A COHERENT CATEGORIFICATION OF THE ASYMPTOTIC AFFINE HECKE ALGEBRA

by

Oron Yehonatan Propp

S.B., Massachusetts Institute of Technology, 2018

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ABSTRACT

Kazhdan–Lusztig categorified the affine Hecke algebra H in terms of equivariant coherent sheaves on the Steinberg variety. Recently, Ben-Zvi–Chen–Helm–Nadler have applied the formalism of categorical traces to construct a "coherent Springer sheaf" on a moduli stack of Deligne–Langlands parameters whose endomorphism algebra recovers H. In this thesis, we extend these results to Lusztig's asymptotic affine Hecke algebra J. Using work of Bezrukavnikov–Ostrik, we construct an "asymptotic coherent Springer sheaf" on an "asymptotic" moduli stack of Deligne–Langlands parameters whose endomorphism algebra identifies with J. We show that a certain restriction of the coherent Springer sheaf identifies with this asymptotic coherent Springer sheaf and induces Lusztig's homomorphism ϕ on endomorphism algebras. Next, following a conjecture of Qiu–Xi, we consider a category of equivariant coherent sheaves on the square of the $\mathbb{G}_m$-fixed points in a Springer fiber. We identify its 2-categorical class with a summand of the asymptotic coherent Springer sheaf, and deduce that it categorifies the corresponding block of J. We then construct a family of functors from the mixed affine Hecke category categorifying ϕ. Finally, we show that the universal trace functor for the mixed affine Hecke category is right t-exact with respect to an exotic t-structure, and sends monoidal duals of connective objects to coconnective objects. To this end, we construct an explicit complex computing the 2-categorical class map for certain monoidal categories over quotient stacks. We then deduce a (co)connectivity statement for the 2-categorical classes associated to Bezrukavnikov–Riche's braid group action for the Springer resolution. In particular, we obtain that the coherent Springer sheaf lies in cohomological degree 0 (i.e., is a sheaf rather than a complex), resolving a conjecture of Ben-Zvi–Chen–Helm–Nadler and Zhu. As a consequence of the proof, we partially resolve another conjecture of Qiu–Xi, showing that J embeds in a K-group of equivariant vector bundles on the square of a finite set.

Thesis supervisor: Roman Bezrukavnikov
Title: Professor of Mathematics

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CHAPTER 1

Introduction

1.1. The affine Hecke algebra

Fix a split connected reductive group $\check{G}$ over a nonarchimedean local field F of residual characteristic q_F, and let G be its Langlands dual group over an algebraically closed field k of characteristic 0. The local Langlands program seeks to classify the irreducible k-linear representations of $\check{G}$ in terms of certain "Langlands parameters" arising from G. This is a deep and difficult problem, only fully understood for certain $\check{G}$.[1]

The classical theory of cuspidal components makes the problem somewhat more tractable. A well-known fact, due to Borel and Casselman, states that smooth representations of $\check{G}$ are equivalent to modules over its Hecke algebra, which is the convolution algebra comprising compactly supported, locally constant functions on $\check{G}$ [**Bor76**]. Bernstein showed that this Hecke algebra decomposes into blocks indexed by the supercuspidal representations of $\check{G}$ and its Levi subgroups [**Ber84**]. We may therefore parameterize the irreducible $\check{G}$-representations one block at a time.

The simplest such block, known as the "Iwahori–Hecke algebra," corresponds to the trivial representation of a maximal torus $\check{T} \subset \check{G}$; we denote this subalgebra by $\mathcal{H}_{q_F^{1/2}}$ (for reasons which will soon become clear). Equivalently, $\mathcal{H}_{q_F^{1/2}}$ consists of those functions which are invariant under left or right translations by an Iwahori subgroup of $\check{G}$. The simple $\mathcal{H}_{q_F^{1/2}}$-modules are known as the "unramified principal series" of $\check{G}$, and are precisely those $\check{G}$-representations appearing in the parabolic induction of unramified characters of $\check{T}$. These representations admit an elegant geometric classification in the setting of Springer theory owed to Kazhdan–Lusztig, and later generalized by Reeder [**KL87, Ree02**].

To state this classification, let $\mathfrak{g}$ denote the Lie algebra of G, and $\mathcal{N} \subset \mathfrak{g}$ its nilpotent cone. We write $\mathcal{B}$ for the flag variety of G, and $\pi\colon \widetilde{\mathcal{N}} := T^*\mathcal{B} \to \mathcal{N}$ for the Springer resolution, i.e., the moment map of the cotangent bundle of $\mathcal{B}$. Both $\widetilde{\mathcal{N}}$ and $\mathcal{N}$ carry actions of G by conjugation, and of the multiplicative group $\mathbb{G}_m$ by square inverse dilations. The Springer resolution is then equivariant for the action of $\widetilde{G} := G \times \mathbb{G}_m$. In particular, for any $(s, q) \in \widetilde{G}$, we may consider the "(s, q)-Springer sheaf" $\pi_* \underline{k}_{\widetilde{\mathcal{N}}^{(s,q)}}$, which is a complex of constructible sheaves on the (s, q)-fixed points $\mathcal{N}^{(s,q)}$, equivariant with respect to the centralizer G^s. When s is semisimple, $\mathcal{N}^{(s,q)}$ comprises only finitely many G^s-orbits, and the Beilinson–Bernstein–Deligne decomposition theorem shows that the (s, q)-Springer sheaf is a sum of shifts of intersection cohomology sheaves extended from these orbits [**BBD82**]. These sheaves then classify the simple $\mathcal{H}_{q_F^{1/2}}$-modules:

1.1.1. Theorem ([**KL87, Ree02**]). *The simple $\mathcal{H}_{q_F^{1/2}}$-modules are in bijection with G-conjugacy classes of q_F-commuting pairs of nilpotent and semisimple elements in G, i.e.,*

$$\{(e, s) \in \mathcal{N} \times G^{\mathrm{ss}} : ses^{-1} = q_F e\}/G, \tag{1.1.1.1}$$

together with a simple G^s-equivariant local system on the orbit of $e \in \mathcal{N}^{(s,q_F^{1/2})}$ appearing in the decomposition of the $(s, q_F^{1/2})$-Springer sheaf.

We refer to such pairs (e, s) as "$q_F^{1/2}$-Deligne–Langlands parameters." In fact, this notion makes sense for any $q \in \mathbb{G}_m$ in place of $q_F^{1/2}$. The corresponding deformation of $\mathcal{H}_{q_F^{1/2}}$ is known as the "affine Hecke algebra"; it is a $\mathbb{Z}[v^{\pm 1}]$-algebra $\mathcal{H}$ whose specialization at $v = q_F^{1/2}$ recovers $\mathcal{H}_{q_F^{1/2}}$, and whose specialization at $v = 1$ recovers the group ring of the extended affine Weyl group W^{aff} of $\check{G}$. Theorem 1.1.1 then extends

[1]For but a few examples, see [**Bor79, Vog93, HT01, Hen00, GT11, GT10, Art13**].

to analogous classification of the simple $\mathcal{H}_q$-modules in terms of q-Deligne–Langlands parameters, for any $q \in \mathbb{G}_m$.

The proof of Kazhdan–Lusztig's classification relies on a remarkable precursor to geometric Langlands duality, later generalized by Chriss–Ginzburg, Lusztig, and Ben-Zvi–Chen–Helm–Nadler:

1.1.2. THEOREM ([**KL87, CG10, Lus98, BZCHN22**]). *The affine Hecke algebra is isomorphic to the 0th K-group of $\widetilde{G}$-equivariant coherent sheaves on the Steinberg variety under convolution, i.e.,*

$$\mathcal{H} \cong K_0(\widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}}/\widetilde{G}). \tag{1.1.2.1}$$

This algebra isomorphism is moreover compatible with the Bernstein isomorphism

$$Z(\mathcal{H}) \cong R(\widetilde{G})$$

between the center of $\mathcal{H}$ and the representation ring of $\widetilde{G}$.

In particular, central characters of $\mathcal{H}$ are identified with semisimple $(s,q) \in \widetilde{G}$. Specializing at such a character yields the Borel–Moore homology of the (s,q)-fixed points of the Steinberg variety, which by a simple base-change calculation, identifies with (the cohomology of) the derived endomorphism algebra of the (s,q)-Springer sheaf:

$$\mathcal{H}_{(s,q)} \cong \mathrm{H}_*^{\mathrm{BM}}(\widetilde{\mathcal{N}}^{(s,q)} \times_{\mathcal{N}^{(s,q)}} \widetilde{\mathcal{N}}^{(s,q)}) \cong \mathrm{H}^* \operatorname{End}_{\mathcal{N}^{(s,q)}}(\pi_* \underline{k}_{\widetilde{\mathcal{N}}^{(s,q)}})^{\mathrm{op}}. \tag{1.1.2.2}$$

The simple $\mathcal{H}_{(s,q)}$-modules are then given by the multiplicity spaces of the local systems appearing in the decomposition of $\pi_* \underline{k}_{\widetilde{\mathcal{N}}^{(s,q)}}$. Moreover, their standard covers and costandard hulls are obtained as indecomposable summands of the !- and $*$-fibers of $\pi_* \underline{k}_{\widetilde{\mathcal{N}}^{(s,q)}}$ at e, respectively. Equivalently, the former are isotypic components of $\mathrm{H}_*^{\mathrm{BM}}(\mathcal{B}_e^{(s,q)})$, which carries commuting actions of (1.1.2.2) and the common centralizer $G^{s,e}$; here $\mathcal{B}_e := \widetilde{\mathcal{N}} \times_{\mathfrak{g}} \{e\}$ denotes the Springer fiber above e [**CG10**].

1.2. Coherent Springer theory

Ben-Zvi–Chen–Helm–Nadler have recently "upgraded" Kazhdan–Lusztig's classification to the entire derived category of $\mathcal{H}$-modules, i.e., to all unramified principal series representations of $\check{G}$ [**BZCHN22**]. To do this, we may no longer work one central character at a time; rather, we require a "moduli space" interpolating between all $\mathcal{N}^{(s,q)}$, and a "family of sheaves" interpolating between all (s,q)-Springer sheaves.

The necessary framework is provided by the theory of "higher traces," as developed by Gaitsgory–Kazhdan–Rozenblyum–Varshavsky and Campbell–Ponto [**GKRV22, CP22**]. Namely, to any monoidal k-linear dg-category $\mathcal{A}$ (see §1.8 for our precise assumptions), one may associate two traces. On the one hand, the Hochschild homology $\mathrm{HH}(\mathcal{A})$ is the trace of $\mathcal{A}$ in the category of dg-categories, whose symmetric monoidal structure is provided by the Lurie tensor product. On the other hand, the *categorical* trace

$$\mathrm{Tr}(\mathcal{A}) := \mathcal{A} \underset{\mathcal{A} \otimes \mathcal{A}}{\otimes} \mathcal{A}$$

is the trace of the category of $\mathcal{A}$-modules in the "2-Morita category" comprising module categories over monoidal dg-categories. The categorical trace is itself a dg-category, and carries a "universal trace functor"

$$[-] \colon \mathcal{A} \to \mathrm{Tr}(\mathcal{A}) \tag{1.2.0.1}$$

coequalizing left and right multiplication in $\mathcal{A}$. In particular, letting $\mathbf{1}_{\mathcal{A}}$ denote the monoidal unit of $\mathcal{A}$, we obtain a distinguished object $[\mathbf{1}_{\mathcal{A}}] \in \mathrm{Tr}(\mathcal{A})$. If the monoidal structure on $\mathcal{A}$ is furthermore *rigid*, then the Hochschild homology of $\mathcal{A}$ inherits a k-algebra structure, and we have a canonical algebra isomorphism

$$\mathrm{HH}(\mathcal{A}) \simeq \operatorname{End}_{\mathrm{Tr}(\mathcal{A})}([\mathbf{1}_{\mathcal{A}}])^{\mathrm{op}} \tag{1.2.0.2}$$

paralleling (1.1.2.2).

Ben-Zvi et al. apply this framework to the natural categorification of (1.1.2.1), i.e., the *mixed affine Hecke category*

$$\mathcal{H}^{\mathrm{coh}} := \mathrm{QC}^!(\widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}}/\widetilde{G})$$

consisting of ind-coherent sheaves on the derived Steinberg stack. More specifically, they identify the categorical trace of $\mathcal{H}^{\mathrm{coh}}$ with ind-coherent sheaves on the derived loop space (i.e., the derived self-intersection of the diagonal) of the quotient stack $\mathfrak{g}/\widetilde{G}$, supported on the nilpotent cone:

$$\mathrm{Tr}(\mathcal{H}^{\mathrm{coh}}) \simeq \mathrm{QC}^!(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})).$$

At the level of k-points, this loop space is simply the classical inertia

$$\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})(k) \cong \{(e,s,q) \in \mathcal{N} \times \widetilde{G} : ses^{-1} = q^2 e\}/\widetilde{G},$$

so it may be regarded as a moduli stack of "all" Deligne–Langlands parameters as in (1.1.1.1). However, to obtain a good geometric structure, we have had to drop the semisimplicity assumption on s. A clear candidate for our desired "family of (s,q)-Springer sheaves" is now the *coherent Springer sheaf*

$$\mathcal{S} := \mathcal{L}\pi_* \mathcal{O}_{\mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G})} \in \mathrm{QC}^!(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})), \tag{1.2.0.3}$$

that is, the pushforward of the structure sheaf under the loop space-analog of the Springer resolution. Ben-Zvi et al. confirm this expectation, identifying the universal trace functor (1.2.0.1) with a specific correspondence; in particular, they show that the trace of the monoidal unit identifies with $\mathcal{S}$. Finally, they show that the mixed affine Hecke category is rigid, and that its Hochschild homology identifies with the k-linearization $\mathcal{H}_k$ via (1.1.2.1) and the Chern character on K-theory. Combining these results with (1.2.0.2) yields:

1.2.1. Theorem ([**BZCHN22**]). *The k-linearization of $\mathcal{H}$ is isomorphic to the endomorphism algebra of the coherent Springer sheaf, i.e.,*

$$\mathcal{H}_k \simeq \mathrm{HH}(\mathcal{H}^{\mathrm{coh}}) \simeq \mathrm{End}_{\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})}(\mathcal{S})^{\mathrm{op}}, \tag{1.2.1.1}$$

and all nonzero cohomology groups vanish.

This result is closely reminiscent of (1.1.2.2). In fact, the analogy between the coherent Springer sheaf and the (s,q)-Springer sheaves can be made precise: the completion of $\mathcal{S}$ at any semisimple $(s,q) \in \widetilde{G}$ is "Koszul dual" to $\pi_* k_{\widetilde{\mathcal{N}}^{(s,q)}}$ [**Che23**].

Finally, Theorem 1.2.1 implies that the category of $\mathcal{H}$-modules identifies with the full subcategory of $\mathrm{QC}^!(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))$ generated by $\mathcal{S}$, that is,

$$\mathcal{H}_k\text{-mod} \simeq \langle \mathcal{S} \rangle \hookrightarrow \mathrm{QC}^!(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})). \tag{1.2.1.2}$$

The functor exhibits many other desirable properties, such as compatibility with parabolic induction, which we will not discuss further. More broadly, the embedding (1.2.1.2) realizes the conjectural "categorical local Langlands correspondence" formulated by Zhu, Hellmann, and Fargues–Scholze for the unramified principal series of $\check{G}$ [**Zhu21, Hel23, FS21**].

1.3. The asymptotic affine Hecke algebra

The affine Hecke algebra satisfies a curious property, first observed by Lusztig: namely, ignoring roots of unity, the set of simple $\mathcal{H}_q$-modules is canonically independent of q. This phenomenon may be seen quite explicitly at the level of Deligne–Langlands parameters. Given any $(e,s,q) \in \mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})(k)$, we may extend the nilpotent e to an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ by means of the Jacobson–Morozov theorem. This $\mathfrak{sl}_2$-triple lifts to a homomorphism $\varphi_e \colon \mathrm{SL}_2 \to G$ sending $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ to the exponential of e, which yields a cocharacter

$$\begin{aligned} \check{\lambda}_e \colon \mathbb{G}_m &\to \widetilde{G} \\ t &\mapsto (\varphi_e(\left[\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right]), t) \end{aligned}$$

centralizing e. The triple $(e, (s,q) \cdot \check{\lambda}_e(t))$ is therefore another point of $\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})(k)$, and it is not hard to see that this procedure gives a bijection between q- and qt-Deligne–Langlands parameters.

To explain this phenomenon, Lusztig constructed an *asymptotic affine Hecke algebra* $\mathcal{J}$, defined over $\mathbb{Z}$, along with an injective homomorphism

$$\phi \colon \mathcal{H} \hookrightarrow \mathcal{J} \otimes_{\mathbb{Z}} \mathbb{Z}[v^{\pm 1}]$$

closely relating the two $\mathbb{Z}[v^{\pm 1}]$-algebras [**Lus87a, Lus87b, Lus89**]. More precisely:

(1) The algebra $\mathcal{J}$ is equipped with a distinguished basis indexed by W^{aff}, an analog of the Kazhdan–Lusztig basis of $\mathcal{H}$. The relations between the basis elements of $\mathcal{J}$ are suitable "truncations" of the relations between the corresponding Kazhdan–Lusztig basis elements. Thus, $\mathcal{J}$ may be regarded as a sort of combinatorial "limit" of $\mathcal{H}_q$ as the parameter q tends to 0.
(2) Using these bases, Lusztig defined "completions" of $\mathcal{H}$ and $\mathcal{J}[v^{\pm 1}]$, and showed that ϕ induces an isomorphism between these completions. In particular, elements of $\mathcal{J}$ admit power series expressions in the elements of $\mathcal{H}$.

(3) The algebra $\mathcal{J}$ splits into blocks $\mathcal{J}_e$ indexed by the G-orbits in $\mathcal{N}$, giving a spectral "refinement" of Bernstein's decomposition. Moreover, the simple $\mathcal{J}_e[v^{\pm 1}]$-modules are canonically indexed by Deligne–Langlands parameters with nilpotent e, and restrict along the component $\phi_e\colon \mathcal{H} \to \mathcal{J}_e[v^{\pm 1}]$ to the corresponding standard modules of $\mathcal{H}$ (for q sufficiently generic). Thus, the map ϕ explicitly realizes the bijections between sets of simple $\mathcal{H}_q$-modules for varying q.

The algebra $\mathcal{J}$ therefore captures much of the structure and representation theory of $\mathcal{H}$, and has found many applications in its study [**HN14, Daw22, BDD22**].

Two other perspectives on $\mathcal{J}$ have since appeared in the literature. First, Braverman–Kazhdan have interpreted $\mathcal{J}$ in terms of harmonic analysis on $\check{G}$ [**BK18**]. Given any smooth $\check{G}$-representation V, the invariants of V under an Iwahori subgroup of $\check{G}$ carry an action of the Iwahori–Hecke algebra $\mathcal{H}_{q_F^{1/2}}$ (as in §1.1). Braverman–Kazhdan proved that this $\mathcal{H}_{q_F^{1/2}}$-action extends uniquely to a $\mathcal{J}$-action if V is tempered, and moreover, if V is parabolically induced from the twist of a tempered representation by a positive character. Note that such representations generate all finite-length $\check{G}$-representations by the Langlands classification; Braverman–Kazhdan's result thus provides an additional sense in which $\mathcal{J}$ captures the representation theory of $\mathcal{H}$ [**Sil78, BW00**]. Finally, Braverman–Kazhdan showed that $\mathcal{J}$ arises as the Iwahori bi-invariants of a larger algebra, intermediate between the Hecke algebra of $\check{G}$ and the convolution algebra of Schwartz functions on $\check{G}$.

