On the Combinatorics of Representations of $Sp(2n, \mathbb{C})$

by

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Abstract

In this thesis we study the finite-dimensional representations of the symplectic group $Sp(2n, \mathbf{C})$ from a combinatorial viewpoint. It is known that the irreducible representations of $Sp(2n, \mathbf{C})$ are indexed by partitions λ with at most n parts, and also that to each irreducible representation $\tilde{\phi}_\lambda$ corresponding to the partition λ, one can associate a finite set of semi-standard tableaux (which we call symplectic tableaux)of shape λ whose cardinality is precisely the dimension of $\tilde{\phi}_\lambda$. On passing from the representation $\tilde{\phi}_\lambda$ to its character one obtains a polynomial $sp_\lambda(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})$ (in the $2n$ variables $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$) which can be expressed as an integral linear combination of monomials in $\{x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}\}$, each monomial arising from a symplectic tableau of shape λ. From the tableau description or otherwise it follows that the characters $sp_\lambda(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})$ are invariant polynomials under the action of the hyperoctahedral group $\mathbf{B_n}$, the Weyl group of $Sp(2n, \mathbf{C})$. The finite-dimensional irreducible representations of $Sp(2n, \mathbf{C})$ may therefore be studied by means of the "symplectic" Schur functions sp_λ where λ runs over all partitions with at most n parts. For $Sp(2n, \mathbf{C})$ a Schensted-type algorithm which produces symplectic tableaux was recently developed by Allan Berele (1984). Here the standard tableaux occurring as the right tableaux of the Knuth-Schensted insertion process are replaced by sequences of shapes $S_\mu^k = (\emptyset = \mu^0, \mu^1, \ldots, \mu^k = \mu)$ where any two consecutive shapes differ by exactly one box. We refer to such a sequence as an up-down tableau.

Our first result is a reformulation of the Berele correspondence, showing that its output may be encoded as a triple consisting of

1) a symplectic tableau of shape μ,
2) a standard Young tableau of shape λ, $\lambda \supseteq \mu$, and
3) a lattice permutation which fits the skew-shape $\lambda \supseteq \mu$ and has weight β, for some shape β with columns of even length.

This leads to the discovery of an interesting connection with the Knuth- Schensted algorithm. One of the main results following from this is a combinatorial description of the decomposition of an irreducible character of $Gl(2n, \mathbf{C})$ when restricted to $Sp(2n, \mathbf{C})$. We also characterise equivalence classes of words giving the same symplectic tableaux under Berele insertion: the result in this direction is that Knuth-equivalent words give rise to the same symplectic tableau and the same lattice permutation, and conversely.

We present the Cauchy identity for $Sp(2n, \mathbf{C})$

$$\prod_{1\le i<j\le n}(1-t_it_j)\prod_{i=1}^{n}\prod_{j=1}^{n}(1-t_ix_j)^{-1}(1-t_ix_j^{-1})^{-1}$$

$$=\sum_{\mu,\ell(\mu)\le n} sp_\mu(x_1,x_1^{-1},\ldots,x_n,x_n^{-1})s_\mu(t_1,\ldots,t_n)$$

together with a bijective proof using the Berele correspondence. There is also a dual identity, which we prove via a new "dual" Berele insertion scheme. These two identities may be interpreted as giving the decomposition into irreducibles of the action of $Sp(V)\times Gl(W)$ on the symmetric algebra $S(V\otimes W)$ and the exterior algebra $\bigwedge(V\otimes W)$, respectively, for vector spaces V and W of appropriate dimensions.

The symplectic Schur functions sp_λ are in fact an integral basis for the ring $\tilde{\Lambda}_n = \mathbf{Z}[x_1,x_1^{-1},\ldots,x_n,x_n^{-1}]^{\mathbf{B_n}}$, and an inner product may be defined on this ring by making this basis self-dual. We show how to characterise dual bases in $\tilde{\Lambda}_n$ by means of the Cauchy identity.

Thesis Supervisor: Richard P. Stanley
Title: Professor of Applied Mathematics.

On the Combinatorics of Representations of $Sp(2n, C)$

This work is dedicated to $\mathcal{E}.\mathcal{R}.\mathcal{A}$.

Contents

Introduction

This thesis studies the finite-dimensional representations of the symplectic group $Sp(2n, \mathbf{C})$ from the combinatorial viewpoint of semistandard tableaux and the associated polynomials. The ultimate aim of such an investigation would be to produce a theory analogous to the well-developed theory of Schur functions s_λ indexed by partitions λ. For $\ell(\lambda) \le n$, the Schur functions $\{s_\lambda\}$ are in fact the characters of the irreducible finite-dimensional polynomial representations of $Gl(n, \mathbf{C})$. From general representation theory it follows that the characters are polynomials invariant under the action of the symmetric group S_n, the Weyl group of $Gl(n, \mathbf{C})$. One of the remarkable properties of the representation ϕ_λ is the existence of a set of objects, semistandard tableaux of shape λ, whose cardinality is the dimension of the representation. This leads to a purely combinatorial definition of s_λ in terms of semi-standard tableaux:

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \text{ tableau of shape } \lambda} wt(T)$$

Here the weight $wt(T)$ of T is a certain monomial.

The symplectic Schur function $sp_\lambda(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})$ may be defined as the character of the irreducible representation $\tilde{\phi}_\lambda$ of $Sp(2n, \mathbf{C})$; again from general theory it follows that the polynomials sp_λ (in the $2n$ variables $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$) are invariant under the action of the hyperoctahedral group $\mathbf{B}_n$, the Weyl group of $Sp(2n, \mathbf{C})$. As in the case of $Gl(n, \mathbf{C})$, for every irreducible representation $\tilde{\phi}_\lambda$, $\ell(\lambda) \le n$, there is a finite set of objects, symplectic tableaux, whose cardinality is the dimension of the representation. This, as before, gives a combinatorial definition of the characters sp_λ:

$$sp_\lambda(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}) = \sum_{\tilde{T} \text{ symplectic tableau of shape } \lambda} wt(\tilde{T})$$

where $wt(\tilde{T})$ is an appropriate monomial.

The above definition will be the point of departure of our study. The subject of this thesis may in fact be said to be the Berele algorithm for $Sp(2n, \mathbf{C})$, which plays the role of the Knuth-Schensted-Robinson correspondence for $Gl(n, \mathbf{C})$ and the Schur functions s_λ.

In an attempt to make this thesis somewhat self-contained, and accessible to a general mathematical audience, we have devoted the first two chapters to developing the requisite background material.

Chapter 1 presents an overview of the theory of symmetric functions, and collects some of the more specialised properties of the Knuth correspondence which play an especial role in the development of the results of this thesis.

Chapter 2 begins with an exposition of basic facts concerning the classical Lie groups, including the connections between the representation theory of $Gl(n, \mathbf{C})$ and Schur functions. We then go on to discuss the group $Sp(2n, \mathbf{C})$ and its irreducible characters, and introduce the Berele insertion algorithm. In our presentation we have attempted to point out the similarities between the combinatorial settings for the general linear group and the symplectic group.

In Chapter 3 we present our first results concerning the "up-down" tableaux which play the rôle of the Q−tableaux in Schensted insertion. The crucial reformulation of the Berele algorithm, introducing the notion of "n-symplectic" lattice permutations, appears here (Section 9). This enables us to show that the Berele algorithm possesses remarkable properties. The main result of this chapter (Section 10) uncovers the connection between Berele insertion and Schensted insertion, which in effect says that the Schensted Q-tableau partially encodes the up-down tableau of Berele insertion. We also investigate the effect of Knuth transformations on the Berele algorithm.

In Chapter 4 we use these results to derive a generalisation of a classical decomposition formula for the restriction of an irreducible character of $Gl(2n, \mathbf{C})$ down to its symplectic subgroup. As a special case we obtain the classically known expressions for s_λ in terms of sp_λ and vice versa, as functions in the ring $\tilde{\Lambda}_n = \mathbf{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]^{\mathbf{B_n}}$. The power of the Berele algorithm is also evident in the discovery of a Pieri rule for symplectic Schur functions (Section 13).

Among the main results of the latter sections is a presentation of a "Cauchy" identity for $Sp(2n, \mathbf{C})$, together with a bijective proof, using ideas developed in Chapter 3. We also give a dual Berele algorithm, which in turn provides a bijective proof of a dual Cauchy identity.

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Chapter I

The Combinatorics of Symmetric functions

This chapter provides the reader with the necessary background on the combinatorics of symmetric functions and partitions. We make no claims to completeness; there are many excellent references ([An], [Macd], [Stal]) on the subject, upon which we could not possibly improve. Our main goal will be to familiarise the reader with the combinatorial techniques relevant to this thesis. Consequently our approach will be to state major results without proofs, and supply what we hope is an adequate number of examples.

1 Partitions

We begin by defining the most basic unit in the combinatorial theory of symmetric functions.

Definition 1.1 A **partition** λ of a nonnegative integer n is a non-increasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ such that

(1) there is a $\ell \geq 0$ such that $\lambda_k = 0$ for all $k > l$

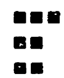
(2) $\sum_i \lambda_i = n$.

We write $\lambda \vdash n$, or $|\lambda| = n$, and refer to n as the size of λ. We call $\ell = \ell(\lambda)$ the **length** of λ. Also, each λ_i is a **part** of λ. Write Par for the set of all partitions, and Par_n for the set of all partitions of n.

By convention, the partition of 0 is the **empty** partition $\emptyset$.

Definition 1.2 The **Ferrers diagram** of a partition λ is a geometric representation of λ obtained by drawing λ_1 boxes in the first row, λ_2 boxes in the second row below the first, ..., etc., with the rows left-justified.

Example 1.3 The partition $(3, 2, 2)$ of 7 has Ferrers diagram



With the Ferrers diagram in mind, we may also refer to a partition λ as a shape. We will find it convenient to coordinatise the Ferrers diagram, so that by the (i, j)th position in λ, we shall mean the box of the diagram in row i and column j, reading the rows from top to bottom, and the columns from left to right.

Definition 1.4 The **conjugate** of $\lambda \vdash n$ is the partition, denoted λ', of n, obtained by reflecting the Ferrers diagram of λ about its main diagonal. Thus λ'_i is the length of ith column of the Ferrers diagram of λ.

It is clear from the Ferrers diagram that $\lambda'_j = |\{i : \lambda_i \geq j\}|$.
Let $m_i(\lambda)$= number of parts of λ of length i. Clearly

$$\lambda'_j - \lambda'_{j-1} = |\{i : \lambda_i = j\}| = m_j(\lambda).$$

Definition 1.5 The **rank** of a partition λ is the number of nodes (boxes) in the main diagonal of the Ferrers diagram of λ.

Definition 1.6 The **Frobenius notation** for a partition λ of rank r specifies λ as follows:

Let α_i be the number of nodes to the right of the diagonal node (i, i) in row i, β_i the number of nodes below the node (i, i), $i = 1, \ldots, r$.

Then $(\alpha_1, \ldots, \alpha_r | \beta_1, \ldots, \beta_r)$ is a description of λ in Frobenius notation.

Notice that $\alpha_i = \lambda_i - i$, $\beta_i = \lambda'_i - i$, so that $\alpha_1 > \ldots > \alpha_r$, $\beta_1 > \ldots > \beta_r$, and

$$|\lambda| = \sum_{i=1}^{r} \alpha_i + \sum_{i=1}^{r} \beta_i + r.$$

Definition 1.7 For λ, μ, in *Par*, we say λ **contains** μ $(\lambda \supseteq \mu)$, if the Ferrers diagram of λ contains the diagram of μ, or equivalently, if $\lambda_i \geq \mu_i$ for all i.

Definition 1.8 If λ, μ are partitions such that $\lambda \supseteq \mu$, the **skew-shape** λ/μ is the set of boxes which are in the Ferrers diagram of λ but not in μ, i.e., the set-theoretic difference between the two Ferrers diagrams.

Example 1.9 The skew-shape $(4,3,3)/(3,2)$ has the Ferrers diagram occupied by the solid squares

$$\begin{array}{llll} \circ & \circ & \circ & \blacksquare \\ \circ & \circ & \blacksquare & \\ \blacksquare & \blacksquare & \blacksquare & \end{array}$$

Definition 1.10 A skew-shape λ/μ is a **horizontal strip** (respectively, a **vertical strip**) if λ/μ has no two boxes in a column (respectively, a row).

Thus the previous example is neither a horizontal nor a vertical strip.

Definition 1.11 Given partitions λ, μ, a **tableau** (or **reverse plane partition** $T_{\lambda/\mu}$ of shape λ/μ is a filling of the squares in the Ferrers diagram of λ/μ with the integers $\{1,2,\ldots\}$ such that the rows and columns are weakly increasing. Equivalently, we may think of a tableau as a strictly increasing sequence of shapes

$$\mu = \lambda^0 \subset \lambda^1 \subset \ldots \subset \lambda^k = \lambda$$

(Simply fill each skew-shape λ^i/λ^{i-1} with the integer i).

Example 1.12 The tableau

$$T = \begin{array}{llll} 1 & 1 & 2 & 3 \\ 1 & 2 & 2 & \\ 3 & & & \end{array}$$

corresponds to the sequence of shapes

$$\emptyset \subset \begin{array}{ll} \blacksquare & \blacksquare \\ \blacksquare & \end{array} \subset \begin{array}{lll} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{array} \subset \begin{array}{llll} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \\ \blacksquare & & & \end{array}$$

Definition 1.13 A tableau $T_{\lambda/\mu}$ is **column-strict** (or **semistandard**) if it is strictly increasing down columns. Equivalently, the strictly increasing sequence of shapes

$$\mu = \lambda^0 \subset \lambda^1 \subset \ldots \subset \lambda^k = \lambda$$

is a column-strict tableau iff each skew-shape λ^i/λ^{i-1} is a horizontal strip.

Example 1.14

The column-strict tableau

$$T = \begin{array}{lll} 1 & 1 & 3 \\ 2 & 3 & \end{array}$$

corresponds to the sequence of shapes

$$\emptyset \subset \begin{array}{ll} \blacksquare & \blacksquare \end{array} \subset \begin{array}{ll} \blacksquare & \blacksquare \\ \blacksquare & \end{array} \subset \begin{array}{lll} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \end{array}$$

Definition 1.15 A tableau $T_{\lambda/\mu}$ of skew-shape λ/μ is **standard** if all the entries in the filling of the Ferrers diagram are distinct. Equivalently, the strictly increasing sequence

$$\mu = \lambda^0 \subset \lambda^1 \subset \ldots \subset \lambda^k = \lambda$$

is a standard skew-tableau if for all $i \leq k$, $\lambda^i/\lambda^{i-1} = (1)$, (i.e., the skew-Ferrers diagram consists of a single box).

In particular, when $\mu = \emptyset$, so that $\lambda/\mu = \lambda$, we refer to a standard tableau of shape λ as a **standard Young tableau.** We will often use the abbreviation SYT for a standard Young tableau.

Notation. We write $f^{\lambda/\mu}$ to denote the number of standard tableaux of skew-shape λ/μ; in particular, f^λ denotes the number of standard Young tableaux of shape λ.

Definition 1.16 The **type** of a column-strict tableau T is the (finite) nonnegative integer vector whose ith component is the number of entries i in T. Note that $sh(T)$ is a partition of $|type(T)| = \sum_i type(T)_i$.

In Example 1.14, the type of T is $(2,1,2)$.

Definition 1.17 For partitions λ, μ, and a nonnegative integer vector α such that $|\lambda/\mu| = |\alpha|$, the **Kostka number** $K_{\lambda/\mu,\alpha}$ is the number of column-strict tableaux of shape λ/μ and type α.

2 Symmetric Functions

Definition 2.1 A function $f(x_1 \ldots x_n)$ is **symmetric** if

$$f(x_{\sigma(1)}, \ldots x_{\sigma(n)}) = f(x_1, \ldots x_n)\ \forall \sigma \in S_n.$$

We denote the ring of symmetric formal power series with integer coefficients in the variables $x_1, \ldots x_n$, $Z(x_1, \ldots x_n)^{S_n}$, by $\Lambda_n(x)$, or simply Λ_n, when there is no ambiguity about the variables involved.

We single out four important symmetric functions:

Definition 2.2 *[Macd][Sta1]*For $\lambda \in Par$,

(1) the **monomial** symmetric function is defined to be

$$m_\lambda(x_1, \ldots, x_n) = \sum_{\substack{\alpha \\ \text{all distinct} \\ \text{permutations of} \\ (\lambda_1, \lambda_2, \ldots)}} x^\alpha$$

where $x^\alpha = \prod_{i\geq 1} x_i^{\alpha_i}$. Thus $m_\lambda(x_1,\ldots,x_n) = 0$ if $\ell(\lambda) > n$.

(2) the **complete homogeneous** symmetric function h_λ is defined to be

$$h_\lambda = \prod_{i=1}^{\ell(\lambda)} h_{\lambda_i},$$

where for $r > 0$,

$$h_r = \sum_{\lambda \vdash r} m_\lambda(x)$$

and $h_0 = 1$ by definition.

(3) the **elementary** symmetric function e_λ is defined to be

$$e_\lambda = \prod_{i=1}^{\ell(\lambda)} e_{\lambda_i},$$

where for $r > 0$,

$$e_r = \sum_{1\leq i_1<\ldots<i_r\leq n} x_{i_1}\ldots x_{i_r},$$

and $e_0 = 1$.

(4) the **power sum** symmetric function is defined to be

$$p_\lambda = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}$$

where for $r > 0$,

$$p_r = \sum_{i=1}^{n} x_i{}^r$$

and $p_0 = 1$.

Theorem 2.3 *The following are* $\mathbf{Z}$*-bases for* Λ_n*:*

(1) $\{m_\lambda(x)\}_{\lambda \in Par}$

(2) $\{h_\lambda(x)\}_{\lambda \in Par}$

(3) $\{e_\lambda(x)\}_{\lambda \in Par}$

while

$\{p_\lambda(x)\}_{\lambda \in Par}$ *is a basis for the ring* $\Lambda_n \otimes \mathbf{Q}$ *of symmetric functions with rational coefficients.*

For single-part partitions $\lambda = (r)$, the homogeneous and elementary symmetric functions have nice generating functions:

Theorem 2.4 *[Stal][Macd]*

$$\prod_{i=1}^{n}(1 - x_i t)^{-1} = \sum_{r\geq 0} h_r(x_1, \ldots, x_n)\, t^r. \tag{1.1}$$

$$\prod_{i=1}^{n}(1 + x_i t) = \sum_{r\geq 0} e_r(x_1, \ldots, x_n)\, t^r. \tag{1.2}$$

There is also a useful automorphism of the algebra Λ_n:

Theorem 2.5 *[Stal][Macd]Define $\omega : \Lambda_n \to \Lambda_n$ by*

$$\omega(h_r) = e_r.$$

Then ω is an involutive algebra automorphism.

Observe that we can extend the notion of symmetric function to functions in infinitely many variables $x_1, \ldots, x_m, \ldots$, by defining $f(x_1, \ldots, x_m, \ldots)$ to be symmetric if

$$f(x_{\sigma(1)}, \ldots x_{\sigma(n)},\ x_{n+1}, \ldots) = f(x_1, \ldots x_n,\ x_{n+1}, \ldots)\ \forall \sigma \in S_n, \text{and for all } n \geq 1.$$

In keeping with the previous notation, we denote the ring of formal power series in infinitely many variables $\{x_i\}$ $\mathbf{Z}(x_1, \ldots, x_m, \ldots)$ by $\Lambda(x)$, or simply Λ when there is no ambiguity about the set of variables in question.

We now define the all-important Schur functions, which form the essential link between combinatorics and representation theory. There are many equivalent definitions; for obvious reasons we prefer the following combinatorial one:

Definition 2.6 For any partitions λ, $\mu, \lambda \supseteq \mu$, the Schur function $s_{\lambda/\mu}$ corresponding to the skew-shape λ/μ is defined to be

$$s_{\lambda/\mu}(x_1, \ldots, x_m, \ldots) = \sum_{\substack{T \text{ column-strict tableau} \\ shape(T)=\lambda/\mu}} weight(T), \tag{1.3}$$

where

$$\begin{aligned} weight(T) &= \prod_i x_i^{\text{number of } i\text{'s in } T} \\ &= x^{type(T)}. \end{aligned}$$

Notice that we made the above definition for possibly infinitely many variables $\{x_i\}$. For finitely many variables $\{x_1, \ldots x_n\}$, we make the natural definition

$$s_{\lambda/\mu}(x_1, \ldots, x_n) = s_{\lambda/\mu}(x_1, \ldots, x_m, \ldots)|_{x_i=0,\, i>n}.$$

Hence, it immediately follows that

Corollary 2.7

$$s_\lambda(x_1, \ldots, x_n) = 0 \text{ if } \ell(\lambda) > n.$$

The reader who desires a more formal justification of this passage between a finite number of variables and infinitely many variables, should consult the first few pages of [Macd].

It is a remarkable fact, not at all obvious from the definition above, that

Theorem 2.8 *The skew-Schur function $s_{\lambda/\mu}(x_1, \ldots, x_n)$ is a symmetric function in $x_1, \ldots, x_n$.*

We postpone the elegant proof due to Bender and Knuth [BKn] until the next chapter, where it is presented in the course of establishing a similar result for the symplectic Schur functions.

The preceding theorem immediately gives the following elegant formula:

Theorem 2.9

$$s_{\lambda/\mu}(x_1, \ldots, x_n) = \sum_{\nu \vdash |\lambda/\mu|} K_{\lambda/\mu,\nu}\, m_\nu(x). \tag{1.4}$$

Proof: Recalling the definition of Kostka numbers in Section 1, it is clear that

$$s_{\lambda/\mu}(x_1, \ldots, x_n) = \sum_{\substack{\alpha \\ |\alpha|=|\lambda/\mu|}} K_{\lambda/\mu,\alpha}\, x^\alpha. \tag{1.5}$$

The statement of the theorem follows, from the definition of the monomial symmetric functions $m_\nu(x)$.•

We state the following combinatorial fact about the Kostka numbers:

Lemma 2.10 *Let $\lambda, \mu \in Par_n$. Then*

$$K_{\lambda,\mu} \neq 0 \iff \sum_{i=1}^r \lambda_i \geq \sum_{i=1}^r \mu_i, \quad r \geq 1$$

Proof: [Macd, p.57] We need only observe that in a column-strict tableau of shape λ, the integers $1, \ldots, r$ must appear in the first r rows. Thus if T is a column-strict tableau of shape λ and type μ, then there must be at least $\sum_{i=1}^r \mu_i$ squares available in the first r rows. •

Now order Par_n in reverse lexicographic order, so that (n) comes first and (1^n) comes last. This is clearly a total order on Par_n. Then we have

Corollary 2.11 *The matrix $(K_{\lambda,\mu})$, whose rows and columns are indexed by Par_n with the reverse lexicographic order, is strictly upper triangular, with 1's on the diagonal.*

Proof: Use Lemma 2.10 above, together with the fact [Macd, p.6] that

$$\sum_{i=1}^{r} \lambda_i \geq \sum_{i=1}^{r} \mu_i, \quad r \geq 1$$

implies λ precedes μ in reverse lexicographic order. •

Finally, we have

Theorem 2.12 *The Schur functions $\{s_\lambda(x_1, \ldots, x_n)\}_{\lambda,\ell(\lambda)\leq n}$ form a basis for the ring Λ_n.*

Proof: By the preceding corollary, the matrix of coefficients $(K_{\lambda,\mu})$ in Theorem 2.9 is invertible with determinant 1, so its inverse is an integer matrix. Hence the monomial symmetric functions are an integer combination of the Schur functions. But the $\{m_\lambda\}_{\ell(\lambda)\leq n}$ form a basis for Λ_n. •

Several remarkable expressions for the Schur function s_λ exist, which are by no means obvious from our combinatorial definition. For the proofs, we refer the reader to [Macd,ch.1], [Sta1].

Theorem 2.13

(1) (The Weyl determinant formula for A_{n-1})[Li1,p.87][Sta1]

$$s_\lambda(x_1, \ldots, x_n) = \frac{det(x_j{}^{\lambda_i+n-i})}{det(x_j{}^{n-i})} \tag{1.6}$$

(2) (Jacobi-Trudi)[Li1,p.87][Sta1]

$$s_\lambda(x_1, \ldots, x_n) = det(h_{\lambda_i-i+j})_{1\leq i,j\leq \ell(\lambda)} \tag{1.7}$$

(3) (Naegelsbach)[Li1,p.89][Sta1]

$$s_\lambda(x_1, \ldots, x_n) = det(e_{\lambda'_i-i+j})_{1\leq i,j\leq \ell(\lambda)} \tag{1.8}$$

We end this section with a few words on one of the more famous combinatorial objects in the theory of Schur functions:

Definition 2.14 A **lattice permutation** is a word of positive integers $w_1 \dots w_n$ such that, reading the word from left to right, the number of occurrences of any integer i is greater than or equal to the number of occurrences of $i+1$. The **weight** of a lattice permutation w is the finite nonnegative integer vector α where $\alpha_i = |\{j : 1 \leq j \leq n, w_j = i\}|$.

Example 2.15 The lattice permutation 12113231 has weight $(4, 2, 2)$.

Remark 2.16

(1) Notice that a lattice permutation always has **partition weight**, i.e., its weight α is a partition.

(2) If the word w is a lattice permutation , so is any initial segment $w_1 \dots w_i$, $i \leq \ell(w)$.

(3) There is a natural bijection between lattice permutations w of length n and weight α, and standard Young tableaux of shape α, defined as follows: for $i = 1, \dots, n$, if $w_i = r$, place an i in the rth row of the Ferrers diagram of α; conversely, given a standard Young tableau of shape α, set $w_i = r$ for all i appearing in row r of α. It is clear that the lattice permutation condition, which seems rather *ad hoc* at first glance, is precisely equivalent to guaranteeing that the tableau associated to the lattice permutation in this manner is standard.

Example 2.17 The lattice permutation 12113231

$$\text{corresponds to} \quad \begin{array}{llll} 1 & 3 & 4 & 8 \\ 2 & 6 & & \\ 5 & 7 & & \end{array}$$

Definition 2.18 Let T be a tableau of skew-shape λ/μ. The **word of T, read in lattice permutation fashion,** is the word obtained by reading the filling T row by row, from **top to bottom** and **right to left.**

Definition 2.19 A lattice permutation w **fits** a skew-shape λ/μ, if there is a column-strict tableau T of shape λ/μ such that the word of T, read in lattice permutation fashion, is w.

Example 2.20

The lattice permutation 12113231 fits the skew-shape $(5,4,3,1)/(3,1,1)$ as follows:

■ ■ ■ 1 2
■ 1 1 3
■ 2 3
1

.

Definition 2.21 For partitions λ, μ, ν, the **Littlewood-Richardson coefficient, which we denote by** $c^{\lambda}_{\mu,\nu}$, is the number of lattice permutations of shape λ/μ and weight ν.

Clearly $c^{\lambda}_{\mu,\nu} = 0$ unless $\lambda \supseteq \mu, \nu$, and $\lambda \vdash (|\mu| + |\nu|)$.

Theorem 2.22

(1) (The Littlewood-Richardson rule)

$$s_{\mu}(t)s_{\nu}(t) = \sum_{\lambda \vdash (|\mu|+|\nu|)} c^{\lambda}_{\mu,\nu}\, s_{\lambda}(t) \tag{1.9}$$

(2) (Schützenberger)

$$s_{\lambda/\mu}(t) = \sum_{\nu} c^{\lambda}_{\mu,\nu}\, s_{\nu}(t) \tag{1.10}$$

Proof: (1) Littlewood and Richardson discovered the rule, and published an incomplete proof, in [LR]; Macdonald gives a completed version of their proof in [Macd], and remarks that other correct proofs were provided by Lascoux and Schützenberger, and Thomas.

(2) Schützenberger[Schu] was apparently the first to prove this formula using the combinatorial definition of the Schur functions, although it was probably known to Littlewood, who seems to have worked primarily with determinantal expressions analogous to equations (1.6)-(1.8). •

Remark 2.23 In Chapter 3 we present a bijection (Theorem 16.11) which in effect establishes (1). We have chosen to postpone the discussion of this bijection because we feel the techniques are more relevant to the development of our other results in Chapter 3.

Corollary 2.24

$$c^{\lambda}_{\mu,\nu} = c^{\lambda}_{\nu,\mu}$$

Proof: Immediate from (1) of Theorem 3.44 and the commutativity in the ring $\Lambda(t)$. •

3 The Knuth-Schensted correspondence

In this section we discuss a procedure of paramount importance in the combinatorics of tableaux and symmetric functions; we shall find that this process appears naturally with startling ubiquity in the theory of symmetric functions, yielding elegant bijective proofs of many classical identities.

Notation. For a positive integer n, we shall denote by $[n]$ the set of integers $\{1, \ldots, n\}$.

Definition 3.1 (Row-insertion)(Schensted) [S] Given a column-strict tableau T of shape λ, we define **row-insertion** of a number x into T, denoted $(T \leftarrow x)$, to mean the following process which results in a new column-strict tableau T^1 of shape λ^1 containing λ, where $\lambda^1/\lambda = (1)$:

(1) Insert x into the first row of T either by displacing ("bumping") the left-most number which is larger than x, or, if no number is larger than x, by adding x to the end of the first row (note that if the row is empty, this means x is added to the first column, thereby creating a new row);

(2) If x did "bump" an element x' from the first row, then insert x' into the second row exactly as above, either by displacing the left-most number which is larger than x', or if no number is larger than x', by adding x' to the end of the row.

(3) Continue in this manner, row by row, until some number is added to the end of a row.

We define **column insertion** of a number x into a tableau T, written $(x \rightarrow T)$, similarly, by replacing the word "row" in the above description by the word "column", and vice versa.

Remark 3.2 We remark that Schensted, in [S], defined these algorithms in the context of permutations, so that his tableaux were standard, i.e., had distinct entries.

Example 3.3 Row-inserting 1 into the tableau

$$T = \begin{array}{l} 1\,1\,2 \\ 3\,3 \end{array}$$

gives the sequence of steps

$$\begin{array}{l} 1\,1\,2 \leftarrow 1 \\ 3\,3 \end{array}, \quad \begin{array}{l} 1\,1\,1 \\ 3\,3 \leftarrow 2 \end{array}, \quad \begin{array}{l} 1\,1\,1 \\ 2\,3 \\ \leftarrow 3 \end{array}, \quad \begin{array}{l} 1\,1\,1 \\ 2\,3 \\ 3 \end{array} = (T \leftarrow 1)$$

Theorem 3.4 (Knuth [Kn1], Schensted [S]) *If T is a column-strict tableau, then $(T \leftarrow x)$ and $(T \rightarrow x)$ are also column-strict tableaux.*

Now consider the effect of successively row-inserting the sequence of letters in a word $w_1 \dots w_n$. If we keep track of the sequence of shapes, we find

Lemma 3.5 $((\dots(\emptyset \leftarrow w_1) \leftarrow w_2) \leftarrow \dots \leftarrow w_n)$ *yields a pair*

$$(P_\lambda, (\lambda^1 = (1) \subset \lambda^2 \dots \subset \lambda^n = \lambda))$$

where $\lambda^i \vdash i$, $i = 1, \dots n$, and P_λ is a column-strict tableau of shape $\lambda \vdash n$.

Definition 3.6 *(* essentially Schensted[S])

The P-tableau corresponding to a sequence of integers $w_1 \dots w_n$ is the column-strict tableau $((\dots(\emptyset(\leftarrow w_1(\leftarrow w_2)\dots \leftarrow w_n)$. The Q-tableau corresponding to the same sequence is the array obtained by putting i in the square which is added to the shape of the P-tableau when w_i is inserted in the P-tableau.

By Lemma 3.5 above, the Q-tableau of any word w is always a standard Young tableau. We can now state

Theorem 3.7 *(essentially* Schensted */S])*

Row-insertion defines a bijection between the set of all words w of length k on the alphabet [n] and the set of all pairs (P_λ, Q_λ) of tableaux of the same shape

$$\lambda \vdash k, \; \ell(\lambda) \leq n,$$

where P_λ is a column-strict tableau and Q_λ is a standard Young tableau .

Proof: (See Schensted [S], Knuth [Kn1]) We need only take care of the reverse direction, i.e., given a pair (P_λ, Q_λ) as described in the statement of the theorem, to retrieve the word w such that row-inserting w produces the pair $(P_\lambda\ ,\ Q_\lambda)$. This essentially amounts to reversing the row-insertion at each step.

The position in Q_λ containing the largest digit n indicates the culmination of the "bumping path" of the insertion of the last letter, w_n, into the P-tableau. If the tableau P_λ contains x where Q_λ contains n, x must have been bumped down into its present position from the preceding row by y, the right-most letter in the preceding row which is smaller than x. Likewise, y was in turn bumped down from the preceding row. We can thus work our way up the tableau P_λ to the first row, where we ultimately "bump out" the letter w_n. (In doing this we are traversing, in reverse order, the bumping path created by inserting w_n). Note that this leaves us with a new P-tableau P', of shape $\lambda^{n-1} \vdash (n-1)$. Now repeat with P', using Q_λ restricted to $[n-1]$ as the new Q-tableau.

To understand the "unbumping" procedure, we recommend working backwards through the sequence in Example 3.8 below. •

We shall refer to the process of traversing a bumping path in reverse order, beginning at a corner square (of the tableau) containing an entry x, as "(row−)unbumping" x, or, less familiarly, as **row-removal** of x.

Notation. In the situation of the Theorem 3.7 above, we write

$$(\emptyset \leftarrow w) = (P_\lambda, Q_\lambda).$$

Also, we shall frequently write

$$(\emptyset \leftarrow w)_1 = P_\lambda, \quad \textit{and} \quad (\emptyset \leftarrow w)_2 = Q_\lambda.$$

Example 3.8

$(\emptyset \leftarrow 31121)$ gives the following sequence of $P-$ and $Q-$tableaux :

$$\left(\begin{array}{cccccc}
3 & \begin{array}{l}1\\3\end{array} & \begin{array}{l}11\\3\end{array} & \begin{array}{l}112\\3\end{array} & \begin{array}{l}111\\2\\3\end{array} & = P \\[2ex]
1 & \begin{array}{l}1\\2\end{array} & \begin{array}{l}13\\2\end{array} & \begin{array}{l}134\\2\end{array} & \begin{array}{l}134\\2\\5\end{array} & = Q
\end{array}\right)$$

so

$$(\emptyset \leftarrow 31121) = \left(\begin{array}{l}111\\2\\3\end{array} , \begin{array}{l}134\\2\\5\end{array}\right)$$

All of the above being true for column-insertion, we have in particular the analogue of Theorem 3.7:

Theorem 3.9 *(essentially* Schensted[S]*) Column-insertion defines a bijection between the set of all words w of length k on the alphabet $[n]$ and the set of all pairs (P_λ, Q_λ) of tableaux of the same shape*

$$\lambda \vdash k,\ \ell(\lambda) \le n,$$

where P_λ is a column-strict tableau and Q_λ is a standard Young tableau .

Notation. In the situation of the Theorem 3.9, we write

$$(w \to \emptyset) = (P_\lambda, Q_\lambda).$$

Also, we shall frequently write

$$(w \to \emptyset)_1 = P_\lambda, \quad \textit{and} \quad (w \to \emptyset)_2 = Q_\lambda.$$

Example 3.10

$(31221 \to \emptyset)$ gives the sequence of column-strict tableaux

$$1, \quad \begin{matrix}1\\2\end{matrix}, \quad \begin{matrix}1\,2\\2\ \ \end{matrix}, \quad \begin{matrix}1\,1\,2\\2\ \ \ \ \ \end{matrix}, \quad \begin{matrix}1\,1\,2\\2\ \ \ \ \ \\3\ \ \ \ \ \end{matrix}$$

so

$$(31221 \to \emptyset) = \left(\begin{matrix}1\,1\,2\\2\ \ \ \ \ \\3\ \ \ \ \ \end{matrix}, \begin{matrix}1\,3\,4\\2\ \ \ \ \ \\5\ \ \ \ \ \end{matrix} \right).$$

As a first application of Schensted insertion, we have, by restricting either of Theorems 3.7 and 3.9 to words which are permutations on $[n]$, a bijective proof of the following enumerative result:

Theorem 3.11

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2 \tag{1.11}$$

It is natural to ask whether there is any relation between the results of row-inserting a word w and column-inserting w. The first step in this direction is the following remarkable fact:

Lemma 3.12 *(essentially* Schensted*[S])Let S be a column-strict tableau, x, y any two positive integers. Then*

$$((x \to S) \leftarrow y) = (x \to (S \leftarrow y)).$$

An easy induction now gives

Theorem 3.13 *Let w be any word in the positive integers $\mathbf{P}$. Then*

$$(\emptyset \leftarrow w)_1 = (w \to \emptyset)_1.$$

That is, the P-tableau corresponding to row-insertion of w coincides with the P-tableau corresponding to column-insertion of w.

We shall be able to say something about the two Q-tableaux in Section 5.

Motivated by the observation that Schensted insertion of a row (respectively, column) produces a column-strict tableau consisting of a single row (respectively, column), we have

Definition 3.14 (Schützenberger) Call a word $w = w_1 \ldots w_n$ a **row-word** if $w_1 \leq \ldots \leq w_n$. Likewise, w is a **column-word** if $w_1 > \ldots > w_n$.

By analysing bumping paths we find:

Lemma 3.15 *Let T be a column-strict tableau, x, y positive integers.*

(1) In the sequence of insertions $(T \leftarrow x \leftarrow y)$, the bumping path of x ends strictly to the left of the bumping path of y

$$\iff x \leq y.$$

(In particular, if $x \leq y$ then the bumping path of x lies to the left of the bumping path of y. That is,

- *the bumping path of x is at least as long as the bumping path of y, and*
- *in each row of $((T \leftarrow x) \leftarrow y)$, the entry bumped into the row by the insertion of x lies strictly to the left of the entry bumped into the row by the subsequent insertion of y.)*

(2) In the sequence of insertions $(x \to y \to T)$, the bumping path of x ends strictly to the left of the bumping path of y

$$\iff x \leq y.$$

(In particular, if $x \leq y$ then the bumping path of x lies above the bumping path of y.)

An immediate consequence of this is:

Lemma 3.16 *Let T be a column-strict tableau, w a row-word. Then*

(1) $sh(T \leftarrow w)/sh(T)$ is a horizontal strip; furthermore, if we number the new squares consecutively (with the numbers $1 \ldots \ell(w)$) as they are added to $sh(T)$, then the resulting filling of the skew-shape $sh(T \leftarrow w)/sh(T)$ is a column-strict tableau (in which i appears to the left of $i+1$ for all i.)

(2)$sh(w \to T)/sh(T)$ is a horizontal strip; furthermore, if we number the new squares consecutively (with the numbers $1 \ldots \ell(w)$) as they are added to $sh(T)$, then the resulting filling of the skew-shape $sh(T \leftarrow w)/sh(T)$ is a column-strict tableau (in which i appears to the left of $i+1$ for all i.)

