DISTRIBUTED DETECTION WITH SELECTIVE COMMUNICATIONS

by

JASON D. PAPASTAVROU

BS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY (1984)

SUBMITTED IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE

DEGREE OF MASTER OF SCIENCE

IN ELECTRICAL ENGINEERING

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

MAY 1986

Signature of Author .

Department of Electrical Engineering
and Computer Science, May 1986

Certified by ______

Professor Michael Athans
Thesis Supervisor

Accepted by ________________________________

Professor A. C. Smith, Chairman,
Department Committee on Graduate Studies

DISTRIBUTED DETECTION WITH SELECTIVE COMMUNICATIONS

by
Jason D. Papastavrou

Submitted the Department of Electrical Engineering
on May 23, 1986 in partial fulfillment of the
requirements for the Degree of Master of Science in
Electrical Engineering

ABSTRACT

In this paper we formulate and solve a distributed binary hypothesis-testing problem. We consider a cooperative team that consists of two decision makers (DM's); one is refered to as the primary DM and the other as the consulting DM. The team objective is to carry out binary hypothesis testing based upon uncertain measurements. The primary DM can declare his decision based only on his own measurements; however, in ambiguous situations the primary DM can ask the consulting DM for an opinion and he incurs a communications cost. Then the consulting DM transmits either a definite recommendation or pleads ignorance. The primary DM has the responsibility of making a final definitive decision. The team objective is the minimization of the probability of error, taking into account different costs for hypothesis misclassification and communication costs. Numerical results are included to demonstrate the dependence of the different decision thresholds on the problem parameters, including different perceptions of the prior information.

Thesis Supervisor : Michael Athans
Professor of Systems Science and Engineering

ACKNOWLEDGMENTS

I would like to express my deepest thanks and gratitude to my parents for the love and constant support, they have been providing me with, all through my life and especially through my five years in MIT.

I am greatly thankful to Professor John Tsitsiklis for many helpful discussions and for constantly providing me with reading material. His ideas and suggestions helped in shaping the proofs of this thesis.

Special thanks to Fifa Monserrate for her typing and help in puting this thesis together.

Last, and certainly not least, I am deeply grateful to my thesis supervisor Professor Michael Athans for everything he has done for me. He was not just a thesis supervisor, but also a friend. He definetely contributed in making my years under his supervision not only educating (he even succeeded in familiarizing me with computers) and productive, but also happy.

Research conducted at the MIT Laboratory for Information and Decision Systems with support provided by the Office of Naval Research under contract ONR/N00014-84-k-0519 (NR 649-003).

TABLE OF CONTENTS

LIST OF FIGURES

1. INTRODUCTION AND MOTIVATION

In this paper we formulate, solve, and analyze a distributed hypothesis-testing problem which is an abstraction of a wide class of team decision problems. It represents a normative version of the "second-opinion" problem in which a primary decision maker (DM) has the option of soliciting, at a cost, the opinion of a consulting DM when faced with an ambiguous interpretation of uncertain evidence.

1.1 Motivating Examples.

Our major motivation for this research is provided by generic hypothesis-testing problems in the field of Command and Control. To be specific, consider the problem of target detection formalized as a binary hypothesis testing problem (H_0 means no target, while H_1 denotes the presense of a target). Suppose that independent noisy measurements are obtained by two geographically distributed sensors (Figure 1). One sensor, the primary DM, has final responsibility for declaring the presense or absence of a target, with different costs associated with the probability of false alarm versus the probability of missed detection. If the primary DM relied only on the measurements of his own sensor, then we have a classical centralized detection problem that has been extensively analyzed; see, for example, Van Trees [1]. If the actual measurements of the second sensor were communicated to the primary DM, we have once more a classical

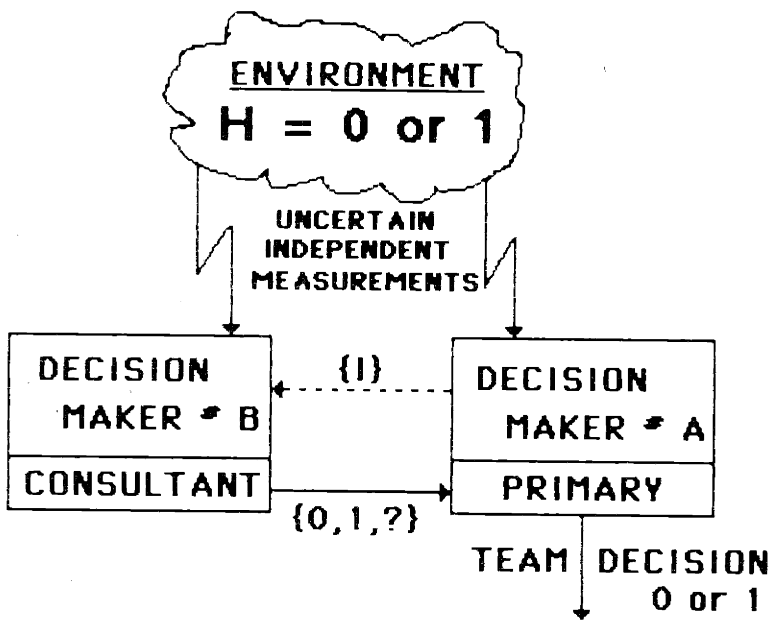


Figure 1. Problem Formulation

centralized detection problem in which we have two independent measurements on the same hypothesis; in this case, we require communication of raw data and this is expensive both from a channel bandwidth point of view and, perhaps more importantly, because radio or acoustic communication can be intercepted by the enemy.

Continuing with the target detection problem, we can arrive at the model that we shall use in the sequel by making the following assumptions which model the desire to communicate as little as possible. The primary DM can look at the data from his own sensor and attempt to arrive at a decision using a likelihood-ratio test (lrt), which yields a threshold test in the linear-Gaussian case. Quite often the primary DM can be confident about the quality of his decision. However, we can imagine that there will be instances that the data will be close to the decision threshold, corresponding to an ambiguous situation for the primary DM. In such cases it may pay off to incur a communications cost and seek some information from the other available sensor. It remains to establish what is the nature of the information to be transmitted back to the primary DM.

In our model, we assume the existence of a consulting DM having access to the data from the other sensor. We assume that the consulting DM has the ability to map the raw data from his sensor into decisions. The consulting DM is "activated" only at the request of the primary DM. It is natural to speculate that his advise will be ternary in nature: YES, I think there is a target; NO, I do not think there is a target; and, SORRY, NOT SURE MYSELF. Note that these transmitted decisions in general require less bits than the raw sensor data, hence the communication is cheap and more likely to escape enemy interception. Then, the primary DM based upon the message received from the consulting DM has the responsibility of making the final binary team decision on whether the target is present or absent.

The need for communicating with small-bit messages can be appreciated if we think of detecting an enemy submarine using passive sonar(Figure 2). We associate the primary DM with an attack submarine, and the consulting DM with a surface destroyer. Both have towed-array sonar capable of long-range enemy submarine detection. Request for information from the submarine to the destroyer can be initiated by having the sub sonar emit a lower power sonar pulse. A short active sonar pulse can be used to transmit the recommendation from the destroyer to the submarine. Thus, the submarine has the choice of obtaining a "second opinion" with minimal compromise of its covert mission.

Of course, target detection is only an example of more general binary hypothesis-testing problems. Hence, one can readily extend the basic distributed team decision problem setup to other situations. For example, in the area of medical diagnosis we imagine a primary physician interpreting the outcomes of several tests. In case of doubt, he sends the patient to another consulting physician for other tests (at a dollar cost), and seeks his recommendation. However, the primary physician has the final diagnostic responsibility. Similar scenarios occur in the intelligence field where the "compartmentalization" of sensitive data, or the protection of a spy, dictate infrequent and low-bit communications. In

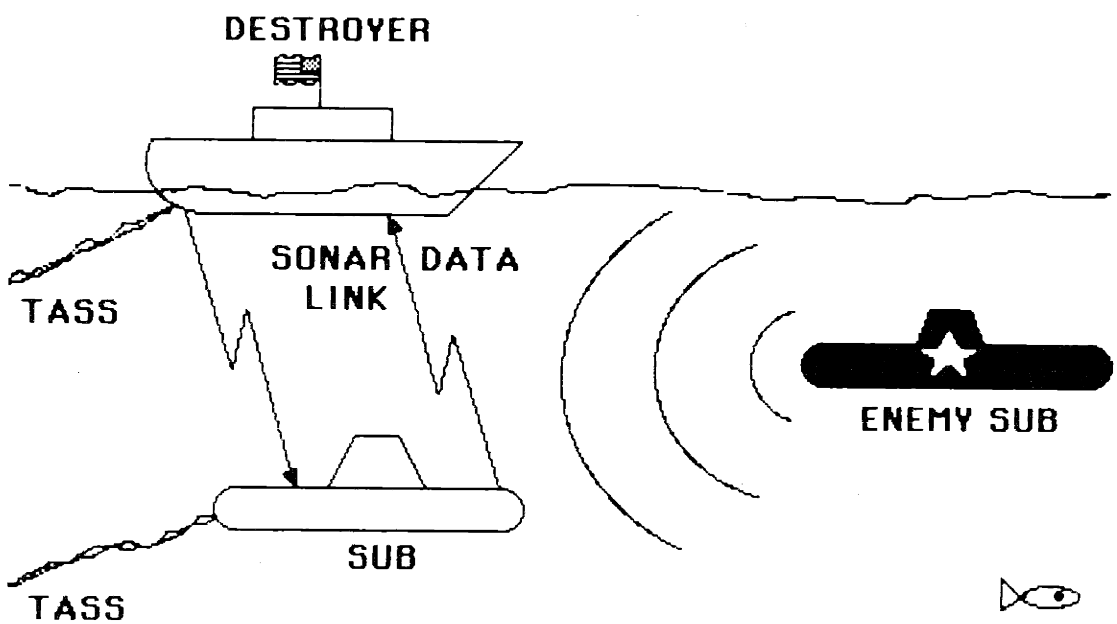


Figure 2. Anti-Submarine Warfare (ASW) Example

more general military Command and Control problems, we seek insight on formalizing the need to break EMCON, and at what cost, to resolve tactical situation assessment ambiguities.

1.2 Literature Review.

The solution of distributed decision problems is quite a bit different, and much more difficult, as compared to their centralized counterparts. Indeed there is only a handful of papers that deal with solutions to distributed hypothesis-testing problems. The first attempt to illustrate the difficulties of dealing with distributed hypothesis-testing problems was published by Tenney and Sandell [2]; they point out that the decision thresholds are in general coupled. Ekchian [3] and Ekchian and Tenney [4] deal with detection networks in which downstream DM's make decisions based upon their local measurements and upstream DM decisions. Kushner and Pacut [5] introduced a delay cost (somewhat similar to the communications cost in our model) in the case that the observations have exponential distributions, and performed a simulation study. Recently, Chair and Varshney [6] have pointed out how the results in [2] can be extended in more general settings. Boettcher [7] and Boettcher and Tenney [8], [9], have shown how to modify the normative solutions in [4] to reflect human limitation constraints, and arrive in at normative/descriptive model that captures the constraints of human implementation in the presense of decision deadlines and increasing human workload; experiments using human subjects showed close agreement with the predictions of their normative/descriptive model. Finally, Tsitsiklis [10] and Tsitsiklis and Athans [11] demonstrate that such distributed hypothesis-testing problems are NP-complete; their research provides theoretical evidence regarding the inherent complexity of solving optimal distributed decision problems as compared to their centralized counterparts (which are trivially solvable).

1.3 Contributions of this Research.

The main contribution of this thesis relates to the formulation and optimal solution of the team decision problem described above. Under the assumption that the measurements are conditionally independent, we show that the optimal decision rules for both the primary and the consulting DM are deterministic and are expressed as likelihood-ratio tests with constant thresholds which are tightly coupled (see Section 3 and the Appendix).

When we specialize the general results to the case that the observations are linear and the statistics are Gaussian, then we are able to derive explicit expressions for the decision thresholds for both the primary and consulting DM's (see Section 4). These threshold equations are tightly coupled, thereby necessitating an iterative solution. They provide clear-cut evidence that the DM's indeed operate as team members; their optimal thresholds are very different from those that they would use in isolation, i.e. in a non-team setting. This, of course, was the case in other versions of the distributed hypothesis-testing problem, e.g. [2].

The numerical sensitivity results (summarized in Section 5) for the linear-Gaussian case provide much needed intuitive understanding of the problem and concrete evidence that the team members operate in a more-or-less intuitive manner, especially after the fact. We study the impact of changing the communications cost and the measurement accuracy of each DM upon the decision thresholds and the overall team performance. In this manner we can obtain valuable insight on the optimal communication frequency between the DM's. As to be expected, as the communication cost increases, the frequency of communication (and asking for a second opinion) decreases, and the team performance approaches that of the primary DM operating in isolation. In addition, we compare the overall distributed team performance to the centralized version of the problem in which the primary DM had access, at no cost, to both sets of observations. In this manner, we can study the degree of inherent performance degradation to be expected as a consequence of enforcing the distributed decision architecture in the overall decision making process.

Finally, we study the team performance degradation when one of the team members, either the primary or the consulting DM, has an erroneous estimate of the hypotheses prior probabilities. This corresponds to mildly different mental models of the prior situation assesment; see Athans [12]. As expected the team performance is much more sensitive to misperceptions by the primary DM as compared to similar misperceptions by the consulting DM. This implies that, if team training reduces misperceptions on the part of the DM's, the greatest payoff is obtained in training the primary DM.

2. PROBLEM DEFINITION

2.1 Problem Description

The problem we study is one of hypothesis testing. The team has to choose among two alternative hypotheses H_0 and H_1, with a priori probabilities

$$P(H_0)=p_0 \qquad\qquad P(H_1)=p_1 \tag{1}$$

Each of two DM's, one called primary (DM A) and one consulting (DM B), receives an uncertain measurement y_α and y_β respectively (Figure 1), distributed with known joint probability density functions

$$P(y_\alpha,y_\beta \mid H_i) \qquad ; \qquad i=0,1. \tag{2}$$

The final decision of the team u_f (0 or 1, indicating H_0 or H_1 to be true) is the responsibility of the primary DM. DM A initially makes a preliminary decision u_α where it can either decide (0 or 1) on the basis of its own data (i.e. y_α), or <u>at a cost</u> ($C\geq0$) can solicit DM B's opinion (u_α=I), prior to making the commital decision.

The consulting DM's decision u_β consists of three distinct messages (call them x,v and z) and is activated only when asked by DM A. We decided to assign three messages to DM B, because we wanted to have one message indicating each of the two hypotheses and one message indicating that the consulting DM is 'not sure.' In fact, we proved that the optimal content for the messages of DM B is the one mentioned above.

When the message from DM B is received,the burden shifts back to the primary DM, which is called to make the commital decision of the team based on his own data and the information from the consulting DM.

2.2 Cost Function

We now define the following cost function :

$$J : \{0,1\} x \{H_0, H_1\} \rightarrow \mathbf{R} \tag{3}$$

with $J(u_f, H_i)$ being the cost incurred by the team choosing u_f, when H_i is true.

Then, the optimality criterion for the team is a function

$$J^*: \{0,1,I\} x \{0,1\} x \{H_0, H_1\} \rightarrow \mathbf{R} \tag{4}$$

with

$$J^*(u_\alpha,u_f,H_i)= \begin{cases} J(u_f,H_i)+C & ; \quad u_\alpha=I\,(\text{information requested}) \\ J(u_f,H_i) & ; \quad \text{otherwise} \end{cases} \tag{5}$$

The cost structure of the problem is the usual cost structure used in Hypothesis Testing problems, but also includes the non-negative communication cost, which the team incurs when the DM A decides to obtain the consulting DM's information.

<u>Remark</u> : According to the rules of the problem, when the preliminary decision u_α of the primary DM is 0 or 1, then the final team decision is 0 or 1 respectively (i.e. $P(u_f=i \mid u_\alpha=i)=1$ for $i=0,1$).

The objective of the decision strategies will be to minimize the expected cost incurred

$$\min \; E[J^*(u_\alpha,u_f,H)] \tag{6}$$

where the minimization is over the decision rules of the two DMs. Note that the decision rule of the consulting DM is implicitly included in the cost function, through the final team decision u_f (which is a function of the decision of the consulting DM).

2.3 Prior Knowledge

All the prior information is known to both DMs. The only information they do not share is their observations. Each DM knows only its own observation and, because of the conditional independence assumption, nothing about the other DM's observation.

2.4 Problem Statement

The problem can now be stated as follows :

Problem 2.1 : Given p_0, p_1, the distributions $P(y_\alpha, y_\beta \mid H_i)$ for $i=0,1$ with $y_\alpha \in Y_\alpha$, $y_\beta \in Y_\beta$, and the cost function J^*, find the decision rules u_α, u_β and u_f as functions

$$\gamma_\alpha : Y_\alpha \rightarrow \{0,1,I\} \tag{7}$$

$$\gamma_\beta : Y_\beta \rightarrow \{x,v,z\} \tag{8}$$

and

$$\gamma_f : Y_\alpha \times \{x,v,z\} \rightarrow \{0,1\} \tag{9}$$

subject to : $P(u_f=i \mid u_\alpha=i)=1$ for $i=0,1$, which minimize the expected cost.

2.5 Centralized Version of *Problem 2.1*

The centralized counterpart of the problem, where a single DM receives both observations is a well known problem. The solution is deterministic and given by a likelihood ratio test (lrt). That is :

$$\gamma_c : Y_\alpha \times Y_\beta \rightarrow \{0,1\} \tag{10}$$

with

$$\gamma_c(y_\alpha, y_\beta) = \begin{cases} 0 & ; \quad \Lambda(y_\alpha, y_\beta) \geq t \\ 1 & ; \quad \text{otherwise} \end{cases} \tag{11}$$

where

$$\Lambda(y_\alpha, y_\beta) = [P(y_\alpha, y_\beta \mid H_0)p_0] / [P(y_\alpha, y_\beta \mid H_1)p_1]$$

$$= P(H_0 \mid y_\alpha, y_\beta) / P(H_1 \mid y_\alpha, y_\beta) \tag{12}$$

and t is a precomputed threshold given by

$$t = [J(0,H_1) - J(1,H_1)] / [J(1,H_0) - J(0,H_0)] \tag{13}$$

provided $J(1,H_0) > J(0,H_0)$. Thus, the difficulty of our problem arises because of its decentralized nature.

We will show that, under certain assumptions, the most restrictive of which is conditional independence of the observations, the optimal decision rules for *Problem 2.1* are deterministic and given by lrt's with constant thresholds. The thresholds of the two DMs are coupled, indicating that the DMs work as a team rather than individuals.

3. THE SOLUTION TO THE GENERAL PROBLEM

In this section we summarize the main theoretical contributions of this thesis as they relate to the solution of *Problem 2.1.*

Proofs can be found in the Appendix.

3.1 Assumptions

ASSUMPTION 1 : $J(1,H_0) > J(0,H_0)$; $J(0,H_1) > J(1,H_1)$ (14)

or it is more costly for the team to err than to be correct.

This logical assumption is made in order to motivate the team members to avoid erring and in order to enable us to put the optimal decisions in lrt form.

ASSUMPTION 2 : $P(y_\alpha | y_\beta, H_i) = P(y_\alpha | H_i)$; $P(y_\beta | y_\alpha, H_i) = P(y_\beta | H_i)$; $i=0,1$ (15)

or the observations y_α and y_β are conditionally independent.

This assumption removes the dependence of the one observation on the other and thus allows us, as we are about to show, to write the optimal decision rules as lrt's with <u>constant</u> thresholds.

ASSUMPTION 3 : Without loss of generality assume that :

$$\frac{P(u_\beta=x \mid u_\alpha=I,H_0)}{P(u_\beta=x \mid u_\alpha=I,H_1)} \geq \frac{P(u_\beta=v \mid u_\alpha=I,H_0)}{P(u_\beta=v \mid u_\alpha=I,H_1)} \geq \frac{P(u_\beta=z \mid u_\alpha=I,H_0)}{P(u_\beta=z \mid u_\alpha=I,H_1)} \qquad (16)$$

This assumption is made in order to be able to distinguish between the messages of DM B.

3.2 Results

The optimal decision rule for the final decision of the primary DM is given in the following theorem.

THEOREM 1 : Given decision rules u_α and u_β, and that information is requested by the primary DM for some $y_\alpha \in Y_\alpha$ (i.e. $P(u_\alpha=I)>0$), then the optimal final decision of the primary DM after the information has been received, can be expressed as a deterministic function

$$\gamma_f : Y_\alpha \times \{x,v,z\} \rightarrow \{0,1\}$$

which is defined by likelihood ratio tests

$$\gamma_f(y_\alpha,u_\beta)=\begin{cases} 0 \; ; & \text{if } u_\beta=i \text{ and } \Lambda_\alpha(y_\alpha)\geq \alpha_i \\ 1 \; ; & \text{otherwise}\end{cases} \qquad \text{for } i=x,v,z \tag{17}$$

where

$$\Lambda_\alpha(y_\alpha) = \frac{p_0\, P(y_\alpha \mid H_0)}{p_1\, P(y_\alpha \mid H_1)} \tag{18}$$

and

$$\alpha_i = \frac{P(u_\beta=i \mid u_\alpha=I\,, H_1)\,[J(1,H_1)-J(0,H_1)]}{P(u_\beta=i \mid u_\alpha=I\,, H_0)\,[J(1,H_0)-J(0,H_0)]} \quad ; \quad i=x,v,z \tag{19}$$

The optimal decision rule for the consulting DM, when the primary DM requests for information, is given in the following theorem.

THEOREM 2 : Given the optimal decision rule u_f (derived in Theorem 1), a decision rule u_α and that information is requested for some $y_\alpha \in Y_\alpha$ (i.e. $P(u_\alpha=I)>0$),the optimal decision rule of the consulting DM is a deterministic function

$$\gamma_\beta : Y_\beta \rightarrow \{x,v,z\}$$

defined by the following likelihood ratio tests

$$\gamma_\beta(y_\beta) = \begin{cases} x & \text{if } \Lambda_\beta(y_\beta)\geq b_1 \text{ and } \Lambda_\beta(y_\beta)\geq b_2 \\ v & \text{if } \Lambda_\beta(y_\beta)< b_1 \text{ and } \Lambda_\beta(y_\beta)\geq b_3 \\ z & \text{if } \Lambda_\beta(y_\beta)< b_2 \text{ and } \Lambda_\beta(y_\beta)< b_3 \end{cases} \tag{20}$$

where

$$\Lambda_\beta(y_\beta) = \frac{p_0\, P(y_\beta \mid H_0)}{p_1\, P(y_\beta \mid H_1)} \tag{21}$$

and

$$b_1 = \frac{P(u_\alpha{=}I \mid H_1) \sum_{u_f} [P(u_f \mid u_\alpha{=}I, u_\beta{=}v, H_1) - P(u_f \mid u_\alpha{=}I, u_\beta{=}x, H_1)] J(u_f, H_1)}{P(u_\alpha{=}I \mid H_0) \sum_{u_f} [P(u_f \mid u_\alpha{=}I, u_\beta{=}x, H_0) - P(u_f \mid u_\alpha{=}I, u_\beta{=}v, H_0)] J(u_f, H_0)} \tag{22}$$

$$b_2 = \frac{P(u_\alpha{=}I \mid H_1) \sum_{u_f} [P(u_f \mid u_\alpha{=}I, u_\beta{=}z, H_1) - P(u_f \mid u_\alpha{=}I, u_\beta{=}x, H_1)] J(u_f, H_1)}{P(u_\alpha{=}I \mid H_0) \sum_{u_f} [P(u_f \mid u_\alpha{=}I, u_\beta{=}x, H_0) - P(u_f \mid u_\alpha{=}I, u_\beta{=}z, H_0)] J(u_f, H_0)} \tag{23}$$

$$b_3 = \frac{P(u_\alpha{=}I \mid H_1) \sum_{u_f} [P(u_f \mid u_\alpha{=}I, u_\beta{=}z, H_1) - P(u_f \mid u_\alpha{=}I, u_\beta{=}v, H_1)] J(u_f, H_1)}{P(u_\alpha{=}I \mid H_0) \sum_{u_f} [P(u_f \mid u_\alpha{=}I, u_\beta{=}v, H_0) - P(u_f \mid u_\alpha{=}I, u_\beta{=}z, H_0)] J(u_f, H_0)} \tag{24}$$

Equivalently, we can write

$$\gamma_\beta(y_\beta) = \begin{cases} x & \text{if } \Lambda_\beta(y_\beta) \geq \beta_1 \\ v & \text{if } \Lambda_\beta(y_\beta) < \beta_1 \text{ and } \Lambda_\beta(y_\beta) \geq \beta_2 \\ z & \text{if } \Lambda_\beta(y_\beta) < \beta_2 \end{cases} \tag{25}$$

where

$$\beta_1 = \max\{b_1, b_2\} \tag{26}$$

and

$$\beta_2 = \min\{b_2, b_3\}\,. \tag{27}$$

We proceed to derive the optimum decision rule for the preliminary decision of the primary DM.

LEMMA 1 : Given the decision rule u_β of the consulting DM and the final decision rule u_f of the primary DM, the preliminary decision rule u_α of the primary DM can be expressed as a deterministic function

$$\gamma_\alpha : Y_\alpha \rightarrow \{0,1,I\}$$

defined by the following *degenerate* (because the thresholds are functions of y_α) lrts

$$\gamma_\alpha(y_\alpha) = \begin{cases} 0 & \text{if } \Lambda_\alpha(y_\alpha) \geq a_1 \text{ and } \Lambda_\alpha(y_\alpha) \geq a_2 \\ I & \text{if } \Lambda_\alpha(y_\alpha) < a_2 \text{ and } 1/\Lambda_\alpha(y_\alpha) < 1/a_3 \\ 1 & \text{if } \Lambda_\alpha(y_\alpha) < a_1 \text{ and } 1/\Lambda_\alpha(y_\alpha) \geq 1/a_3 \end{cases} \tag{28}$$

where $\Lambda_\alpha(y_\alpha)$ is defined in (18) and

$$a_1 = \frac{J(0,H_1) - J(1,H_1)}{J(1,H_0) - J(0,H_0)} \tag{29}$$

$$a_2 = \frac{\sum_{u_f,u_\beta} P(u_f \mid u_\alpha{=}I,u_\beta,y_\alpha)\, P(u_\beta \mid u_\alpha{=}I,H_1)[J(u_f,H_1) + C] - J(0,H_1)}{J(0,H_0) - \sum_{u_f,u_\beta} P(u_f \mid u_\alpha{=}I,u_\beta,y_\alpha)\, P(u_\beta \mid u_\alpha{=}I,H_0)[J(u_f,H_0) + C]} \tag{30}$$

$$a_3 = \frac{\sum_{u_f,u_\beta} P(u_f \mid u_\alpha=I,u_\beta,y_\alpha)\, P(u_\beta \mid u_\alpha=I,H_1)[J(u_f,H_1) + C] - J(1,H_1)}{J(1,H_0) - \sum_{u_f,u_\beta} P(u_f \mid u_\alpha=I,u_\beta,y_\alpha)\, P(u_\beta \mid u_\alpha=I,H_0)[J(u_f,H_0) + C]} \qquad (31)$$

We proceed to show that the thresholds derived above are *independent* of y_α.

COROLLARY 1 : If for some y_α information is requested, according to the rule of Lemma 1 and u_β=x (or z) is returned, then the optimal final decision u_f of the primary DM is <u>always</u> 0 (or 1); that is :

$$P(u_f=0 \mid u_\alpha=I,u_\beta=x,y_\alpha) = 1 \quad \text{for all } y_\alpha \in \{y_\alpha \mid P(u_\alpha=I \mid y_\alpha)=1,\ y_\alpha \in Y_\alpha\} \qquad (32)$$

and

$$P(u_f=1 \mid u_\alpha=I,u_\beta=z,y_\alpha) = 1 \quad \text{for all } y_\alpha \in \{y_\alpha \mid P(u_\alpha=I \mid y_\alpha)=1,\ y_\alpha \in Y_\alpha\} \qquad (33)$$

<u>Remark</u> : From Corrolary 1 we can now give another interpretation to the team procedure: the primary DM can decide 0 or 1 using his own observation or can decide,because of uncertainty, to incur the communication cost (C) and shift the burden of the decision to the consulting DM. Then it is the consulting DM's turn to choose between deciding 0 or 1, or, because of uncertainty, shifting the burden back (at <u>no</u> cost) to the primary DM, which is required to make the final decision

given his observation <u>and</u> the fact that the consulting DM's observation is not good enough for the consulting DM to make the final decision.

According to the above, we can simplify our notation of the consulting DM's messages by changing x to 0, z to 1 and v to ? (which is interpreted as the consulting DM saying "I am not sure").

Define the following secondary variables :

$$\Delta J_0 = J(1,H_0) - J(0,H_0) \tag{34}$$

$$\Delta J_1 = J(0,H_1) - J(1,H_1) \tag{35}$$

$$W^1 = \frac{\Delta J_0 \, \Delta J_1 \, [P(u_\beta=1 \mid H_1) - P(u_\beta=1 \mid H_0)]}{\Delta J_0+\Delta J_1} \tag{36}$$

$$W^2 = \frac{\Delta J_0 \, \Delta J_1 [P(u_\beta=? \mid H_0)P(u_\beta=1 \mid H_1) - P(u_\beta=? \mid H_1)P(u_\beta=1 \mid H_0)]}{\Delta J_0 \, P(u_\beta=? \mid H_0) + \Delta J_1 \, P(u_\beta=? \mid H_1)} \tag{37}$$

$$W^3 = \frac{\Delta J_0 \, \Delta J_1 [P(u_\beta=? \mid H_1)P(u_\beta=0 \mid H_0) - P(u_\beta=? \mid H_0)P(u_\beta=0 \mid H_1)]}{\Delta J_0 \, P(u_\beta=? \mid H_0) + \Delta J_1 \, P(u_\beta=? \mid H_1)} \tag{38}$$

$$W^4 = \frac{\Delta J_0 \, \Delta J_1 \, [P(u_\beta=0 \mid H_0) - P(u_\beta=0 \mid H_1)]}{\Delta J_0+\Delta J_1} \tag{39}$$

$$a_{2,1} = \frac{P(u_\beta=1 \mid H_1) \, \Delta J_1 - C}{P(u_\beta=1 \mid H_0) \, \Delta J_0 + C} \tag{40}$$

$$a_{2,2} = \frac{[P(u_\beta=1 \mid H_1) + P(u_\beta=? \mid H_1)]\,\Delta J_1 - C}{[P(u_\beta=1 \mid H_0) + P(u_\beta=? \mid H_0)]\,\Delta J_0 + C} \tag{41}$$

$$a_{3,1} = \frac{P(u_\beta=0 \mid H_1)\,\Delta J_1 + C}{P(u_\beta=0 \mid H_0)\,\Delta J_0 - C} \tag{42}$$

$$a_{3,2} = \frac{[P(u_\beta=0 \mid H_1)+P(u_\beta=? \mid H_1)]\,\Delta J_1 + C}{[P(u_\beta=0 \mid H_0)+P(u_\beta=? \mid H_0)]\,\Delta J_0 - C} \tag{43}$$

THEOREM 3 : Given the optimum final decision rule u_f of the primary DM (derived in Theorem 1) and the optimum decision rule u_β of the consulting DM, the optimum decision rule for the preliminary decision of the primary DM is given by a deterministic function

$$\gamma_\alpha : Y_\alpha \rightarrow \{0,1,I\}$$

defined by the following likelihood ratio tests

$$\gamma_\alpha(y_\alpha) = \begin{cases} 0 & \text{if } \Lambda_\alpha(y_\alpha) \geq \alpha_1 \\ I & \text{if } \Lambda_\alpha(y_\alpha) < \alpha_1 \text{ and } \Lambda_\alpha(y_\alpha) \geq \alpha_2 \\ 1 & \text{if } \Lambda_\alpha(y_\alpha) < \alpha_2 \end{cases} \tag{44}$$

where

$$\alpha_1 = \begin{cases} a_{2,1} & \text{if } 0 \leq C \leq \min\{W^1, W^2\} \\ a_{2,2} & \text{if } W^2 < C \leq W^4 \\ a_1 & \text{otherwise} \end{cases} \tag{45}$$

and

$$\alpha_2 = \begin{cases} a_{3,1} & \text{if} \quad 0 \le C \le \min\{W^3, W^4\} \\ a_{3,2} & \text{if} \quad W^3 < C \le W^1 \\ a_1 & \text{otherwise} \end{cases} \tag{46}$$

4. THE GAUSSIAN CASE

We now present detailed threshold equations for the case where the probability distributions of the two observations are Gaussian. We selected the Gaussian distribution, despite its cumbersome algebraic formulae, because of its generality. Our eventual objective is to perform numerical sensitivity analysis to the solution of the Gaussian case, in order to gain insight on the team decision-making structure.

