frac{1}{2} v_e^2 k_{\|}^2 \left(\widetilde{\phi} - \frac{\omega}{k_{\|} c} \left(1 + \frac{1}{\tau} \frac{\omega_{*e}}{\omega} \right) \widetilde{A}_{\|} \right) + \omega \omega_{*e} \frac{1}{\tau} \widetilde{\phi} \right\} \tag{5.36}$$

where $\tau = T_e / T_i$.

The terms in the above equation can be rearranged to read

$$\begin{aligned} \frac{\widetilde{n}_i}{n_o} = \frac{e}{T_i} \Bigg\{ & -\widetilde{\phi} + \left(\omega^2 - \frac{1}{2} v_e^2 k_{\|}^2 \right)^{-1} \\ & \times \left[\omega \left(\omega + \frac{\omega_{*e}}{\tau} \right) \widetilde{\phi} - \frac{1}{2} v_e^2 k_{\|}^2 \frac{\omega}{k_{\|} c} \left(1 + \frac{\omega_{*e}}{\tau \omega} \right) \widetilde{A}_{\|} \right] \Bigg\} . \end{aligned} \tag{5.37}$$

Recall that this fluid theory is based on moments of the drift kinetic equation (DKE). The DKE is a good description for electrons, but this description neglects key finite Larmor radius (FLR) effects [71] when used to study ions. To correctly account for these FLR effects, recall that in the kinetic theory, the electrostatic potential $\widetilde{\phi}$ was expanded to second order yielding a second spatial derivative

of $\tilde{\phi}$ in the calculation of the response. Also recall, that in order for such an expansion to be valid, it was required that $x/x_i < 1$. To correctly account for this FLR effect, the resonance operator, $\left(\omega^2 - \frac{1}{2}v_e^2 k_{||}^2\right)^{-1}$, appearing in the above fluid equation must be expanded as follows:

$$\left(\omega^2 - \frac{1}{2}v_e^2 k_{||}^2\right)^{-1}\tilde{\phi} \simeq \frac{1}{\omega^2}\left[1 + \frac{\frac{1}{2}v_e^2 k_{||}'^2 x^2}{\omega^2}\right]\tilde{\phi}$$
$$\rightarrow \frac{1}{\omega^2}\left[\left(1 + \frac{1}{2}\frac{x^2}{x_i^2}\right)\tilde{\phi} + \frac{d^2}{dx^2}\tilde{\phi}\right].$$

Hence, to leading order, including the second derivative of $\tilde{\phi}$ due to FLR effects, the ion density perturbation becomes

$$\frac{\tilde{n}_i}{n_o} \simeq \frac{e}{T_i}\left\{-\tilde{\phi} + \left(1 + \frac{\omega_{*e}}{\tau\omega}\right)\left[\left(1 + \frac{1}{2}\frac{x^2}{x_i^2}\right)\tilde{\phi} + \frac{d^2}{dx^2}\tilde{\phi}\right]\right.$$
$$\left. - \frac{1}{2}\frac{x^2}{x_i^2}\frac{\omega}{k_{||}c}\left(1 + \frac{\omega_{*e}}{\tau\omega}\right)\tilde{A}_{||}\right\}, \qquad (5.38)$$

which is the kinetic theory result. (See Sec. 2.2.)

Similarly, setting $V_{||_o} = 0$ and $\omega_c = 0$ and expanding Eq. (5.26) for $x/x_i < 1$ gives the following expression for the perturbed ion current:

$$\frac{\tilde{J}_{||_i}}{e} \simeq \frac{en_o v_i}{T_i}\left\{\frac{1}{2}\frac{x}{x_i}\left(1 + \frac{\omega_{*e}}{\tau\omega}\right)\left(\tilde{\phi} - \frac{\omega}{k_{||}c}\tilde{A}_{||}\right)\right\}. \qquad (5.39)$$

The above expression is the leading order kinetic theory result for $x/x_i < 1$.

To derive a set of closed, coupled equations for $\tilde{\phi}$ and $\tilde{A}_{||}$, the above expressions for $\tilde{n}$ and $\tilde{J}_{||}$ are used along with quasineutrality and Ampere's law. Quasineutrality, $\tilde{n}_e = \tilde{n}_i$, becomes, using Eqs. (5.31) and (5.38)

$$\left[-\frac{d^2}{dx^2} + \Lambda + \chi^2\right]\tilde{\phi} - \left[\frac{i}{d}(\omega - \omega_{*e})I_1 + x_e\chi^2\right]\frac{v_e}{c}\tilde{A}_{||} = 0 \qquad (5.40)$$

where

$$\Lambda = \frac{1}{d}\left(1 - \frac{\omega_{*e}}{\omega}\right), \quad d = \left(\tau + \frac{\omega_{*e}}{\omega}\right) \quad \text{and} \quad \chi^2 = -\frac{x^2}{2x_i^2}.$$

Similarly, Ampere's law, $\nabla^2 \tilde{A}_{\|} + (4\pi/c)\left(\tilde{J}_{\|_e} + \tilde{J}_{\|_i}\right) = 0$, becomes, using Eqs. (5.35) and (5.39)

$$\frac{\tau v_A^2}{dc^2}\left[\frac{d^2}{dx^2} - b + \alpha_i^2 + \frac{v_e^2}{\tau v_A^2} i(\omega - \omega_{*e}) I_2 - \frac{v_e^2}{\tau v_A^2} i\omega_{*e}\eta_J I_3\right]\tilde{A}_{\|} - \left[\frac{i}{d}(\omega - \omega_{*e}) I_1 + x_e \chi^2\right]\frac{v_e}{c}\tilde{\phi} = 0 \qquad (5.41)$$

where $v_A^2 = B^2/4\pi m_i n_i$, $b = k_y^2 \rho_i^2$ and $\alpha_i^2 = -(v_i^2/2\tau v_A^2)(\tau + \omega_{*e}/\omega)$. In the above the equations, x has been normalized in units of the ion gyro radius ρ_i. Equations (5.40) and (5.41) are equivalent to the leading order terms in Eqs. (4.11) and (4.12).

As in the kinetic theory, Eqs. (5.40) and (5.41) form a coupled self-adjoint set. Hence, a variational integral can be formed. This is done by multiplying Eq. (5.40) by $\tilde{\phi}$ and Eq. (5.41) by $\tilde{A}_{\|}$ and integrating over the plasma. This variational form becomes

$$S = S_\phi + \frac{\tau v_A^2}{dc^2} S_A \qquad (5.42)$$

where

$$S_\phi = \int_{-a}^{a} dx \left[-\left(\frac{d\tilde{\phi}}{dx}\right)^2 - \Lambda\tilde{\phi}^2 + \frac{1}{2}\frac{x^2}{x_i^2}\left(\tilde{\phi} - \frac{\omega}{k_{\|}c}\tilde{A}_{\|}\right)^2 + 2\frac{i}{d}(\omega - \omega_{*e}) I_1 \frac{v_e}{c}\tilde{\phi}\tilde{A}_{\|}\right] + \tilde{\phi}\tilde{\phi}'\Big|_{-a}^{a} \qquad (5.43)$$

and

$$S_A = \int_{-a}^{a} dx \left\{ \left(\frac{d\widetilde{A}_{\|}}{dx} \right)^2 + \left[b - \frac{v_e^2}{\tau v_A^2} i(\omega - \omega_{*e}) I_2 + \frac{v_e^2}{\tau v_A^2} i\omega_{*e} \eta_J I_3 \right] \widetilde{A}_{\|}^2 \right\} - \widetilde{A}_{\|} \widetilde{A}'_{\|} \Big|_{a}^{-a} . \tag{5.44}$$

This variational form is identical to that obtained in Ch. 4 using kinetic theory. To determine the dispersion relation, a procedure identical to that used previously can be performed. A simpler heuristic procedure, however, will be presented in Sec. 5.4.

5.3 Runaway Tearing

Runaway tearing is composed of a three fluid plasma containing 1) bulk electrons with no net flows $V_{\|_o} = 0$, $n = n_o$; 2) beam electrons which carry the current $V_{\|_o} = v_b$, $n = n_b$ with $v_b/v_e \simeq 10$, $n_b/n_0 \simeq 10^{-3}$; and 3) bulk ions with no net flow $V_{\|_o} = 0$.

To analyze the bulk electrons, Eqs. (5.25) and (5.26) are used with $V_{\|_o} = 0$ and $\omega_{m_0} = \omega_{n_o} = \omega + i\omega_c$. The result is identical to that analyzed previously in Eqs. (5.27)–(5.35), only now $V_{\|_o} = J_{\|_0} = 0$. Hence, for the bulk electrons

$$\left.\frac{\tilde{n}_e}{n_o}\right|_{bulk} \cong \frac{e}{T_e}\left\{\tilde{\phi} - i(\omega - \omega_{*e})I_1\frac{v_e}{c}\tilde{A}_{\|}\right\} \tag{5.45}$$

and

$$\left.\frac{\tilde{J}_{\|_o}}{-e}\right|_{bulk} \simeq \frac{ie}{T_e}n_o v_e\left\{-(\omega - \omega_{*e})I_2\frac{v_e}{c}\tilde{A}_{\|} + (\omega - \omega_{*e})I_1\tilde{\phi}\right\}. \tag{5.46}$$

To analyze the beam electrons, Eqs. (5.25) and (5.26) are analyzed for $V_{\|_o} = v_b$, $v_b/v_e \sim 10$ and for $n = n_b$, $n_b/n_o \sim 10^{-3}$. The equation for the perturbed beam density becomes

$$\begin{aligned}\left.\frac{\tilde{n}_e}{n_o}\right|_{beam} &= \left[(\omega_{m_o} - k_{\|}v_b)(\omega_{n_o} - k_{\|}v_b)\right]^{-1} \\ &\times \frac{en_b}{T_e n_o}\left\{-\frac{1}{2}v_e^2k_{\|}^2\left(\tilde{\phi} - \frac{\omega}{k_{\|}c}\left(1 - \frac{\omega_{*e}}{\omega}\right)\tilde{A}_{\|}\right) - ik_{\|}v_b\omega_n^T\tilde{\phi}\right. \\ &\left. + (\omega_{m_o} - k_{\|}v_b)\left[\omega_{*e}\left(\tilde{\phi} + \tilde{A}_{\|}\frac{J_{\|_0}}{en_bc}\frac{L_n}{L_J}\right) + i\omega_n^T\tilde{\phi}\right]\right\}. \end{aligned} \tag{5.47}$$

Since one is concerned with finding the overall electron perturbed density, $\tilde{n}_e = \tilde{n}_e|_{bulk} + \tilde{n}_e|_{beam}$, the terms proportional to n_b in the above equation can be neglected due to $\tilde{n}_b/n_o \sim 10^{-3}$. Hence,

$$\left.\frac{\widetilde{n}_e}{n_o}\right|_{beam} \simeq \frac{e}{n_o T_e}\left(\frac{J_{\|_o}}{ec}\frac{L_n}{L_J}\frac{\omega_{*e}\widetilde{A}_{\|}}{(\omega_{n_o}-k_{\|}v_b)}\right). \tag{5.48}$$

Using Eq. (5.26) the perturbed beam current is given by

$$\begin{aligned}\left.\frac{\widetilde{J}_{\|e}}{-e}\right|_{beam} \simeq & \left[(\omega_{n_0}-k_{\|}v_b)(\omega_{m_o}-k_{\|}v_b)\right]^{-1} \\ & \times \frac{en_b}{T_e}\left\{-\omega_{n_o}\left[\frac{1}{2}v_e^2k_{\|}^2\left(\widetilde{\phi}-\frac{\omega}{k_{\|}c}\left(1-\frac{\omega_{*e}}{\omega}\right)\widetilde{A}_{\|}\right)-iv_b\omega_n^T\widetilde{\phi}\right]\right. \\ & \left.+\left[k_{\|}v_b(\omega_{m_o}-k_{\|}v_b)\right]\frac{\omega_{*e}}{k_{\|}}\left[\left(\widetilde{\phi}+\widetilde{A}_{\|}\frac{J_{\|_0}}{en_bc}\frac{L_n}{L_J}\right)+i\frac{\omega_n^T}{k_{\|}}\widetilde{\phi}\right]\right\}. \end{aligned} \tag{5.49}$$

Similarly, since the end concern is with the total electron perturbed current, $\widetilde{J}_{\|_e} = \widetilde{J}_{\|_e}|_{bulk} + \widetilde{J}_{\|_e}|_{beam}$, any terms in the above equation proportional to n_b can be neglected. Hence,

$$\left.\frac{\widetilde{J}_{\|_e}}{-e}\right|_{beam} \simeq \frac{e}{T_e}\left(\frac{J_{\|_o}}{ec}\frac{L_n}{L_J}\frac{v_b\omega_{*e}\widetilde{A}_{\|}}{(\omega_{n_o}-k_{\|}v_b)}\right). \tag{5.50}$$

To find the total electron perturbations, the result for the bulk electrons and beam electrons are summed. For the density perturbations, the beam contribution, Eq. (5.48), is smaller than the bulk contribution by v_D/v_e, and hence can be neglected [as could the remaining terms in Eq. (5.29)]. Hence, as before

$$\frac{\widetilde{n}_e}{n_o} \simeq en_oT_e\left\{\widetilde{\phi}-i(\omega-\omega_{*e})I_1\frac{v_e}{c}\widetilde{A}_{\|}\right\}. \tag{5.51}$$

The total electron perturbed current is given by

$$\begin{aligned}\frac{\widetilde{J}_{\|_e}}{-e} = \frac{ie}{T_e}n_ov_e\left\{-(\omega-\omega_{*e})I_2\frac{v_e}{c}\widetilde{A}_{\|}+(\omega-\omega_{*e})I_1\widetilde{\phi}\right. \\ \left.-\frac{i\omega_{*e}}{\omega_{n_o}-k_{\|}v_b}\frac{v_b}{v_e}\frac{J_{\|_0}}{en_oc}\frac{L_n}{L_J}\widetilde{A}_{\|}\right\}. \end{aligned} \tag{5.52}$$

Note that

$$\frac{v_b}{v_e}\frac{1}{\omega_{n_{\parallel_o}} - k_{\parallel}v_b}\frac{J_{\parallel_{o_o}}}{en_oc}\frac{L_n}{L_J} = i\frac{n_b}{n_o}\frac{L_n}{L_J}K_2.$$

where K_2 is the diffusive resonance function for the beam given by

$$K_2 \equiv \left(\frac{v_b}{v_e}\right)^2 \frac{i}{(\omega + i\omega_b - k_{\parallel}v_b)} \simeq \frac{v_b^2}{v_e^2}\frac{1}{\omega_b}\frac{1 - ix/x_b}{1 + x^2/x_b^2}.$$

Here, $\omega_b^3 \simeq \frac{1}{3}(k'_{\parallel}v_b)^2 D$, $x_b = \omega_b/k'_{\parallel}v_b$, and $J_{\parallel_{o_o}} = -en_bv_b$. Thus,

$$\frac{\tilde{J}_{\parallel_e}}{-e} \simeq \frac{ie}{T_e}n_ov_e\left\{-(\omega - \omega_{*e})I_2\frac{v_e}{c}\tilde{A}_{\parallel} - \omega_{*e}\frac{n_b}{n_o}\frac{L_n}{L_J}K_2\frac{v_e}{c}\tilde{A}_{\parallel} + (\omega - \omega_{*e})I_1\tilde{\phi}\right\} \quad (5.53)$$

which is the kinetic theory result derived in Ch. 4.

The ion response is identical to that for the regular tearing mode, Eqs. (5.38) and (5.39).

The coupled equations for $\tilde{\phi}$ and $\tilde{A}_{\parallel}$ obtained through quasineutrality and Ampere's law for the case of runaway tearing is the same as that for regular tearing with the replacement

$$\eta_J I_3 \rightarrow \frac{n_b}{n_o}\frac{L_n}{L_J}K_2$$

in Ampere's law. Hence, quasineutrality is given by

$$\left[-\frac{d^2}{dx^2} + \Lambda + \chi^2\right]\tilde{\phi} - \left[\frac{i}{d}(\omega - \omega_{*e})I_1 + x_e\chi^2\right]\frac{v_e}{c}\tilde{A}_{\parallel} = 0 \quad (5.54)$$

and Ampere's law is

$$\frac{\tau v_A^2}{dc^2}\left[\frac{d^2}{dx^2} - b + \alpha_i^2 + \frac{v_e^2}{\tau v_A^2} i(\omega - \omega_{*e}) I_2 - \frac{v_e^2}{\tau v_A^2} i\omega_{*e} \frac{n_b}{n_o} \frac{L_n}{L_J} K_2\right] \tilde{A}_{\|}$$
$$- \left[\frac{i}{d}(\omega - \omega_{*e}) I_1 + x_e \chi^2\right] \frac{v_e}{c} \tilde{\phi} = 0 \tag{5.55}$$

Similarly, the variational principle for this system is given by

$$S = S_\phi + \frac{\tau v_A^2}{dc^2} S_A$$

where

$$S_\phi = \int_{-a}^{a} dx \left[- \left(\frac{d\tilde{\phi}}{dx}\right)^2 - \Lambda \tilde{\phi}^2 + \frac{1}{2} \frac{x^2}{x_i^2} \left(\tilde{\phi} - \frac{\omega}{k_{\|} c} \tilde{A}_{\|}\right)^2 + 2 \frac{i}{d} (\omega - \omega_{*e}) I_1 \frac{v_e}{c} \tilde{\phi} \tilde{A}_{\|} \right]$$
$$+ \tilde{\phi} \tilde{\phi}' \Big|_{-a}^{a} \tag{5.56}$$

and

$$S_A = \int_{-a}^{a} dx \left\{ \left(\frac{d\tilde{A}_{\|}}{dx}\right)^2 + \left[b - \frac{v_e^2}{\tau v_A^2} i(\omega - \omega_{*e}) I_2 + \frac{v_e^2}{\tau v_{\|}^2} i\omega_{*e} \frac{n_b}{n_o} \frac{L_n}{L_J} K_2 \right] \tilde{A}_{\|}^2 \right\}$$
$$- \tilde{A}_{\|} \tilde{A}_{\|}' \Big|_{a}^{-a} \tag{5.57}$$

An analysis of the above system was presented in Ch. 4 which yielded the dispersion relation through a variational calculation. This calculation will not be repeated here, but a simpler heuristic derivation of the dispersion relation will be presented in the next section for the magnetic tearing mode (which neglects the S_ϕ contribution).

5.4 Simplified Fluid Theory

In this section, the above fluid model is reduced to a simplified fluid theory, retaining only the "bare basics" necessary to describe the fundamental tearing mode physics. In particular, the electrostatic potential $\widetilde{\phi}$ is neglected and only the magnetic potential $\widetilde{A}_{||}$ is retained [13]. Since the fluctuations are described by a single potential, $\widetilde{A}_{||}$, the dispersion relation is determined only through the use of Ampere's law. Hence, the aim is to calculate the perturbed parallel current $\widetilde{J}_{||}$ via the continuity and momentum equations. Also, to simplify the analysis, it will be assumed that the ions form a stationary background and, hence, the perturbed parallel current is carried solely by the electrons. This amounts to neglecting the ion inertial stabilization [10,14] discussed above. Hence, the fluid equations need only be developed for electrons.

