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Regularity of Solutions to the Muskat Equation

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Abstract

In this paper, we show that if a solution to the Muskat problem in the case of different densities and the same viscosity is sufficiently smooth, then it must be analytic except at the points where a turnover of the fluids happens.

1. Introduction

The Muskat problem is a free boundary problem studying the interface between fluids in the porous media [30]. It can also describe the Hele–Shaw cell [34]. The density function ρ follows the active scalar equation

$$\frac{\mathrm{d}\rho}{dt} + (v \cdot \nabla)\rho = 0, \tag{1.1}$$

with

$$\rho(x,t) = \begin{cases} \rho_1 & x \in D_1(t), \\ \rho_2 & x \in D_2(t). \end{cases}$$

Here $D_1(t)$ and $D_2(t)$ are open domains with $D_1(t) \cup D_2(t) \cup \partial D_1(t) = \mathbb{R}^2$. The velocity field v in (1.1) satisfies Darcy's law,

$$\frac{\mu}{\kappa} v = -\nabla p - (0, g\rho), \tag{1.2}$$

and the incompressibility condition

$$\nabla \cdot v = 0,$$

where p is the pressure and μ is the viscosity. κ, g are the permeability constant and the gravity force.

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We focus on the problem where two fluids have different densities ρ_1, ρ_2 and the same viscosity μ.

After scaling, the equation for the boundary $\partial D_1(t)$ in the periodic setting read as

$$\frac{\partial f_i}{\partial t}(\alpha,t)=\frac{\rho_2-\rho_1}{2}\int_{-\pi}^{\pi}\frac{\sin(f_1(\alpha)-f_1(\alpha-\beta))(\partial_\alpha f_i(\alpha)-\partial_\alpha f_i(\alpha-\beta))}{\cosh(f_2(\alpha)-f_2(\alpha-\beta))-\cos(f_1(\alpha)-f_1(\alpha-\beta))}\mathrm{d}\beta \tag{1.3}$$

for $i=1,2$ (see [10]). Here $f(\alpha,t)=(f_1(\alpha,t),f_2(\alpha,t))$ is a parameterization of the boundary curve. $f(\alpha,t)-(\alpha,0)$ is periodic in α.

Given an initial interface at time 0, (1.3) is divided into three regimes. When the interface is a graph and the heavier fluid is on the bottom as in Fig. 1a, it is in a stable regime. When heavier fluid is above the boundary as in Fig. 1b, it is in a stable regime when time flows backward. Thus, given any initial data, (1.3) can be solved for small negative time t. In both regimes, shown in Fig. 1a, b, (1.3) can not be solved in the wrong direction unless the initial interface is real analytic. The third regime, shown in Fig. 1c, it highly unstable because the heavier fluid lies on top near point S_1 while the lighter fluid lies on top near point S_2. Note two turnover points T_1 and T_2 where the interface has a vertical tangent. For generic initial data in the turnover regime, (1.3) has no solutions either as time flows forward or backward.

In the third regime, there are several examples from the literature (eg. [9,10,19,20]), but they are all real analytic solutions. Without the real analytic assumption, due to the spatially non-consistent parabolic behavior, the existence is usually false and the uniqueness is unknown. To address this gap, this paper studies to what extent the solution of (1.3) is analytic.

Moreover, for the analytic solutions, one can prove an energy estimate on an analyticity region that shrinks when time increases. That energy estimate implies uniqueness in the class of analytic solutions. [10]. Therefore, the investigation towards analyticity can serve as a first step to deal with the uniqueness.

We introduce a new way to prove that any sufficiently smooth solution is analytic except at the turnover points. Here is our main theorem:

Theorem 1.1. *Let* $f(\alpha,t)=(f_1(\alpha,t),f_2(\alpha,t))\in C^1([-t_0,t_0],H^6[-\pi,\pi]\times H^6[-\pi,\pi]))$ *be a solution of the Muskat equation* (1.3) *satisfying the arc-chord condition. If* $\partial_\alpha f_1(\alpha_0,t)\neq 0$, *and* $-t_0<t<t_0$, *then* $f(\cdot,t)$ *is analytic at* α_0.

Our method concerning the analyticity is not limited to the Muskat problem. A simplified version of our method can be used to show the analyticity of the solution to a kind of non-local differential equations (see Section 10). This approach is new to our best knowledge.

In our forthcoming work [35], we focus on the degenerate analyticity near the turnover points. The existence and uniqueness are crucially related to the way the real-analyticity degenerates at those points. Given an extra assumption, we have the following theorem in [35]:

Theorem 1.2. *Let* $f(\alpha,t)=(f_1(\alpha,t),f_2(\alpha,t))\in C^1([-t_0,t_0],C^{100}([-\pi.\pi])$ *be a solution of the Muskat equation* (1.3) *with two turnover points.* $Z_1(t)$, $Z_2(t)$ *are*

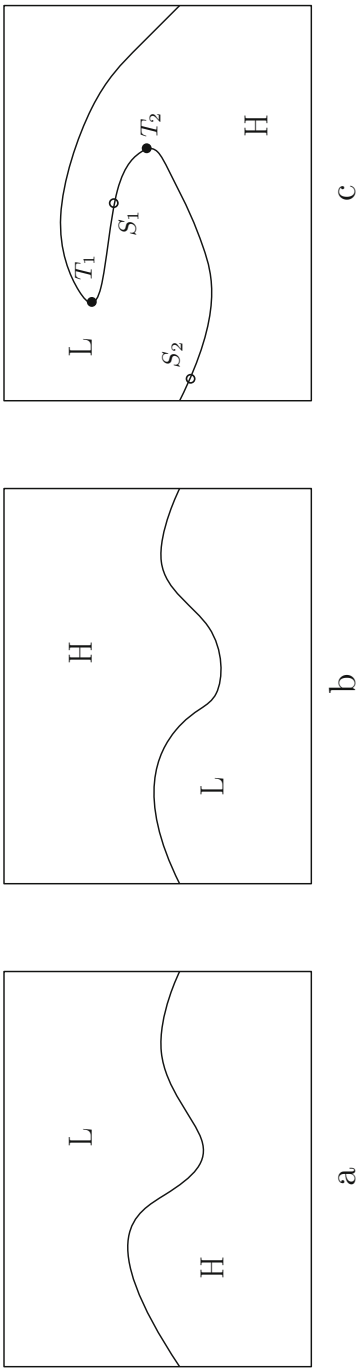

Fig. 1. Three regimes of the Muskat equation

values of α of these two turnover points. If we assume that the solution satisfies the following three conditions:

$$\partial_\alpha^2 f_1(Z_1(t), t) \neq 0, \tag{1.4}$$

$$\partial_\alpha f_1(\alpha, t) \neq 0 \text{ except at } Z_1(t), Z_2(t), \tag{1.5}$$

and

$$\left(\frac{dZ_1}{dt}(t) + \frac{\rho_2 - \rho_1}{2} p.v. \int_{-\pi}^{\pi} \frac{\sin(f_1(\alpha) - f_1(\alpha - \beta))}{\cosh(f_2(\alpha) - f_2(\alpha - \beta)) - \cos(f_1(\alpha) - f_1(\alpha - \beta))} d\beta\right)$$
$$\frac{\rho_2 - \rho_1}{2} \partial_\alpha^2 f_1(Z_1(t), t) < 0, \tag{1.6}$$

then when $-t_0 < t < t_0$, $f(\cdot, t)$ can be analytically extended to region $\Omega = \{x + iy| - \epsilon_1(t) + Z_1(t) \leq x \leq Z_1(t) + \epsilon_1(t), |y| \leq \epsilon_2(t)(x - Z_1(t))^2\}$.

1.1. Background

In order to make the equation well-defined, the arc-chord condition is introduced, saying that

$$F(f) = \left| \frac{\beta^2}{\cosh(f_2(\alpha) - f_2(\alpha - \beta)) - \cos(f_1(\alpha) - f_1(\alpha - \beta))} \right|$$

is in L^∞.

The Rayleigh-Taylor coefficient σ is used to characterize the three regimes in Fig. 1 and is defined as

$$\sigma = \frac{\rho_2 - \rho_1}{2} \frac{\partial_\alpha f_1(\alpha, t)}{(\partial_\alpha f_1(\alpha, t))^2 + (\partial_\alpha f_2(\alpha, t))^2}. \tag{1.7}$$

$\sigma \geq 0$ is corresponding to the stable regime and $\sigma \leq 0$ the backward stable regime. When σ changes sign, it is in the unstable regime.

In the stable regime (heavier liquid is below the lighter liquid), local well-posedness and the global well-posedness with constraints on the initial data have been widely studied, with the lowest space $H^{\frac{3}{2}}$ ([1–7,13–17,21–25,29,31,32,36, 38,39]). The existence of self-similar solutions has also been proved [28]. Interesting readers can see [13,28] for detailed reviews. Due to the parabolic behavior, instant analyticity has been proved in the stable regime. Castro–Córdoba–Fefferman–Gancedo–López-Fernández [10] proved the H^4 solutions become instantly analytic if the solutions remain to be in the stable region for a short time. In [29], also in the stable region, Matioc improved the instant analyticity to H^s, where $s \in (\frac{3}{2}, 3)$. In [27], Gancedo–García-Juárez–Patel–Strain showed that in the stable regime, a medium size initial data in $\dot{\mathcal{F}}^{1,1} \cap L^2$ with $\|f\|_{\mathcal{F}^{1,1}} = \int |\zeta||\hat{f}(\zeta)|\mathrm{d}\zeta$ becomes instantly analytic. Their result also covers the different viscosities case and the 3D case.

When the heavier liquid is above the lighter liquid, the equation is ill-posed when time flows forward [22].

A solution that starts from a stable regime and develops turnover points was first discovered in [10]. That solution still exists for a short time after turnover due to the analyticity when the turnover happens. Moreover, breakdown of smoothness can happen [9]. There are also examples where the solutions transform from stable to unstable and go back to stable [19] and vice versa [20].

Weak solutions and a special kind of weak solutions: mixing solutions of (1.1) have also been studied. They do not satisfy (1.3) and can develop a mixing zone. Weak solutions do not have uniqueness [18]. In all three regimes, there are infinitely many mixing solutions ([8,11,12,26,33,37]).

1.2. The outline of the Proof of Theorem 1.1

Inspired by the instant analyticity results in the stable case [10,27,29], our first idea is localization. If locally the lighter liquid is over the heavier one, we let the time go forward, and if locally the heavier one is over the lighter one, we let the time go backward.

Since it leads to lots of difficulties by the standard method due to the localization, we use a new idea to prove analyticity except at turnover points. The idea is to make a C^1 continuation of the parametrized interface $\alpha \to (f_1(\alpha, t), f_2(\alpha, t))$ to complex α and then prove the C^1 continuation satisfies the Cauchy-Riemann equation. To do so, we break the complex region into curves $\alpha + ic(\alpha)\gamma t$ with $\gamma \in [-1, 1]$. On each such curve, we solve an equation for (f_1, f_2). We then show that when γ varies, our solutions on the curve fit together into an C^1 function of $\alpha + i\beta$. Finally, we prove that C^1 function satisfies the Cauchy-Riemann equation, thus producing the desired analytic continuation.

In Section 3, we define a cut off function $\lambda(\alpha)$ and focus on $f^c(\alpha, t) = \lambda(\alpha) f(\alpha, t)$. We then localize the equation such that the modified R-T condition has a fixed sign. In order to make use of the sign, if the sign is positive, we let the time go forward. If the sign is negative, we let the time go backward.

In Section 4, we introduce $c(\alpha)$ with $\operatorname{supp} c(\alpha) \subset \{\alpha | \lambda(\alpha) = 1\}$ (Fig. 2). With the assumption that $f^c(\alpha, t)$ is analytic in domain $D_A = \{\alpha + i\beta | - c(\alpha)t \le \beta \le c(\alpha)t\}$, we derive the equation on the curve $\{(\alpha + ic(\alpha)\gamma t) | \alpha \in [-\pi, \pi]\}$ for fixed $\gamma \in [-1, 1]$. Then we obtain the equation

$$\frac{d}{dt} z(\alpha, \gamma, t) = T(z(\alpha, \gamma, t), t), \tag{1.8}$$

with $z(\alpha, \gamma, 0) = f^c(\alpha, 0)$. The analyticity assumption on f^c is dropped after we get (1.8).

In Section 5, for each fixed γ, we use the energy estimate and the Galerkin method to show the existence of the solution $z(\alpha, \gamma, t)$. The main term is controlled by Gårding's inequality, where we use a lemma from [9]. This part is similar as to [9,10].

In Sections 6, 7, and 8, we verify that the $z(\alpha, 0, t)$ coincides with the $f^c(\alpha, t)$ and that $z(\alpha, \gamma, t)$ is also smooth enough with respect to γ.

In Section 9, we derive some lemmas about the Cauchy-Riemann operator and use those lemmas to show analyticity of $z(\alpha, \frac{\beta}{c(\alpha)t}\gamma, t)$ by checking that it satisfies the Cauchy-Riemann equations.

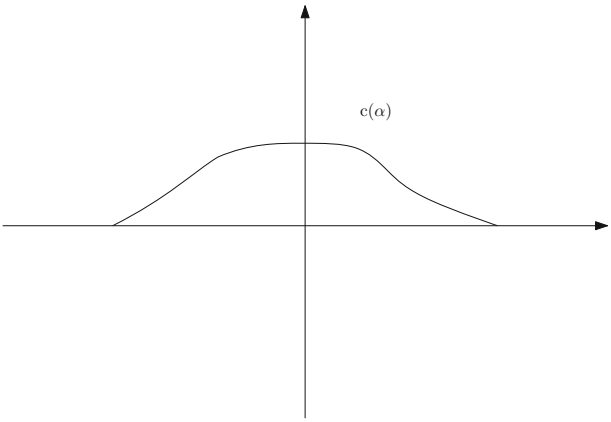

Fig. 2. The curve $c(\alpha)$ for Theorem 1.1

Remark 1.3. In [10], the analyticity domain can be chosen as a strip, and the analyticity follows directly from existence. Since our $c(\alpha)$ is supported in a small region, we do not have such good behavior.

2. Notation

In the paper we will use the following notations:
δ: a sufficiently small number.
$\lambda(\alpha)$: $\lambda(\alpha) \geq 0$ and in $C^{100}(-\infty, \infty)$, satisfying

$$\lambda(\alpha) = \begin{cases} 1 & |\alpha| \leq \delta, \\ 0 & |\alpha| \geq 2\delta. \end{cases}$$

δ_c: sufficiently small number depending on δ.
$c(\alpha)$:

$$\begin{cases} c(\alpha) = \delta_c, \text{ when } |\alpha| \leq \frac{\delta}{32}, \\ \operatorname{supp} c(\alpha) \subset [-\frac{\delta}{8}, \frac{\delta}{8}], \\ c(\alpha) \geq 0, c(\alpha) \in C^{100}(-\infty, \infty), \|c(\alpha)\|_{C^{100}(-\infty,\infty)} \leq \delta. \end{cases}$$

$f(\alpha, t) = (f_1(\alpha, t), f_2(\alpha, t))$: the original solution of the Muskat equation.
$f^c(\alpha, t)$, $\tilde{f}(\alpha, t)$:

$$\begin{aligned} f^c(\alpha, t) &= \lambda(\alpha) f(\alpha, t), \\ \tilde{f}(\alpha, t) &= (1 - \lambda(\alpha)) f(\alpha, t). \end{aligned}$$

t_0: the original solution exists when $t \in [-t_0, t_0]$.
D_A: $D_A = \{(\alpha + i\beta) | -\infty < \alpha < \infty, -c(\alpha)t \leq \beta \leq c(\alpha)t\}$.
For any vector function $z = (z_1, z_2) \in H^k$: $z_1 \in H^k$ and $z_2 \in H^k$.

3. The Localization

This step is to localize the equation such that the R-T coefficient has a fixed sign. Without loss of generality, we study the behavior at origin and let $\frac{\rho_2-\rho_1}{2}=1$. Let $\lambda(\alpha)\in C^{100}(-\infty,\infty)$ satisfying $\lambda(\alpha)\geq 0$ and

$$\lambda(\alpha)=\begin{cases}1 & |\alpha|\leq\delta,\\ 0 & |\alpha|\geq 2\delta,\end{cases}$$

and $f^c(\alpha,t)=f(\alpha,t)\lambda(\alpha)$, $\tilde{f}(\alpha,t)=f(\alpha,t)(1-\lambda(\alpha))$. Here δ is a sufficiently small number such that when $\alpha\in[-2\delta,2\delta]$, $\partial_\alpha f_1(\alpha,0)$ has a fixed sign. Without loss of generality, we assume

$$\partial_\alpha f_1(\alpha,0)>0. \tag{3.1}$$

Then we have

$$\begin{aligned}\frac{\partial f_\mu^c}{\partial t}=&\lambda(\alpha)\int_{-\pi}^{\pi}\frac{\sin(f_1^c(\alpha)-f_1^c(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))(\partial_\alpha f_\mu^c(\alpha)-\partial_\alpha f_\mu^c(\beta))}{\cosh(f_2^c(\alpha)-f_2^c(\beta)+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))-\cos(f_1^c(\alpha)-f_1^c(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta)))}\mathrm{d}\beta\\ &+\lambda(\alpha)\int_{-\pi}^{\pi}\frac{\sin(f_1^c(\alpha)-f_1^c(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))(\partial_\alpha\tilde{f}_\mu(\alpha)-\partial_\alpha\tilde{f}_\mu(\beta))}{\cosh(f_2^c(\alpha)-f_2^c(\beta)+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))-\cos(f_1^c(\alpha)-f_1^c(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta)))}\mathrm{d}\beta.\end{aligned} \tag{3.2}$$

We have $f^c\in C^1([0,t_0],(H^6(\mathbb{T}))^2)$, $\tilde{f}-(\alpha,0)\in C^1([0,t_0],(H^6(\mathbb{T}))^2)$. Here $\mathbb{T}$ is the torus of 2π.

4. The Equation on the Complex Plane

4.1. Change the contour

Let $c(\alpha)$ satisfy

$$\begin{cases}c(\alpha)=\delta_c, \text{ when } |\alpha|\leq\frac{\delta}{32},\\ \operatorname{supp}c(\alpha)\subset[-\frac{\delta}{8},\frac{\delta}{8}],\\ c(\alpha)\geq 0,\ c(\alpha)\in C^{100}(-\infty,\infty),\ \|c(\alpha)\|_{C^{100}(-\infty,\infty)}\leq\delta.\end{cases} \tag{4.1}$$

Here $c(\alpha)$ is defined such that $\tilde{f},\lambda$ can be analytically extended to the complex domain $D_A=\{(\alpha+i\beta)|-\infty<\alpha<\infty,-c(\alpha)t\leq\beta\leq c(\alpha)t\}$ and satisfy

$$\tilde{f}(\alpha+ic(\alpha)\gamma t,t)=\tilde{f}(\alpha,t), \tag{4.2}$$

and

$$\lambda(\alpha+ic(\alpha)\gamma t)=\lambda(\alpha), \tag{4.3}$$

for any $\gamma\in[-1,1]$.