4.1 Notation

We assume that the observations are distributed with the following Gaussian distributions :

$$y_\alpha \sim N(\mu,\sigma_\alpha^2) \quad ; \quad y_\beta \sim N(\mu,\sigma_\beta^2) \tag{47}$$

The two alternative hypotheses are characterized by

$$H_0 : \mu=\mu_0 \quad \text{or} \quad H_1 : \mu=\mu_1 \tag{48}$$

Without loss of generality, assume that

$$\mu_0 < \mu_1 \tag{49}$$

The rest of the notation is the same as in the general problem described above.

4.2 Decision Thresholds

We can show that the optimum decision rules for this example are given by thresholds on the *observation* axes, as shown in Figure 3. The values of the thresholds were obtained for the baseline parameters of Table 1. Before presenting the equations of the thresholds, we define some variables.

Y_α^l : lower threshold of DM A Y_α^u : upper threshold of DM A

Y_α^f : threshold for the final decision of DM A

Y_β^l : lower threshold of DM B Y_β^u : upper threshold of DM B

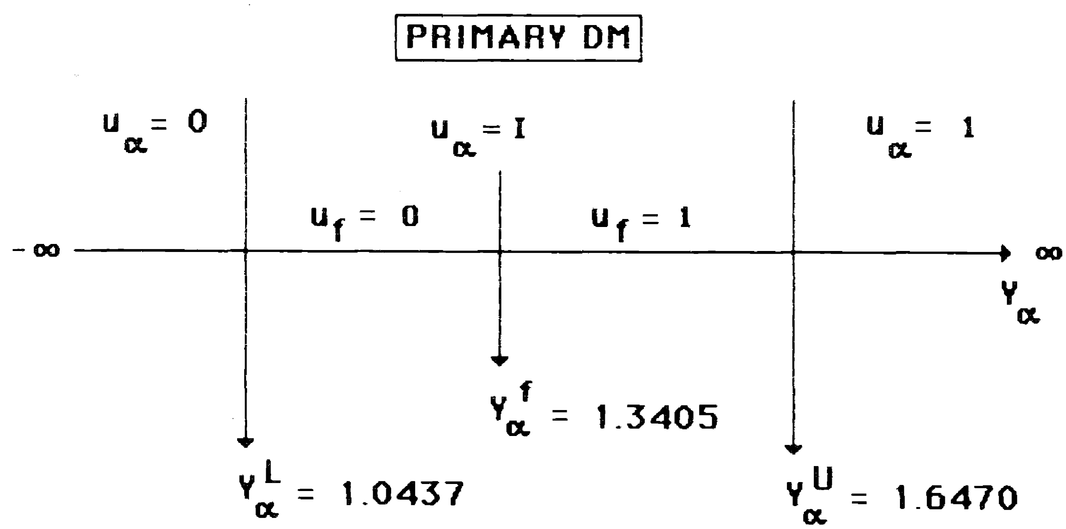
$$\Phi_i^j(k) = \int_{-\infty}^{\frac{Y_i^j - \mu_k}{\sigma_i}} (2\pi)^{-0.5} \exp\left(-0.5\, x^2\right)\, dx \qquad \text{for } i=\alpha,\beta \;;\; j=l,f,u \;;\; k=0,1$$

Note that the above function is the well-known error function, presented with notational modifications to fit the purposes of the problem.

The variables W^i (see eqs. (36) to (39)) are now given by :

FIGURE 3. The Gaussian Example

* OPTIMAL POLICIES ARE DEFINED BY THRESHOLDS



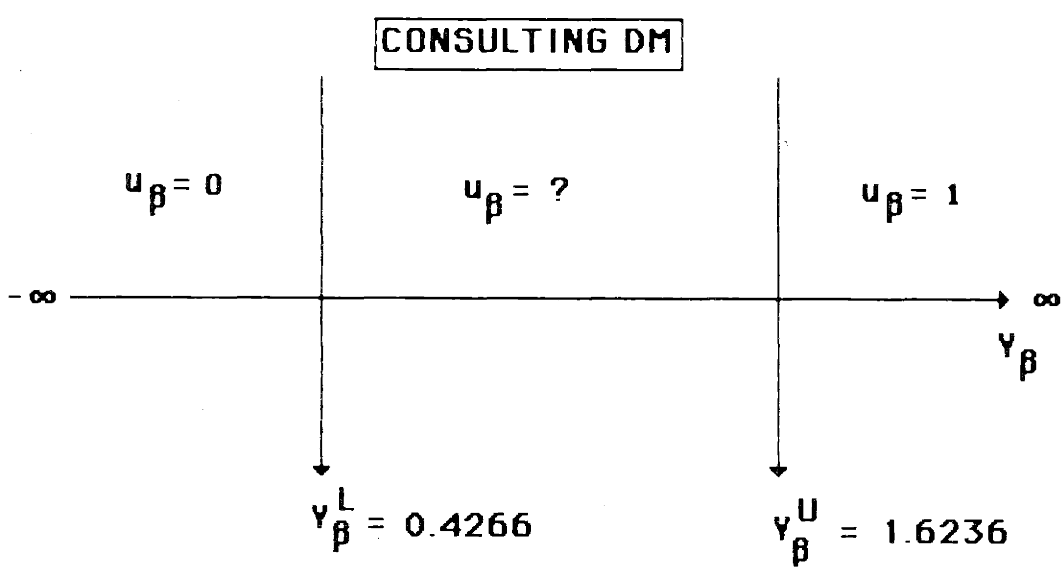


TABLE 1

BASELINE PARAMETER VALUES FOR THE GAUSSIAN EXAMPLE

$H_0: \mu = \mu_0 = -1$

$H_1: \mu = \mu_1 = +3$

$\sigma_\alpha^2 = 1 \quad ; \quad \sigma_\beta^2 = 4$

$p_0 = 0.8$

$C = 0.1$

In Cases 5.4 (Effects of varrying the a priori probabilities of the hypotheses) and 5.5 (Effects of a priori imperfect information) we change σ_α^2 to

$\sigma_\alpha^2 = 8$

$$W^1 = 0.5\,[\Phi_\beta^u(0)-\Phi_\beta^u(1)] \tag{50}$$

$$W^2 = \frac{\Phi_\beta^u(0)-\Phi_\beta^l(0)-\Phi_\beta^u(1)+\Phi_\beta^l(1)+\Phi_\beta^l(0)\Phi_\beta^u(1)-\Phi_\beta^u(0)\Phi_\beta^l(1)}{\Phi_\beta^u(0)-\Phi_\beta^l(0)+\Phi_\beta^u(1)-\Phi_\beta^l(1)} \tag{51}$$

$$W^3 = \frac{\Phi_\beta^l(0)\Phi_\beta^u(1)-\Phi_\beta^l(1)\Phi_\beta^u(0)}{\Phi_\beta^u(0)-\Phi_\beta^l(0)+\Phi_\beta^u(1)-\Phi_\beta^l(1)} \tag{52}$$

$$W^4 = 0.5\,[\Phi_\beta^l(0)-\Phi_\beta^l(1)] \tag{53}$$

$$Y_\alpha^* = (\mu_0+\mu_1)/2 + [\sigma_\alpha^2/(\mu_1-\mu_0)]\,\ln[p_0/(1-p_0)] \tag{54}$$

$$Y_\beta^* = (\mu_0+\mu_1)/2 + [\sigma_\beta^2/(\mu_1-\mu_0)]\,\ln[p_0/(1-p_0)] \tag{55}$$

In (54) and (55), the (centralized) maximum likelihood estimators for each DM are defined.

COROLLARY 2 : If $P(u_\alpha=I)>0$ (i.e. information is requested for some y_α) and if $P(u_\beta=?\,|\,u_\alpha=I)>0$ (i.e. "I am not sure" is returned for some y_β, when information is requested), then the optimal final decision rule of the primary DM is a deterministic function defined by

$$\gamma_f(y_\alpha) = \begin{cases} 0 & \text{if} \quad y_\alpha \le Y_\alpha^f \\ 1 & \text{if} \quad y_\alpha > Y_\alpha^f \end{cases} \tag{56}$$

where

$$Y_\alpha^f = Y_\alpha^* + \frac{\sigma_\alpha^2}{\mu_1-\mu_0} \ln\left(\frac{\Phi_\beta^u(0)-\Phi_\beta^l(0)}{\Phi_\beta^u(1)-\Phi_\beta^l(1)} \right) \tag{57}$$

Remark : Eq. (19) is the corresponding threshold equation for the general case.

COROLLARY 3 : If $P(u_\alpha=I)>0$ (i.e. information is requested for some y_α) and the primary DM's final decision rule is the one given by Corollary 2, then the optimal decision rule of the consulting DM is a deterministic function defined by

$$\gamma_\beta(y_\beta) = \begin{cases} 0 & \text{if} \quad y_\beta \le Y_\beta^l \\ ? & \text{if} \quad Y_\beta^l < y_\beta \le Y_\beta^u \\ 1 & \text{if} \quad Y_\beta^u < y_\beta \end{cases} \tag{58}$$

where :

$$Y_\beta^l = Y_\beta^* + \frac{\sigma_\beta^2}{\mu_1-\mu_0} \min\left\{ \ln\left(\frac{\Phi_\alpha^u(0)-\Phi_\alpha^f(0)}{\Phi_\alpha^u(1)-\Phi_\alpha^f(1)} \right), \ln\left(\frac{\Phi_\alpha^u(0)-\Phi_\alpha^l(0)}{\Phi_\alpha^u(1)-\Phi_\alpha^l(1)} \right) \right\} \tag{59}$$

and :

$$Y_\beta^u = Y_\beta^* + \frac{\sigma_\beta^2}{\mu_1-\mu_0} \max\left\{ \ln\left(\frac{\Phi_\alpha^u(0)-\Phi_\alpha^l(0)}{\Phi_\alpha^u(1)-\Phi_\alpha^l(1)} \right), \ln\left(\frac{\Phi_\alpha^f(0)-\Phi_\alpha^l(0)}{\Phi_\alpha^f(1)-\Phi_\alpha^l(1)} \right) \right\} \tag{60}$$

Remark : Equations (26) and (27) are the corresponding equations for the general case.

COROLLARY 4 : Given that the final decision rule employed by the primary DM is the one of Corollary 2 and that the decision rule employed by the consulting DM is the one of Corollary 3, then the optimal decision rule for the preliminary decision u_α of the primary DM is a deterministic function defined by :

$$\gamma_\alpha(y_\alpha) = \begin{cases} 0 & \text{if} \quad y_\alpha \leq Y_\alpha^l \\ I & \text{if} \quad Y_\alpha^l < y_\alpha \leq Y_\alpha^u \\ 1 & \text{if} \quad Y_\alpha^u < y_\alpha \end{cases} \tag{61}$$

where

$$Y_\alpha^l = \begin{cases} Y_\alpha^* + \dfrac{\sigma_\alpha^2}{\mu_1-\mu_0} \ln\left(\dfrac{1-\Phi_\beta^l(0)+C}{1-\Phi_\beta^l(1)-C} \right) & ; \quad 0 \leq C < \min\{W^1, W^2\} \\ Y_\alpha^* + \dfrac{\sigma_\alpha^2}{\mu_1-\mu_0} \ln\left(\dfrac{1-\Phi_\beta^u(0)+C}{1-\Phi_\beta^u(1)-C} \right) & ; \quad W^2 < C \leq W^4 \\ Y_\alpha^* & ; \quad \text{otherwise} \end{cases} \tag{62}$$

and :

$$Y_\alpha^u = \begin{cases} Y_\alpha^* + \dfrac{\sigma_\alpha^2}{\mu_1-\mu_0} \ln\left(\dfrac{\Phi_\beta^l(0)-C}{\Phi_\beta^l(1)+C} \right) & ; \quad 0 \leq C < \min\{W^3, W^4\} \\ Y_\alpha^* + \dfrac{\sigma_\alpha^2}{\mu_1-\mu_0} \ln\left(\dfrac{\Phi_\beta^u(0)-C}{\Phi_\beta^u(1)+C} \right) & ; \quad W^3 < C \leq W^1 \\ Y_\alpha^* & ; \quad \text{otherwise} \end{cases} \tag{63}$$

<u>Remark</u> : Observe that the equations of all the thresholds include (and

possibly reduce to) a "centralized" part (Y_i^*) indicating the relation of our problem to its centralized counterpart.

<u>Remark</u> : Equations (45) and (46) are the corresponding equations for the general case.

<u>Remark</u> : In the subsequent Section 5, sensitivity analysis is performed. The numerical solutions are obtained by use of a computer algorithm. We present to the algorithm initial estimates for the values of the thresholds for the decision of the secondary DM, and the algorithm yields the optimum thresholds for the decision of both DMs by solving for a fixed point of the threshold eqs. (57), (59), (60), (62) and (63).

5. NUMERICAL SENSITIVITY ANALYSES

We now perform sensitivity studies to the solution of the Gaussian example. Our objective is to analyze the effects on the team performance, when we vary the parameters of our problem, in order to obtain better understanding of the decentralized team decision mechanism. We vary the quality of the observations of each DM (the variance of each DM), the a priori likelihood of the hypotheses and the communication cost. Finally, we study the effects of different a priori knowledge (prior perception errors) for each DM.

We use the following 'minimum error' cost function :

$$J(u_f,H_i) = \begin{cases} 0 & ; \quad u_f = i \\ 1 & ; \quad u_f \neq i \end{cases} \tag{64}$$

We do not need to vary the cost function, because this would be mathematically equivalent to varying the a priori probabilities of the two hypotheses.

The baseline values of the parameters employed in the sensitivity analysis are those presented in Table 1.

5.1 Effects of varying the quality of the observations of the Primary DM

In Figures 4 to 19, the effects of varying the variance of the observations of the Primary DM on all the relevant variables are presented.

Denote :

$C1^*$ = cost incured if the consulting DM makes the decision alone

C = communication cost

We distinguish two cases depending on the cost associated with the information (i.e. the of quality of information)

CASE 1 : $\min(p_0, 1-p_0) \leq C1^* + C$

As the variance of the primary DM increases, it becomes less costly for the team to have the primary DM always decide the more likely hypothesis, than request for information. This occurs because the observation of DM A becomes increasingly worthless. Thus, the primary DM progressively ignores his observation and in order to minimize cost has to choose between "de facto" deciding the more likely hypothesis (and incuring cost equal to the probability of the least likely hypothesis) or "de facto" requesting for information (and thus incuring the communication cost C plus the cost of the consulting DM). In this case, the prior is less than the latter and so the optimum decision of the primary DM, as his variance tends to infinity is to always decide the more likely hypothesis (see Figures 8 and 9, p_0= .8). Recall that Y_α^* is the maximum likelihood estimation threshold, that is the decision threshold that the primary DM representing the probabilities of the DM's decisions, since the decision regions are characterized by the thresholds. For example :

$$P(u_\alpha = I) = \sum_H \int_{y_\alpha :\, Y_\alpha^l < y_\alpha < Y_\alpha^u} P(y_\alpha \mid H)\, P(H)$$

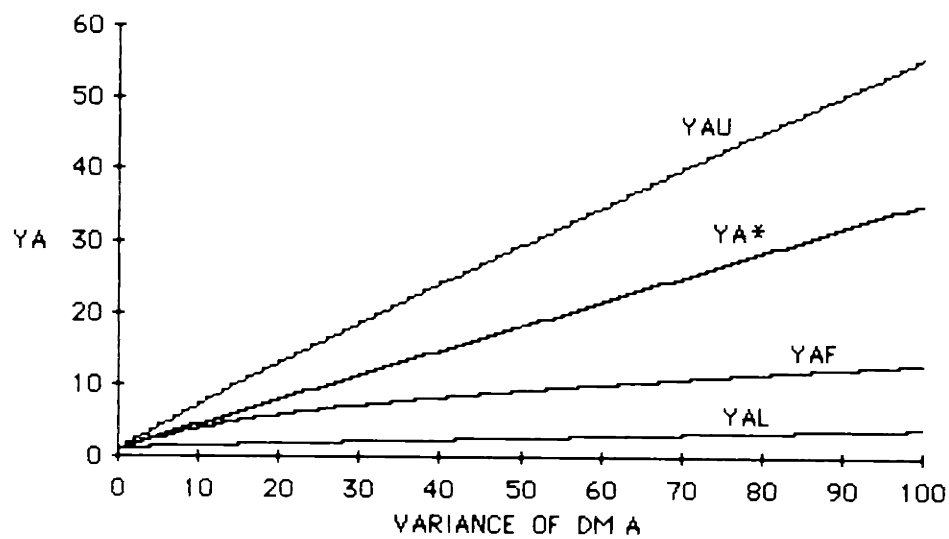
The thresholds of the consulting DM demonstrate some interesting aspects of the team behavior (see Figures 21 and 23). For small values of the variance σ_β^2 they are very close together, as the quality of the observations is very good and so the consulting DM is willing to make the final team decision. As his variance increases, DM B becomes more willing to return u_β=? (i.e. "I am not sure") and let DM A make the final team decision. As the variance continues to increase, the thresholds of the consulting DM converge again. This might seem counter-intuitive, but there is a simple explanation. The consulting DM recognizes that the primary DM, despite knowing that the quality of the consulting DM's information is bad, is willing to incur the communication cost to obtain the information. This indicates that the primary DM is 'confused', that is, the a posteriori probabilities of the two hypotheses (given DM A's observation) are very close together. Hence, the consulting DM becomes more willing to make the final decision. After a certain point ($\sigma_\beta^2 \approx 62.4$) the primary DM does not find it worthwhile to request information from the consulting DM.

<u>Remark</u> : Note (see Figure 21) that the thresholds of the consulting DM

FIGURE 4

DECISION THRESHOLDS OF DM A
AS A FUNCTION OF σ_{α}^{2}

P(Ho)= .8



P(Ho)= .5

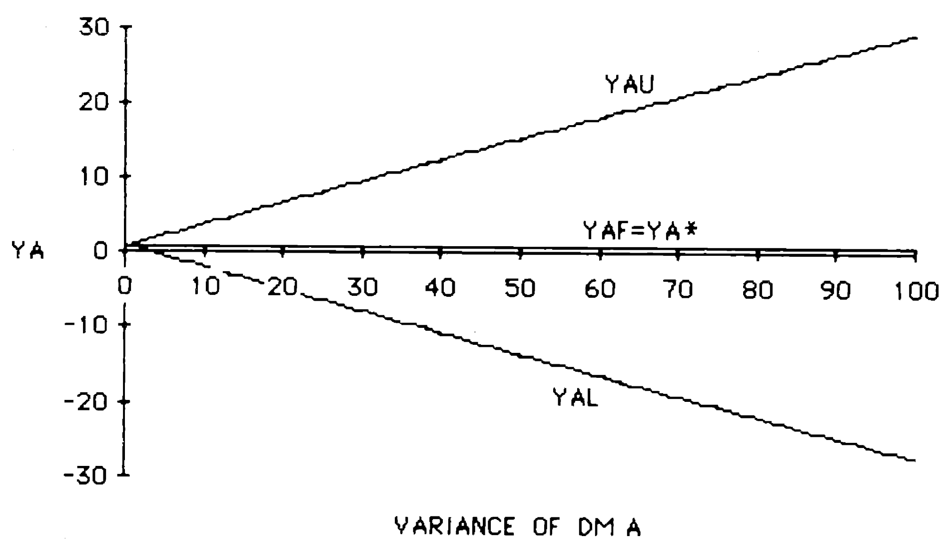
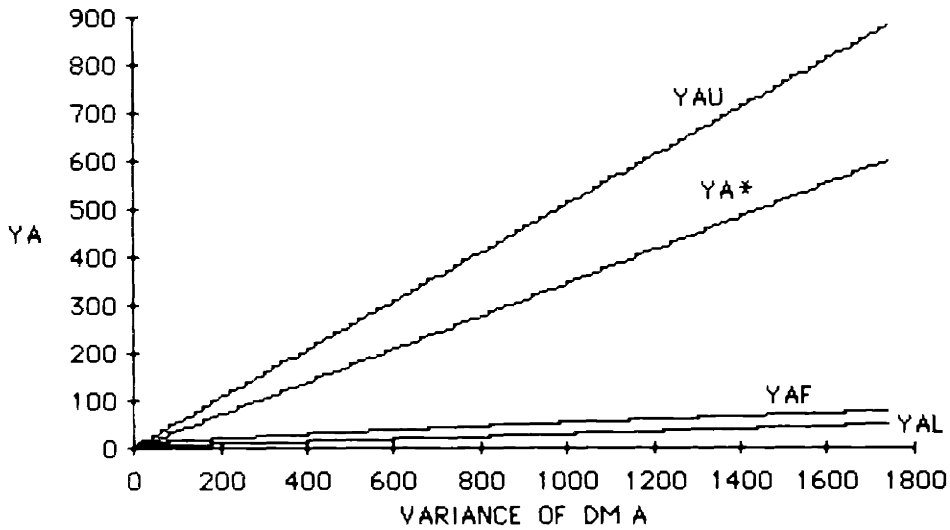


FIGURE 5

DECISION THRESHOLDS OF DM A AS A FUNCTION OF σ_α^2

P(Ho)= .8



P(Ho)= .5

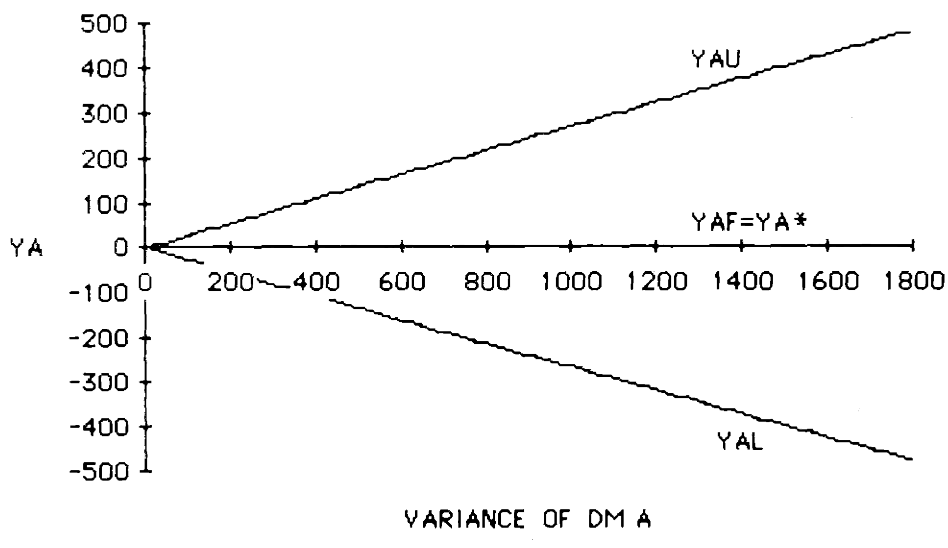
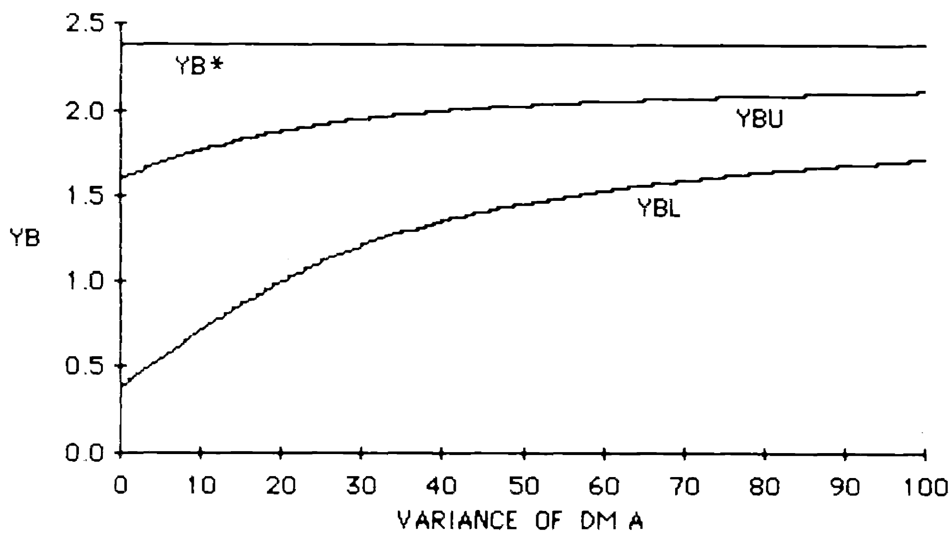


FIGURE 6

DECISION THRESHOLDS OF DM B
AS A FUNCTION OF σ_{α}^2

P(Ho)= .8



P(Ho)= .5

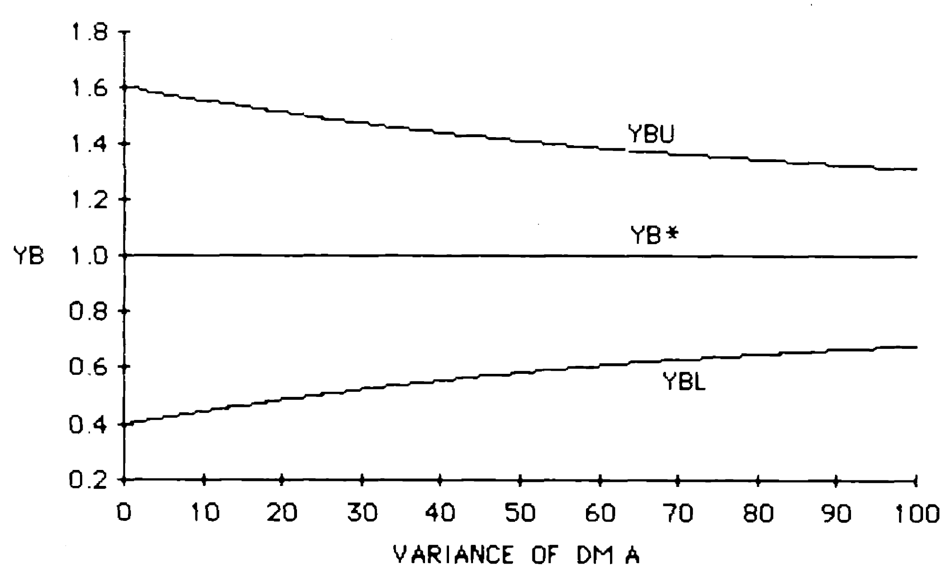
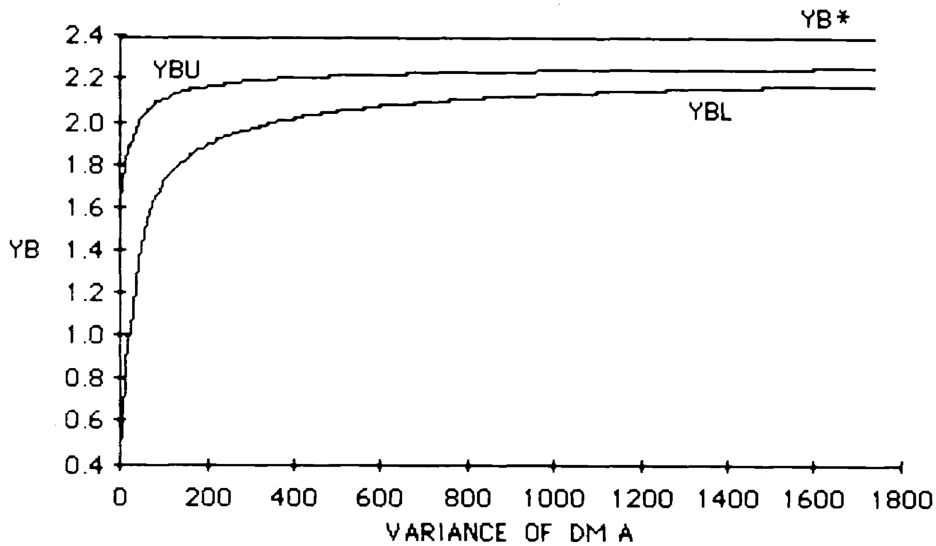


FIGURE 7

DECISION THRESHOLDS OF DM B
AS A FUNCTION σ_α^2

P(Ho)= .8



P(Ho)= .5

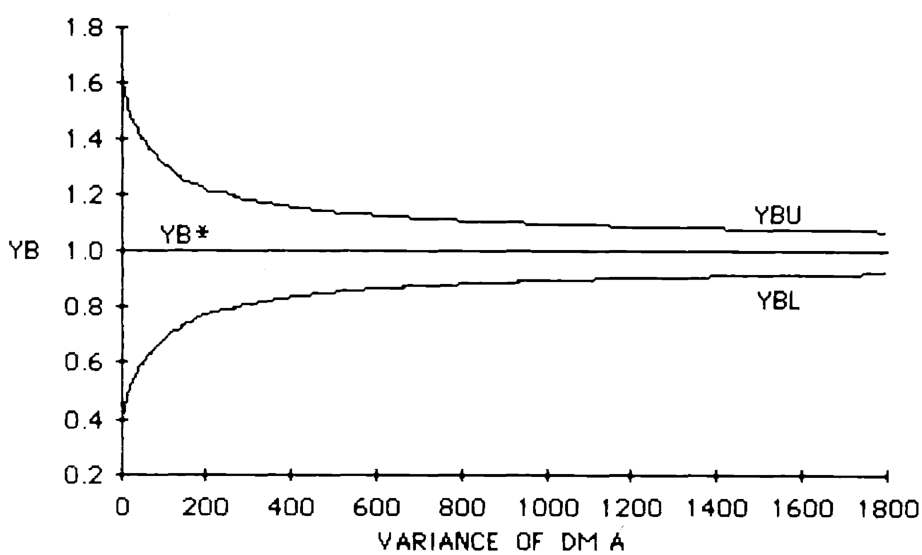


FIGURE 8

PROBABILITIES ASSOCIATED WITH THE
PRELIMINARY DECISION OF DM A
AS A FUNCTION OF σ_α^2

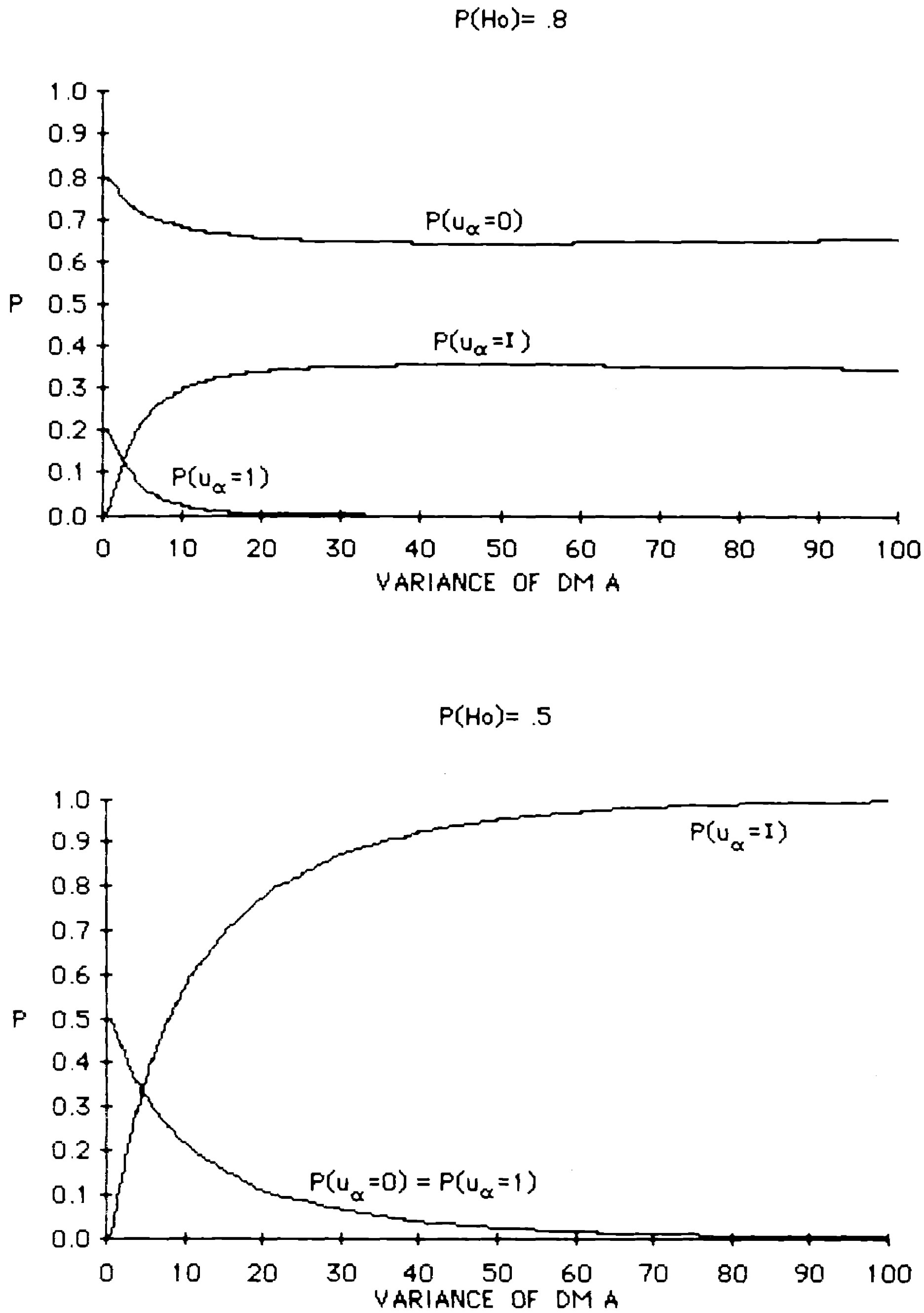
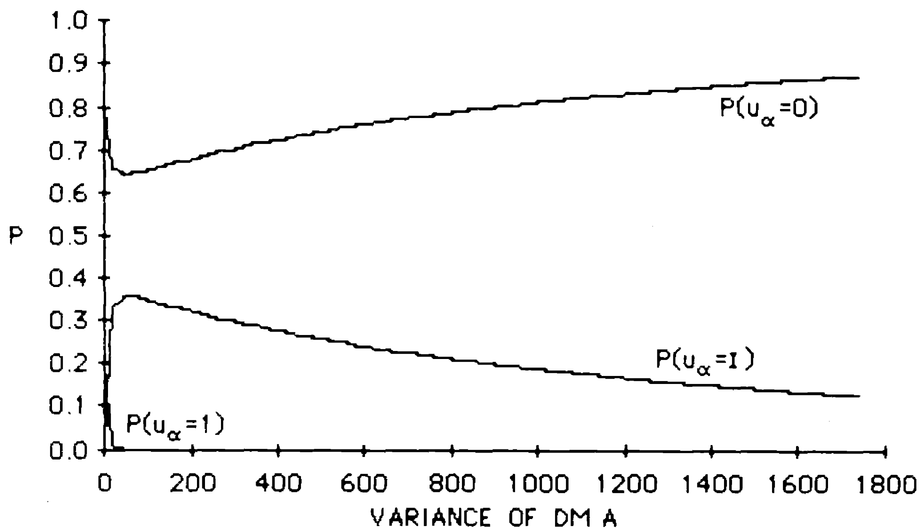