In the previous chapters, this mode is termed the magnetic tearing mode. The magnetic tearing mode neglects the stabilizing effects of particle inertia, which is valid approximation at low plasma beta (as can be seen from the dispersion relation of the electromagnetic tearing mode). The point in exploring this simplified model is that the mathematical analysis can be reduced to a bare minimum and the physics can be illuminated much more clearly.

The fluid equations for this model are derived in a identical fashion as in the full fluid model discussed above, only now the electrostatic potential $\widetilde{\phi}$ is set to zero. The starting point is the drift kinetic equation for electrons where the sole perpendicular guiding center motion is described by $\mathbf{v}_{D\perp} = v_{||}\widetilde{\mathbf{B}}_{\perp}/B$,

$$\left[\frac{\partial}{\partial t} + v_{||}\mathbf{b}\cdot\nabla + v_{||}\frac{\widetilde{\mathbf{B}}_{\perp}}{B}\cdot\nabla - \frac{e}{m}E_{||}\frac{\partial}{\partial v_{||}}\right] f = C(f) \qquad (5.58)$$

where $\widetilde{E}_{||} = \widetilde{A}_{||}i\omega/c$ and $C(f)$ is the total collision operator discussed previously.

Performing the appropriate parallel velocity moments of the above equation yields the continuity equation

$$\left[\frac{\partial}{\partial t} + \nabla \cdot \left(V_{||}\mathbf{b} + V_{||}\frac{\widetilde{\mathbf{B}}_{\perp}}{B} \right) \right] n = C_0 \tag{5.59}$$

and the momentum equation

$$mn\left[\frac{\partial}{\partial t} + \left(V_{||}\mathbf{b} + V_{||}\frac{\widetilde{\mathbf{B}}_{\perp}}{B} \right) \cdot \nabla \right] V_{||}$$
$$= -enE_{||} - \nabla_{||}P_{||} - \nabla_{||} \cdot \frac{\widetilde{\mathbf{B}}_{\perp}}{B} P_{||} + C_1 - mV_{||}C_0, \tag{5.60}$$

where the moments of the collision operator, C_0 and C_1, were discussed in Sec. 5.1.

Linearizing the above equations to first order in the fluctuations yields the following equations describing the perturbed density $\widetilde{n}$ and the perturbed fluid velocity $\widetilde{V}_{||}$:

$$-i\left(\omega_{n_a} - k_{||}V_{||_a}\right)\widetilde{n} + \nabla \cdot \left(\widetilde{V}_{||}\mathbf{b} + V_{||_a}\frac{\widetilde{\mathbf{B}}_{\perp}}{B} \right) n_0 = 0 \tag{5.61}$$

and

$$-imn_0(\omega_{m_0} - k_{||}V_{||0})\widetilde{V}_{||} = -en_0\widetilde{E}_{||} - ik_{||}\widetilde{P}_{||} - \frac{\widetilde{\mathbf{B}}_{\perp}}{B} \cdot \nabla_{\perp}P_{||0}. \tag{5.62}$$

where $\omega_{n_a} = \omega + i\omega_n^T$, $\omega_{m_0} = \omega + i\omega_m^T$ with $\omega_m^T = \omega_n^T = \omega_c$. In deriving the above expressions, the turbulent collision operators acting on the fluctuations were replaced by the effective dissipation frequency, ω_c, as is discussed in the previous sections. Also, for simplicity, the effects of Coulomb collisions are neglected, and it is assumed that the effective dissipation frequencies for density and momentum diffusion are equal, $\omega_m^T = \omega_n^T = \omega_c$. (The notation ω_{n_a} and ω_{m_a} is retained, however, so that comparison to the purely collisional "resistive" limit can be obtained easily. In the resistive limit, $\omega_{m_a} = \omega + i\nu$ whereas $\omega_{n_a} = \omega$.)

Equation (5.61) is the typical form of a perturbed continuity equation for a fluid with equilibrium flow $V_{\|0}$ and perturbed flows $\tilde{V}_{\|}$ and $\tilde{V}_{\perp} = V_{\|0}\tilde{\mathbf{B}}_{\perp}/B$. Notice, however, that the perturbed density is being lost at the rate ω_c due to turbulent diffusion. Equation (5.62) is the perturbed parallel momentum equation for a fluid with equilibrium flow $V_{\|0}$ and a perturbed parallel equilibrium field $\tilde{E}_{\|}$. The last two pressure terms on the right side of Eq. (5.62) represent the parallel component of the divergence of the perturbed pressure tensor. Notice that in this fluid theory

$$\begin{aligned}
&\mathbf{b}\cdot\left[\nabla\cdot\left(\tilde{P}_{\|}\mathbf{bb}+P_{\|0}\tilde{\mathbf{b}}_{\perp}\mathbf{b}+P_{\|0}\mathbf{b}\tilde{\mathbf{b}}_{\perp}\right)\right.\\
&\qquad=\mathbf{b}\cdot\left[\left(\nabla\cdot\tilde{P}_{\|}\mathbf{b}\right)\mathbf{b}+\left(\nabla\cdot P_{\|0}\tilde{\mathbf{b}}_{\perp}\right)\mathbf{b}+\left(\nabla\cdot P_{\|0}\mathbf{b}\right)\tilde{b}_{\perp}\right]\\
&\qquad=\nabla_{\|}\tilde{P}_{\|}+\tilde{\mathbf{b}}_{\perp}\cdot\nabla_{\perp}P_{\|0},
\end{aligned}$$

where $\tilde{\mathbf{b}}_{\perp} = \tilde{\mathbf{B}}_{\perp}/B$. As with Eq. (5.61), the perturbed parallel momentum is being lost at rate ω_c by turbulent diffusion.

Equations (5.61) and (5.62) can be combined to give expressions for $\tilde{n}$ and $\tilde{J}_{\|}$ solely in terms of $\tilde{A}_{\|}$. The perturbed density is given by

$$\begin{aligned}
&\left[\left(\omega_{m_o}-k_{\|}V_{\|0}\right)\left(\omega_{n_o}-k_{\|}V_{\|_o}\right)-\frac{T}{m}k_{\|}^2\right]\tilde{n}\\
&\quad=-i\left(\omega_{m_o}-k_{\|}V_{\|_o}\right)\nabla_{\perp}\cdot V_{\|_o}\frac{\tilde{\mathbf{B}}_{\perp}}{B}n_o-ik_{\|}\frac{qn}{m}\tilde{E}_{\|}-ik_{\|}\frac{T}{m}\frac{\tilde{\mathbf{B}}_{\perp}}{B}\cdot\nabla_{\perp}n_o. \qquad (5.63)
\end{aligned}$$

The perturbed current, $\tilde{J}_{\|} = q(\tilde{n}V_{\|_o} + n_o\tilde{V}_{\|})$, is given by

$$\begin{aligned}
&\left[\left(\omega_{m_o}-k_{\|}V_{\|0}\right)\left(\omega_{n_o}-k_{\|}V_{\|_o}\right)-\frac{T}{m}k_{\|}^2\right]\frac{\tilde{J}_{\|}}{q}\\
&\quad=\frac{\omega_{n_o}}{k_{\|}}\left[ik_{\|}\frac{qn}{m}\tilde{E}_{\|}-ik_{\|}\frac{T}{m}\frac{\tilde{\mathbf{B}}_{\perp}}{B}\cdot\nabla_{\perp}n_o\right]\\
&\qquad-\frac{i}{k_{\|}}\nabla_{\perp}\cdot V_{\|_o}\frac{\tilde{\mathbf{B}}_{\perp}}{B}n_o\left[k_{\|}V_{\|_o}\left(\omega_{m_o}-k_{\|}V_{\|_o}\right)-\frac{T}{m}k_{\|}^2\right]. \qquad (5.64)
\end{aligned}$$

Rewriting the above equation in terms of $\widetilde{A}_{\|}$ gives

$$\left[\left(\omega_{m_0}-k_{\|}V_{\|0}\right)\left(\omega_{n_{\|}}-k_{\|}V_{\|_{\|}}\right)-\frac{T}{m}k_{\|}^2\right]\frac{\widetilde{J}_{\|}}{-e}$$
$$=\frac{en_o}{T}\left\{\frac{T\omega_{n_{\|}}}{mc}(\omega-\omega_{*e})\widetilde{A}_{\|}\right.$$
$$\left.+\left(\frac{T}{m}k_{\|}^2+k_{\|}V_{\|_{\|}}\left(\omega_{m_{\|}}-k_{\|}V_{\|_{\|}}\right)\right)\frac{\omega_{*e}}{k_{\|}}\frac{J_{\|_0}}{enc}\frac{L_n}{L_J}\widetilde{A}_{\|}\right\}. \qquad (5.65)$$

Eq. (5.65) describes the perturbed parallel current $\widetilde{J}_{\|_0}$ as a function of $\widetilde{A}_{\|}$. The above expression includes the effects of turbulent electron diffusion and allows for an equilibrium fluid flow $V_{\|_{\|}}$. A closed system for $\widetilde{A}_{\|}$ is found by combining Eq. (5.65) with the linearized Ampere's law. From this a dispersion relation is determined. In Sec. 5.4.1 the perturbed form of Ampere's law is presented in an inner and outer region for the regular tearing mode. This is repeated in Sec. 5.4.2 for the runaway tearing mode. The dispersion relations for these modes are calculated in Sec. 5.4.3.

5.4.1 Regular Tearing

In the "regular" tearing limit, the equilibrium current is carried by the majority of the electron population, as in the case of a drifted Maxwellian distribution. In the fluid picture, the regular tearing mode is described by a single fluid of electrons, with a fluid velocity $V_{\|_{\|}} = v_D \ll v_e$ and with the equilibrium current given by $J_{\|_0} = -ev_Dn_o$. Using the scaling $v_D \ll v_e$, then Eq. (5.65) for the perturbed current becomes

$$\left(\omega_{m_{\|}}\omega_{n_{\|}}-\frac{T}{m}k_{\|}^2\right)\frac{\widetilde{J}_{\|}}{-e}=\frac{e}{T}n_o\left\{\frac{T\omega_{n_{\|}}}{mc}(\omega-\omega_{*e})\widetilde{A}_{\|}+\frac{T}{m}k_{\|}\omega_{*e}\frac{J_{\|_0}}{enc}\frac{L_n}{L_J}\widetilde{A}_{\|}\right\}. \qquad (5.66)$$

To determine the dispersion relation, Eq. (5.66) is used in Ampere's law. The resulting equation is then equivalent of the expression $L_2\widetilde{A}_{\|}=0$, where the operator L_2 is discussed in Sec. 2.3. In Sec. 3.4, the dispersion relation was then determined

variationally according to the relation $S_A = 0$, where S_A is the variational form for the magnetic tearing mode.

To avoid doing such a detailed variational calculation, a simpler less rigorous derivation of the dispersion relation is given here. This is the matching procedure, also discussed earlier in Sec. 3.3. In this approximation, the plasma is divided into two regions, $|x| < x_c$ and $|x| > x_c$. The perturbed parallel current is approximated in these two regions and an inner and outer solution for $\widetilde{A}_{||}$ is then calculated via Ampere's law. The dispersion relation is then determined by requiring the inner and outer solution, as well as their first derivative, to match at the boundary, $|x| = x_c$, where $x_c \equiv \omega_c / k'_{||} v_e$.

Assuming $|x/x_c| \ll 1$ in the inner region and $|x/x_c| \gg 1$ in the outer region, then Eq. (5.66) gives the following expressions for $\widetilde{J}_{||}$:

$$|x| < x_c, \quad \widetilde{J}_{||} \simeq \frac{-e^2 n_o}{mc} \frac{(\omega - \omega_{*e})}{\omega_{m_{..}}} \widetilde{A}_{||} \tag{5.67}$$

$$|x| > x_c, \quad \widetilde{J}_{||} \simeq \frac{e^2}{T} \frac{\omega_{*e}}{k_{||}} \frac{J_{||_{..}}}{ec} \frac{L_n}{L_J} \widetilde{A}_{||}$$

$$= -\frac{k_y J'_{||_{..}}}{k_{||} B_0} \widetilde{A}_{||} \tag{5.68}$$

Using Ampere's law, $\nabla^2 \widetilde{A}_{||} + (4\pi/c)\widetilde{J}_{||} = 0$, then gives the following approximate equations for $\widetilde{A}_{||}$ in the inner and outer regions:

$$|x| < x_c, \quad \left[\frac{d^2}{dx^2} + \frac{v_e^2}{\tau v_A^2} \frac{i(\omega - \omega_{*e})}{2(\omega_c - i\omega)} \right] \widetilde{A}_{||} = 0 \tag{5.69a}$$

$$|x| > x_c, \quad \left[\frac{d^2}{dx^2} - b - \frac{4\pi}{c} \rho_i^2 \frac{k_y J'_{||_{..}}}{k_{||} B_0} \right] \widetilde{A}_{||} = 0 \tag{5.69b}$$

In the above equations, $b = k_y^2 \rho_i^2$ and x has been normalized in units of the ion gyroradius ρ_i. These equations will be returned to later in Sec. 5.4.3 where the dispersion relation is determined.

5.4.2 Runaway Tearing

In the "runaway" tearing limit, the equilibrium current is carried by an electron beam superimposed on Maxwellian bulk electron distribution. The bulk electrons are characterized by a thermal speed v_e and a fluid density and velocity given by $n = n_o$ and $V_{||_o} = 0$. The beam electrons are characterized by $n = n_b$ and $V_{||_o} = v_b$ where $n_b/n_o \sim 10^{-3}$ and $v_b/v_e \sim 10$. The total current is to be held constant when comparing the runaway to the regular tearing mode, $J_{||_o} = -en_o v_D = -en_b v_b$. To analyze the runaway tearing mode from the fluid point of view, the plasma is considered to be composed of two fluids: 1) bulk electrons and 2) beam electrons.

The perturbed parallel current due to the bulk electrons is given by Eq. (5.65) when $V_{||_o} = 0$,

$$\left.\frac{\tilde{J}_{||}}{-e}\right|_{bulk} \simeq \left(\omega_{m_o}\omega_{n_o} - \frac{T}{m}k_{||}^2\right)^{-1} \frac{e\omega_{n_o}}{n_o mc}(\omega - \omega_{*e})\tilde{A}_{||}. \tag{5.70}$$

Similarly, the perturbed parallel current due to the beam electrons is given by Eq. (5.65) when $V_{||_o} = v_b$ and using $v_b^2/v_e^2 \gg 1$,

$$\left.\frac{\tilde{J}_{||}}{-e}\right|_{beam} \simeq \frac{e}{T}\frac{k_{||}v_b}{\omega_{n_o} - k_{||}v_b}\frac{\omega_{*e}}{k_{||}}\frac{J_{||_o}}{ec}\frac{L_n}{L_J}\tilde{A}_{||}. \tag{5.71}$$

The total perturbed current is given by adding the contribution from the bulk and beam electrons. In the inner region where $|x| < x_c$, x_b (where $x_b = \omega_b/k'_{||}v_b$, and $\omega_{n_o} = \omega + i\omega_b$ for the beam electrons) the perturbed parallel current is given by

$$\begin{aligned} |x| < x_c, \quad \tilde{J}_{||} &\simeq \frac{-e^2 n_o}{mc}\frac{(\omega - \omega_{*e})}{\omega_{m_o}}\tilde{A}_{||} - \frac{ie^2}{Tc}\frac{\omega_{*e}}{\omega_b}v_b^2 n_b \frac{L_n}{L_J}\tilde{A}_{||} \\ &\simeq \frac{-ie^2 n_o}{mc}\left[\frac{(\omega - \omega_{*e})}{(\omega_c - i\omega)} - \frac{2\omega_{*e}}{\omega_b}\frac{v_b^2}{v_e^2}\frac{n_b}{n_o}\frac{L_n}{L_J}\right]\tilde{A}_{||}. \end{aligned} \tag{5.72}$$

In the above expression, the first term on the right is due to the bulk electrons and is the same as the response in the inner region for the regular tearing mode. The second term on the right is due to the beam electrons and represents a shift in the real frequency of the mode away from ω_{*e}. In this term, the approximation $\omega_{n_\alpha} = \omega + i\omega_b \simeq i\omega_b$ was used, thus neglecting a correction of order $\omega/\omega_b \sim (\omega/\omega_c)(v_e/v_b)^{2/3}$ where only corrections of order $\omega/\omega_c \sim x_e/x_c$ are retained in this calculation.

In the outer region, $|x| > x_c$, only the contribution from the beam electron is significant (the bulk electrons carry no equilibrium current). Hence, in the outer region

$$|x| > x_c, \quad \tilde{J}_\parallel \simeq \frac{e}{Tc}\frac{\omega_{*e}}{k_\parallel} J_{\parallel_\alpha} \frac{L_n}{L_J}\tilde{A}_\parallel = -\frac{k_y J'_{\parallel_\alpha}}{k_\parallel B_0}\tilde{A}_\parallel. \tag{5.73}$$

In the outer region, the perturbed current for the runaway case is the same as with the regular tearing case.

Using Ampere's law as well as the above two expressions, equations can be determined which described $\tilde{A}_\parallel$ in the inner and outer regions.

$$|x| < x_c, \quad \left[\frac{d^2}{dx^2} + \frac{iv_e^2}{2\tau v_A^2}\left(\frac{(\omega-\omega_{*e})}{(\omega_c - i\omega)} - \frac{2\omega_{*e}}{\omega_c}\frac{x_c}{x_b}\frac{v_b}{v_e}\frac{n_b}{n_o}\frac{L_n}{L_J}\right)\right]\tilde{A}_\parallel = 0 \tag{5.74}$$

$$|x| > x_c, \quad \left[\frac{d^2}{dx^2} - b - \frac{4\pi}{c}\rho_i^2\frac{k_y J'_{\parallel_\alpha}}{k_\parallel B_0}\right]\tilde{A}_\parallel = 0 \tag{5.75}$$

Again, in the outer region, $|x| > x_c$, the response is the same in both the runaway and regular tearing mode. As will be discussed below, the response in the outer region is simply the ideal MHD response. In the next section, these equations are used to determined the dispersion relation.

5.4.3 Dispersion Relation

The dispersion relation is determined using the following simplified procedure [13]. The approximate form of $\tilde{A}_{||}$ is solved for in an inner region, $|x| < x_c$, in the limit $|x| \ll x_c$, and in an outer region $|x| > x_c$ in the limit $|x| \gg x_c$. The inner and outer solution, as well as their first derivatives, are required to be continuous at $|x| = x_c$. This simple matching procedure at the boundary at $\pm x_c$ then determines the dispersion relation. Although this procedure is very approximate, it is at the same time mathematically simple, and it reproduces the basic dispersion relation for the magnetic tearing mode (in agreement with the variational calculation of the earlier section).