Second, Lusztig conjectured an algebraic description of $\mathcal{J}$ as a direct sum of certain "matrix" algebras. More precisely, for any $e \in \mathcal{N}$, let Z_e denote the reductive part of the centralizer G^e. Lusztig conjectured that there exists a finite Z_e-set $\mathbf{B}_e$ such that $\mathcal{J}_e \cong K_0(\mathbf{B}_e \times \mathbf{B}_e/Z_e)$ as a *based* convolution algebra [**Lus89**]. The latter may be regarded as a kind of "equivariant" matrix algebra over the representation rings $R(Z_e^b)$, where $b \in \mathbf{B}_e$ and Z_e^b denotes its stabilizer; in particular, its specialization to any semisimple $s \in Z_e$ is a semisimple algebra. Several computations in the literature have now established this expectation to be false [**QX23, QX22, BDD22**]. In general, Bezrukavnikov–Ostrik show that each $b \in \mathbf{B}_e$ must be twisted by a certain 2-cocycle in the Schur multiplier of Z_e^b [**BO04**]. Moreover, Bezrukavnikov–Losev show that the set $\mathbf{B}_e$ appears naturally as the "canonical basis" of $K_0(\mathcal{B}_e/\mathbb{G}_m)$, whose existence was originally conjectured by Lusztig; here $\mathbb{G}_m$ acts on $\mathcal{B}_e$ via the cocharacter $\check{\lambda}_e$ [**BL23, Lus99**]. Bezrukavnikov–Ostrik's 2-cocycles then "measure" the failure of each basis element to be Z_e^b-equivariant. Most recently, Qiu–Xi have conjectured that $\mathcal{J}_e$ may be embedded in a ring of the form conjectured by Lusztig, when Z_e is replaced by a certain finite cover whose identity component has simply-connected derived subgroup [**QX22**].

However, a spectral geometric perspective on $\mathcal{J}$, like those in Theorems 1.1.2 and 1.2.1, has not yet emerged. Only in the case $e = 0$ has there been progress: when G is simple and simply-connected, Xi has shown that

$$\mathcal{J}_0 \cong K_0(\mathcal{B} \times \mathcal{B}/G)$$

as based convolution algebras [**Xi16**]. Moreover, the homomorphism ϕ_0 agrees (up to a canonical inner automorphism of $\mathcal{J}_0$) with that given by pushforward and pullback along the correspondence

$$\widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}}/\widetilde{G} \xrightarrow{\mathrm{id}_{\widetilde{\mathcal{N}}} \times p_{\widetilde{\mathcal{N}}}} \widetilde{\mathcal{N}} \times \mathcal{B}/\widetilde{G} \xleftarrow{i_{\mathcal{B}} \times \mathrm{id}_{\mathcal{B}}} \mathcal{B} \times \mathcal{B}/\widetilde{G}, \tag{1.3.0.1}$$

where $p_{\widetilde{\mathcal{N}}}\colon \widetilde{\mathcal{N}} \rightleftarrows \mathcal{B}\colon i_{\mathcal{B}}$ denote the natural projection and inclusion. This map was originally used by Chriss–Ginzburg to prove Theorem 1.1.2 [**CG10**]. Intriguingly, at the categorical level, this functor is *not* monoidal; to this end, Dawydiak has constructed an intermediate monoidal category receiving a monoidal functor from $\mathcal{H}^{\mathrm{coh}}$, along with lifts of the distinguished basis of $\mathcal{J}_0$ to this category in types A_1 and A_2 [**Daw21**]. Finally, Qiu–Xi have conjectured a generalization of (1.3) to all e, namely

$$\mathcal{J}_e \cong K_0(\mathcal{B}_e^{\mathbb{G}_m} \times \mathcal{B}_e^{\mathbb{G}_m}/Z_e). \tag{1.3.0.2}$$

Here the fixed-point variety $\mathcal{B}_e^{\mathbb{G}_m}$ is smooth and projective, so we again obtain a well-behaved convolution product [**QX23**]. However, it is no longer clear how to generalize Chriss–Ginzburg's map to this setting.

The main goal of this thesis is to provide just such a perspective on $\mathcal{J}$. In the ensuing sections, we describe our extension of Theorem 1.2.1 to $\mathcal{J}$. We then "upgrade" this extension to a categorification of $\mathcal{J}$, partially resolving (1.3.0.2). Finally, we give an explicit description of the universal trace functor (1.2.0.1) for $\mathcal{H}^{\mathrm{coh}}$, and study its relationship to certain t-structures on either category.

1.4. Asymptotic coherent Springer theory

The asymptotic version of Ben-Zvi–Chen–Helm–Nadler's theory stems from two elementary observations. First, in §1.3, we saw a sense in which the moduli stack $\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})$ of Deligne–Langlands parameters was "independent of q." However, this independence only holds set-theoretically, and not in any *geometric* sense. In fact, there are two obstructions to the latter:

(1) The $\mathfrak{sl}_2$-triples provided by the Jacobson–Morozov theorem cannot be made to depend algebraically on e. In particular, the cocharacter $\check{\lambda}_e$ may vary discontinuously as e varies between nilpotent orbits.
(2) Even for fixed $e \in \mathcal{N}$, a choice of $\mathfrak{sl}_2$-triple is only determined up to conjugation by the centralizer G^e. In particular, the cocharacter $\check{\lambda}_e$ is not necessarily stable under the action of G^e.

Instead, to obtain $\mathcal{J}$, we must "force" $\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})$ to be geometrically independent of q. The first obstruction suggests that we must separate $\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})$ into nilpotent orbits; choosing representatives for these orbits, we obtain the disjoint union $\bigsqcup_e \widetilde{G}^e/\widetilde{G}^e$ of adjoint quotient stacks. The second obstruction suggests that we must replace $\widetilde{G}^e$ by the stabilizer of our choice of $\mathfrak{sl}_2$-triple, which is isomorphic to $\widetilde{Z}_e = Z_e \times \mathbb{G}_m$; here the latter factor is given by $\check{\lambda}_e$ (whereas the former factor does not depend on our choice of $\check{\lambda}_e$). Altogether, we obtain the stack

$$\bigsqcup_e \widetilde{Z}_e/\widetilde{Z}_e \cong \bigsqcup_e Z_e/Z_e \times \mathbb{G}_m/\mathbb{G}_m \tag{1.4.0.1}$$

which we may regard as a "q-independent" or "asymptotic" stack of Deligne–Langlands parameters.

Second, Bezrukavnikov–Ostrik's result [**BO04**] immediately implies the existence of an "asymptotic coherent Springer sheaf" on this stack satisfying the same property as in Theorem 1.2.1. This sheaf admits an explicit construction, which we now describe. For any algebraic group K, there is a well-known canonical bijection (in fact, group isomorphism)

$$\{\text{Schur multiplier of } K\} \longleftrightarrow \{\text{multiplicative line bundles on } K\}/\cong,$$

originally noted by Elagin [**Ela09**]. The latter are line bundles $\mathcal{C} \in \mathrm{Pic}(K)$ equipped with an isomorphism $\alpha\colon \mathcal{C} \boxtimes \mathcal{C} \cong m^*\mathcal{C}$, where $m\colon K \times K \to K$ denotes the group multiplication. We require α to satisfy the cocycle condition on the triple product $K \times K \times K$, and consider these data up to the obvious notion of isomorphism.

Thus, for each $b \in \mathbf{B}_e$, we take $(\mathcal{C}_b, \alpha_b)$ to be the multiplicative line bundle corresponding to the 2-cocycle of Z_e^b appearing in the canonical basis. Letting $i_b\colon Z_e^b \hookrightarrow Z_e$ denote the inclusion, we form the coherent sheaf

$$\mathcal{S}^{\mathbf{B}_e} := \bigoplus_{b \in \mathbf{B}_e} i_{b,*}\mathcal{C}_b^\vee \tag{1.4.0.2}$$

on Z_e. The Z_e-action on $\mathbf{B}_e$ then extends to a natural conjugation-equivariant structure on $\mathcal{S}^{\mathbf{B}_e}$; that is, $\mathcal{S}^{\mathbf{B}_e}$ descends to a coherent sheaf on the adjoint quotient stack Z_e/Z_e. We may then reinterpret Bezrukavnikov–Ostrik's result as giving an isomorphism

$$\mathcal{J}_e \simeq \mathrm{End}_{Z_e/Z_e}(\mathcal{S}^{\mathbf{B}_e})^{\mathrm{op}}. \tag{1.4.0.3}$$

Moreover, the latter is evidently concentrated in cohomological degree 0, as in Theorem 1.2.1. In particular, we obtain a fully faithful inclusion

$$\mathcal{J}_{e,k}\text{-mod} \simeq \langle \mathcal{S}^{\mathbf{B}_e} \rangle \hookrightarrow \mathrm{QC}^!(Z_e/Z_e)$$

as in (1.2.1.2). We refer to the sheaf $\mathcal{S}^{\mathbf{B}_e}$ as the *asymptotic coherent Springer sheaf* at e.

We are now almost ready to state our first main result. Let $S_e \subset \mathfrak{g}$ denote the Slodowy slice associated to our choice of $\mathfrak{sl}_2$-triple; it is a transversal slice to the G-orbit of e (at e), and carries a natural action of $\widetilde{Z}_e$. In particular, the action of $\mathbb{G}_m$ repels S_e from the point e. Consider the diagram

$$\begin{tikzcd} \{e\}/\widetilde{Z}_e \arrow[rr, bend right, "i_e"'] & S_e/\widetilde{Z}_e \arrow[l, "p_{S_e}"'] \arrow[r, "i_{S_e}"] & \mathfrak{g}/\widetilde{G} \end{tikzcd}$$

whose maps are the natural projection and "inclusions." Taking loop spaces, we obtain a diagram

$$\begin{tikzcd} \widetilde{Z}_e/\widetilde{Z}_e & \mathcal{L}(S_e/\widetilde{Z}_e) \arrow[l, "\mathcal{L}p_{S_e}"'] \arrow[r, "\mathcal{L}i_{S_e}"] & \mathcal{L}(\mathfrak{g}/\widetilde{G}), \\ \arrow[from=1-1, to=1-3, bend right, "\mathcal{L}i_e"'] \end{tikzcd}$$

with the first stack as in (1.4.0.1). The following result appears as Proposition 5.1.2 and Corollary 5.3.14 in the main text:

1.4.1. Theorem. *For any $e \in \mathcal{N}$, there is a canonical isomorphism*

$$(\mathcal{L}p_{S_e,*}\mathcal{L}i^*_{S_e}\mathcal{S})^{\mathbb{G}_m} \simeq \mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m}, \tag{1.4.1.1}$$

where the superscript $(-)^{\mathbb{G}_m}$ denotes the weight-0 component with respect to the cocharacter $\check{\lambda}_e$. The induced homomorphism

$$(\mathcal{L}p_{S_e,*}\mathcal{L}i^*_{S_e})^{\mathbb{G}_m}\colon \operatorname{End}_{\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})}(\mathcal{S})^{\mathrm{op}} \to \operatorname{End}_{\widetilde{Z}_e/\widetilde{Z}_e}(\mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m})^{\mathrm{op}} \tag{1.4.1.2}$$

then identifies with $\phi_{e,k}$ via (1.2.1.1) and (1.4.0.3). Finally, the functors of extension and restriction of scalars along $\phi_{e,k}$ identify with the sheaf-theoretic operations

$$\begin{tikzcd} \mathcal{H}_k\text{-mod} \arrow[r, shift left, "\mathrm{Ind}_{\phi_{e,k}}"] \arrow[d, "\wr"] & \mathcal{J}_{e,k}[v^{\pm 1}]\text{-mod} \arrow[l, shift left, "\mathrm{Res}_{\phi_{e,k}}"] \arrow[d, "\wr"] \\ \langle \mathcal{S} \rangle \arrow[r, shift left, "(\mathcal{L}p_{S_e,*}\mathcal{L}i^*_{S_e})^{\mathbb{G}_m}"] & \langle \mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m} \rangle, \arrow[l, shift left, "\mathrm{pr}_{\mathcal{S}}\mathcal{L}i_{e,*}"] \end{tikzcd} \tag{1.4.1.3}$$

where $\mathrm{pr}_{\mathcal{S}}$ denotes the right-adjoint to (1.2.1.2).

The weight truncation $(-)^{\mathbb{G}_m}$ may be regarded as enforcing the "truncated" relations defining $\mathcal{J}$ as in §1.3; or informally, as further enforcing "q-independence" at the sheaf-theoretic level. Note that in place of $\mathcal{L}p_{S_e,*}\mathcal{L}i^*_{S_e}$, we can "almost" write $\mathcal{L}i^*_e$. Indeed, we can make rudimentary sense of $\mathcal{L}i^*_e$ as a functor on categories of ind-coherent sheaves (even though it does not preserve left t-boundedness), and there is then a canonical isomorphism $(\mathcal{L}i^*_e\mathcal{S})^{\mathbb{G}_m} \simeq (\mathcal{L}p_{S_e,*}\mathcal{L}i^*_{S_e}\mathcal{S})^{\mathbb{G}_m}$. However, this functor $\mathcal{L}i^*_e$ is not in general left-adjoint to $\mathcal{L}i_{e,*}$; moreover, the functor $\mathcal{L}p_{S_e,*}\mathcal{L}i^*_{S_e}$ is the one which arises naturally in the course of our proof (as we shall explain). Nonetheless, when we restrict to the subcategory $\langle \mathcal{S} \rangle \subset \mathrm{QC}^!(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))$, we do obtain an adjunction as in (1.4.1.3).

The proof of the remainder of Theorem 1.4.1 relies on three technical tools. The first is an explicit complex computing the universal trace functor for $\mathcal{H}^{\mathrm{coh}}$ (and in particular, the sheaf $\mathcal{S}$), which we term the *Block–Getzler sheaf*. Indeed, it is a natural enhancement of the "Block–Getzler complex" introduced in [**BG94**] (and fruitfully applied in [**Che20, BZCHN22**]) for computing Hochschild homology in equivariant settings. Here is a special case of our construction; for more precise statements, see Proposition 3.4.11 and the discussion following Definition 3.4.3.

1.4.2. Proposition. Let X be a derived scheme, let G be a reductive group, and suppose the quotient stack X/G is perfect[2]. Let $\mathcal{A}$ be a compactly generated rigid monoidal category admitting a central functor $\Psi\colon \mathrm{QC}(X/G) \to \mathcal{A}$ from the category of quasi-coherent sheaves on X/G, and let $\underline{\mathrm{Hom}}_{X/G}$ and $\underline{\mathrm{Hom}}_{\mathrm{B}G}$ denote the internal Hom spaces of $\mathcal{A}$ in the categories $\mathrm{QC}(X/G)$ and $\mathrm{QC}(\mathrm{B}G)$, respectively. For any compact object $a \in \mathcal{A}^c$, consider the simplicial complex of sheaves on $(X \times G)/G$ whose n-simplices are given by

$$\bigoplus_{a_0,\dots,a_n \in \mathcal{A}^c} \underline{\mathrm{Hom}}_{\mathrm{B}G}(a_0,a_1) \otimes_k \cdots \otimes_k \underline{\mathrm{Hom}}_{\mathrm{B}G}(a_{n-1},a_n) \otimes_k \underline{\mathrm{Hom}}_{X/G}(a_n, a_0 \otimes a) \boxtimes \mathcal{O}_G,$$

and whose face maps $d_0, \dots, d_n$ are the natural extensions of those for the Block–Getzler complex. The totalization of this complex can be lifted to an object $\mathsf{BG}_{X/G}(\mathcal{A}, a) \in \mathrm{QC}(\mathcal{L}(X/G))$ using an explicit homotopy. Moreover, we have a natural isomorphism

$$\mathsf{BG}_{X/G}(\mathcal{A}, a) \simeq \mathrm{Tr}(\Psi)^R([a]),$$

[2] In the sense of Ben-Zvi–Francis–Nadler [**BZFN10**].

where $\mathrm{Tr}(\Psi)^R$ is right-adjoint to the natural functor

$$\mathrm{Tr}(\Psi)\colon \mathrm{QC}(\mathcal{L}(X/G)) \simeq \mathrm{Tr}(\mathrm{QC}(X/G)) \to \mathrm{Tr}(\mathcal{A}). \tag{1.4.2.1}$$

More generally, we allow for the categorical traces (1.4.2.1) to be taken with respect to certain monoidal endofunctors of $\mathrm{QC}(X/G)$ and $\mathcal{A}$, and for $[a]$ to be replaced by the "2-categorical class" of an $\mathcal{A}$-module category in the sense of [**GKRV22**]. In particular, regarding $\mathcal{S}$ as a quasi-coherent sheaf, we obtain an expression $\mathcal{S} \simeq \mathsf{BG}_{\mathfrak{g}/\widetilde{G}}(\mathcal{H}^{\mathrm{coh}}, \mathbf{1}_{\mathcal{H}^{\mathrm{coh}}})$. Better yet, using (1.2.0.3), we obtain the simpler expression

$$\mathcal{L}i^*_{S_e}\mathcal{S} \simeq \mathsf{BG}_{S_e/\widetilde{Z}_e}(\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e), \mathcal{O}_{\widetilde{S}_e/\widetilde{Z}_e}) \tag{1.4.2.2}$$

for any Slodowy slice S_e, where $\widetilde{S}_e := \widetilde{\mathcal{N}} \times_{\mathfrak{g}} S_e$ denotes its resolution.

Our second technical tool is the "noncommutative Springer resolution" constructed by Bezrukavnikov–Mirković [**BM13**]. This is a remarkable tilting vector bundle $\mathcal{E} \in \mathrm{QC}(\widetilde{\mathcal{N}}/G)$, admitting many desirable properties. Most saliently, for any Slodowy slice S_e, the restriction $\mathcal{E}|_{\widetilde{S}_e}$ admits a graded lift (canonical up to twisting) such that the endomorphism algebra

$$\mathcal{A}_{S_e} := \mathrm{End}_{\widetilde{S}_e/\widetilde{Z}_e}(\mathcal{E}|_{\widetilde{S}_e})$$

is concentrated in non-negative weights. Moreover, the weight-0 component $\mathcal{A}_{S_e}^{\mathbb{G}_m}$ is semisimple, and its simple modules identify with the canonical basis of $K_0(\mathcal{B}_e/\mathbb{G}_m)$ discussed in §1.3. Combining (1.4.2.2) with the equivalence

$$\mathrm{Hom}_{\widetilde{S}_e/\widetilde{Z}_e}(\mathcal{E}|_{\widetilde{S}_e}, -)\colon \mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e) \xrightarrow{\sim} \mathcal{A}_{S_e}^{\mathrm{op}}\text{-mod}^{\widetilde{Z}_e} \tag{1.4.2.3}$$

thus yields an explicit complex

$$\cdots \to \mathcal{A}_{S_e} \otimes_k \mathcal{A}_{S_e} \otimes_k \mathcal{A}_{S_e} \boxtimes \mathcal{O}_{\widetilde{Z}_e} \xrightarrow{d_0 - d_1 + d_2} \mathcal{A}_{S_e} \otimes_k \mathcal{A}_{S_e} \boxtimes \mathcal{O}_{\widetilde{Z}_e} \xrightarrow{d_0 - d_1} \mathcal{A}_{S_e} \boxtimes \mathcal{O}_{\widetilde{Z}_e} \tag{1.4.2.4}$$

for $\mathcal{L}i^*_{S_e}\mathcal{S}$, concentrated in non-negative weights. Applying the functor $(\mathcal{L}p_{S_e,*})^{\mathbb{G}_m}$ thus yields a complex

$$\cdots \to \mathcal{A}_{S_e}^{\mathbb{G}_m} \otimes_k \mathcal{A}_{S_e}^{\mathbb{G}_m} \otimes_k \mathcal{A}_{S_e}^{\mathbb{G}_m} \otimes_k \mathcal{O}_{\widetilde{Z}_e} \xrightarrow{d_0 - d_1 + d_2} \mathcal{A}_{S_e}^{\mathbb{G}_m} \otimes_k \mathcal{A}_{S_e}^{\mathbb{G}_m} \otimes_k \mathcal{O}_{\widetilde{Z}_e} \xrightarrow{d_0 - d_1} \mathcal{A}_{S_e}^{\mathbb{G}_m} \otimes_k \mathcal{O}_{\widetilde{Z}_e},$$

and it is not hard to show using (1.4.0.2) that this complex computes $\mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m}$ as in (1.4.1.1).