FIGURE 9

PROBABILITIES ASSOCIATED WITH THE
PRELIMINARY DECISION OF DM A
AS A FUNCTION OF σ_{α}^{2}

P(Ho)= .8



P(Ho)= .5

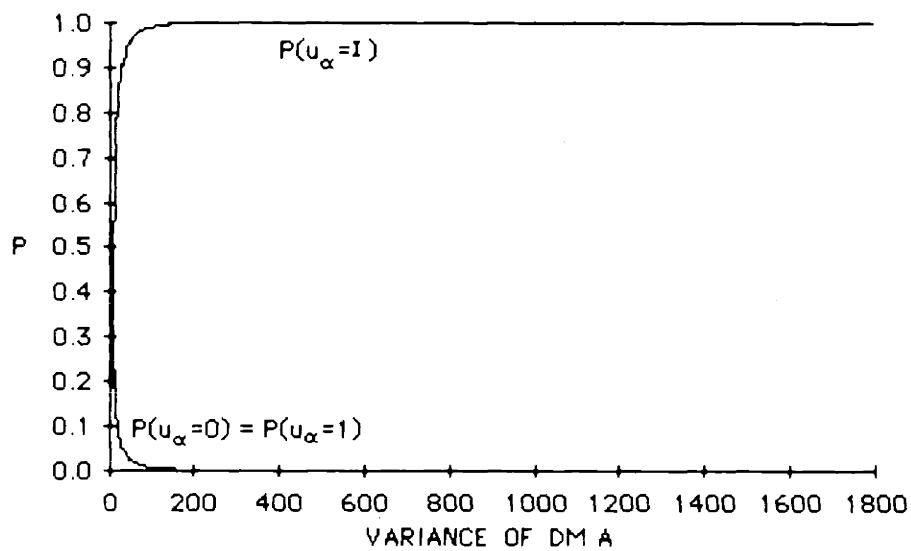


FIGURE 10

PROBABILITIES ASSOCIATED WITH THE
DECISION OF DM B
AS A FUNCTION OF σ_α^2

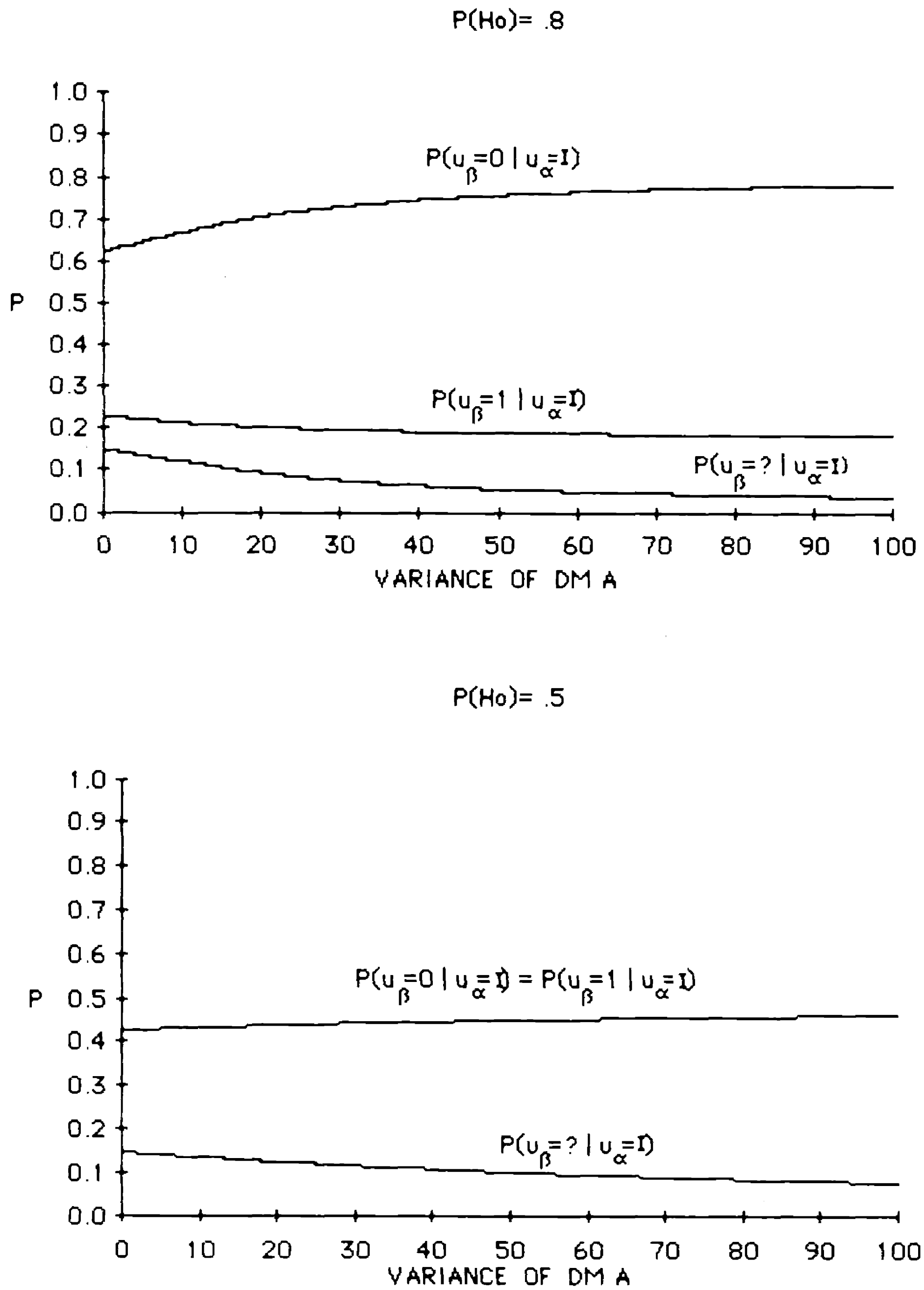


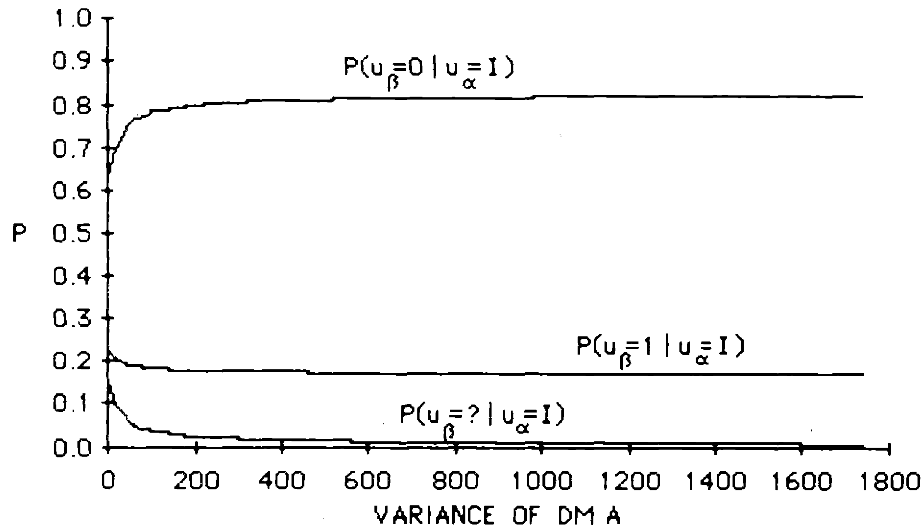
FIGURE 11

PROBABILITIES ASSOCIATED WITH THE

DECISION OF DM B

AS A FUNCTION OF σ_α^2

P(Ho)= .8



P(Ho)= .5

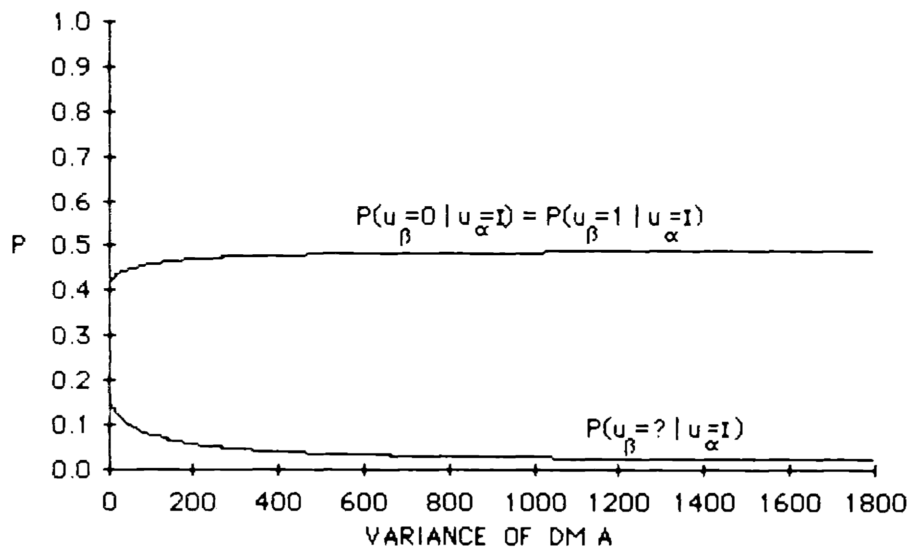
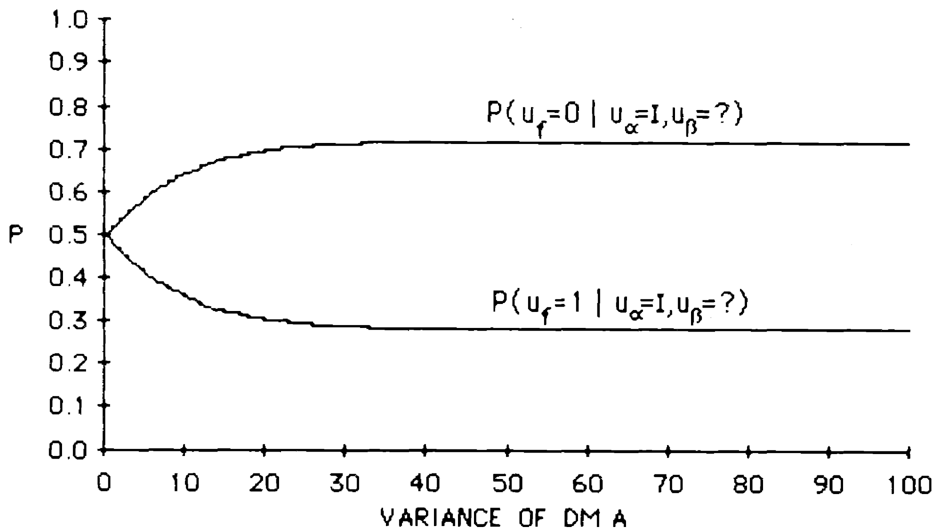


FIGURE 12

PROBABILITIES ASSOCIATED WITH THE FINAL DECISION OF DM A AS A FUNCTION σ_α^2

P(Ho)= .8



P(Ho)= .5

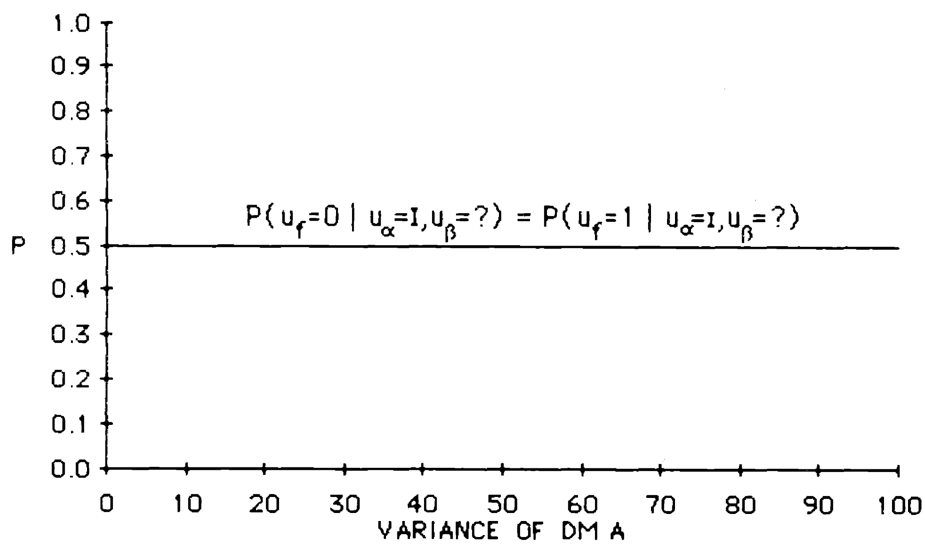
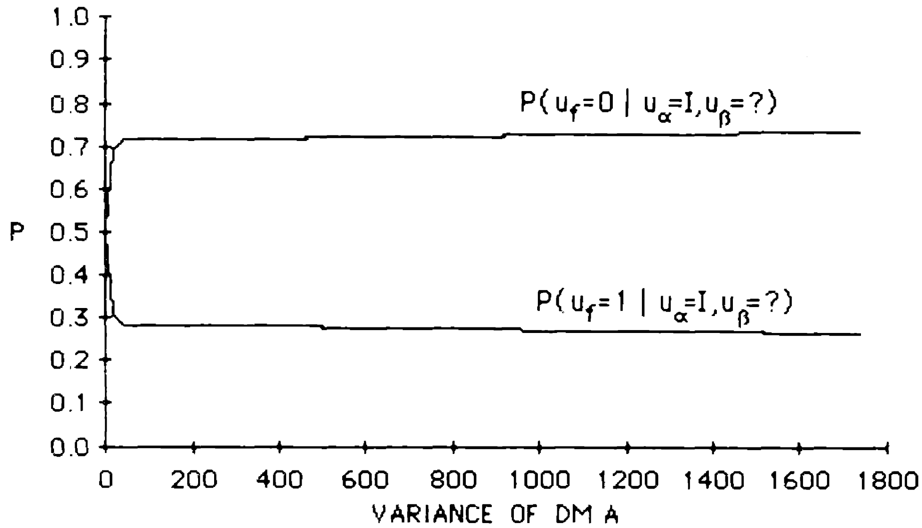


FIGURE 13

PROBABILITIES ASSOCIATED WITH THE
FINAL DECISION OF DM A
AS A FUNCTION OF σ_α^2

P(Ho)= .8



P(Ho)= .5

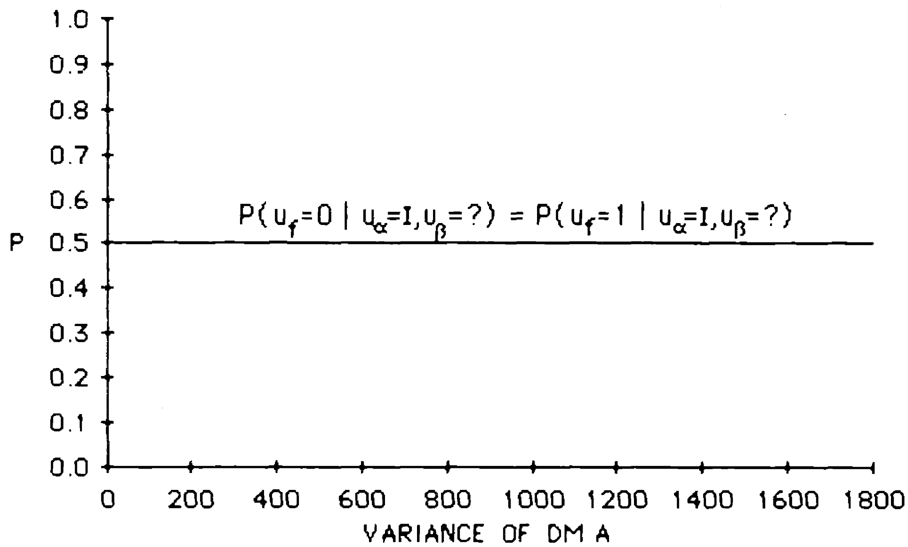
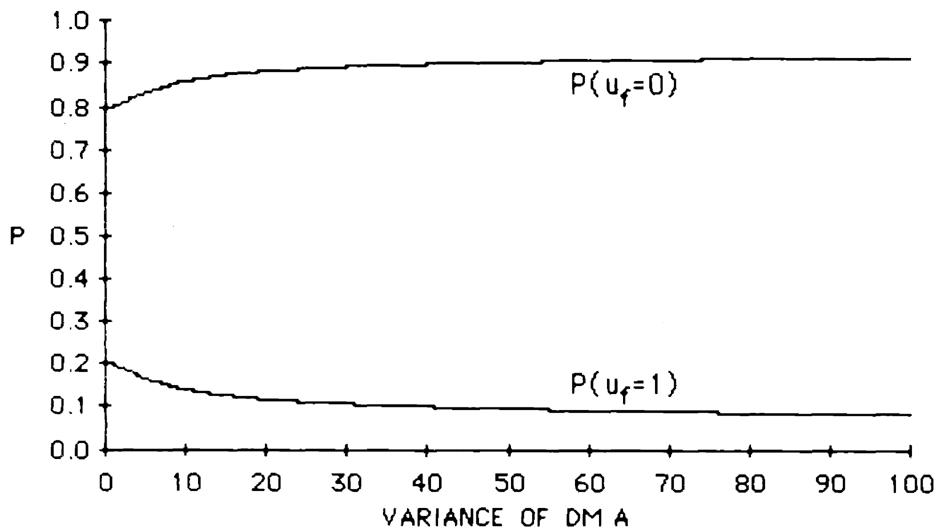


FIGURE 14

PROBABILITIES ASSOCIATED WITH THE
FINAL TEAM DECISION
AS A FUNCTION OF σ_α^2

P(Ho)= .8



P(Ho)= .5

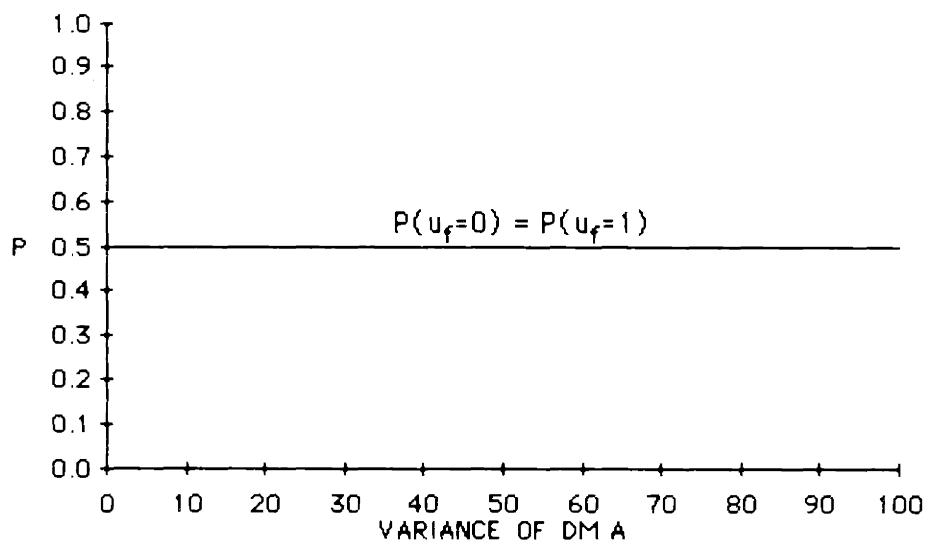
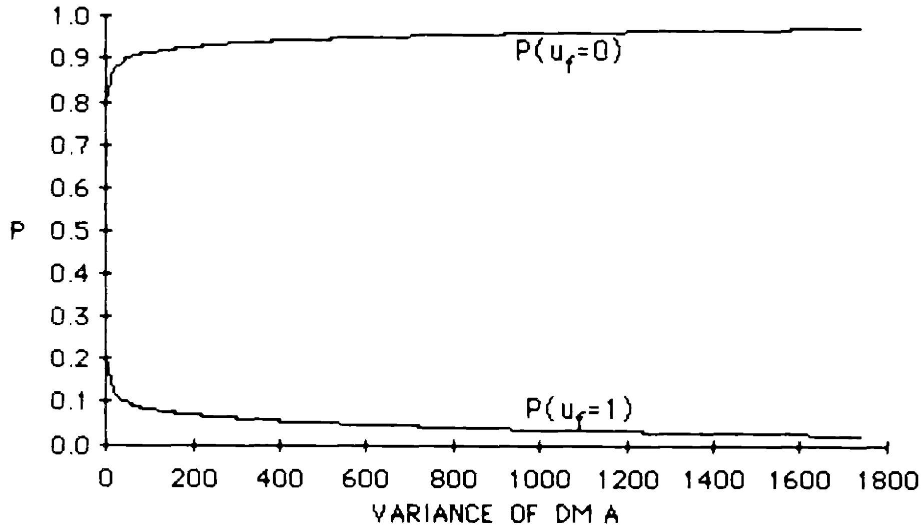


FIGURE 15

PROBABILITIES ASSOCIATED WITH THE
FINAL TEAM DECISION
AS A FUNCTION OF σ_α^2

P(Ho)= .8



P(Ho)= .5

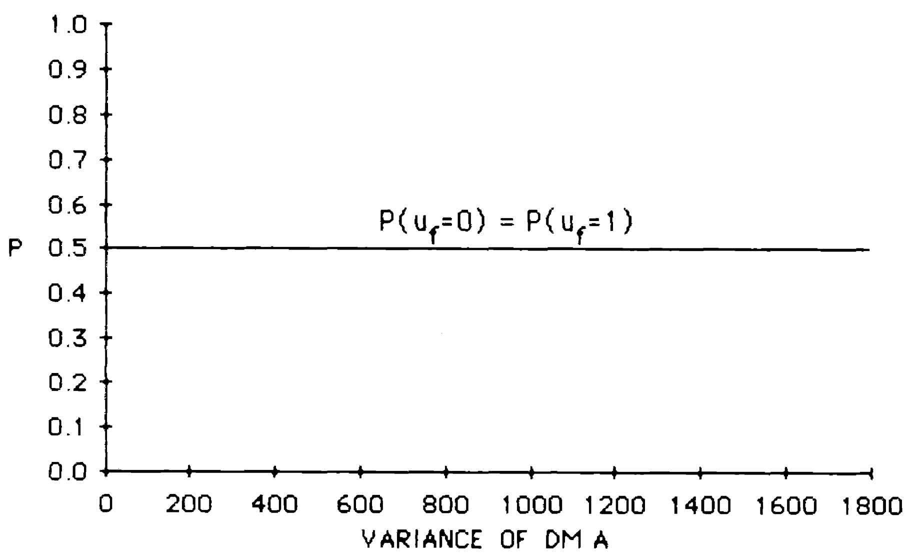
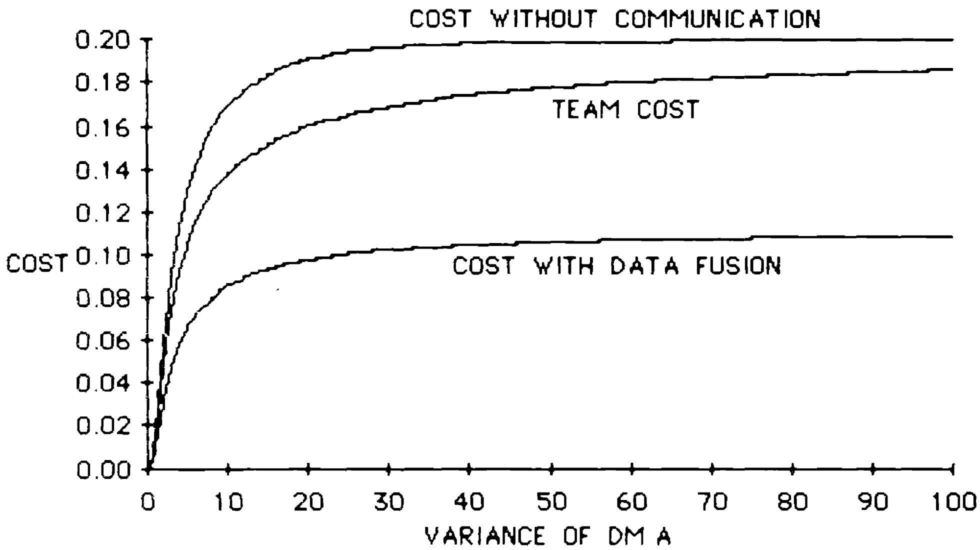


FIGURE 16

COSTS vs. σ_{α}^{2}

P(Ho)=.8



P(Ho)= .5

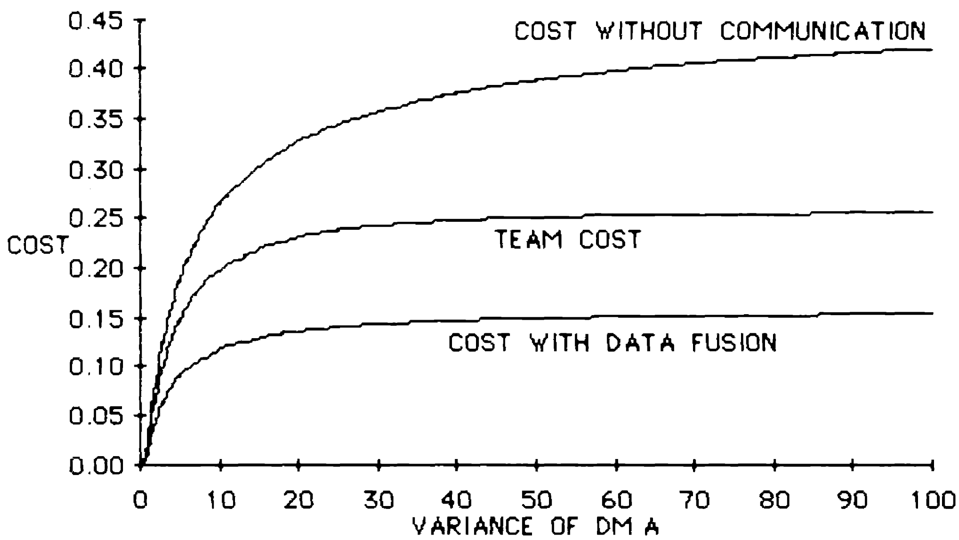
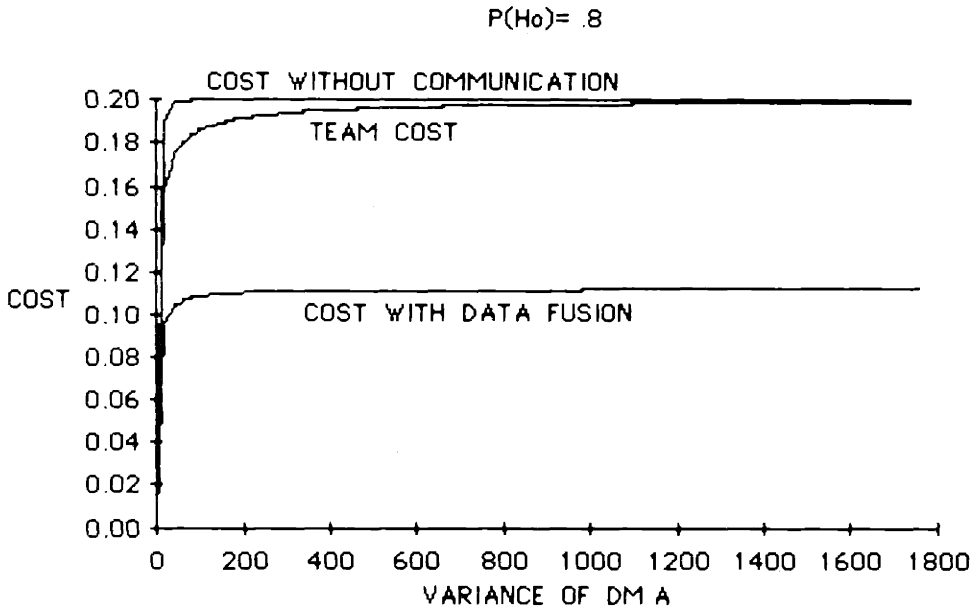