In the outer region $|x| > x_c$ for either the regular or runaway tearing mode, $\tilde{A}_{||}$ satisfies the ideal MHD equation for marginal stability, Eq. (5.75). The solution to this equation is quite complicated, particularly for realistic geometries and equilibrium current profiles. However, the solution to Eq. (5.75) will be assumed to be known through computation techniques, and will designated $\tilde{A}_{||}^{MHD}$. Actually, it is not the magnitude of $\tilde{A}_{||}^{MHD}$ itself which will be assumed known, but rather the ratio of the first derivative of $\tilde{A}_{||}^{MHD}$ to the magnitude. The following shorthand notation will be used.

$$\Delta_+ \equiv \left.\frac{\tilde{A}_{||}'}{\tilde{A}_{||}}\right|_{MHD} \quad \text{for} \quad x > x_c$$

and

$$\Delta_- \equiv \left.\frac{\tilde{A}_{||}'}{\tilde{A}_{||}}\right|_{MHD} \quad \text{for} \quad x < -x_c \tag{5.76}$$

In practice, Δ_+ and Δ_- can be determined from integrating Eq. (5.75) numerically from the edge of the plasma inward to within a specified distance of the rational surface.

In the inner region, $\widetilde{A}_{||}$ satisfies the following equation:

$$|x| < x_c, \quad \left[\frac{d^2}{dx^2} + \frac{iv_e^2}{2\tau v_A^2}\left(\frac{(\omega - \omega_{*e})}{(\omega_c - i\omega)} - \frac{\delta\omega_b}{\omega_c}\right)\right]\widetilde{A}_{||} = 0 \tag{5.77}$$

where

$$\delta\omega_b = \begin{cases} 2\omega_{*e}\dfrac{x_c}{x_b}\dfrac{v_b}{v_e}\dfrac{n_b}{n_o}\dfrac{L_n}{L_J}, & \text{runaway case} \\ 0, & \text{regular case} \end{cases}$$

The solution to this equation is given by

$$\widetilde{A}_{||} = c_+ e^{\alpha x} + c_- e^{-\alpha x}, \quad \text{for} |x| < x_c \tag{5.78}$$

where c_+ and c_- are constants of integration and

$$\alpha^2 = -\frac{iv_e^2}{2\tau v_A^2}\left(\frac{(\omega - \omega_{*e})}{(\omega_c - i\omega)} - \frac{\delta\omega_b}{\omega_c}\right).$$

The inner solution along with its first derivative can be matched at $\pm x_c$ to the outer MHD solution. Matching at $x = +x_c$ yields

$$\Delta_+(x_c) = \frac{\alpha\left(c_+ e^{+\alpha x_c} - c_- e^{-\alpha x_c}\right)}{c_+ e^{\alpha x_c} + c_- e^{-\alpha x_c}}. \tag{5.79}$$

Matching at $x = -x_c$ gives

$$\Delta_-(-x_c) = \frac{\alpha\left(c_+ e^{-\alpha x_c} - c_- e^{\alpha x_c}\right)}{c_+ e^{-\alpha x_c} + c_- e^{\alpha x_c}}. \tag{5.80}$$

Equations (5.79) and (5.80) can be solved to give two equations for the ratio c_+/c_-. Setting these two equations for c_+/c_- equal to one another yields the following dispersion relation:

$$\frac{\Delta_+ + \alpha}{\Delta_+ - \alpha} e^{-2\alpha x_c} = \frac{\Delta_- + \alpha}{\Delta_- - \alpha} e^{2\alpha x_c}. \tag{5.81}$$

The above expression can be rearranged to read

$$\left(1+\frac{\Delta_+}{\alpha}\right)\left(1-\frac{\Delta_-}{\alpha}\right)=\left(1+\frac{\Delta_-}{\alpha}\right)\left(1-\frac{\Delta_+}{\alpha}\right)e^{4\hat{\alpha}x_c}. \tag{5.82}$$

Numerical calculations using typical Alcator C profiles indicate $\Delta_\pm \sim 10^{-2}$. Assuming the scaling $x_c\alpha^2 \sim \Delta_\pm \sim \epsilon^2$ and assuming that $x_c\alpha \sim \epsilon$, then Eq. (5.82) can be expanded to yield

$$\Delta'(x_c) \equiv \Delta_+(x_c) - \Delta_-(-x_c) \cong 2x_c\alpha^2. \tag{5.83}$$

Using the definition of α^2 gives the following dispersion relation:

$$\Delta'(x_c) = i\frac{v_e^2}{\tau v_A^2}x_e\left[\left(1-\frac{\omega}{\omega_{*e}}\right)\left(1+\frac{i\omega_{*e}}{\omega_c}\right)+\frac{\delta\omega_b}{\omega_{*e}}\right]. \tag{5.84}$$

where the expansion $(\omega_c - i\omega)^{-1} \simeq \omega_c^{-1}(1 + i\omega_{*e}/\omega_c)$ has been used.

Setting the imaginary part of Eq. (5.84) equal to zero gives the real frequency of the mode,

$$\omega_{real} = \omega_{*e}(1 + \delta\omega_b/\omega_{*e}). \tag{5.85}$$

Setting the real part of Eq. (5.84) equal to zero gives the growth rate of the mode,

$$\frac{\gamma}{\omega_{*e}} \simeq \frac{1}{\beta_i}\left(\frac{m_e}{\tau m_i}\right)^{1/2}\frac{L_n}{L_s}\Delta'(x_c) - \frac{\omega_{*e}}{\omega_c}\frac{\delta\omega_b}{\omega_{*e}}, \tag{5.86}$$

where $\delta\omega_b = 0$ for the regular tearing mode and is given by

$$\frac{\delta\omega_b}{\omega_{*e}} = 2\frac{x_c}{x_b}\frac{v_b}{v_e}\frac{n_b}{n_o}\frac{L_n}{L_J} \tag{5.87}$$

for the runaway case.

Numerical calculations for Alcator C parameters indicates $\Delta'(x)$ to be a monotonically decreasing function of x passing through zero at $x = W$. A convenient form is to approximate $\Delta'(x)$ as a linear function of x:

$$\Delta'(x) \simeq \Delta_0 \left(1 - \frac{x}{W}\right) \quad . \tag{5.88}$$

Physically, W represents the width at which a magnetic island saturates nonlinearly and, consequently, the width outside of which no free magnetic energy remains to drive the mode.

The physical interpretation of the dispersion relation given by Eq. (5.84) along with a discussion of the resulting growth rate is given in the following section.

5.4.4 Physical Interpretation

Before discussing the physical interpretation of the dispersion relation itself, it is useful to examine the nature of the perturbed parallel currents in the inner and outer regions. In the outer region, $\tilde{J}_{||}$ was the same for both the runaway and regular tearing modes. It is easy to show that this is the ideal MHD response at marginal stability. Specifically, Eq. (5.73) can be derived in the following way. In general, the divergence of the perturbed current must be zero, as is implied by $\nabla \times \tilde{\mathbf{B}} = (4\pi/c)\tilde{\mathbf{J}}$ in MHD or by quasineutrality in kinetic theory. Hence,

$$\nabla \cdot (\tilde{\mathbf{J}}_{\perp} + \tilde{\mathbf{J}}_{||}) = 0 \quad . \tag{5.89}$$

To find $\tilde{\mathbf{J}}_{\perp}$ in ideal MHD, the perturbed form of the momentum equation is used,

$$i\gamma\rho\tilde{\mathbf{v}} = \tilde{\mathbf{J}} \times \mathbf{B}_0 + \mathbf{J}_0 \times \tilde{\mathbf{B}} - \nabla\tilde{P} \tag{5.90}$$

The left hand side of the above equation is zero at marginal stability, and the pressure gradient can be eliminated by taking the curl of Eq. (5.90). The z-component of the curl of Eq. (5.90) can be written as

$$0 = (\mathbf{B}_0 \cdot \nabla)\tilde{J}_{||} + (\tilde{\mathbf{B}} \cdot \nabla)J_{||_o}$$

or

$$\tilde{J}_{||} = -\frac{k_y J'_{||_o}}{k_{||} B_0}\tilde{A}_{||} \tag{5.91}$$

which is identical to Eq. (5.73) for the perturbed parallel current in the outer region.

By comparing Eq. (5.91) and Eq. (5.89), then it is clear that the perturbed perpendicular current in ideal MHD is given by

$$\tilde{\mathbf{J}}_\perp = J_{||_o}\frac{\tilde{\mathbf{B}}_\perp}{B_0}. \tag{5.92}$$

Equation (5.92) makes sense with the intuitive picture of ideal MHD which states that the plasma and field lines move together. Equation (5.92) states that the perturbed current in ideal MHD which results from the presence of a perturbed magnetic field is simply due to the equilibrium current flowing along the total magnetic field, $\mathbf{B}_0 + \tilde{\mathbf{B}}_\perp$.

At small x, the perturbed parallel current has two significant components. The first results from the bulk electrons and the second results from the runaway beam electrons. In general, the perturbed parallel current can be written as

$$\tilde{J}_{||} = q(n_o\tilde{V}_{||} + \tilde{n}V_{||_o}). \tag{5.93}$$

For the bulk electrons where $n = n_o$ and $V_{||_o} = v_D \sim 10^{-2}v_e$ ($V_{||_o} = 0$ for the runaway case), the first term of the right dominates (by v_e/v_D). For the beam electrons when $n = n_b \sim 10^{-3}n_o$ and $V_{||_o} = v_b \sim 10v_e$, the second term on the right dominates (by v_b/v_e). Hence, $\tilde{J}_{||}$ can be approximate in the inner region as

$$\widetilde{J}_{\|}\Big|_{bulk} \simeq q n_o \widetilde{V}_{\|} \qquad (5.94a)$$

$$\widetilde{J}_{\|}\Big|_{beam} \simeq q v_b \widetilde{n}. \qquad (5.94b)$$

Note that in the inner region, the effects of the fluid drift of the bulk electrons (the equilibrium current) is neglectable for the regular tearing mode.

To calculate $\widetilde{J}_{\|}$ in the inner region from the bulk electrons for either the regular or runaway case, the perturbed fluid velocity need only be calculated. For the small x limit ($k_{\|} \to 0$), the perturbed parallel momentum equation, Eq. (5.61), reduces to the following in the inner region:

$$\begin{aligned} -imn_o\omega_{m_{\|}}\widetilde{V}_{\|} &\simeq q n_o \widetilde{E}_{\|} - \frac{\widetilde{\mathbf{B}}_{\perp}}{B}\cdot\nabla_{\perp}P_{\|_o} \\ &= -in_o\omega\frac{e}{c}\left(1-\frac{\omega_{*e}}{\omega}\right)\widetilde{A}_{\|}, \end{aligned} \qquad (5.95)$$

where $\omega_{m_{\|}} = \omega + i\omega_c$.

The first term on the left (proportional to ω) of Eq. (5.95) represents the change in parallel momentum due to the fluid element oscillating at the real frequency of the mode (an "inertial" term). The second term on the left (proportional to ω_c) represents the loss of parallel momentum due to turbulent radial transport (a "dissipative" term). The first term on the right represents the force on the fluid element due to the presence of a perturbed parallel electric field. The second term on the right represents the force on the fluid element due to the gradient of the perpendicularly convected equilibrium parallel pressure.

Combining Eq. (5.94a) and Eq. (5.95) then give an expression for $\widetilde{J}_{\|}$ due to the bulk electrons in the inner region, $|x| < x_c$:

$$\left.\widetilde{J}_{\|}\right|_{bulk} \simeq \frac{-e^2 n_o}{c m_e}\frac{\omega}{\omega_{m_{\|}}}\left(1-\frac{\omega_{*e}}{\omega}\right)\widetilde{A}_{\|}$$
$$\simeq i\frac{e^2 n_o}{c m_e}\frac{\omega}{\omega_c}\left(1-\frac{\omega_{*e}}{\omega}\right)\left(1+\frac{i\omega}{\omega_c}\right)\widetilde{A}_{\|}. \qquad (5.96)$$

The imaginary term (or "dissipative" term) in the above expression is a result of balancing the dissipative loss of momentum due to turbulent diffusion against the driving forces due to the electric field and pressure gradient in the momentum equation, Eq. (5.95). The factor $(\omega-\omega_{*e})$ arises from the Doppler shift of $\tilde{E}_{\|}$ as seen by the electron fluid. The real term (or "reactive" term) in the above expression is a correction term (of order ω_*/ω_c) which arises since the electron fluid has finite inertia and oscillates at the real frequency of the mode, $\omega \sim \omega_{*e}$. This reactive term results from the inclusion of the inertial term on the left hand side of Eq. (5.95).

To calculate $\widetilde{J}_{\|}$ in the inner region from the runaway beam electrons, the perturbed density need only be calculated, as stated in Eq. (5.94b). For the small x limit ($k_{\|} \to 0$), the perturbed density equation, Eq. (5.62), reduces to the following in the inner region:

$$-i\omega_{n_{\|}}\widetilde{n} + \nabla\cdot V_{\|_{\|}}\frac{\widetilde{\mathbf{B}}_{\perp}}{B}n_b = 0, \qquad (5.97)$$

where $\omega_{n_{\|}} = \omega + i\omega_b$, and $\omega_b^3 = (k_{\|}{}'v_b)^2 D/3$ is the effective dissipative frequency for turbulent diffusion of the beam. Since $\omega/\omega_b \sim (\omega_*/\omega_c)(v_e/v_b)^{2/3}$, the term proportional to real frequency, ω, can be neglected in the above equation (only corrections of order ω_*/ω_c are retained). Eq. (5.97) becomes

$$\omega_b\widetilde{n} \simeq -\nabla\cdot V_{\|_{\|}}\frac{\widetilde{\mathbf{B}}_{\perp}}{B}n_b$$
$$= -\frac{ik_y}{B}\frac{J'_{\|_{\|}}}{e}\widetilde{A}_{\|}. \qquad (5.98)$$

The above equation represents a balance between the dissipative loss of density due to turbulent diffusion with the loss of density due to the parallel flow of density along the total field line, $\mathbf{B}_0 + \widetilde{\mathbf{B}}_\perp$.

Combining Eq. (5.98) with Eq. (5.94b) gives $\widetilde{J}_{||}$ in the inner region from the beam electrons:

$$\begin{aligned}\widetilde{J}_{||}\Big|_{beam} &\simeq i\frac{v_b}{\omega_b}\frac{k_y}{B}J'_{||_o}\widetilde{A}_{||} \\ &= -\left(\frac{ie^2 n_o}{m_e c}\right)2\frac{\omega_{*e}}{\omega_b}\frac{v_b^2}{v_e^2}\frac{n_b}{n_o}\frac{L_n}{L_J}\widetilde{A}_{||} \\ &= -\left(\frac{ie^2 n_o}{m_e c}\right)\frac{\delta\omega_b}{\omega_c}\tilde{A}_{||} \qquad (5.99)\end{aligned}$$

where

$$\delta\omega_b = 2\omega_{*e}\frac{x_c}{x_b}\frac{v_b}{v_e}\frac{n_b}{n_o}\frac{L_n}{L_J}.$$

The above expression of the beam contribution to the perturbed current is imaginary, and, hence, dissipative. This term arises from balancing the fluid loss per unit volume due to turbulent diffusive dissipation with that loss due to fluid flow along the total magnetic field lines.

In order to give a physical interpretation of the dispersion relation, Eq. (5.84), it is helpful to consider an alternate heuristic derivation of the dispersion relation based on an energy integral formed from Ampere's law. To form this integral, S, Ampere's law is multiplied by $\widetilde{A}^*_{||}$ and then integrated over the plasma:

$$S = \int_{-a}^{a} dx \left\{ \left|\frac{d\widetilde{A}_{||}}{dx}\right|^2 + b|\widetilde{A}_{||}|^2 - \frac{4\pi}{c}\widetilde{A}^*_{||}\widetilde{J}_{||} \right\}, \qquad (5.100)$$

where an integration of parts has been performed on the second derivative term and where $\pm a$ represents the edge of the plasma. It is assumed that $\widetilde{A}_{||}$ and its first derivative vanish at the plasma edge.

The first two terms in the integral S represent the energy required to bend the equilibrium magnetic field lines. Notice that

$$|\tilde{\mathbf{B}}_\perp|^2 = |\tilde{B}_x|^2 + |\tilde{B}_y|^2 = k_y^2|\tilde{A}_\parallel|^2 + \left|\frac{d\tilde{A}_\parallel}{dx}\right|^2$$

which is the usual expression for line bending energy in the ideal MHD energy principle. The last term in the integral S represents the energy contained in the interaction of the perturbed parallel electric field and current. Recall that the power dissipated by the interaction of $\tilde{E}_\parallel$ and $\tilde{J}_\parallel$ scales as

$$P \sim \tilde{E}_\parallel^* \tilde{J}_\parallel \sim -\frac{i\omega}{c}\tilde{A}_\parallel^* \tilde{J}_\parallel.$$

Thus, $\tilde{A}_\parallel^* \tilde{J}_\parallel / c$ has the units of energy and represents the time integrated power.

To find the dispersion relation, the integral S is broken into two parts, one for the outer region, $|x| > x_c$, and one for the inner region, $|x| < x_c$. The appropriate approximate expressions for $\tilde{J}_\parallel$ are then substituted into the integrals in the inner and outer regions. This gives

$$S = S_{out} + S_{in},$$

where

$$S_{out} = \int_{|x|>x_c} dx \left\{ \left(\frac{d\tilde{A}_\parallel}{dx}\right)^2 + b\tilde{A}_\parallel^2 + \frac{4\pi}{c}\rho_i^2 \frac{k_y J'_{\parallel_o}}{k_\parallel B_0}\tilde{A}_\parallel^2 \right\} \tag{5.101}$$

and

$$S_{in} = \int_{|x|<x_c} dx \left\{ \left(\frac{d\tilde{A}_\parallel}{dx}\right)^2 - \frac{iv_e^2}{2\tau v_A^2}\left(\frac{(\omega-\omega_{*e})}{(\omega_c - i\omega)} - \frac{\delta\omega_b}{\omega_c}\right)\tilde{A}_\parallel^2 \right\} \tag{5.102}$$

where the absolute value signs have been dropped for convenience. Equations (5.101) and (5.102) are simply the integrals resulting from multiplying by $\widetilde{A}_{||}$ the approximate versions of Ampere's law in the inner and outer regions, Equations (5.74) and (5.75).