Now we assume f^c is also analytic in this complex domain D_A. For any fixed $\gamma \in [-1, 1]$, we want to find the new equation on the contour $\{\alpha + ic(\alpha)\gamma t | \alpha \in [-\pi, \pi]\}$. Let $\alpha_\gamma^t = \alpha + ic(\alpha)\gamma t$. We have

$$\begin{aligned}
&\frac{df_\mu^c(\alpha_\gamma^t, t)}{dt} = ic(\alpha)\gamma(\partial_\alpha f_\mu^c)(\alpha_\gamma^t, t) + (\partial_t f_\mu^c)(\alpha_\gamma^t, t) \\
&= ic(\alpha)\gamma(\partial_\alpha f_\mu^c)(\alpha_\gamma^t, t) \\
&\quad + \lambda(\alpha_\gamma^t)\int_{-\pi}^{\pi} \frac{\sin(f_1^c(\alpha_\gamma^t, t) - f_1^c(\beta_\gamma^t, t) + \tilde{f}_1(\alpha_\gamma^t, t) - \tilde{f}_1(\beta_\gamma^t, t))((\partial_\alpha f_\mu^c)(\alpha_\gamma^t, t) - (\partial_\beta f_\mu^c)(\beta_\gamma^t, t))}{\cosh(f_2^c(\alpha_\gamma^t, t) - f_2^c(\beta_\gamma^t, t) + \tilde{f}_2(\alpha_\gamma^t, t) - \tilde{f}_2(\beta_\gamma^t, t)) - \cos(f_1^c(\alpha_\gamma^t, t) - f_1^c(\beta_\gamma^t, t) + \tilde{f}_1(\alpha_\gamma^t, t) - \tilde{f}_1(\beta_\gamma^t, t))} \\
&\quad (1 + ic'(\beta)\gamma t)\mathrm{d}\beta \\
&\quad + \lambda(\alpha_\gamma^t)\int_{-\pi}^{\pi} \frac{\sin(f_1^c(\alpha_\gamma^t, t) - f_1^c(\beta_\gamma^t, t) + \tilde{f}_1(\alpha_\gamma^t, t) - \tilde{f}_1(\beta_\gamma^t, t))((\partial_\alpha \tilde{f}_\mu)(\alpha_\gamma^t, t) - (\partial_\beta \tilde{f}_\mu)(\beta_\gamma^t, t))}{\cosh(f_2^c(\alpha_\gamma^t, t) - f_2^c(\beta_\gamma^t, t) + \tilde{f}_2(\alpha_\gamma^t, t) - \tilde{f}_2(\beta_\gamma^t, t)) - \cos(f_1^c(\alpha_\gamma^t, t) - f_1^c(\beta_\gamma^t, t) + \tilde{f}_1(\alpha_\gamma^t, t) - \tilde{f}_1(\beta_\gamma^t, t))} \\
&\quad (1 + ic'(\beta)\gamma t)\mathrm{d}\beta.
\end{aligned} \tag{4.4}$$

4.2. The equation on the curve

Let $z(\alpha, \gamma, t)$ be the solution of the equation (4.4) with initial data $z(\alpha, \gamma, 0) = f^c(\alpha, 0)$. Our motivation is to set $z(\alpha, \gamma, t) = f^c(\alpha + ic(\alpha)\gamma t, t)$. Since

$$(\partial_\alpha f_\mu^c)(\alpha_\gamma^t, t) = \frac{\partial_\alpha(f_\mu^c(\alpha_\gamma^t, t))}{1 + ic'(\alpha)\gamma t},$$

we have

$$\begin{aligned}
&\frac{dz_\mu(\alpha, \gamma, t)}{dt} = \frac{ic(\alpha)\gamma}{1 + ic'(\alpha)\gamma t}\partial_\alpha z_\mu(\alpha, \gamma, t) \\
&\quad + \lambda(\alpha_\gamma^t)\int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha, \gamma, t) - z_1(\beta, \gamma, t) + \tilde{f}_1(\alpha_\gamma^t, t) - \tilde{f}_1(\beta_\gamma^t, t))(\frac{\partial_\alpha z_\mu(\alpha, \gamma, t)}{1+ic'(\alpha)\gamma t} - \frac{\partial_\beta z_\mu(\beta, \gamma, t)}{1+ic'(\beta)\gamma t})(1 + ic'(\beta)\gamma t)\mathrm{d}\beta}{\cosh(z_2(\alpha, \gamma, t) - z_2(\beta, \gamma, t) + \tilde{f}_2(\alpha_\gamma^t, t) - \tilde{f}_2(\beta_\gamma^t, t)) - \cos(z_1(\alpha, \gamma, t) - z_1(\beta, \gamma, t) + \tilde{f}_1(\alpha_\gamma^t, t) - \tilde{f}_1(\beta_\gamma^t, t))} \\
&\quad + \lambda(\alpha_\gamma^t)\int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha, \gamma, t) - z_1(\beta, \gamma, t) + \tilde{f}_1(\alpha_\gamma^t, t) - \tilde{f}_1(\beta_\gamma^t, t))((\partial_\alpha \tilde{f}_\mu)(\alpha_\gamma^t, t) - (\partial_\beta \tilde{f}_\mu)(\beta_\gamma^t, t))(1 + ic'(\beta)\gamma t)\mathrm{d}\beta}{\cosh(z_2(\alpha, \gamma, t) - z_2(\beta, \gamma, t) + \tilde{f}_2(\alpha_\gamma^t, t) - \tilde{f}_2(\beta_\gamma^t, t)) - \cos(z_1(\alpha, \gamma, t) - z_1(\beta, \gamma, t) + \tilde{f}_1(\alpha_\gamma^t, t) - \tilde{f}_1(\beta_\gamma^t, t))}
\end{aligned} \tag{4.5}$$

with

$$z(\alpha, \gamma, 0) = f(\alpha, 0).$$

We drop the analyticity assumption of f^c from now. Notice that $\tilde{f}$ and λ can still be analytically extended to D_A as in (4.2) and (4.3).

5. The Existence of z for Fixed γ

5.1. Energy estimate

We first assume z is of finite Fourier modes here and do the energy estimate. The idea of the energy estimate is similar as in [9,10].

Since $\tilde{f}(\alpha + ic(\alpha)\gamma t, t) = \tilde{f}(\alpha, t)$, $\lambda(\alpha + ic(\alpha)\gamma t) = \lambda(\alpha)$, we have

$$\begin{aligned}\frac{dz_\mu(\alpha,\gamma,t)}{dt} = T(z) &= \frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha z_\mu(\alpha)\\ &+\lambda(\alpha)\int_{-\pi}^{\pi}\frac{\sin(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))\left(\frac{\partial_\alpha z_\mu(\alpha)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z_\mu(\beta)}{1+ic'(\beta)\gamma t}\right)(1+ic'(\beta)\gamma t)}{\cosh(z_2(\alpha)-z_2(\beta)+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))-\cos(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))}\mathrm{d}\beta\\ &+\lambda(\alpha)\int_{-\pi}^{\pi}\frac{\sin(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))((\partial_\alpha\tilde{f}_\mu)(\alpha)-(\partial_\beta\tilde{f}_\mu)(\beta))(1+ic'(\beta)\gamma t)}{\cosh(z_2(\alpha)-z_2(\beta)+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))-\cos(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))}\mathrm{d}\beta.\end{aligned} \tag{5.1}$$

Here we omit the dependency of z on γ and t, and the dependency of $\tilde{f}$ on t for the sake of simplicity. Let

$$U = (H^5(\mathbb{T}))^2,$$

where $\mathbb{T}$ is the torus of length 2π and

$$\|z\|_{Arc} = \sup_{\alpha\in[-2\delta,2\delta],\beta\in[-\pi,\pi]}\frac{(\alpha-\beta)^2}{|\cosh(z_2(\alpha)-z_2(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))-\cos(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))|}.$$

For the L^2 norm, we have

$$\|T(z)\|_{L^2[-\pi,\pi]} \lesssim \|T(z)\|_{L^\infty[-\pi,\pi]} \lesssim C(\|z\|_U)\|z\|_{Arc}.$$

Here C is a bounded function depending on δ, δ_c and $\|f\|_{C^1([0,t],(H^6[-\pi,\pi])^2)}$. We will keep using the same notation C in the following proof.

Now we take 5th derivative and have

$$\begin{aligned}\partial_\alpha^5 T(z) &= \frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha^6 z_\mu(\alpha)\\ &+\lambda(\alpha)\int_{-\pi}^{\pi}\frac{\sin(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))(\frac{\partial_\alpha^6 z_\mu(\alpha)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta^6 z_\mu(\beta)}{1+ic'(\beta)\gamma t})(1+ic'(\beta)\gamma t)}{\cosh(z_2(\alpha)-z_2(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))-\cos(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))}\mathrm{d}\beta\\ &+\lambda(\alpha)\int_{-\pi}^{\pi}\frac{\sin(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))((\partial_\alpha^6\tilde{f}_\mu)(\alpha)-(\partial_\beta^6\tilde{f}_\mu)(\beta))(1+ic'(\beta)\gamma t)}{\cosh(z_2(\alpha)-z_2(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))-\cos(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))}\mathrm{d}\beta\\ &+\sum_i O^i = T_1(z)+T_2(z)+T_3(z)+\sum_i O^i.\end{aligned}$$

Here O^i terms contain at most 5th derivative on both z and $\tilde{f}$.

Before we show the explicit form of O^i, we introduce some notations. Let

$$\begin{aligned}V_g^k(\alpha) &= (g_1(\alpha,t), g_2(\alpha,t), \partial_\alpha g_1(\alpha,t), \partial_\alpha g_2(\alpha,t), ..., \partial_\alpha^k g_1(\alpha,t), \partial_\alpha^k g_2(\alpha,t)),\\ \tilde{V}_g^k(\alpha) &= \left(\frac{g_1(\alpha,t)}{1+ic'(\alpha)\gamma t}, \frac{g_2(\alpha,t)}{1+ic'(\alpha)\gamma t}, ..., \partial_\alpha^k g_1(\alpha,t)\partial_\alpha^5\left(\frac{1}{1+ic'(\alpha)\gamma t}\right),\right.\\ &\left.\partial_\alpha^k g_2(\alpha,t)\partial_\alpha^5\left(\frac{1}{1+ic'(\alpha)\gamma t}\right)\right).\end{aligned} \tag{5.2}$$

$V_{g,i}^k(\alpha)$ is the ith component in $V_g^k(\alpha)$ and $\tilde{V}_{g,i}^k(\alpha)$ the ith component in $\tilde{V}_g^k(\alpha)$.

When we write $X_i(\alpha, t)$, we mean

$$X_i(\alpha, t) = \partial_\alpha^{l_i}\left(\frac{1}{1 + ic'(\alpha)\gamma t}\right), \tag{5.3}$$

with $l_i \leq 5$.

A function $K^j_{-\sigma}(A, B)$, $K^j_{-\sigma}(A, B, C)$ is of $-\sigma$ type if, for A, B, C in R^n, it has the form

$$c_j \frac{\sin(A_1 + B_1)^{m_1}\cos(A_1 + B_1)^{m_2}}{(\cosh(A_2 + B_2) - \cos(A_1 + B_1))^{m_0}} \times (\sinh(A_2 + B_2))^{m_3}(\cosh(A_2 + B_2))^{m_4}\Pi_{j=1}^{m_5}(A_{\lambda_j})\Pi_{j=1}^{m_6}(B_{\lambda_{j,2}})\Pi_{j=1}^{m_7}(C_{\lambda_{j,3}}), \tag{5.4}$$

with $m_1 + m_3 + m_5 + m_6 + m_7 - 2m_0 \geq -\sigma$. c_j is a constant.

We claim that we can write O^i as following three types, by separating the highest order term in the derivative. Here we omit the dependency on γ and t.

$$O^{1,i} = \partial_\alpha^{b_i}\lambda(\alpha)\int_{-\pi}^{\pi} K^i_{-1}(V_z^3(\alpha) - V_z^3(\beta), V_{\tilde{f}}^3(\alpha) - V_{\tilde{f}}^3(\beta), \tilde{V}_z^3(\alpha) - \tilde{V}_z^3(\beta))X_{i'}(\beta)(\tilde{z}^3(\alpha) - \tilde{z}^3(\beta))\mathrm{d}\beta,$$

where $\tilde{z}^3 \in V_z^3 \cup \tilde{V}_z^3 \cup V_{\tilde{f}}^3$, $1 \leq b_i \leq 5$.

$$O^{2,i} = \partial_\alpha^{b_i}\lambda(\alpha)\int_{-\pi}^{\pi} K^i_{-1}(V_z^2(\alpha) - V_z^2(\beta), V_{\tilde{f}}^2(\alpha) - V_{\tilde{f}}^2(\beta), \tilde{V}_z^2(\alpha) - \tilde{V}_z^2(\beta))X_{i'}(\beta)(\tilde{z}^5(\alpha) - \tilde{z}^5(\beta))\mathrm{d}\beta,$$

where $\tilde{z}^5 \in V_z^5 \cup \tilde{V}_z^5 \cup V_{\tilde{f}}^5$, $1 \leq b_i \leq 5$.

$$O^{3,i} = \partial_\alpha^{b_i}\left(\frac{ic(\alpha)\gamma}{1 + ic'(\alpha)\gamma t}\right)\partial_\alpha^{b_i'} z_\mu(\alpha),$$

where $1 \leq b_i, b_i' \leq 5$.

Then we have the following lemmas:

Lemma 5.1. *We have*

$$\|O^i\|_{L^2[-\pi,\pi]} \lesssim C(\|z\|_U, \|z\|_{Arc}).$$

Proof. Since K^i_{-1} is of -1 type, we have

$$\|K^i_{-1}(V_z^3(\alpha) - V_z^3(\beta), V_{\tilde{f}}^3(\alpha) - V_{\tilde{f}}^3(\beta), \tilde{V}_z^3(\alpha) - \tilde{V}_z^3(\beta))(\alpha - \beta)\|_{C^0[-2\delta,2\delta]\times[-\pi,\pi]} \lesssim C(\|z\|_U, \|z\|_{Arc}),$$

we could use Lemma 11.4 to get the result for $O^{1,i}$. Moreover, we have

$$\|K^i_{-1}(V_z^2(\alpha) - V_z^2(\beta), V_{\tilde{f}}^2(\alpha) - V_{\tilde{f}}^2(\beta), \tilde{V}_z^2(\alpha) - \tilde{V}_z^2(\beta))(\alpha - \beta)\|_{C^1[-2\delta,2\delta]\times[-\pi,\pi]} \lesssim C(\|z\|_U, \|z\|_{Arc}),$$

we then use Lemma 11.2 to get the estimate for $O^{2,i}$. $O^{3,i}$ can be bounded easily. □

Lemma 5.2. *We have*

$$\|T_3(z)\|_{L^2[-\pi,\pi]} \lesssim C(\|z\|_U, \|z\|_{Arc}).$$

Proof. Let

$$K(z(\alpha)-z(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta)) = \frac{\sin(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))}{\cosh(z_2(\alpha)-z_2(\beta)+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))-\cos(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))}. \tag{5.5}$$

It is also of K_{-1} type. We have

$$\|K(z(\alpha)-z(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta))(\alpha-\beta)\|_{C^1[-2\delta,2\delta]\times[-\pi,\pi]} \lesssim C(\|z\|_U, \|z\|_{Arc}).$$

Then the result follows from Lemma 11.2. □

Then we are left to deal with $T_1 + T_2$.

By using the same notation as in Lemma 5.2, we have

$$\begin{aligned} T_2(z(\alpha)) &= \lambda(\alpha) p.v. \int_{-\pi}^{\pi} K(z(\alpha)-z(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta))(1+ic'(\beta)\gamma t)\mathrm{d}\beta \frac{\partial_\alpha^6 z_\mu(\alpha)}{(1+ic'(\alpha)\gamma t)} \\ &\quad - \lambda(\alpha) p.v. \int_{-\pi}^{\pi} K(z(\alpha)-z(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta))\partial_\beta^6 z_\mu(\beta)\mathrm{d}\beta \\ &= T_{2,1}(z)(\alpha) + T_{2,2}(z)(\alpha). \end{aligned}$$

Moreover, we could further split the $T_{2,2}$ and have

$$\begin{aligned} T_{2,2}(z)(\alpha) &= -\lambda(\alpha) p.v. \int_{-\pi}^{\pi} K(z(\alpha)-z(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta))\partial_\beta^6 z_\mu(\beta)\mathrm{d}\beta \\ &= -\lambda(\alpha) \lim_{\beta\to\alpha} \left(K(z(\alpha)-z(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta)) \tan\left(\frac{\alpha-\beta}{2}\right)\right) p.v. \\ &\quad \int_{-\pi}^{\pi} \cot\left(\frac{\alpha-\beta}{2}\right) \partial_\beta^6 z(\beta)\mathrm{d}\beta \\ &\quad - \lambda(\alpha) \int_{-\pi}^{\pi} \left(K(z(\alpha)-z(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta)) \tan\left(\frac{\alpha-\beta}{2}\right) \right. \\ &\quad \left. - \lim_{\beta\to\alpha} \left(K(z(\alpha)-z(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta)) \tan\left(\frac{\alpha-\beta}{2}\right)\right)\right) \\ &\quad \cot\left(\frac{\alpha-\beta}{2}\right) \partial_\beta^6 z(\beta)\mathrm{d}\beta \\ &= T_{2,2,1}(z) + T_{2,2,2}(z). \end{aligned}$$

Since $K(z(\alpha)-z(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta))$ is of -1 type, we have

$$\begin{aligned} &\|K(z(\alpha)-z(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta)) \tan\left(\frac{\alpha-\beta}{2}\right)\|_{C^3_{\alpha,\beta}([-2\delta,2\delta]\times[-\pi,\pi])} \\ &\lesssim C(\|z\|_{Arc} + \|z\|_U). \end{aligned} \tag{5.6}$$

Let

$$\tilde{K}(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta)) = \frac{K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\tan\left(\frac{\alpha-\beta}{2}\right)-\lim_{\beta\to\alpha}\left(K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\tan\left(\frac{\alpha-\beta}{2}\right)\right)}{\tan\left(\frac{\alpha-\beta}{2}\right)}.$$

Then

$$\begin{aligned}&\|\tilde{K}(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\|_{C^2_{\alpha,\beta}([-2\delta,2\delta]\times[-\pi,\pi])}\\&\lesssim C(\|z\|_{Arc}+\|z\|_U).\end{aligned}$$

We can do the integration by parts in $T_{2,2,2}(z)$ to get that

$$\begin{aligned}T_{2,2,2}(z)(\alpha)&=-\lambda(\alpha)\int_{-\pi}^{\pi}\tilde{K}(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\partial_\beta^6 z(\beta)\mathrm{d}\beta\\&=\lambda(\alpha)\int_{-\pi}^{\pi}\partial_\beta\tilde{K}(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\partial_\beta^5 z(\beta)\mathrm{d}\beta.\end{aligned}$$

Therefore we have

$$\|T_{2,2,2}(z)(\alpha)\|_{L^2[-\pi,\pi]}\lesssim C(\|z\|_{Arc}+\|z\|_U)\|\partial_\alpha^5 z(\alpha)\|_{L^2[-\pi,\pi]}.$$

In conclusion, we have

$$\begin{aligned}\frac{\mathrm{d}\partial_\alpha^5 z_\mu(\alpha,\gamma,t)}{dt}&=T_1(z)+T_{2,1}(z)+T_{2,2,1}(z)+(T_{2,2,2}(z)+T_3(z)+\sum_i O^i(z))\\&=\Bigg(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}+\frac{\lambda(\alpha)}{1+ic'(\alpha)\gamma t}p.v.\\&\quad\int_{-\pi}^{\pi}K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(1+ic'(\beta)\gamma t)\mathrm{d}\beta\Bigg)\partial_\alpha^6 z_\mu(\alpha)\\&\quad-\lambda(\alpha)\lim_{\beta\to\alpha}\left(K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\tan\left(\frac{\alpha-\beta}{2}\right)\right)2\pi\Lambda(\partial_\alpha^5 z)(\alpha)\\&\quad+(T_{2,2,2}(z)+T_3(z)+\sum_i O^i(z)),\end{aligned}\tag{5.7}$$

where Λ is $(-\Delta)^{\frac{1}{2}}$ on the Torus $\mathbb{T}$ of length 2π and

$$\left\|(T_{2,2,2}(z)+T_3(z)+\sum_i O^i)\right\|_{L^2[-\pi,\pi]}\lesssim C(\|z\|_{Arc}+\|z\|_U).\tag{5.8}$$

Then we have

$$\begin{aligned}
\frac{d}{dt}\|\partial_\alpha^5 z(\alpha,\gamma,t)\|^2_{L^2_\alpha[-\pi,\pi]} &= 2\Re\int_{-\pi}^{\pi}\partial_\alpha^5 z(\alpha,\gamma,t)\cdot\overline{\partial_\alpha^5\frac{d}{dt}z(\alpha,\gamma,t)}\mathrm{d}\alpha\\
&= \sum_{\mu=1,2}\underbrace{(2\Re\int_{-\pi}^{\pi}\partial_\alpha^5 z_\mu(\alpha)\overline{\left(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}+\frac{\lambda(\alpha)}{1+ic'(\alpha)\gamma t}p.v.\int_{-\pi}^{\pi}K(z(\alpha)-z(\beta),\tilde f(\alpha)-\tilde f(\beta))(1+ic'(\beta)\gamma t)\mathrm{d}\beta\right)}}_{\text{main term}}\\
&\quad\underbrace{\overline{\cdot\partial_\alpha^6 z_\mu(\alpha)}\mathrm{d}\alpha}_{\text{main term}}\\
&\quad\underbrace{-2\Re\int_{-\pi}^{\pi}\partial_\alpha^5 z_\mu(\alpha)\lambda(\alpha)\overline{\lim_{\beta\to\alpha}\left(K(z(\alpha)-z(\beta),\tilde f(\alpha)-\tilde f(\beta))\tan\left(\frac{\alpha-\beta}{2}\right)\right)2\pi\Lambda(\partial_\alpha^5 z_\mu)(\alpha)\mathrm{d}\alpha)}}_{\text{main term}}\\
&\quad+B.T,
\end{aligned}\tag{5.9}$$

where $B.T. \le C(\|z\|_{Arc}+\|z\|_U)$. Next we show a lemma for controlling the main terms.