Finally, our third technical tool is a more general notion of "monoidal functor," which is exactly adapted for identifying induced homomorphisms such as (1.4.1.2). The following later appears as Definition 5.2.4 (without the compact-generation assumptions):

1.4.3. Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be compactly generated rigid monoidal categories. A *homomorphism datum* from $\mathcal{A}$ to $\mathcal{B}$ is a triple $(\mathcal{M}, \beta, \check{\beta})$ consisting of a compactly generated right-dualizable $(\mathcal{B}, \mathcal{A})$-bimodule $\mathcal{M}$, along with left $\mathcal{B}$-module homomorphisms $\beta\colon \mathcal{B} \to \mathcal{M}$ and $\check{\beta}\colon \mathcal{M} \to \mathcal{B}$ which

(1) preserve compact objects; and
(2) induce inverse equivalences on Hochschild homology.

We say that a homomorphism datum is *unital* if additionally $\check{\beta} \circ \beta \simeq \mathrm{id}_{\mathcal{B}}$.

Given a homomorphism datum $(\mathcal{M}, \beta, \check{\beta})$, we may form the composite functor

$$F_{(\mathcal{M},\beta,\check{\beta})}\colon \mathcal{A} \xrightarrow{\mathrm{act}_{\mathcal{M}}} \mathrm{End}_{\mathcal{B}}(\mathcal{M})^{\mathrm{rev}} \xrightarrow{\check{\beta} \circ - \circ \beta} \mathrm{End}_{\mathcal{B}}(\mathcal{B})^{\mathrm{rev}} \simeq \mathcal{B}, \tag{1.4.3.1}$$

where the superscript $(-)^{\mathrm{rev}}$ denotes the reverse monoidal structure, and $\mathrm{act}_{\mathcal{M}}$ denotes the right $\mathcal{A}$-action on $\mathcal{M}$. Explicitly, for $a \in \mathcal{A}$, we have

$$F_{(\mathcal{M},\beta,\check{\beta})}(a) = \check{\beta}(\beta(\mathbf{1}_{\mathcal{B}}) \otimes a).$$

In general, $F_{(\mathcal{M},\beta,\check{\beta})}$ is *not* monoidal. However, it preserves the monoidal units if $(\mathcal{M}, \beta, \check{\beta})$ is unital, and it induces an algebra homomorphism

$$\mathrm{HH}(F_{(\mathcal{M},\beta,\check{\beta})})\colon \mathrm{HH}(\mathcal{A}) \to \mathrm{HH}(\mathcal{B})$$

provided that $\mathrm{HH}(\mathcal{A})$ is concentrated in degree 0 and the k-linearized Chern character $K_0(\mathcal{A})_k \to \mathrm{HH}_0(\mathcal{A})$ is surjective (as in Theorem 1.2.1). This closely parallels the structures we saw for Chriss–Ginzburg's homomorphism in §1.3. Finally, on categorical traces, we have a composite functor

$$F^{\mathrm{Tr}}_{(\mathcal{M},\beta,\check{\beta})}\colon \mathrm{Tr}(\mathcal{A}) \xrightarrow{\mathrm{Tr}(\mathcal{M})} \mathrm{Tr}(\mathcal{B}) \xrightarrow{\mathrm{pr}_{[\mathcal{B}]}} \langle [\mathbf{1}_{\mathcal{B}}] \rangle$$

sending $[\mathbf{1}_{\mathcal{A}}]$ to $[\mathbf{1}_{\mathcal{B}}]$, and the induced homomorphism

$$F^{\mathrm{Tr}}_{(\mathcal{M},\beta,\check{\beta})}\colon \operatorname{End}_{\mathrm{Tr}(\mathcal{A})}([\mathbf{1}_{\mathcal{A}}])^{\mathrm{op}} \to \operatorname{End}_{\mathrm{Tr}(\mathcal{B})}([\mathbf{1}_{\mathcal{B}}])^{\mathrm{op}} \tag{1.4.3.2}$$

identifies with $\mathrm{HH}(F_{(\mathcal{M},\beta,\check{\beta})})$ via (1.2.0.2).

We may fit (1.4.1.2) into this framework as follows. On the one hand, as shown in [**BL23**], we have a monoidal equivalence

$$\operatorname{Hom}_{\mathcal{H}^{\mathrm{coh}}}(\mathcal{E}^{\vee}\boxtimes\mathcal{E},-)\colon \mathcal{H}^{\mathrm{coh}} \xrightarrow{\sim} \mathcal{A}_{\mathfrak{g}}\otimes_{\mathcal{O}(\mathfrak{g})}\mathcal{A}_{\mathfrak{g}}^{\mathrm{op}}\text{-mod}^{\widetilde{G}}_{\mathrm{ren}} =: \mathcal{H}^{\mathrm{mod}}, \tag{1.4.3.3}$$

where the latter denotes the category of $\widetilde{G}$-equivariant $\mathcal{A}_{\mathfrak{g}}$-bimodules, "renormalized" so that its compact objects are cohomologically bounded complexes with finitely generated cohomology (rather than perfect complexes). This equivalence is moreover compatible with the right module structures on (1.4.2.3). On the other hand, for a Slodowy slice S_e, we may consider the monoidal category

$$\mathcal{J}^{\mathrm{mod}}_e := \mathcal{A}^{\mathbb{G}_m}_{S_e}\otimes_k \mathcal{A}^{\mathbb{G}_m,\mathrm{op}}_{S_e}\text{-mod}^{Z_e} \tag{1.4.3.4}$$

of Z_e-equivariant $\mathcal{A}^{\mathbb{G}_m}_{S_e}$-bimodules. By Bezrukavnikov–Losev's result in §1.3, its Hochschild homology identifies canonically with $\mathcal{J}_e$; moreover, we show that it is rigid, and that there are natural identifications $\mathrm{Tr}(\mathcal{J}^{\mathrm{mod}}_e)\simeq \mathrm{QC}^!(Z_e/Z_e)$ and $[\mathbf{1}_{\mathcal{J}^{\mathrm{mod}}_e}]\simeq \mathcal{S}^{\mathbf{B}_e}$, exactly recovering (1.4.0.3). Finally, the triple

$$\Phi^{\mathrm{mod}}_{S_e} := (\mathcal{A}^{\mathbb{G}_m}_{S_e}\otimes_k\mathcal{A}^{\mathrm{op}}_{S_e}\text{-mod}^{\widetilde{Z}_e}, -\underset{\mathcal{A}^{\mathbb{G}_m}_{S_e}}{\otimes}\mathcal{A}_{S_e}, -\underset{\mathcal{A}_{S_e}}{\otimes}\mathcal{A}^{\mathbb{G}_m}_{S_e}) \tag{1.4.3.5}$$

carries a natural structure of unital homomorphism datum from $\mathcal{H}^{\mathrm{mod}}$ to $\mathcal{J}^{\mathrm{mod}}_e\otimes\mathrm{Rep}(\mathbb{G}_m)$; here the latter two functors are given by extension of scalars along the natural inclusion and projection $\mathcal{A}^{\mathbb{G}_m}_{S_e}\hookrightarrow\mathcal{A}_{S_e}\twoheadrightarrow\mathcal{A}^{\mathbb{G}_m}_{S_e}$. It is then straightforward to identify the functor $F^{\mathrm{Tr}}_{\Phi^{\mathrm{mod}}_{S_e}}|_{\langle\mathcal{S}\rangle}$ with $(\mathcal{L}p_{S_e,*}\mathcal{L}i^*_{S_e})^{\mathbb{G}_m}$ and the homomorphism $\mathrm{HH}(F_{\Phi^{\mathrm{mod}}_{S_e}})$ with $\phi_{e,k}$. In fact, seminal results of Bezrukavnikov on the structure of the affine Hecke category [**Bez16**] essentially reduce the latter identification to Lusztig's original definition of ϕ_e. Our general formalism then implies the desired identification of (1.4.1.2).

1.5. Asymptotic Kazhdan–Lusztig theory

We now explain how to upgrade Theorem 1.4.1 to a categorification of $\mathcal{J}_e$, as suggested by Qiu–Xi's conjecture (1.3.0.2). More broadly, given the expression (1.2.0.3) for $\mathcal{S}$, we might hope that $\mathcal{S}^{\mathbf{B}_e}$ is likewise obtained by pushing forward the structure sheaf of a "q-independent" analog of the stack $\mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G})$. Qiu–Xi's conjecture suggests that this stack should be given by the loop space $\mathcal{L}(\mathcal{B}^{\mathbb{G}_m}_e/Z_e)$, according with the general "asymptotic" philosophy of the previous section. We therefore define

$$\mathcal{J}^{\mathrm{coh}}_e := \mathrm{QC}^!(\mathcal{B}^{\mathbb{G}_m}_e\times\mathcal{B}^{\mathbb{G}_m}_e/Z_e)$$

as in (1.4.3.4). General results of Ben-Zvi–Nadler–Preygel yield $\mathrm{Tr}(\mathcal{J}^{\mathrm{coh}}_e)\simeq\mathrm{QC}^!(Z_e/Z_e)$. Moreover, the object $[\mathbf{1}_{\mathcal{J}^{\mathrm{coh}}_e}]$ identifies with the pushforward $\mathcal{L}\pi^{\mathbb{G}_m}_{e,*}\mathcal{O}_{\mathcal{L}(\mathcal{B}^{\mathbb{G}_m}_e/Z_e)}$, where $\pi^{\mathbb{G}_m}_e\colon\mathcal{B}^{\mathbb{G}_m}_e/Z_e\to \mathrm{B}Z_e$ denotes the projection [**BZNP17a**].

In general, the categories $\mathcal{J}^{\mathrm{coh}}_e$ and $\mathcal{J}^{\mathrm{mod}}_e$ are inequivalent. However, their categorical *traces* are equivalent; if we could further identify $[\mathbf{1}_{\mathcal{J}^{\mathrm{coh}}_e}]$ with $[\mathbf{1}_{\mathcal{J}^{\mathrm{mod}}_e}]$, then (1.2.0.2) would yield a monoidal identification $\mathrm{HH}(\mathcal{J}^{\mathrm{coh}}_e)\simeq\mathrm{HH}(\mathcal{J}^{\mathrm{mod}}_e)$. Our strategy is to imitate (1.4.3.5) by endowing the $(\mathcal{J}^{\mathrm{coh}}_e\otimes\mathrm{Rep}(\mathbb{G}_m),\mathcal{H}^{\mathrm{coh}})$-bimodule $\mathrm{QC}^!(\mathcal{B}^{\mathbb{G}_m}_e\times\widetilde{S}_e/\widetilde{Z}_e)$ with the structure of a unital homomorphism datum from $\mathcal{H}^{\mathrm{coh}}$ to $\mathcal{J}^{\mathrm{coh}}_e\otimes\mathrm{Rep}(\mathbb{G}_m)$. This amounts to constructing a pair of $\mathrm{Coh}(\mathrm{B}\widetilde{Z}_e)$-linear functors between the bounded derived categories of coherent sheaves

$$\mathrm{Coh}(\mathcal{B}^{\mathbb{G}_m}_e/\widetilde{Z}_e)\to\mathrm{Coh}(\widetilde{S}_e/\widetilde{Z}_e)\to\mathrm{Coh}(\mathcal{B}^{\mathbb{G}_m}_e/\widetilde{Z}_e), \tag{1.5.0.1}$$

which compose to the identity and induce inverse equivalences on Hochschild homology. One can then use our general formalism to automatically produce comparison maps between $[\mathbf{1}_{\mathcal{J}^{\mathrm{coh}}_e}]$ and $[\mathbf{1}_{\mathcal{J}^{\mathrm{mod}}_e}]$.

For $e=0$, Chriss–Ginzburg's construction (1.3.0.1) suggests the functors $p^*_{\widetilde{\mathcal{N}}}$ and $i^*_{\mathcal{B}}$, respectively. However, for general e, there is no projection $\widetilde{S}_e\to\mathcal{B}^{\mathbb{G}_m}_e$. Instead, we are rescued by Halpern-Leistner's theory of "derived Θ-stratifications" [**HL21**]. The repelling $\mathbb{G}_m$-action on $\widetilde{S}_e$ induces a Białynicki-Birula stratification, which is in particular a derived Θ-stratification of $\widetilde{S}_e$ (with respect to our conventions, which different from

Halpern-Leistner's by a sign). Fix a tuple of integers $\underline{w} := (w_\alpha)_\alpha$ indexed by these strata (equivalently, by Z_e-orbits of connected components in $\mathcal{B}_e^{\mathbb{G}_m}$). Halpern-Leistner's "derived Kirwan surjectivity" provides an infinite semiorthogonal decomposition of $\mathrm{Coh}(\widetilde{S}_e/\widetilde{Z}_e)$, depending on $\underline{w}$, into subcategories whose sum is equivalent to $\mathrm{Coh}(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e)$. As for any semiorthogonal decomposition, we obtain inclusion and projection functors $\mathrm{HL}_{\underline{w}}, \check{\mathrm{HL}}_{\underline{w}}$ as in (1.5.0.1). These functors manifestly compose to the identity, induce inverse equivalences on Hochschild homology, and are $\mathrm{Coh}(\mathrm{B}Z_e)$-linear. However, they are *not* $\mathrm{Coh}(\mathrm{B}\mathbb{G}_m)$-linear (essentially, because the semiorthogonal decomposition depends on the choice of $\mathbb{G}_m$-weights $\underline{w}$).

Nonetheless, we may modify the functors $\mathrm{HL}_{\underline{w}}, \check{\mathrm{HL}}_{\underline{w}}$ to be $\mathrm{Coh}(\mathrm{B}\mathbb{G}_m)$-linear in a canonical fashion. Namely, define $\mathrm{HL}_{\underline{w}}^{\mathrm{gr}}$ via the composition

$$\begin{tikzcd}
\mathrm{Coh}(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e) \arrow[rrr, dashed, "\mathrm{HL}_{\underline{w}}^{\mathrm{gr}}"] \arrow[d, "\wr"] & & & \mathrm{Coh}(\widetilde{S}_e/\widetilde{Z}_e) \\
\mathrm{Coh}(\mathrm{B}\mathbb{G}_m)\otimes\mathrm{Coh}(\mathcal{B}_e^{\mathbb{G}_m}/Z_e) \arrow[r, "\mathrm{id}\otimes\mathrm{in}_{\underline{w}}"] & \mathrm{Coh}(\mathrm{B}\mathbb{G}_m)\otimes\mathrm{Coh}(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e) \arrow[r, "\mathrm{id}\otimes\mathrm{HL}_{\underline{w}}"] & \mathrm{Coh}(\mathrm{B}\mathbb{G}_m)\otimes\mathrm{Coh}(\widetilde{S}_e/\widetilde{Z}_e), \arrow[ur, "-\otimes-"'] &
\end{tikzcd}$$

where $\mathrm{in}_{\underline{w}}$ denotes the weight-twist $\langle -w_\alpha\rangle$ over the connected components in α. This functor is evidently $\mathrm{Coh}(\mathrm{B}\widetilde{Z}_e)$-linear. To construct its retraction $\check{\mathrm{HL}}_{\underline{w}}^{\mathrm{gr}}$, we establish the following "structure theorem" for $\mathrm{Coh}(\widetilde{S}_e/\widetilde{Z}_e)$:

1.5.1. Proposition. Write $\mathrm{HL}_{\underline{w}}^{\mathrm{gr}}$ for the induced functor $\mathrm{QC}^!(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e) \to \mathrm{QC}^!(\widetilde{S}_e/\widetilde{Z}_e)$, and $\mathrm{HL}_{\underline{w}}^{\mathrm{gr},R}$ for its right adjoint. Let

$$T_{\underline{w}} := \mathrm{HL}_{\underline{w}}^{\mathrm{gr},R}\circ \mathrm{HL}_{\underline{w}}^{\mathrm{gr}} \in \mathrm{Alg}(\mathrm{End}_{\mathrm{QC}^!(\mathrm{B}\widetilde{Z}_e)}(\mathrm{QC}^!(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e))) \simeq \mathrm{Alg}(\mathcal{J}_e^{\mathrm{coh}})$$

denote the $\mathrm{QC}^!(\mathrm{B}\widetilde{Z}_e)$-linear monad (or equivalently, algebra object of $\mathcal{J}_e^{\mathrm{coh}}$ acting via convolution). Then there is a canonical equivalence

$$\mathrm{QC}^!(\widetilde{S}_e/\widetilde{Z}_e) \simeq T_{\underline{w}}\text{-mod}_{\mathrm{QC}^!(\mathcal{B}_e^{\mathbb{G}_m})}.$$

Moreover,

(1) the monad $T_{\underline{w}}$ is concentrated in non-negative weights;
(2) the weight-0 component $T_{\underline{w}}^{\mathbb{G}_m}$ is upper-triangular with respect to the standard partial order on Białynicki-Birula strata; and
(3) the diagonal component of $T_{\underline{w}}^{\mathbb{G}_m}$ is isomorphic to the identity monad (equivalently, the monoidal unit $\mathbf{1}_{\mathcal{J}_e^{\mathrm{coh}}}$).

In particular, there are canonical monad homomorphisms

$$\mathrm{id}_{\mathrm{QC}^!(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e)} \to T_{\underline{w}} \to \mathrm{id}_{\mathrm{QC}^!(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e)} \tag{1.5.1.1}$$

composing to the identity.

This result appears as Proposition 6.2.8 and Corollary 6.2.10 in the main text. The proof is essentially a straightforward deduction from Halpern-Leistner's theory, and applies more generally to suitable derived Θ-stratifications over the stack $\mathrm{B}\mathbb{G}_m$. Notably, this proposition gives a precise analogy between the "coherent" categories $\mathrm{QC}^!(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e), \mathrm{QC}^!(\widetilde{S}_e/\widetilde{Z}_e)$ and the "module" categories $\mathcal{A}_{S_e}^{\mathbb{G}_m,\mathrm{op}}\text{-mod}^{\widetilde{Z}_e}, \mathcal{A}_{S_e}^{\mathrm{op}}\text{-mod}^{\widetilde{Z}_e}$. Indeed, the algebra $\mathcal{A}_{S_e}$ satisfies the same three conditions as $T_{\underline{w}}$ with respect to the canonical basis $\mathbf{B}_e$; however, unlike $T_{\underline{w}}$, it has no unipotent tail in weight-0, and it is concentrated in cohomological degree 0. The definition of $\Phi_{S_e}^{\mathrm{mod}}$ now suggests taking extension of scalars along the homomorphisms (1.5.1.1). This recovers $\mathrm{HL}_{\underline{w}}^{\mathrm{gr}}$, and further produces $\check{\mathrm{HL}}_{\underline{w}}^{\mathrm{gr}}$. Finally, we arrive at our second main result, which appears later as Theorem 6.3.5:

1.5.2. Theorem. *The unital homomorphism datum*

$$\Phi_{S_e,\underline{w}}^{\mathrm{coh}} := (\mathrm{QC}^!(\mathcal{B}_e^{\mathbb{G}_m}\times\widetilde{S}_e/\widetilde{Z}_e), \mathrm{id}_{\mathrm{QC}^!(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e)}\otimes\mathrm{HL}_{\underline{w}}^{\mathrm{gr}}, \mathrm{id}_{\mathrm{QC}^!(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e)}\otimes\check{\mathrm{HL}}_{\underline{w}}^{\mathrm{gr}}) \tag{1.5.2.1}$$

from $\mathcal{H}^{\mathrm{coh}}$ to $\mathcal{J}_e^{\mathrm{coh}}\otimes\mathrm{Rep}(\mathbb{G}_m)$ categorifies Theorem 1.4.1. More precisely, the latter functors induce a canonical isomorphism

$$[\mathbf{1}_{\mathcal{J}_e^{\mathrm{coh}}\otimes\mathrm{Rep}(\mathbb{G}_m)}] \simeq \mathcal{S}^{\mathbf{B}_e}\boxtimes\mathcal{O}_{\mathbb{G}_m}, \tag{1.5.2.2}$$

and the functor $F^{\mathrm{Tr}}_{\Phi^{\mathrm{coh}}_{S_e,\underline{w}}}|_{\langle \mathcal{S}\rangle}$ *identifies with* $(\mathcal{L}p_{S_e,*}\mathcal{L}i^*_{S_e})^{\mathbb{G}_m}$*. It follows that the algebra homomorphism*

$$\mathrm{HH}(F_{\Phi^{\mathrm{coh}}_{S_e,\underline{w}}})\colon \mathrm{HH}(\mathcal{H}^{\mathrm{coh}}) \to \mathrm{HH}(\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m))$$

canonically identifies with $\phi_{e,k}$ *(and all nonzero cohomology groups vanish). Finally, the Chern character induces algebra isomorphisms*

$$\mathcal{J}_{e,k}[v^{\pm 1}] \simeq \mathrm{HH}(\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)) \simeq K_0(\mathcal{B}^{\mathbb{G}_m}_e \times \mathcal{B}^{\mathbb{G}_m}_e/\widetilde{Z}_e)_k, \tag{1.5.2.3}$$

depending canonically on the choice of $\underline{w}$*. In particular, the isomorphisms obtained from two weight-vectors* $\underline{w}$ *and* $\underline{w}'$ *differ by a canonical inner automorphism of* $\mathcal{J}_{e,k}[v^{\pm 1}]$*.*

Specializing (1.5.2.3) at any point of $\mathbb{G}_m$ now yields an isomorphism of the form conjectured by Qiu–Xi. The proof of this theorem is straightforward once one establishes (1.5.2.2), which follows from a simple fiber-dimension counting argument. When $e = 0$, we essentially recover Chriss–Ginzburg's construction. Moreover, by results of Ben-Zvi–Nadler–Preygel, the small analog of the intermediate monoidal category of (1.4.3.1) recovers a "Koszul-dual" version of Dawydiak's categorification of $\mathcal{J}_0$ [**BZNP17b, Daw21**]. While the functor to this category is monoidal, its 0th K-theory is generally *not* isomorphic to $\mathcal{J}_0$.

