WAVE EFFECTS OF A FREE SURFACE PIERCING HYDROFOIL

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ABSTRACT

A lifting surface theory is developed for vertically oriented hydrofoils piercing a fluid free surface. The formulation, a singular integral equation, is shown to reduce to known limiting forms for the cases of zero and infinite Froude number. Conditions restricting the mathematical nature of acceptable foil loadings are given for the general case for finite nonzero aspect ratio and Froude number. Two simple foil loadings are numerically investigated, demonstrating the feasibility of a proposed technique for analyzing the dynamics of arbitrarily shaped free surface piercing hydrofoils.

Thesis Supervisor: Professor Jerome H. Milgram
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SYMBOLS

x	coordinate positive upstream
y	coordinate positive downward
z	coordinate positive to port
(x_o, y_o, z_o)	corresponding dummy variables
r	distance from a field point to a singularity in the flow
r'	distance from a field point to an image singularity about the plane at the free surface
C_o	dimensional chord
S_o	dimensional span
g	acceleration of gravity
ρ	the density of water
$AR = S_o/C_o$	Aspect Ratio
U	Free stream velocity in the negative x direction
$\nu = g/U^2$	A parameter of dimension, 1/length
$F = gC_o/U^2$	Inverse of the Froude number squared
$F_n = 1/\sqrt{F}$	Froude number
$\Phi = -Ux + \phi$	total velocity potential
ϕ	perturbation potential
ϕ_s	point source potential
ϕ_d	point dipole potential
$\vec{V}$	total velocity vector
u, v, w	perturbation velocities
$z_{m\ell}$	the dimensionless (fractional chord) foil mean line (surface) distance from the plane $z = 0$
η	the free surface elevation, positive downward

$C_p = \dfrac{p-p_\infty}{\frac{1}{2}\rho U^2}$	pressure coefficient
$\Gamma(x,y)$	The circulation from one point on the foil's surface to the adjacent point on the opposite surface. This also is equal to the dipole sheet strength at (x,y).
$\Gamma(y)$	circulation about a lifting line
$\Delta\phi(x,y)$	potential jump across a foil
$\gamma(x,y)$	bound vorticity density (per unit area)
$\Delta p(x,y)$	pressure jump across a foil
ϕ_{S_o}, ϕ_{C_o}	angles associated with the definitions of the Glauert series loadings
Im	The imaginary part of a complex equation
Re	The real part of a complex equation
I	An arbitrary integral's value
γ'	Euler's constant = .5772156649...
$X(x_o)$	Chordwise foil loading
$Y(y_o)$	Spanwise foil loading
$\overline{X}$	Fourier transform of the chordwise loading
$\overline{Y}$	Laplace transform of the spanwise loading
$-\!\!\!\!\!\!\int$	An integration where the Cauchy Principal value is to be taken
$\underset{\rightarrow}{\int}$	An integration whose path is to be taken below a pole into the lower half of the complex plane
$\times\!\!\!\!\!\!\int$	An integration where the "finite part" is to be taken according to Mangler's "recipe" as set forth in Ashley and Landahl (10) (pp. 130-131)

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INTRODUCTION

A surface piercing hydrofoil may be any of a variety of winglike bodies moving with constant velocity at a fluid free surface and oriented with part of the wing above and part below the free surface. A partial list of examples includes ship's rudders, streamlined vertical struts used to support horizontal hydrofoils and rigid sidewalls of air cushion vehicles. All of the above are, to some extent, thin bodies which experience large transverse forces when yawed a small angle from straight ahead motion.

Were the fluid free surface eliminated by considering the space above it occupied by fluid and the body's mirror image, the physical situation could be mathematically modeled using techniques of classical subsonic wing theory or slender body theory. Very specifically, the present theory for surface piercing hydrofoils is an extension of lifting surface theory to include the effect of a fluid free surface. As in classical lifting surface theory, the analysis for lifting and thickness effects can be treated independently and only lifting effects are considered herein. It is interesting to note that the corresponding thickness effects in free surface flows have been understood since 1898 when J. H. Michell wrote his famous paper on the wave resistance of a thin ship (20).*

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*Numbers in parentheses refer to references listed in the bibliography.

Mathematically, the analysis makes many of the same assumptions as classical lifting surface theory: the body has no thickness and the angle between it and the flow is everywhere small. In addition, the perturbation to the flow by the body must satisfy certain conditions restricting the nature of the waves associated with the flow. These conditions are known collectively as the free surface boundary conditions.

As in lifting surface theory, the ultimate goal of the analysis is to relate the shape of a surface piercing foil to the forces that act upon it when it moves across a free surface. With the present theory the shape of a surface piercing hydrofoil can be obtained directly in terms of the load distribution it is required to achieve. Reversal of the order of solution is a complex but relatively straightforward task, and is often referred to as the inverse problem. The method of the inverse problem in the present theory is simply to superpose solutions to the direct problem in such a way as to generate the desired hydrofoil shape.

Previous known studies of surface piercing hydrofoils have been made by Newman (1), Milgram (2) and Daoud (22). The Newman paper is by far the most comprehensive. Daoud has developed a numerical proceedure and has extensively studied lifting surfaces of relatively low aspect ratio.*

*After this final draft was completed, another paper on this subject by Ismail (24) was brought to the attention of the author. It reports on analytic and experimental work on vertical foils. The experiments included cases in which the foil pierced the free surface, but the analytic work addressed itself primarily to foils which are completely below the free surface.

I STATEMENT OF THE PROBLEM

A surface piercing hydrofoil of rectangular planform is assumed to move with constant speed in the positive x direction. The plane y = 0, is that of an undisturbed free surface and the positive y direction is taken downward in the direction of gravitational acceleration. The transverse coordinate z is taken positive to port. The surface piercing hydrofoil is situated near the plane, z = 0, (because the foil is thin, of small camber and angle of attack) and, establishing the coordinates moving with the foil, the leading edge is taken to be at $x = C_o$ (the chord length), the trailing edge at x = 0 and the lower tip at $y = S_o$ (the span). These conventions are shown on Figure 1.

Letting the lengths be nondimensionalized using the chord length, C_o, and defining the aspect ratio, AR, to be S_o/C_o, the analysis is simplified by a reduction of the number of parameters. Of particular interest is the parameter, gC_o/U^2, which is the inverse of the Froude number squared. This parameter, referred to as F, is central to the analysis. Figure 2 shows the nondimensional formulation. For the purposes of simplicity, the dimensionless variables are assigned the same symbol as their nondimensional counterpart. Further use of the dimensional form is anticipated only when developing the lifting line formulation. The free surface height η is measured in the positive y direction.

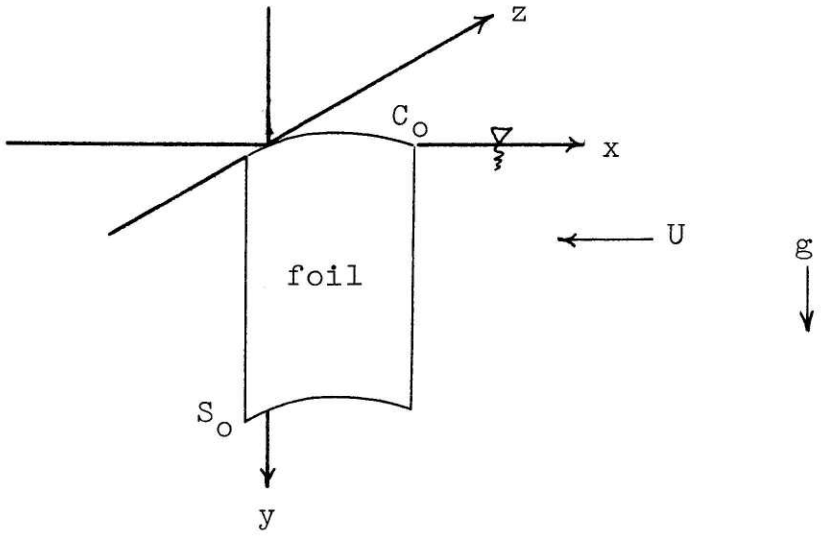


Figure 1 Dimensional coordinate system

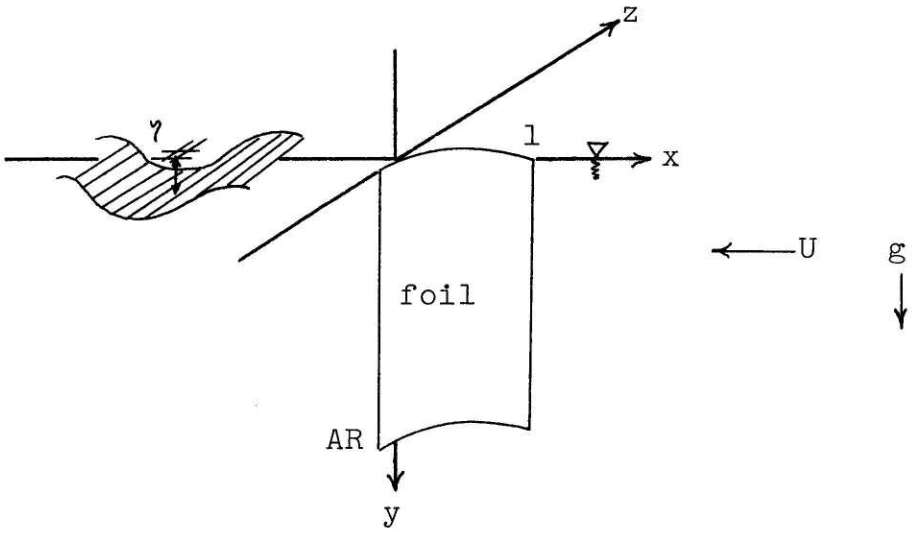


Figure 2 Nondimensional coordinate system

II FUNDAMENTAL EQUATIONS

Both the direct (known loading, unknown shape) and the inverse (unknown loading, known shape) analysis require determination of flow details near the surface piercing foil in terms of other known conditions of the flow at the foil. The inverse analysis specifies that the flow be tangent to the surface of the foil, whereas the direct problem specifies the pressure difference across the foil planform. In both cases other physically motivated restrictive conditions are required of the solution. These are that (i) the flow velocities be finite at the trailing edge (the Kutta condition); (ii) that fluid particles on free surface move with the free surface; (iii) that the pressure at the free surface remain constant; (iv) that any waves generated radiate from the body and decay in amplitude at a rate such that wave energy is conserved. Finally, the flow must satisfy the conservation of mass equation for an inviscid incompressible fluid.

Assuming a velocity potential Φ represents the flow field and that $\vec{V} = \nabla\Phi$, the above conditions can be mathematically formalized for the flow region, $y \geq 0$:

Letting ϕ be a perturbation potential such that $\Phi = -Ux + \phi$, the following conditions are imposed on the solution for ϕ.

(i) The body boundary condition requires that the flow be tangent to the body surface and is written

$$\nabla\Phi \cdot \vec{n} = 0 \quad \text{on } z = z_{m\ell}(x,y) \tag{2.1.1}$$

where $\vec{n}$ is a unit vector normal to the body surface and $z_{m\ell}$ is the location of the body surface (here the mean line of the body).

(ii) The Kutta Condition from airfoil theory requires that pressures be continuous across the wing surface at the trailing edge and, in so doing, affords a method of uniquely setting the circulation and lift for the wing. Requiring only that $\nabla\Phi$ be bounded at the trailing edge suffices to satisfy this condition.

(iii) The Kinematic boundary condition on the free surface requires that a fluid particle on the free surface stay on the free surface. It may be compactly stated as

$$\frac{D(\eta - y)}{Dt} = 0 \quad \text{on} \quad y = \eta(x,y) \tag{2.1.3}$$

(iv) Bernoulli's equation, evaluated on the free surface, is known as the dynamic free surface boundary condition and is written

$$\frac{1}{2}\rho(\nabla\Phi \cdot \nabla\Phi) - \rho g \eta = \text{constant on } y = \eta(x,y) \tag{2.1.4}$$

(v) The governing equation for the flow is Laplace's equation

$$\nabla^2\Phi = 0 \tag{2.1.5}$$

and it applies at all points within the flow $y > 0$ that are not on a boundary of the flow.

Solution of the exact problem as specified cannot be determined using existing analytic techniques. Specifically, the difficulties are the nonlinearity of the free surface conditions and the imposition of all the boundary conditions on surfaces which are not known a priori, but are rather part of the solution with the exception of the known location of the foil (in the inverse problem).

To render the problem tractable, the free surface condition is linearized and all boundary conditions are satisfied on known surfaces assumed arbitrarily close to the unknown surfaces actually specified. Reduction of the complexity of the boundary conditions is based on the assumption that the perturbation velocities $\{\phi_x, \phi_y, \phi_z\} = \{u, v, w\}$ are small (second order) compared to U, the free stream velocity (first order).

Bypassing the usual steps associated with obtaining a formulation for the perturbation potential, ϕ, the results of this linearization are:

(i) The flow tangency condition is specified on the plane z = 0, and the velocity transverse to the flow is

$$\phi_z = -U \frac{\partial z_{mc}}{\partial x} \quad \text{on} \quad z = 0 \qquad (2.2.1)$$

(ii) The Kutta condition requires that the pressure be constant across the foil at its trailing edge. The linearized pressure is given by Bernoulli's equation.

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2} = +2\,\phi_x/U \qquad (2.2.2)$$

The Kutta condition then states that

$\phi_x = 0$ at the trailing edge $x = 0$

(iii) The Kinematic free surface condition is:

$$\frac{D}{Dt}(\eta - y) = \frac{\partial \eta}{\partial t} + (u - U)\frac{\partial \eta}{\partial x} + w\frac{\partial \eta}{\partial z} - v = 0$$

$$v = \frac{\partial \eta}{\partial t} + (u - U)\frac{\partial \eta}{\partial x} + w\frac{\partial \eta}{\partial z} \quad \text{on } y = 0 \qquad (2.2.3)$$

As the solution is independent of time $\eta_t = 0$.

The terms $u\eta_x$ and $w\eta_z$ are second order and dropped leaving:

$$v = \frac{\partial \phi}{\partial y} = -U\frac{\partial \eta}{\partial x} \quad \text{on} \quad y = 0$$

(iv) The dynamic free surface condition; assuming $p = 0$ on $y = 0$, and expanding the $\nabla\Phi \cdot \nabla\Phi$ term is:

$$\frac{1}{2}\left(\Phi_x^2 + \Phi_y^2 + \Phi_z^2\right) - g\eta = 0$$

$$\frac{1}{2}\left((\phi_x - U)^2 + \phi_y^2 + \phi_z^2\right) - g\eta = 0 \quad \text{on } y = 0 \qquad (2.2.4)$$

keeping only first order terms

$$U\phi_x + g\eta = 0 \quad \text{on } y = 0$$

(v) As before, Laplace's equation:

$$\nabla^2\phi = 0 \qquad (2.2.5)$$

must be satisfied at all interior points in the flow. A combined form of the free surface condition may be formed by eliminating η to give:

$$\phi_y + \frac{v^2}{g}\phi_{xx} = 0 \qquad \text{on } y=0 \qquad (2.3)$$

III FORMULATION OF THE SOLUTION IN TERMS OF A DIPOLE SHEET

The development of equations relating to the shape of a foil and the hydrodynamic load acting upon it closely parallels the development of lifting surface theory. As in lifting surface theory, an integral equation is established using Green's theorem which states that a potential (here a velocity potential) may be determined everywhere within and on the boundaries of a fluid domain in terms of the potential and its normal derivative on the boundaries. The development of a theory for a lifting surface of zero thickness is found in many texts on classical incompressible subsonic aerodynamics and is not included in the present paper. (An excellent exposition may be found in Ashley and Landahl (10).) The resulting equation is:

$$\phi = -\frac{1}{2\pi} \iint_{\text{wing \& wake}} \Delta\phi \frac{\partial \phi_s}{\partial z_o} \, dx_o \, dy_o \qquad (3.1)$$

where ϕ is the perturbation velocity potential for the flow on the foil, $\Delta\phi = \phi(x,y,0^+) - \phi(x,y,0^-)$ is the difference in value of the potential on adjacent surfaces of the foil, and ϕ_s is the velocity potential for a point source singularity. Noting that $\partial\phi_s/\partial z_o$ is the potential for a dipole whose axis is transverse to the free stream flow, equation 3.1 may be viewed as the convolution of a potential jump distribution $\Delta\phi$ with the "impulse response" of a point potential jump, the dipole. The potentials for a source and a dipole at

(x_o, y_o, z_o) in an infinite fluid are respectively:

$$\Phi_s = \frac{1}{r} \qquad \Phi_d = \frac{z - z_o}{r^3} \tag{3.2}$$

where

$$r = \sqrt{(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2}$$

When the free surface condition is imposed in the formulation, it is satisfied by utilizing an elemental point dipole potential satisfying the free surface conditions. Many distributions of these dipoles will satisfy the free surface condition as well; a separate section is devoted to developing restrictions which delineate exceptional cases.

Two equivalent formulations for a source in the presence of a free surface are known. In both cases a function, regular in the fluid domain, is added to the potential for a dipole in an infinite fluid, with the two functions collectively satisfying the free surface condition. The first is what may be called the odd image formulation in which the function regular in the fluid domain includes an image sink of equal strength located symmetrically above the plane of the free surface. This potential may be found in many treatises on free surface flows including Wehausen (19) and Peters and Stoker (18). In the present work, a second form is used and may be referred to as the even image formulation as it contains an image source above the plane of the free surface. Following Newman (1) and Wehausen (19)

(equation 13.36), this source potential is:

$$\Phi_s = \frac{1}{r} + \frac{1}{r'} - \frac{4}{\pi} \mathrm{Re} \int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{\lambda}{\lambda - \nu \sec^2\theta} \cdot \cos[\lambda(z-z_o)\sin\theta]\, e^{-\lambda[y+y_o+i(x-x_o)\cos\theta]}\, d\lambda\, d\theta \qquad (3.3)$$

where $\nu = g/U^2$ and x, y, z are temporarily dimensional.

$$r = \sqrt{(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2}$$

$$r' = \sqrt{(x-x_o)^2 + (y+y_o)^2 + (z-z_o)^2}$$

The source potential given by Wehausen (19) is for the even formulation. In Appendix F this form is shown equivalent to that of equation 3.3.

Nondimensionalizing all lengths with the foil's chord C_o and the wave number λ with ν, ϕ_s may be rewritten:

$$\Phi_s = \frac{1}{C_o}\left\{ \frac{1}{r} + \frac{1}{r'} - \frac{4}{\pi} \mathrm{Re} \int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{KF}{K - \sec^2\theta} \cdot \cos[KF(z-z_o)\sin\theta]\, e^{-KF[y+y_o+i(x-x_o)\cos\theta]}\, dK\, d\theta \right\} \qquad (3.4)$$

where $F = gC_o/U^2$.

Finally, the corresponding potential for a dipole with axis transverse to the free stream and situated on the plane $z_o = 0$, is:

$$\phi_d = \frac{1}{C_o}\left\{ \frac{z}{r^3} + \frac{z}{r'^3} - \frac{4}{\pi}\, Re \int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{(KF)^2}{K - \sec^2\theta} \cdot \right.$$

$$\left. \sin(KFz\sin\theta)\, e^{-KF[y+y_o+i(x-x_o)\cos\theta]}\, dK\, d\theta \right\} \tag{3.5}$$

Now, a potential analogous to equation 3.1 for an infinite fluid, satisfying the linearized conditions (equation 2.2) for a surface piercing hydrofoil may be written:

$$\phi(x,y,z) = -\frac{1}{2\pi} \int_0^{AR} \int_0^{1} \Delta\phi(x_o,y_o)\, \phi_d(x,y,z,x_o,y_o,F)\, dx_o\, dy_o \tag{3.6}$$

The functional relation between foil loading and shape may be determined by noting that the induced velocity normal to the foil is $\partial\phi/\partial y$, and that the shape of the foil is:

$$z_{me} = z_{me}(1,y) + \int_1^x \frac{\partial\phi}{\partial z}\, dx_o \tag{3.7}$$

The loading acting on a foil is specified by $\Delta\phi$ since the circulation from a point on one foil surface to the adjacent point on the other is:

$$\Gamma = \int_{x,y,0^-}^{x,y,0^+} \frac{\partial\phi}{\partial q} \cdot dq = -\Delta\phi \tag{3.8.1}$$

$$\Gamma = -\int_{x,y,0^+}^{1,y,0^+} \phi_{x_o}\, dx_o - \int_{1,y,0^-}^{x,y,0^+} \phi_{x_o}\, dx_o = -2\int_{x,y,0^+}^{1,y,0^+} \phi_{x_o}\, dx_o$$

(3.8.2)

Since the perturbation velocity ϕ_x is odd in z, the contribution from each side $z = 0^{\pm}$ is equal.