Lemma 5.3. *If $L_1(\alpha)$, $L_2(\alpha)\in C^2(\mathbb{T})$, $-\Re L_1(\alpha)\ge|\Im L_2(\alpha)|$, $h\in H^1(\mathbb{T})$, then we have*

$$\begin{aligned}
&\Re\left(\int_{-\pi}^{\pi}h(\alpha)\overline{L_1(\alpha)(\Lambda h)(\alpha)}d\alpha+\int_{-\pi}^{\pi}h(\alpha)\overline{L_2(\alpha)\partial_\alpha h(\alpha)}d\alpha\right)\\
&\quad\le C(\|L_1\|_{C^2(\mathbb{T})}+\|L_2\|_{C^2(\mathbb{T})})\|h(\alpha)\|^2_{L^2}.
\end{aligned}$$

Proof. First, we have

$$\begin{aligned}
I_2 &= \Re\left(\int_{-\pi}^{\pi}h(\alpha)\overline{L_2(\alpha)\partial_\alpha h(\alpha)}\mathrm{d}\alpha\right)\\
&= \int_{-\pi}^{\pi}\Re L_2(\alpha)(\Re h(\alpha)\Re h'(\alpha)+\Im h(\alpha)\Im h'(\alpha))\mathrm{d}\alpha\\
&\quad-\int_{-\pi}^{\pi}\Im L_2(\alpha)(\Re h(\alpha)\Im h'(\alpha)-\Im h(\alpha)\Re h'(\alpha))\mathrm{d}\alpha\\
&= I_{2,1}+I_{2,s}.
\end{aligned}$$

We could do the integration by parts to $I_{2,1}$ and have

$$|I_{2,1}| = \left|\frac{1}{2}\int_{-\pi}^{\pi}\frac{\mathrm{d}\Re L_2(\alpha)}{\mathrm{d}\alpha}((\Re h(\alpha))^2+(\Im h(\alpha))^2)\mathrm{d}\alpha\right|\lesssim C(\|L_2(\alpha)\|_{C^1(\mathbb{T})})\|h(\alpha)\|^2_{L^2}.$$

Moreover,

$$\begin{aligned}
I_1 &= \Re\left(\int_{-\pi}^{\pi}h(\alpha)\overline{L_1(\alpha)(\Lambda h)(\alpha)}\mathrm{d}\alpha\right)\\
&= \int_{-\pi}^{\pi}\Re L_1(\alpha)(\Re h(\alpha)\Lambda\Re h(\alpha)+\Im h(\alpha)\Lambda\Im h(\alpha))\mathrm{d}\alpha\\
&\quad-\int_{-\pi}^{\pi}\Im L_1(\alpha)(\Re h(\alpha)\Lambda\Im h(\alpha)-\Im h(\alpha)\Lambda\Re h(\alpha))\mathrm{d}\alpha\\
&= I_{1,M}+I_{1,2}.
\end{aligned}$$

We can still do the integration by parts to the $I_{1,2}$ and have

$$\begin{aligned}&\int_{-\pi}^{\pi} \Im L_1(\alpha)(\Re h(\alpha)\Lambda\Im h(\alpha) - \Im h(\alpha)\Lambda\Re h(\alpha))\mathrm{d}\alpha\\ &= \int_{-\pi}^{\pi} (\Lambda(\Im L_1 \Re h)(\alpha) - \Im L_1(\alpha)\Lambda\Re h(\alpha))\Im h(\alpha)\mathrm{d}\alpha.\end{aligned}$$

Moreover, for any $g_1 \in H^2(\mathbb{T})$, $g_2 \in H^1(\mathbb{T})$, we have

$$\|\Lambda(g_1 g_2) - g_1\Lambda g_2\|_{L^2(\mathbb{T})} \lesssim \|g_2\|_{L^2(\mathbb{T})}\|g_1\|_{H^2(\mathbb{T})}.$$

Hence

$$\begin{aligned}I_{1,2} &\lesssim \|\Im L_1(\alpha)\|_{H^2(\mathbb{T})}\|\Im h(\alpha)\|_{L^2(\mathbb{T})}\|\Re h(\alpha)\|_{L^2(\mathbb{T})}\\ &\lesssim \|\Im L_1(\alpha)\|_{H^2(\mathbb{T})}\|h(\alpha)\|^2_{L^2(\mathbb{T})}.\end{aligned}$$

Now we are left to control $I_{1,M} + I_{2,s}$. We have

$$\begin{aligned}I_{1,M} + I_{2,s} = &\int_{-\pi}^{\pi} \Re L_1(\alpha)(\Re h(\alpha)\Lambda\Re h(\alpha) + \Im h(\alpha)\Lambda\Im h(\alpha))\mathrm{d}\alpha\\ &+ \int_{-\pi}^{\pi} \Im L_2(\alpha)(\Re h(\alpha)\Im h'(\alpha) - \Im h(\alpha)\Re h'(\alpha))\mathrm{d}\alpha.\end{aligned}$$

Now we use a lemma from [9, Section 2.4].

Lemma 5.4. *Let a, b be real valued functions on $\mathbb{T}$, $a(\alpha) \geq |b(\alpha)|$ and satisfying $a, b \in C^2(\mathbb{T})$. Then we have*

$$\Re \int_{\mathbb{T}} \overline{f(x)}(a(x)\Lambda f(x) + b(x)if'(x))dx \geq -C(\|a\|_{C^2(\mathbb{T})} + \|b\|_{C^2(\mathbb{T})}) \int_{\mathbb{T}} |f(x)|^2 dx.$$

Then, from Lemma 5.4, we have

$$I_{1,M} + I_{2,s} \lesssim C(\|L_1\|_{C^2(\mathbb{T})} + \|L_2\|_{C^2(\mathbb{T})})\|h(\alpha)\|^2_{L^2}.$$

Then we get the result. □

Now let

$$\begin{aligned}L^1_z(\alpha,\gamma,t) &= -2\pi \lim_{\beta\to 0}\left(K(z(\alpha,\gamma,t) - z(\beta,\gamma,t), \tilde{f}(\alpha,t) - \tilde{f}(\beta,t))\tan\left(\frac{\alpha-\beta}{2}\right)\right)\\ &= -2\pi\frac{\partial_\alpha z_1(\alpha,\gamma,t) + \partial_\alpha \tilde{f}_1(\alpha,t)}{(\partial_\alpha z_1(\alpha,\gamma,t) + \partial_\alpha \tilde{f}_1(\alpha,t))^2 + (\partial_\alpha z_2(\alpha,\gamma,t) + \partial_\alpha \tilde{f}_2(\alpha,t))^2},\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}L^2_z(\alpha,\gamma,t) = &\left(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t} + \frac{1}{1+ic'(\alpha)\gamma t}p.v.\right.\\ &\left.\int_{-\pi}^{\pi} K(z(\alpha,\gamma,t) - z(\beta,\gamma,t), \tilde{f}(\alpha,t) - \tilde{f}(\beta,t))(1+ic'(\beta)\gamma t)\mathrm{d}\beta\right).\end{aligned} \tag{5.11}$$

Since $\text{supp}\, c(\alpha) \subset \{\alpha | \lambda(\alpha) = 1\}$, from (5.9), we have

$$\begin{aligned}
&\frac{d}{dt}\|\partial_\alpha^5 z(\alpha,\gamma,t)\|^2_{L^2_\alpha[-\pi,\pi]} \\
&= \sum_{\mu=1,2} \underbrace{2\Re \int_{-\pi}^{\pi} \partial_\alpha^5 z_\mu(\alpha)\overline{\lambda(\alpha) L_z^2(\alpha)\partial_\alpha^6 z_\mu(\alpha)}\mathrm{d}\alpha}_{\text{main term}} + \underbrace{2\Re \int_{-\pi}^{\pi} \partial_\alpha^5 z_\mu(\alpha)\overline{\lambda(\alpha) L_z^1(\alpha)\Lambda(\partial_\alpha^5 z_\mu)(\alpha)}\mathrm{d}\alpha}_{\text{main term}} \\
&\quad + B.T.
\end{aligned}$$

From Lemma 5.3, if $-\Re L_z^1(\alpha) \geq |\Im L_z^2(\alpha)|$ for $\alpha \in [-2\delta, 2\delta]$, since $\text{supp}\,\lambda \subset [-2\delta, 2\delta]$, we have

$$\frac{d}{dt}\|\partial_\alpha^5 z(\alpha,\gamma,t)\|^2_{L^2_\alpha[-\pi,\pi]} \lesssim B.T.$$

Moreover, when $t = 0$, $\alpha \in [-2\delta, 2\delta]$, from (5.10), (5.11) and (5.5) we have

$$\begin{aligned}
-\Re L_z^1(\alpha,\gamma,0) &= 2\pi \frac{\partial_\alpha f_1^c(\alpha,0) + \partial_\alpha \tilde{f}_1(\alpha,0)}{(\partial_\alpha f_1^c(\alpha,0) + \partial_\alpha \tilde{f}_1(\alpha,0))^2 + (\partial_\alpha f_2^c(\alpha,0) + \partial_\alpha \tilde{f}_2(\alpha,0))^2} \\
&= 2\pi \frac{\partial_\alpha f_1(\alpha,0)}{(\partial_\alpha f_1(\alpha,0))^2 + (\partial_\alpha f_2(\alpha,0))^2},
\end{aligned}$$

and

$$\begin{aligned}
\Im L_z^2(\alpha,0) &= ic(\alpha)\gamma + \Im(p.v.\int_{-\pi}^{\pi} K(f^c(\alpha,0) - f^c(\beta,0), \tilde{f}(\alpha,0) - \tilde{f}(\beta,0))\mathrm{d}\beta) \\
&= ic(\alpha)\gamma.
\end{aligned}$$

From (3.1), we could choose δ_c in (4.1) to be sufficiently small and have

$$\inf_{\alpha\in[-2\delta,2\delta]} (-\Re L_z^1(\alpha,\gamma,0) - |\Im L_z^2(\alpha,\gamma,0)|) > 0.$$

Then let

$$\|z\|_{RT}(t) = \sup_{\alpha\in[-2\delta,2\delta]} \frac{1}{|\Re L_z^1(\alpha,\gamma,t) + |\Im L_z^2(\alpha,\gamma,t)||}.$$

If $\|z\|_{RT}(t) < \infty$, we have

$$\frac{d}{dt}\|z\|_U^2 \lesssim C(\|z\|_U + \|z\|_{Arc}). \tag{5.12}$$

Therefore, we could let $\|z\|_{\tilde{U}} = \|z\|_U + \|z\|_{Arc} + \|z\|_{RT}$. From (5.12), and the following Lemma 5.5, we have

$$\frac{d}{dt}\|z\|_{\tilde{U}}^2 \lesssim C(\|z\|_{\tilde{U}}).$$

Then $\|z(\alpha,\gamma,t)\|_{\tilde{U}}$ is bounded for sufficiently small time t_1. We claim that the bound and the time can be chosen such that it holds for all $\gamma \in [-1, 1]$.

Lemma 5.5. *We have the following two estimates:*

$$\frac{d}{dt}\|z\|_{Arc} \lesssim C(\|z\|_{\tilde{U}}),$$

and

$$\frac{d}{dt}\|z\|_{RT} \lesssim C(\|z\|_{\tilde{U}}).$$

Proof. For $\|z\|_{RT}$, we have

$$\begin{aligned}
&\frac{d}{dt}\sup_{\alpha\in[-2\delta,2\delta]}\frac{1}{|\Re L_z^1(\alpha)+|\Im L_z^2(\alpha)||}\\
&\quad\le \sup_{\alpha\in[-2\delta,2\delta]}\frac{1}{|\Re L_z^1(\alpha)+|\Im L_z^2(\alpha)||^2}\left(\left\|\frac{d}{dt}L_z^1(\alpha)\right\|_{L_\alpha^\infty[-2\delta,2\delta]}+\|\frac{d}{dt}L_z^2(\alpha)\|_{L_\alpha^\infty[-2\delta,2\delta]}\right).
\end{aligned}$$

From (5.10), we have

$$\begin{aligned}
&\left\|\frac{d}{dt}L_z^1(\alpha)\right\|_{L_\alpha^\infty[-2\delta,2\delta]}\lesssim C(\|z\|_{\tilde{U}})\left(\left\|\frac{d}{dt}z(\alpha,\gamma,t)\right\|_{C^1(\mathbb{T})}+\left\|\frac{d}{dt}\tilde{f}(\alpha,t)\right\|_{C^1(\mathbb{T})}\right)\\
&\quad\lesssim C(\|z\|_{\tilde{U}})(\|T(z)\|_{C^1(\mathbb{T})}+C).
\end{aligned}$$

From (5.1), it is easy to get

$$\|T(z)\|_{C^1(\mathbb{T})}\lesssim C(\|z\|_{\tilde{U}}).$$

Then

$$\left\|\frac{d}{dt}L_z^1(\alpha)\right\|_{L_\alpha^\infty[-2\delta,2\delta]}\lesssim C(\|z\|_{\tilde{U}}).$$

From (5.11), we have

$$\begin{aligned}
&\left\|\frac{d}{dt}L_z^2(\alpha,\gamma,t)\right\|_{L_\alpha^\infty[-2\delta,2\delta]}\lesssim C+C\|\frac{d}{dt}p.v.\int_{-\pi}^{\pi}K(z(\alpha)-z(\beta),\tilde{f}(\alpha)\\
&\quad-\tilde{f}(\beta))(1+ic'(\beta)\gamma t)\mathrm{d}\beta\|_{L_\alpha^\infty[-2\delta,2\delta]}\\
&\quad+\|p.v.\int_{-\pi}^{\pi}K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(1+ic'(\beta)\gamma t)\mathrm{d}\beta\|_{L_\alpha^\infty[-2\delta,2\delta]}\\
&\le\|p.v.\int_{-\pi}^{\pi}\nabla_1K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\cdot\\
&\left(\frac{dz}{dt}(\alpha)-\frac{dz}{dt}(\beta)+\frac{\mathrm{d}\tilde{f}}{dt}(\alpha)-\frac{\mathrm{d}\tilde{f}}{dt}(\beta)\right)(1+ic'(\beta)\gamma t)\mathrm{d}\beta\|_{L_\alpha^\infty[-2\delta,2\delta]}\\
&\quad+\|p.v.\int_{-\pi}^{\pi}K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(ic'(\beta)\gamma)\mathrm{d}\beta\|_{L_\alpha^\infty[-2\delta,2\delta]}\\
&\quad+\|p.v.\int_{-\pi}^{\pi}K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(1+ic'(\beta)\gamma t)\mathrm{d}\beta\|_{L_\alpha^\infty[-2\delta,2\delta]}+C\\
&=Term_{2,1}+Term_{2,2}+Term_{2,3}+C.
\end{aligned}$$

From condition (5.6) and Lemma 11.1, we have

$$Term_{2,2} + Term_{2,3} \lesssim C(\|z\|_{\tilde{U}}).$$

Moreover, $\nabla_1 K(z(\alpha) - z(\beta), \tilde{f}(\alpha) - \tilde{f}(\beta))$ is of -2 type, and

$$\left\| \nabla_1 K(z(\alpha) - z(\beta), \tilde{f}(\alpha) - \tilde{f}(\beta)) \cdot \left(\frac{dz}{dt}(\alpha) - \frac{dz}{dt}(\beta) + \frac{\mathrm{d}\tilde{f}}{dt}(\alpha) - \frac{\mathrm{d}\tilde{f}}{dt}(\beta) \right)(\alpha - \beta) \right\|_{C^1([-2\delta, 2\delta]\times[-\pi,\pi])}$$
$$\lesssim (\|T(z)\|_{C^2(\mathbb{T})} + C)C(\|z\|_{\tilde{U}}).$$

From (5.1), it is easy to get

$$\|T(z)\|_{C^2(\mathbb{T})} \lesssim C(\|z\|_{\tilde{U}}).$$

Then $Term_{2,1} \lesssim C(\|z\|_{\tilde{U}})$. Hence

$$\left\| \frac{d}{dt} L_z^2(\alpha) \right\|_{L_\alpha^\infty[-2\delta, 2\delta]} \lesssim C(\|z\|_{\tilde{U}}).$$

Then we have the estimate

$$\frac{d}{dt}\|z\|_{RT} = \frac{d}{dt} \sup_{\alpha\in[-2\delta,2\delta]} \frac{1}{|\Re L_z^1(\alpha) + |\Im L_z^2(\alpha)||} \lesssim C(\|z\|_{\tilde{U}}).$$

Moreover, we have

$$\frac{d}{dt}\|z\|_{Arc} = \frac{d}{dt} \sup_{\alpha\in[-2\delta,2\delta],\beta\in[-\pi,\pi]} \left| \frac{1}{\left(\frac{\cosh(z_2(\alpha)-z_2(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))-\cos(z_1(\alpha,\gamma,t)-z_1(\beta,\gamma,t)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))}{(\alpha-\beta)^2}\right)} \right|$$
$$\leq \sup_{\alpha\in[-2\delta,2\delta],\beta\in[-\pi,\pi]} \left| \frac{1}{\left(\frac{\cosh(z_2(\alpha)-z_2(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))-\cos(z_1(\alpha,\gamma,t)-z_1(\beta,\gamma,t)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))}{(\alpha-\beta)^2}\right)} \right|^2$$
$$\cdot \left\| \frac{d}{dt} \left(\frac{\cosh(z_2(\alpha) - z_2(\beta) + \tilde{f}_1(\alpha) - \tilde{f}_1(\beta)) - \cos(z_1(\alpha,\gamma,t) - z_1(\beta,\gamma,t) + \tilde{f}_1(\alpha) - \tilde{f}_1(\beta))}{(\alpha-\beta)^2} \right) \right\|_{L^\infty_{\alpha,\beta}[-\pi,\pi]\times[-\pi,\pi]}$$
$$\leq \|z\|_{Arc}^2$$
$$\left(\left\| \frac{\sinh(z_2(\alpha) - z_2(\beta) + \tilde{f}_2(\alpha) - \tilde{f}_2(\beta))}{\alpha-\beta} \frac{\frac{d}{dt}(z_2(\alpha) - z_2(\beta)) + \frac{d}{dt}(\tilde{f}_2(\alpha) - \tilde{f}_2(\beta))}{(\alpha-\beta)} \right\|_{L^\infty_{\alpha,\beta}[-\pi,\pi]\times[-\pi,\pi]} \right.$$
$$\left. + \left\| \frac{\sin(z_1(\alpha) - z_1(\beta) + \tilde{f}_1(\alpha) - \tilde{f}_1(\beta))}{\alpha-\beta} \frac{\frac{d}{dt}(z_1(\alpha) - z_1(\beta)) + \frac{d}{dt}(\tilde{f}_1(\alpha) - \tilde{f}_2(\beta))}{(\alpha-\beta)} \right\|_{L^\infty_{\alpha,\beta}[-\pi,\pi]\times[-\pi,\pi]} \right)$$
$$\lesssim \|z\|_{Arc}^2 \|z\|_U (\|T(z)\|_{C^1_\alpha[-\pi,\pi]} + C)$$
$$\lesssim C(\|z\|_{\tilde{U}}).$$

We also introduce a corollary here to be used in a later section.

