ON THE SECONDARY MOTION INDUCED BY OSCILLATIONS IN A SHEAR FLOW

by

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SECTION I INTRODUCTION

One of the most basic and challenging problems in fluid mechanics is to obtain an understanding of the various physical mechanisms involved during the transition from laminar to turbulent flow. It could be called the missing link between the two regimes of fluid motion. This problem has been subject to a great deal of theoretical and experimental research, especially over the last decade, and considerable progress has been made to bridge the gap. However, there remains a vast amount of work still to be done before our knowledge of the transition phenomenon is complete. One may anticipate that the final answer will include many simultaneous effects.

In order to study the breakdown of laminar flow it is necessary to follow the growth of a disturbance superposed on the basic flow. If this disturbance is of very small amplitude the equations can be linearized and one can develop the so called linear theory of hydrodynamic stability. This theory has been investigated in great detail and a survey of the subject is given in the monograph by Lin [1]. There can be no doubt that the initial trend of a small disturbance will be adequately described by the results of the linearized theory. Indeed there is now ample experimental evidence to support this fact [2]. However as the oscillation grows the non linear terms in the equations become important, and must be included in the investigation.

It has long been recognized that the inclusion of the non linear terms adds two important new features to the problem. Firstly there is the effect of the Reynolds stresses in producing a redistribution of momentum and so a distortion of the original velocity profile, and secondly the excitation of higher harmonics of the original oscillation. For finite amplitude oscillations the modification of the basic flow through the action of the Reynolds stresses can be quite appreciable, and this will in turn modify the rate of extraction of energy from the mean flow to the disturbance, and so the rate of growth of the disturbance. Meksyn and Stuart [3] have calculated these modifications for the case of flow between parallel plates, and also gave some calculations showing that the production of higher harmonics plays a less important role. Their results show that an increase in the amplitude of the oscillation produces a lower critical Reynolds number. More recently Stuart [4] has given a somewhat simpler analysis based on energy methods and has obtained good agreement with experiment for the case of flow between rotating circular cylinders. Both these discussions consider only two dimensional disturbances and for the case of channel flow the modified flow profile does not show a higher velocity gradient near the wall which might be anticipated for a "more turbulent flow." The importance of the critical layer region as the "weak spot" of instability has been stressed by Lin [5]. He has shown that for disturbances in a parallel flow all the harmonic components of the oscillation simultaneously become important around the critical layer, before the amplitude of the fundamental is large enough to cause any significant distortion of the mean flow.

The search for a suitable mechanism to describe the onset of turbulence has aroused much interest. Several very plausible theories have been proposed and all possess some element of truth, although no single one appears to be the complete answer. Landau's concept [6] of successive instabilities seems intuitively very reasonable, and enables one to picture the appearance of additional modes of oscillation corresponding to a sequence of critical Reynolds numbers. Görtler and Witting [7] have proposed a theory, in line with Landau's conjecture, based on the curvature of the streamlines causing a periodic vortex structure. There is experimental evidence to support the existence of secondary vortices although there has not been any definite confirmation of the phase relationships involved.

A horseshoe vortex structure as the fundamental element of transition has been proposed by Theordorson [8], who also suggested that two dimensional disturbances are unimportant during transition. This latter conjecture is strongly supported by the recent experiments of Schubauer, Klebanoff, and Tidstrom, which we shall briefly discuss below [9] [10].

The presence of longitudinal vortices during transition has been reported by many experimenters. These can be observed using dye and china clay techniques [11] [12].

Mention must be made of the relative importance of two and three dimensional disturbances. This is a current issue that has attracted much attention. On the basis of linearized theory, Squire's result [13], namely that three dimensional disturbances are equivalent to two dimensional ones at a lower Reynolds number is applicable, and so to estimate the onset of instability one need only consider two dimensional disturbances. However, no such result can be expected to hold for oscillations of finite amplitude. Indeed one would strongly suspect that as turbulence is an essentially three dimensional phenomenon there must be a stage during development when the three dimensional disturbances tend to dominate. Two dimensional theory cannot be expected to suffice. This simple observation suggests the obvious necessity of a theoretical investigation of three dimensional effects.

Recent experiments also point strongly to the desirability of such an investigation. Notable among the vast array of experiments probing the phenomena of transition is the work of Schubauer, Klebanoff, and Tidstrom, at the National Bureau of Standards. Most of their work has concerned boundary layer transition on a flat plate. Perhaps the most startling and significant fact revealed by these experiments is the almost periodic spanwise variation of intensity with peaks and valleys occupying fixed positions forming streets of high and low intensity. This periodic spanwise variation causes a warping of the velocity profile, the turbulence appearing to originate at these peaks and to spread into the valleys. More recently further experimental work on this spanwise variation has been done and we shall have occasion to refer to it at a later stage.

This brief introduction points to the multitude of effects observed and predicted during transition. If it serves no other purpose at least it does pose the question as to whether there is any advantage in a theoretical approach which does not include all the non linear terms. The complete solution of the non linear equations should automatically include all of these effects. A detailed theoretical consideration of all the non linear terms has been advocated by von Kármán, and it is in this spirit that we have undertaken the present preliminary investigation.

Our task is now quite clear. We require to examine finite amplitude disturbances paying special attention to the three dimensional oscillations. It is to be stressed that this will be done by setting up a systematic perturbation from the linear theory, and that a purely formal mathematical approach is adopted, independent of empirical results.

It is difficult to give a detailed explanation of our conclusions before the actual calculations have been made. Therefore, at this stage only brief comment will be made on the interpretation of the results. A detailed description is given in Section V.

The quantity found to be of prime importance is the mean secondary vorticity in the downstream direction. One term in this vorticity has a periodic spanwise variation and produces a redistribution of momentum in planes perpendicular to the direction of flow. It is this momentum exchange that is responsible for an alternate steepening and flattening of the velocity profile, causing a warping or crumbling effect on the basic flow. Explicit formulas are obtained for the rate of growth of the second order mean motion. Superposed on these secondary vortices there is the vorticity of the primary oscillation itself. This is periodic in the downstream direction and so the two effects combined should produce alternately partial reinforcement and cancellation over each wave length.

The results obtained are applicable to a general parallel flow; but for illustrative purposes we have restricted the detailed calculations to the case of a shear profile. It is to be noted that although our interests are chiefly with three dimensional disturbances we do not discount two dimensional effects. The effects calculated by Meksyn and Stuart are in fact included in our analysis. It is believed, however, that in most situations the spanwise profile distortion will be the more important mechanism during transition.

SECTION II MATHEMATICAL FORMULATION

We now proceed to give the mathematical formulation on which the subsequent calculations will be based. As mentioned in Section I, the calculation of the second order vorticity will play a key role in this development. To this purpose it would suffice to use the vorticity equation, namely,

$$\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial u_i}{\partial x_j} + \frac{1}{R} \frac{\partial^2 \omega_i}{\partial x_j^2}. \tag{2.1}$$

This will be given at the end of this section. But in order to calculate the second order velocities it is convenient to proceed directly from the equations of motion, together with the continuity equation. That is,

$$\frac{\partial u_k}{\partial x_k} = 0, \tag{2.2}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{R} \frac{\partial^2 u_i}{\partial x_j^2}; \tag{2.3}$$

where we have supposed the fluid to be incompressible and all quantities are expressed in dimensionless form.

We consider a basic flow $u = u_0(y)$, and wish to trace the growth of a wave of small amplitude propagating in the x direction and having a z variation in its amplitude, that is standing waves in the spanwise direction. This is not the most general infinitesimal oscillation, as a purely two dimensional component could be superposed. In the present investigation this two dimensional component of the primary oscillation is neglected and we consider only the three dimensional effects. Some justification is desirable in taking this apparently drastic step, and also it is pertinent to add some qualitative remarks as to what the additional characteristics would be if the two dimensional component were not neglected. From a theoretical viewpoint an investigation of purely three dimensional oscillations will yield the effects that originate only with these disturbances. Of course in the practical situation these effects must be present to some degree for undoubtedly there will always be some spanwise irregularity in the amplitude of the oscillation. The question then arises as to which component will be dominant as transition is approached. We believe that the three dimensional wave is the more important one. There is strong experimental support for this conjecture. Schubauer and Klebanoff in their experimental reports say that transition never occurs without first being preceded by a strong warping of the wave. Near the point of breakdown of the linear theory and the onset of finite amplitude effects a typical ratio for the amplitudes of the two and three dimensional waves would appear to be about $\frac{1}{5}$.

What modifications are to be expected if the two dimensional component of the primary oscillation is not neglected? Firstly there will be the purely two dimensional effects, namely a spanwise independent distortion of the basic flow and the generation of higher harmonics, exactly as calculated by Stuart. Also present will be second order interactions of the two and three dimensional oscillations. The latter will produce two components, the first being of high frequency and the second of a quasi steady nature. These interaction effects can be expected to be more important than the purely two dimensional ones; but neither should be comparable to the dominating three dimensional disturbances.