FIGURE 17

COSTS vs. σ_α^2



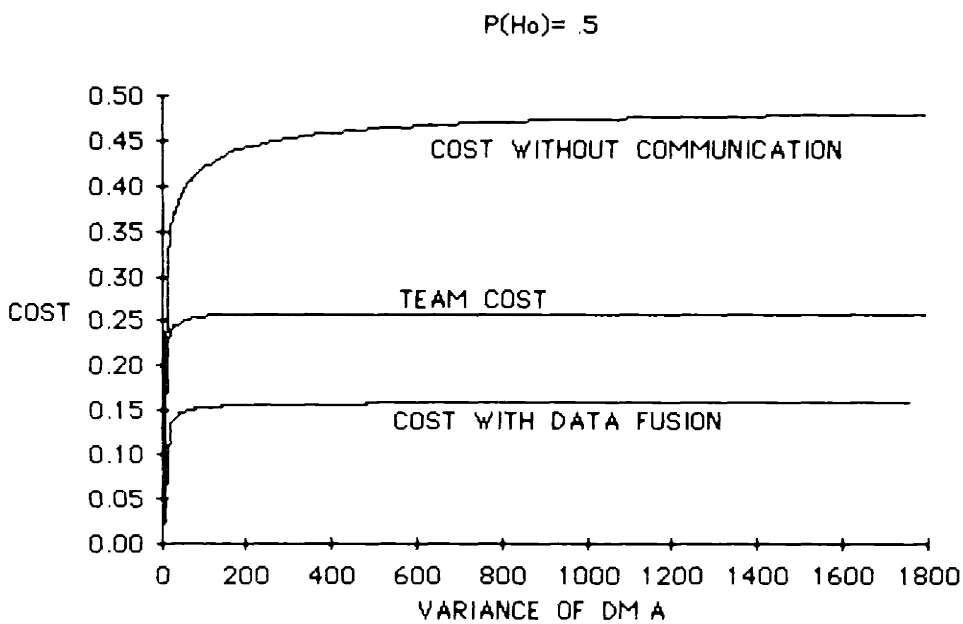


FIGURE 18. PERCENTAGE COST IMPROVEMENT

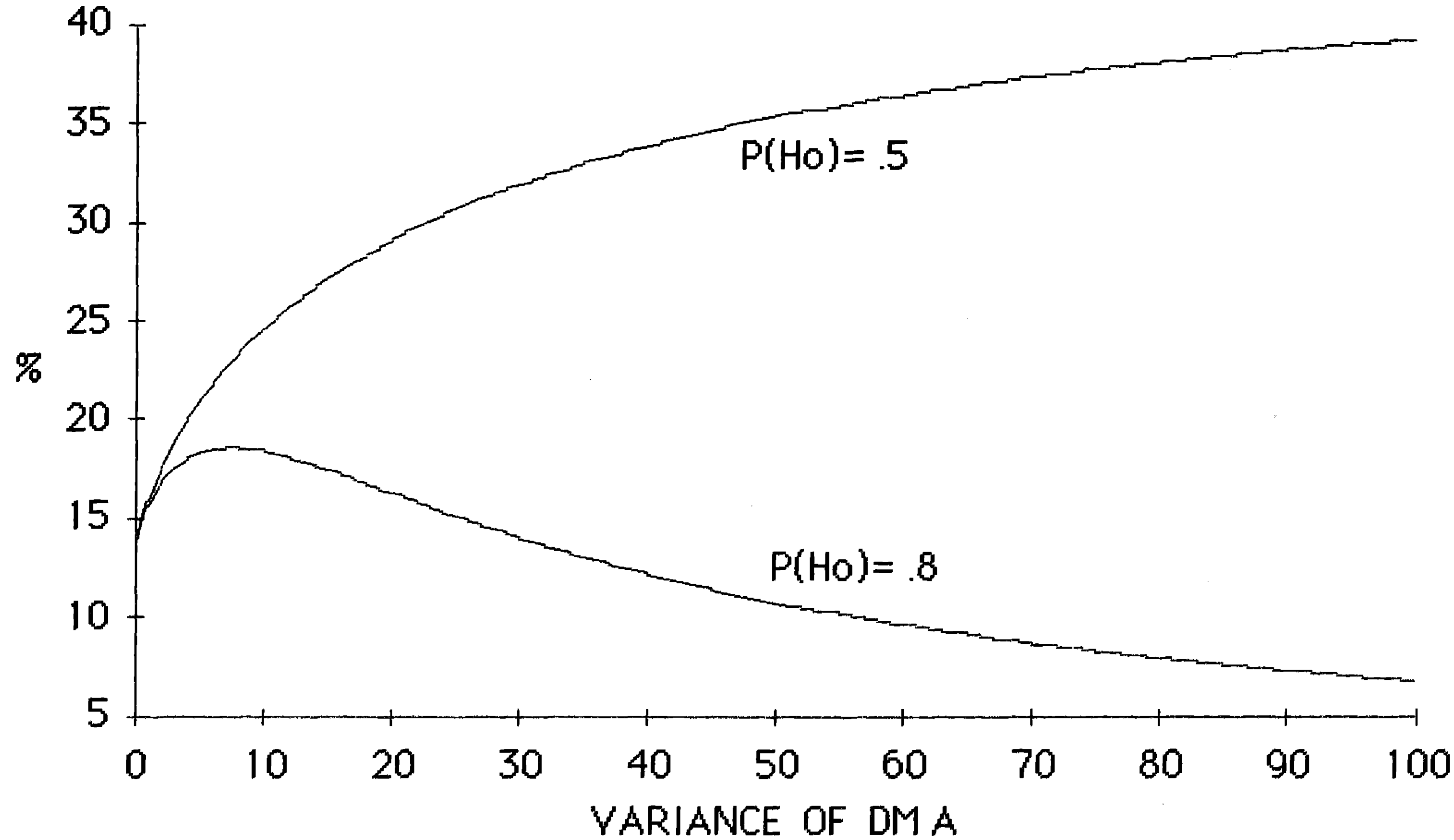
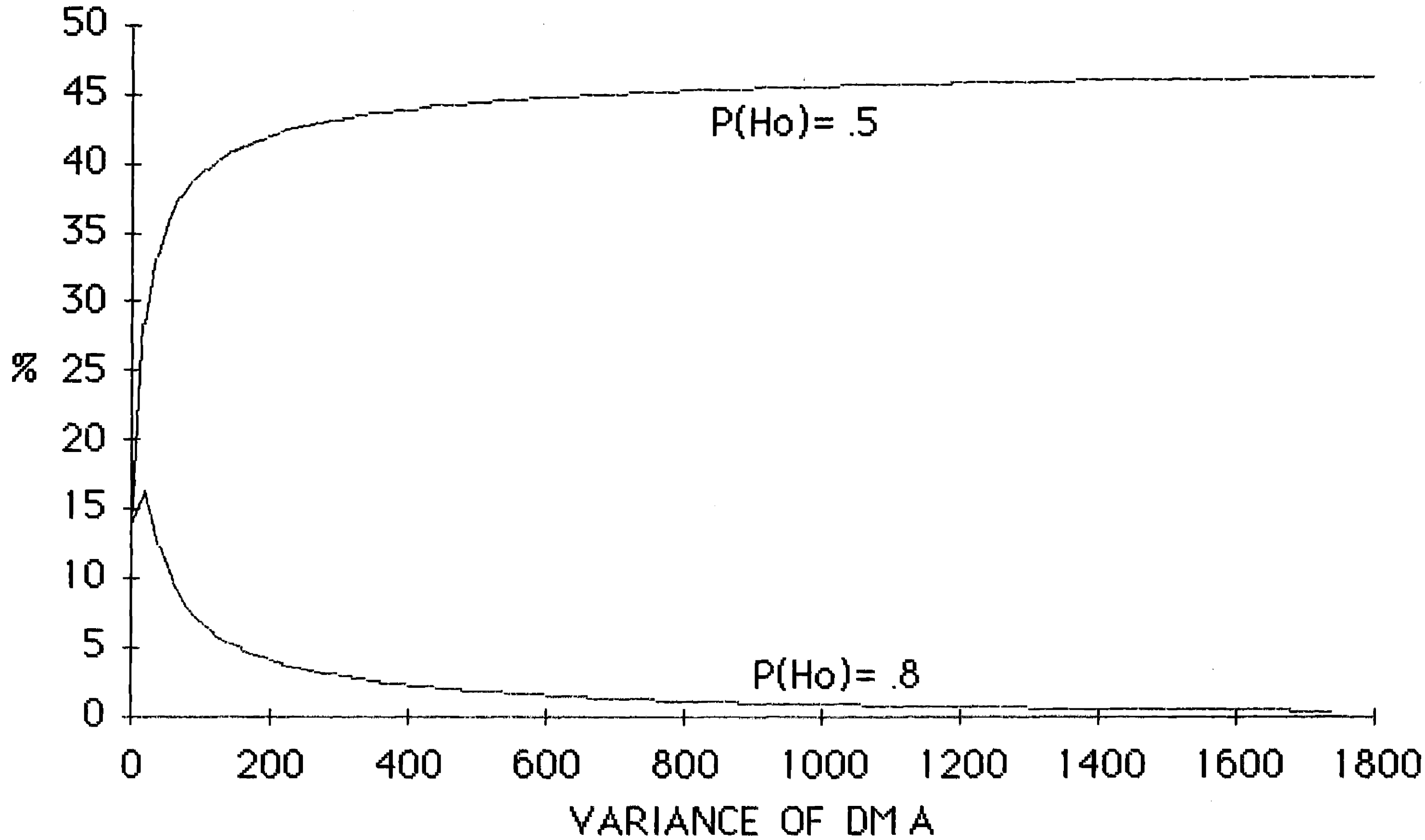


FIGURE 19. PERCENTAGE COST IMPROVEMENT



would employ, if the information option was not available (a well known case). Thus

$$\lim_{\sigma_0^2 \to \infty} P(u_\alpha = I) \to 0$$

Moreover, the percentage gain in cost achieved by the team of DMs, relative to the cost incured by a single DM obtaining a single observation, assymptotically goes to 0, as the variance of the primary DM goes to infinity (Figures 18 and 19, $p_0 = .8$).

An interesting insight can be obtained from Figures 18 and 19 ($p_0 = .8$). As the variance of the primary DM increases the percentage improvement in cost (defined above) is <u>initially increasing</u> and then decreasing assymptotically to zero. The reason for this is that for very small variances, the observations of the primary DM are so good that he does not need the information of the consulting DM. As the variance increases, the primary DM makes better use of the information and so the percentage improvement increases. But, at a certain point as the quality of his observations worsens, the primary DM finds it less costly to start declaring more often the <u>more</u> likely hypothesis (i.e. to bias its decision towards the more likely hypothesis) than requesting for information, for reasons mentioned above, and so the percentage improvement from then on decreases.

<u>CASE 2:</u> $\min(p_0, 1-p_0) > C1^* + C$

With reasoning similar to the above, we obtain that (see Figures 8 and 9,

$p_0 = .5)$:

$$\lim_{\sigma_0^2 \to \infty} P(u_\alpha = I) \to 1$$

Moreover, the percentage improvement is strictly increasing (and keeps increasing to a precomptutable limit ; see Figures 18 and 19, $p_0 = .5$). This reinforces the last point we made in Case 1 above. Since in the present case it is always less costly for the primary DM to request and use the information than to bias its decision towards the more likely hypothesis, the percentage improvement curve does not exhibit the non-monotonic behavior observed in Case 1 above (where $p_0 = .8$).

As can be seen from the Figures 4 to 19, in the case where $p_0 = .5$ the model exhibits symmetry. This is why in Figures 12 to 15 the probabilities of the final decision of the Primary DM and of the final decision of the team are straight lines at 0.5 .

In Figures 16 and 17, the team cost along with an upper and lower bound are shown. The upper bound is obtained (COST WITHOUT COMMUNICATION) from the simple and well known case, where the primary DM is entitled to his own observation only and decides according to the maximum likelihood criterion. The lower bound (COST WITH DATA FUSION) is obtained by assuming that the primary DM is entitled to both obsevations, his own along with the consulting DM's, at no extra cost. The primary DM decides by considering a weighted sum of the observations. The weight of each observation is equal to the variance of the <u>other</u> observation divided by the sum of the variances.

5.2 Effects of varying the quality of the observations of the Consulting DM

In Figures 20 to 27, the effects of varying the variance of the observations of the secondary DM, on all the relevant variables are presented. The baseline parameters of Table 1 are used.

As the variance of the consulting DM's observations increases, less information is requested by the primary DM, that is the primary DM's upper and lower thresholds move closer to each other (Figure 20). This is something we expected, since information of lesser quality is less profitable (more costly) to the team of DMs.

We should note here that the thresholds of a DM is an alternative way of representing the probabilities of the DM's decisions, since the decision regions are characterized by the thresholds. For example :

$$P(u_\alpha = I) = \sum_H \int_{y_\alpha :\, Y_\alpha^l < y_\alpha < Y_\alpha^u} P(y_\alpha \mid H)\, P(H)$$

The thresholds of the consulting DM demonstrate some interesting aspects of the team behavior (see Figures 21 and 23). For small values of the

FIGURE 20. DECISION THRESHOLDS OF DM A AS A FUNCTION OF σ_β^2

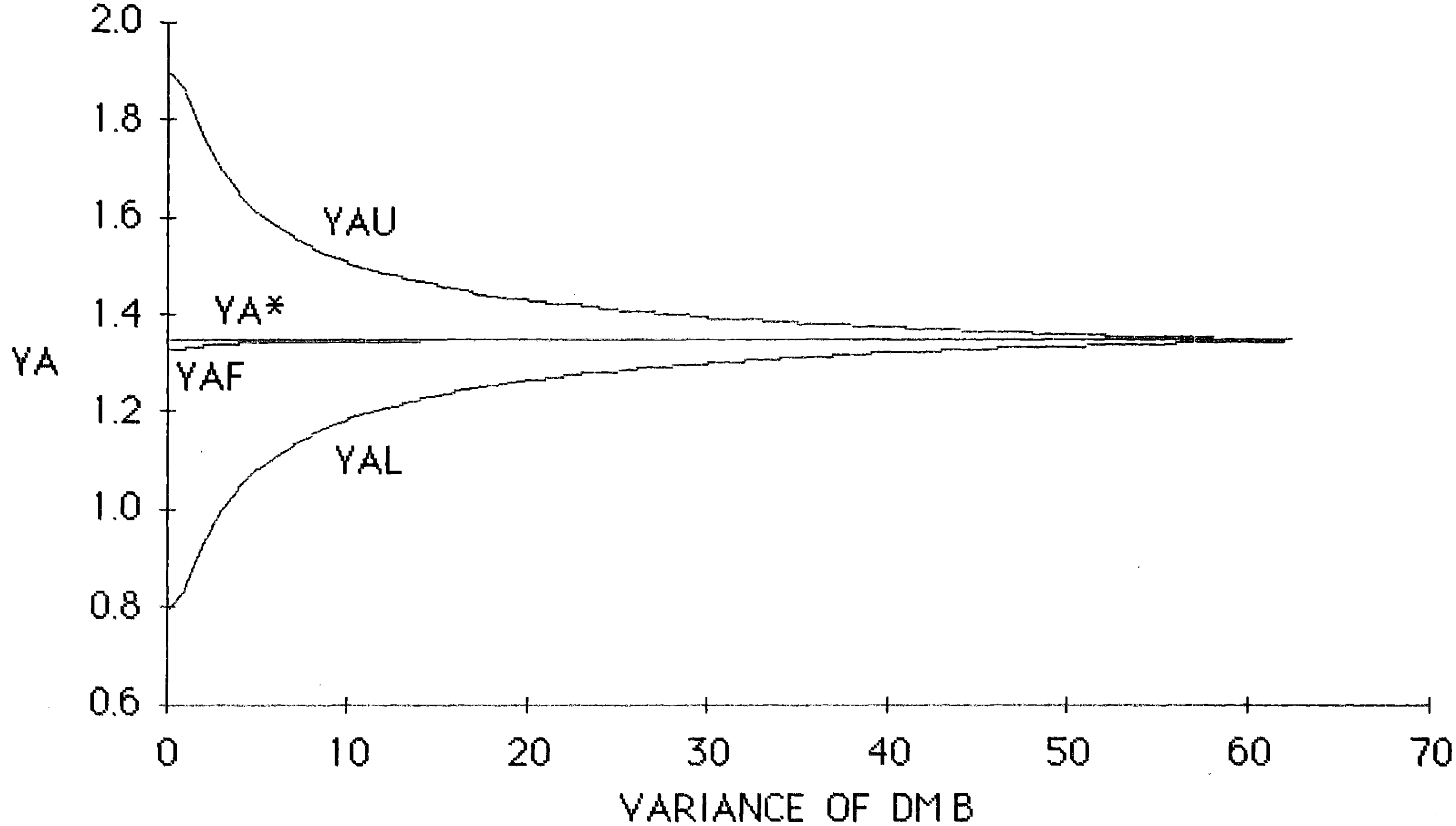
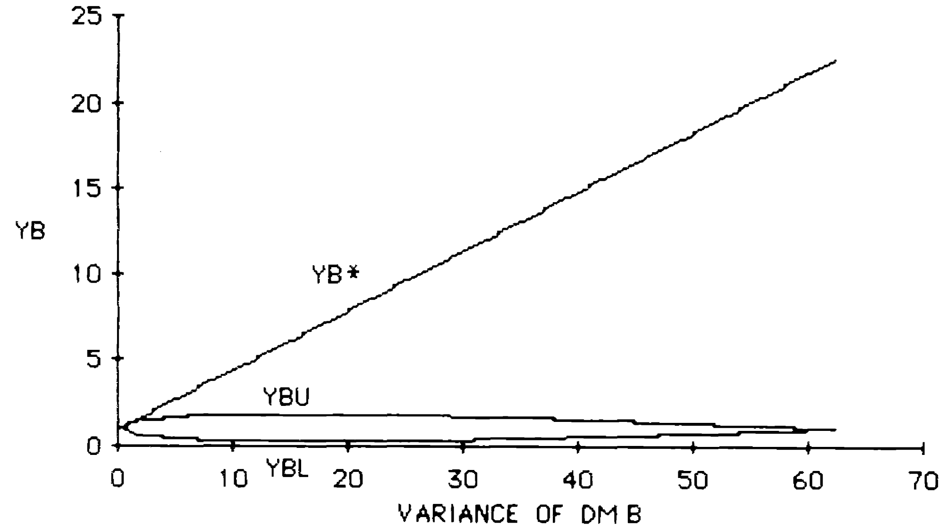