Example 3.17

$$\text{Let } T = \begin{array}{l} 123 \\ 2 \end{array}, \quad w = 1123$$

Computing $(T \leftarrow w)$ gives the sequence of column-strict tableaux

$$\left(\begin{array}{llll} 113 & 111 & 1112 & 11123 \\ 22 & 223 & 223 & 223 \end{array} \right)$$

so, writing in the order in which the new squares are added to $sh(T)$, we have

$$\left(\begin{array}{l} 123 \\ 2 \end{array} \leftarrow 1123 \right) = \left(\begin{array}{l} 11123 \\ 223 \end{array}, \begin{array}{l} \blacksquare\blacksquare\blacksquare 34 \\ \blacksquare 12 \end{array} \right)$$

Likewise,

$$\left(1123 \to \begin{array}{l} 113 \\ 2 \end{array}\right) = \left(\begin{array}{l} 11113 \\ 22 \\ 3 \end{array}, \begin{array}{l} \blacksquare\blacksquare\blacksquare 34 \\ \blacksquare 2 \\ 1 \end{array} \right)$$

Lemma 3.16 provides a bijective proof of formula for the expansion of the product of a Schur function and a homogeneous symmetric function h_n:

Theorem 3.18 (Pieri's rule)

$$s_\mu \, . \, h_n = \sum_{\substack{\lambda \\ \lambda/\mu \text{ is a horizontal} \\ \text{strip of size } n}} s_\lambda \tag{1.12}$$

Donald Knuth [Kn1] introduced what proved to be an extremely useful equivalence relation on words of a fixed length k in the positive integers. By way of motivating the following definition, notice that

$$(\emptyset \leftarrow 211) \text{ and } (\emptyset \leftarrow 121)$$

both have the same P-tableau

$$\begin{matrix} 1\,1 \\ 2 \end{matrix} \; .$$

Definition 3.19 *Let x, x', x'', be positive integers. We define an* **elementary Knuth transformation** *to be either of the following:*

(1) if $x \le x' < x''$, the transformation from the word $x''\,x\,x'$ to the word $x\,x''\,x'$;

(2) if $x < x' \le x''$, the transformation from the word $x'\,x\,x''$ to the word $x'\,x''\,x$.

If two words w, w' differ only by a subword of three (consecutive) letters, with the differing subwords being as in (1) or (2), we say w, w', are **elementary Knuth equivalent**; *if w' can be obtained from w by a series of elementary Knuth transformations, we say w' and w are* **Knuth equivalent**, *and we write*

$$[w] = [w'].$$

Clearly, Knuth-equivalence defines an equivalence relation on the set of all words on a fixed alphabet.

The theorem which makes this notion worthwhile is:

Theorem 3.20 (Knuth)*([Kn1], Theorem 5)*

Let w, w' be words of the same length. Then w and w' are Knuth-equivalent

$$\iff (\emptyset \leftarrow w)_1 = (\emptyset \leftarrow w')_1$$

($\iff$ the P-tableau of w coincides with the P-tableau of w'.)

Thus any column-strict tableau determines a unique word, up to Knuth-equivalence. For instance, given a column-strict tableau P with m rows and n columns, if we denote the row-word of the ith row of P by $w(R_i)$, and the column-word of the jth column of P by $w(C_j)$, then clearly

$$\begin{aligned} P &= (\ldots(\emptyset \leftarrow w(R_m)) \leftarrow w(R_{m-1}) \leftarrow \ldots w(R_1))_1 \\ &= (w(C_1) \rightarrow (w(C_2) \rightarrow \ldots w(C_m) \rightarrow \emptyset)\ldots)_1 \\ &= (\ldots(\emptyset \leftarrow w(C_1)) \leftarrow w(C_2) \leftarrow \ldots w(C_m))_1, \end{aligned}$$

so

$$[w(R_m).w(R_{m-1}).\ldots.w(R_1)] = [w(C_1).w(C_2).\ldots.w(C_m)]$$

Notation. We call $w(R_m).w(R_{m-1}).\ldots.w(R_1)$ the **(row-)word** of the tableau P, denoted $word(P)$, and $w(C_1).w(C_2).\ldots.w(C_m)$ the **column-word** of the tableau P.

Before going on to Knuth's generalisation of Schensted insertion, we make a definition:

Definition 3.21 Call a two-line array

$$\mathcal{T} = \begin{pmatrix} u_1 & \ldots & u_n \\ v_1 & \ldots & v_n \end{pmatrix}$$

a **Knuth two-line array** if

(1) $u_i \leq u_{i+1}$ for all i.

(2) if $u_i = u_{i+1}$ then $v_i \leq v_{i+1}$.

Theorem 3.22 (Knuth)*[Kn1]*

(1) There is a bijection between nonnegative integer matrices $A = (a_{i,j})$ with row sum vector α (i.e., $\alpha_i = \sum_j a_{i,j}$) and column sum vector β (defined likewise) and Knuth two-line arrays

$$\mathcal{T} = \begin{pmatrix} u_1 & \ldots & u_n \\ v_1 & \ldots & v_n \end{pmatrix}$$

where there are α_i i's among the u's, and exactly β_i i's among the v's.

(2) There is a bijection between Knuth two-line arrays

$$\mathcal{T} = \begin{pmatrix} u_1 & \ldots & u_n \\ v_1 & \ldots & v_n \end{pmatrix}$$

where there are α_i i's among the v's, and exactly β_i i's among the u's and pairs of column-strict tableaux (P, Q) of the same shape, with type$(P) = \alpha$, type$(Q) = \beta$. This establishes the Schur function identity (often referred to as the Cauchy *identity (cf. [Macd p.98, Ex. 6])):*

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(x) s_\lambda(y) \tag{1.13}$$

Proof: See Knuth's well-written paper [Kn1].

We content ourselves with a brief description of the bijection here.

We may view the present situation as a generalisation of the setting of Theorem 3.7, (the Knuth-Schensted algorithm), whose input is a set of words $y_{i_1} \dots y_{i_k}$, or equivalently, two-line arrays

$$T = \begin{pmatrix} 1 & \dots & k \\ y_{i_1} & \dots & y_{i_k} \end{pmatrix},$$

where the top row consists of distinct, strictly increasing labels.

Thus our input is now in the form of two-line arrays

$$\mathcal{T} = \begin{pmatrix} x_{i_1} & \dots & x_{i_k} \\ y_{i_1} & \dots & y_{i_k} \end{pmatrix}$$

where the top row has labels which may be repeated, but are still in increasing order; in addition, the array is written so that $x_{i_j} = x_{i_{j+1}}$ implies $y_{i_j} \leq y_{i_{j+1}}$.

We begin by applying the row-insertion algorithm to the word $y_{i_1} \dots y_{i_k}$. Starting with $j = 0$ and a pair of empty tableaux, if at the jth step we have built up a sequence of pairs of tableaux (P_j, Q_j), at step $j + 1$ we set

$$P_{j+1} = P_j \leftarrow y_{i_{j+1}},$$

and Q_{j+1}

$= Q_j$ with $t_{i_{j+1}}$ added in the unique position so as to force $sh(Q_{j+1}) = sh(P_{j+1})$

To go backwards from a pair of column-strict tableau (P, Q) of the same shape, we follow the same idea as in the description of Schensted row-removal:

The position in Q containing the largest and right-most letter y indicates the culmination of the "bumping path" of the insertion of the previous letter, x, into P. If the tableau P contains x' where Q contains y, x' must have been bumped down into its present position from the preceding row by x'', the largest and right-most letter in the preceding row which is smaller than x'. Likewise, x'' was in turn bumped down from the preceding row. We can thus work our way up the tableau P to the first row, where we ultimately "bump out" the letter x. (In doing this we are traversing, in reverse order, the bumping path created by inserting x). Note that this leaves us with a new P', whose shape is the same as that of Q', where Q' is Q without its largest and right-most letter.

See the example which follows. •

Example 3.23

The Knuth two-line array

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 1 & 4 & 1 \end{pmatrix}$$

yields the sequence of column-strict tableaux

$$\left(\begin{array}{ccccccc} 2 & 23 & \begin{matrix} 13 \\ 2 \end{matrix} & \begin{matrix} 134 \\ 2 \end{matrix} & \begin{matrix} 114 \\ 23 \end{matrix} & = & P \\ \\ 1 & 11 & \begin{matrix} 11 \\ 2 \end{matrix} & \begin{matrix} 112 \\ 2 \end{matrix} & \begin{matrix} 112 \\ 23 \end{matrix} & = & Q \end{array} \right)$$

corresponding to the pair of column-strict tableaux

$$\left(\begin{matrix} 114 \\ 23 \end{matrix}, \begin{matrix} 112 \\ 23 \end{matrix} \right)$$

We shall refer to the bijection in (2) of Theorem 3.22 as the **Knuth correspondence.**

Remark 3.24

(1) Observe that if we restrict the Knuth correspondence to Knuth two-line arrays $\mathcal{T}$ where the top row contains each integer at most once, (equivalently, to 0-1 matrices A with at most one nonzero entry in each column) the correspondence reduces to Schensted row-insertion of the word in the bottom row of $\mathcal{T}$, as defined in Theorem 3.7.

(2) In particular, by further restricting to permutation matrices, the correspondence reduces to Schensted's original row-insertion algorithm (which applied to permutations).

In view of the above remark, we introduce the following
Notation. If w is a word on $[n]$, write

$$w \overset{KS}{\longleftrightarrow} (P,Q)$$

if the Knuth correspondence applied to the two-line array

$$\begin{pmatrix} 1 & \dots & k \\ w_1 & \dots & w_k \end{pmatrix},$$

or equivalently, Schensted row-insertion applied to w, yields the pair of column-strict tableaux (P,Q).

Corollary 3.25 *Knuth's correspondence, when restricted to nonnegative integer matrices with row-sum vector μ for a partition μ, gives a bijective proof of the Schur function identity*

$$h_\mu = \sum_{\lambda \vdash |\mu|} K_{\lambda,\mu} \; s_\lambda. \tag{1.14}$$

There is also a **dual Knuth correspondence** which establishes the following enumerative result:

Theorem 3.26 (Knuth) *There is a one-to-one correspondence between 0-1 matrices with row-sum vector α and column-sum vector β and pairs of column-strict tableaux (P,Q) where $sh(P)$ is the conjugate of $sh(Q)$, $type(P) = \alpha$ and $type(Q) = \beta$. This in turn yields the formula (the* dual Cauchy *identity)*

$$\prod_{i,j}(1 + x_i y_j) = \sum_\lambda s_{\lambda'}(x) s_\lambda(y) \tag{1.15}$$

Knuth's correspondence has the following remarkable properties:

Theorem 3.27 (Knuth)

(1) If the nonnegative matrix A corresponds to the pair of column-strict tableaux (P,Q), then the transposed matrix A^t corresponds to (Q,P).

(2) The Knuth correspondence yields a bijection between symmetric *nonnegative integer matrices A with column-sum vector α, and column-strict tableaux P of type α, such that the number of columns of P of odd length is the trace of A. This in turn establishes the following identity:*

$$\prod_i (1 - qx_i)^{-1} \prod_{i<j} (1 - x_i x_j)^{-1} = \sum_\lambda q^{|\{i : \lambda_i' \text{ is odd}\}|} \; s_\lambda(x). \tag{1.16}$$

Corollary 3.28 *(1) Let σ be a permutation on [n], such that σ is an involution. Then $(\emptyset \leftarrow \sigma) = (P, P)$ (where $sh(P) \vdash n$).*

Thus there is a bijection between involutions on [n] and column-strict tableaux of shape λ in Par_n, yielding the formula:

$$\sum_{\lambda \vdash n} f^\lambda = |\{\sigma \in S_n : \sigma = \sigma^{-1}\}| \tag{1.17}$$

(2) If σ is an involution on [n] without fixed points, (so n is even) then

$$\sigma \stackrel{KS}{\longleftrightarrow} (P, P)$$

where $sh(P) \vdash n$ has even columns. This in turn establishes

$$\sum_{\substack{\beta \vdash n \\ \beta' \text{ even}}} f^\beta = (n-1)(n-3)\ldots 5.3.1 \tag{1.18}$$

Proof: (1) follows easily, recalling Remark 3.24(2).

For (2), simply observe that involutions on $[n]$ without fixed points (i.e., products of $\frac{n}{2}$ disjoint transpositions) correspond to n by n permutation matrices with trace zero; hence P has no odd columns. •

We shall be dealing extensively with a particular kind of Knuth two-line array, which we define next:

Definition 3.29 Call a two-line array

$$L = \begin{pmatrix} j_1 & \cdots & j_r \\ i_1 & \ldots & i_r \end{pmatrix}$$

a **Burge** two-line array (after William Burge, cf. [Bu]) if

(1) $j_1 \leq j_2 \ldots \leq j_r$;

(2) $j_k = j_{k+1}$ implies $i_k \leq i_{k+1}$, for all $k = 1, \ldots, r$.

(3) $j_k > i_k$ for all $k = 1, \ldots, r$.

Notice that (1) and (2) are simply the conditions for L to be a Knuth two-line array.

Thus, applying the Knuth correspondence to symmetrised Burge two-line arrays (i.e., Knuth arrays in which each occurrence of a pair $\binom{j}{i}$ is accompanied by the pair $\binom{i}{j}$) establishes a bijection between symmetric nonnegative integer matrices with trace zero and column-strict tableaux of shape with even columns. This in turn is a bijective proof of the Schur function identity:

Theorem 3.30 (Littlewood)

$$\prod_{i<j}(1-t_it_j)^{-1} = \sum_{\beta' \text{ even}} s_\beta(t_1,t_2,\ldots) \tag{1.19}$$

This is one of six identities discovered by Littlewood, for which he supplies algebraic proofs in [Li1,p.235]. Equation 1.14, with the specialisation $q \to 1$, is another of these.

William Burge, in [Bu], gives direct bijective proofs of four of these identities; we shall be especially interested in (1.19) above. We describe what we shall henceforth refer to as the Burge correspondence in

Theorem 3.31 *The following procedure is a bijection between Burge two-line arrays and column-strict tableaux of even-columned shape:*

Given a Burge two-line array

$$L = \begin{pmatrix} j_1 & \ldots & j_r \\ i_1 & \ldots & i_r \end{pmatrix},$$

construct a single column-strict tableau T (instead of a pair), with columns of even length, as follows.

(1) Set $k=0$, $T_k=\emptyset$.

(2) Set $k \to k+1$. T_k is obtained from T_{k-1} in two steps:

(a) $S=(T_{k-1} \leftarrow i_k)$; suppose the square in S containing i_k is in position (s_k,t_k) (i.e.,row s_k, column t_k).
(b) T_k=(S with j_k placed in position (s_k+1,t_k)), i.e., S with j_k placed immediately below the new square created by inserting i_k into T_{k-1}.

(3) If $k=r$, set $T=T_r$ and stop; otherwise, go back to step (2).

Proof: It is not hard to show that this actually produces an even-columned column-strict tableau; we recommend that the reader try to convince himself that this is in fact a bijection. Burge's lucid paper [Bu] supplies all the details.

Since we shall need to know how to recover the two-line array, given an even-columned tableau T, we describe the inverse of above insertion process:

(1) Set $k = r$, $T_k = T$;

(2) Remove largest entry x in T_k; set $j_k = x$.

(3) Row-remove ("unbump") entry $y < x$ immediately above former position of x in T, bumping out an element z (note $z < y < x$) and leaving a tableau S. Set $i_k = z$, $k \leftarrow k - 1$, $T_k = S$.

(4) If $k = 0$, stop; otherwise, go back to step (2). •

Example 3.32 The Burge array

$$L = \begin{pmatrix} 3 & 3 & 3 & 4 & 4 \\ 1 & 2 & 2 & 1 & 1 \end{pmatrix} \mapsto \begin{array}{l} 111 \\ 223 \\ 33 \\ 44 \end{array}$$

via the sequence of insertions

$$\begin{array}{l} 1 \\ 3 \end{array}, \quad \begin{array}{l} 12 \\ 33 \end{array}, \quad \begin{array}{l} 122 \\ 333 \end{array}, \quad \begin{array}{l} 112 \\ 233 \\ 3 \\ 4 \end{array}, \quad \begin{array}{l} 111 \\ 223 \\ 33 \\ 44 \end{array}.$$

Lemma 3.33 *Let*

$$L = \begin{pmatrix} j_1 & \dots & j_r \\ i_1 & \dots & i_r \end{pmatrix}$$

be a Burge two-line array, and let

$$T = \begin{pmatrix} u_{l_1} & \dots & u_{l_r} & \dots & u_{l_{2r}} \\ v_{l_1} & \dots & v_{l_r} & \dots & v_{l_{2r}} \end{pmatrix}$$

be the symmetrisation of L, so that $u_{l_1} \leq \ldots \leq u_{l_r} \leq \ldots \leq u_{l_{2r}}$ is a rearrangement of the multiset $\{i_k, j_k : k = 1 \ldots r\}$, and $\binom{u}{v}$ appears in T once for each occurrence of $\binom{u}{v}$ in L and once for each occurrence of $\binom{v}{u}$ in L. Then

$$L \longleftrightarrow P$$

under the Burge correspondence

$$\Longleftrightarrow T \longleftrightarrow (P, P)$$

under the Knuth correspondence.

Proof: See [Bu]. (It is not too difficult to establish this result by a careful induction argument.) We point out that both L and T, as described above, represent the same monomial in the left-hand side of the identity (1.19).•

We will have occasion to refer to another bijection in [Bu], which establishes the symmetric function identity

Theorem 3.34 (The dual Burge correspondence)

$$\prod_{i\leq j}(1-t_it_j)^{-1} = \sum_{\substack{\lambda \\ \lambda\ even}} s_\lambda(t) \qquad (1.20)$$

Proof: This time the left-hand side counts **dual Burge two-line arrays**

$$L* = \begin{pmatrix} j_1 & \dots & j_r \\ i_1 & \dots & i_r \end{pmatrix}$$

where

(1) $j_1 \leq \ldots \leq j_r$.

(2) $j_k \geq i_k$, all $k = 1, \ldots, r$.

(3) $j_k = j_{k+1}$ implies $i_k > i_{k+1}$.

The insertion process works essentially as in Theorem 3.31, except that it now produces a row-strict tableau with even columns, the conjugate of which then yields the contribution to the right-hand side of equation (1.18).

The algorithm proceeds exactly as in Theorem 3.31, with one change: we replace ordinary row-insertion, where i_k inserted into T_k bumps the left-most element larger than itself, by requiring that each element being inserted into a row bump the first element larger than or equal to itself. (This immediately guarantees row-strictness.) •

There is one final identity, (also belonging to Littlewood's half-dozen, mentioned previously) which we shall need in Chapter 4:

Theorem 3.35 (Burge)*[Bu] There is a bijection establishing the following symmetric function identity due to Littlewood [Li1, p.238, (11.9;1)]:*

$$\prod_{i<j}(1+t_it_j) = \sum_{\substack{r\geq 0 \\ \alpha=(\alpha_1>\ldots>\alpha_r>0)}} s_{\mathcal{F}(\alpha)} \tag{1.21}$$

where $\mathcal{F}(\alpha) = (\alpha_1-1,\ldots,\alpha_r-1|\alpha_1,\ldots,\alpha_r)$ *in Frobenius notation. (Note* $\mathcal{F}(\alpha) = \emptyset$ *if* $\alpha = \emptyset$ *iff* $r = 0$*.)*

Remark 3.36 Macdonald observes [Macd,p.46] that this is essentially Weyl's identity for the root system D_n.

4 Jeu de Taquin

We now describe an operation on skew-tableaux invented by Schützenberger, which turns out to have extraordinary connections with Knuth-Schensted insertion. For an entertaining account of the procedure called *jeu de taquin*, we refer the reader to Chapter 3 of Lynne Butler's thesis [But].

Definition 4.1 The *jeu de taquin* algorithm transforms a column-strict tableau of skew-shape λ/μ into a column-strict tableau of shape ν for some partition ν, by a sequence of slides (Schützenberger's *glissements*, [Schu]). This involves pushing the entries of the skew-tableau T into the north-west corner, or, equivalently, sliding out the empty squares in the north-west corner to the outer boundary of the diagram. A slide is an exchange between an empty square and an adjacent square of the skew-tableau, performed so as to preserve column-strictness and weak increase along rows. In the simplest instance we have the following situation:

$$\begin{matrix} \circ & a \\ b & \end{matrix} \;\rightarrow \begin{cases} \begin{matrix} a & \circ \\ b & \end{matrix}\,, & \text{if } a<b; \\[2ex] \begin{matrix} b & a \\ \circ & \end{matrix}\,, & \text{if } a\geq b. \end{cases}$$

$$\rightarrow \begin{cases} \begin{matrix} a \\ b \end{matrix}\,, & \text{if } a<b; \\[1ex] \begin{matrix} b & a \end{matrix}\,, & \text{if } a\geq b. \end{cases}$$

Observe that the column-strictness requirement forces a unique exchange, in spite of the two possibilities.

In the general case, given a column-strict tableau $T_{\lambda/\mu}$ of skew-shape λ/μ, the algorithm proceeds as follows:

(1) Choose an empty square s in the north-west corner that is adjacent to the skew-tableau $T_{\lambda/\mu}$, such that no square of the skew-tableau lies above or to the left of s. (We call s a **corner square** of $T_{\lambda/\mu}$.)

(2) Move $\{s\}$ south and east by a sequence of slides, until no square of $T_{\lambda/\mu}$ lies east of s, south of s, or in between these two directions, yielding a new column-strict tableau $T^1_{\lambda^1/\mu^1}$ of skew-shape λ^1/μ^1, where $\mu^1 \subset \mu$, and in fact $|\mu^1| = |\mu| - 1$.

(3) Return to step (1) with $T^1_{\lambda^1/\mu^1}$, repeating until there are no empty squares left, and we have a column-strict tableau T_ν of shape ν, for some partition ν.

Theorem 4.2 (Schützenberger)*[Schu]*

(1)(Also Thomas[Th]*)*

Jeu de taquin is well-defined, i.e., for a prescribed skew-tableau $T_{\lambda/\mu}$, the algorithm, as described above, results in the same final tableau, no matter what the choice of corner squares in Step (1).

(2) Given the skew-shape λ/μ, the number of tableaux of shape ν obtained from the set of column-strict tableaux of shape λ/μ via jeu de taquin is the Littlewood-Richardson coefficient $c^\lambda_{\mu,\nu}$, and hence depends only on the shapes λ, μ, ν (and not on (the entries in) the tableaux).

Proof: *For (1), we highly recommend Thomas' extremely lucid paper [Th] on the subject.* ●

Example 4.3

We shall have occasion to use slides primarily in the case of punctured[1] tableaux, viz., tableaux with a single empty square. We illustrate with an example:

Example 4.4

Consider the punctured tableau

$$\begin{array}{cccc} 1 & 1 & 2 & 4 \\ \circ & 3 & 3 & \\ 3 & 4 & 5 & \\ 5 & 6 & & \end{array}$$

[1] This terminology appears in Allan Berele's paper [Be].

The following unique sequence of slides moves the hole out to the boundary of the tableau:

$$\begin{array}{cccc} 1 & 1 & 2 & 4 \\ 3 & 3 & 3 & \\ \circ & 4 & 5 & \\ 5 & 6 & & \end{array} \rightarrow \begin{array}{cccc} 1 & 1 & 2 & 4 \\ 3 & 3 & 3 & \\ 4 & \circ & 5 & \\ 5 & 6 & & \end{array} \rightarrow \begin{array}{cccc} 1 & 1 & 2 & 4 \\ 3 & 3 & 3 & \\ 4 & 5 & \circ & \\ 5 & 6 & & \end{array} ,$$

leaving the column-strict tableau

$$\begin{array}{cccc} 1 & 1 & 2 & 4 \\ 3 & 3 & 3 & \\ 4 & 5 & & \\ 5 & 6 & & \end{array} .$$

A special case of the *jeu de taquin* operation, when performed on a standard Young tableau, defines another standard Young tableau of the same shape :

Definition 4.5 Let Q_λ be a standard Young tableau of shape $\lambda \vdash n$. We define the **evacuated tableau** associated to Q_λ, denoted $Q_\lambda{}^{evac}$, to be the tableau obtained by the following procedure (essentially due to Knuth []):

(1) Set $i = 0$, $Q_i = Q_\lambda$, $\mu^0 = \lambda$.

(2) Set $i \rightarrow i+1$. Remove the square containing i in Q_{i-1}, obtaining a punctured tableau $Q_{i-1}{}^\circ$, with the puncture in position (1,1). Apply *jeu de taquin* to $Q_{i-1}{}^\circ$ to get an ordinary standard Young tableau Q_i.

(3) Set $\mu^i = sh(Q_i)$. Note $|\mu^i| = |\mu^{i-1}| - 1$. In fact, a stronger statement is true: $\mu^i \subset \mu^{i-1}$.

(4) If $i = n = |\lambda|$, stop; otherwise go back to step (2).

Finally, the sequence of shapes ($\mu^n = \emptyset \subset \mu^{n-1} \subset \ldots \subset \mu^0 = \lambda$) is the standard Young tableau $Q_\lambda{}^{evac}$.

Example 4.6

$$\text{If } Q = \begin{array}{cc} 1 & 2 \\ 3 & \\ 4 & \end{array} ,$$

then evacuation gives the sequence of standard tableaux

$$\begin{matrix}12\\3\\4\end{matrix}\quad\begin{matrix}2\\3\\4\end{matrix}\quad\begin{matrix}3\\4\end{matrix}\quad 4\quad \emptyset$$

and hence, reading the sequence of shapes backwards, the evacuated tableau is

$$\begin{matrix}14\\2\\3\end{matrix}$$

A few remarks about this definition of evacuation[2] are in order.

Schützenberger [Schu] defined an operation "J" on standard tableaux, as in

Definition 4.7 *Let P_λ be a standard Young tableau of shape $\lambda \vdash n$. Define $(P_\lambda)^J$ by setting*

$$(\emptyset \leftarrow \rho\sigma\rho)_1 = (P_\lambda)^J,$$

where

(1) σ is any permutation on $[n]$ such that the P-tableau of σ is P_λ, i.e.,

$$(\emptyset \leftarrow \sigma)_1 = P_\lambda,$$

and

(2) $\rho = (n\, n-1\, \dots\, 1)$ is the reverse of the identity permutation.

The following observation was then made[3]:

Theorem 4.8 (Schützenberger)

$$P_\lambda{}^{evac} = P_\lambda{}^J$$

Theorem 4.9 (Schützenberger[Schu])

The operation J is an involution on the set of standard Young tableaux of the same shape.

Proof: The fact that J is an involution follows immediately from the definition of J and the observation that $\rho^2 = 1$. We refer the reader to [] for the fact that J preserves shapes. •

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[2] We thank Professor Adriano Garsia for introducing us to the notion of evacuation; the definition we give above is essentially equivalent to his explanation.

[3] We thank Lynne Butler for bringing this to our attention.

Remark 4.10 We have an induction argument to show that evacuation, as defined in Definition 4.5, is an involution.

Recall Theorem 3.13, in which we were able to establish a partial connection between row- and column-insertion. We now present the promised result concerning the Q-tableaux:

Theorem 4.11 (Schützenberger)*[Schu] Let* σ *be a permutation on* $[n]$*. Let*

$$(\emptyset \leftarrow \sigma) = (P, Q).$$

Then

$$(\sigma \rightarrow \emptyset) = (P, Q^J) = (P, Q^{evac}).$$

Proof: We refrain from presenting our own rather uninspiring induction argument, which is based on Definition 4.5. •

Chapter II

Representations of the symplectic group

In this chapter we explain briefly how the interaction between symmetric functions and the representations of the classical Lie groups comes about. A nodding acquaintance with the representation theory of finite groups on the part of the reader would be helpful, but is not necessary. A more complete exposition can be found in [Ste]. Both [Sta1] and [Ha] are good surveys of the subject.

5 The general linear group

Definition 5.1 Let V be a vector space of dimension n over the complex numbers C. The **general linear group** $Gl(V)$ is the group of all invertible linear transformations $V \mapsto V$. We also write $Gl(n, \mathbf{C})$ for $Gl(V)$, where $Gl(n, \mathbf{C})$ is the group of all invertible n by n matrices over C.

Definition 5.2 A **representation** ϕ of $Gl(n, \mathbf{C})$ is a group homomorphism $\phi : Gl(n, \mathbf{C}) \to Gl(V)$ for some vector space V over C. We say V **affords** the representation ϕ.

Notice that a representaion ϕ afforded by V, makes V into a $Gl(n, \mathbf{C})$-module in a natural way.

We shall only be concerned with finite-dimensional representations:

Definition 5.3 The degree of a finite-dimensional representation $\phi : Gl(n, \mathbf{C}) \to Gl(V)$ is $deg\ \phi = dim\ V$.

Definition 5.4 ϕ is a **polynomial** (respectively **rational**) representation of $Gl(n, \mathbf{C})$ if (there is an ordered basis for V such that) the entries of $\phi(A)$ are polynomial (respectively rational) functions of the entries of A, for all A in $Gl(n, \mathbf{C})$.

Definition 5.5 The representation ϕ is **homogeneous of order** m if each entry of $\phi(A)$ is a homogeneous function of degree m of the entries of A in $Gl(n, \mathbf{C})$, i.e., if $\phi(tA) = t^m\, \phi(A)$, for all t in $\mathbf{C}$.

Example 5.6

(1) The **trivial** representation is defined by

$$\phi : Gl(n, \mathbf{C}) \to \mathbf{C}, \phi\,(A) = 1, \quad \text{all } A;$$

this is a polynomial representation of degree 1.

(2) The **natural or defining** representation

$$\phi\,(A) = A,$$

is polynomial of degree n.

(3) The **contragredient** representation

$$\phi\,(A) = (A^{-1})^t$$

is rational of degree n.

(4) The representation

$$\phi\,(A) = \begin{pmatrix} 1 & ln(|det\ A|) \\ 0 & 1 \end{pmatrix}$$

is not rational.

Definition 5.7 For any representation ϕ of $Gl(n, \mathbf{C})$, the **character** of ϕ is the function

$$char\ \phi : Gl(n, \mathbf{C}) \to \mathbf{C}$$

defined by

$$(char\ \phi)(A) = trace\ (\phi(A)).$$

It is an immediate consequence of this definition that characters are constant on conjugacy classes. Also, recalling that two representations ϕ, ρ, afforded by $Gl(n,\mathbb{C})$-modules V, W, are **equivalent** if there is a $Gl(n,\mathbb{C})$-module isomorphism $T : V \to W$ such that $T\,\rho(A) = \phi(A)\,T$, all $A \in Gl(n,\mathbb{C})$, we observe that *char* ϕ completely determines the representation ϕ up to equivalence.

I. Schur, in his 1901 dissertation[Schur, pp. 68-85], discovered the connections between symmetric functions and the rational representations of $Gl(n,\mathbb{C})$:

Theorem 5.8 (Schur)

(1) Every rational representation of $Gl(n,\mathbb{C})$ is completely reducible.

(2) Irreducible rational representations ϕ are homogeneous, so that the character char $\phi(A)$ is a symmetric Laurent polynomial (an ordinary symmetric polynomial if ϕ is polynomial) in the eigenvalues of A in $Gl(n,\mathbb{C})$.

(3) Write $x_1,\dots,x_n$ for the eigenvalues of an arbitrary A in $Gl(n,\mathbb{C})$; then for each rational representation ϕ there is a multiset M_ϕ of integer n-tuples such that the eigenvalues of $\phi(A)$ are $\{x^\alpha = \prod_{i=1}^n {x_i}^{\alpha_i} : \alpha \in M_\phi\}$. That is, every rational representation ϕ has character

$$(char\ \phi)(x_1,\dots,x_n) = \sum_{\alpha \in M_\phi} x^\alpha,$$

for a uniquely defined multiset M_ϕ of integer n-tuples.

(4) If ϕ is a homogeneous representation of $Gl(n,\mathbb{C})$, of order m for some $m \geq 1$, then

$$char\ \phi = ch\ \chi$$

for some character χ of the symmetric group S_n on n letters, where ch is Frobenius' **characteristic** *map [Macd, p.61]:*

$$ch : \{\text{character ring of } S_n\} \to \Lambda_n, \quad ch(\chi) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma)\ p_{\text{cycle-type of }\sigma}(x_1,\dots,x_n).$$

(p_μ is the power-sum symmetric function, for $\mu \in Par$, cf. Chapter 1.)

(5) The irreducible polynomial representations of $Gl(n,\mathbb{C})$ are indexed by partitions λ of length at most n.

(6) If ϕ^λ is the irreducible polynomial representation indexed by λ, $\ell(\lambda) \leq n$, then char ϕ^λ is the Schur function $s_\lambda(x_1,\dots,x_n)$.

(7) Every rational representation ϕ is of the form

$$\phi(A) = (det(A))^m\ \tilde{\phi}(A)$$

where $(\bar{\phi})$ is a polynomial representation. Also ϕ is irreducible iff $\bar{\phi}$ is irreducible.

Proof: Besides going back to Schur's original paper, we highly recommend the extremely readable and careful account in [Ste, pp.46-55]. •

Example 5.9 (1) The trivial representation corresponds to the empty partition, since its character is

$$1 = s_{\emptyset}(x_1, \ldots, x_n).$$

(2) The defining representation $\phi(A) = A$ has character

$$x_1 + \cdots + x_n = s_{(1)}(x_1, \ldots, x_n),$$

so is indexed by the single-part partition (1) of 1 and is thus irreducible.

(3) The one-dimensional representation $\phi(A) = det(A)$ has character

$$x_1 \cdots x_n,$$

which we observe is the Schur function $s_{(1^n)}$, so this is also an irreducible representation.

(4) The contragredient representation $\phi(A) = (A^{-1})^t$ has character

$$char\ \phi = {x_1}^{-1} + \cdots + {x_n}^{-1} = \left(\prod_{i=1}^{n} {x_i}^{-1} \right) s_{(1^{n-1})}.$$

(5) If $\phi^\lambda : Gl(n, \mathbb{C}) \to Gl(V)$ is the irreducible representation corresponding to the partition λ, $\ell(\lambda) \le n$, then the kth tensor power of ϕ^λ is the representation $: Gl(n, \mathbb{C}) \to Gl(V^{\otimes k})$, defined in the obvious way, and clearly has character

$$charV^{\otimes k} = (char\ \phi^\lambda)^k = {s_\lambda}^k(x_1, \ldots, x_n).$$

Schur also discovered the connection between the representations of $Gl(n, \mathbb{C})$ and those of the symmetric group S_k; his famous "double centraliser theorem" concerning the kth tensor power of the defining representation of $Gl(n, \mathbb{C})$ follows:

Theorem 5.10 (Schur) *Let V be a vector space of dimension n over $\mathbb{C}$. Consider the kth tensor power $V^{\otimes k}$. Then*

(1) the symmetric group S_k and the group $Gl(V)$ each act on $V^{\otimes k}$, and the two actions commute, making $V^{\otimes k}$ an $(S_k \times Gl(V))$-module;

(2) the decomposition of $V^{\otimes k}$ into irreducible sub-modules under this action is

$$V^{\otimes k} = \coprod_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} (S^\lambda \otimes N^\lambda)$$

where S^λ is the irreducible S_k—module indexed by $\lambda \vdash k$, and N^λ is the irreducible $Gl(V) = Gl(n, \mathbb{C})$-module indexed by λ, $(\ell(\lambda) \leq n)$.

Proof: See [Ste]. •

In particular, considering the action of $Gl(V)$ (which is embedded as the subgroup $\{1\} \times Gl(V)$ in $S_k \times Gl(V)$) alone on $V^{\otimes k}$, we have

Theorem 5.11 *Under the action of $Gl(V) = Gl(n, \mathbb{C})$, $V^{\otimes k}$ decomposes into irreducibles as follows:*

$$V^{\otimes k} = \coprod_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} (f^\lambda)\, N^\lambda \tag{2.1}$$

where f^λ, the multiplicity of the irreducible $Gl(n, \mathbb{C})$-module N^λ in the above decomposition, is the number of standard Young tableaux of shape $\lambda \vdash k$.

Proof: We need only remind ourselves that f^λ is in fact the dimension of the irreducible S_k-module S^λ. •

Taking the characters of the representations occurring in (2.1), we have the Schur function identity

$$(x_1 + \cdots + x_n)^k = \sum_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} f^\lambda \; s_\lambda(x_1, \ldots, x_n). \tag{2.2}$$

Remark 5.12 Observe that this identity has already been established combinatorially in Theorem 3.7, via Knuth-Schensted insertion, since clearly the left-hand side of (2.2) enumerates k-words on the alphabet $[n]$, while the right-hand side counts pairs (P_λ, Q_λ) of tableaux of the same shape $\lambda \vdash k$, with P_λ column-strict and Q_λ standard.

6 The symplectic group

For the remainder of this chapter, V will denote a complex vector space of even dimension $2n$. We shall also assume that V is equipped with a non-degenerate bilinear skew-symmetric form $< , >$ (i.e., $< x, y > = - < y, x >$ for all x, y in V).

Definition 6.1 The **symplectic** group $Sp(2n, \mathbf{C}) = Sp(V)$ is the subgroup of $Gl(2n, \mathbf{C})$ which preserves the form $< , >$, i.e.,

$$Sp(2n, \mathbf{C}) = \{A \in Gl(2n, \mathbf{C}) :< Ax, Ay >=< x, y > \text{ for all } x, y \in V\}.$$

We need to introduce a group that will be playing a rôle analogous to that of the symmetric group in the case of $Gl(2n, \mathbf{C})$:

Definition 6.2 The **hyperoctahedral group** is the group B_n of n by n signed permutation matrices, that is, matrices where, in each row and each column, we allow exactly one nonzero entry, which can only be ± 1. Thus the order of the group B_n is $2^n\ n!$.

Remark 6.3

(1) The symplectic group $Sp(2n, \mathbf{C})$ is one of the classical Lie groups (cf. [Hum]), with Lie algebra corresponding to the root system C_n.

(2) The Weyl group (cf. [Hum]) of $Sp(2n, \mathbf{C})$ is the hyperoctahedral group B_n.

As in the previous section, we have

Definition 6.4 A (continuous) **polynomial** representation of $Sp(2n, \mathbf{C})$ is a group homomorphism

$$\phi : Sp(2n, \mathbf{C}) \rightarrow Gl(W)$$

for some complex vector space W, such that the entries of $\phi(A)$ are polynomials in the entries of A ($A \in Sp(2n, \mathbf{C})$).

Note that this makes W into an $Sp(2n, \mathbf{C})$ -module affording the representation ϕ.

Definition 6.5 The **character** of a polynomial representation ϕ of $Sp(2n, \mathbf{C})$ is the function *char* ϕ :$Sp(2n, \mathbf{C}) \rightarrow \mathbf{C}$ such that *char* $\phi(A) =$ *trace* $\phi(A)$.