Corollary 5.6. *For $g(\alpha) \in H^1(\mathbb{T})$, if $z \in H^5(\mathbb{T})$, $\|z\|_{Arc} < \infty$ and $-\Re L_z^1(\alpha) - |\Im L_z^2(\alpha)| > 0$ when $\alpha \in [-2\delta, 2\delta]$, $\gamma \in [-1, 1]$, then we have*

$$\sup_{\gamma\in[-1,1]} \left(\int_{-\pi}^{\pi} g(\alpha)\lambda(\alpha) \overline{\int_{-\pi}^{\pi} K(z(\alpha) - z(\beta), \tilde{f}(\alpha) - \tilde{f}(\beta)) \left(\frac{\partial_\alpha g(\alpha)}{1 + ic'(\alpha)\gamma t} - \frac{\partial_\alpha g(\beta)}{1 + ic'(\beta)\gamma t} \right) d\beta} d\alpha \right.$$
$$\left. + \int_{-\pi}^{\pi} g(\alpha) \overline{\partial_\alpha g(\alpha) \frac{ic(\alpha)\gamma}{1 + ic'(\alpha)\gamma t}} d\alpha \right) \lesssim \|g\|_{L^2}^2.$$

□

5.2. Approximation for the picard theorem

Now we approximate the problem and have the following equations,

$$\begin{aligned}
\frac{dz^n(\alpha,\gamma,t)}{dt} &= \varphi_n * T^n(\varphi_n * z^n) = \varphi_n * \left(\frac{ic(\alpha)\gamma}{1+ic^{'}(\alpha)\gamma t}\partial_\alpha(\varphi_n * z^n)(\alpha)\right)\\
&+ \varphi_n * \left(\lambda(\alpha)\int_{-\pi}^{\pi} K^n((\varphi_n * z^n)(\alpha) - (\varphi_n * z^n)(\beta),\, \tilde{f}(\alpha) - \tilde{f}(\beta))\right.\\
&\left.\left(\frac{\partial_\alpha(\varphi_n * z^n_\mu)(\alpha)}{1+ic'(\alpha)\gamma t} - \frac{\partial_\beta(\varphi_n * z^n_\mu)(\beta)}{1+ic'(\beta)\gamma t}\right)(1+ic'(\beta)\gamma t)\mathrm{d}\beta\right)\\
&+ \varphi_n * \left(\lambda(\alpha)\int_{-\pi}^{\pi} K^n((\varphi_n * z^n)(\alpha) - (\varphi_n * z^n)(\beta),\right.\\
&\left.\tilde{f}(\alpha) - \tilde{f}(\beta))(\tilde{f}_\mu(\alpha) - \tilde{f}_\mu(\beta))(1+ic'(\beta)\gamma t)d\beta\right),
\end{aligned}$$

where

$$\begin{aligned}
&K^n((\varphi_n * z^n)(\alpha) - (\varphi_n * z^n)(\beta),\, \tilde{f}(\alpha) - \tilde{f}(\beta))\\
&\quad = \sin(\Delta((\varphi_n * z_1^n)(\alpha) + \tilde{f}_1(\alpha)))\\
&\quad\cdot \frac{1}{|\cosh(\Delta((\varphi_n * z_2^n)(\alpha) + \tilde{f}_2(\alpha))) - \cos(\Delta((\varphi_n * z_1^n)(\alpha) + \tilde{f}_1(\alpha)))|^2 + \frac{1}{n}\sin\left(\frac{\alpha-\beta}{2}\right)^2}\\
&\quad\cdot \overline{\cosh(\Delta((\varphi_n * z_2^n)(\alpha) + \tilde{f}_2(\alpha))) - \cos(\Delta((\varphi_n * z_1^n)(\alpha) + \tilde{f}_1(\alpha)))},
\end{aligned}$$

with

$$\Delta((\varphi_n * z_2^n)(\alpha) + \tilde{f}_2(\alpha)) = (\varphi_n * z_2^n)(\alpha) - (\varphi_n * z_2^n)(\beta) + \tilde{f}_2(\alpha) - \tilde{f}_2(\beta), \tag{5.13}$$

$$\Delta((\varphi_n * z_1^n)(\alpha) + \tilde{f}_1(\alpha)) = (\varphi_n * z_1^n)(\alpha) - (\varphi_n * z_1^n)(\beta) + \tilde{f}_1(\alpha) - \tilde{f}_1(\beta), \tag{5.14}$$

and initial value $z^n(\alpha,\gamma,0) = \varphi_n * f(\alpha,0)$. Here the convolution of φ_n is the projection to the finite Fourier modes of α.

By the Picard theorem, for any fixed $\gamma \in [-1,1]$, there exists solutions in $C^1([0,t_n], H^5_\alpha(\mathbb{T}))$. Moreover, by the structure of our approximation, we have $z^n = \varphi_n * z^n$, and for $1 \le j \le 5$,

$$\begin{aligned}
&\frac{d}{dt}\int_{-\pi}^{\pi} |\partial_\alpha^j z^n(\alpha,\gamma,t)|^2 d\alpha\\
&\quad = 2\Re\int_{-\pi}^{\pi} \partial_\alpha^j z^n(\alpha,\gamma,t)\overline{\partial_\alpha^j(\varphi_n * T^n(\varphi_n * z^n))}d\alpha\\
&\quad = 2\Re\int_{-\pi}^{\pi} \partial_\alpha^j(\varphi_n * z^n)(\alpha,\gamma,t)\overline{\partial_\alpha^j(T^n(\varphi_n * z^n))}d\alpha.
\end{aligned}$$

Then we can do the energy estimate similar to that the previous section by letting

$$\|z^n\|_{\tilde{U}^n} = \|z^n\|_{H^5[-\pi,\pi]} + \|z^n\|_{RT^n} + \|z^n\|_{Arc^n},$$

where

$$\|z^n\|_{Arc^n} = \sup_{\alpha\in[-2\delta,2\delta],\beta\in[-\pi,\pi]} \frac{|\cosh(\Delta((\varphi_n * z_2^n)(\alpha) + \tilde{f}_2(\alpha))) - \cos(\Delta((\varphi_n * z_1^n)(\alpha) + \tilde{f}_1(\alpha)))|(\alpha-\beta)^2}{|\cosh(\Delta((\varphi_n * z_2^n)(\alpha) + \tilde{f}_2(\alpha))) - \cos(\Delta((\varphi_n * z_1^n)(\alpha) + \tilde{f}_1(\alpha)))|^2 + \frac{1}{n}\sin(\frac{\alpha-\beta}{2})^2},$$

with $\Delta((\varphi_n * z_2^n)(\alpha) + \tilde{f}_2(\alpha))$ and $\Delta((\varphi_n * z_1^n)(\alpha) + \tilde{f}_1(\alpha))$ from (5.13), (5.14), and letting

$$\|z^n\|_{RT^n} = \sup_{\alpha\in[-2\delta,2\delta]} \frac{1}{|\Re L_z^{1,n}(\alpha) + |\Im L_z^{2,n}(\alpha)||},$$

with

$$L_{z^n}^{1,n}(\alpha,\gamma,t) = -2\pi \lim_{\beta\to 0}\left(K^n((\varphi_n * z^n)(\alpha) - (\varphi_n * z^n)(\beta), \tilde{f}(\alpha) - \tilde{f}(\beta))\tan\left(\frac{\alpha-\beta}{2}\right)\right),$$

$$L_{z^n}^{2,n}(\alpha,\gamma,t) = \left(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t} + \frac{1}{1+ic'(\alpha)\gamma t}p.v.\right.$$
$$\left.\int_{-\pi}^{\pi} K^n((\varphi_n * z^n)(\alpha) - (\varphi_n * z^n)(\beta), \tilde{f}(\alpha) - \tilde{f}(\beta))(1+ic'(\beta)\gamma t)d\beta\right).$$

Then we can use the similar energy estimate and the compactness argument to show there exist a solution

$$z(\alpha,\gamma,t) \in L^\infty([0,t_1], H_\alpha^5(\mathbb{T})), \tag{5.15}$$

satisfying

$$z(\alpha,\gamma,t) = \int_0^t T(z)d\tau + f^c(\alpha,0), \tag{5.16}$$

for sufficiently small time t_1. Moreover,

$$\|z\|_{Arc} < \infty, \tag{5.17}$$

and

$$-\Re L_z^1(\alpha) - |\Im L_z^2(\alpha)| > 0, \text{ when } \alpha \in [-2\delta, 2\delta]. \tag{5.18}$$

Since the energy estimate has a bound for all $\gamma \in [-1,1]$, we have a existence time t_1 that holds for all γ.

Now we abuse the notation and write $T(z)$ as $T(z(\alpha,\gamma,t),\gamma,t)$. We have the following lemma:

Lemma 5.7. *For any $g(\alpha), h(\alpha) \in H^{j+1}(\mathbb{T})$, $j = 3,4$, $\|g\|_{Arc} < \infty$ and $\|h\|_{Arc} < \infty$, we have*

$$\|T(g(\alpha),\gamma,t)\|_{H^j(\mathbb{T})} \lesssim 1, \tag{5.19}$$

$$\|T(g(\alpha),\gamma,t) - T(h(\alpha),\gamma,t)\|_{H^j(\mathbb{T})} \lesssim \|g(\alpha) - g(\alpha)\|_{H^{j+1}(\mathbb{T})}, \tag{5.20}$$

$$\|T(g(\alpha),\gamma,t) - T(g(\alpha),\gamma,t')\|_{H^j(\mathbb{T})} \lesssim |t-t'|. \tag{5.21}$$

Proof. We only show (5.19) and the left can be shown in the same way. From (5.1), we have

$$\begin{aligned}
T(g(\alpha),\gamma,t) &= \frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha g(\alpha)\\
&\quad+\lambda(\alpha)\int_{-\pi}^{\pi} K(g(\alpha)-g(\beta),\tilde f(\alpha)-\tilde f(\beta))\left(\frac{\partial_\alpha g(\alpha)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta g(\beta)}{1+ic'(\beta)\gamma t}\right)(1+ic'(\beta)\gamma t)d\beta\\
&\quad+\lambda(\alpha)\int_{-\pi}^{\pi} K(g(\alpha)-g(\beta),\tilde f(\alpha)-\tilde f(\beta))(\partial_\alpha\tilde f(\alpha)-\partial_\beta\tilde f(\beta))(1+ic'(\beta)\gamma t)d\beta\\
&= T_1(g(\alpha),\gamma,t)+T_2(g(\alpha),\gamma,t)+T_3(g(\alpha),\gamma,t).
\end{aligned}$$

It is trivial that T_1 satisfying the (5.19) since $c(\alpha)$ is sufficiently smooth.

Moreover, the $\partial_\alpha\tilde f\in H^5(\mathbb{T})$ and is more regular than $\frac{\partial_\alpha g(\alpha)}{1+ic'(\alpha)\gamma t}$. Hence we only consider T_2. For T_2, we have

$$\|K(g(\alpha)-g(\beta),\tilde f(\alpha)-\tilde f(\beta))(\alpha-\beta)\|_{C^{j-2}([-2\delta,2\delta]\times\mathbb{T})}\lesssim 1. \tag{5.22}$$

Then

$$\|T_2\|_{L^2(\mathbb{T})}\lesssim\|T_2\|_{L^\infty(\mathbb{T})}\lesssim 1.$$

Moreover, we can use the notation from (5.4), (5.2), and get

$$\begin{aligned}
\partial_\alpha^j T_2(g) &= \lambda(\alpha)\int_{-\pi}^{\pi} K(g(\alpha)-g(\beta),\tilde f(\alpha)-\tilde f(\beta))\\
&\quad\left(\frac{\partial_\alpha^{j+1}g(\alpha)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta^{j+1}g(\beta)}{1+ic'(\beta)\gamma t}\right)(1+ic'(\beta)\gamma t)d\beta\\
&\quad+\sum_{j'}\partial_\alpha^{b_{j'}}\lambda(\alpha)\int_{-\pi}^{\pi} K_{-1}^{j'}(V_g^{\left[\frac{j+1}{2}\right]}(\alpha)-V_g^{\left[\frac{j+1}{2}\right]}(\beta),V_f^{\left[\frac{j+1}{2}\right]}(\alpha)-V_f^{\left[\frac{j+1}{2}\right]}(\beta),\\
&\quad\tilde V_g^{\left[\frac{j+1}{2}\right]}(\alpha)-\tilde V_g^{\left[\frac{j+1}{2}\right]}(\beta))\\
&\quad\cdot X_i(\beta)(\tilde z^j(\alpha)-\tilde z^j(\beta))d\beta\\
&= Term_{2,1}+Term_{2,2}.
\end{aligned}$$

Here $\tilde z^j\in V_g^j\cup\tilde V_g^j\cup V_{\tilde f}^j$. $[\frac{j+1}{2}]$ is the biggest integer less than $\frac{j+1}{2}$. Then from (5.22), we could use Lemma 11.2 to bound $Term_{2,1}$. Moreover, since $j+1-[\frac{j+1}{2}]\geq[\frac{j+1}{2}]\geq 2$. We have

$$\begin{aligned}
&\Big\|K_{-1}^{j'}(V_g^{\left[\frac{j+1}{2}\right]}(\alpha)-V_g^{\left[\frac{j+1}{2}\right]}(\beta),V_f^{\left[\frac{j+1}{2}\right]}(\alpha)-V_f^{\left[\frac{j+1}{2}\right]}(\beta),\\
&\qquad\tilde V_g^{\left[\frac{j+1}{2}\right]}(\alpha)-\tilde V_g^{\left[\frac{j+1}{2}\right]}(\beta))(\alpha-\beta)\Big\|_{C^0([-2\delta,2\delta]\times[-\pi,\pi])}\\
&\lesssim C(\|g\|_{H^{j+1}}\|\tilde f\|_{H^{j+1}}).
\end{aligned}$$

Then we could use Lemma 11.4 to bound $Term_{2,2}$. □

Then from Lemma 5.7, (5.15) and (5.16), we have

$$z(\alpha,\gamma,t)\in L^\infty([0,t_1],H^5_\alpha(\mathbb{T}))\cap C^0([0,t_1],H^4_\alpha(\mathbb{T}))\cap C^1([0,t_1],H^3_\alpha(\mathbb{T})). \tag{5.23}$$

6. The Uniqueness

In this section we show there exists sufficiently $0<t_2\le t_1$ such that for $0\le t\le t_2$, we have $z(\alpha,0,t)=f^c(\alpha,t)$.

Let $z^0(\alpha,t)=z(\alpha,0,t)$. From (5.1) and (5.5), we have

$$\begin{aligned}\frac{dz^0_\mu(\alpha,t)}{dt}&=\lambda(\alpha)\int_{-\pi}^{\pi}K(z^0(\alpha)-z^0(\beta),\tilde f(\alpha)-\tilde f(\beta))(\partial_\alpha z^0_\mu(\alpha)-\partial_\beta z^0_\mu(\beta))d\beta\\&\quad+\lambda(\alpha)\int_{-\pi}^{\pi}K(z^0(\alpha)-z^0(\beta),\tilde f(\alpha)-\tilde f(\beta))(\partial_\alpha \tilde f_\mu(\alpha)-\partial_\beta \tilde f_\mu(\beta))d\beta.\end{aligned}$$

Moreover, from (3.2), we have

$$\begin{aligned}\frac{df^c_\mu(\alpha,t)}{dt}&=\lambda(\alpha)\int_{-\pi}^{\pi}K(f^c(\alpha)-f^c(\beta),\tilde f(\alpha)-\tilde f(\beta))(\partial_\alpha f^c_\mu(\alpha)-\partial_\beta f^c_\mu(\beta))d\beta\\&\quad+\lambda(\alpha)\int_{-\pi}^{\pi}K(f^c(\alpha)-f^c(\beta),\tilde f(\alpha)-\tilde f(\beta))(\partial_\alpha \tilde f_\mu(\alpha)-\partial_\beta \tilde f_\mu(\beta))d\beta.\end{aligned}$$

Then we have the equation for the difference:

$$\begin{aligned}&\frac{d(z^0_\mu(\alpha,t)-f^c(\alpha,t))}{dt}=\lambda(\alpha)\int_{-\pi}^{\pi}K(z^0(\alpha)-z^0(\beta),\tilde f(\alpha)-\tilde f(\beta))(\partial_\alpha(z^0_\mu-f^c_\mu)(\alpha)\\&\quad-\partial_\beta(z^0_\mu-f^c_\mu)(\beta))d\beta\\&\quad+\lambda(\alpha)\int_{-\pi}^{\pi}(K(z^0(\alpha)-z^0(\beta),\tilde f(\alpha)-\tilde f(\beta))-K(f^c(\alpha)-f^c(\beta),\tilde f(\alpha)-\tilde f(\beta)))\\&\cdot(\partial_\alpha\tilde f_\mu(\alpha)+\partial_\alpha f^c_\mu(\alpha)-\partial_\beta\tilde f_\mu(\beta)-\partial_\beta f^c_\mu(\beta))d\beta\\&=Term_1+Term_2.\end{aligned}$$

We first control $Term_2$, we have

$$\begin{aligned}Term_2&=\lambda(\alpha)\int_{-\pi}^{\pi}\int_0^1\frac{d}{d\tau}(K(f^c(\alpha)-f^c(\beta)\\&\quad-\tau(f^c(\alpha)-z^0(\alpha)-(f^c(\beta)-z^0(\beta))),\tilde f(\alpha)-\tilde f(\beta)))\\&\cdot(\partial_\alpha\tilde f_\mu(\alpha)+\partial_\alpha f^c_\mu(\alpha)-\partial_\beta\tilde f_\mu(\beta)-\partial_\beta f^c_\mu(\beta))d\tau d\beta\\&=-\lambda(\alpha)\int_{-\pi}^{\pi}\int_0^1\nabla_1K(f^c(\alpha)-f^c(\beta)-\tau(f^c(\alpha)-z^0(\alpha)\\&\quad-(f^c(\beta)-z^0(\beta))),\tilde f(\alpha)-\tilde f(\beta)))\\&\cdot(f^c(\alpha)-z^0(\alpha)-(f^c(\beta)-z^0(\beta)))(\partial_\alpha\tilde f_\mu(\alpha)\\&\quad+\partial_\alpha f^c_\mu(\alpha)-\partial_\beta\tilde f_\mu(\beta)-\partial_\beta f^c_\mu(\beta))d\tau d\beta.\end{aligned}\tag{6.1}$$

Since the component of ∇K is of -2 type, we have

$$\begin{aligned}&\|\nabla_1 K(f^c(\alpha)-f^c(\beta)-\tau(f^c(\alpha)-z^0(\alpha)-(f^c(\beta)-z^0(\beta))),\\&\quad\tilde{f}(\alpha)-\tilde{f}(\beta)))(\alpha-\beta)^2\|_{C^1_{\alpha,\beta}([-2\delta,2\delta]\times[-\pi,\pi])}\\&\lesssim(\|f^c(\alpha)\|_{C^2[-\pi,\pi]}+\|z^0(\alpha)\|_{C^2[-\pi,\pi]}+\|\tilde{f}(\alpha)\|_{C^2[-\pi,\pi]})C(\|f^c-\tau(f^c-z^0)\|_{Arc}).\end{aligned}\tag{6.2}$$

When $t=0$, we have $z^0=f^c$, then

$$\|f^c-\tau(f^c-z^0)\|_{Arc}=\|f^c\|_{Arc}\lesssim 1.\tag{6.3}$$

Moreover, we have the following lemma:

Lemma 6.1. *For $g,h\in C^1(\mathbb{T})$, $\|h\|_{Arc}<\infty$, there exists δ depending on $\|h\|_{Arc}$ and $\|h\|_{C^1(\mathbb{T})}$ such that when $\|g-h\|_{C^1(\mathbb{T})}\le\delta$, we have $\|g\|_{Arc}<\infty$.*

Proof. We have

$$\begin{aligned}&|\cosh(g_2(\alpha)-g_2(\beta)+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))-\cosh(h_2(\alpha)-h_2(\beta)+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))|\\&\le|g_2(\alpha)-g_2(\beta)-(h_2(\alpha)-h_2(\beta))|\int_0^1|\sinh((1-\tau)(h_2(\alpha)-h_2(\beta))\\&\quad+\tau(g_2(\alpha)-g_2(\beta))+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))|d\tau\\&\le(\alpha-\beta)^2\|g-h\|_{C^1(T)}C(\|h\|_{C^1(T)}+\|g-h\|_{C^1(T)}),\end{aligned}$$

and

$$\begin{aligned}&|\cos(g_1(\alpha)-g_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))-\cos(h_1(\alpha)-h_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))|\\&\le|g_1(\alpha)-g_1(\beta)-(h_1(\alpha)-h_1(\beta))|\int_0^1|\sin((1-\tau)(h_1(\alpha)-h_1(\beta))\\&+\tau(g_1(\alpha)-g_1(\beta))+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))|d\tau\\&\le(\alpha-\beta)^2\|g-h\|_{C^1(T)}C(\|h\|_{C^1(T)}+\|g-h\|_{C^1(T)}).\end{aligned}$$

Since

$$\begin{aligned}&|\cosh(h_2(\alpha)-h_2(\beta)+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))-\cos(h_1(\alpha)-h_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))|\\&\quad\ge\|h\|_{Arc}|\alpha-\beta|^2,\end{aligned}$$

we have

$$\begin{aligned}&|\cosh(g_2(\alpha)-g_2(\beta)+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))-\cos(g_1(\alpha)-g_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))|\\&\quad\ge(\|h\|_{Arc}-\|g-h\|_{C^1(T)}C(\|h\|_{C^1(T)}+\|g-h\|_{C^1(T)}))|\alpha-\beta|^2.\end{aligned}$$

Then we have the result. □

Since we have $z^0(\alpha,t) \in C^1([0,t_1], H^3(\mathbb{T}))$, $f^c(\alpha,t) \in C^1([0,t_1], H^6(\mathbb{T}))$, then from (6.3), and Lemma 6.1, there exists t_2, satisfying $0 \le t_2 \le t_1$, such that for $0 \le t \le t_2$,

$$\|f^c - \tau(f^c - z^0)\|_{Arc} \lesssim 1. \tag{6.4}$$

Then from corollary 11.3, (6.1),(6.2), and (6.4), we have

$$\|Term_2\|^2_{L^2(\mathbb{T})} \le \|z^0 - f^c\|^2_{L^2(\mathbb{T})}.$$

Then

$$\begin{aligned}
&\frac{d}{dt}\int_{-\pi}^{\pi} |z^0(\alpha,t) - f^c(\alpha,t)|^2 d\alpha \\
&= 2\Re \int_{-\pi}^{\pi} (z^0(\alpha,t) - f^c(\alpha,t)) \overline{\frac{d}{dt}(z^0(\alpha,t) - f^c(\alpha,t))} d\alpha \\
&= \sum_{\mu=1,2} 2\Re \int_{-\pi}^{\pi} (z^0_\mu(\alpha,t) - f^c_\mu(\alpha,t)) \lambda(\alpha) \overline{\int_{-\pi}^{\pi} K(z^0(\alpha) - z^0(\beta), \tilde{f}(\alpha) - \tilde{f}(\beta))(\partial_\alpha(z^0_\mu - f^c_\mu)(\alpha) - \partial_\beta(z^0_\mu - f^c_\mu)(\beta)) d\beta} d\alpha \\
&\quad + B.T^0,
\end{aligned}$$

where $B.T^0 \lesssim \|z^0(\alpha,t) - f^c(\alpha,t)\|^2_{L^2[-\pi,\pi]}$. Then from corollary 5.6 when $\gamma = 0$, conditions (5.18), (5.15), (5.17), we have

$$\frac{d}{dt}\int_{-\pi}^{\pi} |z^0(\alpha,t) - f^c(\alpha,t)|^2 d\alpha \lesssim \|z^0(\alpha,t) - f^c(\alpha,t)\|^2_{L^2(\mathbb{T})}.$$

Moreover, we have $z^0(\alpha,0) = f^c(\alpha,0)$. Therefore we have

$$z^0(\alpha,t) = f^c(\alpha,t), \tag{6.5}$$

for $0 \le t \le t_2$.