With $\Gamma(x,y)$ known, the local pressure jump can be determined by noting that the local bound vortex density $\gamma(x,y) = -\partial\Gamma/\partial x$. Furthermore, the force on a differential element is $\rho U\gamma(x,y)dxdy$, and the pressure is $\rho U\gamma(x,y)$. For clarity, these results are restated. A positive potential jump $\Delta\phi$ across the foil results when $\phi(x,y,0^+) > \phi(x,y,0^-)$. Integrating $\vec{V} \cdot \vec{d\ell}$ on a path from $(x,y,0^+)$ to $(x,y,0^-)$, the net circulation for positive $\Delta\phi$ is negative (ϕ_{x_o} is positive on $(x,y,0^+)$ in equation 3.8) and the side force is positive owing to the negative free stream ($-U < 0$); the result that the total lift is equal to $\rho U\Gamma_T$, where Γ_T is the total circulation, is negative for $\Delta\phi > 0$, and can be computed by integrating either $\gamma(x,y)$ over the foil or $\Gamma(0,y)$ across the span. Summarizing,

$$\Delta p = p(x,y,0^+) - p(x,y,0^-) = 2\rho U\, \phi_x(x,y,0^+)$$

$$2\,\phi_x(x,y,0^+) = \gamma(x,y) = -\frac{\partial \Gamma}{\partial x}$$

$$\Delta\phi(x,y) = -\Gamma(x,y)$$

(3.9)

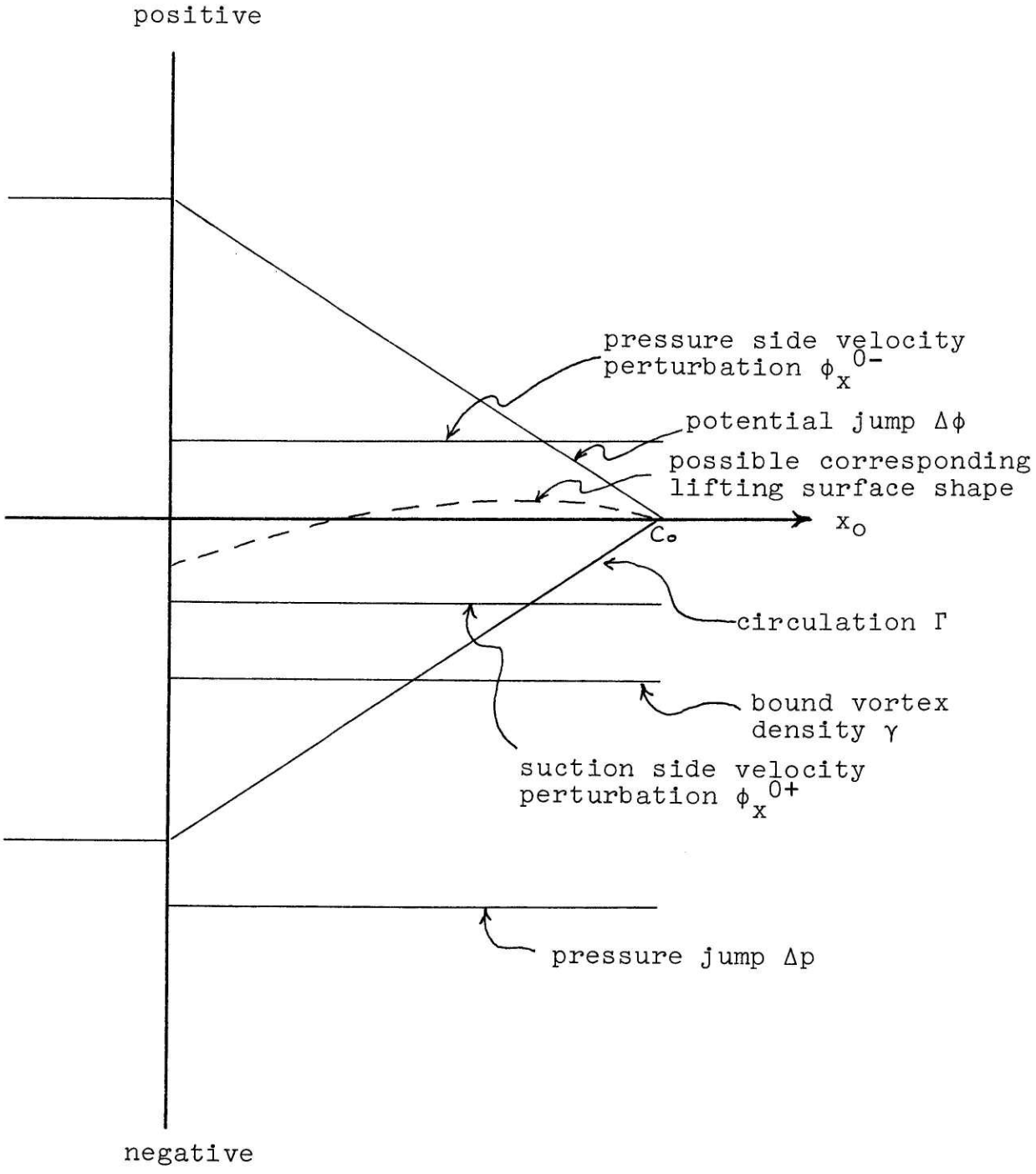


Figure 3 Conventions of positive dipole moment, potential jump, vorticity, side force, circulation and pressure jump

These conventions are illustrated in Figure 3 for the hypothetical case of uniform chordwise loading.

At this point the only modification of the formulation made is to use it to define velocities normal to the foil. Equation 3.6 for ϕ is differentiated with respect to z, and then z is set equal to zero, giving an equation for the "downwash"; a carry over term from airfoil theory where normal velocities on wings are generally in a downward direction. Taking the derivative of ϕ in the z direction within the integral (in this case four integrals) is not always permissible and care must be taken to properly interpret the resulting expression in terms of generalized functions. Formally then:

$$\omega = \Phi_z\Big|_{z=0} = -\frac{1}{2\pi c_o}\int_0^{AR}\int_{-\infty}^{1} \phi(x_o,y_o)\,\frac{\partial \phi_d}{\partial z}\Big|_{z=0}\,dx_o\,dy_o \qquad (3.10)$$

where

$$\frac{\partial \phi_d}{\partial z}\Big|_{z=0} = \left\{\frac{1}{r^3}+\frac{1}{r'^3}-\frac{4}{\pi}\,Re\int_0^{\frac{\pi}{2}}\int_0^{\infty}\frac{K^3F^3}{K-\sec^2\theta}\,\sin^2\theta\cdot e^{-KF[y+y_o+i(x-x_o)\cos\theta]}\,dK\,d\theta\right\}$$

Again following lifting surface theory, the integral equation 3.10, for a surface piercing foil may be reduced to a more compact form by modifying the kernel and reducing the integration limits.

IV THE HORSESHOE VORTEX INTEGRAL EQUATION FORMULATION

The integral equation 3.10 for a dipole sheet is restated with $\Delta\phi(x_o,y_o)$, the local dipole sheet strength, replaced with $\Gamma(x_o,y_o)$ to clarify following developments.

$$\phi(x,y) = \frac{1}{2\pi}\int_0^{AR}\int_0^1 \Gamma(x_o,y_o)\,\phi_d(x,y,x_o,y_o,F)\,dx_o dy_o \tag{4.1}$$

Certain physical arguments may be used to compact the region over which the Green's function ϕ_d must be integrated. As a first step, pressure jumps across the wake ($x < 0$, $0 \le y \le AR$) may be ruled out on the physical ground that an unrestrained pressure discontinuity would result in infinite fluid particle accelerations. A streamwise increase (or decrease) of $\Gamma(x_o,y_o)$ in the wake corresponds to a local increase (or decrease) in the (vertical) vorticity in the flow which must be zero not to produce an unacceptable pressure jump. It can therefore be stated that $\partial\Gamma/\partial x_o$ is zero in the wake region $-\infty < x_o \le 0$. (A possible exception on the plane of the free surface exists and is discussed in Appendix E.) With this result, the integration bounds in equation 4.1 can be reduced to include simply the foil's planform ($0 \le y_o \le AR$, $0 \le x_o \le 1$). This is accomplished by integrating by parts once in the x_o direction; details of the derivation are set

forth in Appendix A. The resulting expression for the downwash is:

$$w(x,y) = \frac{1}{2\pi c_o} \int_0^{AR} \!\!\!\!\!\!\times \int_0^{1} \!\!\!\!\!\!\times \gamma(x_o,y_o)\, G(x,y,x_o,y_o,F)\, dx_o dy_o$$

$$G = -\frac{1}{(y-y_o)^2}\left\{1 - \frac{x-x_o}{\sqrt{(x-x_o)^2+(y-y_o)^2}}\right\}$$

$$-\frac{1}{(y+y_o)^2}\left\{1 - \frac{x-x_o}{\sqrt{(x-x_o)^2+(y+y_o)^2}}\right\}$$

$$+\frac{4}{\pi}\,\mathrm{Im}\int_0^{\frac{\pi}{2}}\!\!\!\!\!\!\times \int_0^{\infty} \frac{K^2F^2}{K-\sec^2\theta}\tan\theta\,\sin\theta\cdot$$

$$e^{-KF[y+y_o+i(x-x_o)\cos\theta]}\, dK d\theta \qquad (4.2)$$

where $\gamma(x_o,y_o)$ is the bound vortex sheet strength and is equal to $-\partial\Gamma(x_o,y_o)/\partial x_o$, and the integral sign with the X signifies that the integral be evaluated using Mangler's "recipe" as set forth in Ashley and Landahl (10), pp. 130-131.

Newman (1) has carried the analysis directly from the dipole sheet form to one which can be obtained by partially integrating the above horseshoe formulation over the y_o direction. The nonkernel part of the integrand is then $\partial^2\Gamma/\partial x_o \partial y_o$ and the kernel is found to represent a point increase in free vorticity. This form is analogous to that of Robinson and Lauerman (6) for a lifting surface in an infinite fluid. It does not, however, yield any further reduction of the integral bounds and slightly increases the complexity of the kernel.

It is included in Appendix C for completeness; no future use of it is made herein.

Before proceeding toward a solution of equation 4.2 for a surface piercing foil of arbitrary loading or shape, three limiting cases are discussed to place the general case in perspective. They are the limits of zero and infinite Froude number and infinite aspect ratio (lifting line).

V THE ZERO FROUDE NUMBER ($F \to \infty$) LIMIT

In his thesis, Letcher (17) makes an observation helpful to those not thoroughly imbued with the concept of Froude numbers. It is that when visualizing a limiting Froude number case, it is often best to imagine that variations of the acceleration of gravity cause the effect rather than variations of the length scale or velocity which are generally more physically apparent and with which one might associate nonFroude number effects. (For example, Reynolds number effects with velocity variation.)

The cases of zero and infinite F can be analyzed with decreasing degrees of mathematical rigor depending upon how early in the formulation the limit is taken. For $F_n \to 0$, g can be imagined to tend to infinity with U and the length scale C_o fixed. From the combined free surface boundary condition it can be seen that as $g \to \infty$ with U held constant, ϕ_y must tend to zero since ϕ_{xx} must remain finite as physical arguments require. Thus, the anticipated limit as $F_n \to 0$ is one satisfying a boundary condition $\phi_y = 0$ on $y = 0$ and physical corresponds to a foil of span $2S_o$ in an infinite fluid. In equation 4.2 this corresponds to the double integral portion of the kernel vanishing as $F_n \to 0$. Letting $KF = \lambda$, this expression becomes

$$-\frac{4}{\pi}\,\mathrm{Im}\int_0^{\frac{\pi}{2}}\int_0^{\infty}\frac{\lambda^2\tan\theta\sin\theta}{\lambda - F\sec^2\theta}\cdot e^{-\lambda\left[y+y_o+i(x-x_o)\cos\theta\right]}\,d\lambda\,d\theta \qquad (5.1)$$

which tends to zero for $F \to \infty$ ($F_n \to 0$). The integral equation 4.2 thus reduces to the familiar integral equation for a lifting surface of span 2AR and may be written:

$$w(x,y) = -\frac{1}{2\pi c_o} \int_{-AR}^{AR} \int_{0}^{1} \gamma(x_o, y_o) \frac{1}{(y-y_o)^2} \left\{ 1 - \frac{x - x_o}{\sqrt{(x-x_o)^2 + (y-y_o)^2}} \right\} dx_o \, dy_o \qquad (5.2)$$

VI THE INFINITE FROUDE NUMBER LIMIT ($F \to 0$)

Following the same lines of thought as in developing the zero Froude number limit, the infinite Froude number limit is considered to be the result of the gravitational acceleration tending to zero. From the combined free surface boundary condition one finds that the new boundary condition on $y = 0$ is that $\phi_{xx} = 0$. Since $\phi \to 0$ for all z at $x = +\infty$, this condition is equivalent to the condition that $\phi = 0$ everywhere on the plane $y = 0$. Considering the formulation for the potential rather than the downwash, it can be seen that the proper limiting form of the Green's function is one in which the nonintegral terms form an odd pair whose contributions to the potential exactly cancel on the surface $y = 0$. Had the present formulation been founded on the Green's function for a source with its odd image, the present task would be to demonstrate that, after having been manipulated into the Green's function for the downwash due to a horseshoe vortex, the double integral part of kernel vanishes in the limit of $F_n \to \infty$.

Conversely, it must now be shown that the double integral term is such that when its infinite Froude number limit is added to the even image, the resulting kernel contains the desired negative image. Using the form of the double integral term in equation 4.2, and separating the residue I_r and Cauchy Principle I_{pv} value contributions:

$$I_R = -\frac{4}{\pi} I_m \int_0^{\frac{\pi}{2}} \pi i F^2 \sec^2\theta \tan\theta \sin\theta \cdot e^{-F\sec^2\theta[(y+y_o)+i(x-x_o)\cos\theta]} d\theta$$

$$= -4F^2 \int_0^{\frac{\pi}{2}} \sec^2\theta \tan\theta \sin\theta \cos[F\sec\theta(x-x_o)] \cdot e^{-F\sec^2\theta(y+y_o)} d\theta \qquad (6.1)$$

Here the order of performing the integration first and then taking the limit is important. For any finite value of $F(y + y_o)$, the integral is convergent as $\theta \to \pi/2$. The subsequent limit as $F \to 0$ then properly assigns a value of zero to the residue contribution. The Cauchy Principle value contribution is:

$$I_{PV} = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \int_0^{\infty} \lambda \tan\theta \sin\theta \sin[\lambda(x-x_o)\cos\theta] \cdot e^{-\lambda(y+y_o)} d\lambda d\theta \qquad (6.2)$$

where the pole has been absorbed by a second order zero at the origin. In Appendix C this integral is shown to equal:

$$I_{PV} = -\frac{2}{(y+y_o)^2}\left\{1 - \frac{x-x_o}{\sqrt{(x-x_o)^2 + (y+y_o)^2}}\right\}$$

The formulation for infinite Froude number may thus be written:

$$w(x,y) = -\frac{1}{2\pi C_o}\int_0^{AR}\int_0^{1}\gamma(x_o,y_o)\left\{\frac{1}{(y-y_o)^2}\left[1-\frac{x-x_o}{\sqrt{(x-x_o)^2+(y-y_o)^2}}\right]-\frac{1}{(y+y_o)^2}\left[1-\frac{x-x_o}{\sqrt{(x-x_o)^2+(y+y_o)^2}}\right]\right\}dx_o dy_o \qquad (6.3)$$

In passing it should be noted that the vanishing contribution from the residue portion of the integral in the kernel might have been anticipated because that part of the integral represents a three dimensional radiated surface wave. Passing to the limit of infinite Froude number, there no longer remains a mechanism for the transmission of gravity waves as to do so there must be a means of storing potential energy. Since in the limit of infinite Froude number, a wave cannot propagate in the medium, it is apparent that no contribution should be made by the residue term.

VII LIFTING LINE THEORY

Lifting line theory is an approximate theory for wings whose chords are small compared to their spans. Physically, the simplifying assumption states that the induced normal velocity on the wing is dominated by the trailing vorticity in the wake and that the effect of bound vorticity can be neglected. The corresponding mathematical statement is that the bound vorticity $\gamma(x_o,y_o)$ is distributed in the chordwise direction as a Dirac delta function. Thus,

$$\gamma(x_o,y_o) = \delta(x_o)\,\Gamma(y_o) \tag{7.1}$$

When substituted into the integral equation 4.2 and integrated with respect to x_o, the delta function selects the value of the integrand at $x_o = 0$. The result is a simplified integral equation for the downwash on the lifting line x = y = 0:

$$w(0,y) = -\frac{1}{2\pi c_o}\int_0^{AR}\Gamma'(y_o)\left\{\frac{1}{(y+y_o)^2} + \frac{1}{(y-y_o)^2}\right. \tag{7.2}$$

$$\left. -\frac{4}{\pi}\,\mathrm{Im}\int_0^{\frac{\pi}{2}}\int_0^{\infty}\frac{(KF)^2}{K-\sec^2\theta}\sin\theta\tan\theta\, e^{-KF(y+y_o)}\,dK\,d\theta\right\}dy_o$$

Because the chord has been set to zero for the lifting line limit, modifications must be made to the nondimensionalization of the equations. Returning to dimensional space variables, taking F as a parameter of dimension (1/length)

and AR as the dimensional span gives the proper interpretation. These all follow if the chord length C_o is taken as dimensionless unity wherever it appears.

The only imaginary contribution from the double integral in the kernel arises from the residue and is:

$$I = -4F^2 \int_0^{\frac{\pi}{2}} \tan^2\theta \sec^3\theta \; e^{-F\sec^2\theta\,(y+y_o)} \, d\theta \qquad (7.3)$$

This can be integrated analytically to give:

$$I = -\frac{F}{y+y_o} e^{-\frac{F}{2}(y+y_o)} K_1\left(\frac{F}{2}(y+y_o)\right) \qquad (7.4)$$

where K_1 is the modified Bessel function of the first kind, order one. The complete equation for a surface piercing lifting line may then be written as:

$$w(o,y) = -\frac{1}{2\pi C_o} \int_0^{AR} \Gamma'(y_o) \left[\frac{1}{(y+y_o)^2} + \frac{1}{(y-y_o)^2} - \frac{F}{y+y_o} e^{-\frac{F}{2}(y+y_o)} K_1\left(\frac{F}{2}(y+y_o)\right)\right] dy_o \qquad (7.5)$$

For small argument q, K_1 may be approximated by

$$K_1(q) \cong \frac{1}{q} + \frac{q}{2} \ln\frac{q}{2} + \cdots$$

and the contribution to the downwash by the loading in the vicinity of $y_o = 0$ is:

$$w(o,y) = -\frac{1}{2\pi c_o}\int_o^{\epsilon} \Gamma(y_o)\left\{\frac{1}{(y-y_o)^2} - \frac{1}{(y+y_o)^2} - \frac{F^2}{4}\ln\left(\frac{F}{4}(y+y_o)\right)\right\} dy_o \qquad (7.6)$$

where ϵ is some small positive number such that for $0 \leq y_o < \epsilon$ the approximation of K_1 is valid.

For the downwash on the lifting line near the free surface due to loading near the free surface, the integral equation indicates that locally the behavior is identical to an odd image or infinite Froude number formulation.

A primary goal of the present research has been to determine necessary conditions on the spanwise (vertical) load distribution which suffice to preclude infinite perturbations of the velocity on the free surface. The semblance between proximity of the free surface and infinite Froude number is central to the resolution of this matter, which is discussed in a later section.

For the integral equation 7.6 above, conditions on the circulation $\Gamma(y_o)$ necessary to insure bounded downwashes on the lifting line are the same as have been obtained by Betz and Peterson (21) for a lifting line and its reversed circulation image. The same result has been obtained by an alternate method in Appendix D and is that:

$$\Gamma(y_o) \sim O(y_o \ln y_o) \text{ as } y_o \to o$$

VIII PERMISSIBLE LOADINGS

With the eventual goal of solving either the direct or inverse problem by means of linearly superposing downwashes associated with assumed loading modes, it is essential to understand precisely the conditions imposed on loadings necessary to insure finite downwash on the free surface. Acceptable loadings must satisfy conditions arising from both the infinite fluid horseshoe vortex (and its image) portion of the kernel and the double integral contribution to the kernel.