We list some general facts about characters of $Sp(2n, \mathbf{C})$ in

Theorem 6.6 *Let* $G = Sp(2n, \mathbf{C})$.

(1) The Lie group $Sp(2n, \mathbf{C})$ *is* **reductive**; *that is, all polynomial representations of* G *are completely reducible.*

(2) If A *is in* G, *its eigenvalues are of the form* $\{x_i^{\pm 1} : x_i \in \mathbf{C}, i = 1, \ldots, n\}$.

(3) With a suitable choice of basis, the subgroup $T = \{diag(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}) : x_i \in \mathbf{C}, x_i \neq 0\}$ *is the set of diagonal matrices in G (the maximal torus (cf. [Hum]) in G).*

(4)

$$\bigcup_{g \in G} gTg^{-1}$$

is dense in G, i.e., the diagonalisable matrices of G are dense in G. Consequently the value of the character char ϕ *(of a representation* ϕ*) at an element A of G, depends only on the eigenvalues of A in G (by the invariance of char* ϕ *on conjugacy classes). Hence char* ϕ *may be viewed as a function of* $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$.

(5) For any polynomial representation ϕ*, the character char* ϕ *is a Laurent polynomial in* $x_1, \ldots, x_n$*, which is invariant under the action of the Weyl group* B_n *of G.*

(6) In fact, for each polynomial representation ϕ*, there is a multiset of integer n-tuples* $\tilde{M}_\phi$ *such that*

$$char\ \phi(x_1, \ldots, x_n) = \sum_{\alpha \in \tilde{M}_\phi} x^\alpha$$

where as before $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$.

Hence char ϕ *is in the ring* $\mathbf{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]^{B_n}$*, the Laurent polynomials in* $x_1, \ldots, x_n$ *which are invariant under the action of* B_n.

(7) The irreducible polynomial representations of G are indexed by partitions λ *of length at most n.*

Definition 6.7 For λ of length at most n, the **symplectic Schur function** sp_λ is the character of the irreducible representation of $Sp(2n, \mathbf{C})$ which is indexed by the partition λ. Thus sp_λ is a function in the ring $\mathbf{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]^{B_n}$.

Notice that $sp_\emptyset = 1$.

Notation. We write $\tilde{\Lambda}_n$ for the ring $\mathbf{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]^{B_n}$.

In order to study the representations of $Sp(2n, \mathbf{C})$ combinatorially, we need an analogue of the combinatorial definition of the ordinary Schur functions. From (6) of the above theorem, for λ in *Par* of length at most n, there is a multiset $\tilde{M}_\lambda$ of nonnegative integer n-tuples such that

$$sp_\lambda(x_1, \ldots, x_n) = \sum_{\alpha \in \tilde{M}_\lambda} \prod_{i=1}^n x_i^{\alpha_i}$$

Hence, we would like to be able to define a set of weighted objects T_λ, such that there is a weight-preserving bijection between $\tilde{M}_\lambda$ and T_λ. Not surprisingly, it transpires that the objects we seek are a subset of the ordinary column-strict tableaux of shape λ.

We choose an alphabet $1 < \bar{1} < 2 < \bar{2} < \ldots < n < \bar{n}$.

Definition 6.8 **A symplectic tableau** $\tilde{T}_\lambda$ of shape λ, $\ell(\lambda) \leq n$, is a filling of the Ferrers diagram of λ with the letters of the alphabet $1 < \bar{1} < 2 < \bar{2} < \ldots < n < \bar{n}$, such that:

(1) the entries are weakly increasing along rows and strictly increasing down the columns

(2) all entries in row i are larger than, or equal to, i.

Observe that (1) simply requires that the tableau be column-strict. We shall often refer to (2) as the **symplectic condition**. We remark that (2) automatically forces the shape of a symplectic tableau in the alphabet $1, \bar{1}, \ldots, n, \bar{n}$, to have length at most n.

Example 6.9

Let $n = 5$, $\lambda = (3, 2, 2) \vdash 7$.

Then

$$\tilde{T}_{(3,2,2)} = \begin{matrix} 1 & \bar{1} & \bar{1} \\ \bar{2} & 3 & \\ 4 & \bar{4} & \end{matrix}$$

is symplectic, but

$$T_{(3,2,2)} = \begin{matrix} 1 & \bar{1} & \bar{1} \\ \bar{1} & 2 & \\ \bar{2} & 3 & \end{matrix}$$

is not.

The next step is to specify a suitable weighting scheme for these tableaux. We do this simply by assigning a weight to each entry of the tableau according to the rule

$$i \rightarrow x_i, \quad \bar{i} \rightarrow x_i^{-1}.$$

Then the weight of a symplectic tableau $\tilde{T}$ is

$$wt(\tilde{T}) = \prod_{i=1}^{n} x_i^{\text{number of } i\text{'s in} \tilde{T}} (x_i^{-1})^{\text{number of } \bar{i}\text{'s in } T},$$

a monomial in $\tilde{\Lambda}_n$.

In Example 6.9 above, $wt((\tilde{T})_{(3,2,2)}) = x_1(x_1{}^{-1})^2 x_2{}^{-1} x_3 x_4 x_4{}^{-1}$.

We are now ready for the combinatorial definition of the symplectic Schur function:

Theorem 6.10 (King)[Ki2] *For all partitions λ of length at most n,*

$$sp_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = \sum_{\substack{\tilde{T}_\lambda \\ \text{symplectic tableau} \\ \text{of shape } \lambda}} wt(\tilde{T}_\lambda) \tag{2.3}$$

Remark 6.11 *The first combinatorial definition (in terms of Gelfand patterns, Gelfand (1950)), of the irreducible characters of $Sp(2n, \mathbb{C})$ essentially followed from the branching rules of Zhelobenko (1962); it is this formulaion which was later converted to semi-standard tableaux by R.C.King (1975). See [Ki2] for more details.*

It is actually possible to show, starting with the above combinatorial definition, that $sp_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$ is invariant under the action of B_n. As promised in Chapter 1, we now give Bender and Knuth's[BKn] elegant proof:

Theorem 6.12 *The expression*

$$\sum_{\substack{\tilde{T}_\lambda \\ \text{symplectic tableau} \\ \text{of shape } \lambda \\ \text{in } \{1,\bar{1},\ldots,n,\bar{n}\}}} wt(\tilde{T}_\lambda) \tag{2.4}$$

defines a function in the ring $\tilde{\Lambda}_n$.

Proof: It suffices to show that the expression (2.4) is invariant under transpositions $(i, i+1)$ and $(i, \bar{i})$. If we represent the weight of a symplectic tableau $\tilde{T}_\lambda$ by the $2n$–tuple $(\alpha_1, \bar{\alpha_1}, \ldots, \alpha_n, \bar{\alpha_n})$, then we must show that

(1) there are exactly as many symplectic tableaux $\tilde{T}_\lambda$ of weight

$$(\alpha_1, \bar{\alpha_1}, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_n, \bar{\alpha_n}),$$

as there are of weight

$$(\alpha_1, \bar{\alpha_1}, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_n, \bar{\alpha_n});$$

(2) there are exactly as many symplectic tableaux $\tilde{T}_\lambda$ of weight

$$(\alpha_1, \bar{\alpha_1}, \ldots, \alpha_i, \bar{\alpha_i}, \ldots, \alpha_n, \bar{\alpha_n}),$$

as there are of weight

$$(\alpha_1, \bar{\alpha_1}, \ldots, \bar{\alpha_i}, \alpha_i, \ldots, \alpha_n, \bar{\alpha_n}).$$

Consider the statement (1) first.

The idea is that, thanks to the column-strictness of the tableau, we can isolate occurrences of i and $i+1$ into segments of the tableau as pictured below, containing a contiguous portion of a row, consisting of r i's followed by s $(i+1)$'s, such that the entries above this strip of length $(r+s)$ are all strictly less than i except for the element above the sth $(i+1)$, which is equal to i; while the entries below the $(r+s)$ strip are all strictly larger than $i+1$, except for the one immediately below the first i, which is equal to $i+1$.

$$\begin{array}{ccccccc} & & & & \{entries\ < i\} & & i \\ i & \ldots & i & (i+1) & \ldots & \ldots & (i+1) \\ (i+1) & \{entries\ > i+1\} & & & & & \end{array}$$

Now simply replace the $(r+s)$-strip by one consisting of s i's followed by r $(i+1)$'s. We have, at least locally, succeeded in exchanging the number of i's and $i+1$'s. It remains to observe that repeating this procedure over the whole tableau establishes (1).

The same trick may be applied to (2): we look for contiguous segments of i's and $\bar{i}$'s. Since our switching takes place within the original row, there is no possibility of violating the symplectic condition. •

Example 6.13 The defining representation of $Sp(2n, \mathbb{C})$ is obtained by considering the natural action of $Sp(2n, \mathbb{C})$ on $Gl(V)$ ($\dim V = 2n$), $\phi : A \mapsto A$.

Clearly, if the eigenvalues of A are $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$, then $char\ \phi(A) = trace\ A = \sum_{i=1}^{n} x_i^{\pm 1} = sp_{(1)}(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$.

Lemma 6.14 (Littlewood)[Li1] *Consider the defining representation of $Sp(2n, \mathbb{C})$, afforded by $V = \tilde{N}^{(1)}$ ($\dim V = 2n$). Let $\tilde{N}^{\lambda}$ be an irreducible module affording the representation of $Sp(2n, \mathbb{C})$ indexed by λ, $\ell(\lambda) \leq n$. Then the tensor product*

$$\tilde{N}^{\lambda} \otimes \tilde{N}^{(1)}$$

decomposes into irreducibles as

$$\coprod_{\substack{\mu \subset \lambda \\ \lambda/\mu = (1)}} \tilde{N}^{\mu} \coprod_{\substack{\nu \supset \lambda \\ \ell(\nu) \leq n \\ \nu/\lambda = (1)}} \tilde{N}^{\nu}$$

We can now state the analogue of the decomposition of the kth tensor power of the defining representation of $Gl(n, \mathbb{C})$:

Theorem 6.15 *Let V afford the defining representation for $Sp(2n, \mathbb{C})$. Then the kth tensor power of V decomposes into irreducible $Sp(2n, \mathbb{C})$ -modules $\tilde{N}^\mu$ as follows:*

$$V^{\otimes k} = \coprod_{\substack{\mu \\ \ell(\mu) \leq n}} \tilde{f}^k_\mu(n) \ \tilde{N}^\mu$$

where $\tilde{f}^k_\mu(n)$ is the number of sequences of shapes $(\emptyset = \mu^0, \mu^1, \ldots, \mu^k = \mu)$ such that

(1) two consecutive shapes differ by exactly one box, i.e., for all $i = 1, \ldots, k$, either $\mu^i/\mu^{(i-1)} = (1)$ or $\mu^{(i-1)}/\mu^i = (1)$.

(2) $\ell(\mu^i) \leq n$, for all $i = 1, \ldots, k$.

Proof: This follows easily by successive applications of Lemma 6.14. •

Taking the characters of the representations involved in the above decomposition, we get the symplectic Schur function identity:

$$(x_1 + x_1{}^{-1} + \cdots + x_n + x_n{}^{-1})^k = \sum_{\substack{\mu \\ \ell(\mu) \leq n}} \tilde{f}^k_\mu(n) \ sp_\mu(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) \qquad (2.5)$$

This in turn begs the question of whether there is a bijective proof.
As we shall see in the next section, the answer is in the affirmative:

7 Berele's algorithm

Theorem 7.1 (Berele)[Be] *There is a bijection between the set of all k-words w on $1, \bar{1}, \ldots, n, \bar{n}$ and the set of all pairs ($\tilde{P}_\mu$, $S^k_\mu(n)$), where $\tilde{P}_\mu$ is a symplectic tableau of shape μ and $S^k_\mu(n)$ is a k-sequence of shapes satisfying the conditions described in Theorem 6.15.*

Proof: We describe Allan Berele's ingenious algorithm: the idea is to follow the usual Knuth-Schensted row-insertion scheme as long as the resulting tableau is symplectic; if it happens that the symplectic condition may be violated, the procedure is suitably modified.

Given a word $w = w_1, \ldots, w_k$ in $1, \bar{1}, \ldots, n, \bar{n}$, to produce a symplectic tableau and a k-sequence of shapes satisfying the conditions of Theorem 6.15:

(1) Set $r = 0$; $\mu^r = \emptyset$; $\tilde{P}^r = \emptyset$.

(2) Set $r \to r + 1$, $P = (\tilde{P}^{r-1} \leftarrow w_r)$.

(3) If P is symplectic, then set $\tilde{P}^r = P$, $\mu^r = sh(P)$.

If P is not symplectic, then there is a symplectic violation in P, i.e., at some point in the insertion process in step (2), an entry z smaller than j was bumped into row j, for some j. It is not hard to see that the first time this happened must have been in the following situation: an $\bar{\imath}$ was bumped out of row i into row $i + 1$, by an i. (For suppose the first time was when x smaller than j got bumped into row j; then the element y which displaced x, was smaller than x, and went into row $j - 1$. Since the latter move did not violate the symplectic condition, we have $y \geq (j - 1)$. On the other hand, $y < x < j$, so we are forced to conclude $y = (j - 1)$ and $x = \overline{(j-1)})$.

Having identified the unique row in the bumping path of w_r prior to which the symplectic condition is satisfied, but at which it is violated, we interrupt the insertion of w_r at this point, where an i is about to displace an $\bar{\imath}$ into row $i + 1$, and instead do the following:

(3.1) Replace the first $\bar{\imath}$ in the (ith) row by the i;

(3.2) Remove the first i in the row (note there is at least one i, thanks to step (3.1); this yields a punctured tableau with a hole in row 1;

(3.3) Slide the hole out via *jeu de taquin* east and south, until a normal tableau $\tilde{P}$ remains.

Observe that (3.3) does not cause any symplectic violations: as the hole migrates to the boundary of the tableau, it is exchanged with elements either to the right or lower, so only elements larger than j move up into any row j ($j \geq i$).

Finally, set $\tilde{P}^i = \tilde{P}, \mu^i = sh(\tilde{P})$.

Notice that the weight of the tableau has been preserved: to check that

$$weight(\tilde{P}) = weight(\tilde{P}^{i-1}) \,.\, weight(w_i),$$

it suffices to observe that at the point where the usual insertion was interrupted, an x_i was about to be added to the weight; but the modification resulted in first contributing $x_i{}^2$ (step (3.1)), then removing an x_i (step (3.3)), which clearly has the same net effect.

(4) If $i < k$, go back to step (2); otherwise, stop.

Finally, $\tilde{P}_\mu = \tilde{P}^k$, and the sequence of shapes is $(\emptyset = \mu^0, \mu^1, \ldots, \mu^k = \mu)$.

Notation. Write

$$(\emptyset \overset{B}{\longleftarrow} w)$$

to denote Berele insertion of the word w.

We illustrate with :

Example 7.2 Take $w = 2\bar{2}\bar{1}211$.

Then $(\emptyset \overset{B}{\longleftarrow} w)$ gives the sequence of symplectic tableaux

$$\left(\begin{array}{cccccccccc}
2 & 2\bar{2} & \begin{matrix}\bar{1}\,\bar{2}\\ 2\end{matrix} & \begin{matrix}\bar{1}\,2\\ 2\,\bar{2}\end{matrix} & \begin{matrix}\circ\,2\\ 2\,\bar{2}\end{matrix} & \rightarrow & \begin{matrix}2\,2\\ \bar{2}\end{matrix} & \begin{matrix}1\,2\\ \circ\end{matrix} & \rightarrow & 1\,2 \\
 & & & & & & & & & \\
\blacksquare & \blacksquare\blacksquare & \begin{matrix}\blacksquare\blacksquare\\ \blacksquare\end{matrix} & \begin{matrix}\blacksquare\blacksquare\\ \blacksquare\blacksquare\end{matrix} & & & \begin{matrix}\blacksquare\blacksquare\\ \blacksquare\end{matrix} & & & \blacksquare\blacksquare
\end{array}\right)$$

Hence

$$(\emptyset \overset{B}{\longleftarrow} w) = \left(1\,2\;, \left(\blacksquare \quad \blacksquare\blacksquare \quad \begin{matrix}\blacksquare\blacksquare\\ \blacksquare\end{matrix} \quad \begin{matrix}\blacksquare\blacksquare\\ \blacksquare\blacksquare\end{matrix} \quad \begin{matrix}\blacksquare\blacksquare\\ \blacksquare\end{matrix} \quad \blacksquare\blacksquare \right)\right)$$

To reverse the bijection, given a pair $(\tilde{P}_\mu\,,\ S^k_\mu(n)\,)$, we need to work our way backwards to reconstruct the sequence of symplectic tableaux, and hence retrieve the letters inserted at each step. We describe the passage from an arbitrary step j to step $j-1$:

We are given shapes μ^{j-1} and μ^j, which differ by exactly one square, and a symplectic tableau $\tilde{P}_j$ of shape μ^j; we have to reconstruct from this information, a symplectic tableau $\tilde{P}_{j-1}$ and a letter w_j such that $(\tilde{P}_{j-1} \overset{B}{\longleftarrow} w_j)$ produces $\tilde{P}_j$. Clearly, two cases arise:

(1) $\mu^{j-1} \subset \mu^j$; this says

$$(\tilde{P}_{j-1} \overset{B}{\longleftarrow} w_j) = (\tilde{P}_{j-1} \leftarrow w_j) = \tilde{P}_j,$$

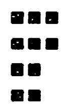
so the $(j-1)$st tableau is obtained by the usual Schensted row-removal, unbumping the letter w_i; or

(2) $\mu^j \subset \mu^{j-1}$; so $(\tilde{P}_{j-1} \leftarrow w_j)$ resulted in a symplectic violation, and thus $(\tilde{P}_{j-1} \overset{B}{\longleftarrow} w_j)$ resulted in a cancellation. We reverse the *jeu de taquin* moves as follows:

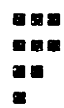
Place a "hole" in the missing square in $\tilde{P}_j$; slide this hole to the first column in the highest possible row, say row i, without violating the symplectic condition. This restores the situation of step (3.2) in the insertion process. Now it is simply a matter of reversing the instructions of (3.2): place an i in the empty square (which is in column 1), change the last i in the row to an $\bar{\imath}$, and bump up an i (row-remove, beginning with row $i-1$). This eventually bumps out w_j, and clearly leaves a symplectic tableau of the required shape μ^{j-1}, by construction. ●

Example 7.3

μ^{j-1}



μ^{j}



$$\tilde{P}_j = \begin{array}{l} 1\,\bar{1}\,2 \\ 2\,3\,4 \\ 3\,\bar{4} \\ \bar{4}\,\circ \end{array}$$

$$\downarrow$$

$$\tilde{P}_{j-1} = \begin{array}{l} 1\,2\,2\nearrow\bar{1} \\ 2\,\bar{2}\,4 \\ 3\,3 \\ \bar{4}\,\bar{4} \end{array} \leftarrow \begin{array}{l} 1\,\bar{1}\,2 \\ 2\,\bar{2}\,4\nearrow 2 \\ 3\,3 \\ \bar{4}\,\bar{4} \end{array} \leftarrow \begin{array}{l} 1\,\bar{1}\,2 \\ \circ\,2\,4 \\ 3\,3 \\ \bar{4}\,\bar{4} \end{array}$$

and thus

$$w_j = \bar{1}.$$

In the next chapter we shall see that Berele's algorithm possesses remarkable properties.

Chapter III

Properties of the Berele algorithm

8 Up-down tableaux

This section studies the combinatorial object which appears as the second component in the image of Berele's correspondence, playing the rôle of the standard Young tableau in the Knuth-Schensted algorithm. Recall that the former objects are sequences of shapes, like the latter, with, in some sense, fewer "growth" restrictions imposed. The aim of this section is to recast these objects in terms of the more familiar standard Young tableaux, leading up to a reformulation of the Berele algorithm which will then hint strongly at possible connections with Knuth-Schensted.

Recall that we write $[n]$ for the set of integers $\{1,\ldots,n\}$, for a positive integer n.

We begin with

Definition 8.1 An **up-down tableau** of length k is a k-sequence of shapes $S^k_\mu = (\emptyset = \mu^0, \mu^1, \ldots, \mu^k = \mu)$ such that two consecutive shapes differ by exactly one box, i.e., for all $i = 1,\ldots,k$, either $\mu^i/\mu^{(i-1)} = (1)$ or $\mu^{(i-1)}/\mu^i = (1)$. We will see shortly that any such sequence of shapes may be encoded as a sequence of tableaux in the entries $[k]$; hence our choice of terminology. We call μ the **shape of the up-down tableau.**

Example 8.2

$$S^5_{(2,1)} = (\emptyset, (1), (2), (1), (1,1), (2,1))$$

is an up-down tableau of length 5 and shape (2,1).

Set F^k_μ ={all up-down k-tableaux of shape μ}.

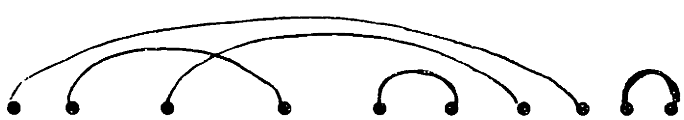
Set $\tilde{f}^k_\mu = \mid F^k_\mu \mid$. Clearly a SYT is a particular case of an up-down tableau, since it may be viewed as a sequence of shapes such that each shape is one box larger than its predecessor.

Lemma 8.3

$$\tilde{f}^{2j}_\emptyset = (2j-1)!! \tag{3.1}$$

Proof: Given $2j$ points on a line, the number of ways of pairing them up (we draw an arc between two points to indicate a pair) into j disjoint pairs is clearly $(2j-1)!!$. Call such a way of pairing up $2j$ points an arc diagram on $[2j]$, denoted $\mathcal{A}_{2j}$. We construct a bijection between the set of $(2j-1)!!$ arc diagrams $\mathcal{A}_{2j}$ and the set $F^{2j}_\emptyset$ as follows[1] At this point we urge the reader to consult the example below.

Example 8.4



										$\mathcal{A}_{2\cdot5}$
4	1	3	$\bar{1}$	2	$\bar{2}$	$\bar{3}$	$\bar{4}$	5	$\bar{5}$	w
4	1 4	1 3 4	○ 3 4	2 3 4	○ 3 4	○ 4	∅	5	∅	
4	1 4	1 3 4	3 4	2 3 4	3 4	4	∅	5	∅	$(T_1, \ldots, T_{10} = \emptyset)$
■	■ ■	■■ ■	■ ■	■ ■ ■	■ ■	■	∅	■	∅	$S^{10}_\emptyset = (\mu^1, \ldots, \mu^{10} = \emptyset)$

If $\mathcal{A}_{2j}$ is an arc diagram on $2j$ points, label the right end-points of the arcs with the barred integers $\bar{1}, \ldots, \bar{j}$ consecutively, reading from left to right. (See Example 8.4). Next, assign the label i to the left end-point of the (unique) arc whose right end-point has label $\bar{i}$. This encodes the arc diagram as a permutation w on the $2j$ letters $1, \ldots, j, \bar{1}, \ldots, \bar{j}$ with the property that every $\bar{i}$ is preceded by an i. We

[1]We thank Professor Richard Stanley for sharing his idea of using arc diagrams for this bijection.

now construct a sequence of shapes from the word $w = w_1, \ldots, w_{2j}$ by forming a sequence of $2j$ tableaux $T_1, \ldots, T_{2j}$:

At the first step, T_1 is the tableau obtained by row-inserting w_1 (note that w_1 is always an unbarred letter).

For the ith step, if w_i is unbarred, then T_i is the tableau obtained by row-inserting w_i into T_{i-1}. Otherwise, we have a situation as in the figure below:

$$w_i = \bar{3}$$

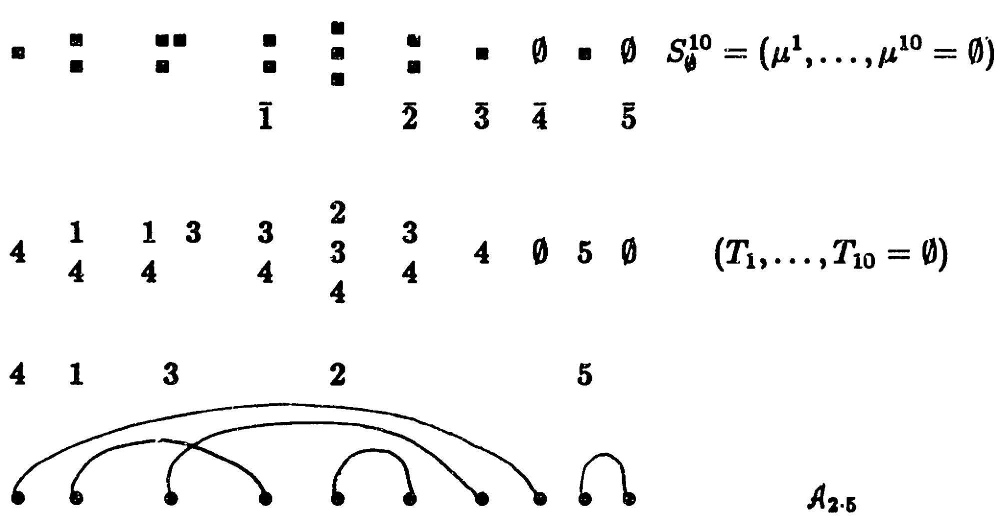
	1 2 5	1 2 5	1 2 5	1 2 5	1 2 5	
$T_{i-1} =$	3 4	∘ 4	4 ∘	4 7	4 7	$= T_i$
	6 7	6 7	6 7	6 ∘	6	

If w_i is barred, say $w_i = \bar{p}$, then p is an entry of T_{i-1}(because $\bar{p}$ must be preceded in w by p). Erase the entry p from T_{i-1} to get a tableau with a hole in it; now play *jeu de taquin* to slide the hole out to the boundary of T_{i-1}. The resulting tableau T_i has one less square than T_{i-1}.

Set $\mu^i = sh(T_i)$ for all $i = 1, \ldots, 2j$. Note that $\mu^{2j} = \emptyset$, since, when we have arrived at the end of the word w, we have gone through all the barred letters $\bar{1}, \ldots, \bar{j}$ and consequently have removed every possible contribution to the tableau. Thus $S_\emptyset^{2j} = (\mu^0, \mu^1, \ldots, \mu^{2j} = \emptyset)$ is an up-down tableau of length $2j$ and shape $\emptyset$.

To reverse the bijection, start with an up-down tableau $S_\emptyset^{2j} = (\mu^1, \ldots, \mu^{2j} = \emptyset)$ of empty shape. We suggest following Example 8.5 along with the text.

Example 8.5



Label each of the $2j$ shapes μ^i, $i = 1, \ldots, 2j$, so as to produce a permutation w as described above (which encodes an arc diagram). Reading the sequence from left to right, label the ith removal of a square with a barred i, placing $\bar{\imath}$ under the shape resulting from the removal. See the example above. Observe that the label assigned to μ^{2j} will always be a barred j since the last shape is empty, and so is the consequence of removing a square. Now we work backwards from the empty shape μ^{2j}, filling each shape to get its associated tableau as constructed in the first half of the bijection. In the process we retrieve the unbarred labels, and hence the whole word. Moving back one step from μ^{2j} to μ^{2j-1} is easy: the associated filling is simply j (i.e., j was removed from μ^{2j-1} to give the empty shape μ^{2j}). Assume μ^i has already been filled, giving a tableau T_i (so the last $(2j - i + 1)$ labels have been determined). To find the filling for μ^{i-1}, we consider the two possible cases:

(i) either μ^i/μ^{i-1} is a box, in which case we simply bump out the extra entry of T_i (invert Schensted row-insertion) to get a tableau T_{i-1} of shape μ^{i-1}; this bumped-out entry is then the label (unbarred) for the shape μ^i;

$$\begin{array}{ccc} \mu^{i-1} & & \mu^i \\ \begin{array}{l} \blacksquare \\ \blacksquare \end{array} & & \begin{array}{l} \blacksquare\blacksquare \\ \blacksquare \end{array} \end{array}$$

$$T_{i-1} = \begin{array}{l} 1 \\ 4 \end{array} \leftarrow T_i = \begin{array}{ll} 1 & 3 \\ 4 & \end{array}$$

$$w_i = 3$$

or,

(ii) μ^i/μ^{i-1} is a box, so there is a barred letter k under μ^i. In terms of the first half of the bijection, this says that the required filling T_{i-1} of μ^{i-1} is such that on evacuating (cf. Section 4) the entry k from it we get the tableau T_i. The reverse process is clearly to place a k in T_i, in the position of the square by which it differs from T_{i-1}, and to perform *jeu de taquin* moves to slide this k into T_i to the unique final position that leaves a valid tableau T_{i-1} of shape μ^{i-1}. See the example which follows.

$$\begin{array}{ccc} \mu^{i-1} & & \mu^i \\ \begin{array}{l} \blacksquare\blacksquare \\ \blacksquare \end{array} & & \begin{array}{l} \blacksquare \\ \blacksquare \end{array} \\ & & \bar{1} \end{array}$$

$$T_{i-1} = \begin{array}{ll} 1 & 3 \\ 4 & \end{array} \leftarrow T_i = \begin{array}{ll} 3 & 1 \\ 4 & \end{array}$$

$\bullet$

Remark 8.6 The procedure described above in fact gives a bijection between $F_\emptyset^{2j}$ and involutions in S_{2j} without fixed points (or, j products of disjoint transpositions), since another way of encoding an arc diagram A_{2j} is easily obtained by simply labelling the $2j$ points with the integers $1,\ldots,2j$ from left to right, and then forming the product of disjoint transpositions $\prod(a_i,a_k)$ over all j arcs with end-points a_i, a_k.

Lemma 8.7

$$\tilde{f}_\mu^k = \binom{k}{|\mu|}(2r-1)!!f^\mu, \quad \mu \vdash (k-2r). \tag{3.2}$$

Proof: We set up a bijection between up-down tableaux S_μ^k and pairs (L,Q_μ) where Q_μ is a standard Young tableau of shape μ and L is a two-line array

$$L = \begin{pmatrix} j_1 & \cdots & j_r \\ i_1 & \cdots & i_r \end{pmatrix}$$

with the j's in the top row written in increasing order; the i's in the bottom row are such that $j_k > i_k$ for each $k = 1,\ldots,r$, and the j's and i's are all distinct and $\{\text{entries in } Q_\mu\} \cup \{\text{entries in } L\} = [k]$. The latter observation will account for the binomial coefficient $\binom{k}{|\mu|}$.

Start with a sequence S_μ^k ; the idea is to build up an associated sequence of tableaux, one for each shape of the sequence. As long as the sequence is increasing, we follow the usual labelling of a standard Young tableau, placing a j in the box of μ^j that was added to μ^{j-1}. (μ^1 is always a single box, so we can start off the process). In general, at step j, given that we have the SYT T_{j-1} associated with μ^{j-1}, and μ^j is one box larger than μ^{j-1}, T_j is simply the SYT obtained by adding a j to T_{j-1} in the position of the added box (in the skew-shape μ^j/μ^{j-1}).

Now suppose μ^j is one box less than μ^{j-1}; let T_i be the SYT corresponding to μ^i. To get T_j, we do the following:

(1) bump out the extra entry of T_{j-1} (the one in the unique square of μ^{j-1} which is not a square of μ^j) by columns (i.e. inverse Schensted column-insertion) to get a tableau T_j of shape μ^j, and a letter x. This means that by column-inserting x into T_j we would retrieve the previous bigger SYT T_{j-1} , and hence its shape μ^{j-1}. See the following example.

$$T_{j-1} = \begin{array}{lll} 1 & 2 & 5 \\ 3 & 4 & \\ 6 & 7 & \end{array} \qquad 3 \leftarrow \begin{array}{ll} 1 & 2 \\ 4 & 5 \\ 6 & 7 \end{array} = T_j$$

$$\left\{ {j \atop 3} \right\} \in L$$

(2) record the fact that a removal occurred at step j by putting the pair (j, x) into a two-line array L, with j on top. Note that since the x was bumped out at step j, it must have been inserted in an earlier step, so $x < j$.

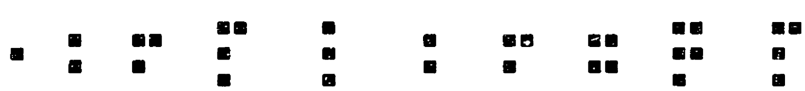
We continue this process to the end of the sequence. Arranging the two-line array L so that the top row is in increasing order, we clearly end up with the requisite two-line array L and a SYT Q_μ of shape μ (the (final) shape of S_μ^k). The process clearly reverses: we work our way backwards from Q_μ , which is the kth step of the sequence, reconstructing the preceding SYT's and hence the sequence of shapes. If we have the SYT T_j for the jth step, and wish to get the SYT T_{j-1}, again two cases arise:

(1) j does not appear in the top row of the two-line array L, indicating that μ^j was not the result of removing a box from μ^{j-1}, but rather came about by adding a box labelled j to μ^{j-1}. Thus deleting the box labelled j from T_j will retrieve an SYT T_j of the correct shape (one box less than μ^j).

(2) j does appear in the top row of L, i.e. a pair (j, i) is in L. This says that T_j was obtained from T_{j-1} as a result of an inverse column-bumping which knocked out the i, or, equivalently, $T_{j-1} = (i \rightarrow T_j)$. (In the previous example, work backwards, i.e., right to left). •

An example should make the bijection transparent:

Example 8.8



$$S^{10}_{(2,1,1)} = \; 1 \quad \begin{array}{l} 1 \\ 2 \end{array} \quad \begin{array}{ll} 1 & 3 \\ 2 & \end{array} \quad \begin{array}{ll} 1 & 3 \\ 2 & \\ 4 & \end{array} \quad \begin{array}{l} 1 \\ 3 \\ 4 \end{array} \quad \begin{array}{l} 1 \\ 3 \end{array} \quad \begin{array}{ll} 1 & 7 \\ 3 & \end{array} \quad \begin{array}{ll} 1 & 7 \\ 3 & 8 \end{array} \quad \begin{array}{ll} 1 & 7 \\ 3 & 8 \\ 9 & \end{array} \quad \begin{array}{ll} 1 & 7 \\ 8 & \\ 9 & \end{array}$$

$$\left\{ {5 \atop 2} \right\} \quad \left\{ {6 \atop 4} \right\} \qquad\qquad \left\{ {10 \atop 3} \right\}$$

$$\mapsto \left(\left\{ \begin{smallmatrix} 5\,6\,10 \\ 2\,4\,3 \end{smallmatrix} \right\},\ \begin{smallmatrix} 1\,7 \\ 8 \\ 9 \end{smallmatrix} \right)$$

Lemma 8.9

$$\tilde{f}^k_\mu = \sum_{\substack{\beta \vdash (k-|\mu|) \\ \beta' \text{even}}} \binom{k}{|\mu|} f^\beta f^\mu \tag{3.3}$$

Proof: We need only refine the bijection of Lemma 8.7, which associates to the up-down k-tableau S^k_μ a pair (L, Q_μ) where L is a two-line array of distinct integers (j, i) such that the j's are written in increasing order in the top row and each j is stricitly greater than the corresponding i below it. L may consequently be viewed as a product of disjoint transpositions (i, j), and as such, Schensted row-insertion applied to the resulting permutation produces a pair of identical SYT's Q_β of shape β where β has even columns (cf. Corollary 3.28); conversely, since Schensted insertion is a bijection, the permutation, and thus the two-line array L, are uniquely recoverable from a SYT Q_β of shape β with even columns. For our purposes it will be more convenient to use the Burge correspondence (cf. Theorem 3.31) between two-line arrays and tableaux with even columns (in this case, restriction to the distinct entries in the two-line array produces the SYT). We remind the reader that (cf. Lemma 3.32) the tableaux produced by the two correspondences coincide, so this does not affect the output of our bijection. We conclude by noting that the binomial coefficient in (3.3) is accounted for by the fact that the construction in effect splits the integers $[k]$ into two disjoint sets, one contributing to the entries of Q_μ and the other to those of Q_β .•

Example 8.10 In the previous example,

$$L = \left\{ \begin{smallmatrix} 5\,6\,10 \\ 2\,4\,3 \end{smallmatrix} \right\} \longleftrightarrow (2\,5)(4\,6)(3\,10) \longleftrightarrow \begin{pmatrix} 2\,3\,4\,5\,6\,10 \\ 5\,10\,6\,2\,4\,3 \end{pmatrix} \overset{KS}{\longleftrightarrow} \begin{smallmatrix} 2 & 3 \\ 4 & 6 \\ 5 & \\ 10 & \end{smallmatrix}$$

Thus

$$S^{10}_{(2,1^2)} \longleftrightarrow \left(\begin{smallmatrix} 2 & 3 \\ 4 & 6 \\ 5 & \\ 10 & \end{smallmatrix} \quad \begin{smallmatrix} 1\,7 \\ 8 \\ 9 \end{smallmatrix} \right)$$

Our next result is a bijection which effectively establishes the Littlewood-Richardson rule for multiplying two Schur functions (see Chapter 1, Theorem 2.22). [2]

[2]We thank Professor Richard Stanley for a comment which gave us the idea of inserting one *tableau* into another.

Theorem 8.11 *Let P_ν , P_μ be tableaux of shapes ν, μ, respectively. Let $w(P_\nu)$ denote the word of P_ν . Then column-insertion of $w(P_\nu)$ into the tableau P_μ (i.e. $(w(P_\nu) \rightarrow P_\mu)$ gives a tableau P_λ of shape $\lambda \supseteq \mu$, and a filling of the skew-shape λ/μ, which may be encoded as a lattice permutation of weight ν by the following scheme:*

(1) column-insert the word $w(P_\nu)$ into the tableau P_μ ;

(2) keep track of the successive additions to the shape μ by starting with a skeleton Ferrers diagram of μ; when w_i of $w(P_\nu)$ is inserted, the shape μ is augmented by a box b, say; we add to the skeleton shape the box b filled with the row number of w_i in P_ν .

Conversely, given a tableau P_λ and a lattice permutation $T_{\lambda/\mu}(\nu)$ of weight ν which fits the skew-shape λ/μ, the lattice permutation specifies a unique order in which the entries in the skew-sub-shape λ/μ of P_λ may be un-bumped, yielding a pair of tableaux (P_μ , P_ν) of shapes μ, ν respectively.