Let α and β be dimensionless wave numbers associated with the x and z directions, these being the downstream and spanwise directions respectively. Limiting ourselves to a primary oscillation having a sinusoidal spanwise variation of amplitude, we may assume the velocity components and the pressure to be of the form,

$$u(x,y,z,t) = \bar{u}(y,z,t) + \sum_{r=1}^{\infty}\left[u^{(r)}(y,t)e^{ir\alpha x} + u^{(r)*}(y,t)e^{-ir\alpha x}\right]\cos r\beta z$$

$$+ \sum_{r=2}^{\infty}\left[U^{(r)}(y,t)e^{ir\alpha x} + U^{(r)*}(y,t)e^{-ir\alpha x}\right], \qquad (2.4)$$

$$v(x,y,z,t) = \bar{v}(y,z,t) + \sum_{r=1}^{\infty}\left[v^{(r)}(y,t)e^{ir\alpha x} + v^{(r)*}(y,t)e^{-ir\alpha x}\right]\cos r\beta z$$

$$+ \sum_{r=2}^{\infty}\left[V^{(r)}(y,t)e^{ir\alpha x} + V^{(r)*}(y,t)e^{-ir\alpha x}\right], \quad (2.5)$$

$$w(x,y,z,t) = \bar{w}(y,z,t) + \sum_{r=1}^{\infty}\left[w^{(r)}(y,t)e^{ir\alpha x} + w^{(r)*}(y,t)e^{-ir\alpha x}\right]\sin r\beta z, \quad (2.6)$$

$$p(x,y,z,t) = \bar{p}(x,y,z,t) + \sum_{r=1}^{\infty}\left[p^{(r)}(y,t)e^{ir\alpha x} + p^{(r)*}(y,t)e^{-ir\alpha x}\right]\cos r\beta z. \quad (2.7)$$

An asterisk has been used to denote a complex conjugate, and the term of zeroth order in $\bar{p}(x,y,z,t)$ is taken to be the prescribed external pressure $p_0(x)$. Bars denote mean values taken with respect to x. Superscripts relate to the order of the harmonic involved and subscripts to the orders of magnitude of the various quantities. Therefore if a is taken as a scale representative of the amplitude of the disturbance, we have,

$$u^{(r)}(y,t) = a^r u_r^{(r)}(y,t) + a^{r+2}u_{r+2}^{(r)}(y,t) + \cdots \quad , \quad (2.8)$$

$$U^{(r)}(y,t) = a^r U_r^{(r)}(y,t) + a^{r+2}U_{r+2}^{(r)}(y,t) + \cdots \quad . \quad (2.9)$$

If the terms of order a^2 and higher are neglected we have the usual linear theory. As this analysis is of an exploratory nature we shall neglect terms of order a^3 and higher. In a complete investigation it would be necessary to include all the higher order terms. This would enable us to trace the growth of a disturbance through to fully developed turbulence. However, the analysis would be greatly complicated by the inclusion of these higher order terms and we shall therefore be content to work to this approximation.

Having set this limitation it is now desirable to drop the superscripts and revert to a less cumbersome notation. Anticipating the z dependence, we write,

$$\bar{u}(y,z,t) = u_0(y) + a^2\big(u_a(y,t)\cos 2\beta z + u_b(y,t)\big) + O(a^4), \quad (2.10)$$

$$\bar{v}(y,z,t) = a^2\big(v_a(y,t)\cos 2\beta z + v_b(y,t)\big) + O(a^4), \quad (2.11)$$

$$\bar{w}(y,z,t) = a^2\big(w_a(y,t)\sin 2\beta z + w_b(y,t)\big) + O(a^4), \quad (2.12)$$

$$\bar{p}(x,y,z,t) = p_0(x) + a^2\big(p_a(y,t)\cos 2\beta z + p_b(y,t)\big) + O(a^4), \quad (2.13)$$

$$u^{(1)}(y,t) = a u_1(y,t) + O(a^3), \tag{2.14}$$

$$v^{(1)}(y,t) = a v_1(y,t) + O(a^3), \tag{2.15}$$

$$w^{(1)}(y,t) = a w_1(y,t) + O(a^3), \tag{2.16}$$

$$p^{(1)}(y,t) = a p_1(y,t) + O(a^3), \tag{2.17}$$

$$u^{(2)}(y,t) = a^2 u_2(y,t) + O(a^4), \tag{2.18}$$

$$v^{(2)}(y,t) = a^2 v_2(y,t) + O(a^4), \tag{2.19}$$

$$w^{(2)}(y,t) = a^2 w_2(y,t) + O(a^4), \tag{2.20}$$

$$p^{(2)}(y,t) = a^2 p_2(y,t) + O(a^4), \tag{2.21}$$

$$U^{(2)}(y,t) = a^2 U_2(y,t) + O(a^4), \tag{2.22}$$

$$V^{(2)}(y,t) = a^2 V_2(y,t) + O(a^4). \tag{2.23}$$

If the assumed forms for u, v, w, p are now substituted into the equations (2.2) and (2.3) we obtain sets of partial differential equations for the determination of the various functions. In a given physical situation there would also be corresponding sets of initial and boundary conditions. The substitution is perfectly straight forward, but tedious, and is therefore omitted. The results yield,

$$0 = -\frac{dp_0}{dx} + \frac{1}{R}\frac{d^2 u_0}{dy^2}. \tag{2.24}$$

$$\left.\begin{aligned}
\frac{\partial v_a}{\partial y} + 2\beta w_a &= 0, \\
\frac{\partial u_a}{\partial t} + v_a\frac{du_0}{dy} + \frac{1}{2}\left(v_1\frac{\partial u_1^*}{\partial y} + v_1^*\frac{\partial u_1}{\partial y}\right) + \frac{\beta}{2}\left(w_1 u_1^* + w_1^* u_1\right) &= \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} - 4\beta^2\right)u_a, \\
\frac{\partial v_a}{\partial t} + \frac{i\alpha}{2}\left(u_1^* v_1 - u_1 v_1^*\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(v_1 v_1^*\right) + \frac{\beta}{2}\left(w_1 v_1^* + w_1^* v_1\right) &= -\frac{\partial p_a}{\partial y} + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} - 4\beta^2\right)v_a, \\
\frac{\partial w_a}{\partial t} + \frac{i\alpha}{2}\left(w_1 u_1^* - w_1^* u_1\right) + \frac{1}{2}\left(v_1\frac{\partial w_1^*}{\partial y} + v_1^*\frac{\partial w_1}{\partial y}\right) + \beta w_1 w_1^* &= 2\beta p_a + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} - 4\beta^2\right)w_a.
\end{aligned}\right\} \tag{2.25}$$

$$
\left.
\begin{aligned}
\frac{\partial v_b}{\partial y} &= 0, \\
\frac{\partial u_b}{\partial t} + v_b \frac{du_0}{dy} + \frac{1}{2}\left(v_1 \frac{\partial u_1^*}{\partial y} + v_1^* \frac{\partial u_1}{\partial y}\right) - \frac{\beta}{2}\left(w_1 u_1^* + w_1^* u_1\right) &= \frac{1}{R}\frac{\partial^2 u_b}{\partial y^2}, \\
\frac{\partial v_b}{\partial t} + \frac{i\alpha}{2}\left(u_1^* v_1 - u_1 v_1^*\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(v_1 v_1^*\right) - \frac{\beta}{2}\left(v_1 w_1^* + v_1^* w_1\right) &= -\frac{\partial p_b}{\partial y} + \frac{1}{R}\frac{\partial^2 v_b}{\partial y^2}, \\
\frac{\partial w_b}{\partial t} &= \frac{1}{R}\frac{\partial^2 w_b}{\partial y^2}.
\end{aligned}
\right\} \quad (2.26)
$$

$$
\left.
\begin{aligned}
i\alpha u_1 + \frac{\partial v_1}{\partial y} + \beta w_1 &= 0, \\
\frac{\partial u_1}{\partial t} + i\alpha u_0 u_1 + v_1 \frac{du_0}{dy} &= -i\alpha p_1 + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} - \alpha^2 - \beta^2\right) u_1, \\
\frac{\partial v_1}{\partial t} + i\alpha u_0 v_1 &= -\frac{\partial p_1}{\partial y} + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} - \alpha^2 - \beta^2\right) v_1, \\
\frac{\partial w_1}{\partial t} + i\alpha u_0 w_1 &= \beta p_1 + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} - \alpha^2 - \beta^2\right) w_1.
\end{aligned}
\right\} \quad (2.27)
$$

$$\left.\begin{aligned}
&2i\alpha u_2 + \frac{\partial v_2}{\partial y} + 2\beta w_2 = 0,\\
&\frac{\partial u_2}{\partial t} + 2i\alpha u_0 u_2 + \frac{i\alpha u_1^2}{2} + v_2\frac{du_0}{dy} + \frac{1}{2}v_1\frac{\partial u_1}{\partial y} + \frac{1}{2}\beta u_1 w_1 = -2i\alpha p_2 + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} - 4\alpha^2 - 4\beta^2\right)u_2,\\
&\frac{\partial v_2}{\partial t} + 2i\alpha u_0 v_2 + \frac{i\alpha u_1 v_1}{2} + \frac{1}{2}v_1\frac{\partial v_1}{\partial y} + \frac{1}{2}\beta v_1 w_1 = -\frac{\partial p_2}{\partial y} + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} - 4\alpha^2 - 4\beta^2\right)v_2,\\
&\frac{\partial w_2}{\partial t} + 2i\alpha u_0 w_2 + \frac{i\alpha u_1 w_1}{2} + \frac{1}{2}v_1\frac{\partial w_1}{\partial y} + \frac{\beta}{2}w_1^2 = 2\beta p_2 + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} - 4\alpha^2 - 4\beta^2\right)w_2.
\end{aligned}\right\}\quad(2.28)$$

$$\left.\begin{aligned}
&2i\alpha U_2 + \frac{\partial V_2}{\partial y} = 0,\\
&\frac{\partial U_2}{\partial t} + 2i\alpha u_0 U_2 + V_2\frac{du_0}{dy} + \frac{i\alpha}{2}u_1^2 + \frac{1}{2}v_1\frac{\partial u_1}{\partial y} - \frac{\beta}{2}u_1 w_1 = \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} - 4\alpha^2\right)U_2,\\
&\frac{\partial V_2}{\partial t} + 2i\alpha u_0 V_2 + \frac{i\alpha}{2}u_1 v_1 + \frac{1}{2}v_1\frac{\partial v_1}{\partial y} - \frac{\beta}{2}v_1 w_1 = \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} - 4\alpha^2\right)V_2.
\end{aligned}\right\}\quad(2.29)$$

The set of equations (2.27) are the familiar ones of the linear theory, and the sets (2.28) and (2.29) are those determining the induced second harmonics, one of these being purely two dimensional. The sets of equations (2.25) and (2.26) decide the way in which the non linear terms will distort the original velocity profile. The velocity components U_b, V_b, W_b, involved in (2.26) result in a purely two dimensional distortion; but are induced by the three dimensional primary oscillation. This is a slightly more general effect than that considered by Stuart. His analysis would correspond essentially to the case $\beta \to 0$. Our interest centres on the three dimensional distortions determined by the functions $U_a, V_a, W_a,$ in the set (1.25). These modifications to the velocity profile will be sinusoidal in z and it might be anticipated that they will cause a spanwise periodic steepening and flattening of the original velocity profile. It is this three dimensional "crumbling effect" that we now investigate.