Finally, the isomorphism (1.5.2.2) can be effectively used to compute the sheaf $\mathcal{S}^{\mathbf{B}_e}$, and hence the structure of $\mathcal{J}_{e,k}$, when $\mathcal{B}^{\mathbb{G}_m}_e$ admits a (suitably equivariant) full exceptional collection. Such a computation is often considerably more laborious using the presentation of $\mathcal{J}_e$ by generators and relations. The existence of such full exceptional collections in many examples suggests existence of a more canonical construction of Qiu–Xi's isomorphism (i.e., not depending on a choice of weights $\underline{w}$). Unfortunately, such a construction is not clear from the present work.

1.6. Bounding the universal trace functor

In particular, Theorem 1.5.2 shows that the universal trace $[\mathbf{1}_{\mathcal{J}^{\mathrm{coh}}_e}]$ is concentrated in cohomological degree 0, i.e., is a *sheaf* rather than a complex. This may be regarded as a "derived enhancement" of De Concini–Lusztig–Procesi's results on the Hodge numbers of $\mathcal{B}^{\mathbb{G}_m}_e$ and its fixed-point loci [**DCLP88**]. Given the parallels we have drawn, one might hope that the coherent Springer sheaf satisfies the same property. Such a statement has been conjectured by Ben-Zvi–Chen–Helm–Nadler and Zhu, and a proof has independently been announced by the latter and Hemo [**BZCHN22, Zhu21**].

More generally, we study the relationship between the universal trace functor

$$[-]\colon \mathcal{H}^{\mathrm{coh}} \to \mathrm{QC}^!(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})) \tag{1.6.0.1}$$

and the t-structures on either category, making essential use of the Block–Getzler sheaf introduced in §1.4. Recall that the category of ind-coherent sheaves on any locally almost of finite type Artin stack carries a standard t-structure [**Gai13**]. However, the mixed affine Hecke category carries an additional "exotic" t-structure coming from the equivalence $\mathcal{H}^{\mathrm{coh}} \simeq \mathcal{H}^{\mathrm{mod}}$ of (1.4.3.3). Our third main result (Theorem 4.3.2 in the sequel) gives general conditions for (co)connectivity of a sheaf $[\mathcal{F}]$ in terms of this exotic t-structure:

1.6.1. Theorem. *The functor* (1.6.0.1) *has cohomological amplitude in* $[-\dim\mathcal{N}, 0]$ *with respect to the exotic t-structure on* $\mathcal{H}^{\mathrm{coh}}$ *and the standard t-structure on* $\mathrm{QC}^!(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))$ *(in particular, it is right t-exact). Moreover, let* $\mathcal{F}$ *be a compact object of* $\mathcal{H}^{\mathrm{coh}}$*, and suppose that the right monoidal dual* $\mathcal{F}^{\vee,R}$ *is connective for the exotic t-structure. Then* $[\mathcal{F}]$ *is coconnective for the standard t-structure.*

In fact, the left and right monoidal duals of $\mathcal{F}$ are canonically isomorphic; we give an explicit formula for these duals using Grothendieck–Serre duality in Remark 3.3.7. Before commenting on the proof of this theorem, we describe some simple consequences. Recall that the category $\mathrm{QC}(\widetilde{\mathcal{N}}/\widetilde{G})$ carries a natural weak action of the affine braid group B^{aff} associated to the extended affine Weyl group W^{aff} [**BR12**]. Specifically, the action of any $a \in B^{\mathrm{aff}}$ is given by a sheaf $\mathcal{K}_a \in \mathcal{H}^{\mathrm{coh}}$, which acts on $\mathrm{QC}(\widetilde{\mathcal{N}}/\widetilde{G})$ via left convolution. Now, the canonical projection $B^{\mathrm{aff}} \twoheadrightarrow W^{\mathrm{aff}}$ admits a section, which sends $w \in W^{\mathrm{aff}}$ to the product of generators of B^{aff} corresponding to any reduced decomposition of w. We denote by B^{aff}_+ the submonoid of B^{aff} generated by the image of this section. Moreover, B^{aff} possesses a "translation subgroup" isomorphic to the weight lattice of G; the corresponding sheaves $\mathcal{K}_a$ are simply given by $\Delta_*\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)$, where $\Delta\colon \widetilde{\mathcal{N}} \hookrightarrow \widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$ denotes the diagonal map, and $\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)$ is the usual $\widetilde{G}$-equivariant line bundle on $\widetilde{\mathcal{N}}$ obtained from the weight λ.

Finally, Bezrukavnikov–Mirković have shown that $\mathcal{K}_a$ is connective for the exotic t-structure on $\mathcal{H}^{\mathrm{coh}}$ whenever $a \in B_+^{\mathrm{aff}}$ [**BM13**]. Theorem 1.6.1 now immediately implies:

1.6.2. Corollary. For any $a \in B_+^{\mathrm{aff}}$, the sheaf $[\mathcal{K}_a]$ is connective, and the sheaf $[\mathcal{K}_{a^{-1}}]$ is coconnective. In particular, for any dominant (resp. anti-dominant) weight λ, the sheaf $[\Delta_* \mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)]$ is connective (resp. coconnective). The coherent Springer sheaf $\mathcal{S} \simeq [\Delta_* \mathcal{O}_{\widetilde{\mathcal{N}}}]$ therefore lies in the heart $\mathrm{Coh}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))^{\heartsuit}$.

This result is Corollary 4.3.4 in the sequel. We now briefly outline the proof of Theorem 1.6.1. The right t-exactness statement is essentially immediate from Proposition 1.4.2. Indeed, the Block–Getzler sheaf over $\mathfrak{g}/\widetilde{G}$ yields a complex of the form (1.4.2.4), with the right-most copy of $\mathcal{A}_{\mathfrak{g}}$ replaced by a connective $\mathcal{A}_{\mathfrak{g}}$-bimodule.

The coconnectivity statement is far more involved. We first reduce coconnectivity of $[\mathcal{F}]$ to coconnectivity of its local cohomology at each nilpotent orbit, using the usual exact triangle for a complementary open and closed subscheme. Next, using a standard cotangent complex argument, we reduce to showing that the local cohomology of $\mathcal{L}i_{S_e}^*[\mathcal{F}]$ along the closed substack $\widetilde{Z}_e/\widetilde{Z}_e \subset \mathcal{L}(S_e/\widetilde{Z}_e)$ is coconnective, for any Slodowy slice S_e. At this point, we require two additional technical tools. First, we construct a cover of Z_e trivializing all cocycles appearing in the canonical basis:

1.6.3. Proposition. There exists a finite cover $Z_e^{\mathrm{cov}} \twoheadrightarrow Z_e$ such that for any $b \in \mathbf{B}_e$, the multiplicative line bundle $(\mathcal{C}_b, \alpha_b)$ on Z_e^b is trivialized after pulling back to $Z_e^{\mathrm{cov},b}$.

This result appears as Proposition 2.3.2 in the main text; its proof makes use of the general existence of a Schur covering of Z_e, combined with a careful type-by-type analysis of the structure of Z_e. Since the projection $Z_e^{\mathrm{cov}} \twoheadrightarrow Z_e$ is faithfully flat, we reduce to showing an analogous coconnectivity statement after pulling back to the stack $\mathcal{L}(S_e/\widetilde{Z}_e^{\mathrm{cov}})$. This essentially amounts to constructing a bounded model for the Block–Getzler sheaf (now over $S_e/\widetilde{Z}_e^{\mathrm{cov}}$) used previously to establish the right t-exactness statement.

To do this, we combine Proposition 1.6.3 with a result of Bezrukavnikov–Mirković (and Kaledin) to show that $\mathcal{A}_{S_e}$ is Z_e^{cov}-equivariantly Morita equivalent to a Koszul quadratic algebra $\mathcal{A}_{S_e}^{\mathrm{cov}}$ [**BM13**]. We may therefore compute the Block–Getzler sheaf using $\mathcal{A}_{S_e}^{\mathrm{cov}}$ in place of $\mathcal{A}_{S_e}$. As our second technical tool, we construct a Koszul bimodule resolution of $\mathcal{A}_{S_e}^{\mathrm{cov}}$, which has length $\dim \widetilde{S}_e$ by Grothendieck–Serre duality (see Proposition 4.1.4). This gives the desired bounded model; we conclude by using Grothendieck local duality and the assumptions on $\mathcal{F}$ to bound its local cohomology.

Separately, the existence of the cover Z_e^{cov} allows us to partially resolve the other conjecture of Qiu–Xi mentioned in §1.3. The following appears as Proposition 2.4.12 and Corollary 2.3.5 in the sequel:

1.6.4. Proposition. For any $e \in \mathcal{N}$, there is a based injection of algebras

$$\mathcal{J}_e \hookrightarrow K_0(\mathbf{B}_e \times \mathbf{B}_e / Z_e^{\mathrm{cov}}).$$

Moreover,

(1) the latter is finitely generated as a (left or right) $\mathcal{J}_e$-module;
(2) the identity component of Z_e^{cov} has simply-connected derived subgroup; and
(3) if all simple factors in G are of classical types, and the identity component of Z_e has simply-connected derived subgroup, then we may take $Z_e^{\mathrm{cov}} = Z_e$.

1.7. Overview

We now give a brief overview of the contents of this thesis. Chapter 2 consists of preliminaries on cocycles, the noncommutative Springer resolution, the canonical basis, the asymptotic affine Hecke algebra, and Bezrukavnikov–Ostrik's structure theorem. The only new content is the construction of the covering group Z_e^{cov} in §2.3, and the partial resolution of Qiu–Xi's conjecture in §2.4. The latter section also contains the construction of the asymptotic coherent Springer sheaf, which is of philosophical importance to the rest of the thesis.

In Chapter 3, we review the formalism of 2-categorical traces from [**GKRV22**], and recall its application to symmetric monoidal categories of quasi-coherent sheaves and convolution categories such as $\mathcal{H}^{\mathrm{coh}}$ (as developed in work of Ben-Zvi, Francis, Nadler, Preygel, and others). Finally, in §3.4, we construct the Block–Getzler sheaf, and show that it computes the 2-categorical class map (under certain assumptions). This result is a key technical tool throughout the rest of the text.

Chapter 4 is dedicated to the results of §1.6. In §4.1, we construct a Koszul resolution for Koszul quadratic algebras with multiple simple modules, which we later use to construct a bounded model for the Block–Getzler sheaf. In §4.2, we construct the exotic t-structure on $\mathcal{H}^{\mathrm{coh}}$ (and its analog over a Slodowy slice) using Preygel's formalism of "regularization." Finally, in §4.3, we combine these technical tools with the Block–Getzler sheaf to prove Theorem 1.6.1. In particular, we deduce the (co)connectivity statement in Corollary 1.6.2 for the braid group action, and conclude that the coherent Springer sheaf lies in the heart.

Chapter 5 is dedicated to the results of §1.4. In §5.1, we use the Block–Getzler sheaf to show that the coherent Springer sheaf "restricts" to the asymptotic coherent Springer sheaf. Next, in §5.2, we develop the formalism of homomorphism data (as described in Definition 1.4.3 and the ensuing discussion). Finally, in §5.3, we construct a homomorphism datum categorifying ϕ_e, and apply this formalism to deduce Theorem 1.4.1.

Chapter 6 corresponds to the content of §1.5. We begin in §6.1 with a review of Halpern-Leistner's theory of "derived Θ-stratifications." The following section (§6.2) is then dedicated to proving Proposition 1.5.1 in this more general setting: we construct the functor $\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}$, establish structural results for the "graded monad" $T_{\underline{w}}$, and deduce existence of the retraction $\check{\mathrm{HL}}^{\mathrm{gr}}_{\underline{w}}$. Finally, in §6.3, we use these functors to construct a family of homomorphism data categorifying ϕ_e (as in Theorem 1.5.2), and deduce Qiu–Xi's K-group conjecture over the field k. We conclude with an example demonstrating how the asymptotic coherent Springer sheaf may be computed from a (suitably equivariant) full exceptional collection in $\mathrm{Coh}(\mathcal{B}_e^{\mathbb{G}_m})$.

1.8. Assumptions and notation

Notational conventions in this thesis are largely drawn from a combination of [**GR17a, GR17b**] and [**BZCHN22**]. Most of these conventions are explained (or at least indicated) throughout the text, but we will collect some of the most salient notions here for the reader's reference.

We work throughout over an algebraically closed field k of characteristic 0. All algebro-geometric objects (schemes, stacks, etc.) are implicitly defined over k, and we sometimes write $* := \operatorname{Spec} k$ for this base scheme. We will mostly work with k-algebras, aside from the rings $\mathcal{H}$ and $\mathcal{J}$, which are defined over $\mathbb{Z}$; we will use the notation $(-)_k := - \otimes_{\mathbb{Z}} k$ to denote the k-linearization. Likewise, all dg-categories will be k-linear. Crucially, unless explicitly stated otherwise, all categories, functors, and Hom-spaces in this thesis are dg-derived, and all limits and colimits are homotopical. Thus, we will write "$(1,1)$-category" for the classical notion of category, and take cohomology to recover non-derived functors from their derived counterparts. To this end, all (co)chain complexes in this thesis are cohomologically indexed. Given a category $\mathcal{C}$ equipped with a t-structure, we let $\mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 0}$ denote its full subcategories of connective and coconnective objects, respectively, and $\mathcal{C}^{\heartsuit}$ denote its heart. We further denote by $\iota^{\leq 0} \dashv \tau^{\leq 0}$ and $\tau^{\geq 0} \dashv \iota^{\geq 0}$ the usual inclusion and truncation functors, and by H^* the functor of (co)chain cohomology. In particular, we use HH to denote the Hochschild *complex*, rather than its cohomology groups $\mathrm{H}^*\,\mathrm{HH}$.

Our conventions for dg-categories follow those of [**GR17a**, Ch. 1], which will be a sufficient reference for our purposes. Thus, we will make frequent use of Lurie's language of ∞-categories and higher algebra, as in [**Lur09, Lur17**]. In particular, we let Vect_k denote the symmetric monoidal, cocomplete, stable ∞-category of chain complexes of vector spaces (obtained by applying the dg-nerve construction to the usual pre-triangulated dg-category, see [**Lur17**, Cons. 1.3.1.6]). We use the term *dg-category* to mean a (presentable) cocomplete stable ∞-category equipped with a Vect_k-module structure. All functors between dg-categories will be continuous (i.e., colimit preserving) unless explicitly stated otherwise. The ∞-category $\mathbf{dgCat}_k$ of dg-categories (and continuous functors) carries a symmetric monoidal structure given by the Lurie tensor product, with unit object Vect_k. For any dg-category $\mathcal{C}$, we let $\mathcal{C}^c$ denote the full subcategory of compact objects (i.e., those $X \in \mathcal{C}$ for which $\mathrm{Hom}_{\mathcal{C}}(X, -)$ preserves countable filtered colimits). If $\mathcal{C}$ is compactly generated, then we may recover it as the Ind-completion $\mathrm{Ind}(\mathcal{C}^c)$. Moreover, given an object $X \in \mathcal{C}$, we write $\langle X \rangle \subset \mathcal{C}$ for the full subcategory weakly generated by X. Finally, given a symmetric monoidal dg-category $\mathcal{C}$ and an algebra object $A \in \mathrm{Alg}(\mathcal{C})$, we let $A\text{-mod}_{\mathcal{C}}$ denote the category of A-module objects in $\mathcal{C}$, and $A\text{-perf}_{\mathcal{C}} := A\text{-mod}_{\mathcal{C}}^c$ the full subcategory of A-perfect objects. We omit the subscript $\mathcal{C}$ when $\mathcal{C} = \mathrm{Vect}_k$. When $\mathcal{C} = \mathbf{dgCat}_k$, we obtain the notion of monoidal dg-category $\mathcal{A}$, and let $\mathcal{A}\text{-}\mathbf{mod}$ denote the $(\infty,2)$-category of $\mathcal{A}$-module categories as in [**GKRV22**, §3.6] (in fact, the $(\infty,2)$-structure on $\mathbf{dgCat}_k \simeq \mathrm{Vect}_k\text{-}\mathbf{mod}$ will be central to this thesis).

We work exclusively with the language of derived algebraic geometry, primarily following [**GR17a**] (though useful references abound). Here, $(1,1)$-functors-of-points from classical commutative rings to sets are replaced by *prestacks*, which are ∞-functors from connective commutative dg-k-algebras to simplicial sets. Unless explicitly specified as "classical," all schemes, stacks, fiber products, etc., are to be understood in the derived sense.[3] We will denote the operation of classical truncation by $(-)^{\mathrm{cl}}$, reserving the notation π_0 for sets of connected components. Given a prestack $\mathcal{X}$, we will sometimes write $x \in \mathcal{X}$ to mean "x is a k-point of $\mathcal{X}$." We write $\mathbb{L}_{\mathcal{X}}$ for the cotangent complex of $\mathcal{X}$ (if it exists). Given a map $f\colon \mathcal{X} \to \mathcal{Y}$, we write $df\colon f^*\mathbb{L}_{\mathcal{Y}} \to \mathbb{L}_{\mathcal{X}}$ for the codifferential, and $\Delta_{\mathcal{X}/\mathcal{Y}}\colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ for the relative diagonal.

Given a prestack $\mathcal{X}$, we let $\mathrm{QC}(\mathcal{X})$ denote the symmetric monoidal dg-category of quasi-coherent sheaves, defined by right Kan extension from the assignment $\mathrm{Spec}(A) \mapsto A\text{-mod}$ on affine derived schemes; it carries a canonical t-structure induced from that on A-mod. We let $\mathrm{Perf}(\mathcal{X}) \subset \mathrm{QC}(\mathcal{X})$ denote the full subcategory spanned by dualizable objects, i.e., those sheaves whose pullback to any affine scheme is quasi-isomorphic to a bounded complex of vector bundles. When $\mathcal{X}$ is perfect (see Definition 3.2.9, as well as [**BZFN10**]), we have $\mathrm{QC}(\mathcal{X}) \simeq \mathrm{Ind}(\mathrm{Perf}(\mathcal{X}))$. When X is a scheme which is almost of finite type, we let $\mathrm{Coh}(X)$ (resp. $\mathrm{Coh}^-(X), \mathrm{Coh}^+(X)$) denote the full subcategory of $\mathrm{QC}(X)$ comprising cohomologically bounded (resp. bounded above, bounded below) complexes with coherent cohomology; if X is eventually coconnective, we have $\mathrm{Perf}(X) \subset \mathrm{Coh}(X)$. When the prestack $\mathcal{X}$ is locally almost of finite type, we let $\mathrm{QC}^!(\mathcal{X})$ denote the dg-category of ind-coherent sheaves, defined by a suitable right Kan extension from the assignment $X \mapsto \mathrm{Ind}(\mathrm{Coh}(X))$; if $\mathcal{X}$ is furthermore an Artin stack, then $\mathrm{QC}^!(\mathcal{X})$ carries a canonical t-structure (see [**Gai13**, Prop. 11.7.5]). When $\mathcal{X}$ is an algebraic stack, we may define a full subcategory $\mathrm{Coh}(\mathcal{X}) \subset \mathrm{QC}^!(\mathcal{X})$ via $*$-pullback to a smooth atlas (and similarly for $\mathrm{Coh}^-(\mathcal{X}), \mathrm{Coh}^+(\mathcal{X})$); when $\mathcal{X}$ is QCA (see Definition 3.3.2, as well as [**DG13**]), we have $\mathrm{QC}^!(\mathcal{X}) \simeq \mathrm{Ind}(\mathrm{Coh}(\mathcal{X}))$. Finally, when $\mathcal{X}$ is smooth, the notions of coherent and perfect, as well as quasi-coherent and ind-coherent, coincide. We work freely with the six-functor formalism (letting $\mathcal{H}om$ denote the internal sheaf-Hom), and use the same notations for all categories of sheaves (e.g., for suitable eventually coconnective $f\colon \mathcal{X} \to \mathcal{Y}$, we write $f^*\colon\ \mathrm{QC}(\mathcal{Y}) \to \mathrm{QC}(\mathcal{X})$ as well as $f^*\colon\ \mathrm{QC}^!(\mathcal{Y}) \to \mathrm{QC}^!(\mathcal{X})$, and likewise for the functors $f_*, f^!$).