Before verifying that this scheme produces the stated objects, we illustrate the mechanics of the procedure with an example:

Example 8.12

$$\text{Let} \quad P_\nu = \begin{array}{l} 1\,3\,3 \\ 2\,4 \\ 3\,5 \\ 4 \end{array} , \quad P_\mu = \begin{array}{l} 2\,2 \\ 5 \end{array}$$

Then $w(P_\nu) \rightarrow P_\mu$ gives

$$4.35.24.133 \rightarrow \begin{array}{l} 2\,2 \\ 5 \end{array}$$

$$= 4.35.24.13 \rightarrow \begin{array}{l} 2\,2 \\ 3\,5 \end{array}, \begin{array}{l} \blacksquare\,\blacksquare \\ \blacksquare\,1 \end{array}$$

$$= 4.35.24.1 \rightarrow \begin{array}{l} 2\,2\,5 \\ 3\,3 \end{array}, \begin{array}{l} \blacksquare\,\blacksquare\,1 \\ \blacksquare\,1 \end{array}$$

$$= 4.35.24 \rightarrow \begin{array}{l} 1\,2\,2\,5 \\ 3\,3 \end{array}, \begin{array}{l} \blacksquare\,\blacksquare\,1\,1 \\ \blacksquare\,1 \end{array}$$

$$= 4.35.2 \rightarrow \begin{array}{l} 1\,2\,2\,5 \\ 3\,3 \\ 4 \end{array}, \begin{array}{l} \blacksquare\,\blacksquare\,1\,1 \\ \blacksquare\,1 \\ 2 \end{array}$$

$$= \dots\dots\dots$$

$$= \left(\begin{array}{l} 1\,2\,2\,5 \\ 2\,3\,3 \\ 3\,4 \\ 4\,5 \end{array} , \begin{array}{l} \blacksquare\,\blacksquare\,1\,1 \\ \blacksquare\,1\,2 \\ 2\,3 \\ 3\,4 \end{array} \right)$$

$$=(P_\lambda\,,\, T_{\lambda/(21)}((3221))),$$

where $T_{\lambda/(21)}((3221))$ is the lattice permutation 11213243.

Proof: v It is clear that $(w(P_\nu) \to P_\mu)$ produces a tableau P_λ of shape $\lambda \supseteq \mu$; we need to show that the filling of the skew-shape λ/μ is indeed a lattice permutation, has weight ν, and that the procedure is reversible.

To show that the skew-filling is a lattice permutation :

First, the word obtained by reading the skew-filling in lattice permutation fashion evidently has weight ν (it has as many i's as there are entries in row i of ν). Working through the example above should convince the reader that it suffices to look at shapes ν of length at most 2: we need only examine the effect of inserting two consecutive rows of a shape in succession (equivalently, the lattice permutation condition need be checked only for consective entied i, $i+1$).

The situation is trivial if ν consists of one row: our filling is simply a string of 1's (a horizontal strip by Lemma 3.16). So assume ν has two rows. We write the word of P_ν as $b_{\nu_2}b_{\nu_2-1}\dots b_1\; a_{\nu_1}a_{\nu_1-1}\dots a_1$, where $\nu_2 \le \nu_1$. See the figure below.

$$P_\nu = \begin{array}{llllllll} a_{\nu_1} & a_{\nu_1-1} & \cdots & a_{\nu_1-\nu_2+2} & a_{\nu_1-\nu_2+1} & \cdots & a_2 & a_1 \\ b_{\nu_2} & b_{\nu_2-1} & \cdots & b_2 & b_1 & & & \end{array}$$

$(b_1 a_{\nu_1} \dots a_1 \to P_\mu)$ adds squares to the shape μ as follows:

$$\begin{array}{llllllll} \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & 1 & \dots & 1_{\nu_1} \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & 1 & & & \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & & & & \\ \blacksquare & \blacksquare & \blacksquare & 1_{\nu_1-\nu_2+1} & & & & & \\ \blacksquare & \blacksquare & \blacksquare & & & & & & \\ \blacksquare & \blacksquare & 2_1 & & & & & & \\ 1_1 & 1_2 & & & & & & & \end{array}$$

By Lemma 3.16, the path traced by the 1's produced by inserting $a_{\nu_1}a_{\nu_1-1}\dots a_1$ proceeds from left to right and upwards, since the row is weakly increasing, and is a horizontal strip. For clarity label the 1's with subscripts, 1_i indicating that it was created by insertion of a_i. Then our previous remark may be restated by the observation that 1_{i+1} appears strictly to the right of, (in the same row or above), 1_i. On the other hand, since $b_1 > a_{\nu_1-\nu_2} \geq a_1$ and b_1 gets inserted immediately after a_{ν_1}, (see the above figure) the first "2" produced in the skew-shape must appear below or to the left of the last 1; i.e. 2_1 appears below or to the left of 1_{ν_1} (cf. Lemma 3.15). The path of ν_2 2's again proceeds upwards and rightwards from this point, since row 2 of P_ν is weakly increasing; we must make sure, then, that at the box containing 2_{ν_2}, there are still at least ν_2 1's above or to the right, or equivalently, that 2_{ν_2} bypasses at most $(\nu_1 - \nu_2)$ 1's on its way up.

Consider first what happens when we insert b_1. We claim 2_1 is to the left of or below $1_{(\nu_1-\nu_2+1)}$ (which is stronger than saying it lies to the left of or below 1_{ν_1}). Let

$$p = \{(i_1,1),(i_2,2),\dots,(i_k,k)\}$$

be the bumping path of $a_{(\nu_1-\nu_2+1)}$, and suppose it bumped z_1 out of position $(i_1,1)$ in column 1, z_1 bumped z_2 out of position $(i_2,2)$ in column 2,..., and finally z_{k-1} bumped z_k out of position $(i_{k-1},k-1)$ in column $(k-1)$ and into position (i_k,k) in column k.

Thus (i_k,k) is labelled $1_{\nu_1-\nu_2+1}$ in the figure. Then $i_1 \geq i_2 \geq \dots \geq i_k$. Now observe that $a_{\nu_1-\nu_2+2} \leq a_{\nu_1-\nu_2+1}$ so its bumping path is, by Lemma 3.16, of the form

$$p' = (i'_1,1),\dots,(i'_{k+r},k+r)\ ,r \geq 1,$$

where $i'_j \leq i_j$ for $j = 1,\dots,k$. Thus insertion of $a_{\nu_1-\nu_2+2}$ does not affect the portion of the tableau below p. If p' and p do intersect at some point, and p' deposits a z'_j , say, in the common position (i_j,j), then clearly $z'_j \leq z_j$. The same may be said for each of the remaining entries of row 1 of P_ν . So after insertion of row 1, the (last bumping) path might contain entries $u_1,\dots,u_k$ where $u_1 \leq a_{\nu_1-\nu_2+1}, u_2 \leq z_1,\dots,u_{k-1} \leq z_k$.

Now $b_1 > a_{\nu_1-\nu_2+1}$ so its bumping path lies (not necessarily strictly) below p (even if $a_{\nu_1-\nu_2+1}$ is no longer in column 1, its position is occupied by a smaller entry) at least up to column k. We claim it cannot go beyond column k, since for any $i \leq k$ the entry v_i, bumped out of column i by insertion of b_1, is $> u_{i+1}$. Thus if the path does get to column $(k-1)$, it bumps out $v_{k-1} > u_k$. But we know u_k is at the foot of a column (subsequent insertions from row 1 went strictly to the right, and possibly above) so v_{k-1} takes its final position under u_k, i.e. the path of b_1 ends below that of $a_{\nu_1-\nu_2+1}$, or to its left.

Now consider b_2. By a similar argument, observing that its bumping path p'' lies

(1) above and to the right of the path p' of its immediate predecessor b_1 ($b_2 \leq b_1$) and that

(2) p' (from above) lies below and to the left of the path of $a_{\nu_1-\nu_2+1}$, hence below and to the left of the higher path p'' of $a_{\nu_1-\nu_2+2}$ which is less than or equal to $a_{\nu_1-\nu_2+1}$ we see that 2_2 does not bypass $1_{\nu_1-\nu_2+2}$.

Repeating the above argument down to b_{ν_2}, the last entry of row 2 of P_ν , we see that 2_{ν_2} lies below or to the left of $1_{\nu_1-\nu_2+1}$, as long as $\nu_2 \leq \nu_1$, which is of course the case if ν is a shape . We have shown that if we read the skew-filling in lattice permutation fashion, every 2 (created by inserting some b_i of row 2) is preceded by a different 1 (the one created by inserting a_i which appears directly above b_i in the tableau P_ν). Thus we have a lattice permutation .

It remains to show that the construction reverses. That is, starting with a tableau P_λ of shape λ and a lattice permutation of partition weight $\nu \subseteq \lambda$ which fits the skew-shape λ/μ, we must recover a tableau P_ν of shape ν and a tableau P_μ of shape μ. As in the above, we represent the lattice permutation by the corresponding filling T_{lp} of the skew-shape λ/μ. Now consider the pair $(P_\lambda\, , T_{lp})$. Again invoking Lemmas 3.15-16, the lattice permutation tells us the order in which the squares were added to the shape μ: the squares with the same entry i, which came from insertion of row i, form a horizontal strip which was created beginning with the lowest and left-most square, and proceeding north-east. See the following example.

Example 8.13 If

$$P_\lambda = \begin{array}{llll} 1 & 2 & 2 & 5 \\ 2 & 3 & 3 & \\ 3 & 4 & & \\ 4 & 5 & & \end{array} , \quad T_{lp} = \begin{array}{llll} \blacksquare & \blacksquare & 1_2 & 1_3 \\ \blacksquare & 1_1 & 2_5 & \\ 2_4 & 3_7 & & \\ 3_6 & 4_8 & & \end{array} ,$$

then T_{lp} specifies the order in which elements of P_λ should be column-removed, as the reverse of the order indicated by the subscripts (so column-remove entry in the square of P_λ with subscript 8 first, then the square with subscript 7, and so on).

We simply use this information to invert Schensted column insertion on the tableau P_λ . Observe that if we unbump the positions corresponding to the ν_i occurrences of i in T_{lp}, starting with the right-most i (which was created the most recently) we get a weakly increasing sequence of numbers of length ν_i. Since the lattice permutation has weight ν, which we know to be a partition, we can fit the unbumped sequences into the rows of the shape ν to get a legal tableau P_ν. Of course what remains of P_λ is the tableau P_μ. •

Theorem 8.14 *There is a bijection from F_μ^k onto the set of pairs $(Q_\lambda, T_{\lambda/\mu}(\beta))$ where $|\lambda| = k$, Q_λ is an SYT of shape λ, β is a shape with even columns, and $T_{\lambda/\mu}(\beta)$ is a lattice permutation of weight β which fits the skew-shape λ/μ.*

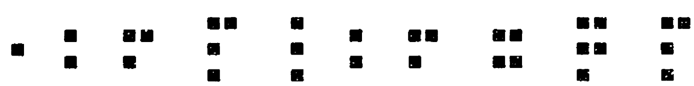
This gives us the formula

$$\tilde{f}_\mu^k = \sum_{|\lambda|=k} f^\lambda \sum_{\substack{\beta \vdash (k-|\mu|) \\ \beta' even}} c_{\mu,\beta}^\lambda \tag{3.4}$$

where $c_{\mu,\beta}^\lambda = |\{$lattice permutation of shape λ/μ, weight $\beta\}|$.

Proof: All the work has essentially been done in Lemmas 8.7, 8.8 and Theorem 8.11. By Lemma 8.8, an up-down k-tableau S_μ^k corresponds to a unique pair (Q_β, Q_μ) where β has even columns and $|\beta| + |\mu| = k$. Theorem 8.11 provides a reversible way of constructing out of (Q_β, Q_μ) a pair $(Q_\lambda, T_{\lambda/\mu}(\beta))$ with the stated properties. •

Example 8.15 Continuing with example 8.8, we have

$S_{(2,1,1)}^{10} =$ 

$$\longleftrightarrow \begin{pmatrix} \begin{matrix} 2 & 3 \\ 4 & 6 \\ 5 & \\ 10 & \end{matrix} & \begin{matrix} 1 & 7 \\ 8 & \\ 9 & \end{matrix} \end{pmatrix}$$

$$\longleftrightarrow \begin{pmatrix} \begin{matrix} 1 & 3 & 7 \\ 2 & 6 & 8 \\ 4 & 9 & \\ 5 & & \\ 10 & & \end{matrix} & \begin{matrix} \blacksquare & \blacksquare & 1 \\ \blacksquare & 1 & 2 \\ \blacksquare & 2 & \\ 3 & & \\ 4 & & \end{matrix} \end{pmatrix}$$

We now return to discuss the arc diagrams $\mathcal{A}_{2j}$, introduced in Lemma 8.3, and the corresponding permutations on unbarred and barred letters in $[2j]$. First:

Definition 8.16 Given an arc diagram $\mathcal{A}_{2j}$ on $2j$ points, the reverse of $\mathcal{A}_{2j}$, denoted $\mathcal{A}_{2j}^{rev}$, is simply the arc diagram obtained by reflecting $\mathcal{A}_{2j}$ about its centre, i.e., by reading $\mathcal{A}_{2j}$ from right to left.

Example 8.17

$A_{2.6} =$ 1 2 3 4 5 6 7 8 9 10 11 12

$A^{rev}_{2.6} =$ 1 2 3 4 5 6 7 8 9 10 11 12

Lemma 8.18 *(1) If $w_1, \ldots, w_{2j}$ is the permutation of barred and unbarred letters $[2j]$ describing the arc diagram A_{2j} , then the reverse A^{rev}_{2j} is described by the permutation $u_1, \ldots, u_{2j}$ obtained as follows: Reading $w_1, \ldots, w_{2j}$ left to right, if w_j is the ith unbarred letter of w (corresponding to the ith left end-point of an arc) write a barred $j+1-i$ under w_j (i.e., we map the unbarred letters of w onto the barred numbers $j, j-1, \ldots, 1$ in that order, reading w from left to right). Let $l_1, \ldots, l_j$ be the subsequence of $[2j]$ for which $w_{l_1}, \ldots, w_{l_j}$ are unbarred; then under w_{l_i} is a barred $(j-i+1)$.*

Example 8.19

w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}	w_{12}
1	4	2	$\bar{1}$	$\bar{2}$	3	$\bar{3}$	5	$\bar{4}$	$\bar{5}$	6	$\bar{6}$
$\bar{6}$	$\bar{5}$	$\bar{4}$			$\bar{3}$		$\bar{2}$			$\bar{1}$	
			6	4		3		5	2		1
u_{12}	u_{11}	u_{10}	u_9	u_8	u_7	u_6	u_5	u_4	u_3	u_2	u_1

Now under each barred i in $w_1, \ldots, w_j$ (these are the right end-points of the arc A_{2j}) write an unbarred $j+1-k$ if $i = w_{l_k}$. In other words, the image of a barred i of w is the unbarred image (determined above) of the unbarred i of w. Finally, read the resulting image word from right to left to get $u_1 \ldots u_{2j} = u$.

(2) Recall that A_{2j} also corresponds to a unique two-line array L formed by writing the numbers of the positions of the right end-points in increasing order in the top row, and the positions of the left end-points in the bottom row, with each left end-point l going under its right end-point r, so that L has the property that every entry in the top row is strictly larger than the corresponding entry in the bottom row. This in turn encodes a permutation σ on $[2j]$ which is a product of j disjoint transpositions, namely the vertical pairs appearing in L. We record here the effect of reversing A_{2j} on its two-line array encoding L and on σ:

A^{rev}_{2j} corresponds to L^{rev} where L^{rev} is obtained by replacing each entry x of L by $2j+1-x$, then switching the top and bottom rows and re-arranging the new

top row so that it is in increasing order. σ is consequently transformed into $\rho\sigma\rho$, where ρ is the reverse of the identity permutation, $\rho = 2j \;\; 2j-1 \;\; \ldots \;\; 1$.

Proof: (1) It is obvious, once an example is worked out, that in forming u we are encoding the reversed arc diagram $\mathcal{A}_{2j}^{rev}$.

(2) Clearly to specify $\mathcal{A}_{2j}^{rev}$ in terms of the positions $1,\ldots,2j$, we need only relabel the points from right to left with $1,\ldots,2j$. This clearly has the stated effect on the entries of L, and evidently changes σ from a product of j transpositions $\prod_{k=1}^{2j}(i_k, j_k)$ to the product $\prod_{k=1}^{j}(2j+1-i_k, 2j+1-j_k)$. We need only observe the following elementary facts about $\rho = 2j \;\; 2j-1 \;\; \ldots \;\; 1)$:

(i) $\rho^{-1} = \rho$

(ii) $\rho(x(i)) = 2j + 1 - x(i) \; \forall s \in S_{2j}$

(iii) if $y(a) = b$ then $xyx^{-1}(x(a)) = x(b)$ ●

Lemma 8.20 *The construction of Lemma 8.3, when restricted to a SYT Q_λ of shape λ, may be used to produce the evacuated tableau $(Q_\lambda)^{evac}$.*

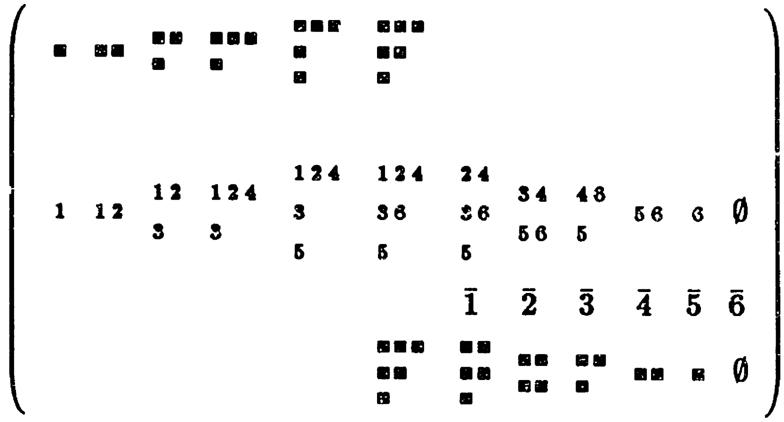
Proof: Restricting the bijection described in Lemma 8.3 to a SYT produces as the image, an arc diagram on $2j$ points such that the last j points are the right end-points of the arcs, or equivalently, in terms of permutations on the $2j$ letters $1,\ldots,j,\bar{1},\ldots,\bar{j}$, those permutations whose last j letters are precisely the barred letters $\bar{1},\ldots,\bar{j}$ in that order. These in turn correspond exactly to permutations on $1,\ldots,j$. The bijection then operates on such a word w by inserting the first j letters into a SYT Q_λ and then successively removing the entries $1,\ldots,j$.

Consequently given a SYT Q_λ , if we successively remove $1,\ldots,j$ and keep track of the shapes, and then read the resulting sequence of shapes in reverse order, it is clear that this sequence is precisely the tableau produced on evacuating Q_λ .
●

Example 8.21

$$\text{Let} \quad Q_\lambda = \begin{matrix} 1\,2\,4 \\ 3\,6 \\ 5 \end{matrix}$$

$\longleftrightarrow S_{\emptyset}^{12} =$



so

Q_λ^{evac}

=tableau formed by sequence of last seven shapes in $S_{\emptyset}^{12}$, read right to left

$$= \begin{array}{l} 1\,3\,6 \\ 3\,4 \\ 5 \end{array}$$

Lemma 8.22

$$\sum_{\substack{\mu \\ |\mu| \le k}} \tilde{f}_\mu^k = \sum_{\substack{\beta \vdash 2k \\ \beta' even}} |\{SYT\, Q_\beta : \, Q_\beta = (Q_\beta)^{evac}\}$$

$$= |\{w \in B_k : w^2 = 1 \text{ and } w \text{ has no fixed points as an element of } S_{2k}\}| \tag{3.5}$$

Proof: Given an up-down k-tableau S_μ^k, append the reverse of S_μ^k to itself so as to get an up-down $2k$-sequence $S_\emptyset^{2k}$ which is symmetric about its middle;

if $S_\mu^k = (\emptyset = \mu^0, \mu^1, \ldots, \mu^k = \mu)$,
then $S_\emptyset^{2k} = (\emptyset = \mu^0, \mu^1, \ldots, \mu^k = \mu, \mu^{k-1}, \ldots, \mu^1, \mu^0 = \emptyset)$.

Now apply the bijection of Lemma 8.3. By the remark at the end of Lm 1, we can associate to $S_\emptyset^{2k}$ a permutation σ on $[2j]$ which is a product of j disjoint transpositions . By Lemma 8.18(2), the reverse sequence is encoded by $\rho\sigma\rho$, where ρ is the reverse of the identity permutation.

Recall from Chapter 1 that if $\sigma \longleftrightarrow (P, Q)$ then $\rho\sigma\rho \longleftrightarrow (P^{evac}, Q^{evac})$.

Clearly

$$\begin{aligned} \{S_\mu^k\} &\longleftrightarrow \{\text{symmetric } S_\emptyset^{2k}\} \\ &\longleftrightarrow \{\sigma : \sigma = \rho\sigma\rho\} \\ &\longleftrightarrow \{\text{all SYT } Q_\beta \text{ of shape } \beta \; : \beta' \text{ even}, Q_\beta = (Q_\beta)^{evac}\} \end{aligned}$$

Finally, observe that as a subgroup of S_{2k}, B_k is characterised by the condition

$$\{w \in B_{2k} \text{ iff } w(2k - i + 1) = 2k - w(i)\}.$$

•

9 n-Symplectic up-down tableaux

Recall that for the group $Sp(2n, \mathbb{C})$, the multiplicity of the representation corresponding to the partition $\mu, (\ell(\mu) \leq n)$, in the k-th tensor power of the defining representation, counts only a subset of S_μ^k (cf. Theorem 6.15). This motivates the following

Definition 9.1 An up-down k-tableau S_μ^k is n-symplectic if $\ell(\mu^i) \leq n \; \forall i, \; 0 \leq i \leq k$.

Set $F_\mu^k(n) = \{S_\mu^k \in F_\mu^k : S_\mu^k \text{ is n-symplectic}\}$.
Let $\tilde{f}_\mu^k(n) = | F_\mu^k(n) |$.

Example 9.2
$\tilde{f}^3_{(1)}(1) = 2$, the 1-symplectic tableaux being $((1), \emptyset, (1))$ and $((1), (2), (1))$;
$\tilde{f}^3_{(1)}(2) = 3$, counting in addition $((1), (1^2), (1))$.

Clearly $\tilde{f}_\mu^k(n) = \tilde{f}_\mu^k$ for $n \geq k$.

We shall find it useful to characterise an n-symplectic up-down tableau $S_\mu^k(n)$ in terms of its encoding (Q_β, Q_μ) $(|\beta| + |\mu| = k)$ of Lemma 8.8, and especially in terms of its encoding $(Q_\lambda, T_{\lambda/\mu}(\beta))$ of Theorem 8.11.

Lemma 9.3
$S_\mu^k \longleftrightarrow (Q_\beta, Q_\mu)$ is n-symplectic iff,
$(i_1 \ldots i_r \rightarrow Q_\mu)$ produces a shape of length at most n, where $i_1, \ldots, i_r$ is the bottom row in the two-line array representation L of Q_β.

Proof: Recall that if Q_β is encoded in the form of a two-line array

$$L = \begin{pmatrix} j_1 & \cdots & j_r \\ i_1 & \cdots & i_r \end{pmatrix}$$

then $j_1, \ldots, j_r$ indicate those stages of the sequence S_μ^k where the shape lost a box, and that at step j_k, i_k was column-bumped out of the previous tableau.

Let $\{T_1, \ldots, T_k = Q_\mu\}$ be the sequence of tableaux associated with S^k_μ via the bijection of Lemma 8.7. Then $T_{s-1} = (i_s \to T_s)$, for $s \in \{j_1, \ldots, j_r\}$. We proceed by induction on the number of removals r.

If $r = 0$, the statement is clear.

If $r = 1$, by definition S^k_μ is n-symplectic iff $\ell(T_i) \leq n \forall\, i = 1, \ldots, k$, so $T_{j_1-1} = (i_1 \to T_{j_1})$ can have length at most n. This ensures that all previous tableaux, (which are smaller in weight), will not exceed n rows. But all succeeding tableaux must also satisfy the same condition, and since the entries added after step j_1 are strictly larger than $j_1 > i_1$, this is true iff $\ell(i_1 \to T_s) \leq n$ for all s between j_1 and k, hence iff $\ell(i_1 \to T_k = Q_\mu) \leq n$, since the tableaux between the unique removal and the final shape are all contained in the final shape μ.

Now assume the statement holds when there are less than r removals, and let $S^k_\mu \longleftrightarrow (L, Q_\mu)$ have r removals $\{(j_1, i_1), \ldots, (j_r, i_r)\}$. Then S^k_μ is n-symplectic
$\iff S^{j_{r-1}}_{sh(T_{j_{r-1}})}$ is n-symplectic and $\ell(T_s) \leq n$ for $s > j_{r-1}$,
$\iff \ell(i_1 \ldots i_{r-1} \to T_{j_{r-1}}) \leq n$ and $\ell(T_s) \leq n$, $s \geq j_r$,
(since the tableaux between $T_{j_{r-1}}$ and T_{j_r} are contained in T_{j_r}, there being only additions in the interim)
$\iff \ell(i_1 \ldots i_r \to T_{j_r}) \leq n$ (since $T_{j_{r-1}} \subset T_{j_r}$)
$\iff \ell(i_1 \ldots i_r \to Q_\mu) \leq n$, since $T_{j_r} \subseteq Q_\mu$. •

Theorem 9.4 *$S^k_\mu \longleftrightarrow (Q_\lambda, T_{\lambda/\mu}(\beta))$ is n-symplectic iff, in the lattice permutation filling $T_{\lambda/\mu}(\beta)$ of the skew-shape λ/μ, the $(2i+1)$'s appear in row $(n+i)$ or above for all $i = 0, 1, \ldots, \frac{1}{2}\ell(\beta)$.*

Proof: Let $Q_{\frac{1}{2}\beta} = (i_1 \ldots i_r \to \emptyset)$. It is clear from the description of the Burge correspondence (Theorem 3.31) that

(1) $Q_{\frac{1}{2}\beta}$ occupies a left-hand sub-shape of Q_β .

(2) If i_k appears in row s, then i_k is smaller than any j_l appearing in a higher row (since such a j_l has an i_l above it; if $i_l > i_k$ we are done, since $j_l > i_l$; if not, i_k must have been inserted prior to i_l, so $j_k < j_l$ implies $i_k (< j_k) < j_l$).

Now suppose the condition on the lattice permutation holds. We argue that inserting $Q_{\frac{1}{2}\beta}$ by rows into Q_μ produces a tableau of length at most n. We write $R_i(T)$ to mean the ith row of the tableau T. $R_1(Q_{\frac{1}{2}\beta})$ is contained in $R_1(Q_\beta)$, and since all the 1's in the lattice permutation are no lower than row n, inserting $R_1(Q_{\frac{1}{2}\beta})$ certainly does not produce more than n rows. $R_2(Q_{\frac{1}{2}\beta})$ is contained in $R_2(Q_\beta)$, and the lowest 2 in the lattice permutation can at worst be in row $n+1$; if this happens, we claim that it must have resulted from inserting a j and not an i. (Otherwise, there are two more rows of the same length as the common length of rows 1 and 2 in Q_β , and these will also produce entries in the lattice permutation in consecutive rows, forcing a contradiction to the hypothesis.) Then all subsequent (reading right to left) i's contribute to squares in rows higher than $(n+1)$.

The argument continues, using fact (2) above: if all 3's in the lattice permutation are in row $n+1$ or higher, when row 3 of $Q_{\frac{1}{2}\beta}$ is inserted into Q_μ, the new squares occur in row $(n+1)-1$ or higher, since each 3 is preceded by at least one 2 in a higher row. This reasoning works as long as Q_β is such that for any i below row 1, there is a j in a higher row. (For then we argue that the bumping path of this i ends in a higher row than that of the j: fact (2) comes in here). Now suppose this is not the case, so that we have the first k rows of Q_β being the same as the first k rows of $Q_{\frac{1}{2}\beta}$. We may assume without loss of generality that we are looking at an odd row $2k+1$ of i's only; now observe that if any of these i's created squares in Q_λ lower than row n, say in row $n+s$ for some $s>0$, then eventually this forces, in the lattice permutation , the occurrence of a $2k+1+2l$ in row $n+s+l>k+l$ since $n\geq k$ ($Q_{\frac{1}{2}\beta}$ certainly has length at most n), contradiction.

Thus the lattice permutation condition implies ℓ $(Q_{\frac{1}{2}\beta}\rightarrow Q_\mu)=\ell(i_1\ldots i_r\rightarrow Q_\mu)\leq n$, which implies S_μ^k is n-symplectic by the preceding lemma.

Conversely suppose S_μ^k is n-symplectic, so that the condition of Lemma 9.3 holds, i.e.,

$$\ell(Q_{\frac{1}{2}\beta}\rightarrow Q_\mu)\leq n.$$

Since row 1 of $Q_{\frac{1}{2}\beta}$ coincides with row 1 of Q_β(cf ch1), clearly all the 1's in $T_{\lambda/\mu}(\beta)$ are no lower than row n. Consequently, all the 2's of the lattice permutation appear no lower than row $(n+1)$. Thus any 3 in the lattice permutation is, at its lowest, in row $(n+2)$, and is then preceded by a 2 in row $(n+1)$. Suppose this was the case.

Then it is clear that when R_2=(row 2 of $Q_{\frac{1}{2}\beta}$) is inserted, the first entry (i.e. the right-most entry in R_2) goes into the position occupied by the 2 above the "bad" 3 in the lattice permutation , i.e., into row $(n+1)$, (since the 2 arose from inserting an entry x strictly larger than the first entry in row 1 of Q_β , viz., the first entry in row 1 of $Q_{\frac{1}{2}\beta}$). But this contradicts the assumption that S_μ^k is n-symplectic. Thus any 3's in the lattice permutation are in row $(n+1)$ or higher, so, as before, the corresponding 4's are no lower than row $(n+2)$. Consequently, any 5's in the lattice permutation can be no lower than row $(n+3)$. However, repeating the above argument we see that a 5 in row $(n+3)$ would mean that the right-most entry in row 3 of $Q_{\frac{1}{2}\beta}$, when inserted into Q_μ , would go in row $((n+3)-2)$=row $(n+1)$, again a contradiction.

The argument clearly extends to the entire length of the lattice permutation , so that $\ell(Q_{\frac{1}{2}\beta} \to Q_\mu) \leq n$ implies that, in the lattice permutation $T_{\lambda/\mu}(\beta)$, $(2i+1)$ appears no lower than row $(n+i)$. •

Since the condition on a lattice permutation which characterises n-symplectic tableaux will prove to be especially important in the next chapter, we make one more definition:

Definition 9.5 A lattice permutation of partition weight β where β has even columns, fits the skew-shape λ/μ ***n*-symplectically**, if in the filling of λ/μ, $(2i+1)$ appears no lower than row $(n+i)$ of $\lambda, \forall\, i = 1, \ldots, \frac{1}{2}\ell(\beta)$.

The above lemma is clearly a bijective proof of the following enumerative result:

Theorem 9.6

$$\tilde{f}^k_\mu(n) = \sum_{|\lambda|=k} f^\lambda \sum_{\substack{\beta \vdash (k-|\mu|) \\ \beta' even}} c^\lambda_{\mu,\beta}(n) \tag{3.6}$$

where $c^\lambda_{\mu,\beta}(n)$
$= |\{\ T_{\lambda/\mu}(\beta)$ *:lattice permutation of shape* λ/μ*, weight* β*, which fits the skew-shape* λ/μ *n-symplectically*$\}|$

As a corollary to the lemma, we immediately deduce

Proposition 9.7

$$\tilde{f}^{n+1}_\mu(n) = \tilde{f}^{n+1}_\mu \tag{3.7}$$

for all μ *of length at most* n.

Equivalently, for shapes μ *of length at most n, all up-down tableaux of length* $(n+1)$ *and shape* μ *are n-symplectic.*

Proof: Let S^{n+1}_μ be an up-down tableau of length (n+1) and shape μ, let its encoding as a pair of SYT's be $(Q_\beta\,, Q_\mu)$ where $\beta \vdash (n+1-|\mu|)$ and has even columns, and corresponds to the two-line array

$$L = \begin{pmatrix} j_1 & \cdots & j_r \\ i_1 & \cdots & i_r \end{pmatrix};$$

let its encoding as a SYT and a lattice permutation be $(Q_\lambda\,, T_{\lambda/\mu}(\beta))$.

Then in any event it is clear that

$$\ell(i_1 \dots i_r \to Q_\mu) \leq n + 1, \text{ since } \ell(\lambda) \leq n+1.$$

Consequently S_μ^{n+1} would fail to be n-symplectic only if $\ell(i_1 \dots i_r \to Q^\mu) \leq n+1$, which in turn means that some 1 in the lattice permutation $T_{\lambda/\mu}(\beta)$ appears in row $n+1$ of λ. But since every 1 in the lattice permutation must be accompanied by a matching 2 in a lower row, and λ has at most $n+1$ rows, this is impossible. ●

Remark 9.8 As a result of the two different encodings of up-down tableaux presented in this section, we may restate Berele's theorem (Theorem 7.1) in either of the following forms:

Berele's algorithm is a bijection between k-words w on $\{1, \bar{1}, \dots, n, \bar{n}\}$ and each of the following:

(1) triples $(\tilde{P}_\mu, Q_\beta, Q_\mu)$
where $\tilde{P}_\mu$ is a symplectic tableau of shape μ,
Q_μ, Q_β are respectively standard Young tableaux of shape μ, β, such that the corresponding up-down tableau $S_\mu^k \longleftrightarrow (Q_\beta, Q_\mu)$ is n-symplectic, and
β is a partition of $(k - |\mu|)$ with even columns;

(2) triples $(\tilde{P}_\mu, Q_\lambda, T_{\lambda/\mu}(\beta))$
where $\lambda \vdash k$, $\lambda \supseteq \mu$, β'even, $\beta \vdash (k - |\mu|)$,
$\tilde{P}_\mu$ is a symplectic tableau of shape μ,
Q_λ is a standard Young tableau of shape λ, and $T_{\lambda/\mu}(\beta)$ is a lattice permutation of weight β which fits λ/μ n-symplectically.

10 The link between Berele and Schensted

Before plunging into the technical aspects of Berele's algorithm, we work out an example computing $(\emptyset \xleftarrow{B} w)$ and $(\emptyset \leftarrow w)$; this will serve to summarise our results so far.

Example 10.1 Pick $w = \bar{3}\bar{1}3\bar{2}123\bar{2}2\bar{1}$. Then computing

$$(\emptyset \xleftarrow{B} w)$$

gives the sequence of symplectic tableaux

$$\begin{array}{c}\bar{3}\end{array}\quad \begin{array}{c}\bar{1}\\ \bar{3}\end{array}\quad \begin{array}{cc}\bar{1}&3\\ \bar{3}&\end{array}\quad \begin{array}{cc}\bar{1}&\bar{2}\\ 3&\\ \bar{3}&\end{array}\quad \begin{array}{c}\bar{2}\\ 3\\ \bar{3}\end{array}\quad \begin{array}{c}2\\ \bar{2}\end{array}\quad \begin{array}{cc}2&3\\ \bar{2}&\end{array}\quad \begin{array}{cc}2&\bar{2}\\ \bar{2}&3\end{array}\quad \begin{array}{cc}2&2\\ \bar{2}&\bar{2}\\ 3&\end{array}\quad \begin{array}{cc}\bar{1}&2\\ \bar{2}&\\ 3&\end{array}$$

Thus

$$(\emptyset \xleftarrow{B} w) = \left(\begin{array}{cc}\bar{1}&2\\ \bar{2}&\\ 3&\end{array} \quad S^{10}_{(211)} \right)$$

where $S^{10}_{(211)}$ is the up-down tableau of Example 8.8.

$$= \left(\begin{array}{cc}\bar{1}&2\\ \bar{2}&\\ 3&\end{array} ,\ \left\{ \begin{array}{ccc}5&6&10\\ 2&4&3\end{array} \right\},\ \begin{array}{cc}1&7\\ 8&\\ 9&\end{array} \right)$$

$$= \left(\begin{array}{cc}\bar{1}&2\\ \bar{2}&\\ 3&\end{array} ,\ \begin{array}{cc}2&3\\ 4&6\\ 5&\\ 10&\end{array} ,\ \begin{array}{cc}1&7\\ 8&\\ 9&\end{array} \right)$$

$$= \left(\begin{array}{cc}\bar{1}&2\\ \bar{2}&\\ 3&\end{array} ,\ \begin{array}{ccc}1&3&7\\ 2&6&8\\ 4&9&\\ 5&&\\ 10&&\end{array} ,\ \begin{array}{ccc}\blacksquare&\blacksquare&1\\ \blacksquare&1&2\\ \blacksquare&2&\\ 3&&\\ 4&&\end{array} \right)$$

while

$$(\emptyset \leftarrow w)$$

gives the sequence of ordinary tableaux

$$\begin{array}{c}\bar{3}\end{array}\quad \begin{array}{c}\bar{1}\\ \bar{3}\end{array}\quad \begin{array}{cc}\bar{1}&3\\ \bar{3}&\end{array}\quad \begin{array}{cc}\bar{1}&\bar{2}\\ 3&\\ \bar{3}&\end{array}\quad \begin{array}{cc}1&\bar{2}\\ \bar{1}&\\ 3&\\ \bar{3}&\end{array}\quad \begin{array}{cc}1&2\\ \bar{1}&\bar{2}\\ 3&\\ \bar{3}&\end{array}\quad \begin{array}{ccc}1&2&3\\ \bar{1}&\bar{2}&\\ 3&&\\ \bar{3}&&\end{array}\quad \begin{array}{ccc}1&2&\bar{2}\\ \bar{1}&\bar{2}&3\\ 3&&\\ \bar{3}&&\end{array}\quad \begin{array}{ccc}1&2&2\\ \bar{1}&\bar{2}&\bar{3}\\ 3&3&\\ \bar{3}&&\end{array}\quad \begin{array}{ccc}1&\bar{1}&2\\ \bar{1}&2&\bar{2}\\ \bar{2}&3&\\ 3&&\\ \bar{3}&&\end{array}$$

so that

$$(\emptyset \leftarrow w) = \left(\begin{array}{ccc}1&\bar{1}&2\\ \bar{1}&2&\bar{2}\\ \bar{2}&3&\\ 3&&\\ \bar{3}&&\end{array} ,\ \begin{array}{ccc}1&3&7\\ 2&6&8\\ 4&9&\\ 5&&\\ 10&&\end{array} \right)$$

Now notice that

$$\begin{matrix} 1 & 3 & 7 \\ 2 & 6 & 8 \\ 4 & 9 & \\ 5 & & \\ 10 & & \end{matrix}$$

appears as the Schensted Q-tableau of w, as well as the second component in the lattice permutation reformulation of the Berele algorithm. This is no accident; the main result of this section states that this connection between the Berele and Knuth-Schensted algorithms holds in general.

We now begin this section in earnest, by presenting some rather technical properties of the Berele algorithm. The reader is warned that the proofs unavoidably consist of checking tedious details about bumping paths and *jeu de taquin* paths; he may consequently wish to skip to the end of the section, where the technicalities obligingly culminate in the rather surprising connection between the reformulation (Remark 9.8(2)) of the Berele algorithm, and the Knuth-Schensted insertion scheme.

