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APPROXIMATE NONLINEAR ESTIMATION

by

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ABSTRACT

The problem of minimum variance estimation is considered for a class of general nonlinear systems. Nonlinear relationships in both the system dynamics, as well as the measurement process are investigated from a new point of view. Two techniques of nonlinear estimation are explored with explicit comparison to the linear, Kalman estimator being made whenever possible.

The first technique uses a series expansion to approximate the nonlinear functions. The required estimation equations are derived in a straightforward manner, and the resulting system is demonstrated in a numerical example with a comparison to two other techniques.

A new approach to nonlinear estimation, which introduces a concept of quasi-linearization, is demonstrated. The proper estimation equations for nonlinear system dynamics, as well as measurement nonlinearities, are derived and some of their special characteristics discussed. The quasi-linear estimator is demonstrated in two examples, both analytically and numerically, and its performance is compared to the series estimator, as well as the linearized estimator.

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CHAPTER I

INTRODUCTION

Making an accurate estimate of a set of physical parameters given information only in the form of measured quantities corrupted by large uncertainties is the problem of concern in this investigation and is one which can be associated with activities of men for countless years. Though perhaps not always recognized as such, examples of early estimation procedures can be found in navigation, at first on land and later on the oceans, utilizing measurements, however crude, of stars, the sun and moon, and familiar landmarks. Such measurement data have been processed in systemmatic ways ranging from simply use of good judgement and previous experience to application of more complex mathematical data processing and geometric tools.

Although the estimation problem as a formal analytic procedure has become a major area of interest to engineering only in relatively recent times, it has by now become somewhat classical. Some of the more recent work which evolved into the estimation theory of today is discussed in the following chapter with emphasis on the developments which are most closely associated with the objectives and results of this thesis.

The early forms of analytical estimation techniques progressed from simple curve fittings, an example of which is the least squares estimator, to the more sophisticated Wiener filter, valid for stationary random processes and time invariant systems.

A more generally applicable technique, one which has found wide application in engineering and is particularly useful in light of the modern day digital computer capability has been developed by Kalman (1960), (1961), (1963) and is often referred to as the Kalman filter or estimator. This revolutionary approach is valid for non-stationary statistics and provides the transient as well as the steady state solution. The Kalman estimator, therefore, considers the dynamics of the system and the statistical nature of any disturbances as well as the measurement quantities. In addition, a priori information about any system parameters may be included in the problem formulation and the Kalman estimator can be constructed in a recursive manner.

All of these features in combination provide a technique for utilizing redundant information for the purpose of estimating the state of some system, and doing so in an optimum fashion. The criterion for optimality can be either maximum likelihood or minimum variance. However, under the restrictions normally imposed upon systems to be estimated with this technique, these two criteria result in the same set of estimation equations. These restrictions are, first, that the statistical nature of the random variable to be estimated can be described by a Gaussian density function and, second, that all the equations associated with the system description be linear. This linearity is required in both the system dynamics as well as the measurement process. The requirement of linear equations does not necessarily restrict the use of Kalman estimators to linear systems; however, it does require some form of linearization before the estimation equations can be applied and therefore an approximation is unavoidably introduced at the start.

The Gaussian restriction mentioned above has been relaxed somewhat with more recent techniques; however, the requirement for linear equations is still basic to Kalman estimation and is part of the motivation for this investigation.

Many practical systems do not readily lend themselves to an accurate linearized description and, therefore, the Kalman estimator is not adequate for these systems. The importance of a nonlinear estimation technique has motivated several individuals to investigate this problem and some specific results of this work are discussed in Chapter II. Most of these techniques depend on a series expansion of the nonlinear function about a reference state. The resulting series does not converge in any predictable manner, if at all, which is a serious limitation of this procedure. Also, there are many nonlinear relationships of interest, such as two or three level switches and quantizers, for which a series expansion cannot be used. These two major difficulties with nonlinear estimation using series expansions, have provided the motivation for this research.

Using a new point of view, two techniques for nonlinear minimum variance estimation are presented in this thesis. Chapter III deals with a Taylor Series expansion procedure which is slightly different from that taken by others and consequently provides a different result. A completely new technique for nonlinear estimation is introduced in Chapter IV, the required estimation equations for the resulting recursive estimator are developed, and some of their more interesting properties discussed. This technique is referred to as a quasi-linearization and is more precisely defined in Chapter IV. Under certain conditions these equations can be simplified for easier computation and in Chapter V an example is used to compare this simplified technique with the series expansion procedure as well as a linearized estimator using the Kalman formulation and the general quasi-linear estimator.

Chapter VI deals with some of the more mechanical problems associated with the implementation of the new approach. A comparison of the complete formulation of the quasi-linear estimator, the simplified estimation equations, and a linearized estimator is then made for a nonlinear second order oscillator in Chapter VII. Finally, Chapter VIII is devoted to a summary of the results and conclusions contained in the preceding chapters as well as some remarks on the general nature of quasi-linear techniques and related problems not considered in this thesis.

Throughout this investigation, a particular type of system has been used as a vehicle for demonstration of the estimation techniques. The system chosen is one which can be described by continuous dynamic equations and a series of discrete measurements at arbitrarily selected points. The reason for this choice is two fold. First, this type of system is most probably the type which will be implemented in present day as well as future systems with high speed digital computers. Although the digital nature of these computers requires that the system equations in the computer for both system dynamics and measurements be discrete in form, the actual equations for motion of the system are described by continuous differential equations. Second, the restriction of discussion to such a system is by no means a limitation of the techniques described, and in most cases the extension to other types of systems is straightforward. The restriction does, however, pinpoint the analysis allowing a more concentrated investigation of the estimation problem.

CHAPTER II

THE GENERAL ESTIMATION PROBLEM

2.1 Problem Definition

In order to facilitate an understanding of the following discussion concerning optimum estimation, both linear and nonlinear, a system of equations describing the general estimation problem is given here. For all further discussions, it will be assumed that a vector quantity known as the state of some dynamical system has a time behavior which can be described by the following first order differential equation

$$\dot{x} = f(x, t) + G(t)\, u \tag{2.1}$$

In Eq. (2.1) $f(x, t)$ is a general nonlinear $n \times 1$ vector function of the n dimensional state vector x, $G(t)$ is an $n \times n$ coefficient matrix and u is an $n \times 1$ vector of independent Gaussian white noise processes. A matrix which represents the variance of this noise process is defined by the following

$$\overline{u(\tau_1)\, u(\tau_2)^T} = Q(\tau_1)\, \delta(\tau_2 - \tau_1) \tag{2.2}$$

where T is used to signify a transpose and the bar represents an ensemble expectation.

Note that by a proper redefinition of state variables, any order system can be expressed in the form of a first order vector differential equation. For example, given the scalar system

$$\ddot{y} = g(y, t) \tag{2.3}$$

Define

$$x_1 = y$$

$$x_2 = \dot{y} = \dot{x}_1$$

then

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ g(x_1, t) \end{bmatrix} = f(x, t) \tag{2.4}$$

Equation (2. 4) represents a two dimensional state and is a first order equation as desired.

Necessary for any of the estimation techniques described here is a measurement process which uses information external to the system itself in order to improve the knowledge of the state vector at any time. This measurement process may take the form of a continuous signal or it may be a discrete set of quantities at selected points. In either case, the general relationship between the measurement quantity and the system state is described as follows

$$\dot{z} = h(x, t) + v \tag{2.5}$$

In the above equation z is the $m \times 1$ measurement quantity with $m \leq n$; $h(x, t)$ is, in general, an $m \times 1$ nonlinear vector function of the state and v is an $m \times 1$ vector of Gaussian white noise processes. In this case, the variance matrix of the noise is defined by the following equation

$$\overline{v(\tau_1)\, v(\tau_2)^T} = R(\tau_1)\, \delta(\tau_2 - \tau_1) \tag{2.6}$$

Important to all of the estimation schemes discussed here are the following two quantities. The first represents the state estimate which is taken to be the conditional mean of the random process. Thus,

$$\overline{x} = \int_{-\infty}^{\infty} x\, p(x \mid Z)\, dx \tag{2.7}$$

In Eq. (2.7), $p(x \mid Z)$ represents the probability density function of the state vector, x, conditioned upon all the measurement quantities in the set, Z. This definition of the estimate minimizes the variance of the estimation error conditioned on all the information in hand:

the a priori information and the measurements. The second quantity of interest is a representation of the mean squared error in the state estimate and is also the second central moment of the above density function.

$$P = \overline{(x - \bar{x})(x - \bar{x})^T} = \int_{-\infty}^{\infty} (x - \bar{x})(x - \bar{x})^T \, p(x|Z) \, dx \qquad (2.8)$$

The above equations will be specialized for the particular type of problem under consideration in the following discussions. In addition, new quantities will be introduced; however, this general definition of the basic problem will be retained.

2.2 Research Related to this Investigation

The problems of estimation and control of general nonlinear systems have been considered for many years with early work being accomplished by Zadeh and Ragazzini (1950), Laning (1951), Booton (1952) and others. Only the special case of stationary statistics and linear systems could actually be carried through to a useful solution. The linear system with nonstationary statistics was treated from a new viewpoint by Kalman (1960), Kalman and Bucy (1961) as well as Stratonovich (1959) and (1960). The work of Stratonovich pioneered the exploration of random process behavior and the evolution of the conditional density function which, coupled with Kalman's fresh approach to estimation triggered a more intensive investigation into the nonlinear problem.

There are also several alternative approaches to nonlinear estimation, suggested by Bryson and Frazier (1962), Detchmendy and Sridhar (1966), Friedland and Bernstein (1966), Ohap and Stubberud (1965) as well as others. These individuals were generally concerned with particular specialized problems and their work is somewhat peripheral to the contents of the following chapters. The work based on the examination of the conditional density function and the conditional mean as the optimum estimate is most closely allied to this work and is therefore discussed further in the following.

As pointed out by Stratonovich, it is sufficient, for a nonlinear minimum variance estimate, to have knowledge of the probability density function of the random process to be estimated conditioned upon the measurement process. The mean of this density function is easily shown to be the optimum estimate. Thus, the estimation problem can be considered as one of determining the time evolution of the above mentioned function, and Stratonovich (1960) attempted to do just that. Although not completely successful, it was the start of a new approach which was also followed by several others. Four years after this attempt, Kushner (1964) provided an essentially correct formulation of the nonlinear problem using this approach. The most important general result of Kushner's paper is the differential equation for the evolution of the conditional density function. Using this equation, an expression for the density function at any time can be used to evaluate any desired moments, the first of which is the minimum variance estimate.

Once this procedure had been outlined by Kushner, several individuals pursued this approach and approximately one year later Bucy (1965) presented a more rigorous derivation and an example. The straightforward and relatively simple procedure proposed by Bucy also resulted in a partial differential equation governing the temporal evolution of the conditional density. The result of applying the Ito (1961) stochastic calculus to the assumed Markov process is the desired result given below.

$$dH = \tilde{A}\, H\, dt + (h - \bar{h})\, R^{-1}\, (dz - h\, dt)\, H \tag{2.9}$$

In the above equation $H(x, t, Z) = p(x \mid Z)$ represents the desired conditional density function, and h and dz are related to the nonlinear measurement function as follows:

$$dz = h\, dt + R^{1/2}\, d\beta \tag{2.10}$$

$d\beta$ being a vector process of independent Brownian motions. The $\tilde{A}$ operator is defined by the following expression:

$$\tilde{A}\,(\cdot) = -\frac{\partial}{\partial x}\,[f(\cdot)] + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i\, \partial x_j}\,[\sigma_i\, \sigma_j\, (\cdot)] \tag{2.11}$$

corresponding to the state differential equation given below

$$dx = f(x, t)\, dt + \sigma(x)\, d\beta \tag{2.12}$$

Using Eq. (2.9), approximate equations of evolution of the mean of the state, $\bar{x}$ and the variance $P = \overline{(x - \bar{x})^2}$ can be determined. Most of the individuals dealing with this problem have then made the assumption that $(x - \bar{x})$ has a symmetric distribution and that a series expansion of the nonlinear functions about $\bar{x}$ can be accomplished. This expansion can be used to evaluate any number of moments; however, as demonstrated in the third and fifth chapters of this thesis, the computations required for moments higher than second are lengthly at best.

Following Bucy's paper, Bass et al (1965), extended the above procedure for approximating the evolution of moment terms to the general vector case in a straightforward and rigorous manner. Under the assumption that central moments of order higher than two are approximately zero, the following set of estimation equations were found:

$$\dot{\bar{x}} = f(\bar{x}) + \frac{1}{2} f_{xx}(\bar{x})\, P + P\, h_x^T(\bar{x})\, R^{-1} \left[z - h(\bar{x}) - \frac{1}{2} h_{xx}(\bar{x})\, P \right] \tag{2.13}$$

In the above equation, the subscripts signify a partial derivative with respect to the vector x. Thus:

$$\begin{aligned} h_x &= \frac{\partial}{\partial x}(h) \\ h_{xx} &= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(h)\right] \end{aligned} \tag{2.14}$$

For the second central moment, the following equation results:

$$\dot{P} = P f_x^T(\bar{x}) + f_x(\bar{x}) P - P [h_x^T(\bar{x}) R^{-1} h_x(\bar{x})] P$$

$$+ G(\bar{x}) Q G(\bar{x})^T + \frac{1}{2} [G(\bar{x}) Q G(\bar{x})^T]_{xx} P \qquad (2.15)$$

$$- \frac{1}{2} \{P h_{xx}^T(\bar{x}) R^{-1} [z - h(\bar{x}) - \frac{1}{2} h_{xx}(\bar{x}) P]\} P$$

The assumption of only two significant moments is not required of this general technique; however, it is again a practical one from a computational viewpoint. In some cases, it may be important to keep an odd moment, for example the third, and a possible procedure for this is discussed in Chapter III. These considerations are also discussed in a later paper by Schwartz and Bass (1966) along with some explanation of the use of a priori information.

One of the most extensive discussions of the use of conditional density functions and moment evolutions for nonlinear estimation can be found in Fisher (1966). The dynamic equations which describe the conditional probability density of the state vector are derived with the detail of each restriction and assumption discussed. Ordinary stochastic differential equations are then derived for central moments of this density function. The particular moments used by Fisher are a special type especially useful for density functions which are near Gaussian. A further description of these moments is found in Appendix A of this thesis. Ultimately, the requirement for a Taylor Series expansion of the nonlinear function is again necessary in order to obtain a solution to the moment equations, as in the case of the other works described above.

All of the work mentioned above represents investigations of continuous systems and continuous measurement processes. Ho and Lee (1964) have considered the discrete problem utilizing a Bayesian approach to solve for the conditional density function and from this the desired estimation updates at measurement points. As discussed in their paper, several difficulties were encountered in the application of this technique to the general nonlinear problem which resulted in very limited use. However, the approach was again explored by Jazwinski (1966) for a general nonlinear discrete measurement with additive Gaussian white noise. The procedure is the following. Given a measurement which can be described as follows

$$z = h(x, t) + v \tag{2.16}$$

where $h(x, t)$ is a general nonlinear measurement function and v is Gaussian noise with variance

$$\overline{v(\tau_1)\, v(\tau_2)^T} = R(\tau_1)\, \delta(\tau_2 - \tau_1) \tag{2.17}$$

the conditional probability density function of the state, x conditioned upon the measurement sequence, Z, is given by

$$p\,[x(t) \,|\, Z(t)] = H(x, t, Z) \tag{2.18}$$

By indicating the measurement sequence up to but not including the present measurement with $Z^-(t)$ and using $Z\ (t)$ to indicate the complete sequence, an expression for the conditional density after measurement incorporation is found to be

$$H(x, t, Z) = \frac{p(z|x)\, H(x, t, Z^-)}{\int_{-\infty}^{\infty} p(z|x)\, H(x, t, Z^-)\, dx} \tag{2.19}$$

The above equality also makes use of Bayes' rule.

Because of the assumption of Gaussian v, the conditional density $p(z|x)$ can be written as follows:

$$p(z|x) = \frac{1}{(2\pi)^{m/2}\, |R|^{1/2}}\, e^{-\frac{1}{2}[z - h(x,t)]^T R^{-1} [z - h(x,t)]} \tag{2.20}$$

The equations for the update of any of the central moments of $H(x, t, Z)$ can then be found as demonstrated by the following expressions for the first two such central moments.

$$\bar{x}^+ = \int_{-\infty}^{\infty} x\, H(x, t, Z)\, dx$$

$$P^+ = \overline{(x - \bar{x}^+)(x - \bar{x}^+)^T} = \int_{-\infty}^{\infty} xx^T\, H(x, t, Z)\, dx - \bar{x}^+ \bar{x}^{+T} \tag{2.21}$$

In order to evaluate the expectations indicated in Eq. (2.21), a Taylor Series expansion about $\bar{x}$ is again employed. The resulting expressions for the updates of the first two central moments are given below

$$\bar{x}^{+} = \bar{x}^{-} + \frac{1}{D} P^{-} H^{T} R^{-1} [z - h(\bar{x}, t)] \tag{2.22}$$

$$P^{+} = \frac{P^{-}}{D} - \frac{1}{D^{2}} P^{-} H^{T} R^{-1} [z - h(\bar{x}, t)] [z - h(\bar{x}, t)]^{T} R^{-1} H P^{-} \tag{2.23}$$

where

$$\begin{aligned} D = 1 + \frac{1}{2} \{ & [I_{d^{T} P^{-} d} \, h(\bar{x}, t)]^{T} R^{-1} [z - h(\bar{x}, t)] \\ & -tr(H^{T} R^{-1} H P^{-}) + [z - h(\bar{x}, t)]^{T} R^{-1} H P^{-} H^{T} R^{-1} [z - h(\bar{x}, t)] \} \end{aligned} \tag{2.24}$$

and

$$H = \left[\frac{\partial h_{i}(\bar{x}, t)}{\partial x_{j}} \right]$$

$$d^{T} = \left[\frac{\partial}{\partial x_{1}}, \frac{\partial}{\partial x_{2}}, \ldots, \frac{\partial}{\partial x_{n}} \right]. \tag{2.25}$$

$$I_{d^T P^- d} = \begin{bmatrix} d^T P^- d & & 0 \\ & \ddots & \\ 0 & & d^T P^- d \end{bmatrix}$$

Some of the characteristics of these results are worthy of special note, particularly as a comparison to other recent results in nonlinear estimation. First, the two update equations, Eqs. (2. 22) and (2. 23), are coupled through their direct dependence on the actual measurement quantity. Thus, P does not represent an ensemble statistic as in the linear estimation problem, but reflects the actual random errors entering the system through the measurement process. As also pointed out by Jazwinski, these update equations will not reduce to the proper linear equations if the nonlinearity is specialized to a linear function. This is a result of the nature of the approximation made. By using a different approximation technique, a result is obtained in Chapter III of this investigation which is different from that above in both respects.

Recently, Kushner (1967) has reported on a technique for achieving approximate nonlinear minimum variance filtering which is most closely related to the work contained in the subsequent chapters here and tends to support the general results as well as those for a particular example. Again, the initial effort of Kushner is aimed at developing differential expressions for the conditional density function of the random process and from this, equations describing the evolution of the first two moments. Each of these latter expressions contains expectation terms which must be evaluated in some approximate manner. As pointed out by Kushner and as also noted in subsequent chapters of this investigation, the procedure of expanding these expectations in a Taylor Series is often not an adequate approximation.

The general procedure therefore is to assume a particular form for the conditional density function and proceed with the evaluation of the required expectations. An example of the van der Pol oscillator is used by Kushner and has provided an interesting comparison for the results of this paper as shown in Chapter VII.

It is worth noting that Kushner is one of the few individuals who has attempted even in a qualitative manner, to show specifically the salient differences and advantages of using nonlinear estimation techniques over simply a Kalman filter linearized about a reference state. One of the original objectives of the investigations described in the following was to examine the conditions which warrant the extra complexity involved in nonlinear estimations. It became evident to the author that linearized estimation was quite adequate in many practical nonlinear cases and that some of the nonlinear schemes which have been proposed offer no significant gains over use of simply a linearized estimator.

CHAPTER III

SERIES ESTIMATION

3.1 Introduction

The focal point of the approximation method discussed in this chapter is the nonlinear functional element which can be isolated in either the system dynamics, assumed to be continuous, or in the measurement geometry where measurements are assumed to be discrete. This approach is better understood by examination of Fig. 3-1. With reference to the figure, the nonlinear functions f(x) and H(x) which appear in the dynamics and in the measurement process, respectively, have corresponding counterparts in the system model and measurement incorporation process. These quantities N_1 and N_2 in some sense approximate the behavior of the corresponding nonlinear function. One approximation method is the subject of this chapter and another is discussed in Chapter IV.

The approach which will be taken in this chapter is based upon the assumption that the probability density function associated with the state variable is adequately described by a finite series of moments. For a great many useful cases, there are several factors which tend to support the assumption that the density function under consideration is, in fact, near Normal. For this reason, it is useful to examine a particular set of moments which relates the expansion of any density function to the central moment expansion of a Normal density function. These quasi-moments are potentially a convenient means of expression for the following development. However, the first three moments in this series are identical to the corresponding central moments of the Normal distribution. For this reason, the discussion which follows

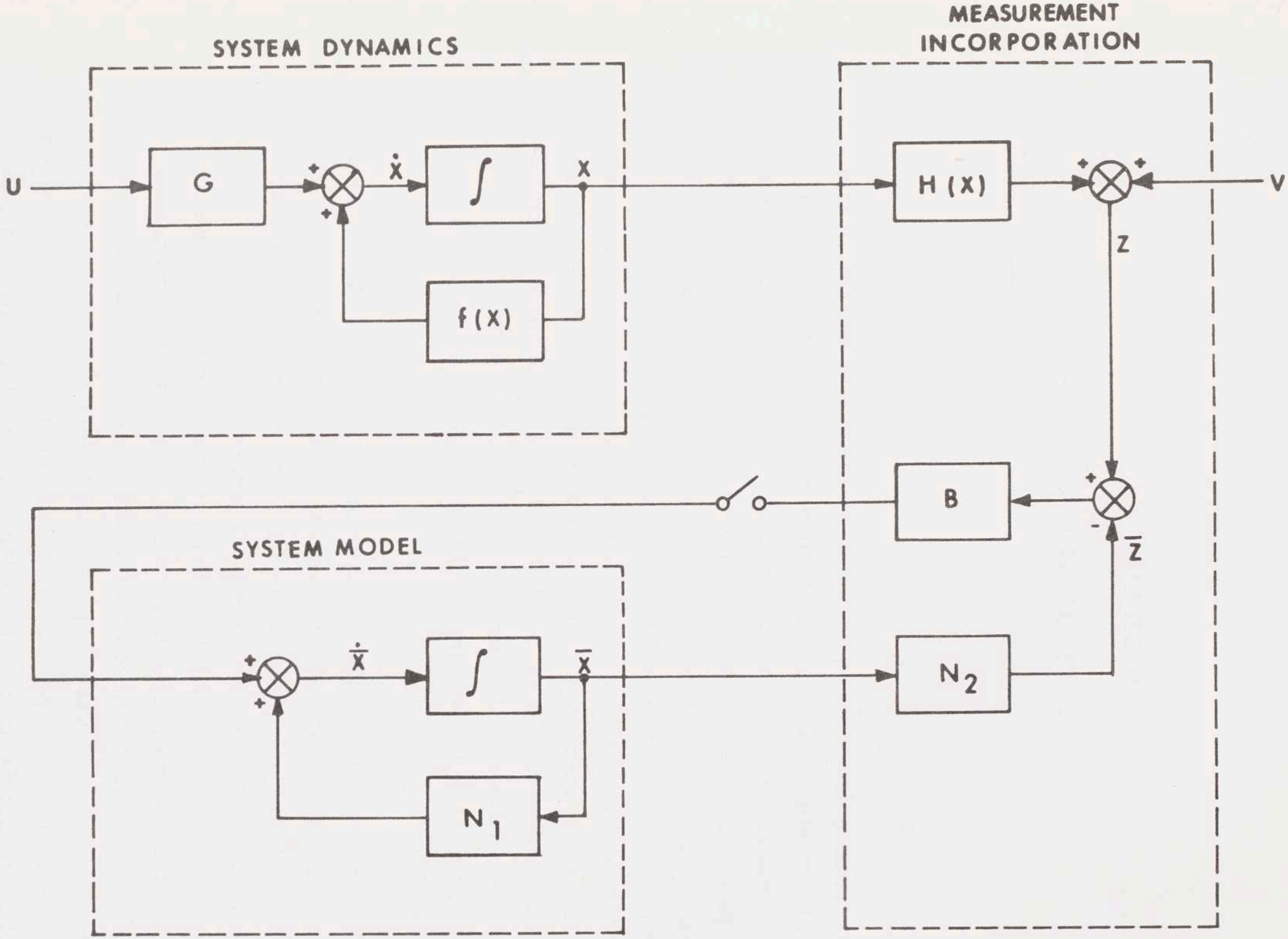


Figure 3-1 Optimum Estimator Block Diagram

will consider only the moments of the Normal density function. For use in higher order expansions, the conversion from one to the other is a simple process discussed in Appendix A.

The density function described by this series is a real time variable which is conditioned upon all of the measurements taken up to the present time. The first moment of the density function is, therefore, the conditional mean, and represents the optimum estimate of the state.

The series expansion which will be employed here is of the form of a Taylor Series expanded about the ensemble expectation of the state such that any function of the state vector, x, can be described by the following equation

$$q_i(x) = q_i(\bar{x}) + \sum_{j=1}^{n} \frac{\partial q_i(\bar{x})}{\partial x_j} (x_j - \bar{x}_j)$$
$$+ \frac{1}{2!} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 q_i(\bar{x})}{\partial x_j \, \partial x_k} (x_j - \bar{x}_j)(x_k - \bar{x}_k) + \ldots \tag{3.1}$$

where, in Eq. (3. 1) and in all of the following equations, the bar over any quantity is used to indicate the ensemble expectation of that quantity. Taking the expectation of Eq. (3. 1) results in a series expression for the expected value of any vector function of the state variable.

$$\overline{q_i(x)} = q_i(\bar{x}) + \sum_{j=1}^{n} \frac{\partial q_i(\bar{x})}{\partial x_j} \overline{(x_j - \bar{x}_j)}$$
$$+ \frac{1}{2!} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 q_i(\bar{x})}{\partial x_j \, \partial x_k} \overline{(x_j - \bar{x}_j)(x_k - \bar{x}_k)} + \ldots \tag{3.2}$$

The second term in the expansion, which represents the ensemble expectation of the error in the estimate, or the average deviation of the state from its mean value is zero. Thus, Eqs. (3. 2) reduces to the following equation, which will be used to expand any vector function of x as it appears in the following derivation of a set of recursive estimation equations.

$$\overline{q_i(x)} = q_i(\bar{x}) + \frac{1}{2!}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 q_i(\bar{x})}{\partial x_j \, \partial x_k} \overline{(x_j - \bar{x}_j)(x_k - \bar{x}_k)} + \ldots \tag{3. 3}$$

Note that if the distribution function is, in fact, symmetric as it would be if nearly Gaussian, the third order moment term

$$\frac{1}{3!}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n} \frac{\partial^2 q_i(\bar{x})}{\partial x_j \, \partial x_k \, \partial x_\ell} \overline{(x_j - \bar{x}_j)(x_k - \bar{x}_k)(x_\ell - \bar{x}_\ell)}$$

and all higher order odd moment terms will be approximately zero; however, even order moments will be nonzero, in general. If the assumption of a near Gaussian density function is valid, these higher order even moments can be expressed in terms of the second order moment using the relationships shown in Appendix D.

For the general density function all the higher order moments will be nonzero. Therefore, in order to select a reasonable number of terms for inclusion in the estimation scheme, a careful examination of the particular nonlinear system and the relative behavior of the moment terms must be made.

In this chapter, a series expression which contains all moments up to and including the third, will be discussed in detail and the recursive estimation equations appropriate to this order will be derived. The extension to expansions containing higher order moments is a straightforward procedure and will also be indicated.

3.2 Nonlinear System Dynamics

Consider a system described by the following vector differential equation

$$\dot{x} = f(x) + G u \tag{3.4}$$

where f(x) is a general $n \times 1$ vector function of x, u is an $n \times 1$ vector of zero mean Gaussian white noise with a variance matrix defined by

$$\overline{u(\tau_1)\, u(\tau_2)^T} = Q\, \delta\, (\tau_2 - \tau_1) \tag{3.5}$$

and G is a constant coefficient matrix independent of x.

Taking the ensemble expectation of Eq. (3.4) results in

$$\overline{\dot{x}} = \overline{f(x)} + G\, \overline{u} = \overline{f(x)} \tag{3.6}$$

The differential equation for the state estimate, therefore, depends solely on the expected value of f(x). With reference to Fig. 3-1, $\overline{f(x)}$ corresponds to N_1, the approximation function used to model the system dynamics.

The expansion about the expected value of x, which is used to approximate this vector nonlinear function is given by the following expression

$$\overline{f(x)} \approx N_1\, (m_1,\ m_2,\ \ldots\ )$$

where the m_i's represent the moment parameters associated with the probability density function of x. Following the format of Eq. (3. 3)

$$\overline{f_i(x)} = f_i(\bar{x}) + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 f_i(\bar{x})}{\partial x_j \partial x_k} \overline{(x - \bar{x})_j (x - \bar{x})_k}$$

$$+ \frac{1}{6} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \frac{\partial^3 f_i(\bar{x})}{\partial x_j \partial x_k \partial x_\ell} \overline{(x - \bar{x})_j (x - \bar{x})_k (x - \bar{x})_\ell} \tag{3. 7}$$

The differential equation for the first moment of the density function and also for the state estimate is thus given by

$$\dot{\bar{x}} = \overline{f(x)} \tag{3. 8}$$

where $\overline{f(x)}$ is approximated with the series defined above. For a complete estimation scheme, the differential equation for all the moments included in Eq. (3. 7) must also be determined. The second and third central moments are defined by

$$P_{ij} = \overline{(x_i - \bar{x}_i)(x_j - \bar{x}_j)} \tag{3. 9}$$

and

$$T_{ijk} = \overline{(x_i - \bar{x}_i)(x_j - \bar{x}_j)(x_k - \bar{x}_k)} \tag{3. 10}$$

The differential equation for P is found as follows.

$$\dot{P}_{ij} = \overline{(\dot{x}_i - \dot{\bar{x}}_i)(x_j - \bar{x}_j)} + \overline{(x_i - \bar{x}_i)(\dot{x}_j - \dot{\bar{x}}_j)} \tag{3.11}$$

From Eqs. (3. 4) and (3. 8)

$$\dot{x}_i = f_i(x) + G_{i\ell} u_\ell$$

$$\dot{\bar{x}}_i = \overline{f_i(x)}$$

In the above expression for x_i and in the following discussion, terms which have subscripts reoccurring such as $G_{i\ell} u_\ell$ imply a summation over the common subscript as in the general tensor notation. Substituting the above equation for $\dot{x}_i$ and $\dot{\bar{x}}_i$ into Eq. (3. 11) results in

$$\begin{aligned} \dot{P}_{ij} &= \overline{[f_i(x) + G_{i\ell} u_\ell - \overline{f_i(x)}]\,[x_j - \bar{x}_j]} \\ &\quad + \overline{[x_i - \bar{x}_i]\,[f_j(x) + u_\ell G_{j\ell} - \overline{f_j(x)}]} \end{aligned} \tag{3.12}$$

$$\begin{aligned} \dot{P}_{ij} &= \overline{f_i(x)\, x_j} - \overline{f_i(x)}\, \bar{x}_j + G_{i\ell}\, \overline{u_\ell\, x_j} - G_{i\ell}\, \bar{u}_\ell\, \bar{x}_j - \overline{f_i(x)}\, \bar{x}_j + \overline{f_i(x)}\, \bar{x}_j \\ &\quad + \overline{x_i\, f_j(x)} - \bar{x}_i\, \overline{f_j(x)} + \overline{x_i\, u_\ell}\, G_{j\ell} - \bar{x}_i\, \bar{u}_\ell\, G_{j\ell} - \bar{x}_i\, \overline{f_i(x)} + \bar{x}_i\, \overline{f_j(x)} \end{aligned} \tag{3.13}$$

Noting that, by definition $\bar{u} = 0$, Eq. (3. 13) becomes

$$\dot{P}_{ij} = \overline{f_i(x)\, x_j} - \overline{f_i(x)}\; \overline{x_j} + G_{i\ell}\, \overline{u_\ell\, x_j} + \overline{x_i\, f_j(x)} - \overline{x_i}\; \overline{f_j(x)} + \overline{x_i\, u_\ell}\, G_{j\ell} \tag{3.14}$$

The terms which involve the correlation between u and x can be further simplified as follows

$$G_{i\ell}\, \overline{u_\ell\, x_j} = G_{i\ell}\, \overline{u_\ell \left[x(0) + \int_0^t \{ f\,[x(s)] + u_m\, G_{mj} \}\, ds \right]} \tag{3.15}$$

The first term on the right hand side of Eq. (3.15) is zero because x(0) is uncorrelated with any noise for $t \geq 0$. Thus,

$$G_{i\ell}\, \overline{u_\ell\, x_j} = \int_0^t \{ G_{i\ell}\, \overline{u_\ell(t)\, f_j\,[x(s)]} \}\, ds + \int_0^t \{ G_{i\ell}\, \overline{u_\ell(t)\, u_m(s)}\, G_{mj} \}\, ds$$

In the first integral above, $u_\ell(t)$ and $f_j\,[x(s)]$ are correlated only for $s = t$ and at that point the correlation is finite; hence, the contribution of $\overline{u_\ell(t)\, f_j\,[x(s)]}$ to the integral for $0 \leq s \leq t$ is zero. Using Eq. (3.5), the second integral reduces to

$$\int_0^t [G_{i\ell}\, \overline{u_\ell(t)\, u_m(s)}\, G_{mj}] = \frac{1}{2} G_{i\ell}\, Q_{\ell m}\, G_{mj} \tag{3.16}$$

A similar result is found for the $\overline{x_i\, u_\ell}\, G_{j\ell}$ term. Thus, Eq. (3.14) becomes

$$\dot{P}_{ij} = \overline{f_i(x)\, x_j} - \overline{f_i(x)}\; \overline{x_j} + \overline{x_i\, f_i(x)} - \overline{x_i}\; \overline{f_j(x)} + G_{i\ell}\, Q_{\ell m}\, G_{mj} \tag{3.17}$$

or using vector notation

$$\dot{P} = \overline{f(x)\, x^T} - \overline{f(x)}\, \overline{x}^T + \overline{x\, f(x)^T} - \overline{x}\, \overline{f(x)}^T + G\, Q\, G^T \tag{3.18}$$

In Eq. (3.17), the terms which involve the expectation of a product of vector random variables may be evaluated using a matrix series expansion of the following form.

$$\begin{aligned} q_{ij}(x) = q_{ij}(\overline{x}) &+ \frac{1}{2} \sum_{\ell} \sum_{m} \frac{\partial^2\, q_{ij}(\overline{x})}{\partial\, x_\ell\, \partial\, x_m}\, P_{\ell m} \\ &+ \frac{1}{6} \sum_{\ell} \sum_{m} \sum_{n} \frac{\partial^3\, q_{ij}(\overline{x})}{\partial\, x_\ell\, \partial\, x_m\, \partial\, x_n}\, T_{\ell mn} \end{aligned} \tag{3.19}$$

The reason for carrying through the derivation of $\dot{P}$ using component notation is for comparison with the differential equations of the higher order moments, discussed below. The vector expression (Eq. (3.18)) can easily be found directly.

It is interesting to note that if, indeed, $f(x)$ is a linear function of x, i.e., $f(x) = Fx$, then

$$\overline{f(x)\, x^T} = F\, \overline{x\, x^T}$$

$$\overline{x\, f(x)^T} = \overline{x\, x^T}\, F^T$$

and Eq. (3.18) reduces to

$$\dot{P} = F\, P + P\, F^T + G\, Q\, G^T \tag{3.20}$$

Equation (3. 20) is the familiar linear estimation equation for extrapolation of P with no measurements.

Before proceeding with the derivation of higher order moment differential equations, it is convenient at this point to make a note of the symmetric nature of each of the moment expressions. In the case of the second order moment

$$P_{ij} = P_{ji}$$

If this relationship had been exploited in the derivation of P, only a small reduction in algebra could have been obtained. However, a reduction of this type becomes increasingly significant as the order of the moments increases.

In order to utilize the symmetry property, the following notation will be employed.

$$N \left\{ \qquad \qquad \right\}_{sem_{i,\,j,\,k}}$$

The above symbolism implies that there are N terms of the form in brackets and that each can be found from the one shown explicitly by a permutation of the indices indicated. For example

$$3 \left\{ x_i \; x_j \; x_k \right\}_{sem_{i,\,j,\,k}} = x_i \, x_j \, x_k + x_j \, x_k \, x_i + x_k \, x_i \, x_j$$

Thus, Eq. (3.17) could also be written in the form

$$\dot{P}_{ij} = 2 \left\{ \overline{f_i(x)\, x_j} - \overline{f_i(x)}\, \bar{x}_j + \frac{1}{2} G_{i\ell}\, Q_{\ell m}\, G_{mj} \right\}_{sem_{i,j}} \tag{3.21}$$

The expression for the derivative of the third moment, T_{ijk}, and by an obvious extension, the derivative of all higher order moments is given in the following.

$$\begin{aligned} \dot{T}_{ijk} = {} & \overline{(\dot{x}_i - \dot{\bar{x}}_i)(x_j - \bar{x}_j)(x_k - \bar{x}_k)} + \overline{(x_i - \bar{x}_i)(\dot{x}_j - \dot{\bar{x}}_j)(x_k - \bar{x}_k)} \\ & + \overline{(x_i - \bar{x}_i)(x_j - \bar{x}_j)(\dot{x}_k - \dot{\bar{x}}_k)} \end{aligned} \tag{3.22}$$

or

$$T_{ijk} = 3 \left\{ \overline{(\dot{x}_i - \dot{\bar{x}}_i)(x_j - \bar{x}_j)(x_k - \bar{x}_k)} \right\}_{sem_{i,j,k}} \tag{3.23}$$

Again using the following differential relationships

$$\begin{aligned} \dot{x}_i &= f_i(x) + G_{i\ell}\, u_\ell \\ \dot{\bar{x}}_i &= \overline{f_i(x)} \end{aligned} \tag{3.24}$$

Equation (3.23) becomes

$$T_{ijk} = 3 \left\{ \overline{f_i(x)\,(x_j - \bar{x}_j)\,(x_k - \bar{x}_k)} + G_{i\ell}\, \overline{u_\ell\,(x_j - \bar{x}_j)\,(x_k - \bar{x}_k)} \right. \tag{3.25}$$

$$\left. - \overline{f_i(x)}\; \overline{(x_j - \bar{x}_j)\,(x_k - \bar{x}_k)} \right\}_{\text{sem}_{i,j,k}}$$

Following the procedure used in the derivation of $\dot{P}$, the terms involving u can be simplified in the following manner. Using Eq. (3.16)

$$- G_{i\ell}\, \overline{u_\ell\, x_j}\; \bar{x}_k = \frac{1}{2} G_{i\ell}\, Q_{\ell m}\, G_{mj}\, \bar{x}_k$$

$$- G_{i\ell}\, \overline{u_\ell\, x_k}\; \bar{x}_j = \frac{1}{2} G_{i\ell}\, Q_{\ell m}\, G_{mj}\, \bar{x}_j \tag{3.26}$$

Terms of the form $G_{i\ell}\, \overline{u_\ell\, x_j\, x_k}$ are treated as follows.

$$G_{i\ell}\, \overline{u_\ell\, x_j\, x_k} = G_{i\ell}\, \overline{u_\ell(t) \left[x_j(0) + \int_0^t \{ f_j\,[x(s)] + u_m(s)\, G_{mj} \}\, ds \right] \times \cdots}$$

$$\overline{\cdots \left[x_k(0) + \int_0^t \{ f_k\,[x(r)] + u_n(r)\, G_{nk} \}\, dr \right]} \tag{3.27}$$

$$G_{i\ell}\overline{u_\ell x_j x_k} = G_{i\ell}\overline{u_\ell(t)\, x_j(0) x_k(0)} + G_{i\ell}\overline{u(t)\, x_j(0) \int_0^t \{f_k\,[x(r) + u_n(r) G_{nk}\}\, dr}$$

$$+ G_{i\ell}\overline{u_\ell(t) \int_0^t \{f_j\,[x(s)] + u_m(s)\, G_{mj}\}\, ds\, x_k(0)} \qquad (3.28)$$

$$+ G_{i\ell}\overline{u_\ell(t) \int_0^t\!\!\int \{f_j\,[x(s)] + u_m(s) G_{mj}\}\,\{f_k\,[x(r)] + u_n(r) G_{nk}\}\, dr\, ds}$$

The first term is zero since $x_j(0)\ x_k(0)$ is uncorrelated with any component of u(t) for $t \geq 0$. As discussed above, the correlation between $u_\ell(t)$ and $f_j\ [x(s)]$ does not contribute to the integral from 0 to t. Therefore, the second and third integrals in Eq. (3.28) become

$$\overline{G_{i\ell}\, u_\ell(t)\, x_j(0) \int_0^t \{f_k\,[x(r) + u_n(r)\, G_{nk}\}\, dr} = \frac{1}{2}\,\overline{x_j(0)}\, G_{i\ell}\, Q_{\ell n}\, G_{nk} \qquad (3.29)$$

and

$$\overline{G_{i\ell}\, u_\ell(t) \int_0^t \{f_j\,[x(s)] + u_m(s)\, G_{mj}\}\, ds\, x_k(0)} = \frac{1}{2}\, G_{i\ell}\, Q_{\ell m}\, G_{mj}\, \overline{x_k(0)} \qquad (3.30)$$

The last term in Eq. (3.28) may be expanded in the following manner.

$$G_{i\ell}\,\overline{u_\ell(t)\int_0^t\!\!\int \{f_j[x(s)] + u_m(s)\,G_{mj}\}\,\{f_k[x(r)] + u_n(r)\,G_{nk}\}\,drds} =$$

$$G_{i\ell}\,\overline{u_\ell(t)\int_0^t\!\!\int f_j[x(s)]\,f_k[x(r)]\,drds} + G_{i\ell}\,\overline{u_\ell(t)\int_0^t\!\!\int f_j[x(s)]\,u_n(r)\,G_{nk}\,drds}$$

$$+\, G_{i\ell}\,\overline{u_\ell(t)\int_0^t\!\!\int u_m(s)\,G_{mj}\,f_k[x(r)]\,drds} + G_{i\ell}\,\overline{u_\ell(t)\int_0^t\!\!\int u_m(s)\,u_n(r)\,G_{mj}\,G_{nk}\,drds}$$

(3.31)

Again $u(t)$ is correlated with $f[x(s)]$ only at the limit of the integral. Therefore, the first term on the right hand side of Eq. (3.31) is zero. The last term is also zero because $\overline{u(t)\,u(s)\,u(r)} = 0$ for all r, s, and t. This latter fact comes from the assumption that u is a Gaussian white noise process. The remaining terms in Eq. (3.31) become

$$\frac{1}{2}\,G_{i\ell}\,Q_{\ell n}\,G_{nk}\,[\bar{x}_j - \overline{x_j(0)}] + \frac{1}{2}\,G_{i\ell}\,Q_{\ell m}\,G_{mj}\,[\bar{x}_k - \overline{x_k(0)}]$$

Combining the above result with Eqs. (3.29) and (3.30) results in

$$G_{i\ell}\,\overline{u_\ell\,x_j\,x_k} = \frac{1}{2}\,G_{i\ell}\,Q_{\ell n}\,G_{nk}\,\bar{x}_j + \frac{1}{2}\,G_{i\ell}\,Q_{\ell m}\,G_{mj}\,\bar{x}_k \qquad (3.32)$$

The above terms cancel with those in Eq. (3.26) and, therefore, the differential equation for $\dot{T}_{ijk}$ is independent of Q. Thus, from Eq. (3.25)

$$\dot{T}_{ijk} = 3\ \{\overline{f_i(x)\,x_j\,x_k} - \overline{f_i(x)\,x_k}\,\bar{x}_i - \overline{f_i(x)\,x_j}\,\bar{x}_k + \overline{f_i(x)}\,\bar{x}_j\,\bar{x}_k$$

$$-\overline{f_i(x)}\,\overline{x_j\,x_k} - \overline{f_i(x)}\,\bar{x}_j\,\bar{x}_k + \overline{f_i(x)}\,\bar{x}_j\,\bar{x}_k + \overline{f_i(x)}\,\bar{x}_j\,\bar{x}_k\}_{\mathrm{sem}_{i,\,j,\,k}}$$

$$\dot{T}_{ijk} = 3\ \{\overline{f_i(x)\,x_j\,x_k} - \overline{f_i(x)\,x_k}\,\bar{x}_i - \overline{f_i(x)\,x_j}\,\bar{x}_k$$

$$+2\,\overline{f_i(x)}\,\bar{x}_j\,\bar{x}_k - \overline{f_i(x)}\,\overline{x_j\,x_k}\}_{\mathrm{sem}_{i,\,j,\,k}} \tag{3.33}$$

In Eq. (3.33), the terms which involve the expectation of the product of three vector terms must be evaluated using the series expression of the following form.

$$\overline{q_{ijk}(x)} = q_{ijk}(\bar{x}) + \frac{1}{2}\sum_{\ell}\sum_{m}\frac{\partial^2 q_{ijk}(\bar{x})}{\partial x_\ell\ \partial x_m}\ P_{\ell m}$$

$$+\frac{1}{6}\sum_{\ell}\sum_{m}\sum_{n}\frac{\partial^3 q_{ijk}(\bar{x})}{\partial x_\ell\,\partial x_m\,\partial x_n}\ T_{\ell mn} \tag{3.34}$$

The terms which involve expectations of one or two random variables can be evaluated in the same manner as for the $\dot{P}_{ij}$ expression (Eq. (3.14)).