Regarding representation theory, we allow reductive groups to be disconnected, instead specifying a group as "connected reductive" when necessary. Given a linear algebraic group G, we let $\mathrm{B}G = */G$ denote its classifying stack, and set $\mathrm{Rep}(G) := \mathrm{QC}(\mathrm{B}G)$. Thus, we use $\mathrm{Rep}(G)^c$ to denote the full subcategory of "finite-dimensional" representations. Given $A \in \mathrm{Alg}(\mathrm{Rep}(G))$, we set $A\text{-mod}^G := A\text{-mod}_{\mathrm{Rep}(G)}$ and $A\text{-perf}^G := A\text{-perf}_{\mathrm{Rep}(G)}$. If G acts on a set S, we let $G^s \subset G$ denote the stabilizer of an element $s \in S$. In particular, for $g \in G$ or $x \in \mathfrak{g}$, we write G^g and G^x for the corresponding centralizers, i.e., their stabilizers under the adjoint actions of G. Likewise, given a $\mathfrak{g}$-representation V, we let $\mathfrak{g}^v \subset \mathfrak{g}$ denote the annihilator of $v \in V$; in particular, for x as above, $\mathfrak{g}^x$ denotes the centralizer. Finally, we use $Z(G), Z(\mathfrak{g})$ to denote the centers of $G, \mathfrak{g}$, respectively.

Throughout the text, we set $\widetilde{G} := G \times \mathbb{G}_m$, and similarly $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus k$ for its Lie algebra $\mathfrak{g}$ (where k denotes the 1-dimensional abelian Lie algebra).[4] Equivariance with respect to the group $\widetilde{G}$ yields an additional weight-grading, which is central to this work. Given any $\mathbb{Z}$-graded object V and $w \in \mathbb{Z}$, we denote by $V_w, V_{\geq w}, V_{\leq w}$ the sum of the components of V lying in weight-w, weights $\geq w$, and weights $\leq w$ (and likewise for $V_{>w}, V_{<w}$). We also let $\langle - \rangle$ denote the grading "twist" (or "shift"); that is, $V\langle w \rangle$ is the $\mathbb{Z}$-graded object for which $(V\langle w\rangle)_{w'} = V_{w+w'}$. The twist $\langle n \rangle$ therefore corresponds to the action of the weight $-n$ character of $\mathbb{G}_m$. Note that we have set $\mathbb{G}_m$ to act on $\mathfrak{g}$ and the fibers of $\widetilde{\mathcal{N}}$ by weight -2 (i.e., $z \cdot x = z^{-2}x$ for $x \in \mathfrak{g}$), and hence on functions by weight 2 (i.e., $z \cdot f(-) = z^2 f(-)$ for $f \in \mathfrak{g}^*$). This convention differs from that of [**BZCHN22**], but all its results continue to apply after straightforward modifications. Our convention instead ensures that the Jacobson–Morozov cocharacter of $\widetilde{G}$ projects to the tautological cocharacter of $\mathbb{G}_m$ for any e; this in turn agrees with Lusztig's conventions for the asymptotic affine Hecke algebra.

[3]The sole exceptions are the notations X^G, X^g for a group G acting on a classical scheme X and $g \in G$; these will denote the *classical* fixed points (as for $\mathcal{B}_e^{\mathbb{G}_m}$). We instead use loop space notation as in Definition 3.2.2 or mapping stack notation as in §6.1 to denote various notions of "derived" fixed points.

[4]Note that the latter conflicts with our notation for the Grothendieck simultaneous resolution; however, this should not cause any confusion, as the latter is only briefly mentioned in §2.2.7, and does not play any essential role in this work.

CHAPTER 2

The asymptotic coherent Springer sheaf

2.1. The Schur multiplier

2.1.1. In this section, we collect various results on the Schur multiplier of a linear algebraic group G. Other discussions of this topic may be found in [**Ela09**], and in [**Ros21**] for the case of connected groups. Most statements in this section are well-known for finite groups; our task is only to show that they carry over to linear algebraic groups. In particular, we show that G admits a Schur covering, and give various criteria for computing its Schur multiplier.

We first review some general properties of cocycles[5] on G that will be used in the sequel. We begin by recalling the definition, owed to Elagin:

2.1.2. Definition ([**Ela09**, Def. 1.4]). Let $m\colon G \times G \to G$ denote the multiplication map. A *cocycle on* G is the data of a line bundle $\mathcal{C}$ on G and an isomorphism $\alpha\colon \mathcal{C} \boxtimes \mathcal{C} \simeq m^*\mathcal{C}$ satisfying the following associativity condition: the isomorphisms

$$(\mathrm{id} \times m)^* \alpha \circ (\mathrm{id} \boxtimes \alpha), (m \times \mathrm{id})^* \alpha \circ (\alpha \boxtimes \mathrm{id})\colon \mathcal{C} \boxtimes \mathcal{C} \boxtimes \mathcal{C} \simeq (m \circ (\mathrm{id} \times m))^* \mathcal{C} \tag{2.1.2.1}$$

of line bundles on $G \times G \times G$ are equal. A *morphism of cocycles* $(\mathcal{C}, \alpha) \to (\mathcal{C}', \alpha')$ is a morphism of line bundles $\mathcal{C} \to \mathcal{C}'$ commuting with α, α'.

2.1.3. Notation. We denote the resulting $(1,1)$-category of cocycles on G by $\mathrm{Coc}(G)$; it carries a natural rigid symmetric monoidal structure under the tensor product of line bundles. We let $\mathrm{M}(G)$ denote the abelian group of isomorphism classes of $\mathrm{Coc}(G)$, and refer to it as the *Schur multiplier* of G. Moreover, given a group homomorphism $\varphi\colon G' \to G$, we have a natural monoidal functor $\varphi^*\colon \mathrm{Coc}(G) \to \mathrm{Coc}(G')$ and corresponding restriction homomorphism $\varphi^*\colon \mathrm{M}(G) \to \mathrm{M}(G')$.

Finally, we let $X^*(G), X_*(G)$ denote the group of characters (i.e., the Pontryagin dual) and cocharacters of G, respectively.

2.1.4. We now recall the relationship between Definition 2.1.2 and the classical notion of Schur multiplier:

2.1.5. Proposition. The $(1,1)$-category $\mathrm{Coc}(G)$ is naturally monoidally equivalent to the 2-group of central extensions[6] of G by $\mathbb{G}_m$ (in the category of linear algebraic groups). Moreover, let $(\mathcal{C}, \alpha) \in \mathrm{Coc}(G)$, and let

$$1 \to \mathbb{G}_m \to G_{(\mathcal{C},\alpha)} \xrightarrow{p_{(\mathcal{C},\alpha)}} G \to 1 \tag{2.1.5.1}$$

be the associated central extension. Then $[(\mathcal{C}, \alpha)]$ lies in the kernel of the restriction map

$$p^*_{(\mathcal{C},\alpha)}\colon \mathrm{M}(G) \to \mathrm{M}(G_{(\mathcal{C},\alpha)}).$$

Finally, if $[(\mathcal{C}, \alpha)]$ has finite order, then (2.1.5.1) canonically descends to a central extension

$$1 \to X^*(\langle [(\mathcal{C}, \alpha)] \rangle) \to \overline{G}_{(\mathcal{C},\alpha)} \xrightarrow{\overline{p}_{(\mathcal{C},\alpha)}} G \to 1. \tag{2.1.5.2}$$

[5]These are sometimes also referred to as "multiplicative" or "translation-invariant" line bundles.

[6]Here, morphisms are commutative diagrams of group homomorphisms

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & G' \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] & G'' \arrow[r] & G \arrow[r] & 1,
\end{tikzcd}$$

which are automatically isomorphisms. Moreover, the monoidal structure is provided by the "Baer sum" of extensions, and monoidal inverses are given by inversion in $\mathbb{G}_m$.

Proof. For the first assertion, we recall the construction, and leave the remaining details (some of which are carried out in [**Ela09**, §1]) to the reader. Given a cocycle $(\mathcal{C}, \alpha) \in \mathrm{M}(G)$, we may form the graded $\mathcal{O}_G$-algebra

$$\mathcal{R}_{(\mathcal{C},\alpha)} := \bigoplus_{n \in \mathbb{Z}} \mathcal{C}^{\otimes n}. \tag{2.1.5.3}$$

The relative spectrum

$$G_{(\mathcal{C},\alpha)} := \underline{\mathrm{Spec}}_G(\mathcal{R}_{(\mathcal{C},\alpha)}) \xrightarrow{p_{(\mathcal{C},\alpha)}} G$$

is then a principal $\mathbb{G}_m$-bundle. Moreover, we may equip $G_{(\mathcal{C},\alpha)}$ with a multiplication via the map of $\mathcal{O}_G$-algebras

$$m^*\mathcal{R}_{(\mathcal{C},\alpha)} \to \mathcal{R}_{(\mathcal{C},\alpha)} \boxtimes \mathcal{R}_{(\mathcal{C},\alpha)}$$

generated by (the inverse of) α. The associativity condition (2.1.2.1) now guarantees associativity of this multiplication, and the remaining properties can be verified similarly. Conversely, given a central extension

$$1 \to \mathbb{G}_m \to G' \to G \to 1,$$

the coordinate ring $\mathcal{O}_{G'}$ carries a grading induced by the right-regular representation of $\mathbb{G}_m$, and it is not difficult to show that the weight-1 component[7] yields a cocycle on G.

For the second assertion, note that we have natural $\mathcal{R}_{(\mathcal{C},\alpha)}$-module isomorphisms

$$p^*_{(\mathcal{C},\alpha)}\mathcal{C} \simeq \mathcal{C} \otimes \mathcal{R}_{(\mathcal{C},\alpha)} \simeq \mathcal{R}_{(\mathcal{C},\alpha)},$$

which clearly trivialize $p^*_{(\mathcal{C},\alpha)}\alpha$.

Finally, for the third assertion, note that when $[(\mathcal{C}, \alpha)]$ has finite order d, we also have a $\mathbb{Z}/d\mathbb{Z}$-graded $\mathcal{O}_G$-algebra

$$\overline{\mathcal{R}}_{(\mathcal{C},\alpha)} := \bigoplus_{0 \le n < d} \mathcal{C}^{\otimes n}$$

via the isomorphism $\mathcal{C}^{\otimes d} \simeq \mathcal{O}_G$. It is now straightforward to verify that this yields a central extension (2.1.5.2) with the desired property. □

2.1.6. In fact, we shall soon see that *every* cocycle on G has finite order. Regardless, in the situation of (2.1.5.1), we obtain a canonical decomposition[8]

$$\mathrm{Rep}(G_{(\mathcal{C},\alpha)}) \simeq \bigoplus_{n \in X^*(\mathbb{G}_m)} \mathrm{Rep}(G_{(\mathcal{C},\alpha)})_n \tag{2.1.6.1}$$

as $\mathrm{Rep}(G)$-module categories (and similarly for $\overline{G}_{(\mathcal{C},\alpha)}$), where we have let n denote the nth power of the tautological character of $\mathbb{G}_m$. This allows us to "twist" any $\mathrm{Rep}(G)$-module category by a cocycle:

2.1.7. Definition. Let $\mathcal{C}$ be a $\mathrm{Rep}(G)$-module category. The $(\mathcal{C}, \alpha)$-*twist* of $\mathcal{C}$ is the $\mathrm{Rep}(G)$-module category

$$\mathcal{C}^{(\mathcal{C},\alpha)} := \mathcal{C} \otimes_{\mathrm{Rep}(G)} \mathrm{Rep}(G_{(\mathcal{C},\alpha)})_1.$$

Likewise, we define the $(\mathcal{C}, \alpha)$-twist of a small category via the corresponding decomposition of $\mathrm{Rep}(G_{(\mathcal{C},\alpha)})^c$.

2.1.8. Equivalently, we may write $\mathcal{C}^{(\mathcal{C},\alpha)} \simeq \mathcal{C} \otimes_{\mathrm{Rep}(G)} \mathrm{Rep}(G)^{(\mathcal{C},\alpha)}$. Note that by Proposition 2.1.5, we have canonical $\mathrm{Rep}(G)$-linear equivalences

$$\mathrm{Rep}(G)^{(\mathcal{C},\alpha)} \otimes_{\mathrm{Rep}(G)} \mathrm{Rep}(G)^{(\mathcal{C}',\alpha')} \simeq \mathrm{Rep}(G)^{(\mathcal{C},\alpha)\cdot(\mathcal{C}',\alpha')} \tag{2.1.8.1}$$

for any cocycles $(\mathcal{C}, \alpha), (\mathcal{C}', \alpha') \in \mathrm{Coc}(G)$ (see also [**Ela09**, Prop. 1.5]). In particular,

$$\mathrm{Rep}(G_{(\mathcal{C},\alpha)})_n \simeq \mathrm{Rep}(G)^{(\mathcal{C},\alpha)^n} \tag{2.1.8.2}$$

for each character n, and similarly for $\overline{G}_{(\mathcal{C},\alpha)}$. Moreover, given a homomorphism $\varphi \colon G' \to G$, the naturality statement yields a $\mathrm{Rep}(G)$-linear functor

$$\mathrm{Res}^G_{G'} \colon \mathrm{Rep}(G)^{(\mathcal{C},\alpha)} \to \mathrm{Rep}(G')^{\varphi^*(\mathcal{C},\alpha)}. \tag{2.1.8.3}$$

[7] Alternatively, we may take the weight-(-1) component with respect to the left-regular representation.
[8] See for instance [**BZCHN22**, Rem. 2.36].

Two cases of Definition 2.1.7 will be of especial interest to us. Letting X be a scheme with an action of G, we refer to the objects of $\mathrm{QC}(X/G)^{(\mathcal{C},\alpha)}$ as $(\mathcal{C},\alpha)$-*equivariant sheaves* on X, and to the objects of $\mathrm{Rep}(G)^{(\mathcal{C},\alpha)}$ as $(\mathcal{C},\alpha)$-*representations* of G.

2.1.9. Remark. An alternative definition of these objects which is less amenable to ∞-categorical methods, though more explicit, was given in [**Ela09**]; we record it for the reader's intuition. There, a $(\mathcal{C},\alpha)$-equivariant sheaf on X was defined as a sheaf $\mathcal{F}$ on X and an isomorphism $\theta\colon \mathcal{C}\boxtimes\mathcal{F}\simeq \mathrm{act}^*\mathcal{F}$, satisfying a cocycle condition given by commutativity of the following diagram:

$$\begin{tikzcd}
 & \mathcal{C}\boxtimes \mathrm{act}^*\mathcal{F} \arrow[r, "(\mathrm{id}\times\mathrm{act})^*\theta"] & (\mathrm{act}\circ(\mathrm{id}\times\mathrm{act}))^*\mathcal{F} \arrow[d, equal] \\
\mathcal{C}\boxtimes\mathcal{C}\boxtimes\mathcal{F} \arrow[ur, "\mathrm{id}\boxtimes\theta"] \arrow[dr, "\alpha\boxtimes\mathrm{id}"'] & & \\
 & m^*\mathcal{C}\boxtimes\mathcal{F} \arrow[r, "(m\times\mathrm{id})^*\theta"] & (\mathrm{act}\circ(m\times\mathrm{id}))^*\mathcal{F}
\end{tikzcd}$$

In particular, an $(\mathcal{O}_G,\mathrm{id})$-equivariant sheaf is just a usual G-equivariant sheaf. Compatibility of these two notions is shown in [**Ela09**, Prop. 1.6].

2.1.10. Finally, we record the following general properties of cocycles on G, which we will use in §2.4 to construct the asymptotic coherent Springer sheaf.

2.1.11. Lemma. Let $(\mathcal{C},\alpha)\in\mathrm{Coc}(G)$. Then $\mathcal{C}$ is canonically equivariant with respect to the adjoint action of G, and α descends to an isomorphism of G-equivariant sheaves.

Proof. The structure sheaf $\mathcal{O}_{G_{(\mathcal{C},\alpha)}}$ is canonically equivariant with respect to the adjoint action of $G_{(\mathcal{C},\alpha)}$. This action factors through G, and commutes with the action of $\mathbb{G}_m$ on $G_{(\mathcal{C},\alpha)}$ by translations. Thus, each summand of (2.1.5.3) is canonically G-equivariant, and the conclusion follows. ☐

2.1.12. Remark. One can construct this equivariance structure more explicitly via α. Specifically, pulling back (2.1.2.1) along the map

$$G\times G\to G\times G\times G$$
$$(g,g')\mapsto(g,g',g^{-1})$$

gives an isomorphism

$$(\mathcal{C}\otimes i^*\mathcal{C})\boxtimes\mathcal{C}\simeq \mathrm{ad}^*\mathcal{C}, \tag{2.1.12.1}$$

where $i\colon G\to G$ denotes the inversion map, and $\mathrm{ad}\colon G\times G\to G$ denotes the adjoint action. Next, pulling back α along the map $G\to G\times G$ given by $g\mapsto(g,g^{-1})$ gives an isomorphism

$$\mathcal{C}\otimes i^*\mathcal{C}\simeq\mathcal{O}_G\otimes_k\mathcal{C}_e, \tag{2.1.12.2}$$

where $e\in G(k)$ denotes the identity element. Finally, pulling back α to (e,e) gives an isomorphism $\mathcal{C}_e\otimes\mathcal{C}_e\simeq\mathcal{C}_e$, so tensoring with $\mathcal{C}_e^\vee$ yields a canonical isomorphism $\mathcal{C}_e\simeq k$. Thus, using (2.1.12.1) and (2.1.12.2), we obtain an isomorphism $\mathcal{O}_G\boxtimes\mathcal{C}\simeq\mathrm{ad}^*\mathcal{C}$, and one can check using the associativity condition (2.1.2.1) that this gives a conjugation-equivariant structure on $\mathcal{C}$ commuting with α.

2.1.13. Lemma. Let $(\mathcal{C},\alpha)\in\mathrm{Coc}(G)$, and let V be a compact $(\mathcal{C},\alpha)$-representation. Then V generates $\mathrm{Rep}(G)^{(\mathcal{C},\alpha)}$ as a $\mathrm{Rep}(G)$-module. Moreover, if G is reductive[9], then we have canonical isomorphisms

$$\Gamma(G/G,\mathcal{C})\simeq K_0(\mathrm{Rep}(G)^{(\mathcal{C},\alpha)})_k\cong R(G)_k\cdot[V]\subset R(G_{(\mathcal{C},\alpha)})_k, \tag{2.1.13.1}$$

where the inclusion into the k-linearized representation ring of $G_{(\mathcal{C},\alpha)}$ is via (2.1.6.1).