To evaluate S_{out}, $\widetilde{A}_{||}$ in this outer region is required to satisfy the ideal MHD version of Ampere's law, Eq. (5.75), and, hence, $\widetilde{A}_{||}^{out} = \widetilde{A}_{||}^{MHD}$. Integrating by parts then gives

$$\begin{aligned} S_{out} &= \widetilde{A}_{||}\widetilde{A}'_{||}\Big|_{x_c}^{a} + \widetilde{A}_{||}\widetilde{A}'_{||}\Big|_{-a}^{-x_c} = -A_{||_0}^2\left(\frac{\widetilde{A}'_{||}(x_c)}{A_{||}} - \frac{\widetilde{A}'_{||}(-x_c)}{A_{||}}\right) \\ &\equiv -A_{||_0}^2\Delta'(x_c). \end{aligned} \tag{5.103}$$

Here, it is assumed $\widetilde{A}_{||}(x_c) = \widetilde{A}_{||}(-x_c) = A_{||_0}$. Hence, $\Delta'(x_c)$ is interpreted as the energy drive for the tearing mode resulting from the interaction of $\widetilde{E}_{||}$ and $\widetilde{A}_{||}$ in the outer, ideal MHD region.

In the inner region, the "constant ψ" approximation of resistive MHD is used, which assumes $\widetilde{A}_{||}$ to be a constant in the inner, "non-ideal" region. Thus, letting $\widetilde{A}_{||} = A_{||_0}$, gives

$$S_{in} = -A_{||_0}^2\frac{iv_e^2}{\tau v_A^2}x_c\left(\frac{(\omega-\omega_{*e})}{(\omega_c - i\omega)} - \frac{\delta\omega_b}{\omega_c}\right). \tag{5.104}$$

Again, this term is interpreted as the energy in the inner region resulting from the interaction of $\widetilde{E}_{||}$ and $\widetilde{J}_{||}$.

The actual dispersion relation is determined by setting $S = S_{out} + S_{in} = 0$. This gives the dispersion relation previously stated by Eq. (5.84):

$$\Delta'(x_c) \simeq i\left(\frac{m_i}{\tau m_e}\right)^{1/2}\beta_i\frac{L_s}{L_n}\left[\left(1-\frac{\omega}{\omega_{*e}}\right)\left(1+\frac{i\omega_{*e}}{\omega_c}\right)+\frac{\delta\omega_b}{\omega_{*e}}\right] \tag{5.105}$$

where $\Delta'(x_c) \simeq \Delta_0(1 - x_c/W)$.

As mentioned previously, $\Delta'(x_c)$ is the ideal MHD energy drive for the tearing mode in the outer region. (Since $S_{out} < 0$, it is a "negative energy" mode.) The source of this energy is the third term in Eq. (5.101) involving the equilibrium current gradient $J'_{\|_o}$. In the limit of a infinitesimally narrow tearing dissipative region $\Delta'(x) \rightarrow \Delta_0$, which is the energy drive in the resistive MHD limit. Notice that the ideal perturbed current (the energy drive term) goes as $J'_{\|_o}/k_\| \sim 1/x$ and, hence, becomes very large as $x \rightarrow 0$ and dominates the energy integral S_{out}. If the effects of turbulent diffusion are included, however, the dissipative region then takes an a significant width. With turbulent diffusion, the perturbed current $\tilde{J}_\|$ no longer acts as a source of free energy within x_c of the rational surface but, rather, becomes dissipative in nature. Hence, the ideal MHD energy drive is reduced from Δ_0 to $\Delta_0(1 - x_c/W)$. In nonlinear calculations of the resistive tearing mode, it is shown that the magnetic islands produce by magnetic tearing saturate at a width W. Here, $\Delta'(x = W) = 0$ and, hence, there no longer exists free magnetic energy in the outer region ($|x| > W$), this free energy being necessary to drive the tearing mode. Hence if $x_c > W$, then $\Delta' < 0$ and there is insufficient free energy outside the dissipation region ($|x| > x_c$), and the mode is stabilized.

The first term on the right of Eq. (5.105), proportional to $(\omega - \omega_{*e})$, is imaginary and, thus, dissipative. This represents the energy dissipated in the inner region due to the interaction of $\tilde{E}_\|$ and $\tilde{J}_{\|_o}$. This dissipation is an irreversible transfer of energy from the tearing mode field $\tilde{A}_\|$ to the bulk electron population in the inner region. (In contrast, Δ' represents a relaxation of the outer magnetic field topology to a lower energy state due to the presence of $\tilde{A}_\|$.) The second term on the right of Eq. (5.105), proportional to ω_*/ω_c, is real and, hence, reactive. This term represents the energy necessary to sustain the particle (electron in this model) oscillation since the mode has a real frequency and, hence, this is an inertial term. The third term on the right of Eq. (5.105), proportional to $\delta\omega_b/\omega_{*e}$, is again imaginary and dissipative. This term represents the energy withdrawn from the wave field and dissipated into random energy of the beam electrons in the inner region.

It is perhaps useful to rewrite the dispersion relation in the following way:

$$\Delta_0\left(1-\frac{x_c}{W}\right) = -\left(\frac{m_i}{\tau m_e}\right)^{1/2}\beta_i\frac{L_s}{L_n}\left[\frac{i(\omega-\omega_{*e}-\delta\omega_b)}{\omega_{*e}} - \frac{(\omega-\omega_{*e})}{\omega_{*e}}\frac{\omega_{*e}}{\omega_c}\right]. \quad (5.106)$$

In this form, it is clear that the dissipation on the right of the above equation is proportional to $(\omega-\omega_{*e}-\delta\omega_b)$. In general, if the dissipation is calculated in a frame at rest with the electron fluid, the dissipation is proportional to the frequency of the mode, ω. Hence, the frequency shifts in the factor multiplying the dissipation in the above expression can be viewed as Doppler frequency shifts due to the electron fluid moving relative to the laboratory. The first frequency shift, ω_{*e}, is due to the bulk electron fluid undergoing a perpendicular motion at the diamagnetic drift speed, v_{*e}, where $\omega_{*e} = k_y v_{*e}$. This frequency shift arises from the second term on the right of the momentum equation for the bulk electrons in the inner region, Eq. (5.95). This term describes the force arising from the gradient of the perpendicularly convected equilibrium pressure due to the presence of $\widetilde{B}_\perp$. The second frequency shift, $\delta\omega_b$, is due to the beam electron fluid undergoing a perpendicular motion at the beam drift speed, δv_b, where $\delta\omega_b = k_y\delta v_b$. This term arises from the second term in the beam continuity equation, Eq. (5.97). This term describes the beam density loss due to the beam fluid streaming at v_b along the total field lines, $\mathbf{B}_0 + \widetilde{\mathbf{B}}_\perp$.

In summary, the real frequency and growth rate of the tearing mode are given by the following relations:

$$\omega_{real} \simeq \omega_{*e}\left(1+\frac{\delta\omega_b}{\omega_{*e}}\right) \quad (5.107)$$

and

$$\frac{\gamma}{\omega_{*e}} \simeq \frac{1}{\beta_i}\left(\frac{m_e}{\tau m_i}\right)^{1/2}\frac{L_n}{L_s}\Delta'(x_c) - \frac{\omega_{*e}}{\omega_c}\frac{\delta\omega_b}{\omega_{*e}} \quad (5.108)$$

where

$$\delta\omega_b = 2\omega_{*e}\frac{x_c}{x_b}\frac{v_b}{v_e}\frac{n_b}{n_o}\frac{L_n}{L_J} \tag{5.109}$$

and where $\delta\omega_b = 0$ for the case of the regular tearing mode.

The real frequency has two contributions. The first is due to the bulk electron fluid, ω_{*e}, and the second is due to the beam electron fluid, $\delta\omega_b$. Both frequencies can be viewed as Doppler shifts arising from the electron fluid undergoing perpendicular oscillations in the presence of the tearing mode magnetic field perturbation.

The growth rate has two significant contributions. The first is proportional to $\Delta'(x_c)$ which represents the free magnetic energy available to drive the tearing mode, the origin of which lies in the region outside the dissipative layer. The expression $\Delta'(x_c)$ for the energy drive contains the stabilizing term due to the effect of turbulent electron diffusion prohibiting the formation of the ideal MHD perturbed current within x_c of the rational surface, $\Delta'(x_c) = \Delta_0(1 - x_c/W)$. The second term in the expression for the growth rate represents a stabilization due to the finite inertia of the bulk electron fluid. Since the bulk electrons are oscillating at $\omega_{*e} + \delta\omega_b$, which is slightly higher than their "natural" oscillation frequency, ω_{*e}, additional energy is required to maintain this increase in the oscillatory motion. The shift in frequency $\omega_{*e} + \delta\omega_b$ is due to the presence of the beam electron fluid. Without the beam electrons, $\omega = \omega_{*e}$, and no additional energy is required. The implication of these results regarding present day tokamak experiments is discussed in Ch. 7.

Chapter 6
NUMERICAL RESULTS

This chapter presents numerical solutions of the eigenmodes and eigenfrequency for the tearing mode. The equations governing the evolution of perturbed particle densities, $\tilde{f}_e$ and $\tilde{f}_i$, are solved numerically; and through the aid of Ampere's law and quasineutrality the eigenmodes, $\tilde{\phi}$ and $\tilde{A}_{||}$, along with the eigenfrequency, ω, are calculated. Two numerical codes are employed in this calculation of the eigenfrequency for two different limits. The first method, which is the more exact method, involves the use of an initial value code, TEDIT [21,22], previously used in the study of the finite beta drift wave [19]. Specifically, TEDIT solves for the electron response utilizing the full diffusion operator, $D\partial^2/\partial x^2$. The second method involves a shooting code, inherently simpler than the initial value code, which solves for the electron response in the Krook approximation: Both codes give qualitatively similar results [14,19] and support the above analytical expression for the dispersion relation arrived at through the variational calculation.

The initial value code, TEDIT, follows the time evolution of all the perturbed quantities. One begins with arbitrary perturbed potentials, $\tilde{\phi}$ and $\tilde{A}_{||}$, and distribution functions, $\tilde{f}_e$ and $\tilde{f}_i$. Regardless of the initial functions, if a growing (unstable) eigenmode exists, it will eventually dominate the long time solution. By definition, an eigenmode exists when all quantities $\tilde{\phi}$, $\tilde{A}_{||}$, $\tilde{f}_e$ and $\tilde{f}_i$ vary as $\exp(-i\omega t)$, where the eigenfrequency ω is constant for all x. The main virtue of this initial value approach is that it allows for the electron response to be evolved with the inclusion of a spatial diffusion operator according to Eq. (2.20) of Sec. 2.1. The ion response evolves according to the linearized Vlasov equation. The time–evolution code TEDIT uses an implicit–iterative scheme to advance the electron and ion kinetic equations in time, with $\tilde{\phi}$ and $\tilde{A}_{||}$ being calculated from the quasineutrality condition and Ampere's Law. The equations for the electron and ion responses are advanced in time until

$$\omega(x,t) = \frac{i}{\tilde{A}_{||}(x,t)} \frac{\partial \tilde{A}_{||}(x,t)}{\partial t} \tag{6.1}$$

becomes independent of both x and t, indicating that an eigenmode of frequency ω has been established. For a given set of parameters, TEDIT yields the most unstable eigenmode.

In the Krook approximation, a shooting code is employed to solve for the eigenfrequency in the case where the electron response evolves according to Eq. (2.20) with $-D\partial^2/\partial x^2$ replaced with a Krook type diffusion frequency, ω_c. As discussed above in Sec. 2.4, this is analytically shown to be valid when $x_c/x_T < 1$, and when $\tilde{A}_{||}$ is approximately a constant (an even function) near the rational surface. Under this approximation, a shooting code can be used to directly solve the coupled equations for $\tilde{\phi}$ and $\tilde{A}_{||}$, given by Eqs. (2.30) in Sec. 2.3. In this case, the resonance operators appearing in L_1, L_2 and L_x can be written in terms of the plasma dispersion function as discussed in Sec. 2.4. The above coupled equations can then be solved using standard shooting methods which is inherently simpler numerically than the initial value method used in the time–evolving code TEDIT.

In either case, the above numerical codes are used to calculate the eigenfunctions, $\tilde{\phi}$ and $\tilde{A}_{||}$, in an intermediate slab region extending approximately thirty ion gyroradii on either side of the rational surface at $x = 0$. At the edges of this slab region, $\tilde{\phi}$ and $\tilde{A}_{||}$ are required to match onto the ideal MHD solutions which obey the marginal stability equation, Eq. (3.18), along with $\tilde{E}_{||} = 0$. Specifically, the ratios $(\tilde{A}'_{||}/\tilde{A}_{||})_{MHD}$ are calculated at each edge of the slab region and their values are chosen such that, to leading order

$$\Delta'(a) \equiv \left.\frac{\tilde{A}'_{||}(a)}{\tilde{A}_{||}(0)}\right|_{MHD} - \left.\frac{\tilde{A}'_{||}(-a)}{\tilde{A}_{||}(0)}\right|_{MHD} = \Delta_0\left(1 - \frac{a}{W}\right). \tag{6.2}$$

Here, $\pm a$ indicates the edges of the intermediate slab region. Hence, the boundary conditions are specified by inputting values of Δ_0 and W. Figure 6.1 shows a typical plot of $\tilde{A}_{||}$ and $\tilde{\phi}$ calculated using the above procedure.

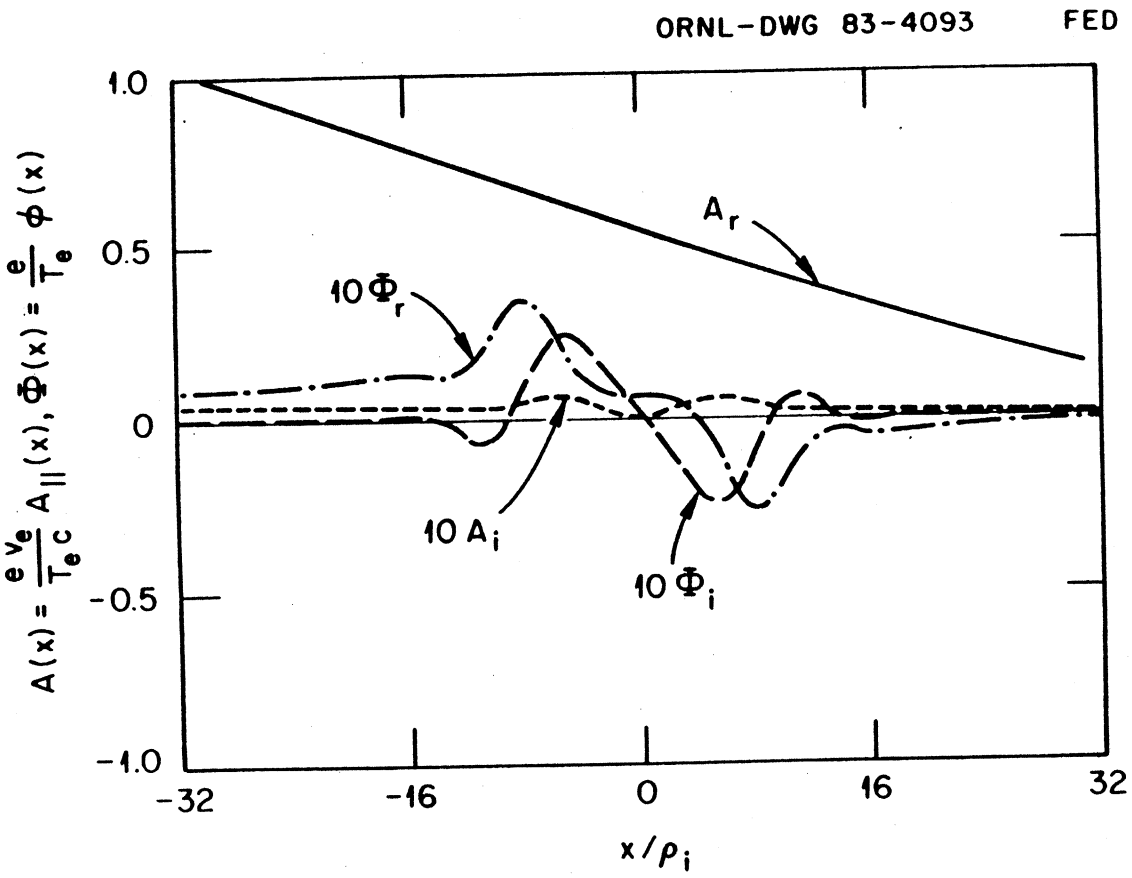

Figure 6.1

Numerical solutions of the perturbed wave potentials, $\tilde{\phi}$ and $\tilde{A}_{\|}$, as a function of x. The subscripts r and i refer to the real and imaginary parts, respectively.

The results from TEDIT and the shooting code agree extremely well, as previously observed for the case of the finite beta drift wave. Figure 6.2 depicts the growth rate as a function of the diffusion coefficient as obtained from the two codes. Since both codes are observed to give the same qualitative results, the shooting code is used to generate the numerical data discussed below due to its faster computational speed compared to that of TEDIT.

A plot of $\Delta_0 \equiv \Delta'(x = 0)$ as a function of x_c at marginal stability ($\gamma = 0$) is shown in Fig. 6.3 for various values of W. These results agree qualitatively with the analytical formula obtained from the variational calculation which specifies marginal stability to occur when Eq. (3.51) is satisfied with an equality sign. Solving the above equation for Δ_0 gives

$$\Delta_0 = 2\beta_i \left(\frac{L_s}{L_n} \frac{\tau}{\sqrt{2}} \right)^{1/2} \left(1 - \frac{x_c}{\sqrt{2}W} \right)^{-1} . \tag{6.3}$$

The plot in Fig. 6.3 of Δ_0 versus x_c shows the qualitative behavior indicated in Eq. (6.3). In particular, note that the slope of the curve increases as the parameter W is decreased. Also notice that the value of Δ_0 at which $x_c = 0$ is independent of W.

Figure 6.4 plots a similar graph of Δ_0 as a function of x_c at marginal stability for several values of β_i. Figure 6.5 shows the same curve of Δ_0 versus x_c for different values of L_s/L_n. Both figures are in qualitative agreement with Eq. (6.3); that is, an increase in either β_i or L_s/L_n leads to a simple vertical displacement of the Δ_0 versus x_c curve.