The practicality of an estimation technique which involves these higher order moments is greatly enhanced by noting the symmetry discussed above. This symmetry greatly reduces the computer storage requirements as well as the computational complexity.

The procedure outlined above can be extended to moments of any order in a straightforward but increasingly complex manner. The important factors to consider when contemplating such an extension are the increased computation time, additional storage locations and the accuracy to be gained by the addition of these higher order terms. The number of unique terms associated with each moment expression is related to the dimension of the state vector and the moment as follows. For the zeroth order moment, the number of unique terms is, of course, equal to the state vector dimension (n) for a vector function, and for terms of the form $f(x)\, x^T$, this number becomes $n \times n$. For the second order moment $P_{ij} = \overline{(x_i - x_i)(x_j - x_j)}$, the total number of terms are n^2, but due to the symmetry property only half of all the terms not on the diagonal of the matrix are unique. Thus, there are

$$\frac{n^2 + n}{2} = \frac{n\,(n + 1)}{2} \tag{3.35}$$

unique terms.

The general expression for the number of unique terms in a symmetric tensor of order, r and dimension n is the binomial coefficient

$$\binom{n + r - 1}{r}$$

which can be seen by examining a table of binary coefficients for a few example cases. The table below illustrates the increase in complexity associated with the higher order moments for some typical vector dimensions.

		MOMENT			
		I	II	III	IV
Vector Dimension	1	1	1	1	1
	2	2	3	4	5
	3	3	6	10	15
	4	4	10	20	35
	5	5	15	35	70
	6	6	21	56	126

The general computational problem associated with the operations which must be performed with these variables is, usually, more complex than reflected simply by the number in the above table. From the point of view of one interested in implementing this technique in a computer, the number of multiplications may be a more interesting parameter to examine. This parameter is, of course, a function of the particular nonlinearity under consideration.

The increased complexity of higher order moments for multidimensional series expansions can only be assessed with a particular problem in mind. However, for practical reasons, the series must usually be truncated after a relatively small number of terms, which means a careful examination of the relative importance of the higher order moments must be made. It is not clear that the series expansion converges in any predictable manner for the general nonlinear estimation problem.

3.3 Measurement Updates

At periodic points, a measurement of data external to, but dependent upon, the system will be used to update the estimation process. The measurement quantity, and the best estimate of this quantity, are used in the following general expression

$$\overline{x}^{+} = A + B\,(z - \overline{z}^{-}) \tag{3.36}$$

where the superscript plus and the bar over a variable indicates that the associated expectation is conditioned upon all measurements up to and including the present one, the superscript minus with the bar signifies an expectation which does not include consideration of the present measurement. The vector A and the matrix B are assumed to be statistical quantities and are determined in the following manner.

Taking the expectation of Eq. (3.36), conditioned upon the set of measurements up to, but not including the present one, results in

$$\overline{(\overline{x}^{+})}^{-} = \overline{A}^{-} + \overline{[B\,(z - \overline{z}^{-}]}^{-} \tag{3.37}$$

The measurement quantities can be expressed in the following way

$$\begin{aligned} z &= H(x) + v \\ \overline{z}^{-} &= H^{-}(x) \end{aligned} \tag{3.38}$$

where H(x) is a general nonlinear vector function of the state and v is a Gaussian white noise process independent of x with variance defined by

$$\overline{v(\tau_1)\, v(\tau_2)^T} = R\, \delta\, (\tau_2 - \tau_1) \tag{3.39}$$

Using Eqs. (3. 37) and (3. 38)

$$\overline{(\overline{x}^+)}^- = \overline{A}^- + B\, \overline{[H(x) - \overline{H^-(x)} + v]}^-$$

$$\overline{(\overline{x}^+)} = \overline{A}^- \tag{3.40}$$

Since A is, by definition, a statistical quantity

$$\overline{(\overline{x}^+)}^- = A$$

The double expectation above will be taken to mean the expectation of $\overline{x}^+$ before the last measurement and so

$$\overline{(\overline{x}^+)}^- = \overline{x}^-$$

Thus:

$$A = \overline{x}^- \tag{3.41}$$

To determine B, the following technique may be used. The expectation of the squared error given all the measurements may be formed as follows:

$$\overline{e^+ e^{+T}} = \overline{(x - \bar{x}^+)(x - \bar{x}^+)^T} \qquad (3.42)$$

$$\overline{e^+ e^{+T}} = \overline{(x - x^-)(x - x^-)^T} - \overline{[x - x^-][H(x) - \overline{H^-(x)} + v]^T} B^T$$

$$- B\overline{[H(x) - \overline{H^-(x)} + v][x - \bar{x}^-]^T} + B\overline{[H(x) - \overline{H^-(x)} - v][H(x) - \overline{H^-(x)} + v]^T} B^T$$

$$(3.43)$$

$$\overline{e^+ e^{+T}} = \overline{e^- e^{-T}} - \overline{e^-[H(x) - \overline{H^-(x)} + v]^T} B^T - \overline{e^- v^T} B^T - B\overline{[H(x) - \overline{H^-(x)} + v] e^{-T}} - B\overline{v e^{-T}}$$

$$+ B\overline{v [H(x) - \overline{H^-(x)}]^T} B^T + B\overline{[H(x) - \overline{H^-(x)}] v^T} B^T$$

$$+ B\overline{[H(x) - \overline{H^-(x)}][H(x) - \overline{H^-(x)}]^T} B^T + B\overline{v v^T} B^T$$

$$(3.44)$$

As defined above, the noise vector, v is a white noise process, and therefore uncorrelated with e^-, x^- and any other expression which does not consider the present measurement. (i. e., a term with a superscript -). In addition, the expectation of the error, e^- is zero. Therefore, Eq. (3. 44) reduces to

$$\overline{e^+ e^{+T}} = \overline{e^- e^{-T}} - \overline{e^- H(x)^T} B^T - B \overline{H(x) e^{-T}}$$

$$+ B [\overline{H(x) H(x)^T} - \overline{H^-(x)}\ \overline{H^-(x)^T}] B^T + B R B^T \tag{3.45}$$

Now define B to be the sum of the gain which minimizes the trace of $\overline{e^+ e^{+T}}$, B_M and a small but otherwise arbitrary variation from that value, δB. Using this definition

$$B = B_M + \delta B \tag{3.46}$$

Equation (3.45) becomes

$$\overline{e^+ e^{+T}} = \overline{e^- e^{-T}} - \overline{e^- H(x)^T} (B_M + \delta B)^T - (B_M + \delta B) \overline{H(x) e^{-T}}$$

$$+ [B_M + \delta B] [\overline{H(x) H(x)^T} - \overline{H^-(x)}\ \overline{H^-(x)^T} + R] [B_M + \delta B]^T \tag{3.47}$$

Collecting all the terms of the first variation (i.e., terms with δB), taking the trace and setting the result equal to zero provides the following expression.

$$2 \operatorname{tr}\left\{\left[B_M [\overline{H(x) H(x)^T} - \overline{H^-(x)}\ \overline{H^-(x)^T} + R] - \overline{e^- H(x)^T}\right] \delta B\right\} = 0 \tag{3.48}$$

Since δB was defined to be an arbitrary variation of B, Eq. (3. 48) can only be true in general, if the following equation is also true

$$B_M \, [\overline{H(x) \, H(x)^T} - \overline{H^-(x)} \; \overline{H^-(x)^T} + R] = \overline{e^- \, H(x)^T}$$

or

$$B_M = \overline{e^- H(x)^T} \; [\overline{H(x) \, H(x)^T} - \overline{H^-(x)} \; \overline{H^-(x)^T} + R]^{-1} \qquad (3.49)$$

An examination of the terms of second variation in Eq. (3. 47) will provide, in conjunction with Eq. (3. 49), both the necessary and sufficient conditions for determining the value of B which minimizes tr $(\overline{e^+ \, e^{+T}})$. The terms involving δB^2 can be grouped as follows.

$$\delta B \, \{ [\overline{H(x) \, H(x)^T} - \overline{H^-(x)} \; \overline{H^-(x)^T} + R] \} \, \delta B^T$$

When rewritten in the following way

$$\delta B \quad \overline{[H(x) - \overline{H^-(x)}] \, [H(x) - \overline{H^-(x)}]^T} + R \quad \delta B^T$$

it is obvious that the quantity in braces is always positive definite if there is measurement noise, as assumed. Therefore, the second variation is always positive, and Eq. (3. 49) does indeed provide a minimizing expression for B.

The evaluation of the expectations in Eq. (3. 49) is again performed with a series expansion of the form of Eq. (3. 7). Thus, the update expression for the state estimate is given by

$$\bar{x}^+ = \bar{x}^- + B\,[z - \overline{H^-(x)}] \tag{3.50}$$

where B is given by Eq. (3. 49).

The update equations for the higher order moments are found in a similar manner. First, the expression used to find P^+ will be found in component form and then the update expression for the third order moment will be outlined using the results of the first derivation.

$$P^+_{ij} = \overline{(x_i - \bar{x}_i)(x_j - \bar{x}_j)}$$

where

$$\begin{aligned} \bar{x}^+_i &= \bar{x}^-_i + B_{i\ell}\,[H_\ell(x) - \overline{H^-_\ell(x)} + v_\ell] \\ \bar{x}^+_j &= \bar{x}^-_j + B_{jm}\,[H_m(x) - \overline{H^-_m(x)} + v_m] \end{aligned} \tag{3.51}$$

Using Eqs. (3. 51) in the expression for P^+_{ij} results in

$$P^+_{ij} = \overline{\{(x_i - \bar{x}^-_i) + B_{i\ell}\,[H_\ell(x) - \overline{H^-_\ell(x)} + v_\ell]\}\,\{(x_j - \bar{x}^-_j) + B_{jm}\,[H_m(x) - \overline{H^-_m(x)} + v_m]\}} \tag{3.52}$$

$$\begin{aligned} P^+_{ij} = P^-_{ij} &- 2\,\{B_{i\ell}\,\overline{[H_\ell(x) - \overline{H^-_\ell(x)} + v_\ell]\,[x_j - \bar{x}_j]}\}_{sem_{i,j}} \\ &+ B_{i\ell}\,\overline{[H_\ell(x) - \overline{H^-_\ell(x)} + v_\ell]\,[H_m(x) - \overline{H^-_m(x)} + v_m]}\,B_{mj} \end{aligned} \tag{3.53}$$

Noting that:

$$\overline{v_i} = 0$$

$$\overline{(x_i - x_i^-)} = 0$$

$$\overline{(x_i - x_i^-)\, v_j} = \overline{(x_i - x_i^-)}\ \overline{v_j} = 0$$

and

$$\overline{v_i\, v_j} = R_{ij}$$

Equation (3. 53) becomes

$$P_{ij}^+ = P_{ij}^- -2\ \{B_{i\ell}\ \overline{[H_\ell(x)]\ [x_j - \overline{x}_j]}\}_{sem_{i,j}}$$

$$+ B_{i\ell}\ [\overline{H_\ell(x)\ H_m(x)} - \overline{H_\ell^-(x)}\ \overline{H_m^-(x)} + R_{\ell m}]\ B_{mj}$$

(3. 54)

From Eq. (3. 49)

$$B_{i\ell} = \overline{e_i^-\ H_r(x)}\ [\overline{H(x)\ H(x)^T} - \overline{H^-(x)}\ \overline{H^-(x)^T} + R]_{r\ell}^{-1}$$

(3. 55)

It is important to note that in Eq. (3.55), the subscripts on the bracketed terms refer to the elements of the indicated inverse matrix. In general

$$[M]^{-1}_{ij} \neq [M_{ij}]^{-1}$$

and consequently care must be exercised in the evaluation of Eq. (3.55) for use in the estimation equations.

Substituting Eq. (3.49) into (3.54) results in

$$P^{+}_{ij} = P^{-}_{ij} - 2 \left\{ \overline{e^{-}_{i} H_{r}(x)} \left[\overline{H(x) H(x)^{T}} - \overline{H^{-}(x)}\ \overline{H^{-}(x)^{T}} + R \right]^{-1}_{r\ell} \overline{[H_{\ell}(x)] [x_{j} - \bar{x}^{-}_{j}]} \right\}_{sem_{i,j}}$$

$$+ \overline{e_{i} H_{r}(x)} \left[\overline{H(x) H(x)^{T}} - \overline{H^{-}(x)}\ \overline{H^{-}(x)^{T}} + R \right]^{-1}_{r\ell} I_{\ell s} \overline{H_{s}(x) e^{-}_{j}}$$

(3.56)

where $I_{\ell s}$ is the identity matrix resulting from

$$\left[\overline{H(x) H(x)^{T}} - \overline{H^{-}(x)}\ \overline{H^{-}(x)^{T}} + R \right]_{\ell m} \left[\overline{H(x) H(x)^{T}} - \overline{H^{-}(x)}\ \overline{H^{-}(x)^{T}} + R \right]^{-1}_{ms}$$

In this case, the two terms implied by $\{ \quad \}_{sem_{i,j}}$ are, in face identical because of the symmetric form of the term. In addition, each of the two terms is equal to the last terms so that Eq. (3.56) can be simplified to

$$P^+_{ij} = P^-_{ij} - \overline{e^-_i H_r(x)} \, [\overline{H(x)\, H(x)^T} - \overline{H^-(x)}\, \overline{H^-(x)^T} + R]^{-1}_{r\ell} \, [\overline{H_\ell(x)\, e^-_j}]$$

(3. 57)

Once again, the terms involving expectations are to be evaluated using series expansions and using caution with the evaluation of the inverse matrix terms.

Equation (3. 57) could have been easily, and more directly, derived using entirely vector notation resulting in

$$P^+ = P^- - \overline{e^- H(x)^T} \, [\overline{H(x)\, H(x)^T} - \overline{H^-(x)}\, \overline{H^-(x)^T} + R]^{-1} \, \overline{H(x)\, e^{-T}}$$

(3. 58)

However, the procedure just used will now prove useful in the extension of this to the derivation of the update equation for the third order moment term

$$T_{ijk} = \overline{(x_i - \bar{x}^+_i)\,(x_j - \bar{x}^+_j)\,(x_k - \bar{x}^+_k)} \qquad (3.59)$$

Using Eq. (3. 51)

$$\dot{T}_{ijk} = \overline{\{(x_i - \bar{x}^-_i) - B_{i\ell}[H_\ell(x) - H^-_\ell(x) + v_\ell\} \; \{(x_j - \bar{x}^-_j) + B_{jm}[H_m(x) - H^-_m(x) + v_m]\}} \times$$

$$\overline{\ldots \{(x_k - \bar{x}^-_k) + B_{kn}\,[H_n(x) - H^-_n(x) + v_n]\}}$$

(3. 60)

Noting that:

$$\overline{v}_i = 0$$

$$\overline{(x_i - x_i^-)} = 0$$

$$\overline{(x_i - x_i^-) v_j} = \overline{(x_i - x_i^-)}\ \overline{v}_j = 0$$

and

$$\overline{H_i(x) v_j v_k} = \overline{H_i(x)}\ \overline{v_j v_k} = \overline{H_i(x)}\ R_{jk}$$

Equation (3. 60) can be reduced to

$$T^+_{ijk} = T^-_{ijk} + 3 \{P^-_{ij} B_{km} \overline{H^-_m(x)} - \overline{(x_i - \overline{x}_i^-)(x_j - \overline{x}_j^-) H_m(x)} B_{mk}\}_{sem_{i,j,k}}$$

$$+ 3 \{\overline{(x_i - \overline{x}_i^-) H_\ell(x) H_m(x)} B_{j\ell} B_{mk} - \overline{(x_i - \overline{x}_i^-) H_m(x)} H^-_\ell(x) B_{j\ell} B_{mk}$$

$$- \overline{(x_i - \overline{x}_i^-) H_\ell(x)}\ \overline{H^-_m(x)} B_{j\ell} B_{mk}\}_{sem_{i,j,k}}$$

$$- 3 \{B_{i\ell} \overline{H_\ell(x) H_m(x)}\ \overline{H^-_n(x)} B_{j\ell} B_{nk}\}_{sem_{i,j,k}}$$

$$+ 3 \{B_{i\ell} \overline{H_\ell(x)}\ \overline{H^-_m(x)}\ \overline{H^-_n(x)} B_{j\ell} B_{nk}\}_{sem_{i,j,k}}$$

$$- B_{i\ell} \overline{H_\ell(x) H_m(x) H_n(x)} B_{j\ell} B_{nk} + B_{i\ell} \overline{H^-_\ell(x)}\ \overline{H^-_m(x)}\ \overline{H^-_n(x)} B_{j\ell} B_{nk}$$

(3. 61)

The value of B given by Eq. (3. 55) can now be used in Eq. (3. 61) to find the update equation for T_{ijk}. Additional algebraic manipulation may further reduce the expression for T^+_{ijk}; however, it is best, before doing so, to also exploit the characteristics of the particular nonlinearity under consideration. For this reason, the more general form of Eq. (3. 61) is retained.

In Eq. (3. 61), expectations of third order tensors can be evaluated by extending Eq. (3. 30) to the following.

$$\overline{q_{ijk}(x)} = q_{ijk}(\bar{x}) + \frac{1}{2} \sum_{\ell} \sum_{m} \frac{\partial^2 q_{ijk}(\bar{x})}{\partial x_\ell \, \partial x_m} P_{\ell m}$$

$$+ \frac{1}{6} \sum_{\ell} \sum_{m} \sum_{n} \frac{\partial^3 q_{ijk}(\bar{x})}{\partial x_\ell \, \partial x_m \, \partial x_n} T_{\ell mn} \qquad (3.\ 62)$$

Expectations of lower order terms are evaluated as shown above for the update of P_{ij}.

The above procedure can be extended to any higher order moment in a straightforward manner. The mth central moment can be written

$$M^+_{ijk\ldots} = \underbrace{\overline{(x_i - x^+_i)(x_j - x^+_j)(x_k - x^+_k)\ldots}}_{\text{m terms}}$$

Making use of the symmetry properties as discussed above will greatly reduce the required computations.

It is important to emphasize that the quantity B was assumed to be only a function of the statistics of the state and noises. Taking a Bayesian point of view, Jazwinski (1966) (see Chapter II) has derived a set of update equations where B depends on the actual measurement quantity as well as state and noise statistics. This, of course, subjects the filter gain to the random actions of noise in the measurement; however, in the absence of good a priori statistical data, a filter gain sensitive to actual measurements may, in fact, be better on any given Monto Carlo run.

3.4 Limitation of Series Expansions

The general technique used in the above procedure is that of expanding the expectation of any function of the state in a series about a reference point. The function of the state may be of any form: a vector, matrix or tensor. The reference state is usually chosen to be the current estimate of the state with the tacit assumption that the series expression has a known behavior with respect to the relative magnitudes of each of the terms. This knowledge is required in order to truncate that series with a reasonable number of terms and still approximate the desired function in an acceptable way.

The Taylor Series was used in the above because it is readily applicable to this problem; however, any series with the desirable properties mentioned above may be used.

The problems associated with using the Taylor Series and generally with most series expressions are:

1) If the series must be truncated, it is necessary to have the terms decreasing in a known manner in order to make an intelligent decision about which terms to include. The complete Taylor Series can be written in the form

$$\overline{q_i(x)} = q_i(\bar{x}) + \frac{1}{2} \sum \sum \frac{\partial^2 q_i(\xi)}{\partial x_j \, \partial x_k} \overline{(x - \bar{x})_j (x - \bar{x})_k} \qquad (3.63)$$

where the second term represents a remainder term with ξ a random variable in the same space as x. In the series, as used, the final term is evaluated about $\bar{x}$ and therefore, the series expression is not complete. It is also not always clear that the series is converging in a desirable manner.

2) A large number of nonlinear functions encountered in the field of estimation and control are such that a series expression is not applicable. In the case of a Taylor Series, the derivatives which appear in all but the first term, must exist. Thus, "hard" nonlinearity such as two or three level switches and quantizers, cannot be adequately described using these techniques.

3) As can be seen from the previous discussion, the complexity involved in the evaluation of higher order moment terms makes the series approach unattractive for systems requiring a large number of terms in the series expansion. The application of series expansion estimation in Chapter V demonstrates this fact. If the series is to be truncated after only two terms, (i.e., only statistics up to the second order are to be considered) a different approach to the nonlinear approximation can be taken. (See Chapter IV.)

CHAPTER IV

A QUASI-LINEAR ESTIMATION TECHNIQUE

4.1 Introduction

One approach to the problem of state estimation for systems with nonlinear functions is to first isolate the nonlinearities and then consider the problem of determining reasonable approximations for the nonlinearities in the system differential equations and / or the measurement process. Once these approximations are found, techniques which are analogous to those used in linear estimation theory may be employed in order to complete the problem. The proper selection of a reasonable approximation to the nonlinearities thus becomes the focal point of this discussion. It is important to make the distinction between an approximation to the nonlinearity and the process of linearization. The approximations contained in this chapter and in Chapter III are not the same as linearizing and once an approximation is developed, the procedure for deriving the recursive estimation equations, although analogous to that of linear estimation techniques is different in several significant ways. One such nonlinear technique has been discussed in Chapter III, and in this chapter a new approach is developed with some discussion of the salient differences between the two approaches.

Consider the class of nonlinear systems, the dynamics of which can be described by the following first order differential equation

$$\dot{x} = f(x) + Gu \tag{4.1}$$

where $f(x)$ is a general nonlinear function of the state vector x, G is a matrix coefficient independent of x, and u is a vector of Gaussian white noise, each component of which is uncorrelated with x. Equation (4.2) defines the covariance of u :

$$\overline{u(\tau_1)\, u(\tau_2)^T} = Q\, \delta\, (\tau_2 - \tau_1) \tag{4.2}$$

The expectation indicated by the bar in Eq. (4.2) and in all the subsequent work in this chapter is taken over an ensemble of random processes.

As an aid to understanding the following approximation technique, the above system can also be presented in a block diagram form as shown below :

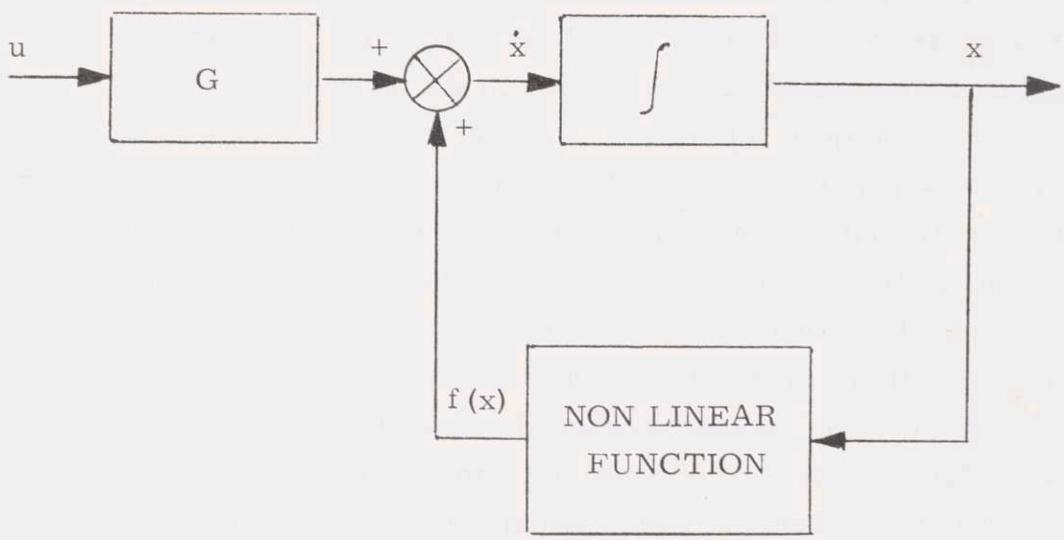


Figure 4-1 System Block Diagram

The success of any estimation scheme hinges on how well the system can be modelled and therefore, in this case, on how well the function f (x), the output of the nonlinearity, can be approximated. The value of the approximation must reflect considerations of both the accuracy with which the model fits the true nonlinear relationship and also the usefulness of the resulting estimation algorithm.

The quasi-linear technique which will be discussed in detail here is a method for finding an expected value for the nonlinear function f (x) to be used in a recursive relationship for an optimum estimate of the state. The expectation involved is conditioned upon the statistical nature of the input to the nonlinear function, the state variable x. In order to proceed with the evaluation of the expection involved, it is therefore necessary at this point to make certain assumptions about the state vector. These assumptions, which are related to the statistics of the state, basically require that the input to the nonlinearity be an ensemble bias vector plus a zero mean multi-dimensional Gaussian process. The bias which is also the expected value of x is the optimum estimate and the Gaussian random variable is, in fact, the deviation of the estimate from the true value, the estimation error.

The following represents some of the more important factors which lend support to the above assumptions. They are also some of the moviation for taking a quasi-linear viewpoint for developing a non-linear estimation scheme.

1) The noise which enters the system either through the dynamics or from external measurements can be assumed to be of Gaussian form. If this noise signal is a significant portion of the random component in x, then a Gaussian description of the random process to be estimated would seem to be reasonable.

2) The statistical behavior of the estimator is, to some degree, a function of the initial conditions used. Generally, the initial conditions for the equations in any estimation technique which are most readily available, and also most meaningful in a wide variety of estimation problems, are of Gaussian form.

3) With reference to the block diagram shown in Fig. 4-1, the integration which takes place immediately before the nonlinear function operates on the state, has a filtering effect which often tends to make the random portion of x appear more nearly like a Gaussian random variable even though $\dot{x}$ may be considerably non-Gaussian in nature due to the direct dependence of $\dot{x}$ on f (x).

4) For a large number of useful nonlinear functions, the unimodal and symmetric properties of a Gaussian function are preserved if the initial conditions are Gaussian. If only first and second order statistics are to be considered, as is true here, then the assumption of a Gaussian form is further justified.

The most serious factor which would tend to undermine the assumption that x is a bias plus a Gaussian random variable would be a system nonlinearity of the type which has extreme skewness or other property in conflict with the general properties of a Gaussian process. This nonlinear effect might arise either in the system dynamics or at a measurement point, if the measurement nonlinearity is again the type which will not preserve the general shape of a Gaussian density function. If for some reason initial conditions on the estimation are known to differ greatly from a Gaussian form, the above required assumption will not be reasonable.

These then are the considerations which influence the confidence associated with the assumptions basic to this quasi-linear estimation technique. It is possible to make any other assumption about the statistical nature of x; however, because of the filtering effect of the system integration, the Gaussian assumption is most widely applicable. As is true with most approximate techniques, numerical studies to demonstrate the effect of the basic assumptions are always desirable, if not necessary. Chapters V and VII are devoted to such studies.

4.2 A Minimum Variance Approximation

As discussed above, the basic requirement for this quasi-linear estimation scheme is a method for approximating a general nonlinear function of the state vector x. The approximator which will be used here, is depicted in block diagram form below.

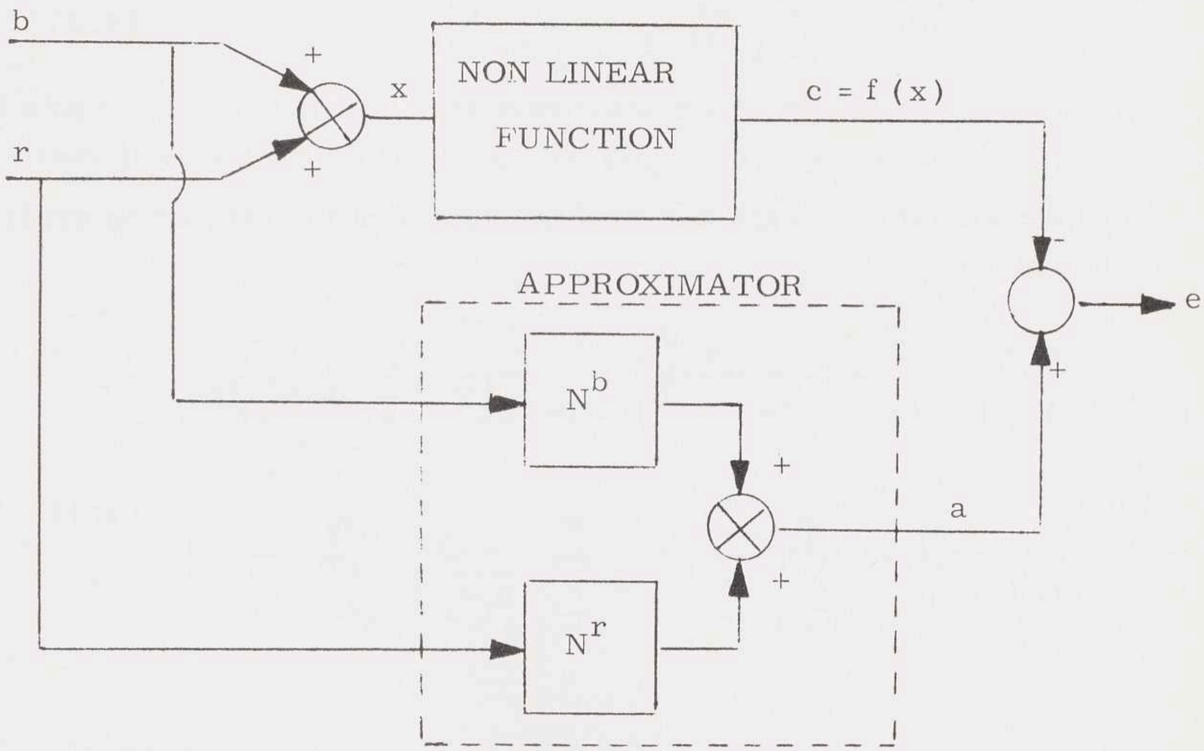


Figure 4-2 Quasi-Linear Approximator

The matrices, N^b and N^r are to be selected such that the expectation of the squared error in the estimate of f(x) is minimized. This property may be expressed as

$$\underset{a}{\text{MIN}} \left\{ \text{trace } [\overline{e e^T}] \right\}$$

where the expectation indicated by the bar is again an ensemble average.

In general, the approximation matrices will be assumed to be simply a set of time varying gains such that the approximate output is given by the following expression

$$a_i = \sum_{j=1}^{n} N_{ij}^b \, b_j + \sum_{j=1}^{n} N_{ij}^r \, r_i \tag{4.3}$$

An expression for the expectation of the square of the estimation error is then given by:

$$\text{tr}\,(\overline{e e^T}) = \text{tr}\,[\overline{(a-c)(a-c)^T}] = \text{tr}\,(\overline{a a^T} - \overline{a c^T} - \overline{c a^T} + \overline{c c^T})$$

$$\text{tr}\,(\overline{e e^T}) = \sum_{i=1}^{m} \overline{a_i^2} - 2 \sum_{i=1}^{m} \overline{a_i c_i} + \sum_{i=1}^{m} \overline{c_i^2} \tag{4.4}$$

Substituting Eq. (4.3) into the above expression results in

$$\operatorname{tr}(\overline{e e^T}) = \sum_{i=1}^{m} \left[\sum_{j=1}^{n} \sum_{k=1}^{n} N_{ij}^b N_{ik}^b \overline{b_j b_k} \right.$$

$$+ 2 \sum_{j=1}^{n} \sum_{k=1}^{n} N_{ij}^b N_{ik}^r \overline{b_j}\, \overline{r_k}$$

$$\left. + \sum_{j=1}^{n} \sum_{k=1}^{n} N_{ij}^r N_{ik}^r \overline{r_j r_k} \right] \tag{4.5}$$

$$- 2 \sum_{i=1}^{m} \left[\sum_{j=1}^{n} N_{ij}^b \overline{b_j}\, \overline{c_i} + \sum_{j=1}^{n} N_{ij}^r \overline{r_j}\, \overline{c_i} \right] + \sum_{i=1}^{m} \overline{c_i^2}$$

Using the fact that r is a zero mean process, and that b represents the ensemble mean and therefore $\overline{b} = b$, Eq. (4.5) reduces to

$$\operatorname{tr}(\overline{e e^T}) = \sum_{i=1}^{m} \left[\sum_{j=1}^{n} \sum_{k=1}^{n} N_{ij}^b N_{ik}^b b_j b_k + N_{ij}^r N_{ik}^r \overline{r_j r_k}) \right.$$

$$\left. - 2 \left(\sum_{j=1}^{n} N_{ij}^b b_j \overline{c_i} + \sum_{j=1}^{n} N_{ij}^r \overline{r_j c_i} \right) + \overline{c_i^2} \right] \tag{4.6}$$

To minimize Eq. (4.6) with respect to each element of the approximation matrices, the following technique is used. It is possible to define the general matrix element as the sum of the corresponding element of the optimum gain matrix and a small, but otherwise arbitrary deviation from that value. Thus

$$N_{ij}^{b} = N_{ij}^{b0} + \delta N_{ij}^{b}$$

$$N_{ij}^{r} = N_{ij}^{r0} + \delta N_{ij}^{r} \tag{4.7}$$

Substituting these expressions into Eq. (4.6) results in

$$\begin{aligned} \operatorname{tr}(\overline{e e^T}) = \sum_{i=1}^{m} \Bigg\{ \sum_{j=1}^{n} \sum_{k=1}^{n} \Bigg[& (N_{ij}^{b0} + \delta N_{ij}^{b})(N_{ik}^{b0} + \delta N_{ik}^{b})\, b_j\, b_k \\ & + (N_{ij}^{r0} + \delta N_{ij}^{r})(N_{ik}^{r0} + \delta N_{ik}^{r})\, \overline{r_j\, r_k} \Bigg] \\ & - 2 \left[\sum_{j=1}^{n} (N_{ij}^{b0} + \delta N_{ij}^{b})\, b_j\, \overline{c_i} + \sum_{j=1}^{n} (N_{ij}^{r0} + \delta N_{ij}^{r})\, \overline{r_j\, c_i} \right] + \overline{c_i^2} \Bigg\} \end{aligned} \tag{4.8}$$

The necessary and sufficient conditions for minimizing $\operatorname{tr}(\overline{e e^T})$ are: 1) the first variation must equal zero; and 2) the second variation must be positive. An expression for the first variation may be found from Eq. (4.8) by collecting all terms of order δN_{ij}.

$$\sum_{i=1}^{m} \left\{ \sum_{j=1}^{n} \sum_{k=1}^{n} \left[(N_{ij}^{b0} \, \delta N_{ik}^{b} + \delta N_{ij}^{b} \, N_{ik}^{b0}) \, b_j \, b_k \right. \right.$$

$$\left. + (N_{ij}^{r0} \, \delta N_{ik}^{r} + \delta N_{ij}^{r} \, N_{ik}^{r0}) \, \overline{r_j \, r_k} \right] \tag{4.9}$$

$$\left. - 2 \left[\sum_{j=1}^{n} \delta N_{ij}^{b} \, b_j \, \overline{c_i} + \sum_{j=1}^{n} \delta N_{ij}^{r} \, \overline{r_j \, c_i} \right] \right\} = 0$$

By a proper redefinition of variables, it is easy to show that Eq. (4.9) can be reduced to the following

$$2 \sum_{i=1}^{m} \sum_{j=1}^{n} \left\{ \delta N_{ij}^{b} \left[\left(\sum_{k=1}^{n} N_{ik}^{b0} \, b_j \, b_k \right) - b_j \, \overline{c_i} \right] \right.$$

$$\left. + \delta N_{ij}^{r} \left[\left(\sum_{k=1}^{n} N_{ik}^{r0} \, \overline{r_j \, r_k} \right) - \overline{r_j \, c_i} \right] \right\} = 0 \tag{4.10}$$

The ensemble mean, b and the random variable, r are independent and as defined above, both δN_{ij}^{b} and δN_{ij}^{r} are completely arbitrary. Therefore, the above equality could only be true for the trival case of b and r identically zero unless the following two expressions are also satisfied.

$$\sum_{k=1} N_{ik}^{b0} \, b_j \, b_k = b_j \, \overline{c_i} \tag{4.11}$$

for all i and j

$$\sum_{k=1} N_{ik}^{r0} \, \overline{r_j \, r_k} = \overline{r_j \, c_i} \tag{4.12}$$

Each of the above equations can be rewritten in the more useful vector notation as shown below

$$N^b \, b b^T = \bar{c} \, b^T \tag{4.13}$$

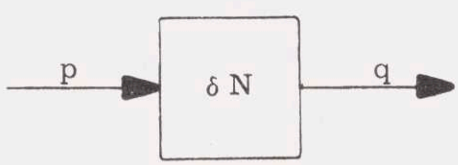
$$N^r \, \overline{r r^T} = \overline{c \, r^T} \tag{4.14}$$

In Eqs. (4.13) and (4.14) and in all of the subsequent work, the superscript 0 indicating the optimum value of the gain matrix has been dropped. Whenever the matrix appears in the following equations it will be interpreted as the optimum value.

Returning to Eq. (4.8) and collecting all the terms of second order variation, provides the following expression

$$\sum_{i=1}^{m} \left[\sum_{j=1}^{n} \sum_{k=1}^{n} \delta N_{ij}^{b} \, \delta N_{ik}^{b} \, b_j \, b_k + \sum_{j=1}^{n} \sum_{k=1}^{n} \delta N_{ij}^{r} \, \delta N_{ik}^{r} \, \overline{r_j \, r_k} \right]$$

Consider the following hypothetical system



Each component of the output, q can be expressed by the following

$$q_i = \sum_{j=1}^{n} \delta N_{ij} \, p_i \tag{4.15}$$

and the components of the mean squared output are given by

$$\overline{q_i^2} = \sum_{j=1}^{n} \sum_{k=1}^{n} \delta N_{ij} \, \delta N_{ik} \, \overline{p_i \, p_k} \tag{4.16}$$

Thus, each of the two terms in the second variation expression which are contained in the brackets represents a mean squared quantity and must be a positive number. Consequently, the second variation of Eq. (4.6) is positive and Eqs. (4.13) and (4.14) do, indeed define necessary conditions for a minimum mean squared error.

Two important properties of the above approximation equations can be demonstrated without any further manipulation of Eqs. (4.13) and (4.14). The approximation to the output of the nonlinear function

$$a = N^b b + N^r r \tag{4.17}$$

has the desirable property of rendering the quasi-linear approximation unbiased. From Eq. (4.13)

$$N^b b = \overline{c} \tag{4.18}$$

and therefore

$$\overline{a} = \overline{c} + N^r \overline{r} = \overline{c} \tag{4.19}$$

or

$$\bar{e} = \overline{(a - c)} = 0 \qquad (4.20)$$

Also, the error in the quasi-linear approximation is uncorrelated with the input to the nonlinearity.

$$\overline{ex^T} = \overline{[N^b b + N^r r - c][b + r]^T} \qquad (4.21)$$

$$\overline{ex^T} = N^b bb^T + N^b b\bar{r}^T + N^r \bar{r} b^T + N^r \overline{rr^T} - \bar{c} b^T - \overline{cr^T} \qquad (4.22)$$

$$\overline{ex^T} = (N^b bb^T - \bar{c} b^T) + (N^r \overline{rr^T} - \overline{cr^T}) \qquad (4.23)$$

The two terms in Eq. (4.23) which are in parenthesis are precisely the defining relations for N^b and N^r and are equal to zero by virtue of Eqs. (4.13) and (4.14).

The general approximation determined above will now be considered in light of the estimation problem. The bias component is, in fact, the ensemble mean of the random state variable and will be denoted by $\bar{x}$, and the random input component is the deviation of the state from that mean and is, therefore, the error in the estimate.

$$e = (x - \bar{x}) \qquad (4.24)$$

Associated with the error and necessary for the estimation procedure is a matrix which represents the mean squared errors in the estimate and is defined by the following equation

$$P = \overline{e\,e^T} = \overline{(x - \bar{x})\,(x - \bar{x})^T} \tag{4.25}$$

Rewriting Eqs. (4.13) and (4.14) in light of the above

$$N^{\bar{x}}\,\overline{\bar{x}\,\bar{x}^T} = \overline{c\,(x)}\;\bar{x}^T \tag{4.26}$$

$$N^e\,\overline{e\,e^T} = N^e P = \overline{c\,(x)\,e^T} \tag{4.27}$$

Equation (4.27) can be solved for N^e because the matrix $\overline{e\,e^T}$ always has a unique inverse.

$$N^e = \overline{c\,(x)\,e^T}\;P^{-1} \tag{4.28}$$

A similar procedure is not possible for Eq. (4.26) because the matrix $\bar{x}\,\bar{x}^T$ is always singular, as is any vector diadic product. A useful definition can be found by defining a pseudo inverse matrix. Such an inverse, as well as the most general solution to Eq. (4.13), is found in Appendix B. Using the results of Appendix B and Eq. (4.28), the gain to $\bar{x}$ can be written as

$$N^{\bar{x}} = \frac{\overline{c\,(x)}\;\bar{x}^T}{\bar{x}^T\,\bar{x}} + N^e\left[\,I - \frac{\bar{x}\,\bar{x}^T}{\bar{x}^T\bar{x}}\,\right] \tag{4.29}$$

where I is the $n \times n$ identity matrix.

Note that with these definitions for N^e and $N^{\bar{x}}$ as c (x) is specialized to a linear function, C x

$$N^{\bar{x}} = N^e = C \tag{4.30}$$

Two additional properties which result from this definition of the quasi-linear approximators are the following. From Eq. (4.18), the product of the gain to $\bar{x}$ and $\bar{x}$ itself is equal to the expectation of the nonlinear output, $\overline{c(x)}$. Taking the derivative of this quantity with respect to x results in the following

$$\frac{d}{d\bar{x}}\left[N^{\bar{x}}\,\bar{x}\right] = \frac{d}{d\bar{x}}\left[\overline{c(x)}\right] \tag{4.31}$$

$$\frac{d}{d\bar{x}}\left[\overline{c(x)}\right] = \frac{1}{(2\pi)^{n/2}\,|P|^{1/2}} \times$$

$$\int \cdot \cdot \int_{-\infty}^{\infty} \frac{d}{d\bar{x}}\left[c(x)\, e^{-n/2\,(x-\bar{x})^T P^{-1}(x-\bar{x})}\right] dx \tag{4.32}$$

The differentiation with respect to $\bar{x}$ can be evaluated as follows

$$\frac{d}{d\bar{x}}\left[c(x)\, e^{-1/2\,(x-\bar{x})^T P^{-1}(x-\bar{x})}\right] =$$

$$\left\{\frac{d}{d\bar{x}}\left[c(x)\right] - 1/2\, c(x)\,\frac{d}{d\bar{x}}\left[(x-\bar{x})^T P^{-1}(x-\bar{x})\right]\right\} e^{-1/2(x-\bar{x})^T P^{-1}(x-\bar{x})} \tag{4.33}$$

The first term is zero because $c(x)$ is not an explicit function of $\bar{x}$ and therefore the following result is found

$$\frac{d}{d\bar{x}} [c(x) e^{-1/2 (x - \bar{x})^T P^{-1} (x - \bar{x})}] =$$

$$c(x)(x - \bar{x})^T P^{-1} e^{-1/2 (x - \bar{x})^T P^{-1} (x - \bar{x})} \qquad (4.34)$$

Substituting this result into Eq. (4.32)

$$\frac{d}{d\bar{x}} [\overline{c(x)}] =$$

$$\frac{1}{(2\pi)^{n/2} |P|^{1/2}} \int_{-\infty}^{\infty} \cdots \int c(x)(x - \bar{x})^T P^{-1} e^{-1/2 (x-\bar{x})^T P^{-1} (x-\bar{x})} dx \qquad (4.35)$$

However this right hand side of Eq. (4.35) is simply another expectation integral. Thus

$$\frac{d}{d\bar{x}} [\overline{c(x)}] = \overline{c(x)(x - \bar{x})^T P^{-1}} \qquad (4.36)$$

The right hand side of Eq. (4.36) is precisely the quasi-linear gain to the random input component, N^e. Therefore, the following property of these estimators is always true.

$$\frac{d}{d\bar{x}} [N^{\bar{x}} \bar{x}] = N^e \qquad (4.37)$$

Another interesting and often useful property of the quasi-linear approximations can be expressed by the following equation. For $f(x)$ equal to an odd function

$$\lim_{\overline{x} \to 0} \left[N^{\overline{x}} \overline{x} \right] = N^{e} \tag{4.38}$$

This result can be shown in the following manner. If a difference matrix, Δ is defined by the following equation

$$\Delta = N^{\overline{x}} - N^{e} = \left[\frac{\overline{f(\overline{x} + e)} - \overline{f(\overline{x} + e) e^{T}} P^{-1} \overline{x}}{|\overline{x}|} \right] \frac{\overline{x}^{T}}{|\overline{x}|} \tag{4.39}$$

then the $(ij)^{th}$ element of Δ can be expressed as follows

$$\Delta_{ij} = v_i u_j$$

where

$$v_i = \left[\frac{\overline{f(\overline{x} + e)} - \overline{f(\overline{x} + e) e^{T}} P^{-1} \overline{x}}{|\overline{x}|} \right]_i$$

and

$$u_j = \frac{\overline{x}_j}{|\overline{x}|}$$

Clearly, the following inequality is also true

$$\Delta_{ij} \leq |v|\,|u| = |v|$$

Now, if $|v| \rightarrow 0$ as $\bar{x} \rightarrow 0$, then each element of $\Delta \rightarrow 0$ and the proof of Eq. (4.38) is complete. Using Eq. (4.37) the following is always found to be true,

$$\frac{d}{d\bar{x}}\left[\,\overline{f(\bar{x}+e)}\,\right] = N^{e}$$

and for $\bar{x} = 0$, $N^{e} = \overline{f(e)e^{T}}\,P^{-1}$. Using a Taylor series expansion for $\overline{f(\bar{x}+e)}$

$$\overline{f(x+e)} = \overline{f(e)e^{T}}\,P^{-1}\,\bar{x} + 0\,(|\bar{x}|^{2}) \qquad (4.40)$$

Therefore, to first order in $\bar{x}$, $\overline{f(\bar{x}+e)} = \overline{f(e)e^{T}}\,P^{-1}$ and consequently $|v| \rightarrow 0$ as $\bar{x} \rightarrow 0$.