The first lemma is an obvious but nonetheless important observation about Berele insertion, which we state separately only because it is invoked so often:

Lemma 10.2 *Suppose Berele insertion* $(\tilde{P}_\mu \xleftarrow{B} x)$ *of a letter* x *in* $1, \bar{1}, \ldots, n, \bar{n}$ *into a symplectic tableau* $\tilde{P}_\mu$ *causes a cancellation. Then Knuth-Schensted insertion* $(\tilde{P}_\mu \leftarrow x)$ *results in augmenting the shape* μ *of* $\tilde{P}_\mu$ *by a square in column 1.*

Lemma 10.3 *Let* w *be a word of length* k *in* $1, \bar{1}, \ldots, n, \bar{n}$ *and let* x, x' *be consecutive letters in* w*, with* x *preceding* x' *in* w*, such that* $x \leq x'$*. Then in applying the Berele algorithm to the word* w*, if inserting* x' *(i.e.,* $((\tilde{P}_\mu \xleftarrow{B} x) \xleftarrow{B} x')$*) causes a cancellation, so did the (prior) insertion* $(\tilde{P}_\mu \xleftarrow{B} x)$ *of* x*.*

Proof: For suppose not; let the symplectic tableau built up prior to inserting x be $\tilde{P}$; then Berele insertion of x into $\tilde{P}$ is the same as Knuth-Schensted row insertion of x into $\tilde{P}$ and produces some larger tableau $\tilde{P}'$. On the other hand Berele insertion of x' into $\tilde{P}'$ results in a cancellation, so that Schensted row insertion of x' into $\tilde{P}'$ must yield a tableau $\tilde{P}''$ whose shape differs from that of $\tilde{P}'$ by an extra square in column one (since the Schensted bumping path of $(\tilde{P}' \leftarrow x')$ must end in column one). But by Lemma 3.15 of Chapter 1, $x \leq x'$ implies that the Schensted (row-insertion) bumping path of x' lies to the right and ends no lower than that of x, so we have a contradiction. •

Lemma 10.4 *Let w be a k-word in $1, \bar{1}, \ldots, n, \bar{n}$ and let x, x' be consecutive letters in w with x preceding x' and $x \leq x'$. Suppose Berele insertion performed on w causes cancellations at x and at x'; then the taquin path followed by the hole created upon inserting x ends in the same row as, or higher than, the taquin path of the hole created by the subsequent insertion of x'.*

Proof: We refer to the figure below in the course of our argument. Suppose $\tilde{P}$ is the symplectic tableau built up prior to insertion of x, and let x cause a cancellation in row i of $\tilde{P}$, so that Berele insertion of x before the *jeu de taquin* step results in the punctured tableau

$$(\tilde{P}_\mu \xleftarrow{B} x) = \begin{array}{llllllll} \bullet & \bullet & \bullet & \circ & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\ \circ & a_1 & \bullet & \bullet & a_r & \bullet & & \\ b_1 & \bullet & \bullet & b_r & \bullet & & & \\ c_1 & \bullet & \bullet & & & & & \\ \bullet & \bullet & & & & & & \\ \bullet & & & & & & & \end{array}$$

Let r be the smallest index such that $b_r \leq a_r$; consider now the effect of *jeu de taquin* on $\tilde{P}\,'$. In row i the hole slides out to the space above b_r, then swaps positions with b_r; again in row $i+1$ it slides rightwards until it stops above c_s, where s is least such that $c_s \leq b_{s+1}$ (clearly $s \geq r$). This continues until it hits the boundary of $\tilde{P}\,'$, leaving a tableau $\tilde{P}\,'$ of smaller shape.

Now consider the Berele insertion of x' into $\tilde{P}\,'$. Since $x' \geq x$, it is clear that the cancellation caused by x' occurs in a row no higher than row i, where the cancellation caused by x occurred (cf. Lemma 3.15). Thus prior to *jeu de taquin*, we have a punctured tableau as in the figure above, with the hole in row i or lower. To perform *jeu de taquin*, in each row we look for the smallest index r in the row below such that the rth element in it is less than or equal to the $(r+1)$st element in the current row, as before. It is clear that the new hole must migrate to a position which is (weakly) to the right of the old hole produced by x, if we are dealing with a row which was affected by the taquin path of insertion of x.

The statement of the Lemma follows. •

The converse of the above lemma is also true:

Lemma 10.5 *Let w be a k-word in $1, \bar{1}, \ldots, n, \bar{n}$ and let x, x' be consecutive letters in w with x preceding x'. Suppose Berele insertion performed on w causes cancellations at x and at x', such that the taquin path followed by the hole created* *upon inserting x ends in the same row as, or higher than, that of the hole created by the subsequent insertion of x'. Then $x \leq x'$.*

Proof: Suppose to the contrary that $x > x'$. Consider the results of ordinary Schensted insertion $(\tilde{P} \leftarrow x \leftarrow x')$, where as usual $\tilde{P}$ is the initial tableau. Then both bumping paths end in column 1, since x must do so, and that of x' must end below it. Also the path of x' lies strictly above that of x until it reaches the first column (Lemma 3.16), so the symplectic violation due to x' is in a row $r(x') \geq r(x)$, if $r(x)$ is the row of the hole produced upon (Berele) insertion of x. But by the arguments in the previous lemma, this contradicts the hypothesis about the culminating squares of the two taquin paths. •

Lemma 10.6 *Let w be a k-word in $1, \bar{1}, \ldots, n, \bar{n}$ and let x, x' be consecutive letters in w with x preceding x'. Suppose Berele insertion performed on w causes a cancellation at x but not at x'. Then $x \leq x'$.*

Proof: Suppose to the contrary that $x' < x$. Then considering the effect of ordinary Schensted insertion of x, x' into the symplectic tableau created up to the insertion of x, the bumping path of x' lies strictly to the left of that of x until it hits column 1 (which must happen since it does for x by Lemma 10.2). Suppose inserting x caused a symplectic violation in row i; thus the (Schensted-insertion) path of x bumps an $\bar{\imath}$ out of row i. Consequently the path of x' bumps the entry a_i out of row i, where $a_i \leq \bar{\imath}$, contradicting the hypothesis that x does not cause a cancellation (the tableau is symplectic up to row i).•

The next lemma is crucial to our proof of the main theorem of this section.

Lemma 10.7 *Let*

$$L = \begin{pmatrix} j_1 & \cdots & j_r \\ i_1 & \cdots & i_r \end{pmatrix}$$

be a Burge two-line array, (cf. Definition 3.29) so that $j_1 < \ldots < j_r$ and $j_k > i_k$, all $k = 1, \ldots, r$. Let Q_β be the SYT produced by applying the Burge correspondence of Theorem 3.31 to L (so β is a shape with even columns). Then

$$\begin{aligned} Q_\beta &= (j_{l_1} \ldots j_{l_r} i_1 \ldots i_r \rightarrow \emptyset) \\ &= (\emptyset \leftarrow j_{l_1} \ldots j_{l_r} i_1 \ldots i_r) \end{aligned}$$

where $l_1, \ldots, l_r$ are such that $i_{l_1} < \ldots < i_{l_r}$.

Proof: Consider the effect of the Knuth-Schensted correspondence on L: this yields a pair of SYT $(P(i_1 \ldots i_r), Q(j_1 \ldots j_r))$ of the same shape $\frac{1}{2}\beta$, say, where $P(i_1 \ldots i_r) = (i_1 \ldots i_r \rightarrow \emptyset)_1 = (\emptyset \leftarrow i_1 \ldots i_r)_1$, and $Q(j_1 \ldots j_r)$ is a SYT with entries $j_1 \ldots j_r$, where j_k occupies the position created by insertion of i_k after $i_1 \ldots i_{k-1}$.

Recall that under the Knuth-Schensted bijection, the pair (Q, P) corresponds to the inverse array L^{-1}, where L^{-1} is obtained from L by switching the top and bottom rows of L, and then re-arranging the vertical pairs so that the (new) top line is in increasing order. Thus,

$$L^{-1} = \begin{pmatrix} i_{l_1} & \ldots & i_{l_r} \\ j_{l_1} & \ldots & j_{l_r} \end{pmatrix},$$

where $i_{l_1} < \ldots < i_{l_r}$. It follows that $Q(j_1 \ldots j_r) = (\emptyset \leftarrow j_{l_1} \ldots j_{l_r})_1$.

Now consider what happens when we reverse the Burge correspondence to retrieve L from Q_β. Recall that this is done as follows:

(1) Set $k = r$. Set $T_k = Q_\beta$.

(2) Set $j_k =$ largest entry in T_k.

(3) Remove j_k from T_k; unbump the element immediately above j_k by rows. This results in i_k being bumped out of the tableau. Call the resulting tableau T_{k-1}.

(3) Set $k = k - 1$.

(4) If $k = 0$, stop, else go to Step (2).

Suppose we modify this process slightly, replacing (2) and (3) by

(2') Set $j_{r+1}' = \infty$; $j_k' =$ largest entry in T_k which is smaller than j_{k+1}.

(3') unbump element a_k immediately above j_k' by rows; slide j_k' up into position occupied by a_k. This results in i_k' being bumped out of the tableau. Call the resulting tableau T_{k-1}.

It is not hard to see that at each step of this new process, the element being bumped out is the same as that in the inverse Burge process: $i_k' = i_k$ all $k = 1 \ldots r$. (Also clearly $j_k' = j_k$). This follows since at step $(k-1)$, we unbump from the element a_{k-1} directly above $j_{k-1} : a_{k-1} < j_{k-1} < j_k$, so a_{k-1} is to the left or above the starting point a_k of the kth bumping path (which was the element directly above j_k). Thus the $(k-1)$st bumping path lies to left or above the kth bumping path, and is unaffected by the presence of j_k in the tableau.

This means that at the end of the modified procedure, we are left with some SYT T in the entries $j_1 \ldots j_r$, and we have bumped out $i_r, i_{r-1} \ldots i_1$ (in that order), so that $Q_\beta = (T \leftarrow i_1 \ldots i_r)$. However, the nature of the modification is such that at each step when i_k is bumped out, j_k moves up one box, i.e., j_k occupies the position that would be created by re-inserting i_k into the tableau. In other words, $T = Q(j_1 \ldots j_r)$.

Consequently

$$P(i_1 \dots i_r) = (i_1 \dots i_r \to \emptyset)_1 = (\emptyset \leftarrow i_1 \dots i_r)_1,$$
$$Q_\beta = (Q(j_1 \dots j_r) \leftarrow i_1 \dots i_r) = (\emptyset \leftarrow j_{l_1} \dots j_{l_r} \leftarrow i_1 \dots i_r)_1$$

where $i_{l_1} < \dots < i_{l_r}$, as required.

Note that it also follows from this argument that the common shape $\frac{1}{2}\beta$ of $P(i_1 \dots i_r)$ and $Q(j_1 \dots j_r)$ is precisely "half" the shape of the even-columned β, i.e., $\frac{1}{2}\beta = (\beta_2, \beta_4, \dots, \beta_{2l})$ if $\ell(\beta) = 2l$ (remembering that $\beta_{2i-1} = \beta_{2i}, \forall\ i = 1 \dots l$). •

We can now present the main result of this section, demonstrating the close connection between the Berele algorithm and Knuth-Schensted insertion:

Theorem 10.8 *Let $S^k_\mu(n)$ be an n-symplectic up-down tableau of length k and shape μ, and let its encoding as a standard Young tableau and a lattice permutation be the pair $(Q_\lambda, T_{\lambda/\mu}(\beta))$ where as usual $\lambda \supseteq \mu$ and β is a shape with even columns. Then*

a k-word w in $1, \bar{1}, \dots, n, \bar{n}$
fits the sequence of shapes $S^k_\mu(n)$ in the sense of Berele
iff w fits Q_λ in the sense of Schensted.

Equivalently, if

$$w \longleftrightarrow (P_\nu, Q_\nu)$$
where Q_ν is an SYT and

$$w \overset{B}{\longleftrightarrow} (\tilde{P}_\mu, Q_\lambda, T_{\lambda/\mu}(\beta))$$
where Q_λ is an SYT and
$T_{\lambda/\mu}(\beta)$ is a lattice permutation of weight β which fits λ/μ n-symplectically,

then

$$\lambda = \nu \text{ and } Q_\lambda = Q_\nu .$$

Proof: Recall that in the Berele process, if

$$w \overset{B}{\longleftrightarrow} (\tilde{P}_\mu, S^k_\mu(n)),$$

then the up-down tableau $S^k_\mu(n)$ corresponding to w may be represented by a pair (L, Q_μ) where L is a Burge two-line array

$$\begin{pmatrix} j_1 & \dots & j_r \\ i_1 & \dots & i_r \end{pmatrix},$$

so that $j_1 < \ldots < j_r$ and $j_k > i_k$, all $k = 1, \ldots, r$, and Q_μ is a standard Young tableau of shape μ, the shape of the symplectic tableau corresponding to w.

We shall first show how the right Schensted tableau of w can be reconstructed from the information contained in Q_μ and L. Our procedure will turn out, (thanks to Lemma 10.7) to be equivalent to column-inserting (the Knuth-equivalent word of) Q_β (the even-columned tableau corresponding to L under the Burge algorithm) into Q_μ, which establishes the statement of the theorem (by Remark 9.8(2)).

Clearly we can partially reconstruct the right Schensted tableau by column re-inserting the i 's in the order

$$(i_1 \ldots i_r \rightarrow Q_\mu).$$

We break up the word w into segments $w^1 x_{j_1} w^2 \ldots x_{j_{s-1}} w^s x_{j_s} \ldots x_{j_r} w^r$ where $\{x_{j_1}, \ldots, x_{j_r}\}$ are the letters whose insertion causes symplectic violations, i.e., cancellations in the Berele scheme.

Now observe that at any cancellation step in the Berele algorithm, say step j_k, where we kept track of the j_kth shape by column-bumping an i_k out of the previous tableau, the effect of ordinary Schensted insertion of x_{j_k} on the right $(Q-)$tableau would have been to add a square labelled j_k to the bottom of the first column of the previous tableau (by Lemma 10.2). Thus at the very first cancellation, the right Schensted tableau is easily recovered by column-inserting $j_1 i_1$ into Q_μ.

Also note that for any $s = 1 \ldots r$, the subword w^s of w between two successive cancellations $x_{j_{s-1}}, x_{j_s}$ consists only of letters strictly larger than x_{j_s} (by Lemma 10.3) and larger than or equal to $x_{j_{s-1}}$ (by Lemma 10.4). This means that the bumping paths of $x_{j_{s-1}}, x_{j_s}$ lie to the left of the portion of the tableau affected by w^s.

Consequently, it must be possible to recreate the right Schensted tableau from the partial reconstruction achieved by $(i_1 \ldots i_r \rightarrow Q_\mu)$, by column-inserting the j_k's in a suitable order.

We now go about determining the correct insertion order for the j's. Observe that if $taq(x_k)$ denotes the row in which the *taquin* path of insertion of x_k ends, then

$$\begin{aligned} i_{s-1} \leq i_s \quad & \Longrightarrow taq(x_{s-1}) \leq taq(x_s) \\ & \text{(by Lemma 3.15(2))} \\ & \Longrightarrow x_{i_{s-1}} \leq x_{i_s} \\ & \text{(by Lemma 10.5)} \\ & \Longrightarrow j_{s-1} \text{ appears to the left of } j_s \text{ in } Q_\nu \\ & \text{(the right Schensted tableau of } w) \end{aligned}$$

Thus in the correct column-insertion order of the two j's in recreating the right Schensted tableau, j_s is inserted before j_{s-1}.

Similarly,

$$\begin{aligned} i_{s-1} > i_s \quad & \Longrightarrow taq(x_{s-1}) > taq(x_s) \\ & \text{(by Lemma 3.15(2))} \\ & \Longrightarrow x_{i_{s-1}} > x_{i_s} \\ & \text{(by Lemma 10.3)} \\ & \Longrightarrow j_{s-1} \text{ appears above } j_s \text{ in } Q_\nu \\ & \text{(the right Schensted tableau of } w) \end{aligned}$$

Thus in the correct column-insertion order of the two j's in recreating the right Schensted tableau, j_{s-1} is inserted before j_s.

Clearly then, if $l_1, \ldots, l_r$ is the permutation of $[r]$ such that $i_{l_1} < \ldots < i_{l_r}$, then the right Schensted tableau Q_ν of w is recreated by column-inserting the j 's in the order $j_{l_1} \to \ldots \to j_{l_r} \to$, i.e.,

$$\begin{aligned} Q_\nu & = (j_{l_1} \ldots j_{l_r}) \to (i_1 \ldots i_r \to Q_\mu)) \\ & = (j_{l_1} \ldots j_{l_r} \to (i_1 \ldots i_r)) \to Q_\mu) \\ & = ((j_{l_1} \ldots j_{l_r} i_1 \ldots i_r) \to Q_\mu) \\ & = (word(Q_\beta) \to Q_\mu) \text{ by Theorem 3.20} \\ & = P_\lambda \text{ (by Remark 9.8(2))} \end{aligned}$$

•

11 Knuth transformations

In this section we study the effect of the Knuth-transformations (described in Chapter 1) on the Berele algorithm. The technical lemmas appearing in the previous section, as well as the main theorem, are invoked to yield a complete characterisation of the notion of Knuth-equivalent words in the symplectic context.

Theorem 11.1 *Let w and w' be words on $1, \bar{1}, \ldots, n, \bar{n}$ of the same length which are Knuth-equivalent (i.e., $(\emptyset \leftarrow w)_1 = (\emptyset \leftarrow w')_1$). Then*

$$(\emptyset \xleftarrow{B} w)_1 = (\emptyset \xleftarrow{B} w')_1$$
$$\text{and } (\emptyset \xleftarrow{B} w)_3 = (\emptyset \xleftarrow{B} w')_3.$$

That is, w, w' give the same symplectic tableau under Berele insertion, as well as the same lattice permutation under the reformulation 9.8(2).

Proof: We must show that, given a symplectic tableau $\tilde{P}$, the following two statements hold:

(1) $x \le x' < x''$ implies $(\tilde{P} \leftarrow x'' \leftarrow x \leftarrow x') = (\tilde{P} \leftarrow x \leftarrow x'' \leftarrow x')$

(2) $x < x' \le x''$ implies $(\tilde{P} \leftarrow x' \leftarrow x \leftarrow x'') = (\tilde{P} \leftarrow x' \leftarrow x'' \leftarrow x)$

For the sake of completeness, we reproduce Knuth's original proof for ordinary tableaux (taking care of the case when the Berele algorithm on w does not result in cancellations), proceeding by induction on $\ell(\tilde{P})$.

Now assume (1) and (2) hold for symplectic tableaux of length $< \ell(\tilde{P})$. Consider (1) with regard to $\tilde{P}$.

If x does not bump x'', we have, assuming no cancellations due to any of the x's occur in row 1,

$$(\tilde{P} \leftarrow x'') = \quad \begin{array}{ccccccccc} \bullet & \bullet & \bullet & \bullet & x'' & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

x'' bumps $y'' > x''$ into row 2

$$((\tilde{P} \leftarrow x'') \leftarrow x) = \quad \begin{array}{ccccccccc} \bullet & x & \bullet & \bullet & x'' & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

x bumps $y < x''$ into row 2, $y > x$

$$((\tilde{P} \leftarrow x'') \leftarrow x) \leftarrow x' = \quad \begin{array}{ccccccccc} \bullet & x & \bullet & x' & \bullet & x'' & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

x' bumps $y' \le x''$,

since $y' =$ first element $> x' \ge x$ so $y' >= y > x$.

Thus we find ourselves inserting into the tableau $\tilde{P}_1$ which has one row less than $\tilde{P}$, the sequence y'', y, y' where $y \le y' \le x'' < y''$.

On the other hand,

$$\begin{array}{rl}
(\tilde{P} \leftarrow x) = & \begin{array}{cccccccc} \bullet & \bullet & \bullet & \bullet & x & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & & & & \end{array} \\
\textit{x bumps } y > x & \\
(\tilde{P} \leftarrow x) \leftarrow x'' = & \begin{array}{cccccccc} \bullet & x & \bullet & \bullet & x'' & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & & & & \end{array} \\
x'' \textit{ bumps } y'' > x'', y'' \geq y & \\
((\tilde{P} \leftarrow x) \leftarrow x'') \leftarrow x' = & \begin{array}{cccccccc} \bullet & x & x' & \bullet & x'' & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & & & & \end{array} \\
x' \textit{ bumps } y' > x' \geq y &
\end{array}$$

so $y <= y' <= y''$; however $x' < x'' < y''$ so x' bumps $y' <= x'' < y''$.

This time we are inserting the sequence y, y'', y' into $\tilde{P}$, and $y <= y' < y''$ as before.

Consequently induction hypothesis applies in this case.

Now suppose x does bump x'':

Then

$$\begin{array}{rl}
(\tilde{P} \leftarrow x'') = & \begin{array}{ccccccc} \bullet & \bullet & \bullet & \bullet & x'' & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & & & \end{array} \\
x'' \textit{ bumps first } y'' > x'' & \\
(\tilde{P} \leftarrow x'') \leftarrow x = & \begin{array}{ccccccc} \bullet & \bullet & \bullet & \bullet & x & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & & & \end{array} \\
\textit{x bumps } x'' & \\
((\tilde{P} \leftarrow x'') \leftarrow x) \leftarrow x' & \begin{array}{ccccccc} \bullet & \bullet & \bullet & \bullet & x & \bullet & x' \\ \bullet & \bullet & \bullet & \bullet & & & \end{array} \\
x' \textit{ bumps first } y' > x', \textit{ so } y' > x &
\end{array}$$

and y' having been to the right of y'' in the original row, is $\geq y''$, so that we are reduced to inserting into the shorter tableau $\tilde{P}_1$ the sequence y'', x'', y' where $x'' < y'' <= y'$.

while

$$\begin{array}{rl}
(\tilde{P} \leftarrow x) = & \begin{array}{ccccccc} \bullet & x & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & & & \end{array} \\
\textit{x bumps the first } y > x &
\end{array}$$

where $y = y''$ above since x bumps x'', so the first thing $> x''$ coincides with the first thing $> x$;

$$\begin{array}{rl}
(\tilde{P} \leftarrow x) \leftarrow x'' = & \begin{array}{ccccccc} \bullet & x & \bullet & \bullet & \bullet & x'' & \bullet \\ \bullet & \bullet & \bullet & \bullet & & & \end{array} \\
x'' \textit{ bumps the first } y' > x'' &
\end{array}$$

so $y' >= y$ (y having been bumped by x);

$$((\tilde{P} \leftarrow x) \leftarrow x'') \leftarrow x' = \qquad \begin{array}{ccccccc} \bullet & x & \bullet & \bullet & \bullet & x' & \bullet \\ \bullet & \bullet & \bullet & \bullet & & & \end{array}$$

x' must bump x'' since x would have done so.

Thus we are reduced to inserting into $\tilde{P}$ the sequence $y = y'', y', x''$ where $x'' < y' <= (y =)y''$.

Again induction hypothesis applies.

We have shown that when there is no possibility of a cancellation, (1) may be reduced to performing the sequence of insertions (1) or (2) into a shorter tableau.

It remains to verify that cancellations do not change the results of the two insertions of (1).

The result holds if $\tilde{P} = \emptyset$, for then

$$(x'' \leftarrow xx') = \begin{cases} \begin{array}{cc} x & x' \\ x'' & \end{array}, & \text{if } (x'' \neq \bar{1} \textit{ or } x \neq 1) \text{ (i.e. no cancellations occur)} \\ x' = 1, & \text{else } (\textit{i.e. if} x'' = \bar{1}, x = 1, \textit{ so } x' = 1) \end{cases}$$

and

$$(x \leftarrow x''x') = \begin{cases} \begin{array}{cc} x & x' \\ x'' & \end{array}, & \text{if } (x'' \neq \bar{1} \textit{ or } x' \neq 1) \\ x = 1, & \text{else } (\textit{i.e. if} x'' = \bar{1}, x' = 1, \textit{ so } x = 1) \end{cases}$$

Similarly

$$(x' \leftarrow x''x) = \begin{cases} \begin{array}{cc} x & x'' \\ x' & \end{array}, & \text{if } (x' \neq \bar{1} \textit{ or } x \neq 1) \\ x'', & \text{else} \end{cases}$$

and

$$(x' \leftarrow x''x) = \begin{cases} \begin{array}{cc} x & x'' \\ x' & \end{array}, & \text{if } (x' \neq \bar{1} \textit{ or } x \neq 1) \\ x'', & \text{else} \end{cases}$$

In the general case it suffices to look at the first instance of a cancellation. It is clear that during the insertion of any element into a tableau, the first cancellation occurs when, at some point of the bumping path, we find ourselves attempting to insert an i into row i which contains at least one $\bar{\imath}$.

Consider the subwords $x''xx'$ and $xx''x'$. We make the following observations:

(1) If x'' causes a cancellation, so does x, in the subword $x''xx'$ by Lemma 10.6, and in $xx''x'$ by Lemma 10.3. Furthermore, Lemma 10.6 says that x' also causes a cancellation in $xx''x'$ if x'' does, so by considering the identical left Schensted tableaux for the two subwords it follows that all three letters cause cancellations if the largest one does, in both subwords.

(2) If x' causes a cancellation, so does x, in $x''xx'$ by Lemma 10.3, and in $xx''x'$ by Lemma 10.3 and Lemma 10.6 (noting that the intermediate insertion of x'' creates a bumping path to the right of the bumping paths of x, x'.)

Thus in both subwords either x alone causes a cancellation, or both x, x'' do, or x, x' do. Suppose the subwords occur in positions $j, j+1, j+2$.

Let $S_\mu^k = (L, Q_\mu)$ be the up-down tableau for w with the subword $x''xx'$ and let $S_\mu^{k\,\prime} = (L', Q_\nu)$ be the up-down tableau for the subword $xx''x'$

If only x causes a cancellation , then in the encoding of the up-down tableau, if an i_x is column-bumped out at step $(j+1)$ for the first subword $x''xx'$, clearly the same entry i_x is column-bumped out when x causes a cancellation in $xx''x'$, since the bumping path of x'' lies to the right of the portion of the tableau affected by x.

Clearly the symplectic tableau is unaffected by the transformation, since x causes a symplectic violation in the same row for both subwords, and thus $(\mu = \nu)$ while the up-down tableaux are affected as follows:

The pair $(j+1, i_x)$ appears in L, as opposed to (j, i_x) in L', while j appears in Q_μ where $j+1$ appears in Q_ν. Since Theorem 10.8 tells us that the shape $\lambda \supseteq \mu$, $(\lambda \vdash \ell(w))$ corresponding to the two up-down tableaux is the same (being the shape of the ordinary Schensted tableaux: recall that the left Schensted tableau is invariant under Knuth transformations), it is clear that the lattice permutation obtained upon inserting (the Burge tableau corresponding to) L into Q_μ is the same as that obtained when L' is inserted into Q_ν .

If x, x'', x' all cause cancellations, column-bumping out i_x, i''_x, i'_x respectively in the subword $x''xx'$, then it follows, again by considering the relative positions of the bumping paths, that the same entries are bumped out of the up-down tableau for the transformed subword $xx''x'$. As before, this implies no changes to the symplectic tableau of w, while the up-down tableaux are affected only in their two-line arrays: the pairs $(j, i_{x''}), (j+1, i_x), (j+2, i_{x'})$ in L' are replaced by $(j, i_x), (j+1, i_{x''}), (j+2, i_{x'})$ in L. But since $i_{x''} > i_x, i_x < i_{x'}$ (by Lemmas 10.4, 10.5) this means the bottom rows of L and L' differ precisely by a Knuth transformation, and consequently the corresponding even-columned Burge tableaux are identical, which in turn implies the lattice permutations coincide.

Finally if x, x' are the only ones causing cancellations, it follows as before that for both subwords, the up-down tableaux encoding results in column-bumning out the same entries $i_x, i_{x'}$ at steps $j+1, j+2$ respectively for the subword $x''xx'$ and at steps $j, j+2$ respectively for the subword $xx''x'$. Once more it is clear that the symplectic tableau is the same in both cases, while the up-down tableaux differ as follows:

The pairs $(j+1, i_x)$, $(j+2, i_{x'})$ in L are replaced by (j, i_x) $(j+2, i_{x'})$ in L', while the entry j in Q_μ is replaced by the entry $j+1$ in Q_ν (recall that $\mu = \nu$ since the symplectic tableaux are the same). Invoking Theorem 10.8 once more, the equality of the two Schensted ($Q-$)tableaux forces the lattice permutations obtained by inserting the (Burge tableaux corresponding to the) two-line arrays L into Q_μ , L' into Q_ν , to be identical.

The argument for the second Knuth transformation being identical in all significant details, we have shown that

If w and w' are words on $1, \bar{1}, \ldots, n, \bar{n}$ of the same length which are Knuth-equivalent then
$$(\emptyset \xleftarrow{B} w)_1 = (\emptyset \xleftarrow{B} w')_1$$
and $$(\emptyset \xleftarrow{B} w)_3 = (\emptyset \xleftarrow{B} w')_3.$$

•

Towards the goal of achieving as complete a characterisation as possible of words (of a fixed length k) which correspond to the same symplectic tableau, we have

Theorem 11.2 *Let w, w' be two words of the same length in $1, \bar{1}, \ldots, n, \bar{n}$. Then*

w and w' are Knuth-equivalent iff
$$(\emptyset \xleftarrow{B} w)_1 = (\emptyset \xleftarrow{B} w')_1$$
and
$$(\emptyset \xleftarrow{B} w)_3 = (\emptyset \xleftarrow{B} w')_3.$$

Proof: The only if part has been demonstrated in the previous theorem.

Since the two lattice permutations are the same, and the corresponding skew-shapes are the same, so that the SYT's

$$(\emptyset \leftarrow w)_2,\ (\emptyset \leftarrow w')_2$$

and have the same shape λ, say. Thus we may assume $(\emptyset \overset{B}{\longleftarrow} w) = (\tilde{P}_\mu\,,\, Q_\lambda\,,\, T_{\lambda/\mu}(\beta))$ and $(\emptyset \overset{B}{\longleftarrow} w') = (\tilde{P}_\mu\,,\, Q_\lambda{}'\,,\, T_{\lambda/\mu}(\beta))$ where only the SYT's Q_λ , $Q_\lambda{}'$ differ. (Note that by Theorem 18.1, this says at least that the two left Schensted tableaux of w and w' have the same shape λ.)

We now claim that the left Schensted tableau of w can be uniquely recovered from $(\tilde{P}_\mu\,,\, T_{\lambda/\mu}(\beta))$. The Berele algorithm essentially removes all "bad" pairs $(i, \bar{\imath})$ from the left Schensted tableau of w. Our strategy will thus be to restore these pairs (an i in row i, and the $\bar{\imath}$ in row $(i+1)$) to $\tilde{P}_\mu$, in the right order, so as to retrieve the left Schensted tableau. We proceed as follows.

First, for convenience, we represent the information contained in the pair $(\tilde{P}_\mu\,,\, T_{\lambda/\mu}(\beta))$ in a single Ferrers diagram of shape λ, where the squares which make up the shape μ are filled to give the symplectic tableau $\tilde{P}_\mu$, and the squares corresponding to the skew-shape λ/μ are filled with the lattice permutation of weight β. Since β is a shape with even columns, the lattice permutation is such that to every $2i+1$ in the lattice permutation there is a matching $2i+2$. We propose to augment the tableau $\tilde{P}_\mu$ by adding back at each step, a pair of squares numbered $(2i+1, 2i+2)$, taking the lowest and left-most square numbered $2i+1$ and the lowest and left-most square marked $2i+2$, for $i = 0, 1, 2, \ldots$.

Thus we order the pairs of squares marked $(2i+1, 2i+2)$ in the lattice permutation by associating to the lowest and left-most $2i+1$, the lowest and left-most $2i+2$ and then repeating with the next occurrence. Also, any pair $(2i+1, 2i+2)$ precedes a $(2i+3, 2i+4)$. Note that this is exactly the reverse of the order in which the cancellations occurred (it corresponds to column-inserting the bottom line in the two-line array scheme.

Example 11.3 Take

$$\tilde{P}_\mu = \begin{array}{ll} 1 & \bar{1} \\ 2 & \end{array}\,, \qquad T_{\lambda/\mu}(\beta) = \begin{array}{lll} \blacksquare & \blacksquare & 1 \\ \blacksquare & 1 & 2 \\ 2 & 3 & \\ 4 & & \end{array}$$

Now order the pairs $(2i+1, 2i+2)$ in the lattice permutation as follows:

$$T_{\lambda/\mu}(\beta) = \begin{array}{lll} \blacksquare & \blacksquare & 2 \\ \blacksquare & 1 & 2 \\ 1 & 3 & \\ 3 & & \end{array}$$

Next, replace each pair of squares in order:

$$\tilde{P}_\mu \to \begin{array}{ll} 1 & \bar{1} \\ 2 & \circ \\ \circ & \end{array} \to \begin{array}{ll} 1 & \bar{1} \\ \circ & 2 \\ \circ & \end{array} \to \begin{array}{ll} 1 & \bar{1} \\ 2 & 2 \\ \bar{2} & \end{array}$$

$$\to \begin{array}{lll} 1 & \bar{1} & \circ \\ 2 & 2 & \circ \\ \bar{2} & & \end{array} \to \begin{array}{lll} \circ & 1 & \bar{1} \\ \circ & 2 & 2 \\ \bar{2} & & \end{array} \to \begin{array}{lll} 1 & 1 & \bar{1} \\ \bar{1} & 2 & 2 \\ \bar{2} & & \end{array}$$

$$\to \begin{array}{lll} 1 & 1 & \bar{1} \\ \bar{1} & 2 & 2 \\ \bar{2} & \circ & \\ \circ & & \end{array} \to \begin{array}{lll} 1 & 1 & \bar{1} \\ \bar{1} & 2 & 2 \\ \circ & \bar{2} & \\ \circ & & \end{array} \to \begin{array}{lll} 1 & 1 & \bar{1} \\ \bar{1} & 2 & 2 \\ 2 & \bar{2} & \\ \bar{2} & & \end{array}$$

Example 11.4 Now take the same symplectic tableau as before, but let the lattice permutation be

$$T_{\lambda/\mu}(\beta) = \begin{array}{lll} \blacksquare & \blacksquare & 1 \\ \blacksquare & 1 & 2 \\ 1 & 2 & \\ 2 & & \end{array}$$

The order of reconstruction is now

$$T_{\lambda/\mu}(\beta) = \begin{array}{lll} \blacksquare & \blacksquare & 3 \\ \blacksquare & 2 & 3 \\ 1 & 2 & \\ 1 & & \end{array}$$

So this time we have:

$$\tilde{P}_\mu \to \begin{array}{ll} 1 & \bar{1} \\ 2 & \\ \circ & \\ \circ & \end{array} \to \begin{array}{ll} 1 & \bar{1} \\ 2 & \\ 3 & \\ \bar{3} & \end{array}$$

$$\to \begin{array}{ll} 1 & \bar{1} \\ 2 & \circ \\ 3 & \circ \\ \bar{3} & \end{array} \to \begin{array}{ll} 1 & \bar{1} \\ \circ & 2 \\ \circ & 3 \\ \bar{3} & \end{array} \to \begin{array}{ll} 1 & \bar{1} \\ 2 & 2 \\ \bar{2} & 3 \\ \bar{3} & \end{array}$$

$$\to \begin{array}{lll} 1 & \bar{1} & \circ \\ 2 & 2 & \circ \\ \bar{2} & 3 & \\ \bar{3} & & \end{array} \to \begin{array}{lll} \circ & 1 & \bar{1} \\ \circ & 2 & 2 \\ \bar{2} & 3 & \\ \bar{3} & & \end{array} \to \begin{array}{lll} 1 & 1 & \bar{1} \\ \bar{1} & 2 & 2 \\ \bar{2} & 3 & \\ \bar{3} & & \end{array}$$

Certainly this reconstructs the correct shape λ. We now need to specify how to label the pair of squares at each step, so as to end up with a column-strict tableau at the end . The idea is essentially to replace the removals in the reverse of the order in which the Berele algorithm caused them to occur. From previous arguments, (as in the proof of the main theorem) we know that adding the first pair of squares, labelled (1,1), to the shape μ restores what would have been the shape of the tableau if ordinary Schensted insertion had been performed instead of a Berele cancellation. But a little reflection on the modus operandi of the Berele algorithm shows us that this must mean that

(*) it is possible to play *jeu de taquin* on these two blank squares to get them to occupy the first column in two consecutive rows.

This really amounts to reversing the Berele cancellation step, so it is clear that what needs to be done is the following: slide the higher empty square corresponding to a '1', via jeu de taquin, to the highest possible row of $\tilde{P}_\mu$, without creating any symplectic violations, and then into column 1, ending in position $(j,1)$,say. Now slide the empty square corresponding to '2' in the lattice permutation into column 1 below it. Thus the two empty squares now occupy the first column in rows j and $j+1$. Clearly the correct labels for these squares are j and $\bar{j}$, giving a tableau $\tilde{P}_{\mu^1}$ with exactly one symplectic violation.

Now repeat with the next pair of squares in the ordering, and with the new tableau $\tilde{P}_{\mu^1}$. This time we are replacing the penultimate Berele cancellation; by Theorem 10.8, this augmented shape μ^2 is precisely the shape we would end up with if the last two Berele cancellations had been replaced by ordinary Schensted insertion, so as before

(**) it should be possible, via *jeu de taquin*, to slide the higher empty square to some row j and column 1, (and again this is to be done without causing any symplectic violations), and then to slide the lower square into row $(j+1)$ and column 1.

Note that since we are now adding to a tableau that already has a symplectic violation, these squares do not get the labels j, $\bar{j}$, but k, $\bar{k}$ where the previous pair in violation is $k-1$, $\overline{k-1}$.

Clearly this may be continued until all the pairs in the lattice permutation are exhausted; the resulting non-symplectic tableau of shape λ is precisely what the outcome of the ordinary Schensted algorithm performed on w would be.

We remind ourselves of the two crucial points of the argument convincing us that this procedure works, and with the desired result:

(1) Theorem 10.8, which says that adding back pairs of squares of the lattice

permutation in the order described above replicates the shape of the Schensted tableau at each step,

(2) The above process essentially replaces each Berele cancellation (in reverse order) by the corresponding Schensted insertion.