The variational calculation performed in Sec. 4.4 gave the result of Eq. (3.50). As is often the case in a variational calculation, one expects the functional dependence of the result with respect to the various parameters involved to be similar to that of the exact solution. The numerical coefficients appearing in the variational solution, however, are only approximations to those of the exact solution whose values can be made more exact by using trial functions closer to the exact

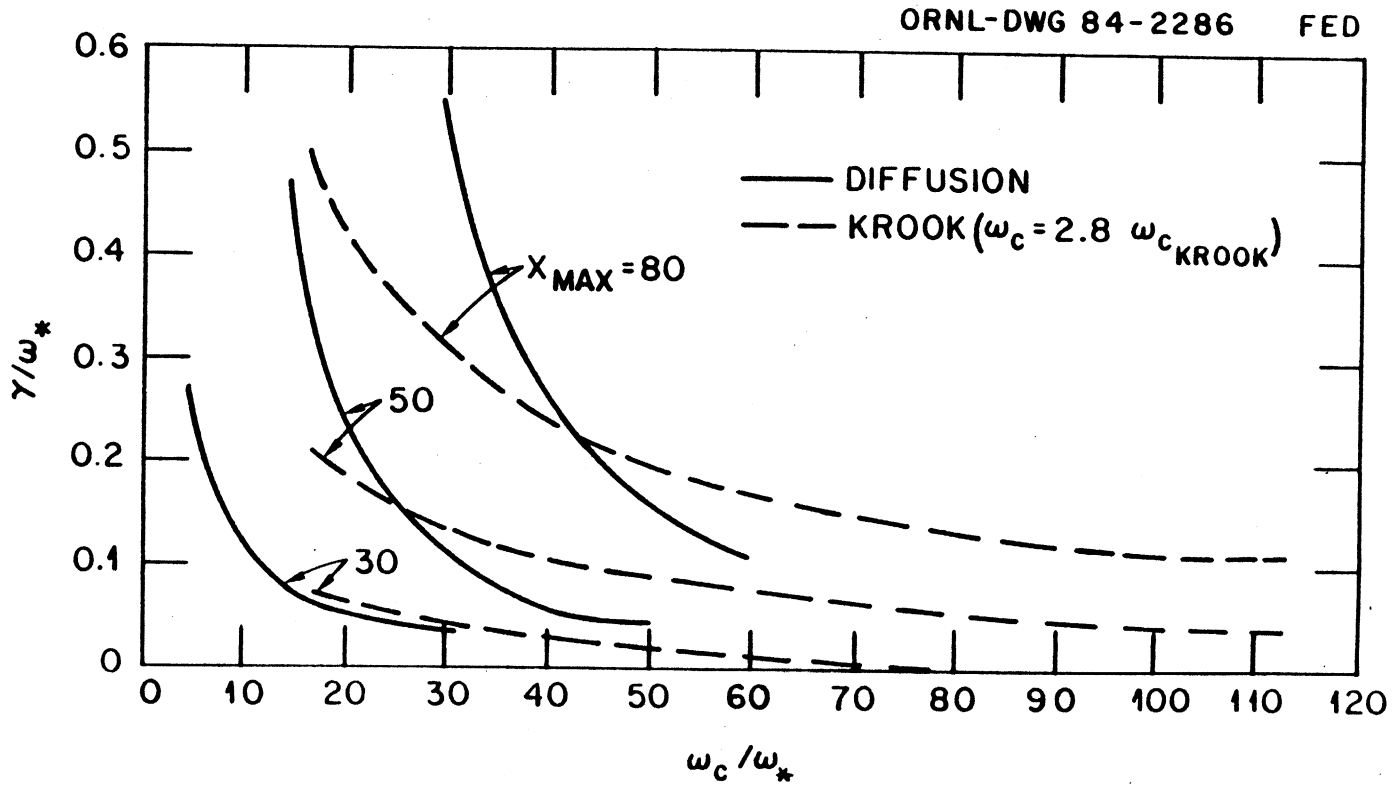

Figure 6.2

Numerical results for the growth rate comparing the initial value code, TEDIT, using the full diffusion operator, to the shooting code, which uses the Krook approximation. Here the growth rate is plotted for increasing values of the diffusion coefficient (increasing ω_c). The parameter x_{max} represents the plasma edge and is a measure of the energy drive, $\Delta'(0)$.

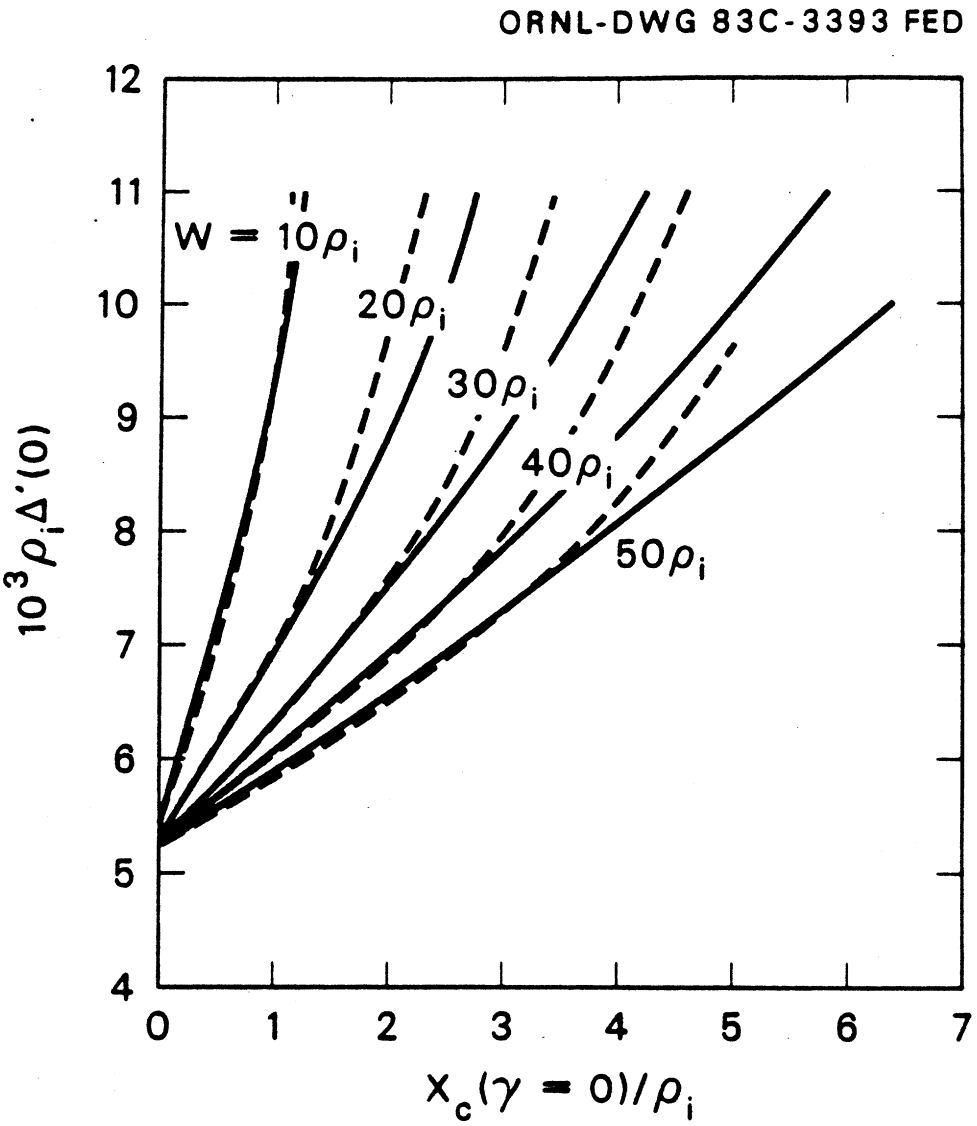

Figure 6.3

The amount of diffusion (x_c) necessary to obtain marginal stability ($\gamma = 0$) vs. given values of free energy ($\Delta'(0)$) for several values of W, where $\Delta'(W) = 0$. The dashed curves are plots of the analytical results indicated by Eq. (6.3).

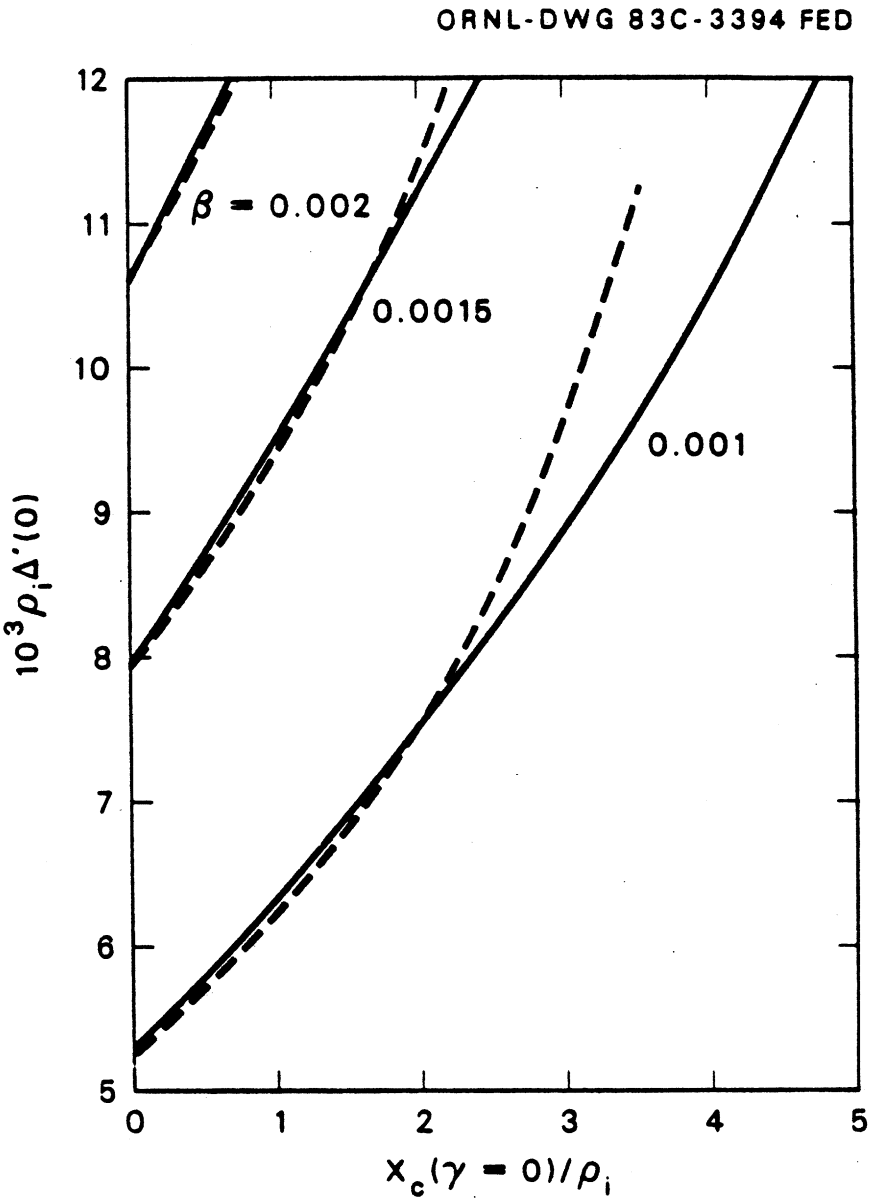

Figure 6.4

The amount of diffusion (x_c) necessary to obtain marginal stability ($\gamma = 0$) vs. given values of free energy ($\Delta'(0)$) for several values of the plasma ion beta, β_i. The dashed curves are plots of the analytical results indicated by Eq. (6.3).

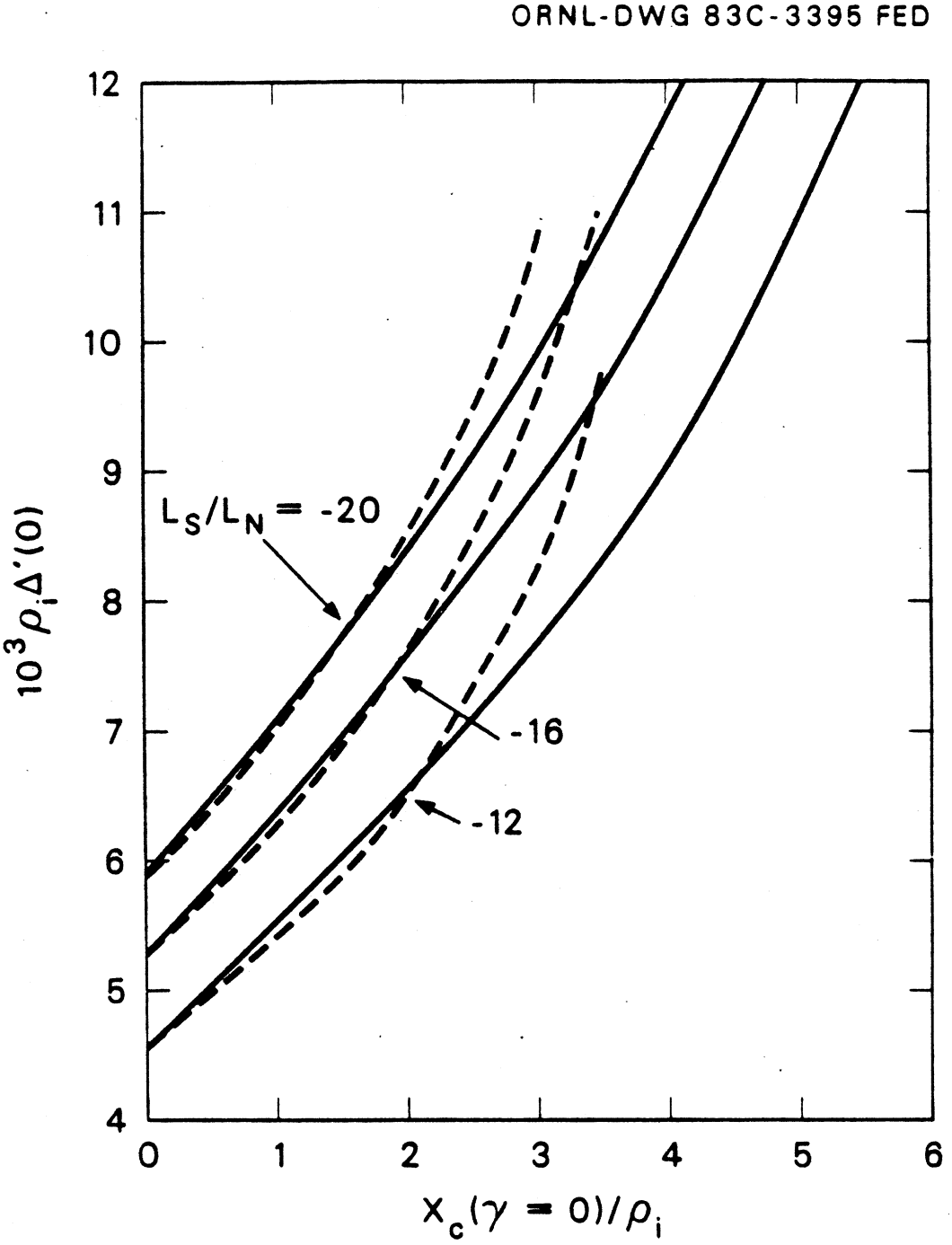

Figure 6.5

The amount of diffusion (x_c) necessary to obtain marginal stability ($\gamma = 0$) vs. given values of the free energy ($\Delta'(0)$) for several values of the magnetic shear, L_s/L_n. The dashed curves are plots of the analytical results indicated by Eq. (6.3).

eigenfunction. In order to reflect this uncertainty in the numerical coefficients of the variational solution, Eq. (3.50), two adjustment parameters, α_1 and α_2, are introduced into the above expression for the growth rate as follows:

$$\gamma = \frac{k'_{\|}\rho_i v_i^2}{\pi v_e}\left\{\frac{1}{\beta_i}\Delta'(\alpha_1 x_c) - \alpha_2\left(\frac{\tau}{\sqrt{2}}\frac{L_s}{L_n}\right)^{1/2}\right\}. \tag{6.4}$$

where $\Delta'(x) = \Delta_0(1 - x/W)$. Here, α_1 reflects the uncertainty in the variational determination of the magnitude of the value of the parameter a (the edge of slab region about the rational surface) as performed in the calculation of the magnetic terms, $\mathcal{L}_A$, appearing in Sec. 4.4. Likewise, α_2 reflects the uncertainty in the overall magnitude of the contribution of the electrostatic terms, $\mathcal{L}_\phi$, to the variational integral due to such approximations as extending the limits of integration to infinity (see Sec. 4.4). By comparing Eq. (6.4) to the numerical data displayed in Figs. 6.3–6.5, one can fit this numerical data [14] to a high degree of accuracy by choosing the values of α_1 and α_2 to be

$$\alpha_1 \simeq 4.5 \quad \alpha_2 \simeq 4/3. \tag{6.5}$$

Results indicate that α_1 is a weakly dependent function of the ratio x_c/W (α_1 decreases as x_c/W increases). This reflects the fact that the function $\Delta'(x)$ is not strictly a decreasing linear function of x, as approximated analytically ($\Delta'(x) = \Delta_0(1 - x/W)$); rather, $\Delta'(x)$ is a function whose slope increases (becomes less negative) as x increases.

Numerical results for the runaway tearing mode are not as extensive as those presented above for the regular tearing mode. The runaway tearing mode was studied using the shooting code with a monoenergetic beam on the tail of Maxwellian for the equilibrium electron distribution, as discussed in Ch. 4. Typical plots for the eigenfunctions $\tilde{\phi}$ and $\tilde{A}_{\|}$ are shown in Fig. 6.6 and 6.7, respectively.

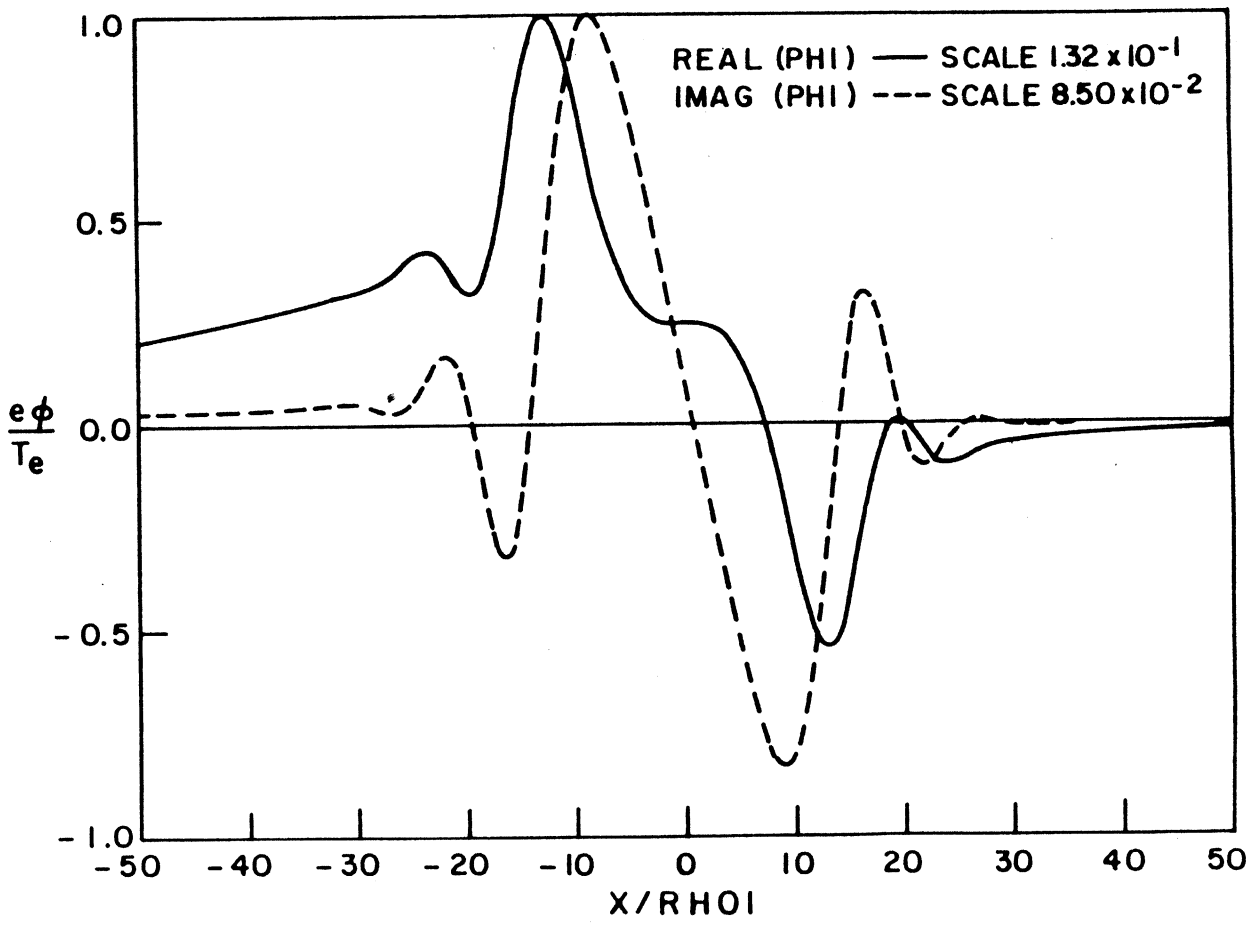

Figure 6.6

Numerical solution of the perturbed electrostatic potential, $\tilde{\phi}$, as a function of x for the runaway tearing mode.