Thus, for a state variable with a "small" mean value, the gain to the mean and the gain to the random component are approximately the same. What constitutes "small" is not obvious in general and will usually depend on several other system parameters. This question will be considered again in Chapter VII and the usefullness of Eq. (4.38) demonstrated in a numerical example at that time.

It is important to note that, if one can assume $N^{e} = N^{\bar{x}}$ and if the limiting characteristic of $N^{\bar{x}}$ as the non-linearity is specialized to a linear function is not particularly important, a great deal of computation complexity can be saved. The equation for $N^{\bar{x}}$ can be simplified to

$$N^{\bar{x}} = \frac{\overline{c(x)}\,\bar{x}^T}{\bar{x}^T\bar{x}} \tag{4.41}$$

because the additional term is arbitrary as shown in Appendix C. The computation of N^e which involves the expectation of the matrix $\overline{c(x)\,x^T}$ need not be performed since the estimation equations will contain only $N^{\bar{x}}$.

The quasi-linear approximation matrices, as defined above, can be used for nonlinear estimation whenever a nonlinear function appears in the estimation equations. The consequences of this approximation for the particular recursive relationships which are used in an optimal estimation scheme are discussed below.

4.3 Extrapolation of a Quasi-Linear Estimator

Consider a system which can be described by the Eq. (4.1) repeated here for convenience

$$\dot{x} = f(x) + G u \tag{4.1}$$

where f (x) is a general nonlinear vector function of the state, G is a coefficient matrix independent of x, and u is a Gaussian white noise process. The differential equation used to model this system and to describe the extrapolation of the optimum estimate is given by the expectation of Eq. (4.1).

$$\overline{\dot{x}} = \overline{f(x)} \tag{4.42}$$

where the bar indicates an ensemble expectation.

The quasi-linear technique described above can now be used to approximate the expectation of the nonlinear function, resulting in a differential equation of the form shown below.

$$\dot{\overline{x}} = N_f^{\overline{x}} \, \overline{x} \tag{4.43}$$

where $N_f^{\overline{x}}$ the approximation to f (x) for the mean input $\bar{x}$, is a function of the first and second order statistics of the state vector and is determined using the procedure discussed in Section 4.2.

The covariance matrix which is a representation of the expected value of the squared error in the estimate of x can be approximated in a similar manner.

$$P = \overline{ee^T} = \overline{(x - \bar{x})(x - \bar{x})^T} \tag{4.44}$$

and

$$\dot{P} = \overline{(\dot{x} - \dot{\bar{x}})(x - \bar{x})^T} + \overline{(x - \bar{x})(\dot{x} - \dot{\bar{x}})^T} \tag{4.45}$$

The quasi-linear approximation for the differential equations of the state and its estimate are as follows:

$$\begin{aligned} \dot{x} &\approx N_f^{\bar{x}}\,\bar{x} + N_f^{e}\,(x - \bar{x}) + Gu \\ \dot{\bar{x}} &\approx N_f^{\bar{x}}\,\bar{x} \end{aligned} \tag{4.46}$$

Substituting Eqs. (4.46) into Eq. (4.45) results in the following expression for $\dot{P}$.

$$\dot{P} = N_f^e\,\overline{(x - \bar{x})(x - \bar{x})^T} + G\,\overline{u(x - \bar{x})^T} + \overline{(x - \bar{x})(x - \bar{x})^T}\,N_f^{e^T} + \overline{(x - \bar{x})u^T}\,G^T \tag{4.47}$$

$$\dot{P} = N_f^e\,P + P\,N_f^{e^T} + G\,\overline{ux^T} + \overline{xu^T}\,G^T \tag{4.48}$$

The last two terms of Eq. (4.48) may be evaluated in the following manner

$$\overline{xu^T}G^T = \overline{x(0)\,u(t)^T}\,G^T + \int_0^t \overline{\left\{ f\left[x(s) \right] + Gu(s) \right\} ds\, u(t)^T}\, G^T$$

As discussed in Chapter III $\overline{x(0)\,u(t)^T} = 0$ and $\overline{f[\,x(s)\,]\,u(t)^T}$ does not contribute to the integral for $0 \quad s \quad t$. Thus:

$$\overline{xu^T}\,G^T = 1/2\,G\,Q\,G^T \tag{4.49}$$

Similarly, the other term of Eq. (4.48) reduces to:

$$\overline{Gux^T} = 1/2\,G\,Q\,G^T \tag{4.50}$$

Using the above results, Eq. (4.48) becomes:

$$\dot{P} = N_f^e\,P + P\,N_f^{e^T} + G\,Q\,G^T \tag{4.51}$$

Equations (4.43) and (4.51) together provide a means of extrapolating the first and second order statistics of the state from one measurement point to the next. At each measurement time, a different set of equations is necessary in order to update the estimate using the information contained in the measurement quantity and the statistics associated with both the present estimate and the measurement process.

4.4 Updating With Nonlinear Measurements

At discrete times, a measurement of some quantity external to the system, but related to the state is necessary in order to update the current state estimate. This measurement quantity will be defined by the following relationship

$$z = H(x) + v \tag{4.52}$$

where $H(x)$ is a general nonlinear function of the state vector x and v is a Gaussian white noise process independent of the state. The random vector v obeys the following relationship

$$\overline{v(\tau_1)\, v(\tau_2)^T} = R\,\delta(\tau_2 - \tau_1) \tag{4.53}$$

In order to update the state estimate, it is necessary to find an approximate expression for the expected value of $H(x)$ which will allow a recursive formulation of the update equations. This problem is exactly analogous to finding the quasi-linear approximation to $f(x)$ in Eq. (4.1) in all but one respect. In the case of measurement updates, because of the added information contained in the measurement quantity z, either of two sets of assumptions about the probability density function of the state may be used to evaluate the expectation involved in the quasi-linear approximation. This conditional expectation may be found using either of the following assumptions.

1) The input to the nonlinear measurement may be assumed to be just the bias plus Gaussian variable available just prior to the measurement. This assumption does not make use of the actual measurement in the evaluation of the required expectation.

2. The probability density function of the input to the nonlinearity may be modified by the measurement quantity z. The expectation required will then be evaluated on the basis of a probability density function conditioned upon z which, in general, will no longer be of Gaussian form. This latter fact naturally presents some additional computational difficulties; however, it would seem that the added information contained in the conditional density function would allow a greater accuracy in the computation of $\overline{H(x)}$.

The first approach is exactly the same as that used in the development of the extrapolation equations for system dynamic nonlinearities. In this case, the evaluation of $\overline{H(x)}$ follows the procedure described in Section 4.2. Using the second approach introduces certain different aspects which increase the complexity of the estimation procedure.

The ensemble expectation of H (x) using the probability density function available immediately before a measurement can be found from:

$$\overline{H(x)} = \int_{-\infty}^{\infty} H(x)p(x)dx \qquad (4.54)$$

where p(x) is the Gaussian probability density of the state. Using the conditional density function discussed above, Eq. (4.54) becomes

$$\overline{H(x)} = \int_{-\infty}^{\infty} H(x)\, p(x \mid z)\, dx \qquad (4.55)$$

where $p(x \mid z)$ is the probability density function of the state before the measurement update, but given the particular measurement quantity for that time. Equation (4.55) may be evaluated as follows.

$$p(x \mid z) = \frac{p(z \mid x)\, p(x)}{p(z)}$$

$$p(x \mid z) = \frac{p(z \mid x)\, p(x)}{\int_{-\infty}^{\infty} p(z \mid x)\, p(x)\, dx} \tag{4.56}$$

Thus, evaluation of the conditional expectation of the non-linear function $H(x)$ focuses on the problem of determining the following relationships

$$\int_{-\infty}^{\infty} p(z \mid x)\, p(x)\, dx$$

and (4.57)

$$\int_{-\infty}^{\infty} H(x)\, p(z \mid x)\, p(x)\, dx$$

Under the assumption that the measurement noise is a Gaussian process, and making use of Eq. (4.52), $p(z \mid x)$ can be written in the following form.

$$p(z \mid x) = \frac{1}{(2\pi)^{n/2}\, |R|^{1/2}} \exp\left\{ -1/2 [z - H(x)]^T R^{-1} [z - H(x)] \right\} \tag{4.58}$$

Using Eqs. (4.58) and (4.56) the following expression for Eq. (4.54) can be found.

$$\overline{H(x)} = \frac{\int_{-\infty}^{\infty} H(x)\, p(x) \exp\left\{-1/2\,[z - H(x)]^T R^{-1} [z - H(x)]\right\} dx}{\int_{-\infty}^{\infty} p(x) \exp\left\{-1/2\,[z - H(x)]^T R^{-1} [z - H(x)]\right\} dx} \tag{4.59}$$

In general, Eq. (4.59) can only be evaluated with numerical techniques. However, in certain cases of interest the expression for $\overline{H(x)}$ can be further modified to gain insight into the selection of a particular numerical procedure. A good example of this type of modification is found for quantized measurements where the conditional density function would seem to have a significant advantage over the unconditional Gaussian density function of the state. This nonlinear measurement relationship is demonstrated in the following figure and can be expressed by Eq. (4.60), where δz is the quantization level assumed be a constant in this case.

$$z = H(x) + v = n\,\delta z + v \tag{4.60}$$

An alternative and perhaps more realistic definition of a quantized measurement is the following

$$z = H(x + v) \tag{4.61}$$

with the function H as defined above.

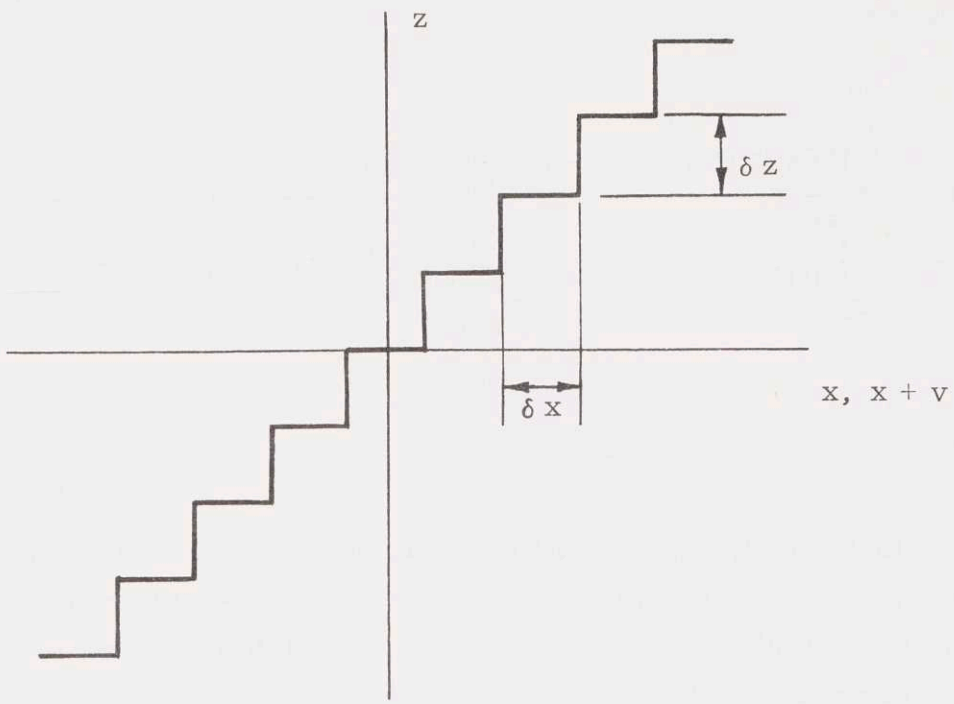


Figure 4-3 Quantized Measurements

Since x and v are independent, the probability density function of x + v is the convolution of the two density functions, both of which are Gaussian. Thus, the procedure is similar, but more computationally complex. For purposes of demonstrating the type of procedure which must be followed, Eq. (4.60) will be used as the definition of the nonlinearity for the following. Specializing Eq. (4.59) to the chosen nonlinear relationship results in each of the integrals being replaced by an infinite series with each term a weighted error function. Thus, Eq. (4.62) represents the expression for $\overline{H(x)}$ for the quantized measurement as defined by Eq. (4.60).

$$\overline{H(x)} = \frac{\displaystyle\sum_{n=-\infty}^{\infty} n\delta z \exp\left[-1/2\,(z-n\delta z)^T R^{-1}(z-n\delta z)\right] \int_{(n-1/2)\delta x}^{(n+1/2)\delta x} \exp\left[-1/2\,(x-\bar{x})^T P^{-1}(x-\bar{x})\right] dx}{\displaystyle\sum_{n=-\infty}^{\infty} \exp\left[-1/2\,(z-n\delta z)^T R^{-1}(z-n\delta z)\right] \int_{(n-1/2)\delta x}^{(n+1/2)\delta x} \exp\left[-1/2\,(x-\bar{x})^T P^{-1}(x-\bar{x})\right] dx} \tag{4.62}$$

Since the above summations have infinite limits, a practical solution would require some judgement on where to truncate the series. Because of the known behavior of error functions, this can be done in a systematic manner by including, in the series, those terms required in order to consider a chosen number of standard deviations of the apriori probability density function of x about its mean value. This number should be selected with consideration of required accuracy and computational time available. A level of confidence can be associated with the selection of n because of the Gaussian nature of the apriori density function of x.

Having thus found an expression for the expectation of the nonlinear measurement function, either by assuming the probability density function of x is Gaussian or by using a probability density function which is conditioned upon the actual measurement, this expression can be used to update the state estimate in the following way. Governed by the quasi-linear nature of this approximation and the familiar form of the update equations used in linear estimation techniques, it is reasonable to define the following form for the quasi-linear measurement update equation.

$$\overline{x^+} = \overline{x^-} + K[z - \overline{H(x)}] \tag{4.63}$$

where $\overline{x^+}$ is the updated state estimate, $\overline{x^-}$ is the estimate just prior to incorporation of the measurement information and K is a statistical gain factor chosen to minimize the ensemble expectation of the squared error in the estimate. To determine the proper value of K, the following procedure is both convenient and straight-forward.

The mean squared error in the estimate after incorporation of the measurement is formed by taking the trace of

$$\overline{e^+ e^{+T}}$$

as defined below.

$$\overline{e^+ e^{+T}} = \overline{(x - \overline{x}^+)(x - \overline{x}^+)^T}$$

$$\overline{e^+ e^{+T}} = \overline{(x - \overline{x^-})(x - \overline{x^-})^T} - K\overline{[z - \overline{H(x)}][x - \overline{x}^-]^T} \tag{4.64}$$

$$-\overline{[x - \overline{x}^-][z - \overline{H(x)}]^T}K^T + K\overline{[z - \overline{H(x)}][z - \overline{H(x)}]^T}K^T$$

Using Eq. (4.51) and the general form of the quasi-linear approximation for the expectation of any nonlinear measurement function, Eq. (4.64) can be rewritten as:

$$\overline{e^+e^{+T}} = \overline{e^-e^{-T}} - \overline{K[H(x) - N_m^{\bar{x}}\,\overline{x^-} + v]\,e^{-T}} - \overline{e^-[H(x) - N_m^{\bar{x}}\,x^- + v]^T K^T}$$

$$+ \overline{K[H(x) - N_m^{\bar{x}}\,\overline{x^-} + v][H(x) - N_m^{\bar{x}}\,\overline{x^-} + v]^T K^T} \tag{4.65}$$

where $N_m^{\bar{x}}$ is the measurement quasi-linear approximation matrix. Using the assumptions associated with the noise and some algebraic expansion, Eq. (4.65) becomes

$$\overline{e^+e^{+T}} = \overline{e^-e^{-T}} - K\overline{H(x)e^{-T}} - \overline{e^-H(x)^T}K^T + K\overline{H(x)H(x)^T}K^T$$

$$- K\overline{H(x)\bar{x}^{-T}}\,N_m^{\bar{x}T}K^T - KN_m^{\bar{x}}\,\overline{\bar{x}^-H(x)^T}K^T$$

$$+ KN_m^{\bar{x}}\,\overline{\bar{x}^-\bar{x}^{-T}}\,N_m^{\bar{x}T}K^T + KRK^T \tag{4.66}$$

To determine the optimum value of K requires that the trace of Eq. (4.44) be minimized with respect to K. This minimization is accomplished by defining K to be the sum of the optimum value, K^0 plus a small, but otherwise arbitrary variation, δK. The necessary and sufficient conditions for minimizing the trace of

$$\overline{e^+e^{+T}}$$

are: 1, the first variation of the trace of Eq. (4.66) is zero; and 2, the second variation is positive. It is convenient to first substitute the following expression into Eq. (4.66) and then to take the trace of the result.

$$K = K^0 + \delta K \tag{4.67}$$

Equation (4.66) becomes:

$$\overline{e^+ e^{+T}} = \overline{e^- e^{-T}} - \overline{e^- H(x)^T}\,(K^0 + \delta K)^T - (K^0 + \delta K)\,\overline{H(x)\, e^{-T}}$$

$$+ (K^0+\delta K)\,\overline{H(x)\, H(x)^T}\,(K^0+\delta K)^T - (K^0+\delta K)\,\overline{H(x)\, x^{-T}}\, N_m^{\bar{x}}{}^T (K^0+\delta K)^T$$

$$- (K^0+\delta K)\, N_m^{\bar{x}}\, \overline{x^- H(x)^T}\,(K^0+\delta K)^T + (K^0+\delta K)\, N_m^{\bar{x}}\, \overline{x^- x^{-T}}\,(K^0+\delta K)^T$$

$$+ (K^0 + \delta K)\, R\, (K^0 + \delta K)^T \tag{4.68}$$

The terms in Eq. (4.68) which contain the first variation are:

$$- \overline{e^- H(x)^T}\, \delta K^T - \delta K\, \overline{H(x)\, e^{-T}} + \delta K\, \overline{H(x) H(x)^T}\, K^{0T} + K^0\, \overline{H(x) H(x)^T}\, \delta K^T$$

$$+ \delta K\, N_m^{\bar{x}}\, \overline{x^- x^{-T}}\, N_m^{\bar{x}T}\, K^{0T} + K^0 N_m^{\bar{x}}\, \overline{x^- x^{-T}}\, N_m^{\bar{x}T}\, \delta K^T - K^0\, \overline{H(x)\, x^{-T}}\, N_m^{\bar{x}T}\, \delta K^T$$

$$- \delta K\, \overline{H(x)\, x^{-T}}\, N_m^{\bar{x}T}\, K^{0T} - K^0 N_m^{\bar{x}}\, \overline{x^- H(x)^T}\, \delta K^T - \delta K\, N_m^{\bar{x}}\, \overline{x^- H(x)^T}\, K^{0T}$$

$$+ K^0 R\, \delta K^T + \delta K\, R\, K^{0T}$$

Taking the trace of these terms and setting the resulting expression equal to zero

$$2\,\mathrm{tr}\left\{\delta K\left[-\overline{H(x)\tilde{e}^{T}}+\overline{H(x)H(x)^{T}}\,K^{0^{T}}+\bar{N}_{m}^{\tilde{x}}\,\overline{\tilde{x}\tilde{x}^{T}}\,\bar{N}_{m}^{\tilde{x}^{T}}K^{0^{T}}\right.\right.$$

$$\left.\left.-\overline{H(x)\tilde{x}^{T}}\,\bar{N}_{m}^{\tilde{x}^{T}}K^{0^{T}}-\bar{N}_{m}^{\tilde{x}}\,\overline{\tilde{x}H(x)^{T}}\,K^{0^{T}}+RK^{0^{T}}\right]\right\}=0$$

$$(4.69)$$

Since δK was defined to be an arbitrary variation of the gain factor K, Eq. (4.69) will be satisfied, in general, only if the term in brackets is identically zero.

$$\left[\overline{H(x)H(x)^{T}}+\bar{N}_{m}^{\tilde{x}}\,\overline{\tilde{x}\tilde{x}^{T}}\,\bar{N}_{m}^{\tilde{x}^{T}}-\overline{H(x)\tilde{x}^{T}}\,\bar{N}_{m}^{\tilde{x}^{T}}-\bar{N}_{m}^{\tilde{x}}\,\overline{\tilde{x}H(x)^{T}}+R\right]K^{0^{T}}$$

$$-\overline{H(x)\tilde{e}^{T}}=0 \qquad (4.70)$$

The equation which defines the optimum gain is therefore given by the following

$$K^{0}=\overline{\tilde{e}H(x)^{T}}\left[\overline{H(x)H(x)^{T}}+\bar{N}_{m}^{\tilde{x}}\,\overline{\tilde{x}\tilde{x}^{T}}\,\bar{N}_{m}^{\tilde{x}^{T}}-\overline{H(x)\tilde{x}^{T}}\,\bar{N}_{m}^{\tilde{x}^{T}}-\bar{N}_{m}^{\tilde{x}}\,\overline{\tilde{x}H(x)^{T}}+R\right]^{-1}$$

$$(4.71)$$

Returning to Eq. (4.68), an examination of the terms involving the second variation of K will provide, in conjunction with Eq. (4.71), both the necessary and sufficient conditions for an optimum gain. The terms of second variation are:

$$\delta K\,\overline{H(x)H(x)^T}\,\delta K^T - \delta K\,\overline{H(x)\bar{x}^T}\,N_m^{x\,T}\,\delta K^T$$

$$- \delta K N_m^{\bar{x}}\,\overline{\bar{x}\,H(x)^T}\,\delta K^T + \delta K N_m^{\bar{x}}\,\overline{\bar{x}\,\bar{x}^T}\,N_m^{\bar{x}\,T}\,\delta K^T + \delta K R\,\delta K^T \tag{4.72}$$

These terms can be rewritten in the following form

$$\delta K\left[\overline{H(x)H(x)^T} - \overline{H(x)\bar{x}^T}\,N_m^{\bar{x}\,T} - N_m^{\bar{x}}\,\overline{\bar{x}\,H(x)^T} + N_m^{\bar{x}}\,\overline{\bar{x}\,\bar{x}^T}\,N_m^{\bar{x}\,T} + R\right]\delta K^T$$

$$= \delta K\left[\overline{(H(x) - N_m^{\bar{x}}\bar{x})\,(H(x) - N_m^{\bar{x}}\bar{x})^T} + R\right]\delta K^T \tag{4.73}$$

The first term in the brackets on the right hand side of Eq. (4.73) is always a positive semidefinite matrix and R is always positive definite if there is noise in the measurement process as postulated. Therefore, the second variation terms of Eq. (4.68) are always positive and Eq. (4.71) is, indeed the equation which defines the value of K required to minimize the expected value of the squared error.

Having derived Eq. (4.71), it now becomes necessary to examine more closely the evaluation of each term and to discuss the use of K in a recursive estimation scheme. First, it is both interesting and reassuring to note that if H (x) is, in fact, a linear function (i.e., H (x) = h x) then the following simplifications also result.

$$N_m^{\bar{x}} = H$$

$$\overline{e^- H(x)^T} = \overline{e^- x^{-T}}\, H^T = \overline{e^- e^{-T}}\, H^T$$

$$\overline{e^- H(x)^T} = P^- H^T$$

Making the above substitutions into Eq. (4.71) results in Eq. (4.74)

$$K^0 = P^- H^T \left[H \overline{x^- x^{-T}} H^T - H \bar{x}^- \bar{x}^{-T} H^T + R \right]^{-1}$$

$$K^0 = P^- H^T \left[H P^- H^T + R \right]^{-1} \tag{4.74}$$

The above expression is identical to that obtained in several other ways for the now well-known linear filtering problem.

In the general nonlinear case, Eq. (4.71) can be further simplified by noting that the quasi-linear approximation provides the following relationships.

$$\overline{H(x)} \approx N_m^{\bar{x}}\, \bar{x}^-$$

$$H(x) \approx N_m^{\bar{x}}\, \bar{x}^- + N_m^{e}\, e^- \tag{4.75}$$

The above Eq. can be used to evaluate the terms in the expression for K^0.

$$\overline{e^{-}H(x)^{T}} = \overline{e^{-}\bar{x}^{-T}}\,\bar{N}_m^{xT} + \overline{e^{-}e^{-T}}\,N_m^{eT} = P^{-}N_m^{eT} \tag{4.76}$$

$$\begin{aligned}\overline{H(x)H(x)^{T}} &= \overline{(\bar{N}_m^{x}\bar{x}^{-} + N_m^{e}e^{-})(\bar{N}_m^{x}\bar{x}^{-} + N_m^{e}e^{-})^{T}} \\ &= \bar{N}_m^{x}\,\overline{\bar{x}^{-}\bar{x}^{-T}}\,\bar{N}_m^{xT} + N_m^{e}P^{-}N_m^{eT}\end{aligned} \tag{4.77}$$

$$\overline{H(x)\bar{x}^{-T}}\,\bar{N}_m^{xT} = \bar{N}_m^{x}\,\overline{\bar{x}^{-}\bar{x}^{-T}}\,\bar{N}_m^{xT} \tag{4.78}$$

$$\bar{N}_m^{x}\,\overline{\bar{x}^{-}H(x)^{T}} = \bar{N}_m^{x}\,\overline{\bar{x}^{-}\bar{x}^{-T}}\,\bar{N}_m^{xT} \tag{4.79}$$

Using these two equations in Eq. (4.71) provides the following equation for the optimum gain.

$$K^{0} = P^{-}N_m^{eT}\,[\,N_m^{e}P^{-}N_m^{eT} + R\,]^{-1} \tag{4.80}$$

Equation (4.80) provides an expression for K^0 which is of the same form as Eq. (4.74) for a linear measurement except that the linear geometry matrix, H is replaced by a quasi-linear matrix N_m^e which is generally a function of the first and second order statistics of the state vector.

The update equation for the covariance matrix given the optimum gain factor, K^0 can be found by returning to Eq. (4.68) and examining the terms of zeroth order in δK.

$$\overline{e^+ e^{+T}} = \overline{e^- e^{-T}} - \overline{e^- H(x)^T}\, K^{0T} - K^0\, \overline{H(x)\, e^{-T}} + K^0\, \overline{H(x) H(x)^T}\, K^{0T}$$

$$- K^0\, \overline{H(x)\, \bar{x}^{-T}}\, N_m^{\bar{x}T} K^{0T} - K^0 N_m^{\bar{x}}\, \overline{\bar{x}^- H(x)^T}\, K^{0T}$$

$$+ K^0 N_m^{\bar{x}}\, \overline{\bar{x}^- \bar{x}^{-T}}\, N_m^{\bar{x}T} K^{0T} + K^0 R K^{0T} \qquad (4.81)$$

As with Eq. (4.80), $\overline{e^- H(x)^T}$ can be replaced by $P^- N_m^{eT}$ and

$$\overline{H(x) H(x)^T}$$

by

$$N_m^{\bar{x}}\, \overline{\bar{x}^- \bar{x}^{-T}}\, N_m^{\bar{x}T} + N_m^e P^- N_m^{eT}$$

resulting in

$$P^+ = P^- - P^- N_m^{eT} K^{0T} - K^0 N_m^e P^- + K^0 N_m^{\bar{x}}\, \overline{\bar{x}^- \bar{x}^{-T}}\, N_m^{\bar{x}T} K^{0T} + K^0 N_m^e P^- N_m^{eT} K^{0T}$$

$$- K^0 N_m^{\bar{x}}\, \overline{\bar{x}^- \bar{x}^{-T}}\, N_m^{xT} K^{0T} - K^0 N_m^{\bar{x}} K^{0T} + K^0 N_m^{\bar{x}}\, \overline{\bar{x}^- \bar{x}^{-T}}\, N_m^{\bar{x}T} K^{0T}$$

$$+ K^0 R K^{0T} \qquad (4.82)$$

As can readily be seen, the terms above which do not cancel may be combined to give the following equation.

$$P^{+} = P^{-} - P^{-} N_m^{e^T} K^{0^T} - K N_m^e P^{-} + K^0 [N_m^e P^{-} N_m^{e^T} + R] K^{0^T} \tag{4.83}$$

Substituting the expression for K^0 given by Eq. (4.80) into the above expression and noting the symmetry of the bracketed term results in the update equation for the covariance matrix P.

$$\begin{aligned} P^{+} = P^{-} & - P^{-} N_m^{e^T} [N_m^e P^{-} N_m^{e^T} + R]^{-1} N_m^e P^{-} \\ & - P^{-} N_m^{e^T} [N_m^e P^{-} N_m^{e^T} + R]^{-1} N_m^e P^{-} \\ & + P^{-} N_m^{e^T} [N_m^e P^{-} N_m^{e^T} + R]^{-1} [N_m^e P^{-} N_m^{e^T} + R] \times \\ & [N_m^e P^{-} N_m^{e^T} + R]^{-1} N_m^e P^{-} \end{aligned}$$

$$P^{+} = P^{-} - P^{-} N_m^{e^T} [N_m^e P^{-} N_m^{e^T} + R]^{-1} N_m^e P^{-} \tag{4.84}$$

Once again, the form of the resulting expression is similar to that found in linear estimation except for the appearance of N_m^e instead of H.

Summarizing, for the assumed continuous system with discrete measurements the following equations provide a recursive estimate of the state vector x.

Between measurements (Extrapolation):

$$\dot{\bar{x}} = N_f^{\bar{x}} \bar{x}$$

$$\dot{P} = N_f^e P + P N_f^{e^T} + G Q G^T \tag{4.85}$$

At a measurement point (Updating):

$$\bar{x}^+ = \bar{x}^- + P^- N_m^{e^T} [N_m^e P^- N_m^{e^T} + R]^{-1} [z - N_m^{\bar{x}} \bar{x}^-]$$

$$P^+ = P^- - P^- N_m^{e^T} [N_m^e P^- N_m^{e^T} + R]^{-1} N_m^e P^- \tag{4.86}$$

It is interesting to note that the complicated looking expressions for $N_f^{\bar{x}}$ and $N_m^{\bar{x}}$ need not actually be computed for the general case since they both appear multiplied by $\bar{x}$.

$$N_f^{\bar{x}} \bar{x} = \overline{f(x)}$$

$$N_m^{\bar{x}} \bar{x} = \overline{H(x)}$$

Also, N_f^e appears only in the form

$$N_f^e P = \overline{f(x) e^T}$$

Only N_m^e need be evaluated explicitly for use in the measurement incorporation equations.

As pointed out earlier the evaluation of $\overline{H(x)}$ can be based upon either of two different probability density functions. The results summarized above are valid if $\overline{H(x)}$ is computed with the density function of x prior to the measurement. Or one might use the more complicated procedure in some cases.

Finally, it is interesting to note that the basic character of a nonlinear operation, namely, that behavior depends on the amplitude of the input, is reflected in the above results. The estimation equations are related to a series of gains which depend, in general, on the mean value and covariance matrix of x and hence the name quasi-linear.

4.5 Some Illustrations of Quasi-Linear Approximation

A demonstration of the techniques described above is presented here for some simple cases in order to provide some insight into the use of the quasi-linear estimation Eqs. and also to illustrate the procedure involved in formulating a set of recursive Eqs. In order to better illustrate the nature of the approximation, the discussion here will be limited only to nonlinear functions for which an analytical solution for the quasi-linear approximation is possible.

4.5.1 A Scalar Quadratic Nonlinearity

Given the following differential Eq. for the dynamics of a scalar state

$$\dot{x} = ax + bx^2 \tag{4.87}$$

the first task is to find the quasi-linear approximation to this nonlinear function. Equation (4.28) specialized to this scalar case is given below

$$N_f^e = \frac{\overline{c(x)x} - \overline{c(x)}\,\overline{x}}{P} \tag{4.88}$$

and Eq. (4.29) becomes

$$N_f^{\overline{x}} = \frac{\overline{c(x)}}{\overline{x}}$$

where $\overline{x}$ is the state estimate, P is the mean square error in the estimate and

$$\overline{c(x)} = a\overline{x} + b\overline{x^2}$$

$$\overline{c(x)x} = a\overline{x^2} + b\overline{x^3} \tag{4.89}$$

To find the expectation of c given that x is a bias plus a zero mean Gaussian random variable, the following integral must be evaluated.

$$\bar{c} = \int_{-\infty}^{\infty} (ax + bx^2)\, p(x)\, dx$$

$$\bar{c} = \int_{-\infty}^{\infty} [\, a(\bar{x} + r) + b(\bar{x} + r)^2 p(r)\, dr\,] \tag{4.90}$$

Using a Normal probability density function for

$$\bar{c} = \frac{1}{\sqrt{2\pi}\, P^{1/2}} \int_{-\infty}^{\infty} [\, a(\bar{x} + r) + b(\bar{x} + r)^2\,]\, e^{-\frac{r^2}{2P}}\, dr$$

$$\bar{c} = \frac{1}{\sqrt{2\pi P}} \left\{ \int_{-\infty}^{\infty} (a\bar{x} + b\bar{x}^{2})\, e^{-\frac{r^2}{2P}}\, dr + \int_{-\infty}^{\infty} (a + 2b\bar{x})\, r\, e^{-\frac{r^2}{2P}}\, dr + \int_{-\infty}^{\infty} b r^2 e^{-\frac{r^2}{2P}}\, dr \right. \tag{4.91}$$

Each of the above three integrals is easily evaluated resulting in Eq. (4.64)

$$\bar{c} = a\bar{x} + b\bar{x}^2 + bP \tag{4.92}$$

In a similar manner $\overline{cx}$ is found to be

$$\overline{cx} = a(\bar{x}^2 + P) + b(\bar{x}^3 + 3\bar{x}P) \tag{4.93}$$

Thus, the quasi-linear approximation matrices are given by

$$\begin{aligned} N_f^{\bar{x}} &= a + b\bar{x} + b\frac{P}{x} \\ N_f^{e} &= a + 2b\bar{x} \end{aligned} \tag{4.94}$$

and in this simple case, the differential Eqs. required for the estimate are

$$\begin{aligned} \dot{\bar{x}} &= N_f^{\bar{x}}\bar{x} = a\bar{x} + b\bar{x}^2 + bP \\ \dot{P} &= 2N_f^{e}P = 2(aP + 2b\bar{x}P) \end{aligned} \tag{4.95}$$

It is interesting at this point to compare the quasi-linear estimation given by Eq. (4. 95) and the analogous expression which results from the Taylor Series expansion techniques described in Chapter III. For this simple scalar nonlinear function, the Taylor Series up to and including second order terms would be of the form.

$$\dot{\bar{x}} = \overline{f(x)} \approx f(\bar{x}) + 1/2 \frac{\partial^2 f(\bar{x})}{\partial x^2} P \tag{4.96}$$

Substituting for f (x), the nonlinear function of Eq. (4. 87), the above equation becomes.

$$\dot{\bar{x}} = a\bar{x} + b\bar{x}^2 + 1/2\,(2b)P$$

also (4. 97)

$$\dot{P} = 2\frac{\partial f(\bar{x})}{\partial x} P = 2\,(aP + 2b\bar{x}P)$$

In this case the Taylor Series estimation procedure results in precisely the same estimation equations. If the series expansion is restricted to only second order terms and also if the nonlinearity is quadratic in form, these two techniques will always agree.

The difference between the two approximation techniques will manifest itself in different differential equations for the state estimate as well as the variance for certain other types of nonlinearities as demonstrated in the following example.

4. 5. 2 A Trigonometic Nonlinearity

If the following equation represents the dynamics of some system

$$\dot{x} = \sin x \tag{4.98}$$

then the quasi-linear estimation equation is found in the following way. The expectation of the output of the nonlinear function is given by

$$c = \frac{1}{\sqrt{2\pi P}} \int_{-\infty}^{\infty} \sin(\bar{x} + r)\, e^{-\frac{r^2}{2P}}\, dr$$

$$\bar{c} = \frac{}{\sqrt{2\pi P}} \left[\int_{-\infty}^{\infty} \sin \bar{x} \cos r\, e^{-\frac{r^2}{2P}}\, dr + \int_{-\infty}^{\infty} \cos \bar{x} \sin r\, e^{-\frac{r^2}{2P}}\, dr \right]$$

(4.99)

The integrand in the second term on the right hand side of Eq. (4.99) is an odd function of r and therefore when it is integrated over the interval $-\infty \leq r \leq \infty$ the result is equal to zero. The first term may be integrated as follows.

$$\bar{c} = \frac{1}{\sqrt{2\pi P}} \sin \bar{x} \int_{-\infty}^{\infty} \cos r\, e^{-\frac{r^2}{2P}}\, dr$$

(4.100)

$$\bar{c} = \sin \bar{x}\, e^{-P/2}$$

Thus, the quasi-linear gain used to approximate the nonlinear function for the state estimation is given by Eq. (4.101)

$$N_f^{\bar{x}} = \frac{\sin \bar{x}}{\bar{x}} e^{-P/2} \tag{4.101}$$

and the differential Eq. for the estimate is

$$\dot{\bar{x}} = \sin \bar{x}\, e^{-P/2} \tag{4.102}$$

Using a Taylor Series expansion to approximate this same non-linear system results in the following

$$\dot{\bar{x}} = \sin \bar{x} + 1/2\,(-\sin \bar{x})\, P$$

or (4.103)

$$\dot{\bar{x}} = \sin \bar{x}\,(1 - 1/2\, P)$$

Returning to Eq. (4.102) and expanding the exponential function in a series results in

$$\dot{\bar{x}} = \sin \bar{x}\,(1 - 1/2\, P + 1/8\, P^2 - \ldots) \tag{4.104}$$

Thus the Taylor Series provides the same information as the first two terms in the expansion of Eq. 4.102. This result is clearly a demonstration of the difference between a second order series expansion and the assumption of a Gaussian distribution function. Indeed, if the fourth order moment term of the Taylor Series were computed, it would be $1/8\, P^2$. Each additional term in the series is considerably more difficult to compute than the previous one and, therefore, it is generally not desirable to include terms of order higher than two. The equations for $\dot{P}$ are given below. For the quasi-linear estimator

$$N_f^e = \cos(\bar{x})\, e^{-P/2}$$

$$\dot{P} = 2P\cos(\bar{x})\, e^{-P/2} \qquad (4.105)$$

For the series estimator

$$\frac{\partial f(\bar{x})}{\partial x} = \cos(\bar{x})$$

$$\dot{P} = 2P\cos(\bar{x}) \qquad (4.106)$$

In this case only the first term in the series for $e^{-P/2}$ is included in the series estimator equation.

4.5.3 An Exponential Nonlinearity

For a scalar system with the following differential equation

$$\dot{x} = K e^{x} \qquad (4.107)$$

the quasi-linear approximation matrices are found in the following way.

$$N_f^e = \frac{K}{P}\left(\overline{x e^{x}} - \bar{x}\,\overline{e^{x}}\right) = \frac{K}{P}\, e^{\bar{x}}\, \overline{r e^{r}} \qquad (4.108)$$

The expectation above can be rewritten as

$$\overline{re^r} = \frac{1}{\sqrt{2\pi P}} \int_{-\infty}^{\infty} r\, e^r s^{-\frac{r^2}{2P}}\, dr$$

$$\overline{re^r} = \frac{1}{\sqrt{2\pi P}} \int_{-\infty}^{\infty} r\, e^{-\left(\frac{r^2}{2P} - r\right)}\, dr \tag{4.109}$$

The exponent appearing in Eq. (4.109) can be redefined in terms of a perfect square as follows

$$\frac{r^2}{2P} - r = y^2 - P/2 \tag{4.110}$$

where

$$y = \frac{1}{\sqrt{2}} \left(\frac{r - P}{\sqrt{P}}\right)$$

and

$$dr = \sqrt{2P}\, dy$$

Equation (4.109) then becomes

$$\overline{re^r} = \frac{1}{\sqrt{2\pi P}} \int_{-\infty}^{\infty} (\sqrt{2P}\, y + P)\, e^{P/2}\, e^{-y^2} \sqrt{2P}\, dy$$

$$\overline{re^r} = P e^{P/2} \tag{4.111}$$

and Eq. (4.108) is the following

$$N_f^e = \frac{K}{P}\, e^{\bar{x}}\, P e^{P/2} = K e^{(\bar{x} + P/2)} \tag{4.112}$$

The quasi-linear gain to the bias component is found from

$$N_f^{\bar{x}} = \frac{\overline{K e^x}}{\bar{x}} \tag{4.113}$$

Using a procedure analogous to that above, the expectation in Eq. (4.113) is clearly

$$\overline{e^x} = e^{\bar{x} + P/2} \tag{4.114}$$

Thus,

$$N_f^{\bar{x}} = \frac{K e^{\bar{x} + P/2}}{\bar{x}} \tag{4.115}$$

The extrapolation equations for this estimate then become

$$\dot{\bar{x}} = N_f^{\bar{x}} \bar{x} = K e^{\bar{x} + P/2}$$

$$\dot{P} = 2 N_f^{e} P = 2 K e^{\bar{x} + P/2} P \tag{4.116}$$

Again by way of comparison, the corresponding estimation equations for a series estimator of second order are the following:

$$\dot{\bar{x}} = K e^{\bar{x}} + 1/2\, K\, (e^{\bar{x}})\, P = K e^{\bar{x}} (1 + 1/2\, P) \tag{4.117}$$

$$\dot{P} = 2 \frac{\partial f(\bar{x})}{\partial x} P = 2 K e^{\bar{x}} P \tag{4.118}$$

One further example will serve to illustrate the estension of both approximation techniques to a simple case with a two dimensional state vector and one which is not always amenable to linearization techniques.

4.5.4 A System Identification Problem

Consider a system the dynamics of which can be described by the following differential equation

$$\dot{Y} = A Y \tag{4.119}$$

where A is a constant but unknown parameter. It is possible, in this instance, to define a two dimensional state vector in the following way

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} Y \\ A \end{bmatrix} \tag{4.120}$$

The differential equation for this vector is

$$\dot{x} = f(x) = \begin{bmatrix} x_1 x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 & 0 \\ 0 & 0 \end{bmatrix} x \tag{4.121}$$

Thus, the parameter A as well as the original state variable Y are combined in a single state vector differential equation, and both variables can be estimated jointly.