Proof. Let $V'\in\mathrm{Rep}(G)^{(\mathcal{C},\alpha)}$. Then $V'\otimes V^*$ is a G-representation by (2.1.8.1), and V' is a summand of $(V'\otimes V^*)\otimes V$ as the characteristic of k is 0. The latter assertion is now immediate from the Peter–Weyl theorem and the proof of Proposition 2.1.5. ☐

2.1.14. We now give a series of criteria for computing Schur multipliers under various assumptions on G. We begin by recalling how to compute the Schur multiplier of a connected group (in characteristic 0). This is a very specific case of the main result of [**KN06**], or alternatively, an (unpublished) result of Gabber.

[9] In this thesis, we use "reductive" to mean "linearly reductive"; in particular, G is not assumed to be connected.

2.1.15. Notation. We let $\pi_0(G), \pi_1(G)$ denote the component group and fundamental group of G, respectively. Moreover, we let $G^\circ, [G,G]$ denote its identity component and derived subgroup, as usual.

2.1.16. Proposition ([**KN06**, Thm. 0.1], [**Ros21**, Thm. 1.3]). *Suppose G is connected. Then*

$$\mathrm{M}(G) \cong X^*(\pi_1([G,G])) \cong \mathrm{Pic}(G),$$

functorially in G. In particular, it is finite.

2.1.17. From this, we deduce the following finiteness result, which will allow us to construct Schur coverings in Proposition 2.1.25:

2.1.18. Corollary. *The group $\mathrm{M}(G)$ is finite.*

Proof. Consider the restriction homomorphism

$$\mathrm{M}(G) \to \mathrm{M}(G^\circ). \tag{2.1.18.1}$$

By Proposition 2.1.16, the latter is finite, so it suffices to show that the kernel of (2.1.18.1) is finite.
Central extensions in the kernel of (2.1.18.1) are in bijection with short exact sequences

$$1 \to \mathbb{G}_m \times G^\circ \to G' \to \pi_0(G) \to 1 \tag{2.1.18.2}$$

such that the outer action

$$\pi_0(G) \to \mathrm{Out}(\mathbb{G}_m \times G^\circ)$$

is of the form

$$\begin{pmatrix} \mathrm{const}_{\mathrm{id}_{\mathbb{G}_m}} & c \\ 0 & \psi_G \end{pmatrix}, \tag{2.1.18.3}$$

where ψ_G is the original outer action of the extension

$$1 \to G^\circ \to G \to \pi_0(G) \to 1,$$

and $c\colon \pi_0(G) \to X^*(G^\circ)$ is a 1-cocycle (the action of $\pi_0(G)$ on $X^*(G^\circ)$ is given by pullback along ψ_G). Moreover, the short exact sequences (2.1.18.2) are considered up to splittings of the extension

$$1 \to \mathbb{G}_m \to G'^\circ \to G^\circ \to 1,$$

i.e., up to conjugation of (2.1.18.3) by

$$\begin{pmatrix} \mathrm{id}_{\mathbb{G}_m} & a \\ 0 & \mathrm{id}_{G^\circ} \end{pmatrix} \tag{2.1.18.4}$$

for $a \in X^*(G^\circ)$. This operation corresponds to subtracting the coboundary corresponding to a from c; thus, such outer actions are classified by the cohomology group

$$\mathrm{H}^1(\pi_0(G), X^*(G^\circ)). \tag{2.1.18.5}$$

Since $\pi_0(G)$ is finite, (2.1.18.5) is torsion; moreover, $X^*(G^\circ)$ is a finite-rank lattice, and therefore (2.1.18.5) is finitely generated, hence finite. Finally, since $\mathbb{G}_m \times G^\circ$ is linear and k is of characteristic 0, the set of short exact sequences (2.1.18.2) corresponding to any element of (2.1.18.5) is finite by [**LA17**, Thm. 4.1], and so the kernel of (2.1.18.1) is finite, as desired. □

2.1.19. We now recall several useful criteria for computing the Schur multipliers of products, semidirect products, and central extensions, respectively. All of these statements are well-known in the case of finite groups (see for instance [**HKY19**, §2]).

2.1.20. Notation. Given linear algebraic groups G, H, we let $X^*(G,H)$ denote the group of *bimultiplicative* morphisms $G \times H \to \mathbb{G}_m$ (which in particular, necessarily factor through the abelianizations $G^{\mathrm{ab}} \times H^{\mathrm{ab}}$). Equivalently, we have

$$\mathrm{Hom}_{\mathrm{gp}}(H, X^*(G)) \cong X^*(G,H) \cong \mathrm{Hom}_{\mathrm{gp}}(G, X^*(H)),$$

i.e., the group of homomorphisms from one group to the Pontryagin dual of the other.

2.1.21. Lemma. Let G, H be linear algebraic groups. There is a functorial isomorphism

$$\mathrm{M}(G\times H)\simeq \mathrm{M}(G)\times \mathrm{M}(H)\times X^*(G,H). \tag{2.1.21.1}$$

In particular, if $|\pi_0(G^{\mathrm{ab}})|$ and $|\pi_0(H^{\mathrm{ab}})|$ are coprime (e.g., if either G or H is connected), then

$$\mathrm{M}(G\times H)\simeq \mathrm{M}(G)\times \mathrm{M}(H).$$

Proof. Consider the restriction map

$$\mathrm{M}(G\times H)\to \mathrm{M}(G)\times \mathrm{M}(H).$$

Taking the external tensor product of cocycles immediately impies that it is split surjective. Moreover, a central extension in the kernel is clearly given by a semidirect product

$$(\mathbb{G}_m\times G)\rtimes H \tag{2.1.21.2}$$

such that the action

$$H\to \mathrm{Aut}(\mathbb{G}_m\times G)$$

is of the form

$$\begin{pmatrix} \mathrm{const}_{\mathrm{id}_{\mathbb{G}_m}} & c \\ 0 & \mathrm{const}_{\mathrm{id}_G}\end{pmatrix}, \tag{2.1.21.3}$$

where $c\colon H\to X^*(G)$ is a homomorphism. Since the group of homomorphisms of the form (2.1.21.3) is abelian, quotienting by splittings of the restriction of (2.1.21.2) to G (as in (2.1.18.4)) does nothing. Thus, we obtain (2.1.21.1).

To show that the latter condition implies that $X^*(G,H)$ is trivial, recall that a commutative linear algebraic group is the product of its semisimple and unipotent parts. Thus, we may assume that G^{ab} is the product of a torus and a finite abelian group. Since any homomorphism from H to a free group is trivial, we may assume that G^{ab} is finite. Thus, it suffices to classify homomorphisms $\pi_0(H^{\mathrm{ab}})\to X^*(\pi_0(G^{\mathrm{ab}}))$, which are all trivial if $|\pi_0(G^{\mathrm{ab}})|$ and $|\pi_0(H^{\mathrm{ab}})|$ are coprime. $\square$

2.1.22. Lemma. Let $G\rtimes\Gamma$ be a semidirect product, with either Γ finite or $X^*(G)$ trivial. Then we have an exact sequence

$$0\to \mathrm{H}^1(\Gamma, X^*(G))\to \ker\big(\mathrm{M}(G\rtimes\Gamma)\twoheadrightarrow \mathrm{M}(\Gamma)\big)\to \mathrm{M}(G)^{\Gamma}\to \mathrm{H}^2(\Gamma, X^*(G)),$$

which is suitably functorial in both G and Γ.[10]

Proof. We begin by showing that the natural restriction map

$$\ker\big(\mathrm{M}(G\rtimes\Gamma)\twoheadrightarrow \mathrm{M}(\Gamma)\big)\to \mathrm{M}(G) \tag{2.1.22.1}$$

lands in $\mathrm{M}(G)^{\Gamma}$. Suppose we have a central extension

$$1\to\mathbb{G}_m\to G'\to G\rtimes\Gamma\to 1 \tag{2.1.22.2}$$

whose pullback to Γ is split. Then the projection $G'\to G\rtimes\Gamma\to\Gamma$ admits a section, and we obtain an adjoint action of Γ on G' lifting that on $G\rtimes\Gamma$. In particular, this action fixes $G'\times_{G\rtimes\Gamma} G$, and so the restriction of (2.1.22.2) to $\mathrm{M}(G)$ is fixed by Γ.

Moreover, as in (2.1.21.2), the kernel of (2.1.22.1) is given by semidirect products

$$(\mathbb{G}_m\times G)\rtimes\Gamma$$

with outer action as in (2.1.18.3); as in (2.1.18.5), these are classified by the group cohomology $\mathrm{H}^1(\Gamma, X^*(G))$.

To construct the final map, let

$$1\to\mathbb{G}_m\to G'\to G\to 1 \tag{2.1.22.3}$$

be a central extension fixed under the action of Γ, and consider the group A of automorphisms of this central extensions, i.e., commutative diagrams

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \downarrow\wr & & \downarrow\wr & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

[10] In the latter case, we interpret the group cohomology on either end as the trivial group.

There is an evident exact sequence

$$1 \to X^*(G) \to A \to \mathrm{Aut}(G), \tag{2.1.22.4}$$

and since Γ fixes (2.1.22.3), pulling back along $\Gamma \to \mathrm{Aut}(G)$ gives a short exact sequence

$$1 \to X^*(G) \to A \times_{\mathrm{Aut}(G)} \Gamma \to \Gamma \to 1 \tag{2.1.22.5}$$

with the obvious outer action of Γ on $X^*(G)$. Thus, (2.1.22.5) represents a class in $\mathrm{H}^2(\Gamma, X^*(G))$, and it is straightforward to verify that it depends only on the isomorphism class of (2.1.22.3).

Suppose now that (2.1.22.5) is a semi-direct product, i.e., we have a splitting homomorphism $\Gamma \to A$. Then (2.1.22.3) lifts to a central extension

$$1 \to \mathbb{G}_m \to G' \rtimes \Gamma \to G \rtimes \Gamma \to 1,$$

whose restriction to $\mathrm{M}(\Gamma)$ is evidently trivial.

We leave the verification of functoriality for maps $G' \to G$ commuting with the Γ-actions, and for maps $\Gamma' \to \Gamma$, as an exercise. □

2.1.23. Lemma. Let $Z \subset G$ be central. Then we have a functorial exact sequence

$$0 \to X^*([G,G] \cap Z) \to \mathrm{M}(G/Z) \to \mathrm{M}(G) \to X^*(G,Z).$$

Proof. Suppose we are given a central extension

$$1 \to \mathbb{G}_m \to G' \to G/Z \to 1 \tag{2.1.23.1}$$

whose pullback to G is trivial. Then we have a short exact sequence

$$1 \to Z \to \mathbb{G}_m \times G \to G' \to 1,$$

and we see that (2.1.23.1) is determined by an element of $X^*(Z)$. Moreover, two such elements determine isomorphic central extensions if and only if they differ by an element of the image of $X^*(G) \to X^*(Z)$. The exact sequence

$$1 \to [G,G] \cap Z \to Z \to G^{\mathrm{ab}}$$

of abelian groups then shows that this quotient is isomorphic to $X^*([G,G] \cap Z)$.

To construct the final map, let

$$1 \to \mathbb{G}_m \to G' \to G \to 1 \tag{2.1.23.2}$$

be a central extension, let

$$1 \to \mathbb{G}_m \to Z' \to Z \to 1 \tag{2.1.23.3}$$

be its pullback to Z, and let A be as in (2.1.22.4). The action of Z' on G' via inner automorphisms gives a homomorphism $Z' \to A$ that is trivial on $\mathbb{G}_m$; moreover, the induced homomorphism $Z \to \mathrm{Aut}(G)$ is trivial by assumption, so we obtain a homomorphism $Z \to X^*(G)$, i.e., an element of $X^*(G,Z)$. It is easy to see that this element depends only on the isomorphism class of (2.1.23.2).

Finally, suppose that this element of $X^*(G,Z)$ is trivial. Then $Z' \subset G'$ is central, and in particular, commutative. Since any injective homomorphism from a torus to a commutative linear algebraic group admits a retract, the extension (2.1.23.3) is split, and so Z lifts to a central subgroup of G'. In particular, (2.1.23.2) is pulled back from the central extension

$$1 \to \mathbb{G}_m \to G'/Z \to G/Z \to 1,$$

as desired.

We leave the verification of functoriality as an exercise. □

2.1.24. We now turn to the construction of Schur coverings of linear algebraic groups; these are (non-canonical) central extensions of G by $\mathrm{M}(G)$ which trivialize all cocycles of G.

2.1.25. Proposition. The group G admits a *Schur covering*, i.e., there exists a central extension[11]

$$1 \to X^*(\mathrm{M}(G)) \to G^{\mathrm{sch}} \xrightarrow{p_{\mathrm{sch}}} G \to 1 \tag{2.1.25.1}$$

[11]Note that, unlike some authors, we do not require $X^*(\mathrm{M}(G))$ to be contained in the derived subgroup of G^{sch}.

such that the map

$$\mathrm{M}(G) \to \mathrm{M}(G^{\mathrm{sch}}) \tag{2.1.25.2}$$

is trivial. Moreover, we have a canonical $\mathrm{Rep}(G)$-linear decomposition

$$\mathrm{Rep}(G^{\mathrm{sch}}) \simeq \bigoplus_{[(\mathcal{C},\alpha)]\in \mathrm{M}(G)} \mathrm{Rep}(G)^{(\mathcal{C},\alpha)}, \tag{2.1.25.3}$$

where the summand $\mathrm{Rep}(G)^{(\mathcal{C},\alpha)}$ is equivalent to the full subcategory of representations on which $X^*(\mathrm{M}(G))$ acts through the character given by evaluation at $[(\mathcal{C},\alpha)]$.

Proof. Since $\mathrm{M}(G)$ is finite abelian by Corollary 2.1.18, we may choose a finite set of generators $[(\mathcal{C}_1,\alpha_1)],\dots,[(\mathcal{C}_r,\alpha_r)]$, each of finite order. By Proposition 2.1.5, we have central extensions

$$1 \to X^*(\langle[(\mathcal{C}_i,\alpha_i)]\rangle) \to \overline{G}_{(\mathcal{C}_i,\alpha_i)} \to G \to 1.$$

for each i. Thus, taking fiber products over G gives a central extension

$$1 \to X^*(\langle[(\mathcal{C}_1,\alpha_1)]\rangle) \times \cdots \times X^*(\langle[(\mathcal{C}_r,\alpha_r)]\rangle) \to \overline{G}_{(\mathcal{C}_1,\alpha_1)} \times_G \cdots \times_G \overline{G}_{(\mathcal{C}_r,\alpha_r)} \to G \to 1,$$

which is precisely (2.1.25.1). Applying Proposition 2.1.5 to the factorizations

$$G^{\mathrm{sch}} \twoheadrightarrow \overline{G}_{(\mathcal{C}_i,\alpha_i)} \xrightarrow{p_{(\mathcal{C}_i,\alpha_i)}} G$$

of p_{sch} then gives (2.1.25.2). Finally, applying (2.1.8.1) and (2.1.8.2) to the decomposition

$$\mathrm{Rep}(G^{\mathrm{sch}}) \simeq \mathrm{Rep}(\overline{G}_{(\mathcal{C}_1,\alpha_1)}) \otimes_{\mathrm{Rep}(G)} \cdots \otimes_{\mathrm{Rep}(G)} \mathrm{Rep}(\overline{G}_{(\mathcal{C}_r,\alpha_r)})$$

yields (2.1.25.3). □

2.1.26. The following lemma allows us to construct "small" $(\mathcal{C},\alpha)$-representations of connected semisimple groups, and will be used in §2.2 to remove the simply-connectedness assumption for the noncommutative Springer resolution.

2.1.27. Lemma. If G is connected and semisimple, then any Schur covering G^{sch} is isomorphic to the universal cover of G. Moreover, for each $(\mathcal{C},\alpha) \in \mathrm{Coc}(G)$, there exists a minuscule weight λ of G^{sch} such that the irreducible G^{sch}-representation $L(\lambda)$ descends to a $(\mathcal{C},\alpha)$-representation of G.

Proof. By Proposition 2.1.16, we have $X^*(\mathrm{M}(G)) \simeq \pi_1(G)$, from which the first assertion is immediate. For the latter, let $T \subset G$ be a maximal torus, and recall that

$$\pi_1(G) \cong X^*(T)^\vee/\langle\Phi^\vee\rangle,$$

where $\langle\Phi^\vee\rangle$ denotes the coroot lattice. Thus,

$$\mathrm{M}(G) \cong X^*(\pi_1(G)) \cong \Lambda/X^*(T), \tag{2.1.27.1}$$

where Λ denotes the abstract weight lattice (of the root system of G), and for any dominant $\lambda \in \Lambda^+$, the character of $\pi_1(G)$ on the G^{sch}-representation $L(\lambda)$ is given by the image of λ in (2.1.27.1). Recall that $\Lambda/\langle\Phi\rangle$ is canonically isomorphic to the set of minuscule weights, where $\langle\Phi\rangle$ denotes the root lattice (namely, take the unique minimal dominant weight lifting any element of the former set). Since $\Lambda/\langle\Phi\rangle \twoheadrightarrow \Lambda/X^*(T)$, it suffices by (2.1.25.3) to choose any lift of $[(\mathcal{C},\alpha)]$ to $\Lambda/\langle\Phi\rangle$. □

2.1.28. Finally, the following two lemmas will be used in §2.3 to show that a certain covering group has simply-connected derived subgroup.

2.1.29. Lemma. If G is reductive (resp., connected), then so is any Schur covering G^{sch}. When both hold, G^{sch} has simply-connected derived subgroup.

Proof. The first assertion holds as any extension of reductive groups is reductive. For the second assertion, we first reduce to the case where G is reductive. By [**Bor91**, Cor. 14.11], the surjection $p_{\mathrm{sch}}\colon G^{\mathrm{sch},\circ} \twoheadrightarrow G$ induces a surjection $p_{\mathrm{sch}}\colon \mathrm{R}_u G^{\mathrm{sch},\circ} \twoheadrightarrow \mathrm{R}_u G$ of unipotent radicals. Since the kernel of this map is finite, it is an isomorphism. Note that both are normal subgroups; we claim that

$$G^{\mathrm{sch}}/\mathrm{R}_u G^{\mathrm{sch},\circ} \to G/\mathrm{R}_u G$$

is a Schur covering. Denote these groups by $G^{\mathrm{sch,red}}$ and G^{red}, respectively, and consider the commutative diagram

$$\begin{tikzcd} \mathrm{M}(G^{\mathrm{red}}) \arrow[r] \arrow[d] & \mathrm{M}(G^{\mathrm{sch,red}}) \arrow[d] \\ \mathrm{M}(G) \arrow[r, "0"] & \mathrm{M}(G^{\mathrm{sch}}). \end{tikzcd} \tag{2.1.29.1}$$

It suffices to show that the upper horizontal map is trivial. By [**Con14**, Prop. 5.4.1], we have

$$[G,G] \cong \mathrm{R}_u([G,G]) \rtimes [G^{\mathrm{red}}, G^{\mathrm{red}}],$$

hence $\pi_1([G,G]) \cong \pi_1([G^{\mathrm{red}}, G^{\mathrm{red}}])$ (as the two spaces are related via an affine fibration). Proposition 2.1.16 then implies that the left-most vertical map in (2.1.29.1) is an isomorphism. Moreover, Proposition 2.1.22 and a further application of [**Con14**, Prop. 5.4.1] imply that the right-most vertical map is also an isomorphism (note that $\mathrm{M}(\mathrm{R}_u G)$ is trivial by Proposition 2.1.16). This proves the claim.