7. The Continuity of z with Respect to γ

We first show $\|z(\alpha,\gamma,t) - z(\alpha,\gamma',t)\|_{H^3(\alpha)} \lesssim |\gamma - \gamma'|$.

For the sake of simplicity, we further shrink the time t_1 to $\tilde{t}_1$ such that for all $0 \le t \le \tilde{t}_1$, $\gamma, \gamma' \in [-1,1]$, $\tau \in [0,1]$, we have $\|\tau z(\alpha,\gamma,t) + (1-\tau)z(\alpha,\gamma',t) - \tau z(\alpha,\gamma,0) - (1-\tau)z(\alpha,\gamma',0)\|_{C^1(\mathbb{T})} = \|\tau z(\alpha,\gamma,t) + (1-\tau)z(\alpha,\gamma',t) - f^c(\alpha,0)\|_{C^1(\mathbb{T})}$ is sufficiently small. Then from Lemma 6.1, we have

$$\|\tau z(\alpha,\gamma,t) + (1-\tau)z(\alpha,\gamma',t)\|_{Arc} < \infty.$$

This is not necessary but helps to simplify our estimate in this section.

Now we estimate the difference, we have

$$\begin{aligned}
\frac{dz(\alpha,\gamma)}{dt} - \frac{dz(\alpha,\gamma')}{dt} &= T(z(\alpha,\gamma,t),\gamma,t) - T(z(\alpha,\gamma',t),\gamma',t) \\
&= (T(z(\alpha,\gamma,t),\gamma,t) - T(z(\alpha,\gamma',t),\gamma,t)) + \left(\int_{\gamma}^{\gamma'} (\partial_\eta T)(z(\alpha,\gamma',t),\eta,t) d\eta\right) \\
&= Term_1 + Term_2.
\end{aligned} \tag{7.1}$$

For $Term_2$, we have

$$\begin{aligned}(\partial_\gamma T)(z(\alpha),\gamma,t) &= \frac{d}{d\gamma}\left(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\right)\partial_\alpha z(\alpha)\\ &+\lambda(\alpha)\int_{-\pi}^{\pi} K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\frac{d}{d\gamma}\left(\frac{1+ic'(\beta)\gamma t}{1+ic'(\alpha)\gamma t}-1\right)d\beta\partial_\alpha z(\alpha)\\ &+\lambda(\alpha)\int_{-\pi}^{\pi} K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(\partial_\alpha\tilde{f}(\alpha)-\partial_\beta\tilde{f}(\beta))ic'(\beta)td\beta\\ &= Term_{2,1}+Term_{2,2}+Term_{2,3}. \end{aligned} \tag{7.2}$$

Since $z(\alpha,\gamma,t)\in L_t^\infty([0,t_0],H^5(\mathbb{T}))$, we have

$$\|Term_{2,1}\|_{H^3(\mathbb{T})}\lesssim 1.$$

Moreover,

$$\begin{aligned}Term_{2,2} = \lambda(\alpha)\int_{-\pi}^{\pi} & K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\\ &\left[\frac{(ic(\beta)-ic(\alpha))t}{1+ic'(\alpha)\gamma t}-\frac{(ic(\beta)-ic(\alpha))\gamma t ic'(\alpha)t}{(1+c'(\alpha)\gamma t)^2}\right]d\beta\partial_\alpha z(\alpha).\end{aligned}$$

Since K is of -1 type, we have

$$K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(\alpha-\beta)\in C^3([-2\delta,2\delta]\times[-\pi,\pi]). \tag{7.3}$$

Therefore $\|Term_{2,2}\|_{H^3(\mathbb{T})}\lesssim 1$. Moreover, $Term_{2,3}$ can be bounded in the similar way since $|\frac{\partial_\alpha\tilde{f}(\alpha)-\partial_\beta\tilde{f}(\beta)}{\alpha-\beta}|\in C^3([-2\delta,2\delta]\times[-\pi,\pi])$ and we get

$$\|Term_{2,3}\|_{H^3(\mathbb{T})}\lesssim 1.$$

Then we have

$$\|(\partial_\gamma T)(z(\alpha),\gamma,t)\|_{H^3(\mathbb{T})}\lesssim 1, \tag{7.4}$$

and

$$\|Term_2\|_{H^3(\mathbb{T})}\lesssim|\gamma-\gamma'|. \tag{7.5}$$

For $Term_1$, notice that $\tilde{f}(\alpha + ic(\alpha)\gamma t, t) = \tilde{f}(\alpha, t)$, we have

$$\begin{aligned}
Term_1 = & \frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha(z(\alpha,\gamma)-z(\alpha,\gamma')) \\
& + \lambda(\alpha)\int_{-\pi}^{\pi} K(z(\alpha,\gamma)-z(\beta,\gamma), \tilde{f}(\alpha)-\tilde{f}(\beta))\left(\frac{\partial_\alpha z(\alpha,\gamma)-\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t}\right. \\
& \left.-\left(\frac{\partial_\beta z(\beta,\gamma)-\partial_\beta z(\beta,\gamma')}{1+ic'(\beta)\gamma t}\right)\right) \\
& \cdot (1+ic'(\beta)\gamma t)d\beta \\
& + \lambda(\alpha)\int_{-\pi}^{\pi}(K(z(\alpha,\gamma)-z(\beta,\gamma), \tilde{f}(\alpha)-\tilde{f}(\beta)) - K(z(\alpha,\gamma') \\
& - z(\beta,\gamma'), \tilde{f}(\alpha)-\tilde{f}(\beta))) \\
& \left(\frac{\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t} - \frac{\partial_\beta z(\beta,\gamma')}{1+ic'(\beta)\gamma t} + \partial_\alpha\tilde{f}(\alpha) - \partial_\beta\tilde{f}(\beta)\right)(1+ic'(\beta)\gamma t)d\beta \\
= & \, Term_{1,1} + Term_{1,2} + Term_{1,3}. \qquad (7.6)
\end{aligned}$$

It is easy to get that

$$\|Term_{1,1}\|_{L^2(T)} + \|Term_{1,2}\|_{L^2(T)} \lesssim \|z(\alpha,\gamma)-z(\alpha,\gamma')\|_{H^3(\mathbb{T})}. \qquad (7.7)$$

Moreover,

$$\begin{aligned}
Term_{1,3} = & \lambda(\alpha)\int_{-\pi}^{\pi}\int_0^1 \nabla_1 K(\tau(z(\alpha,\gamma)-z(\beta,\gamma)) + (1-\tau)(z(\alpha,\gamma')-z(\beta,\gamma')), \\
& \tilde{f}(\alpha)-\tilde{f}(\beta))d\tau \\
& \cdot (z(\alpha,\gamma)-z(\beta,\gamma)-(z(\alpha,\gamma')-z(\beta,\gamma'))) \\
& \cdot \left(\frac{\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t} - \frac{\partial_\beta z(\beta,\gamma')}{1+ic'(\beta)\gamma t} + \partial_\alpha\tilde{f}(\alpha) - \partial_\beta\tilde{f}(\beta)\right)(1+ic'(\beta)\gamma t)d\beta. \qquad (7.8)
\end{aligned}$$

Since the component of $\nabla_1 K$ is of -2 type, we have

$$\begin{aligned}
& \sup_\tau \|\nabla_1 K(\tau(z(\alpha,\gamma)-z(\beta,\gamma)+(1-\tau)(z(\alpha,\gamma') \\
& -z(\beta,\gamma')), \tilde{f}(\alpha)-\tilde{f}(\beta))(\alpha-\beta)^2\|_{C^3([-2\delta,2\delta]\times[-\pi,\pi])} \lesssim 1,
\end{aligned}$$

then we have

$$\|Term_{1,3}\|_{L^2(T)} \lesssim \|z(\alpha,\gamma)-z(\alpha,\gamma')\|_{H^3(T)}. \qquad (7.9)$$

Now we control $\partial_\alpha^3 Term_1$. For $\partial_\alpha^3 Term_{1,1}$, we have

$$\begin{aligned}
\partial_\alpha^3 Term_{1,1} = & \frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha^4(z(\alpha,\gamma)-z(\alpha,\gamma')) \\
& + \sum_{1\le j\le 3} C_{1,j}\partial_\alpha^j\left(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\right)\partial_\alpha^{4-j}(z(\alpha,\gamma)-z(\alpha,\gamma')) \\
= & \, Term^3_{1,1,M} + Term^3_{1,1,2}. \qquad (7.10)
\end{aligned}$$

Here $\|Term^3_{1,1,2}\|_{L^2(\mathbb{T})} \lesssim \|z(\alpha,\gamma) - z(\alpha,\gamma')\|_{H^3(\mathbb{T})}$. For $\partial^3_\alpha Term_{1,2}$, from Lemma 11.6 and (7.3), we have

$$\begin{aligned}\partial^3_\alpha Term_{1,2} &= \lambda(\alpha)\int_{-\pi}^{\pi} K(z(\alpha,\gamma)-z(\alpha-\beta,\gamma),\tilde{f}(\alpha)-\tilde{f}(\alpha-\beta))\\ &\cdot\left(\partial^3_\alpha\left(\frac{\partial_\alpha(z(\alpha,\gamma)-z(\alpha,\gamma'))}{1+ic'(\alpha)\gamma t}\right)-\partial^3_\alpha\left(\frac{\partial_\alpha(z(\alpha-\beta,\gamma)-z(\alpha-\beta,\gamma'))}{1+ic'(\alpha-\beta)\gamma t}\right)\right)\\ &(1+ic'(\alpha-\beta)\gamma t)d\beta + Term^3_{1,2,1},\end{aligned} \tag{7.11}$$

where $\|Term^3_{1,2,1}\|_{L^2(\mathbb{T})} \lesssim \|z(\alpha,\gamma)-z(\alpha,\gamma')\|_{H^3(\mathbb{T})}$. Moreover, we have

$$\begin{aligned}&\lambda(\alpha)\int_{-\pi}^{\pi} K(z(\alpha,\gamma)-z(\alpha-\beta,\gamma),\tilde{f}(\alpha)-\tilde{f}(\alpha-\beta))\\ &\cdot\left(\partial^3_\alpha\left(\frac{\partial_\alpha(z(\alpha,\gamma)-z(\alpha,\gamma'))}{1+ic'(\alpha)\gamma t}\right)-\partial^3_\alpha\left(\frac{\partial_\alpha(z(\alpha-\beta,\gamma)-z(\alpha-\beta,\gamma'))}{1+ic'(\alpha-\beta)\gamma t}\right)\right)\\ &(1+ic'(\alpha-\beta)\gamma t)d\beta\\ &=\lambda(\alpha)\int_{-\pi}^{\pi} K(z(\alpha,\gamma)-z(\alpha-\beta,\gamma),\tilde{f}(\alpha)-\tilde{f}(\alpha-\beta))\\ &\cdot\left(\frac{\partial^4_\alpha(z(\alpha,\gamma)-z(\alpha,\gamma'))}{1+ic'(\alpha)\gamma t}-\frac{\partial^4_\alpha(z(\alpha-\beta,\gamma)-z(\alpha-\beta,\gamma'))}{1+ic'(\alpha-\beta)\gamma t}\right)(1+ic'(\alpha-\beta)\gamma t)d\beta\\ &+\sum_{0\le j\le 2} C_j\lambda(\alpha)\int_{-\pi}^{\pi} K(z(\alpha,\gamma)-z(\alpha-\beta,\gamma),\tilde{f}(\alpha)-\tilde{f}(\alpha-\beta))(1+ic'(\alpha-\beta)\gamma t)\\ &\cdot\left(\partial^{1+j}_\alpha(z(\alpha,\gamma)-z(\alpha,\gamma'))\partial^{3-j}_\alpha\left(\frac{1}{1+ic'(\alpha)\gamma t}\right)-\partial^{1+j}_\alpha(z(\alpha-\beta,\gamma)\right.\\ &\left.-z(\alpha-\beta,\gamma'))\partial^{3-j}_\alpha\left(\frac{1}{1+ic'(\alpha-\beta)\gamma t}\right)\right)d\beta\\ &=Term^3_{1,2,M}+Term^3_{1,2,3}.\end{aligned} \tag{7.12}$$

Then from Lemma 11.2, we have

$$\|Term^3_{1,2,3}\|_{L^2(\mathbb{T})} \lesssim \|z(\alpha,\gamma)-z(\alpha,\gamma')\|_{H^3(\mathbb{T})}. \tag{7.13}$$

For $\partial^3_\alpha Term_{1,3}$, we use equation (7.8). Since

$$\|\nabla K(\tau(z(\alpha,\gamma)-z(\beta,\gamma))+(1-\tau)(z(\alpha,\gamma')-z(\beta,\gamma')),\tilde{f}(\alpha)-\tilde{f}(\beta))(\alpha-\beta)^2\|_{C^3([-2\delta,2\delta]\times[-\pi,\pi])} \lesssim 1,$$

and

$$\frac{\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t}+\partial_\alpha\tilde{f}(\alpha)\in H^4(\mathbb{T}),$$

from Lemma 11.7 we have

$$\|\partial^3_\alpha Term_{1,3}\|_{L^2(\mathbb{T})} \lesssim \|z(\alpha,\gamma)-z(\alpha,\gamma')\|_{H^3(\mathbb{T})}. \tag{7.14}$$

In conclusion, from (7.7), (7.9), and (7.5), we have

$$\begin{aligned}\frac{d}{dt}\int_{-\pi}^{\pi}|z(\alpha,\gamma)-z(\alpha,\gamma')|^2d\alpha &= 2\Re\int_{-\pi}^{\pi}(z(\alpha,\gamma)-z(\alpha,\gamma'))\overline{Term_1+Term_2}d\alpha\\ &\lesssim |\gamma-\gamma'|^2+\|z(\alpha,\gamma)-z(\alpha,\gamma')\|^2_{H^3(\mathbb{T})}.\end{aligned}$$

From (7.10), (7.11), (7.12), (7.13), (7.14) and (7.5), we have

$$\begin{aligned}&\frac{d}{dt}\int_{-\pi}^{\pi}|\partial_\alpha^3(z(\alpha,\gamma)-z(\alpha,\gamma'))|^2d\alpha\\ &=2\Re\int_{-\pi}^{\pi}\partial_\alpha^3(z(\alpha,\gamma)-z(\alpha,\gamma'))\cdot\overline{\partial_\alpha^3 Term_1+\partial_\alpha^3 Term_2}d\alpha\\ &=2\Re\int_{-\pi}^{\pi}\partial_\alpha^3(z(\alpha,\gamma)-z(\alpha,\gamma'))\cdot\overline{Term^3_{1,1,M}+Term^3_{1,2,M}}d\alpha+B.T^0\\ &=\sum_{\mu=1,2}2\Re\int_{-\pi}^{\pi}\partial_\alpha^3(z_\mu(\alpha,\gamma)-z_\mu(\alpha,\gamma'))\overline{\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha^4(z_\mu(\alpha,\gamma)-z(\alpha,\gamma'))}d\alpha\\ &\quad+\sum_{\mu=1,2}2\Re\int_{-\pi}^{\pi}\partial_\alpha^3(z_\mu(\alpha,\gamma)-z_\mu(\alpha,\gamma'))\overline{\lambda(\alpha)\int_{-\pi}^{\pi}K(z(\alpha,\gamma)-z(\alpha-\beta,\gamma),\tilde f(\alpha)-\tilde f(\alpha-\beta))}\\ &\quad\cdot\overline{\left(\frac{\partial_\alpha^4(z(\alpha,\gamma)-z(\alpha,\gamma'))}{1+ic'(\alpha)\gamma t}-\frac{\partial_\alpha^4(z(\alpha-\beta,\gamma)-z(\alpha-\beta,\gamma'))}{1+ic'(\alpha-\beta)\gamma t}\right)(1+ic'(\alpha-\beta)\gamma t)d\beta}d\alpha\\ &\quad+B.T^0,\end{aligned}$$

where

$$|B.T^0|\lesssim\|z(\alpha,\gamma)-z(\alpha,\gamma')\|^2_{H^3(\mathbb{T})}+|\gamma-\gamma'|^2.$$

Then from corollary 5.6, we have

$$\frac{d}{dt}\int_{-\pi}^{\pi}|\partial_\alpha^3(z(\alpha,\gamma)-z(\alpha,\gamma'))|^2d\alpha\lesssim\|z(\alpha,\gamma)-z(\alpha,\gamma')\|^2_{H^3(\mathbb{T})}+|\gamma-\gamma'|^2.$$

Moreover, the initial date $\|z(\alpha,\gamma)-z(\alpha,\gamma')\|^2_{H^3(\mathbb{T})}|_{t=0}=0$. Therefore we have

$$\left\|\frac{z(\alpha,\gamma)-z(\alpha,\gamma')}{\gamma-\gamma'}\right\|_{H^3(\mathbb{T})}\lesssim 1. \tag{7.15}$$

8. The Differentiability of z with Respect to γ

Now we show the differentiability. We define a new function $w(\alpha,\gamma,t)$. It satisfies the equation that $\frac{dz}{d\gamma}$ would satisfy if it is differentiable.

Let w be the solution of the equation

$$\frac{dw(\alpha,\gamma,t)}{dt}=\tilde T(w)=D_zT(z(\alpha,\gamma,t),\gamma,t)[w]+\partial_\gamma T(\alpha,\gamma,t), \tag{8.1}$$

with initial value $w(\alpha,\gamma,0)=0$. Here $D_zT(z(\alpha,\gamma,t),\gamma,t)[w]$ is the Gateaux derivative.

As in the existence of $z(\alpha,\gamma,t)$, we first show the energy estimate. First, from (7.4), we have

$$\|(\partial_\gamma T)(z(\alpha),\gamma,t)\|_{H^3(\mathbb{T})} \lesssim 1.$$

Moreover,

$$\begin{aligned} D_z T(z(\alpha,\gamma,t),\gamma,t)[w] &= \frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha w(\alpha,\gamma) \\ &+ \lambda(\alpha)\int_{-\pi}^{\pi} K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde f(\alpha)-\tilde f(\beta))\left(\frac{\partial_\alpha w(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta w(\beta,\gamma)}{1+ic'(\beta)\gamma t}\right) \\ &(1+ic'(\beta)\gamma t)d\beta \\ &+ \lambda(\alpha)\int_{-\pi}^{\pi}(\nabla_1 K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde f(\alpha)-\tilde f(\beta))\cdot(w(\alpha,\gamma)-w(\beta,\gamma))) \\ &\cdot\left(\partial_\alpha\tilde f(\alpha)-\partial_\beta\tilde f(\beta)+\frac{\partial_\alpha z(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z(\beta,\gamma)}{1+ic'(\beta)\gamma t}\right)(1+ic'(\beta)\gamma t)d\beta. \end{aligned} \tag{8.2}$$

It has the similar structure as (7.6) and (7.8). The only difference between the first two terms in (7.6) and (8.2) is that $\partial_\alpha w(\alpha,\gamma)$ takes the place of $\frac{z(\alpha,\gamma)-z(\alpha,\gamma')}{\gamma-\gamma'}$. In (7.8), and the third term of (7.6), $w(\alpha,\gamma)$ takes the place of $\frac{z(\alpha,\gamma)-z(\alpha,\gamma')}{\gamma-\gamma'}$ and $\int_0^1 \nabla_1 K(\tau(z(\alpha,\gamma)-z(\beta,\gamma))+(1-\tau)(z(\alpha,\gamma')-z(\beta,\gamma')),\tilde f(\alpha)-\tilde f(\beta))d\tau$ is replaced by $\nabla_1 K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde f(\alpha)-\tilde f(\beta))$. Therefore we could use the similar estimate and have

$$\frac{d}{dt}\|w\|^2_{H^3(\mathbb{T})} \lesssim C(\|w\|^2_{H^3(\mathbb{T})}).$$

As in the existence of $z(\alpha,\gamma,t)$, we could do the similar energy estimate to the approximation of the equation

$$\frac{dw^n(\alpha,\gamma,t)}{dt} = \tilde T(w) = \varphi_n * (D_z T(z(\alpha,\gamma,t),\gamma,t)[\varphi_n * w^n]) + \varphi_n * (\partial_\gamma T(\alpha,\gamma,t)), \tag{8.3}$$

with initial value $w^n(\alpha,\gamma,0)=0$. Then from the Picard theorem and compactness argument, there exists $0 \le t_3 \le t_1$, such that

$$w(\alpha,\gamma,t) = \int_0^t \tilde T(w(\alpha,\gamma,\tau))d\tau,$$

and

$$\|w(\alpha,\gamma,t)\|_{L^\infty([0,t_3],H^3_\alpha(\mathbb{T}))} \lesssim 1. \tag{8.4}$$

We claim there is an uniform t_3 holds for all $\gamma \in [-1,1]$. Moreover, we have the following lemma:

Lemma 8.1. *For any $g(\alpha), h(\alpha) \in H^{j+1}(\mathbb{T})$, $j \leq 2$, we have*

$$\|\tilde{T}(g(\alpha), \gamma, t)\|_{H^j(\mathbb{T})} \lesssim 1, \tag{8.5}$$

$$\|\tilde{T}(g(\alpha), \gamma, t) - \tilde{T}(h(\alpha), \gamma, t)\|_{H^j(\mathbb{T})} \lesssim \|g(\alpha) - h(\alpha)\|_{H^{j+1}(\mathbb{T})}, \tag{8.6}$$

$$\lim_{t \to t'} \|\tilde{T}(g(\alpha), \gamma, t) - \tilde{T}(g(\alpha), \gamma, t')\|_{H^j(\mathbb{T})} = 0. \tag{8.7}$$

Proof. It is easy to get these bounds since $z(\alpha, \gamma, t) \in L^\infty([0, t_0], H^5_\alpha(\mathbb{T})) \cap C^0([0, t_0], H^4_\alpha(\mathbb{T}))$. □

Then we have $w(\alpha, \gamma, t) \in L^\infty([0, t_3], H^3_\alpha(\mathbb{T})) \cap C^0([0, t_3], H^2_\alpha(\mathbb{T})) \cap C^1([0, t_3], H^1_\alpha(\mathbb{T}))$.