The quasi-linear matrix which represents the gain to the error may be found in the following way.

$$N_f^e = \overline{f(x)x^T} - \overline{f(x)}\,\bar{x}^T \; P^{-1}$$

$$N_f^e = \begin{bmatrix} x_1^2 x_2 - x_1 x_2 x_1 & x_1 x_2^2 - x_1 x_2 x_2 \\ 0 & 0 \end{bmatrix} P^{-1} \tag{4.122}$$

The expectations in Eq. (4.122) are evaluated in the usual way with the following result

$$\overline{x_1 x_2} = \bar{x}_1 \bar{x}_2 + P_{12}$$

$$\overline{x_1^2 x_2} = \bar{x}_1^2 \bar{x}_2 + \bar{x}_2 P_{11} + 2 \bar{x}_1 P_{21} \tag{4.123}$$

$$\overline{x_1 x_2^2} = \bar{x}_1 \bar{x}_2^2 + \bar{x}_1 P_{22} + 2 \bar{x}_2 P_{12}$$

Thus

$$N_f^e = \begin{bmatrix} \bar{x}_2 P_{11} + \bar{x}_1 P_{21} & \bar{x}_1 P_{22} + \bar{x}_2 P_{12} \\ 0 & 0 \end{bmatrix} P^{-1} \tag{4.124}$$

In order to compute the gain to the bias component, the following expressions are needed

$$U = \frac{\overline{f(x)}\,\bar{x}^T}{\bar{x}^T \bar{x}} = \frac{1}{\bar{x}_1^2 + \bar{x}_2^2} \begin{bmatrix} (\bar{x}_1 \bar{x}_2 + P_{12}) \bar{x}_1 & (\bar{x}_1 \bar{x}_2 + P_{12}) \bar{x}_2 \\ 0 & 0 \end{bmatrix} \tag{4.125}$$

$$V = \left[I - \frac{\bar{x}\bar{x}^T}{\bar{x}^T\bar{x}} \right] = \begin{bmatrix} 1 - \dfrac{\bar{x}_2^{\,2}}{\bar{x}_1^{\,2} + \bar{x}_2^{\,2}} & - \dfrac{\bar{x}_1 \bar{x}_2}{\bar{x}_1^{\,2} + \bar{x}_2^{\,2}} \\ - \dfrac{\bar{x}_2 \bar{x}_1}{\bar{x}_1^{\,2} + \bar{x}_2^{\,2}} & 1 - \dfrac{\bar{x}_1^{\,2}}{\bar{x}_1^{\,2} + \bar{x}_2^{\,2}} \end{bmatrix} \tag{4.126}$$

Equations (4.124), (4.125) and (4.126) provide the required expressions for the computation of $N_f^{\bar{x}}$ where

$$N_f^{\bar{x}} = U + N_f^e \, V \tag{4.127}$$

Again, as an interesting comparison, a Taylor Series expansion may be used for this type of nonlinearity in the following way. In two dimensions, the series approximation is of the form

$$\overline{f_i(x)} \approx f_i(\bar{x}) + 1/2 \sum_{j=1}^{2} \sum_{k=1}^{2} \frac{\partial^2 f_i(x)}{\partial x_j \partial x_k} P_{jk} \tag{4.128}$$

For this problem, the following expressions follow from Eq. (4.128)

$$\overline{f_1(x)} \approx \bar{x}_1 \bar{x}_2 + 1/2 \, (P_{12} + P_{21}) = \bar{x}_1 \bar{x}_2 + P_{12}$$

$$\overline{f_2(x)} \approx 0 \tag{4.129}$$

Using these results, the differential equations for the series estimator are found to be

$$\dot{\bar{x}}_1 = \overline{f_1(x)} = \bar{x}_1\bar{x}_2 + P_{12}$$

$$\dot{\bar{x}}_2 = \overline{f_2(x)} = 0 \tag{4.130}$$

and

$$\frac{\partial f(\bar{x})}{\partial x} = \begin{bmatrix} \bar{x}_2 & \bar{x}_1 \\ 0 & 0 \end{bmatrix}$$

$$\dot{P} = \begin{bmatrix} \bar{x}_2 & \bar{x}_1 \\ 0 & 0 \end{bmatrix} P + P \begin{bmatrix} \bar{x}_2 & 0 \\ \bar{x}_1 & 0 \end{bmatrix} \tag{4.131}$$

$$\dot{P}_{11} = 2\,(P_{11}\bar{x}_2 + P_{12}\bar{x}_1)$$

$$\dot{P}_{12} = P_{12}\bar{x}_2 + P_{22}\bar{x}_1$$

$$\dot{P}_{21} = P_{21}\bar{x}_2 + P_{22}\bar{x}_1 \tag{4.132}$$

$$\dot{P}_{22} = 0$$

In a similar way, the quasi-linear estimation equations can be found from the above results

$$\dot{\bar{x}}_1 = \bar{x}_1 \bar{x}_2 + P_{12}$$
$$\dot{\bar{x}}_2 = 0 \tag{4.133}$$

and

$$N_f^e P = \begin{bmatrix} \bar{x}_2 P_{11} + \bar{x}_1 P_{21} & \bar{x}_1 P_{22} + \bar{x}_2 P_{12} \\ 0 & 0 \end{bmatrix} \tag{4.134}$$

Using

$$\dot{P} = N_f^e P + P N_f^{eT}$$

then

$$\dot{P}_{11} = 2\,(\bar{x}_2 P_{11} + \bar{x}_1 P_{21})$$
$$\dot{P}_{12} = \bar{x}_1 P_{22} + \bar{x}_2 P_{12}$$
$$\dot{P}_{21} = \bar{x}_1 P_{22} + \bar{x}_2 P_{21} \tag{4.135}$$
$$\dot{P}_{22} = 0$$

The series estimator described by Eqs. (4.130) and (4.132) is identical to that found using the quasi-linear technique, Eqs. (4.133) and (4.135). In this case, the two techniques are the same because the nonlinear function is quadratic in nature. As was shown in Sections 4.5.2 and 4.5.3, the two estimators will significantly differ for other types of nonlinear functions. For polynomial nonlinearities of orders higher than quadratic, the resulting equations are also different even if the series expansion is limited to only first and second order statistics as in the case of the quasi-linear estimator.

Summarizing, there are three important factors concerning the two approximation methods discussed in Chapter III and in this chapter.

1) There is a basic difference in the underlying assumptions. In one case, the probability density function of the state is assumed to be Gaussian while in the other case, the ensemble expection of any function of the state is described by a Taylor Series up to and including second order terms. Although higher order terms may be included, the increased computational complexity as well as the computer storage requirements become rapidly impractical.

2) As shown above, quadratic nonlinearities result in the two approaches producing the same approximate estimation equations. However, for some types of nonlinearities, the quasi-linearization technique contains more information about the system dynamics. This is a result of the basic difference in assumptions discussed above.

3) For nonlinearities with nonanalytic derivatives, only the quasi-linear technique can be used. An example of this, a quantizer, was discussed in Section 4.4.

Further discussion of the differences between series estimation techniques and the quasi-linear estimator (Chapter V) as well as a demonstration of some of the features of the later (Chapter VII) can be found in the following chapters.

CHAPTER V

A COMPARISON OF THREE ESTIMATION TECHNIQUES

5.1 Introduction and Problem Definition

The purpose of this chapter is that of comparing three basically different approaches to solving an estimation problem which is non-linear. A series expansion technique, the subject of Chapter III, will be demonstrated with expansions up to 2nd, 3rd and 4th order. These three estimators coupled with the quasi-linear estimation scheme discussed in Chapter IV and the familiar linear estimator will provide five different systems for comparison. The comparison will be made on three levels. First, the development of recursive estimation equations will be compared with emphasis on the required assumptions and necessary analytic complexity. Second, the preparation of a digital computer program and problems of a numerical nature which must be delt with are discussed. Finally the three estimation schemes, which result in five different estimators, will be compared with respect to performance.

As a vehicle for all of the above objectives, a simple estimation problem has been selected which is complex enough to reveal the salient differences among the five estimators and yet simple enough to make the algebraic manipulations required for the series expansion procedure tractible. The system dynamics can be described by the following differential equation.

$$\dot{x} = -\sin(x) + u \tag{5.1}$$

where x is the state variable and u is a white noise process independent of x and defined by Eq. (5.2)

$$\overline{u(\tau_1)\,u(\tau_2)} = Q\,\delta\,(\tau_2 - \tau_1) \tag{5.2}$$

Here, as in previous chapters, the bar over a quantity is used to represent an ensemble expectation and the δ, as used above, is a Dirac delta function.

Equation (5.1) with the driving noise, u set equal to zero can be solved analytically as follows.

$$\frac{dx}{dt} = -\sin(x)$$

$$\int_{x_0}^{x} \csc(x)\,dx = -\int_0^t dt$$

$$\ln\tan\left(\frac{x}{2}\right)\Bigg|_{x_0}^{x} = -t \tag{5.3}$$

$$\tan\left(\frac{x}{2}\right) = \tan\left(\frac{x_0}{2}\right)e^{-t}$$

$$x = 2\tan^{-1}\left[\tan\left(\frac{x_0}{2}\right)e^{-t}\right]$$

Figure 5-1 shows the behavior of Eq. (5.3) for various initial conditions.

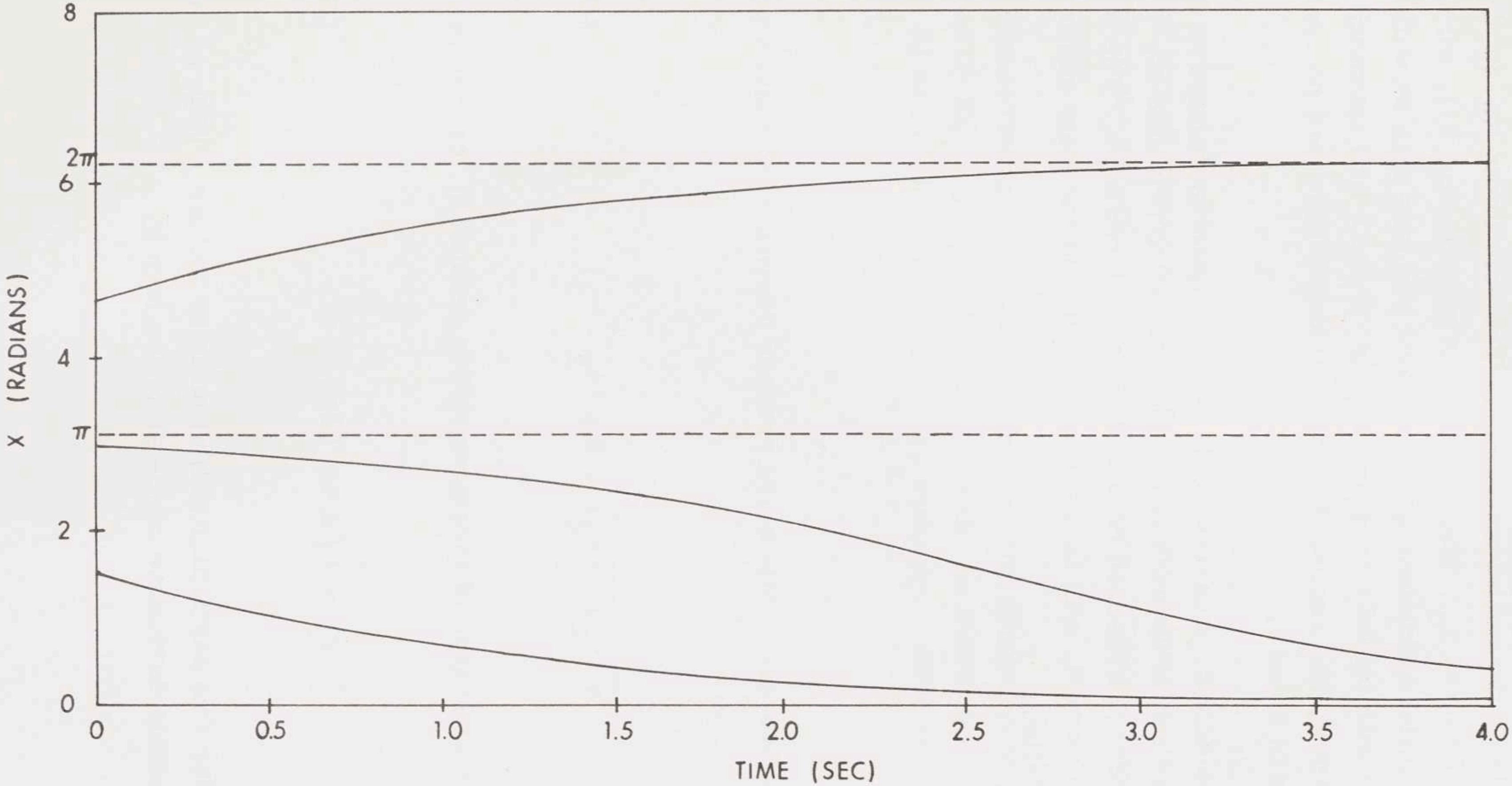


Figure 5-1 System Dynamics

As can be seen from Eq. (5.3), the asymptotic behavior of the state depends on the initial conditions. The solution to Eq. (5.3) as $t \to \infty$ is a function of x_0. Thus, with reference to Fig. 5-1, all states which have initial conditions greater than π and less than 2π will approach 2π and those with initial conditions between 0 and π will approach 0, etc. For convenience, the region of study considered here will be restricted to that between 0 and π.

Periodically, measurements of some quantity related to the state will be used to update each of the estimation procedures. Two different types of measurements will be considered here. The first, a linear measurement, will be a direct measure of the state and is expressed as follows

$$z_1 = x + v \tag{5.4}$$

where v is a Gaussian white noise process dependent of x and u and is defined by Eq. (5.5)

$$\overline{v(\tau_1)\, v(\tau_2)} = R\,\delta(\tau_2 - \tau_1) \tag{5.5}$$

The second type of measurement is defined by the following equation

$$z_2 = 1/2 \sin(2x) + v \tag{5.6}$$

The reason for this particular definition is as follows. With no driving noise, the second derivative of the state is given by

$$\ddot{x} = \frac{d}{dt} [- \sin (x)] = \cos (x) \sin (x) \tag{5.7}$$

$$\ddot{x} = 1/2 \sin (2x) \tag{5.8}$$

Thus, the second measurement type, a nonlinear function of the state, is related to the second derivative of the state, if there is no system driving noise.

Summarizing, a continuous system, the dynamics of which are expressed by Eq. (5.1), along with discrete measurements given by Eqs. (5.4) and (5.6) will be examined with three different estimation schemes.

5.2 Linear Estimation

The simplest and most familiar approach to state estimation is that of linearizing all the system equations and using linear estimation techniques, the Kalman filter. The technique of linearization which is generally used, defines a reference state, x_R and considers the problem of estimating deviations from this reference, defined by

$$\delta x = x - x_R \tag{5.9}$$

The success of this technique depends on the accuracy of the assumption that the deviation state remains small enough to allow a linear approximation to the dynamics of δx. In order to help insure success, it is possible to periodically redefine the reference to be the most recent estimate. This can be done as often as every measurement or at any frequency less than that depending on the particular system under consideration. This process, often called retification, will result in zero initial conditions for the differential equation of δx. In order to make an honest comparison of linear and nonlinear estimation, the above procedure will be employed and also a comparison made to a linear estimator with no retification.

From the definition of δx (Eq. (5.9)) and using Eq. (5.1), the following procedure can be used to find the differential equation for the reference state and the deviation state.

$$\dot{x} = \dot{x}_R + \dot{\delta x} = -\sin(x_R + \delta x) + u$$

$$\dot{x}_R + \dot{\delta x} = -[\sin(x_R)\cos(\delta x) + \cos(x_R)\sin(\delta x)] + u \tag{5.10}$$

Using the assumption that δx is small

$$\dot{x}_R + \dot{\delta x} = -\sin(x_R) - \cos(x_R)\,\delta x + u \tag{5.11}$$

By defining the reference state to be

$$\dot{x}_R = -\sin(x_R) \tag{5.12}$$

the deviation state dynamics are given by

$$\delta\dot{x} = -\cos(x_R)\;\delta x + u \tag{5.13}$$

The linear estimation procedure now involves the integration of Eq. (5.12) and the integration of the expectation of Eq. (5.13), given below

$$\dot{\overline{\delta x}} = -\cos(x_R)\,\overline{\delta x} \tag{5.14}$$

At any time, the linear state estimate is found as the sum of the reference state plus the deviation estimate.

$$\bar{x}_L = x_R + \overline{\delta x} \tag{5.15}$$

In order to properly consider measurement updates, it is also necessary to know a quantity which represents the mean squared error in the estimate given by Eq. (5.16).

$$P = \overline{(\delta x - \overline{\delta x})^2} \quad . \tag{5.16}$$

where

$$\delta x = x - x_R$$

$$\overline{\delta x} = \bar{x}_L - x_R$$

The differential equation for P is easily shown to be

$$\dot{P} = -2 \cos (x_R) P + Q \tag{5.17}$$

Using the same linearization procedure for the measurement updates results in the following equations. Defining a measurement deviation as follows:

$$\begin{aligned} \delta z &= z - z_R \\ \overline{\delta z} &= \bar{z}_L - z_R \end{aligned} \tag{5.18}$$

The linear measurement is incorporated in the following way:

$$\overline{\delta z_1} = \overline{\delta x}$$

$$\overline{\delta x}^+ = \overline{\delta x}^- + B \; [\delta z_1 - \overline{\delta z_1}] \tag{5.19}$$

$$B = \frac{P^-}{P^- + R} \tag{5.20}$$

$$P^+ = P^- - BP^- = (1 - B)^2 P^- + B^2 R \tag{5.21}$$

In the case of the second type of measurement (Eq. (5.6)) the equations which correspond to Eqs. (5.19) - (5.21) are the following:

$$\overline{\delta z_2} = \cos (2x_R)\, \overline{\delta x}$$

$$\overline{\delta x}^+ = \overline{\delta x}^- + B\, [\,\delta z_2 - \overline{\delta z}_2\,] \tag{5.22}$$

$$P^+ = [1 - \cos (2x_R)\, B]^2\, P^- + B^2\, P \tag{5.23}$$

where

$$B = \frac{P^- \cos (2x_R)}{P^- \cos^2 (2x_R) + R} \tag{5.24}$$

Equation (5.23) may also be written in the following form

$$P^+ = P^- - \cos (2x_R)\, B\, P^- \tag{5.25}$$

as in Eq. (5.21). However, the form of (5.23) is preferred in order to insure that P be a positive number as is dictated by its physical definition.

After a measurement update, the optimum linear estimate is once again found with the aid of Eq. (5.15).

5.3 Nonlinear Estimation Using Series Expansion Techniques

The series expansion procedure which was discussed in Chapter III will first be specialized to a scalar state with terms up to and including the fourth central moment. Then the equations appropriate to this particular example can be found as a special case of the scalar estimation equations.

The differential equations of the first four central moment, used to extrapolate the series estimator between measurements are found as follows. The nonlinear function

$$\dot{\bar{x}} = \overline{f(x)} \tag{5.26}$$

is expanded in the following Taylor Series

$$\overline{f(x)} = f(\bar{x}) + \frac{1}{2}\frac{\partial^2 f(\bar{x})}{\partial x^2} P + \frac{1}{6}\frac{\partial^3 f(\bar{x})}{\partial x^3} T + \frac{1}{24}\frac{\partial^4 f(\bar{x})}{\partial x^4} S \tag{5.27}$$

In Eq. (5.27), the third and fourth order central moment terms are designated by T and S, respectively. They have the following definitions for this scalar case.

$$\begin{aligned} T &= \overline{(x - \bar{x})^3} \\ S &= \overline{(x - \bar{x})^4} \end{aligned} \tag{5.28}$$

Using Eq. (5.27) in Eq. (5.26) provides the desired extrapolation equation for the first moment, $\bar{x}$. Similarly

$$\dot{P} = 2\,\overline{(x - \bar{x})\,(\dot{x} - \dot{\bar{x}})} = 2\,\overline{[\,x - \bar{x}\,]\;[\,f(x) - \overline{f(x)} + u\,]} \qquad (5.29)$$

$$\dot{P} = 2\,[\,\overline{x\,f(x)} - \bar{x}\,\overline{f(x)} + \overline{xu}\,] \qquad (5.30)$$

where

$$\overline{x\,f(x)} = \bar{x}\,f(\bar{x}) + 1/2\,[\,\bar{x}\,\frac{\partial^2 f(\bar{x})}{\partial x^2} + 2\,\frac{\partial f(\bar{x})}{\partial x}\,]\;P$$

$$+ \frac{1}{6}\,[\,\bar{x}\,\frac{\partial^3 f(\bar{x})}{\partial x^3} + 3\,\frac{\partial^2 f(\bar{x})}{\partial x}\,]\;T + \frac{1}{24}\,[\,\bar{x}\,\frac{\partial^4 f(\bar{x})}{\partial x^4} + 4\,\frac{\partial^3 f(\bar{x})}{\partial x^3}\,]\;S$$

$$(5.31)$$

and

$$\bar{x}\,\overline{f(x)} = \bar{x}\,f(\bar{x}) + \frac{1}{2}\bar{x}\,\frac{\partial^2 f(\bar{x})}{\partial x^2}\,P + \frac{1}{6}\bar{x}\,\frac{\partial^3 f(\bar{x})}{\partial x^3}\,T + \frac{1}{24}\,\bar{x}\,\frac{\partial^4 f(\bar{x})}{\partial x^4}\;S$$

$$(5.32)$$

The last term in Eq. (5.30) may be evaluated as follows

$$\overline{xu} = \overline{x(0)\,u(t)} + \int_0^t \overline{\left\{ f[x(s)] + u(s) \right\} ds\, u(t)} \qquad (5.33)$$

The initial value of x is uncorrelated with the driving noise u (t) for any $t \geq 0$. Thus,

$$\overline{xu} = \int_0^t \overline{f[x(s)]\; u(t)}\, ds + \int_0^t \overline{u(s)\, u(t)}\, ds \qquad (5.34)$$

In the interval $0 \leq s \leq t$, f[x (s)] and u (t) are correlated only at s = t and at that point the correlation is finite. Therefore, the term $\overline{f[x(s)]\; u(t)}$ does not contribute to the integral over the indicated region and consequently

$$\overline{xu} = 1/2\, Q \qquad (5.35)$$

Thus, Eqs. (5.31) (5.32) and (5.35) used in conjunction with Eq. (5.30) provide the differential equations for the second central moment.

Following the same procedure

$$\dot{T} = 3\, \overline{(x - \overline{x})^2\, (\dot{x} - \dot{\overline{x}})} = 3\, \overline{[\, x - \overline{x}\,]^2\, [\, f(x) - \overline{f(x)} + u\,]} \qquad (5.36)$$

$$\dot{T} = 3[\, \overline{x^2 f(x)} - \overline{x^2}\; \overline{f(x)} + \overline{x^2 u} - 2\overline{x}\; \overline{x f(x)} + 2\,\overline{x}^2\; \overline{f(x)} - 2\overline{x}\; \overline{xu}\,]$$

$$(5.37)$$

In Eq. (5.37), each of the expectations indicated by a bar are again expanded in a series using the following equations.

$$\overline{x^2 f(x)} = \bar{x}^2 f(\bar{x}) + 1/2\,[\,\bar{x}^2 \frac{\partial^2 f(\bar{x})}{\partial x^2} + 4\bar{x}\,\frac{\partial f(\bar{x})}{\partial x} + 2\,f(\bar{x})\,]\,P$$

$$+ \frac{1}{6}\,[\,\bar{x}^2 \frac{\partial^3 f(\bar{x})}{\partial x^3} + 6\bar{x}\,\frac{\partial^2 f(\bar{x})}{\partial x^2} + 6\,\frac{\partial f(\bar{x})}{\partial x}\,]$$

$$+ \frac{1}{24}\,[\,\bar{x}^2 \frac{\partial^4 f(\bar{x})}{\partial x^2} + 8\bar{x}\,\frac{\partial^3 f(\bar{x})}{\partial x^3} + 12\,\frac{\partial^2 f(\bar{x})}{\partial x^2}\,]\,S \qquad (5.38)$$

$$\overline{x^2}\;\overline{f(x)} = [\,\bar{x}^2 + P\,]\,[\,f(\bar{x}) + 1/2\,\frac{\partial^2 f(\bar{x})}{\partial x^2}P + \frac{1}{6}\,\frac{\partial^3 f(\bar{x})}{\partial x^3}T$$

$$+ \frac{1}{24}\,\frac{\partial^4 f(\bar{x})}{\partial x^4}\,S\,] \qquad (5.39)$$

$$\overline{x^2 u} = \overline{[\,x(0) + \int_0^t \left\{ f[x(s)] + u(s) \right\} ds\,] \times \cdots}$$

$$\overline{\cdots\left[x(0) + \int_0^t \left\{ f\left[x(r)\right] + u(r) \right\} dr\right] u(t)}$$

$$(5.40)$$

$$\overline{x^2 u} = \overline{x(0)^2 u} + 2\,\overline{x(0)\int_0^t f[x(s)]\, u(t)\, ds} + 2\,\overline{x(0)\int_0^t u(s) u(t)\, ds}$$

$$+ \iint_0^t \overline{f[x(s)]\, f[x(r)]\, u(t)}\, dr\, ds + 2\iint_0^t \overline{f[x(s)]\, u(r) u(t)}\, dr\, ds$$

$$+ \iint_0^t \overline{u(s) u(r) u(t)}\, dr\, ds \qquad (5.41)$$

As discussed above, $x(0)$ and similarly $x(0)^2$ are uncorrelated with $u(t)$, and $\overline{f[x(s)]\, u(t)}$ and similarly $\overline{f[x(s)]\, f[x(r)]\, u(t)}$ do not contribute to the integrals over the region shown. In addition, because u is assumed to be a zero mean Gaussian white noise process

$$\overline{u(s)\, u(r)\, u(t)} = 0 \qquad \text{for all } r,\ s \text{ and } t$$

Thus, Eq. (5.41) becomes.

$$\overline{x^2 u} = 2\,\overline{x(0)}\,(1/2\,Q) + 2\int_0^t \overline{f[x(s)]}\, ds\,(1/2\,Q)$$

$$\overline{x^2 u} = \overline{x(0)}\,Q + Q[\,\bar{x} - \overline{x(0)}\,]$$

$$\overline{x^2 u} = Q\bar{x} \qquad (5.42)$$

The remaining terms in Eq. (5.37) can be evaluated using the expressions given previously. It is interesting to note explicitly that the terms in Eq. (5.37) which involve u are as follows

$$\overline{x^2 u} - 2x\,\overline{xu} = Q\bar{x} - 2\bar{x}\,(1/2\,Q) = 0 \tag{5.43}$$

Thus, as also shown in Chapter III the extrapolation of the third central moment, T is independent of Q.

$$\dot{T} = 3[\,\overline{x^2 f(x)} - \overline{x^2}\,\overline{f(x)} - 2\bar{x}\,\overline{xf(x)} + 2\bar{x}^2\,\overline{f(x)}\,] \tag{5.44}$$

Finally, the fourth central moment is extrapolated in the following way.

$$\dot{S} = 4\overline{(x-\bar{x})^3\,(\dot{x}-\dot{\bar{x}})} = 4\overline{[\,x-\bar{x}\,]\,[\,f(x)-\overline{f(x)}+u\,]} \tag{5.45}$$

$$\dot{S} = 4[\,\overline{x^3 f(x)} - \overline{x^3}\,\overline{f(x)} + \overline{x^3 u} - 3\bar{x}\,\overline{x^2 f(x)} + 3\bar{x}\,\overline{x^2}\,\overline{f(x)}$$

$$-3\bar{x}\,\overline{x^2 u} + 3\bar{x}^2\,\overline{xf(x)} - 3\bar{x}^3\,\overline{f(x)} + 3\bar{x}^2\,\overline{xu}\,] \tag{5.46}$$

The terms which involve u explicitly will be examined first. They are

$$\overline{x^3 u} - 3\bar{x}\,\overline{x^2 u} + 3\,\bar{x}^2\,\overline{xu}$$

The first may be expanded as in the case of $\overline{x^2 u}$.

$$\overline{x^3 u} = \overline{\left[x(0) + \int_0^t \left\{ f[x(s)] + u(s) \right\} ds \right] \times \left[x(0) + \int_0^t \left\{ f[x(r)] + u(r) \right\} dr \right] \times \left[x(0) + \int_0^t \left\{ f[x(v)] + u(v) \right\} dv \right] u(t)} \qquad (5.47)$$

Keeping in mind the discussion given above, Eq. (5.47) can be reduced to the following

$$\overline{x^3 u} = 3\,\overline{x(0)^2}\,(1/2\,Q) + 6\,\overline{x(0)\int_0^t f[x(s)]}\,ds\,(1/2\,Q) + 3\iint_0^t \overline{f[x(s)]\,f[x(r)]}\,dr\,ds\,(1/2\,Q) + \iiint_0^t \overline{u(r)\,u(s)\,u(v)\,u(t)}\,dr\,ds\,dv \qquad (5.48)$$

The last term above may be expanded into three terms using the relationship between fourth and second order central moments of a Gaussian random variable discussed in Appendix D.

$$\int\int\int_0^t \overline{u(r)\,u(s)\,u(v)\,u(t)}\;dr\,ds\,dv$$

$$= \int\int\int_0^t \left[\, \overline{u(r)\,u(v)}\;\overline{u(s)\,u(t)} + \overline{u(r)\,u(s)}\;\overline{u(v)\,u(t)} + \overline{u(r)\,u(t)}\;\overline{u(s)\,u(v)} \,\right] dr\,ds\,dv \qquad (5.49)$$

Integrating this expression three times results in

$$\int\int_0^t \left[\, Q\delta\,(v-r)\overline{u(s)\,u(t)} + 1/2\;Q\,\overline{u(r)\,u(s)} + Q\delta\,(v-s)\overline{u(r)u(t)} \,\right] dr\,ds$$

$$= \int_0^t \left[\, 1/2\;Q^2\delta\,(v-r) + 1/2\;Q^2\delta\,(s-v) + Qt\,\overline{u(r)\,u(t)} \,\right]\; dr$$

$$= 1/2\;Q^2t + 1/2\;Q^2t + 1/2\;Q^2t = 3/2\;Q^2t \qquad (5.50)$$

Returning to Eq. (5.48), the third term on the right hand side of the equation does not have a clear evaluation; however, consider the following expansion for $\overline{x^2}$.

$$\overline{x^2} = \overline{\left[x(0) + \int_0^t \left\{ f[\,x(s)\,] + u(s) \right\} ds \right]^2} \qquad (5.51)$$

$$\overline{x^2} = \overline{x(0)^2} + 2\,\overline{x(0)}\left[\overline{x} - \overline{x(0)}\right]$$

$$+ \int\!\!\int_0^t \overline{f[x(s)]\, f[x(r)]}\, dr\, ds + \int\!\!\int_0^t \overline{u(s)\, u(r)}\, dr\, ds \qquad (5.52)$$

The last term above is similar to one evaluated in order to obtain Eq. (5.50). Thus, Eq. (5.52) can be rearranged to give

$$\int\!\!\int_0^t \overline{f[x(s)]\, f[x(r)]}\, dr\, ds = \overline{x^2} - \overline{x(0)^2} - 2\,\overline{x(0)}[\,\overline{x} - \overline{x(0)}\,] \qquad (5.53)$$

Substituting this result, as well as the result of Eq. (5.50) into Eq. (5.48)

$$\overline{x^3 u} = 3/2\,\overline{x(0)^2}Q + 3\,\overline{x(0)}Q[\,\overline{x} - \overline{x(0)}\,] + 3/2\,Q[\,\overline{x^2} - \overline{x(0)^2}$$

$$- 2\,\overline{x(0)}\,(\overline{x} - \overline{x(0)}) - Qt] + 3/2\,Qt \qquad (5.54)$$

$$\overline{x^3 u} = 3/2\,Q\overline{x^2}$$

Using Eq. (5.54) and previous results, the terms of Eq. (5.46) involving u can be summarized as follows

$$3/2\,Q\,\overline{x^2} - 3\bar{x}\,(Q\bar{x}) + 3\bar{x}^2\,(1/2\,Q) = 3/2\,Q\,(\overline{x^2} - \bar{x}^2)$$

or (5.55)

$$= 3/2\,QP$$

Returning to Eq. (5.46), the first term may be expanded with the following equation.

$$\overline{x^3 f(x)} = \bar{x}^3 f(\bar{x}) + 1/2\,[\,\bar{x}^3 \frac{\partial^2 f(\bar{x})}{\partial x^2} + 6\bar{x}^2 \frac{\partial f(\bar{x})}{\partial x} + 6\bar{x} f(\bar{x})\,]\,P$$

$$+ 1/6\,[\,\bar{x}^3 \frac{\partial^3 f(\bar{x})}{\partial x^3} + 9\bar{x}^2 \frac{\partial^2 f(\bar{x})}{\partial x^2}\; 18\bar{x} \frac{\partial f(\bar{x})}{\partial x} + 6\,f(\bar{x})\,]\,T$$

$$+ 1/24\,[\,\bar{x}^3 \frac{\partial^4 f(\bar{x})}{\partial x^4} + 12\bar{x}^2 \frac{\partial^3 f(\bar{x})}{\partial x^3} + 36\bar{x} \frac{\partial^2 f(\bar{x})}{\partial x^2}$$

$$+ 24 \frac{\partial f(\bar{x})}{\partial x}\,]\,S \qquad (5.56)$$

and the second with Eq. (5.57).

$$\overline{x^3}\,\overline{f(x)} = \left[\overline{x}^3 + 3\overline{x}P + T\right]\left[f(\overline{x}) + 1/2\,\frac{\partial^2 f(\overline{x})}{\partial x^2}P + 1/2\,\frac{\partial^3 f(\overline{x})}{\partial x^3}T\right.$$

$$\left. + 1/24\,\frac{\partial^4 f(\overline{x})}{\partial x^4}S\right] \tag{5.57}$$

The remaining terms may be evaluated using equations found in the development of $\dot{T}$ or $\dot{P}$.

By making all the indicated expansions and carrying out the required algebra the following extrapolation equations are found.

$$\dot{\overline{x}} = f(\overline{x}) + 1/2\,\frac{\partial^2 f(\overline{x})}{\partial x^2}P + 1/6\,\frac{\partial^3 f(\overline{x})}{\partial x^3}T + 1/24\,\frac{\partial^4 f(\overline{x})}{\partial x^4}S \tag{5.58}$$

$$\dot{P} = 2\left[\frac{\partial f(\overline{x})}{\partial x}P + 1/2\,\frac{\partial^2 f(\overline{x})}{\partial x^2}T + 1/6\,\frac{\partial^3 f(\overline{x})}{\partial x^3}S + 1/2\,Q\right] \tag{5.59}$$

$$\dot{T} = 3\left[\frac{\partial f(\overline{x})}{\partial x}T + 1/2\,\frac{\partial^2 f(\overline{x})}{\partial x^2}S - 1/2\,\frac{\partial^2 f(\overline{x})}{\partial x^2}P^2 - 1/6\,\frac{\partial^3 f(\overline{x})}{\partial x^3}PT\right.$$

$$\left. - 1/24\,\frac{\partial^4 f(\overline{x})}{\partial x^4}PS\right] \tag{5.60}$$

$$\dot{S} = 4\left[\frac{\partial f(\overline{x})}{\partial x}S - 1/2\,\frac{\partial^2 f(\overline{x})}{\partial x^2}PT - 1/6\,\frac{\partial^3 f(\overline{x})}{\partial x^3}T^2 - 1/24\,\frac{\partial^4 f(\overline{x})}{\partial x^r}TS\right.$$

$$\left. + 3/2\,QP\right] \tag{5.61}$$

Specializing Eqs. (5.58) through (5.61) for the nonlinear function

$$f(x) = -\sin(x) \tag{5.62}$$

results in the following.

$$\dot{\bar{x}} = \sin(\bar{x})\,[\,1/2\,P - 1/24\,S - 1\,] + 1/6\cos(\bar{x})\,T \tag{5.63}$$

$$\dot{P} = \sin(\bar{x})\,T + \cos(\bar{x})\,[\,1/3\,S - 2P\,] + Q \tag{5.64}$$

$$\dot{T} = \sin(\bar{x})\,[\,3/2\,S + 1/8\,PS - 3/2\,P^2\,] - \cos(\bar{x})\,[\,3T + 1/2\,PT\,] \tag{5.65}$$

$$\dot{S} = \sin(\bar{x})\,[\,1/6\,TS - 2PT\,] - \cos(\bar{x})\,[\,4S + 2/3\,T^2\,] + 6QP] \tag{5.66}$$

Updating the estimator with periodic measurements requires a set of equations for each type of measurement. The general form of measurement update for each of the four moments is found as follows.

For the state estimate

$$\overline{x^+} + \overline{x^-} + B\,[\,z - \bar{z}\,] \tag{5.67}$$

where z is the actual measurement, z is the ensemble expectation of the measurement written, in general, as

$$
\begin{aligned}
z &= H(x) + v \\
\bar{z} &= \overline{H(x)}
\end{aligned}
\tag{5.68}
$$

and B is the optimum gain factor given by

$$
B = \frac{\overline{(x - \bar{x})\, H(x)}}{\overline{H(x)^2} - \overline{H(x)}^2 + R} \tag{5.69}
$$

Proceeding as in the case of nonlinear system dynamics, a series expansion for $\overline{H(x)}$ is defined by:

$$
\overline{H(x)} = H(\bar{x}) + 1/2\, \frac{\partial^2 H(\bar{x})}{\partial x^2} P + 1/6\, \frac{\partial^3 H(\bar{x})}{\partial x^3} T + 1/24\, \frac{\partial^4 H(\bar{x})}{\partial x^4} S \tag{5.70}
$$

In order to evaluate Eq. (5.69) $\overline{xH(x)}$ and $\overline{H(x)^2}$ are required:

$$\overline{xH(x)} = \bar{x}H(\bar{x}) + 1/2\,[\,\bar{x}\frac{\partial^2 H(\bar{x})}{\partial x^2} + 2\frac{\partial H(\bar{x})}{\partial x}\,]\,P$$

$$+ 1/6\,[\,\bar{x}\frac{\partial^3 H(\bar{x})}{\partial x^3} + 3\frac{\partial^2 H(\bar{x})}{\partial x^2}\,]\,T + 1/24\,[\,\bar{x}\frac{\partial^4 H(\bar{x})}{\partial x^4}$$

$$+ 4\frac{\partial^3 H(\bar{x})}{\partial x^3}\,]\,S \qquad (5.71)$$

$$\overline{H(x)^2} = H(\bar{x})^2 + [\,H(\bar{x})\frac{\partial^2 H(\bar{x})}{\partial x^2} + \left(\frac{\partial H(\bar{x})}{\partial x}\right)^2\,]\,P$$

$$+ 1/6\,[\,2H(\bar{x})\frac{\partial^3 H(\bar{x})}{\partial x^3} + 6\frac{\partial H(\bar{x})}{\partial x}\,\frac{\partial^2 H(\bar{x})}{\partial x^2}\,]\,T \qquad (5.72)$$

$$+ 1/24\,[\,2H(\bar{x})\frac{\partial^4 H(\bar{x})}{\partial x^4} + 8\frac{\partial H(\bar{x})}{\partial x}\,\frac{\partial^3 H(\bar{x})}{\partial x^3} + 6\left(\frac{\partial^2 H(\bar{x})}{\partial x^2}\right)^2\,]\,S$$

Equations (5.70) (5.71) and (5.72) can thus be used to evaluate Eq. (5.69).

The general update expression for the second central moment is found as follows.

$$P^+ = \overline{(x - \overline{x}^+)^2} = \overline{\left\{(x - \overline{x}^-) - B[\,H(x) - \overline{H(x)} + v\,]\right\}^2} \qquad (5.73)$$

$$P^+ = P^- - 2B\overline{[\,x - \overline{x}^-\,][\,H(x) - \overline{H(x)} + v\,]} + B^2\overline{[\,H(x) - \overline{H(x)} + v\,]^2}$$

(5.74)

$$P^+ = P^- - 2B[\,\overline{x\,H(x)} - \overline{x}\,\overline{H(x)}\,] + B^2[\,\overline{H(x)^2} - \overline{H(x)}^2 + R\,] \qquad (5.74)$$

Equation (5.74) can be further simplified by making use of the Eq. for B.

$$P^+ = P^- - B\,\overline{(x - \overline{x}^-)\,H(x)} \qquad (5.75)$$

However, the more general form of Eq. (5.74) is useful for comparison to the update equations for T and S. All of the series expansions required for P^+ are given above. (Note that the expression in Eq. (5.75) with the bar over it is just the numerator of B.)

Following this same procedure, the corresponding equation for T is found

$$T^+ = \overline{(x - \overline{x}^+)^3} = \overline{\left\{(x - x^-) - B[\,H(x) - H(x) + v\,]\right\}^3} \qquad (5.76)$$

$$T^{+} = T^{-} - 3B\,\overline{[\,x - \bar{x}^{-}\,]^{2}\,[\,H(x) - \overline{H(x)} + v\,]}$$

$$+\,3B^{2}\,\overline{[\,x - \bar{x}^{-}\,]\,[\,H(x) - \overline{H(x)} + v\,]^{2}} - B^{3}\,\overline{[\,H(x) - \overline{H(x)} + v\,]^{3}}$$

(5.77)

$$T^{+} = T^{-} - 3B\,[\,\overline{(x - \bar{x}^{-})^{2}\,H(x)} - P^{-}\,\overline{H(x)}\,]$$

$$+\,3B^{2}\,[\,\overline{(x - \bar{x}^{-})H(x)^{2}} - 2\,\overline{(x - \bar{x}^{-})H(x)}\;\overline{H(x)}\,]$$

$$-\,B^{3}\,[\,\overline{H(x)^{3}} + 2\,\overline{H(x)}^{3} - 3\,\overline{H(x)^{2}}\;\overline{H(x)}\,] \qquad (5.78)$$

In order to evaluate Eq. (5.78), the following expansions are required in addition to those given above:

$$\overline{(x - \bar{x}^{-})^{2}\,H(\bar{x})} = H(x)\,P^{-} + \frac{\partial H(\bar{x})}{\partial x}\,T^{-} + 1/2\,\frac{\partial^{2} H(\bar{x})}{\partial x^{2}}\,S^{-}$$

(5.79)

$$\overline{(x - x^-)\, H(x)^2} = 2\, H(\bar{x}) \frac{\partial H(\bar{x})}{\partial x}\, P^- + [\, H(\bar{x}) \frac{\partial^2 H(\bar{x})}{\partial x^2} + \left(\frac{\partial H(\bar{x})}{\partial x} \right)^2]\; T^-$$

$$+ [\; 1/3\; H(\bar{x}) \frac{\partial^3 H(\bar{x})}{\partial x^3} + \frac{\partial H(\bar{x})}{\partial x} \; \frac{\partial^2 H(\bar{x})}{\partial x^2}]\; S^- \qquad (5.80)$$

$$\overline{H(x)^3} = H(\bar{x})^3 + 1/2\; [\; 3H\,(\bar{x})^2 \; \frac{\partial^2 H(\bar{x})}{\partial x^2} + 6\, H(\bar{x}) \left(\frac{\partial H(\bar{x})}{\partial x} \right)^2]\; P^-$$

$$+ 1/6\; [\; 3H(\bar{x})^2 \; \frac{\partial^3 H(\bar{x})}{\partial x^3} + 18\; H(x)\; \frac{\partial H(\bar{x})}{\partial x} \; \frac{\partial^2 H(\bar{x})}{\partial x^2} + 6 \left(\frac{\partial H(\bar{x})}{\partial x} \right)^3 T^-$$

$$+ 1/24 [\; 3H(\bar{x})^2 \; \frac{\partial^4 H(\bar{x})}{\partial x^4} + 24\; H(\bar{x})\; \frac{\partial H(\bar{x})}{\partial x} \frac{\partial^3 H(\bar{x})}{\partial x^3}$$

$$+ 18\; H(\bar{x}) \left(\frac{\partial^2 H(\bar{x})}{\partial x^2} \right)^2 + 36 \left(\frac{\partial H(\bar{x})}{\partial x} \right)^2 \frac{\partial^2 H(\bar{x})}{\partial x^2}]\; S^- \qquad (5.81)$$

Finally the update for S is found as follows:

$$S^+ = \overline{(x - x^+)^4} = \overline{\left\{ (x - \bar{x}^-) - B\, [\, H(x) - \overline{H(x)} + v\,] \right\}^4} \qquad (5.82)$$

Carrying out the expansion, noting the properties of the Gaussian random variable, v, the update equation becomes:

$$
\begin{aligned}
S^{+} = S^{-} &- 4B\left[\,\overline{(x - \bar{x}^{-})^{3} H(x)} - T^{-}\,\overline{H(x)}\,\right] \\
&+ 6B^{2}\left[\,\overline{(x - \bar{x}^{-})^{2}} - 2\,\overline{(x - \bar{x}^{-})^{2} H(x)}\;\overline{H(x)} + P^{-}\,\overline{H(x)}^{2} + P^{-} R\,\right] \\
&- 4B^{3}\left[\,\overline{(x - \bar{x}^{-}) H(x)^{3}} - 3\,\overline{(x - \bar{x}^{-}) H(x)^{2}}\;\overline{H(x)} \right. \\
&\qquad \left. + 3\,\overline{(x - \bar{x}^{-}) H(x)}\;\overline{H(x)}^{2} + 3\,\overline{(x - \bar{x}^{-}) H(x)}\,R\,\right] \\
&+ B^{4}\left[\,\overline{H(x)^{4}} - 3\,\overline{H(x)}^{4} - 4\,\overline{H(x)^{3}}\;\overline{H(x)} + 6\,\overline{H(x)^{2}}\;\overline{H(x)}^{2} \right. \\
&\qquad \left. + 6\,\overline{H(x)^{2}}\,R - 6\,\overline{H(x)}^{2}\,R + 3R^{2}\,\right]
\end{aligned}
\tag{5.83}
$$

The additional expansions required for the evaluation of Eq. (5.83) are:

$$
\overline{(x - \bar{x}^{-})^{3} H(x)} = H(\bar{x})\,T^{-} + \frac{\partial H(\bar{x})}{\partial x}\,S^{-} \tag{5.84}
$$

$$\overline{(x - \bar{x}^-)^2 H(x)^2} = H(\bar{x})^2 P^- + 2 H(\bar{x}) \frac{\partial H(\bar{x})}{\partial x} T^- + \left[H(\bar{x}) \frac{\partial^2 H(\bar{x})}{\partial x^2} + \left(\frac{\partial H(\bar{x})}{\partial x} \right)^2 \right] S^- \tag{5.85}$$

$$\overline{(x - \bar{x}^-) H(x)^3} = 3 H(\bar{x})^2 \frac{\partial H(\bar{x})}{\partial x} P^- + \left[3/2\, H(\bar{x})^2 \frac{\partial^2 H(\bar{x})}{\partial x^2} + 3 H(\bar{x}) \left(\frac{\partial H(\bar{x})}{\partial x} \right)^2 \right] T^- + \left[1/2\, H(\bar{x})^2 \frac{\partial^3 H(\bar{x})}{\partial x^3} + 3 H(\bar{x}) \frac{\partial H(\bar{x})}{\partial x} \frac{\partial^2 H(\bar{x})}{\partial x^2} + \left(\frac{\partial H(\bar{x})}{\partial x} \right)^3 \right] S^- \tag{5.86}$$

and

$$
\begin{aligned}
\overline{H(x)^4} = {} & H(\bar{x})^4 + 1/2\,[\,4\,H(\bar{x})^3\,\frac{\partial^2 H(\bar{x})}{\partial x^2} + 12\,H(\bar{x})^2 \left(\frac{\partial H(\bar{x})}{\partial x}\right)^2]\; P^- \\
& + 1/6\,[\,4\,H(\bar{x})^3\,\frac{\partial^3 H(\bar{x})}{\partial x^3} + 36\,H(\bar{x})^2\,\frac{\partial H(\bar{x})}{\partial x}\,\frac{\partial^2 H(\bar{x})}{\partial x^2} \\
& \quad + 24\,H(\bar{x}) \left(\frac{\partial H(\bar{x})}{\partial x^4}\right)^3]\; T^- \\
& + 1/24\,[\,4\,H(\bar{x})^3\,\frac{\partial^4 H(\bar{x})}{\partial x^4} + 48\,H(\bar{x})^2\,\frac{\partial H(\bar{x})}{\partial x}\,\frac{\partial^3 H(\bar{x})}{\partial x^3} \\
& \quad + 36\,H(\bar{x})^2 \left(\frac{\partial^2 H(\bar{x})}{\partial x^2}\right)^2 + 144\,H(\bar{x}) \left(\frac{\partial H(\bar{x})}{\partial x}\right)^2 \frac{\partial^2 H(\bar{x})}{\partial x^2} \\
& \quad + 24 \left(\frac{\partial H(\bar{x})}{\partial x}\right)^4]\; S^- \qquad (5.87)
\end{aligned}
$$

Thus Eqs. (5.84) through (5.87) coupled with the previous results provide the means for evaluation of Eq. (5.83).