First, consider the contribution of the algebraic terms in the kernel corresponding to the zero Froude number, double aspect ratio formulation. By considering the downwash far downstream in the wake, it can be shown that the loading at the plane of the free surface must have zero first derivative with respect to y_o. Noting first that loading below the free surface plane is evenly mirrored above the plane and second, that the derivative of an even function is an odd function, it can be seen that if the derivative of the vertical (spanwise) loading does not tend to zero as $y \to 0^+$, there will be a discontinuity in the derivative of the spanwise loading across the plane of the free surface. Since the strength of the trailing vortex sheet is equal to the derivative of the spanwise loading, it also will be discontinuous at the plane of the free surface if the vertical load's derivative is nonzero. It can easily be seen that the discontinuity of this

vortex sheet strength precludes the usual principal value integral cancellation at the discontinuity and gives rise to a logarithmic singularity in the downwash

$$w(y) = \int_{-AR}^{AR} \frac{\partial \Gamma}{\partial y_o} \cdot \frac{1}{y - y_o} \, dy_o \qquad (8.1)$$

if at the surface $y = 0$, $\partial\Gamma/\partial y_o$ is discontinuous. The contribution to the downwash from the immediate ε vicinity of the free surface is

$$w(o) = -2 \int_o^{\epsilon} \frac{\partial \Gamma}{\partial y_o} \frac{1}{y - y_o} \, dy_o = -2 \frac{\partial \Gamma(o^+)}{\partial y_o} \ln(y)$$

$$\text{if } \frac{\partial \Gamma(o^+)}{\partial y_o} = -\frac{\partial \Gamma(o^-)}{\partial y_o} \qquad (8.2)$$

In summary, the portion of the kernel consisting of the horseshoe and its positive image imposes a restriction of zero derivative of the loading (circulation or bound vortex sheet strength) at the plane of the free surface.

The contribution to the downwash from the double integral term in the kernel cannot be properly assessed far downstream (as was done for the other terms) because three dimensional wave radiation effects preclude the downstream reduction to two dimension (Trefftz plane) flow. (Far downstream the downwash from the double integral term in fact vanishes due to cancellation effects of the highly oscillatory integrand.)

Proper assessment of the double integral's contribution can be made by reversing the orders of integration to first integrate over the foil's planform. Defining the Laplace transform of the spanwise loading and the Fourier transform of the chordwise loading as,

$$\bar{Y}(\lambda) = \int_0^{AR} Y(y_o)\, e^{-\lambda y_o}\, dy_o \qquad (8.3.1)$$

$$\bar{X}(p\lambda) = \int_0^{1} X(x_o)\, e^{i\lambda p x_o}\, dx_o \qquad (8.3.2)$$

$$\gamma(x_o, y_o) = Y(y_o)\, X(x_o) \qquad (8.3.3)$$

the double integral contribution in equation 4.2 (with $\lambda = KF$ and $p = \cos\theta$) becomes

$$w_w(x,y) = -\frac{2}{\pi^2}\,\frac{I_m}{C_o}\int_0^{1}\int_0^{\infty} \bar{X}(p\lambda)\,\bar{Y}(\lambda)\,\frac{\lambda^2}{p^2\lambda - F}\cdot p\sqrt{1-p^2}\; e^{-\lambda[y+ixp]}\, d\lambda\, dp \qquad (8.4)$$

For the downwash to remain finite for all finite F and y (and in the limit as $y \to 0$) a sufficient condition is that the product $\bar{X}\bar{Y}$ be $o(1/\lambda^2)$ as $\lambda \to \infty$. This may be shown by considering the absolute convergence of the λ integral in equation 8.4. Considering only the contribution to the λ integral from some large $\lambda = B >> F/p^2$ to $\lambda = \infty$;

$$I = \int_B^{\infty} \bar{X}(p\lambda)\,\bar{Y}(\lambda)\,\frac{\lambda^2}{p^2\lambda - F}\, e^{-\lambda[y+ixp]}\, d\lambda \qquad (8.5.1)$$

$$I \approx \frac{1}{p^2}\int_B^\infty \bar{X}(p\lambda)\bar{Y}(\lambda)\,\lambda\, e^{-\lambda[y+ix_p]}\,d\lambda \qquad (8.5.2)$$

If the product $\bar{X}\bar{Y}$ tends to zero as $\lambda^{-\beta}$ for $\lambda \to \infty$, the integral will converge only if $\beta > 2$ such that the integrand will be $0(\lambda^{-\beta+1})$ and its improper integral will decay like $\lambda^{-\beta+2}$. Since for any $\beta > 2$, the integral (8.5.2) converges, the loading $\bar{X}(p\lambda)\bar{Y}(\lambda)$ must be $o(1/\lambda^2)$. In the delimiting case, $\bar{X}(p\lambda)\bar{Y}(\lambda) \sim 0(1/\lambda^2)$ as $\lambda \to \infty$, the integrand tends to zero as $1/\lambda$ as $\lambda \to \infty$ and a logarithmic singular in the downwash at the free surface results.

It seems highly unlikely that the independent logarithmic singularities in the downwash at the free surface from the double integral term and from the infinite fluid and image terms cancel because the double integral term's contribution must decay far downstream for any nonzero finite F whereas the infinite fluid terms tend toward two dimensional flow. In addition, the magnitude of the double integral term contribution is dependent on F while the other contributions are not, and exact cancellation of the logarithmic singularities cannot be generally expected.

In summary, for surface piercing hydrofoils of arbitrary finite aspect ratio operating at any finite nonzero Froude number, two restrictions on load distributions are imposed. First, the derivative of the load at the free surface must be

zero; second, the product of the loading transforms $\bar{X}(p\lambda)\bar{Y}(\lambda)$ must be $o(1/\lambda^2)$ as $\lambda \to \infty$.

It will be recalled that the previously obtained formulation for the lifting line case indicated that the downwash on the lifting line would remain finite if the loading was $0(y_o \ell n y_o)$ as $y_o \to 0$. This clearly violates the conditions imposed on lifting surfaces since not only is the vertical derivative of the load infinite, but product of the loading transforms:

$$\bar{X}(p\lambda) = \int_0^1 \delta(x_o) e^{i\lambda p x_o} dx_o = 1 \qquad (8.6.1)$$

$$\bar{Y}(\lambda) = \int_0^\infty y_o \ln y_o e^{-\lambda y_o} dy_o = \frac{1}{\lambda^2}\left[2 - \ln(\gamma'\lambda)\right] \qquad (8.6.2)$$

γ' = Euler's constant

decays only as $0(\ell n\lambda/\lambda^2)$ as $\lambda \to \infty$. (Here the spanwise transform has been integrated to infinity for simplicity as the result is sought only for large λ, and as $\lambda \to \infty$ only the behavior of the loading in the immediate vicinity of the free surface affects the transform.)

This apparent contradiction may be resolved by noting that in the development for a lifting line, only the downwash on the lifting line itself is examined. It was found that on the lifting line the only double integral term contribution to the downwash arises from the residue and is a radiated wave effect. The downwash resulting from the near field (Cauchy principle value integral) contribution, although zero on the lifting line is unbounded as the free surface is approached elsewhere on the track z = 0 for some loadings.

In lifting surface theory this infinity of the downwash could not be tolerated because the foil occupied a finite portion of the track on the free surface and an infinite downwash (excepting integrable infinities at discrete points across the chord) would imply an unrealizable foil shape. For lifting line theory, however, this difficulty does not arise because the theory does not (and is not expected to) satisfy body boundary conditions on a foil. It is rather a theory for an outer region far from the foil, whose inner limit gives the downwash on a high aspect ratio foil induced by trailing vorticity in its wake. Whether the infinite velocity induced on the free surface track $z = 0$ extends over the entire free surface is not known. Furthermore, the acceptability of this infinity in terms of the amount of kinetic energy per unit distance downstream (and therefore work done by the foil) remains to be investigated.

In summary, the lifting line high aspect ratio theory need not require finite transverse perturbation velocities to assure attainable foil shapes, but must not permit infinities which lead to infinite kinetic energy per unit distance downstream in the wake as this corresponds to a physically unacceptable infinite drag force. It is apparent that the conditions imposed in lifting surfaces theory (the transform of the load being $o(1/\lambda^2)$ as $\lambda \to \infty$ and the derivative of the vertical load being zero) are stronger than

required for lifting line theory, but further investigations are needed to ascertain whether vertical loadings as singular as $y\ell ny$ are permissible.

IX LOADING MODES

A standard lifting surface theory technique for the inverse problem is to assume a series of loading modes, calculate the downwash from each mode at selected points on the planform, and solve a linear matrix equation for the strengths of the various modes required to produce the known downwash (the boundary condition on the wing). Experience has shown that a series of chordwise and spanwise modes known as the Glauert Series rapidly converge for most wing shapes. The loadings in this series are:

$$X(x_o) = a_o \sqrt{\frac{x_o}{1-x_o}} + \sum_{n=1}^{\infty} a_n \sin n\phi_{c_o} \qquad (9.1.1)$$

$$Y(y_o) = \sum_{n=1}^{\infty} b_n \sin n\phi_{sp} \qquad (9.1.2)$$

where the angles ϕ_{sp} and ϕ_{C_o} are related to y_o and x_o and are shown on Figure 4. The first chordwise mode, the a_o term, corresponds to a two dimensional flat plate; the second, a_1, gives a parabolic meanline. Higher chordwise modes do not affect the total load and serve only to shift the distribution of chordwise load fore and aft. Likewise, the first spanwise mode (elliptical) gives all the lift and higher modes tend only to modify the spanwise lift distribution. The transforms of the first two Glauert chordwise modes and the first Glauert spanwise mode are given below. In addition, several modes whose transforms are particularly well behaved are given.

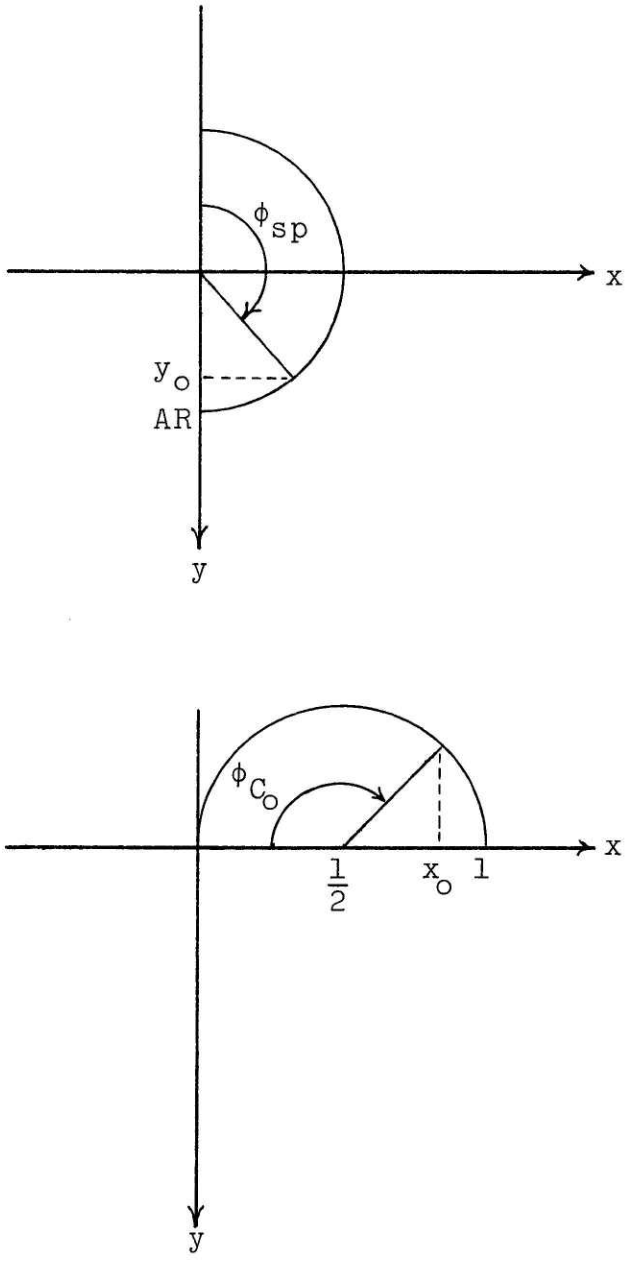


Figure 4 Glauert series mode angle definitions

TABLE 1

Spanwise Modes

1. Elliptical: first Glauert series term

$$Y(y_o) = \frac{4}{\pi AR^2} \sqrt{AR^2 - y_o^2}$$

$$\bar{Y}(\lambda) = 2/AR\lambda \left[I_1(AR\lambda) - L_1(AR\lambda)\right]$$

$$\bar{Y}(\lambda) \sim O(1/\lambda^{3/2}) \quad \text{as } \lambda \to \infty$$

$$\bar{Y}(o) = 1$$

2. Modified Elliptical:

$$Y(y_o) = \frac{16}{AR^3\pi} \, y_o \sqrt{y_o(AR - y_o)}$$

$$\bar{Y}(\lambda) = \frac{4}{AR\lambda} \left[I_1\left(\frac{AR}{2}\lambda\right) + I_2\left(\frac{AR}{2}\lambda\right)\right] e^{-\frac{AR}{2}\lambda}$$

$$\bar{Y}(\lambda) \sim O(1/\lambda^{3/2}) \quad \text{as } \lambda \to \infty$$

$$\bar{Y}(o) = 1$$

3. Sinusoid squared:

$$Y(y_o) = \frac{2}{AR} \sin^2 \left(\frac{\pi}{AR} y_o\right)$$

$$\bar{Y}(\lambda) = \frac{4\pi^2}{AR^3} (1 - e^{-AR\lambda}) \frac{1}{\lambda(\lambda^2 + 4\pi^2/AR^2)}$$

$$\bar{Y}(\lambda) \sim O(1/\lambda^3) \quad \text{as } \lambda \to \infty$$

$$\bar{Y}(o) = 1$$

4. Cubic spanwise:

$$Y(y_o) = 12y_o^2(AR - y_o)$$

$$\bar{Y}(\lambda) = \frac{12}{\lambda} \left[e^{-AR\lambda}\left(\frac{AR^2}{\lambda} + \frac{4AR}{\lambda^2} + \frac{6}{\lambda^3}\right) + \frac{2AR}{\lambda^2} - \frac{6}{\lambda^3}\right]$$

$$\bar{Y}(\lambda) \sim O(1/\lambda^3) \quad \text{as } \lambda \to \infty$$

$$\bar{Y}(o) = 1$$

TABLE 2

Chordwise Modes

1. Flat plate loading:

$$X(x_o) = \frac{2}{\pi} \sqrt{\frac{x_o}{1 - x_o}}$$

$$\bar{X}(\lambda p) = e^{-i\lambda p}[J_o(\lambda p) + i\, J_1(\lambda p)]$$

$$\bar{X}(\lambda p) \sim O(1/\sqrt{\lambda p}) \quad \text{as } \lambda p \to \infty$$

$$\bar{X}(o) = 1$$

2. Elliptical loading:

$$X(x_o) = \frac{8}{\pi} \sqrt{x_o}\, \sqrt{1 - x_o}$$

$$\bar{X}(\lambda p) = \frac{2}{\lambda p}\, J_1(\lambda p) e^{i\lambda p}$$

$$\bar{X}(\lambda p) \sim O(1/(\lambda p)^{3/2}) \quad \text{as} \quad \lambda p \to \infty$$

$$\bar{X}(o) = 1$$

3. Parabolic loading:

$$X(x_o) = 6x_o(1 - x_o)$$

$$\bar{X}(\lambda p) = \frac{3}{\lambda p} e^{i\lambda p} \left[\frac{\sin \lambda p}{(\lambda p)^2} - \frac{\cos \lambda p}{\lambda p} \right]$$

$$\bar{X}(\lambda p) \sim O(1/(\lambda p)^2) \quad \text{as} \quad \lambda p \to \infty$$

$$\bar{X}(o) = 1$$

X MODIFICATION OF THE FORMULATION FOR NUMERICAL CALCULATIONS

Rewriting the integral equation for the downwash 4.2 to include the developments (for the double integral) of equation 8.4, the formulation becomes:

$$w(x,y) = -\frac{1}{2\pi c_o}\int_0^{AR}\int_0^{1}\gamma(x_o,y_o)\left\{\frac{1}{(y-y_o)^2}\left[1 - \frac{x-x_o}{\sqrt{(x-x_o)^2+(y-y_o)^2}}\right] + \frac{1}{(y-y_o)^2}\left[1 - \frac{x-x_o}{\sqrt{(x-x_o)^2+(y+y_o)^2}}\right]\right\} dx_o dy_o$$

$$+ \frac{2}{\pi^2}\,\frac{Im}{c_o}\int_0^{1}\int_0^{\infty}\bar{X}(p\lambda)\,\bar{Y}(\lambda)\,\frac{\lambda^2 p}{p^2\lambda - F}\sqrt{1-p^2}\cdot e^{-\lambda(y+ixp)}\,d\lambda\,dp \qquad (10.1)$$

where as before $\gamma(x_o,y_o) = X(x_o)Y(y_o)$ and $\{\bar{X},\bar{Y}\}$ are the transforms of the loadings as defined in equations 8.3.

The first integral in the above equation can be seen to be the equation for a lifting surface of aspect ratio 2AR in an infinite fluid and may be rewritten:

$$w_{2AR}(x,y) = -\frac{1}{2\pi c_o}\int_{-AR}^{AR}\int_0^{1}\gamma(x_o,y_o)\cdot \frac{1}{(y-y_o)^2}\left\{1 - \frac{x-x_o}{\sqrt{(x-x_o)^2+(y-y_o)^2}}\right\} dx_o dy_o \qquad (10.2)$$

To calculate the downwash $W_{2AR}(x,y)$ due to a loading $\gamma(x_o,y_o)$, the continuous loading is approximated by taking the loading to be concentrated at equally spaced chordwise positions.

The resulting "vortex lattice" formulation, proposed by Faulkner (3), has been successfully employed by Milgram (4) in the study of yacht sails. Accurate results near the center of a wing's span and thus near the plane of the free surface are attained using this method. A short computer program has been written to calculate downwashes W_{2AR} using this vortex lattice technique.

The second integral contribution to the downwash in equation 10.1 gives the perturbation from the zero Froude number limit representing both bound and radiated wave effects. A change of variables and a factorization of the load transforms facilitates the numerical integration. Modified transforms are assumed such that:

$$\bar{X}(p\lambda) = \frac{1}{p\lambda}\ \bar{X}_m(p\lambda) \qquad (10.3.1)$$

$$\bar{Y}(\lambda) = \frac{1}{\lambda}\ \bar{Y}_m(\lambda) \qquad (10.3.2)$$

Letting $p = \sin\theta$, the second expression in 10.1 becomes:

$$w_w(x,y) = \frac{2}{\pi^2 c_o}\ \mathrm{Im}\int_0^{\frac{\pi}{2}}\int_0^{\infty} \frac{\bar{X}_m(\lambda\sin\theta)\,\bar{Y}_m(\lambda)}{\lambda\sin^2\theta - F}\cdot \cos^2\theta\ e^{-\lambda(y + i x\sin\theta)}\, d\lambda\, d\theta \qquad (10.4)$$

Taking the imaginary part,

$$w_w(x,y) = \frac{2}{\pi^2 c_o} \int_0^{\frac{\pi}{2}} \cos^2\theta \left\{ \int_0^{\infty} \frac{e^{-\lambda y}\, \bar{Y}_m(\lambda)}{\lambda \sin^2\theta - F} \cdot \right.$$

$$\left[\cos(x\lambda\sin\theta) \cdot \mathrm{Im}\, \bar{X}_m(\lambda\sin\theta) - \sin(x\lambda\sin\theta) \cdot \right.$$

$$\left. \mathrm{Re}\, \bar{X}_m(\lambda\sin\theta) \right] d\lambda + \pi e^{-\frac{Fy}{\sin^2\theta}} \bar{Y}_m\left(\frac{F}{\sin^2\theta}\right) \cdot$$

$$\left[\cos\left(\frac{Fx}{\sin\theta}\right) \cdot \mathrm{Re}\, \bar{X}_m\left(\frac{F}{\sin^2\theta}\right) + \sin\left(\frac{Fx}{\sin\theta}\right) \cdot \right.$$

$$\left. \left. \mathrm{Im}\, \bar{X}_m\left(\frac{F}{\sin^2\theta}\right) \right] \right\} d\theta \qquad (10.5)$$

A computer program has been written to evaluate this expression for finite nonzero F and any permissible loading. The inner integral over λ is truncated at a value of λ beyond which the contribution to the integral can be shown to be negligible. This follows for points below the free surface from the effect of the decaying exponential and for points on the free surface from the decay of the pole and the loading's transform (under the present notation $\bar{X}_m(\lambda)\bar{Y}_m(\lambda) \sim 0(1)$ as $\lambda \to \infty$).