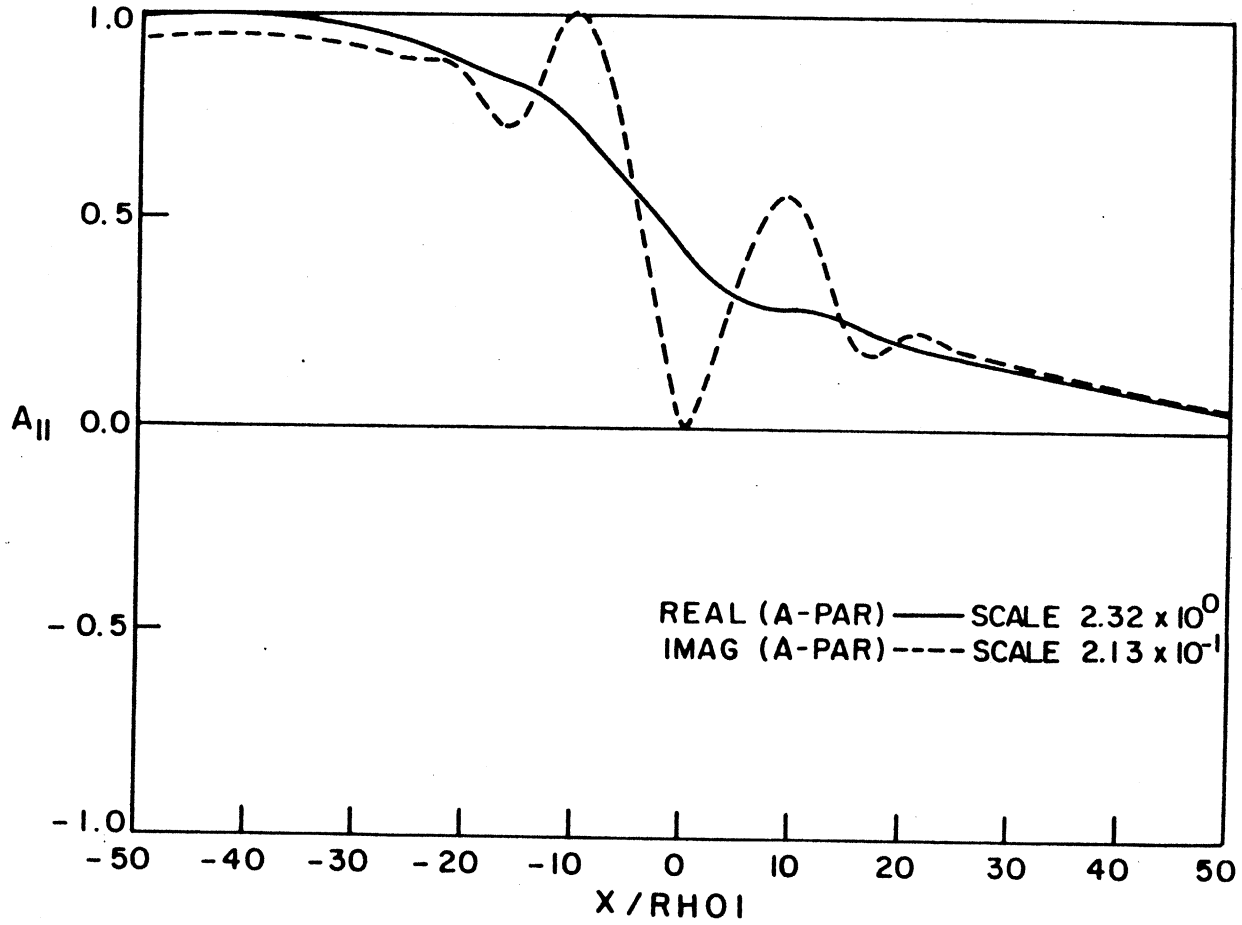

Figure 6.7

Numerical solution of the perturbed magnetic potential, $\tilde{A}_{||}$, as a function of x for the runaway tearing mode.

As discussed in Ch 4, the runaway electrons produce a frequency shift above ω_{*e} given by

$$\frac{\delta\omega_b}{\omega_{*e}} \simeq \sqrt{2}\frac{n_b v_b L_n}{n_0 v_e L_J}. \tag{6.6}$$

(For the regular tearing mode, $\delta\omega \simeq 0$.) Figure 6.8 is a plot of the frequency shift $\delta\omega$ for both the runaway and regular tearing mode as a function of v_b/v_e. The total current was held equal when comparing the regular and runaway modes at the value $J_{\|0} = -en_b v_b$. Figure 6.8 shows that $\delta\omega$ for the runaway mode is significantly greater than $\delta\omega$ for the regular mode, which agrees with the basic analytic theory. Also notice from Fig. 6.8 that the slope of $\delta\omega/\omega_{*e}$ vs. v_b/v_e is linearly increasing, which agrees with Eq. (6.6).

The growth rate for the runaway mode is given approximately by

$$\frac{\gamma}{\omega_{*e}} \simeq \frac{4}{\pi}\left(\frac{\tau m_e}{2m_i}\right)^{1/2}\frac{\tau L_s}{\beta_i L_n}\left\{\Delta'\left(\frac{\sqrt{2}x_c}{2W}\right)\right.$$
$$\left. -\pi\sqrt{\frac{2}{3}}\frac{\beta_i}{\tau}\left(\frac{\tau L_s}{\sqrt{3}L_n}\right)^{1/2} - \left(\frac{2m_i}{\tau m_e}\right)^{1/2}\beta_i\frac{\tau L_s}{2L_n}\frac{\delta\omega_b}{\omega_c}\right\} \tag{6.7}$$

as discussed in Ch. 4. The third term on the right of the above equation is the result of including the beam electrons and this term is absent for the regular tearing mode. Figure 6.9 shows a plot of the growth rate as a function of L_s/L_n for the regular and runaway tearing modes. These results indicate the runaway mode is more stable than the regular mode. Figure 6.9 also indicates that the runaway tearing mode is stabilized more rapidly for increasing L_s/L_n than is the regular tearing mode, which is in qualitative agreement with Eq. (6.7).

In summary, the numerical results for the regular tearing obtained from the shooting code agree remarkably well with the analytical expression for the growth rate, Eq. (6.4), where the parameters α_1 and α_2 are given by Eq. (6.5). Numerical

results for the runaway tearing are less extensive, but qualitatively support the analytical formulae given by Eqs. (6.6) and (6.7). In particular, the frequency shift for the runaway mode is significantly larger than that for the regular mode, and the stability of the tearing mode is shown to be enhanced by the fast electrons.

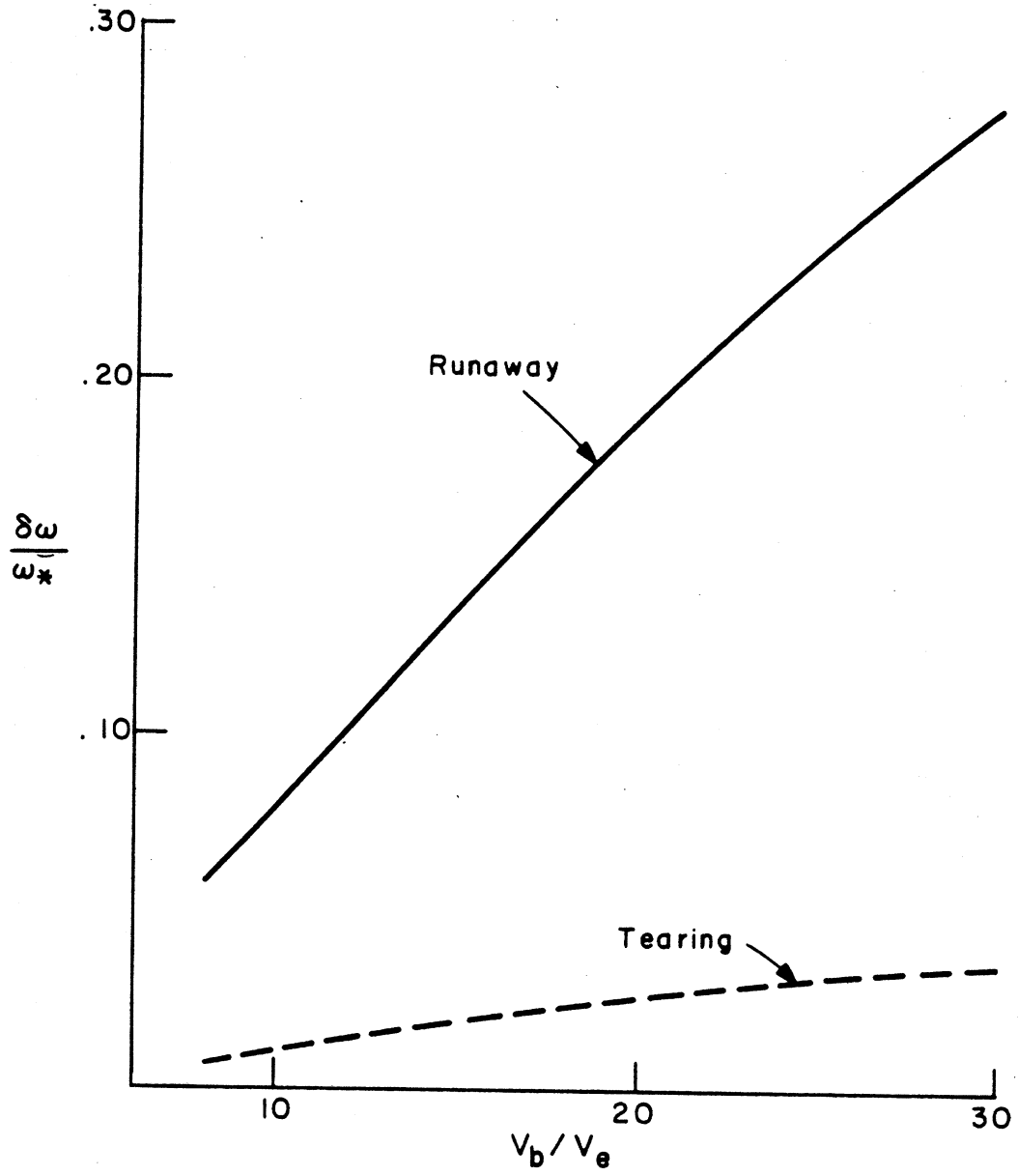

Figure 6.8

The real frequency shift, $\delta\omega = \omega - \omega_{*e}$, vs. the beam electron speed, v_b, for the regular and runaway tearing modes. The equilibrium current is determined by $J_{\|0} = -en_b v_b$, where the beam density n_b is held constant.

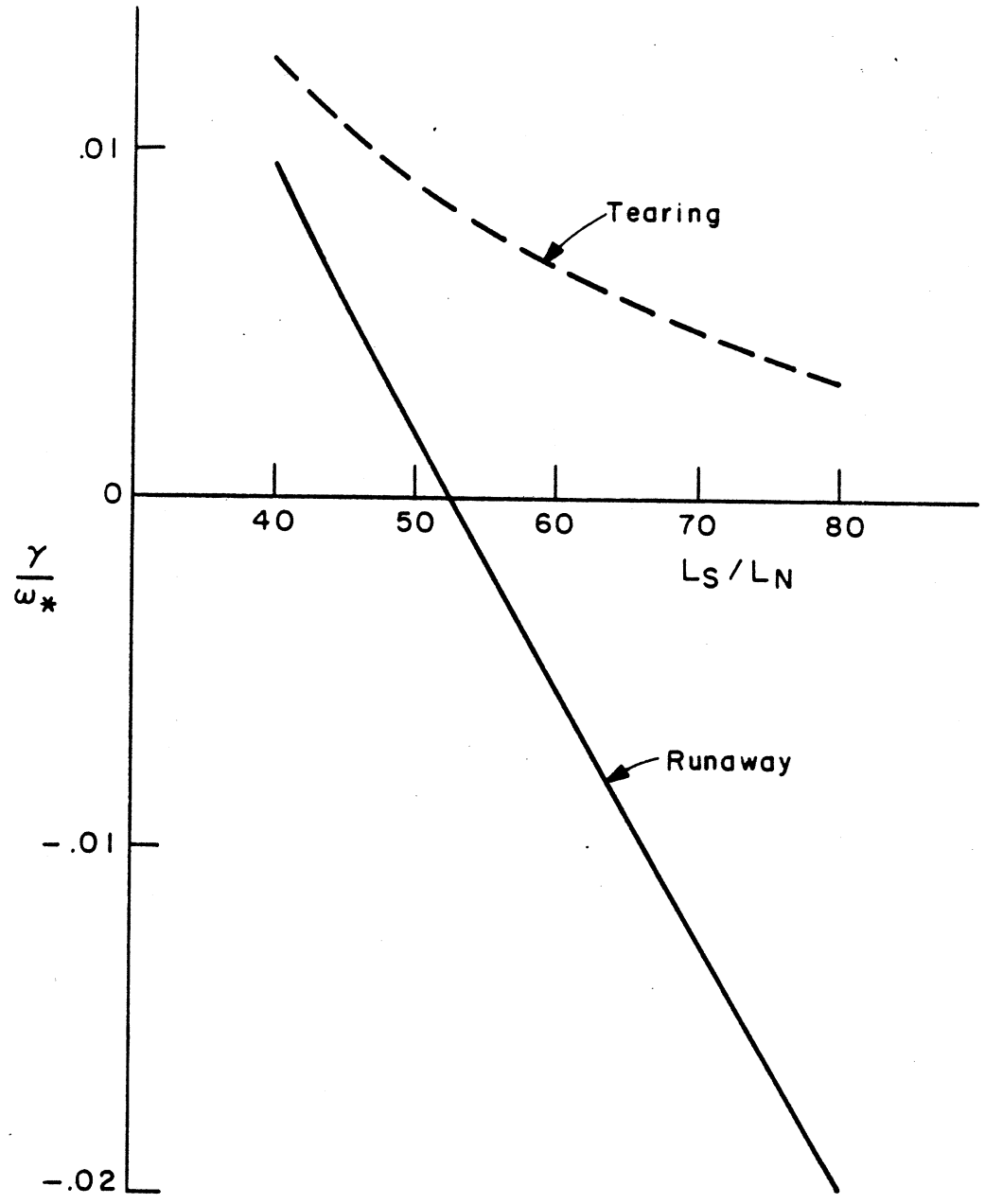

Figure 6.9

The growth rate γ as a function of magnetic shear L_s/L_n for the regular and runaway tearing modes as calculated by the shooting code.

Chapter 7
SUMMARY

A set of coupled, self-adjoint equations were derived for the potentials $\tilde{\phi}$ and $\tilde{A}_{||}$ describing low-beta, low-frequency fluctuations in a tokamak-type plasma with a sheared, slab geometry. This set of equations is globally valid over the entire plasma, includes the effects of turbulent electron diffusion, and allows the use of various current-carrying equilibrium electron distributions. This system describes unstable finite-β drift waves when analyzed for high m modes [16,18,19] and describes tearing modes when analyzed for low m modes [13,14]. By recasting this system into cylindrical geometry, the stability of the $m = 1$ mode can be studied. Similarly, by including toroidal curvature drift effects, the stability of ballooning modes can be studied. The effects of turbulent diffusion and runaway electrons on the $m = 1$ mode and ballooning modes has not yet been determined. The present study has been concerned solely with the $m = 2$ (or higher) tearing mode.

The effects of turbulent electron diffusion were incorporated into the model by applying the NSA [17] to the collisionless DKE for electrons. The tearing mode was implicitly assumed to exist in a plasma containing a background of drift wave turbulence. The presence of these turbulent drift waves produced overlapping phase space islands which, in turn, led to the production of stochastic electron orbits. The effect of these stochastic orbits was to produce very fine scale spatial structure in the perturbed electron distribution function. The NSA procedure amounted to performing a "coarse-grain" spatial averaging over this fine microscale structure resulting from stochastic orbits. Essentially, the NSA replaced the nonlinear terms in the orbit operator of the DKE with a spatial diffusion operator, $-D\partial^2/\partial x^2$. This reduced the DKE to a linear equation which can then be solved for the perturbed electron distribution. The qualitative behavior of the NSA was preserved by replacing the diffusion operator in the DKE with an effective diffusive frequency, $-D\partial^2/\partial x^2 \rightarrow \omega_c = [(k'_{||}v_e)^2 D/3]^{1/3}$. This was referred to as the Krook approximation. This approximation greatly simplified the mathematical treatment.

The ion response was given according to the linearized Vlasov equation. Quasineutrality and the perturbed parallel Ampere's law were used to give a coupled set of equations for $\tilde{\phi}$ and $\tilde{A}_{\|}$. Since these equations were self-adjoint, a variational integral was formed. This variational integral was used in the calculation of the dispersion relation of the tearing mode.

Moments of the NSA version of the DKE were performed in order to derive a fluid model which was equivalent to the NSA kinetic theory. This gave a fluid model in which radial turbulent diffusion of the density appeared in the continuity equation (zeroth moment) and radial turbulent diffusion of the momentum appeared in the momentum equation (first moment). Essentially, under the Krook approximation, the fluid theory (and, hence, the NSA kinetic theory) diffuses both the perturbed density and the perturbed momentum at equal rates given by the effective diffusion frequency, ω_c. This fluid theory gave the same results as the kinetic theory when the dispersion relations for the tearing mode were calculated.