Thus we have shown that

$$\tilde{P}_\mu = (\emptyset \xleftarrow{\beta} w)_1 \text{ and}$$

$$T_{\lambda/\mu}(\beta) = (\emptyset \xleftarrow{\beta} w)_3$$

uniquely determine the left Schensted tableau of w', and hence two words w and w' giving the same symplectic tableau and the same lattice permutation of the same skew-shape, must be Knuth-equivalent. ●

Remark 11.5 *Note that it does not suffice to assume that w and w' give the same symplectic tableau and the same lattice permutation; we also need to know that the skew-shapes are the same, or equivalently, that the shapes of the Schensted tableaux of the two words are the same.*

Example 11.6

The pair

$$\tilde{P}_\mu = \begin{array}{ll} 1 & \bar{1} \\ 2 & \end{array}, \quad T_{lp} = \begin{array}{lll} \blacksquare & \blacksquare & 1 \\ \blacksquare & 2 & \\ 1 & & \\ 2 & & \end{array}$$

determines the Knuth class of the tableau

$$\begin{array}{lll} 1 & 1 & \bar{1} \\ \bar{1} & 2 & \\ 3 & & \\ \bar{3} & & \end{array}$$

while the pair

$$\tilde{P}_\mu = \begin{array}{ll} 1 & \bar{1} \\ 2 & \end{array}, \quad T'_{lp} = \begin{array}{lll} \blacksquare & \blacksquare & 1 \\ \blacksquare & 1 & 2 \\ 2 & & \end{array}$$

determines the Knuth class of the tableau

$$\begin{array}{lll} 1 & 1 & \bar{1} \\ \bar{1} & 2 & \bar{2} \\ \bar{2} & & \end{array}.$$

Using the invariance of the symplectic tableau of a word under Knuth transformations, we can establish the following

Proposition 11.7 *Let w be a word of length $2k$ on $1, \bar{1}, \ldots, n, \bar{n}$ such that its associated symplectic tableau (under Berele insertion) is empty. Then the left Schensted tableau is of the form*

$$\begin{array}{lllll} 1 & \ldots & \ldots & \ldots & 1 \\ \bar{1} & \ldots & \ldots & \ldots & \bar{1} \\ 2 & \ldots & \ldots & 2 & \\ \bar{2} & \ldots & \ldots & \bar{2} & \\ \vdots & \vdots & \ldots & \vdots & \\ k & \ldots & k & & \\ \bar{k} & \ldots & \bar{k} & & \end{array}$$

In particular, the resulting shape under Schensted insertion has even columns.

Proof: Observe first that if $\mathcal{B}(w)_1 = \emptyset$ then every occurrence of i in w is accompanied by an $\bar{i}$. This is clear from the nature of the algorithm and the fact that every letter of w eventually must succumb to a cancellation.

Let $\bar{n}$ be the largest letter appearing in w. That n and $\bar{n}$ ultimately disappear signifies that at some point in the insertion a column of length n is constructed , with n and $\bar{n}$ at the foot of the column, so that the next insertion affects this column in such a way as to knock the $\bar{n}$ into row $(n+1)$, and thus annihilate the pair $(n, \bar{n})$. We conclude that either i or $\bar{i}$ appears in this column,for each $i = 1, \ldots, n$, and therefore that $i, \bar{i}$ are letters of w for each $i = 1, \ldots, n$. Now consider a slight modification to the Berele algorithm, in the form of allowing the first occurrence of a symplectic violation in each row. Then clearly a parallel Schensted algorithm produces a left tableau P whose first column has length $2n$ and consists of the entries $1, \bar{1}, \ldots, n, \bar{n}$.

Denote the columns of P by $C_1, \ldots, C_r$, say. Let $w(C_i)$ be the word obtained by reading the ith column C_i from the bottom up. Then by Theorem 3.20, w is Knuth-equivalent to

$$w(C_1)w(C_2)\ldots w(C_r).$$

We proceed by induction on r, the number of columns of the left Schensted tableau of w. The result clearly holds if $r = 1$, by the preceding remarks.

Assume it is true for words whose left Schensted tableau has $< r$ columns. Then by Theorem 11.1, $w(C_1).\{w(C_2)\ldots w(C_r)\}$ also gives the empty symplectic tableau under Berele insertion. But clearly so does $w(C_1) = \bar{n}n\ldots\bar{1}1$. Thus the symplectic tableau corresponding to the word $w(C_2)\ldots w(C_r)$ must also be empty. But now induction hypothesis applies, since the left Schensted tableau P' of this word by construction has $(r-1)$ columns. Since P consists of C_1 pasted to the left edge of P', we are done. ●

Chapter IV

Symplectic Schur functions

In this chapter we use the combinatorial results developed so far to derive identities involving the characters of the symplectic group. The formulas (4.2) and (4.7) are classically known, although we believe our combinatorial approach to be new. Equation (4.1), which gives the complete decomposition of an arbitrary irreducible character of $Gl(2n, \mathrm{C})$ restricted to $Sp(2n, \mathrm{C})$, is, to our knowledge, new.

12 Restriction from $Gl(2n, \mathrm{C})$ to $Sp(2n, \mathrm{C})$

We begin this section with a combinatorial presentation of a classical result on the restriction of irreducible representations of $Gl(2n, \mathrm{C})$ to $Sp(2n, \mathrm{C})$. Littlewood[Li1](ch XI, pp 233-240;eq II, p.240 (11.9)) algebraically develops parallel formulas for the orthogonal groups, using symmetric function identities. A subcase of equation (4.2) is stated in the appendix of his book(p.295), as is equation (4.7). In [Li2] Littlewood uses Corollary (11.5) below in his computation of the Poincaré polynomial of $Sp(2n, \mathrm{C})$. (Later in this chapter we present our own calculation, which does not differ substantially.)

We recall the following definitions from Chapter 2:

For any partition λ, the Schur function s_λ corresponding to λ is defined to be

$$s_\lambda(x_1, \ldots, x_m, \ldots) = \sum_{\substack{T \text{ column-strict tableau} \\ shape(T)=\lambda}} wt(T)$$

where

$$wt(T) = \prod_i x_i^{\text{number of } i\text{'s in } T}.$$

For any partition μ of length at most n, the symplectic Schur function $sp_\mu(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$ is defined to be

$$sp_\mu(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = \sum_{\substack{\tilde{T} \text{ column-strict} \\ \text{symplectic tableau} \\ \text{shape}(\tilde{T}) = \mu}} wt(\tilde{T})$$

where

$$wt(\tilde{T}) = \prod_{i=1}^{n} x_i^{\text{number of } i\text{'s in } \tilde{T} - \text{number of } \bar{i}\text{'s in } \tilde{T}}.$$

Theorem 12.1 *For all partitions λ of length at most $2n$,*

$$s_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = \sum_{\substack{\mu \subseteq \lambda \\ \ell(\mu) \leq n}} sp_\mu(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) \left(\sum_{\beta' \text{even}} c^\lambda_{\mu,\beta}(n) \right) \tag{4.1}$$

Proof: We give what is essentially a combinatorial proof.

The left-hand side enumerates the set
$\mathcal{L} = \{P_\lambda$: column-strict tableau of shape $\lambda\}$, while the right-hand side counts pairs in the set
$\mathcal{R} = \{(\tilde{P}_\mu, T_{\lambda/\mu}(\beta))$: $\tilde{P}_\mu$ is a symplectic tableau of shape μ in $1, \bar{1}, \ldots, n, \bar{n}$ and $T_{\lambda/\mu}(\beta)$ is a lattice permutation of weight β which fits the skew-shape λ/μ n-symplectically$\}$.

Fix any standard Young tableau Q_λ of shape λ. Define a mapping $\phi : \mathcal{L} \mapsto \mathcal{R}$ by

$$\phi(P_\lambda) = ((\emptyset \xleftarrow{B} w)_1, (\emptyset \xleftarrow{B} w)_3)$$

where

$$(P_\lambda, Q_\lambda) \xleftrightarrow{KS} w$$

By Theorem (10.8), if $w \xleftrightarrow{KS} (P_\lambda, Q_\lambda)$ then $(\emptyset \xleftarrow{B} w)_2 = Q_\lambda$, so ϕ is well-defined and in fact a bijection. •

If N^λ is the irreducible $Gl(2n, \mathbf{C})$ -module corresponding to the partition λ, and $\tilde{N}^\mu$ is the irreducible $Sp(2n, \mathbf{C})$ -module corresponding to the partition μ, then in representation-theoretic terms, the above theorem says that

$$N^\lambda \downarrow \begin{matrix} Gl(2n,\mathbf{C}) \\ Sp(2n,\mathbf{C}) \end{matrix} = \bigoplus_{\substack{\mu \\ \ell(\mu) \leq n}} \left(\sum_{\beta' \text{even}} c^\lambda_{\mu,\beta}(n) \right) \tilde{N}^\mu$$

i.e., the multiplicity of the irreducible representation $\tilde{N}^{\mu}$ of $Sp(2n, \mathbf{C})$ in the representation N^{λ} restricted from $Gl(2n, \mathbf{C})$ to $Sp(2n, \mathbf{C})$ is

$$\left(\sum_{\beta' even} c^{\lambda}_{\mu,\beta}(n)\right)$$

We now give an example to illustrate the computational value of the preceding theorem:

Example 12.2 $n = 2$; $Sp(4, \mathbf{C}) \prec Gl(4, \mathbf{C})$.
Take $\lambda = (2^2, 1^2)$.
We have, listing each occurrence of an sp_{μ} by enumerating all possible lattice permutations of shape λ/μ and even-columned partition weight, and then discarding the non-2-symplectic ones:

$$s_{(2^2,1^2)}(x_1^{\pm 1}, x_2^{\pm 1}) = sp_{\begin{smallmatrix} 1 & 1 \\ 2 & 2 \\ 3 & \\ 4 & \end{smallmatrix}} + sp_{\begin{smallmatrix} \blacksquare & 1 \\ \blacksquare & 2 \\ 3 & \\ 4 & \end{smallmatrix}} + sp_{\begin{smallmatrix} \blacksquare & 1 \\ \blacksquare & 2 \\ 1 & \\ 2 & \end{smallmatrix}} .0 + sp_{\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \\ 1 & \\ 2 & \end{smallmatrix}} .0$$

$$= sp_{\emptyset} + sp_{1^2};$$

(the last two lattice permutations fail to be 2-symplectic because of the 1 in row 3).

Corollary 12.3 *If $\ell(\lambda) \leq n + 1$, then*

$$s_{\lambda}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = \sum_{\substack{\mu \subseteq \lambda \\ \ell(\mu) \leq n}} sp_{\mu}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) \left(\sum_{\beta' even} c^{\lambda}_{\mu,\beta}\right) \qquad (4.2)$$

Proof: If $\ell(\lambda) \leq n$ then any lattice permutation (of even-columned partition weight) which fits a skew-shape λ/μ, trivially fits it n-symplectically; for $\ell(\lambda) = n + 1$ the same is true by Proposition (9.7) of Chapter 3. •

The equivalent of formula (4.2) above for shapes λ of length at most n seems to appear in [We] (p.222, eqn(8.15). See also (8.8)).

Remark 12.4 Our proof of Theorem (11.1) above is not as combinatorial as one could hope for, in the sense that we have not provided a direct bijection between the objects enumerated by each side. Note however that such a bijection must necessarily exist, since by our results on Knuth-equivalence, specifically Thm (10.1), the mapping ϕ defined in the proof of Theorem (11.1) above is independent of the choice of the standard Young tableau Q_{λ}.

Corollary 12.5 *The multiplicity of the trivial representation of $Sp(2n, \mathbb{C})$ in the irreducible representation N^λ of $Gl(2n, \mathbb{C})$ ($\ell(\lambda) \leq 2n$) is nonzero iff λ' is even, in which case the multiplicity is 1.*

Proof: The multiplicity of the trivial representation of $Sp(2n, \mathbb{C})$ in N^λ is the coefficient of $sp_\emptyset$ in $s_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$, which is $\sum_{\beta' even} c^\lambda_{\emptyset,\beta}$.

Clearly there is exactly one lattice permutation fitting the shape $\lambda/\emptyset = \lambda$, and it has even columns iff λ does. So assume $\lambda = \beta$ has even columns. It remains to observe that the unique lattice permutation filling of β is n-symplectic as long as $\ell(\lambda) \leq 2n$, since the only way it can fail to be so is if for some $i = 1, \ldots, \frac{1}{2}\ell(\lambda), 2i + 1 > n + i$. But then $\ell(\lambda) \geq 2i + 2 > 2(n-1) + 2 = 2n$, contradiction. •

Corollary 12.6

$$s_{(k^{2n})}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = sp_\emptyset(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = 1 \tag{4.3}$$

That is, the irreducible representations (k^{2n}) of $Gl(2n, \mathbb{C})$ restrict to the trivial representation of $Sp(2n, \mathbb{C})$.

Proof: This is of course immediate if we use Theorem 10.8 and the fact that Knuth transformations preserve symplectic tableaux under Berele insertion (Theorem 11.1), since there is a unique column-strict tableau of shape (k^{2n}). It is also quite trivial character-theoretically, since $s_{(k^{2n})}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = 1$. Finally, it is a pleasantly easy exercise to consider the possible n-symplectic lattice permutations of even-columned weight:

To show that $c^{(k^{2n})}_{\mu,\beta}(n) = 0$ unless $\mu = \emptyset$, simply observe that if $\ell(\mu) = r \leq n$, then the first column of (k^{2n}) must contain the entries $1, \ldots, 2n - r$ in the lattice permutation filling, (and $2n - r$ must be even, being the last letter in the lattice permutation). Observing that n-symplectic failures must appear in the first column if they appear at all, any lattice permutation filling is n-symplectic iff $(r + 2i + 1) \leq (n + i)$ for all i such that $i \leq \frac{(2n-r-2)}{2}$, (since $2i + 2$ must be in $[2n - r]$).

But for i such that $2i + 2 = 2n - r$,

$$r + 2i + 1 \leq n + i \iff i \leq n - r - 1$$

$$\iff 2n - r - 2 \leq 2(n - r - 1) \iff r \leq 0 \iff \mu = \emptyset.$$

•

Corollary 12.7

1.

$$s_{(k)}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = sp_{(k)}(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$$

Equivalently, $h_k(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = \tilde{h}_k(x)$.

2.

$$s_{(1^k)}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} sp_{(1^{k-2i})}(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$$

for $k \leq n$.

Equivalently, $e_k(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) - e_{k-2}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = sp_{(1^k)} = \tilde{e}_k$.

3.

$$s_{(1^{n+r})}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = s_{(1^{n-r})}(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$$

for $r = 1, \ldots, n$.

Proof: Only *3.* requires comment. Let $\lambda = (1^l) \subseteq (1^{2n})$. Clearly for $k \leq n$, there is a unique lattice permutation c of shape $(1^l)/(1^k)$, whose column is even iff $l - k$ is even. We claim that c is n-symplectic iff $k \leq 2n - l$. Consider the filling T_c of c into the skew-shape $(1^l)/(1^k)$. In checking that this is n-symplectic, it suffices to look at the lowest odd entry of T_c. This is $k + 2i + 1$ where i is determined by the observation that

$$l - k = 2i + 2.$$

Thus T_c is n-symplectic
iff $l - 1 \leq n + i$
iff $l - 1 \leq n + \frac{(l-k-2)}{2}$
iff $k \leq 2n - l$.

Consequently we have

$$\begin{aligned} s_{(1^{n+r})}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) &= \sum_{\substack{k \leq n, k \leq (n-r) \\ n+r-k \text{ even}}} sp_{(1^k)} \\ &= \sum_{\substack{k \leq (n-r) \\ n-r-k \text{ even}}} sp_{(1^k)} \\ &= s_{(1^{n-r})}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) \end{aligned}$$

•

Corollary 12.8

$$\tilde{h}_\mu(x_1^{\pm 1},\ldots,x_n^{\pm 1}) = \sum_{\substack{\nu \\ \ell(\nu)\le n}} sp_\nu \left(\sum_{\lambda\supseteq\nu} K_{\lambda\,\mu} \sum_{\beta \text{ even}} c^\lambda_{\nu,\beta}(n) \right) \tag{4.4}$$

Proof: This is immediate from (4.1), and the well-known expansion of the homogeneous symmetric function h_μ in terms of Schur functions, in the ring $\Lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1})$, (cf. equation (1.14)).

However, a direct combinatorial proof in the style of Chapter 1, Corollary 3.25, is also possible. We shall discuss this viewpoint in section 14, where it appears naturally as a special case of the bijection establishing the Cauchy identity. •

We now consider the coefficients appearing in the expansion (4.2) a little more closely. Recall from Chapter 1, equation (1.19) that

$$\begin{aligned} F(t) &= \prod_{i<j}(1-t_it_j)^{-1} \\ &= \sum_{\substack{\beta \\ \beta' \text{even}}} s_\beta(t_1,t_2,\ldots). \end{aligned}$$

Taking the inner product of the symmetric function $F(t)s_\mu(t)$ with the Schur function $s_\lambda(t)$, and using the expansion above, we get:

$$\begin{aligned} < s_\lambda, F(t)s_\mu > &= \sum_{\substack{\beta \\ \beta' \text{even}}} < s_\lambda, s_\beta s_\mu > \\ &= \sum_{\substack{\beta \\ \beta' \text{even}}} c^\lambda_{\mu,\beta} \\ &= \textit{coefficient of } sp_\mu(x_1^{\pm 1},\ldots,x_n^{\pm 1}) \textit{ in } s_\lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1}), \\ &\quad \textit{for } \ell(\lambda),\ \ell(\mu) \le n \end{aligned}$$

Let us call this coefficient d^λ_μ. Let $D(n)$ be the matrix whose rows and columns are indexed by partitions of length at most n, ordered lexicographically, say (any total order will do), such that the (λ,μ)-entry of $D(n)$ is

$$\begin{cases} d^\lambda_\mu & \text{if } \lambda \supseteq \mu; \\ 0 & \text{else.} \end{cases}$$

The following result will enable us to write down the inverse $D(n)^{-1}$ of $D(n)$:

Proposition 12.9 *Let $F(t)$ be in $\bigwedge(t)$, the ring of symmetric formal power series (over $\mathbf{Z}$) in the n variables $t_1, \ldots, t_n$, such that $F(0) = 1$ (so $F^{-1}(t)$ is also in $\bigwedge(t)$). Then the matrices $(< s_{\lambda/\mu}, F(t) >)_{\lambda,\mu}$ and $(< s_{\lambda/\mu}, F^{-1}(t) >)_{\lambda,\mu}$ (rows and columns indexed by $Par_n = \{\nu \in Par : \ell(\nu) \leq n\}$ with some total ordering) are inverses of each other.*

Proof: We compute

$$\sum_{\nu} < s_{\lambda/\nu}, F(t) >< s_{\nu/\mu}, F^{-1}(t) > .$$

Recall that in the ring $\bigwedge(t) \otimes \bigwedge(s)$ of formal power series (over $\mathbf{Z}$) in two sets of variables $\{t\}$ and $\{s\}$ which are separately symmetric in $\{t\}$ and $\{s\}$, the following identity holds [MacD, Chapter 1, p.41, (5.10)]:

$$s_{\lambda/\mu}(t, s) = \sum_{\lambda \supseteq \nu \supseteq \mu} s_{\lambda/\nu}(s) s_{\nu/\mu}(t). \tag{4.5}$$

Consequently, in $\bigwedge(t) \otimes \bigwedge(s)$,

$$\begin{aligned}
< s_\lambda(t, s), s_\mu(s) s_\nu(t) > &= \sum_{\substack{\tau \\ \lambda \supseteq \tau}} < s_{\lambda/\tau}(s) s_\tau(t), s_\mu(s) s_\nu(t) > \\
&= \sum_{\substack{\tau \\ \lambda \supseteq \tau}} < s_{\lambda/\tau}(s), s_\mu(s) > \ < s_\tau(t), s_\nu(t) > \\
&=< s_{\lambda/\nu}(s), s_\mu(s) > \\
&\quad (\{s_\lambda(t)\} \ \textit{being an orthonormal basis for} \ \textstyle\bigwedge(t)) \\
&=< s_{\lambda/\nu}, s_\mu > \\
&=< s_\lambda, s_\mu s_\nu > \\
&= c^\lambda_{\mu,\nu}
\end{aligned}$$

Since the functions $\{s_\mu(t) s_\nu(s)\}_{\mu,\nu}$ clearly form a basis for the ring $\bigwedge(t) \otimes \bigwedge(s)$, this means

$$< h(t, s), f(t) g(s) >=< h, fg > \tag{4.6}$$

for all symmetric functions f, g and all $h(t, s)$ in $\bigwedge(t, s)$(jointly symmetric in {t}, {s}).

Consider $< s_{\lambda/\mu}(t, s), F(t) F^{-1}(s) >$.

By (4.6), this equals

$$< s_{\lambda/\mu}, FF^{-1} >=< s_{\lambda/\mu}, s_{\emptyset} >= \delta_{\lambda\mu}$$

On the other hand, by (4.5) we have

$$\begin{aligned} < s_{\lambda/\mu}(t,s), F(t)F^{-1}(s) > &= \sum_{\substack{\tau \\ \lambda\supseteq\tau\supseteq\mu}} < s_{\lambda/\tau}(t)s_{\tau/\mu}(s), F(t)F^{-1}(s) > \\ &= \sum_{\substack{\tau \\ \lambda\supseteq\tau\supseteq\mu}} < s_{\lambda/\tau}(t), F(t) > < s_{\tau/\mu}(s), F^{-1}(s) > \end{aligned}$$

•

We can now write down the expansion of $sp_\lambda(x)$ in terms of $s_\lambda(x_1^{\pm1}, \ldots, x_n^{\pm1})$, for partitions λ of length at most n: (An analogous expression for the characters of the orthogonal group appears in [Lil p.240, eqn II]; see also the appendix, p.295 of the same reference.)

Theorem 12.10

$$sp_\lambda(x_1^{\pm1}, \ldots, x_n^{\pm1}) = \sum_{\mu\subseteq\lambda} s_\mu(x_1^{\pm1}, \ldots, x_n^{\pm1}) \left(\sum_{\substack{r\geq0 \\ \alpha=(\alpha_1>\ldots>\alpha_r>0)}} (-1)^{|\alpha|} c^\lambda_{\mu,\mathcal{F}(\alpha)} \right) \quad (4.7)$$

where $\mathcal{F}(\alpha) = (\alpha_1 - 1, \ldots, \alpha_r - 1|\alpha_1, \ldots, \alpha_r)$ in Frobenius notation (cf. Definition 1.6),
and $|\alpha| = \sum_{i=1}^r \alpha_i$. (Note $\mathcal{F}(\alpha) = \emptyset$ if $\alpha = \emptyset$ iff $r = 0$.)

Proof: We need only recall a well-known identity [Lil p.238], [Macd pp. 46-47] (see also Theorem 3.34):

$$\prod_{i<j}(1 - t_i t_j) = \sum_{\substack{r\geq0 \\ \alpha=(\alpha_1>\ldots>\alpha_r>0)}} (-1)^{|\alpha|} s_{\mathcal{F}(\alpha)} \quad (4.8)$$

Now observe that for the matrix $D(n) = (d^\lambda_\mu)$, $(\ell(\lambda), \ell(\mu) \leq n)$, taking

$$F(t) = \prod_{1\leq i<j\leq n} (1 - t_i t_j)^{-1}$$

gives

$$d^\lambda_\mu =< s_{\lambda/\mu}, F(t) >,$$

so that, applying Proposition (11.9) to this choice of F, we get

$$\begin{aligned}(D(n)^{-1})_{\lambda,\mu} &= < s_{\lambda/\mu}, F^{-1}(t) > \\ &= \sum_{\substack{r \geq 0 \\ \alpha=(\alpha_1>\ldots>\alpha_r>0)}} (-1)^{|\alpha|} c^{\lambda}_{\mu,\mathcal{F}(\alpha)} \quad \text{by (4.7)}\end{aligned}$$

Treating (4.1) as a matrix equation, inverting the matrix on the right-hand side gives the desired result. •

Corollary 12.11 *The symplectic Schur functions $\{sp_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) : \ell(\lambda) \leq n\}$ form an integral basis for the ring $\tilde{\Lambda}_n$ of Laurent polynomials with integer coefficients in $\{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$ which are invariant under the action of the hyperoctahedral group $\mathbf{B_n}$.*

Proof: (1) shows the $\{sp_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) : \ell(\lambda) \leq n\}$ span $\tilde{\Lambda}_n$, since the ordinary Schur functions certainly do;

(2) shows they are linearly independent, since the Schur functions $s_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$ are so. •

Definition 12.12 *For λ of length at most $2n$, define $sp_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$ to be*

$$sp_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = \sum_{\mu \subseteq \lambda} s_\mu(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) \left(\sum_{\substack{r \geq 0 \\ \alpha=(\alpha_1>\ldots>\alpha_r>0)}} (-1)^{|\alpha|} c^{\lambda}_{\mu,\mathcal{F}(\alpha)} \right) \quad (4.9)$$

Lemma 12.13 *If $\ell(\lambda) = n+1$, then $sp_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = 0$.*

Proof: From Corollary (11.3) above, for any μ of length at most $n+1$, the partitions ν for which sp_ν appears in the decomposition of s_μ have length at most n. That is, we have

$$s_\mu(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = \sum_{\substack{\nu \subseteq \mu \\ \ell(\nu) \leq n}} d^{\mu}_{\nu}\, sp_\nu(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$$

•

Consequently, substituting in (4.9):

$$\begin{aligned} sp_\lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1}) &= \sum_{\substack{\mu\subseteq\lambda \\ \ell(\mu)\le n}} s_\mu(x_1^{\pm 1},\ldots,x_n^{\pm 1})(D(n)^{-1})_{\lambda,\mu} \\ &= \sum_{\mu\subseteq\lambda}\sum_{\substack{\nu\subseteq\mu \\ \ell(\nu)\le n}} (D(n)^{-1})_{\lambda,\mu}\, d_\nu^\mu\, sp_\nu(x_1^{\pm 1},\ldots,x_n^{\pm 1}) \\ &= \sum_{\substack{\nu \\ \ell(\nu)\le n}} \left(\sum_{\substack{\mu\subseteq\lambda \\ \mu\supseteq\nu}} (D(n)^{-1})_{\lambda,\mu}\, d_\nu^\mu\right) sp_\nu(x_1^{\pm 1},\ldots,x_n^{\pm 1}) \\ &= \sum_{\substack{\nu \\ \ell(\nu)\le n}} \delta_{\lambda\nu}\, sp_\nu(x_1^{\pm 1},\ldots,x_n^{\pm 1}) \\ &= 0 \quad \text{since} \quad \ell(\lambda) > n \end{aligned}$$

•

We would now like to mention R.C. King's "modification rule" [1] for defining $sp_\lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1})$ when $\ell(\lambda) > n$, for $Sp(2n, \mathrm{C})$. The rule has the effect of setting sp_λ to be 0 or $\pm sp_\mu$ for some μ of length at most n, and is apparently derived [Ki2] from the Weyl character formula for sp_λ (the analogue of the Jacobi-Trudi identity (cf. Theorem 2.13) for ordinary Schur functions s_λ).

We state the latter below:

Theorem 12.14 *Weyl[We p.219,* THEOREM (7.8.E)*](The Weyl character formula for $Sp(2n, \mathrm{C})$) If $\ell(\lambda) \le n$,*

$$sp_\lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1}) = \frac{1}{2} det(\tilde{h}_{\lambda_i-i-j-2} + \tilde{h}_{\lambda_i-i+j}) \tag{4.10}$$

where $\tilde{h}_k(x_1^{\pm 1},\ldots,x_n^{\pm 1})$ is the kth homogeneous symmetric function in 2n variables, specialised to $x_1^{\pm 1},\ldots,x_n^{\pm 1}$.

Proof: *[Li p.233]* Littlewood's derivation of the analogue of this formula for the character of the orthogonal group, goes through for the symplectic group, mutatis mutandis. •

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[1] We thank Professor Richard Stanley for drawing our attention to this topic.

By manipulating rows and columns in the above determinant, and in the equivalent expression (which is the analogue of equation (1.8))

$$sp_\lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1}) = \frac{1}{2}det(e_{\lambda'_i-i-j-2} + e_{\lambda'_i-i+j}) \tag{4.11}$$

where $e_k(x_1^{\pm 1},\ldots,x_n^{\pm 1})$ is the kth elementary symmetric function in $2n$ variables, specialised to $x_1^{\pm 1},\ldots,x_n^{\pm 1}$, and λ' denotes the conjugate of λ, King [Ki2] obtains the following rule:

Theorem 12.15 *King[Ki2] Suppose that for $\ell(\lambda) > n$, $sp_\lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1})$ is defined by the right-hand side of the Weyl character formula (4.10). Then the following combinatorial rule for computing sp_λ holds : Let $h = 2\{\ell(\lambda) - (n+1)\}$. Then*

$$sp_\lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1})$$
$$= \begin{cases} (-1)^{x+1} sp_\mu(x_1^{\pm 1},\ldots,x_n^{\pm 1}), \\ \quad \text{if, starting from the bottom row of } \lambda, \text{removing a connected} \\ \quad \text{border strip of length } h, \text{ occupying } x \text{ rows, leaves the shape } \mu; \\ 0, \quad \text{otherwise.} \end{cases} \tag{4.12}$$

Successive applications of this rule clearly reduce any "virtual" character sp_λ to a legal one sp_μ, i.e., one with $\ell(\mu) \le n$.

It is also immediate from this definition that

Corollary 12.16 *With $sp_\lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1})$ defined as in Theorem (11.5), $sp_\lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1}) = 0$ if $\ell(\lambda) = n+1$.*

Example 12.17 Take $n = 2, \lambda = (3,2^4)$, *so* $\ell(\lambda) = 5$ *and* $h = 4$. Then

$$sp_{(3,2^4)} = +sp_{(3,2,1^2)} = -sp_{(3,2)}.$$

Remark 12.18 We should also mention the work of Koike and Terada in this regard; in [KT] they derive what seems to be a different combinatorial rule for the value of sp_λ, $\ell(\lambda) > n$, defined as in (4.10). Their techniques, however, do not seem to diverge substantially from King's, in that they, too, use both the definition (4.10) in terms of the h's, and the expression (4.11) in terms of the e's.

With this extended definition of the sp_λ's, another uniform formula for decomposing the ordinary Schur function s_λ for all partitions λ of length at most $2n$ holds:

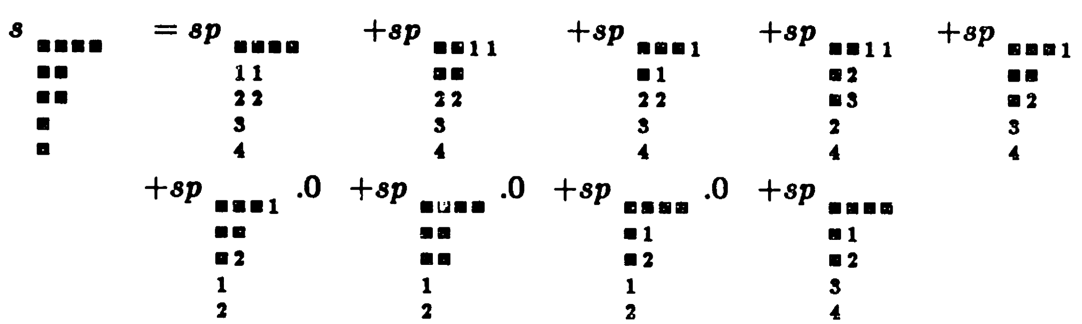
$$s_\lambda(x_1^{\pm1},\ldots,x_n^{\pm1}) = \sum_{\mu\subseteq\lambda} sp_\mu(x_1^{\pm1},\ldots,x_n^{\pm1}) \left(\sum_{\beta' even} c^\lambda_{\mu,\beta}\right) \tag{4.13}$$

Recall our extension (4.9) of the definition of the symplectic Schur function to shapes of length between n and $2n$, via our inversion of (4.1). We have verified computationally that (4.1) agrees with (4.13) and King's "modification" for a large number of examples, one of which we work out below. Since our extension of the definition of sp_λ can probably be shown (by purely algebraic means) to coincide with the extension provided naturally by the Weyl character formula, this immediately suggests the problem of uncovering the precise combinatorial connections between the formula (4.9) and King's rule (Theorem 12.15).

Example 12.19 In this example we compute the decomposition of the character $s_{(4,2^2,1^2)}$ of $Gl(6,\mathbb{C})$ when restricted to $Sp(6,\mathbb{C})$ in two ways: first using our result (4.1), which involves identifying the 3-symplectic lattice permutations to compute the multiplicities of valid symplectic characters, and then using King's modification rules and equation (4.13), which allows the occurrence of the virtual characters,and does not discriminate against any lattice permutation of the right weight and skew-shape. Clearly for the two methods to yield the same outcome, some cancellations must take place; as we shall see at least in this example, the correct number of virtual characters, corresponding to negative symplectic Schur functions, occurs in King's scheme, so as to cancel out precisely those contributions to the multiplicities made by allowing every lattice permutation, instead of just the n-symplectic ones. Note that in both calculations we can exploit Lemma (11.13) and Corollary (11.16) to ignore symplectic Schur functions corresponding to partitions of length $n+1$.

Set $\lambda = (4,2,2,1,1)$, so $\ell(\lambda) = 5$.

We choose to enumerate every even-columned lattice permutation which fits a skew-sub-shape of λ, and discard (in our schematic representation, multiply by 0) those which are not 3-symplectic. According to (4.1), we have



$$= sp_{(4)} + sp_{(2,2)} + sp_{(3,1)} + sp_{(2,1,1)} + sp_{(3,2,1)} + sp_{(4,1,1)}$$

On the other hand, using the formula (4.13) and King's modification rule we get

$$
\begin{aligned}
s_{\begin{array}{l}\blacksquare\blacksquare\blacksquare\blacksquare\\ \blacksquare\blacksquare\\ \blacksquare\blacksquare\\ \blacksquare\\ \blacksquare\end{array}}
&= sp_{\begin{array}{l}\blacksquare\blacksquare\blacksquare\blacksquare\\ 1\,1\\ 2\,2\\ 3\\ 4\end{array}}
+ sp_{\begin{array}{l}\blacksquare\blacksquare 1\,1\\ \blacksquare\blacksquare\\ 2\,2\\ 3\\ 4\end{array}}
+ sp_{\begin{array}{l}\blacksquare\blacksquare\blacksquare 1\\ \blacksquare 1\\ 2\,2\\ 3\\ 4\end{array}}
+ sp_{\begin{array}{l}\blacksquare\blacksquare 1\,1\\ \blacksquare 2\\ \blacksquare 3\\ 2\\ 4\end{array}} \\
&+ sp_{\begin{array}{l}\blacksquare\blacksquare\blacksquare 1\\ \blacksquare\blacksquare\\ \blacksquare 2\\ 3\\ 4\end{array}}
+ sp_{\begin{array}{l}\blacksquare\blacksquare\blacksquare 1\\ \blacksquare\blacksquare\\ \blacksquare 2\\ 1\\ 2\end{array}} (*)
+ sp_{\begin{array}{l}\blacksquare\blacksquare\blacksquare\blacksquare\\ \blacksquare\blacksquare\\ \blacksquare\blacksquare\\ 1\\ 2\end{array}} (**)
+ sp_{\begin{array}{l}\blacksquare\blacksquare\blacksquare\blacksquare\\ \blacksquare 1\\ \blacksquare 2\\ 1\\ 2\end{array}} (***) \\
&+ sp_{\begin{array}{l}\blacksquare\blacksquare\blacksquare\blacksquare\\ \blacksquare 1\\ \blacksquare 2\\ 3\\ 4\end{array}}
+ sp_{\begin{array}{l}\blacksquare\blacksquare\blacksquare 1\\ \blacksquare\blacksquare\\ \blacksquare 2\\ \blacksquare\\ \blacksquare\end{array}} (*)
+ sp_{\begin{array}{l}\blacksquare\blacksquare\blacksquare\blacksquare\\ \blacksquare 1\\ \blacksquare 2\\ \blacksquare\\ \blacksquare\end{array}} (***)
+ sp_{\begin{array}{l}\blacksquare\blacksquare\blacksquare\blacksquare\\ \blacksquare\blacksquare\\ \blacksquare\blacksquare\\ \blacksquare\\ \blacksquare\end{array}} (**)
\end{aligned}
$$

which has the same result, since the terms with the same number of asterisks cancel out according to King's rule; here $h = 2\{\ell(\lambda) - (3+1)\}$, so from the shapes of length > 3 we try to remove a connected border-strip of length 2:

$$sp_{\begin{array}{l}\blacksquare\blacksquare\blacksquare\\ \blacksquare\blacksquare\\ \blacksquare\\ \blacksquare\\ \blacksquare\end{array}} (*) = -sp_{(3,2,1)}$$

$$sp_{\begin{array}{l}\blacksquare\blacksquare\blacksquare\blacksquare\\ \blacksquare\\ \blacksquare\\ \blacksquare\\ \blacksquare\end{array}} (***) = -sp_{(4,1,1)}$$

$$sp_{\begin{array}{l}\blacksquare\blacksquare\blacksquare\blacksquare\\ \blacksquare\blacksquare\\ \blacksquare\blacksquare\\ \blacksquare\\ \blacksquare\end{array}} (**) = -sp_{(4,2,2)}$$

13 A Pieri rule for $Sp(2n,\mathbb{C})$

In this section we appeal once more to the technical properties of Berele insertion developed in Chapter 3 to derive a rule for decomposing the product

$$sp_\lambda(x_1^{\pm1},\ldots,x_n^{\pm1})sp_{(k)}(x_1^{\pm1},\ldots,x_n^{\pm1}) = sp_\lambda(x_1^{\pm1},\ldots,x_n^{\pm1})\tilde{h}_k(x_1^{\pm1},\ldots,x_n^{\pm1})$$

as an integer combination of symplectic Schur functions[2]. Ultimately we hope to be able to use this result, in conjuction with the techniques of Theorem (8.11), towards the larger goal of discovering a combinatorial rule for multiplying two symplectic Schur functions.

Theorem 13.1 *(A Pieri rule for symplectic Schur functions) Let λ be any partition of length at most n; k any nonnegative integer. Then*

$$sp_\lambda(x_1^{\pm1},\ldots,x_n^{\pm1})\ \tilde{h}_k(x_1^{\pm1},\ldots,x_n^{\pm1})$$

$$= \sum_{\substack{\nu \\ \ell(\nu)\le n}} sp_\nu(x_1^{\pm1},\ldots,x_n^{\pm1}) \left(\sum_{r=0}^{k} \sum_{\substack{\mu \\ \mu\subseteq\lambda,\ \mu\subseteq\nu \\ |\lambda/\mu|=r, |\nu/\mu|=k-r}} c^{\lambda}_{\mu,(1^r)} c^{\nu}_{\mu,(1^{k-r})} \right) \qquad (4.14)$$

Proof: The left-hand side enumerates pairs $(\tilde{P}_\lambda, w)$ where $\tilde{P}_\lambda$ is a symplectic tableau of shape λ and w is a k-word in $x_1^{\pm1},\ldots,x_n^{\pm1}$ which is a *row-word*, i.e., (cf. Definition 3.14) $w_1 \le \ldots \le w_k$. Recall from Chapter 1, Theorem 3.18, that the Pieri rule for ordinary Schur functions is essentially a combinatorial statement about the effect of Schensted insertion of a row-word into a column-strict tableau: the added squares form a horizontal strip. It is a tribute to the power of the Berele algorithm that the effect of Berele insertion of a row-word into a symplectic tableau may be described just as elegantly.