The two types of measurements under consideration here can now be used to specialize the above. For the linear measurement:

$$z_1 = x + v \tag{5.4}$$

$$H(\bar{x}) = \bar{x} \tag{5.88}$$

$$\frac{\partial H(\bar{x})}{\partial x} = 1 \tag{5.89}$$

Equation (5.70) becomes

$$\overline{H(x)} = \bar{x} \tag{5.90}$$

Similarly

$$\begin{aligned} \overline{x\,H(x)} &= \bar{x}^2 + P^- \\ \overline{H(x)^2} &= \bar{x}^2 + P^- \end{aligned} \tag{5.91}$$

and

$$B_1 = \frac{P^-}{P^- + R} \tag{5.92}$$

thus

$$\overline{x^+} = \overline{x^-} + B_1 [z_1 - \overline{x^-}]$$

$$P^+ = P^- - B_1 P^- = (1 - B_1)^2 P^- + B_1^2 R \tag{5.93}$$

Specializing Eqs. (5.79), (5.80) and (5.81)

$$\overline{(x - \overline{x^-})^2 H(x)} = \bar{x} P^- + T^- \tag{5.94}$$

$$\overline{(x - \overline{x^-}) H(x)^2} = 2 \bar{x} P^- + T^- \tag{5.95}$$

$$\overline{H(x)^3} = \bar{x}^3 + 3 \bar{x} P^- + T^- \tag{5.96}$$

and therefore;

$$T^+ = T^- - 3B_1[\bar{x}P^- + T^- - P^-\bar{x}] + 3B_1^2[2\bar{x}P^- + T^- - 2\bar{x}P^-]$$

$$- B_1^3[\bar{x}^3 + 3\bar{x}P^- + T^- + 2\bar{x}^3 - 3\bar{x}(\bar{x}^2 + P^-)]$$

$$T^+ = T^-[1 - 3B_1 + 3B_1^2 - B_1^3] = (1 - B_1)^3 T^- \tag{5.97}$$

Specializing Eq. (5.84), (5.85), (5.86) and (5.87)

$$\begin{aligned}
\overline{(x - \bar{x}^-)^3 H(x)} &= \bar{x}T^- + S^- \\
\overline{(x - \bar{x}^-)^2 H(x)^2} &= \bar{x}^2 P^- + 2\bar{x}T^- + S^- \\
\overline{(x - \bar{x}^-) H(x)^3} &= 3\bar{x}^2 P^- + 3\bar{x}T^- + S^- \\
\overline{H(x)^4} &= \bar{x}^4 + 6\bar{x}^2 P^- + 4\bar{x}T^- + S^-
\end{aligned} \tag{5.98}$$

and using previous results

$$S^+ = S^- - 4B_1[\bar{x}T^- + S^- - T^-\bar{x}] + 6B_1^2[\bar{x}^2P^- + 2\bar{x}T^- + S^-$$

$$- 2\bar{x}^2P^- - 2\bar{x}T^- + \bar{x}^2P^- + P^-R] - 4B_1^3[3\bar{x}^2P^- + 3\bar{x}T^- + S^-$$

$$- 6\bar{x}^2P^- - 3\bar{x}T^- + 3\bar{x}^2P^- + 3P^-R] + B_1^4[\bar{x}^4 + 6\bar{x}^2P^-$$

$$+ 4\bar{x}T^- + S^- - 3\bar{x}^4 - 4\bar{x}^4 - 12\bar{x}^2P^- - 4\bar{x}T^- + 6\bar{x}^4$$

$$+ 6\bar{x}^2P^- + 6\bar{x}^2R + 6P^-R - 6\bar{x}^2R + 3R^2 \qquad (5.99)$$

$$S^+ = S^-(1 - 4B_1 + 6B_1^2 - 4B_1^3 + B_1^4) + B_1^2R(6P^- - 12B_1P^- + 6B_1^2P^- + 3B_1^2R)$$

$$S^+ = (1 - B_1)^4S^- + 3B_1^2R(2P^- - 4B_1P^- + 2B_1^2P^- + B_1^2R) \qquad (5.100)$$

The second measurement type

$$z_2 = 1/2 \sin(2x) + v \qquad (5.8)$$

has all terms non-zero and therefore does not lend itself to any significant simplification. Therefore, only the series expansions required for the update expressions will be specialized for z_2.

$$\overline{H(x)} = \sin(2\bar{x})[1/2 - P^- + 1/3\, S^-] - 2/3 \cos(2\bar{x})\, T^- \qquad (5.101)$$

$$\overline{(x - \bar{x}^-)\, H(x)} = \sin(2\bar{x})[-T^-] + \cos(2\bar{x})[P^- - 2/3\, S^-] \qquad (5.102)$$

$$\overline{H(x)^2} = 1/4 \sin(2\bar{x}) + \cos(4\bar{x})[P^- - 4/3\, S^-] - 4/3 \sin(4\bar{x})\, T^- \qquad (5.103)$$

$$\overline{(x - \bar{x}^-)^2\, H(x)} = \sin(2\bar{x})[1/2\, P^- - S^-] + \cos(2\bar{x})\, T^- \qquad (5.104)$$

$$\overline{(x - \bar{x}^-)\, H(x)^2} = \sin(4\bar{x})[1/2\, P^- - 4/3\, S^-] + \cos(4\bar{x})\, T^- \qquad (5.105)$$

$$\begin{aligned}\overline{H(x)^3} = {} & \sin^3(2\bar{x})[1/8 - 3/4\, P^- + 7/4\, S^-] + \cos^3(2\bar{x})\, T^- \\ & + \cos^2(2\bar{x}) \sin(2\bar{x})[3/2\, P^- - 5S^-] \\ & - \sin^2(2\bar{x}) \cos(2\bar{x})[7/2\, T^-]\end{aligned} \qquad (5.106)$$

$$\overline{(x - \bar{x}^-)^3} = \sin(2\bar{x})[1/2\, T^-] + \cos(2\bar{x})\, S^- \qquad (5.107)$$

$$\overline{(x - \bar{x}^-)^2} = \sin(2\bar{x})[1/4\, P^-] + \sin(4\bar{x})[1/2\, T^-] + \cos(4\bar{x})\, S^- \qquad (5.108)$$

$$\overline{(x - \bar{x}^-) H(x)^3} = \sin^3(2\bar{x}) [- 3/4 \, T^-] + \cos^3(2\bar{x}) S^-$$

$$+ 3/2 \cos^2(2\bar{x}) \sin(2\bar{x}) T^-$$

$$+ \sin^2(2\bar{x}) \cos(2\bar{x}) [3/4 \, P^- - 2 S^-] \qquad (5.109)$$

$$\overline{H(x)^4} = \sin^4(2\bar{x}) [1/16 - 1/2 \, P^- + 5/3 \, S^-] + \cos^4(2\bar{x}) S^-$$

$$+ \sin^2(2\bar{x}) \cos^2(2\bar{x}) [3/2 \, P^- - 8 S^-]$$

$$- \sin^3(2\bar{x}) \cos(2\bar{x}) [7/3 \, T^-] + \cos^3(2\bar{x}) \sin(2\bar{x}) [2 T^-]$$

$$(5.110)$$

Equations (5.110) when properly substituted into Eqs. (5.69), (5.75), (5.78), and (5.83) provide the update equations for the non-linear measurement, z_2.

Some of the required algebra has been carried out in this chapter to provide evidence of the straightforward but lengthly calculation required of the series estimation procedure. Keeping in mind the system under consideration is a scalar state, a serious deficiency of the series approach is evident.

By way of contrast, the quasi-linear estimation equations for the same system are derived below.

5.4 Nonlinear Estimation Using a Quasi-Linear Approximation

The third type of estimation scheme to be considered here is based upon the discussion of Chapter IV and is referred to as a quasi-linear estimator. For the nonlinear function

$$f(x) = -\sin(x) \tag{5.111}$$

the quasi-linear approximator is given by:

$$\begin{aligned} N_f^{\bar{x}} &= -\frac{\sin(\bar{x})}{\bar{x}} e^{-P/2} \\ N_f^{e} &= -\cos(\bar{x})\, e^{-P/2} \end{aligned} \tag{5.112}$$

where $\bar{x}$ is the quasi-linear estimate and the ensemble expectation of the state variable, x; and P is the variance defined by

$$P = \overline{(x - \bar{x})^2} \tag{5.113}$$

The differential equations required by the quasi-linear estimator for extrapolation are therefore:

$$\dot{\bar{x}} = N_f^{\bar{x}}\, \bar{x} \tag{5.114}$$

$$\dot{P} = 2\, N_f^{e}\, P + Q \tag{5.115}$$

In order to update the estimate with measurement data, the following equations are used. For the linear measurement,

$$z_1 = x + v$$

$$N_{1m}^{\bar{x}} = N_{1m}^{e} = 1 \tag{5.116}$$

and

$$\bar{x}^{+} = \bar{x}^{-} + B_1 [z_1 - \bar{z}_1] \tag{5.117}$$

where

$$\bar{z}_1 = N_{1m}^{\bar{x}} \bar{x} = \bar{x} \tag{5.118}$$

and

$$B_1 = \frac{P^{-}}{P^{-} + R} \tag{5.119}$$

the variance is updated with the following equation.

$$P^{+} = P^{-} - B_1 P^{-} = (1 - B_1)^2 P^{-} + B_1^2 R \tag{5.120}$$

For the nonlinear measurement

$$z_2 = 1/2 \sin(2x) + v$$

$$N_{2m}^{\bar{x}} = 1/2 \frac{\sin(2\bar{x})}{\bar{x}} e^{-2P} \tag{5.121}$$

$$N_{2m}^{e} = \cos(2\bar{x}) e^{-2P}$$

and

$$\bar{x}^{+} = \bar{x}^{-} + B_2 [z_2 - \bar{z}_2] \tag{5.122}$$

where

$$z_2 = N_{2m}^{x} \bar{x} = 1/2 \sin(2x) e^{-2P} \tag{5.123}$$

$$B_2 = \frac{P^{-} N_{2m}^{e}}{P^{-} (N_{2m}^{e})^2 + R} \tag{5.124}$$

and

$$P^{+} = P^{-} - B_2 N_{2m}^{e} P^{-} = (1 - B_2 N_{2m}^{e})^2 P^{-} + B_2^{2} R \tag{5.125}$$

Using the equations found in Section 5.2, 5.3 and 5.4, a numerical simulation is now used to compare the performance of these various estimators.

5.5 Digital Simulation and Results

A digital simulation of the above nonlinear estimation problem was performed using an IBM 360/75 computer and the MAC compiler language of the M. I. T. Instrumentation Laboratory. Each case was a Monte Carlo run with the random noise being generated by a random number generator using a Normal Distribution of numbers with a zero mean and specified standard deviation.

In each of the cases considered, the initial error in the estimate was one radian and the measurement interval was 0.2 seconds. The parameters which were varied for each of five estimators were the measurement noise variance and the driving noise variance. Both linear and nonlinear measurements, z_1 and z_2 were also considered for each system.

The results of using the linear measurement, z_1 are summarized in Fig. 5-2 for all of the estimators considered. As can be readily seen from the figure, each of the estimators performs well. For this estimation problem, the effect of the nonlinear system dynamics is minor compared to the nonlinear effects in the measurement process. There is no apparent advantage to using either of the nonlinear techniques for this system if a direct measurement of the state is available. However, this is only true if the linear estimate is rectified at each measurement point. If this rectification procedure is not used, the linear estimator is not at all comparable to any of the nonlinear techniques because of the inaccurate approximation to the system dynamics, a result also verified numerically.

In Figs. 5-3 and 5-4, the performance of each of the five estimators: three series estimators, a quasi-linear estimator and the linear estimator, is shown for a measurement noise variance of 0.02 sq. rad. and a system driving noise with variance 0.01 sq. rad. In this case, and in the remaining cases, only the nonlinear measurement z is considered. As can be seen from Fig. 5-4, the linear errors and the quasi-linear errors generally bracket the errors of all the series

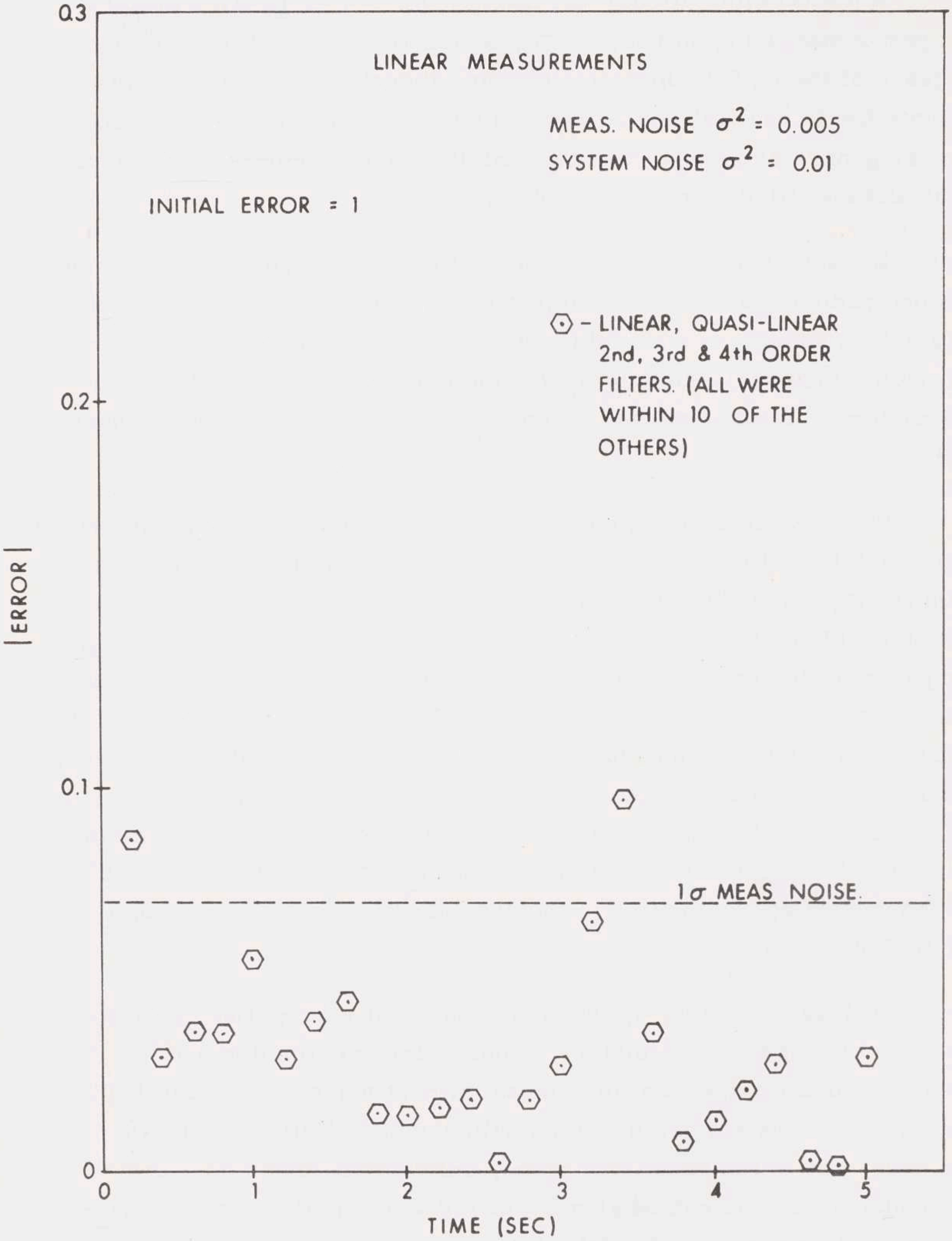


Figure 5-2 Estimation Errors - Linear Measurements

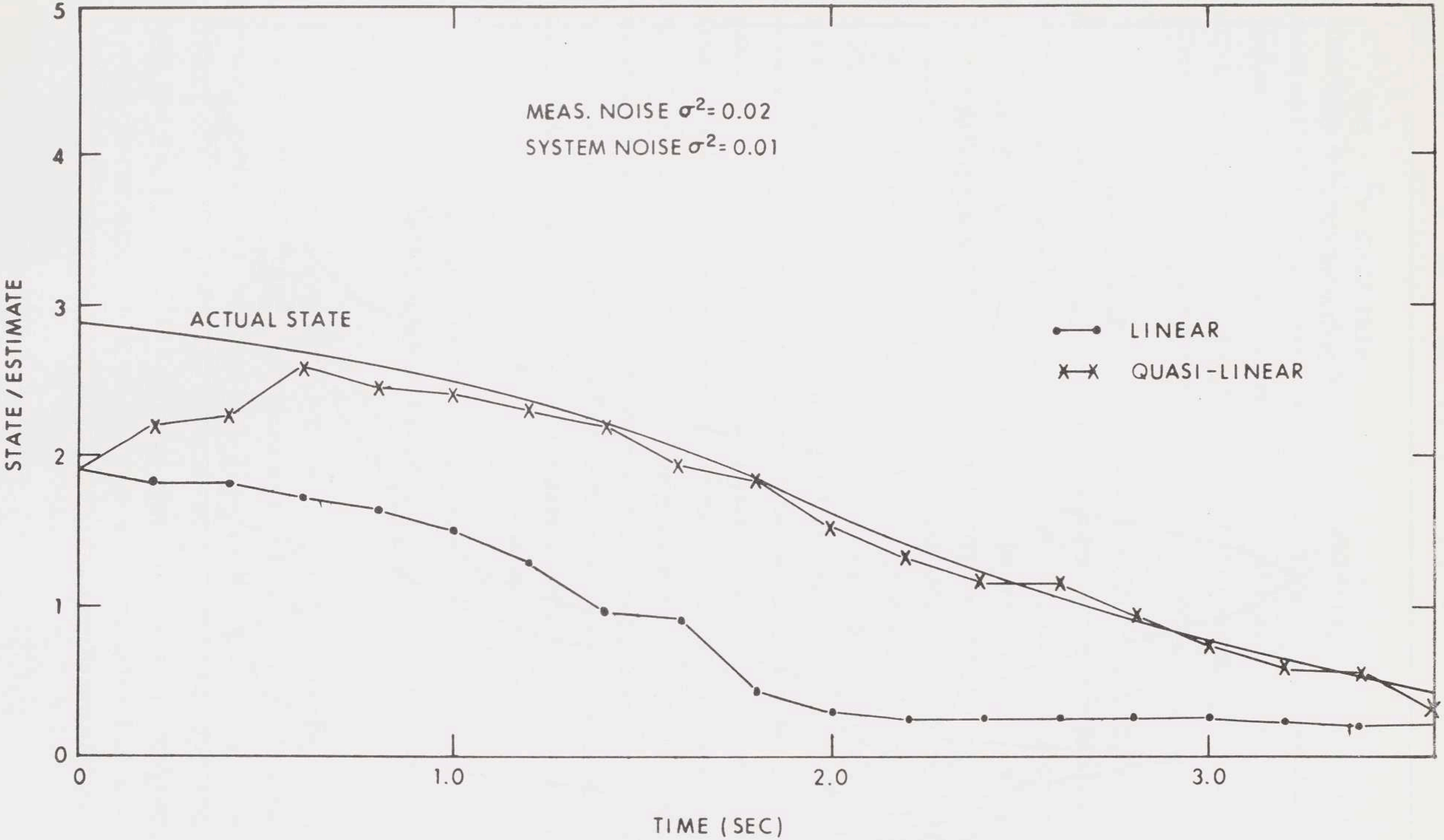


Figure 5-3 Linear and Quasi-Linear - State Estimates

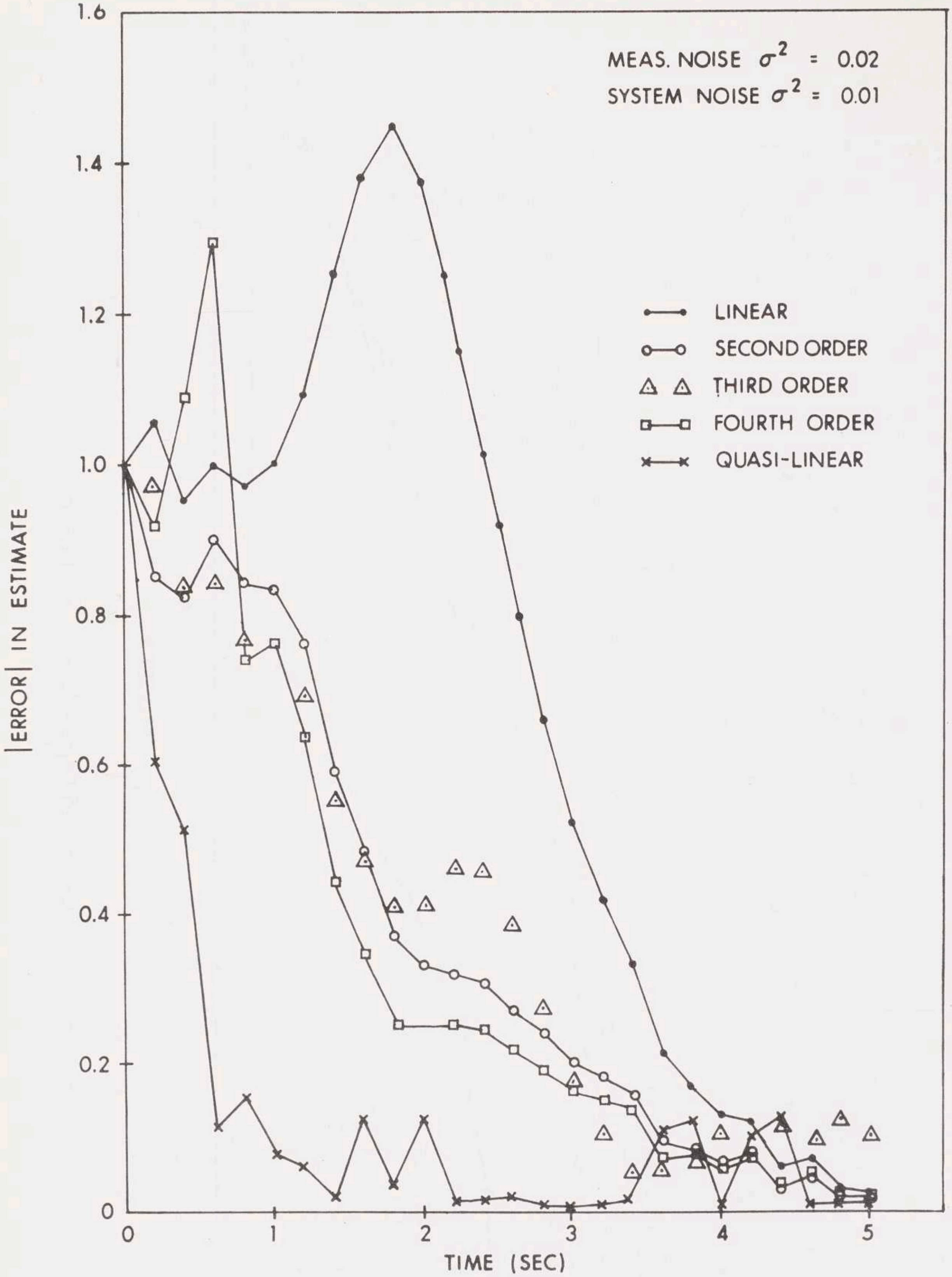


Figure 5-4 Estimation Errors for Five Estimators

estimators in this case. From Fig. 5-3, which depicts the time behavior of the linear and quasi-linear estimators, it is clear that the relatively large initial error does not allow effective linearization. With reference to both figures for this simulation run, the quasi-linear estimator results in the smallest estimation error.

For a smaller measurement noise variance of 0.005 sq. rad., the performance of the quasi-linear and linear estimator is shown in Figs. 5-5 and 5-6. All of the series estimators, not shown for purposes of clarity, again demonstrated an improvement over the linearized estimator but did not equal the performance of the quasi-linear technique. An important difference between this case and the one above is, of course, that random numbers entering the measurement process in this run are only half those which would occur for a variance of 0.02 sq. rad. More important than the actual random numbers, however, is the relative magnitude between R and P in the measurement update equations. This situation manifests itself as an increased sensitivity of the nonlinear estimators to measurements. Because of the nonlinear nature of the estimation equations, the excursions of the estimate, as can be seen in Fig. 5-5, are much larger than would occur with just the addition of a random number from the distribution specified. This amplification of random affects occurs for each of the nonlinear estimators considered.

Again, it is clear that the linear estimator cannot be used for initial errors of this magnitude even if the measurements are relatively accurate.

Another case was considered with a relatively large system driving noise. All of the estimators were examined with the measurement noise variance equal to 0.02 sq. rad. and a driving noise variance of 0.10 sq. rad. The behavior of the state estimate using a linear and a quasi-linear technique is shown in Fig. 5-7. Although, the quasi-linear approach again provides a distinct advantage over the linear estimator, the increased value of Q results in a better

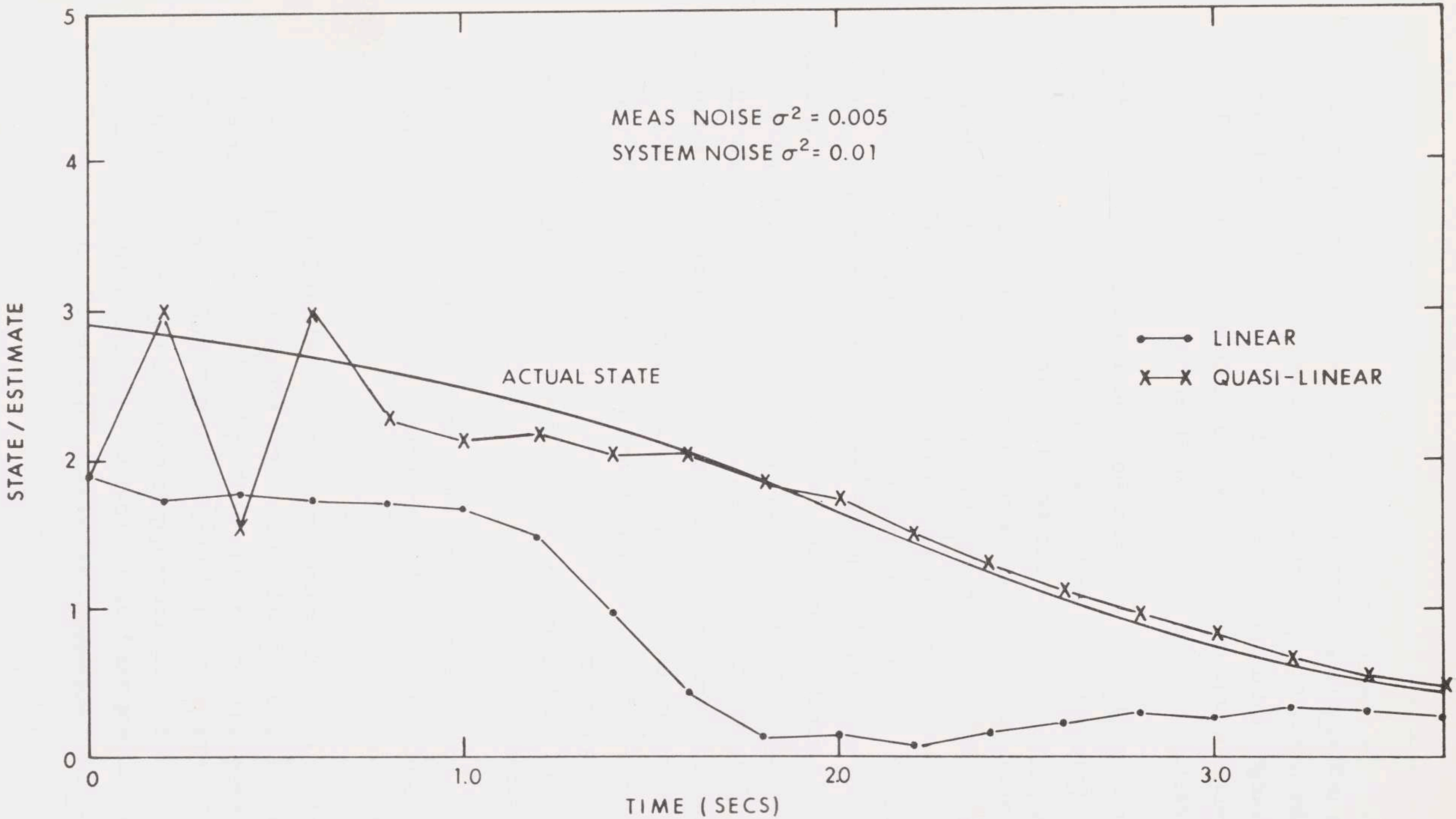


Figure 5-5 Linear and Quasi-Linear State Estimates

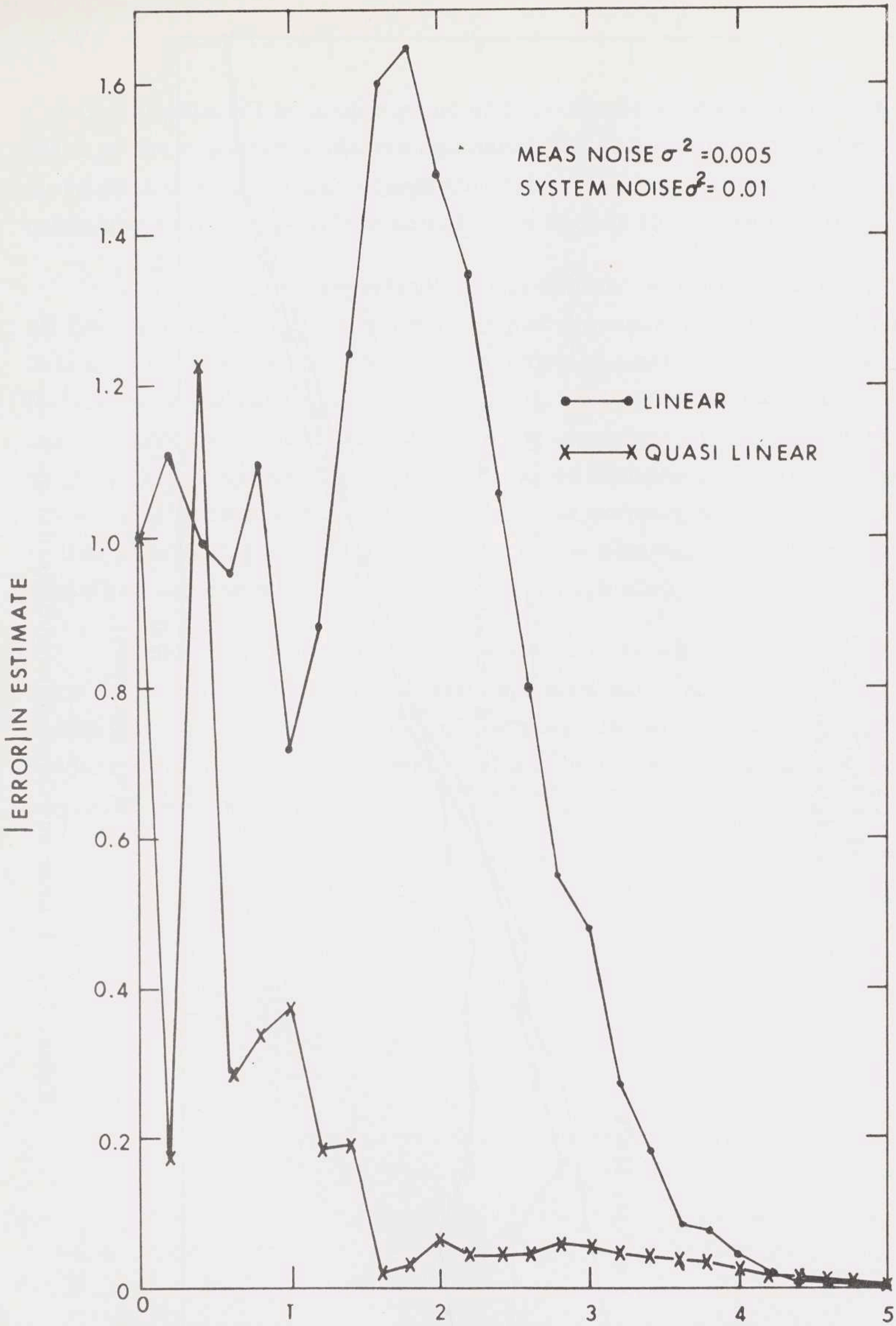


Figure 5-6 Linear and Quasi-Linear Estimation Errors

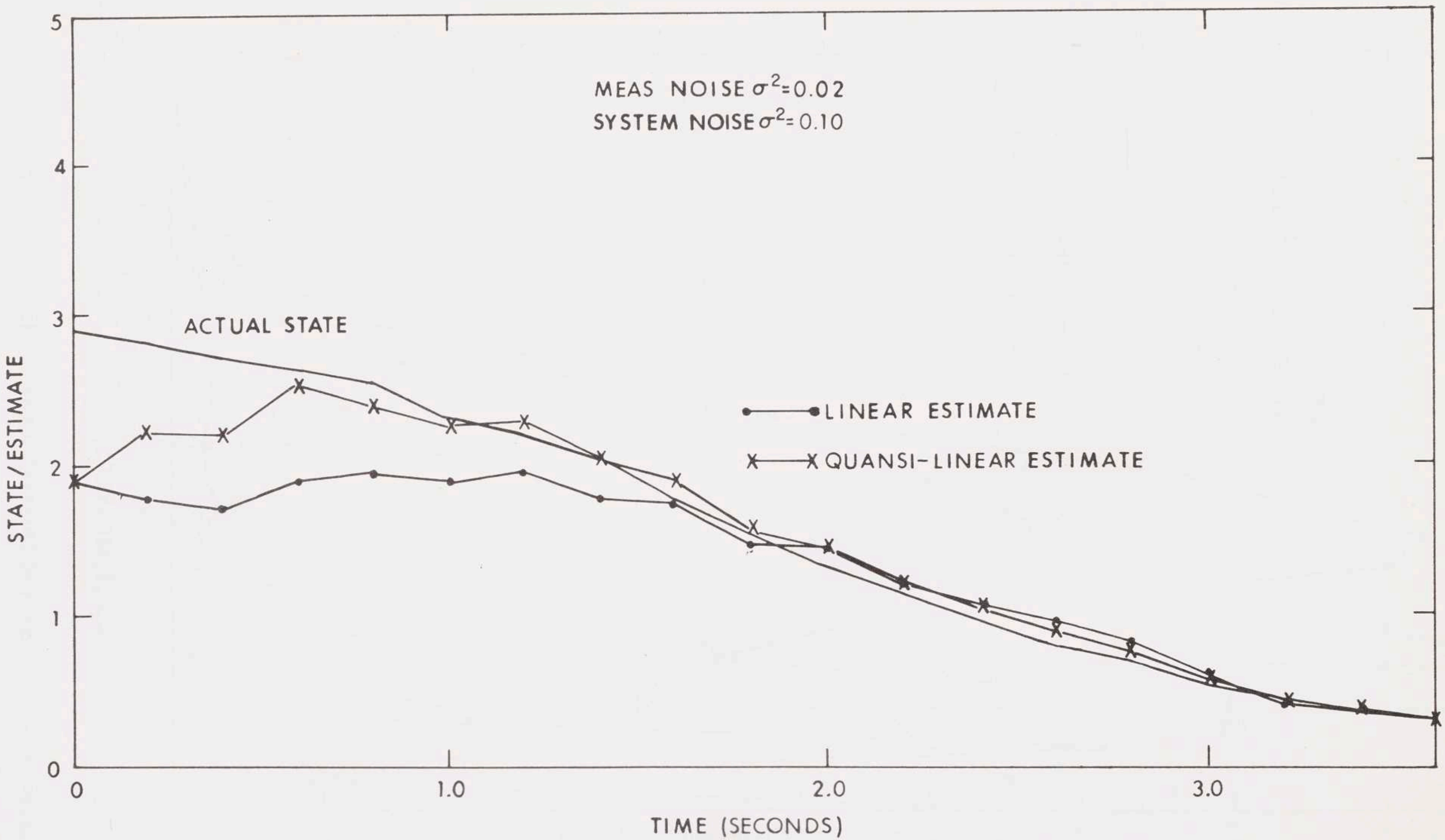


Figure 5-7 Linear and Quasi-Linear State Estimate

linear estimate. The larger value of Q produces a relatively larger value of the covariance matrix and hence the estimator tends to be more sensitive to measurements throughout the run, thus allowing the estimator to converge to the actual state sooner than in the earlier cases.

In Fig. 5-8, the magnitude of the estimation errors is shown for all five estimators for the same case just discussed. From this figure, it is obvious that the third and fourth order estimators are not as effective as in the case summarized by Fig. 5-4. The second order estimator again shown an advantage over the linear estimator, but displays slightly larger errors than those associated with the quasi-linear estimate. The series terms are apparently not converging asymptotically in this case, a fact which points up one of the serious difficulties of this technique and one which other individuals have also noted.

Finally, in Fig. 5-9, the results of a case with a large measurement noise variance of 0.10 sq. rad. and no system noise is shown. The consistently better performance of the quasi-linear estimator is again evident; however, the performance of all the estimators is somewhat degraded as expected.

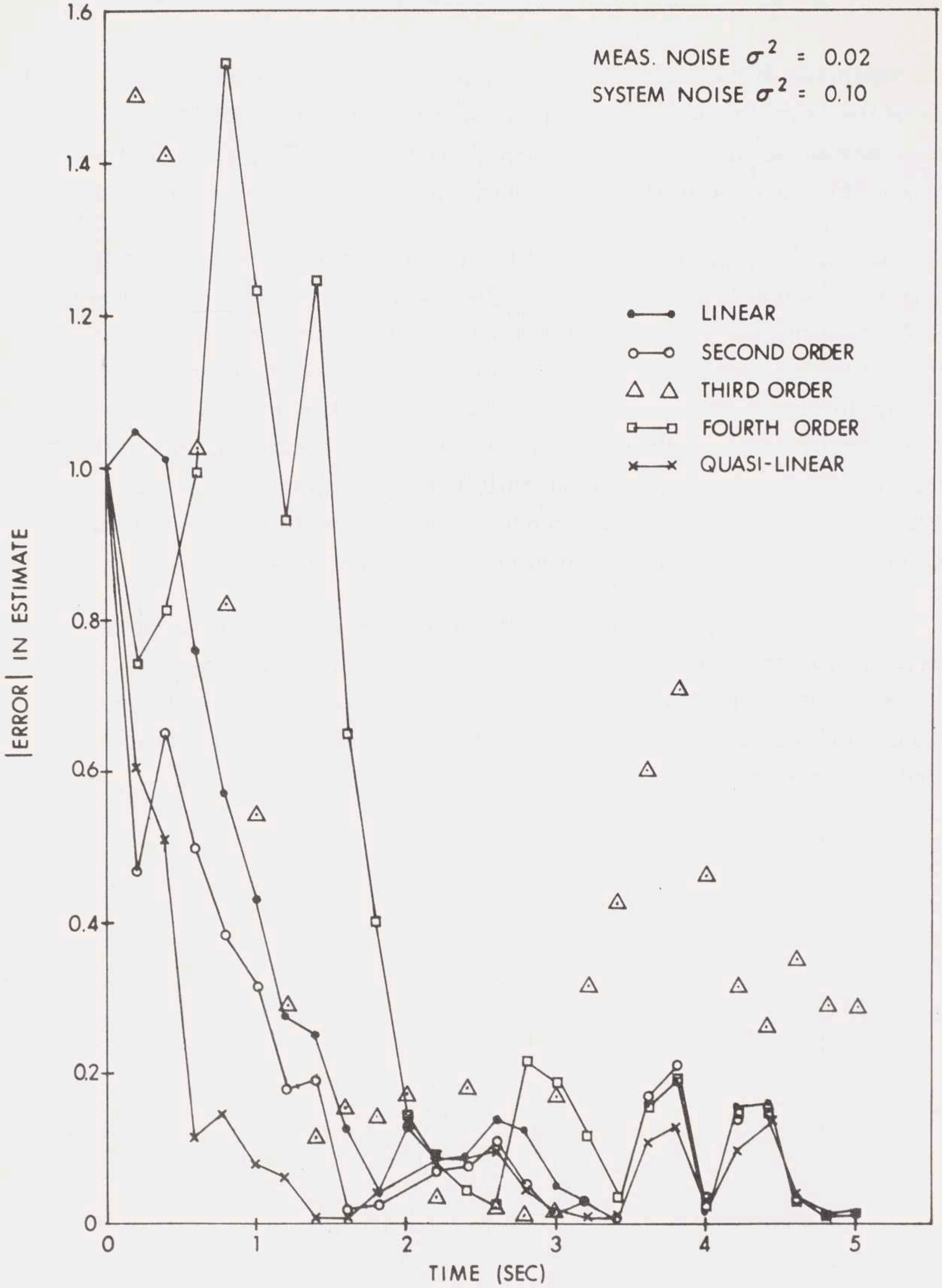


Figure 5-8 Estimation Errors for Five Estimators

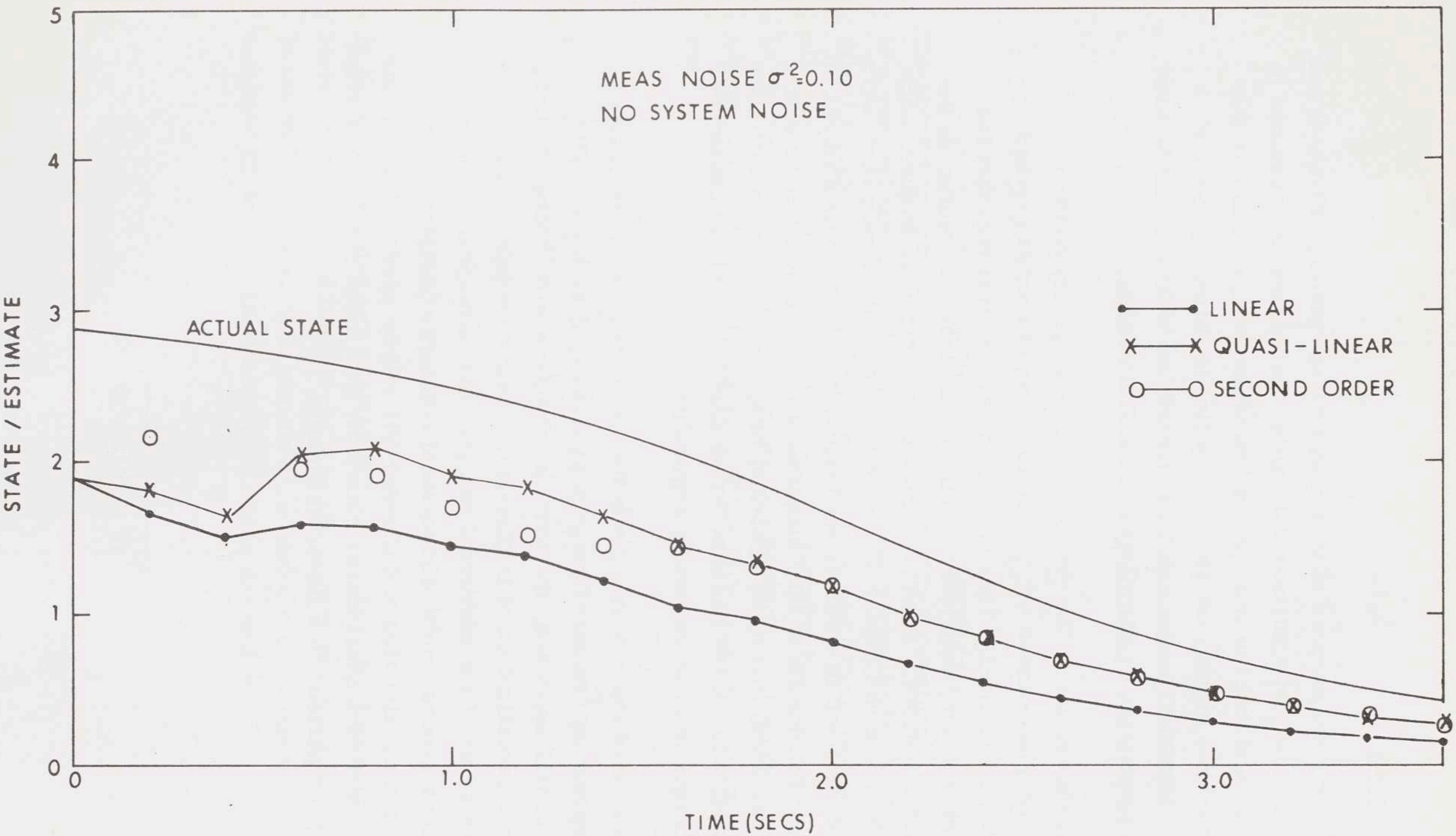


Figure 5-9 State Estimate for Three Estimators

5.6 Summary

The original purpose of this chapter was twofold. First, it was presented to show that nonlinear estimation techniques are required in certain cases and that the two approaches under consideration in this work do provide a significant gain over a linear estimate. Second, a comparison between the two nonlinear techniques was to be made with respect to estimation equations and numerical results.