For this purpose it is convenient to use the mean vorticity components $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ where,

$$(\bar{\xi}, \bar{\eta}, \bar{\zeta}) = \left(\frac{\partial \bar{w}}{\partial y} - \frac{\partial \bar{v}}{\partial z}, \frac{\partial \bar{u}}{\partial z} - \frac{\partial \bar{w}}{\partial x}, \frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} \right). \qquad (2.30)$$

Thus we have,

$$\left.\begin{aligned} \bar{\xi} &= a^2(\xi_a \sin 2\beta z + \xi_b) + O(a^4), \\ \bar{\eta} &= a^2(\eta_a \sin 2\beta z + \eta_b) + O(a^4), \\ \bar{\zeta} &= -\frac{du_0}{dy} + a^2(\zeta_a \cos 2\beta z + \zeta_b) + O(a^4). \end{aligned}\right\} \quad (2.31)$$

where on referring to equations (2.10) (2.11) and (2.12),

$$\xi_a = \frac{\partial w_a}{\partial y} + 2\beta v_a \;;\quad \eta_a = -2\beta u_a \;;\quad \zeta_a = -\frac{\partial u_a}{\partial y} \;; \qquad (2.32)$$

$$\xi_b = \frac{\partial w_b}{\partial y} \;;\quad \eta_b = 0 \;;\quad \zeta_b = -\frac{\partial u_b}{\partial y}. \qquad (2.33)$$

The differential equation governing ξ_a is determined by eliminating p_a from the set (1.25). We find,

$$\frac{\partial \xi_a}{\partial t} + 2\beta \frac{\partial}{\partial y}\left(v_1 v_1^* + w_1 w_1^*\right) + \left(\frac{1}{2}\frac{\partial^2}{\partial y^2} + 2\beta^2\right)\left(v_1 w_1^* + v_1^* w_1\right)$$

$$= \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} - 4\beta^2\right)\xi_a. \qquad (2.34)$$

This last equation (and two similar ones) can, as we remarked earlier, be obtained from the vorticity equation (2.1) on taking averages in x. In fact a very much more general form can be obtained without approximation, or assumption on the spanwise dependence of the oscillation. If u', v', w', are the components of the oscillation it is easily verified that,

$$\frac{\partial \overline{\xi}}{\partial t} + \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)\overline{v'w'} + \frac{\partial^2}{\partial y \partial z}\left(\overline{w'^2} - \overline{v'^2}\right) = \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\overline{\xi}. \qquad (2.35)$$

SECTION III CALCULATION OF DISTORTION EFFECTS AT HIGH REYNOLDS NUMBERS

For flows at high Reynolds numbers the velocity field will conform to the inviscid equations as a first approximation, except within the critical layer region, where viscous corrections would be needed. A discussion of the subtleties involved in this limiting process can be found in Chapter 8 of Reference [1]. Indeed the inviscid limit is justifiable for the case of amplified disturbances and for neutral disturbances as a limiting case. If the complex wave velocity is $c = c_r + i c_i$, then c_r represents the wave speed, and $c_i \gtreqless 0$ implies amplified, neutral, or damped disturbances respectively. It is convenient to introduce the amplitude functions for the primary oscillations; these are denoted by a circumflex. Thus, $u_1(y,t) = \hat{u}_1(y) e^{-i\alpha c t}$, and similarly for v_1, w_1, and p_1.

In studying the growth or decay of an oscillation we are interested in the case when c_i is close to zero. For the case of large Reynolds numbers we can find quite simple explicit formulas for the rates of growth of the second order velocity modifications. For this purpose it is convenient to rewrite equations (2.25) and (2.34). Dropping the viscous terms we have,

$$\frac{\partial v_a}{\partial y} + 2\beta w_a = 0, \qquad (3.1)$$

$$\frac{\partial w_a}{\partial y} + 2\beta v_a = \mathcal{E}_a, \tag{3.2}$$

$$\frac{\partial \mathcal{E}_a}{\partial t} + \hat{X}_a(y)\, e^{2\alpha c_i t} = 0, \tag{3.3}$$

$$\frac{\partial u_a}{\partial t} + v_a \frac{du_0}{dy} + \hat{Y}_a(y)\, e^{2\alpha c_i t} = 0, \tag{3.4}$$

where, $$\hat{X}_a(y) = 2\beta \frac{d}{dy}\left(\hat{v}_1 \hat{v}_1^* + \hat{w}_1 \hat{w}_1^*\right) + \left(\frac{1}{2}\frac{d^2}{dy^2} + 2\beta^2\right)\left(\hat{v}_1 \hat{w}_1^* + \hat{v}_1^* \hat{w}_1\right), \tag{3.5}$$

and, $$\hat{Y}_a(y) = \frac{1}{2}\left(\hat{v}_1 \frac{d\hat{u}_1^*}{dy} + \hat{v}_1^* \frac{d\hat{u}_1}{dy}\right) + \frac{\beta}{2}\left(\hat{w}_1 \hat{u}_1^* + \hat{w}_1^* \hat{u}_1\right). \tag{3.6}$$

Integrating (3.3) we have,

$$\mathcal{E}_a = -\hat{X}_a(y)\left(\frac{e^{2\alpha c_i t} - 1}{2\alpha c_i}\right) + \hat{C}(y). \tag{3.7}$$

The form of the solution is so written that the limiting case $c_i \to 0$ can be readily discussed. Clearly the arbitrary function of integration gives the initial value of ξ_a, which we may take as zero for the purpose at hand. We note in passing that a more general result would be obtained if use were made of equation (2.35).

Using this result for ξ_a the time dependence of the velocity components can easily be found. The results are,

$$u_a = -\frac{du_0}{dy}\hat{v}_a(y)\left(\frac{e^{2\alpha c_i t}-1-2\alpha c_i t}{(2\alpha c_i)^2}\right) - \hat{Y}_a(y)\left(\frac{e^{2\alpha c_i t}-1}{2\alpha c_i}\right), \tag{3.8}$$

$$v_a = \hat{v}_a(y)\left(\frac{e^{2\alpha c_i t}-1}{2\alpha c_i}\right), \tag{3.9}$$

$$w_a = \hat{w}_a(y)\left(\frac{e^{2\alpha c_i t}-1}{2\alpha c_i}\right), \tag{3.10}$$

$$\xi_a = -\hat{X}_a(y)\left(\frac{e^{2\alpha c_i t}-1}{2\alpha c_i}\right), \tag{3.11}$$

where,

where, $$\frac{d\hat{v}_a}{dy} + 2\beta\hat{w}_a = 0, \tag{3.12}$$

and, $$\frac{d\hat{w}_a}{dy} + 2\beta \hat{v}_a = -\hat{X}_{a}(y). \tag{3.13}$$

These equations clearly show that the second order mean velocities (which are sinusoidal in z) grow exponentially in time, or in the case of neutral primary oscillations as powers of t. Similar growth rates can be found for the purely two dimensional distortions. The most interesting feature of this analysis is that it shows that the non linearity induces a second order mean vorticity having a component ξ_a in the downstream direction. It is this mechanism which produces a spanwise periodic momentum exchange and causes a periodic warping of the original velocity profile.

In the case of neutral disturbances, which can be properly treated in the limit as $c_i \to 0$, we also get a build up in the second order velocity and vorticity components. Using, $\lim\limits_{c_i \to 0}\left(\dfrac{e^{2\alpha c_i t} - 1}{2\alpha c_i}\right) = t$, and that

$\lim\limits_{c_i \to 0} \dfrac{\left(e^{2\alpha c_i t} - 1 - 2\alpha c_i t\right)}{(2\alpha c_i)^2} = \dfrac{t^2}{2}$, we have for a neutral disturbance,

$$\left.\begin{aligned} u_a &= -\frac{du_0}{dy}\hat{v}_a(y)\frac{t^2}{2} - \hat{Y}_a(y)\,t \;; \quad v_a = \hat{v}_a(y)\,t\,; \\ w_a &= \hat{w}_a(y)\,t \;; \quad \xi_a = -\hat{X}_a(y)\,t\,. \end{aligned}\right\} 3.14$$

Therefore, even for neutral disturbances the secondary flow will build up and eventually dominate.

The above results can be applied to a quite general parallel flow problem. A knowledge of the solutions based on linearized theory is sufficient to obtain results for the rate of distortion of the basic flow. It is convenient to list these mean second order quantities for future reference,

$$\left.\begin{aligned}
U_a \cos 2\beta z &= \left[-\frac{dU_0}{dy}\hat{V}_a(y)\frac{\left(e^{2\alpha c_i t}-1-2\alpha c_i t\right)}{(2\alpha c_i)^2} - \hat{Y}_a(y)\left(\frac{e^{2\alpha c_i t}-1}{2\alpha c_i}\right)\right]\cos 2\beta z, \\
V_a \cos 2\beta z &= \left[\hat{V}_a(y)\left(\frac{e^{2\alpha c_i t}-1}{2\alpha c_i}\right)\right]\cos 2\beta z, \\
W_a \sin 2\beta z &= \left[\hat{W}_a(y)\left(\frac{e^{2\alpha c_i t}-1}{2\alpha c_i}\right)\right]\sin 2\beta z, \\
\xi_a \sin 2\beta z &= \left[-\hat{X}_a(y)\left(\frac{e^{2\alpha c_i t}-1}{2\alpha c_i}\right)\right]\sin 2\beta z.
\end{aligned}\right\} \quad (3.15)$$

We now perform these calculations for the shear profile $U_0(y) = \tanh y$. Such a calculation will shed considerable light on the mechanism involved.

SECTION IV APPLICATION TO THE CASE OF A SHEAR PROFILE

In this section detailed calculations are given for the case of the shear profile $U_0(y) = \tanh y$. Such a shear flow can readily be approximated experimentally by mixing two parallel streams, and so the theoretical predictions should be capable of direct confirmation. At the time of performing these calculations no experimental results, for a shear flow with special attention to spanwise variations, appear to be available. Such experiments would be highly desirable.

If the equations (2.27) for the primary oscillation are rewritten in terms of amplitude functions, we have, on neglecting the viscous terms,

$$i\alpha\hat{u}_1 + \frac{d\hat{v}_1}{dy} + \beta\hat{w}_1 = 0, \tag{4.1}$$

$$i\alpha(U_0 - c)\hat{u}_1 + \frac{dU_0}{dy}\hat{v}_1 = -i\alpha\hat{p}_1, \tag{4.2}$$

$$i\alpha(U_0 - c)\hat{v}_1 = -\frac{d\hat{p}_1}{dy}, \tag{4.3}$$

$$i\alpha(U_0 - c)\hat{w}_1 = \beta\hat{p}_1, \tag{4.4}$$

and so by elimination,

$$(U_0 - c)\left(\frac{d^2}{dy^2} - \alpha^2 - \beta^2\right)\hat{v}_1 - \frac{d^2 U_0}{dy^2}\hat{v}_1 = 0. \qquad (4.5)$$

For $U_0(y) = \tanh y$ we have a point of inflexion at $y = 0$, and there is a neutral oscillation [14] given by,

$$\hat{v}_1 = \operatorname{sech} y\,; \quad \alpha^2 + \beta^2 = 1\,; \quad c = 0 \qquad (4.6)$$

In fact one can set up a regular perturbation in c, about this solution, using the method of variation of constants [1]. As our interest centres on the case c_i small, we shall content ourselves with the first term of this perturbation.