So let us consider the situation $\tilde{P}_\lambda \xleftarrow{B} w$ where $w_1 \le \ldots \le w_k$. Observe that

(1) The letters of w which cause cancellations must form an initial segment of w, so there is some $r \ge 0$ such that $w_1,\ldots,w_r$ cause cancellations, but w_i does not for $i > r$. This follows from Lemma (10.2).

(2) If for $i = 1,\ldots,r$, inserting w_i results in the loss of a square in row p_i, then $p_1 \le \ldots \le p_r$, by Lemma (10.4).

---

[2] We thank Professor Adriano Garsia for a discussion which set us thinking about this problem.

(3) By Lemma (10.6), if r is the cut-off point as above, so that w_r is the last cancellation and w_{r+1} starts off a sequence of additions to the tableau, then $w_r \le w_{r+1}$.

(4) If the letters whose insertion results in additions to the tableau are, as above, $w_{r+1}, \ldots, w_k$, with respective added squares in rows $q_1, \ldots, q_{k-r}$, then $q_1 \ge \ldots \ge q_{k-r}$. This is precisely the statement of Lemma 3.8 of Chapter 1 which ensures that the squares so added form a horizontal strip (i.e., no two in a column).

Thus Berele insertion of a row into a symplectic tableau $\tilde{P}_\lambda$ produces a pair $(\tilde{P}_\nu, S^k_{(\lambda \to \nu)})$, $\ell(\nu) \le n$, where the second component is a k-sequence of shapes of the form

$$(\lambda = \mu^0 \supset \mu^1 \supset \ldots \supset \mu^r \subset \mu = \nu^1 \subset \nu^2 \subset \ldots \nu^{k-r} = \nu),$$

such that:

• the difference between two consecutive shapes is exactly one box;

• if we label, in order, the boxes lost in the shape λ by $1, \ldots, r$ (see eg) then these filled boxes form a column-strict skew-plane partition of shape λ/μ, (since $i+1$ appears strictly to the left of i (by (2) above)), so in particular λ/μ is a horizontal strip;

• if we label the boxes added to μ in the remaining $k-r$ steps, these form a column-strict skew-tableau of shape ν/μ, so in particular ν/μ is a horizontal strip.

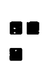
Conversely, given a symplectic tableau of shape ν and a k-sequence of shapes starting at λ and ending in ν, both shapes having length at most n, with the properties described in the preceding paragraph, it is clear that reversing the Berele insertion results in a row-word of length k: this follows from Lemma 3.8, observation (2) above (Lemma 10.4), and Lemma (10.6).

Thus the coefficient of sp_ν in $sp_\lambda\, sp_{(k)}$ is

$$|\{\mu : \mu \subseteq \nu, \mu \subseteq \lambda, \nu/\mu, \lambda/\mu \textit{ are horizontal strips, and } |\nu/\mu| + |\lambda/\mu| = k\}|.$$

We now remind the reader that the Littlewood-Richardson coefficient $c^\lambda_{\mu,(1^r)}$ is nonzero only when λ/μ is a horizontal strip, in which case it is 1. •

Example 13.2 We compute

sp  $(x_1{}^{\pm}, x_2{}^{\pm}, x_3{}^{\pm}, x_4{}^{\pm}) \,.\, \tilde{h}_3,$

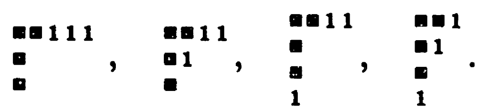
by listing all the possible up-down sequences of shapes of length 3 which begin with the shape $\lambda = (2,1,1)$.

We need to find all sequences ending in some shape ν, characterised by $\mu \subseteq \lambda, \mu \subseteq \nu$, such that

$$\lambda/\mu \text{ and } \nu/\mu \text{ are horizontal strips, and } |\nu/\mu| + |\lambda/\mu| = 3.$$

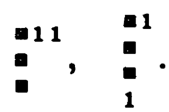
For $|\lambda/\mu| = 0$:

The shapes ν are simply the ones which are 3-Pieri over λ, and these are

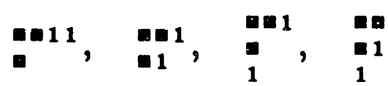


For $|\lambda/\mu| = 1$:

For $\mu = (1^3)$, the shapes ν which are 2-Pieri over μ are:



For $\mu = (2,1)$, the shapes ν which are 2-Pieri over μ are:

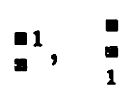


For $|\lambda/\mu| = 2$: The only μ such that λ/μ is 2-Pieri is $\mu = (1^2)$

the sequence 

is not allowed, since condition 3. in the proof of Theorem 13.1 would be violated.)

The shapes ν which are 1-Pieri over $\mu = (1^2)$ are



Finally, observe that there are no shapes μ such that λ/μ is 3-Pieri.

Thus

$$sp_{(2,1,1)}(x_1^{\pm}, x_2^{\pm}, x_3^{\pm}, x_4^{\pm}) \ . \ \tilde{h}_3,$$

$$= sp_{(5,1^2)} + sp_{(4,2,1)} + sp_{(4,1^3)} + sp_{(3,2,1^2)}$$
$$+ sp_{(3,1^2)} + sp_{(4,1)} + sp_{(3,2)} + sp_{(3,1^2)} + sp_{(2^2,1)}$$
$$+ sp_{(2,1)} + sp_{(1^3)}.$$

As a special case which follows by a trivial computation, we have:

Corollary 13.3 *If $m \geq k$ then*

$$sp_{(m)}(x) \ \ sp_{(k)}(x) = \sum_{i=1}^{k} \sum_{j=0}^{i} sp_{(m+k-i-j,i-j)}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) \tag{4.15}$$

As a corollary, we deduce

Corollary 13.4 *In the ring $\tilde{\Lambda}_n$, the coefficient of sp_λ in the expression of $\tilde{h}_2^k$ $(k \geq 0)$ as a linear combination of the basis elements $\{sp_\mu\}_{\ell(\mu) \leq n}$ is zero unless $\lambda \vdash 2m$ for some $m \geq 0$.*

(In representation-theoretic terms, this says the following: Suppose the irreducible $Sp(2n, \mathbf{C})$ -module $\tilde{N}^\lambda$ appears in the decomposition of the kth tensor power of the adjoint action of $Sp(2n, \mathbf{C})$. Then $|\lambda|$ is a multiple of 2.)

Proof: Let $\tilde{P}_\mu$ be any symplectic tableau of shape μ. By the Pieri rule, it is clear that if the up-down sequence produced by

$$(\tilde{P}_\mu \xleftarrow{B} w)$$

for an increasing word w of length 2, has shape λ, then $|\lambda|$ is equal to $|\mu|$ or $|\mu| \pm 2$. Thus if μ is even, so is $|\lambda|$.●

14 The Poincaré series for $Sp(2n, \mathbf{C})$

In this section we consider the character of the adjoint action of $Sp(2n, \mathbf{C})$ on its Lie algebra $\varsigma p(2n)$, and give an easy computation of the Hilbert series of the symmetric and exterior algebra invariants.

Recall

Definition 14.1 The **adjoint action** of $Sp(2n, \mathbf{C})$ on $\varsigma p(2n)$ is defined by

$$g.X = gXg^{-1} \, \forall \, g \in Sp(n, \mathbf{C}), \ X \in \varsigma p(2n).$$

We first record the computation giving the character of the adjoint representation of $Sp(2n, \mathbb{C})$:

Proposition 14.2 *The character of the adjoint representation of $Sp(2n,\mathbb{C})$ is the symmetric function $\tilde{h}_2(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$ in $\tilde{\Lambda}_n$.*

Proof: Let $e_{i,j}$ be the $2n$ by $2n$ matrix with a 1 in position (i,j) and 0's elsewhere; then these $\{e_{i,j}\}$ form a basis for the Lie algebra of all $2n$ by $2n$ matrices $\mathcal{G}l(2n)$. Recall [Hum, p.4] that the $(2n^2+n)$-dimensional Lie algebra $\varsigma p(n)$ has basis

$$\begin{aligned}
X_i &= e_{i,i} - e_{n+i,n+i}, \quad i = 1, \ldots, n \\
X_{ij} &= e_{i,j} - e_{n+j,n+i}, \quad 1 \le i \ne j \le n \\
Y_i &= e_{i,n+i}, \quad i = 1, \ldots, n \\
Y_{ij} &= e_{i,n+j} + e_{j,n+i}, \quad 1 \le i < j \le n \\
Z_i &= e_{n+i,i}, \quad i = 1, \ldots, n \\
Z_{ij} &= e_{n+i,j} + e_{n+j,i}, \quad 1 \le i < j \le n.
\end{aligned}$$

To compute the character of this action, it suffices to compute the eigenvalues of an arbitrary diagonal matrix $g = \text{diag}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) \in Sp(2n, \mathbb{C})$: We have

$$\begin{aligned}
gX_ig^{-1} &= X_i, \quad i = 1, \ldots, n \\
gX_{ij}g^{-1} &= (x_i x_j{}^{-1})X_{ij}, \quad 1 \le i \ne j \le n \\
gY_ig^{-1} &= x_i{}^2 Y_i, \quad i = 1, \ldots, n \\
gY_{ij}g^{-1} &= (x_i x_j)Y_{ij}, \quad 1 \le i < j \le n \\
gZ_ig^{-1} &= x_i{}^{-2} Z_i, \quad i = 1, \ldots, n \\
gZ_{ij}g^{-1} &= (x_i{}^{-1} x_j{}^{-1})Z_{ij}, \quad 1 \le i < j \le n
\end{aligned}$$

The character of the adjoint representation of $Sp(2n, \mathbb{C})$ is thus

$$\begin{aligned}
&char(ad)(x) \\
&= n + \sum_{1 \le i \ne j \le n} x_i x_j{}^{-1} + \sum_{1 \le i \le n} (x_i{}^2 + x_i{}^{-2}) + \sum_{1 \le i < j \le n} (x_i x_j + x_i{}^{-1} x_j{}^{-1}) \\
&= \sum_{1 \le i \le j \le n} (x_i x_j{}^{-1} + x_i x_j + x_i{}^{-1} x_j{}^{-1}) + \sum_{1 \le i < j \le n} x_i{}^{-1} x_j \\
&= \tilde{h}_2(x) \\
&(= h_2(x_1^{\pm 1}, \ldots, x_n^{\pm 1}))
\end{aligned}$$

•

Recall [Macd, Chapter 1, Section 7]

Definition 14.3 If f, g are symmetric functions in $\bigwedge(t)$ and g is written as a sum of monomials, the **plethysm** of f with g is defined to be the symmetric function obtained by substituting the monomials of g in f:

$$f[g] = f(\{x^\alpha\}_{\alpha \in M_g})$$

if $g = \sum_{\alpha \in M_g} u_\alpha \, x^\alpha$.

We need one more fact from representation theory :

Theorem 14.4 *[cf. Sta2] If G is a Lie group, V a G-module affording the representation ρ with character char(ρ), and $\theta_1, \ldots, \theta_m$ are the eigenvalues of ρ (i.e., the eigenvalues of $\rho(g)$ for an arbitrary $g \in G$), then*

(1) the character char$(S^k \rho)$ of the kth symmetric power $S^k(V)$ is $h_k(\theta_1, \ldots, \theta_m)$, where h_k is the complete homogeneous symmetric function of degree k;

(2) the character of the kth exterior power $\bigwedge^k(V)$ is char$(\bigwedge^k \rho) = e_k(\theta_1, \ldots, \theta_m)$, where e_k is the elementary symmetric function of degree k.

Corollary 14.5 *For the adjoint representation ad of $Sp(2n, C)$:*

(1) The character char $S^k(ad)$ is the plethysm $h_k[\tilde{h}_2(x_1^{\pm 1}, \ldots, x_n^{\pm 1})]$.

(2) The character char $\bigwedge^k(ad)$ is the plethysm $e_k[\tilde{h}_2(x_1^{\pm 1}, \ldots, x_n^{\pm 1})]$.

Proof: This follows immediately from the preceding theorem, Proposition 13.2, and the definition of plethysm, since the eigenvalues of the character of *ad* are precisely the monomials of $\tilde{h}_2$. •

Definition 14.6 The space of **invariants** of a G-module V affording the representation ρ of the group G is the $\emptyset$-isotypic component of G:

$$Inv\ V = Inv\ \rho = \{v \in V : g.v = v \ \forall\, g \in G\}$$

Clearly if $V = \coprod_{k \geq 0} V^k$ is a graded ring, so is $Inv\ V$,

$$Inv(V) = \coprod_{k \geq 0} Inv^k(V).$$

Definition 14.7 Let $R = \coprod_{k\geq 0} R^k$ be any graded ring. The **Hilbert series (Poincaré series)** of R is the power series defined by

$$F(R,q) = \sum_{k\geq 0} q^k \dim R^k.$$

Remark 14.8 We have

$$F(Inv\ \rho, q) = \sum_{k\geq 0} q^k \dim (Inv^k\ \rho)$$

$$= \sum_{k\geq 0} q^k \dim(\emptyset\text{ th} - \text{isotypic component of } k\text{th graded piece } \rho^k \text{ of } \rho)$$

$$= \sum_{k\geq 0} q^k \ < char(\rho^k), char(\emptyset) >=< \sum_{k\geq 0} q^k\ char\ \rho^k, char(\emptyset) >$$

Proposition 14.9

(1)

$$F(Inv\ S(ad)), q) = \prod_{k=1}^{n} (1 - q^{2k})^{-1}$$

(2)[We,p.] [Li](The Poincaré series for $Sp(2n, \mathbb{C})$ *)*

$$F(Inv\ \bigwedge(ad), q) = \prod_{i=1}^{n} (1 + q^{4i-1})$$

Proof:

(1):

$$F(char\ S(ad), q) = \sum_{k\geq 0} q^k \dim char\ S^k(ad)$$

But

$$\sum_{k\geq 0} q^k\ char\ S^k(ad) = \sum_{k\geq 0} q^k\ h_k[\tilde{h}_2] = \prod_{1\leq i\leq j\leq n} (1 - qy_iy_j)^{-1},$$

if $y_i = x_i$, $y_{n+i} = x_i^{-1}$, $i = 1, \ldots, n$

$$= \sum_{\substack{\gamma \text{ even} \\ \ell(\gamma)\leq 2n}} s_\gamma(q^{\frac{1}{2}}y_1, \ldots, q^{\frac{1}{2}}y_{2n})$$

$$= \sum_{\substack{2\lambda \\ \ell(\lambda)\le 2n}} q^{|\lambda|}\, s_{2\lambda}(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$$

where for λ in Par, 2λ is the even-rowed partition such that $(2\lambda)_i = 2.\ \lambda_i$. By Corollary 12.5, the coefficient of $sp_\emptyset$ in $s_{2\lambda}(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$

$$(= \text{dimension of } (\emptyset - \text{isotypic component}) \text{ in}$$

$$\text{the representation } \mathrm{N}^{2\lambda} \downarrow {}^{Gl(2n,\mathbb{C})}_{Sp(2n,\mathbb{C})}\ )$$

$$\begin{cases} 1, & \text{if } 2\lambda \text{ has even columns;} \\ 0, & \text{else} \end{cases}$$

Thus the Hilbert series of $S(ad) = \coprod_{k\ge 0} S^k(ad)$ is

$$\sum_{\substack{2\lambda\ even \\ \ell(2\lambda)\le 2n}} q^{|\lambda|} = \sum_{k\ge 0} q^k\, |\{\lambda \vdash k : \lambda = (\lambda_1, \lambda_1, \ldots, \lambda_l, \lambda_l), l = \ell(\lambda) \le n\}|$$

$$= \sum_{k\ge 0} q^{2k}\, |\{\lambda \vdash k, \ell(\lambda) \le n\}| = \sum_{\substack{\lambda \\ \ell(\lambda)\le n}} q^{2|\lambda|} = \prod_{i=1}^{n} (1 - q^{2i})^{-1}.$$

(2) $\bigwedge^k(ad)$ has character $e_k[\tilde{h}_2]$; thus

$$\sum_{k\ge 0} q^k\ char\ \bigwedge^k(ad) = \sum_{k\ge 0} q^k\, e_k[\tilde{h}_2] = \prod_{1\le i\le j\le 2n} (1 + q y_i y_j)$$

if $y_i = x_i$, $y_{n+i} = x_i^{-1}$, $i = 1, \ldots, n$,

$$= \sum_{\alpha=(\alpha_1>\ldots>\alpha_r>0)} s_{\mathcal{F}(\alpha)'}(q^{\frac{1}{2}} y_1, \ldots, q^{\frac{1}{2}} y_{2n})$$

(recall $\mathcal{F}(\alpha) = (\alpha_1 - 1, \ldots, \alpha_r - 1 | \alpha_1, \ldots, \alpha_r)$)

$$= \sum_{\alpha=(\alpha_1>\ldots>\alpha_r>0)} q^{\frac{1}{2}|\mathcal{F}(\alpha)'|}\, s_{\mathcal{F}(\alpha)'}(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$$

$$= \sum_{2n\ge r\ge 0}\ \sum_{\alpha=(\alpha_1>\ldots>\alpha_r>0)} q^{|\alpha|}\, s_{\mathcal{F}(\alpha)'}(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$$

As before, the coefficient of $sp_\emptyset$ in $s_{(\alpha_1,\ldots,\alpha_r|\alpha_1-1,\ldots,\alpha_r-1)}$ is nonzero

$$\iff (\alpha_1, \ldots, \alpha_r | \alpha_1 - 1, \ldots, \alpha_r - 1) \text{ has even columns}$$

$$\iff (a)\ 2|(\alpha_i - 1) + i\ \forall\, i \text{ and } (b)\ |(2i-1)st\ row| = |(2i)th\ row| \forall\, i$$

$$\iff \alpha_{2i-1} + (2i-1) = \alpha_{2i} + 2i \text{ and } \alpha_{2i-1} \text{ even, } \alpha_{2i} \text{ odd}$$

$\iff r = 2s$ is even, (since $\alpha_{2i-1} = \alpha_{2i} + 1$ is even implies $\alpha_{2i} \geq 1$), and α is such that $2n > \alpha_2 > \alpha_4 > \ldots > \alpha_{2s} > 0$, $s \geq 0$, α_{2i} odd $\forall\, i$.

Note that for such α, $|\alpha| = 2\sum_{i=1}^{s} \alpha_{2i} + s$.

Thus

$$F(Inv \bigwedge(ad), q) = \sum_{s=0}^{n} \sum_{\substack{(2n>\alpha_2>\alpha_4>\ldots>\alpha_{2s}>0) \\ \alpha_{2i}\ odd}} q^{2\sum_{i=1}^{s}\alpha_{2i}+s}$$

$$= \sum_{s=0}^{n} \sum_{\substack{\mu\in Par \\ \ell(\mu)=s \\ \mu \text{ has distinct odd parts} \\ \text{between } 1 \text{ and } (2n-1)}} q^{2|\mu|+\ell(\mu)}$$

$$= \sum_{\substack{\mu\in Par \\ \ell(\mu)\leq n \\ \mu \text{ has distinct odd parts} \\ \text{between } 1 \text{ and } (2n-1)}} q^{2|\mu|} q^{\ell(\mu)}$$

$$= \prod_{i=1}^{n}(1 + (q^2)^{2i-1}.q) = \prod_{i=1}^{n}(1 + q^{4i-1}).$$

•

15 The Cauchy identity for $Sp(2n, \mathbf{C})$

In this section we state a symplectic analogue of the Cauchy identity (1.13)

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(x) s_\lambda(y)$$

of Chapter 1, and give a bijective proof. The identity, which holds in the ring $\mathbf{Z}(t_1 \ldots t_n)^{S_n} \otimes \mathbf{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{B_n}$, is as follows:

Theorem 15.1

$$\prod_{1\leq i<j\leq n} (1 - t_i t_j) \prod_{i,j=1}^{n} (1 - t_i x_j)^{-1} (1 - t_i x_j{}^{-1})^{-1}$$

$$= \sum_{\substack{\mu \\ \ell(\mu)\leq n}} sp_\mu(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) s_\mu(t_1, \ldots, t_n) \tag{4.16}$$

Proof: We remark that this may be obtained algebraically by using Theorem 12.1 to substitute for $s_\lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1})$ in the Cauchy identity for ordinary Schur functions; writing $s_\lambda(t)$ for $s_\lambda(t_1,\ldots,t_n)$ and $sp_\mu(x)$ for $sp_\mu(x_1^{\pm 1},\ldots,x_n^{\pm 1})$, we have

$$\prod_{i,j=1}^{n}(1-t_ix_j)^{-1}(1-t_ix_j{}^{-1})^{-1}$$

$$= \sum_{\substack{\lambda \\ \ell(\lambda)\le n}} s_\lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1})s_\lambda(t)$$

$$= \sum_{\substack{\lambda \\ \ell(\lambda)\le n}} \sum_{\substack{\mu\subseteq\lambda \\ \ell(\mu)\le n}} sp_\mu(x)\left(\sum_{\beta' even} c^\lambda_{\mu,\beta}\right)s_\lambda(t)$$

$$= \sum_{\substack{\mu \\ \ell(\mu)\le n}} sp_\mu(x)\sum_{\beta' even}\left(\sum_{\substack{\lambda \\ \ell(\lambda)\le n}} c^\lambda_{\mu,\beta}s_\lambda(t_1\ldots t_n)\right)$$

$$= \sum_{\substack{\mu \\ \ell(\mu)\le n}} sp_\mu(x)\sum_{\beta' even} s_\mu(t)s_\beta(t)$$

$$= \sum_{\substack{\mu \\ \ell(\mu)\le n}} sp_\mu(x)s_\mu(t)\left(\sum_{\beta' even} s_\beta(t)\right)$$

$$= \sum_{\substack{\mu \\ \ell(\mu)\le n}} sp_\mu(x)s_\mu(t)\prod_{1\le i<j\le n}(1-t_it_j)^{-1} \quad \text{by equation (1.19)}$$

•

In order to give a bijective proof of this identity, it clearly simplifies matters to re-write it as (cf. above proof)

$$\prod_{i,j=1}^{n}(1-t_ix_j)^{-1}(1-t_ix_j{}^{-1})^{-1} = \sum_{\substack{\mu \\ \ell(\mu)\le n}} sp_\mu(x_1^{\pm 1},\ldots,x_n^{\pm 1})s_\mu(t)\left(\sum_{\beta' even} s_\beta(t)\right)$$

We may enumerate the left-hand side by means of Knuth two-line arrays (cf. Definition 3.21)

$$\mathcal{T} = \begin{pmatrix} t_{i_1} & \ldots & t_{i_k} \\ y_{i_1} & \ldots & y_{i_k} \end{pmatrix}$$

where the y_{i_j}'s are in the set $\{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$; such a two-line array would correspond to the term $(t_{i_1} \ldots t_{i_k} y_{i_1} \ldots y_{i_k})$ in the expansion of the left side as a formal power series. We impose the usual lexicographic ordering (cf. Definition 3.21) on the arrays, viz., $t_{i_1} \leq \ldots \leq t_{i_k}$, and $t_{i_j} = t_{i_{j+1}}$ implies $y_{i_j} \leq y_{i_{j+1}}$.

The right-hand side clearly counts the set of all triples $(\tilde{P}_\mu(x), P_\mu(t), P_\beta(t))$, where $\tilde{P}_\mu(x)$ is a symplectic tableau of shape μ with entries in $1, \bar{1}, \ldots, n, \bar{n}$, corresponding to the variables $\{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$, $P_\mu(t)$ is an ordinary tableau with entries in $[n]$ of the same shape μ as P_μ, and $P_\beta(t)$ is an ordinary tableau of shape β with entries in $[n]$, where β has even columns.

We now demonstrate a correspondence between these two sets of objects, in the proof of

Theorem 15.2 *There is a bijection establishing the identity :*

$$\prod_{i,j=1}^{n} (1 - t_i x_j)^{-1} (1 - t_i x_j{}^{-1})^{-1}$$

$$= \sum_{\substack{\mu \\ \ell(\mu) \leq n}} sp_\mu(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) s_\mu(t_1, \ldots, t_n) \left(\sum_{\beta' even} s_\beta(t_1, \ldots, t_n) \right) \tag{4.17}$$

Proof: The basic idea is to use the Berele algorithm on the bottom row of x's and x^{-1}'s in the two-line array $\mathcal{T}$, thereby obtaining a symplectic tableau P_μ which contributes to the symplectic Schur function $sp_\mu(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$, and keeping track appropriately with the t's in the top row of the array. The remaining output of the original Berele algorithm was an up-down tableau $S_\mu^k(n)$. However, one of the reformulations presented in Chapter 3 (Lemma 8.7) encoded this as a pair (Q_μ, L) where Q_μ is a standard Young tableau of shape μ and L is a two-line array

$$\begin{pmatrix} j_1 & \cdots & j_r \\ i_1 & \cdots & i_r \end{pmatrix}$$

with $j_1 < \ldots < j_r$ and $j_k > i_k$, all $k = 1, \ldots, r$. L was in turn converted into a SYT Q_β of shape β with even columns (Lemma 8.9).

By following exactly the same algorithm as in Lemma 8.7, we can go from the two-line array $\mathcal{T}$ to a triple $(\tilde{P}_\mu, P_\mu, \mathcal{L})$, where P_μ, which takes the place of Q_μ in Lemma 8.7, is no longer necessarily standard, but is an ordinary tableau of shape μ in $[n]$, and $\mathcal{L}$ is a two-line array with repeated entries, coming from the top row of t's, $\mathcal{L} =$

$$\begin{pmatrix} j_1 & \cdots & j_r \\ i_1 & \cdots & i_r \end{pmatrix}$$

such that $j_1 \leq \ldots \leq j_r$. To be able to apply the general form of the Burge correspondence to $\mathcal{L}$, with the aim of converting it into a tableau of shape β with even columns, $\mathcal{L}$ must be a Burge two-line array, or equivalently, must have the following properties:

(1) $j_k > i_k$, all $k = 1, \ldots, r$.

(2) $\mathcal{L}$ is in lexicographic order, i.e., $j_k = j_{k+1}$ implies $i_k \leq i_{k+1}$.

Before verifying that (1) and (2) hold, we review the procedure of Lemma 8.7 by describing it in the present context of repeated labels (the t's).

We may view the present situation as a generalisation of the setting of the Berele algorithm, whose input is a set of words $y_{i_1} \ldots y_{i_k}$, in $1, \bar{1}, \ldots, n, \bar{n}$, or equivalently, two-line arrays

$$T = \begin{pmatrix} 1 & \ldots & k \\ y_{i_1} & \ldots & y_{i_k} \end{pmatrix},$$

where the top row consists of distinct, strictly increasing labels. Thus our input is now in the form of two-line arrays

$$\mathcal{T} = \begin{pmatrix} t_{i_1} & \ldots & t_{i_k} \\ y_{i_1} & \ldots & y_{i_k} \end{pmatrix}$$

where the bottom row is a word in $1, \bar{1}, \ldots, n, \bar{n}$ and the top row has labels which may be repeated, but are still in increasing order; in addition, the array is written so that $t_{i_j} = t_{i_{j+1}}$ implies $y_{i_j} \leq y_{i_{j+1}}$.

We begin by appplying the Berele algorithm to the word $y_{i_1} \ldots y_{i_k}$. As long as there are no cancellations, this is the same as the Knuth-Schensted process; consequently we can mimic the Knuth correspondence (Theorem 3.22) for the two-line array up to this point, so that if at the jth step we have built up a sequence of pairs of tableaux $(\tilde{P}_j, P_j)$, at step $j+1$ we set

$$\tilde{P}_{j+1} = \tilde{P}_j \overset{B}{\longleftarrow} y_{i_{j+1}} \quad (\textit{Berele insertion}).$$

If Berele insertion does not result in a cancellation, set

$$P_{j+1} = P_j$$

with $t_{i_{j+1}}$ added in the unique position so as to force $sh(P_{j+1}) = sh(\tilde{P}_{j+1})$;

otherwise, $sh(\tilde{P}_{j+1}) = \mu^{j+1}$, say, has one box less than $sh(\tilde{P}_j) = \mu^j$, and we get P_{j+1} from P_j as follows:

• bump out the extra entry of P_j (the one in the unique square of μ^j which is not a square of μ^{j+1}) by columns (i.e. inverse Schensted column insertion) to get a tableau P_{j+1} of shape μ^{j+1}, and a letter x. This means that by column-inserting x into P_{j+1} we would retrieve the previous larger tableau P_j of shape μ^j.

• We record the fact that a removal has occurred at step j by putting the pair (t_{i_j}, x) into a two-line array $\mathcal{L}$, with t_{i_j} on top. Note that since x was already present in the tableau at step $t_{i_j}, x \leq t_{i_j}$.

We continue this process to the end of the word $y_{i_1} \dots y_{i_k}$, arranging the two-line array $\mathcal{L}$ so that the top row is weakly increasing.

Example 15.3 The word $t_1 x_1{}^{-1} t_1 x_2 t_2 x_1{}^{-1} t_2 x_2 (t_4 x_1)^2 t_4 x_1{}^{-1} t_5 x_1$ corresponds to the two-line array

$$\mathcal{T} = \begin{pmatrix} 1 & 1 & 2 & 2 & 4 & 4 & 4 & 5 \\ \bar{1} & 2 & \bar{1} & 2 & 1 & 1 & \bar{1} & 1 \end{pmatrix}$$

We proceed with Berele insertion of the word in the bottom row of $\mathcal{T}$, from left to right as usual. In the schematic that follows, the computation is arranged so that:

• the first row contains the successive symplectic tableaux, ending in the final tableau $\tilde{P}_\mu$,

• the second row encodes the up-down tableau resulting from the Berele insertion, ending in the final tableau P_μ ,

• the third row encodes the removals in the form of pairs which, at the end of the process,may be put together into a Burge two-line array $\mathcal{L}$, so that ultimately the pair $(P_\mu , \mathcal{L})$ contains all the information to completely and uniquely specify the up-down tableau.

$$\mathcal{T} \to \begin{pmatrix} \bar{1} & \bar{1}2 & \substack{\bar{1}\bar{1} \\ 2\phantom{\bar{1}}} & \substack{\bar{1}\bar{1}2 \\ 2\phantom{\bar{1}2}} & \substack{\bar{1}2 \\ 2} & 22 & \substack{\bar{1}2 \\ 2} & 22 & \tilde{P}_\mu \\ & & & & & & & & \\ 1 & 11 & \substack{11 \\ 2} & \substack{112 \\ 2} & \substack{12 \\ 2} & 12 & \substack{12 \\ 4} & 12 & P_\mu \\ & & & & & & & & \\ & & & & \{\substack{4 \\ 1}\} & \{\substack{4 \\ 2}\} & & \{\substack{5 \\ 4}\} & \mathcal{L} \end{pmatrix}$$

$$\to \left(22\,,\ \ 12\,,\ \ \{\substack{445 \\ 124}\} \right)$$

$$\to \left(22\,,\ \ 12\,,\ \ \substack{124 \\ 445} \right) = (\tilde{P}_{(2)}, P_{(2)}, P_{(3,3)}).$$

We have thus shown that a two-line array $\mathcal{T}$ above may be mapped to a triple $(\tilde{P}_\mu, P_\mu, \mathcal{L})$ where $\tilde{P}_\mu$ is a symplectic tableau of shape μ, P_μ is an ordinary tableau of the same shape, and $\mathcal{L}$ is a two-line array

$$\begin{pmatrix} j_1 & \dots & j_r \\ i_1 & \dots & i_r \end{pmatrix}$$

such that $j_1 \leq \dots \leq j_r$ and $j_k \geq i_k$, all $k = 1, \dots, r$, with entries from the top row of labels t, such that $j_k \geq i_k$. (We will see shortly that in fact $j_k > i_k$.)

If $\mathcal{L}$ has no repeated entries in its top row, we could apply the argument of Lemma 8.7 to conclude that this mapping is reversible, that is, that the pair $(P_\mu, \mathcal{L})$ contains all the information necessary to recover the up-down tableau arising from applying Berele to $y_{i_1} \dots y_{i_k}$. Recall that the essential observation in this case is that $\mathcal{L}$ enables us to locate uniquely the labels (= steps of the Berele algorithm) at which the removals occurred.

If, on the other hand, there are repeated entries z among the top row of $\mathcal{L}$, and there are more z's in the original two-line array, then we need to know which of the labels z resulted in removals. This ambiguity is conveniently resolved by Lemma 10.3, which tells us the following:

If a letter y in the bottom row, corresponding to some repeated label z, in the top row of $\mathcal{T}$ caused a cancellation, so did all letters preceding y and having the same label z in the top row, since, in the chosen ordering for $\mathcal{T}$, all such letters are $\leq y$.

Thus a segment of $\mathcal{T}$ consisting of equal labels z in the top row, whose corresponding y's cause cancellations upon Berele insertion, is an initial segment of the sub-array of $\mathcal{T}$ consisting of all pairs with the label z, thereby enabling us to associate the removals with the correct labels uniquely, and consequently to retrieve the associated up-down tableau.

Example 15.4 Conversely, given $(\tilde{P}_{(2)}, P_{(2)}, P_{(3,3)})$ as above, to retrieve $\mathcal{T}$: First observe that the top row of $\mathcal{T}$ is distributed between the entries of P_μ and those of $\mathcal{L} \longleftrightarrow P_\beta$. Also from the arguments in the text, we know where to place each removal pair (j, i) of $\mathcal{L}$; thus we have

$$\begin{matrix} 1 & 1 & 2 & 2 & 4 & & 4 & & 4 & & 5 \\ & & & & & \{ {4 \atop 1} \} & & \{ {4 \atop 2} \} & & \{ {5 \atop 4} \} & \end{matrix}$$

We can therefore reconstruct the up-down tableau, working backwards from P_μ and the removal pairs above:

$$\begin{array}{ccccccccc}
1 & 1 & 2 & 2 & 4 & 4 & 4 & 5 & \\
 & & & & \left\{ {4 \atop 1} \right\} & \left\{ {4 \atop 2} \right\} & & \left\{ {5 \atop 4} \right\} & \\
 & & & & & & & & \\
1 & 11 & {11 \atop 2} & {112 \atop 2} & {12 \atop 2} & 12 & {12 \atop 4} & 12 & P_\mu
\end{array}$$

In general, to go from

$$\begin{pmatrix} t_{i+1} \\ T_{i+1} \end{pmatrix} \text{ to } \begin{pmatrix} t_i \\ T_i \end{pmatrix},$$

- if $T_{i+1} \supseteq T_i$, then T_i is the tableau obtained by deleting right-most entry t_{i+1} in T_{i+1},
- otherwise, there is a pair

$$\left\{ {t_{i+1} \atop x} \right\}$$

under this t_{i+1}, so $T_i = (x \to T_{i+1})$.

This gives the up-down tableau

Finally the word forming the bottom row of $\mathcal{T}$,

$$\bar{1}2\bar{1}211\bar{1}1,$$

is recovered from the pair

$$(\tilde{P}_{(2)} = \ 22\ , S^8_{(2)}).$$

Lemma 10.3 also establishes property (1) for $\mathcal{T}$: no j_k can equal an i_k, since all labels equal to j_k and preceding it also caused cancellations and therefore are already in the top row of $\mathcal{L}$ before j_k.

To verify that property (2) holds, we invoke Lemma 10.4. Suppose x, x' are two consecutive letters in the bottom row of $\mathcal{T}$, both of which have the same label z, say, in the top row, and suppose $x \le x'$ and both x, x' result in cancellations on Berele insertion, contributing labels i_p, i_{p+1} to the bottom row of the two-line array $\mathcal{L}$ (with the same top label z). Suppose the ordinary tableau prior to insertion of x was P_μ, and became $P_\mu{}'$ after inserting x, and $P_\mu{}''$ after inserting x'. (Thus $(i_{p+1} \to P_\mu{}'') = P_\mu{}'$; $i_p \to P_\mu{}' = P_\mu$). By Lemma 10.4 , the taquin path for x' ends in a weakly lower row than that of x, so that the bumping path of $(i_{p+1} \to P_\mu{}'')$ ends in a weakly lower row than that of $(i_p \to P_\mu{}') = (i_p \to i_{p+1} \to P_\mu{}'')$, and thus Lemma 3.15 of Chapter 1 immediately yields $i_{p+1} \geq i_p$.

We have shown how to construct, from the two-line array $\mathcal{T}$, a triple $(\tilde{P}_\mu, P_\mu, \mathcal{L})$. Because of properties (1) and (2) of $\mathcal{L}$, it is clear that $(P_\mu, \mathcal{L})$ encodes the up-down tableau resulting from the Berele algorithm applied to the lower row of $\mathcal{T}$; thus this up-down tableau is uniquely recoverable from $(P_\mu, \mathcal{L})$. Now it is simply a matter of reversing the Berele bijection, starting with the symplectic tableau P_μ, to retrieve the word forming the lower row of $\mathcal{T}$. The top row of $\mathcal{T}$, of course, is just the set of entries in P_μ and $\mathcal{L}$, arranged in increasing order. We note that the two-line array $\mathcal{T}$ recovered in this manner obeys the ordering defined above, as can be seen by reversing the preceding arguments.

The final step is to exhibit a bijection between two-line arrays $\mathcal{L}$ and ordinary tableaux with even-columned shapes. But this is precisely what the Burge correspondence of Chapter 1 achieves: $\mathcal{L}$ is the right type of two-line array because of (1) and (2). •

As a special case of this bijection, we can give a purely combinatorial proof of Corollary 12.8:

Corollary 15.5 *There is a bijection establishing*

$$\tilde{h}_\mu(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) = \sum_{\substack{\nu \\ \ell(\nu) \leq n}} sp_\nu \left(\sum_{\lambda \supseteq \nu} K_{\lambda\mu} \sum_{\beta \text{ even}} c^\lambda_{\nu,\beta}(n) \right)$$

Proof: If $\ell = \ell(\mu)$,

$$\tilde{h}_\mu = \tilde{h}_{\mu_1} \tilde{h}_{\mu_2} \ldots \tilde{h}_{\mu_\ell}$$

thus we may think of the left-hand side as counting Knuth two-line arrays

$$\mathcal{T} = \begin{pmatrix} 1 \ldots 1 & \ldots & \ell \ldots \ell \\ w^1 & \ldots & w^\ell \end{pmatrix}$$

where each w^i is an increasing word in $1, \bar{1}, \ldots, n, \bar{n}$ of length μ_i, which, when weighted under the usual symplectic scheme, contributes a monomial to the symmetric function $\tilde{h}_{\mu_i}$.