We claim that we could do the similar argument as in the estimate of $\|z(\alpha, \gamma) - z(\alpha, \gamma')\|_{H^3_\alpha(\mathbb{T})} \lesssim |\gamma - \gamma'|$ to get

$$\|w(\alpha, \gamma) - w(\alpha, \gamma)\|_{H^1_\alpha(\mathbb{T})} \lesssim |\gamma - \gamma'|.$$

Then from (8.4), we have

$$\lim_{\gamma' \to \gamma} \|w(\alpha, \gamma') - w(\alpha, \gamma)\|_{H^2_\alpha(\mathbb{T})} = 0. \tag{8.8}$$

Now we show w is the derivative of z with respect of γ. Let

$$v(\alpha, \gamma, \gamma', t) = \frac{z(\alpha, \gamma, t) - z(\alpha, \gamma', t)}{\gamma - \gamma'} - w(\alpha, \gamma, t).$$

From (8.1) and (7.1), we have

$$\begin{aligned}
\frac{dv(\alpha, \gamma, \gamma', t)}{dt} &= \frac{T(z(\alpha, \gamma, t), \gamma, t) - T(z(\alpha, \gamma', t), \gamma, t)}{\gamma - \gamma'} + \frac{\int_\gamma^{\gamma'} (\partial_\eta T)(z(\alpha, \gamma', t), \eta, t) d\eta}{\gamma - \gamma'} \\
&\quad - D_z T(z(\alpha, \gamma, t), \gamma, t)[w(\alpha, \gamma, t)] - (\partial_\gamma T)(z(\alpha, \gamma, t), \gamma, t) \\
&= \left(\frac{T(z(\alpha, \gamma, t), \gamma, t) - T(z(\alpha, \gamma', t), \gamma, t)}{\gamma - \gamma'} - D_z T(z(\alpha, \gamma, t), \gamma, t)[w(\alpha, \gamma, t)] \right) \\
&\quad + \left(\frac{\int_\gamma^{\gamma'} (\partial_\eta T)(z(\alpha, \gamma', t), \eta, t) d\eta}{\gamma - \gamma'} - (\partial_\gamma T)(z(\alpha, \gamma, t), \gamma, t) \right) \\
&= Term_1 + Term_2.
\end{aligned}$$

We have

$$\begin{aligned}
Term_2 &= \frac{1}{\gamma - \gamma'} \int_\gamma^{\gamma'} (\partial_\eta T)(z(\alpha, \gamma', t), \eta, t) d\eta - (\partial_\gamma T)(z(\alpha, \gamma, t), \gamma, t) \\
&= \int_\gamma^{\gamma'} \frac{(\partial_\eta T)(z(\alpha, \gamma', t), \eta, t) - (\partial_\gamma T)(z(\alpha, \gamma', t), \gamma, t)}{\gamma - \gamma'} d\eta \\
&\quad + ((\partial_\gamma T)(z(\alpha, \gamma', t), \gamma, t) - (\partial_\gamma T)(z(\alpha, \gamma, t), \gamma, t)).
\end{aligned}$$

From (7.2) and (7.15), we have

$$\begin{aligned}&\|((\partial_\gamma T)(z(\alpha,\gamma',t),\gamma,t)-(\partial_\gamma T)(z(\alpha,\gamma,t),\gamma,t))\|_{L^2(\mathbb{T})}\\&\lesssim \|z(\alpha,\gamma,t)-z(\alpha,\gamma',t)\|_{H^2(\mathbb{T})}\lesssim|\gamma-\gamma'|,\end{aligned}$$

and

$$\|(\partial_\eta T)(z(\alpha,\gamma',t),\eta,t)-(\partial_\gamma T)(z(\alpha,\gamma',t),\gamma,t)\|_{L^2(\mathbb{T})}\lesssim|\eta-\gamma|,$$

Then

$$\|Term_2\|_{L^2(\mathbb{T})}\lesssim|\gamma-\gamma'|.$$

Moreover, for $Term_1$, from (8.2), (7.1), (7.6) and (7.8), we have

$$\begin{aligned}&\frac{T(z(\alpha,\gamma,t),\gamma,t)-T(z(\alpha,\gamma',t),\gamma,t)}{\gamma-\gamma'}-D_zT(z(\alpha,\gamma,t),\gamma,t)[w(\alpha,\gamma,t)]\\&=\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha(v(\alpha,\gamma,\gamma'))\\&\quad+\lambda(\alpha)\int_{-\pi}^{\pi}K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde f(\alpha)-\tilde f(\beta))\left(\frac{\partial_\alpha v(\alpha,\gamma,\gamma')}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta v(\beta,\gamma,\gamma')}{1+ic'(\beta)\gamma t}\right)\\&(1+ic'(\beta)\gamma t)d\beta\\&\quad+\lambda(\alpha)\int_{-\pi}^{\pi}\int_0^1\nabla_1K(\tau(z(\alpha,\gamma)-z(\beta,\gamma))\\&+(1-\tau)(z(\alpha,\gamma')-z(\beta,\gamma')),\tilde f(\alpha)-\tilde f(\beta))\\&\cdot\left(\frac{z(\alpha,\gamma)-z(\beta,\gamma)-(z(\alpha,\gamma')-z(\beta,\gamma'))}{\gamma-\gamma'}\right)\\&\quad\left(\frac{\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z(\beta,\gamma')}{1+ic'(\beta)\gamma t}+\partial_\alpha\tilde f(\alpha)-\partial_\beta\tilde f(\beta)\right)(1+ic'(\beta)\gamma t)d\tau d\beta\\&\quad-\lambda(\alpha)\int_{-\pi}^{\pi}\nabla_1K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde f(\alpha)-\tilde f(\beta))\cdot(w(\alpha,\gamma)-w(\beta,\gamma))\\&\cdot\left(\partial_\alpha\tilde f(\alpha)-\partial_\beta\tilde f(\beta)+\frac{\partial_\alpha z(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z(\beta,\gamma)}{1+ic'(\beta)\gamma t}\right)(1+ic'(\beta)\gamma t)d\beta\\&=Term_{1,1}+Term_{1,2}+Term_{1,3}+Term_{1,4}.\end{aligned}$$

Here

$$
\begin{aligned}
&Term_{1,3}+Term_{1,4}=\lambda(\alpha)\int_{-\pi}^{\pi}\int_0^1 \nabla_1 K(\tau(z(\alpha,\gamma)-z(\beta,\gamma))+(1-\tau)(z(\alpha,\gamma')-z(\beta,\gamma')),\tilde{f}(\alpha)\\
&-\tilde{f}(\beta))d\tau\cdot(v(\alpha,\gamma,\gamma')-v(\beta,\gamma,\gamma'))\\
&\cdot\left(\partial_\alpha\tilde{f}(\alpha)-\partial_\beta\tilde{f}(\beta)+\frac{\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z(\beta,\gamma')}{1+ic'(\beta)\gamma t}\right)(1+ic'(\beta)\gamma t)d\beta\\
&+\lambda(\alpha)\int_{-\pi}^{\pi}[\int_0^1 \nabla_1 K(\tau(z(\alpha,\gamma)-z(\beta,\gamma))+(1-\tau)(z(\alpha,\gamma')-z(\beta,\gamma')),\tilde{f}(\alpha)-\tilde{f}(\beta))d\tau\\
&-\nabla_1 K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde{f}(\alpha)-\tilde{f}(\beta))]\cdot(w(\alpha,\gamma)-w(\beta,\gamma))\\
&\left(\frac{\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z(\beta,\gamma')}{1+ic'(\beta)\gamma t}+\partial_\alpha\tilde{f}(\alpha)-\partial_\beta\tilde{f}(\beta)\right)(1+ic'(\beta)\gamma t)d\beta\\
&+\lambda(\alpha)\int_{-\pi}^{\pi}\nabla_1 K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde{f}(\alpha)-\tilde{f}(\beta))\cdot(w(\alpha,\gamma)-w(\beta,\gamma))\\
&\left(\frac{\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z(\beta,\gamma')}{1+ic'(\beta)\gamma t}-(\frac{\partial_\alpha z(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z(\beta,\gamma)}{1+ic'(\beta)\gamma t})\right)(1+ic'(\beta)\gamma t)d\beta\\
&=Term_{1,5}+Term_{1,6}+Term_{1,7}.
\end{aligned}
$$

Since the component of $\nabla_1 K$ is of -2 type, we could use Lemma 11.2 to bound $Term_{1,5}$ and have

$$\|Term_{1,5}\|^2_{L^2(\mathbb{T})}\lesssim\|v(\alpha,\gamma,\gamma')\|^2_{L^2(\mathbb{T})}.$$

For $Term_{1,6}$, we have

$$
\begin{aligned}
&\|\left[\int_0^1 \nabla_1 K(\tau(z(\alpha,\gamma)-z(\beta,\gamma))+(1-\tau)(z(\alpha,\gamma')-z(\beta,\gamma')),\tilde{f}(\alpha)-\tilde{f}(\beta))d\tau\right.\\
&\quad\left.-\nabla_1 K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde{f}(\alpha)-\tilde{f}(\beta))\right](\alpha-\beta)^2\|_{C^1([-2\delta,\delta]\times[-\pi,\pi])}\\
&\lesssim\|z(\alpha,\gamma)-z(\alpha,\gamma')\|_{H^3(\mathbb{T})}.
\end{aligned}
$$

Then from Lemma 11.3, and (7.15), we have

$$\|Term_{1,6}\|_{L^2(\mathbb{T})}\lesssim|\gamma-\gamma'|.$$

From Lemma 11.3, we again have

$$
\begin{aligned}
&\|Term_{1,7}\|_{L^2(\mathbb{T})}\\
&\lesssim\|\nabla_1 K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde{f}(\alpha)-\tilde{f}(\beta))(\alpha-\beta)^2\|_{C^1([-2\delta,\delta]\times[-\pi,\pi])}\|w(\alpha,\gamma)\|_{C^2(\mathbb{T})}\\
&\left\|\frac{\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t}-\frac{\partial_\alpha z(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}\right\|_{L^2(\mathbb{T})}\\
&\lesssim\|z(\alpha,\gamma)-z(\alpha,\gamma')\|_{H^1(\mathbb{T})}\lesssim|\gamma-\gamma'|,
\end{aligned}
$$

where we use (8.4) and (7.15). Therefore we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{-\pi}^{\pi}|v(\alpha,\gamma,\gamma')|^2d\alpha\\
&=2\Re\int_{-\pi}^{\pi}v(\alpha,\gamma,\gamma')\overline{\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha(v(\alpha,\gamma,\gamma'))}d\alpha\\
&\quad+2\Re\int_{-\pi}^{\pi}v(\alpha,\gamma,\gamma')\overline{\lambda(\alpha)\int_{-\pi}^{\pi}K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde{f}(\alpha)-\tilde{f}(\beta))\left(\frac{\partial_\alpha v(\alpha,\gamma,\gamma')}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta v(\beta,\gamma,\gamma')}{1+ic'(\beta)\gamma t}\right)(1+ic'(\beta)\gamma t)d\beta d\alpha}\\
&\quad+B.T^0,
\end{aligned}
$$

where

$$B.T^0 \lesssim |\gamma - \gamma'|^2 + \|v(\alpha, \gamma, \gamma')\|^2_{L^2(\mathbb{T})}.$$

Then by corollary 5.6, and the initial value $v(\alpha, \gamma, \gamma', 0) = 0$, from the Gronwall's inequality we have $\lim_{\gamma' \to \gamma} \|v(\alpha, \gamma, \gamma', t)\|_{L^2(\mathbb{T})} = 0$ when $t \leq t_3$.

Form (8.4), and (7.15), we have

$$\|v(\alpha, \gamma, \gamma', t)\|_{H^3(\mathbb{T})} \lesssim 1.$$

Moreover from the interpolation theorem, we have

$$\lim_{\gamma' \to \gamma} \|v(\alpha, \gamma, \gamma', t)\|_{H^2(\mathbb{T})} = 0.$$

Then from (8.8), we have

$$z(\alpha, \gamma) \in C^1_\gamma([-1, 1], H^2_\alpha(\mathbb{T})),$$

with

$$\frac{d}{d\gamma} z(\alpha, \gamma, t) = w(\alpha, \gamma).$$

From (8.1), we also have

$$\frac{d}{dt}\frac{d}{d\gamma} z = \frac{dw}{dt} = \frac{dT(z)}{d\gamma} = \frac{d}{d\gamma}\frac{d}{dt} z.$$

In conclusion we have

$$\begin{cases} z(\alpha, \gamma, t) \in C^1_t([0, t_1], H^3_\alpha(\mathbb{T})), \\ z(\alpha, \gamma, t) \in C^1_\gamma([-1, 1], H^2_\alpha(\mathbb{T})), \\ \frac{dz}{d\gamma} \in C^0_t([0, t_3], H^2_\alpha(\mathbb{T})) \cap C^1_t([0, t_3], H^1_\alpha(\mathbb{T})), \\ \frac{d}{dt}\frac{d}{d\gamma} z = \frac{d}{d\gamma}\frac{d}{dt} z, \\ z(\alpha, 0, t) = f^c(\alpha, t), \text{ when } 0 \leq t \leq t_2. \end{cases} \tag{8.9}$$

9. The Analyticity

In this section, we want to show $f(\alpha, t)$ is a real analytic function near 0 for each fixed t, $0 < t < t_3$. We first show that it is enough to prove that

$$\frac{ic(\alpha)t}{1 + ic'(\alpha)\gamma t}\frac{d}{d\alpha} z(\alpha, \gamma, t) - \frac{d}{d\gamma} z(\alpha, \gamma, t) = 0. \tag{9.1}$$

Lemma 9.1. *If z satisfies (9.1), then $f(x)$ can be analytically extended to $D_A = \{\alpha + iy | -\infty < \alpha < \infty, -c(\alpha)t \leq y \leq c(\alpha)t\}$.*

Proof. From the uniqueness (6.5), we have $z(\alpha, 0, t) = f^c(\alpha, t)$. Then

$$f^c(\alpha + iy, t) = \begin{cases} z\left(\alpha, \frac{y}{c(\alpha)t}, t\right), & c(\alpha) \neq 0, \\ z(\alpha, 0, t), & c(\alpha) = 0, \end{cases}$$

is a extension of $f^c(\alpha, t)$ on D_A. Moreover, when $c(\alpha) \neq 0$, we have

$$\frac{d}{d\alpha}\left(z\left(\alpha, \frac{y}{c(\alpha)t}, t\right)\right) = (\partial_\alpha z)\left(\alpha, \frac{y}{c(\alpha)t}, t\right) - (\partial_\gamma z)\left(\alpha, \frac{y}{c(\alpha)t}, t\right)\left(\frac{yc'(\alpha)}{c(\alpha)^2 t}\right),$$
$$\frac{d}{dy}\left(z\left(\alpha, \frac{y}{c(\alpha)t}, t\right)\right) = (\partial_\gamma z)\left(\alpha, \frac{y}{c(\alpha)t}, t\right)\left(\frac{1}{c(\alpha)t}\right).$$

Then we have

$$\begin{aligned} &\frac{d}{d\alpha} f^c(\alpha + iy, t) + i\frac{d}{dy} f^c(\alpha + iy, t) \\ &= (\partial_\alpha z)\left(\alpha, \frac{y}{c(\alpha)t}, t\right) - (\partial_\gamma z)\left(\alpha, \frac{y}{c(\alpha)t}, t\right)\left(\frac{yc'(\alpha)}{c(\alpha)^2 t}\right) \\ &+ i(\partial_\gamma z)\left(\alpha, \frac{y}{c(\alpha)t}, t\right)\left(\frac{1}{c(\alpha)t}\right). \end{aligned}$$

Now let $\frac{y}{c(\alpha)t} = \gamma$. Then

$$\begin{aligned} &\frac{d}{d\alpha} f^c(\alpha + iy, t) + i\frac{d}{dy} f^c(\alpha + iy, t) = \partial_\alpha z(\alpha, \gamma, t) - \left(\frac{c'(\alpha)\gamma}{c(\alpha)} - \frac{i}{c(\alpha)t}\right)\partial_\gamma z(\alpha, \gamma, t) \\ &= \partial_\alpha z(\alpha, \gamma, t) - \left(\frac{ic'(\alpha)\gamma t + 1}{ic(\alpha)t}\right)\partial_\gamma z(\alpha, \gamma, t) \\ &= 0. \end{aligned}$$

Moreover, $z(\alpha, \gamma, t) \in C^1_\gamma([-1, 1], H^2_\alpha(\mathbb{T}))$. Then $\partial_\alpha f^c$, $\partial_\gamma f^c$ are continuous. Therefore we have the analyticity of f^c near 0. We also have $f(\alpha, t) = f^c(\alpha, t)$ when $|\alpha| \leq \delta$. Then we have the result. □

Let

$$A_0(h)(\alpha, \gamma, t) = \left(\frac{ic(\alpha)t}{1 + ic'(\alpha)\gamma t}\partial_\alpha - \partial_\gamma\right) h(\alpha, \gamma, t).$$

Before we prove $A_0(z) = 0$, we introduce some general lemmas.