A complete solution of the linear system corresponding to the neutral oscillation with $c = 0$, $\alpha^2 + \beta^2 = 1$, is,

$$\hat{u}_1 = \frac{i}{\alpha}\left[\beta^2 \operatorname{cosech} y - \operatorname{sech} y \tanh y\right], \qquad (4.7)$$

$$\hat{v}_1 = \operatorname{sech} y, \qquad (4.8)$$

$$\hat{w}_1 = \beta \operatorname{cosech} y, \qquad (4.9)$$

$$\hat{p}_1 = \frac{i\alpha}{\beta} \operatorname{sech} y. \qquad (4.10)$$

Using these solutions in equations (3.5) and (3.6) we find,

$$\hat{X}_a(y) = \frac{16\beta}{\sinh^3 2y}\left[(1-\beta^2)\cosh 2y - \beta^2\right], \tag{4.11}$$

$$\hat{Y}_a(y) = 0. \tag{4.12}$$

As these equations have been obtained for an inviscid fluid, and for large Reynolds numbers (which is the case under consideration), they will be reliable solutions outside the critical layer. The singularities induced in $\hat{u}_1$, $\hat{w}_1$, and $\hat{X}_a$, at $y=0$, are artificial ones in the sense that they would not be present in any real fluid. On a closer examination the singularities are seen to arise from the cross wave component of the oscillation, and it can be remedied by including the viscous terms, as we shall describe below.

Corresponding to the neutral oscillation $\hat{v}_1 = \operatorname{sech} y$, the two dimensional component of this oscillation is given by $i\alpha\hat{u}_1 + \beta\hat{w}_1 = \operatorname{sech} y \tanh y$. However the cross wave component is $\hat{\phi} = \beta\hat{u}_1 + i\alpha\hat{w}_1 = \frac{i\beta}{\alpha}\operatorname{sech}^3 y \coth y$, and has a pole at $y=0$. The complete differential equation for $\hat{\phi}$ is obtained by taking a suitable combination from the set (2.27). It is,

$$\frac{d^2\hat{\phi}}{dy^2} - \left[i\alpha R \tanh y + 1\right]\hat{\phi} = \beta R \operatorname{sech}^3 y. \tag{4.13}$$

Formally putting R infinite we have the inviscid solution,

$$\hat{\phi} = -\frac{\beta}{i\alpha}\operatorname{sech}^3 y \coth y. \tag{4.14}$$

It is well known that the critical layer has a thickness of order $(\alpha R)^{-\frac{1}{3}}$, and it is within this region that the solution $\hat{\phi}$ will be modified by a boundary layer type correction.

A formal method of dealing with a homogeneous second order differential equation, involving a large parameter whose coefficient has a turning point, has been given by Langer [15]. The extension to the non homogeneous problem is very simple. We sketch the method as it applies to the problem at hand. For convenience we write $\lambda^2 = \alpha R$; and first consider the general solution $\hat{\phi}_1$ of the homogeneous equation:

$$\frac{d^2\hat{\phi}_1}{dy^2} - \left[i\lambda^2 \tanh y + 1\right]\hat{\phi}_1 = 0. \tag{4.15}$$

Let, $$\omega(y) = \int_0^y \sqrt{\tanh y}\, dy = \tanh^{-1}\sqrt{\tanh y} - \tan^{-1}\sqrt{\tanh y}, \tag{4.16}$$

$$z(y,\lambda) = ix = i\lambda^{\frac{2}{3}}\left(\frac{3}{2}\omega\right)^{\frac{2}{3}}, \tag{4.17}$$

$$Q(y) = \left(\tfrac{3}{2}\omega\right)^{\frac{1}{6}}\left(\frac{d\omega}{dy}\right)^{-\frac{1}{2}}, \tag{4.18}$$

$$\hat{\psi}_1(y) = Q(y)\,h(z), \tag{4.19}$$

where, $$\frac{d^2h}{dz^2} + zh = 0. \tag{4.20}$$

A straightforward substitution then shows that $\hat{\psi}_1$ satisfies the equation,

$$\frac{d^2\hat{\psi}_1}{dy^2} - \left[i\lambda^2\tanh y + \frac{Q''}{Q}\right]\hat{\psi}_1 = 0, \tag{4.21}$$

$\frac{Q''}{Q}$ being regular and non zero in a neighborhood of $y=0$. Equation (4.21) is considered the approximating differential equation for (4.15), and correct to order λ^{-2} we may write $\hat{\phi}_1 = \hat{\psi}_1$.

The solutions of equation (4.20) are the modified Hankel functions of order one-third, $h_1(z)$ and $h_2(z)$,

where, $$h_1(z) = \left(\tfrac{2}{3}z^{\frac{3}{2}}\right)^{\frac{1}{3}} H^{(1)}_{\frac{1}{3}}\left(\tfrac{2}{3}z^{\frac{3}{2}}\right), \tag{4.22}$$

and $$h_2(z) = \left(\tfrac{2}{3}z^{\frac{3}{2}}\right)^{\frac{1}{3}} H^{(2)}_{\frac{1}{3}}\left(\tfrac{2}{3}z^{\frac{3}{2}}\right). \tag{4.23}$$

Therefore as $Q(y)$ is a known function of y, we can write the general solution of equation (4.15) as,

$$\hat{\phi}_1(y) = \left(A_1 h_1(z) + A_2 h_2(z)\right) Q(y), \tag{4.24}$$

where A_1 and A_2 are arbitrary constants.

Returning now to the non homogeneous problem the solution $\hat{\phi}(y)$ is required to be zero at $y=0$, and to approach the inviscid solution for $(\alpha R)^{\frac{1}{3}} y$ large. An application of the method of variation of constants readily shows that this solution is,

$$\hat{\phi}(y) = -\frac{\beta R}{(\alpha R)^{\frac{2}{3}}} Q^4(y) \operatorname{sech}^3 y \, L(z), \tag{4.25}$$

where $L(z)$ is the Lommel function defined by,

$$L(z) = \frac{\int_0^z \left(h_1(t) h_2(z) - h_2(t) h_1(z)\right) dt}{W\left(h_1(z), h_2(z)\right)}, \tag{4.26}$$

$W\left(h_1(z), h_2(z)\right)$ being the Wronskian of the functions $h_1(z)$ and $h_2(z)$. In the usual normalization for the Hankel functions,

$$W\left(h_1(z), h_2(z)\right) = -\frac{4i}{\pi}\left(\frac{3}{2}\right)^{\frac{1}{3}}. \tag{4.27}$$

The Lommel function $L(z)$ behaves like $\frac{1}{z}$ for $|z|$ large and satisfies the differential equation,

$$\frac{d^2L}{dz^2} + zL = 1. \tag{4.28}$$

The method of obtaining the precise correction using the Lommel function has been included for two reasons. Firstly its use would be required in an exact treatment of the problem, and secondly the form of the viscous modification near $y=0$ is apparent. Indeed, to obtain the most important results of the subsequent analysis it is necessary to consider only the form of the modified quantities. Subscripts m will be used to denote quantities with the viscous correction included. Clearly from equation (4.25) the modified solution for $\hat{X}_a(y)$ will be of the form,

$$\hat{X}_m(y) = \hat{X}_a(y)\, f(h), \quad \text{where} \quad y = \frac{\varepsilon h}{2}, \tag{4.29}$$

and ε is considered as a small parameter of order $(\alpha R)^{-\frac{1}{3}}$. Here $\hat{X}_a(y)$, calculated on the basis of an inviscid fluid, has a pole of order three at $y=0$, and behaves like $e^{-4|y|}$ for large $|y|$ (see equation (4.11)). $f(h)$ is of the nature of a boundary layer correction which is required to have the following properties. $f(h)$ is to be a regular even monotone function of h, to have a zero of order at least four at $h=0$, and to tend to unity in a strong exponential manner as $\eta \to \infty$. For the present, these are the only properties assumed for $f(\eta)$.

We now calculate the second order velocities induced by the vorticity distribution $\hat{X}_m(y)$, defined above. The differential equations to determine $\hat{v}_m(y)$, $\hat{w}_m(y)$ are obtained from (3.12) and (3.13). These are,

$$\frac{d\hat{v}_m}{dy} + 2\beta\hat{w}_m = 0, \qquad (4.30)$$

$$\frac{d\hat{w}_m}{dy} + 2\beta\hat{v}_m = -\hat{X}_m. \qquad (4.31)$$

Therefore by elimination we have,

$$\frac{d^2\hat{v}_m}{dy^2} - 4\beta^2\hat{v}_m = 2\beta\hat{X}_m, \qquad (4.32)$$

and
$$\hat{w}_m = -\frac{1}{2\beta}\frac{d\hat{v}_m}{dy}. \qquad (4.33)$$

We require the solution of (4.32), for $y>0$, subject to the conditions that $\hat{v}_m(0)=0$, and $\hat{v}_m(y)\to 0$ as $y\to\infty$. $\hat{v}_m(y)$ and $\hat{w}_m(y)$ are defined for $y<0$ as odd and even functions respectively, namely,

$$\hat{v}_m(-y) = -\hat{v}_m(y) \quad ; \hat{w}_m(-y) = \hat{w}_m(y). \tag{4.34}$$

It is to be noted in passing that the solution of (4.32) with $\hat{X}_m = 0$ gives two dimensional potential type solutions.

The solution of (4.32), satisfying the conditions at infinity, can be written down in explicit form by the method of variation of constants. We have,

$$\hat{v}_m(y) = -Ae^{-2\beta y} + \int_\infty^y \hat{X}_m(t) \sinh 2\beta(y-t)\, dt, \tag{4.35}$$

and,

$$\hat{w}_m(y) = -Ae^{-2\beta y} - \int_\infty^y \hat{X}_m(t) \cosh 2\beta(y-t)\, dt. \tag{4.36}$$

where both integrals are clearly convergent.