The discussion in the previous sections clearly points up the advantages of quasi-linear estimation in terms of complexity and implementation difficulties. This is due, in part, to that fact that the nonlinear relationship considered, permits an analytic solution to the quasi-linear equations. However, even if numerical techniques, such as those discussed in Chapter VI, were necessary algebraic complexity such as that involved in a series estimation technique would still not be in evidence. In addition to the calculations required by a series estimator, it was found that many practical problems associated with the numerical behavior of the moment terms must be carefully considered if this technique is to be generally applicable.

Because of the consistently better performance of the quasi-linear estimator as evidenced numerically, as well as the analytic simplicity of this approach, the next two chapters will be devoted to some further investigation of implementing this technique. Chapter VI will discuss some of the additional analytic and numerical techniques required for a general multi-dimensional nonlinear problem using a quasi-linear approximation and Chapter VII will be used to demonstrate the performance of a quasi-linear estimator in a more complex situation including a comparison to a linear estimator.

CHAPTER VI

COMPUTATIONAL TECHNIQUES ASSOCIATED WITH QUASI-LINEAR ESTIMATION

6.1 Introduction

The use of quasi-linear estimation techniques, as discussed in the preceding chapters, hinges on the ability to evaluate the expectation of a general nonlinear function of the state vector under the assumption that the state vector can be described by a particular probability density function. The form of this density function has been assumed to be Gaussian for reasons discussed in Chapter IV. Therefore, in the general case, a multiple integral of the following form must be evaluated.

$$I = \frac{1}{(2\pi)^{n/2}\,|P|^{1/2}} \int_{-\infty}^{\infty}\!\!\cdots\!\int dx_1 \ldots dx_n \, f(x)\, e^{-1/2\,[x-\bar{x}]^T P^{-1} [x-\bar{x}]} \tag{6.1}$$

In Eq. (6.1), the number of integrals is equal to the dimension of the state vector, n, and the matrix P is the covariance matrix of errors in the estimate of x. The matrix P is also the covariance of the assumed Gaussian density function and is defined by:

$$P = \overline{(x - \bar{x})\,(x - \bar{x})^T} \tag{6.2}$$

In a number of useful cases, the function, $f(x)$ is such that the integral in Eq. (6.1) can be analitically evaluated. In general, however, it is necessary to use some approximation technique to find $\overline{f(x)}$. The simplest approach to this problem, and also the most generally applicable in light of the present-day high speed digital computer capability, is to utilize one of the many numerical integration techniques which are available. The majority of useful techniques, however, are designed for the solution of a single integral. There are a great many practical systems, for which the nonlinear vector function $f(x)$ is actually a collection of one or more scalar nonlinear functions, each a function of just one variable. In such cases, each element of $\overline{f(x)}$ can be evaluated separately using a single integral.

The general nonlinear vector function is not as simply treated. Therefore, the first problem which will be discussed here will be the separation of Eq. (6.1) into the product of n single integrals. Some of the techniques for numerical evaluation of the resulting single integrals will then be discussed, keeping in mind the kinds of nonlinear functions which typically arise in estimation problems.

6.2 Multiple Expectation Integrals

The general expectation integral of Eq. (6.1) can be rewritten as:

$$I = K \int_{-\infty}^{\infty} \cdots \int dr_1 \ldots dr_n \; f\,[x(\bar{x}, r)]\; e^{-1/2\, r^T P^{-1} r} \tag{6.3}$$

where K is the scalar coefficient of the integral in Eq. (6.1), redefined simply for convenience, and r is defined by the following

$$r = x - \bar{x} \tag{6.4}$$

The matrix P^{-1} will not generally be diagonal; however, because of its physical definition, it will always be real and symmetric and therefore, a transformation which will result in a diagonal matrix is always possible. By defining a new variable which is related to r by the following expression

$$r = T\, s \tag{6.5}$$

such that

$$T^T P^{-1} T = D \tag{6.6}$$

where D is a diagonal matrix, the exponent of Eq. (6.3) will become

$$-\frac{1}{2}\ [\alpha_1\, s_1^{\,2} + \alpha_2\, s_2^{\,2} + \ldots + \alpha_n\, s_n^{\,2}]$$

The α_i in the above expression are the diagonal elements of D, and are also the Eigenvalues of (P^{-1}). The matrix T can be formed by making each column of T a unit Eigenvector of (P^{-1}). The differentials, $ds_1 \ldots ds_n$ are related to the original variables $dr_1 \ldots dr_n$ in the following way

$$ds_1 \ldots ds_n = J\ dr_1 \ldots dr_n \tag{6.7}$$

where the Jacobian, J is given by:

$$J = \begin{vmatrix} \dfrac{\partial s_1}{\partial r_1} & \dfrac{\partial s_1}{\partial r_2} & \cdots & \dfrac{\partial s_1}{\partial r_n} \\ \dfrac{\partial s_2}{\partial r_1} & \dfrac{\partial s_2}{\partial r_2} & \cdots & \dfrac{\partial s_2}{\partial r_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial s_n}{\partial r_1} & \dfrac{\partial s_n}{\partial r_2} & \cdots & \dfrac{\partial s_n}{\partial r_n} \end{vmatrix} \tag{6.8}$$

Using the above relationships, the integral of Eq. (6.3) can now be rewritten as:

$$I = K \int_{-\infty}^{\infty} \cdots \int J (ds_1 \ldots ds_n)\, g(\bar{x}, s)\, e^{-1/2\,[\alpha_1 s_1^2 + \alpha_2 s_2^2 + \ldots + \alpha_n s_n^2]} \tag{6.9}$$

$$I = K\, J \int_{-\infty}^{\infty} e^{-1/2\,\alpha_1 s_1^2}\, ds_1 \int_{-\infty}^{\infty} e^{-1/2\,\alpha_2 s_2^2}\, ds_2 \ldots \int_{-\infty}^{\infty} g(\bar{x}, s)\, e^{-1/2\,\alpha_n s_n^2}\, ds_n \tag{6.10}$$

where $g(\bar{x}, s) = f(\bar{x}, Tr)$.

The series of single integrations indicated in Eq. (6.10) can be a formidable task even after the nontrivial procedure of diagonalizing has been accomplished. For example, consider the following double integral.

$$I = \int_{-\infty}^{\infty} e^{-1/2\,\alpha_1 s_1^2}\, ds_1 \int_{-\infty}^{\infty} e^{-1/2\,\alpha_2 s_2^2}\, ds_2\; g(\bar{x}, s_1, s_2)\, ds_2 \tag{6.11}$$

Assume that a numerical procedure has been selected in order to evaluate each of the above integrals. A general relationship for the approximation in such a procedure may be written in the following way.

$$F = \sum_{i=1}^{m} K(\xi_i)\, f(\xi_i) \tag{6.12}$$

Thus,

$$\int_{-\infty}^{\infty} e^{-1/2\,\alpha_2 s_2^2}\, g(\bar{x}, s_1, s_2)\, ds_2 \approx \sum_{i=1}^{m} K(\zeta_i)\, e^{-1/2\,\alpha_2 \zeta_i^2}\, g(\bar{x}, s_1, \zeta_i) \tag{6.13}$$

From Eq. (6.13), a series of m terms which are weighted functions of $\bar{x}$ and s_1 results. Each of these terms then becomes a new single integral to be evaluated. Again an approximation such as that in Eq. (6.12) may be used to evaluate each term. Thus, there are m terms resulting from the first integration and each of these is approximated by another m term series. Therefore, m^2 numbers must be calculated. In general there are m^n such numbers for an expectation involving n integrations.

A further demonstration of this procedure is discussed in the following. Consider a function of two state vector components which can be expressed by the following

$$f(x) = x_1^m \, x_2^n \tag{6.14}$$

$$f\,[x\;(\bar{x}\,,\; r)] = (\bar{x}_1 + r_1)^m\,(\bar{x}_2 + r_2)^n \tag{6.15}$$

After expansion of Eq. (6.15), one of the resulting terms will be

$$r_1^m \, r_2^n$$

Examining this term in detail, the integral corresponding to Eq. (6. 3) is

$$I = K \int_{-\infty}^{\infty}\int dr_1\, dr_2\, r_1^{\,m}\, r_2^{\,n}\, e^{-1/2\, r^T P^{-1} r} \tag{6.16}$$

Making the substitution

$$r = T s$$

results in equations of the following general form

$$\begin{aligned} r_1 &= T_{11}\, s_1 + T_{12}\, s_2 \\ r_2 &= T_{21}\, s_1 + T_{22}\, s_2 \end{aligned} \tag{6.17}$$

Thus,

$$r_1^{\,m}\, r_2^{\,n} = (T_{11}\, s_1 + T_{12}\, s_2)^m\, (T_{21}\, s_1 + T_{22}\, s_2)^n \tag{6.18}$$

and

$$I = K J \int_{-\infty}^{\infty}\int ds_1\, ds_2\, (T_{11}\, s_1 + T_{12}\, s_2)^m\, (T_{21}\, s_1 + T_{22}\, s_2)^n\, e^{-1/2\,[\alpha_1 s_1^2 + \alpha_2 s_2^2]} \tag{6.19}$$

where J, α_1 and α_2 are as defined above. The double integration can now be rewritten as

$$I = K J \int_{-\infty}^{\infty} e^{-1/2\,\alpha_1 s_1^2} ds_1 \int_{-\infty}^{\infty} e^{-1/2\,\alpha_2 s_2^2} ds_2 (T_{11} s_1 + T_{12} s_2)^m (T_{21} s_1 + T_{22} s_2)^n \tag{6.20}$$

Before computing any of the actual values of the α_i's and T_{ij}'s it is often useful to examine more closely some of the characteristics of the terms in Eq. (6.20). Because the exponential weighting function is even and the integration interval is symmetric about zero, all odd terms in the integrand do not contribute any net value to the integral. In the example, the integrand in Eq. (6.20) can be expanded with the r + 1st term of the first expansion times the t + 1st term of the second expansion being given by the following

$$T_{11}^{m-r}\ T_{12}\ T_{21}^{n-t}\ T_{22}\ \frac{n!\ m!}{(n-r)!\ (m-t)!\ r!\ t!}\ s_1^{m+n-(r+t)}\ s_2^{r+t}$$

From this expression and the particular type of integral involved, it is obvious that for m+n equal to an odd integer, the double integration is zero. For m+n even, r+t must also be even in order to have a non zero contribution to the overall expectation. Thus, for $r_1^m\ r_2^n$ if m+n is odd, the expectation is always zero, and for m+n even, only the r + t even terms need be considered. These latter terms can be evaluated in closed form with the following general expression

$$\int_{-\infty}^{\infty} z^{2n}\, e^{-az^2}\, dz = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n\, a^n} \sqrt{\frac{\pi}{a}} \tag{6.21}$$

Because of the assumption that r is a multi-dimensional Gaussian variable, any term which is the expectation of products of the components of r, raised to a power represents a term of the moments of the Gaussian density function. Such terms, if they are even, can always be expressed as a function of the second order moment terms as discussed in Appendix D. If odd, these terms are always zero. The above example is therefore provided to demonstrate the procedure of separating integrals and, in fact, need not be used for functions which are of polynomial form. However, the same steps and examinations of even and odd terms can be followed for any functions, $f[x(\overline{x}, r)]$ which appear in Eq. (6. 3).

As can be seen from this example, the procedure of expanding the nonlinear function may increase the resulting computational complexity considerably. Therefore, each particular case should be examined to determine whether or not the original form should be approximated directly.

In the general case, after the required expectation is redefined in the form of Eq. (6. 10), the evaluation of the integrals must be performed in some approximate manner. It is possible to approximate the nonlinear function by some other function such as a series, polynomial or straight line segments; however, this procedure must be tailored to a particular nonlinearity and is therefore not general. For this reason, as well as the numerical accuracy attainable on high speed computers, only numerical techniques will be considered in the following discussion.

6.3 Numerical Approximation to Expectation Integrals

All of the expectation integrals which arise in the computation of the quasi-linear approximation, under the Gaussian assumption can be reduced to the following general form as shown above.

$$I = \int ds \; e^{-as^2} \; f(s) \tag{6.22}$$

Several techniques have been developed for numerical evaluation of a single integral with an integrand composed of a general function of the independent variable multiplied by a weighting function. These are generally called Gaussian quadrature formulae. A specialization of this general procedure for the particular weighting function

$$w(s) = e^{-s^2} \tag{6.23}$$

and for an integral with infinite limits is known as the Hermite-Gauss quadrature. Note that Eq. (6.22) can always be rewritten such that Eq. (6.23) is the appropriate weighting function, simply by the proper definition of a new variable.

The Hermite-Gauss quadrature technique is briefly summarized in the following. A polynomial function of the independent variable, s is defined by the following

$$\pi(s) = 2^{-m} \; H_m(s) \tag{6.24}$$

where $H_m(s)$ is the mth Hermite polynomial. (See Appendix C for a definition of Hermite polynomials.) The value of m corresponds to the number of abscissa points to be used in the approximation and is therefore a function of the desired accuracy and acceptable computational complexity.

According to the Hermite-Gauss procedure the appropriate formula for numerical integration is then

$$\int_{-\infty}^{\infty} e^{-s^2} f(s)\, ds = \sum_{r=1}^{m} K_r f(s_r) + E \qquad (6.25)$$

where s_r is the r^{th} zero of the polynomial $\pi(s)$. The r^{th} coefficient is evaluated using the following relationship.

$$K_r = \frac{2^{m+1}\, m!\, \sqrt{\pi}}{[H_{m+1}(s_r)]^2} \qquad (6.26)$$

and the error term, E which represents the remainder when the summation is truncated at a finite value of m, can be expressed as follows.

$$E = \frac{m!\, \sqrt{\pi}}{2^m\, (2m)!} f^{(2m)}(\xi) \qquad (6.27)$$

In the above equations $f^{(2m)}(\xi)$ is evaluated at some appropriate (but unknown) ξ in the same region as s.

As an example of the evaluation of an integral of the form in Eq. (6. 22), consider the following 4 point approximation.

$$\int_{-\infty}^{\infty} e^{-s^2} f(s)\, ds \approx \sum_{r=1}^{4} K_r f(s_r) \tag{6.28}$$

Equation (6. 24), specialized to this example is:

$$\pi(s) = 1/16\, H_4(s) = s^4 - 3\, s^2 + 3/4 \tag{6.29}$$

The zeros of $\pi(s)$ are easily found to be

$$x_1 = \sqrt{\frac{3 + \sqrt{6}}{2}} = 1.651; \qquad x_2 = \sqrt{\frac{3 - \sqrt{6}}{2}} = 0.525$$

$$x_3 = -\sqrt{\frac{3 + \sqrt{6}}{2}} = -1.651; \qquad x_4 = -\sqrt{\frac{3 - \sqrt{6}}{2}} = -0.525$$

$$(6.30)$$

and the corresponding values of the coefficients found from Eq. (6. 26) are given below

$$H_1 = H_3 = 0.081; \qquad H_2 = H_4 = 0.805 \tag{6.31}$$

Thus the 4 point numerical approximation to Eq. (6. 28) is the following

$$\int_{-\infty}^{\infty} e^{-s^2} f(s)\, ds \approx .081\, [\, f(1.651) + f(-1.651)] + .805\, [f(.525) + f(-.525)] \tag{6.32}$$

Another numerical approximation procedure which is also tailored to integrals of the form which arise for the expectation of functions of Gaussian random variables has been suggested by Gelb and Vander Velde (1968). The general form of integral to be evaluated is the following:

$$I = \frac{1}{\sqrt{2\pi}\, \sigma} \int_{-\infty}^{\infty} f(\overline{x} + r)\, e^{-1/2 \frac{r^2}{\sigma^2}}\, dr \tag{6.33}$$

Note that evaluating the integral in Eq. (6. 33) is equivalent to evaluating the integral in Eq. (6. 22). However, the form shown in Eq. (6. 33) is more convenient for this particular discussion.

The Gaussian probability density function is expressed by

$$p(r, \sigma) = \frac{1}{\sqrt{2\pi}\, \sigma}\, e^{-1/2 \frac{r^2}{\sigma^2}} \tag{6.34}$$

and therefore by definition a differential of the associated probability distribution function, F is

$$d\,[F(v)] = \frac{1}{\sqrt{2\pi}}\; e^{-v^2/2}\, dv \tag{6.35}$$

where a new Gaussian random variable with unit variance has been introduced and is defined by

$$v = \frac{r}{\sigma} \tag{6.36}$$

The integral in Eq. (6. 33) can, therefore be expressed as

$$I = \int_0^1 f(\overline{x} + \sigma r)\, d\,[F(v)] \tag{6.37}$$

and the approximation to this integral is given by the following

$$I \approx \delta\,[F(v)] \sum_i f(\overline{x} + \sigma\, v_i) \tag{6.38}$$

where the increments of F(v) are assumed to be equal. The values of v_i at which the nonlinear function is to be evaluated, can be chosen is a number of ways, one of which (suggested by Vander Velde) is a trapezoidal integration rule which approximates each portion of the integral by the area under a straight line segment. This linear function is selected such that it is equal to the nonlinear function at each of the end points of the integration interval.

The approximation given by Eq. (6.38) has the build-in advantage of selecting more sample points where the probability density function is greatest. This is a result of the choice of equal $\delta[F(v)]$ for $0 \leq F(v) \leq 1$.

Using this technique for a cubic nonlinearity, which resprsents a smooth nonlinear function, as well as a relay, chosen to represent a discontinuous nonlinear relationship, Gelb and Vander Velde (1968) have shown that maximum errors do not exceed 10% for a 10 point approximation procedure and 5% for 20 points.

6.4 General Considerations

For many of the estimation problems which can be solved using a quasi-linear technique, there are singularities which can arise in the expression for the nonlinear function. A familiar example of this is the expression for the gravitational acceleration of a spherically symmetric planet given by

$$g = - \frac{\mu}{|r|^3} r \tag{6.39}$$

where μ is the appropriate gravitational constant and r is the position vector of the second body, perhaps a spacecraft. Inherent in the definition of the gravity using Eq. (6.39), is the assumption that the spacecraft state vector always satisfies the following inequality constraint.

$$|r| \geq R$$

where R is the radius of the planet under consideration.

It is possible to define a six-dimensional state vector as follows.

$$x = \begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix} = \begin{bmatrix} r \\ \\ v \end{bmatrix} \tag{6.40}$$

where r and v represent the position and velocity vectors of a body, respectively. Then the differential equation describing the free fall motion of the spacecraft about the planet can be expressed as

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \\ -\mu \dfrac{x_1}{|x_1|^3} \end{bmatrix} \tag{6.41}$$

The associated quasi-linear estimation procedure requires the evaluation of the ensemble expectation of Eq. (6. 41)

$$\hat{\dot{x}} = \begin{bmatrix} x_2 \\ \\ -\mu \overline{\left(\dfrac{x_1}{|x_1|^3}\right)} \end{bmatrix} \tag{6.42}$$

Thus, one of the integrals to be evaluated is the following

$$\int_{-\infty}^{\infty} \cdots \int \frac{x_1}{|x_1|^3} \; e^{-1/2 \, [x-\bar{x}] \, P^{-1} \, [x-\bar{x}]} \; dx$$

where the vector x_1 represents the position component of the state, and P is the covariance matrix of estimation errors. As is obvious from Eq. (6. 42), $x_1 = 0$ is a singular point and would result in an infinite value of the integral if the infinite limits were used. However, as pointed out in the above discussion, the chosen expression for the gravitation acceleration is only valid for $x_1 > R$ and in fact, that is the only region of interest for this problem. The expectation involved in the quasi-linear approximation should therefore be taken over a restricted region.

By way of illustration, consider the following one dimensional example. Suppose the position of a body on a line were to be estimated, but it was known that the body could not lie within a fixed distance of a point defined to be the coordinate origin. (See Fig. 6-1.)

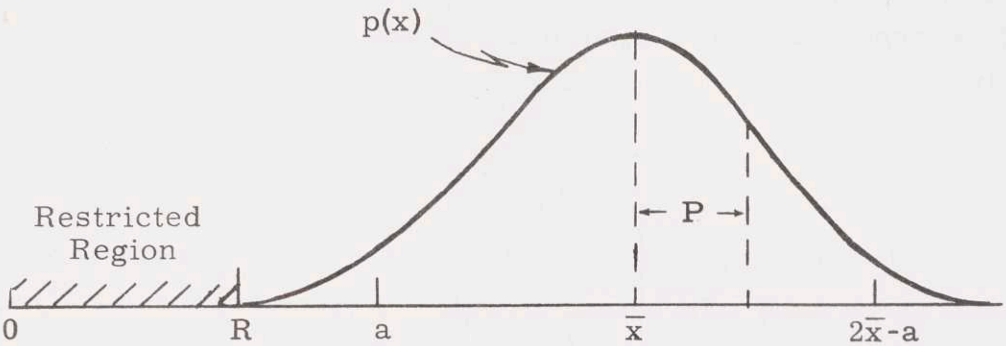


Figure 6-1

Under the quasi-linear assumption, the position of the body is a Gaussian random variable with mean $\bar{x}$ and variance P. The density function of such a random variable is depicted in the figure. If one were to select the region of integration to be from a to $2\bar{x} - a$, where $a \geq R$, then any numerical difficulty in the region 0 to R, if it existed, would be avoided. The inaccuracy which might be introduced by the above procedure can be quickly evaluated by examining the magnitude of P with respect to the interval $\bar{x} - a$ and the point, R. If, for example, the density function of x were to look similar to that shown on the following page.

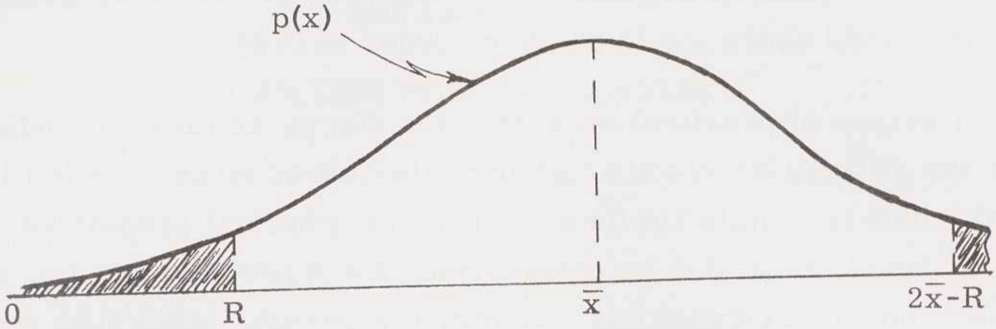


Figure 6-2

Then even choosing the largest acceptable and symmetric region of integration, from R to $2\bar{x} - R$ would still result in the neglect of the shaded regions.

Because of the Gaussian form of p(x), knowledge of P will allow an exact determination of the area of the probability density function which has been neglected and consequently an evaluation of the inaccuracy introduced by restricting the region of integration is straightforward.

In many practical situations, the loss of accuracy as a result of using finite integration limits is not significant as illustrated by the following. Consider a spacecraft circling the Earth in an orbit the altitude of which has an expected value equal to 100 n. m. The region of integration used to evaluate the expected value of the gravitational acceleration is restricted to

$$R_E \leq |x_1| \leq 2\,|\bar{x}_1| - R_E$$

where R_E is the radius of the Earth and $\bar{x}_1$ the expected value of the position vector. Then a one sigma error in the estimate of the spacecraft altitude can be as large as 33 n. m. and the area of the density function which would be neglected as a result of the finite integration limits described above would be of the order of 0.3%.

The type of practical consideration discussed above is indicative of both the difficulties which arise in evaluation of an expectation integral over doubly infinite limits as well as the physical properties of a problem, which must also be considered. Each problem has its own particular physical constraints and very few general techniques can be developed. Also, it is useful to note that even though an expectation integral constains no singularities, one may take advantage of the physical interpretation of such an expression to possibly simplify the numerical or analytic integration and also to perhaps increase the accuracy of a numerical approximation for a given computational complexity.

CHAPTER VII

THE VAN DER POL OSCILLATOR

7.1 Introduction

The purpose of this chapter is to investigate in some depth the behavior of the linear and quasi-linear estimation techniques. The example system chosen is a second order oscillator with highly nonlinear behavior allowing a demonstration of the multi-dimensional form of the estimation equations as well as a comparison of the two estimators in a truly nonlinear environment. In contrast to the example in Chapter V, this investigation will deal with only two estimation schemes and will consider some of the sensitivities of each to various system parameters as well as the stability of the estimated quantities.

The nonlinear oscillator to be estimated is described by the classic van der Pol equation given below

$$\ddot{x} = -x + \epsilon \dot{x} (1 - x^2) \tag{7.1}$$

This second order differential equation may be rewritten as two first order equations in the following way

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \epsilon x_2 (1 - x_1^2) \end{aligned} \tag{7.2}$$

By defining a vector

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{7.3}$$

Equation (7. 2) can be written in vector form as follows

$$\dot{x} = f(x) = \begin{bmatrix} x_2 \\ -x_1 + \epsilon\, x_2\,(1 - x_1^2) \end{bmatrix} \tag{7.4}$$

The behavior of the van der Pol oscillator is depicted in Figs. 7-1 and 7-2. In the first figure, the behavior of x_1 is plotted for three different values of ϵ. As is obvious from Eq. (7. 1) as $\epsilon \to 0$, $x_1(t) \to \cos(t)$ and for relatively large ϵ, x_1 switches sign rapidly. Figure 7-2 shows the behavior of x_2 as a function of time, demonstrating the short periods of large velocity. As can be seen from Eq. (7. 2), these will occur when $|x_1| < 1$ and are the regions of greatest interest as far as estimation behavior is concerned. Based on these figures, a value of $\epsilon = 3$ has been selected for the remaining investigation.

It will be assumed that the actual state derivatives are the ideal, described by Eqs. (7. 2), plus additive Gaussian white noise as shown below:

$$\begin{aligned} \dot{x}_1 &= x_2 + u_1 \\ \dot{x}_2 &= -x_1 + \epsilon\, x_2\,(1 - x_1^2) + u_2 \end{aligned} \tag{7.5}$$

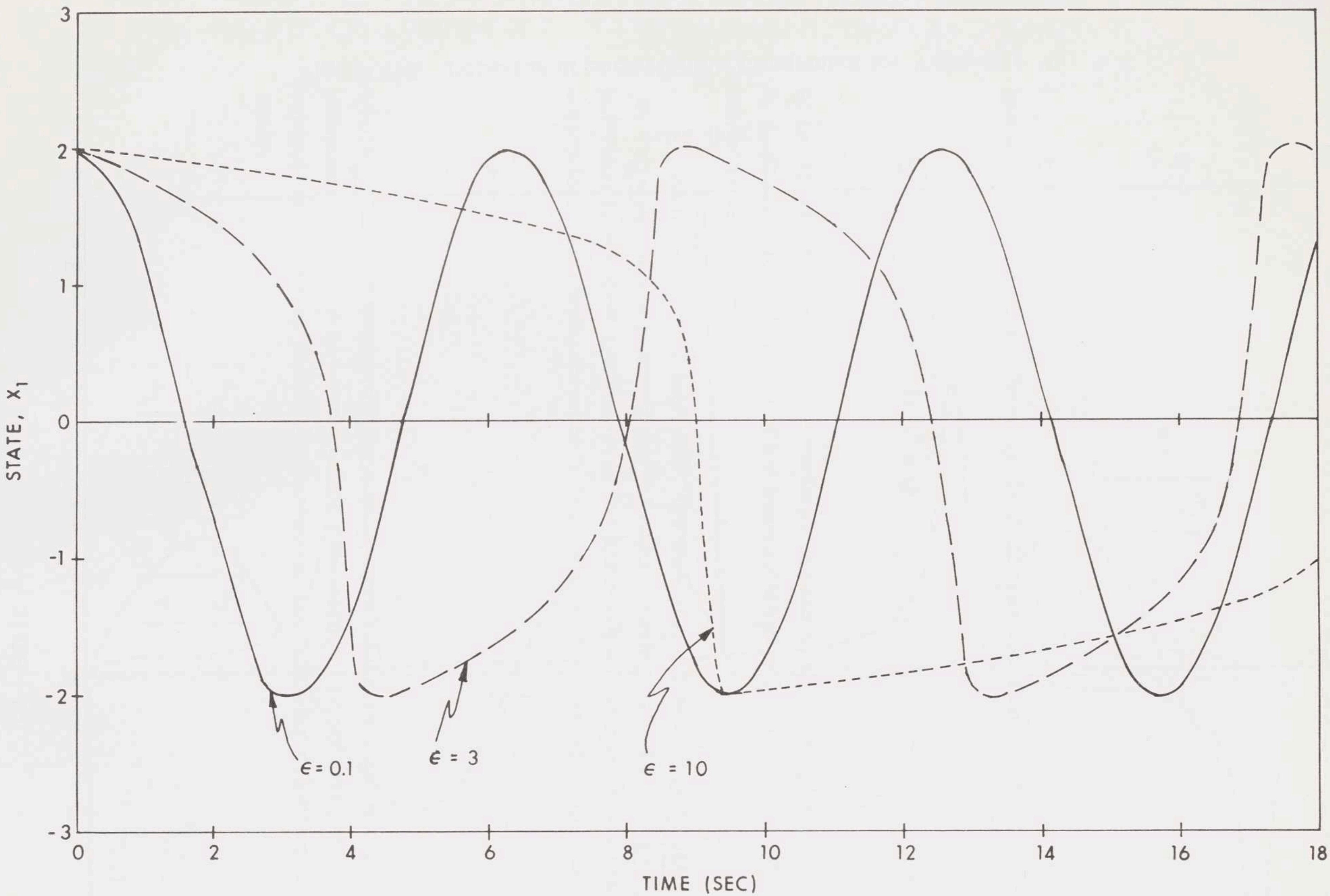


Figure 7-1 Behavior of First State Component for Various ϵ

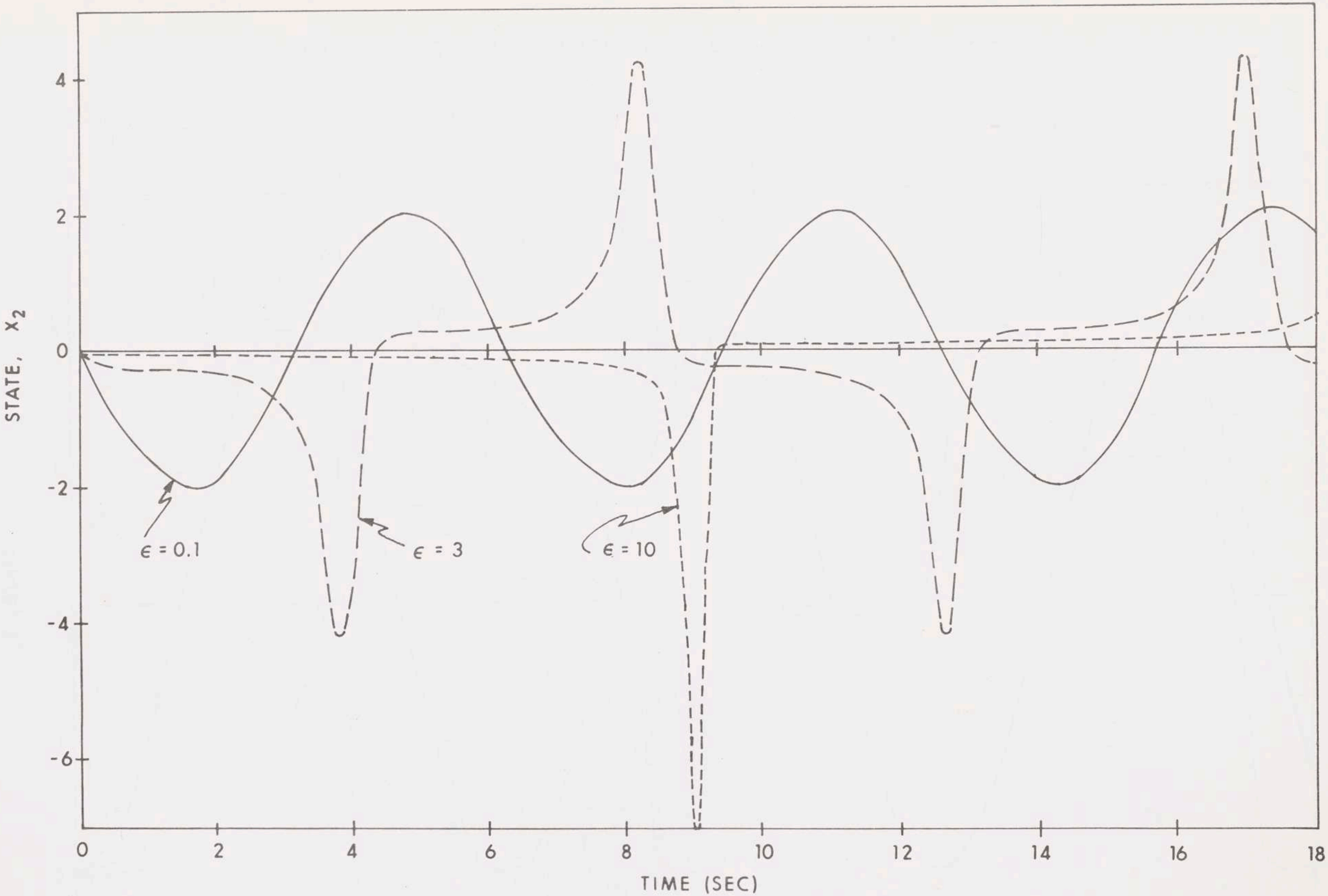


Figure 7-2 Behavior of Second State Component for Various ϵ

where

$$u = \begin{bmatrix} u_1 \\ \\ u_2 \end{bmatrix} \tag{7.6}$$

has a variance matrix defined by the following equation.

$$\overline{u(\tau_2)\, u(\tau_1)^T} = Q\, \delta\, (\tau_2 - \tau_1) \tag{7.7}$$

In Eq. (7. 7), Q represents the variance matrix of the noise process and is diagonal in form because of the assumption of independent white noise components in u.

At discrete time intervals, the size of which will remain one of the variable system parameters, a scalar measurement directly proportional to the first component of the state will be used to update the estimates. This measurement is described by the following equation

$$z_1 = x_1 + v_1 \tag{7.8}$$

where v_1 is again assumed to be a zero mean Gaussian white noise process independent of x and u. For convenience in comparing this example to a general vector measurement, the following definition is made.

$$z = \begin{bmatrix} z_1 \\ \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \\ 0 \end{bmatrix} + \begin{bmatrix} v_1 \\ \\ 0 \end{bmatrix}$$

$$z = H\, x + v \tag{7.9}$$

In Eq. (7. 9), the geometry matrix, H is given by

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{7.10}$$

and

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{7.11}$$

The Gaussian process, v has the following variance matrix.

$$\overline{v(\tau_2)\, v(\tau_1)^T} = R\, \delta\, (\tau_2 - \tau_1)$$

Thus, Eq. (7. 5) describes the system dynamics and Eq. (7. 9) the measurement process associated with the actual system to be estimated with linear and quasi-linear estimation schemes.

7.2 Linear Estimation Equations

As discussed in Chapter V, the linear estimation procedure is based upon the definition of a reference state and deviations from that reference.

$$\delta x = \begin{bmatrix} \delta x_1 \\ \\ \delta x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_{1R} \\ \\ x_2 - x_{2R} \end{bmatrix} \tag{7.12}$$

Using this definition, the differential equations for the actual state may be expanded as follows

$$\dot{x}_1 = \dot{x}_{1R} + \delta\dot{x}_1 = x_{2R} + \delta x_2 + u_1$$

$$\dot{x}_2 = \dot{x}_{2R} + \delta\dot{x}_2 = -(x_{1R} + \delta x_1) + \epsilon\,[x_{2R} + \delta x_2]\,[1 - (x_{1R} + \delta x_1)^2] + u_2 \tag{7.13}$$

Expanding the second equation above and dropping all variational terms of order 2 or higher results in

$$\dot{x}_{2R} + \delta\dot{x}_2 = -(x_{1R} + \delta x_1) + \epsilon\, x_{2R}(1 - x_{1R}^2) + \epsilon\,[(1 - x_{1R}^2)\,\delta x_2 - 2\,x_{1R}\,x_{2R}\,\delta x_1] + u_2 \tag{7.14}$$

By defining the reference state differential equations to be

$$\dot{x}_{1R} = x_{2R}$$

$$\dot{x}_{2R} = -x_{1R} + \epsilon\, x_{2R}\,(1 - x_{1R}^2) \tag{7.15}$$

the deviation state dynamics are then described by the following:

$$\delta\dot{x}_1 = \delta x_2 + u_1$$

$$\delta\dot{x}_2 = -(1 + 2\epsilon\, x_{1R}\, x_{2R})\,\delta x_1 + \epsilon\,(1 - x_{1R}^2)\,\delta x_2 + u_2 \tag{7.16}$$

For convenience in defining the estimation equations in vector form, Eqs. (7.16) can be rewritten as follows

$$\delta\dot{x} = F\,\delta x + u \tag{7.17}$$

where the matrix F is defined by

$$F = \begin{bmatrix} 0 & 1 \\ -1 - 2\,\epsilon\, x_{1R}\, x_{2R} & \epsilon\,(1 - x_{1R}^2) \end{bmatrix} \tag{7.18}$$

and

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{7.19}$$

At any time, the linear state estimate is found from Eq. (7.20)

$$\bar{x}_L = x_R + \overline{\delta x} \tag{7.20}$$

where $\overline{\delta x}$ is the ensemble expectation of the deviation state and is the solution to the following differential equation

$$\dot{\overline{\delta x}} = F\,\overline{\delta x} \tag{7.21}$$

In addition to the state estimate, a statistical measurement of the estimation errors is also required for proper measurement incorporation. This quantity defined by the following matrix

$$P = \overline{(\delta x - \overline{\delta x})\,(\delta x - \overline{\delta x})^T} \tag{7.22}$$

is extrapolated from one measurement to the next with the following equation

$$\dot{P} = FP + PF^T + Q \tag{7.23}$$

The measurement incorporation procedure for the linear filter again utilizes a deviation vector defined as follows

$$\begin{aligned} \delta z &= z - z_R = H(x - x_R) = H\,\delta x + v \\ \overline{\delta z} &= H\,\overline{\delta x} \end{aligned} \tag{7.24}$$

where H is given by Eq. (7.10).

Defining an optimum gain matrix by the following expression

$$B = P^{-} H^{T} [H P^{-} H^{T} + R]^{-1} \tag{7.25}$$

the update equations for the state estimate and the covariance matrix are:

$$\overline{\delta x^{+}} = \overline{\delta x^{-}} + B [\delta z - H \overline{\delta x^{-}}] \tag{7.26}$$

$$P^{+} = (I - BH) P^{-} (I - BH)^{T} + BRB^{T} \tag{7.27}$$

where I is the identity matrix, in this case two-dimensional.

After each measurement incorporation, the linear estimate is found by employing Eq. (7.20).

The rectification procedure which was discussed in Chapter V will be employed here, and the effect of redefining the reference at intervals greater than the measurement interval will also be examined.

7.3 Quasi-Linear Estimation Equations

The results of Chapter IV have indicated two possible sets of estimation equations for the quasi-linear technique. The simplified set is only rigorously true if $\bar{x}$ is "small". The particular estimation problem considered in this chapter has an associated $|\bar{x}|$ which ranges up to approximately 2.5, however, it does have the interesting property that, for an accurate estimator the time average of $\bar{x}$ is close to zero. Therefore, the simplified set of equations may possibly result in a useful filter. Because of this, both sets of estimation equations will be derived here and each resulting filter will be simulated in order to compare the performance.

7.3.1 General Formulation

The general set of quasi-linear estimation equations which were derived in Chapter IV requires the use of two gain matrices which are defined by the following equations

$$N_f^e = \left[\overline{f(x)\, x^T} - \overline{f(x)}\, \bar{x}^T \right] P^{-1}$$

$$N_f^{\bar{x}} = \frac{\overline{f(x)}\, \bar{x}^T}{\bar{x}^T \bar{x}} + N_f^e \left[I - \frac{\bar{x}\bar{x}^T}{\bar{x}^T \bar{x}} \right] \tag{7.28}$$

However, it is important to note that the matrices themselves need never actually be computed because they always appear in the estimation equations as follows. The state estimation is given by

$$\dot{\bar{x}} = N_f^{\bar{x}}\, \bar{x} = \overline{f(x)} \tag{7.29}$$

Thus, only $\overline{f(x)}$ need be evaluated and not the entire $N_f^{\bar{x}}$ matrix. Similarly N_f^e appears as follows

$$\dot{P} = N_f^e \, P + P \, N_f^{e^T} + G Q G^T \tag{7.30}$$

For this particular case, with G = I, Eq. (7.30) reduces to

$$\dot{P} = [\overline{f(x) \, x^T} - \overline{f(x)} \, \bar{x}^T] + [\overline{x \, f(x)^T} - \bar{x} \, \overline{f(x)}^T] + Q \tag{7.31}$$

Therefore, the only new quantity to be evaluated is $\overline{f(x) \, x^T}$.