FIGURE 21

DECISION THRESHOLDS OF DM B
AS A FUNCTION OF σ_β^2



DETAIL OF ABOVE CURVES

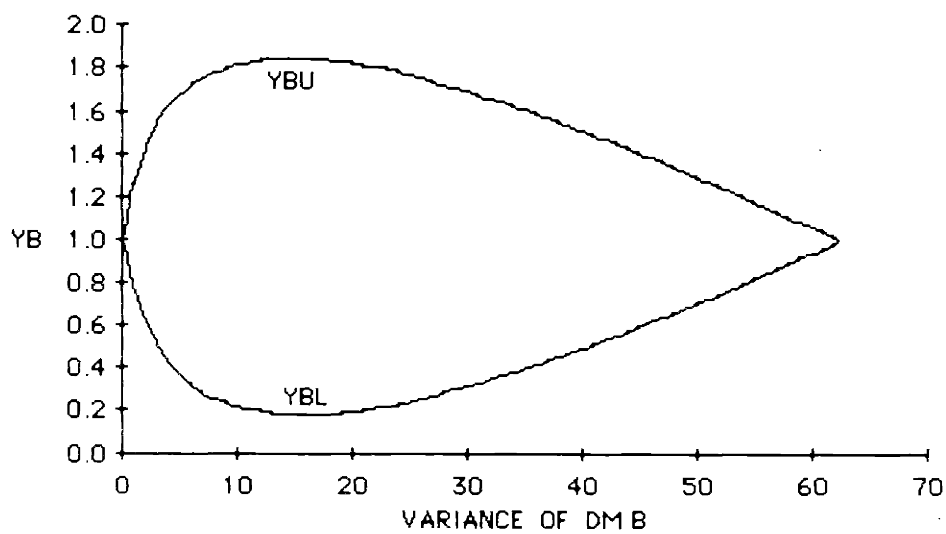


FIGURE 22. PROBABILITIES ASSOCIATED WITH THE PRELIMINARY DECISION OF DM A AS A FUNCTION OF σ_β^2

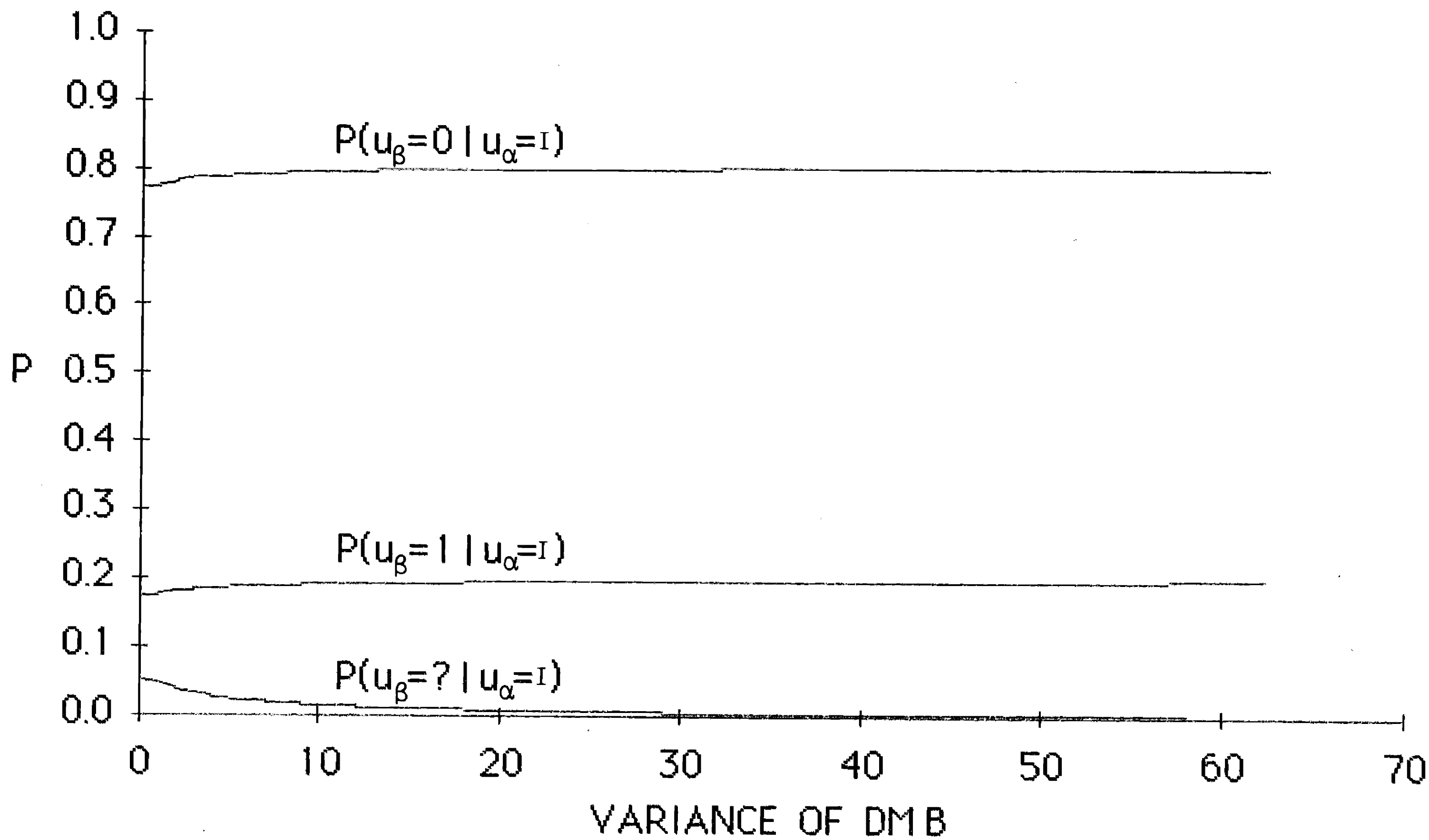


FIGURE 23. PROBABILITIES ASSOCIATED WITH THE DECISION OF DM B AS A FUNCTION OF σ_β^2

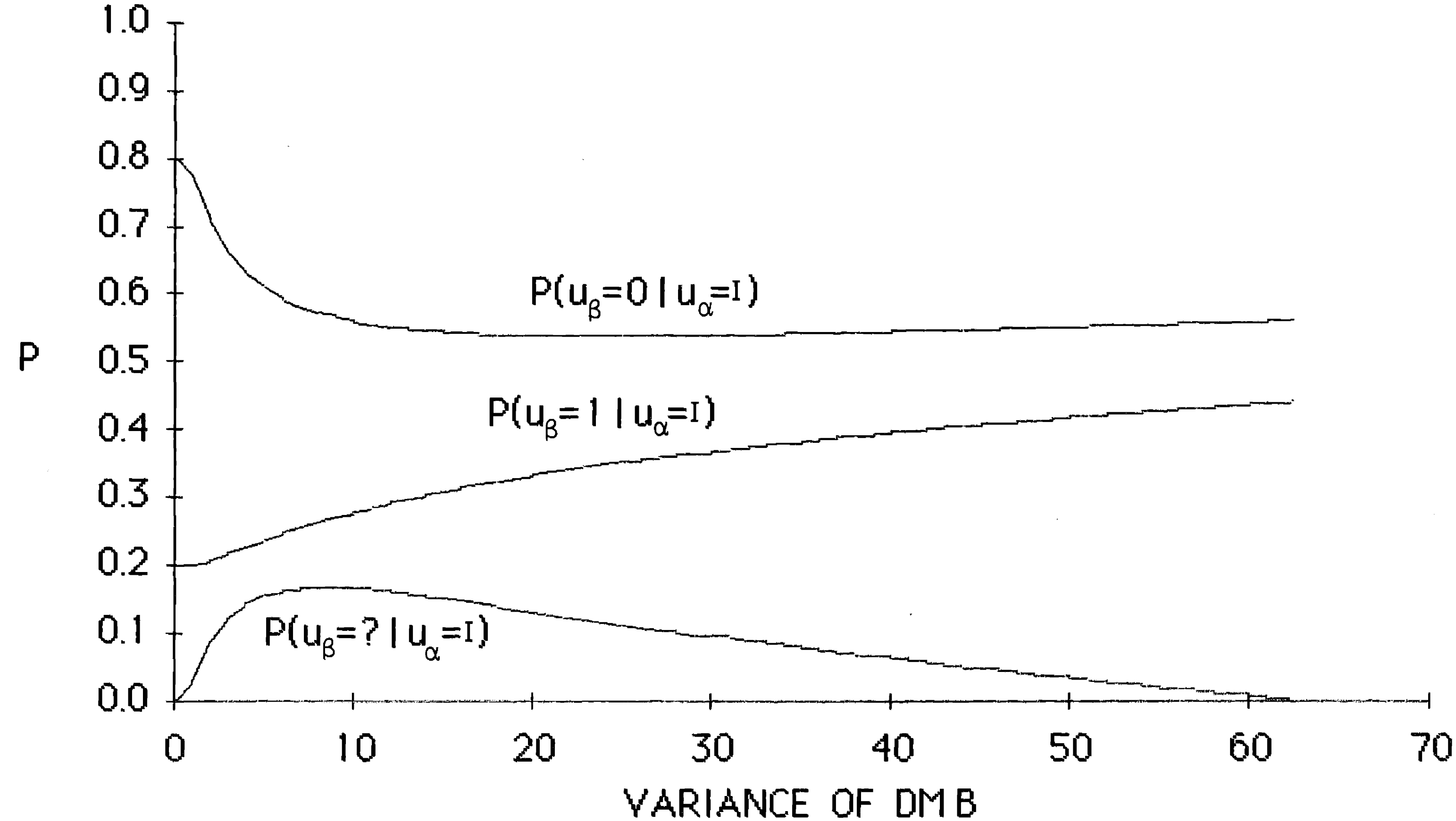


FIGURE 24. PROBABILITIES ASSOCIATED WITH THE FINAL DECISION OF DM A AS A FUNCTION OF σ_β^2

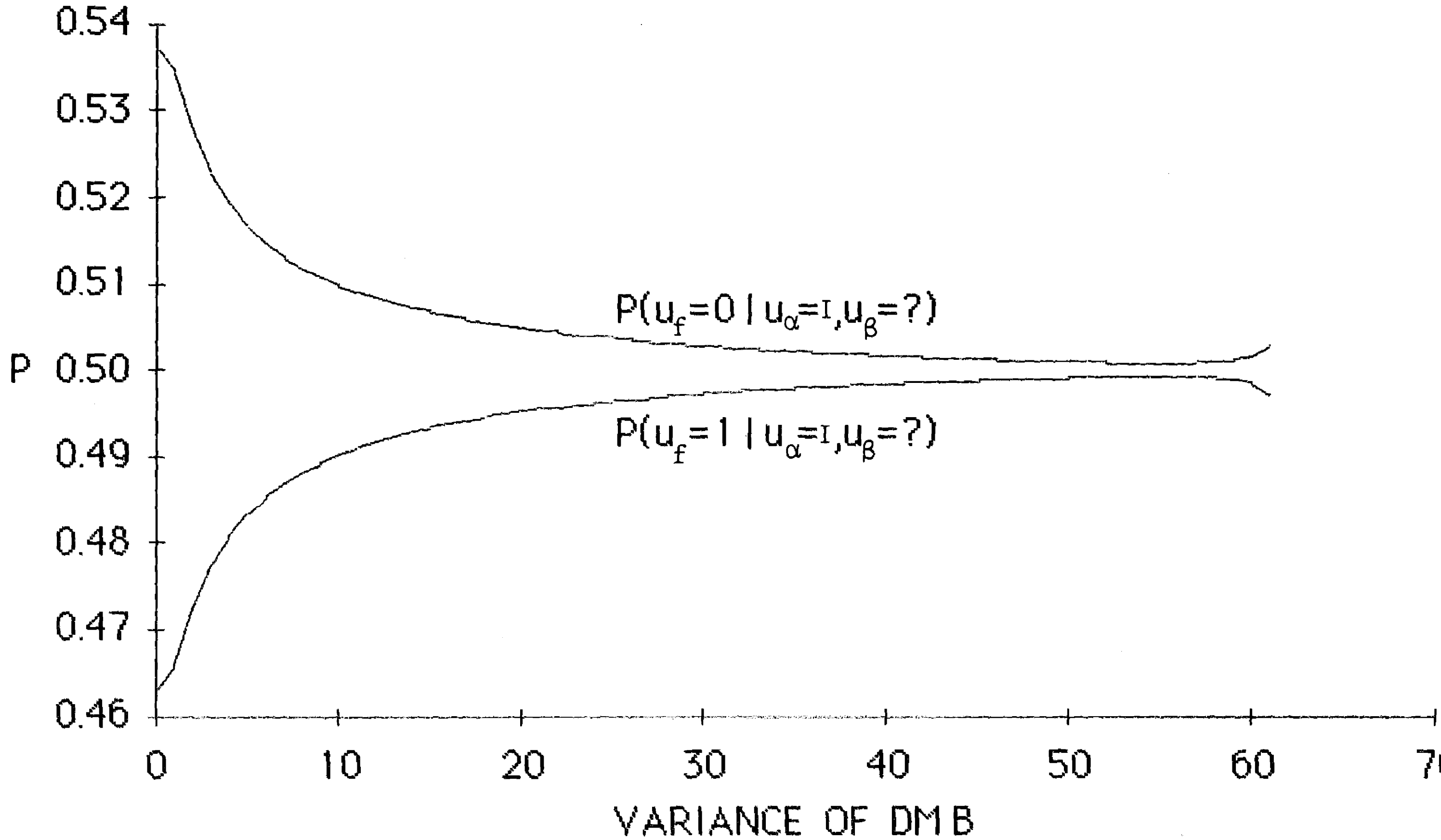


FIGURE 25. PROBABILITIES ASSOCIATED WITH THE FINAL TEAM DECISION AS A FUNCTION OF σ_{β}^2

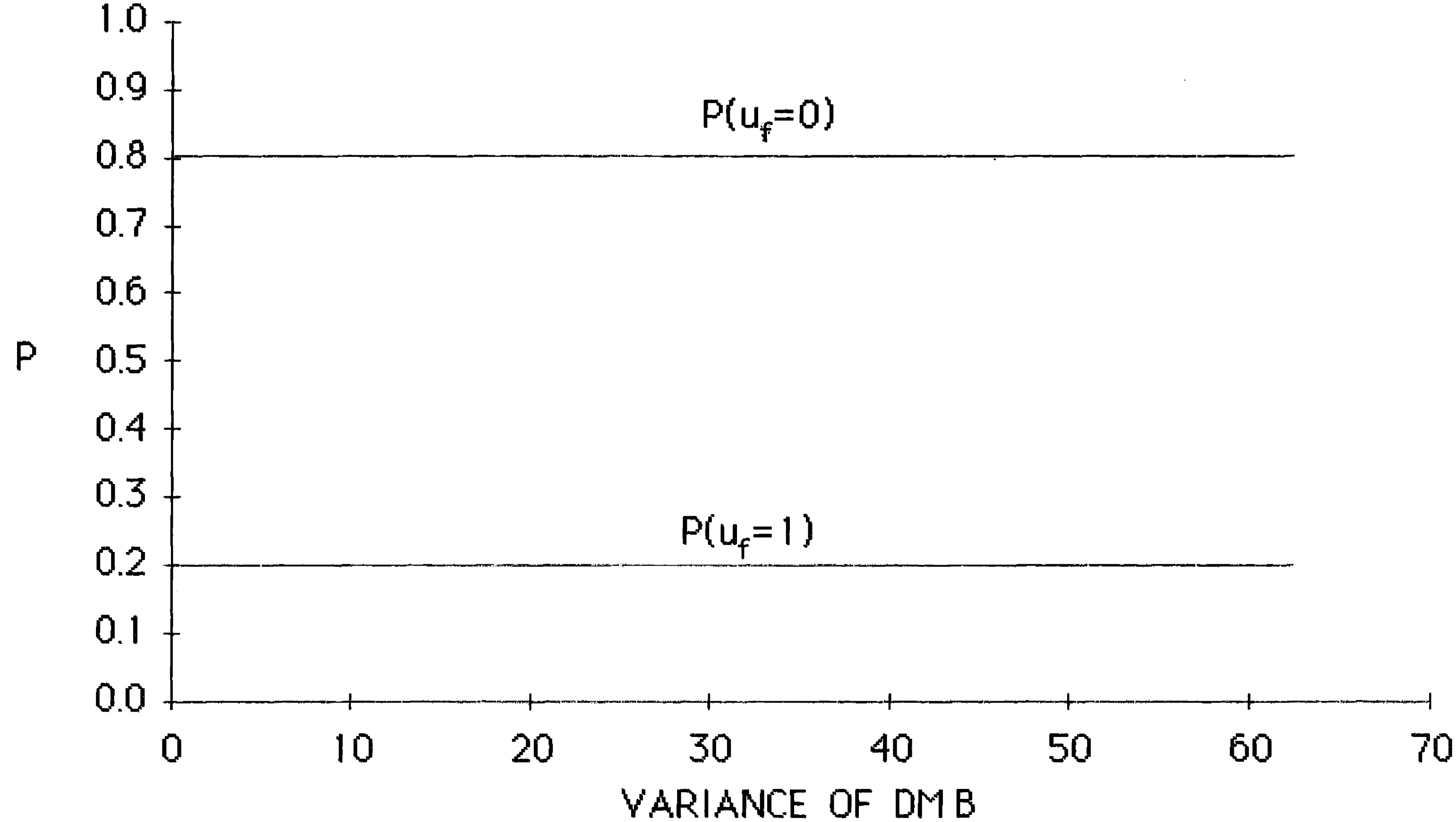


FIGURE 26. COSTS vs. σ_β^2

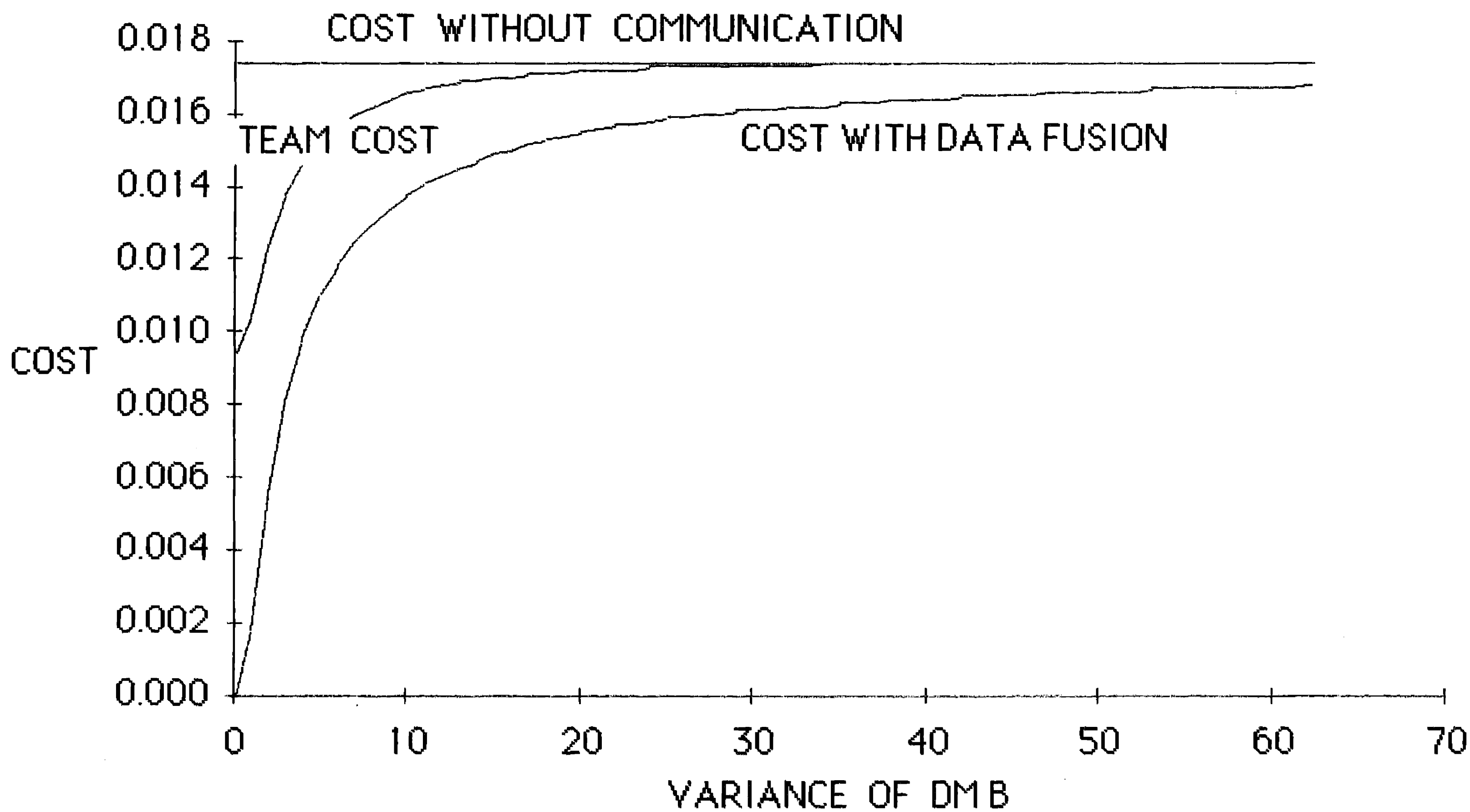
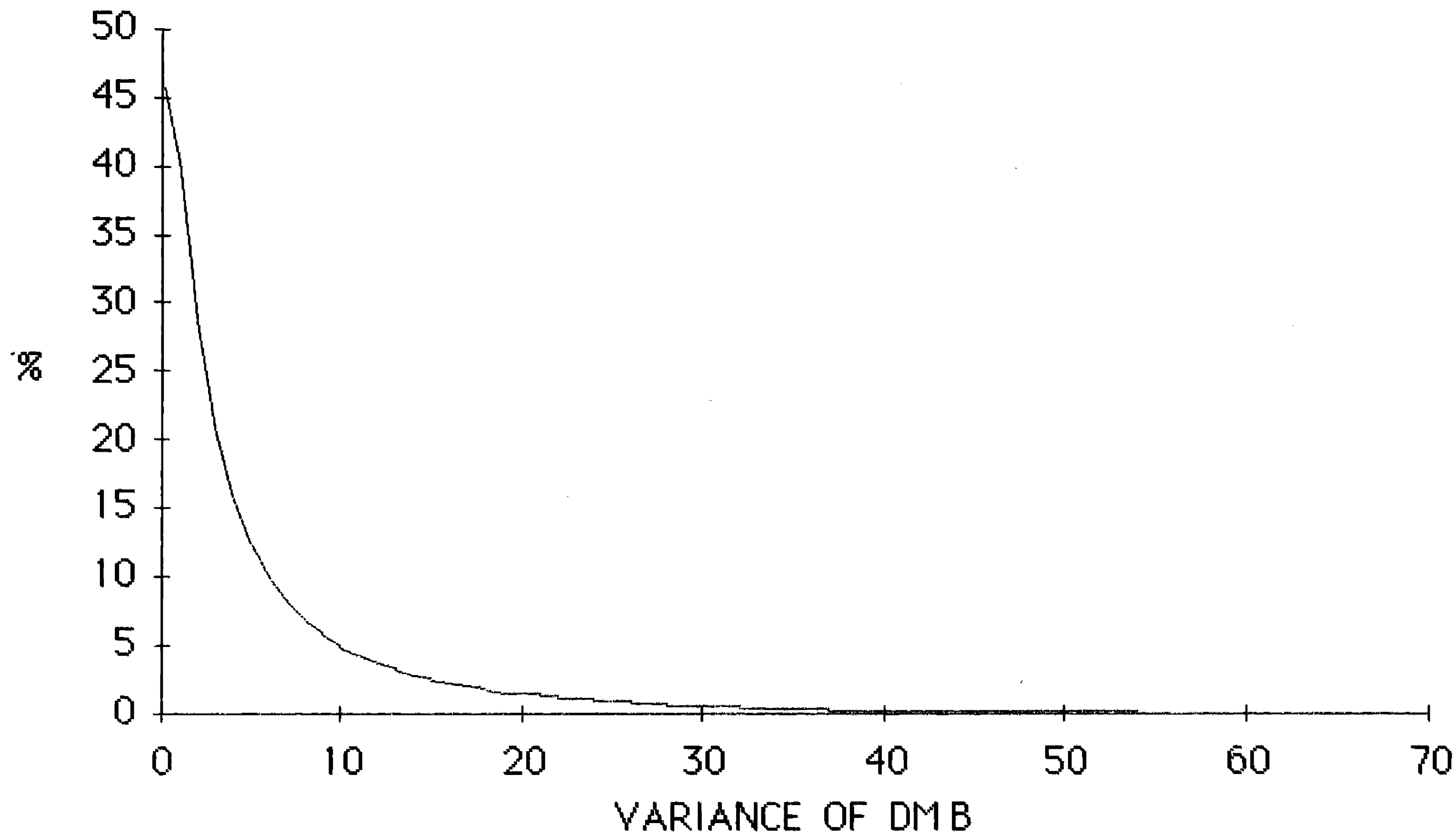


FIGURE 27. PERCENTAGE COST IMPROVEMENT vs. σ_β^2



variance σ_β^2 they are very close together, as the quality of the observations is very good and so the consulting DM is willing to make the final team decision. As his variance increases, DM B becomes more willing to return u_β=? (i.e. "I am not sure") and let DM A make the final team decision. As the variance continues to increase, the thresholds of the consulting DM converge again. This might seem counter-intuitive, but there is a simple explanation. The consulting DM recognizes that the primary DM, despite knowing that the quality of the consulting DM's information is bad, is willing to incur the communication cost to obtain the information. This indicates that the primary DM is 'confused', that is, the a posteriori probabilities of the two hypotheses (given DM A's observation) are very close together. Hence, the consulting DM becomes more willing to make the final decision. After a certain point ($\sigma_\beta^2 \approx 62.4$) the primary DM does not find it worthwhile to request information from the consulting DM.

Remark : Note (see Figure 21) that the thresholds of the consulting DM converge to 1 which is the maximum likelihood threshold if the a priori probabilities of the two hypotheses were equal. But, the a priori probabilities *which the consulting DM uses in its calculations* are functions of the given a priori probabilities (i.e. p_i) and the fact that the primary DM has requested information (i.e. $P(u_\alpha = \mathrm{I} \mid H_i)$). In fact, the consulting DM uses as his a priori probabilities its own estimates of the primary DM's a posteriori probabilities. That is :

$$P(H_0 \mid u_\alpha = I) = \frac{P(H_0)\, P(u_\alpha = I \mid H_0)}{\sum_H P(H)\, P(u_\alpha = I \mid H)} \qquad (65)$$

From the above, we deduce that for large variances ($\sigma_\beta^2 \approx 62$) the estimates, of the consulting DM, for the a posteriori probabilities of the primary DM are very close to .5, reinforcing the point we made about the primary DM "being confused."

From Figures 26 and 27 it is clear that, as the variance of the consulting DM increases, the team percentage gain in cost decreases to 0, since the primary DM eventually makes all the decisions alone as in the centralized case.

Finally in Figure 25, the probabilities associated with the final team decision are presented. The probability of the team deciding 0 (or 1) is almost equal to the a priori probability of H_0 (or H_1 respectively). This is a more general observation: whenever the quality of the observations of the primary DM is relatively good (i.e. σ_α^2 small), then the final team decision is unbiased. This reinforces the point made with the two Cases in 5.1 above, where it was shown that as the quality of the primary DM's observations decreases, the primary DM biases his preliminary decision, and consequently the final team decision, towards the a priori more likely hypothesis.

5.3 Effects of varying the Communication Cost

In Figures 28 to 35, the effects of varying the communication cost C on all the relevant variables are presented. The baseline parameters of Table 1 are used.

Increasing the communication cost is very similar to increasing the variance of the consulting DM, since in both cases the team "gets less for its money" (because the team has to incur an increased cost, either in the form of an increased communication cost, or in the form of the final cost, because of the worse performance of the consulting DM).

The thresholds of the primary DM (see Figure 28), exhibit the same behavior as in 5.2 above (converging together at C≈.35). The thresholds of the consulting DM (see Figure 29) converge together for the same reasons as in 5.2 above; that is the consulting DM realizes that the primary DM is confused, since the primary DM is willing to incur the communication cost, despite the high variance of the consulting DM's observation, and so the consulting DM tries to make the commital decision himself. Of course, the thresholds do not start together for small values of the communication cost (as in 5.2), because low communication cost does not imply ability for the consulting DM to make accurate decisions. In fact, for small values of the communication cost, DM A is apt to request information more often than what is really needed and so the consulting DM returns more often u_{β}=? (i.e. "I am not sure") and lets DM A make the team final decision.