So suppose that G is reductive. By the previous assertion, $G^{\mathrm{sch},\circ}$ is reductive; thus, $[G^{\mathrm{sch},\circ}, G^{\mathrm{sch},\circ}]$ is connected and semisimple, and its projection to $[G,G]$ is an isogeny. Lemma 2.1.27 therefore gives a unique factorization

$$\begin{tikzcd} {[G,G]^{\mathrm{sch}}} \arrow[rr, dashed, two heads] \arrow[dr, two heads] & & {[G^{\mathrm{sch},\circ}, G^{\mathrm{sch},\circ}]} \arrow[dl, two heads] \\ & {[G,G].} & \end{tikzcd} \tag{2.1.29.2}$$

Moreover, by Proposition 2.1.16, we have a commutative diagram

$$\begin{tikzcd} \mathrm{M}([G,G]) \arrow[r] \arrow[d, "\wr"] & \mathrm{M}([G^{\mathrm{sch},\circ}, G^{\mathrm{sch},\circ}]) \arrow[d, "\wr"] \\ \mathrm{M}(G) \arrow[r, "0"] & \mathrm{M}(G^{\mathrm{sch}}). \end{tikzcd}$$

In particular, the map $\pi_1([G^{\mathrm{sch},\circ}, G^{\mathrm{sch},\circ}]) \to \pi_1([G,G])$ is trivial, so by (2.1.29.2), we have $[G,G]^{\mathrm{sch}} \cong [G^{\mathrm{sch},\circ}, G^{\mathrm{sch},\circ}]$. In particular, the kernel of the isogeny (2.1.29.2) has cardinality at least $|\pi_1([G,G])| = |X^*(\mathrm{M}(G))|$, so the same is true of the kernel of $p_{\mathrm{sch}}|_{G^{\mathrm{sch},\circ}}$, and therefore $G^{\mathrm{sch}} \cong G^{\mathrm{sch},\circ}$, as desired. The final assertion now follows from (2.1.29.2). ☐

2.1.30. Lemma. Let G, H be linear algebraic groups. Then for any Schur covering $(G \times H)^{\mathrm{sch}}$, there exist Schur coverings $G^{\mathrm{sch}}, H^{\mathrm{sch}}$ and a commutative diagram

$$\begin{tikzcd} (G \times H)^{\mathrm{sch}} \arrow[rr, dashed] \arrow[dr, "p_{\mathrm{sch}}"'] & & G^{\mathrm{sch}} \times H^{\mathrm{sch}} \arrow[dl, "p_{\mathrm{sch}} \times p_{\mathrm{sch}}"] \\ & G \times H. & \end{tikzcd}$$

Proof. By Lemma 2.1.21, we have a central extension

$$1 \to X^*(\mathrm{M}(G)) \times X^*(\mathrm{M}(H)) \to (G \times H)^{\mathrm{sch}} / X^*(X^*(G,H)) \to G \times H \to 1.$$

The proof of Proposition 2.1.5 now easily implies that this splits as a product of central extensions (2.1.25.1) for G and H. ☐

2.2. The noncommutative Springer resolution

2.2.1. This section consists of recollections on Bezrukavnikov–Mirković's noncommutative Springer resolution and Lusztig's canonical basis of the K-theory of a Springer fiber. We begin by describing several key properties of this resolution that will be used throughout the sequel (in particular, the Koszul grading and braid positivity properties). We then explain how to extend these properties to any reductive group, and construct the cocycles appearing in the canonical basis.

2.2.2. We begin by briefly reviewing the constructions of the Springer resolution and of Slodowy slices. For further details, see [**CG10**]. Fix a connected reductive group G, and let $\mathfrak{g}$ denote its Lie algebra. We henceforth identify $\mathfrak{g} \cong \mathfrak{g}^*$ via a non-degenerate form $\langle -, - \rangle$. Let $\mathcal{B}$ denote the flag variety of G. Then (the total space of) the cotangent bundle $\widetilde{\mathcal{N}} := T^*\mathcal{B}$ carries a canonical symplectic structure and Hamiltonian G-action. The moment map $\pi\colon \widetilde{\mathcal{N}} \to \mathfrak{g}$ is then a resolution of singularities of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$. Moreover, it is $\widetilde{G} := G \times \mathbb{G}_m$ equivariant, where G acts on $\mathfrak{g}$ adjointly, and $\mathbb{G}_m$ scales $\mathfrak{g}$ and the fibers of $\widetilde{\mathcal{N}}$ by weight -2. We refer to π as the *Springer resolution* (of $\mathcal{N}$).

Recall moreover that $\mathcal{N}$ is a union of finitely many G-orbits, each of which is a conical symplectic subvariety, hence even-dimensional. These G-orbits are equipped with a standard partial order via closures; the unique minimal orbit $\mathbb{O}_0$ consists only of the 0 nilpotent, and the unique maximal orbit $\mathbb{O}_{\mathrm{reg}}$ consists of all regular nilpotent elements. Fix a nilpotent element $e \in \mathcal{N}$. We denote the fiber of π over e by $\mathcal{B}_e := \widetilde{\mathcal{N}} \times_{\mathfrak{g}} \{e\}$, and refer to it as the *(derived) Springer fiber* above e. In particular, it carries a $\widetilde{G}^e$-action, and its classical truncation is the classical Springer fiber $\mathcal{B}_e^{\mathrm{cl}} \subset \mathcal{B}$, consisting of all Borel subgroups of G whose Lie algebra contains e.

Now, recall that by the Jacobson–Morozov theorem, we may extend e (non-uniquely) to an $\mathfrak{sl}_2$-triple $\{e, h, f\}$ in $\mathfrak{g}$. Fixing such a choice, the adjoint action of h yields a grading $\mathfrak{g} \cong \bigoplus_{w \in \mathbb{Z}} \mathfrak{g}_w$, i.e., a decomposition of $\mathfrak{g}$ into weight-spaces. This grading is additive with respect to the Lie bracket $[-, -]$, and the actions of e and f raise and lower the grading by 2, respectively. In particular, the centralizers $\mathfrak{g}^e$ and $\mathfrak{g}^f$ lie in non-negative and non-positive weights, respectively. Set $S_e := e + \mathfrak{g}^f \subset \mathfrak{g}$ (here and onward, we misuse notation slightly by only indicating the dependence on e). This is an affine subspace which intersects the orbit $G \cdot e$ transversally at e (in $\mathfrak{g}$); we refer to it as a *Slodowy slice* at e.

We now wish to modify the $\mathbb{G}_m$-action on S_e coming from ad_h to be *repelling*. To this end, let $\varphi_e\colon \mathrm{SL}_2 \to G$ be the group homomorphism associated to our choice of $\mathfrak{sl}_2$-triple, and define a cocharacter $\check{\lambda}_e \in X_*(\widetilde{G})$ via the formula

$$\check{\lambda}_e(t) := (\varphi_e(\left[\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right]), t) \in G \times \mathbb{G}_m$$

for any $t \in \mathbb{G}_m$. We refer to this as the *Jacobson–Morozov cocharacter* of $\widetilde{G}$ for e (though it of course depends on the choice of $\mathfrak{sl}_2$-triple). In particular, the adjoint action of $\check{\lambda}_e$ on S_e fixes e, and *repels* S_e from this point (i.e., the vector space underlying S_e lies in *strictly* negative weights; equivalently, the coordinate ring $\mathcal{O}(S_e)$ lies in strictly *positive* weights).

We now extend this $\mathbb{G}_m$-action to an action of a larger reductive subgroup of $\widetilde{G}$. Set $Z_e := G^{e,h,f} \cong G^{e,\mathrm{red}}$, i.e., the common centralizer of the $\mathfrak{sl}_2$-triple $\{e, h, f\}$, or equivalently, the reductive part of the centralizer of e. Then Z_e commutes with the cocharacter $\check{\lambda}_e$, and we have an action of $\widetilde{Z}_e$ on S_e, with $\mathbb{G}_m$ acting via $\check{\lambda}_e$. Equivalently, we have $\widetilde{Z}_e \cong \widetilde{G}^{\mathfrak{sl}_2} \cong \widetilde{G}^{e,\mathrm{red}}$ (i.e., the centralizer of the $\mathfrak{sl}_2$-subalgebra generated by $\{e, h, f\}$), and $\widetilde{G}^e \cong G^e \rtimes \mathbb{G}_m$, with $\mathbb{G}_m$ acting on the unipotent radical of G^e via strictly positive weights. We set $\mathfrak{z}_e$ to be the Lie algebra of Z_e, and continue to use the notation $\tilde{\mathfrak{z}}_e \cong \mathfrak{z}_e \oplus k$ for the Lie algebra of $\widetilde{Z}_e$.

Finally, transversality yields (derived) pullback squares

$$\begin{tikzcd} \mathcal{B}_e/\widetilde{Z}_e \arrow[r, hook, "i_{\mathcal{B}_e}"] \arrow[d, "\pi_e"] & \widetilde{S}_e/\widetilde{Z}_e \arrow[r, "i_{\widetilde{S}_e}"] \arrow[d, "\pi_{S_e}"] & \widetilde{\mathcal{N}}/\widetilde{G} \arrow[d, "\pi"] \\ e/\widetilde{Z}_e \arrow[r, hook, "i_e"] & S_e/\widetilde{Z}_e \arrow[r, "i_{S_e}"] & \mathfrak{g}/\widetilde{G}, \end{tikzcd} \tag{2.2.2.1}$$

where the variety $\widetilde{S}_e$ is the (classical, smooth) *resolution to the Slodowy slice* (note that the map i_{S_0} is the identity). The map π_{S_e} is again a symplectic resolution, hence semismall. Moreover, the subvariety $\mathcal{B}_e \subset \widetilde{S}_e$ is Lagrangian; in particular, we have $2 \dim \mathcal{B}_e = \dim \widetilde{S}_e$.

2.2.3. Assume for the moment that G is semisimple and simply-connected (we shall remove these hypotheses shortly). Then there exists a remarkable $\widetilde{G}$-equivariant vector bundle $\mathcal{E}$ on $\widetilde{\mathcal{N}}$ known as the *Bezrukavnikov–Mirković tilting bundle* [**BM13, BL23**]. We recall some of its salient properties in the following paragraphs.

2.2.4. The pullback of $\mathcal{E}$ to the resolution of any Slodowy slice S_e is a tilting generator. Thus,

$$\mathcal{A}_{S_e} := \mathrm{End}_{\widetilde{S}_e}(i_{\widetilde{S}_e}^* \mathcal{E})$$

is an $\mathcal{O}(S_e)$-algebra in cohomological degree 0 with a compatible $\widetilde{Z}_e$-action; we refer to this as the *noncommutative resolution to the Slodowy slice*, or in the case $e = 0$, as the *noncommutative Springer resolution.*[12] We therefore have a $\mathrm{Rep}(\widetilde{Z}_e)$-linear equivalence[13]

$$\mathrm{Hom}_{\widetilde{S}_e}(i^*_{\widetilde{S}_e}\mathcal{E}, -)\colon\ \mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e) \xrightarrow{\sim} \mathcal{A}^{\mathrm{op}}_{S_e}\text{-mod}^{\widetilde{Z}_e}, \tag{2.2.4.1}$$

and we refer to the t-structure on $\mathrm{Coh}(\widetilde{S}_e/\widetilde{Z}_e)$ corresponding to the usual t-structure on the right-hand side as the *exotic t-structure*.

2.2.5. Write

$$\{\mathcal{E}^{S_e}_b : b \in \mathbf{B}_e\} \tag{2.2.5.1}$$

for the set of indecomposable summands of $i^*_{\widetilde{S}_e}\mathcal{E} \in \mathrm{Coh}(\widetilde{S}_e)$. Then there exists a graded lift of each vector bundle $\mathcal{E}^{S_e}_b$ to $\mathrm{Coh}(\widetilde{S}_e/\mathbb{G}_m)$ (unique up to simultaneous twists of (2.2.5.1)) such that the induced grading on

$$\mathcal{A}^{\mathrm{cov}}_{S_e} := \mathrm{End}_{\widetilde{S}_e}\Big(\bigoplus_{b\in\mathbf{B}_e} \mathcal{E}^{S_e}_b\Big) \tag{2.2.5.2}$$

is Koszul (and compatible with the grading on $\mathcal{O}(S_e)$).[14] That is, for each $b, b' \in \mathbf{B}_e$, the vector space $\mathrm{Hom}_{\widetilde{S}_e}(\mathcal{E}^{S_e}_b, \mathcal{E}^{S_e}_{b'})$ is concentrated in non-negative $\mathbb{G}_m$-weights; moreover, the weight-0 component is spanned by the identity map if $b = b'$, and is 0 otherwise.

2.2.6. Let $\mathcal{L}^{S_e}_b \in \mathrm{Coh}(\widetilde{S}_e/\mathbb{G}_m)$ denote the simple object in the heart of the exotic t-structure with projective cover $\mathcal{E}^{S_e}_b$, so that

$$\mathrm{Hom}_{\widetilde{S}_e}(\mathcal{E}^{S_e}_{b'}, \mathcal{L}^{S_e}_b) \simeq \begin{cases} k\langle 0\rangle & \text{if } b = b', \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.6.1}$$

In particular, the $\mathcal{O}(S_e)$ action on (2.2.6.1) factors through the maximal ideal of e, so $\mathcal{L}^{S_e}_b$ is supported on the Springer fiber $\mathcal{B}_e$. The set $\{[\mathcal{L}^{S_e}_b] : b \in \mathbf{B}_e\}$ then forms a $\mathbb{Z}$-basis for the equivariant K-theory group $K_0(\mathcal{B}_e/\mathbb{G}_m)$; we refer to it as the *canonical basis*. We will often misuse terminology slightly and refer to the set $\mathbf{B}_e$ itself as "the canonical basis."

Moreover, as in [**BM13**, Prop. 5.5], Grothendieck–Serre duality implies that $\mathrm{Ext}^n_{\widetilde{S}_e}(\mathcal{L}^{S_e}_{b'}, \mathcal{L}^{S_e}_b)$ is concentrated in weight-n (and vanishes for $n \geq \dim \widetilde{S}_e$); thus, $\mathcal{A}^{\mathrm{cov}}_{S_e}$ is in fact a Koszul quadratic algebra, and is generated over its weight-0 subalgebra $\mathcal{A}_{S_e,0}$ in weight-1.

Finally, we let $E^{S_e}_b$ and $L^{S_e}_b$ denote the corresponding objects on the right-hand side of (2.2.4.1) (after forgetting the Z_e-equivariance). These are the indecomposable projective and simple right $\mathcal{A}_{S_e}$-modules, respectively. Moreover, we set

$$E^{S_e,\ell}_b := \mathrm{Hom}_{\widetilde{S}_e}(\mathcal{E}^{S_e}_b, i^*_{\widetilde{S}_e}\mathcal{E}) \simeq \mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}_{S_e}}(E^{S_e}_b, \mathcal{A}_{S_e}) \tag{2.2.6.2}$$

to be the corresponding indecomposable projective *left* $\mathcal{A}_{S_e}$-modules.

2.2.7. Let $\check{G}$ denote the *Langlands dual* reductive group of G. Recall that the *extended affine Weyl group* of $\check{G}$ is given by the semidirect product

$$W^{\mathrm{aff}} := W \ltimes X_*(\check{T}) = W \ltimes X^*(T),$$

where $T \subset G$ is a maximal torus, $\check{T} \subset \check{G}$ is the dual torus, and W denotes the finite Weyl group. Then the category $\mathrm{QC}(\widetilde{\mathcal{N}}/\widetilde{G})$ carries a compact object-preserving weak action of the *affine braid group* B^{aff} associated to[15] W^{aff}, owed to [**BR12**]. More precisely, recall that the (*mixed*) *affine Hecke category*[16] $\mathcal{H}^{\mathrm{coh}}$ is the

[12] Note that this is the *opposite* algebra to that considered in [**BM13**], and instead agrees with the algebra of [**BL23**]. We will continue to work with right modules in place of left modules, and vice versa.

[13] See for instance [**BZCHN22**, Lem. 2.10] for the dg-categorical perspective.

[14] This is the algebra "$\mathcal{A}_e$" of [**BM13**, §5.5]; our notation differs in order to emphasize its relation to the Slodowy slice, and its Morita equivalence to $\mathcal{A}_{S_e}$, which can be made equivariant once all cocycles are trivialized as in Corollary 2.3.7.

[15] More specifically, the *non-extended* affine Weyl group $W \ltimes \langle\Phi\rangle \subset W^{\mathrm{aff}}$ admits a Coxeter presentation associated to the affine Dynkin diagram of $\check{G}$. The group B^{aff} is then the analogous extension of the Artin–Tits braid group associated to this Coxeter presentation.

[16] This non-standard notation is intended to distinguish $\mathcal{H}^{\mathrm{coh}}$ from its "module" incarnation, which will be introduced in §4.2.

category of $\widetilde{G}$-equivariant ind-coherent sheaves on the (derived) Steinberg variety, i.e.,

$$\mathcal{H}^{\mathrm{coh}} := \mathrm{QC}^!(\widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}}/\widetilde{G}). \tag{2.2.7.1}$$

This is a compactly generated monoidal category acting on $\mathrm{QC}(\widetilde{\mathcal{N}}/\widetilde{G})$, where the algebra and module structures are both given by (left) convolution (which we denote by "$*$"). The weak action of B^{aff} on $\mathrm{QC}(\widetilde{\mathcal{N}}/\widetilde{G})$ is then given by a "homomorphism" $B^{\mathrm{aff}} \to \mathcal{H}^{\mathrm{coh}}$, i.e., by *coherent* sheaves $\mathcal{K}_a \in \mathcal{H}^{\mathrm{coh}}$ admitting isomorphisms $\mathcal{K}_a * \mathcal{K}_{a'} \simeq \mathcal{K}_{aa'}$ for each $a, a' \in B^{\mathrm{aff}}$.

Let us give a unique characterization of these sheaves. Given $w \in W^{\mathrm{aff}}$, we may consider a minimal decomposition of w as a product of simple reflections, and take the product of the corresponding generators of B^{aff}. This product is independent of the choice of decomposition of w, and hence yields a set-theoretic section of the canonical surjection $B^{\mathrm{aff}} \to W^{\mathrm{aff}}$, which we denote by $w \mapsto \widetilde{w}$. We further denote the submonoid of B^{aff} generated by the image of this section by B^{aff}_+. It suffices to construct the sheaves $\mathcal{K}_{\widetilde{w}}$ for generators $w \in W^{\mathrm{aff}}$.

First, given $\lambda \in X^*(T)$, there is an associated $\widetilde{G}$-equivariant line bundle $\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)$ on $\widetilde{\mathcal{N}}$ obtained by pullback along the composition

$$\widetilde{\mathcal{N}}/\widetilde{G} \to \mathcal{B}/\widetilde{G} \cong \mathrm{B}\widetilde{B} \to \mathrm{B}T.$$

If λ is *dominant*, then $\mathcal{K}_{\widetilde{\lambda}} \simeq \Delta_{\widetilde{\mathcal{N}}/\mathfrak{g},*}\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda) \in \mathcal{H}^{\mathrm{coh}}$, where $\Delta_{\widetilde{\mathcal{N}}/\mathfrak{g}} \colon \widetilde{\mathcal{N}}/\widetilde{G} \hookrightarrow \widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}}/\widetilde{G}$ denotes the diagonal map. More generally, B^{aff} has a "translation subgroup" isomorphic to $X^*(T)$, which acts by the sheaves $\Delta_{\widetilde{\mathcal{N}}/\mathfrak{g},*}\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)$.

Next, for a simple reflection $s_\alpha \in W$, recall that the *Grothendieck simultaneous resolution* $\pi^{\mathfrak{g}} \colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is the variety of pairs $(x, \mathfrak{b})$, where $x \in \mathfrak{g}$ and $\mathfrak{b} \in \mathcal{B}$ is a Borel subalgebra containing x; the Springer resolution is the (reduced) closed subvariety of $\widetilde{\mathfrak{g}}$ given by requiring x to be nilpotent. Let $\mathfrak{h}$ denote the universal Cartan algebra of $\mathfrak{g}$ (which carries a natural action of W), and consider the map $\widetilde{\mathfrak{g}} \to \mathfrak{h}$ given by $(x, \mathfrak{b}) \mapsto x \bmod [\mathfrak{b}, \mathfrak{b}]$. The latter induces a resolution of singularities $\widetilde{\mathfrak{g}} \to \mathfrak{g} \times_{\mathfrak{h}//W} \mathfrak{h}$, which is an isomorphism over the open subscheme $\mathfrak{g}^{\mathrm{rs}} \subset \mathfrak{g}$ comprising all regular semi-simple elements. In particular, the map $\pi^{\mathfrak{g}}$ is a principal W-torsor over $\mathfrak{g}^{\mathrm{rs}}$, so we may consider the closure of the graph of s_α on $(\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}) \times_{\mathfrak{g}} \mathfrak{g}^{\mathrm{rs}}$, which is a closed $\widetilde{G}$-stable (classical, smooth) subscheme $\Gamma^{\mathfrak{g}}_{s_\alpha} \subset \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$ (note that the latter fiber product is in fact classical). The classical scheme-theoretic intersection

$$\Gamma_{s_\alpha} := \big(\Gamma^{\mathfrak{g}}_{s_\alpha} \underset{\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}}{\times} \widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}}\big)^{\mathrm{cl}} \subset \widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$$

is then a $\widetilde{G}$-stable closed subscheme of the Steinberg variety, and $\mathcal{K}_{\tilde{s}_\alpha} \simeq \mathcal{O}_{\Gamma_{s_\alpha}} \in \mathcal{H}^{\mathrm{coh}}$ is its structure sheaf.