For $\hat{v}_m(0) = 0$, A is determined as,

$$A = \int_0^\infty \hat{X}_m(t) \sinh 2\beta t\, dt, \tag{4.37}$$

and so, $\hat{w}_m(0) = \int_0^\infty \hat{X}_m(t) e^{-2\beta t}\, dt. \quad (4.38)$

Using this value for the constant A, $\hat{v}_m(y)$ and $\hat{w}_m(y)$ can be rewritten in an alternative form, convenient for small values of y, namely,

$$\hat{v}_m(y) = \int_0^y \hat{X}_m(t)\sinh 2\beta(y-t)\,dt - \left(\int_0^\infty \hat{X}_m(t)\, e^{-2\beta t}\,dt\right)\sinh 2\beta y, \quad (4.39)$$

$$\hat{w}_m(y) = -\int_0^y \hat{X}_m(t)\cosh 2\beta(y-t)\,dt + \left(\int_0^\infty \hat{X}_m(t)\, e^{-2\beta t}\,dt\right)\cosh 2\beta y. \quad (4.40)$$

Approximate values for the constants A and $\hat{w}_m(0)$ can be obtained in the following way.

$\hat{X}_a(t)$ can be expanded as a Laurent Series about of the form,

$$\hat{X}_a(t) = \frac{a_0}{t^3} + \frac{a_1}{t} + \cdots, \quad (4.41)$$

where for the case at hand,

$$a_0 = 2\beta(1-2\beta^2). \quad (4.42)$$

Putting $t = \frac{\varepsilon\tau}{2}$, and $y = \frac{\varepsilon\eta}{2}$, we have from equation (4.37),

$$A = \int_0^\infty \frac{\varepsilon}{2}\,\hat{X}_a\!\left(\frac{\varepsilon\tau}{2}\right) f(\tau)\sinh(\varepsilon\beta\tau)\,d\tau, \quad (4.43)$$

and so,

$$A = \frac{4a_0\beta}{\varepsilon}\int_0^\infty \frac{1}{\tau^2} f(\tau)\,d\tau + O(\varepsilon). \tag{4.44}$$

In a similar way,

$$\hat{w}_m(0) = \frac{4a_0}{\varepsilon^2}\int_0^\infty \frac{1}{\tau^3} f(\tau)\,d\tau - \frac{4\beta a_0}{\varepsilon}\int_0^\infty \frac{1}{\tau^2} f(\tau)\,d\tau + O(1). \tag{4.45}$$

Also from equation (4.39) using the same transformation,

$$\hat{v}_m(y) = \int_0^\eta \frac{\varepsilon}{2}\hat{X}_a\left(\frac{\varepsilon\tau}{2}\right) f(\tau)\sinh\varepsilon\beta(\eta-\tau)\,d\tau - \left(\int_0^\infty \frac{\varepsilon}{2}\hat{X}_a\left(\frac{\varepsilon\tau}{2}\right) f(\tau)\,e^{-\varepsilon\beta\tau}\,d\tau\right)\sinh\varepsilon\beta\eta, \tag{4.46}$$

and thus,

$$\hat{v}_m(y) = \frac{4a_0\beta}{\varepsilon}\int_0^\eta \frac{(\eta-\tau)}{\tau^3} f(\tau)\,d\tau - \frac{4a_0\beta}{\varepsilon}\int_0^\infty \frac{\eta}{\tau^3} f(\tau)\,d\tau + O(1). \tag{4.47}$$

Thus for η large,

$$\hat{v}_m(y) \to -\frac{4a_0\beta}{\varepsilon}\int_0^\infty \frac{1}{\tau^2} f(\tau)\,d\tau + O(1). \tag{4.48}$$

Therefore $\hat{v}_m(y)$ tends to a constant value of order $\frac{1}{\varepsilon}$ as we approach the outer edge of the critical layer. On comparing equations (4.44) and (4.48) it is seen that it is this term that induces the external potential motion, $Ae^{-2\beta y}$.

This analysis shows very clearly the importance of the critical layer. Although viscous forces are negligible outside the small region $y = O(\varepsilon)$, we see that they are indeed responsible for inducing a potential component to the secondary motion, which is dominant far away from the critical layer. This two dimensional potential flow in the $y-z$ plane may be described as being due to a source distribution at the critical layer of strength $\sigma(z) = \sigma_0 \cos 2\beta z$ per unit length. Here $\sigma_0 = -\frac{A}{\pi}$, and A is given by equation (4.44). This source distribution is of $O\left(\frac{a^2}{\varepsilon}\right)$, and determines the direction of the circulation at infinity.

For the purposes of calculation we have taken $f(\eta) = 1 - e^{-\eta^4}$. This is used in place of the exact viscous correction, as given previously in terms of the Lommel function. The previous analysis shows that the only possible discrepancy that can arise from this assumption is in the numerical values within the critical layer. No qualitative disagreement can result. With this choice of $f(\eta)$, an integration by parts shows that,

$$\int_0^\infty \frac{1}{\tau^3}\left(1 - e^{-\tau^4}\right) d\tau = \frac{1}{2}\Gamma\left(\frac{1}{2}\right), \tag{4.49}$$

and

$$\int_0^\infty \frac{1}{\tau^2}\left(1 - e^{-\tau^4}\right) d\tau = \Gamma\left(\frac{3}{4}\right). \tag{4.50}$$

On using these results we have from equations (4.42), (4.43) and (4.44),

$$A = \frac{8\beta^2(1-2\beta^2)}{\varepsilon}\Gamma\left(\tfrac{3}{4}\right) + O(\varepsilon), \tag{4.51}$$

$$\hat{w}_m(0) = \frac{4\beta(1-2\beta^2)}{\varepsilon^2}\Gamma\left(\tfrac{1}{2}\right) - \frac{8\beta^2(1-2\beta^2)}{\varepsilon}\Gamma\left(\tfrac{3}{4}\right) + O(1), \tag{4.52}$$

and so the final expressions for the second order velocities become,

$$\hat{U}_m = -\operatorname{sech}^2 y\, \hat{v}_m, \tag{4.53}$$

$$\hat{v}_m = -Ae^{-2\beta y} + \int_\infty^y \frac{16\beta[(1-\beta^2)\cosh 2t - \beta^2][1-e^{-(\frac{2t}{\varepsilon})^4}]}{\sinh^3 2t}\sinh 2\beta(y-t)\,dt, \tag{4.54}$$

$$w_m = -Ae^{-2\beta y} - \int_\infty^y \frac{16\beta[(1-\beta^2)\cosh 2t - \beta^2][1-e^{-(\frac{2t}{\varepsilon})^4}]}{\sinh^3 2t}\cosh 2\beta(y-t)\,dt. \tag{4.55}$$

These are convenient forms for y large. Equations (4.39) and (4.40) are more suitable for $y = O(\varepsilon)$.

It is to be noted that the x component of the secondary vorticity $-\hat{X}_m$, will change sign if $\beta^2 > \frac{1}{2}$, (and so $\alpha^2 < \frac{1}{2}$). We confine our attention to the case $\beta^2 < \frac{1}{2}$, corresponding to fairly slow spanwise variations of amplitude. This case seems more likely to fit the practical situation. If $\beta^2 > \frac{1}{2}$ and the secondary vorticity changes sign as y increases, the physical picture would have to be modified. It is to be remembered that $\frac{\alpha}{\beta}$ is half the ratio of the wavelengths in the z and x directions respectively.

The numerical calculations have been performed for the typical case $\beta = \frac{1}{2}$. ε has been taken as $\frac{1}{5}$ corresponding to $R = O(10^2)$. A smaller ε would merely confine the rapid variations to a closer band near $y=0$, and induce a larger potential component for the external flow. The choice $\beta = \frac{1}{2}$ is also convenient in that for $y > \varepsilon$, where we may take $f(\tau) \simeq 1$, the integrals can be evaluated by elementary methods. We have, (for $y > \varepsilon$),

$$\hat{v}_m(y) = -Ae^{-y} + \frac{1}{4}\left[\coth y + 2\tanh y + 3\sinh y \log\left|\tanh\frac{y}{2}\right|\right], \qquad (4.56)$$

$$\hat{w}_m(y) = -Ae^{-y} + \frac{1}{4}\left[\operatorname{cosech}^2 y - 2\operatorname{sech}^2 y - 3 - 3\cosh y \log\left|\tanh\frac{y}{2}\right|\right]. \qquad (4.57)$$

Numerical integration methods with steps of $y = \cdot 01$, were used to calculate $\hat{v}_m$ and $\hat{w}_m$ near $y = 0$. The resulting amplitude functions $\hat{u}_m$, $\hat{v}_m$, and $\hat{w}_m$, are shown in Fig. 1, for $y > 0$. $\hat{u}_m$ and $\hat{v}_m$ are odd, and $\hat{w}_m$ is an even function of y It is to be remembered that the actual second order velocities also have a periodic z dependence and an exponential dependence on time (see equation (3.15)).

Some clarification of the physical situation is obtained by finding the projections in the y-z plane of the streamlines for the secondary flow. These are found from,

$$\frac{dy}{\hat{v}_m \cos 2\beta z} = \frac{dz}{\hat{w}_m \sin 2\beta z} . \qquad (4.58)$$

On using equation (4.30) we have,

$$\frac{d\hat{v}_m}{\hat{v}_m} = -2\beta \cot 2\beta z \, dz, \qquad (4.59)$$

and so the streamlines can be obtained from the equation,

$$\hat{v}_m \sin 2\beta z = \text{constant}. \qquad (4.60)$$

For the case $\beta = \frac{1}{2}$, $\varepsilon = \frac{1}{5}$, these streamlines are shown in Fig. 2. The actual streamlines will be, of course, in the form of distorted spirals.

Lastly we shall require the projections on the $y-z$ plane of the streamlines of the primary oscillation. For this purpose it is necessary to include a viscous modification for $\hat{w}_1(y)$. On assuming $\hat{w}_{1m} = \beta \operatorname{cosech} y \, g(\eta)$, and recalculating $\hat{\chi}_a$ using equation (3.5) it is found that taking $g(\eta) = 1 - e^{-\eta^4}$, gives good agreement with $\hat{\chi}_m$ over the whole range of y.