FIGURE 28. THRESHOLDS OF DM A vs. C

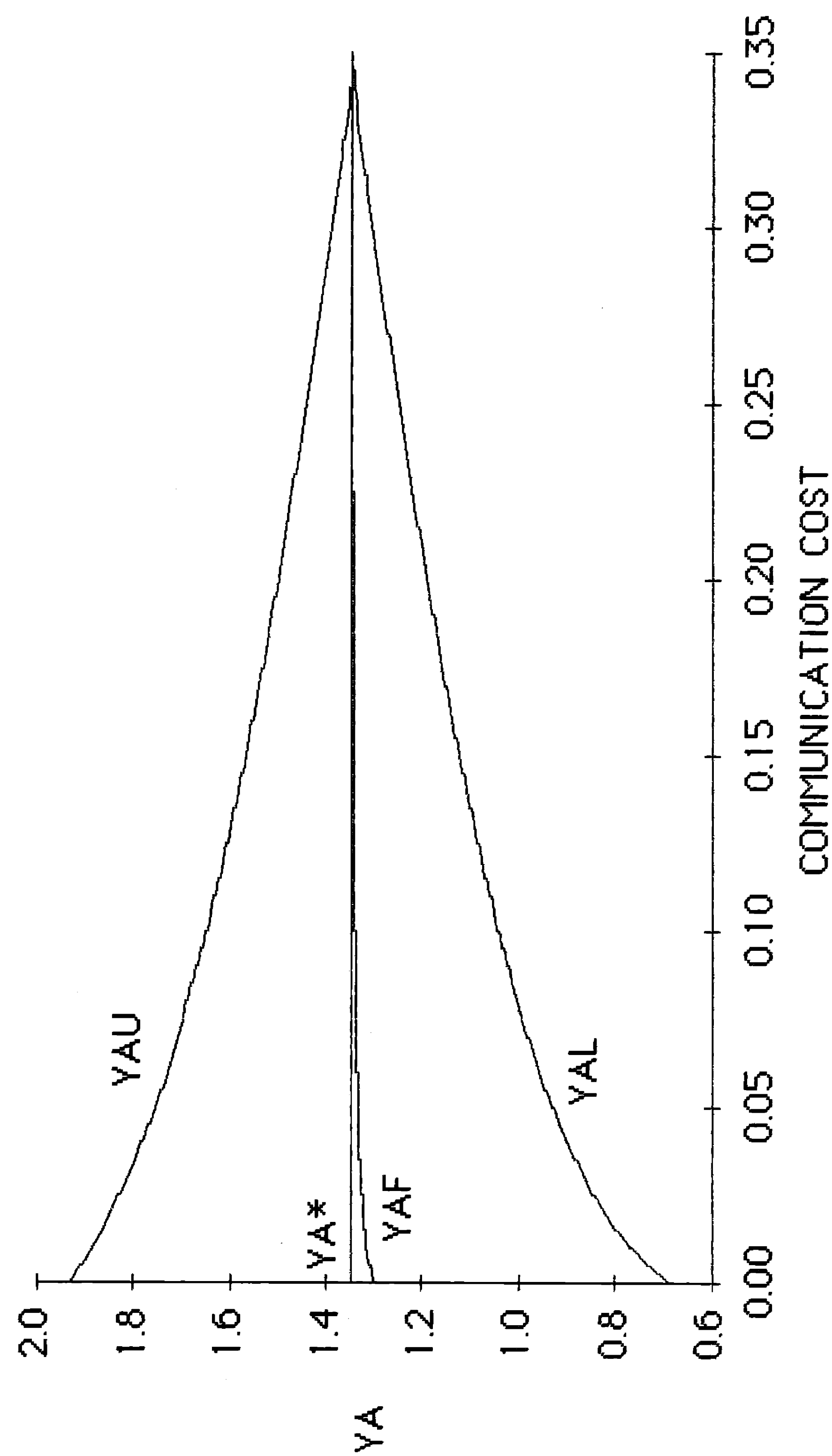


FIGURE 29. THRESHOLDS OF DM B vs. C

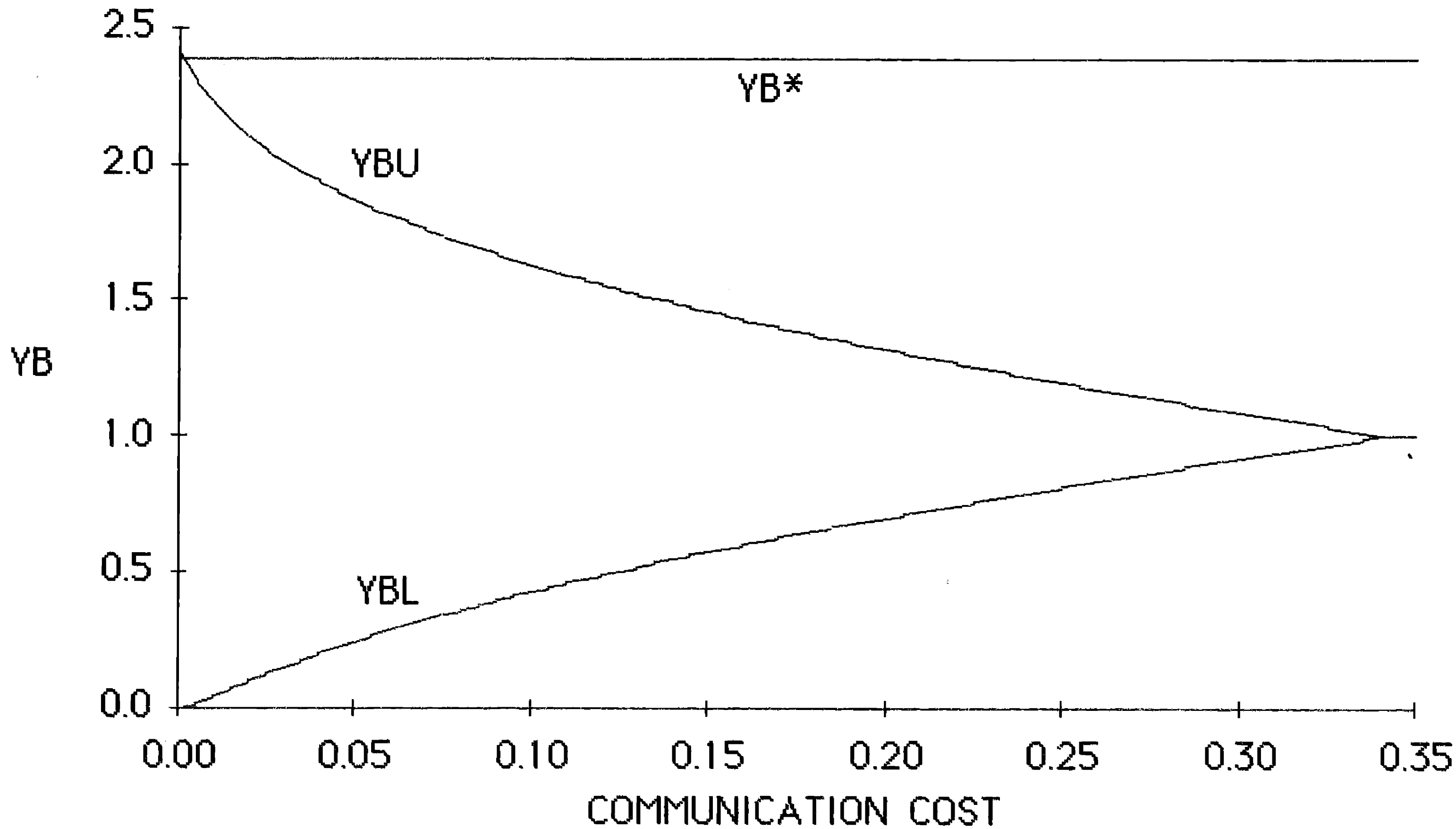


FIGURE 30. PROBABILITIES ASSOCIATED WITH THE PRELIMINARY DECISION OF DM A vs. C

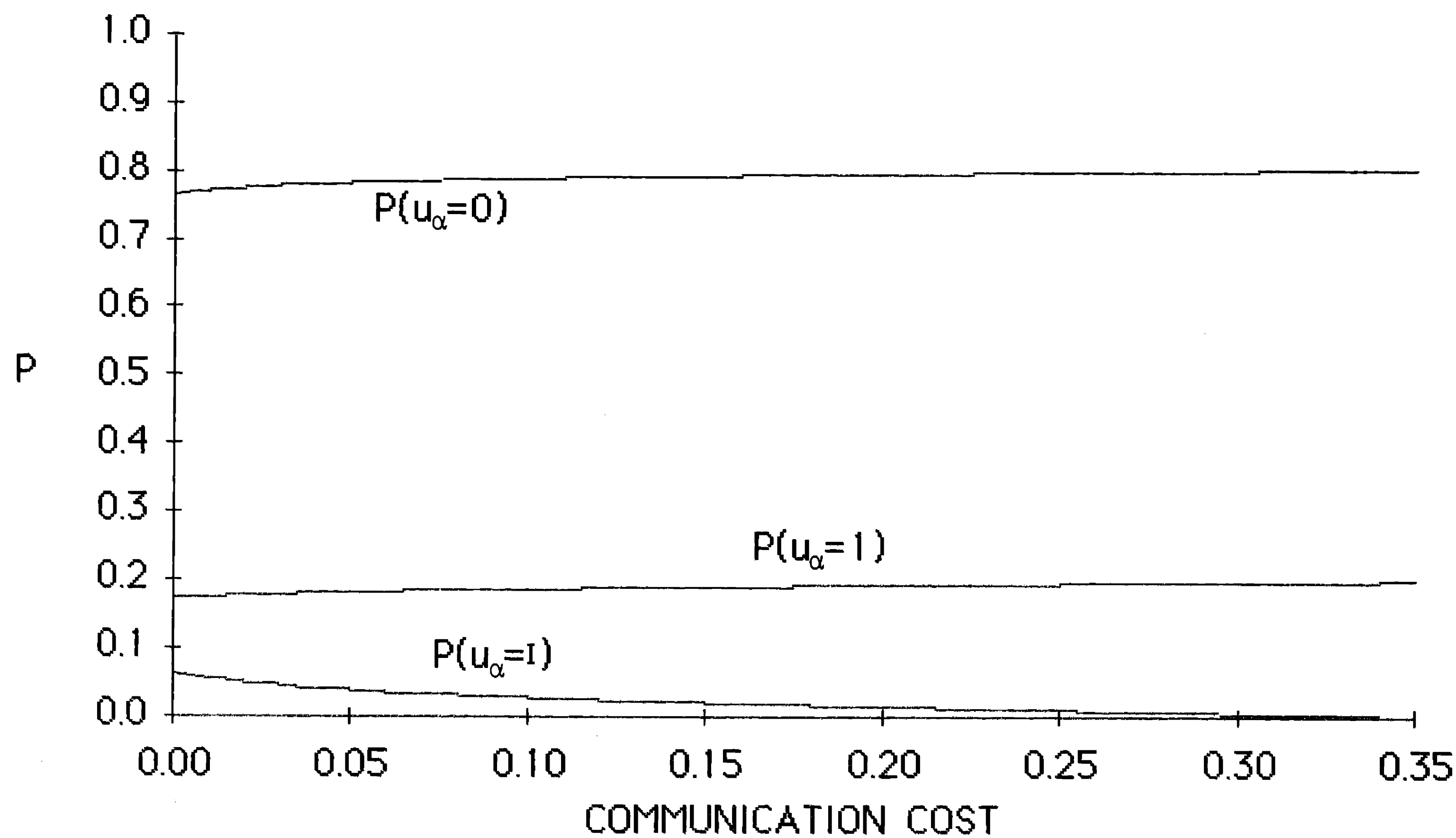


FIGURE 31. PROBABILITIES ASSOCIATED WITH THE DECISION OF DM B vs. C

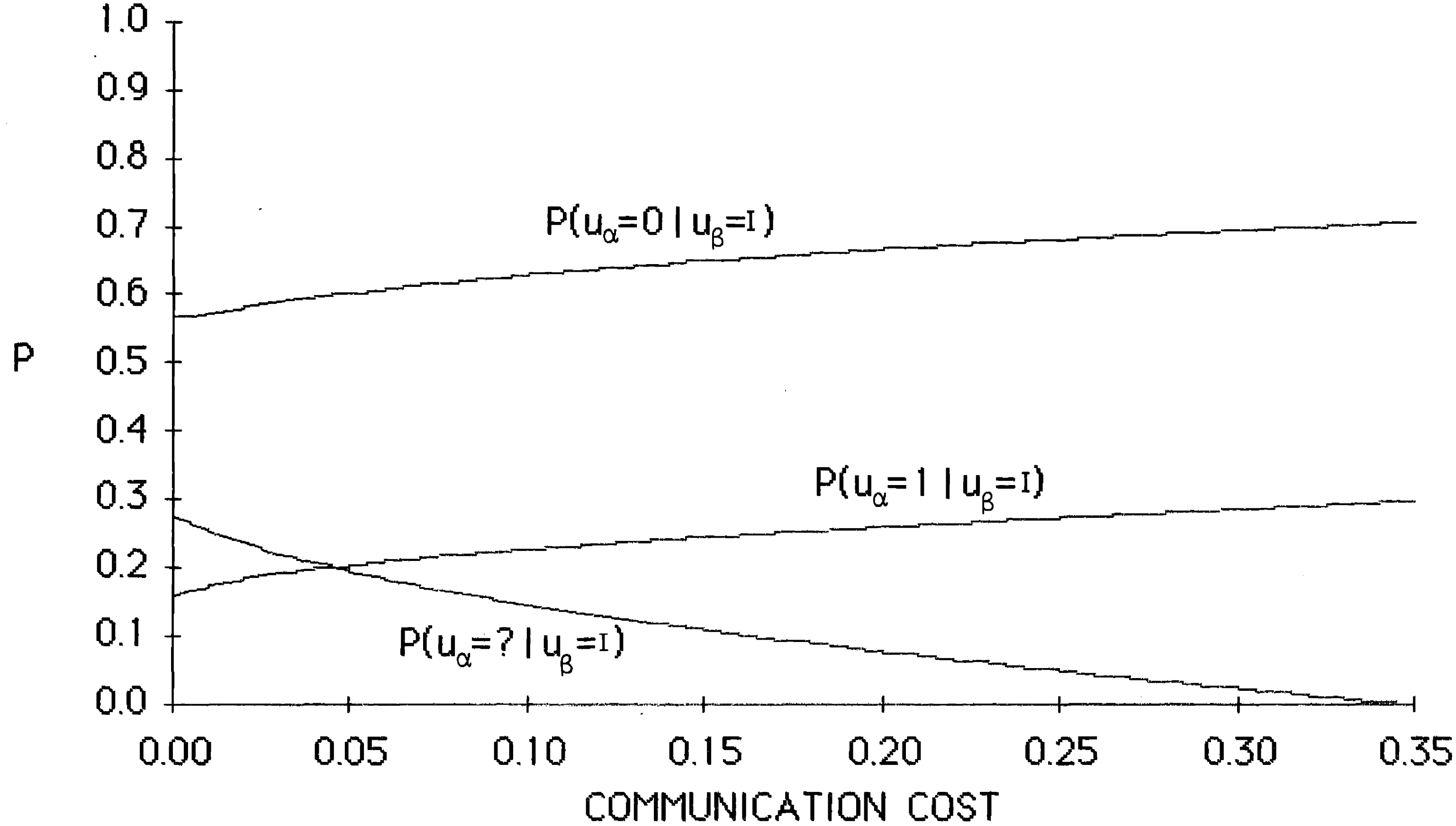


FIGURE 32. PROBABILITIES ASSOCIATED WITH THE FINAL DECISION OF DM A vs. C

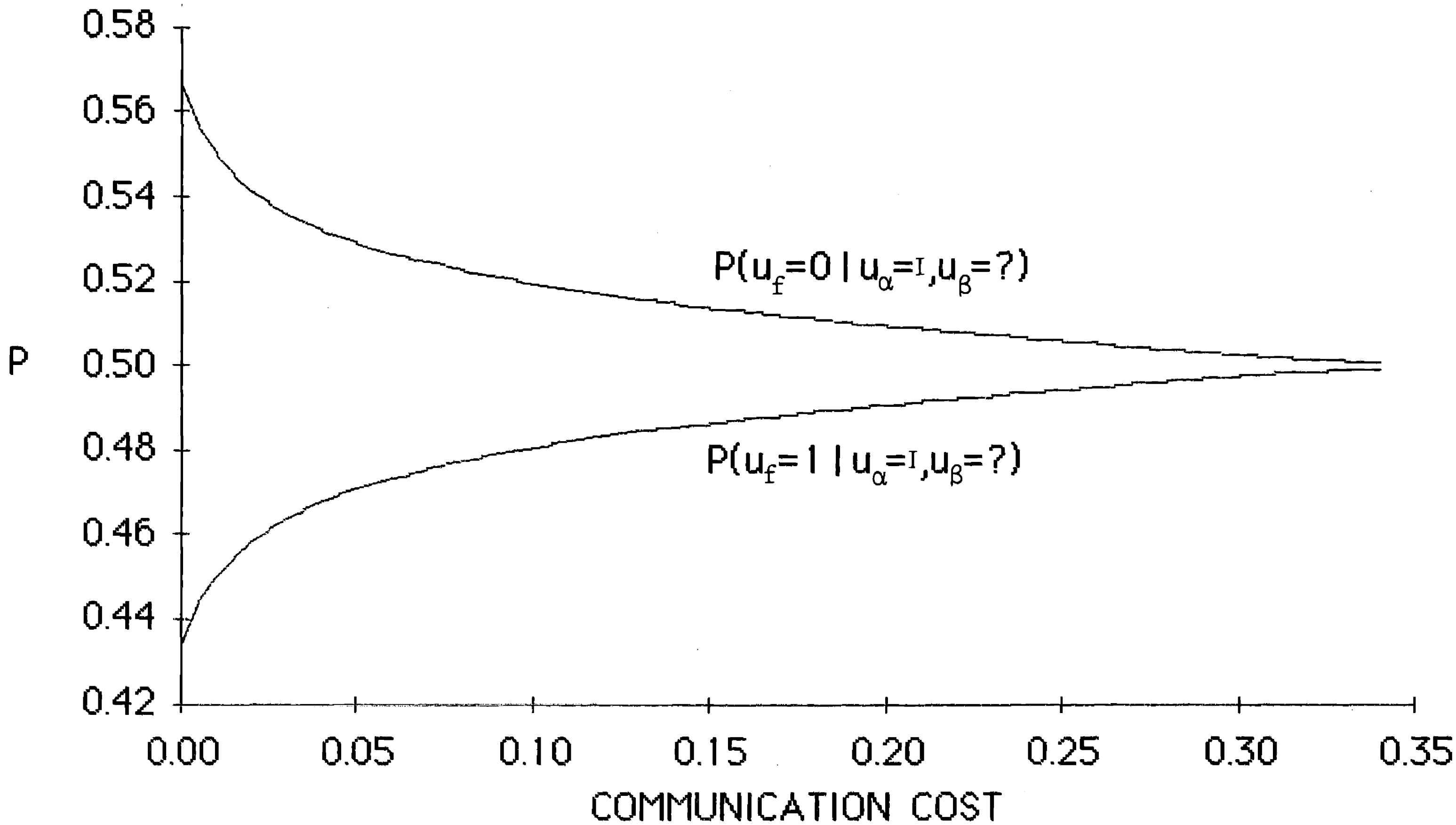


FIGURE 33. PROBABILITIES ASSOCIATED WITH THE FINAL TEAM vs. C

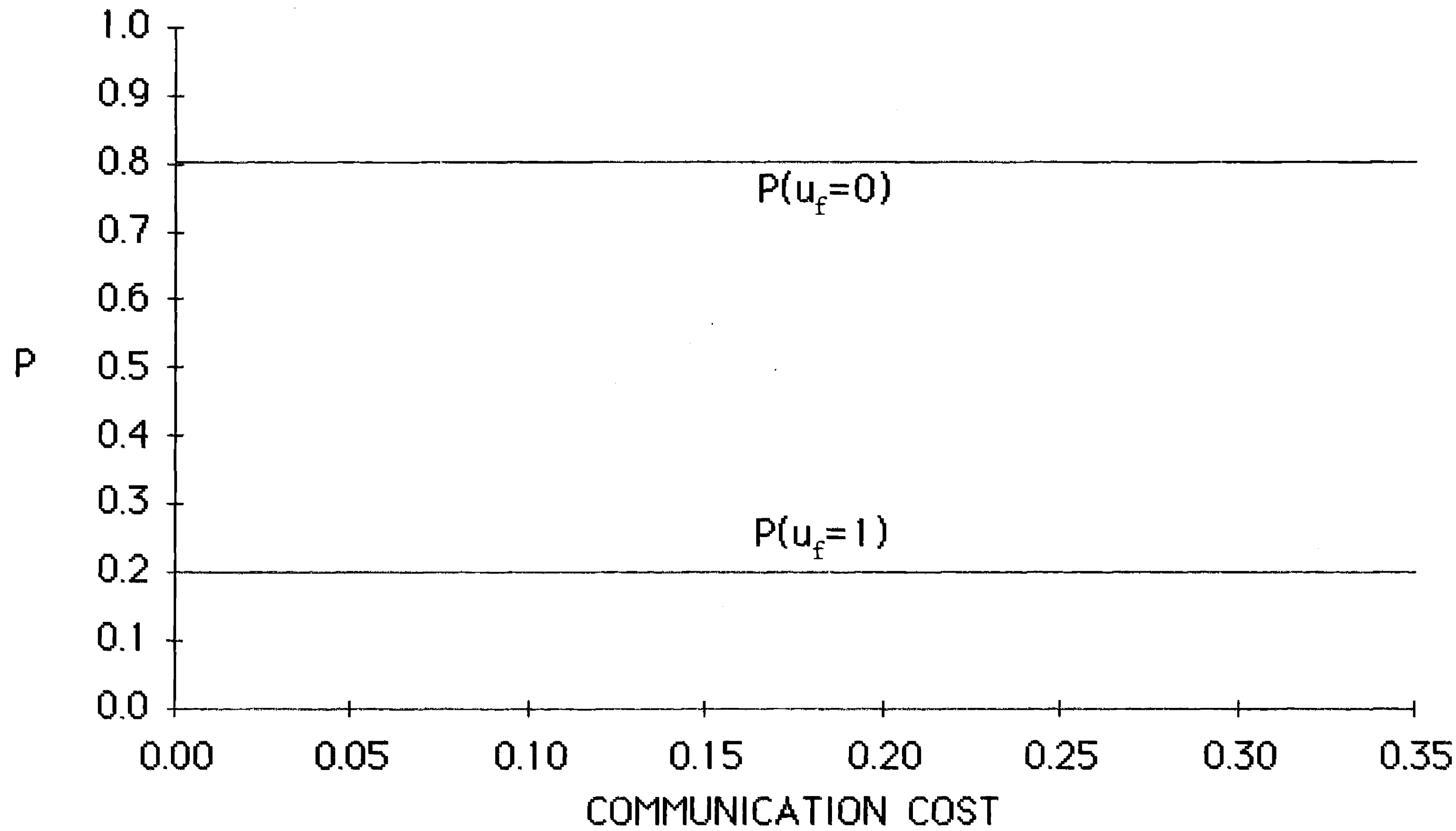


FIGURE 34. COSTS vs. C

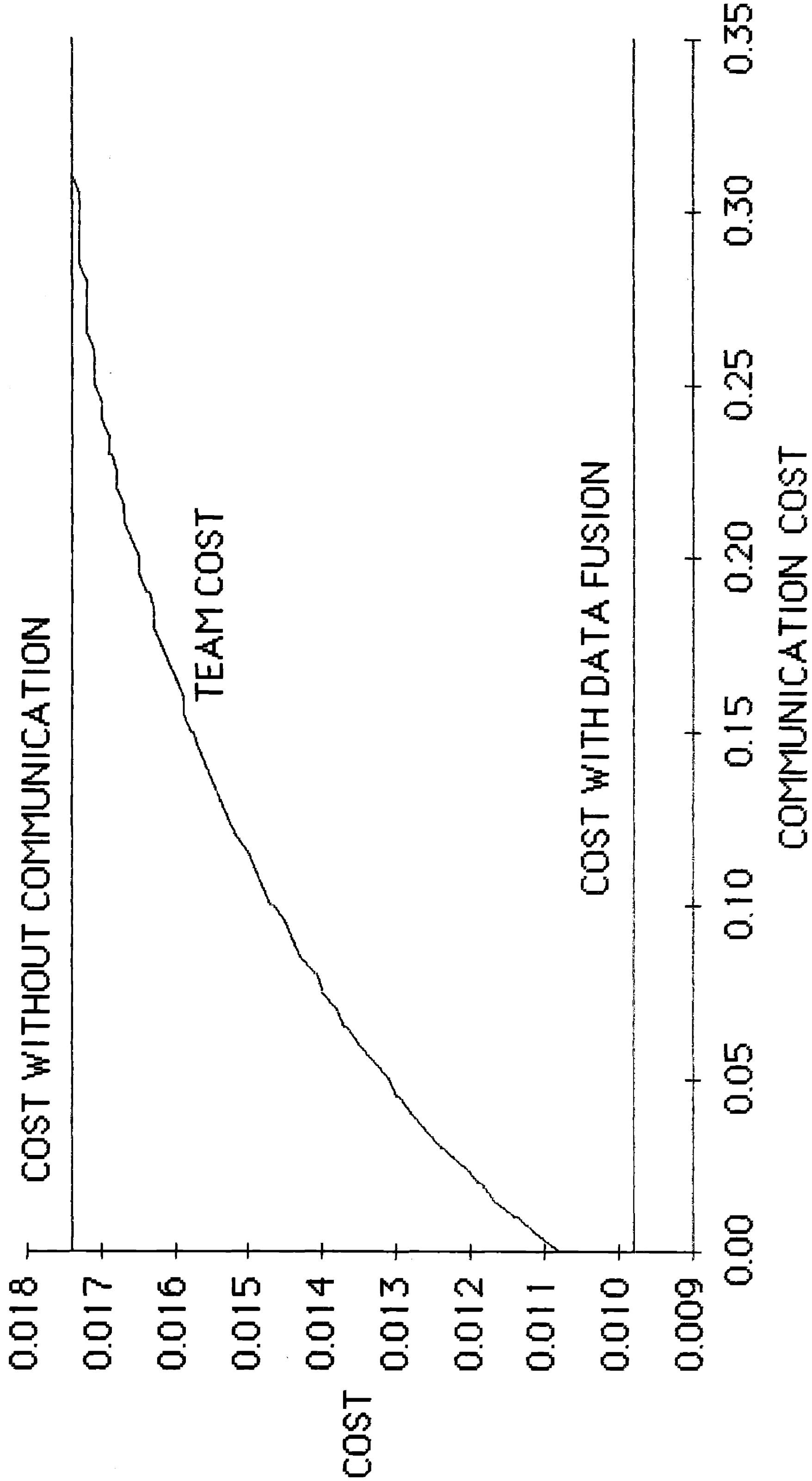
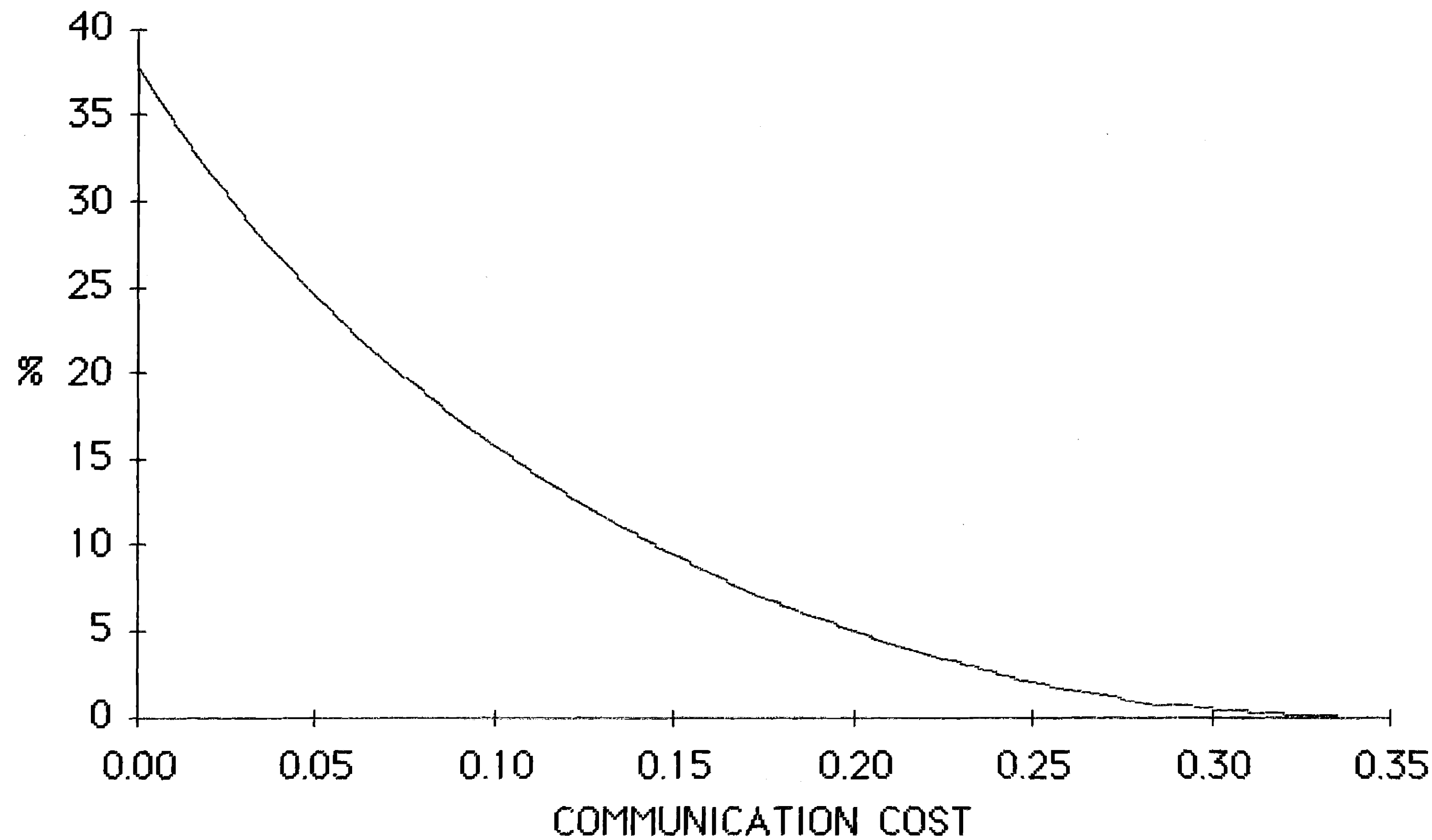


FIGURE 35. PERCENTAGE COST IMPROVEMENT vs. C



As can be seen from Figure 33, the probabilities associated with the final team decision are equal to the a priori probabilities of the hypotheses. That is the final team decision is unbiased. As previously discussed, this happens because the variance of the primary DM is small ($\sigma_\alpha^2=1$).

Again it is clear from Figures 34 and 35 that, as the communication cost increases, the percentage gain achieved by the team of the DMs decreases to zero as the communication becomes more costly, and less frequent, until we reach the centralized case solution.

5.4 Effects of varying the a priori probabilities of the hypotheses

In Figures 36 to 43, we see the effects of varying the a priori probability of H_0 on all the relevant variables. The baseline values of Table 1 are used, except that $\sigma_\alpha^2=8$.

This case does not present many interesting points. As expected, there is symmetry in the performance of the team about the value $p_0= 0.5$. The closer p_0 is to 0.5 information is requested more often by DM A (Figure 38) and the more often "I am not sure" is returned by DM B (Figure 39). This is understandable, because the closer p_0 is to 0.5, the bigger the prior uncertainty. Consequently, the percentage improvement achieved by the team of the DMs is monotonically increasing with p_0 from 0 to 0.5

FIGURE 36. DECISION THRESHOLDS OF DM A vs. P(Ho)

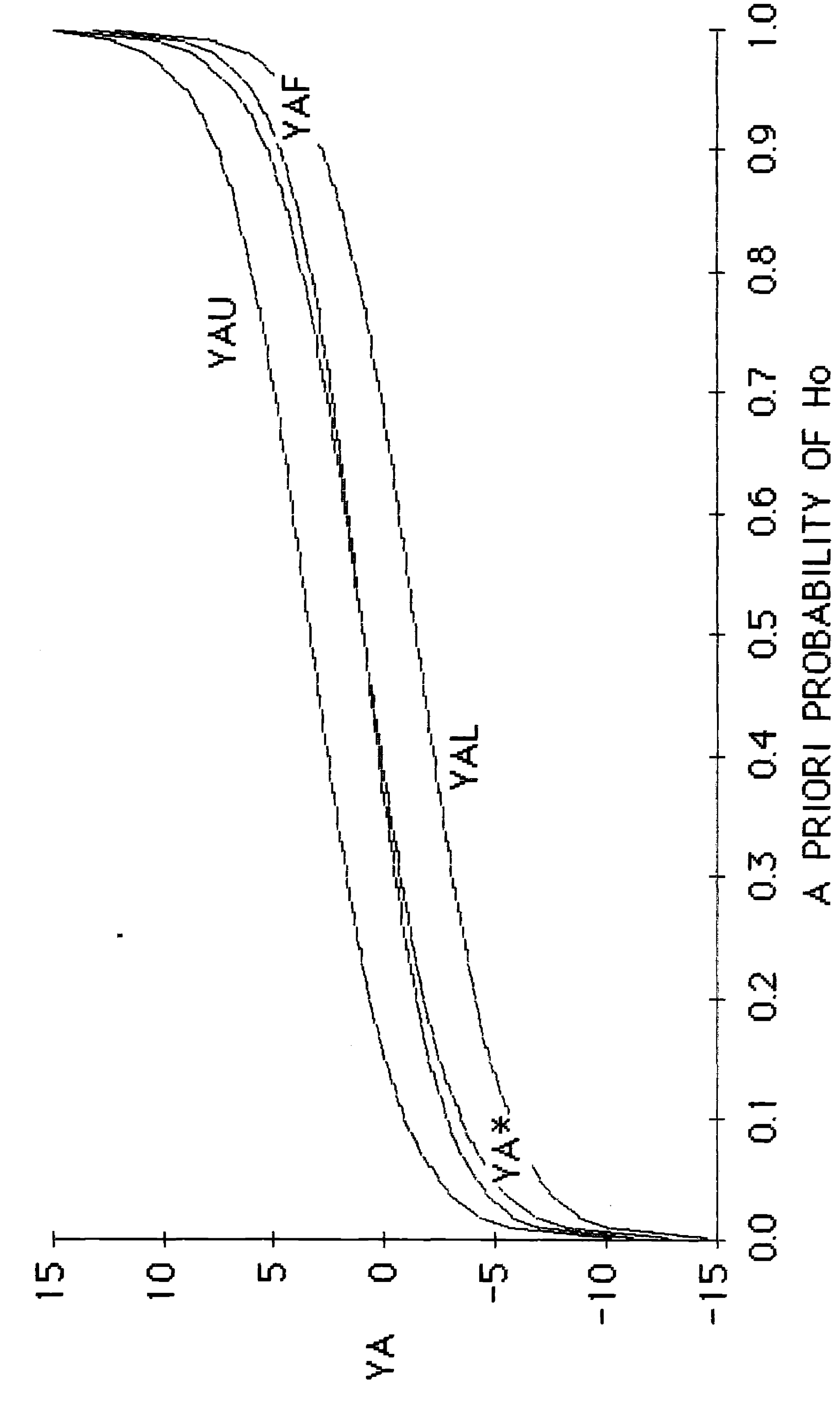
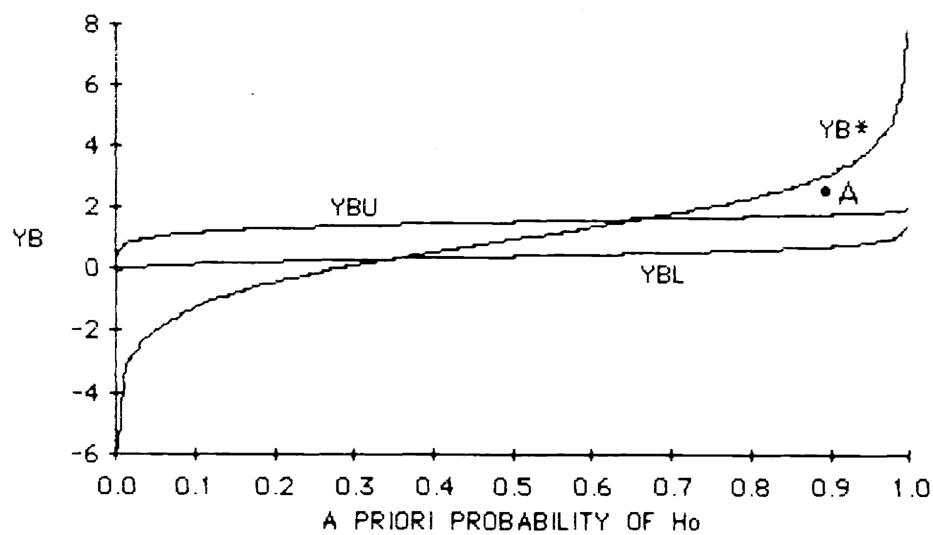


FIGURE 37

DECISION THRESHOLDS OF DM B vs. P(Ho)



DETAIL OF THE ABOVE CURVES

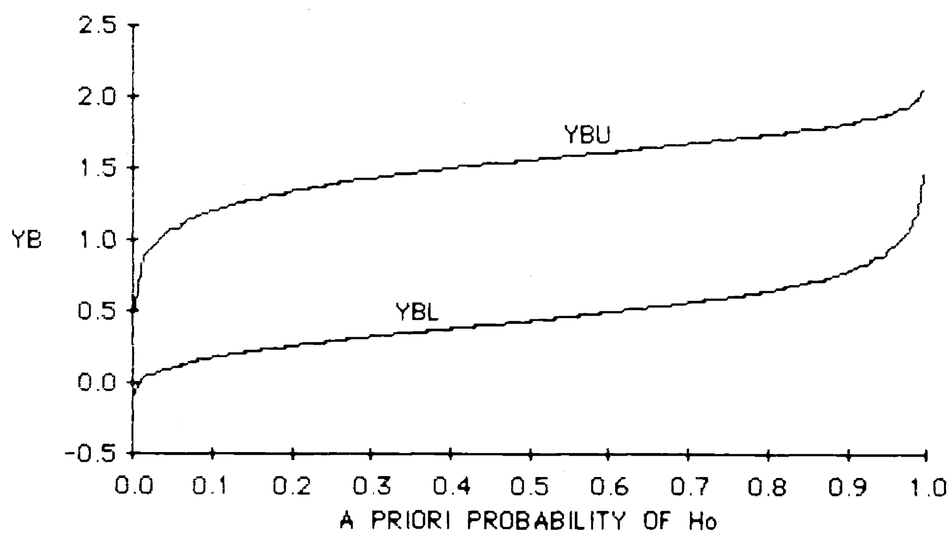


FIGURE 38. PROBABILITIES ASSOCIATED WITH THE PRELIMINARY DECISION OF DM A vs. P(Ho)

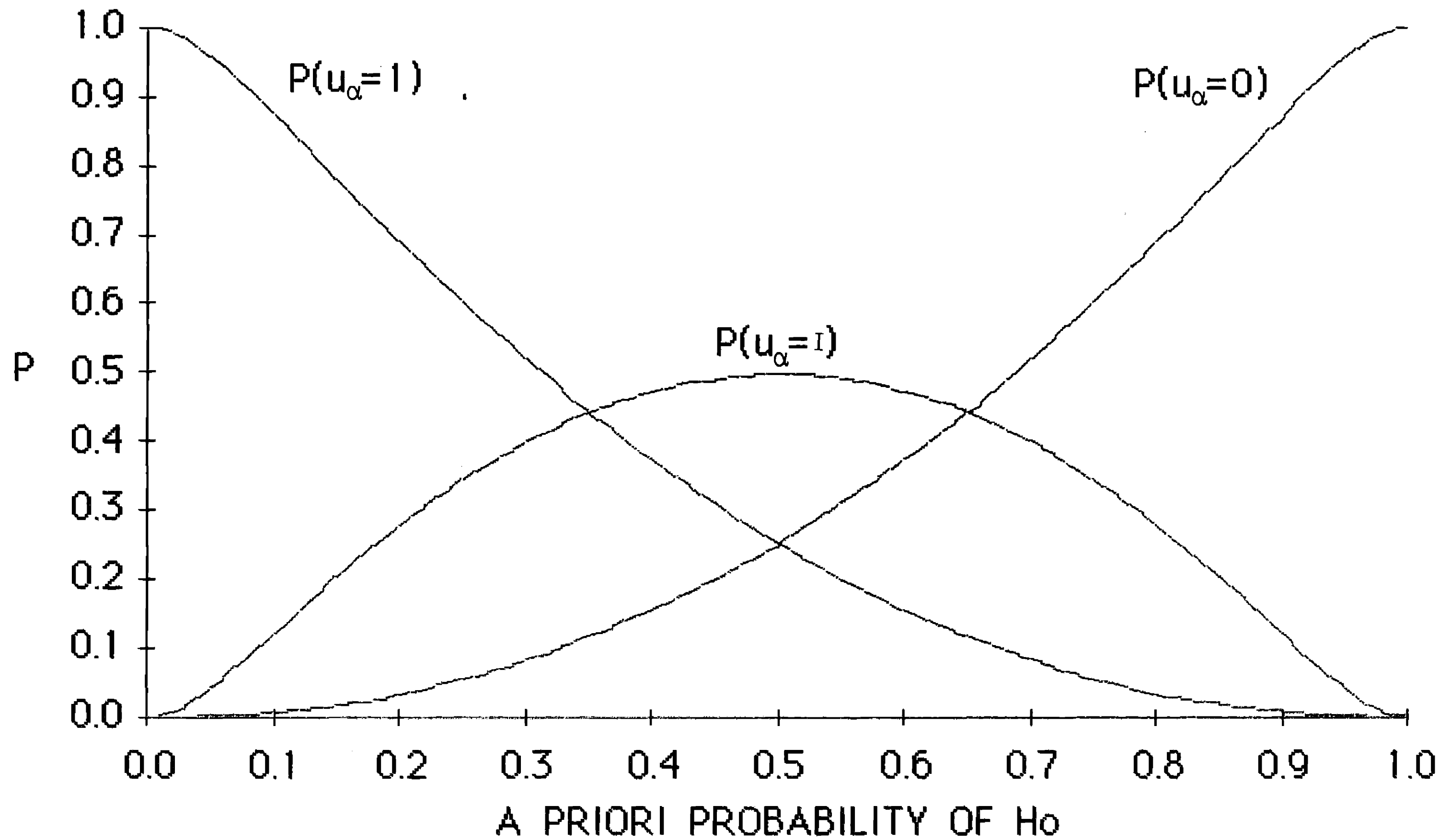


FIGURE 39. PROBABILITIES ASSOCIATED WITH THE DECISION OF DM B vs. P(Ho)

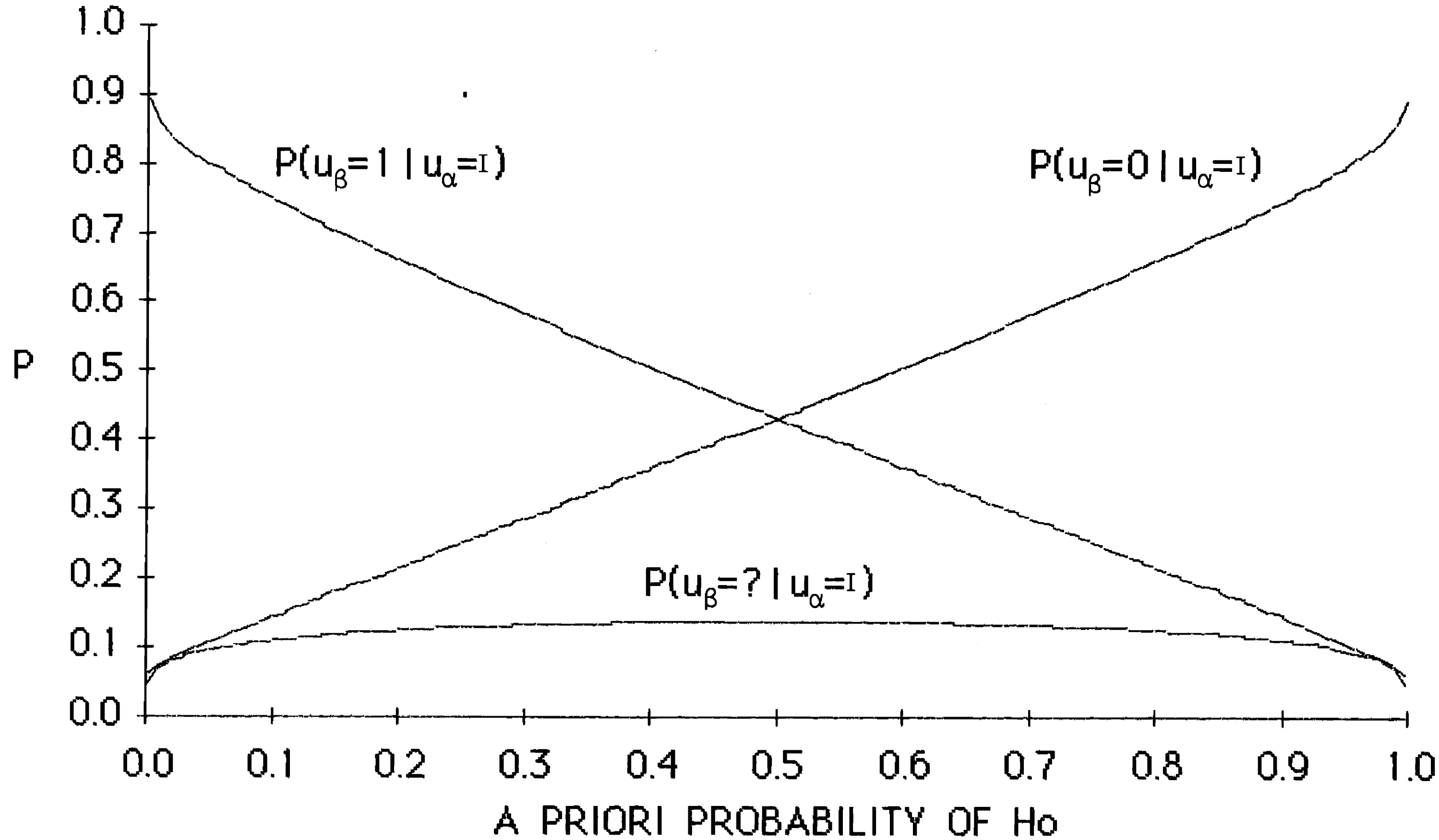


FIGURE 40. PROBABILITIES ASSOCIATED WITH THE FINAL DECISION OF DM A vs. P(Ho)

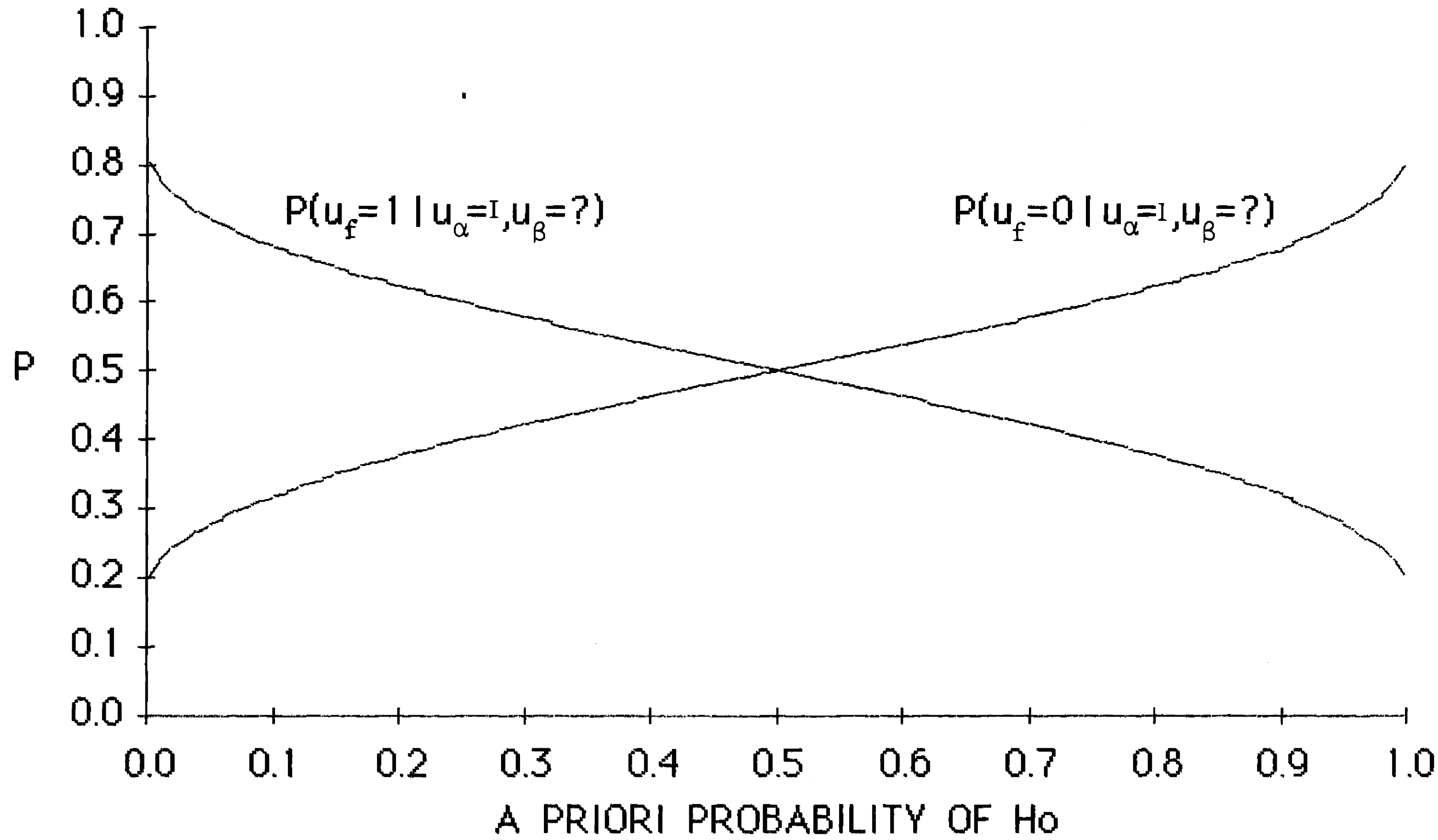


FIGURE 41. PROBABILITIES ASSOCIATED WITH THE FINAL TEAM DECISION vs. P(Ho)

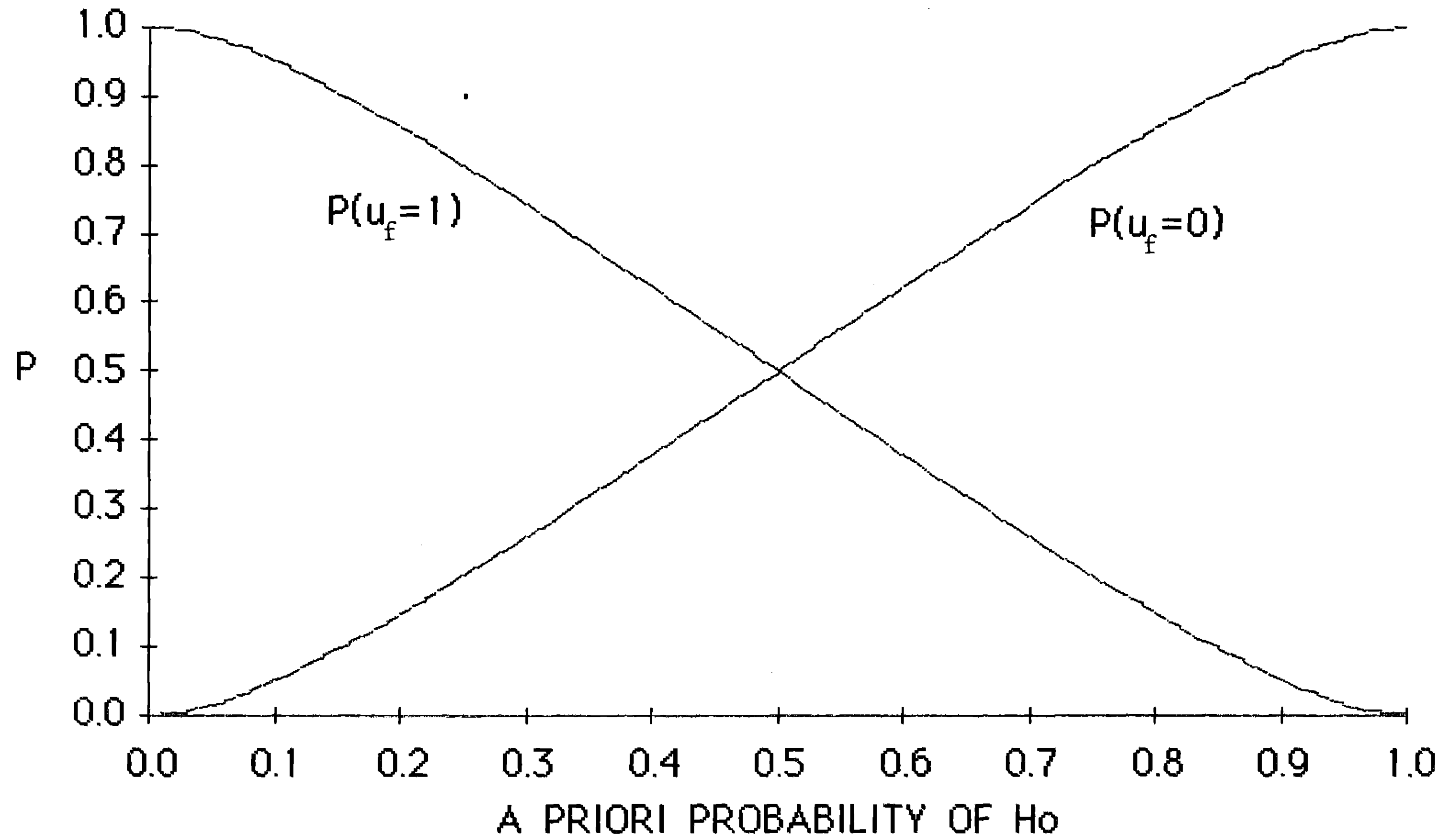


FIGURE 42. COSTS vs. P(Ho)