XI NUMERICAL RESULTS

Two loadings have been numerically investigated. They were chosen as representative of two basic loading forms: those with and those without loading at the plane of the free surface. In both cases the chordwise loading was taken to be elliptical. For load extending to the surface, the elliptical spanwise loading was chosen, and for no load at the surface the cubic spanwise mode was used. (See definitions in Tables 1 and 2.) In all cases the calculations are for a negative unit total circulation such that the lift force acts to port (positive z direction) and is of magnitude ρU. The choice of aspect ratio for all loadings was somewhat arbitrarily taken as three, and the effect of varying aspect ratio has not been studied. Finally, the effect of the free surface was anticipated to be significant only near the free surface and has been studied only in this vicinity.

Figures 5 and 6 show the computed downwashes and shapes for an elliptical spanwise and chordwise loading. At zero Froude number the shape can be seen to contain both the camber effect of the bound vorticity and the induced angle of attack effect of the trailing and free vorticity systems. As the Froude number increases from 0.0 to 0.2, the most dramatic effect is the reversal of the sign of the induced angle of attack. The effect continues to grow as the Froude number increases to 0.5, where a reduction in camber becomes apparent.

Both effects continue to develop up to a Froude number of 0.8. Beyond a Froude number of .8, however, the induced angle increases with Froude number and returns to values of the same sign as occur in the zero Froude number limit. In terms of the known infinite Froude number limit, this result could have been anticipated since in this limit the effect of the free surface is to increase the magnitude of the downwash on the foil. (In the infinite Froude number limit the proper image above the plane of the free surface is a foil lifting in the direction opposite that of the actual foil, and the actual foil operates in the "upwash" of the image foil.)

The second loading, cubic spanwise with zero load and zero derivative of vertical load distribution at the free surface, generates a downwash whose variations with Froude number are similar to those for the elliptical spanwise loading. This is particularly interesting since at zero Froude number, the downwash from this loading differs significantly from that of the elliptical loading case. Here, at zero Froude number, there is an upwash near the plane of the free surface due to dominate free and trailing vortex effects of the load concentration near the foil's tip. As may be seen in Figures 7 and 8, the velocities induced at the same depth below the free surface by this loading are an order of magnitude greater than those induced by elliptical spanwise loading. Like the zero Froude number case, it

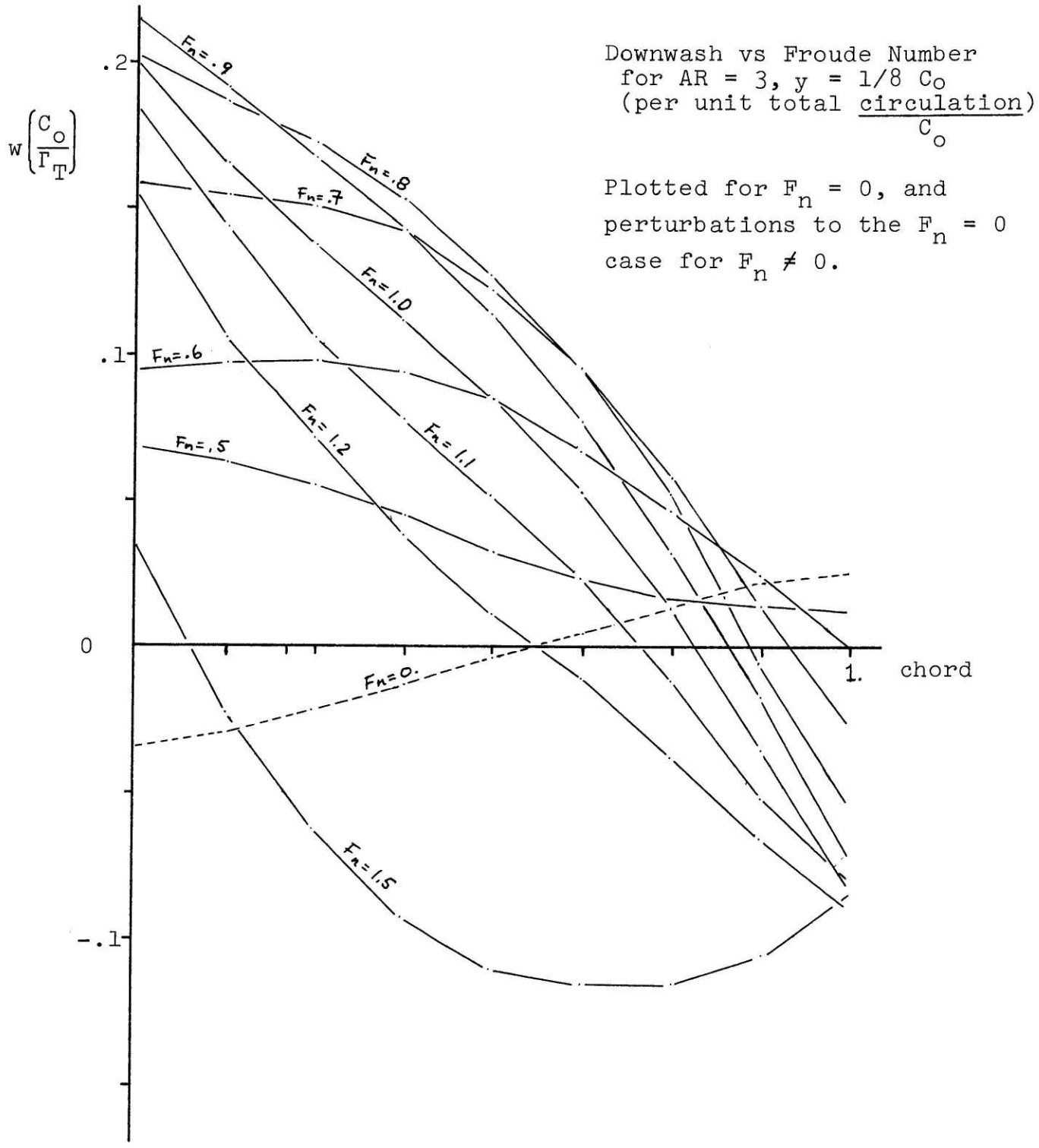


Figure 5 Downwash for elliptical spanwise and chordwise loading

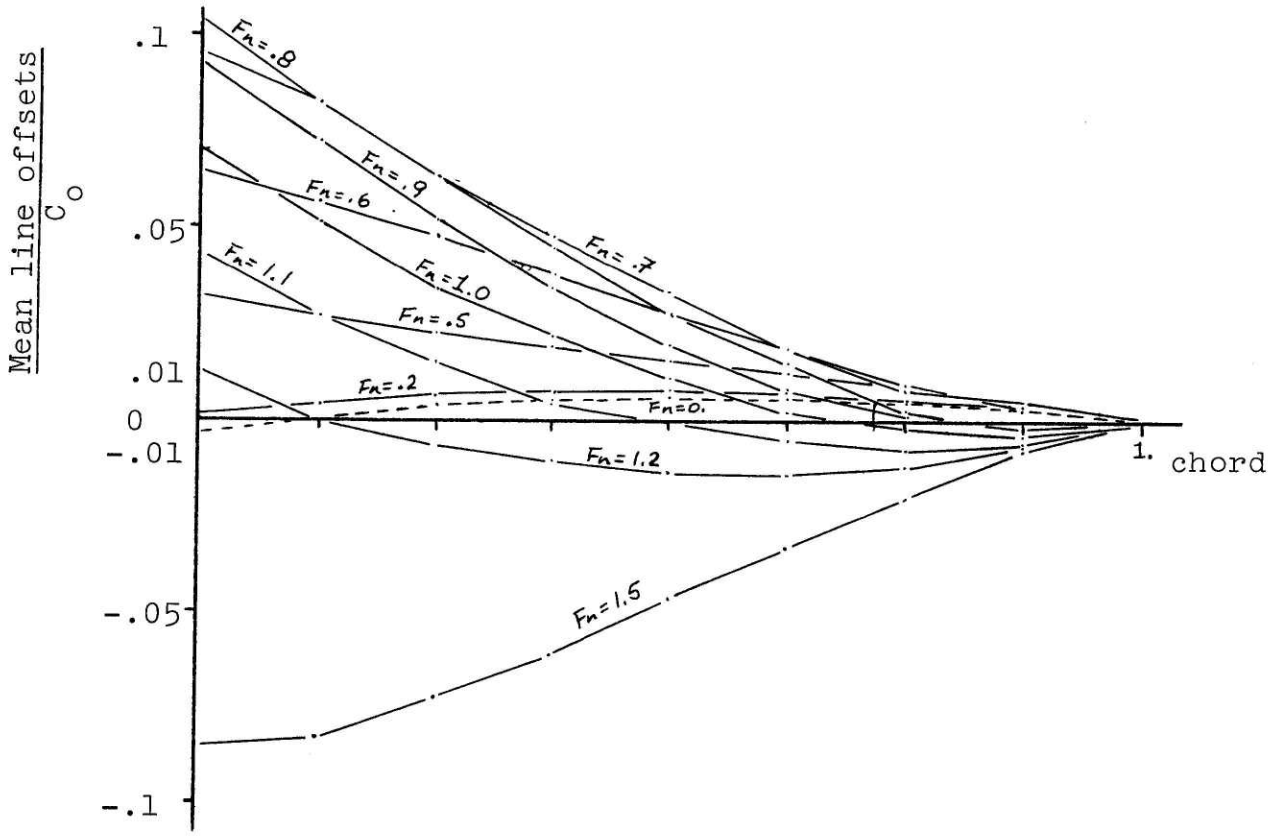


Figure 6 Mean line for elliptical spanwise and chordwise loading
(Conditions identical to those for Figure 5)

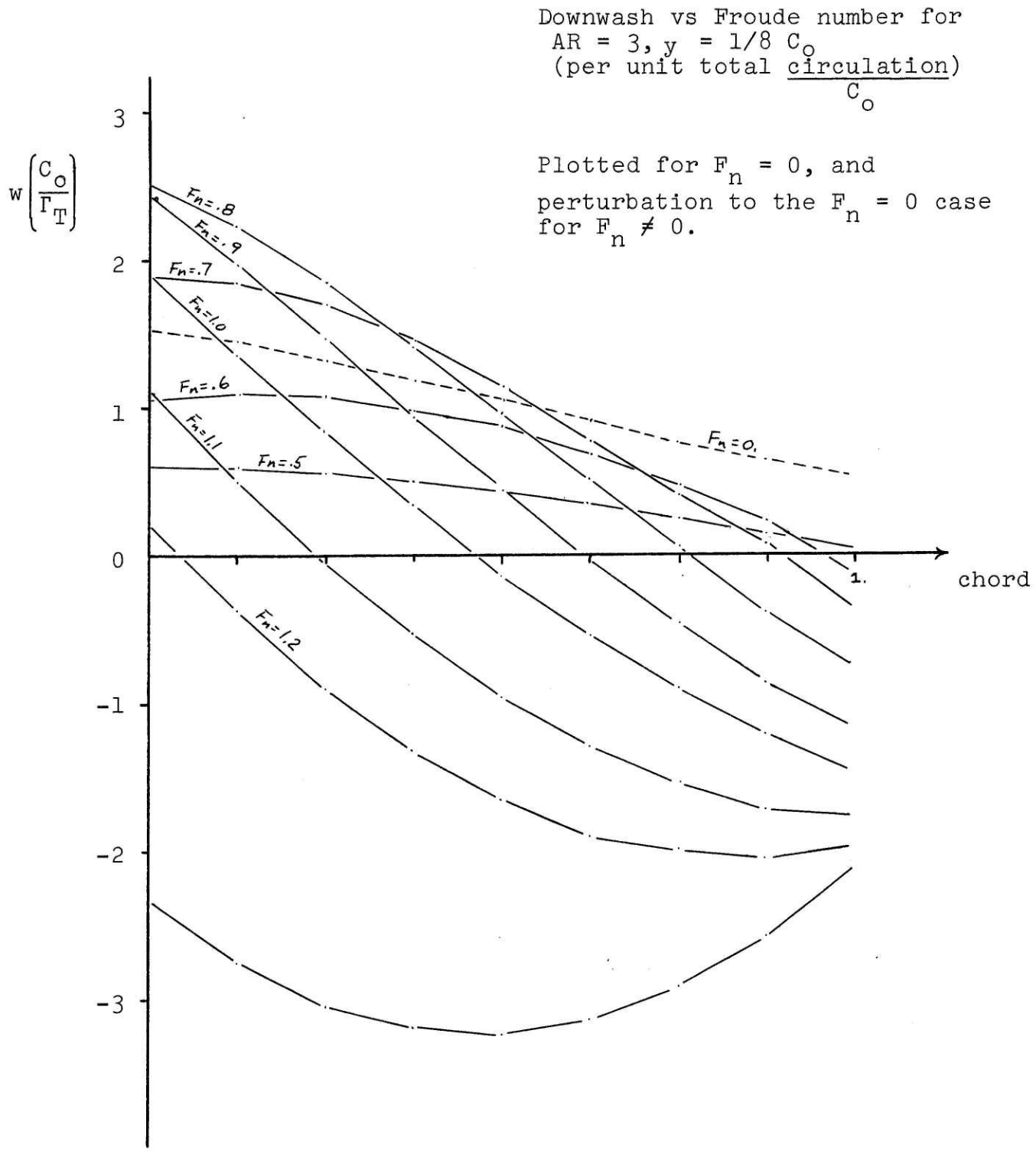


Figure 7 Downwash for cubic spanwise and elliptical chordwise loading

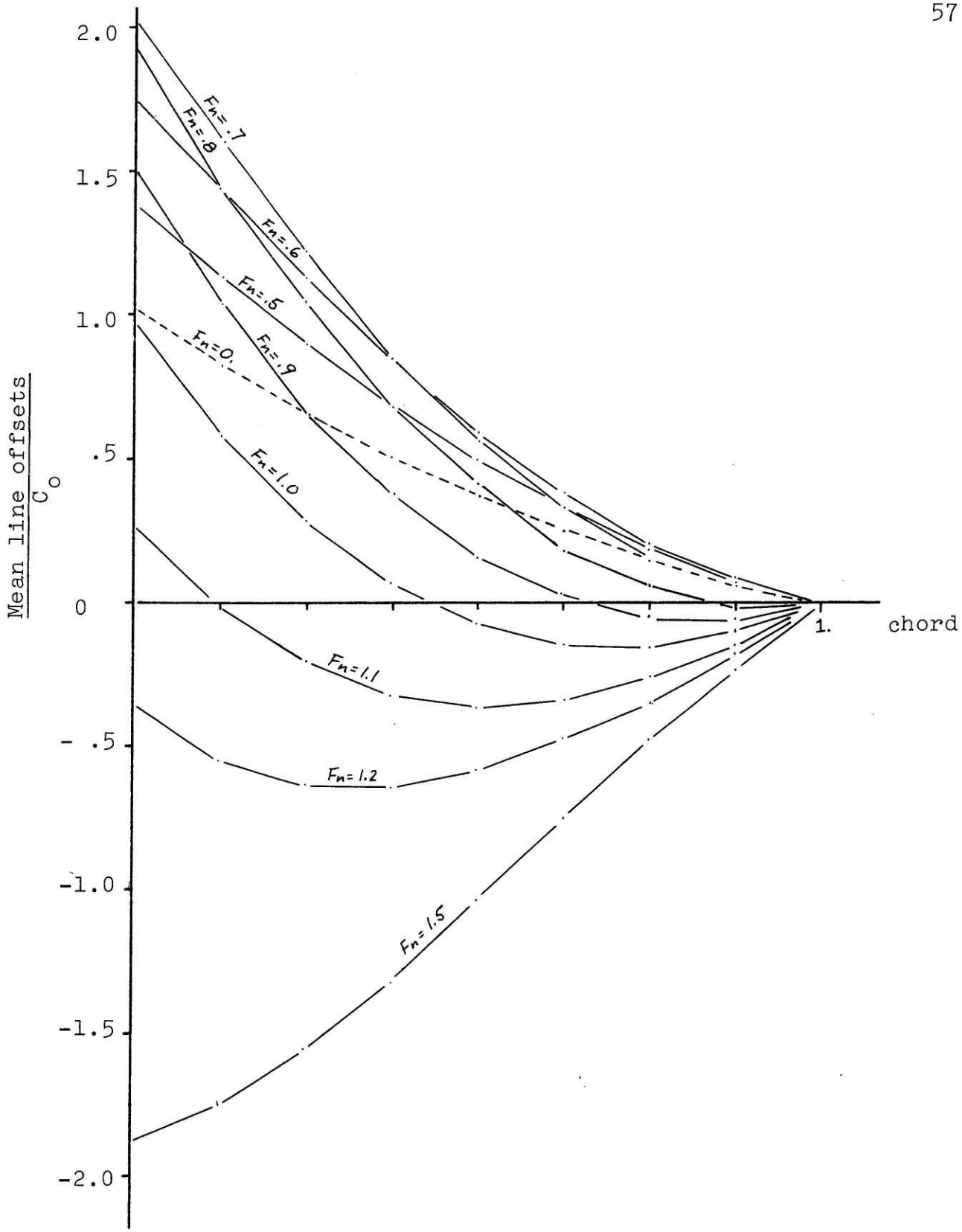


Figure 8 Mean line for cubic spanwise and elliptical chordwise loading (Conditions identical to those for Figure 7)

appears that larger magnitude wave effects result from the less smooth spanwise loading and the consequential variations in the strength of the free and trailing vortex systems. Because of the significantly larger velocities induced by this second mode, it appears that it's strength would be less than the first mode's in any superposition scheme, and therefore, represent a small variation in the overall dynamics of a foil being studied.

For both loadings, the reversal of the downwash reaches a maximum at a Froude number of 0.8. Considering the ratio of the foil's chord to the length of a plane wave propagating in the direction of the foil at a phase velocity equal to the foil's velocity, it can be shown that a Froude number of 0.8 corresponds to a wavelength equal to four chords.

$$V_p = \sqrt{g L_w / 2\pi}$$

$$F_n = U / \sqrt{g c_o}$$

then $V_p = U$ implies

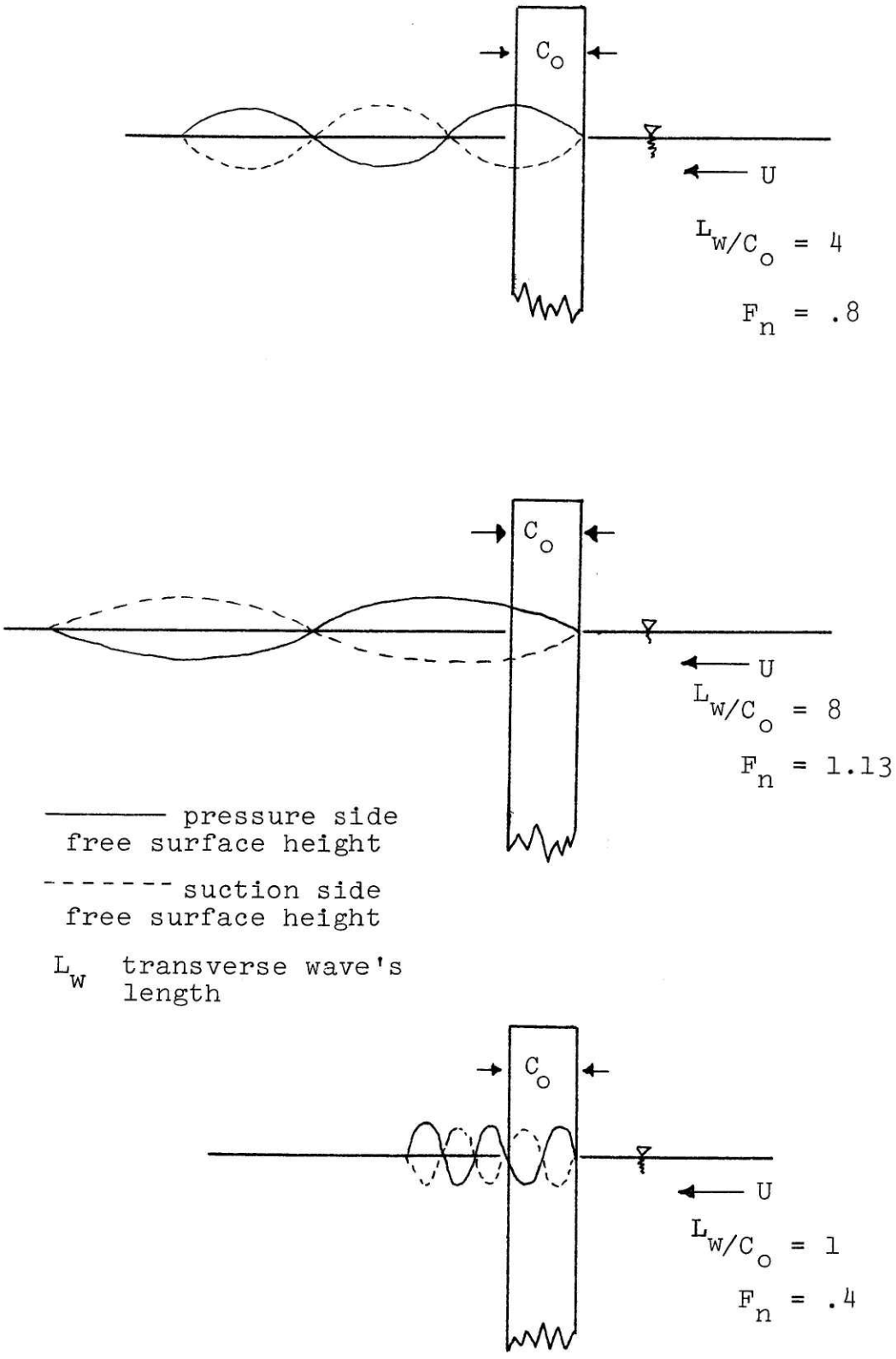
$$L_w / c_o = 2\pi F_n^2$$

where L_w is the length and V_p is the phase velocity of a two dimensional plane progressive wave. For $F_n = 0.8$, $L_w/C_o = 4.02$ as stated above. Recalling that both the streamwise velocity perturbations ϕ_x and the free surface elevation η are equal in magnitude and opposite in sign at adjacent points on the two sides of the foil, a crude physical

explanation for the connection between $L_w/C_o = 4$, and the maximum reversed downwash may be offered. If the wave pattern near the foil is dominated by a transverse wave of length $L_w = 4C_o$ with no elevation jump at the leading edge, there will be a wave crest at the trailing edge on the pressure side and a trough on the suction side. This clearly violates the Kutta condition and the condition requiring continuity of pressure across the wake. One suspects, therefore, that other waves included in the solution cancel this pressure jump at the trailing edge and in the wake. These waves need not be transverse and will, in general, give both streamwise and transverse velocity perturbations. It is these transverse velocities that are suspected of causing the maximum upwash for a Froude number of 0.8. To some extent this effect is anticipated at all Froude numbers, but it will be relatively stronger wherever the transverse wave system has a crest and trough at opposite sides of the trailing edge. Finally, in the $F_n = 0.8$ case, this effect might be expected to reach a maximum since at this Froude number the total side force due to this transverse wave system (per unit wave amplitude) is maximized. At higher Froude numbers the transverse wave's crest is downstream of the foil and at lower Froude numbers the shorter waves on either side will tend to display cancellation effects. Figure 9 illustrates some of these ideas.