With $\hat{w}_{1m} = \beta \operatorname{cosech} y \left(1 - e^{-\left(\frac{2y}{\varepsilon}\right)^4}\right)$, the streamlines for the primary oscillation are found by solving,

$$\frac{dy}{\hat{v}_1 \cos\beta z} = \frac{dz}{\hat{w}_{1m} \sin\beta z}. \qquad (4.61)$$

Integrating we find,

$$\sin\beta z = \sin\beta z_0 \, e^{\beta^2 \int_0^y \coth t \left(1 - e^{-\left(\frac{2t}{\varepsilon}\right)^4}\right) dt}, \qquad (4.62)$$

where $z = z_0$, when $y = 0$. For the case $\beta = \frac{1}{2}$, $\varepsilon = \frac{1}{5}$, these streamlines are shown in Fig. 3.

SECTION V PHYSICAL INTERPRETATION OF THE RESULTS

In Sections II, III and IV, calculations have been made for the secondary velocities induced by oscillations of finite amplitude. It is clear that this mechanism in which the secondary vorticity produces a spanwise redistribution of momentum will apply to non linear oscillations in any parallel flow, although the phase relationships may differ from case to case. Therefore in this discussion we shall restrict ourselves to an interpretation in the case of the shear profile for which detailed results were obtained in Section IV.

For the x independent secondary flow we have velocity components given by, (see (3.15)).

$$u_m \cos 2\beta z = \hat{u}_m(y) \cos 2\beta z \, \frac{\left(e^{2\alpha c_i t} - 1 - 2\alpha c_i t\right)}{(2\alpha c_i)^2}, \tag{5.1}$$

$$v_m \cos 2\beta z = \hat{v}_m(y) \cos 2\beta z \left(\frac{e^{2\alpha c_i t} - 1}{2\alpha c_i}\right), \tag{5.2}$$

$$w_m \sin 2\beta z = \hat{w}_m(y) \sin 2\beta z \left(\frac{e^{2\alpha c_i t} - 1}{2\alpha c_i}\right). \tag{5.3}$$

Also the x component of vorticity is given by,

$$\xi_m \sin 2\beta z = -\hat{X}_m(y) \sin 2\beta z \left(\frac{e^{2\alpha c_i t} - 1}{2\alpha c_i} \right). \quad (5.4)$$

Here $\hat{U}_m(y)$, $\hat{V}_m(y)$, $\hat{X}_m(y)$, are odd, and $\hat{W}_m(y)$ is an even function of y.

The signs of $u_m \cos 2\beta z$, $v_m \cos 2\beta z$, and $w_m \sin 2\beta z$, in the cell $y > 0$, $0 < z < \frac{\pi}{\beta}$, (where $\xi_m \sin 2\beta z$ is negative) are indicated by the following:

At $z = 0$, $u_m \cos 2\beta z > 0$; $v_m \cos 2\beta z < 0$; $w_m \sin 2\beta z = 0$.

At $z = \frac{\pi}{2\beta}$, $u_m \cos 2\beta z = 0$; $v_m \cos 2\beta z = 0$; $w_m \sin 2\beta z \gtrless 0$, for small y. for large y.

At $z = \frac{\pi}{\beta}$, $u_m \cos 2\beta z < 0$; $v_m \cos 2\beta z > 0$; $w_m \sin 2\beta z = 0$.

Therefore in this region we do have a consistent physical picture, with a large scale eddying motion as indicated by the streamlines in Fig. 2. The nett effect of the secondary flow is to induce a two dimensional cellular structure on the motion (Fig. 4), with momentum being fed in toward the viscous region at $z=0, \pm\frac{\pi}{\beta}, \pm\frac{2\pi}{\beta}, \ldots$ (where the amplitude of the oscillation is maximum), and extracted at the intermediate spanwise positions $z = \pm\frac{\pi}{2\beta}, \pm\frac{3\pi}{2\beta} \ldots$ (where the amplitude of the oscillation is zero). This large scale exchange process produces an alternate excess and defect of x momentum at these points. In turn, it is responsible for a spanwise alternate bulging and thinning of the original velocity profile. From Fig. 2 it can be seen that this effect should be most pronounced at the outer edges of the critical layer. The flow profile gradually becomes more and more warped until it eventually crumbles completely into turbulence.

This picture is somewhat modified by the effects due to the primary oscillation which is periodic in x and has a spanwise periodicity $\frac{2\pi}{\beta}$ (twice that of the secondary flow). The y and z components are in phase and produce the streamline pattern indicated in Fig. 3, the sign of the vorticity reversing itself every half wavelength in the downstream direction. This x-vorticity of the primary oscillation is an even function of y and is of maximum effect at values of x where the amplitude of the downstream component of the oscillation is zero. This vortex structure is indicated in Fig. 5.

The steady vortex structure of the secondary flow will therefore be modified by the periodic switching on and off of the primary oscillation. Superposing these two mechanisms there results an alternate partial reinforcement and cancellation of the two motions, each half wavelength as we move downstream. This superposition is shown diagrammatically in Fig. 6, where for illustrative purposes it has been supposed that the reinforcement and cancellation is complete. Initially at the onset of instability the effects of the primary oscillation will dominate; but eventually as the instability becomes more violent the secondary flow, having a faster growthrate, should triumph.

It is now possible to summarize the most important features of the mechanism discussed with reference to the shear profile. If one proceeds downstream taking observations over one spanwise period of the primary oscillation (say $z = -\frac{\pi}{\beta}$ to $z = +\frac{\pi}{\beta}$), the theory would predict the following effects to be prominent.

(i) A gradual bulging of the profile at $z = 0$ and $z = \pm\frac{\pi}{\beta}$, and a thinning of the profile at $z = \pm\frac{\pi}{2\beta}$.

(ii) The appearance of two systems of four vortices intensifying once each half wavelength, as indicated in Fig. 6; their effect being strongest when the x component of the primary oscillation has zero amplitude.

(iii) When this vortex structure is most apparent (twice each wavelength) it should be accompanied by an increased rate of bulging of the profile at $z = 0$, alternately above and below the point of inflexion.

(iv) The points where there is an excess of momentum, such as $z = 0$, should show up more sharply than the points, such as $z = \pm \frac{\pi}{2\beta}$, where there is a defect.

(v) The maximum effect of this alternate bulging and thinning should be most prominent just outside the critical layer.

(vi) Turbulence would be expected to first appear at the points of maximum bulging.

(vii) Although the vortex structure should be strongest at the outer edges of the critical layer, the large scale eddies should be detectable well away from this region.

SECTION VI CONCLUDING REMARKS

In this thesis we have used formal mathematical methods to investigate finite amplitude oscillations during the breakdown of laminar flow. It is to be emphasized that the results have evolved from a straightforward perturbation of the linear theory and that no physical assumptions have been necessary. While we do not discount ordinary two dimensional distortions, it is felt that in many situations the formation of secondary vortices and the associated crumbling of the profile will be more strongly evident. Certainly this mechanism must be present to some degree during transition. In recent experimental literature there is repeated reference to the formation of this secondary vortex structure.

During completion of this work Dr. G.B. Schubauer and Mr. P.S. Klebanoff, in a private communication, have very kindly forwarded the results of some recent experiments on a Blasius profile, performed at the National Bureau of Standards. Despite the vast difference between the case of shear flow and boundary layer instability, there is a very distinct qualitative agreement with the present theory. One marked difference is that the direction of the secondary vortices is reversed. This discrepancy in the phase relations is believed to be due to the very different nature of the problem. An application of the present theory to the case of boundary layer transition is planned in the near future.

Two further experimental papers have been brought to the attention of the author during the preparation of this thesis [16] [17].

The great importance of non linear effects during transition makes it a problem clearly warranting detailed study. The complexity of the situation makes it all the more desirable that any theoretical approach should be a systematic one. It is believed that the present investigation is a step in this direction.

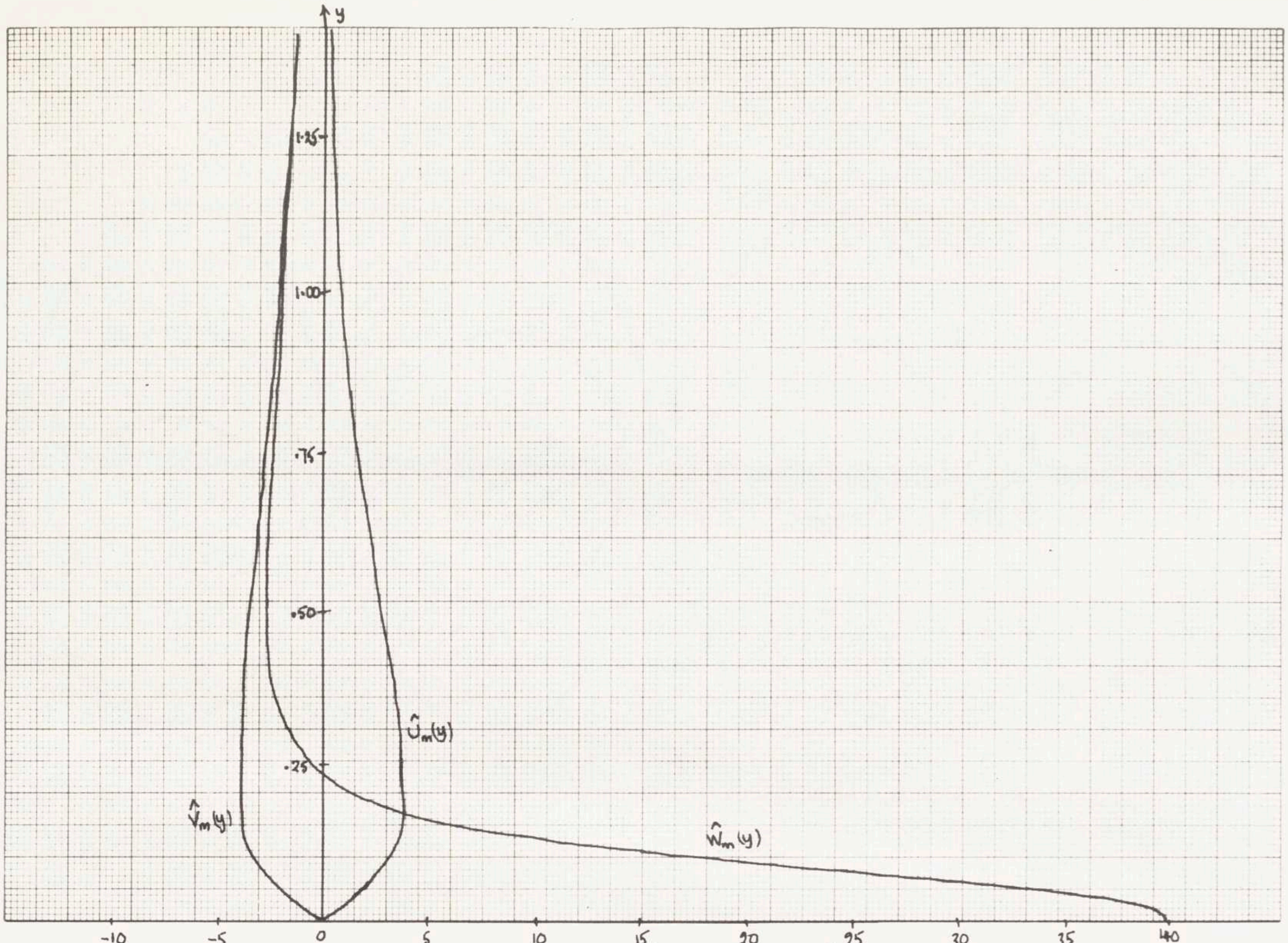

Fig 1. Velocity amplitude functions for secondary flow.