Lemma 9.2. *If all the derivatives are well-defined and* $\partial_\alpha \partial_\gamma h(\alpha, \gamma, t) = \partial_\gamma \partial_\alpha h(\alpha, \gamma, t)$, *we have*

$$\left(\frac{ic(\alpha)t}{1 + ic'(\alpha)\gamma t}\partial_\alpha - \partial_\gamma\right)\frac{\partial_\alpha h(\alpha, \gamma)}{1 + ic'(\alpha)\gamma t} = \frac{\partial_\alpha}{1 + ic'(\alpha)\gamma t}\left(\frac{ic(\alpha)t}{1 + ic'(\alpha)\gamma t}\partial_\alpha - \partial_\gamma\right) h(\alpha, \gamma),$$

Proof. First, for the right hand side, we have

$$(RHS) = \frac{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha^2 h(\alpha,\gamma)}{1+ic'(\alpha)\gamma t} + \frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\left(\frac{-ic''(\alpha)\gamma t}{(1+ic'(\alpha)\gamma t)^2}\partial_\alpha h(\alpha,\gamma)\right) - \frac{\partial_\alpha\partial_\gamma h(\alpha,\gamma)}{1+ic'(\alpha)\gamma t} + \frac{ic'(\alpha)t\partial_\alpha h(\alpha,\gamma)}{(1+ic'(\alpha)\gamma t)^2}.$$

Also

$$(LHS) = \left[\frac{ic'(\alpha)t}{(1+ic'(\alpha)\gamma t)^2} - \frac{ic(\alpha)tic''(\alpha)\gamma t}{(1+ic'(\alpha)\gamma t)^3}\right]\partial_\alpha h(\alpha,\gamma) + \frac{ic(\alpha)t}{(1+ic'(\alpha)\gamma t)^2}\partial_\alpha^2 h(\alpha,\gamma) - \frac{1}{1+ic'(\alpha)\gamma t}\partial_\alpha\partial_\gamma h(\alpha,\gamma).$$

From the two equalities above, we have the result. □

Lemma 9.3. *If all the derivatives are well-defined and we have*

$$\frac{d}{dt}h = \frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha h + \tilde{T}(h),$$

and $\frac{d}{d\alpha}\frac{d}{dt}h = \frac{d}{dt}\frac{d}{d\alpha}h$, $\frac{d}{dt}\frac{d}{d\gamma}h = \frac{d}{d\gamma}\frac{d}{dt}h$, $\frac{d}{d\alpha}\frac{d}{d\gamma}h = \frac{d}{d\gamma}\frac{d}{d\alpha}h$, *then we have*

$$\frac{d}{dt}A_0(h) = \frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha A_0(h) + A_0(\tilde{T}(h)).$$

Proof. First,

$$\begin{aligned}
&\frac{d}{dt}A_0(h)(\alpha,\gamma,t)\\
&= \frac{d}{dt}\left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\right)\partial_\alpha h + \frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha\frac{d}{dt}h - \partial_\gamma\frac{d}{dt}h\\
&= \frac{d}{dt}\left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\right)\partial_\alpha h + \left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha - \partial_\gamma\right)\left(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha h\right) + A_0(\tilde{T}(h))\\
&= \frac{d}{dt}\left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\right)\partial_\alpha h + \frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha\left(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\right)\partial_\alpha h + \frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha^2 h\\
&\quad - \partial_\gamma\left(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\right)\partial_\alpha h - \left(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\gamma\partial_\alpha h\right) + A_0(\tilde{T}(h))\\
&= \left(\underbrace{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha\left(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\right)}_{Term_1} + \underbrace{\frac{d}{dt}\left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\right) - \partial_\gamma\left(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\right)}_{Term_2}\right)\partial_\alpha h\\
&\quad + \underbrace{\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha - \partial_\gamma\right)\partial_\alpha h}_{Term_3} + A_0(\tilde{T}(h)).
\end{aligned}$$

Moreover, we have

$$\underbrace{\frac{d}{dt}\left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\right)-\partial_\gamma\left(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\right)}_{Term_2}=0,$$

$$\underbrace{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha\left(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\right)}_{Term_1}=\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha\left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\right),$$

and

$$\begin{aligned}&\underbrace{\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma\right)\partial_\alpha h}_{Term_3}\\&=\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha\left(\left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma\right)h\right)\\&-\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha\left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\right)\partial_\alpha h.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d}{dt}A_0(h)(\alpha,\gamma,t)&=\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha\left(\left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma\right)h\right)+A_0(\tilde{T}(h))\\&=\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha A_0(h)+A_0(\tilde{T}(h)).\end{aligned}$$

□

Lemma 9.4. *Let $\tilde{K}$ be meromorphic. $\partial_\alpha X(\alpha,\gamma)$, $\partial_\gamma X(\alpha,\gamma)$ are well-defined and in $C^0_\alpha[-\pi,\pi]$, $\partial_\alpha h(\alpha,\gamma)$ and $\partial_\gamma h(\alpha,\gamma)$ are well-defined vector functions with components in $C^0_\alpha[-\pi,\pi]$. If for fixed α, there is no singular point in the integrals below and $c(\pi)=c(-\pi)=0$, $c(\alpha)\in W^{2,\infty}$, then we have*

$$\begin{aligned}&A_0\Big(\int_{-\pi}^{\pi}\tilde{K}(h(\alpha,\gamma)-h(\beta,\gamma))X(\beta,\gamma)(1+ic'(\beta)\gamma t)d\beta\Big)\\&=\int_{-\pi}^{\pi}\nabla\tilde{K}(h(\alpha,\gamma)-h(\beta,\gamma))\cdot(A_0(h)(\alpha,\gamma)-A_0(h)(\beta,\gamma))X(\beta,\gamma)(1+ic'(\beta)\gamma t)d\beta\\&+\int_{-\pi}^{\pi}\tilde{K}(h(\alpha,\gamma)-h(\beta,\gamma))A_0(X)(\beta,\gamma)(1+ic'(\beta)\gamma t)d\beta\\&=D_h\Big(\int_{-\pi}^{\pi}\tilde{K}(h(\alpha,\gamma)-h(\beta,\gamma))X(\beta,\gamma)(1+ic'(\beta)\gamma t)d\beta\Big)[A_0(h)]\\&+\int_{-\pi}^{\pi}\tilde{K}(h(\alpha,\gamma)-h(\beta,\gamma))A_0(X)(\beta,\gamma)(1+ic'(\beta)\gamma t)d\beta.\end{aligned}$$

Here D_h is the Gateaux derivative.

Proof. We have

$$
\begin{aligned}
&A_0\left(\int_{\alpha-\pi}^{\alpha+\pi}\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta\right)\\
&=\underbrace{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(\tilde{K}(h(\alpha,\gamma)-h(-\pi,\gamma))X(-\pi,\gamma)(1+ic'(-\pi)\gamma t)-\tilde{K}(h(\alpha,\gamma)-h(\pi,\gamma))X(\pi,\gamma)(1+ic'(\pi)\gamma t))}_{Term_1}\\
&\quad+\underbrace{\int_{\alpha-\pi}^{\alpha+\pi}\nabla\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))\cdot\left(\left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma\right)h(\alpha,\gamma)-(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)h(\alpha-\beta,\gamma,t)\right)}_{Term_2}\\
&\quad\underbrace{X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta}_{Term_2}\\
&\quad+\underbrace{\int_{\alpha-\pi}^{\alpha+\pi}\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))\left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma\right)X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta}_{Term_3}\\
&\quad+\underbrace{\int_{\alpha-\pi}^{\alpha+\pi}\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))X(\alpha-\beta,\gamma)\left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}ic''(\alpha-\beta)\gamma t-ic'(\alpha-\beta)t\right)d\beta}_{Term_4}\\
&=Term1+\\
&\quad+\underbrace{\int_{\alpha-\pi}^{\alpha+\pi}\nabla\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))\cdot\left(\left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma\right)h(\alpha,\gamma)-\left(\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha-\partial_\gamma\right)h(\alpha-\beta,\gamma,t)\right)}_{Term_{2,1}}\\
&\quad\underbrace{X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta}_{Term_{2,1}}\\
&\quad+\underbrace{\int_{\alpha-\pi}^{\alpha+\pi}\nabla\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))\cdot((\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha)h(\alpha-\beta,\gamma,t))}_{Term_{2,2}}\\
&\quad\underbrace{X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta}_{Term_{2,2}}\\
&\quad+\underbrace{\int_{\alpha-\pi}^{\alpha+\pi}\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))\left(\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha-\partial_\gamma\right)X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta}_{Term_{3,1}}\\
&\quad+\underbrace{\int_{\alpha-\pi}^{\alpha+\pi}\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))\left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha\right)X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta}_{Term_{3,2}}\\
&\quad+\underbrace{\int_{\alpha-\pi}^{\alpha+\pi}\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))X(\alpha-\beta,\gamma)\left(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}ic''(\alpha-\beta)\gamma t-ic'(\alpha-\beta)t\right)d\beta}_{Term_4}\\
&=Term_1+Term_2+Term_{2,2}+Term_{3,1}+Term_{3,2}+Term_4.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&Term_{2,2} + Term_{3,2} + Term_4\\
&= \int_{\alpha-\pi}^{\alpha+\pi} \frac{d}{d\beta}(\tilde{K}(h(\alpha,\gamma) - h(\alpha-\beta,\gamma))X(\alpha-\beta,\gamma)(ic(\alpha-\beta)t\\
&\quad - \frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(\alpha-\beta)\gamma t)))d\beta\\
&= \tilde{K}(h(\alpha,\gamma) - h(-\pi,\gamma))X(-\pi,\gamma)(ic(-\pi)t - \frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(-\pi)\gamma t))\\
&\quad - \tilde{K}(h(\alpha,\gamma) - h(\pi,\gamma))X(\pi,\gamma)(ic(\pi)t - \frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(\pi)\gamma t)).
\end{aligned}
$$

We use the condition that $c(-\pi) = c(\pi) = 0$ and we could get the $Term_1 + Term_{2,2} + Term_{3,2} + Term_4 = 0$. Then we have the result. □

Now we use Lemmas 9.2, 9.3, 9.4, to show the result. From (4.5) and (5.5), we have

$$
\begin{aligned}
&\frac{dz_\mu(\alpha,\gamma,t)}{dt} = \frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha z_\mu(\alpha,\gamma,t)\\
&+\lambda(\alpha_\gamma^t)\int_{-\pi}^{\pi} K(z(\alpha,\gamma,t) - z(\beta,\gamma,t), \tilde{f}(\alpha_\gamma^t,t) - \tilde{f}(\beta_\gamma^t,t))\\
&\left(\frac{\partial_\alpha z_\mu(\alpha,\gamma,t)}{1+ic'(\alpha)\gamma t} - \frac{\partial_\beta z_\mu(\beta,\gamma,t)}{1+ic'(\beta)\gamma t}\right)(1+ic'(\beta)\gamma t)d\beta\\
&+\lambda(\alpha_\gamma^t)\int_{-\pi}^{\pi} K(z(\alpha,\gamma,t) - z(\beta,\gamma,t), \tilde{f}(\alpha_\gamma^t,t) - \tilde{f}(\beta_\gamma^t,t))((\partial_\alpha\tilde{f}_\mu)(\alpha_\gamma^t,t)\\
&-(\partial_\beta\tilde{f}_\mu)(\beta_\gamma^t,t))(1+ic'(\beta)\gamma t)d\beta.
\end{aligned}
$$

Since we have $A_0(\lambda(\alpha_\gamma^t)) = 0$, from Lemma 9.3, we get

$$
\begin{aligned}
&\frac{dA_0(z_\mu)}{dt} = \frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha A_0(z_\mu)(\alpha,\gamma,t)\\
&+\lambda(\alpha_\gamma^t)A_0(\int_{-\pi}^{\pi} K(z(\alpha,\gamma,t) - z(\beta,\gamma,t), \tilde{f}(\alpha_\gamma^t,t) - \tilde{f}(\beta_\gamma^t,t))\\
&\left(\frac{\partial_\alpha z_\mu(\alpha,\gamma,t)}{1+ic'(\alpha)\gamma t} - \frac{\partial_\beta z_\mu(\beta,\gamma,t)}{1+ic'(\beta)\gamma t} + (\partial_\alpha\tilde{f}_\mu)(\alpha_\gamma^t,t) - (\partial_\beta\tilde{f}_\mu)(\beta_\gamma^t,t)\right)(1+ic'(\beta)\gamma t)d\beta).
\end{aligned}
$$

Let

$$
\begin{aligned}
h(\alpha,\gamma) &= (z(\alpha,\gamma,t), \tilde{f}(\alpha_\gamma^t,t), \frac{\partial_\alpha z_\mu(\alpha,\gamma,t)}{1+ic'(\alpha)\gamma t} + (\partial_\alpha\tilde{f}_\mu)(\alpha_\gamma^t,t)),\\
\tilde{K}(h(\alpha,\gamma) - h(\beta,\gamma)) &= K(h_1(\alpha,\gamma) - h_1(\beta,\gamma), h_2(\alpha,\gamma)\\
&\quad -h_2(\beta,\gamma))(h_3(\alpha,\gamma) - h_3(\beta,\gamma)),
\end{aligned}
$$

and

$$
X(\alpha,\gamma) = 1.
$$

then by Lemmas 9.4 and 9.2 and $A_0((\partial_\alpha \tilde{f}_\mu)(\alpha_\gamma^t, t)) = 0$, and $A_0(\tilde{f}_\mu(\alpha_\gamma^t, t)) = 0$, we have

$$\begin{aligned}
\frac{dA_0(z_\mu)}{dt} &= \frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha A_0(z_\mu)(\alpha,\gamma,t)\\
&\quad + \lambda(\alpha_\gamma^t)\int_{-\pi}^{\pi} K(z(\alpha,\gamma,t)-z(\beta,\gamma,t),\tilde{f}(\alpha_\gamma^t,t)-\tilde{f}(\beta_\gamma^t,t))\\
&\quad \left(\frac{\partial_\alpha A_0(z_\mu)(\alpha,\gamma,t)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta A_0(z_\mu)(\beta,\gamma,t)}{1+ic'(\beta)\gamma t}\right)\\
&\quad \cdot(1+ic'(\beta)\gamma t)d\beta\\
&\quad + \lambda(\alpha_\gamma^t)\int_{-\pi}^{\pi} \nabla_1 K(z(\alpha,\gamma,t)-z(\beta,\gamma,t),\tilde{f}(\alpha_\gamma^t,t)-\tilde{f}(\beta_\gamma^t,t))\\
&\quad \cdot(A_0(z)(\alpha,\gamma,t)-A_0(z)(\beta,\gamma,t))\\
&\quad \cdot\left(\frac{\partial_\alpha z_\mu(\alpha,\gamma,t)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z_\mu(\beta,\gamma,t)}{1+ic'(\beta)\gamma t}+(\partial_\alpha\tilde{f}_\mu)(\alpha_\gamma^t,t)-(\partial_\beta\tilde{f}_\mu)(\beta_\gamma^t,t)\right)\\
&\quad (1+ic'(\beta)\gamma t)d\beta\\
&= Term_1 + Term_2 + Term_3.
\end{aligned}$$

Since

$$\begin{aligned}
&\|\nabla_1 K(z(\alpha,\gamma,t)-z(\beta,\gamma,t)+\tilde{f}(\alpha_\gamma^t,t)-\tilde{f}(\beta_\gamma^t,t))(\alpha-\beta)^2\|_{C^1([-2\delta,2\delta]\times[-\pi,\pi])}\lesssim 1,\\
&\left\|\frac{\partial_\alpha z_\mu(\alpha,\gamma,t)}{1+ic'(\alpha)\gamma t}+(\partial_\alpha\tilde{f}_\mu)(\alpha_\gamma^t,t)\right\|_{C^2([-2\delta,2\delta]\times[-\pi,\pi])}\lesssim 1,
\end{aligned}$$

by Lemma 11.3, we have

$$\|Term_3\|_{L^2(\mathbb{T})}\lesssim \|A_0(z)\|_{L^2(\mathbb{T})}.$$

Then by corollary 5.6, we have

$$\frac{d}{dt}\|A_0(z)\|_{L^2(\mathbb{T})}^2\lesssim \|A_0(z)\|_{L^2(\mathbb{T})}^2.$$

Moreover when $t=0$, $A_0(z) = -\partial_\gamma z(\alpha,\gamma,0) = -\partial_\gamma f(\alpha,0) = 0$. Therefore $A_0(z)=0$, when $t\le t_3$.

10. Using the Energy Estimate to Show the Analyticity

Following an idea similar to that in the previous sections, we introduce a way to study the analyticity of the solution to some differential equations, which is, to our knowledge, a new method.

Theorem 10.1. *Let $T(f)$ be an operator satisfying the conditions below. We assume that there exists $\epsilon>0$, $k\ge 1$, $f_0\in H^k(\mathbb{T})$, such that when $\|f-f_0\|_{H^k}\lesssim\epsilon$,*

(a) (Boundedness) $T(f): H^k(\mathbb{T})\to H^k(\mathbb{T})$ with $\|T(f)\|_{H^k(\mathbb{T})}\lesssim 1$,

(b) (Existence and boundedness of the Fréchet derivative) $\|D_f(T(f))[h]\|_{H^k(\mathbb{T})} \lesssim \|h\|_{H^k(\mathbb{T})}$,
(c) (L^2 boundedness of the Fréchet derivative) $\|D_f(T(f))[h]\|_{L^2(\mathbb{T})} \lesssim \|h\|_{L^2(\mathbb{T})}$,
(d) $\frac{d}{dx}T(f) = D_f(T(f))[\frac{df}{dx}]$,
(e) $iD_f(T(f))[h] = D_f(T(f))[ih]$.

Here $\mathbb{T}$ *is the torus of length* 2π. *If* $f_0(x)$ *also satisfies the equation*

$$f_0'(x) = T(f_0), \tag{10.1}$$

$f_0(x)$ *must be real analytic.*

Proof. First, we assume f_0 to be an analytic function with analytic continuation $f(x,t)$. Then through the Cauchy-Riemann equation and (10.1), we have

$$\begin{cases} \frac{d}{dt}f(x,t) = iT(f(x,t)), \\ f(x,0) = f_0(x). \end{cases} \tag{10.2}$$

Our goal is to show this solution $f(x,t)$ does exist and is analytic.

Through (a), (b), we have

$$\|T(f) - T(g)\|_{H^k(\mathbb{T})} \lesssim \|f - g\|_{H^k(\mathbb{T})}.$$

We can use the Picard theorem to show there is a solution satisfying

$$f(x,t) - f(x,0) = \int_0^t iT(f(x,\tau))d\tau.$$

with $|t| < t_0$ for some $t_0 > 0$. Moreover

$$f(x,t) \in W^{2,\infty}((-t_0,t_0), H^k(\mathbb{T})). \tag{10.3}$$

By (10.3), we have

$$\lim_{\Delta t\to 0}\left\| \frac{f(x,t) - f(x,t+\Delta t)}{\Delta t} - \frac{d}{dt}f(x,t)\right\|_{H^k(\mathbb{T})} = 0.$$

Hence

$$\lim_{\Delta t\to 0}\left\| \frac{\frac{d}{dx}f(x,t) - \frac{d}{dx}f(x,t+\Delta t)}{\Delta t} - \frac{d}{dx}\frac{d}{dt}f(x,t)\right\|_{H^{k-1}(\mathbb{T})} = 0.$$

Therefore we have

$$\frac{d}{dt}\frac{d}{dx}f(x,t) = \frac{d}{dx}\frac{d}{dt}f(x,t) \in C^0((-t_0,t_0), H^{k-1}(\mathbb{T})) \subset C^0((-t_0,t_0), L^2(\mathbb{T})), \tag{10.4}$$

and

$$\frac{d}{dx}f(x,t) + i\frac{d}{dt}f(x,t) \in W^{1,\infty}((-t_0,t_0), L^2(\mathbb{T})). \tag{10.5}$$

Then we can control $\|\frac{d}{dx}f(x,t)+i\frac{d}{dt}f(x,t)\|_{L^2(\mathbb{T})}$. We have

$$\begin{aligned}
&\left|\frac{d}{dt}\int_{-\pi}^{\pi}\left|\left(\frac{d}{dx}+i\frac{d}{dt}\right)f(x,t)\right|^2dx\right| \\
&=\left|2Re\int_{-\pi}^{\pi}\left(\frac{d}{dx}+i\frac{d}{dt}\right)f(x,t)\overline{\frac{d}{dt}\left(\frac{d}{dx}+i\frac{d}{dt}\right)f(x,t)dx}\right| \\
&=\left|2Re\int_{-\pi}^{\pi}\left(\frac{d}{dx}+i\frac{d}{dt}\right)f(x,t)\overline{\left(\frac{d}{dx}+i\frac{d}{dt}\right)\frac{d}{dt}f(x,t)dx}\right| \\
&=\left|2Re\int_{-\pi}^{\pi}\left(\frac{d}{dx}+i\frac{d}{dt}\right)f(x,t)\overline{\left(\frac{d}{dx}+i\frac{d}{dt}\right)iT(f(x,t))dx}\right| \\
&=\left|2Re\int_{-\pi}^{\pi}\left(\frac{d}{dx}+i\frac{d}{dt}\right)f(x,t)\overline{iD_fT(f(x,t))[\left(\frac{d}{dx}+i\frac{d}{dt}\right)[f(x,t)]dx}\right| \\
&\lesssim\int_{-\pi}^{\pi}\left|\left(\frac{d}{dx}+i\frac{d}{dt}\right)f(x,t)\right|^2dx. \qquad (10.6)
\end{aligned}$$

Here the first equality follows from (10.5), the second from (10.4).

Through (10.1), (10.2), we have

$$\left\|\frac{d}{dx}f(x,t)+i\frac{d}{dt}f(x,t)\right\|_{L^2(\mathbb{T})}|_{t=0}=0.$$

Moreover, from(10.5), $\|\frac{d}{dx}f(x,t)+i\frac{d}{dt}f(x,t)\|^2_{L^2(\mathbb{T})}\in W^{1,\infty}(-t_0,t_0)$. Then

$$\left\|\frac{d}{dx}f(x,t)+i\frac{d}{dt}f(x,t)\right\|^2_{L^2(\mathbb{T})}=\int_0^t\frac{d}{d\tau}\left\|\frac{d}{dx}f(x,\tau)+i\frac{d}{d\tau}f(x,\tau)\right\|^2_{L^2(\mathbb{T})}d\tau.$$

Hence we can use the Gronwall inequality and get

$$\left\|\frac{d}{dx}f(x,t)+i\frac{d}{dt}f(x,t)\right\|_{L^2(\mathbb{T})}=0. \qquad (10.7)$$

Moreover, since $k\geq 1$, from (10.3), we have $\frac{d}{dt}f(x,t)\in W^{1,\infty}((-t_0,t_0),H^1(\mathbb{T}))$. Then $\frac{d}{dt}f(x,t)$ is continuous in x and t. Therefore $\partial_x f(x,t)$ is continuous in x and t.