Both numerical and analytical results indicate the tearing mode to have a real frequency equal to the electron diamagnetic frequency and a growth rate given by

$$\gamma \simeq \frac{k'_{\|}\rho_i v_i^2}{\pi v_e}\left\{\frac{1}{\beta_i}\Delta'(\alpha_1 x_c) - \alpha_2\left(\frac{L_s}{L_n}\right)^{1/2}\right\}. \tag{7.1}$$

where α_1 and α_2 are numerical constants on the order of unity. Since the real frequency is given by $\omega_r = \omega_{*e}$ indicates that the tearing mode is, in fact, an electron drift wave driven unstable by the equilibrium current gradient. Here, the first term on the right of Eq. (7.1) is the contribution from the electron magnetic terms and is similar to the basic collisionless tearing mode [8] result ($\gamma \sim \Delta'(0)$) modified to include diffusive electron effects. The second term on the right is the contribution from ion electrostatic terms. Physically, the first term represents the free magnetic energy in the outer region ($|x| > x_c$) available to drive the tearing mode. The second term represents the energy required to maintain the ion motion. Since this mode now has a finite real frequency, energy is needed to sustain the ion

motion, whereas in resistive MHD, the mode is purely growing. At low densities the first term dominates, indicating that the tearing mode can be stabilized for sufficiently large values of the electron diffusion coefficient. At high densities, the second term becomes important and consequently the tearing mode can be stabilized for sufficiently high β_i.

Numerical calculations of $\Delta'(x)$ for Alcator C profiles indicate that $\Delta'(x)$ is a monotonically decreasing function of x with $\Delta'(0) > 0$. Hence, for low densities, stability is obtained for $\alpha_1 x_c > W$, where $\Delta'(W) = 0$ and α_1 is a numerical constant. This stability criterion can be written as $D_e > 3k'_{||} v_e (W/\alpha_1)^3$ with $k'_{||} = mq'/Rq^2$, where the functional dependence of W on the profile quantities must be determined numerically. This equation indicates that increased turbulent electron diffusion stabilizes the tearing mode. Consequently, if $D_e \sim 1/n$, then there exists some critical density below which the tearing mode is stabilized. Theoretically predicted values of the critical density are in approximate agreement with experimental values; however, the experimental scaling [7] of $n_c \sim B^2$ has not been explicitly derived unless $D_e \sim B^2$. (Note that for the parameters $a \simeq 20$ cm, $L_s/L_n \simeq 20$, $W/a \simeq .05$ and $T_e = 1$ Kev, then stabilization occurs for $D_e \geq 10^4 \text{cm}^2/\text{sec}$.)

A related analysis reported by Meiss et al. [72] also treated the problem of the effect of electron diffusion on the tearing mode. They arrived at a very different conclusion, however; namely, that diffusion has virtually no effect on the tearing mode nor did they find the additional stabilizing term due to ion inertial effects. The results of the present work for the "magnetic" tearing mode (neglecting the effects of a finite electrostatic potential) differ only by the inclusion of the additional physical effect of turbulent smearing of the perturbed current which thus reduces the available energy to the value $\Delta'(\alpha_1 x_c)$. The analysis by Meiss et al., by asymptotically matching an inner solution to an ideal MHD solution at large x, intrinsically contained the full MHD energy, $\Delta'(0)$, and could not consider this effect. Besides the difference that Meiss et al. ignored the the stabilizing effect of ion inertia, their results agree with the above results of the "magnetic" tearing mode except for the phenomena of turbulent smearing of the perturbed current.

The effects of runaway electrons were introduced into this model by replacing the drifted Maxwellian electron equilibrium distribution with a runaway-type equilibrium distribution. For simplicity, the runaway distribution was modelled by a monoenergetic beam of fast electrons on the tail of a Maxwellian. Analytical results indicate that the presence of runaway electrons shift the real frequency of the mode from ω_{*e} to $\omega_{*e} + \delta\omega_b$, where

$$\frac{\delta\omega_b}{\omega_{*e}} \simeq \sqrt{2}\frac{n_b}{n_0}\frac{v_b}{v_e}\frac{L_n}{L_J}. \tag{7.2}$$

This frequency shift can be viewed as a Doppler shift of the dissipative response of the electrons resulting from the motion of the beam electrons relative to the laboratory. In turn, this frequency shift led to the appearance of an additional stabilizing term in the expression of the growth rate. The growth rate is approximately modified to read

$$\gamma \simeq \frac{k'_{\|}\rho_i v_i^2}{\pi v_e}\left\{\frac{1}{\beta_i}\Delta'(\alpha_1 x_c) - \alpha_2\left(\frac{L_s}{\tau L_n}\right)^{1/2} - \alpha_3\left(\frac{\tau m_i}{m_e}\right)^{1/2}\frac{L_s}{L_n}\frac{\delta\omega_b}{\omega_c}\right\}, \tag{7.3}$$

where α_3 is a constant of order unity. This stabilizing term resulting from the beam electrons represents the energy required to maintain the particle oscillation at the shifted frequency $\omega_{*e} + \delta\omega_b$. The above expression is qualitatively supported by the numerical calculations.

In a tokamak plasma, runaway electrons (or a significant population of fast electrons) is only observed at low densities. As the density increases, the fast electron population relaxes back to the bulk population due to the increase in collisionality. Hence, the additional stability provided by the fast electrons is a low density phenomenon, and this effect should diminish as the bulk plasma density is raised.

The growth rate as expressed by Eqs. (7.1) and (7.3) also indicates that at high densities, the tearing mode can also be stabilized due to the effects of ion inertia.

Assuming the effect of electron diffusion and runaway electrons can be ignored, this stabilization takes the form of specifying a critical ion β, above which the tearing mode is stabilized. Setting $x_c = 0$, this critical β is given by

$$\beta_i > \beta_{i_c} \equiv \frac{3}{4}\Delta_0\left(\sqrt{2}\tau\frac{L_n}{L_s}\right)^{1/2}, \quad \Delta_0 \equiv \Delta'(0). \tag{7.4}$$

Here, Δ_0 is normalized in units of the ion gyroradius. (Note that for the parameters $\Delta_0 a \simeq 20$, $\rho_i/a \simeq .002$, $L_s/L_n \simeq 20$ and $\tau \simeq 1$, then $\beta_c \simeq 10^{-2}$.)

The result of stabilization at ion betas above some critical value, $\beta_c \sim \Delta_0(L_n/L_s)^{1/2}$, was calculated previously by Basu and Coppi [11] through a kinetic treatment utilizing asymptotic analysis. The work of Basu and Coppi [11] and of Coppi et al. [10] also modified this result to include the effects of finite temperature gradients in which they found stabilization to occur when the ion beta exceeded a critical value of $\beta_c \simeq \Delta_0(L_n/L_s)^2 I_e$. Here I_e is a function of the electron temperature gradient whose magnitude is on the order of unity. This critical ion beta is much lower than that occurring in the absence of temperature gradients, indicating that the effects of finite temperature gradients on the $m = 2$ mode are strongly stabilizing. A recent analysis by Drake et al. [12] based on the Braginskii fluid equations [36] including the effects of finite temperature gradients also gave a result of the form $\beta_c \simeq \Delta_0(L_n/L_s)^2 I_e$ for the onset of stabilization. (Note that for the parameters $\Delta_0 a \simeq 20$, $\rho_i/a \simeq .002$, $L_s/L_n \simeq 20$, then $\beta_c \simeq 10^{-4}$.)

In comparing the above expression for the growth rate for the tearing mode, Eq. (7.3), to present tokamak experiments, it is important to note the following: The energy drive for the tearing mode, and hence its stability, is determined largely by the parameters Δ'_0 and W. The actual values of Δ'_0 and W are fairly sensitive functions of the equilibrium current profile, the magnetic shear and the position of the rational surface. In general, Δ'_0 increases for more "rounded" current profiles and decreases as the rational surface moves toward the edge of the plasma [30–32]. Generally, $\Delta' a < 100$ and more typically $\Delta' a \simeq 20$–30 [12]. The island saturation

width W also increases for more "rounded" profiles and the value of W is maximum in the middle region between $r = 0$ and $r = a$ [33,34]. Generally, $W/a < .5$ and more typically $W/a \sim .1$ [34]. The stabilization criterion for turbulent electron diffusion occurs for $a_1 x_c > W$ and is independent of Δ'_0. This criterion can be written as

$$D > 3m \left(\frac{W}{\alpha_1 a}\right)^3 \frac{L_n}{L_s} \frac{a^2}{L_n^2} a v_e. \tag{7.5}$$

Using the parameters $\rho_i/a \sim .002$, $a \sim 20$ cm, $\alpha_1 \approx 4$, $W/a \sim .05$, $L_s/L_n \sim 20$ and $T \simeq 1$ keV indicates stabilization for $D > 10^4$ cm^2/sec. This in turn gives values for the density threshold on order of those observed in Alcator C. Notice that the value of D necessary for stabilization scales as W^3; hence, diffusion can only suppress "small" islands. Ion inertia provides stabilization for values of β given by

$$\beta > (\Delta'_0 a) \frac{\rho_i}{a} \left(\frac{L_n}{L_s}\right)^{1/2} \Big|_{\nabla T = 0} \tag{7.6a}$$

in the absence of temperature gradients [11]; and for values of β given by

$$\beta > (\Delta'_0 a) \frac{\rho_i}{a} \left(\frac{L_n}{L_s}\right)^{2} \Big|_{\nabla T \neq 0} \tag{7.6b}$$

including finite temperature gradient effects [12]. For a value of shear given by $L_s/L_n \simeq 20$, notice that the β threshold for stability is smaller when finite temperature gradients are included by the factor 10^{-2}. Also notice that the β threshold is linearly proportional to Δ'_0. Since tearing modes are observed in Alcator C when $\beta \simeq 5 \times 10^{-4}$, $\rho_i/a \simeq .002$ and $L_s/L_n \sim 20$ indicates that the equilibrium profiles must be strongly MHD unstable with $\Delta'_0 a \sim 100$. To examine the effect of runaway electrons, notice that stability is obtained, according to Eq. (4.101), for

$$\Delta'(\alpha_1 x_c) < \alpha_2 \hat{\beta} \left[1 + \frac{\alpha_3}{\alpha_2} \left(\frac{m_i}{m_e}\right)^{1/2} \left(\frac{L_s}{L_n}\right)^{1/2} \frac{\omega_{*e}}{\omega_c} \frac{\delta\omega_b}{\omega_{*e}}\right] \tag{7.7}$$

where

$$\hat{\beta} = \begin{cases} \beta \left(\frac{L_s}{L_n} \right)^{1/2} & \text{for } \nabla T = 0; \\ \beta \left(\frac{L_s}{L_n} \right)^{2} & \text{for } \nabla T \neq 0. \end{cases}$$

Recall from Ch. 4 that the effect of runaway electrons is to provide a stabilizing correction to the particle inertia term, as indicated above. For $\delta\omega_b/\omega_{*e} \simeq 10^{-2}$, $\omega_{*e}/\omega_c \simeq 10^{-1}$ and $L_s/L_n \simeq 20$, then the runaway term is a correction of order $\omega_{*e}/\omega_c \sim 10^{-1}$ to the particle inertia stabilization. Hence, the runaway correction is only significant in a low density discharge when the leading order contribution from $\Delta'(\alpha_1 x_c)$ nearly cancels the leading order ion inertia term; that is, when $\Delta'(\alpha_1 x_c) - \alpha_2 \hat{\beta} = \delta\Delta' > 0$, where $\delta\Delta'/\Delta' \sim \omega_{*e}/\omega_c$. In this case, runaway electrons will stabilize the mode provided $\delta\Delta' < (m_i L_s/m_e L_n)^{1/2} \alpha_3 \hat{\beta} \delta\omega_b/\omega_c$.

In summary, the tearing mode is stabilized at low densities for sufficiently large values of the turbulent electron diffusion coefficient, D_e, and at high densities stabilization is obtained for sufficiently large values of β_i. For low β_i, in which one can approximate $\gamma \sim \Delta'(\alpha_1 x_c)$, then stability is obtained when $x_c > W$, where $\Delta'(W) = 0$. Provided $D_e \sim 1/n$, this implies that a density threshold must be surpassed before the $m = 2$ tearing mode is observed. Physically, turbulent electron diffusion prevents a perturbed current from forming within a correlation distance, x_c, of the rational surface. Hence, turbulent diffusion cuts into the available magnetic driving energy, Δ'. At high plasma β, the effects of ion inertia become important. At high densities in which the effects of electron diffusion become negligible, then this ion inertia effect implies that the tearing mode is again stabilized for β_i above some critical value. In addition, the existence of a fast electron population at low densities further enhances stability. The fast electrons provide an additional frequency shift $\delta\omega_b$ and more energy is required to maintain the particle oscillation. These results indicate that it may be possible for a tokamak experiment to operate in a parameter regime such that the tearing mode is stabilized at all densities. This results from the combined effects of turbulent diffusive stabilization and runaway electron stabilization at low densities, and ion inertial stabilization at high densities. Making this work in practice would be an important step toward the elimination of major disruptions in tokamaks.

Chapter 8

SUGGESTIONS FOR FUTURE STUDY

8.1 Tokamak Plasma

This study has examined the stability of the $m = 2$ tearing mode in the slab geometry equivalent of a tokamak plasma. The results for the effect of turbulent electron diffusion on the tearing mode are quite complete. A relatively simple expression for the growth rate was determined from the leading order terms from the variational calculation of the dispersion relation. This expression for the growth rate is easily interpreted in physical terms and it has been verified in detail by numerical calculation.

The results for the effect of runaway electrons on the tearing mode are not as clear as those examining the turbulent diffusive stabilization. From an analytical point of view, in order to determined an expression for the growth rate including the effects of the fast electrons, it was necessary to calculate higher order corrective terms to the basic turbulent diffusive dispersion relation. That is, the effects of the fast electrons are seen as order ω_{*e}/ω_c corrections to the dispersion relationship. Since the growth rate was determined via a variational calculation, there is a greater amount of uncertainty in the runaway results than there is for the basic turbulent diffusive growth rate. Hence, it is important to verify the results for the runaway growth rate expression numerically. This has not yet been done in detail. Thus, the first step in a future study of the runaway tearing mode is to continue solving the coupled equations numerically using the shooting code in order to verify the parameter scaling appearing in the analytical expression. Once the analytical expression has been suitably verified, the effect of runaway electrons on the tearing mode could be further improved by using a more realistic runaway distribution (and including relativistic effects) for the equilibrium than the simple monoenergetic electron beam used in this study.

The $m = 2$ tearing mode is only one of many low-frequency, low-beta fluctuations that the coupled equations of Ch. 2 for $\tilde{\phi}$ and $\tilde{A}_{||}$, Eqs. (2.30), are capable of describing. Originally, these coupled equations were used to study finite-beta drifted waves in the absence of an equilibrium current [16,18,19]. By recasting these equations in cylindrical geometry, it would be quite easy to study the effects of turbulent diffusion and runaway electrons on the $m = 1$ internal kink mode. Furthermore, if the toroidal curvature guiding center drift was included in the model, the effect of turbulent diffusion and runaway electrons on the stability of the ballooning mode could be determined. The ideal starting point for a first calculation of these effects on the $m = 1$ mode and the ballooning mode would be the fluid model present in Ch. 5. This would provide the simplest and fastest way of determining an approximate dispersion relation for these instabilities, including the effects of turbulent diffusion and runaway electrons.

8.2. Magnetotail Plasma

The Earth's magnetotail develops due to the interaction of the magnetic dipole field of the Earth with a southward pointing interplanetary magnetic field (IMF) carried by the solar wind [73]. Since the IMF is southward, an X–point (a null point) forms in the magnetic field on the dayside of the Earth. Due to the momentum of the solar wind, the magnetic lines, which have torn to form an X–point, are carried with the solar wind to the nightside of the Earth where they for an elongated magnetotail (see Fig. 8.1). As more and more of the dayside magnetic flux is torn and wrapped around to the nightside of the Earth, the magnetotail becomes more elongated and narrower. This process continues until the magnetotail becomes too "stretched", thus developing a topology which becomes unstable and subject to spontaneous reconnection [1,2,26]. Such an elongated and unstable magnetotail is somewhat similar to an one–dimensional current sheet, the latter being always unstable to the tearing mode [74]. Figure 8.2 depicts the reconnection process in a current sheet and in a magnetotail geometry.

The question as to how elongated and how stretched the magnetotail must become before there exists sufficient free magnetic energy to drive a tearing process has been studied in detail by Schindler [26]. Consider a coordinate frame such that the magnetotail is elongated in the x–direction and whose current flows in the y–direction as shown in Fig. 8.3. As is discussed above in the case of magnetic tearing in a tokamak, the bulk of the plasma in the magnetotail can adequately be described by ideal MHD — it is only in a small region about the neutral sheet (where $B_z = 0$) where the non–ideal effects of dissipation become important. As is mentioned previously, the available free energy, which is necessary to drive the tearing instability, is determined by the global magnetic topology in the regions away from the neutral sheet. Hence, a particular magnetic structure can be analyzed using ideal MHD to determine if there exists free energy which is necessary to drive a spontaneous reconnection process. As is pointed out by Schindler [26], the existence of global free magnetic energy is not enough to assure that a reconnection process will occur–there must also exist a sufficiently strong dissipation mechanism within a finite region about the neutral sheet. The existence of and the exact nature of

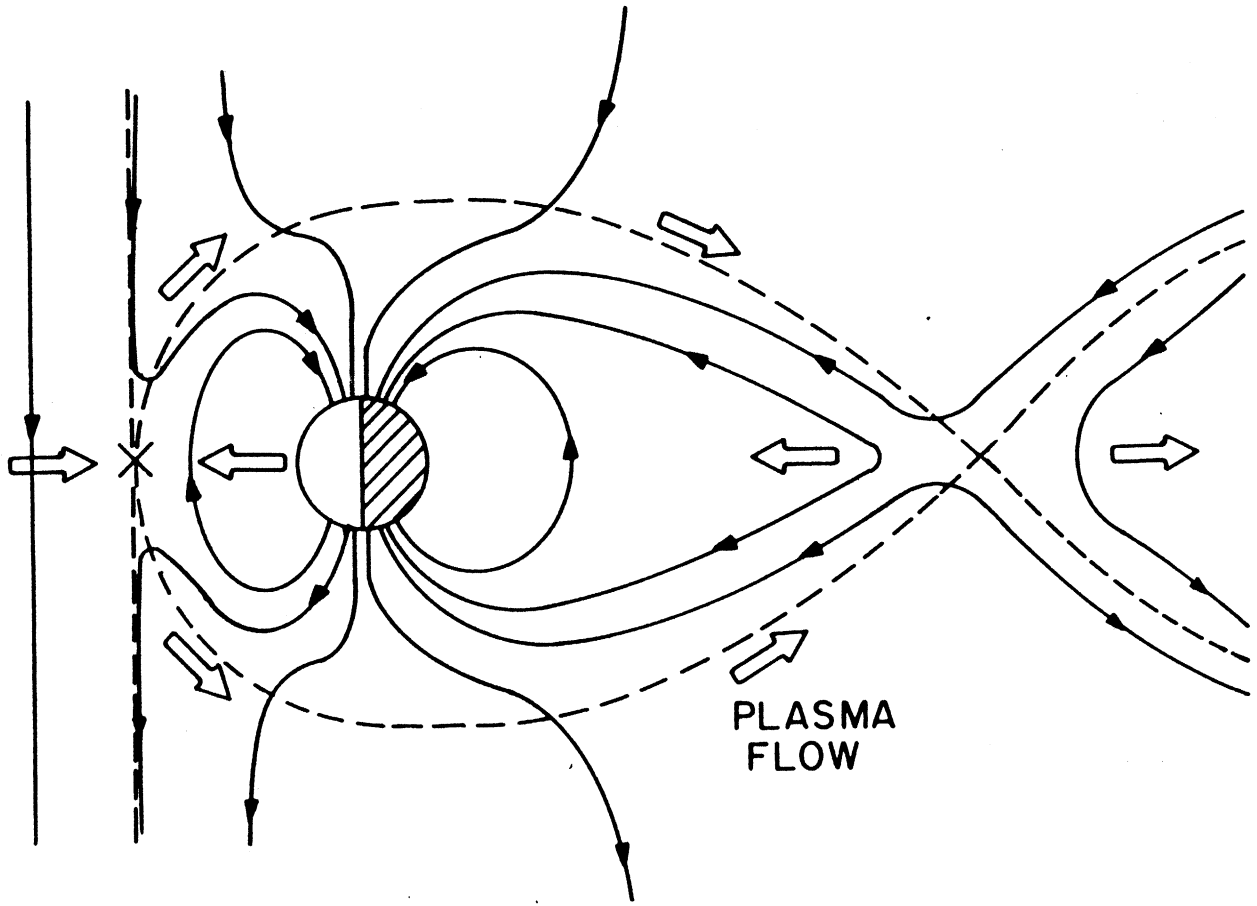

Figure 8.1

Schematic of the development of the Earth's magnetotail for a southward interplanetary magnetic field.