The decay of the wave effects below the free surface was investigated in the case of elliptical spanwise and

Figure 9 Idealized transverse wave system of a surface piercing hydrofoil



chordwise loading. Figure 10 illustrates this behavior for depths of $1C_o$, $2C_o$ and $3C_o$ below the surface and generally confirms the anticipated exponential decay below the free surface that is associated with deep water gravity waves.

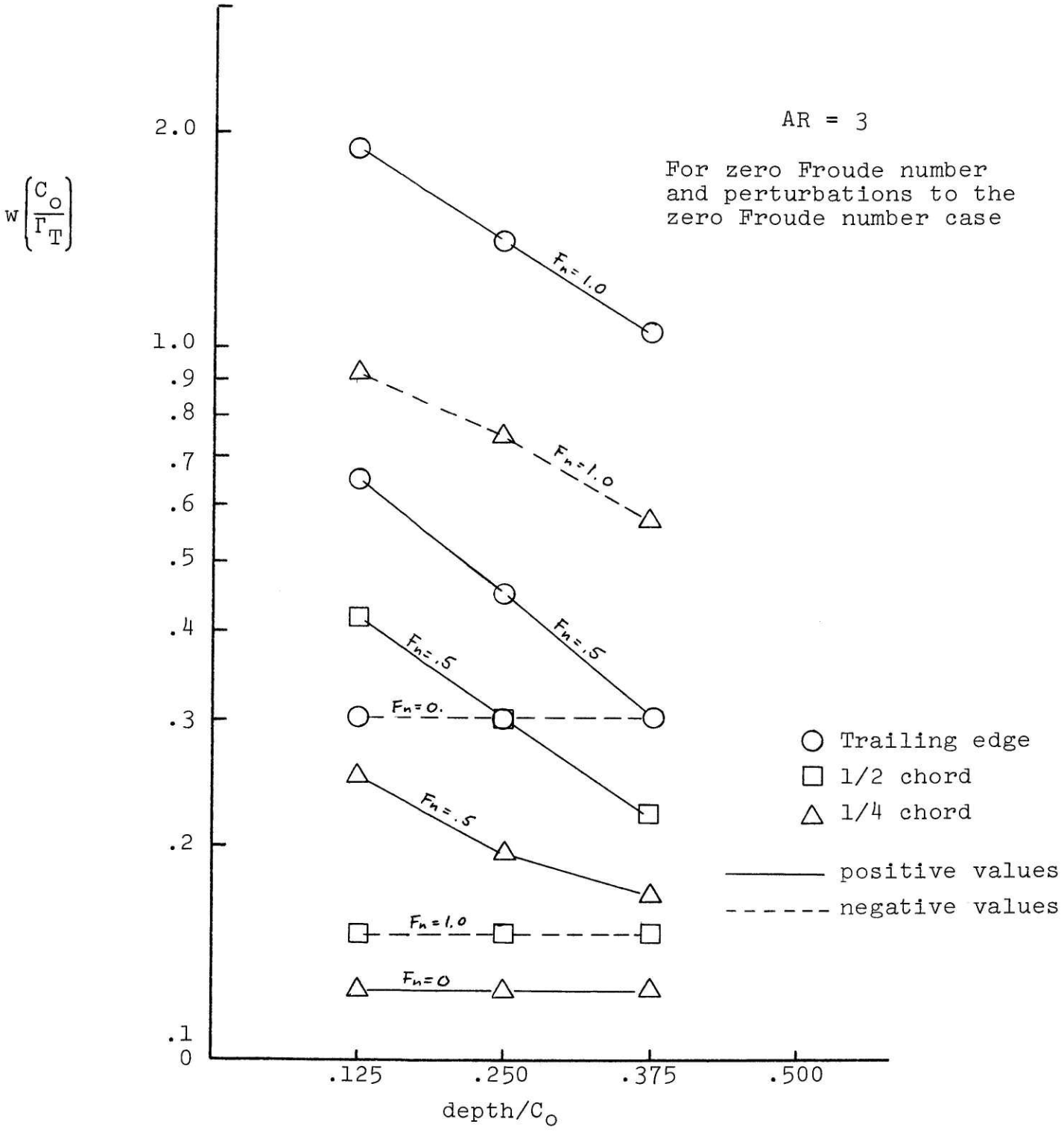


Figure 10 Downwash variation with depth under a free surface for elliptical spanwise and chordwise loading

XII CONCLUSIONS AND FUTURE DIRECTIONS

Two significant results have been achieved. First, a formulation of the surface piercing hydrofoil problem has been developed to represent a foil of arbitrary shape in terms of a distribution of elemental horseshoe vortex singularities. Conditions sufficient to preclude infinities of the free surface elevation (or perturbation velocities at the free surface) have been determined and a variety of possible loading modes have been given along with information sufficient to determine their satisfaction of these conditions.

The second result is the proposed and demonstrated technique of numerically calculating foil shapes for arbitrary loadings. Using a vortex lattice scheme combined with a numerical calculation of a double improper integral, shapes due to any permissible loading can be determined. It appears that using this technique for determining the side forces acting on a known surface peircing foil shape is possible using linear superposition. To reduce the expense of computing a large number of downwash (shape) modes, the existing routines could possibly be modified to more expeditiously perform the several numerical integrations.

A remaining difficulty with the present theoretical results is the apparent inability of linear theory to account for the frequently observed jump in the height of the

free surface at the trailing edge. A discussion of this matter is undertaken in Appendix E. Extension of the analysis to investigate an "inner" region of the flow near the intersection of the track of the foil and the free surface might prove most rewarding.

In summary, a formulation for surface piercing hydrofoils has been developed and the relation between loading and shape has been investigated in several instances to demonstrate the viability of using this formulation and associated numeric techniques to analyze free surface piercing hydrofoils. Future efforts should be directed on two fronts. First, the physics of the trailing edge free surface jump should be studied in greater detail. Second, direct comparisons between theoretical and experimental results should be made for some Froude number, aspect ratio and simple foil shape to determine the applicability of this linear theory to real physical problems.

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Appendix A

TRANSFORMATION FROM THE DIPOLE TO HORSESHOE VORTEX FORMULATION

As given in equation 3.10, the dipole formulation is:

$$w(x,y) = -\frac{1}{2\pi C_o} \int_{o}^{AR} \int_{-\infty}^{1} \Gamma(x_o,y_o) \left.\frac{\partial \phi_d}{\partial z}\right|_{z=o} dx_o\, dy_o \qquad (A.1)$$

$$\left.\frac{\partial \phi_d}{\partial z}\right|_{z=o} = \frac{1}{r^3} + \frac{1}{r'^3} - \frac{4}{\pi} Re \int_{0}^{\frac{\pi}{2}} \int_{0}^{\infty} \frac{K^3 F^3 \sin^2\theta}{K - \sec^2\theta} \cdot e^{-KF[y+y_o + i(x-x_o)\cos\theta]}\, dk\, d\theta \qquad (A.2)$$

The development for the algebraic terms in the Green's function A.2, is identical to that for classical lifting surface theory and is carried out by returning to the form of the potential with the differentiation, with respect to z_o, and the limit as z_o tends to zero taken outside the integrals over x_o and y_o. In this form, an integration by parts is performed in the x_o coordinate leaving an evaluated portion and an integral equation for unknown, $\partial\Gamma/\partial x_o$, which is the negative of the local bound vortex strength, $\gamma(x_o,y_o)$. The steps may be found in many texts including Ashley and Landahl (10).

The double integral contribution may be evaluated by integrating with respect to x_o inside the κ and θ integrals.

The x_o integration from equation 3.10 is:

$$I = \int_{-\infty}^{1} \Gamma(x_o, y_o)\, e^{iKFx_o\cos\theta}\, dx_o \qquad (A.3)$$

$$I = \frac{\Gamma(x_o, y_o)\, e^{iKFx_o\cos\theta}}{iKF\cos\theta}\Bigg|_{-\infty}^{1} + i\int_{o}^{1} \frac{\partial\Gamma}{\partial x_o}\, \frac{e^{iKFx_o\cos\theta}}{KF\cos\theta}\, dx_o \qquad (A.4)$$

The evaluated portion is zero at $x_o = 1$, because $\Gamma(1,y_o) = 0$ (no load ahead of the leading edge) and as $x_o \to -\infty$, when subsequently integrated over the wave number K, the expression with $x_o \to -\infty$ will make no contribution due to its highly oscillatory integrand, as is formally demonstrable using the Reimann-Lesbeque Lemma. Again reversing the order of integration, the double integral contribution is:

$$\frac{4}{\pi} I_m \int_{o}^{\frac{\pi}{2}} \int_{o}^{\infty} \frac{K^2F^2\tan\theta}{K - \sec^2\theta}\, \sin\theta \cdot e^{-KF[y+y_o+i(x-x_o)\cos\theta]}\, dK\, d\theta \qquad (A.5)$$

The complete expression for $w(x,y)$ in terms of $\gamma(x_o,y_o)$ is given as equation 4.2. Owing to the sign convention $\gamma(x_o,y_o) = -\partial\Gamma/\partial x_o$, a sign change is introduced with the substitution.

Appendix B

TRANSFORMATION FROM HORSESHOE VORTEX TO NEWMAN'S AND ROBINSON AND LAURMANN'S FORM

As in Appendix A, only the double integral term is treated. Development of the algebraic terms can be found in Robinson and Laurmann (6).

Taking the y_o integration inside the κ and θ integrals,

$$I = \int_0^{AR} \gamma(x_o,y_o)\, e^{-KFy_o}\, dy_o$$

$$= -\frac{\gamma(x_o,y_o)\, e^{-KFy_o}}{KF}\Bigg|_0^{AR} + \int_0^{AR} \frac{e^{-KFy_o}}{KF}\, \frac{\partial\gamma}{\partial y_o}\, dy_o$$

$$= \frac{\gamma(x_o,0)}{KF} + \int_0^{AR} \frac{e^{-KFy_o}}{KF}\, \frac{\partial\gamma}{\partial y_o}\, dy_o$$

$$= \frac{1}{KF} \int_0^{AR} \frac{\partial\gamma}{\partial y_o} \left(e^{-KFy_o} - 1 \right) dy_o \qquad \text{(B.1)}$$

Returning to a form with the y_o integration carried out after the κ and θ integrals, the complete formulation is:

$$w(x,y) = -\frac{1}{2\pi C_o} \int_0^1 \int_0^{AR} \frac{\partial\gamma(x_o,y_o)}{\partial y_o} \left\{ \frac{\sqrt{(x-x_o)^2+(y-y_o)^2}}{(x-x_o)(y-y_o)} \right.$$

$$\left. - \frac{1}{y-y_o} - \frac{\sqrt{(x-x_o)^2+(y+y_o)^2}}{(x-x_o)(y+y_o)} + \frac{1}{y+y_o} \right.$$

$$-\frac{4}{\pi}\,\mathrm{Im}\int_{0}^{\frac{\pi}{2}}\int_{0}^{\infty}\frac{KF}{K-\sec^2\theta}\,\sin\theta\tan\theta\cdot$$

$$e^{-KF(y+i(x-x_0)\cos\theta)}\left(e^{-KFy_0}-1\right)dK\,d\theta\Big\}dx_0\,dy_0$$

(B.2)

Appendix C

THE INFINITE FROUDE NUMBER LIMIT OF THE DOUBLE INTEGRAL TERM IN THE HORSESHOE VORTEX KERNEL

Letting $\lambda = KF$, the double integral term in the horseshoe vortex kernel may be written:

$$I = -\frac{4}{\pi} \operatorname{Im} \int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{\lambda^2 \tan\theta \sin\theta}{\lambda - F \sec^2\theta} \cdot e^{-\lambda[y+y_o + i(x-x_o)\cos\theta]} \, d\lambda \, d\theta \tag{C.1}$$

Assuming for the moment that the proper interpretation of C.1, as $F \to 0$ can be made setting $F = 0$,

$$I = -\frac{4}{\pi} \operatorname{Im} \int_0^{\frac{\pi}{2}} \int_0^{\infty} \lambda \tan\theta \sin\theta \cdot e^{-\lambda[y+y_o + i(x-x_o)\cos\theta]} \, d\lambda \, d\theta \tag{C.2}$$

Defining $y + y_o = a$, $x - x_o = b$;

$$I = +\frac{4}{\pi} \int_0^{\frac{\pi}{2}} \sin\theta \tan\theta \int_0^{\infty} \lambda \sin(\lambda b \cos\theta) e^{-\lambda a} \, d\lambda \, d\theta$$

$$= +\frac{4}{\pi} \int_0^{\frac{\pi}{2}} \sin\theta \tan\theta \frac{2ab\cos\theta}{[a^2 + b^2\cos^2\theta]^2} \, d\theta$$

$$= +\frac{8ab}{\pi} \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{[a^2 + b^2\cos^2\theta]^2} \, d\theta$$

$$I = \frac{4ab}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2\theta}{[a^2 + b^2\cos^2\theta]^2}\, d\theta$$

letting $\xi = \tan\theta$, $d\xi = \sec^2\theta\, d\theta$

$$I = \frac{4ab}{\pi} \int_{-\infty}^{\infty} \frac{\xi^2}{\xi^2+1} \frac{1}{\left[a^2 + b^2 \frac{1}{\xi^2+1}\right]^2} \frac{d\xi}{\xi^2+1}$$

$$= \frac{4ab}{\pi} \frac{1}{a^4} \int_{-\infty}^{\infty} \frac{\xi^2}{[\xi^2 + 1 + b^2/a^2]^2}\, d\xi$$

letting $c^2 = 1 + b^2/a^2$

$$I = \frac{4b}{\pi a^3} \int_{-\infty}^{\infty} \frac{\xi^2}{[\xi^2 + c^2]^2}\, d\xi$$

$$I = \frac{4b}{\pi a^3} \int_{-\infty}^{\infty} \frac{\xi^2}{[\xi + ic]^2 [\xi - ic]^2}\, d\xi \qquad (C.3)$$

This may be evaluated using residue theory. The residue at $\zeta = ic$, is found by expanding $\zeta^2/[\zeta+ic]^2$ about $\zeta = ic$, giving:

$$\frac{\xi^2}{[\xi+ic]^2} = \frac{1}{4} + (\xi - ic)\cdot\frac{-i}{4c} + \cdots$$

The residue is then $-\frac{i}{4c}$, the coefficient of the first order pole of the Laurant expansion for the entire integrand.

The integral may then be evaluated.

$$I = \frac{4b}{\pi a^3}\cdot 2\pi i\cdot\left(-\frac{i}{4c}\right)$$

$$= \frac{2\,(x-x_o)}{(y+y_o)^2\sqrt{(x-x_o)^2+(y+y_o)^2}} \qquad (C.4)$$

which is the proper result for the bound vorticity, but doesn't include the effect of the trailing vorticity in the wake. This might have been anticipated since the partial integration carried out in the wake in arriving at the horse-shoe formulation is invalid for the case F = 0. From equation A.4, with the negative infinite limit replaced by -B,

$$\lim_{B\to-\infty} -\frac{\Gamma(B,y_o)\,e^{iKFB\cos\theta}}{iKF\cos\theta} \qquad (C.5)$$

Noting that for a unit horseshoe $-\Gamma(+B,y_o) \to -1$ as $B \to -\infty$ and reintroducing C.5 into the double integral A.2,

$$\lim_{B\to-\infty} \frac{4}{\pi} \operatorname{Re} \int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{(KF)^3 \sin^2\theta}{K - \sec^2\theta} \frac{1}{iKF\cos\theta} \cdot e^{-KF[y+y_o+i(x-B)\cos\theta]} dk d\theta$$

$$\lim_{B\to-\infty} \frac{4}{\pi} \operatorname{Im} \int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{(KF)^2}{K - \sec^2\theta} \sin\theta \tan\theta \cdot e^{-KF[y+y_o+i(x-B)\cos\theta]} dk d\theta$$

Again, letting $\lambda = KF$, this is identical, except for sign, to the integral in C.1, and the result may be written directly.

$$\lim_{B\to-\infty} -\frac{2(x-B)}{(y+y_o)^2} \frac{1}{\sqrt{(x-B)^2+(y+y_o)^2}} = -\frac{2}{(y+y_o)^2} \tag{C.7}$$

Thus, in the limit as $F \to 0$, the double integral term gives

$$-\frac{2}{(y+y_o)^2} + \frac{2(x-x_o)}{(y+y_o)^2\sqrt{(x-x_o)^2+(y+y_o)^2}} \tag{C.8}$$

and, in effect, transforms the Green's function into that which satisfies $\phi = 0$ on $y = 0$, the infinite Froude number limit of the free surface boundary condition.

Appendix D

VERTICAL LOAD RESTRICTIONS FOR LIFTING LINES

For small $(y \pm y_o)$, the poles in the kernel of equation 7.6 dominate the logarithmic singularity. Thus, for small $(y \pm y_o)$

$$w(o,y) \approx \frac{1}{2\pi c_o} \int_o^{\epsilon} \Gamma'(y_o) \left[\frac{1}{(y-y_o)^2} - \frac{1}{(y+y_o)^2} \right] dy_o \tag{D.1}$$

where ε has been chosen such that the approximation is valid for $0 < y_o < \varepsilon$. Returning to the form preceding differentiation of the potential with respect to z and taking the limit of z tends to zero,

$$w(o,y) = \lim_{z \to o} \frac{\partial}{\partial z} \frac{1}{2\pi c_o} \int_o^{\epsilon} \Gamma'(y_o) \left\{ \frac{z}{(y-y_o)^2 - z^2} - \frac{z}{(y+y_o)^2 - z^2} \right\} dy_o \tag{D.2}$$

Integrating D.2 twice by parts and taking the limit $z \to 0$ and derivative with respect to z

$$w(o,y) = \frac{-1}{2\pi c_o} \left\{ \Gamma'(\epsilon) \frac{2y}{\epsilon^2 - y^2} + \frac{1}{2} \frac{\partial \Gamma(\epsilon)}{\partial y_o} \ln \left(\frac{\epsilon + y}{\epsilon - y} \right) + \frac{2}{y} \Gamma'(o) + \frac{\pi}{2} \left(y \frac{\partial \Gamma(\epsilon)}{\partial y_o} - \Gamma'(y) - \lim_{q \to o} q \frac{\partial \Gamma(q)}{\partial q} \right) \right\} \tag{D.3}$$

For $w(0,y)$ to remain bounded as $y \to 0$, the term $\frac{2}{y}\Gamma(0)$ requires $\Gamma(0) = 0$, and the term

$$\lim_{q \to 0} q \frac{\partial \Gamma(q)}{\partial q} \tag{D.4}$$

requires that $\Gamma(q)$ tend to zero at least as fast as $q\ell nq$ as $q \to 0$.