Specializing f(x) to the problem under consideration,

$$f(x) = \begin{bmatrix} x_2 \\ \\ -x_1 + \epsilon \, x_2 \, (1 - x_1^2) \end{bmatrix} \tag{7.32}$$

Taking the expectation of Eq. (7.32) results in

$$\overline{f(x)} = \begin{bmatrix} \bar{x}_2 \\ \\ -x_1 + \epsilon \, (\bar{x}_2 - \overline{x_2 \, x_1^2}) \end{bmatrix} \tag{7.33}$$

The expectation, $\overline{x_2 x_1^2}$ may be further simplified as follows. Expanding each state variable in terms of a mean and random component

$$\overline{x_2 x_1^2} = \overline{(\bar{x}_2 + r_2)(\bar{x}_1 + r_1)^2} \tag{7.34}$$

$$\overline{x_2 x_1^2} = \bar{x}_2 \bar{x}_1^2 + 2\bar{x}_1 \bar{x}_2 \bar{r}_1 + \bar{x}_2 \overline{r_1^2} + \bar{x}_1^2 \bar{r}_2 + 2\bar{x}_1 \overline{r_1 r_2} + \overline{r_1^2 r_2} \tag{7.35}$$

Using the fact that r is treated as a zero mean Gaussian random variable, Eq. (7.35) reduces to the following:

$$\overline{x_2 x_1^2} = \bar{x}_2 \bar{x}_1^2 + \bar{x}_2 P_{11} + 2\bar{x}_1 P_{12}$$

Thus, Eq. (7.33) becomes

$$\overline{f(x)} = \begin{bmatrix} \bar{x}_2 \\ \\ -\bar{x}_1 + \epsilon (\bar{x}_2 - \bar{x}_1^2 \bar{x}_2 - \bar{x}_2 P_{11} - 2\bar{x}_1 P_{12}) \end{bmatrix} \tag{7.36}$$

The matrix $\overline{f(x)\, x^T}$ may be evaluated in a similar manner.

$$\overline{f(x)\,x^T} = \begin{bmatrix} \overline{x_1 \, x_2} & \overline{x_2^{\,2}} \\[2em] -\overline{x_1^{\,2}} + \epsilon\, \overline{x_1 \, x_2 \,(1 - x_1^{\,2})} & -\overline{x_1 \, x_2} + \epsilon\, \overline{x_2^{\,2}\,(1 - x_1^{\,2})} \end{bmatrix} \tag{7.37}$$

The expectations of products of state components in Eq. (7.37) may be reduced in the same way as shown for $\overline{f(x)}$. The result is

$$\overline{f(x)\,x^T} = \begin{bmatrix} \bar{x}_1\,\bar{x}_2 + P_{12} & \bar{x}_2^{\,2} + P_{22} \\[2em] \begin{array}{c} -(\bar{x}_1^{\,2} + P_{11}) + \epsilon\,(\bar{x}_1\,\bar{x}_2 + P_{12}) \\[1em] -\epsilon\,(\bar{x}_1^{\,3}\,\bar{x}_2 + 3\,\bar{x}_1^{\,2}\,P_{12} + 3\,\bar{x}_1\,\bar{x}_2\,P_{11} + P_{1112}) \end{array} & \begin{array}{c} -(\bar{x}_1\,\bar{x}_2 + P_{12}) + \epsilon\,(\bar{x}_2^{\,2} + P_{22}) \\[1em] -\epsilon\,(\bar{x}_1^{\,2}\,\bar{x}_2^{\,2} + \bar{x}_1^{\,2}\,P_{22} + 4\,\bar{x}_1\,\bar{x}_2\,P_{12} + \bar{x}_2^{\,2}\,P_{11} + P_{1122}) \end{array} \end{bmatrix} \tag{7.38}$$

where P_{1112} and P_{1122} represent the fourth moment terms $\overline{r_1^{\,3}\, r_2}$ and $\overline{r_1^{\,2}\, r_2^{\,2}}$ repsectively. As shown in Appendix D, because of the assumed Gaussian nature of r, these fourth moment terms can be expressed as functions of second moment terms. In this case

$$
\begin{aligned}
P_{1112} &= 3\, P_{11}\, P_{12} \\
P_{1122} &= P_{11}\, P_{22} + 2\, P_{12}^{\,2}
\end{aligned}
\tag{7. 39}
$$

Each element of the $\overline{f(x)\, x^T}$ matrix is therefore given by the following:

$$
\begin{aligned}
[\overline{f(x)\, x^T}]_{11} &= \bar{x}_1\, \bar{x}_2 + P_{12} \\
[\overline{f(x)\, x^T}]_{12} &= \bar{x}_2^{\,2} + P_{22} \\
[\overline{f(x)\, x^T}]_{21} &= -(\bar{x}_1^{\,2} + P_{11}) + \epsilon(\bar{x}_1\, \bar{x}_2 + P_{12} - \bar{x}_1^{\,3}\, \bar{x}_2 - 3\bar{x}_1^{\,2}\, P_{12} \\
&\qquad - 3\bar{x}_1\, \bar{x}_2\, P_{11} - 3\, P_{11}\, P_{12}) \\
[\overline{f(x)\, x^T}]_{22} &= -(\bar{x}_1\, \bar{x}_2 + P_{12}) + \epsilon(\bar{x}_2^{\,2} + P_{22} - \bar{x}_1^{\,2}\, \bar{x}_2^{\,2} - \bar{x}_1^{\,2}\, P_{22} \\
&\qquad - 4\bar{x}_1\, \bar{x}_2\, P_{12} - \bar{x}_2^{\,2}\, P_{11} - P_{11}\, P_{22} - 2\, P_{12}^{\,2})
\end{aligned}
\tag{7. 40}
$$

The expressions in Eq. (7. 40) along with Eq. (7. 36) are used in Eq. (7. 29) and (7. 31) for extrapolation of the quasi-linear estimate. .

7.3.2 Simplified Formulation

As discussed in Chapter IV, for an odd nonlinearity, the quasi-linear approximation matrix for $\overline{x}$ approaches the corresponding matrix for e as $\overline{x} \rightarrow 0$. The example under consideration in this chapter, in addition to having a nonlinearity which is odd, also has the interesting property that the time average of x is zero, and consequently for a good estimator, the time average of $\overline{x}$ should also be close to zero. Because of these special characteristics, it is of interest to also consider here the simplified set of quasi-linear estimation equations which require the computation of only one approximation matrix. This matrix is defined by the following equation.

$$N_f = \frac{\overline{f(x)}\,\overline{x}^T}{\overline{x}^T\,\overline{x}} \tag{7.41}$$

The matrix N_f is employed in the extrapolation of both the state estimate and the covariance matrix using the following equations:

$$\dot{\overline{x}} = N_f\,\overline{x} \tag{7.42}$$

$$\dot{P} = N_f\,P + P\,N_f^T + Q \tag{7.43}$$

In order to evaluate N_f as defined in Eq. (7.41), the ensemble expectation of f(x) is required; however, Eq. (7.36) provides an expression for $\overline{f(x)}$ and N_f is thus found to be

$$N_f = \frac{1}{\bar{x}_1^{\,2} + \bar{x}_2^{\,2}} \begin{bmatrix} \bar{x}_2\,\bar{x}_1 & \bar{x}_2^{\,2} \\ -\bar{x}_1^{\,2} + \epsilon\,\bar{x}_1\,\bar{x}_2 & -\bar{x}_1\,\bar{x}_2 + \epsilon\,\bar{x}_2^{\,2} \\ -\epsilon\,\bar{x}_1\,(\bar{x}_1^{\,2}\bar{x}_2 + \bar{x}_2 P_{11} + 2\bar{x}_1 P_{12}) & -\epsilon\,\bar{x}_2(\bar{x}_2\,\bar{x}_1^{\,2} + \bar{x}_2 P_{11} + 2\bar{x}_1 P_{12}) \end{bmatrix}$$

(7.44)

7.3.3 Measurement Incorporation

The measurement incorporation procedure for each of the two quasi-linear estimators is the same. Because the measurement is a linear one, the following linear update equations are used. If the measurement is defined by

$$z = Hx + v = \begin{bmatrix} x_1 + v_1 \\ 0 \end{bmatrix} \qquad (7.45)$$

then the state update is given by the following

$$\bar{x}^{+} = \bar{x}^{-} + K\,[z - H\bar{x}^{-}] \qquad (7.46)$$

where

$$K = P^{-} H^{T}\,[H P^{-} H^{T} + R]^{-1} \qquad (7.47)$$

The proper update equation for the covariance matrix is given by

$$P^{+} = P^{-} - K\,H\,P^{-} = (I - K\,H)\,P^{-}\,(I - K\,H)^{T} + K\,R\,K^{T} \qquad (7.48)$$

where K is defined by Eq. (7.47).

Using the three sets of estimation equations, one for a linear estimator and two for quasi-linear techniques, a numerical study is used to demonstrate some of the performance characteristics of each estimator.

7.4 Digital Simulation and Results

As in Chapter V, an IBM 360/75 computer and the MAC compiler language of the M. I. T. Instrumentation Laboratory was used to simulate the above estimation equations. Again, a series of Monte Carlo runs were performed with actual random numbers selected from a Gaussian distribution with specified variance.

The rectification procedure discussed in the above description of the linearized estimator was examined numerically to determine the importance of redefining the reference state to be the current estimate. The two extremes of this procedure are, of course, to redefine the reference after each measurement or to use the initial reference throughout the run. These two situations are depicted in Fig. 7-3. The need for rectification for this particular nonlinear system is clear; however, it was found that redefining the reference after every two or even five measurement incorporations did not significantly degrade the linear estimator performance. The measurement intervals for all of the simulation runs were 0.5 seconds. The performance of the estimators was also not very sensitive to this parameter. As can be seen from the figure the initial error in the estimate is 1 and is therefore not small enough for a linear assumption to be valid. The requirement for rectification is also a function of this error and must, therefore, be assessed for each set of initial conditions.

The discussion of the remaining simulation runs is divided into two parts. The first part concerns a set of three runs all of which have the same initial error of 1 unit and an initial variance estimate of 1 sq. unit indicating that this error represents a 1σ value. In this set of runs two levels of measurement noise variance and two levels of system driving noise variance are examined. The second part of the discussion is concerned with two sets of Monte Carlo runs, each representing a ten sample case from the same distribution of random processes. For each Monte Carlo set the initial conditions as well as the measurement errors are random numbers selected from a Gaussian

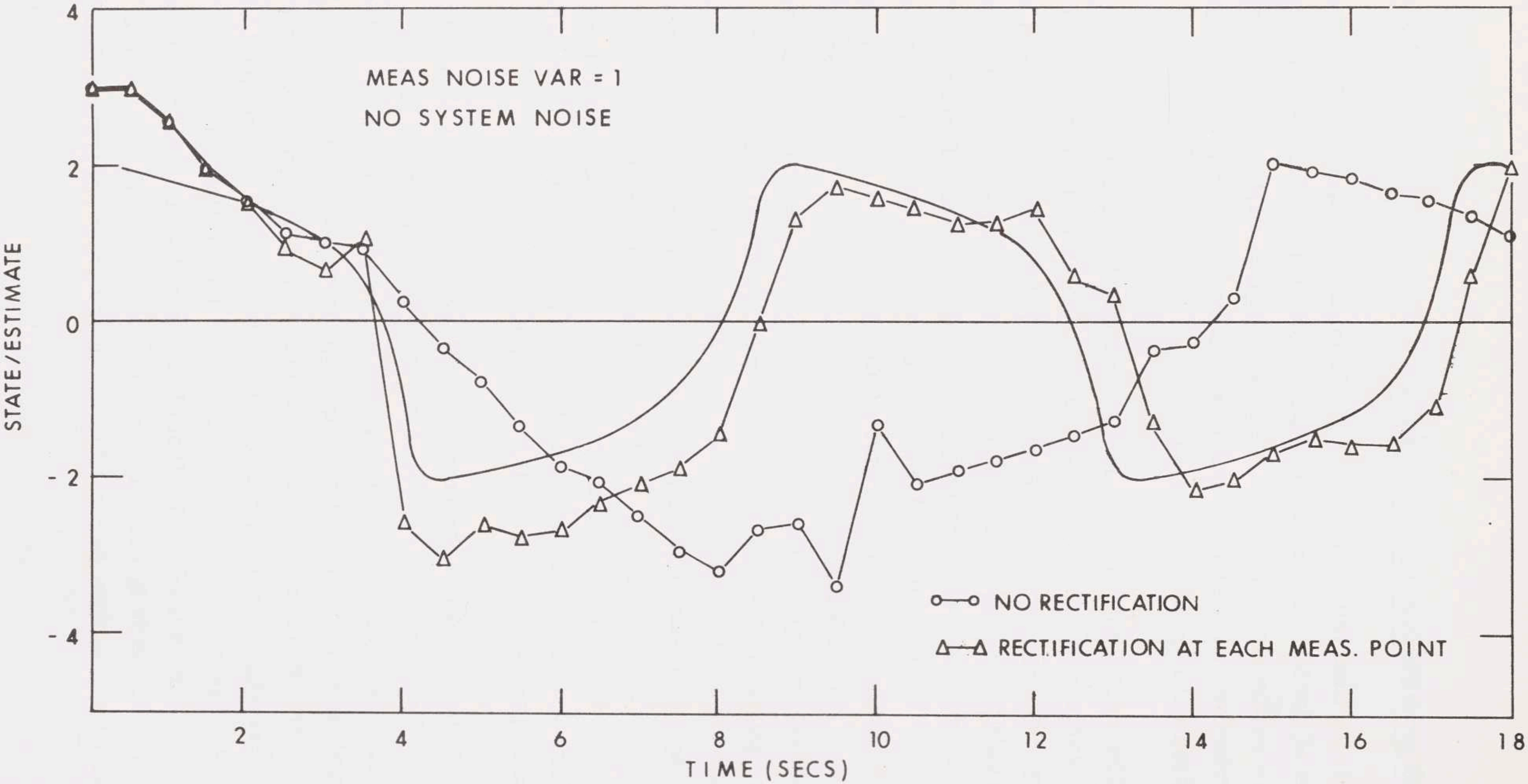


Figure 7-3 Effect of Rectification in Linear Estimation

distribution. Each set of runs contained initial errors with a variance of 1 sq. unit. Each set did not have any system driving noise, however, the measurement noise variance was 1 sq. unit in the first case and 4 sq. units in the second case.

Examining the individual runs in detail, the first case considered did not have any system driving noise and the measurement noise variance was 1 sq. unit. Figure 7-4 demonstrates the behavior of the state estimate as determined by a linear estimator and the general form of the quasi-linear estimator. A comparison of the two quasi-linear estimation formulations for this same case is shown in Fig. 7-5. Associated with each estimation scheme is an estimate of the mean squared error in the state estimate. This quantity, hereafter referred to as the variance estimate, is determined from the appropriate equations for the quantity, P discussed in the previous sections. The behavior of the variance estimate for each of the three types of estimators which corresponds to this particular run is depicted in Fig. 7-6. As can be seen from these three figures, both quasi-linear estimators have slightly better performance than the linear estimate and are comparable to each other. In Fig. 7-6, the relatively sharp peaks which occur for the linear estimate of the variance indicate some instability in this quantity. This problem as well as similar difficulties with the estimate of the state are more pronounced with larger measurement noise.

The next set of three figures shows the behavior of these same three estimators for a run with measurement noise variance equal to 4 sq. units. The other system parameters and initial conditions are the same as those used for the run discussed above. Figures 7-7 through 7-9 correspond, in content, to the first set of figures with the first two depicting estimator behavior and the third showing the estimated variance as a function of time.

From Fig. 7-7, it is clear that the linear estimator exhibits periods of relative instability particularly at the points where the estimate changes algebraic sign as demonstrated by the large "overshoot" in the dynamics of the estimator. Figure 7-9 reveals a similar erratic

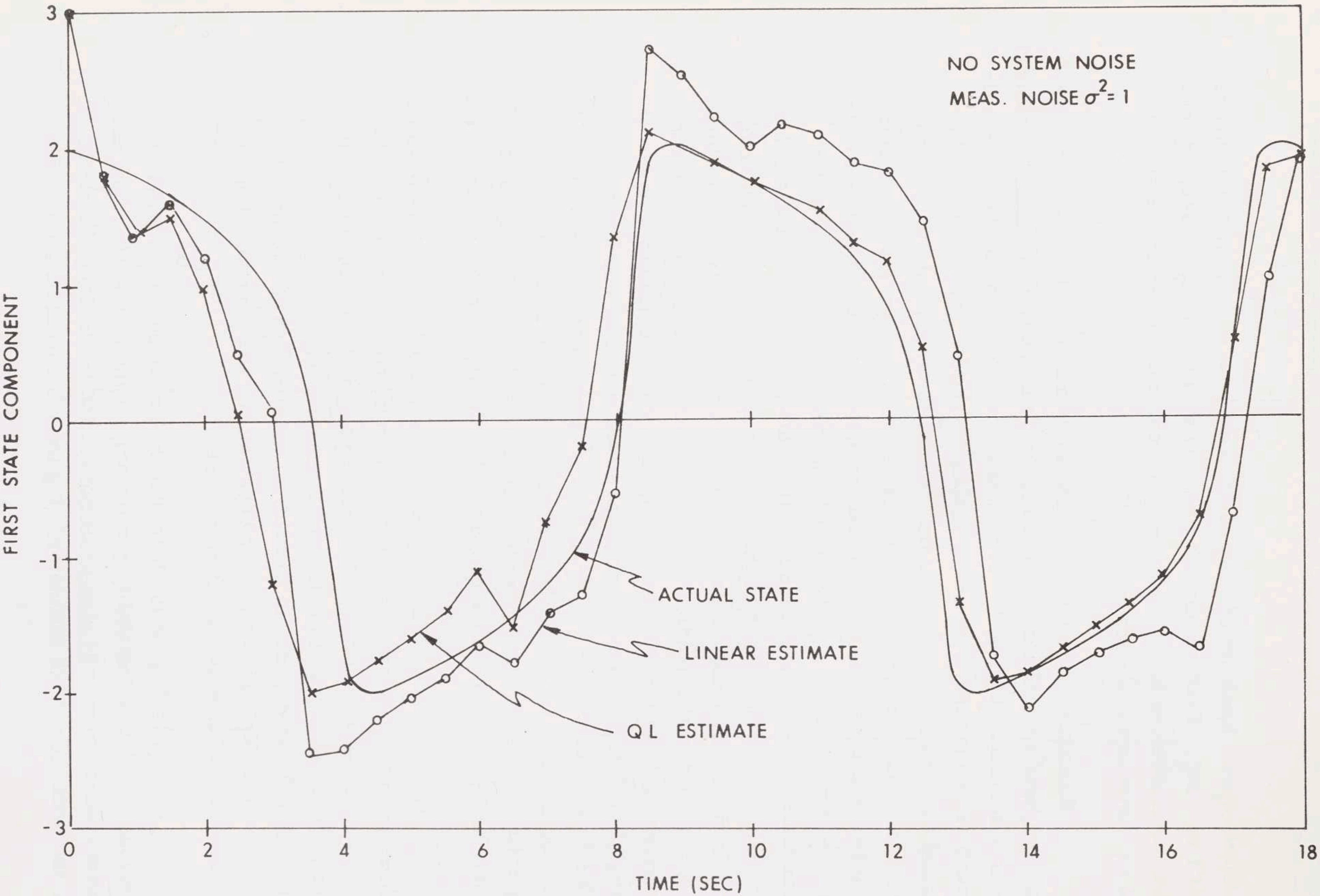


Figure 7-4 State Estimates - Linear, Quasi-Linear

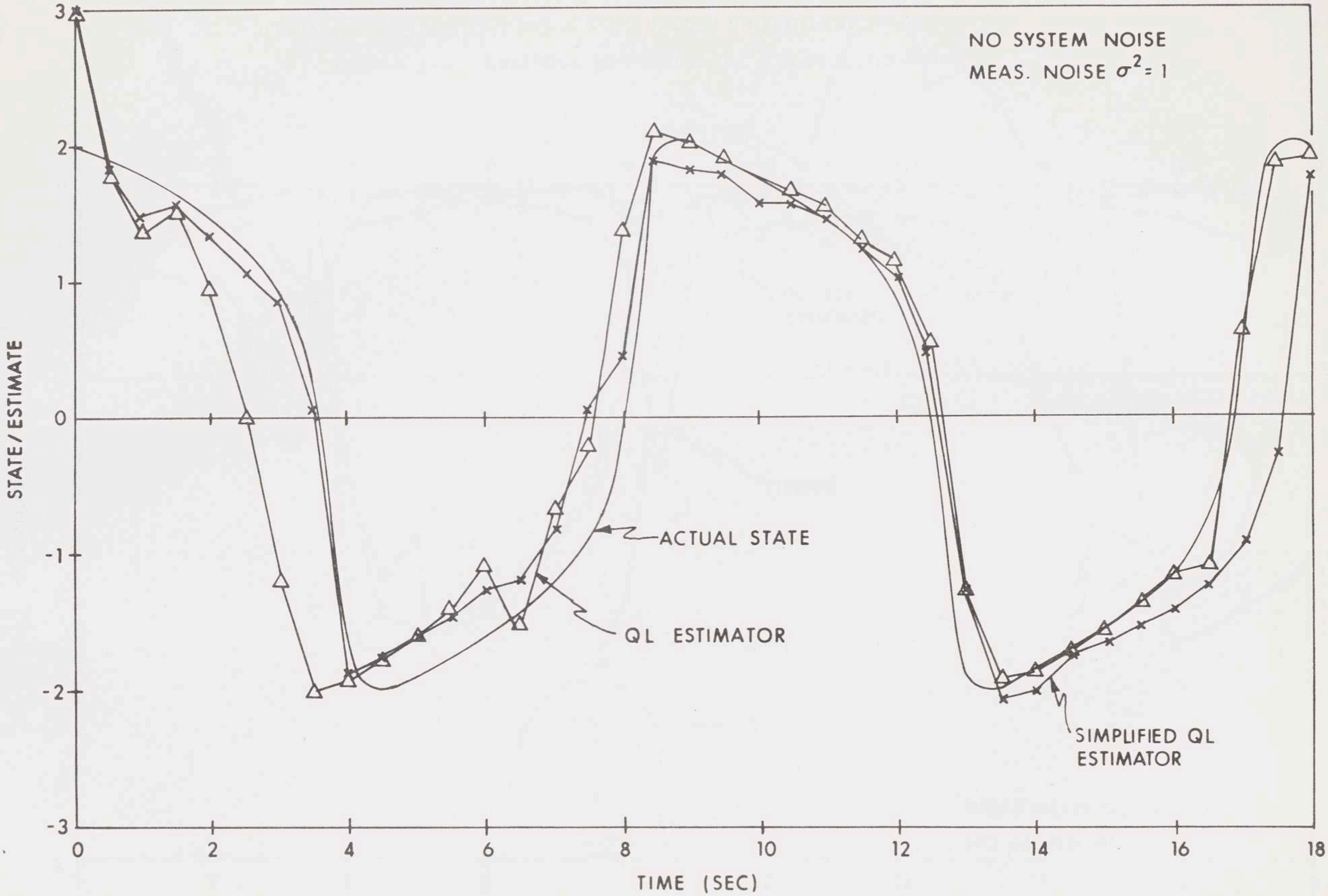


Figure 7-5 State Estimates - Two Quasi-Linear Estimators

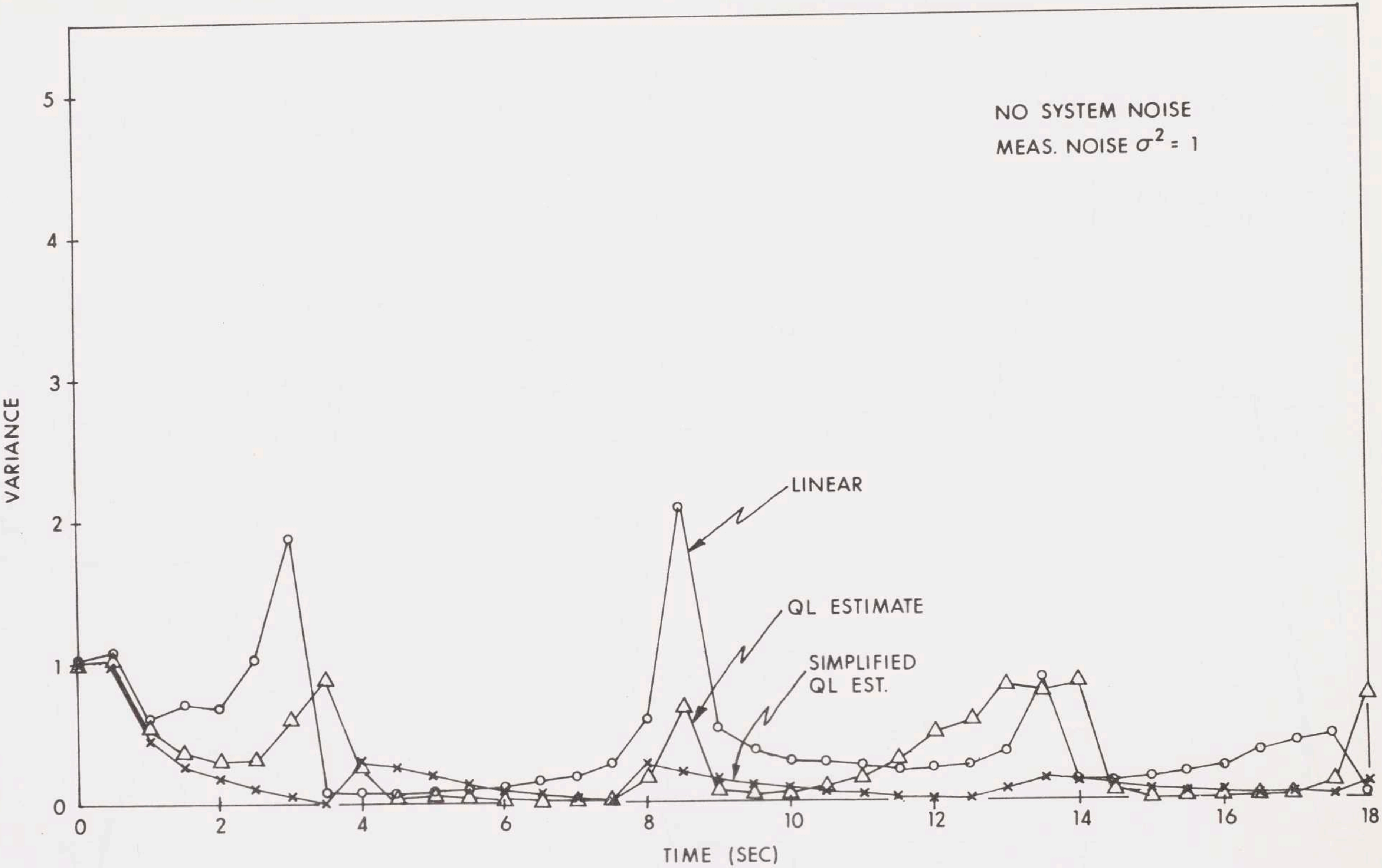


Figure 7-6 Variance Estimates for Three Estimators

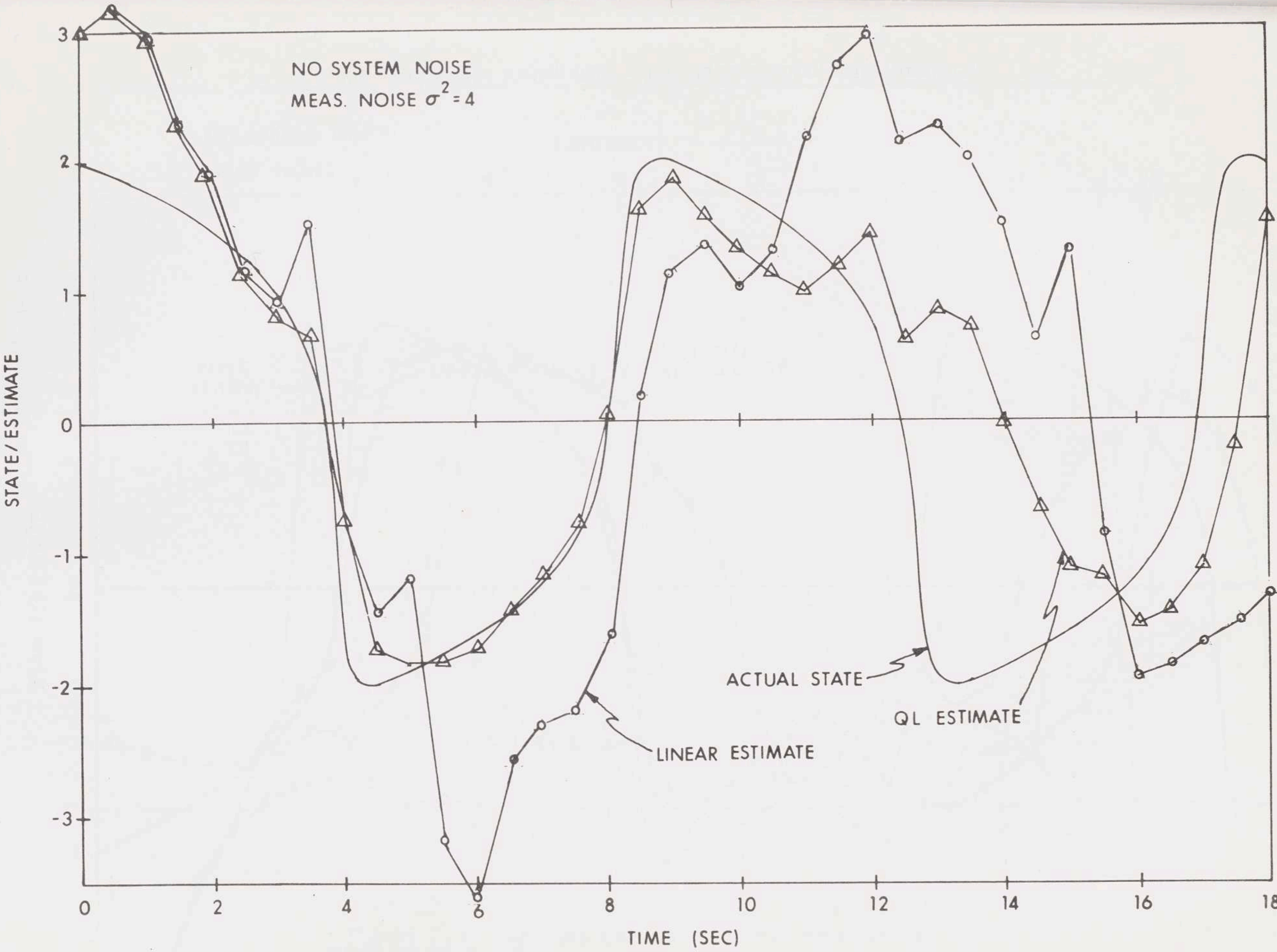


Figure 7-7 State Estimates - Linear, Quasi-Linear

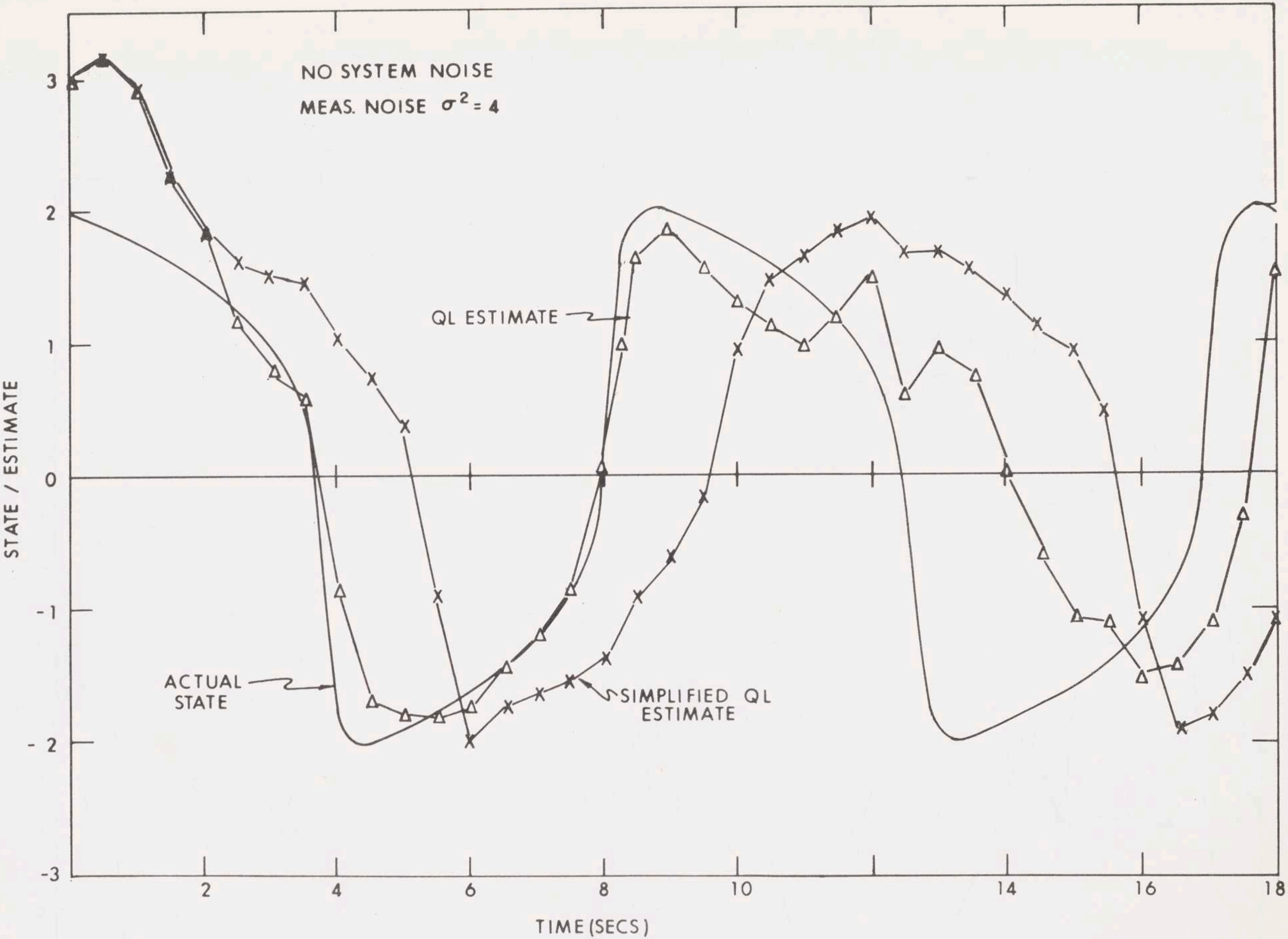


Figure 7-8 State Estimates - Two Quasi-Linear Estimators

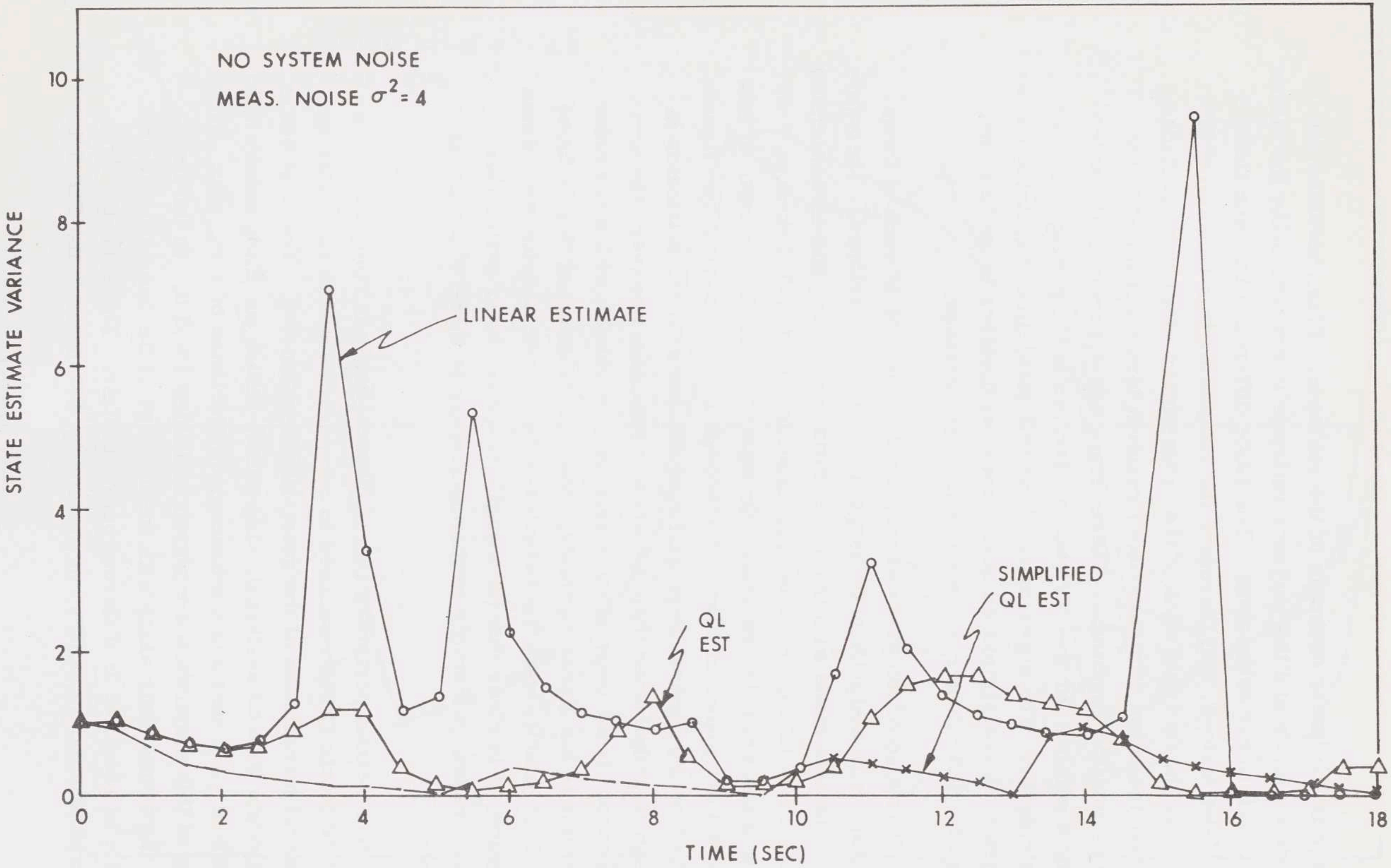


Figure 7-9 Variance Estimate - Three Estimators

behavior in the linear estimate of the variance. This characteristic of the state estimate and the variance estimate is not noticeable for either of the quasi-linear estimators. The basic difference between the two types of estimators, and therefore the reason for this behavioral difference, is the fact that each of the quasi-linear estimators includes some consideration of higher order moment terms not found in the linear estimation equations. These terms are apparent in the extrapolation equations for the variance as derived in the previous section of this chapter. The superiority of the full quasi-linear estimator over the linearized and simplified quasi-linear estimators is more evident in this case with larger measurement noise variance.

It is also of interest to consider the behavior of each of these estimators with a significantly large driving noise variance. Therefore, another run was made with the same initial conditions and measurement parameters as in the first case (summarized in Figs. 7-4 through 7-6) but with a system noise variance, Q, equal to 0. 5. The results of this simulation are shown in Figs. 7-10 through 7-12. The first two figures indicate that the general formulation of the quasi-linear estimator is still significantly better than the linear estimator; however, the simplified quasi-linear estimation equations give results which are quite comparable to the linear estimate. Further evidence of this is found in Fig. 7-12 which shows the behavior of the variance estimate. From this figure, it is clear that the simplified quasi-linear estimation equations result in stability problems similar to those of the linear estimator.

The second portion of this numerical study is composed of two sets of 10 Monte Carlo runs used to compare the linear estimator and the general formulation of the quasi-linear estimator. The first set is for a system with no system driving noise, initial condition errors with variance of 1 sq. unit and a measurement variance of 1 sq. unit. The results of this simulation are shown in Tables I and II. In the first table, the actual state along with the average of the two estimates of the state is shown at 21 different points in time. The average error is also shown and a quantity not previously discussed is used to help

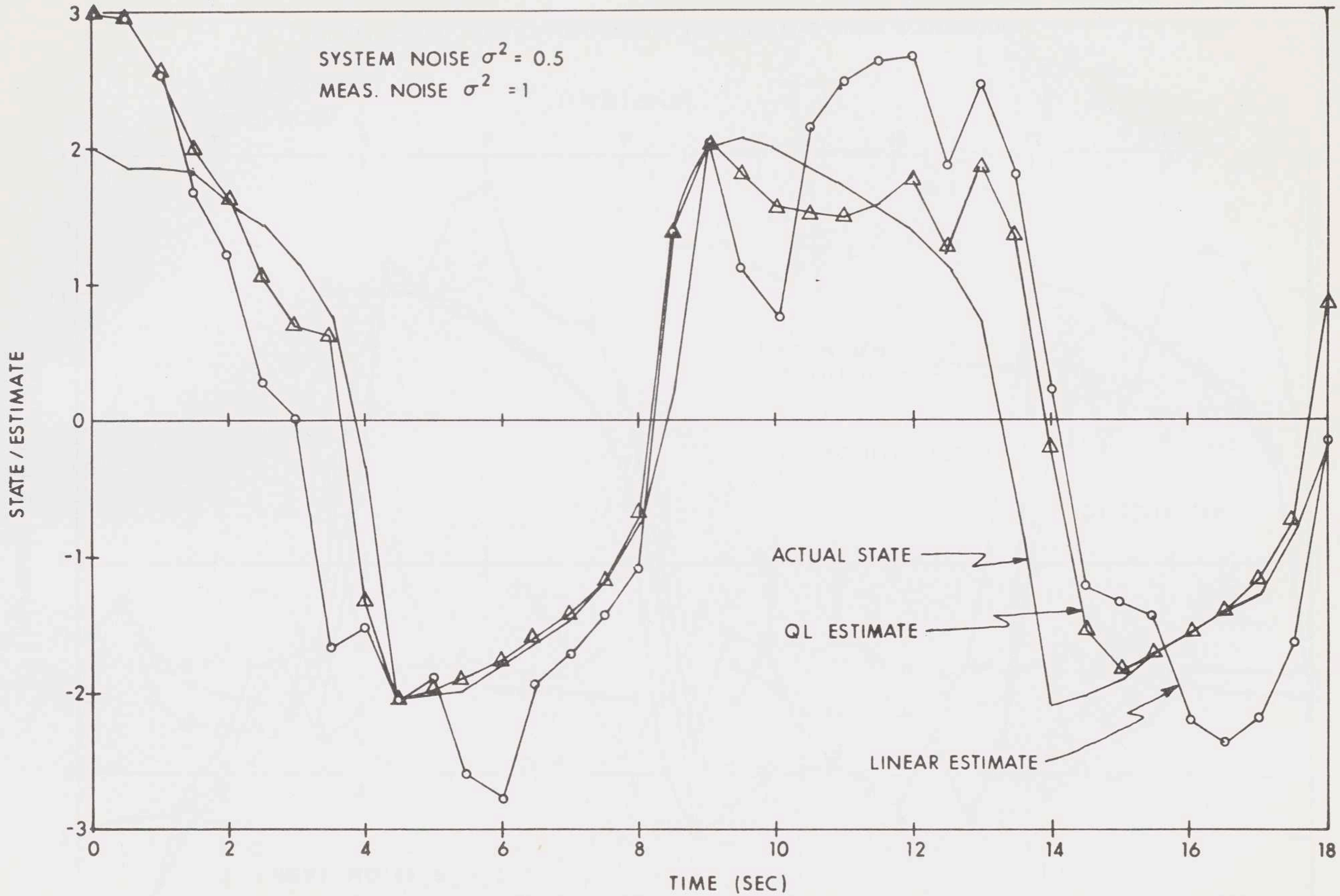


Figure 7-10 State Estimates - Linear, Quasi-Linear

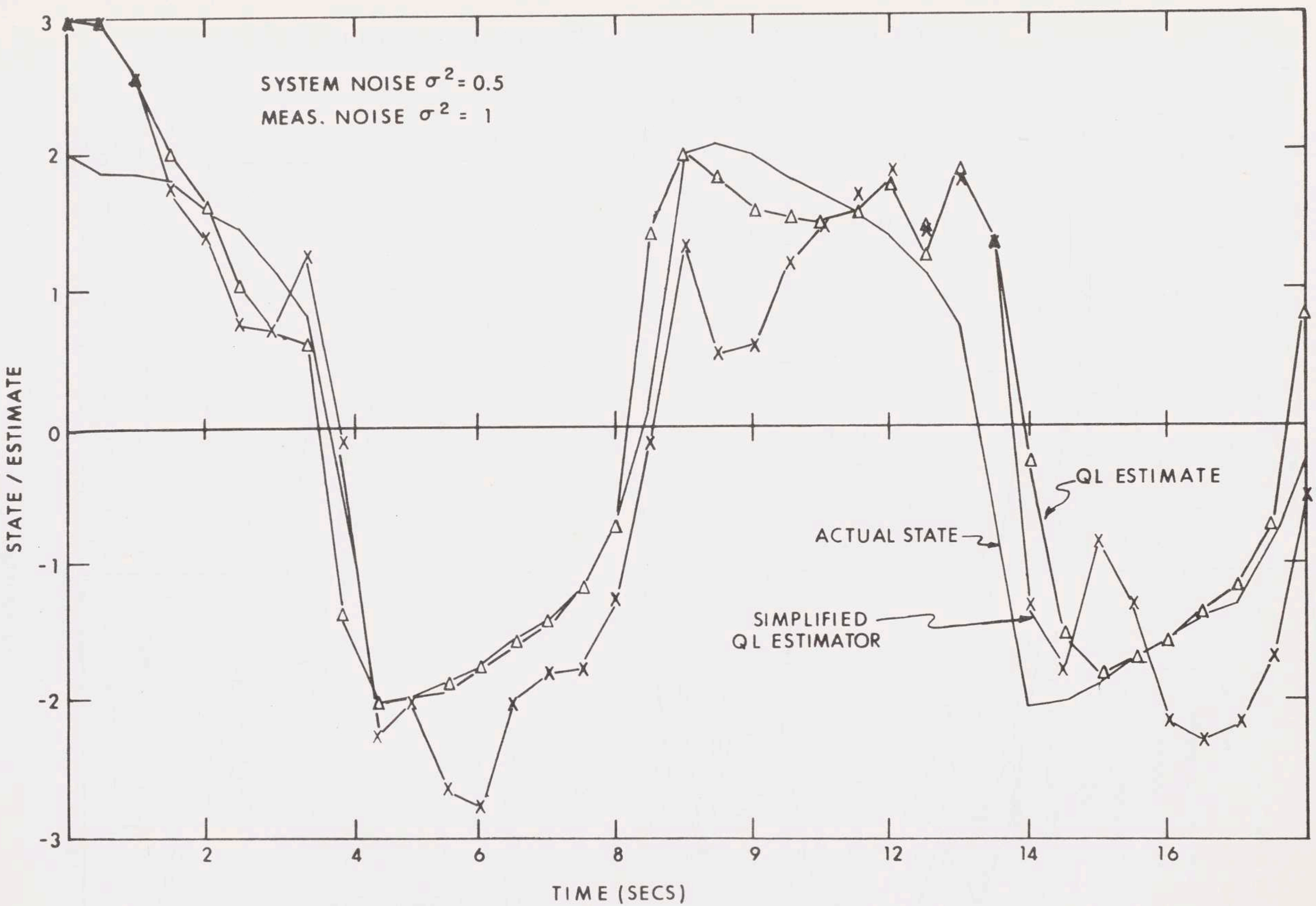


Figure 7-11 State Estimates - Two Quasi-Linear Estimators

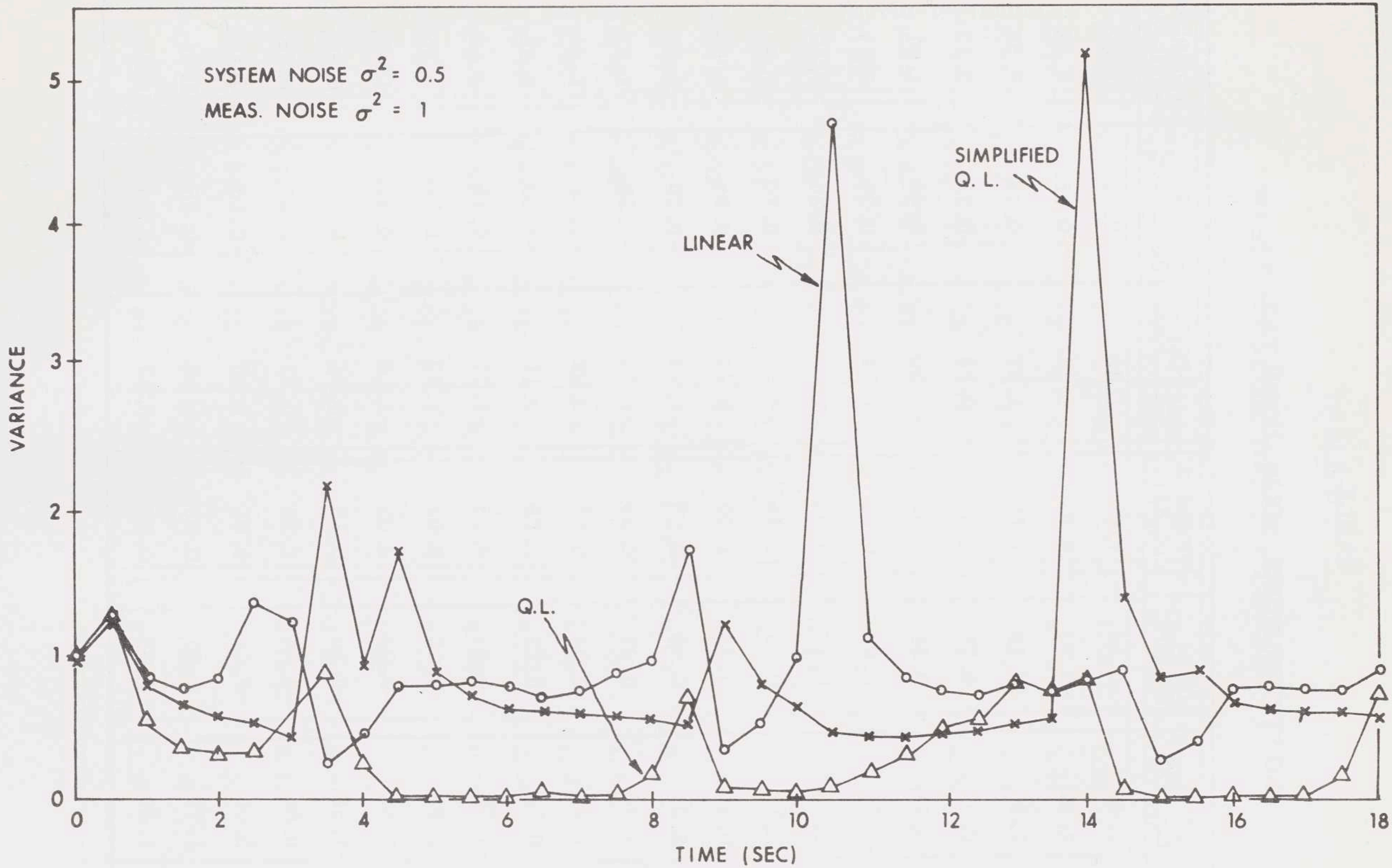


Figure 7-12 Variance Estimates - Three Estimators

TABLE I

NO SYSTEM NOISE, MEAS. NOISE VAR. = 1

Time	State	Avg. Lin. Est.	Avg. Lin. Err.	Computed Lin. Err.	Avg. Q. L. Est.	Avg. Q. L. Err.	Computed Q. L. Var.
0	2.00	1.48	-0.52	1.09	1.48	-0.52	1.09
1	1.79	1.57	-0.22	0.62	1.63	-0.15	0.43
2	1.48	1.30	-0.19	1.60	1.37	-0.12	0.46
3	0.96	0.97	0.01	1.59	0.18	-0.79	2.80
4	-1.69	-0.71	0.98	1.59	-1.20	0.49	1.01
5	-1.90	-1.81	0.09	0.02	-1.74	0.24	0.14
6	-1.63	-1.50	0.13	0.08	-1.29	0.34	0.59
7	-1.24	-1.11	0.13	0.26	-0.54	0.70	1.46
8	-0.11	0.41	0.52	1.70	0.74	0.85	1.65
9	1.99	2.07	0.08	0.15	1.79	-0.20	0.27
10	1.75	1.78	0.03	0.05	1.24	-0.52	1.67
11	1.43	1.39	-0.04	0.11	0.82	-0.61	1.45
12	0.84	0.75	-0.09	0.66	-0.16	-1.00	2.38
13	-1.96	-1.55	0.41	0.95	-1.77	0.19	0.12
14	-1.86	-1.92	-0.06	0.01	-1.62	0.24	0.23
15	-1.58	-1.62	-0.04	0.05	-1.10	0.48	1.41
16	-1.16	-1.36	-0.20	0.15	-0.71	0.59	0.90
17	0.38	0.04	-0.34	1.14	0.97	0.59	0.88
18	1.96	2.00	0.04	0.01	1.80	-0.16	0.14
19	1.71	1.76	0.05	0.03	1.25	-0.46	1.41
20	1.37	1.40	0.03	0.04	0.87	-0.50	1.07