Then by the (10.7) and (10.2), we have the analyticity. □

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Declarations

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11. Appendix

Lemma 11.1. *For $G(\alpha,\beta)\in C^1([-2\delta,2\delta]\times[-\pi,\pi])$, we have*

$$\|p.v.\int_{-\pi}^{\pi}\frac{G(\alpha,\beta)}{(\alpha-\beta)}d\beta\|_{L^\infty[-2\delta,2\delta]}\lesssim\|G(\alpha,\beta)\|_{C^1([-2\delta,2\delta]\times[-\pi,\pi])}.$$

Proof. We have

$$\begin{aligned}&\left|p.v.\int_{-\pi}^{\pi}\frac{G(\alpha,\beta)}{(\alpha-\beta)}d\beta\right|\\&\quad=\left|\int_{-\pi}^{\pi}\frac{G(\alpha,\beta)-G(\alpha,\alpha)}{(\alpha-\beta)}d\beta\right|\\&\qquad+\left|p.v.\int_{-\pi}^{\pi}\frac{1}{\alpha-\beta}d\beta G(\alpha,\alpha)\right|\lesssim\|G(\alpha,\beta)\|_{C^1([-2\delta,2\delta]\times[-\pi,\pi])}.\end{aligned}$$

□

Lemma 11.2. *For $g(\alpha)\in L^2[-\pi,\pi]$, $G(\alpha,\beta)\in C^1([-2\delta,2\delta]\times[-\pi,\pi])$, we have*

$$\left\|\int_{-\pi}^{\pi}G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{(\alpha-\beta)}d\beta\right\|_{L^2[-2\delta,2\delta]}\lesssim\|G(\alpha,\beta)\|_{C^1([-2\delta,2\delta]\times[-\pi,\pi])}\|g\|_{L^2[-\pi,\pi]}.$$

Proof. We have

$$\begin{aligned}&\int_{-\pi}^{\pi} G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{(\alpha-\beta)}d\beta\\&=\int_{-\pi}^{\pi}\frac{G(\alpha,\beta)-G(\alpha,\alpha)}{\alpha-\beta}(g(\alpha)-g(\beta))d\beta+G(\alpha,\alpha)p.v.\int_{-\pi}^{\pi}\frac{g(\alpha)-g(\beta)}{\alpha-\beta}d\beta\\&=Term_1+Term_2.\end{aligned}$$

Here

$$\|Term_1\|_{L^2[-2\delta,2\delta]}\lesssim\|G(\alpha,\beta)\|_{C^1([-2\delta,2\delta]\times[-\pi,\pi])}\|g(\alpha)\|_{L^2[-\pi,\pi]},$$

and

$$\|Term_2\|_{L^2[-2\delta,2\delta]}\lesssim\|G(\alpha,\alpha)\|_{C^0([-2\delta,2\delta])}\|g(\alpha)\|_{L^2[-\pi,\pi]}.$$

□

Corollary 11.3. *For $g(\alpha)\in L^2[-\pi,\pi]$, $h(\alpha)\in C^2[-\pi,\pi]$, $G(\alpha,\beta)\in C^1([-2\delta,2\delta]\times[-\pi,\pi])$, we have*

$$\begin{aligned}&\left\|\int_{-\pi}^{\pi}G(\alpha,\beta)\frac{(g(\alpha)-g(\beta))(h(\alpha)-h(\beta))}{(\alpha-\beta)^2}d\beta\right\|_{L^2[-2\delta,2\delta]}\\&\lesssim\|G(\alpha,\beta)\|_{C^1([-2\delta,2\delta]\times[-\pi,\pi])}\|g\|_{L^2[-2\delta,2\delta]}\|h\|_{C^2[-\pi,\pi]}.\end{aligned}$$

Proof. We can use Lemma 11.2 and let $\tilde{G}(\alpha,\beta)=G(\alpha,\beta)\frac{h(\alpha)-h(\beta)}{(\alpha-\beta)}$. □

Lemma 11.4. *For $g(\alpha)\in H^1[-\pi,\pi]$, $G(\alpha,\beta)\in C^0([-2\delta,2\delta]\times[-\pi,\pi])$, we have*

$$\left\|\int_{-\pi}^{\pi}G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{(\alpha-\beta)}d\beta\right\|_{L^2[-2\delta,2\delta]}\lesssim\|G(\alpha,\beta)\|_{C^0([-2\delta,2\delta]\times[-\pi,\pi])}\|g\|_{H^1[-\pi,\pi]}.$$

Proof. We have

$$\begin{aligned}&\left|\int_{-\pi}^{\pi}G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{(\alpha-\beta)}d\beta\right|\\&\leq\|G(\alpha,\beta)\|_{C^0([-2\delta,2\delta]\times[-\pi,\pi])}\int_{-\pi}^{\pi}\frac{|g(\alpha)-g(\beta)|}{(\alpha-\beta)}d\beta\\&\leq\|G(\alpha,\beta)\|_{C^0([-2\delta,2\delta]\times[-\pi,\pi])}\int_{-\pi}^{\pi}\int_0^1|g'(\tau(\alpha)+(1-\tau)(\beta))|d\tau d\beta\\&\lesssim\|G(\alpha,\beta)\|_{C^0([-2\delta,2\delta]\times[-\pi,\pi])}\|g\|_{H^1[-\pi,\pi]}.\end{aligned}$$

□

Lemma 11.5. *For $g(\alpha)\in H^k(\mathbb{T})$, $G(\alpha,\beta)\in C^1([-2\delta,2\delta]\times\mathbb{T})\cap C^k([-2\delta,2\delta]\times\mathbb{T})$, for $k\geq 0$, we have*

$$\begin{aligned}&\left\|\int_{-\pi}^{\pi}G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{\alpha-\beta}d\beta\right\|_{H^k[-2\delta,2\delta]}\\&\lesssim(\|G(\alpha,\beta)\|_{C^1([-2\delta,2\delta]\times\mathbb{T})}+\|G(\alpha,\beta)\|_{C^k([-2\delta,2\delta]\times\mathbb{T})})\|g\|_{H^k(\mathbb{T})}.\end{aligned}$$

Proof. From Lemma 11.2, when $k = 0$, we have the L^2 norm. Moreover, when $k \geq 1$, we have

$$\begin{aligned}
&\partial_\alpha^k \int_{-\pi}^{\pi} G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{\alpha-\beta}d\beta \\
&= \partial_\alpha^k \int_{-\pi}^{\pi} G(\alpha,\alpha-\beta)\frac{g(\alpha)-g(\alpha-\beta)}{\beta}d\beta \\
&= \sum_{0\leq j\leq k} C_j \int_{-\pi}^{\pi} \partial_\alpha^j G(\alpha,\alpha-\beta)\frac{\partial_\alpha^{k-j}g(\alpha)-\partial_\alpha^{k-j}g(\alpha-\beta)}{\beta}d\beta.
\end{aligned}$$

For $j \leq k-1$, we could use Lemma 11.2 to get the estimate. For $j = k$, we could use Lemma 11.4 to get the estimate. □

Lemma 11.6. *For $k \geq 2$, $g(\alpha) \in H^k(\mathbb{T})$, $G(\alpha,\beta) \in C^k([-2\delta, 2\delta]\times\mathbb{T})$, we have*

$$\begin{aligned}
&\left\| \partial_\alpha^k \left(\int_{-\pi}^{\pi} G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{\alpha-\beta}d\beta \right) - \int_{-\pi}^{\pi} G(\alpha,\beta)\frac{\partial_\alpha^k g(\alpha)-\partial_\beta^k g(\beta)}{\alpha-\beta}d\beta \right\|_{L^2[-2\delta,2\delta]} \\
&\lesssim \|G(\alpha,\beta)\|_{C^k([-2\delta,2\delta]\times\mathbb{T})}\|g\|_{H^{k-1}(\mathbb{T})}
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&\partial_\alpha^k \left(\int_{-\pi}^{\pi} G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{\alpha-\beta}d\beta \right) - \int_{-\pi}^{\pi} G(\alpha,\beta)\frac{\partial_\alpha^k g(\alpha)-\partial_\beta^k g(\beta)}{\alpha-\beta}d\beta \\
&= \sum_{1\leq j\leq k} C_j \int_{-\pi}^{\pi} \partial_\alpha^j (G(\alpha,\alpha-\beta))\frac{\partial_\alpha^{k-j}g(\alpha)-\partial_\beta^{k-j}g(\alpha-\beta)}{\beta}d\beta.
\end{aligned}$$

Then for $j \leq k-1$. we could use Lemma 11.2 to get the estimate. Then for $j = k$. we could use Lemma 11.4 to get the estimate. □

Lemma 11.7. *For $g(\alpha) \in H^3(\mathbb{T})$, $h(\alpha) \in H^3(\mathbb{T})$, $G(\alpha,\beta) \in C^3([-2\delta, 2\delta]\times\mathbb{T})$, we have*

$$\begin{aligned}
&\left\| \partial_\alpha^3 \left(\int_{-\pi}^{\pi} G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{\alpha-\beta}\frac{h(\alpha)-h(\beta)}{\alpha-\beta}d\beta \right) \right\|_{L^2[-2\delta,2\delta]} \\
&\lesssim \|G(\alpha,\beta)\|_{C^3([-2\delta,2\delta]\times\mathbb{T})}\|g\|_{H^3(\mathbb{T})}\|h\|_{H^3(\mathbb{T})}.
\end{aligned}$$

Proof. We take the derivative and have

$$\begin{aligned}
&\partial_\alpha^3 \Big(\int_{-\pi}^{\pi} G(\alpha,\alpha-\beta)\frac{g(\alpha)-g(\alpha-\beta)}{\beta}\frac{h(\alpha)-h(\alpha-\beta)}{\beta}d\beta \Big) \\
&= \sum_{j_1+j_2+j_3=3} \int_{-\pi}^{\pi} \partial_\alpha^{j_1}(G(\alpha,\alpha-\beta))\frac{\partial_\alpha^{j_2}g(\alpha)-\partial_\alpha^{j_2}g(\alpha-\beta)}{\beta}\frac{\partial_\alpha^{j_3}h(\alpha)-\partial_\alpha^{j_3}h(\alpha-\beta)}{\beta}d\beta.
\end{aligned}$$

If $j_2 \leq 1$, $j_3 \leq 1$, then

$$\begin{aligned}
&\left\| \int_{-\pi}^{\pi} \partial_\alpha^{j_1}(G(\alpha,\alpha-\beta))\frac{\partial_\alpha^{j_2}g(\alpha)-\partial_\alpha^{j_2}g(\alpha-\beta)}{\beta}\frac{\partial_\alpha^{j_3}h(\alpha)-\partial_\alpha^{j_3}h(\alpha-\beta)}{\beta}d\beta \right\|_{L^2[-2\delta,2\delta]} \\
&\lesssim \|\partial_\alpha^{j_1}(G(\alpha,\alpha-\beta))\|_{C^0([-2\delta,2\delta]\times\mathbb{T})}\|g\|_{C^2(\mathbb{T})}\|h\|_{C^2(\mathbb{T})} \\
&\lesssim \|G(\alpha,\beta)\|_{C^3([-2\delta,2\delta]\times\mathbb{T})}\|g\|_{H^3(\mathbb{T})}\|h\|_{H^3(\mathbb{T})}.
\end{aligned}$$

If $j_2 = 3$, then $j_1 = j_3 = 0$, by Lemma 11.3, we have

$$\begin{aligned}
&\left\| \int_{-\pi}^{\pi} \partial_\alpha^{j_1}(G(\alpha, \alpha-\beta)) \frac{\partial_\alpha^{j_2} g(\alpha) - \partial_\alpha^{j_2} g(\alpha-\beta)}{\beta} \frac{\partial_\alpha^{j_3} h(\alpha) - \partial_\alpha^{j_3} h(\alpha-\beta)}{\beta} d\beta \right\|_{L^2[-2\delta,2\delta]} \\
&\lesssim \|(G(\alpha, \alpha-\beta))\|_{C^1([-2\delta,2\delta]\times\mathbb{T})} \|g\|_{H^3(\mathbb{T})} \|h\|_{C^2(\mathbb{T})} \\
&\lesssim \|G(\alpha,\beta)\|_{C^3([-2\delta,2\delta]\times\mathbb{T})} \|g\|_{H^3(\mathbb{T})} \|h\|_{H^3(\mathbb{T})}.
\end{aligned}$$

If $j_2 = 2$, then $j_1 = 1$ or $j_3 = 1$, by Lemma 11.4, we have

$$\begin{aligned}
&\left\| \int_{-\pi}^{\pi} \partial_\alpha^{j_1}(G(\alpha, \alpha-\beta)) \frac{\partial_\alpha^{j_2} g(\alpha) - \partial_\alpha^{j_2} g(\alpha-\beta)}{\beta} \frac{\partial_\alpha^{j_3} h(\alpha) - \partial_\alpha^{j_3} h(\alpha-\beta)}{\beta} d\beta \right\|_{L^2[-2\delta,2\delta]} \\
&\lesssim \|\partial_\alpha^{j_1}(G(\alpha, \alpha-\beta))\|_{C^0([-2\delta,2\delta]\times\mathbb{T})} \|g\|_{H^3(\mathbb{T})} \|\partial_\alpha^{j_3} h\|_{C^1(\mathbb{T})} \\
&\lesssim \|G(\alpha,\beta)\|_{C^3([-2\delta,2\delta]\times\mathbb{T})} \|g\|_{H^3(\mathbb{T})} \|h\|_{H^3(\mathbb{T})}.
\end{aligned}$$

When $j_3 = 3$ or $j_3 = 2$, it can treated similarly as $j_2 = 3$ and $j_2 = 2$. □

References

1. ABELS, H., MATIOC, B.-V.: Well-posedness of the muskat problem in subcritical lp-sobolev spaces. *Eur. J. Appl. Math.* **33**(2), 224–266, 2022
2. ALAZARD, T., LAZAR, O.: Paralinearization of the Muskat equation and application to the Cauchy problem. *Arch. Ration. Mech. Anal.* **237**(2), 545–583, 2020
3. ALAZARD, T., NGUYEN, Q.-H.: Endpoint sobolev theory for the muskat equation. arXiv: Analysis of PDEs, 2020
4. ALAZARD, T., NGUYEN, Q.-H.: On the cauchy problem for the muskat equation with non-lipschitz initial data. *Commun. Partial Differ. Equ.* **46**(11), 2171–2212, 2021
5. ALAZARD, T., QUOC, H.: On the cauchy problem for the muskat equation. ii: Critical initial data. *Ann. PDE* **7**, 06, 2021
6. AMBROSE, D.M.: Well-posedness of two-phase hele-shaw flow without surface tension. *Eur. J. Appl. Math.* **15**, 597–607, 2004
7. CAMERON, S.: Global well-posedness for the two-dimensional muskat problem with slope less than 1. *Anal. PDE* **12**, 997–1022, 2018 **01**
8. CASTRO, Á. CÓRDOBA, D., FARACO, D.: Mixing solutions for the muskat problem. arXiv preprint arXiv:1605.04822, 2016
9. CASTRO, Á., CÓRDOBA, D., FEFFERMAN, C., GANCEDO, F.: Breakdown of smoothness for the muskat problem. *Arch. Ration. Mech. Anal.* **208**(3), 805–909, 2013
10. CASTRO, Á., CÓRDOBA, D., FEFFERMAN, C., GANCEDO, F., LÓPEZ-FERNÁNDEZ, M.: Rayleigh-taylor breakdown for the Muskat problem with applications to water waves. *Ann. Math.* **2**(175), 909–948, 2012
11. CASTRO, Á., FARACO, D., MENGUAL, F.: Degraded mixing solutions for the muskat problem. *Calc. Var. Partial. Differ. Equ.* **58**, 1–29, 2019
12. CASTRO, Á., FARACO, D., MENGUAL, F.: Localized mixing zone for muskat bubbles and turned interfaces. *Ann. PDE* **8**(1), 7, 2022
13. CHEN, K., QUOC, H., XU, Y.: The muskat problem with C^1 data. Trans. Am. Math. Soc. **9** 2021
14. CHENG, C.A., GRANERO-BELINCHÓN, R., SHKOLLER, S.: Well-posedness of the muskat problem with h2 initial data. *Adv. Math.* **286**, 32–104, 2016
15. CONSTANTIN, P., CÓRDOBA, D., GANCEDO, F., STRAIN, R.M.: On the global existence for the muskat problem. *J. Eur. Math. Soc.* **15**, 201–227, 2013

16. Constantin, P., Córdoba, D., Gancedo, F., Strain, R.: On the muskat problem: Global in time results in 2d and 3d. *Am. J. Math.* **138**, 10, 2013
17. Constantin, P., Gancedo, F., Shvydkoy, R., Vicol, V.: Global regularity for 2d muskat equations with finite slope. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire* **34**(4), 1041–1074, 2017
18. Cordoba, D., Faraco, D., Gancedo, F.: Lack of uniqueness for weak solutions of the incompressible porous media equation. *Arch. Ration. Mech. Anal.* **200**, 725–746, 2011
19. Córdoba, D., Gómez-Serrano, J., Zlatoš, A.: A note on stability shifting for the Muskat problem. *Philos. Trans. R. Soc. Lond.* **373**(2050), 20140278, 10, 2015
20. Córdoba, D., Gómez-Serrano, J., Zlatoš, A.: A note on stability shifting for the Muskat problem, II: From stable to unstable and back to stable. *Anal. PDE* **10**(2), 367–378, 2017
21. Córdoba, A., Córdoba, D., Gancedo, F.: Interface evolution: the hele-shaw and muskat problems. *Ann. Math.* **173**(1), 477–542, 2011
22. Córdoba, D., Gancedo, F.: Contour dynamics of incompressible 3-d fluids in a porous medium with different densities. *Commun. Math. Phys.* **273**, 445–471, 2007 **06**
23. Córdoba, D., Gancedo, F.: A maximum principle for the muskat problem for fluids with different densities. *Commun. Math. Phys.* **286**, 12, 2007
24. Córdoba, D., Lazar, O.: Global well-posedness for the 2d stable muskat problem in $h^{3/2}$. *Annales scientifiques de l'École Normale Supérieure* **54**, 03, 2018
25. Deng, F., Lei, Z., Lin, F.: On the two-dimensional muskat problem with monotone large initial data. *Commun. Pure Appl. Math.***70**, 2016
26. Förster, C., Székelyhidi, L.: Piecewise constant subsolutions for the muskat problem. *Commun. Math. Phys.* **363**, 1051–1080, 2018
27. Gancedo, F., García-Juárez, E., Patel, N., Strain, R.: On the muskat problem with viscosity jump: Global in time results. *Adv. Math.* **345**, 552–597, 2019
28. García-Juárez, E., Gómez-Serrano, J., Nguyen, H.Q., Pausader, B.: Self-similar solutions for the muskat equation. 2021
29. Matioc, B.-V.: The Muskat problem in 2d: equivalence of formulations, well-posedness, and regularity results. Anal. PDE, 2018. To appear
30. Muskat, M.: Two fluid systems in porous media. the encroachment of water into an oil sand. *Physics* **5**(9), 250–264, 1934
31. Nguyen, H., Pausader, B.: A paradifferential approach for well-posedness of the muskat problem. *Arch. Ration. Mech. Anal.* **237**, 07, 2020
32. Nguyen, H.Q.: Global solutions for the muskat problem in the scaling invariant besov space $\dot{B}^1_{\infty,1}$. *Adv. Math.* **394**, 108122, 2022
33. Noisette, F., Székelyhidi, L.: Mixing solutions for the muskat problem with variable speed. *J. Evol. Equ.* **21**, 3289–3312, 2021
34. Shaw, H.H.: On the motion of a viscous fluid between two parallel plates. *Nature* **58**, 34–36, 1898
35. Shi, J.: Regularity of solutions to the muskat equation ii: the degenerated regularity near the turnover points. 2022. In preparation
36. Siegel, M., Caflisch, R., Howison, S.: Global existence, singular solutions, and ill-posedness for the muskat problem. *Commun. Pure Appl. Math.* **57**, 1374–1411, 2004 **10**
37. Székelyhidi, L., Jr.: Relaxation of the incompressible porous media equation. *In Annales scientifiques de l'Ecole normale supérieure* **45**, 491–509, 2012
38. Yi, F.: Global classical solution of muskat free boundary problem. *J. Math. Anal. Appl.* **288**(2), 442–461, 2003
39. Yi, F.: Local classical solution of muskat free boundary problem. *J. Math. Anal. Appl.* **288**, 442–461, 2003 **12**

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