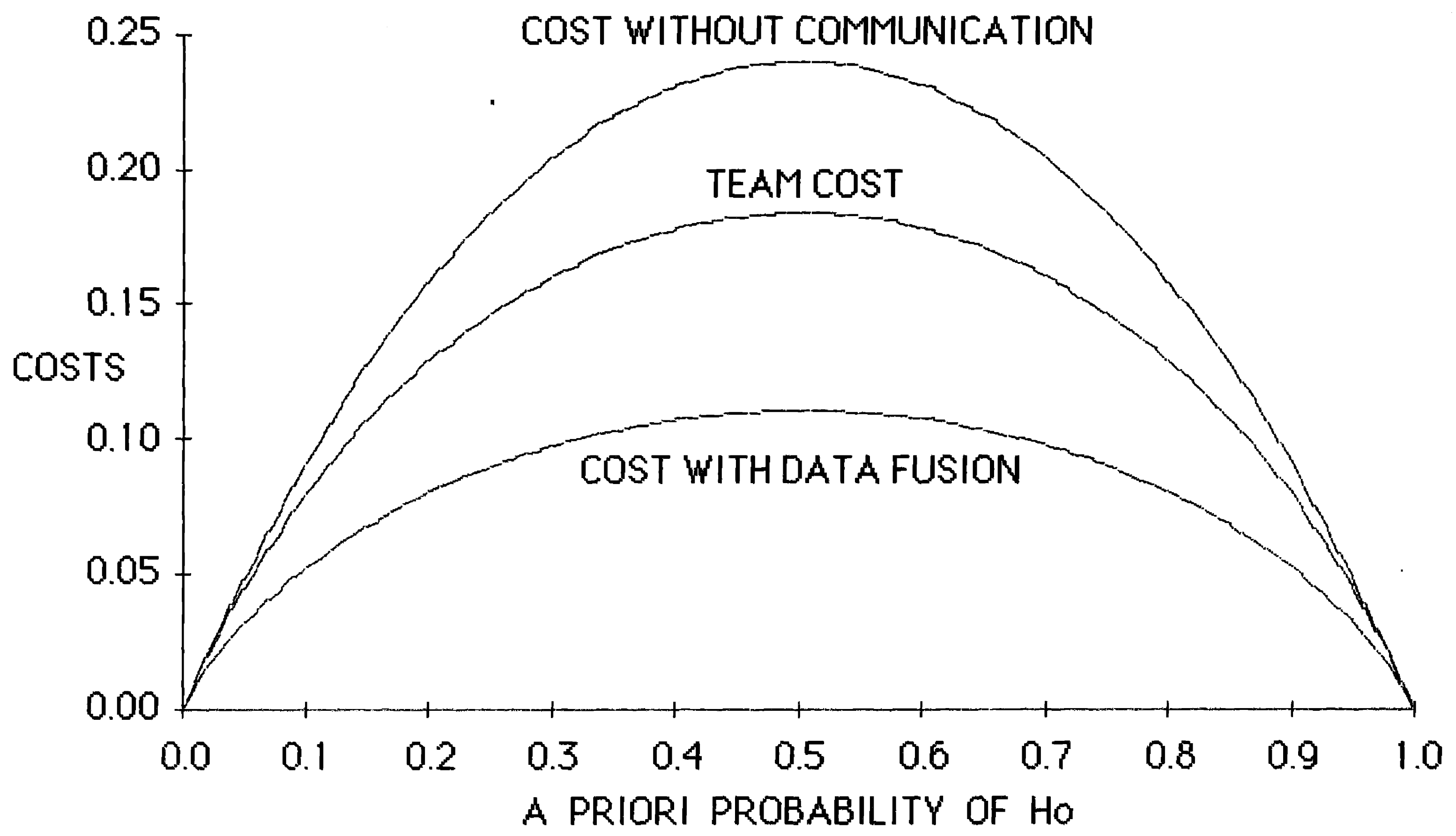
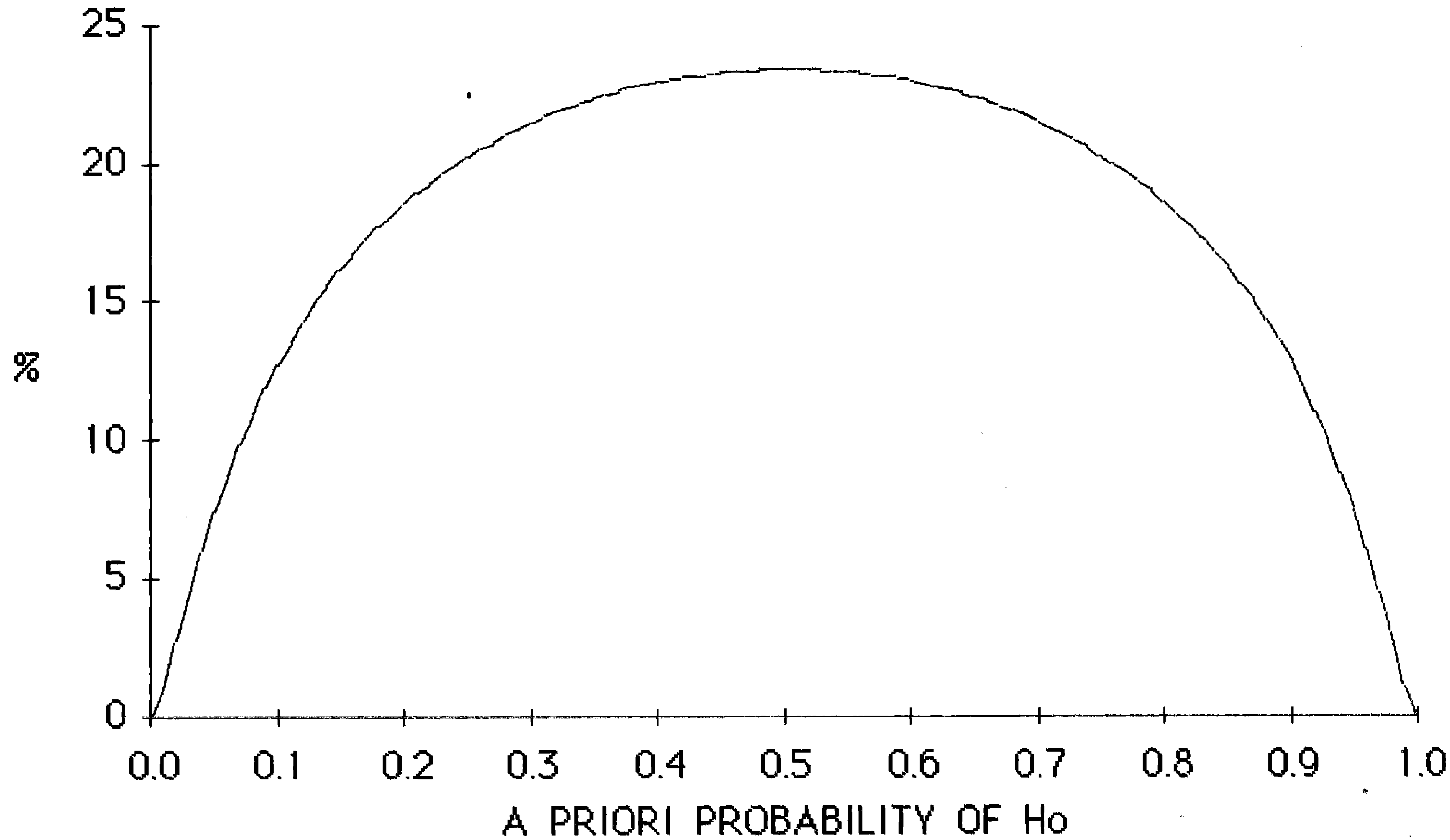


FIGURE 43. PERCENTAGE COST IMPROVEMENT vs. P(Ho)



and monotonically decreasing from 0.5 to 1.

In the limiting cases, $p_0 \approx 0$ or $p_0 \approx 1$, there is less incentive for extra information as to be expected. Hence, the benefits of team decision making are best when the prior uncertainty is large, even if team communication is relatively expensive.

In Figure 37, the decision thresholds of DM B, along with the maximum likelihood threshold of DM B (Y_β^*) are presented. Consider the points lying lower than the Y_β^* and higher than Y_β^u (for example point A for p_0=.9). For observations in this region, the optimum decision of the consulting DM, whenever the primary DM requests for information, is $u_\beta=1$ (because the observation is larger then the "upper" threshold). But, if the consulting DM were to decide alone, as an individual and not as a part of a team, he would decide $u_\beta=0$ (because the observation is lower than Y_β^u). The reasons for this have been analyzed extensively in 5.2 above and are that the consulting DM uses as his a priori probabilities his own <u>estimates</u> of the primary DM's a posteriori probabilities. The important conclusion is that a decision maker can, with the same input, make a totally different decision if he is a part of a team of decision makers, than if he were to decide alone.

Finally, from Figure 41, we obtain that the final decision of the team is very close to being unbiased, although a little biased towards the more likely hypothesis (the $P(u_f=0)$ lies a little below the 45° line for p_0 between 0 and 0.5, and lies a little over the 45° line for p_0 between 0.5 and 1).

5.5 Effects of imperfect a priori information

Up to now we have assumed that both DMs have identical knowledge of all parameters that characterize the environment that the decision makers operate in. Also, that they are perfectly rational.

If we want to endow the DMs with some "human" qualities, then it is conceivable that each DM may have a different perception of "ground truth." In particular, we may assume that only one of the DMs knows the true prior probability p_0 associated with the hupotheses, while the other does not. This may be the consequence of incomplete "team training." Obviously the quality of the team decision process will suffer as a function of the misperception of the prior uncertainty. The sensitivity studies described below quantify the degree of erroneous knowledge of p_0 upon the degradation in team performance.

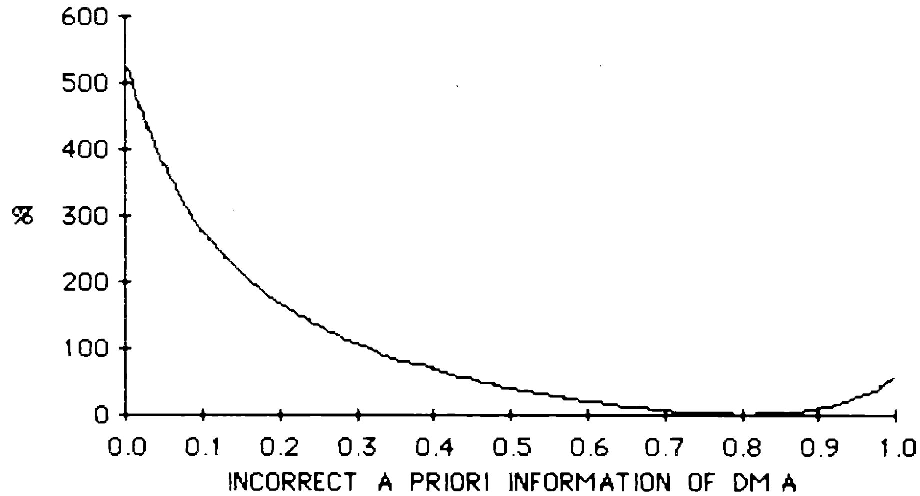
The baseline values in Table 1 are used, except that $\sigma_\alpha^2=0.8$

CASE 1 : Only the consulting DM knows the true p_0

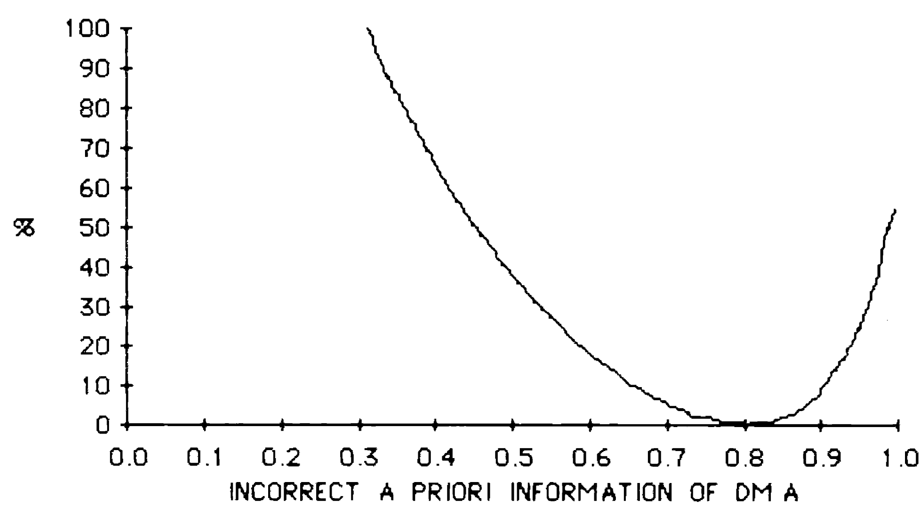
FIGURE 44

PERCENTAGE LOSS IN COST

TRUE VALUE KNOWN TO DM B : P(Ho)= .8



DETAIL OF ABOVE



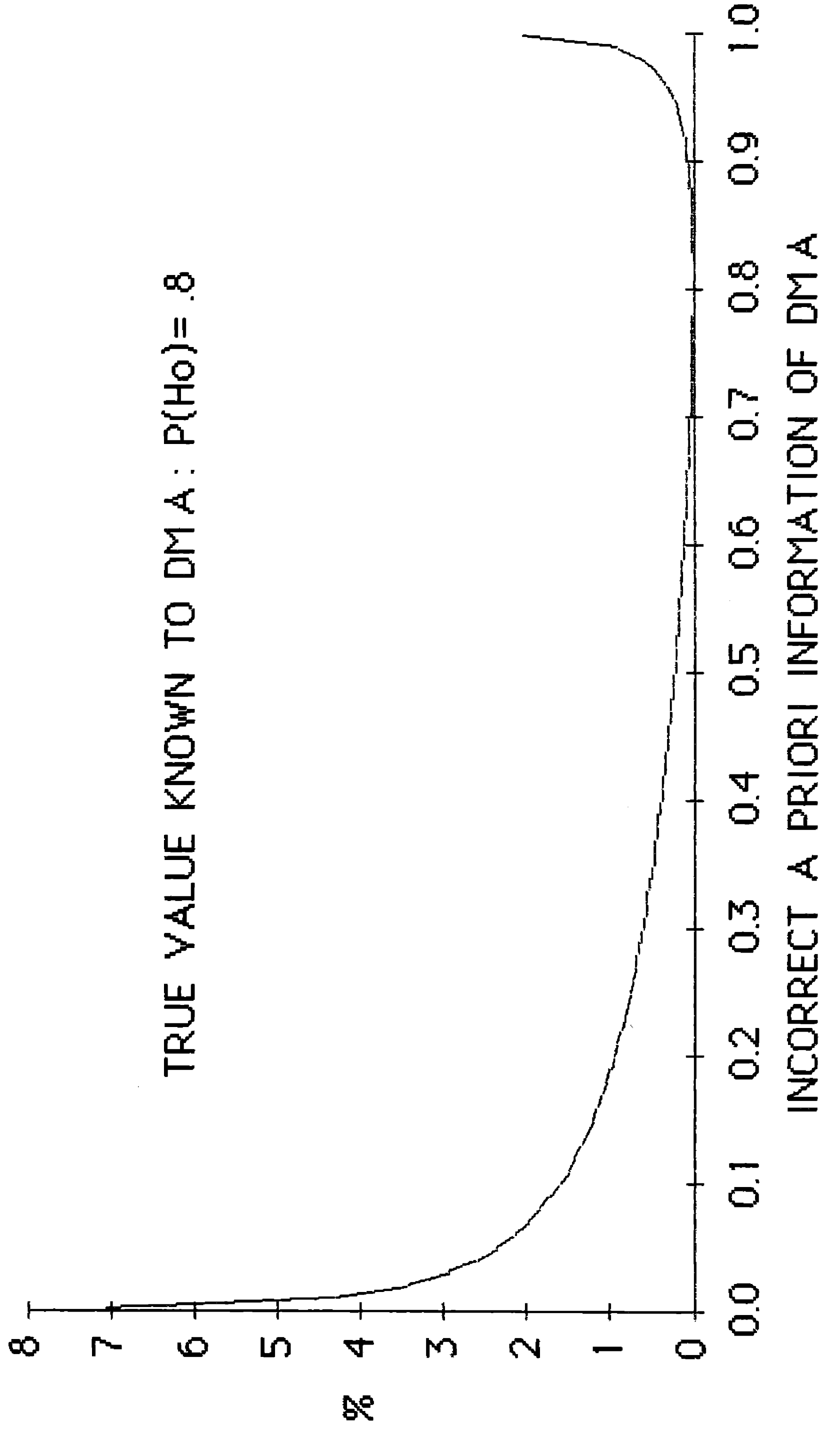


FIGURE 45. PERCENTAGE LOSS IN COST

From Figure 44, where the true p_0 is 0.8, we deduce that our model is relatively robust for small errors. If the primary DM's erroneous p_0 is anywhere between 0.7 and 0.9, performance of the team will be not more than 10% away from the optimum. On the other hand, if DM A believes that the hypotheses are equally likely (p_0=0.5) then the team performance degrades about 40%.

<u>CASE 2 :</u> Only the primary DM knows the true p_0

As we see in Figure 45, where the true p_0 is 0.8, our model exhibits remarkable robustness qualities. If the consulting DM's erroneous p_0 is as far out as 0.01, the performance of the team will not be further than 7% away from the optimal. This can be explained by looking at the consulting DM's thresholds as functions of p_0 (Figure 37). We observe that for values of p_0 between 0.01 and 0.99, the thresholds do not change by much. This occurs because, as explained in detail in 5.2 above, the consulting DM knows that the primary DM requests for information when its a posteriori probabilities of the two hypotheses are roughly equal, which is the case indeed. As already stated, the consulting DM uses as its a priori probabilities its estimates of the a posteriori probabilities of the primary DM. Therefore, the consulting DM's estimates of the primary DM's a posteriori probabilities are good, besides the discrepancy in p_0, and the team's performance is not influenced by much.

The bottom line is that if we need invest time and money to correct misperceptions of the team members, we should invest it to train the primary DM. This conclusion is of course valid for the numerical values used in this study.

6. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

6.1 Conclusions

Team detection is the easiest form of decentralized decision making. A problem of team detection with communications cost was studied extensively, in which the team consists of two decision makers, the Primary and the Consulting, each receiving an observation. The Primary is responsible for the team decision, selecting one of two alternative hypotheses, and can solicit, at a cost, the Consultant's opinion. The Consultant's opinion consists of one and a half bits of information. The team objective is to minimize a cost function, which depends on the team decision and the true hypothesis.

The Consultant's opinion is activated only when asked. We proved that the optimal decision of the his optimal decision should be either definitive (i.e. indicating one of the hypotheses to be true) and thus accepted by the Primary decision maker, or "I am not sure" in which case the burden of the final decision shifts back to the Primary.

We showed that by invoking the conditional independence assumption the optimal decision rules of both decision makers are given by deterministic functions, expressed as likelihood ratio tests with constant thresholds. The optimal decision thresholds of the two decision makers are coupled and can not be expressed in closed form.

In the linear Gaussian case the optimal decision rules reduce to threshold tests on the observations axes. This case was used to perform sensitivity analyses, in order to enhance our knowledge of the team decision process.

Deterioration of the quality of the observations of the Primary decision maker result in two very different situations. If the cost of declaring "de facto" the a priori most probable hypothesis is less than the sum of the communication cost plus the cost of the consulting decision maker making the final decision, then the Primary decision maker declares the more probable hypothesis to be true. Otherwise, the Primary decision maker decides to incur the communication cost and passes the responsibility of the team decision to the Consulting decision maker.

When the quality of the observations of the Consulting decision maker decreases, less information is requested by the Primary decision maker. Moreover, the Consultant becomes more willing to make the final decision, as he realizes that the Primary must be really confused, since the Primary is willing to incur the communication cost for information of lesser quality.

The effects of increasing communication cost are very similar to the effects of decreasing quality of the observations of the Consulting decision maker, since in both cases the Consultant's information becomes less helpful for the team.

By varying the a priori probabilities of the hypotheses, we observed that our model exhibits symmetry and that the final decision tends to biased towards the a priori more likely hypothesis. Moreover, we found that the team performance deteriorates, if one of the decision makers has imperfect a priori information. The degradation is much bigger when the Primary decision maker has the imperfect a priori information.

Finally, we would like to emphasize the two most important conclusions of this thesis. First, the optimal decision rules of the two decision makers are coupled. That is the optimal decision rule of each team member depends on the decision rules of the rest members of the team. Second, because of this, a team member can make decisions which are in total contrast with the decisions that the team member would make, if he were make the final decision alone and not as the part of a team.

6.2 Suggestions for Future Research

Our model could be extended in several different ways, besides the obvious ones of studying more complex organizations. We could allocate to the Consulting decision maker more than one and a half bits of information. It would be interesting to obtain a graph of the optimum number of half bits to be allocated to the Consultant decision maker as a function of the cost per half bit.

Another possible extention would be to allocate communication capacity to the Primary decision maker. That is, whenever the Primary decision maker requests information to send a message concerning his observation to the Consultant. It would be interesting to study the impact on team performance of such a communication scheme.

Finally, another interesting extention would be to introduce the concept of the "novice" decision maker. This decision maker would not have accurate a priori knowledge. The Consultant decision maker would then try not only to minimize the team cost, but also to restore the misperceptions of the "novice" decision maker. A feed back mechanism should be devised in order to allow the Consultant to "train" the Primary decision maker.

APPENDIX : PROOFS

Proof of Theorem 1 :

$$\min_{\gamma_f} \sum_{H,u_\alpha,u_\beta,u_f} \int_{y_\alpha,y_\beta} P(u_\alpha,u_\beta,u_f,y_\alpha,y_\beta,H)\, J^*(u_\alpha,u_f,H) =$$

$$= \sum_{H,u_\alpha,u_\beta,u_f} \int_{y_\alpha,y_\beta} P(H)\, P(u_\alpha,u_\beta,u_f \mid y_\alpha,y_\beta,H)\, P(y_\alpha,y_\beta \mid H)\, J^*(u_\alpha,u_f,H)$$

None of the decisions u_α, u_β and u_f depend on the (unknown) hypothesis H.

Recalling the definition of $J^*(u_\alpha,u_f,H)$ (eq. (5)) and summing explicitly over u_α we obtain

$$\sum_{H,u_\beta,u_f} \int_{y_\alpha,y_\beta} P(H) \Big[P(u_\alpha{=}0,u_\beta,u_f \mid y_\alpha,y_\beta)\, J(u_f,H) + P(u_\alpha{=}1,u_\beta,u_f \mid y_\alpha,y_\beta)\, J(u_f,H) +$$

$$+ P(u_\alpha = I,u_\beta,u_f \mid y_\alpha,y_\beta)\, [J(u_f,H)+C] \Big]\, P(y_\alpha,y_\beta \mid H) =$$

$$\sum_{H,u_f} \int_{y_\alpha,y_\beta} P(H) \Big[P(u_\alpha{=}0,u_f \mid y_\alpha,y_\beta)\, J(u_f,H) + P(u_\alpha{=}1,u_f \mid y_\alpha,y_\beta)\, J(u_f,H) +$$

$$+ \sum_{u_\beta} P(u_\alpha = I,u_\beta,u_f \mid y_\alpha,y_\beta)\, [J(u_f,H)+C] \Big]\, P(y_\alpha,y_\beta \mid H) \quad (66)$$

because by definition, u_β is considered only when information is requested (i.e. $u_\alpha = I$) and thus when $u_\alpha=0$ or 1, u_f is independent of u_β and y_β. Hence

$$\sum_{u_\beta} P(u_\alpha=i,u_\beta,u_f \mid y_\alpha,y_\beta)\, J(u_f,H) = J(u_f,H) \sum_{u_\beta} P(u_\alpha=i,u_\beta,u_f \mid y_\alpha,y_\beta) =$$

$$= J(u_f,H)\, P(u_\alpha=i,u_f \mid y_\alpha) \qquad \text{for } i=0,1$$

Again by definition, when the preliminary decision u_α of the primary DM is deterministic (0 or 1), the final decision is $u_f = u_\alpha$; that is :

$$P(u_f=i \mid u_\alpha=i,y_\alpha) = 1 \qquad \text{for } i=0,1$$

Therefore ,

$$P(u_\alpha=i,u_f \mid y_\alpha) = P(u_f \mid u_\alpha=i,y_\alpha)P(u_\alpha=i \mid y_\alpha) = \begin{cases} P(u_\alpha=i \mid y_\alpha) & \text{if } u_f=i \\ & \text{for } i=0,1 \\ 0 & \text{otherwise} \end{cases} \quad (67)$$

Substituting (67) into (66) we obtain

$$\sum_H \int_{y_\alpha,y_\beta} P(H) \left[P(u_\alpha=0 \mid y_\alpha)\, J(0,H) + P(u_\alpha=1 \mid y_\alpha)\, J(1,H) \right] P(y_\alpha,y_\beta \mid H) +$$

$$+ \sum_{H,u_\beta,u_f} P(H)\, P(u_\alpha = I,u_\beta,u_f \mid y_\alpha,y_\beta)\, [J(u_f,H)+C]\, P(y_\alpha,y_\beta \mid H) \quad (68)$$

The first summation is independent of u_f. Thus, it suffices to find the decision rule for u_f that minimizes the second summation; this yields

$$\min_{\gamma_f} \sum_{H,u_\beta,u_f} \int_{y_\alpha,y_\beta} P(H)\, P(u_\alpha = I, u_\beta, y_\alpha, y_\beta)\, P(u_\beta \mid u_\alpha = I, y_\alpha, y_\beta)\, P(u_\alpha = I \mid y_\alpha) \left[J(u_f,H)+C\right] P(y_\alpha, y_\beta \mid H)$$

$$= \sum_{H,u_\alpha,u_\beta,u_f} \int_{y_\alpha,y_\beta} P(H)\, P(u_f \mid u_\alpha = I, u_\beta, y_\alpha, y_\beta)\, P(u_\beta \mid u_\alpha = I, y_\alpha, y_\beta)\, P(u_\alpha = I \mid y_\alpha) \left[J(u_f,H)+C\right] P(y_\alpha, y_\beta \mid H)$$

The last equality holds because u_f is independent of y_β when u_β is known, u_β is independent of y_α, and u_α is independent of y_β (due to the conditional independence assumption).

Summing explicitly over u_f substituting for

$$P(u_f=1 \mid u_\alpha=I, u_\beta, y_\alpha) = 1 - P(u_f=0 \mid u_\alpha=I, u_\beta, y_\alpha) \;,$$

invoking Assumption 2, and ignoring a constant term we deduce that

$$\min_{\gamma_f} \sum_{H,u_\beta} \int_{y_\alpha} P(H)\, P(u_f=0 \mid u_\alpha = I, u_\beta, y_\alpha)\, P(u_\alpha = I \mid y_\alpha)\, P(y_\alpha \mid H) \left[J(0,H) - J(1,H) \right] \int_{y_\beta} P(u_\beta \mid u_\alpha = I, y_\beta)\, P(y_\beta \mid H) =$$

$$+ \sum_{H} \int_{y_\alpha} P(H)\, P(u_f \mid u_\alpha = I, u_\beta = x, y_\alpha)\, P(u_\alpha = I \mid y_\alpha)\, P(u_\beta = x \mid u_\alpha = I, H)\, P(y_\alpha \mid H) \left[J(0,H) - J(1,H) \right] +$$

$$+ \sum_{H} \int_{y_\alpha} P(H)\, P(u_f=0 \mid u_\alpha = I, u_\beta = v, y_\alpha)\, P(u_\alpha = I \mid y_\alpha)\, P(u_\beta = v \mid u_\alpha = I, H)\, P(y_\alpha \mid H) \left[J(0,H) - J(1,H) \right] +$$

$$+ \sum_{H} \int_{y_\alpha} P(H)\, P(u_f=0 \mid u_\alpha = I, u_\beta = z, y_\alpha)\, P(u_\alpha = I \mid y_\alpha)\, P(u_\beta = z \mid u_\alpha = I, H)\, P(y_\alpha \mid H) \left[J(0,H) - J(1,H) \right]$$

To derive the optimal final decision u_f (=0 or 1) of the primary DM, when information is requested and $u_\beta = x$ is received we must solve for

$$\min_{u_f} \int_{y_\alpha} P(u_f=0 \mid u_\alpha = I, u_\beta = x, y_\alpha)\, P(u_\alpha = I \mid y_\alpha) \sum_{H} P(H)\, P(u_\beta = x \mid u_\alpha = I, H)\, P(y_\alpha \mid H) \left[J(0,H) - J(1,H) \right] \quad (69)$$

This is minimized by choosing

$$P(u_f=0 \mid u_\alpha = I, u_\beta = x, y_\alpha) = \begin{cases} 0\,; & \sum_{H} P(H)\, P(u_\beta = x \mid u_\alpha = I, H)\, P(y_\alpha \mid H) \left[J(0,H) - J(1,H) \right] \leq 0 \\ 1\,; & \text{otherwise} \end{cases} \quad (70)$$

and so the optimal decision rule can be expressed as a deterministic function:

$$\gamma_f(x,y_\alpha) = \begin{cases} 0 \;; & P(u_f=0 \mid u_\alpha=I, u_\beta=x, y_\alpha) = 1 \\ 1 \;; & \text{otherwise} \end{cases} \tag{71}$$

From (70) we obtain

$$\sum_H P(H)\, P(u_\beta=x \mid u_\alpha=I,H)\, P(y_\alpha \mid H) \left[J(0,H)-J(1,H)\right] \underset{u_f=0}{\overset{u_f=1}{\gtrless}} 0 \tag{72}$$

where the notation

$$f(x) \underset{u=1}{\overset{u=0}{\gtrless}} t$$

means

$$\text{choose} \begin{cases} u=1 & \text{if } f(x)>t \\ \text{either} & \text{if } f(x)=t \\ u=0 & \text{if } f(x)<t \end{cases}$$

Expanding (72) over H and invoking Assumption 1 we obtain

$$\Lambda_\alpha(y_\alpha) \underset{u_f=1}{\overset{u_f=0}{\gtrless}} \frac{P(u_\beta=x \mid u_\alpha=I,H_1)\left[J(1,H_1)-J(0,H_1)\right]}{P(u_\beta=x \mid u_\alpha=I,H_0)\left[J(0,H_0)-J(1,H_0)\right]} = \alpha_x \tag{73}$$

Thus, the first part of the Theorem is proved.

Proceeding in a similar manner for the cases when $u_\beta = v$ and when $u_\beta = z$ is received, the rest of the Theorem is proved.□

<u>Remark</u> : It was shown that the optimal decision rule for u_f is deterministic and given by likelihood ratio tests, regardless of the forms of J, $P(u_\alpha \mid y_\alpha)$ and $P(u_\beta \mid u_\alpha = I, y_\beta)$, as long as the conditional independence assumption holds.

We now proceed to prove three Corollaries, which will be helpful in the subsequent proofs.