TABLE II

NO SYSTEM NOISE, MEAS. NOISE VAR. = 1

Time	State	Avg. Lin. Var. Est.	Computed Lin. Var.	Variance of Lin. Var. Est.	Avg. Q. L. Var. Est.	Computed Q. L. Var.	Variance of Q. L. Var. Est.
0	2.00	1.00	1.09	0.08	1.00	1.09	0.08
1	1.79	0.71	0.62	0.06	0.57	0.43	0.02
2	1.48	1.83	1.60	13.23	0.59	0.46	0.10
3	0.96	3.35	1.59	81.93	0.38	2.80	5.98
4	-1.69	0.42	1.59	3.83	0.57	1.01	0.63
5	-1.90	0.07	0.02	0.02	0.04	0.14	0.01
6	-1.63	0.09	0.08	0.02	0.10	0.59	0.28
7	-1.24	0.50	0.26	0.43	0.21	1.46	1.64
8	-0.11	0.62	1.70	1.98	0.10	1.65	2.20
9	1.99	0.11	0.15	0.04	0.08	0.27	0.06
10	1.75	0.08	0.05	0.01	0.04	1.67	2.67
11	1.43	0.07	0.11	0.01	0.09	1.45	1.86
12	0.84	0.43	0.66	0.43	0.22	2.38	5.03
13	-1.96	0.70	0.95	0.88	0.06	0.12	0.01
14	-1.86	0.08	0.01	0.02	0.10	0.23	0.07
15	-1.58	0.08	0.05	0.01	0.06	1.41	1.84
16	-1.16	0.15	0.15	0.03	0.33	0.90	0.55
17	0.38	0.37	1.14	0.79	0.14	0.88	0.58
18	1.96	0.02	0.01	0.00	0.09	0.14	0.06
19	1.71	0.03	0.03	0.00	0.02	1.41	1.93
20	1.37	0.05	0.04	0.00	0.08	1.07	0.99

evaluate each estimate. This quantity, the computed variance, is found by taking the average of the square of the actual errors in the estimate for each of the 10 runs. It, therefore, represents a true variance in the estimate for a 10 sample case. In the second table, the estimated variance for each type of estimator is compared to this computed variance. In addition, another new parameter is introduced. This parameter is a measure of the variance in the estimate of the variance and is found by taking the difference between the estimated variance for each Monte Carlo run and the computed variance for the entire 10 sample case, squaring this quantity and averaging over the ten runs. Therefore, it represents the mean squared deviation of the estimated variance from the computed variance.

From Table I, it can be seen that use of the quasi-linear estimation technique does not provide any significant advantage over using a simple linear estimator, for a measurement noise variance of 1 sq. unit. This fact was also evident in the individual run, the results of which were shown in Fig. 7-4. In Table II there is some evident of instability in the variance estimate at the time points 2 and 3, however, the overall behavior of the two estimators is again comparable.

For a set of runs with measurement noise variance of 4 sq. units, a summary of estimator performance is depicted in Tables III and IV. Again the linear estimator and the general formulation of the quasi-linear estimator are compared. From Table III, there is evidence of some advantage to using the quasi-linear estimator from the point of view of estimator accuracy and also instability in the estimate is more pronounced for the linear estimate as seen at time points 4 and 13, by examination of the computed variance. In Table IV, the instability associated with the estimate of the variance is more pronounced as can be seen by examining the variance of the linear variance estimate.

TABLE III

NO SYSTEM NOISE, MEAS. NOISE $\sigma^2 = 4$

Time	State	Avg. Lin. Est.	Avg. Lin. Err.	Computed Lin. Var.	Avg. Q. L. Est.	Avg. Q. L. Err.	Computed Q. L. Var.
0	2.00	1.48	-0.52	1.09	1.48	-0.52	1.09
1	1.79	1.41	-0.38	1.18	1.47	-0.32	0.76
2	1.48	1.06	-0.42	2.98	1.16	-0.32	1.07
3	0.96	0.73	-0.23	2.45	0.28	-0.68	2.74
4	-1.69	-0.36	1.33	5.22	-0.56	1.13	2.64
5	-1.90	-1.58	0.32	0.77	-1.07	0.83	1.06
6	-1.63	-1.49	0.14	1.64	-0.84	0.79	1.56
7	-1.24	-1.07	0.17	1.88	-0.02	1.22	3.61
8	-0.11	-0.57	-0.46	2.14	0.71	0.82	2.29
9	1.99	1.03	-0.96	3.00	1.52	-0.47	0.38
10	1.75	1.64	-0.11	1.15	1.06	-0.69	1.25
11	1.43	1.32	-0.11	1.46	0.34	-1.09	2.99
12	0.84	1.14	0.30	1.79	-0.12	-0.99	2.82
13	-1.96	0.07	2.03	5.90	-1.09	0.87	1.60
14	-1.86	-1.19	0.67	2.17	-1.25	0.61	0.89
15	-1.58	-1.39	0.19	0.60	-0.72	0.86	2.67
16	-1.16	-1.70	-0.54	1.04	-0.24	0.92	2.20
17	0.38	-0.72	-1.10	1.98	1.06	0.68	1.54
18	1.96	1.66	-0.30	0.77	1.58	-0.38	0.62
19	1.71	1.88	0.17	0.14	0.82	-0.89	2.55
20	1.37	1.30	-0.07	0.72	0.23	-1.14	3.30

TABLE IV

NO SYSTEM NOISE, MEAS. NOISE $\sigma^2 = 4$

Time	State	Avg. Lin. Var. Est.	Computed Lin. Var.	Variance of Lin. Var. Est.	Avg. Q. L. Var. Est.	Computed Q. L. Var.	Variance of Q. L. Var. Est.
0	2.00	1.00	1.09	0.01	1.00	1.09	0.01
1	1.79	1.21	1.18	0.22	0.88	0.76	0.03
2	1.48	1.20	2.98	4.06	0.85	1.07	0.16
3	0.96	0.72	2.45	3.22	0.71	2.74	4.39
4	-1.69	1.34	5.22	18.41	0.58	2.64	5.47
5	-1.90	1.15	0.77	1.33	0.38	1.06	0.53
6	-1.63	1.17	1.64	1.62	0.40	1.56	1.53
7	-1.24	0.91	1.88	1.21	0.23	3.61	11.65
8	-0.11	1.29	2.14	2.08	0.30	2.29	4.18
9	-1.99	0.58	3.00	6.32	0.30	0.38	0.10
10	1.75	0.35	1.15	0.84	0.30	1.25	1.10
11	1.43	0.57	1.46	1.52	0.38	2.99	7.16
12	0.84	0.56	1.79	1.78	0.40	2.82	6.14
13	-1.96	0.71	5.90	27.50	0.25	1.60	1.99
14	-1.86	1.68	2.17	4.02	0.22	0.89	0.61
15	-1.58	1.59	0.60	12.21	0.16	2.67	6.44
16	-1.16	0.84	1.04	1.07	0.20	2.20	4.01
17	0.38	3.81	1.98	44.33	0.09	1.54	2.14
18	1.96	1.13	0.77	1.59	0.04	0.62	0.34
19	1.71	1.09	0.14	0.13	0.09	2.55	6.11
20	1.37	0.95	0.72	2.22	0.07	3.30	10.48

Referring back to Figs. 7-7 and 7-9, this degraded performance of the linear estimate is also evident. The averaging affect of the 10 sample statistics provides some smoothing of the behavior as seen in all these Tables. A look at the individual runs reveals even larger peaks in the variance of the linear estimates. On the other hand, the averages for the quasi-linear estimate as shown in the Tables are also typical of the individual runs.

Finally, a phase plane plot of the position x_1 and velocity x_2 of the van der Pol oscillator is depicted in Fig. 7-13. Also included on the figure is the estimate generated by the general form of the quasi-linear estimator for two complete cycles. The particular case shown in Fig. 7-13 is for a run, with no system driving noise and a measurement noise variance of 1 sq. unit also summarized by Figs. 7-4 through 7-6.

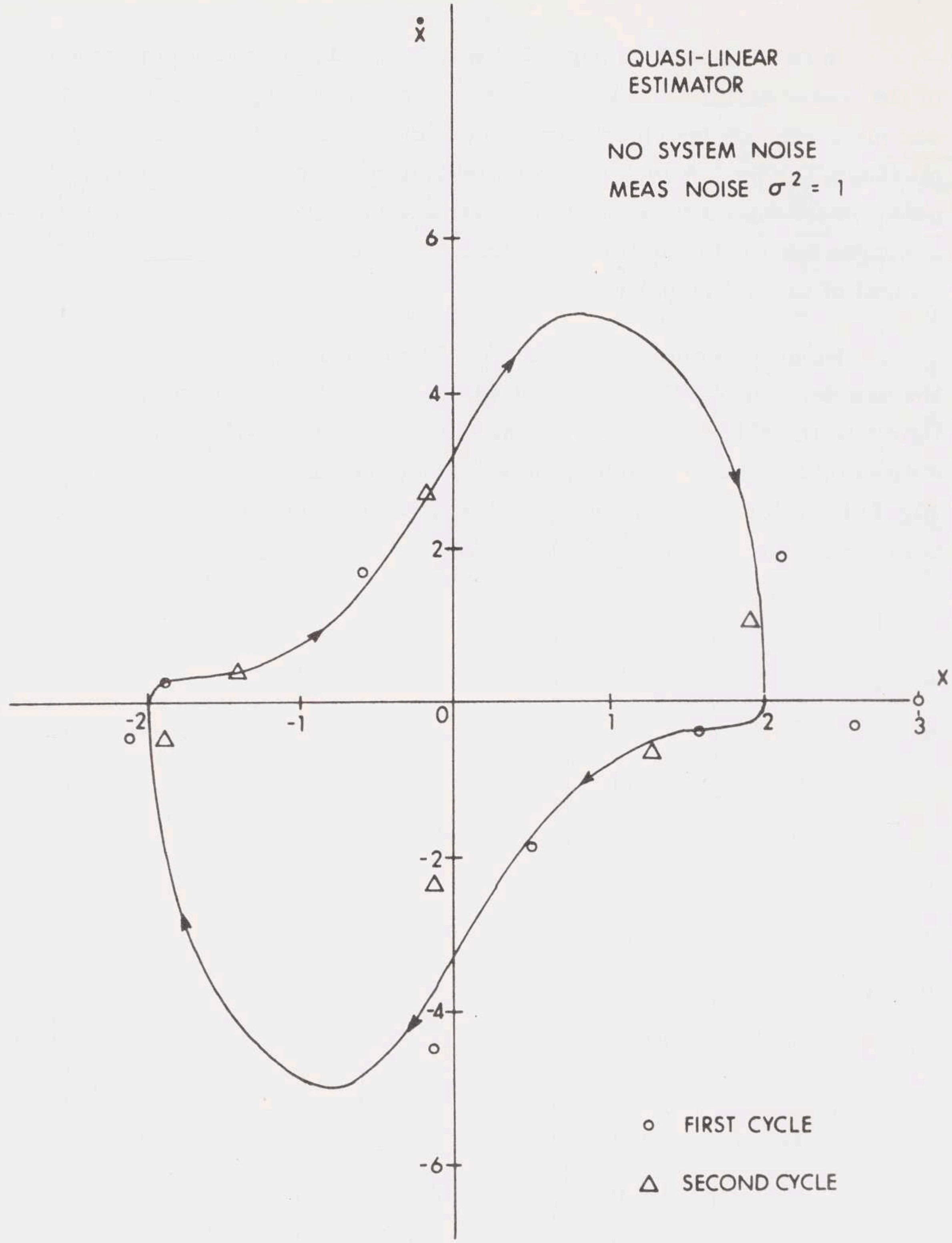


Figure 7-13 Phase-Plane Behavior

7.5 Summary

The van der Pol oscillator has provided a system which exhibits regions of highly nonlinear behavior and therefore presents a useful vehicle for comparing the linearized estimator and the quasi-linear estimator. Also, because of the zero mean value of the system state when averaged over time, the simplified quasi-linear estimation equations give good results in some cases, namely when the driving noise is small.

The estimation equations, shown in the earlier sections of this chapter, can all be written in algebraic form allowing a direct comparison of each estimation scheme. As was noted earlier, both quasi-linear schemes consider the fourth order moments of the density function; however, the general formulation provides a more exact relationship. It is of interest to note that this general formulation produces exactly the same set of estimation equations as was found by Kushner (1967) when he specialized his problem to a Gaussian case.

In the case of the simplified quasi-linear estimator, the accuracy involved in the approximation required for the simplified result is degraded for a large driving noise variance. However, large Q does not seem to degrade the performance of the general quasi-linear estimator more than would seem reasonable.

The general remarks and conclusions concerning the performance of the linear and general quasi-linear estimation equations are substantiated by the statistical data obtained from 10 sample Monte Carlo runs. A great deal more data is required, however, before any general quantitative information can be stated.

CHAPTER VIII

SUMMARY AND GENERAL COMMENTS

8.1 Summary of Results

The problem of nonlinear estimation has been viewed in a slightly different manner in the preceding chapters than that which has been recently discussed by other authors. The evolution of a conditional density function, as the fundamental basis for minimum variance estimation, which is the focal point of most of the published research in recent years, need not be considered explicitly. As shown in this study, it is also possible to directly examine the nonlinear function as it appears in estimation equations and to develop methods for directly approximating this nonlinearity. Several individuals have shown that a differential equation for the temporal evolution of the desired conditional density function can be found: however, the solution for the general nonlinear estimation problem also requires some technique of approximation. Therefore, the two approaches have similar difficulties and often produce similar results, as discussed in Chapter VII. The advantage of simply determining an approximation to the nonlinear element, in addition to not requiring the somewhat involved stochastic calculus necessary for finding the partial differential equation of the conditional density function, is also that the resulting analysis much of the well known work in nonlinear control theory, thereby allowing further insight into the behavior of a given nonlinear system.

The series expansion technique, as developed in Chapter III, provides one straightforward procedure for replacing any expectation of a nonlinear function by a Taylor series, expanded about the current estimated quantities. By assuming a form for the measurement update, several of the desirable properties of an optimum estimator are retained. First, the equations for the state and the higher order moments are uncoupled. Second, all the moments are, indeed, ensemble statistics and do not depend on random numbers. Third, the estimation equations reduce to the Kalman filter equations when the nonlinearity is specialized to a linear function. This assumed form for incorporating measurements is more restrictive than is actually required and therefore, may cause some loss in accuracy. This possible problem, versus the above mentioned advantages, must be considered for the specific system of interest. For the one discussed in Chapter V, the restricted form of measurement incorporation did provide a useful set of nonlinear estimation equations. An additional problem of nonlinear estimation with any series expansion technique, and one which is difficult to analyze quantitatively, is concerned with the term by term behavior of the Taylor series. It is not clear that the moment terms converge rapidly, if at all, and once again, a particular series estimator must be investigated for a selected nonlinear system in order to answer the questions of convergence. The behavior of the higher order moments was found to depend not only on the nonlinear system under investigation, but also on various other system parameters, such as driving noise, measurement noise and the measurement interval. In theory, an asymptotically converging set of moments should result in an increasingly accurate estimator as the number of moments is increased. However, the discussion in Chapter V shows a large increase in computational complexity as this number is increased even for a system which can be described by scalar equations. In addition, it may be necessary to include only some of the higher order moments (e.g., all even moments). The selection of which moments to include is again problem dependent.

Finally, although the series estimation procedure does produce a significant improvement over a linearized estimator for nonlinear systems, it cannot be used to solve problems with "hard" nonlinearities; that is, nonlinear functions for which some of the derivatives are not finite. A well known example of a nonlinearity of this type is the quantizer.

The quasi-linear estimation technique introduced in Chapter IV, and demonstrated numerically in Chapters V and VII, shows a marked improvement over linear estimation in each of the example problems. It also has the advantages of: 1) being useful for any type of nonlinear relationships; and 2) being identical to the optimum linear estimation when all of the nonlinear functions are specialized to linear relationships. As discussed in Chapter VII, the higher order moment terms, which are implicitly contained in the quasi-linear formulations, provide solutions for the state estimate and variance estimate which do not exhibit some of the stability problems associated with a linearized estimator. In the scalar example of Chapter V, the quasi-linear estimator consistently provided a better estimate than the linear or any of the series estimators considered.

Two significant limitations of this quasi-linear approach are: 1) it is not always possible to find an analytic solution to the expectation integral required by this technique, and, consequently, numerical procedures, some of which are discussed in Chapter VI, are required; and 2) nonlinear functions which result in a state with a highly skewed probability density function are not adequately described by the quasi-linear formulation of Chapter IV. However, this formulation may be generalized to somewhat alleviate this problem as discussed in Section 8. 2.

In Chapter IV, it was demonstrated analytically that under certain circumstances, a simplified quasi-linear estimator could be developed. Such circumstances are that the conditional expectation of the state be close to zero. In Chapter VII, the characteristic of having the time average of the state be zero was demonstrated as an adequate approximation to this requirement and, thus, the simplified quasi-linear estimator provided useful results. However, the simplified estimator performance was adequate only for small system driving noise.

8.2 Generalization of the Quasi-Linear Approach

The technique of using a quasi-linear approximation for the nonlinearity in the system or measurement process requires the evaluation of certain expectations of functions of a random variable given the probability density function of that variable. The expectations required are the result of requiring that the quasi-linear approximation produce the minimum mean squared error in the approximation. Such a mean is to be taken over the ensemble of random processes. Important to the entire quasi-linear procedure, then, is the expression for the above mentioned probability density function. For several reasons enumerated in Chapter IV, it is logical to assume that this density function is of Gaussian form. Once this assumption has been made, the entire random process can be described by the evolution of two parameters which represent the mean and variance of the random variable.

The assumption of a Gaussian process is not, however, a requirement of this general quasi-linear approach. If a convenient, preferably analytic, expression for a probability density function can be developed which is a function of three or more parameters, a similar procedure can be devised with the required number of parameters included in the estimation equations. It seems likely that a third order term would be most useful, if it is a measure of skewness, a property not expressible with the Gaussian assumption. Having such an expression, the quasi-linear estimator can be developed in a manner exactly analogous to that described in Chapter IV.

A technique for describing a general probability density function in a convenient series has been suggested by Cramér (1946). For the normalized random variable

$$x = \frac{\xi - m}{\sigma} \tag{8.1}$$

where ξ is any random variable, m is the mean value of ξ and σ is the standard deviation of ξ, the probability density function p(x) can be written in the following series.

$$
\begin{aligned}
p(x) = &\ \phi(x) \\
& - \frac{1}{3!}\frac{\mu_3}{\sigma^3}\phi^{(3)}(x) \\
& + \frac{1}{4!}\left(\frac{\mu_4}{\sigma^4} - 3\right)\phi^{(4)}(x) + \frac{10}{6!}\left(\frac{\mu_3}{\sigma^3}\right)^2\phi^{(6)}(x) \\
& - \cdots
\end{aligned}
\tag{8. 2}
$$

In the above equation, the terms on each line are of the same order of magnitude in the original variable of expansion. For a detailed derivation of Eq. (8. 2), which is somewhat involved, the reader is referred to Cramér. In Eq. (8. 2) the first term, $\phi(x)$, is a Normal density function, the μ's are central moments of the general density function p(x), and $\phi^{(n)}(x)$ is the n^{th} derivative of $\phi(x)$ with respect to x. Note that if p(x) is actually Normal, $\frac{\mu_4}{\sigma^4} = 3$ and $\mu_3 = 0$ resulting in $p(x) = \phi(x)$.

Clearly, Eq. (8. 2) can be used in a quasi-linear estimation technique in the same manner as the normal distribution was used in the preceding chapters. For example, if the nonlinear problem under consideration were such that the distribution were significantly skewed, a third central moment may be included in the estimator with all expectations being taken with respect to the following density function:

$$
p(x) \approx \phi(x) - \frac{1}{3!}\frac{\mu_3}{\sigma^3}\phi^{(3)}(x) \tag{8. 3}
$$

If

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-1/2\, x^2} \tag{8.4}$$

then

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-1/2\, x^2} \left[1 - \frac{\mu_3}{6\sigma^3} x (3 - x^2)\right] \tag{8.5}$$

8.3 Suggestions for Further Work

There are several aspects of nonlinear estimation using series expansion techniques which require additional studies. First, the problem of analytically determining the relative magnitudes of successive moments has not been solved, and is one that is extremely important to the successful application of this technique. In order to properly select a truncation point, the above information is required.

The more general problem of series convergence and asymptotic behavior is one which needs additional work also. It is highly desirable to select a series expansion with desirable convergence characteristics. This series may not be in terms of central moments or of a Taylor series form.

As demonstrated in Chapter V, even a simple scalar nonlinear system may require a very complex algorithm for series estimation. Techniques for simplifying the required estimation equations should be developed, exploiting some of the special characteristics of moment terms, such as their symmetry.

The quasi-linear approach to nonlinear estimation has shown a great deal of promise in this investigation, and further studies both analytic and numerical are necessary to fully exploit this promise. Specifically, the class of nonlinear systems for which this technique is most useful should be identified, and numerical problems associated with implementation requires further study. The general problem of evaluating the multiple integral expressions which are required has been discussed in Chapter VI. However, as shown, a considerable effort is required in order to separate the integral. This fact is sufficient motivation for a closer examination of the possibility of approximating the multiple integral directly.

More generally, the question of stability of the estimation equations can be considered as was done for a specific problem briefly in Chapter VII. Also, the general technique of quasi-linear approximation, as described in Section 8. 2, suggests a larger class of problems for which this approach might be useful. The series expression for a general density function, which is given there, provides a means of extending the quasi-linear approach; however, the problems associated with a vector state and the associated evaluation of expectations given a density function in terms of a vector mean, a covariance matrix and a third order moment tensor are considerable. This area would seem most worthy of investigation.

APPENDIX A

QUASI-MOMENT FUNCTIONS

It is often possible in engineering applications to justify the assumption that random variables tend to be near-Gaussian. It is therefore useful to consider a set of moment functions which are tailored to provide a description of such a distribution.

If $p(x, t)$ is any n-dimensional probability density function and $g(x, t)$ is a Gaussian probability density function with the same mean and covariance matrix, the following function can then be formed as the ratio of these density functions.

$$f(x, t) = \frac{p(x, t)}{g(x, t)} \tag{A. 1}$$

The characteristic functions which correspond to $p(x, t)$ and $g(x, t)$ are

$$\phi(\lambda, t) = \int_{-\infty}^{\infty} p(x, t)\, e^{i \lambda^T x}\, dx \tag{A. 2}$$

and

$$\psi(\lambda, t) = \int_{-\infty}^{\infty} g(x, t)\, e^{i \lambda^T x}\, dx \tag{A. 3}$$

respectively.

Because of the Gaussian form of g(x, t) the characteristic function can also be written as

$$\psi(\lambda, t) = e^{(i \lambda^T \bar{x} - 1/2 \lambda^T P \lambda)} \tag{A. 4}$$

where $\bar{x}$ and P are the mean and covariance of the Gaussian density function, respectively. The ratio of $\phi(\lambda, t)$ and $\psi(\lambda, t)$ can therefore be written in the following manner.

$$\frac{\phi(\lambda, t)}{\psi(\lambda, t)} = \phi(\lambda, t)\, e^{-i \lambda^T \bar{x} + 1/2 \lambda^T P \lambda} \tag{A. 5}$$

The expression in Eq. (A. 5) can now be expanded in an n-dimensional Maclaurin series.

$$\frac{\phi(\lambda, t)}{\psi(\lambda, t)} = 1 + i \sum_{j=1}^{n} K_j(t)\, \lambda_j + \frac{(i)^2}{2!} \sum_{j=1}^{n} \sum_{k=1}^{n} K_{jk}(t)\, \lambda_i\, \lambda_k + \ldots \tag{A. 6}$$

where

$$\underbrace{K_{ij\ldots}(t)}_{M} = \frac{1}{(i)^M} \frac{\partial^M}{\partial\lambda_i\,\partial\lambda_i\cdots}\left[\frac{\phi(\lambda, t)}{\psi(\lambda, t)}\right]_{\lambda=0} \tag{A. 7}$$

Equation (A. 7) defines the desired moment functions.

Using the mean and covariance along with the moment functions defined by Eq. (A. 7), any probability density function can be expressed by a series of moments which are referenced to the Gaussian probability density function.

The moments given by Eq. (A. 7) are closely related to the central moments as can be shown by comparing each term of a Maclaurin series for $\phi(\lambda, t)\, e^{-i\lambda^T \bar{x}}$ and

$$\phi(\lambda, t) = e^{(i\,\lambda^T \bar{x} - 1/2\,\lambda^T P\lambda)}\left[1+\sum_{n=3}^{\infty}\frac{(i)^M}{M!}\sum_{j,k,\ldots}^{n} K_{jk\ldots}\,\lambda_j\,\lambda_k\cdots\right] \tag{A. 8}$$

The results of the first four moment terms are given below, with $P_{ij\ldots}$ used to signify central moments.

$$\begin{aligned}
P_i &= K_i \\
P_{ij} &= K_{ij} \\
P_{ijk} &= K_{ijk} \\
P_{ijk\ell} &= K_{ijk\ell} + P_{ij}\,P_{k\ell} + P_{ik}\,P_{j\ell} + P_{i\ell}\,P_{jk}
\end{aligned} \tag{A. 9}$$

APPENDIX B

THE PSEUDO-INVERSE MATRIX

There are many practical situations which arise in engineering requiring the inverse of a matrix which is singular or perhaps not even square. In order to handle these situations, a pseudo-inverse may be defined in a simple and useful way. According to Penrose, the following four equations will uniquely define a pseudo-inverse denoted by A'.

$$\begin{aligned} AA'A &= A \\ A'AA' &= A' \\ (AA')^T &= AA' \\ (A'A)^T &= A'A \end{aligned} \qquad (B.1)$$

The first equation taken alone can be used as the definition of a pseudo-inverse and provides a useful result in many cases; however, the non-unique quality of such a definition presents problems in the solution to certain matrix equations discussed below.

The particular problem of interest in this paper arises from the optimality condition for the quasi-linear approximation in Chapter IV. In this case, the matrix which requires an inverse is the diadic product of a vector with itself, a quantity which can easily be shown to have a zero determinant. Using the above equations as the definition, the pseudo-inverse of the diadic product $\bar{x}\bar{x}^T$ is found to be

$$(\bar{x}\,\bar{x}^T)' = \frac{\bar{x}\,\bar{x}^T}{(\bar{x}^T\,\bar{x})^2} \tag{B. 2}$$

Of particular interest to the discussion of Chapter IV is a further result of Penrose. As a special case of his first theorem, the following important result is found. A necessary and sufficient condition for the equation

$$X\,B = C \tag{B. 3}$$

to have a solution is that

$$C\,B'\,B = C \tag{B. 4}$$

where X, B, C can be any rectangular matrices. If Eq. (B. 4) is satisfied, then the most general solution to Eq. (B. 3) is the following

$$X = C\,B' + Y\,[I - B\,B'] \tag{B. 5}$$

where B' is the pseudo-inverse of B as defined by equation set (B. 1) and Y is completely arbitrary. The freedom in choice of Y is easily verified by substituting the expression for X into (B. 3) and noting that because of the definition of B' the term which Y multiplies is always zero.

Specializing these results to the equations for the quasi-linear estimator derived in Chapter IV.

$$N^{\bar{x}}\,\bar{x}\,\bar{x}^T = \overline{c(x)}\;\bar{x}^T \qquad \text{(B. 6)}$$

the general form of the gain matrix is the following

$$N^{\bar{x}} = \frac{\overline{c(x)}\;\bar{x}^T}{\bar{x}^T\,\bar{x}} + Y\left[I - \frac{\bar{x}\,\bar{x}^T}{\bar{x}^T\bar{x}}\right] \qquad \text{(B. 7)}$$

The simplest solution to N is, of course, found by setting Y = 0; however, the freedom in choice of Y can also be exploited in the following manner. Suppose Y were defined to be

$$Y = \left[\overline{c(x)\,x^T} - \overline{c(x)}\;\bar{x}^T\right] P^{-1} \qquad \text{(B. 8)}$$

then if the general nonlinear function c(x) is specialized to the linear form Cx

$$Y = C\,(\overline{xx^T} - \bar{x}\,\bar{x}^T)\,P^{-1} = C \qquad \text{(B. 9)}$$

and the special case of $N^{\bar{x}}$ becomes

$$N^{\bar{x}} = \frac{C\,\bar{x}\,\bar{x}^T}{\bar{x}^T\,\bar{x}} + C\left[I - \frac{\bar{x}\,\bar{x}^T}{\bar{x}^T\bar{x}}\right] = C \qquad \text{(B. 10)}$$

Note that if Y were chosen to be zero, the limit of $N^{\bar{x}}$ would be

$$N^{\bar{x}} = \frac{C\,\bar{x}\,\bar{x}^T}{\bar{x}^T\,\bar{x}} \tag{B. 11}$$

From the derivation in Chapter IV, Eq. (B. 8) is recognized as the definition of the gain to the error, N^e. Thus, a particularly useful definition for $N^{\bar{x}}$ is the following

$$N^{\bar{x}} = \frac{\overline{c(x)\,x^T}}{\bar{x}^T\,\bar{x}} + N^e \left[I - \frac{\bar{x}\,\bar{x}^T}{\bar{x}^T\,\bar{x}} \right] \tag{B. 12}$$

For the special case of $N^{\bar{x}}$ equal to a scalar Eq. (B. 7) reduces to

$$N^{\bar{x}} = \frac{\overline{c(x)}}{\bar{x}} \tag{B. 13}$$

which eliminates the freedom in selection of a Y value. However, note that as $\overline{c(x)}$ in Eq. (B. 13) is specialized to $C\bar{x}$, $N^{\bar{x}}$ becomes C as desired.

APPENDIX C

HERMITE POLYNOMIALS

The Hermite polynomial is generally defined by the following expression

$$H_n(s) = (-1)^n e^{s^2} \frac{d^n}{ds^n} (e^{-s^2}) \qquad \text{(C. 1)}$$

Such a definition is motivated by the fact that integrals over the interval $-\infty \leq s \leq \infty$ with the weighting function

$$w(s) = e^{-s^2} \qquad \text{(C. 2)}$$

occur often and the definition given by Eq. (C. 1) has the following desirable property.

$$\int_{-\infty}^{\infty} e^{-s^2} H_n(s) H_m(s)\, ds = 0 \quad n \neq m \qquad \text{(C. 3)}$$

That is, over the infinite interval Eq. (C. 1) defines a set of polynomials orthogonal with respect to the weighting function of Eq. (C. 2).

The Hermite polynomials obey the following recursive relationship

$$H_{n+1}(s) = 2s\, H_n(s) - 2_n\, H_{n-1}(s) \tag{C. 4}$$

with

$$H_0(s) = 1; \qquad H_1(s) = 2s \tag{C. 5}$$

A list of the first ten hermite polynomials defined by Eq. (C. 1) is given below

$H_0(s) = 1$

$H_1(s) = 2s$

$H_2(s) = 4s^2 - 2$

$H_3(s) = 8s^3 - 12s$

$H_4(s) = 16s^4 - 48s^2 + 12$

$H_5(s) = 32s^5 - 160s^2 + 120s$

$H_6(s) = 64s^6 - 480s^4 + 720s^2 - 120$

$H_7(s) = 128s^7 - 1344s^5 + 3360s^3 - 1680s$

$H_8(s) = 256s^8 - 3584s^6 + 13440s^4 - 13440s^2 + 1680$

$H_9(s) = 512s^9 - 9216s^7 + 48384s^5 - 80640s^3 + 30240s$

$H_{10}(s) = 1024s^{10} - 23040s^8 + 161280s^6 - 403200s^4 + 302400s^2 - 30240$

APPENDIX D

CHARACTERISTICS OF GAUSSIAN MOMENT TERMS

The multi-dimensional Gaussian random variable has associated with it a characteristic function which is defined as the following expectation.

$$\phi(\lambda) = \mathcal{E}[e^{i\lambda^T x}] = \frac{1}{(2\pi)^{n/2}\,|P|^{1/2}} \int_{-\infty}^{\infty}\cdots\int e^{i\lambda^T x}\, e^{-1/2\, x^T P^{-1} x}\, dx \tag{D.1}$$

where $i = \sqrt{-1}$ and P is the $n \times n$ covariance matrix. The integration indicated above can easily be performed to give the following expression for the characteristic function

$$\phi(\lambda) = e^{-1/2\, \lambda^T P \lambda} \tag{D.2}$$

The function can now be used to generate expressions for the central moments of the Gaussian density function by making use of the fact that partial derivatives of $\phi(\lambda)$ evaluated at $\lambda = 0$ are directly related to these moments. This relationship can be expressed as follows

$$\left\{ \frac{\partial^{(a+b+\dots)}}{\partial\lambda_1^{\,a}\,\partial\lambda_2^{\,b}\dots} \left[\phi(\lambda)\right] \right\}_{\lambda=0} = i^{(a+b+\dots)}\, M_{ab\dots} \tag{D. 3}$$

Thus, in order to find the moment $M_{ab\dots}$ one must take the $(a+b+\dots)^{th}$ derivative of $\phi(\lambda)$ with respect to the proper λ's and evaluate the result for all λ's = 0. For example

$$M_{123} = \left\{ \frac{1}{(i)^6} \frac{\partial^6}{\partial\lambda_1\,\partial\lambda_2^{\,2}\,\partial\lambda_3^{\,3}} \left[\phi(\lambda)\right] \right\}_{\substack{\lambda_1=0\\ \lambda_2=0\\ \lambda_3=0}} \tag{D. 4}$$

Take the partial derivatives of $\phi(\lambda)$ is straightforward as shown in the following equations using differential notation

$$\phi(\lambda) = e^{-1/2\,\lambda^T P \lambda} \tag{D. 5}$$

$$\delta\,\phi(\lambda) = -\,(\delta\lambda_1^{\,T} P \lambda)\, e^{-1/2\,\lambda^T P \lambda} \tag{D. 6}$$

$$\begin{aligned} \delta^2\phi(\lambda) &= [(\delta\lambda_1^{\,T} P \lambda)(\lambda^T P\, \delta\lambda_2) - \delta\lambda_1^{\,T} P\, \delta\lambda_2]\, e^{-1/2\,\lambda^T P \lambda} \\ &= \delta\lambda_1^{\,T}\,[P \lambda \lambda^T P - P]\, \delta\lambda_2\, e^{-1/2\,\lambda^T P \lambda} \end{aligned} \tag{D. 7}$$

$$\delta^3\phi(\lambda) = \delta\lambda_1^T \left[P\,\delta\lambda_3\,\lambda^T P + P\lambda\,\delta\lambda_3^T P - (P\lambda\lambda^T P - P)\,\delta\lambda_3^T P\lambda \right] \delta\lambda_2\, e^{-1/2\,\lambda^T P\lambda}$$

(D. 8)

$$\delta^4\phi(\lambda) = \delta\lambda_1^T \left[P\,\delta\lambda_3\,\delta\lambda_4^T P + P\,\delta\lambda_4\,\delta\lambda_3^T P - (P\,\delta\lambda_4\,\lambda^T P + P\lambda\,\delta\lambda_4^T P)\,\delta\lambda_3^T P\lambda \right.$$

$$\left. - (P\lambda\lambda^T P - P)\,\delta\lambda_3^T P\,\delta\lambda_4 \right] \delta\lambda_2\, e^{-1/2\,\lambda^T P\lambda}$$

$$-\delta\lambda_1^T \left[P\,\delta\lambda_3\,\lambda^T P + P\lambda\,\delta\lambda_3^T P \right.$$

$$\left. - (P\lambda\lambda^T P - P)\,\delta\lambda_3^T P\lambda \right] \delta\lambda_2\,(\delta\lambda_4^T P\lambda)\, e^{-1/2\,\lambda^T P\lambda}$$

(D. 9)

This procedure can be carried out to as many terms as desired.

Using Eq. (D. 4), the following results are obtained

$$M_a = \frac{1}{i}\,\frac{\partial\phi(\lambda)}{\partial\lambda}\bigg|_{\lambda_a = 0} = 0 \qquad \text{(D. 10)}$$

$$M_{ab} = \frac{1}{(i)^2}\,\frac{\partial^2\phi(\lambda)}{\partial\lambda_a\,\partial\lambda_b}\bigg|_{\lambda = 0} = P_{ab} \qquad \text{(D. 11)}$$

$$M_{abc} = \frac{1}{(i)^3} \left. \frac{\partial^3 \phi(\lambda)}{\partial\lambda_a \, \partial\lambda_b \, \partial\lambda_c} \right|_{\lambda = 0} = 0 \qquad \text{(D. 12)}$$

$$M_{abcd} = \frac{1}{(i)^4} \left. \frac{\partial^4 \phi(\lambda)}{\partial\lambda_a \, \partial\lambda_b \, \partial\lambda_c \, \partial\lambda_d} \right|_{\lambda = 0} = P_{ac} \, P_{bd} + P_{ad} \, P_{bc} + P_{ab} \, P_{cd} \qquad \text{(D. 13)}$$

In general, for n Gaussian random variables

$$v_1, \; v_2, \; v_3, \; \ldots, \; v_n$$

the n^{th} moment is given by

$$\overline{v_1 \, v_2 \, v_3} \ldots \, v_n = \sum \prod v_i \, v_j \ldots \qquad \text{(D. 14)}$$

where $\sum \prod$ symbol is to be interpreted as the summation of all terms of distinct products of the random variables taken two at a time. There are

$$\frac{n!}{(n/2)\,!\;2^{n/2}}$$

such distinct products.

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BIOGRAPHICAL SKETCH

Roger J. Phaneuf was born November 4, 1940 in Bristol, Connecticut. He attended public schools in Bristol and was graduated from Bristol High School in June, 1958.

Mr. Phaneuf entered M. I. T. in September, 1958 where he received his Bachelor of Science degree from the Aeronautics and Astronautics Department in June, 1962. As an undergraduate, he was a member of the cooperative program and was employed by the Lockheed Missiles and Space Division in Sunnyvale, California from August, 1960 to February, 1961.

Following graduation in June 1962, Mr. Phaneuf was employed by Douglas Aircraft Company in Santa Monica, California until his return to M. I. T. in September, 1962 for graduate studies. At that time, he also joined the Experimental Astronomy Laboratory as a Research Assistant and worked on an inertial star tracking system under the supervision of Prof. Winston Markey. In January, 1963 he joined the Instrumentation Laboratory working on the Apollo Project under the general supervision of Dr. Richard Battin, and in June, 1964 he was awarded the degree of Master of Science from the Department of Aeronautics and Astronautics. He has continued to work both full and part time on various aspects of the Apollo guidance system under the direct supervision of Mr. Norman E. Sears. He has also assisted in the teaching of graduate courses in inertial guidance and nonlinear control systems.

Mr. Phaneuf is married to the former Andrea Kramer of Cambridge, Massachusetts. They have a son, Peter, and reside in Waltham, Massachusetts. He is a member of Sigma Gamma Tau and Sigma Xi honorary fraternities.

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