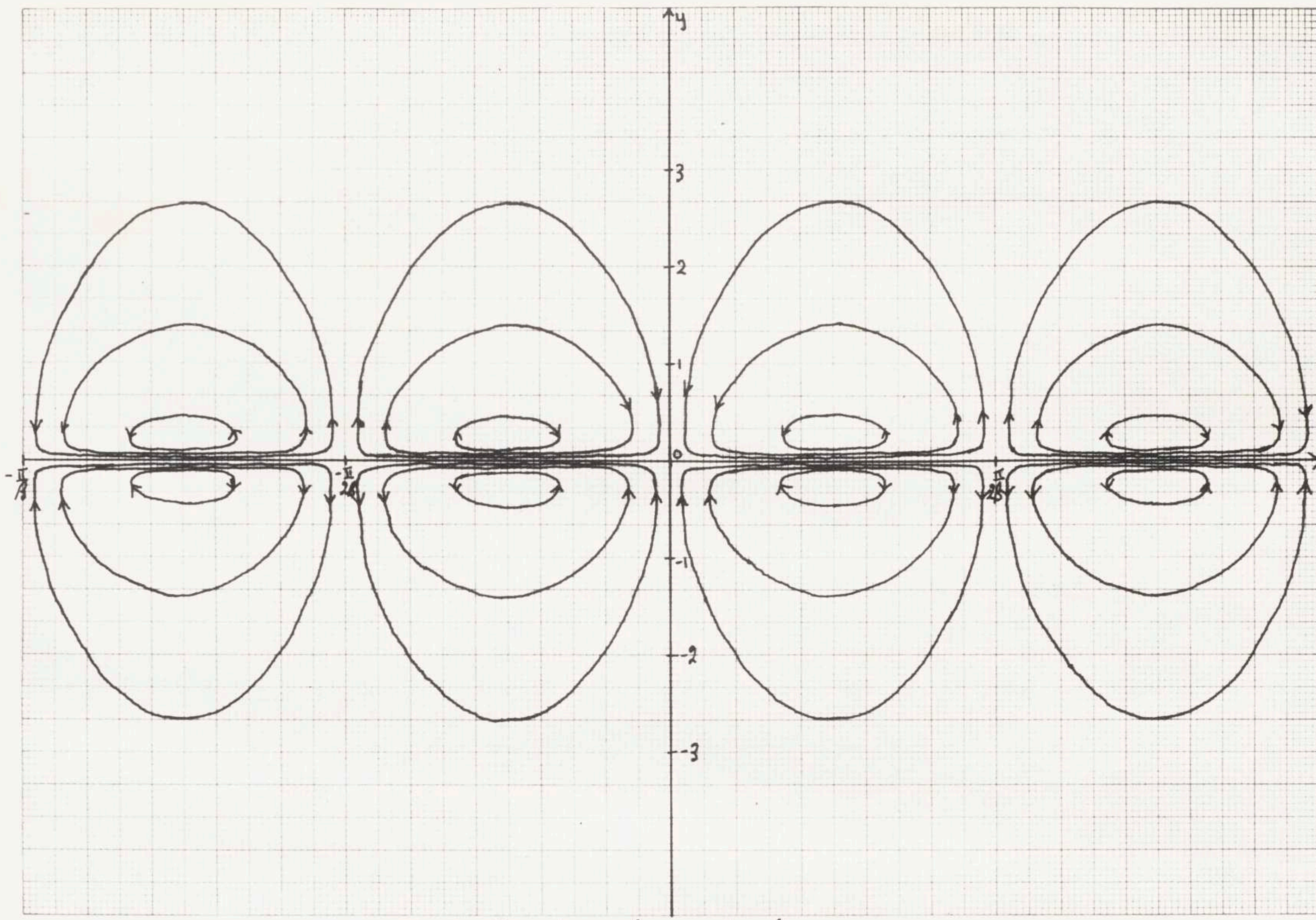

Fig 2. Streamlines of secondary flow.

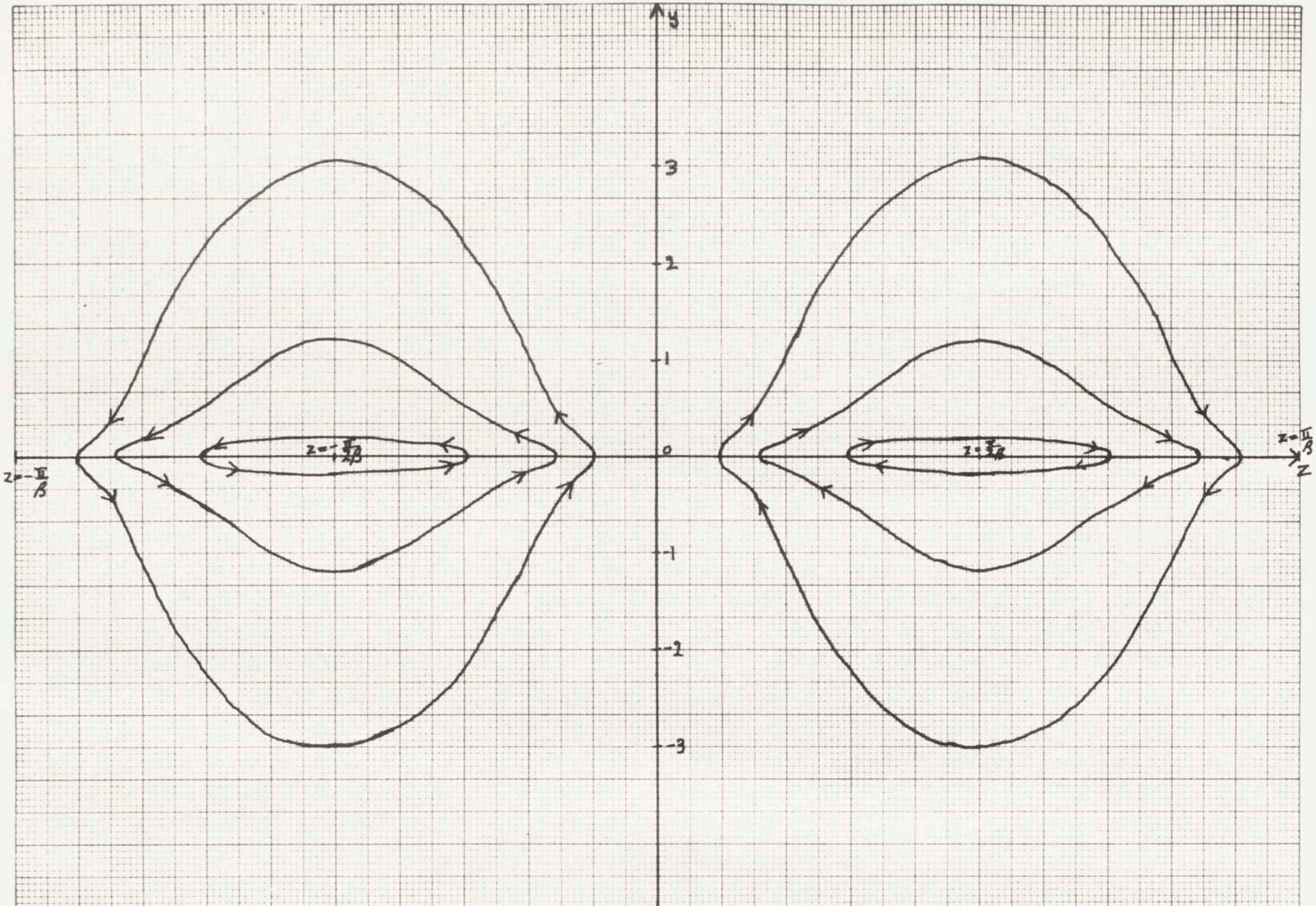

Fig 3. Streamlines of primary oscillation.