COROLLARY 5 : If the optimal decision rule presented in Theorem 1 is employed for u_f by the primary DM, then, whenever the following conditional probabilities are defined, we have

$$P(u_f=0 \mid u_\alpha = I, u_\beta = x, y_\alpha) \geq P(u_f=0 \mid u_\alpha = I, u_\beta = v, y_\alpha) \geq P(u_f=0 \mid u_\alpha = I, u_\beta = z, y_\alpha) \quad (74)$$

Proof : From Assumption 3 and (43) :

$$\alpha_x \leq \alpha_v \leq \alpha_z \quad (75)$$

From Theorem 1 it follows that

$$P(u_f=0 \mid u_\alpha = I, u_\beta = i, y_\alpha) = \frac{\displaystyle\int_{y_\alpha : \Lambda_\alpha(y_\alpha) \geq \alpha_i \,;\, P(u_\alpha = I \mid y_\alpha) > 0} \sum_H P(H)\, P(y_\alpha \mid H)}{\displaystyle\int_{y_\alpha : P(u_\alpha = I \mid y_\alpha) > 0} \sum_H P(H)\, P(y_\alpha \mid H)} \qquad (76)$$

Note that $P(H)\, P(y_\alpha \mid H)$ is always non-negative. This together with (75) and (76) yields (74).□

COROLLARY 6 : If the optimal decision rule , derived in Theorem 1, is employed for u_f, then, when the following conditional probabilities are defined, we have

$$\sum_{u_f} \left[P(u_f \mid u_\alpha = I, u_\beta = x, y_\alpha) - P(u_f \mid u_\alpha = I, u_\beta = z, y_\alpha) \right] J(u_f, H_0) \leq 0 \qquad (77)$$

$$\sum_{u_f} \left[P(u_f \mid u_\alpha = I, u_\beta = x, y_\alpha) - P(u_f \mid u_\alpha = I, u_\beta = v, y_\alpha) \right] J(u_f, H_0) \leq 0 \qquad (78)$$

$$\sum_{u_f} \left[P(u_f \mid u_\alpha = I, u_\beta = v, y_\alpha) - P(u_f \mid u_\alpha = I, u_\beta = z, y_\alpha) \right] J(u_f, H_0) \leq 0 \qquad (79)$$

Proof : Only (77) will be proved, as the proofs of (78) and (79) are very similar.

$$\sum_{u_f} \left[P(u_f \mid u_\alpha = I, u_\beta = x, y_\alpha) - P(u_f \mid u_\alpha = I, u_\beta = z, y_\alpha) \right] J(u_f, H_0) =$$

$$= P(u_f=0 \mid u_\alpha = I, u_\beta = x, y_\alpha)\, J(0,H_0) + P(u_f=1 \mid u_\alpha = I, u_\beta = x, y_\alpha)\, J(1,H_0) -$$

$$- P(u_f=0 \mid u_\alpha = I, u_\beta = I, y_\alpha)\, J(0,H_0) - P(u_f=1 \mid u_\alpha = I, u_\beta = z, y_\alpha)\, J(1,H_0) =$$

$$= P(u_f=0 \mid u_\alpha = I, u_\beta = x, y_\alpha)\, J(0,H_0) + \left[1 - P(u_f=0 \mid u_\alpha = I, u_\beta = x, y_\alpha) \right] J(1,H_0) -$$

$$- P(u_f=0 \mid u_\alpha = I, u_\beta = z, y_\alpha)\, J(0,H_0) + \left[1 - P(u_f=0 \mid u_\alpha = I, u_\beta = z, y_\alpha) \right] J(1,H_0) =$$

$$= \left[P(u_f=0 \mid u_\alpha = I, u_\beta = x, y_\alpha) - P(u_f=0 \mid u_\alpha = I, u_\beta = z, y_\alpha) \right] \left[J(0,H_0) - J(1,H_0) \right] \leq 0$$

The above hold because of Assumption 1 and Corollary 5.□

COROLLARY 7 : If the optimal decision rule, derived in Theorem 1, is employed for u_f, then, whenever the following conditional probabilities are defined, we have

$$\sum_{u_f} \left[P(u_f \mid u_\alpha = I, u_\beta = x, H) - P(u_f \mid u_\alpha = I, u_\beta = z, H) \right] J(u_f, H_0) \leq 0 \qquad (80)$$

$$\sum_{u_f} \left[P(u_f \mid u_\alpha = I, u_\beta = x, H) - P(u_f \mid u_\alpha = I, u_\beta = v, H) \right] J(u_f, H_0) \leq 0 \qquad (81)$$

$$\sum_{u_f} \left[P(u_f \mid u_\alpha = I, u_\beta = v, H) - P(u_f \mid u_\alpha = I, u_\beta = z, H) \right] J(u_f, H_0) \leq 0 \qquad (82)$$

Proof : The proof follows from Corollary 6, by multiplying (77), (78) and (79) respectively by $P(y_\alpha \mid H)$ and intergrating with respect to y_α. Note that H may be either H_0 or H_1.□

Proof of Theorem 2 :

$$\min_{\gamma_\beta} \sum_{H,u_\alpha,u_\beta,u_f} \int_{y_\alpha,y_\beta} P(u_\alpha,u_\beta,u_f,y_\alpha,y_\beta,H)\, J^*(u_\alpha,u_f,H) =$$

$$= \sum_{H,u_\alpha,u_\beta,u_f} \int_{y_\alpha,y_\beta} P(H)\, P(u_\alpha,u_\beta,u_f \mid y_\alpha,y_\beta)\, P(y_\alpha,y_\beta \mid H)\, J^*(u_\alpha,u_f,H) =$$

$$= \sum_{H,u_\beta,u_f} \int_{y_\alpha,y_\beta} P(H) \left[P(u_\alpha{=}0 \mid y_\alpha)\, J(0,H) + P(u_\alpha{=}1 \mid y_\alpha)\, J(1,H) \right] P(y_\alpha,y_\beta \mid H) +$$
$$+ \sum_{H,u_\beta,u_f} \int_{y_\alpha,y_\beta} P(H)\, P(u_\alpha{=}I,u_\beta,u_f \mid y_\alpha,y_\beta) \left[J(u_f,H)+C \right] P(y_\alpha,y_\beta \mid H)$$

where all the steps were explained in the proof of Theorem 1.

The first summation is independent of u_β. Thus, it suffices to find the function $u_\beta = \gamma(y_\beta)$ that minimizes the following :

$$\min_{\gamma_\beta} \sum_{H,u_\beta,u_f} \int_{y_\alpha,y_\beta} P(H)\, P(u_\alpha{=}I,u_\beta,u_f \mid y_\alpha,y_\beta) \left[J(u_f,H)+C \right] P(y_\alpha,y_\beta \mid H) =$$

$$= \sum_{H,u_\beta,u_f} \int_{y_\alpha,y_\beta} P(H)\, P(u_f \mid u_\alpha = I, u_\beta, y_\alpha)\, P(u_\beta \mid u_\alpha = I, y_\beta)\, P(u_\alpha = I \mid y_\alpha)$$

$$\left[J(u_f,H)+C\right] P(y_\alpha, y_\beta \mid H) =$$

$$= \sum_{u_\beta} \int_{y_\beta} P(u_\beta \mid u_\alpha = I, y_\beta) \sum_{H,u_f} P(H)\, P(y_\beta \mid H) \left[J(u_f,H)+C\right] \int_{y_\alpha} P(u_f \mid u_\alpha = I, u_\beta, y_\alpha)$$

$$P(u_\alpha = I \mid y_\alpha)\, P(y_\alpha \mid H) =$$

$$= \sum_{u_\beta} \int_{y_\beta} P(u_\beta \mid u_\alpha = I, y_\beta) \sum_{H,u_f} P(H)\, P(u_\alpha = I, u_f \mid u_\beta, H)\, P(y_\beta \mid H) \left[J(u_f,H)+C\right] =$$

$$= \sum_{u_\beta} \int_{y_\beta} P(u_\beta \mid u_\alpha = I, y_\beta) \sum_{H,u_f} P(H)\, P(u_f \mid u_\alpha = I, u_\beta, H)\, P(u_\alpha = I \mid H)$$

$$P(y_\beta \mid H) \left[J(u_f,H)+C\right] \qquad (83)$$

For $i=x,v,z$ denote

$$p^i = \sum_{H,u_f} P(H)\, P(u_f \mid u_\alpha = I, u_\beta, H)\, P(u_\alpha = I \mid H)\, P(y_\beta \mid H) \left[J(u_f,H)+C\right] \qquad (84)$$

Then, in order to minimize (83) we find that

$$P(u_\beta = i \mid u_\alpha = I, y_\beta) = \begin{cases} 1 & \text{if} \quad p^i = \{p^x, p^v, p^z\} \\ 0 & \text{otherwise} \end{cases} \qquad (85)$$

Thus, the optimal decision rule can be expressed as a deterministic function :

$$\gamma_\beta(y_\beta) = i \qquad \text{if } P(u_\beta=i \mid u_\alpha=I, y_\beta)=1 \qquad \text{for } i=x,v,z \qquad (86)$$

Putting the above in likelihood ratio form and invoking Corollary 7, equations (20)-(27) are obtained completing the proof of Theorem 2.□

Remark : The optimal decision rule for u_β is deterministic and can be expressed as likelihood ratio tests with constant thresholds, regardless of the form of J and $P(u_f \mid u_\alpha = I, u_\beta, y_\alpha)$.

Proof of Lemma 1 :

$$\min_{\gamma_\alpha} \sum_{H,u_\alpha,u_\beta,u_f} \int_{y_\alpha,y_\beta} P(u_\alpha,u_\beta,u_f,y_\alpha,y_\beta,H)\, J^*(u_\alpha,u_f,H) =$$

$$= \sum_{H,u_\alpha,u_\beta,u_f} \int_{y_\alpha,y_\beta} P(H)\, P(u_\alpha,u_\beta,u_f \mid y_\alpha,y_\beta)\, P(y_\alpha,y_\beta \mid H)\, J^*(u_\alpha,u_f,H) =$$

$$= \int_{y_\alpha} \Big[P(u_\alpha=0 \mid y_\alpha) \sum_H P(H)\, P(y_\alpha \mid H)\, J(0,H) + P(u_\alpha=1 \mid y_\alpha) \sum_H P(H)\, P(y_\alpha \mid H)\, J(1,H) +$$

$$+ P(u_\alpha = I \mid y_\alpha) \sum_{H,u_\beta,u_f} \int_{y_\beta} P(H)\, P(u_f \mid u_\alpha = I, y_\alpha)\, P(u_\beta \mid u_\alpha = I, y_\beta)\, [J(u_f,H)+C]\, P(y_\alpha,y_\beta \mid H) \Big]$$

where all the steps were explained in the proof of Theorem 1.

Invoking the conditional independence assumption and integrating over y_β, we obtain :

$$\int_{y_\alpha}\Big[P(u_\alpha=0 \mid y_\alpha)\sum_H P(H)\,P(y_\alpha \mid H)\,J(0,H) + P(u_\alpha=1 \mid y_\alpha)\sum_H P(H)\,P(y_\alpha \mid H)\,J(1,H) +$$

$$+ P(u_\alpha=1 \mid y_\alpha)\sum_{H,u_\beta,u_f} P(H)\,P(u_f \mid u_\alpha=I,u_\beta,y_\alpha)\,P(u_\beta \mid u_\alpha=I,H)\,P(y_\alpha \mid H)\,[J(u_f,H)+C] \quad (87)$$

Set

$$p^i = \sum_H P(H)\,P(y_\alpha \mid H)\,J(i,H) \qquad i=0,1 \qquad (88)$$

$$p^I = \sum_{H,u_\beta,u_f} P(H)\,P(u_f \mid u_\alpha=I,u_\beta,y_\alpha)\,P(u_\beta \mid u_\alpha=I,H)\,P(y_\alpha \mid H)\,[J(u_f,H)+C] \qquad (89)$$

Then, in order to minimize (87) we use

$$P(u_\alpha=i \mid y_\alpha) = \begin{cases} 1 & \text{if } p^i = \{p^0,p^1,p^I\} \\ 0 & \text{otherwise} \end{cases} \qquad (90)$$

Thus, the optimal decision rule can be expressed as a deterministic function :

$$\gamma_\alpha(y_\alpha) = i \qquad \text{if} \quad P(u_\alpha = i \mid y_\alpha) \quad \text{for} \quad i=0,1,I \tag{91}$$

Then, invoking Assumption 1, it is a matter of simple, but tedious algebraic manipulations to put the decision rule in the form of (28)-(31).□

Proof of Corollary 1 : Only (33) will be proved, since the proof of (32) is very similar.

Suppose that for u_α, the optimal policy derived in Lemma 1 is employed and that information is requested for some y_α. Then from (28) :

$$\Lambda_\alpha(y_\alpha) < \alpha_2 \qquad \text{and} \qquad 1/\Lambda_\alpha(y_\alpha) < \alpha_3$$

According to the optimal decision rule u_f, derived in Theorem 1, whenever u_β=z is returned

$$\Lambda_\alpha(y_\alpha) \underset{u_f=1}{\overset{u_f=0}{\gtrless}} \alpha_z$$

Thus, it suffices to show that

$$\alpha_2 \le \alpha_z \tag{92}$$

because it implies that

$$\Lambda_\alpha(y_\alpha) < \alpha_2 \le \alpha_z \tag{93}$$

and so $u_f=1$ will always be the optimal final decision.

We now prove that (92) is indeed true :

$$\alpha_2 \le \alpha_z \quad \Rightarrow$$

$$\Rightarrow \frac{\sum_{u_\beta,u_f} P(u_f \mid u_\alpha = I, u_\beta, y_\alpha)\, P(u_\beta \mid u_\alpha = I, H_1)\, [J(u_f,H_1)+C] - J(0,H_1)}{J(0,H_0) - \sum_{u_\beta,u_f} P(u_f \mid u_\alpha = I, u_\beta, y_\alpha)\, P(u_\beta \mid u_\alpha = I, H_0)\, [J(u_f,H_0)+C]} \le$$

$$\le \frac{P(u_\beta = z \mid u_\alpha = I, H_1)\, [J(1,H_1) - J(0,H_1)]}{P(u_\beta = z \mid u_\alpha = I, H_0)\, [J(0,H_0) - J(1,H_0)]} \quad \Rightarrow$$

$$\Rightarrow \frac{\sum_{u_\beta} P(u_f=1 \mid u_\alpha = I, u_\beta, y_\alpha)\, P(u_\beta \mid u_\alpha = I, H_1)\, [J(1,H_1) - J(0,H_1)] + C}{\sum_{u_\beta} P(u_f=1 \mid u_\alpha = I, u_\beta, y_\alpha)\, P(u_\beta \mid u_\alpha = I, H_0)\, [J(0,H_0) - J(1,H_0)] - C} \le$$

$$\le \frac{P(u_\beta = z \mid u_\alpha = I, H_1)\, [J(1,H_1) - J(0,H_1)]}{P(u_\beta = z \mid u_\alpha = I, H_0)\, [J(0,H_0) - J(1,H_0)]} \tag{94}$$

Recalling Assumption 1 and that $C \geq 0$, multiply with the product of the denominators and simplify to get :

$$C\Big[P(u_\beta=z \mid u_\alpha=I,H_0)[J(0,H_0)-J(1,H_1)] + P(u_\beta=z \mid u_\alpha=I,H_1)[J(1,H_1)-J(0,H_1)]\Big] +$$

$$+ \Big[\big[P(u_\beta=x \mid u_\alpha=I,H_1)P(u_\beta=z \mid u_\alpha=I,H_0) - P(u_\beta=x \mid u_\alpha=I,H_0)P(u_\beta=z \mid u_\alpha=I,H_1)\big]$$

$$P(u_f=1 \mid u_\alpha=I,u_\beta=x,y_\alpha) +$$

$$+ \big[P(u_\beta=v \mid u_\alpha=I,H_1)P(u_\beta=z \mid u_\alpha=I,H_0) - P(u_\beta=v \mid u_\alpha=I,H_0)P(u_\beta=z \mid u_\alpha=I,H_1)\big]$$

$$P(u_f=1 \mid u_\alpha=I,u_\beta=v,y_\alpha)\Big]$$

$$\Big[J(0,H_0) - J(1,H_0)\Big]\Big[J(1,H_1) - J(0,H_1)\Big] \leq 0 \qquad (95)$$

(95) is valid because of Assumptions 1 and 3. Hence, (92) is valid and thus Corollary 1 is proved.□

Proof of Theorem 3 : The proof of this theorem is tedious, yet easy. It is broken up to various cases and subcases, the optimal decision thresholds for each one are obtained and then everything is combined to obtain equations (44), (45) and (46).

In the proof, it will be considered whether :

CASE A : $P(u_\beta=v \mid u_\alpha= I,H_0) \le P(u_\beta=v \mid u_\alpha= I,H_1)$

in which case with simple, but tedious algebra yields

$$W_2 \le W_1 \le W_4 \le W_3 \qquad \text{and} \qquad \alpha_v \ge a_1$$

or :

CASE B : $P(u_\beta=v \mid u_\alpha= I,H_0) > P(u_\beta=v \mid u_\alpha= I,H_1)$

in which case we obtain

$$W_3 < W_4 < W_1 < W_2 \qquad \text{and} \qquad \alpha_v < a_1$$

In the end, it will be shown that the decision thresholds are independent of whether Case A or Case B is true.

Start by considering Corollary 1 and by assuming to know what the final decision of DM A, given his own observation, will be even in the case when information is requested and $u_\beta= v$ is returned. That is, assume to know the position of $\Lambda(y_\alpha)$ relative to α_V. It will be then shown that the thresholds are independent of the position of $\Lambda(y_\alpha)$.

<u>CASE 1</u> : $\Lambda(y_\alpha) < \alpha_v$

In this case it is known, that whenever information is requested, DM A decides 1 if u_β= v or z is returned and 0 if u_β= x is returned. Then, the thresholds a_2 and a_3 (eq. (30) and (31)) reduce to :

$$a_2 = a_{2,2} \qquad (96)$$
$$a_3 = a_{3,1}$$

<u>Subcase 1.1.A</u> : If $C \geq P(u_\beta=0 \mid u_\alpha= I,H_0)\,\Delta J_0$

Then

$$C \geq P(u_\beta=v \mid u_\alpha= I,H_0)\,\Delta J_0 \quad \Leftrightarrow \quad 1/a_{3,1} \leq 0$$

and

$$C \geq P(u_\beta=v \mid u_\alpha= I,H_0)\,\Delta J_0 \quad \Leftrightarrow \quad C \geq W_4 \quad \Leftrightarrow \quad a_1 \geq a_{2,2}$$

Also

$$\alpha_v \geq a_1$$

Consequently the decision rule of (28) reduces to :

$$\Lambda(y_\alpha) \underset{u_\alpha=1}{\overset{u_\alpha=0}{\gtrless}} a_1 \qquad (97)$$

<u>Subcase 1.2.A</u> : If $W_3 \leq C < P(u_\beta=0 \mid u_\alpha= I,H_0)\,\Delta J_0$

$$W_3 \leq C < P(u_\beta=0 \mid u_\alpha= I,H_0)\,\Delta J_0 \quad \Leftrightarrow \quad a_{2,2} < a_1 \leq \alpha_v \leq a_{3,1}$$

The decision rule of (28) reduces to the decision rule of (97).

<u>Subcase 1.3.A</u>: If $W_4 \le C < W_3$

$$W_4 \le C < W_3 \quad \Leftrightarrow \quad a_{2,2} \le a_1 \le a_{3,1} < \alpha_v$$

The decision rule of (28) reduces again to the decision rule of (97).

<u>Subcase 1.4.A</u>: If $W_2 \le C < W_4$

$$W_2 \le C < W_4 \quad \Leftrightarrow \quad a_{3,1} < a_1 < a_{2,2} \le \alpha_v$$

The decision rule of (28) reduces to

$$u_\alpha = \begin{cases} 0 \; ; & a_{2,2} < \Lambda(y_\alpha) \\ I \; ; & a_{3,1} \le \Lambda(y_\alpha) \le a_{2,2} \\ 1 \; ; & \Lambda(y_\alpha) < a_{3,1} \end{cases} \tag{98}$$

<u>Subcase 1.5.A</u>: If $0 \le C < W_2$

$$0 \le C < W_2 \quad \Leftrightarrow \quad a_{3,1} < a_1 \le \alpha_v < a_{2,2}$$

The decision rule of (28) reduces to

$$\Lambda_\alpha(y_\alpha) \underset{u_\alpha = 1}{\overset{u_\alpha = I}{\gtrless}} a_{3,1} \tag{99}$$

<u>Subcase 1.1.B</u>: If $C \ge P(u_\beta = 0 \mid u_\alpha = I, H_0)\, \Delta J_0$

Then

$$C \ge P(u_\beta = v \mid u_\alpha = I, H_0)\, \Delta J_0 \quad \Leftrightarrow \quad 1/a_3 \le 0$$

Also

$$\alpha_v < a_1$$

Consequently the decision rule of (28) reduces to

$$u_\alpha = 1 \qquad (100)$$

Subcase 1.2.B : If $W_2 \leq C < P(u_\beta=0 \mid u_\alpha = I, H_0)\,\Delta J_0$

$$W_2 \leq C < P(u_\beta=0 \mid u_\alpha = I, H_0)\,\Delta J_0 \quad \Leftrightarrow \quad a_{2,2} \leq \alpha_v < a_1 < a_{3,1}$$

The decision rule of (28) reduces to the decision rule (100).

Subcase 1.3.B : If $W_4 \leq C < \min\{P(u_\beta=0 \mid u_\alpha = I, H_0)\,\Delta J_0\,,\, W_2\}$

$$W_4 \leq C < \min\{P(u_\beta=0 \mid u_\alpha = I, H_0)\,\Delta J_0\,,\, W_2\} \quad \Leftrightarrow \quad \alpha_v < a_{2,2} \leq a_1 \leq a_{3,1}$$

The decision rule of (28) reduces to the decision rule of (100).

Subcase 1.4.B : If $W_3 \leq C < W_4$

$$W_3 \leq C < W_4 \quad \Leftrightarrow \quad \alpha_v \leq a_{3,1} < a_1 < a_{2,2}$$

The decision rule of (28) reduces to the decision rule of (100).

Subcase 1.5.B : If $0 \leq C < W_3$

$$0 \leq C < W_3 \quad \Leftrightarrow \quad a_{3,1} < \alpha_v < a_1 < a_{2,2}$$

The decision rule of (28) reduces to the decision rule of (99).

<u>CASE 2</u>: $\Lambda(y_\alpha) \geq \alpha_v$

In this case, whenever information is requested, DM A decides 1 if $u_\beta = z$ is returned and decides 0 if $u_\beta = x$ or v is returned. Then, the thresholds a_2 and a_3 (eq. (30) and (31)) reduce to :

$$a_2 = a_{2,1} \qquad (101)$$

$$a_3 = a_{3,2}$$

<u>Subcase 2.1.A</u>: If $C \geq \left[1 - P(u_\beta=1 \mid u_\alpha = I, H_0)\right] \Delta J_0$

Then

$$C \geq \left[1 - P(u_\beta=1 \mid u_\alpha = I, H_0)\right] \Delta J_0 \quad \Leftrightarrow \quad W_1 \leq C \quad \Leftrightarrow \quad a_{2,1} \leq a_1$$

Also

$$\alpha_v \geq a_1$$

Consequently the decision rule of (28) reduces to

$$u_\alpha = 0 \qquad (102)$$

<u>Subcase 2.2.A</u>: If $W_3 \leq C < \left[1 - P(u_\beta=1 \mid u_\alpha = I, H_0)\right] \Delta J_0$

$$W_3 \leq C < \left[1 - P(u_\beta=1 \mid u_\alpha = I, H_0)\right] \Delta J_0 \quad \Leftrightarrow \quad a_{2,1} < a_1 \leq \alpha_v \leq a_{3,2}$$

The decision rule of (28) reduces to the decision rule of (102).

Subcase 2.3.A : If $W_1 \le C < W_3$

$$W_1 \le C < W_3 \quad \Leftrightarrow \quad a_{2,1} \le a_1 \le a_{3,1} < \alpha_v$$

The decision rule of (28) reduces to the decision rule of (102).

Subcase 2.4.A : If $W_2 \le C < W_1$

$$W_2 \le C < W_1 \quad \Leftrightarrow \quad a_{3,2} < a_1 < a_{2,1} \le \alpha_v$$

The decision rule of (28) reduces to the decision rule of (102).

Subcase 2.5.A : If $0 \le C < W_2$

$$0 \le C < W_2 \quad \Leftrightarrow \quad a_{3,2} < a_1 < \alpha_v < a_{2,1}$$

The decision rule of (28) reduces to

$$\Lambda_\alpha(y_\alpha) \underset{u_\alpha = I}{\overset{u_\alpha = 0}{\gtrless}} a_{2,1} \tag{103}$$

Subcase 2.1.B : If $C \ge \left[1 - P(u_\beta = 1 \mid u_\alpha = I, H_0) \right] \Delta J_0$

Then

$$C \ge \left[1 - P(u_\beta = v \mid u_\alpha = I, H_0) \right] \Delta J_0 \quad \Leftrightarrow \quad 1/a_{3,2} \le 0$$

and

$$C \ge \left[1 - P(u_\beta = 1 \mid u_\alpha = I, H_0) \right] \Delta J_0 \quad \Leftrightarrow \quad W_1 \le C \quad \Leftrightarrow \quad a_{2,1} \le a_1$$

Also

$$\alpha_v < a_1$$

Consequently the decision rule of (28) reduces to the decision rule of (97).

<u>Subcase 2.2.B</u> : If $W_2 \le C < \left[1 - P(u_\beta=1 \mid u_\alpha = I, H_0)\right] \Delta J_0$

$$W_2 \le C < \left[1 - P(u_\beta=1 \mid u_\alpha = I, H_0)\right] \Delta J_0 \quad \Leftrightarrow \quad a_{2,1} \le \alpha_v < a_1 < a_{3,2}$$

The decision rule of (28) reduces to the decision rule of (97).

<u>Subcase 2.3.B</u> : If $W_1 \le C < \min\{W_2, \left[1 - P(u_\beta=1 \mid u_\alpha = I, H_0)\right] \Delta J_0$

$$W_1 \le C < \min\{W_2, \left[1 - P(u_\beta=1 \mid u_\alpha = I, H_0)\right] \Delta J_0\} \quad \Leftrightarrow \quad \alpha_v < a_{2,1} \le a_1 \le a_{3,2}$$

The decision rule of (28) reduces to the decision rule of (97).

<u>Subcase 2.4.B</u> : If $W_3 \le C < W_1$

$$W_3 \le C < W_1 \quad \Leftrightarrow \quad \alpha_v \le a_{3,2} < a_1 < a_{2,1}$$

The decision rule of (28) reduces to

$$u_\alpha = \begin{cases} 0 \ ; & a_{2,1} < \Lambda(y_\alpha) \\ I \ ; & a_{3,2} \le \Lambda(y_\alpha) \le a_{2,1} \\ 1 \ ; & \Lambda(y_\alpha) < a_{3,2} \end{cases} \qquad (104)$$

<u>Subcase 2.5.B</u> : If $0 \le C < W_3$

$$0 \le C < W_3 \quad \Leftrightarrow \quad a_{3,2} < \alpha_v < a_1 < a_{2,1}$$

The decision rule of (28) reduces to

$$\Lambda_\alpha(y_\alpha) \underset{u_\alpha = I}{\overset{u_\alpha = 0}{\gtrless}} a_{2,1} \qquad (105)$$

We can now summarize the twenty cases as follows :

$$\gamma_\alpha(y_\alpha) = \begin{cases} 0 & \text{if } \Lambda_\alpha(y_\alpha) \geq \alpha_1 \\ I & \text{if } \Lambda_\alpha(y_\alpha) < \alpha_1 \text{ and } \Lambda_\alpha(y_\alpha) \geq \alpha_2 \\ 1 & \text{if } \Lambda_\alpha(y_\alpha) < \alpha_2 \end{cases} \qquad (44)$$

where

$$\alpha_1 = \begin{cases} a_1 \; ; & P(u_\beta = v \mid u_\alpha = I, H_0) \leq P(u_\beta = v \mid u_\alpha = I, H_1) \text{ and } C \geq W_4 \\ & \text{or } P(u_\beta = v \mid u_\alpha = I, H_0) > P(u_\beta = v \mid u_\alpha = I, H_1) \text{ and } C \geq W_1 \\ a_{2,2} \; ; & P(u_\beta = v \mid u_\alpha = I, H_0) \leq P(u_\beta = v \mid u_\alpha = I, H_1) \text{ and } W_4 > C \geq W_2 \\ a_{2,1} \; ; & P(u_\beta = v \mid u_\alpha = I, H_0) \leq P(u_\beta = v \mid u_\alpha = I, H_1) \text{ and } W_2 > C \geq 0 \\ & \text{or } P(u_\beta = v \mid u_\alpha = I, H_0) > P(u_\beta = v \mid u_\alpha = I, H_1) \text{ and } W_1 > C \geq 0 \end{cases} \qquad (106)$$

and

$$\alpha_2 = \begin{cases} a_1 \; ; & P(u_\beta = v \mid u_\alpha = I, H_0) \leq P(u_\beta = v \mid u_\alpha = I, H_1) \text{ and } C \geq W_4 \\ & \text{or } P(u_\beta = v \mid u_\alpha = I, H_0) > P(u_\beta = v \mid u_\alpha = I, H_1) \text{ and } C \geq W_1 \\ a_{3,1} \; ; & P(u_\beta = v \mid u_\alpha = I, H_0) > P(u_\beta = v \mid u_\alpha = I, H_1) \text{ and } W_1 > C \geq W_3 \\ a_{3,2} \; ; & P(u_\beta = v \mid u_\alpha = I, H_0) \leq P(u_\beta = v \mid u_\alpha = I, H_1) \text{ and } W_4 > C \geq 0 \\ & \text{or } P(u_\beta = v \mid u_\alpha = I, H_0) > P(u_\beta = v \mid u_\alpha = I, H_1) \text{ and } W_3 > C \geq 0 \end{cases} \qquad (107)$$

The above equations of the thresholds are independent of the position of $\Lambda_\alpha(y_\alpha)$ relative to α_v (Cases 1 and 2).

In the discussion of Cases A and B above, it was shown, using simple but tedious algebra, that :

$$P(u_\beta = v \mid u_\alpha = I, H_0) \le P(u_\beta = v \mid u_\alpha = I, H_1) \quad \Leftrightarrow \quad W_2 \le W_1 \le W_4 \le W_3$$

The parts of the above relation, which are going to be employed in the further simplification of the threshold equations, are presented below:

$$P(u_\beta = v \mid u_\alpha = I, H_0) \le P(u_\beta = v \mid u_\alpha = I, H_1) \quad \Leftrightarrow \quad W_2 \le W_4 \qquad (108)$$

$$P(u_\beta = v \mid u_\alpha = I, H_0) \le P(u_\beta = v \mid u_\alpha = I, H_1) \quad \Leftrightarrow \quad W_2 \le W_1 \qquad (109)$$

$$P(u_\beta = v \mid u_\alpha = I, H_0) > P(u_\beta = v \mid u_\alpha = I, H_1) \quad \Leftrightarrow \quad W_1 > W_3 \qquad (110)$$

$$P(u_\beta = v \mid u_\alpha = I, H_0) \le P(u_\beta = v \mid u_\alpha = I, H_1) \quad \Leftrightarrow \quad W_4 \le W_3 \qquad (111)$$

It is now straightforward to show that equations (108) to (111) together with equations (106) and (107), yield equations (45) and (46) completing the proof of the Theorem.□

Proof of Corollaries 2,3 and 4 : No actual proof will presented. By substituting directly the Gaussian probability density functions on the threshold equations derived for the general case and solving for y_α or y_β, the corresponding thresholds for the Gaussian case are obtained.

Whenever DM A requests information and u_β=x (or z) is returned, the final team decision is always 0 (or 1). So, the notation is changed from x (or z) to 0 (or 1). Moreover, whenever u_β=v is returned, the final decision can either 0 or 1. So, the notation is changed from v to ? indicating that DM B is not sure.

Moreover, the subscripts of the thresholds indicate the decision maker whose decision they characterize and the superscripts of the thresholds indicate the content of the decision:

u (upper) : for any observation greater than the upper threshold the optimal decision is 1 (since we assumed $\mu_0 < \mu_1$).

l (lower) : for any observation smaller than the lower threshold the optimal decision is 0.

f (final) : characterizes the final decision

This completes the discussion of Corollaries 2,3 and 4.

REFERENCES

[1] Van Trees, H.L., "Detection, Estimation and Modulation Theory, vol. I" New York : Wiley, 1969

[2] Tenney, R.R., and Sandell, N.R., "Detection with Distributed Sensors" IEEE Trans. of Aerospace and Electronic Systems, AES-17, July 1981, pp. 501-509

[3] Ekchian, L.K., "Optimal Design of Distributed Detection Networks" Ph.D. Dissertation, Dept. of Elect. Eng. and Computer Science Mass. Inst. of Technology, Cambridge, Mass., 1982

[4] Ekchian, L.K., and Tenney, R.R., "Detection Networks" Proceedings of the 21st IEEE Conference on Decision and Control 1982, pp. 686-691

[5] Kushner, H.J., and Pacut, A., "A Simulation Study of a Decentralized Detection Problem" IEEE Trans. on Automatic Control, 27, Oct. 1982, pp. 1116-1119

[6] Chair, Z., and Varshney, P.K., "Optimal Data Fusion in Multiple Sensor Detection Systems" IEEE Trans. on Aerospace and Electronic Systems, 21, January 1986, pp. 98-101

[7] Boettcher, K.L., "A Methodology for the Analysis and Design of Human Information Processing Organizations" Ph.D. Dissertation, Dept. of Elect. Eng. and Computer Science, Mass. Inst. of Technology, Cambridge, Mass., 1985

[8] Boettcher, K.L., and Tenney, R.R., "On the Analysis and Design of Human Information Processing Organizations" Proceedings of the 8th MIT/ONR Workshop on C^3 Systems, 1985, pp. 69-74

[9] Boettcher, K.L., and Tenney, R.R., "Distributed Decisionmaking with Constrained Decision Makers : A Case Study" Proceedings of the 8th MIT/ONR Workshop on C^3 Systems, 1985, pp. 75-79

[10] Tsitsiklis, J.N., "Problems in Decentralized Decision Making and Computation" Ph.D. Dissertation, Dept. of Elec. Eng. and Computer Science Mass. Inst. of Technology, Cambridge, Mass., 1984

[11] Tsitsiklis, J.N., and Athans, M., "On the Complexity of Decentralized Decision Making and Detection Problems" IEEE Trans. on Automatic Control, 30, May 1985, pp.440-446