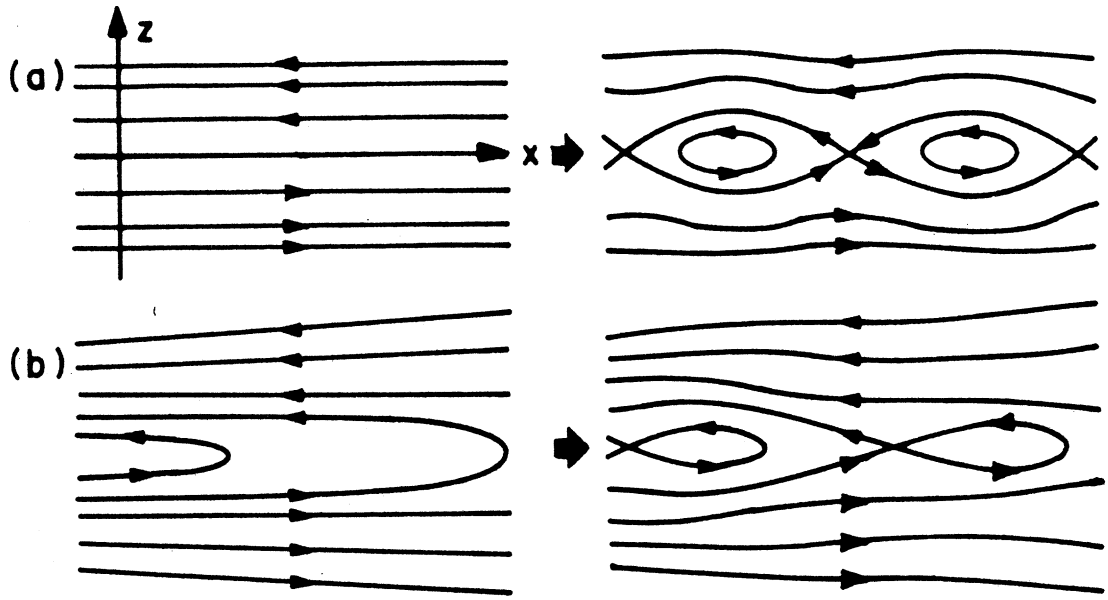

Figure 8.2

Basic forms of spontaneous reconnection: (a) the tearing instability in a plasma current sheet, (b) generalized tearing in a magnetrotail configuration.

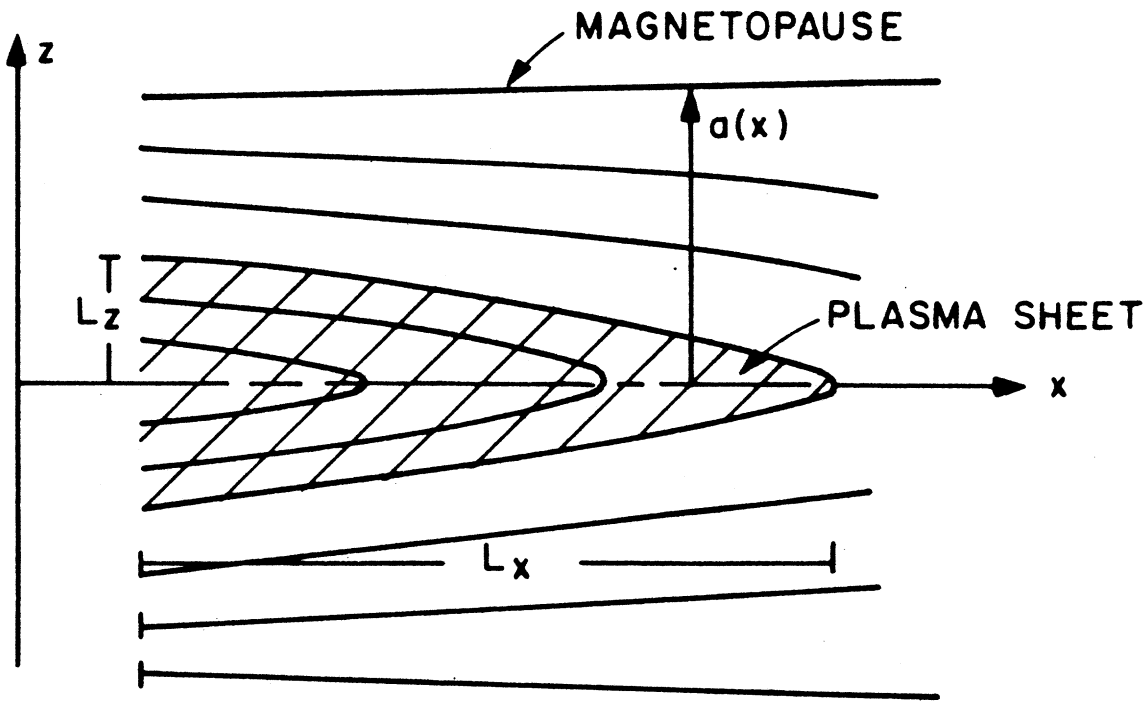

Figure 8.3

Schematic illustration of the magnetotail. The x-coordinate is positive in the anti-solar direction.

this dissipation mechanism, even in magnetic structures which provide global free magnetic energy, is currently a question of debate [2].

Most studies concerned with the details of the dissipation within the region about the neutral sheet assume that the global equilibrium has already developed into a state which provides free magnetic energy [1,62,75–77]. As done in previous chapters, it is possible to define a quadratic form from Ampere's law to represent the energy drive for the reconnection mechanism [8,24,25,62]:

$$\delta W = \int d^3x \left[|\nabla \tilde{A}|^2 - \frac{4\pi}{c} \tilde{J} \tilde{A} \right], \tag{8.1}$$

where $\tilde{J}$ is the ideal MHD perturbed current. Here, $\delta W < 0$ indicates that there is sufficient free energy in the outer region, which is necessary for the existence of the tearing instability.

A common model often used for the equilibrium which will provide free energy is given by [1,9]

$$B_{0x} = B_0 \tanh(z/\lambda). \tag{8.2}$$

In ideal MHD, the equilibrium current is given by $J_0 = J_0(A_0)$, where $\mathbf{B}_0 = \nabla \times A_0 \mathbf{e}_y$. The perturbed current given by ideal MHD is simply $\tilde{J} = (dJ_0/dA_0)\tilde{A}$, which for the above equilibrium model is then

$$\frac{4\pi}{c} \tilde{J} = \frac{4\pi}{c} \frac{dJ_0}{dA_0} \tilde{A} = \frac{2}{\lambda^2} \mathrm{sech}^2(z/\lambda) \tilde{A}, \tag{8.3}$$

where the form for the fluctuation potential is assumed to be

$$\tilde{\mathbf{A}} = \tilde{A}(z) \exp^{\gamma t} \cos(kx) \mathbf{e}_y. \tag{8.4}$$

The linearized form of Ampere's law for $\tilde{A}$ can then be solved [20,77] to give

$$\tilde{A}(z) = \frac{B_1}{k} \exp(-k|z|) \left[1 + \frac{\tanh(|z|/\lambda)}{k\lambda} \right]. \tag{8.5}$$

Using the above expression, the quadratic form, given by Eq. (8.3), can be evaluated. One finds that there exists free energy, $\delta W < 0$, provided $k^2\lambda^2 < 1$. Hence, the equilibrium given by Eq. (8.2), when analyzed globally using ideal MHD, will provide free energy to perturbations with sufficiently long wavelengths in the x–directions along the tail.

Coppi et al. [1] considered such an equilibrium along with a dissipation mechanism in the region about the neutral sheet. They argued that in a thin region about the neutral sheet, of thickness $d = (r_L\lambda)^{1/2}$, where r_L is the mean particle Larmor radius, the plasma particles behave as if they were unmagnetized [1,78]. The particles in this thin layer then provide collisionless dissipation due to the wave–particle resonance interaction (Landau damping). Since $\delta W < 0$ for $k^2\lambda^2 < 1$, this mode is a negative energy mode. Hence, this mode will become unstable in the presence of positive dissipation provided by Landau damping of particles in the unmagnetized region near the neutral sheet. Characteristic growth rates [1,76] for this instability are given by

$$\gamma \sim k v_{th} (r_L/\lambda)^{3/2}, \tag{8.6}$$

where v_{th} is the particle thermal speed. The mode resulting from electron (ion) dissipation is referred to as the electron (ion) tearing mode.

This situation changes, however, if the component of the equilibrium magnetic field normal to the current sheet, B_n, is considered [62,76]. In this case, the electrons in the thin region about the neutral sheet no longer behave as unmagnetized particles; rather, they gyrate about this normal field. The primary response of the electrons is then an adiabatic response, and the dissipative wave–particle interaction is suppressed. The adiabatic response of the electrons adds a large stabilizing

term to the quadratic form, δW. Coroniti [62] approximated δW, including the effects of $B_n \neq 0$, to be

$$\delta W = C \int dA_0 |\tilde{A}|^2 \left[k^2\lambda^2 - 1 + \frac{\pi k^2 \lambda^2 B_0}{B_n} \right]. \tag{8.7}$$

The first two terms in the above expression represent the global ideal MHD energy, as in Eq. (8.1). The last term represents the energy required to sustain the adiabatic motion of the electrons and is strongly stabilizing.

Galeev and Zelenyi [76] have studied in detail the effect of B_n on the electron response near the neutral sheet. When non-ideal MHD effects are included, the linearized form of Ampere's law for the fluctuations can be written as

$$\left[\frac{d^2}{dz^2} + \left(k^2 + V_0(z) + V_1(z, \omega, k) \right) \right] \tilde{A}(z) = 0. \tag{8.8}.$$

Here, $V_0 = (4\pi/c) dJ_0/dA_0 = -(2/\lambda^2)\mathrm{sech}^2(z/\lambda)$ and $V_1 \tilde{A} = (4\pi/c)\tilde{J}_1$, where $\tilde{J}_1$ represents the contribution to the perturbed current from particles in the region near the current sheet.

Equation (8.8) is a Schrodinger equation and describes the behavior of the wave function, $\tilde{A}$, in the presence of the potentials V_0 and V_1, with the energy eigenvalue $E = -k^2$. Here, V_0 represents a potential well of width λ. When $V_1 = 0$, wave functions can exist within this well and, hence, instability is possible. On the other hand, the potential resulting from the adiabatic electrons near the neutral sheet, V_1, takes the form of a potential peak or barrier centered about the neutral sheet. The width [76] of this barrier is given by

$$\Delta_j = \delta_j \lambda \tag{8.9}$$

where $\delta_j = (r_{Lj}/\lambda)^{1/2}$ or B_n/B_0, whichever is larger. (The subscript j refers either to ions or electrons).

Provided that the potential barrier, V_1, is large enough, it is possible to exclude the wave function, $\tilde{A}$, from existing within the potential well. In this case, the mode is stabilized. Galeev and Zelenyi [76] conclude that the electron tearing mode is completely stabilized by the normal magnetic field, whereas the ion tearing mode may only be destabilized in a certain restricted range of values of the normal magnetic field.

At present, theoretical studies of reconnection in the magnetotail find that the tearing mode is stabilized by the effects of the normal field near the neutral sheet. This may explain the observed "quiet times" during the presence of an elongated tail structure. It is still a question of debate, however, as to the exact mechanism triggering the tearing mode instability and the onset of substorms [2].

A possible effect of turbulent electron diffusion on the magnetotail tearing mode is to provide a destabilization mechanism even if a normal magnetic field component, B_n, is present. As discussed above, the normal field B_n introduces a large stabilizing barrier V_1 in a narrower region of order $\Delta_e \simeq (r_{L_e}\lambda)^{1/2}$ about the neutral sheet. Here, r_{L_e} is the mean electron Larmor radius and λ is the width of the current sheet. In the magnetotail, $r_{L_e} \sim 4$ km and $\lambda \sim 6 \times 10^3$ km and, hence, $\Delta_e \sim 150$ km. By comparison, the diffusion layer from the observed electrostatic noise is on the order of 3000 km. Turbulent diffusion would prohibit any structure from forming on any scale length less than this diffusive layer width. Hence, the potential barrier V_1 would be eliminated. It is clear that turbulent diffusion may have a strong effect on the tearing mode in the Earth's magnetotail.

Appendix A

Nomenclature

a	plasma edge
A	magnetic potential
$\tilde{A}_{\|}$	perturbed parallel magnetic potential
b	line bending term, $k_y^2\rho_i^2$
$\mathbf{b}$	unit vector along the magnetic field
$\mathbf{B}$	magnetic field
$\tilde{\mathbf{B}}_{\perp}$	perturbed magnetic field
c	speed of light
c_n	various constants, $n = 0, 1, 2 \cdots$
$C_{\pm}$	constant characterizing the ideal MHD form of $\tilde{A}_{\|}$
d	ion term in the coupled equations for $\tilde{\phi}$ and $\tilde{A}_{\|}$
D	diffusion coefficient
e	electron charge
$\mathbf{E}$	electric field
$\tilde{E}_{\|}$	perturbed parallel electric field
f	function characterizing the ideal MHD solution for $\tilde{A}_{\|}$
f_α	particle distribution function of species α
g	function characterizing the ideal MHD solution for $\tilde{A}_{\|}$
$\tilde{g}_i$	non-adiabatic ion response
G	diffusive Green's function characterizing the electron response
$\tilde{h}_e$	non-adiabatic electron response
I_n	electron resonance function, $n = 0, 1, 2 \cdots$
J	particle (plasma) current
k	wavenumber
K_n	beam electron resonance function, $n = 0, 1, 2 \cdots$
$L_{\pm}$	slope of the trial function $\tilde{A}_{\|}$

L_1, L_2, L_x	electrostatic, magnetic and coupling operators in the coupled equations for $\tilde{\phi}$ and $\tilde{A}_{\|}$
L_s, L_n, L_J	length scales for the shear, equilibrium density and equilibrium current
$\mathcal{L}_{A,\phi}$	magnetic and electric integral forms
m	poloidal mode number
m_α	mass of species α
n	toroidal mode number
n_α	density of species α
$P_{\|}$	parallel pressure
q	ionic charge
r_L	average Larmor radius
R_n	electron resonance operators, $n = 0, 1, 2 \cdots$
S	variational form
$S(x)$	source term
$S_{k\omega}$	fluctuation power spectrum
t	time
T_α	temperature of species α
v	particle velocity
v_b	beam velocity
v_A	Alfven velocity
v_D	drift velocity
v_α	thermal velocity of species α
V	fluid velocity
W	island saturation width
x	spatial coordinate
x_α	particle resonance point of species α
x_c	diffusive correlation distance
Z	plasma disperison function
α	variational parameter of trial function $\tilde{\phi}$

α_i^2	ion term in coupled equations for $\tilde{\phi}$ and $\tilde{A}_{\|}$
β_α	plasma beta of species α
$\beta_{A,\phi}$	magnetic and electric boundary terms
γ	growth rate
Γ	normalized growth rate
Γ_n	exponential times modified Bessel functions, $n = 0, 1, 2 \cdots$
δ	small variation
Δ'	ideal MHD energy drive
ϵ	small parameter
ζ	argument of plasma dispersion function
η	plasma resistivity
η_J	normalized current gradient parameter
θ	poloidal angle
κ_n	beam electron resonance kernel
λ	current gradient parameter characterizing the ideal MHD solutions
Λ	eigenvalue drift wave linear dispersion relation
ν	collision frequency
ξ	argument of the plasma disperison function
ρ_α	gyroradius of particle α
σ_n	corrective factor for finite ω_{*e}/ω_c, $n = 1$ and 2
$\sum$	summation
τ	time variable
τ_{ac}	autocorrelation time
τ_k	Kolmogorov time
ϕ	toroidal angle
$\tilde{\phi}$	perturbed electrostatic potential
χ^2	ion term in coupled equations for $\tilde{\phi}$ and $\tilde{A}_{\|}$
ω	eigenfrequency

ω_c	diffusive correlation frequency
ω_t	transit frequency
$\omega_{p\alpha}$	plasma frequency of species α
$\omega_{*\alpha}$	drift frequency of species α
Ω_c	cyclotron frequency
$\parallel$	parallel component
$\perp$	perpendicular component
∇	gradient operator
$[Q]_0$	microscale ensemble average
$\langle Q \rangle$	angular average of Q
Q^*	complex conjugate of Q
$\hat{Q}$	normalized quantity Q
$\tilde{Q}$	perturbed quantity
$\overline{Q}$	equilibrium quantity
Q_e	electron quantity
Q_i	ion quantity
Q_+	expression for Q for $x > 0$
Q_-	expression for Q for $x < 0$

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