Appendix E

FREE SURFACE ELEVATION DISCONTINUITIES

A frequently observed phenomenon is a jump in the elevation of the free surface at the trailing edge of a surface piercing foil. Although this effect appears to be of first order magnitude (crude experiments indicate a linear relation between angle of attack and this elevation jump), it has not been possible to predict it as part of the solution to the linearized problem.

The difficulty arises from the imposition of the pressure continuity condition across the wake. Without a free surface (at zero Froude number), the pressure everywhere in the flow is given by the linearized Bernoulli equation:

$$p = +\rho U \phi_x \quad , \quad y > 0 \qquad \text{(E.1)}$$

This relation also applies at points below a free surface, but does not hold on the free surface where the linearized dynamic free surface boundary condition gives:

$$p = +\rho U \phi_x - \rho g \eta \quad , \text{ on } y = 0 \qquad \text{(E.2)}$$

and permits an elevation jump while maintaining the imposed pressure continuity providing there is a corresponding discontinuity in the streamwise velocity perturbation, ϕ_x.

Although a plausible explanation for the free surface elevation jump can be argued, based on line singularities

of presently unknown form and strength, to do so without first thoroughly investigating a possible inner region of the flow at the intersection of the free surface and the foil would be pure conjecture. It should also be noted that equally plausible nonlinear effects can be argued as the mechanism producing the free surface jump.

Appendix F

CONVERSION BETWEEN FORMS OF THE FREE SURFACE SOURCE POTENTIAL

The familiar expression,

$$\frac{1}{r} = \frac{1}{(x^2+y^2+z^2)^{1/2}} = \frac{1}{2\pi}\int_0^{\infty} dk \int_{-\pi}^{\pi} d\theta \left\{ e^{-ky} \cdot e^{ik(x\cos\theta + z\sin\theta)} \right\} \tag{F.1}$$

may be reduced to:

$$\frac{1}{r} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} dk \; e^{-ky} \cos(kz\sin\theta)\cos(kx\cos\theta) \tag{F.2}$$

From Wehausen (19)(equation 13.36), using our coordinates, the source potential is:

$$\phi_s = \frac{1}{r} - \frac{1}{r'} - \frac{4\nu}{\pi}\int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{e^{-\lambda(y+y_o)}}{\lambda\cos^2\theta - \nu} \cdot \cos[\lambda(z-z_o)\sin\theta]\cos[\lambda(x-x_o)\cos\theta]\, d\lambda\, d\theta \tag{F.3}$$

adding and subtracting 2/r' in its two forms,

$$\phi_s = \frac{1}{r} + \frac{1}{r'} - \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \int_0^{\infty} \left[\frac{\nu}{\lambda \cos^2\theta - \nu} + 1 \right] \cdot$$

$$e^{-\lambda(y+y_0)} \cos\left(\lambda(z-z_0)\sin\theta\right) \cos\left(\lambda(x-x_0)\cos\theta\right) \cdot$$

$$d\lambda\, d\theta \qquad \text{(F.4)}$$

which reduces to the form stated in equation 3.3,

$$\phi_s = \frac{1}{r} + \frac{1}{r'} - \frac{4}{\pi} \mathrm{Re} \int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{\lambda}{\lambda - \nu \sec^2\theta} \cdot$$

$$\cos\left[\lambda(z-z_0)\sin\theta\right] e^{-\lambda\left[y+y_0+i(x-x_0)\cos\theta\right]} \cdot$$

$$d\lambda\, d\theta \qquad \text{(F.5)}$$

Appendix G

WAVE EFFECTS OF THE HORSESHOE VORTEX SINGULARITY

At one stage in the development of the present formulation, it was believed that the downwash contributed by the double integral term of the horseshoe vortex kernel (equation 4.2) could be tabulated as a function of $(x - x_o, y + y_o, F)$ and used in computing the downwashes corresponding to various loadings. This approach, however, improperly restricted the permitted loadings near the free surface (zero loading at the surface) and was abandoned.

One aspect of this work is significant, and this is the relation between the forms of the various terms in the horseshoe vortex kernel. Figure 10 shows the downwash contributed by the double integral term of the kernel for various finite values of the parameter, ν, and the limiting case $\nu = 0$. It can be seen that the downwash tends toward the proper limit quite rapidly upstream and less so downstream where anticipated radiating waves give an oscillatory downwash.

Figure 11 shows the effect of approaching the free surface, and the resulting increases in the downwash which are seen to be singular roughly as $1/(y+y_o)^2$ as $(y+y_o) \to 0$. Again this result might have been anticipated as this is the manner in which the other terms in the kernel behave for $(y-y_o) \to 0$ and $(y+y_o) \to 0$, respectively.

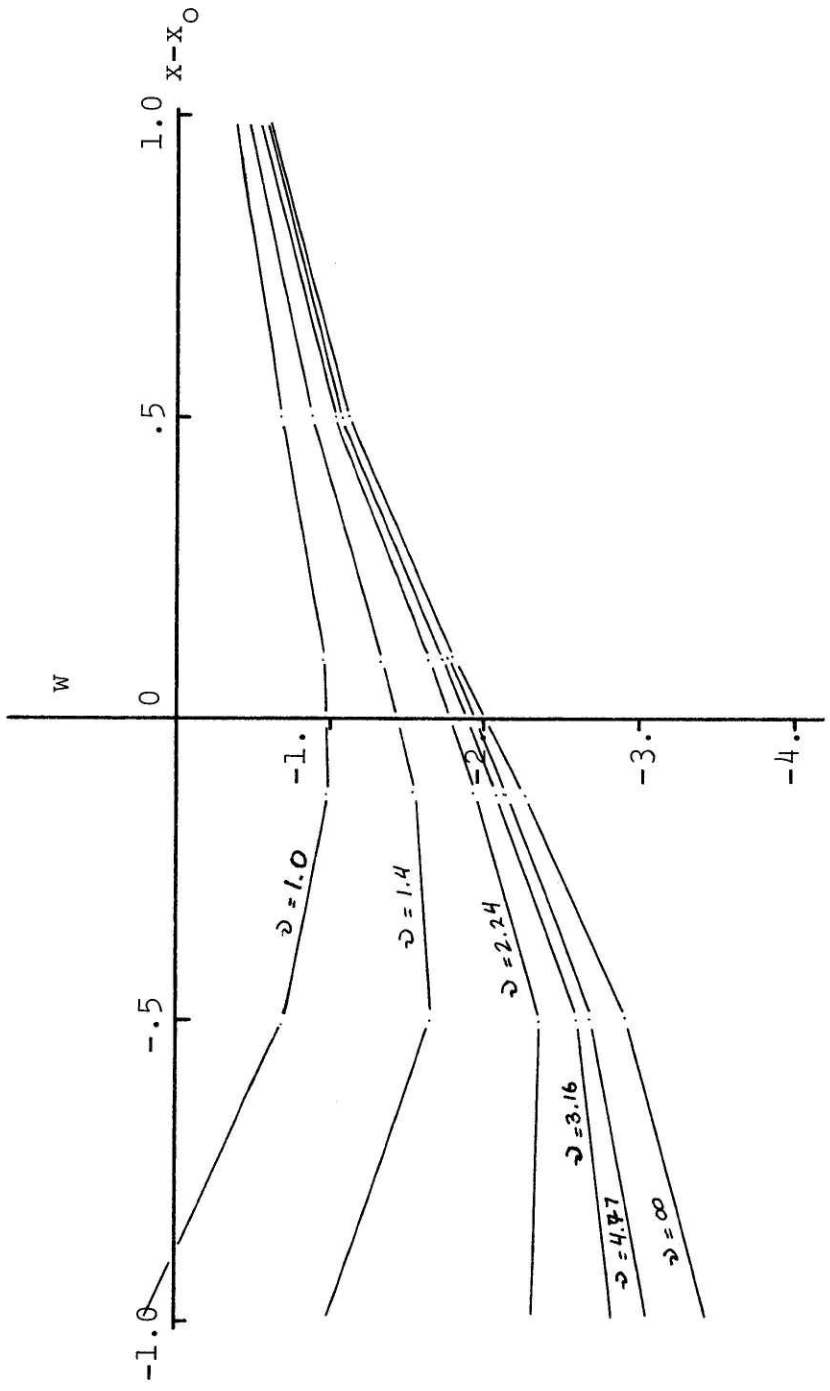
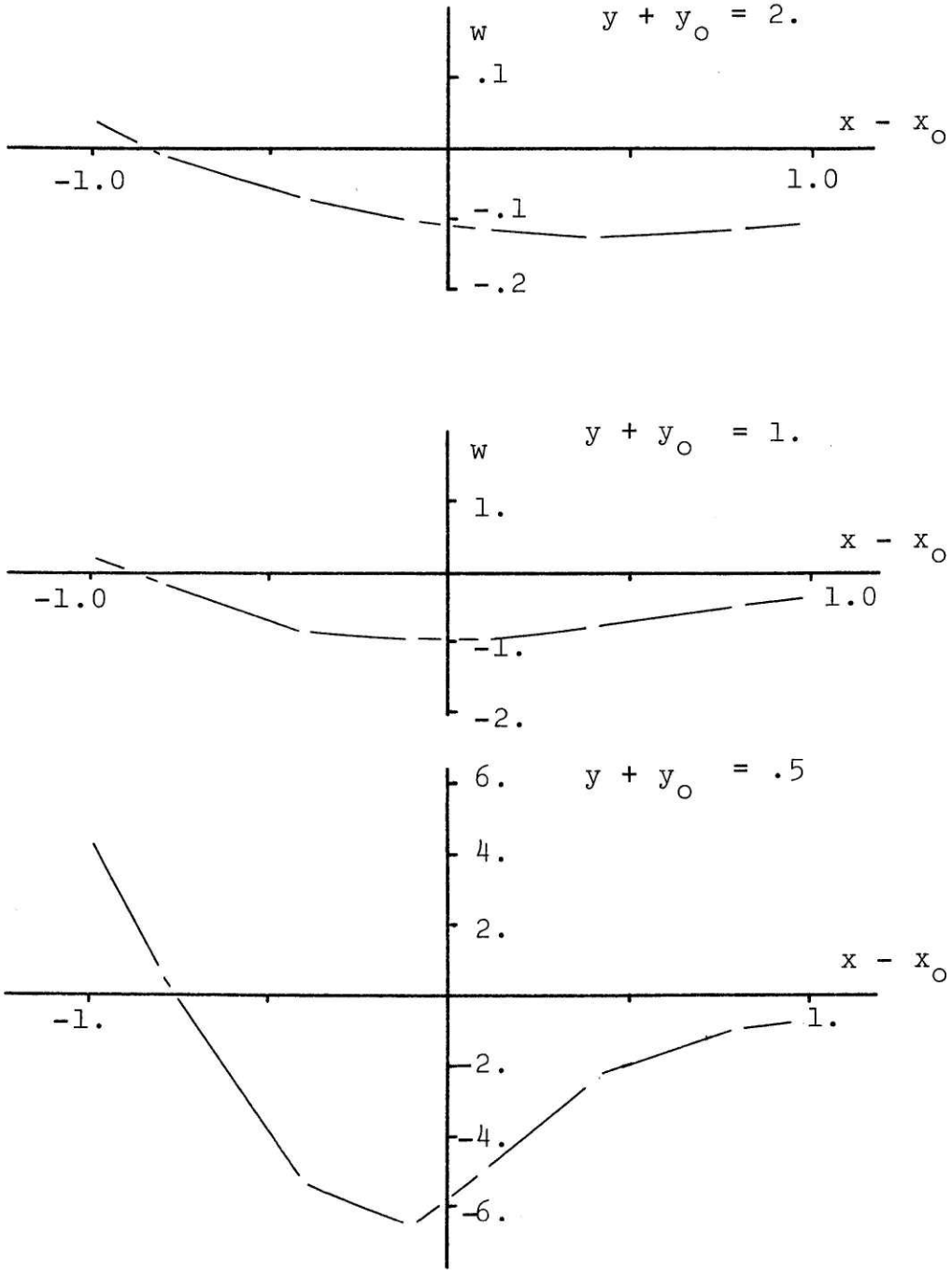


Figure 11 Point horseshoe vortex singulariy: downwash versus $(x - x_o)$ for various Froude numbers* at fixed $y + y_o = 1$.
(*) In this case, the dimensional parameter ν

Figure 12 Point horseshoe vortex singularity: downwash versus $(x - x_o)$ for varying $(y + y_o)$ at fixed $\nu = 1$.



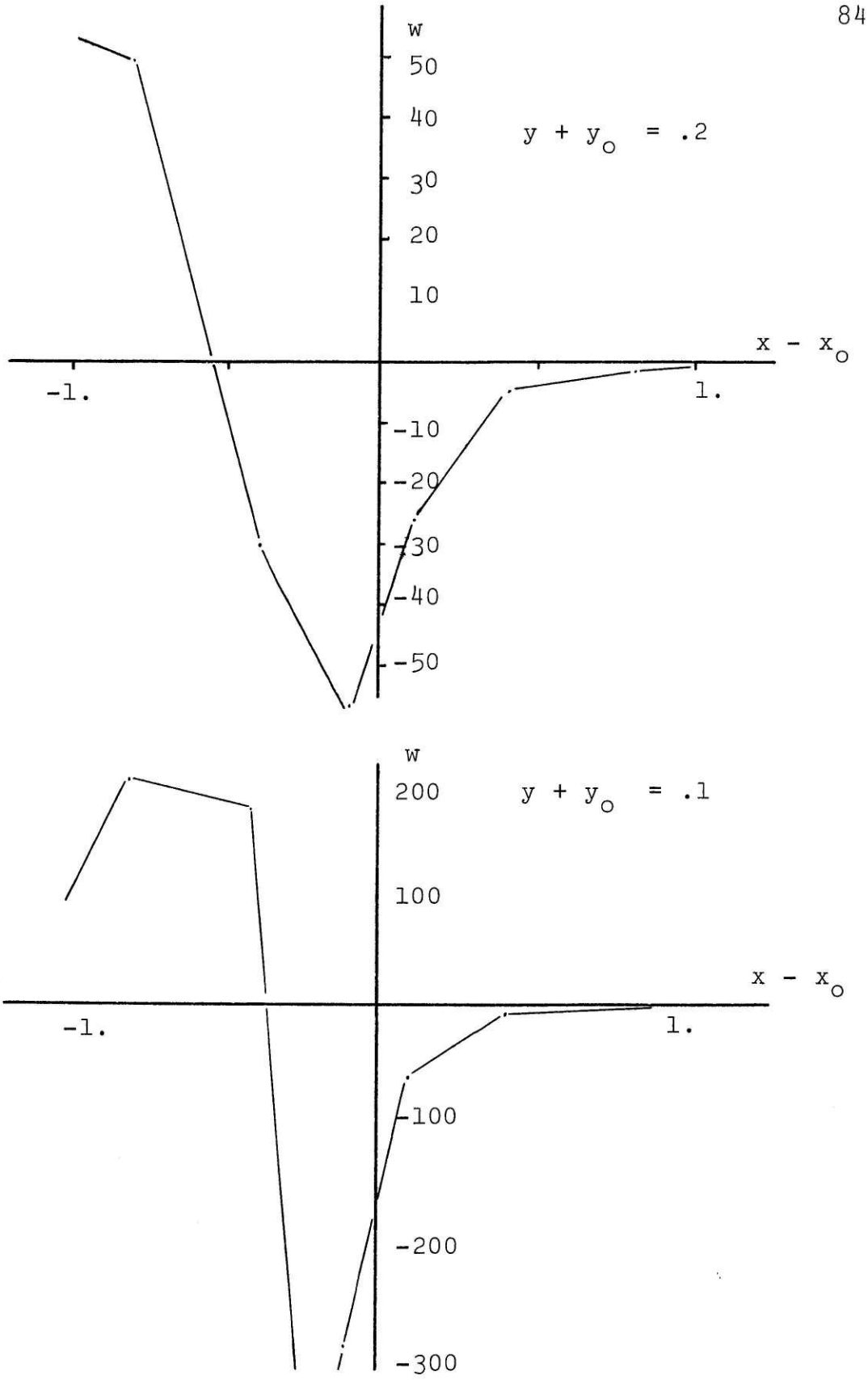


Figure 12
(continued)