Applying the bijection of the previous theorem to such a two-line array, we obtain a triple $(\tilde{P}_\nu, P_\nu, P_\beta)$, where as usual $|\nu| + |\beta| = |\mu|$ (the length of the word), and β is a partition with even columns. The pertinent observation at this point is the following:

$$\{entries\ in\ P_\nu\} \cup \{entries\ in\ P_\beta\} = \{1^{\mu_1}, \ldots, \ell^{\mu_\ell}\}(\text{ as a multiset}).$$

Consequently, if we take the correspondence one step further by using Theorem 8.11, we have

$$\mathcal{T} \mapsto (\tilde{P}_\nu, P_\nu, P_\beta) \mapsto (\tilde{P}_\nu, P_\lambda, T_{\lambda/\nu}(\beta))$$

where P_λ is a column-strict tableau of shape $\lambda \vdash |\mu|$, and weight μ by the above remark, and $T_{\lambda/\nu}(\beta)$ is, as usual, an n-symplectic lattice permutation of shape λ/ν and weight β. We now remind the reader that the Kostka number $K_{\lambda\,\mu}$ counts precisely the number of column-strict tableaux P_λ of shape λ and weight μ. ●

By using the Littlewood-Richardson rule, we may further rewrite the Cauchy identity as

$$\prod_{i,j=1}^{n} (1 - t_i x_j)^{-1} (1 - t_i x_j{}^{-1})^{-1}$$

$$= \sum_{\substack{\mu \\ \ell(\mu) \le n}} \sum_{\lambda} sp_\mu(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) s_\lambda(t_1, \ldots, t_n) \left(\sum_{\beta' even} c^{\lambda}_{\mu,\beta} \right)$$

Now consider the product

$$\prod_{i=1}^{m} \prod_{j=1}^{n} (1 - t_i x_j)^{-1} (1 - t_i x_j{}^{-1})^{-1}$$

where m, the number of t's, is not necessarily equal to n. From our work in Chapter 3, it is clear that the above bijection, when applied to two-line arrays enumerating this product, produces triples $(\tilde{P}_\mu, P_\lambda, T_{\lambda/\mu}(\beta))$ where $(P_\lambda, T_{\lambda/\mu}(\beta))$ corresponds to an n-symplectic up-down tableau, with the added restriction: $\ell(\mu), \ell(\lambda) \le m$, the maximum number of distinct t's that can appear as labels. Thus for general m and n we have the identity

$$\prod_{i=1}^{m} \prod_{j=1}^{n} (1 - t_i x_j)^{-1} (1 - t_i x_j{}^{-1})^{-1}$$

$$= \sum_{\substack{\lambda \\ \ell(\lambda) \le m}} \sum_{\substack{\mu \subseteq \lambda \\ \ell(\mu) \le n}} sp_\mu(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) s_\lambda(t_1, \ldots, t_m) \left(\sum_{\beta' even} c^{\lambda}_{\mu,\beta}(n) \right) \tag{4.18}$$

In this form the identity immediately lends itself to a representation- theoretic interpretation:

If V and W are vector spaces over C of dimensions m and $2n$, respectively, the left side of the identity is precisely the character of the action of $Gl(V) \times Sp(W)$ on the symmetric algebra $S(V \otimes W)$, while the right side gives the decomposition of this action into irreducible modules, each irreducible representation $N^\lambda \otimes \tilde{N}^\mu$,(where N^λ is the irreducible representation of $Gl(m, C)$ corresponding to the partition $\lambda, \ell(\lambda) \leq m$, and $\tilde{N}^\mu$ is the irreducible representation of $Sp(2n, C)$ corresponding to the partition $\mu, \ell(\mu) \leq n$)) occurring with multiplicity

$$\sum_{\beta' even} c^\lambda_{\mu,\beta}(n).$$

If $m \leq n+1$, by Proposition 9.7, all up-down tableaux that arise in the bijection of Theorem 15.2 must be n-symplectic, (since $\ell(\lambda) \leq m \leq n+1$) and we have

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(1-t_ix_j)^{-1}(1-t_ix_j{}^{-1})^{-1}$$

$$= \sum_{\substack{\lambda \\ \ell(\lambda)\leq m}} \sum_{\substack{\mu\subseteq\lambda \\ \ell(\mu)\leq n}} sp_\mu(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) s_\lambda(t_1, \ldots, t_m) \left(\sum_{\beta' even} c^\lambda_{\mu,\beta} \right),$$

Note that various specialisations in the formula (4.8) are now possible. Equating coefficients of the square-free term $t_1 \ldots t_m$ in the t's we get, for $m \leq (n+1)$,

$$\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor}(-1)^k(x_1+x_1{}^{-1}+\cdots+x_n+x_n{}^{-1})^{m-2k}\binom{m}{2k}\binom{2k}{k}\frac{k!}{2^k}$$

$$= \sum_{\substack{\lambda\vdash m \\ \ell(\lambda)\leq n}} f^\lambda sp_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}), \tag{4.19}$$

and further substitution of $x_i = 1$, for all i, in this gives, for $m \leq (n+1)$,

$$\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor}(-1)^k(2n)^{m-2k}\binom{m}{2k}\binom{2k}{k}\frac{k!}{2^k}$$

$$= \sum_{\substack{\lambda\vdash m \\ \ell(\lambda)\leq n}} f^\lambda \dim \tilde{N}^\lambda \tag{4.20}$$

where $\tilde{N}^\lambda$ is the irreducible $Sp(2n,\mathrm{C})$ -module for λ.
Specialising to $x_i = 1$ in (4.19) gives, for $m \leq (n+1)$

$$\prod_{1\leq i<j\leq m}(1-t_it_j)\prod_{i=1}^{m}(1-t_i)^{-2n}$$
$$= \sum_{\substack{\lambda \\ \ell(\lambda)\leq n}} s_\lambda(t_1,\ldots,t_m)\dim \tilde{N}^\lambda, \tag{4.21}$$

a formula involving the dimensions of the irreducible representations of $Sp(2n,\mathrm{C})$.

16 A dual Berele algorithm

By using the formula of Theorem 12.1 in the dual Cauchy identity

$$\prod_{i,j}(1+x_iy_j) = \sum_\lambda s_\lambda(x)s_{\lambda'}(y)$$

of Chapter 1 (equation (1.15)) we get an analogous dual Cauchy identity for $Sp(2n,\mathrm{C})$:

$$\prod_{1\leq i,j\leq n}(1+t_ix_j)(1+t_ix_j{}^{-1}) = \sum_{\substack{\mu \\ \ell(\mu')\leq n}} sp_{\mu'}(x)s_\mu(t)\sum_{\gamma\, even} s_\gamma(t) \tag{4.22}$$

Using the identity (1.20) of Chapter 1, we obtain

Theorem 16.1 *(The dual Cauchy identity for $Sp(2n,\mathrm{C})$)*

$$\prod_{1\leq i\leq j\leq n}(1-t_it_j)\prod_{1\leq i,j\leq n}(1+t_ix_j)(1+t_ix_j{}^{-1})$$
$$= \sum_{\substack{\mu \\ \ell(\mu')\leq n}} sp_{\mu'}(x_1^{\pm 1},\ldots,x_n^{\pm 1})s_\mu(t_1,\ldots,t_n) \tag{4.23}$$

Equation (4.23) immediately suggests the existence of a dual to the Berele correspondence, in the spirit of the dual Knuth correspondence (cf. Theorem 3.26). Before proceeding to present this, we adopt the following
Notation. In this section we denote the conjugate of a tableau P, by P^t, and the conjugate of a shape μ by μ^t.
First, a definition:

Definition 16.2 *Call a tableau P_λ of shape λ* **row-symplectic** *if*

(1) P_λ is strictly increasing along rows, weakly increasing down columns

(2) every entry in column i of P_λ is larger than or equal to i.

Clearly P_λ is row-symplectic iff its conjugate P_λ^t is symplectic.

Theorem 16.3 *There is a bijection $\mathcal{B}*$ between words of length k in the alphabet $1, \bar{1}, \ldots, n, \bar{n}$ and pairs $(P_\mu *,\ S_\mu^k(n) *)$ where $P_\mu *$ is a row-symplectic tableau of shape μ, and $S_\mu^k(n) *$ is an up-down k-tableau of shape μ , $S_\mu^k(n) * = (\emptyset, \mu^1, \ldots, \mu^k = \mu)$, such that each μ^i has at most n columns.*

Proof: Let $w = w_1 \ldots w_k$ be a word of length k. The bijection is nothing but a column-insertion version of the Berele algorithm: Column-insert w $(w \to \emptyset)$ from the right end leftwards, with the bumping rule being as follows: an element x about to be inserted into column i bumps the first (highest) element in column i which is strictly larger than itself. (This ensures row-strictness). If at some stage of the column-insertion, an i is about to displace an $\bar{i}$ from column i into column $i+1$,

• replace the first (highest) $\bar{i}$ in column i with an i; remove the first (highest) i in column i (which must be in row 1), thereby creating a punctured tableau with a hole in position $(1, i)$;

• slide the hole in row 1, column i out to the right boundary of the tableau via jeu de taquin, leaving a row-symplectic tableau whose shape has one box less than the previous shape.

It is clear that continuing in this manner produces a row-symplectic tableau of some shape μ, where μ has $\leq n$ columns, and a k-sequence of shapes culminating in the shape μ, such that two consecutive shapes differ by exactly one box, and each shape has at most n columns.

It is not difficult to see that the process reverses, in complete analogy with the Berele algorithm. Example 16.4 should make this clear.•

We shall refer to this bijection as the **dual** Berele insertion algorithm, writing $(w \xrightarrow{\mathcal{B}*} \emptyset)$ to signify applying the insertion to the word w.

Example 16.4 We compute

$$(\bar{3}12311\bar{1}\bar{2} \xrightarrow{\mathcal{B}*} \emptyset):$$

Recall that this means column-insertion, beginning with the right-most letter and working leftwards.) We obtain the sequence of row-symplectic tableaux

$$\bar{2} \qquad 1\bar{2} \qquad \circ\bar{2} = \bar{2} \qquad 1\bar{2} \qquad \begin{matrix}1\bar{2}\\ 3\phantom{\bar{2}}\end{matrix} \qquad \begin{matrix}1\bar{2}\\ 23\end{matrix} \qquad \begin{matrix}1\circ\\ 13\end{matrix} = \begin{matrix}13\\ 1\end{matrix} \qquad \begin{matrix}13\\ 1\\ \bar{3}\end{matrix}$$

$$= \left(\begin{matrix}13\\ 1\\ \bar{3}\end{matrix}, \left(\blacksquare \quad \blacksquare\blacksquare \quad \blacksquare \quad \blacksquare\blacksquare \quad \begin{matrix}\blacksquare\blacksquare\\ \blacksquare\end{matrix} \quad \begin{matrix}\blacksquare\blacksquare\\ \blacksquare\blacksquare\end{matrix} \quad \begin{matrix}\blacksquare\blacksquare\\ \blacksquare\end{matrix} \quad \begin{matrix}\blacksquare\blacksquare\\ \blacksquare\\ \blacksquare\end{matrix} \right)\right)$$

In the reverse direction, in general we have a row-symplectic tableau T_{i+1} and a shape μ^i which differs from $sh(T_{i+1})$ by exactly one box; we need to retrieve the tableau T_i and the letter w_i such that $(w_i \xrightarrow{B*} T_i)$ yields T_{i+1}. As usual, the two cases are:

- $\mu^i \subset sh(T_{i+1})$, in which case simply column-remove the entry in T_{i+1} in the extra box (remembering to bump equal elements); the letter which is bumped out is w_i.
- $\mu^i \supset sh(T_{i+1})$; place an empty box (circle) in the missing position in T_{i+1}; slide this via *jeu de taquin* moves (preserving row-strictness) to the left-most column, say column j, in row 1, without violating the row-symplectic condition; now put a $\bar{j}$ in position $(1, j)$ and column-bump out a j, remembering to bump equal elements. Again, the bumped-out letter is w_i.

To illustrate, consider going from

$$\left(\begin{matrix}\blacksquare\blacksquare\\ \blacksquare\\ \\ 13\\ 1\end{matrix} \right)$$

to a row-symplectic tableau of shape

$$\begin{matrix}\blacksquare\blacksquare\\ \blacksquare\blacksquare\end{matrix}$$

We have

$$\begin{matrix}13\\ 1\end{matrix} \to \begin{matrix}13\\ 1\circ\end{matrix} \to \begin{matrix}1\circ\\ 13\end{matrix} \to \begin{matrix}1^{\swarrow 2}\,\bar{2}\\ 1\quad 3\end{matrix} \to \left(\begin{matrix}1\bar{2}\\ 1_{\swarrow}23\end{matrix} \right)$$

so w_i=1 in this case.

In order to apply this to obtain a bijective proof of (4.22), it is clear that we need an encoding of up-down tableaux similar in spirit to Lemma 8.7:

Lemma 16.5 *There is a another bijection $S*$ between up-down k-tableaux S_μ^k of shape μ and the set of all pairs $(Q_\mu, L*)$ such that Q_μ is a standard Young tableau of shape μ and $L*$ is a two-line array with entries*

$$\begin{pmatrix} j_1 & \dots & j_r \\ i_1 & \dots & i_r \end{pmatrix}$$

such that $j_1 < \dots < j_r$ and $j_k > i_k$, all $k = 1, \dots, r$, and $\{$entries in $Q_\mu\} \cup \{$entries in $L*\} = [k]$. The image of an up-down k-tableau $S_\mu^k = (\emptyset = \mu^0 \dots \mu^k = \mu)$ under S^* is as follows:

Build up a sequence of SYT's, one for each shape in S_μ^k, as in Lemma 8.7: as long as the shapes are increasing, follow the usual labelling of an SYT. In general, at step $j+1$, given that we have the SYT T_j associated with μ^j, and μ^{j+1} is one box larger than μ^j, T_{j+1} is simply the SYT obtained by adding a $j+1$ to T_j in the position of the added box (in the skew-shape μ^{j+1}/μ^j).

Now suppose μ^{j+1} is one box less than μ^j; let T_j be the SYT corresponding to μ^j. To get T^{j+1}, we do the following (see Example 16.6):

(1) bump out the extra entry of T_j (the one in the unique square of μ^j which is not a square of μ^{j+1}) by rows (i.e. inverse Schensted row insertion) to get a tableau T_{j+1} of shape μ^{j+1}, and a letter x. This means that by row-inserting x into T_{j+1} we would retrieve the previous bigger SYT $T_j(T_{j+1} = (T_j \leftarrow x))$, and hence its shape μ^j.

Example 16.6

$$\mu^j = \begin{matrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{matrix} \qquad \mu^{j+1} = \begin{matrix} \blacksquare & \blacksquare \\ \blacksquare & \end{matrix}$$

$$T_j = \begin{matrix} 1 & 2 \\ 3 & 5 \end{matrix} \qquad T_{j+1} = \begin{matrix} 1 & 5 \\ 3 & \end{matrix}$$

$$\left\{ \begin{matrix} j+1 \\ 2 \end{matrix} \right\}$$

(2) We record the fact that a removal occurred at step j by putting the pair (j, i) into a two-line array $L*$, with j on top. Note that since the i was bumped out at step j, it must have been inserted in an earlier step, so $i < j$.

We continue this process to the end of the sequence. Arranging the two-line array $L*$ so that the top row is in increasing order, we clearly end up with the requisite two-line array $L*$ and a SYT Q_μ of shape μ (the (final) shape of S_μ^k).

Proof: That this is a bijection follows exactly as in Lemma 8.7. Example 16.7 should make this clear.●

Example 16.7

$$S^{10}_{(2,1,1)} = \quad 1 \quad \begin{matrix}1\\2\end{matrix} \quad \begin{matrix}1\,3\\2\end{matrix} \quad \begin{matrix}1\,3\\2\\4\end{matrix} \quad \begin{matrix}1\\2\\4\end{matrix} \quad \begin{matrix}2\\4\end{matrix} \quad \begin{matrix}2\,7\\4\end{matrix} \quad \begin{matrix}2\,7\\4\,8\end{matrix} \quad \begin{matrix}2\,7\\4\,8\\9\end{matrix} \quad \begin{matrix}2\,8\\4\\9\end{matrix}$$

$$\left\{\begin{matrix}5\\3\end{matrix}\right\} \quad \left\{\begin{matrix}6\\1\end{matrix}\right\} \qquad\qquad \left\{\begin{matrix}10\\7\end{matrix}\right\}$$

$$\mapsto \left(\left\{\begin{matrix}5\,6\,10\\3\,1\,7\end{matrix}\right\}, \quad \begin{matrix}2\,8\\4\\9\end{matrix} \right)$$

By using the dual Burge correspondence on the two-line array $L*$, we can convert it into an even-rowed SYT Q_γ of shape γ, as illustrated in Example 16.8 below.

Example 16.8 In the preceding example, the array

$$L* = \left\{\begin{matrix}5\,6\,10\\3\,1\,7\end{matrix}\right\} \longleftrightarrow \begin{pmatrix}1\ 7\\3\ 10\\5\\6\end{pmatrix}^t \longleftrightarrow \begin{matrix}1\ 3\ 5\ 6\\7\ 10\end{matrix}$$

Consequently we have

Lemma 16.9 *There is a bijection between up-down tableaux S^k_μ of length k and shape μ, and pairs (Q_μ, Q_γ) of standard tableaux where Q_μ has shape μ and Q_γ has shape γ, γ with even rows, and $k = |\mu| + |\gamma|$.*

We now state an analogue of Theorem 8.11:

Theorem 16.10 *Let P_μ, P_ν be column-strict tableaux of shapes μ, ν, respectively. Let $wc(P_\nu)$ be the word of P_ν read column-wise, i.e. $wc(P_\nu)$ is the word obtained by reading P_ν by columns, from bottom to top and left to right. (Thus $\emptyset \leftarrow wc(P_\nu)$ gives P_ν). Now row-insert $wc(P_\nu)$ into P_μ, producing a tableau P_λ of shape $\lambda \supseteq \mu$, and a filling of the skew-shape $\lambda \supseteq \mu$ obtained as follows: (see eg) each time an entry is added to the tableau P_μ as a result of row-inserting a letter x of $wc(P_\nu)$, add the corresponding square to the Ferrers diagram of μ, and label the square with the* **column-number** *of x in P_ν.*

Then this filling of λ/μ, when read from bottom to top *and* left to right, *is in fact a lattice permutation $T_{\lambda^t/\mu^t}(\nu^t)$ of weight ν^t which fits the skew-shape λ^t/μ^t.*

Conversely given the pair $(P_\lambda\ ,\ T_{\lambda^t/\mu^t}(\nu^t))$, the lattice permutation $T_{\lambda^t/\mu^t}(\nu^t)$ uniquely specifies an order in which to row-unbump elements on the boundary of P_λ so as to retrieve a pair of tableaux $(P_\mu\ ,P_\nu\ )$.

Hence the above process establishes a bijection between pairs of column-strict tableaux $(P_\mu\ ,\ P_\nu\ )$ and pairs $(P_\lambda\ ,T_{\lambda^t/\mu^t}(\nu^t))$, where P_λ is a column-strict tableau of shape $\lambda \supseteq \mu$, and the second component is a lattice permutation which fits the skew-shape λ^t/μ^t and has weight ν^t.

Proof: We spare ourselves what is essentially a repetition of the tedious details in the proof of Theorem 8.11. •

Finally we have the analogue of Theorem 8.14:

Theorem 16.11 *There is a bijection between up-down k-tableaux S_μ^k of shape μ and pairs $(Q_\lambda\ ,\ T_{\lambda^t/\mu^t}(\gamma^t))$ where $\lambda \vdash k, \lambda \supseteq \mu$, $\gamma \vdash (k - |\mu|)$ and γ has all rows of even length and $T_{\lambda^t/\mu^t}(\gamma^t)$ is a lattice permutation of weight γ^t which fits the skew-shape λ^t/μ^t.*

Definition 16.12 *Call an up-down k-tableau* dual n-symplectic *if no shape in the sequence has a row of length larger than n.*

These are precisely the up-down tableaux appearing in the dual algorithm for $Sp(2n, \mathbf{C})$ as stated in Theorem 16.3. Recall the work done in section 8, characterising n-symplectic tableaux.

In similar fashion, we have

Theorem 16.13 *The pair $(Q_\lambda\ ,\ T_{\lambda^t/\mu^t}(\gamma^t))$ as in Theorem 16.11 represents an n-symplectic tableau iff, in the representation of $T_{\lambda^t/\mu^t}(\gamma^t)$ as in Theorem 16.10, (i.e. so that the skew-tableau of shape λ/μ, when read from left to right and bottom to top, gives a lattice permutation of weight γ^t), the $(2i+1)$'s do not appear to the right of column $(n+i)$, for any occurrence of $(2i+1)$ in the lattice permutation , $i \geq 0$.*

We also have

Theorem 16.14 *Let w be a k-word in $1, \bar{1}, \ldots, n, \bar{n}$, and suppose the dual Knuth correspondence applied to w (i.e., $(w \stackrel{K*}{\rightarrow} \emptyset)$ where we bump equal elements to produce a row-strict tableau) yields the pair $(P_\nu\ ,Q_\nu\ )$ where $\lambda \vdash k$, P_ν is row-strict and Q_ν is a standard Young tableau . Suppose the dual Berele correspondence gives $w \longleftrightarrow (\tilde{P}_\mu\ ^*,\ Q_\lambda\ ,\ T_{\lambda^t/\mu^t}(\gamma^t))$. Then*

$$\nu = \lambda \text{ and } Q_\nu = Q_\lambda .$$

We now present the bijective proof of the dual Cauchy identity:

Theorem 16.15 *(The dual Cauchy identity for $Sp(2n, \mathbb{C})$) There is a bijection establishing*

$$\prod_{1\le i,j\le n} (1 + t_i x_j)(1 + t_i x_j{}^{-1})$$
$$= \sum_{\substack{\mu \\ \ell(\mu')\le n}} sp_{\mu'}(x_1^{\pm 1}, \dots, x_n^{\pm 1}) s_\mu(t_1, \dots, t_n) \left(\sum_{\gamma even} s_\gamma(t_1, \dots, t_n) \right) \tag{4.24}$$

Proof: The right side of the identity counts triples $(\tilde{P}_\mu *, P_\mu *, P_\gamma)$ where $\tilde{P}_\mu *$ is a row-symplectic tableau (so that its conjugate is a symplectic tableau of shape μ^t), P_μ is a column-strict tableau of shape μ and P_γ is a column-strict tableau of shape γ where γ has even rows.

We may enumerate the left-hand side as two-line arrays $\mathcal{T}*$ consisting of vertically arranged pairs

$$\mathcal{T}* = \begin{pmatrix} t_{i_1} & \dots & t_{i_k} \\ y_{i_1} & \dots & y_{i_k} \end{pmatrix}$$

where the y's are in $1, \bar{1}, \dots, n, \bar{n}$; clearly each pair occurs at most once. We choose to write $\mathcal{T}*$ by ordering the top row of t's in decreasing order from left to right, and then placing the corresponding y's so that if $t_{i_j} = t_{i_{j+1}}$ then $y_{i_j} < y_{i_{j+1}}$. For instance, the word $t_1 x_1{}^{-1} t_1 x_2{}^{-1} t_2 x_1 t_3 x_3 t_4 x_2{}^{-1} t_4 x_2 t_5 x_2{}^{-1}$ is represented by

$$\mathcal{T}* = \begin{pmatrix} 5 & 4 & 4 & 3 & 2 & 1 & 1 \\ \bar{2} & 2 & \bar{2} & 3 & 1 & \bar{1} & \bar{2} \end{pmatrix}$$

Now we apply the dual Berele correspondence of Theorem 16.3 to the bottom row of x's and x^{-1}'s, replacing the resulting up-down tableau by a sequence of column-strict tableaux and a two-line array

$$L* = \begin{pmatrix} \dot{j}_1 & \dots & \dot{j}_r \\ \dot{i}_1 & \dots & \dot{i}_r \end{pmatrix}$$

constructed as in Lemma 16.5, and arranged so that the top row is increasing from left to right. Note that the only difference from the situation of Lemma 16.5 is that instead of using distinct labels we now have possibly repeated labels.

Example 16.16 *Consider the two-line array*

$$\mathcal{T}* = \begin{array}{cccccccccccc} 5&5&5&4&4&3&3&2&2&2&1&1\\ 1&\bar{1}&2&\bar{1}&2&\bar{1}&3&1&\bar{2}&\bar{4}&\bar{1}&\bar{2}\end{array}$$

Working from right-to-left, dual Berele-insertion on the bottom row gives:

$$\begin{array}{cccccccccccc}
\begin{matrix}5\\1\end{matrix} & \begin{matrix}5\\\bar{1}\end{matrix} & \begin{matrix}5\\2\end{matrix} & \begin{matrix}4\\\bar{1}\end{matrix} & \begin{matrix}4\\2\end{matrix} & \begin{matrix}3\\\bar{1}\end{matrix} & \begin{matrix}3\\3\end{matrix} & \begin{matrix}2\\1\end{matrix} & \begin{matrix}2\\\bar{2}\end{matrix} & \begin{matrix}2\\\bar{4}\end{matrix} & \begin{matrix}1\\\bar{1}\end{matrix} & \begin{matrix}1\\\bar{2}\end{matrix} \\[2ex]
\begin{matrix}\bar{1}3\bar{4}\\\bar{1}\end{matrix} & \begin{matrix}\bar{1}3\bar{4}\\\bar{1}\\\bar{1}\end{matrix} & \begin{matrix}\bar{1}\bar{2}\bar{4}\\\bar{1}3\\2\end{matrix} & \begin{matrix}\bar{1}\bar{2}\bar{4}\\\bar{1}\\3\end{matrix} & \begin{matrix}\bar{1}\bar{2}\bar{4}\\2\bar{2}\\3\end{matrix} & \begin{matrix}\bar{1}\bar{2}\bar{4}\\\bar{2}\\3\end{matrix} & \begin{matrix}\bar{2}\bar{4}\\\bar{2}\\3\end{matrix} & \begin{matrix}\bar{2}\bar{4}\\\bar{2}\end{matrix} & \begin{matrix}\bar{1}\bar{2}\\\bar{2}\bar{4}\end{matrix} & \begin{matrix}\bar{1}\bar{2}\\\bar{4}\end{matrix} & \bar{1}\bar{2} & \bar{2} \\[2ex]
\begin{matrix}225\\3\end{matrix} & \begin{matrix}125\\2\\3\end{matrix} & \begin{matrix}124\\25\\3\end{matrix} & \begin{matrix}124\\2\\3\end{matrix} & \begin{matrix}123\\24\\3\end{matrix} & \begin{matrix}123\\2\\3\end{matrix} & \begin{matrix}12\\2\\3\end{matrix} & \begin{matrix}12\\2\end{matrix} & \begin{matrix}11\\22\end{matrix} & \begin{matrix}11\\2\end{matrix} & 11 & 1 \\[2ex]
\left\{\begin{matrix}5\\1\end{matrix}\right\} & \left\{\begin{matrix}5\\4\end{matrix}\right\} & & \left\{\begin{matrix}4\\3\end{matrix}\right\} & & & & \left\{\begin{matrix}2\\1\end{matrix}\right\} & & & &
\end{array}$$

$$\mapsto \left(\begin{matrix}\bar{1}3\bar{4}\\\bar{1}\end{matrix},\ \begin{matrix}225\\3\end{matrix},\ \left\{\begin{matrix}2455\\1341\end{matrix}\right\} \right)$$

$$\mapsto \left(\begin{matrix}\bar{1}3\bar{4}\\\bar{1}\end{matrix},\ \begin{matrix}225\\3\end{matrix},\ \begin{pmatrix}134\\145\\2\\5\end{pmatrix}^{t} \right)$$

$$\mapsto \left(\begin{matrix}\bar{1}3\bar{4}\\\bar{1}\end{matrix},\ \begin{matrix}225\\3\end{matrix},\ \begin{matrix}1125\\34\\45\end{matrix} \right)$$

Let us first observe that the procedure of Lemma 16.5 will indeed replace the sequence of shapes with column-strict tableaux: since the possibility of equal elements one above the other can only arise if, in the course of building up the row-symplectic tableau, we inserted first a y, then a $y' \geq y$ (take conjugates in Lemma 3.15). But then the associated labels t, t' are unequal, by our ordering of $\mathcal{T}*$.

We now claim that the two-line array $L*$ (whose entries are a subset of the top row of $\mathcal{T}*$) is such that

(1) if an entry y with label t causes a cancellation upon dual Berele insertion, so do all subsequent y's with the same label t. (Observe that "subsequent" now means "to the left" in $\mathcal{T}*$). (2) $L*$ is in "dual Burge" lexicographic order: if $j_p = j_{p+1}$ then $i_p \geq i_{p+1}$; note that $j_p \leq j_{p+1}$ by construction.

As in the proof of Theorem 15.2, to show (1) we need:

Lemma 16.17 *Let $\tilde{P}_\mu*$ be a row-symplectic tableau of shape μ, let x, x' be letters in $1, \bar{1}, \ldots, n, \bar{n}$ such that $x < x'$. Suppose dual Berele insertion of x' into $\tilde{P}_\mu*$ causes a cancellation. Then so does the subsequent insertion of x.*

Proof: Suppose not. Since dual Berele insertion of x' into $\tilde{P}_\mu*$ causes a cancellation, this means that when replaced by ordinary Schensted dual column-insertion, the bumping path of x' ends in row 1. But (by taking conjugates in Lemma 3.15) subsequent insertion of the strictly smaller x must produce a bumping path that ends strictly above that of x', contradiction. •

Now (1) follows easily, since a y' following a y with the same label t is, by our ordering of $\mathcal{T}*$, strictly less than y.

To show (2) we need:

Lemma 16.18 *Let $\tilde{P}_\mu*$ be a row-symplectic tableau of shape μ, and let x, x' be such that $x < x'$ and dual Berele insertion of x' into $\tilde{P}_\mu*$ causes a cancellation, as does the subsequent insertion of x. Then the taquin path of x ends in the same row as, or lower than, that of x'.*

Proof: Simply take conjugates in the proof of Lemma 10.4. •

To see why this gives us (2), observe that if $j_p = j_{p+1}$, and the label j_p was for an element x', and j_{p+1} was for x, then since insertion of x' preceded insertion of x, we must have $x < x'$ according to the ordering chosen for $\mathcal{T}*$. Also both x and x' caused cancellations (j_p, j_{p+1} both appear in the top row of $L*$), so Lemma above applies. But this means that if j_p resulted in row-unbumping an i_p from the column-strict tableau $P_\mu*$ constructed up to this point, and then j_{p+1} resulted in row-unbumping i_{p+1}, then re-inserting i_{p+1} row-wise recovers the end of the taquin path of x, and subsequently re-inserting i_p reproduces the end of the taquin path of x'. Consequently the bumping path of i_{p+1} ends below or in the same row as that of i_p, which implies $i_p + 1 \geq i_p$.

It remains to observe that properties (1) and (2) of $L*$ ensure that the labels corresponding to cancellations can be uniquely identified from $L*$, and hence, using P_μ, the up-down tableau is recoverable as before, thereby leading to the retrieval of the word forming the bottom row of $\mathcal{T}*$. The top row consists of the entries in P_μ and $L*$, written in the pre-determined order. (Note that reversing the above arguments shows that the recovered two-line array $\mathcal{T}*$ will obey the correct ordering relations).

Example 16.19 Continuing with Example 16.8, given the triple

$$\left(\begin{matrix} \bar{1}\,3\,\bar{4} \\ \bar{1} \end{matrix}\;,\;\; \begin{matrix} 2\,2\,5 \\ 3 \end{matrix}\;,\;\; \begin{matrix} 1\,1\,2\,5 \\ 3\,4 \\ 4\,5 \end{matrix} \right),$$

we know the top row of $\mathcal{T}$ and we know precisely at which of these labels the removal pairs corresponding to the even-rowed tableau are located. Thus, working backwards (i.e., **left-to-right**) from the tableau

$$\begin{matrix} 2\,2\,5 \\ 3 \end{matrix}\;,$$

we get the up-down tableau whose shapes are those of the sequence below:

5	5	5	4	4	3	3	2	2	2	1	1
$\{{5 \atop 1}\}$	$\{{5 \atop 4}\}$		$\{{4 \atop 3}\}$				$\{{2 \atop 1}\}$				
225 3	125 2 3	124 25 3	124 2 3	123 24 3	123 2 3	12 2 3	12 2	11 22	11 2	11	1
(←1)	(←4)	remove right-most 5	(←3)	remove right-most 4	remove right-most 3	remove right-most 5	(←1)	remove right-most 2	remove right-most 2	remove right-most 1	

Finally, because of (1) and (2), $L*$ is precisely the type of two-line array to which the dual Burge correspondence applies, producing an even-columned row-strict tableau, and thus (by taking conjugates) an even-rowed column-strict tableau. ●

Using Theorem 16.10 and Theorem 16.13 in the dual Cauchy identity (4.24), for general m and n we have the identity

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(1+t_ix_j)v(1+t_ix_j{}^{-1})$$
$$= \sum_{\substack{\lambda \\ \ell(\lambda)\leq m}} \sum_{\substack{\mu\subseteq\lambda \\ \ell(\mu')\leq n}} sp_{\mu'}(x_1^{\pm 1},\ldots,x_n^{\pm 1})s_\lambda(t_1,\ldots,t_m)\left(\sum_{\gamma even} c^{\lambda'}_{\mu',\gamma'}(n)\right) \tag{4.25}$$

In this form the identity lends itself to a representation-theoretic interpretation:

If V and W are vector spaces over C of dimensions m and $2n$, respectively, the left side of the identity is precisely the character of the action of $Gl(V)\times Sp(W)$ on the exterior algebra $\bigwedge(V\otimes W)$, while the right side gives the decomposition of this action into irreducible modules, each irreducible representation $N^\lambda\otimes\tilde{N}^{\mu'}$ (where N^λ is the irreducible representation of $Gl(m,\mathbf{C})$ corresponding to the partition λ, $\ell(\lambda)\leq m$, and $\tilde{N}^{\mu'}$ is the irreducible representation of $Sp(2n,\mathbf{C})$ corresponding to the partition μ', $\ell(\mu')\leq n$)), occurring with multiplicity

$$\sum_{\gamma even} c^{\lambda'}_{\mu',\gamma'}(n).$$

17 An inner product on $\tilde{\Lambda}_n$

Define an inner product on $\tilde{\Lambda}_n$ by making the basis $\{sp_\mu(x_1^{\pm 1},\ldots,x_n^{\pm 1})\}$ orthonormal, i.e., set

$$<< sp_\mu, sp_\nu >>= \delta_{\mu,\nu}$$

The mapping

$$\phi : sp_\lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1}) \mapsto s_\lambda(t_1,\ldots,t_n)$$

defines a linear transformation of vector spaces in $\mathrm{Hom}(\tilde{\Lambda}_n,\Lambda_n)$. The following result characterises dual bases in $\tilde{\Lambda}_n$:(cf.[Macd, p.34, (4.6)] for the Schur function analogue)

Proposition 17.1 *The following are equivalent:*

(1) The bases $\{f_\lambda\}_{\ell(\lambda)\leq n},\{g_\mu\}_{\ell(\mu)\leq n}$ are dual with respect to the inner product defined above.

(2)

$$\prod_{1\le i<j\le n}(1-t_it_j)\prod_{i,j=1}^{n}(1-t_ix_j)^{-1}(1-t_ix_j^{-1})^{-1}$$
$$=\sum_{\substack{\lambda\\ \ell(\lambda)\le n}} f_\lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1})\phi(g_\lambda)(t_1\ldots t_n)$$

Proof: Let

$$f_\lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1}) = \sum_{\substack{\nu\\ \ell(\nu)\le n}} a_\nu{}^\lambda sp_\nu(x);$$
$$g_\lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1}) = \sum_{\substack{\nu\\ \ell(\nu)\le n}} b_\nu{}^\lambda sp_\nu(x) \quad ;$$

Since the $\{sp_\lambda(x)\}$ are orthonormal,

$$<< f_\lambda, g_\mu >>= \sum_{\substack{\nu\\ \ell(\nu)\le n}} a_\nu{}^\lambda\, b_\nu{}^\mu$$

which is clearly also the coefficient of $sp_\lambda(x)s_\mu(t)$ in the right-hand side of (2). But by the Cauchy identity (4.16), (2) is equivalent to

$$\sum_{\substack{\lambda\\ \ell(\lambda)\le n}} f_\lambda(x)(\phi(g_\lambda))(t) = \sum_{\substack{\lambda\\ \ell(\lambda)\le n}} sp_\lambda(x)s_\lambda(t),$$

and hence to

$$\sum_{\substack{\nu\\ \ell(\nu)\le n}} a_\nu{}^\lambda\, b_\nu{}^\mu = \delta_{\lambda,\mu},$$

which is (1). ●

Open Problems

We conclude with a discussion of some open problems suggested by this thesis.

PROBLEM 1: Recall our "essentially" combinatorial proof of equation (12.1) in Theorem 11.1: For all partitions λ of length at most $2n$,

$$s_\lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1}) = \sum_{\substack{\mu\subseteq\lambda \\ \ell(\mu)\le n}} sp_\mu(x_1^{\pm 1},\ldots,x_n^{\pm 1})\left(\sum_{\beta' \, even} c^\lambda_{\mu,\beta}(n)\right)$$

We would like to find a direct bijection ϕ between the set of column-strict tableaux P_λ of shape λ and the set of pairs $(\tilde{P}_\mu\,,\,T_{\lambda/\mu}(\beta))$, where $\mu(\subseteq\lambda)$ is a partition of length at most n, $\beta\vdash|\lambda|-|\mu|$ and has even columns, and the lattice permutation $T_{\lambda/\mu}(\beta)$ is n-symplectic. Theorem 11.2 shows that such a bijection must coincide with the result of Berele insertion on the word of the tableau P_λ. ϕ will probably involve a selective *jeu de taquin* process on P_λ.

PROBLEM 2: Another open problem already mentioned in Chapter 4 is that of combinatorially reconciling our extension (equation(4.9)) of the definition of sp_λ with King's modification rule (4.12).

PROBLEM 3: Is there a "nice" combinatorial rule for multiplying two symplectic Schur functions? The simplicity of the Pieri rule (Section 13) suggests that the answer is in the affirmative.

PROBLEM 4: Find a combinatorial proof of the Weyl character formula (4.10). Note that this is related to Problem 2.

PROBLEM 5: Is there a naturally occurring, useful notion of symplectic skew-Schur functions? Recalling [Macd, p.40 (5.4)] the determinantal expression for the ordinary skew-Schur functions in terms of the homogeneous symmetric functions, perhaps an elucidation of Problem 4 might lead to the formulation of such a notion. After considering a possible definition for a "skew-symplectic" tableau, we are inclined to think that a symplectic Schur function of skew-shape λ/μ would depend on both λ and μ.

PROBLEM 6: The characterisation of the inner product of Section 17 deserves to be further investigated; we hope to exploit it to discover new identities.

PROBLEM 7: Do all the work in this thesis for the orthogonal groups! The immediate open problem here is to find an insertion algorithm analogous to Berele's scheme for the orthogonal groups. We point out that at least two definitions of "orthogonal" tableaux exists. The first is due to R.C. King; the second we were able to obtain by converting a formulation in terms of Gelfand patterns due to Robert Proctor.

The insertion scheme should give the right decomposition when restricting from the general linear group to its orthogonal subgroup, i.e., the analogue of Theorem 10.8 must hold. Clearly, as a combinatorial tool, the power of such an algorithm cannot be overstated.

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