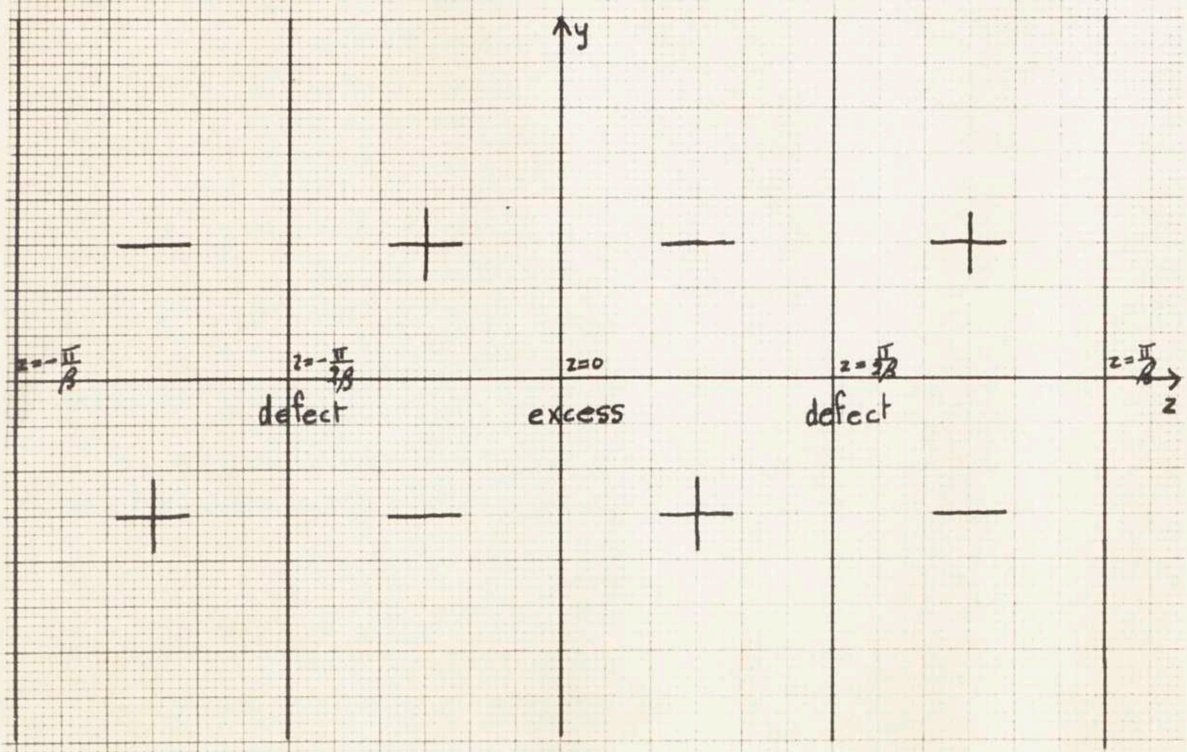

Fig 4. Cellular vortex structure of secondary flow.
Signs indicate direction of circulation.

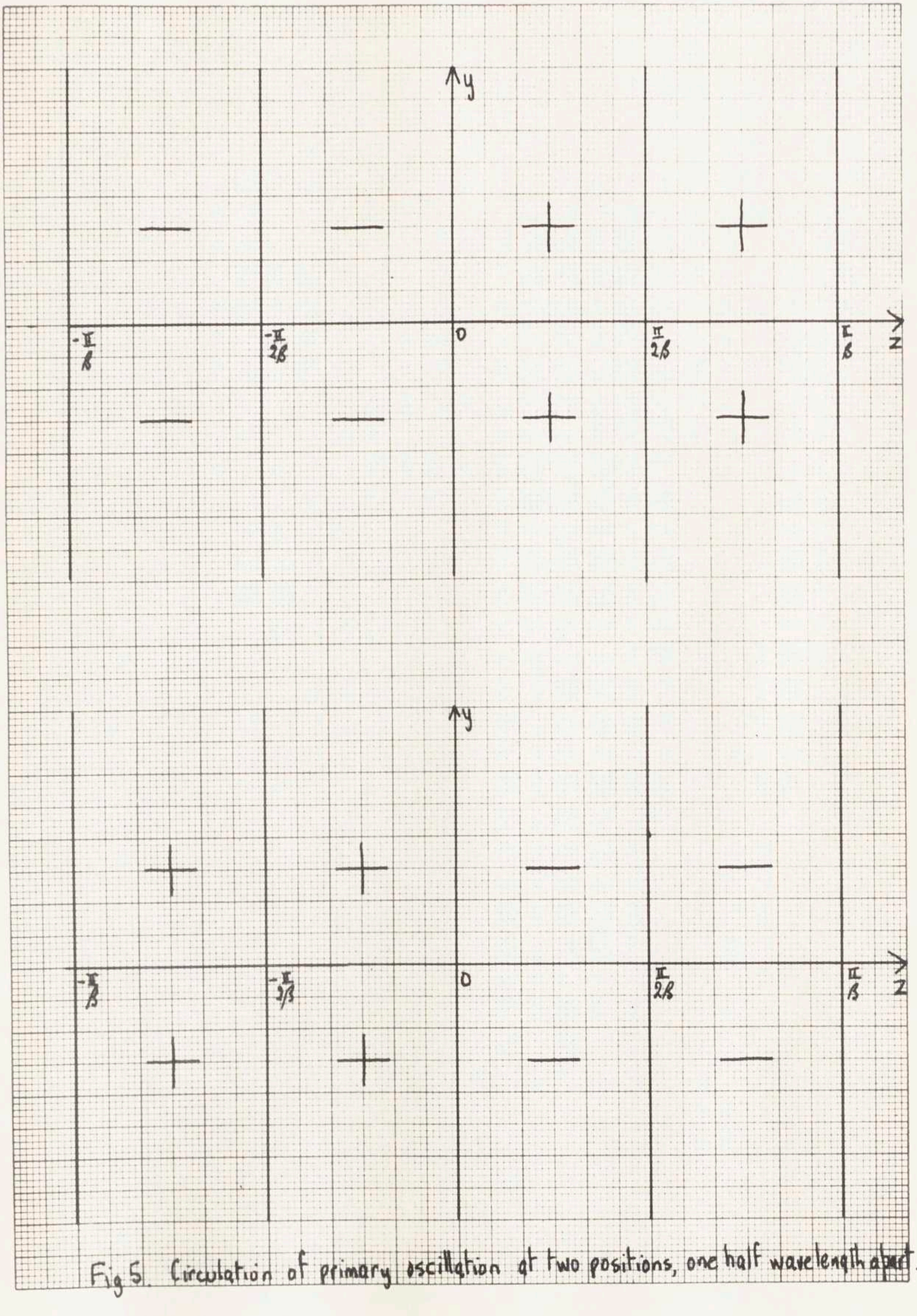

Fig 5. Circulation of primary oscillation at two positions, one half wavelength apart.

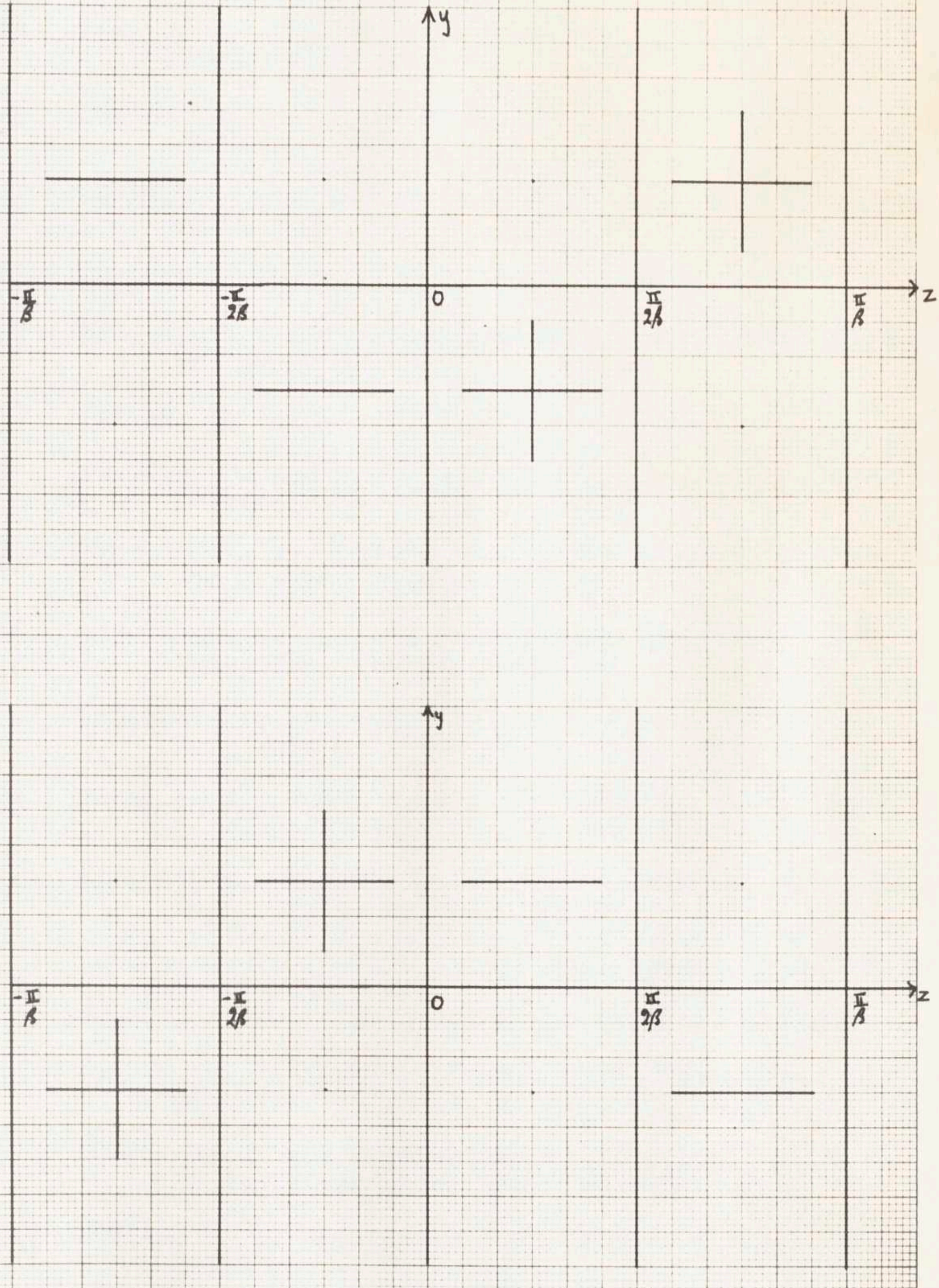

Fig 6. Superposition of primary and secondary flows, showing cells of partial cancellation and reinforcement at two positions, one half wave length apart.

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BIOGRAPHICAL NOTE

David John Benney was born in Wellington, New Zealand on April 8, 1930. He attended Wellington College and Victoria University College, Wellington, where he received a B. Sc. in 1951 and an M. Sc. in 1952. Between 1952 and 1954 he was at Emmanuel College, Cambridge, England, receiving a B. A. in 1954, and subsequently an M. A. in 1958. He returned to New Zealand in 1954 as a lecturer in mathematics at Canterbury University College, where he remained until August 1957. In September 1957 he began graduate work in mathematics at the Massachusetts Institute of Technology. He has published the following articles.

On the Limiting Equilibrium of n Masses. American Mathematical Monthly, January 1958

Escape from a Circular Orbit using Tangential Thrust. Jet Propulsion, March 1958