Appendix H

LISTINGS OF COMPUTER PROGRAMS USED TO EVALUATE DISTRIBUTED LOAD DOWNWASH CONTRIBUTIONS FROM THE DOUBLE INTEGRAL IN THE HORSESHOE VORTEX KERNEL USING LOADING TRANSFORMS

```
C         MAIN PROGRAM FOR CALCULATING DOWNWASH THRU LOAD TRANSFORMS
      COMMON ER
      DIMENSION W(10)
      DATA IR,IW/5,6/
1     FORMAT(8F10.5)
2     FORMAT('1DOWNWASH ALONG CHORD FOR UNIT CIRCULATION:')
3     FORMAT('0ASPECT RATIO:',F10.5)
4     FORMAT('0FROUDE NUMBER:',F10.5)
5     FORMAT('0CHORD:',F10.5)
6     FORMAT('0FREE STREAM:',F10.5)
7     FORMAT(1H ,10E11.3)
8     FORMAT(1H1)
9     FORMAT('0DOWNWASH WITH LEADING EDGE AT RIGHT:',//)
11    FORMAT('0DEPTH:',F10.5)
      READ(IR,1) CHD,AR,ER
10    READ(IR,1) Y,F
      IF(Y.LT.0.0) GOTO 40
      UINF=SQRT(32.2*CHD/F)
      DO 100 I=1,9
      X=.125*FLOAT(I-1)
      WRITE(IW,1) X,Y,F,CHD,AR,ER
      CALL WAVES(X,Y,F,CHD,AR,ANS)
100   W(I)=ANS
      WRITE(IW,2)
      F=SQRT(1./F)
      WRITE(IW,3) AR
      WRITE(IW,4) F
      WRITE(IW,5) CHD
      WRITE(IW,6) UINF
      WRITE(IW,11) Y
      WRITE(IW,9)
      WRITE(IW,7) (W(I),I=1,9)
      WRITE(IW,8)
      GOTO 10
40    STOP
      END
```

```
      SUBROUTINE WAVES(X,Y,F,CHD,AR,ANS)                                WAVEC001
      COMMON ER                                                         WAVEC002
      DIMENSION Q(1CCC)                                                 WAVEC003
      DATA PI/3.14159/                                                  WAVEC004
      DATA IR,IW/5,6/                                                   WAVEC005
      IER=0                                                             WAVEC006
      IB=2                                                              WAVEC007
      IE=10                                                             WAVEC008
      IS=1                                                              WAVEC009
      NI=10                                                             WAVEC010
      PANS=1C                                                           WAVEC011
      Q(1)=0.0                                                          WAVEC012
80    H=1./FLOAT(NI)                                                    WAVEC013
      Q(NI+1)=0.0                                                       WAVEC014
      DO 100 I=IB,IE,IS                                                 WAVEC015
      D3=H*(I-1)                                                        WAVEC016
      TH=PI*(D3**2)/2.                                                  WAVEC017
      P=SIN(TH)                                                         WAVEC018
      CALL WVNBR(P,F,AR,X,Y,ANS,IER)                                    WAVEC019
      IF(IER.NE.0) GCTC 2C0                                             WAVEC020
      CALL GAMMA(F/(P**2),P,AR,GAMR,GAMI)                               WAVEC021
      R1=F*X/P                                                          WAVEC022
      R2=GAMR*CCS(R1)+GAMI*SIN(R1)                                      WAVEC023
      IF(R2.EQ.0.0) GCTC 90                                             WAVEC024
      R=-F*Y/(P**2)+ALCG(ABS(PI*R2))                                    WAVEC025
      IF(R+180.) 90,9C,91                                               WAVEC026
9C    R=0.0                                                             WAVEC027
      GOTO 1C0                                                          WAVEC028
91    R=(R2/ABS(R2))*EXP(R)                                             WAVEC029
1C0   Q(I)=(1.-P**2)*(ANS+R)*PI*D3                                      WAVEC030
      CALL SIMR(IE,H,Q,ANS)                                             WAVEC031
      TEST=ABS((ANS-PANS)/ANS)                                          WAVEC032
      IF(TEST.LT.ER) GCTC 15C                                           WAVEC033
      IF(IE.GT.499) GCTC 14C                                            WAVEC034
      CALL SPRD(Q,NI+1)                                                 WAVEC035
      NI=NI*2                                                           WAVEC036
```

```
      IE=NI                                                              WAVEC037
      IS=2                                                               WAVEC038
      PANS=ANS                                                           WAVEC039
      GOTO 80                                                            WAVEC040
140   WRITE(IW,1) X,Y,F,CHD,AR,ANS,TEST                                  WAVE0041
1     FORMAT(' WAVES:',7E12.3)                                           WAVE0042
      WRITE(IW,4)                                                        WAVE0043
4     FORMAT(' WAVES NOT CONVERGENT')                                    WAVE0044
150   ANS=ANS*(-2./(CHD*(PI**2)))                                        WAVE0045
      WRITE(IW,1) ANS                                                    WAVE0046
      J=NI+1                                                             WAVE0047
      WRITE(IW,3) (Q(I),I=1,J)                                           WAVE0048
      RETURN                                                             WAVE0049
200   WRITE(IW,2) IER,ANS                                                WAVE0050
2     FORMAT(' WVNBR NOT CONVERGENT',I10,E20.6)                          WAVE0051
      J=NI+1                                                             WAVE0052
      WRITE(IW,3) (Q(I),I=1,J)                                           WAVE0053
3     FORMAT(1H ,10E11.3)                                                WAVE0054
      STOP                                                               WAVE0055
      END                                                                WAVE0056
      SUBROUTINE SPRD(S,N)                                               WAVE0057
      DIMENSION S(1)                                                     WAVE0058
      NN=N-1                                                             WAVE0059
      DO 10 I=1,NN                                                       WAVE0060
      J=N-I+1                                                            WAVE0061
      K=2*J-1                                                            WAVE0062
10    S(K)=S(J)                                                          WAVE0063
      RETURN                                                             WAVE0064
      END                                                                WAVE0065
```

```
      SUBROUTINE TRAP(NI,H,F,ANS)                                  TRAPC001
      DIMENSION F(1)                                               TRAP0002
      ANS=0.0                                                      TRAPC003
      IF(NI.EQ.0) GOTO 20                                          TRAPC004
      ANS=(F(1)+F(NI+1))*0.5                                       TRAPC005
      DO 10 I=2,NI                                                 TRAPC006
10    ANS=ANS+F(I)                                                 TRAPC007
      ANS=ANS*H                                                    TRAPC008
20    RETURN                                                       TRAPC009
      END                                                          TRAPC10
```

```
      SUBROUTINE FREQ(P,X,AR,A)                                          FREQC001
      A=P*(2.-X)                                                         FREQC002
C        MAXIMUM RADIAN FREQUENCY OF INTEGRANDS SINUSOIDS                FREQC003
      RETURN                                                             FREQC004
      END                                                                FREQC005
```

```
      SUBROUTINE SIMR(NI,H,FUNC,ANS)                                    SIMRC001
      DIMENSION FUNC(1)                                                 SIMRC002
      K=NI+1                                                            SIMRC003
      ANS=FUNC(1)+FUNC(K)                                               SIMRC004
      DO 10 J=2,NI,2                                                    SIMRC005
10    ANS=ANS+4*FUNC(J)                                                 SIMRC006
      IF(NI.EQ.2)GOTO 30                                                SIMRC007
      K=NI-1                                                            SIMRC008
      DO 20 J=3,K,2                                                     SIMRC009
20    ANS=ANS+2*FUNC(J)                                                 SIMRC010
30    ANS=ANS*H/3.                                                      SIMRC011
      RETURN                                                            SIMRC012
      END                                                               SIMRC013
```

```
      SUBROUTINE WVNBR(P,F,AR,X,Y,TOTAL,IER)                            WVNR0001
      COMMON ER                                                         WVNR0002
      DIMENSION F1(5000)                                                WVNR0003
      DATA IR,IW/5,6/                                                   WVNR0004
      DATA PI/3.14159/                                                  WVNR0005
      T1=0.0                                                            WVNR0006
      T2=0.0                                                            WVNR0007
      T3=0.0                                                            WVNR0008
      IER=0                                                             WVNR0009
      TOTAL=0.0                                                         WVNR0010
      RS=F/(P**2)                                                       WVNR0011
      CALL FREQ(P,X,AR,RADF)                                            WVNR0012
C        RADF IS MAXIMUM RADIAN FREQUENXY                               WVNR0013
C        TEN INTERVALS PER MINIMUM PERIID                               WVNR0014
      DR=PI/(5.*RADF)                                                   WVNR0015
      DRX=.25/Y                                                         WVNR0016
      DRS=RS/4.                                                         WVNR0017
       D2=DR                                                            WVNR0018
      IF(D2.GT.DRX) D2=DRX                                              WVNR0019
      IF(D2.GT.DRS) D2=DRS                                              WVNR0020
      DR=.2*DR                                                          WVNR0021
      D2=.2*D2                                                          WVNR0022
C        CIW IS CAUCHY INTEGRAL GAP HALF WIDTH                          WVNR0023
      CIW=.5*D2                                                         WVNR0024
      IF(CIW.GT..1) CIW=.1                                              WVNR0025
      UPLS=RS-CIW                                                       WVNR0026
      UPLX=170./Y                                                       WVNR0027
      UPLL=100./AR                                                      WVNR0028
      UL=UPLS                                                           WVNR0029
      IF(UL.GT.UPLL) UL=UPLL                                            WVNR0030
      IF(UL.GT.UPLX) UL=UPLX                                            WVNR0031
      NI=UL/D2+1.                                                       WVNR0032
      I=NI/2                                                            WVNR0033
      IF(I*2.NE.NI) NI=NI+1                                             WVNR0034
      D2=UL/FLOAT(NI)                                                   WVNR0035
C        PORTION OUT THE SINGULARITY                                    WVNR0036
```

```
      F1(1)=0.0                                                          WVNRC037
      DO 100 I=1,NI                                                      WVNRC038
      R=I*D2                                                             WVNRC039
      J=I+1                                                              WVNRC040
      CALL GAMMA(R,P,AR,GAMR,GAMI)                                       WVNRC041
100   F1(J)=EXP(-Y*R)*(GAMI*COS(R*X*P)-GAMR*SIN(R*X*P))/((P**2)*R-F)     WVNRC042
      CALL TRAP(NI,D2,F1,T1)                                             WVNRC043
      IF(UL.LT.UPLS) GOTO 300                                            WVNRC044
C        PRINCIPAL VALUE PART ASSUMING FUNCTION LESS POLE BEHAVES        WVNRC045
C        LINEARLY FROM RS-CIW TO RS+CIW                                  WVNRC046
      R=RS+CIW                                                           WVNRC047
      CALL GAMMA(R,P,AR,GAMR,GAMI)                                       WVNRC048
      F1(1)=EXP(-Y*R)*(GAMI*COS(R*X*P)-GAMR*SIN(R*X*P))                  WVNRC049
      R=RS-CIW                                                           WVNRC050
      F1(2)=EXP(-Y*R)*(GAMI*COS(R*X*P)-GAMR*SIN(R*X*P))                  WVNRC051
      T2=F1(1)-F1(2)                                                     WVNRC052
C        SINGULASRITY TO INFINITY                                        WVNRC053
      TOTAL=T1+T2                                                        WVNRC054
      J=0                                                                WVNRC055
      RL=RS+CIW                                                          WVNRC056
225   DO 250 I=1,16                                                      WVNRC057
      R=RL+(I-1)*DR                                                      WVNRC058
      CALL GAMMA(R,P,AR,GAMR,GAMI)                                       WVNRC059
250   F1(I)=EXP(-Y*R)*(GAMI*COS(R*X*P)-GAMR*SIN(R*X*P))/((P**2)*R-F)     WVNRC060
      CALL TRAP(15,DR,F1,T3)                                             WVNRC061
      TEST=ABS(T3/(TOTAL+T3))                                            WVNRC062
      TOTAL=TOTAL+T3                                                     WVNRC063
      IF(TEST.LT.ER) GOTO 275                                            WVNRC064
      J=J+1                                                              WVNRC065
      IF(J.GT.60) GOTO 410                                               WVNRC066
      RL=RL+15.*DR                                                       WVNRC067
      GOTO 225                                                           WVNRC068
275   RETURN                                                             WVNRC069
300   TOTAL=T1                                                           WVNRC070
      RETURN                                                             WVNRC071
400   IER=-NI                                                            WVNRC072
```

```
      TOTAL=R1                                                  WVNRC073
      RETURN                                                    WVNRC074
410   IER=1                                                     WVNRC075
      TOTAL=RL                                                  WVNRC076
      RETURN                                                    WVNRC077
      END                                                       WVNRC078
```

```
      SUBROUTINE GAMMA(S,P,AR,GAMR,GAMI)                                GAM10001
C        CUBIC SPANWISE AND ELLIPTICAL CHORDWISE LOADING                GAM10002
      DATA IW,IR/5,6/                                                   GAM10003
      DATA PI/3.14159/                                                  GAM10004
      IF(AR*S.GT.100.) GOTO 10                                          GAM10005
      Z=S*P                                                             GAM10006
      IF(Z.GT..82355E+06) GOTO 10                                       GAM10007
C        COMPUTE BESSEL FUNCTION OF THE FIRST KIND ORDER ONE USING      GAM10008
C        POLYNOMIAL APPROXIMATION GIVEN IN A AND S HANDBOOK             GAM10009
      IF (Z) 10,10,15                                                   GAM10010
10    R1=0.0                                                            GAM10011
      GAMR=0.0                                                          GAM10012
      GAMI=0.0                                                          GAM10013
      RETURN                                                            GAM10014
15    IF(Z-3.) 16,18,18                                                 GAM10015
16    Z1=Z/3.                                                           GAM10016
      IF(Z1.LT.1.E-09) GOTO 12                                          GAM10017
      R1=0.5-.56250*(Z1**2)+.21094*(Z1**4)-.03954*(Z1**6)+.00443*(Z1**8)GAM10018
      GOTO 17                                                           GAM10019
12    R1=0.5                                                            GAM10020
17    R1=R1*Z                                                           GAM10021
      GOTO 20                                                           GAM10022
18    Z1=3./Z                                                           GAM10023
      IF(Z1.LT.1.E-50) GOTO 10                                          GAM10024
      IF(Z1.LT.1.E-10) GOTO 13                                          GAM10025
      F1=.79788+.00000156*Z1+.01659667*(Z1**2)+.00017105*(Z1**3)-.02495 GAM10026
     111*(Z1**4)+.00113653*(Z1**5)                                      GAM10027
      T1=Z-2.35619+.125*Z1+.0000656*(Z1**2)-.00637879*(Z1**3)+.00074350*(GAM10028
     1Z1**4)+.000798*(Z1**5)                                            GAM10029
      GOTO 19                                                           GAM10030
13    T1=Z                                                              GAM10031
      F1=.79788                                                         GAM10032
19    R1=F1*COS(T1)/SQRT(Z)                                             GAM10033
20    CONTINUE                                                          GAM10034
      R=S                                                               GAM10035
      R2=     (EXP(-AR*R)*((AR**2)/R+4.*AR/(R**2)+6./(R**3))+2.*AR/(R**2)GAM10036
```

```
     1-6./(R**3))                                                    GAM1C037
150   GAMR=24.*R1*R2*COS(S*P)                                        GAM1C038
      GAMI=24.*R1*R2*SIN(S*P)                                        GAM1C039
      RETURN                                                         GAM1C040
      END                                                            GAM1C041
```

```
      SUBROUTINE GAMMA(S,P,AR,GAMR,GAMI)                                GAM20001
C        ELLIPTICAL SPANWSIE AND CHORDWISE LOADING                      GAM20002
      DIMENSION G(51)                                                   GAM20003
      DATA PI/3.14159/                                                  GAM20004
      DATA IW,IR/5,6/                                                   GAM20005
      DATA G/0.0,.047939,.091990,.132480,.169710,.203952,.235457,.264454GAM20006
     1,.291151,.315740,.338395,.359276,.378530,.396290,.412679,.427810,.GAM20007
     2441783,.454694,.466629,.477666,.487877,.497329,.506083,.514194,.52GAM20008
     31712,.528625,.535156,.541164,.546746,.551933,.556757,.561246,.5654GAM20009
     426,.569319,.572948,.576333,.579492,.582442,.585199,.587776,.590187GAM20010
     5,.592445,.594560,.596542,.598402,.600147,.601787,.603328,.604777,.GAM20011
     6606142,.607426/                                                   GAM20012
      Z=S*P                                                             GAM20013
      IF(Z.GT..82355E+06) GOTO 10                                       GAM20014
C        COMPUTE BESSEL FUNCTION OF THE FIRST KIND ORDER ONE USING      GAM20015
C        POLYNOMIAL APPROXIMATION GIVEN IN A AND S HANDBOOK             GAM20016
      IF (Z) 10,10,15                                                   GAM20017
10    R1=0.0                                                            GAM20018
      GAMR=0.0                                                          GAM20019
      GAMI=0.0                                                          GAM20020
      RETURN                                                            GAM20021
15    IF(Z-3.) 16,18,18                                                 GAM20022
16    Z1=Z/3.                                                           GAM20023
      IF(Z1.LT.1.E-09) GOTO 12                                          GAM20024
      R1=0.5-.56250*(Z1**2)+.21094*(Z1**4)-.03954*(Z1**6)+.00443*(Z1**8)GAM20025
      GOTO 17                                                           GAM20026
12    R1=0.5                                                            GAM20027
17    R1=R1*Z                                                           GAM20028
      GOTO 20                                                           GAM20029
18    Z1=3./Z                                                           GAM20030
      IF(Z1.LT.1.E-50) GOTO 10                                          GAM20031
      IF(Z1.LT.1.E-10) GOTO 13                                          GAM20032
      F1=.79788+.00000156*Z1+.01659667*(Z1**2)+.00017105*(Z1**3)-.002495GAM20033
     111*(Z1**4)+.00113653*(Z1**5)                                      GAM20034
      T1=Z-2.35619+.125*Z1+.0000656*(Z1**2)-.00637879*(Z1**3)+.0007435*(GAM20035
     1Z1**4)+.000798*(Z1**5)                                            GAM20036
```

```
      GOTO 19                                                                  GAM20037
13    T1=Z                                                                     GAM20038
      F1=.79788                                                                GAM20039
19    R1=F1*COS(T1)/SQRT(Z)                                                    GAM20040
20    CONTINUE                                                                 GAM20041
C        MODIFIED STRUVE FUNCTION LESS MODIFIED BESSEL FUNCTION OF             GAM20042
C        THE SECOND KIND: BOTH ORDER ONE: FROM TABLES FOR SMALL                GAM20043
C        ARGUMENT AND VIA ASYMPTOTIC EXPANSION FOR LARGE                       GAM20044
      Z=AR*S                                                                   GAM20045
      IF(Z.GT.5.) GOTO 100                                                     GAM20046
      IF(Z.EQ.5.) GOTO 90                                                      GAM20047
      I=10.*Z+1.                                                               GAM20048
      J=I+1                                                                    GAM20049
      R2=G(I)+(G(J)-G(I))*(10.*Z-I+1.)                                         GAM20050
      GOTO 150                                                                 GAM20051
90    R2=G(51)                                                                 GAM20052
      GOTO 150                                                                 GAM20053
100   R2=-1.+1./(Z**2)                                                        GAM20054
      IF(Z.LT.6.93) GOTO 140                                                   GAM20055
      R2=R2+48./(Z**4)+180./(Z**6)                                             GAM20056
140   R2=-R2*2./PI                                                             GAM20057
150   GAMR=(4./AR)*R1*R2*COS(S*P)                                              GAM20058
      GAMI=(4./AR)*R1*R2*SIN(S*P)                                              GAM20059
      RETURN                                                                   GAM20060
      END                                                                      GAM20061
```

Appendix I

LISTINGS OF COMPUTER PROGRAMS USED TO EVALUATE DISTRIBUTED LOAD DOWNWASH CONTRIBUTIONS FROM THE ZERO FROUDE NUMBER TERMS IN THE HORSESHOE VORTEX KERNEL

```
C          DOUBLE ASPECT RATIO VORTEX VATTICE PROGRAM                                    VCRLC001
      DIMENSICN SPL(3C),CHL(10)                                                          VCRLC002
      DIMENSICN W(10)                                                                    VCRLC003
      DATA IR,IW/5,6/                                                                    VCRLC004
      READ(IR,1) CHC,AR,NC,NS                                                            VCRLC005
1     FORMAT(2F10.5,2I5)                                                                 VCRLCC06
      WRITE(IW,1) CHC,AR,NC,NS                                                           VCRLC007
C          GIVES DOWNWASH PER UNIT CIRCULATION                                           VCRLC008
      F=.5*AR*(CHC**2)                                                                   VCRLCCC9
2     FORMAT(' FOR CIRCULATICN USE:',F10.5,'*UINF*LIFT CCEFFICIENT')                     VCRLCC10
      WRITE(IW,2) F                                                                      VCRLC011
      HC=CHC/FLOAT(NC)                                                                   VCRLC012
      HS=(CHC*AR/FLOAT(NS))                                                              VCRLC013
      HS2=HS/2.                                                                          VCRLC014
      NP=NC+1                                                                            VCRLC015
      NS1=2*NS-1                                                                         VCRLC016
      CALL LCAD(SPL,CHL,NC,NS)                                                           VCRLC017
      FAC=1./FLOAT(NC)                                                                   VCRLC018
      DC 10 I=1,NC                                                                       VCRLC019
10    CHL(I)=CHL(I)*FAC                                                                  VCRLC02C
      FAC=AR/FLOAT(NS)                                                                   VCRLC021
      DC 20 I=1,NS                                                                       VCRLC022
20    SPL(I)=SPL(I)*FAC                                                                  VCRLC023
      DO 200 L=7,19,6                                                                    VCRLC024
      Y=HS*(L-1)                                                                         VCRLC025
      DC 100 M=1,NP                                                                      VCRLC026
      DW=C.O                                                                             VCRLC027
      X=(CHC/FLOAT(NC))*(M-1)                                                            VCRLC028
      K=-1                                                                               VCRLC029
      KOAC=NS+1                                                                          VCRLC030
      B=1.                                                                               VCRLC031
      DC 50 IY=1,NS1                                                                     VCRLC032
      KOAC=KOAC+K                                                                        VCRLC033
      IF(KOAC.EC.1) K=1                                                                  VCRLC034
      YC=-HS*NS+HS*IY                                                                    VCRLC035
      Y1=YC+HS2                                                                          VCRLC036
```

```
      Y2=YC-HS2                                                          VCRL0037
      DO 50 IX=1,NC                                                      VCRL0038
      X1=HC*(IX-.5)                                                      VCRL0039
50    DW=DW+SPL(KCAD)*CHL(IX)*HORVOR(X,Y,X1,Y1,Y2)*B                     VCRL0040
100   W(M)=DW                                                            VCRL0041
      WRITE(IW,3) (W(M),M=1,NP)                                          VCRL0042
200   CONTINUE                                                           VCRL0043
3     FORMAT(' ',10E11.3)                                                VCRL0044
      STOP                                                               VCRL0045
      END                                                                VCRL0046
```

```
      FUNCTION HORVOR(X,Y,X1,Y1,Y2)                                          HSVRC001
C        FINITE HORSESHOE VORTEX DOWNWASH                                    HSVRC002
      DX=X1-X                                                                HSVRC003
      DY1=Y1-Y                                                               HSVRC004
      DY2=Y2-Y                                                               HSVRC005
      DEN1=SQRT(DX**2+DY1**2)                                                HSVRC006
      DEN2=SQRT(DX**2+DY2**2)                                                HSVRC007
      HORVOR=(1./DY2)*(1.+DX/DEN2)-(1./DY1)*(1.+DX/DEN1)+(1./DX)*(DY2/DE     HSVRC008
     1N2-DY1/DEN1)                                                           HSVRC009
      HORVOR=HORVOR/(4.*3.14159)                                             HSVRC010
      RETURN                                                                 HSVRC011
      END                                                                    HSVRC012
```

```
      SUBROUTINE LOAD(SPL,CHL,NC,NS)                                    LD#10001
C        SPANWISE CUBIC CHORDWISE ELLIPTICAL                            LD#10002
      DIMENSION SPL(30),CHL(10)                                         LD#10003
      HC=1./FLOAT(NC)                                                   LD#10004
      AR=FLOAT(NS)/FLOAT(NC)                                            LD#10005
      HS=AR/FLOAT(NS)                                                   LD#10006
      DO 100 I=1,NC                                                     LD#10007
      X=HC*(I-.5)                                                       LD#10008
100   CHL(I)=8.*SQRT(X-X**2)/3.14159                                    LD#10009
      DO 200 I=1,NS                                                     LD#10010
      Y=FLOAT(I-1)*HS                                                   LD#10011
200   SPL(I)=12.*(Y**2)*(AR-Y)                                          LD#10012
      RETURN                                                            LD#10013
      END                                                               LD#10014
```

```
      SUBROUTINE LOAD(SPL,CHL,NC,NS)                                    LD#20001
C        FOR ELLIPTICAL SPANWISE AND ELLIPTICAL CHORDWISE               LD#20002
      DIMENSION SPL(30),CHL(10)                                         LD#20003
      HC=1./FLOAT(NC)                                                   LD#20004
      AR=FLOAT(NS)/FLOAT(NC)                                            LD#20005
      HS=AR/FLOAT(NS)                                                   LD#20006
      DO 100 I=1,NC                                                     LD#20007
      X=HC*(I-.5)                                                       LD#20008
100   CHL(I)=8.*SQRT(X-X**2)/3.14159                                    LD#20009
      DO 200 I=1,NS                                                     LD#20010
      Y=FLOAT(I-1)*HS                                                   LD#20011
200   SPL(I)=4.*SQRT(AR**2-Y**2)/(3.14159*(AR**2))                      LD#20012
      RETURN                                                            LD#20013
      END                                                               LD#20014
```

Appendix J

LIST OF COMPUTER PROGRAMS USED TO EVALUATE THE DOWNWASH OF A POINT HORSESHOE VORTEX SINGULARITY

```
C         POINT HORSESHOE VORTEX: FREE SURFACE EFFECTS                    KRNLC001
C         PROGRAM ONE MODIFIED                                            KRNLC002
      COMMON CTH,CPH,STH,C2,DX,Y,F                                        KRNLC003
      COMMON THD,SPH,ER                                                   KRNLC004
      DIMENSION DUM(1CCC),XN(20),YN(20)                                   KRNLC005
      DATA IR,IW/5,6/                                                     KRNLC006
      DATA PI/3.14159/                                                    KRNLC007
4     FORMAT(3F6.3,2X,4E15.7)                                             KRNLC008
5     FORMAT(' RESULTS ABOVE * SQRT Y: CAUCHY',E14.6,1X,'RESIDUE',E14.6,  KRNLC009
     11X,'UPSTREAM',E14.6,1X,'DOWNSTREAM',E14.6,////)                     KRNLC010
7     FORMAT(1H ,4E2C.6,I10)                                              KRNLC011
9     FORMAT(1HO,9HABS(X-XO),1X,F10.5,1CX,11HCAUCHY PART,4X,E16.6,10X,17  KRNLC012
     1HDOWNWASH UPSTREAM,4X,E16.6)                                        KRNLC013
10    FORMAT(1H ,4HY+YC,6X,F10.5,10X,12HRESIDUE PART,3X,E16.6,10X,19HDOW  KRNLC014
     1NWASH DOWNSTREAM,2X,E16.6,///)                                      KRNLC015
14    FORMAT(2I5,2F10.5)                                                  KRNLC016
15    FORMAT(I10,2E2C.6)                                                  KRNLC017
11    FORMAT(8F10.5)                                                      KRNLC018
20    FORMAT(1H ,53HTWICE INFINITE FLUID REVERSED IMAGE DOWNWASH UPSTREA  KRNLC019
     1M,3X,E15.6,3X,10HDOWNSTREAM,3X,E15.6)                               KRNLC020
13    READ(IR,14) NX,NY,F,ER                                              KRNLC021
      IF(NX.LE.0) GOTO 4000                                               KRNLC022
      WRITE(IW,14)NX,NY,F,ER                                              KRNLC023
      READ(IR,11) (XN(I),I=1,NX)                                          KRNLC024
      READ(IR,11) (YN(I),I=1,NY)                                          KRNLC025
      DO 20000 IYY=1,NY                                                   KRNLC026
      DO 20000 IXX=1,NX                                                   KRNLC027
      DX=XN(IXX)                                                          KRNLC028
      Y=YN(IYY)                                                           KRNLC029
      PANS=0.0                                                            KRNLC030
      NI=10                                                               KRNLC031
      IB=2                                                                KRNLC032
      IE=10                                                               KRNLC033
      IS=1                                                                KRNLC034
      DUM(1)=0.0                                                          KRNLC035
50    NP=NI+1                                                             KRNLC036
```

```
      H=1./FLOAT(NI)                                                    KRNL0037
      DO 100 I=IB,IE,IS                                                 KRNL0038
      D=H*(I-1)                                                         KRNL0039
      TH=PI*(1.-D**2)/2.                                                KRNL0040
      THD=90.*TH/1.5708                                                 KRNL0041
      CALL WNINT(TH,ANS)                                                KRNL0042
      DUM(I)=ANS*D*PI                                                   KRNL0043
100   CONTINUE                                                          KRNL0044
      DUM(NP)=0.0                                                       KRNL0045
      CALL SIMR(NI,H,DUM,ANS)                                           KRNL0046
      IF(ABS(ANS).LE.1.E-50) GOTO 200                                   KRNL0047
      TEST=ABS((ANS-PANS)/ANS)                                          KRNL0048
      IF (TEST.LE.ER) GOTO 202                                          KRNL0049
      PANS=ANS                                                          KRNL0050
      NI=NI*2                                                           KRNL0051
      IF(NI.GE.1000) GOTO 200                                           KRNL0052
      CALL SPRD(DUM,NP)                                                 KRNL0053
      IB=2                                                              KRNL0054
      IE=NI                                                             KRNL0055
      IS=2                                                              KRNL0056
      GOTO 50                                                           KRNL0057
200   WRITE(IW,15) NI,TEST,ANS                                          KRNL0058
202   CONTINUE                                                          KRNL0059
      CINT=ANS                                                          KRNL0060
C     RESIDUE PART OF KERNEL                                            KRNL0061
      SUM=0.0                                                           KRNL0062
      N=-1                                                              KRNL0063
      T1=0.0                                                            KRNL0064
2800  N=N+1                                                             KRNL0065
      NS=4                                                              KRNL0066
      T2=F*Y*((3.14159*(N+.5)/(F*DX))**2-1.)                            KRNL0067
      IF (T2.LE.0.0) GOTO 2800                                          KRNL0068
      IF (T2.GE.170.) T2=170.                                           KRNL0069
      PANS=0.0                                                          KRNL0070
2900  DT=(T2-T1)/FLOAT(NS)                                              KRNL0071
      NS2=NS+1                                                          KRNL0072
```

```
      DO 3000 I=1,NS2                                                    KRNL0073
      J=I-1                                                              KRNL0074
      Q=J*DT+T1                                                          KRNL0075
3000  DUM(I)=RFUN(Q)*EXP(-Q)                                             KRNL0076
      CALL SIMR(NS,DT,DUM,ANS)                                           KRNL0077
      IF(ABS(ANS).LT.1.E-50) GOTO 3500                                   KRNL0078
      TEST=ABS((ANS-PANS)/ANS)                                           KRNL0079
      IF (TEST.LT.ER) GOTO 3500                                          KRNL0080
      IF(ABS(ANS).LT.ABS(SUM)/ER) GOTO 3500                              KRNL0081
      IF (NS.GT.256) GOTO 3500                                           KRNL0082
      PANS=ANS                                                           KRNL0083
      NS=NS*2                                                            KRNL0084
      GOTO 2900                                                          KRNL0085
3500  CONTINUE                                                           KRNL0086
      SUM=SUM+ANS                                                        KRNL0087
      IF (T2.GE.170.) GOTO 3700                                          KRNL0088
      IF(ABS(SUM).LE.1.E-50) GOTO 3700                                   KRNL0089
      TEST=ABS(ANS/SUM)                                                  KRNL0090
      IF (TEST.LT.ER) GOTO 3800                                          KRNL0091
      T1=T2                                                              KRNL0092
      GOTO 2800                                                          KRNL0093
3700  WRITE(IW,7) T1,T2,ANS,TEST,NS                                      KRNL0094
3800  SUM=SUM*(-EXP(-F*Y)*4*F/Y)                                         KRNL0095
      UPST=CINT                                                          KRNL0096
      DOWN=SUM-CINT                                                      KRNL0097
      WRITE(IW,9) DX,CINT,UPST                                           KRNL0098
      WRITE(IW,10) Y,SUM,DOWN                                            KRNL0099
      WRITE(7,4) DX,Y,F,SUM,CINT,UPST,DOWN                               KRNL0100
      HV1=1./(Y**2)                                                      KRNL0101
      HV2=-DX/((Y**2)*SQRT(DX**2+Y**2))                                  KRNL0102
      HVUP=-2.*(HV1+HV2)                                                 KRNL0103
      HVDN=-2.*(HV1-HV2)                                                 KRNL0104
      WRITE(IW,20) HVUP,HVDN                                             KRNL0105
      FY=SQRT(Y)                                                         KRNL0106
      CINT=CINT*FY                                                       KRNL0107
      UPST=UPST*FY                                                       KRNL0108
```

```
      SUM=SUM*FY                                                        KRNL0109
      DOWN=DOWN*FY                                                      KRNL0110
      WRITE(IW,5) CINT,SUM,UPST,DOWN                                    KRNL0111
20000 CONTINUE                                                          KRNL0112
      GOTO 13                                                           KRNL0113
4000  STOP                                                              KRNL0114
      END                                                               KRNL0115
```

```
      SUBROUTINE WNINT(TH,ANS)                                          WVNIC001
C        INTEGRAL OVER WAVE NUMBERS                                     WVNIC002
      COMMON CTH,CPH,STH,C2,DX,Y,F                                      WVNIC003
      COMMON THD,SPH,ER                                                 WVNIC004
      DIMENSION V(7),W(7)                                               WVNIC005
      DIMENSION DUM(1CCC)                                               WVNIC006
      DATA V/.19304,1.0266,2.56788,4.90035,8.18215,12.73418,19.39573/   WVNIC007
      DATA W/4.09319E-01,4.21831E-01,1.47126E-01,2.06335E-02,1.07401E-03 WVNIC008
     1,1.58655E-05,3.17032E-08/                                         WVNIC009
1     FORMAT(I5,6E17.6)                                                 WVNIC010
      IR=5                                                              WVNIC011
      IW=6                                                              WVNIC012
      CTH=COS(TH)                                                       WVNIC013
      STH=SIN(TH)                                                       WVNIC014
      A1=0.0                                                            WVNIC015
      A2=0.0                                                            WVNIC016
      A4=0.0                                                            WVNIC017
      A3=0.0                                                            WVNIC018
      DEN=SQRT(Y**2+(DX*CTH)**2)                                        WVNIC019
      CPH=Y/DEN                                                         WVNIC020
      SPH=-DX*CTH/DEN                                                   WVNIC021
      DS=DX*CTH/Y                                                       WVNIC022
      PP=DS**2                                                          WVNIC023
      DS=1.732*DS                                                       WVNIC024
      IF (PP.GT.3.0) GOTO 411                                           WVNIC025
      SZC=0.5*(3.-PP)                                                   WVNIC026
      CP=1.                                                             WVNIC027
      IF(ABS(SZC-1.).GT.DS) GOTO 70                                     WVNIC028
      CP=SZC                                                            WVNIC029
      DS=2.*DS                                                          WVNIC030
C        INTEGRATE REGION NEAR THE NU MERICAL SINGULARITY               WVNIC031
70    PANS=0.0                                                          WVNIC032
      TEST=0.0                                                          WVNIC033
      IB=1                                                              WVNIC034
      IE=3                                                              WVNIC035
      IS=1                                                              WVNIC036
```

```
      IF(DS.GT.CP) DS=CP                                                WVNIC037
      NI=2                                                              WVNIC038
90    NP=NI+1                                                           WVNIC039
      H=2.*DS/FLOAT(NI)                                                 WVNIC040
      DO 100 I=IB,IE,IS                                                 WVNIC041
      S=CP-DS+(I-1)*H                                                   WVNIC042
100   DUM(I)=FUNS(S)                                                    WVNIC043
      CALL SIMR(NI,H,DUM,ANS)                                           WVNIC044
      IF(ABS(ANS).LE.1.E-50) GOTO 110                                   WVNIC045
      TEST=ABS((ANS-PANS)/ANS)                                          WVNIC046
      IF (TEST.LE.ER) GOTO 110                                          WVNIC047
      IF (NI.GT.256) GOTO 105                                           WVNIC048
      CALL SPRD(DUM,NP)                                                 WVNIC049
      NI=NI*2                                                           WVNIC050
      IB=2                                                              WVNIC051
      IE=NI                                                             WVNIC052
      IS=2                                                              WVNIC053
      PANS=ANS                                                          WVNIC054
      GOTO 90                                                           WVNIC055
105   WRITE(IW,1) NI,DS,H,THD,ANS,TEST                                  WVNIC056
110   A1=ANS                                                            WVNIC057
C        INTEGRATE REGION ZERO TO PEAK REGION                           WVNIC058
      PANS=0.0                                                          WVNIC059
      IF(DS.GE.CP) GOTO 211                                             WVNIC060
      IB=2                                                              WVNIC061
      IE=3                                                              WVNIC062
      IS=1                                                              WVNIC063
      DUM(1)=0.0                                                        WVNIC064
      TEST=0.0                                                          WVNIC065
      NI=2                                                              WVNIC066
190   NP=NI+1                                                           WVNIC067
      H=(CP-DS)/FLOAT(NI)                                               WVNIC068
      DO 200 I=IB,IE,IS                                                 WVNIC069
      S=(I-1)*H                                                         WVNIC070
200   DUM(I)=FUNS(S)                                                    WVNIC071
      CALL SIMR(NI,H,DUM,ANS)                                           WVNIC072
```

```
      IF (ABS(ANS).LE.1.E-50) GOTO 210                                  WVNIC073
      TEST=ABS((ANS-PANS)/ANS)                                          WVNIC074
      IF (TEST.LE.ER) GOTO 210                                          WVNIC075
      IF (NI.GT.256) GOTO 205                                           WVNIC076
      CALL SPRD(DUM,NP)                                                 WVNIC077
      NI=NI*2                                                           WVNIC078
      IB=2                                                              WVNIC079
      IE=NI                                                             WVNIC080
      IS=2                                                              WVNIC081
      PANS=ANS                                                          WVNIC082
      GOTO 190                                                          WVNIC083
205   WRITE(IW,1) NI,DS,H,A1,THD,ANS,TEST                               WVNIC084
210   A2=ANS                                                            WVNIC085
211   CONTINUE                                                          WVNIC086
C        INTEGRATE REGION JUST BEYOND SINGULAR POINT                    WVNIC087
      IF(CP+DS.LT.SZC) GOTO 385                                         WVNIC088
      SUL=CP+DS                                                         WVNIC089
      GOTO 411                                                          WVNIC090
385   SUL=2.*SZC-CP-DS                                                  WVNIC091
      PANS=0.0                                                          WVNIC092
      IB=1                                                              WVNIC093
      IE=3                                                              WVNIC094
      IS=1                                                              WVNIC095
      TEST=0.0                                                          WVNIC096
      NI=2                                                              WVNIC097
390   NP=NI+1                                                           WVNIC098
      H=2.*(SZC-CP-DS)/FLOAT(NI)                                        WVNIC099
      DO 400 I=IB,IE,IS                                                 WVNIC100
      S=CP+DS+H*(I-1)                                                   WVNIC101
400   DUM(I)=FUNS(S)                                                    WVNIC102
      CALL SIMR(NI,H,DUM,ANS)                                           WVNIC103
      IF (ABS(ANS).LE.1.E-50) GOTO 410                                  WVNIC104
      TEST=ABS((ANS-PANS)/ANS)                                          WVNIC105
      IF (TEST.LE.ER) GOTO 410                                          WVNIC106
      IF (NI.GT.256) GOTO 405                                           WVNIC107
      CALL SPRD(DUM,NP)                                                 WVNIC108
```

```
      NI=NI*2                                                          WVNI0109
      IB=2                                                             WVNI0110
      IE=NI                                                            WVNI0111
      IS=2                                                             WVNI0112
      PANS=ANS                                                         WVNI0113
      GOTO 390                                                         WVNI0114
405   WRITE(IW,1) NI,SUL,H,A1,THD,ANS,TEST                             WVNI0115
410   A4=ANS                                                           WVNI0116
411   CONTINUE                                                         WVNI0117
C        LAGUERRE POLYNOMIAL INTERPOLATION QUADRATURE TO INFINUTY      WVNI0118
      SS=SUL                                                           WVNI0119
      DO 300 I=1,7                                                     WVNI0120
       S=SS+V(I)                                                       WVNI0121
       T1=FUNS(S)                                                      WVNI0122
      BF=1.                                                            WVNI0123
      IF(T1.LT.0.0) BF=-1.                                             WVNI0124
      IF(T1.EQ.0.0) GOTO 300                                           WVNI0125
      T1=ABS(T1)                                                       WVNI0126
      E1=ALOG(T1)                                                      WVNI0127
      E2=V(I)                                                          WVNI0128
         E3=ALOG(W(I))                                                 WVNI0129
      E=E1+E2+E3                                                       WVNI0130
      IF (ABS(E).GT.170.) GOTO 300                                     WVNI0131
      A3=A3+BF*EXP(E)                                                  WVNI0132
300   CONTINUE                                                         WVNI0133
      ANS=A1+A2+A3+A4                                                  WVNI0134
      RETURN                                                           WVNI0135
      END                                                              WVNI0136
```

```
      FUNCTION FUNS(S)                                                  FUNS0001
      COMMON CTH,CPH,STH,C2,DX,Y,F                                      FUNS0002
      COMMON THD,SPH,ER                                                 FUNS0003
      IF (S.LE.0.0) GOTO 20                                             FUNS0004
      PP=(DX*CTH/Y)**2                                                  FUNS0005
      T1=(4./3.14159)*(F**2)                                            FUNS0006
      T2=STH**2                                                         FUNS0007
      T3=(DX/Y)*(S**2)                                                  FUNS0008
      T4=2.*S-3.+PP                                                     FUNS0009
      T5=-F*Y*S/(CTH**2)                                                FUNS0010
      T6=(1.+PP)**2                                                     FUNS0011
      T7=S**2-2.*S+1.+PP                                                FUNS0012
      T8=CTH**4                                                         FUNS0013
      IF (T8.LE.0.0) GOTO 20                                            FUNS0014
      TEST=T5-ALOG(T8)                                                  FUNS0015
      IF(ABS(TEST).GT.160.) GOTO 20                                     FUNS0016
      FUNS=(T1*T2*T3*T4/(T6*T7))*EXP(TEST)                              FUNS0017
      RETURN                                                            FUNS0018
20    FUNS=0.0                                                          FUNS0019
      RETURN                                                            FUNS0020
      END                                                               FUNS0021
```

```
      FUNCTION RFUN(Q)                                                  RFUNC001
      COMMON CTH,CPH,STH,C2,DX,Y,F                                      RFUNC002
      COMMON THD,SPH,ER                                                 RFUNC003
      X1=Q/(F*Y)                                                        RFUNC004
      RFUN=SQRT(X1*(X1+1))*COS(F*DX*SQRT(X1+1))                         RFUNC005
      RETURN                                                            RFUNC006
      END                                                               RFUNC007
```

```
      SUBROUTINE SIMR(NI,H,FUNC,ANS)                                    SIMRC001
      DIMENSION FUNC(1000)                                              SIMRC002
      K=NI+1                                                            SIMRC003
      ANS=FUNC(1)+FUNC(K)                                               SIMRC004
      DO 10 J=2,NI,2                                                    SIMRC005
10    ANS=ANS+4*FUNC(J)                                                 SIMRC006
      IF(NI.EQ.2)GOTO 30                                                SIMRC007
      K=NI-1                                                            SIMRC008
      DO 20 J=3,K,2                                                     SIMRC009
20    ANS=ANS+2*FUNC(J)                                                 SIMRC010
30    ANS=ANS*H/3.                                                      SIMRC011
      RETURN                                                            SIMRC012
      END                                                               SIMRC013
```

BIOGRAPHICAL NOTE

Edward C. Kern, Jr. was born on June 1, 1945 and attended the Lawrenceville School in Lawrenceville, New Jersey. He received the degree Artium Baccalaureus from Dartmouth College in 1967 and the degree Bachelor of Engineering from the Thayer School of Engineering at Dartmouth College in 1968. He then entered the Graduate School at Massachusetts Institute of Technology and received the S.M. and Ocean Engineer's degrees in 1971. While at Massachusetts Institute of Technology he has held appointments as teaching assistant, research assistant and DSR engineer. He is a member of Sigma Xi.