Finally, we may state the *braid positivity* property of the noncommutative Springer resolution. Given any Slodowy slice S_e, we have a monoidal category

$$\mathcal{H}^{\mathrm{coh}}_{S_e} := \mathrm{QC}^!(\widetilde{S}_e \times_{S_e} \widetilde{S}_e/\widetilde{Z}_e) \tag{2.2.7.2}$$

and a monoidal functor $i^*_{S_e} \colon \mathcal{H}^{\mathrm{coh}} \to \mathcal{H}^{\mathrm{coh}}_{S_e}$, hence an affine braid group action on $\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e)$. In particular, for $e = 0$, we recover the usual action on $\mathrm{QC}(\widetilde{\mathcal{N}}/\widetilde{G})$. The action of any $a \in B^{\mathrm{aff}}_+$ on $\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e)$ is then right t-exact with respect to the *exotic* t-structure constructed in (2.2.4.1). We will in fact be interested in the action of $\mathcal{H}^{\mathrm{coh}}_{S_e}$ on $\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e)$ via *right* convolution. Note that pullback along the "swap" map $\sigma \colon \widetilde{S}_e \times_{S_e} \widetilde{S}_e \to \widetilde{S}_e \times_{S_e} \widetilde{S}_e$ interchanging the two copies of $\widetilde{S}_e$ intertwines right and left convolution. Moreover, σ^* preserves the sheaves $\Delta_{\widetilde{\mathcal{N}}/\mathfrak{g},*}\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)$ and $\mathcal{O}_{\Gamma_{s_\alpha}}$, hence induces an anti-involution $\sigma^* \colon B^{\mathrm{aff}} \to B^{\mathrm{aff,op}}$ (which sends any $\widetilde{w}$ to $\widetilde{w^{-1}}$). Thus, the braid positivity property also holds for right convolution.

2.2.8. We now aim to remove the assumptions on G; so suppose once again that G is connected and reductive. We shall need the following lemma:

2.2.9. Lemma. Set $\overline{G} := G/Z(G)^\circ$. Then the functor

$$\mathrm{Res}^G_{[G,G]} \colon \mathrm{Rep}(G) \to \mathrm{Rep}([G,G]) \tag{2.2.9.1}$$

admits a (non-canonical) $\mathrm{Rep}(\overline{G})$-linear section. In particular, applying $- \otimes_{\mathrm{Rep}(\overline{G})} \mathrm{QC}(\widetilde{\mathcal{N}}/\overline{G})$, we obtain a $\mathrm{Rep}(G)$-linear section of the restriction functor

$$\mathrm{QC}(\widetilde{\mathcal{N}}/G) \to \mathrm{QC}(\widetilde{\mathcal{N}}/[G,G]).$$

Proof. Set $Z := Z([G,G]) \cap Z(G)^\circ$, and recall that we have a short exact sequence

$$1 \to Z \xrightarrow{g \mapsto (g, g^{-1})} [G,G] \times Z(G)^\circ \to G \to 1. \tag{2.2.9.2}$$

In particular, we have $\overline{G} \cong [G,G]/Z$. Since $Z(G)^\circ$ is a torus, we have $\mathrm{Rep}(\overline{G})$-linear decompositions

$$\mathrm{Rep}(G) \simeq \bigoplus_{\chi \in X^*(Z(G)^\circ)} \mathrm{Rep}(\overline{G})_\chi, \qquad \mathrm{Rep}([G,G]) \simeq \bigoplus_{\chi \in X^*(Z)} \mathrm{Rep}(\overline{G})_\chi, \tag{2.2.9.3}$$

as in (2.1.6.1). Moreover, the restriction functor (2.2.9.1) is determined by the canonical restriction map

$$X^*(Z(G)^\circ) \to X^*(Z), \tag{2.2.9.4}$$

which is a surjection. Thus, choosing any set-theoretic splitting of (2.2.9.4) yields the result. □

2.2.10. Now, consider the homomorphisms

$$[G,G]^{\mathrm{sch}} \twoheadrightarrow [G,G] \hookrightarrow G.$$

By Lemma 2.1.27 and the previous discussion, we have a Bezrukavnikov–Mirković tilting bundle $\mathcal{E}$ on $\widetilde{\mathcal{N}}/[G,G]^{\mathrm{sch}}$. For each nontrivial $(\mathcal{C}, \alpha) \in \mathrm{M}([G,G])$, choose a minuscule weight $\lambda_{(\mathcal{C},\alpha)}$ of $[G,G]^{\mathrm{sch}}$ as in Lemma 2.1.27, and set $\lambda_{(\mathcal{O}_{[G,G]}, \mathrm{id})} := 0$.[17] Thus, by (2.1.25.3) and Lemma 2.2.9, we have functors

$$\mathrm{QC}(\widetilde{\mathcal{N}}/[G,G]^{\mathrm{sch}}) \simeq \bigoplus_{(\mathcal{C},\alpha) \in \mathrm{M}([G,G])} \mathrm{QC}(\widetilde{\mathcal{N}}/[G,G])^{(\mathcal{C},\alpha)} \xrightarrow{\bigoplus L(\lambda_{(\mathcal{C},\alpha)})^\vee \otimes -} \mathrm{QC}(\widetilde{\mathcal{N}}/[G,G]) \to \mathrm{QC}(\widetilde{\mathcal{N}}/G).$$

Misusing notation, we also denote the image of $\mathcal{E}$ under this composition by $\mathcal{E}$; though it is not uniquely determined, it is evidently also a tilting bundle, and it is a compact generator of $\mathrm{QC}(\widetilde{\mathcal{N}}/G)$ under the action of $\mathrm{QC}(\mathrm{B}G)$ by Lemma 2.1.13. Thus, the properties in §2.2.4, §2.2.5, and §2.2.6 also hold for $\mathcal{E}$ in this more general setting, and we carry over all notations from these paragraphs. Moreover, we may extend the braid group action of §2.2.7 to this setting using the same constructions[18], and the braid positivity property evidently carries over.

2.2.11. Our goal now is to lift the properties in §2.2.5 and §2.2.6 to $\mathcal{A}_{S_e}$, i.e., to construct $\widetilde{Z}_e$-equivariant analogs of the simple and indecomposable projective $\mathcal{A}_{S_e}^{\mathrm{cov,op}}$-modules satisfying analogous grading properties. Our first step will be to show that these modules admit equivariance structures with respect to certain canonical cocycles of their stabilizers in Z_e.

2.2.12. Notation. Let $\overline{Z}_e := Z_e / Z_e^\circ \cdot Z(G)$. The Z_e-action on $\mathcal{A}_{S_e}$ induces a canonical action of Z_e on the set $\mathbf{B}_e$ of simple modules, which evidently factors through $\overline{Z}_e$. For each $b \in \mathbf{B}_e$, let $Z_e^b \subset Z_e$ denote the stabilizer of b. We fix a set $\mathbf{B}_e^{\mathrm{orb}}$ of orbit representatives for the action of $\overline{Z}_e$ on $\mathbf{B}_e$.

2.2.13. Proposition. There exists a canonical cocycle $(\mathcal{C}_b, \alpha_b) \in \mathrm{Coc}(Z_e^b)$ for which the modules $L_b^{S_e}$ and $E_b^{S_e}$ admit canonical $(\mathcal{C}_b, \alpha_b)$-equivariant structures, i.e., canonical lifts to $\mathcal{A}_{S_e}^{\mathrm{op}}\text{-mod}^{Z_e^b, (\mathcal{C}_b, \alpha_b)}$.

2.2.14. Note that by [GR17a, Cor. 8.5.7], we may regard $\mathcal{A}_{S_e}^{\mathrm{op}}\text{-mod}^{Z_e^b, (\mathcal{C}_b, \alpha_b)}$ as the category of modules for the monad $(- \otimes \mathcal{A}_{S_e}) \in \mathrm{End}(\mathrm{Rep}(Z_e^b))$ on the $\mathrm{Rep}(Z_e^b)$-module category $\mathrm{Rep}(Z_e^b)^{(\mathcal{C}_b, \alpha_b)}$. Thus, it is equivalent to show that the vector spaces underlying these modules carry compatible $(\mathcal{C}_b, \alpha_b)$-representation structures.

Proof. We begin by showing that $L_b^{S_e}$ carries a canonical projective representation of Z_e^b. Since Z_e acts on $\mathcal{A}_{S_e}$ by k-algebra automorphisms, it preserves the Jacobson radical $\mathrm{J}(\mathcal{A}_{S_e})$; since the quotient $\mathcal{A}_{S_e}/\mathrm{J}(\mathcal{A}_{S_e})$ is finite-dimensional, it is Artinian, hence semisimple, and

$$\mathcal{A}_{S_e}/\mathrm{J}(\mathcal{A}_{S_e}) \simeq \bigoplus_{b \in \mathbf{B}_e} \mathrm{End}_k(L_b^{S_e}). \tag{2.2.14.1}$$

[17] We do not actually need these weights to be minuscule (in fact, all we really need is to choose nonzero compact objects of $\mathrm{Rep}(G)^{(\mathcal{C},\alpha)}$). However, this is in a sense the "simplest" way to modify $\mathcal{E}$ to be $[G,G]$-equivariant.

[18] Note that the de-equivariantized action on $\mathrm{QC}(\widetilde{\mathcal{N}})$ factors through the affine braid group for $[G,G]^{\mathrm{sch}}$ as in (2.2.9.2). We may thus regard the de-equivariantized affine braid group action for G as obtained from the pushforward of that for $[G,G]^{\mathrm{sch}}$ under the evident map $\widetilde{\mathcal{N}} \times_{[\mathfrak{g},\mathfrak{g}]} \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$ (which is an isomorphism on classical truncations). Since this gives a monoidal functor between the de-equivariantized affine Hecke categories, it is now straightforward to deduce the requisite relations between the sheaves $\mathcal{K}_a$.

Moreover, the Z_e-action preserves the summands corresponding to each $\overline{Z}_e$-orbit in $\mathbf{B}_e$, and Z_e^b preserves the summand corresponding to b. Thus, by the Skolem–Noether theorem, Z_e^b acts by inner automorphisms of this summand, i.e., via a projective representation

$$Z_e^b \to \mathrm{Aut}(\mathrm{End}_k(L_b^{S_e})) \simeq \mathrm{PGL}(\underline{L}_b^{S_e}), \tag{2.2.14.2}$$

where $\underline{L}_b^{S_e}$ denotes the underlying vector space of the module $L_b^{S_e}$. Pulling back the canonical cocycle of $\mathrm{PGL}(\underline{L}_b^{S_e})$ (corresponding to the central extension given by $\mathrm{GL}(\underline{L}_b^{S_e})$) then gives a cocycle $(\mathcal{C}_b, \alpha_b) \in \mathrm{M}(Z_e^b)$, and (2.1.8.3) shows that $L_b^{S_e}$ admits a $(\mathcal{C}_b, \alpha_b)$-equivariant structure.

Next, we claim that we have a decomposition

$$\mathcal{A}_{S_e} \simeq \bigoplus_{b \in \mathbf{B}_e} \underline{L}_b^{S_e,*} \otimes_k E_b^{S_e} \tag{2.2.14.3}$$

of right $\mathcal{A}_{S_e}$-modules, such that Z_e permutes the summands according to the $\overline{Z}_e$-action on $\mathbf{B}_e$. By the argument of [**BL23**, Rem. 4.6], there exists a $\widetilde{Z}_e$-stable choice of Koszul grading on $\mathcal{A}_{S_e}$ (which is not necessarily the same as that provided by the $\mathbb{G}_m$-action on $\mathcal{A}_{S_e}$); choosing such a grading, we obtain a $\widetilde{Z}_e$-stable subspace

$$\bigoplus_{b \in \mathbf{B}_e} \mathrm{End}_k(L_b^{S_e}) \subseteq \mathcal{A}_{S_e} \tag{2.2.14.4}$$

given by the 0th graded piece. In particular, we obtain a decomposition of the unit element into orthogonal idempotents $e_b := \mathrm{id}_{L_b^{S_e}}$, on which Z_e acts by permutation according to the $\overline{Z}_e$-action on $\mathbf{B}_e$. Thus, it suffices to show that $e_b\mathcal{A}_{S_e} \simeq \underline{L}_b^{S_e,*} \otimes_k E_b^{S_e}$. Since the latter is a projective cover of $\underline{L}_b^{S_e,*} \otimes_k L_b^{S_e}$, the right $\mathcal{A}_{S_e}$-module surjection $e_b\mathcal{A}_{S_e} \twoheadrightarrow \underline{L}_b^{S_e,*} \otimes_k L_b^{S_e}$ furnished by (2.2.14.1) shows that it is a direct summand of $e_b\mathcal{A}_{S_e}$. Moreover, the complementary submodule is contained in $J(\mathcal{A}_{S_e})$, hence is trivial.

It follows that the module $\underline{L}_b^{S_e,*} \otimes_k E_b^{S_e}$ carries a Z_e^b-equivariant structure. Moreover, choosing a basis of $\underline{L}_b^{S_e}$, we obtain a further decomposition of e_b into primitive orthogonal idempotents, and these decompositions are permuted according to the projective action of Z_e^b on $\underline{L}_b^{S_e}$. It follows that the Z_e^b-representation $\underline{L}_b^{S_e,*} \otimes_k E_b^{S_e}$ splits as a tensor product of projective representations, i.e., $E_b^{S_e}$ carries a projective representation of Z_e^b. Since $\underline{L}_b^{S_e,*}$ is a $(\mathcal{C}_b^\vee, \alpha_b^\vee)$-representation of Z_e^b, this must be a $(\mathcal{C}_b, \alpha_b)$-representation by (2.1.8.1). Moreover, it is easy to see that this projective action is compatible with the right $\mathcal{A}_{S_e}$-module structure, which yields the conclusion. □

2.2.15. Note that by §2.1.8, the functor (2.2.4.1) also induces an equivalence

$$\mathrm{QC}(\widetilde{S}_e/Z_e^b)^{(\mathcal{C}_b,\alpha_b)} \xrightarrow{\sim} \mathcal{A}_{S_e}^{\mathrm{op}}\text{-mod}^{Z_e^b,(\mathcal{C}_b,\alpha_b)}.$$

Thus, the sheaves $\mathcal{L}_b^{S_e}$ and $\mathcal{E}_b^{S_e}$ also admit canonical $(\mathcal{C}_b, \alpha_b)$-equivariant structures. It follows that the decomposition (2.2.14.3) also gives rise to a decomposition

$$i_{\widetilde{S}_e}^* \mathcal{E} \simeq \bigoplus_{b \in \mathbf{B}_e} \underline{L}_b^{S_e,*} \otimes_k \mathcal{E}_b^{S_e} \tag{2.2.15.1}$$

of Z_e-equivariant vector bundles. Moreover, equipping the left-hand side with the $\mathbb{G}_m$-equivariant structure arising from that on each $\mathcal{E}_b^{S_e}$, we obtain a $\mathbb{G}_m$-equivariant structure on $i_{\widetilde{S}_e}^* \mathcal{E}$ inducing a Koszul grading on $\mathcal{A}_{S_e}$. In particular, the "Koszul dual" algebra $\mathrm{End}_{\mathcal{A}_{S_e}^{\mathrm{op}}}(\mathcal{A}_{S_e,0})$ has its degree-n cohomology concentrated in weight $-n$.

Henceforth, when we refer to $i_{\widetilde{S}_e}^* \mathcal{E}$, we implicitly equip it with this particular $\widetilde{Z}_e$-equivariant structure, and likewise for $\mathcal{A}_{S_e}$. Finally, we set

$$\mathrm{in}_0 \colon \mathcal{A}_{S_e,0} \hookrightarrow \mathcal{A}_{S_e}, \qquad \mathrm{pr}_0 \colon \mathcal{A}_{S_e} \twoheadrightarrow \mathcal{A}_{S_e,0}, \tag{2.2.15.2}$$

to be the ($\widetilde{Z}_e$-equivariant) algebra homomorphisms induced by (2.2.14.4) and (2.2.14.1), respectively; the latter is a retraction of the former.

2.3. A covering group of the reductive centralizer

2.3.1. In this section, we establish existence of a finite cover of Z_e which trivializes all cocycles appearing in the canonical basis. This will be a key technical tool in the sequel. For instance, it will assist in later homological computations by allowing us to replace $\mathcal{A}_{S_e}$ with the Koszul quadratic algebra $\mathcal{A}_{S_e}^{\mathrm{cov}}$.

2.3.2. PROPOSITION. There exists a finite cover $p_{\mathrm{cov}}\colon Z_e^{\mathrm{cov}} \twoheadrightarrow Z_e^{\mathrm{sch}}$ such that for any $b \in \mathbf{B}_e$, the class of $(\mathcal{C}_b, \alpha_b)$ lies in the kernel of the restriction map

$$\mathrm{M}(Z_e^b) \to \mathrm{M}(Z_e^{\mathrm{cov},b}). \tag{2.3.2.1}$$

2.3.3. REMARK. In fact, over the course of the proof, we shall show that we may take $Z_e^{\mathrm{cov}} = Z_e^{\mathrm{sch}}$, unless the adjoint group of G contains either

(1) a simple factor of type E_6 for which the corresponding block of e has weighted Dynkin diagram

$$\begin{array}{ccccccccc} 0 & — & 0 & — & 2 & — & 0 & — & 0; \\ & & & & | & & & & \\ & & & & 0 & & & & \end{array} \tag{2.3.3.1}$$

(2) a simple factor of type E_8 for which the corresponding block of e has weighted Dynkin diagram

$$\begin{array}{ccccccccccccc} 0 & — & 0 & — & 0 & — & 0 & — & 0 & — & 0 & — & 0. \\ & & & & & & & & | & & & & \\ & & & & & & & & 2 & & & & \end{array} \tag{2.3.3.2}$$

If this occurs, we may take Z_e^{cov} to be a (non-central, non-split) extension of Z_e^{sch} by

$$(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^{n_1} \times (\mathbb{Z}/3\mathbb{Z})^{n_2},$$

where n_1, n_2 are the multiplicities of the nilpotents (2.3.3.1), (2.3.3.2) in e, respectively.

PROOF. Let $b \in \mathbf{B}_e$. It suffices to show that either $[(\mathcal{C}_b, \alpha_b)]$ lies in the image of the restriction map

$$\mathrm{M}(Z_e) \to \mathrm{M}(Z_e^b), \tag{2.3.3.3}$$

or there exists a finite cover ${}^bZ_e^{\mathrm{cov}} \twoheadrightarrow Z_e$ such that $[(\mathcal{C}_b, \alpha_b)]$ lies in the kernel of

$$\mathrm{M}(Z_e^b) \to \mathrm{M}({}^bZ_e^{\mathrm{cov}}). \tag{2.3.3.4}$$

Indeed, let $b_1, \dots, b_r \in \mathbf{B}_e$ denote those canonical basis elements whose cocycles do not lie in the image of (2.3.3.3). Consider the fiber square

$$\begin{tikzcd} Z_e^{\mathrm{cov}} \arrow[r, two heads] \arrow[d, two heads] & Z_e^{\mathrm{sch}} \arrow[d, two heads] \\ {}^{b_1}Z_e^{\mathrm{cov}} \times_{Z_e} \cdots \times_{Z_e} {}^{b_r}Z_e^{\mathrm{cov}} \arrow[r, two heads] & Z_e, \end{tikzcd} \tag{2.3.3.5}$$

and the corresponding diagram of stabilizers of b. If $[(\mathcal{C}_b, \alpha_b)]$ lies in the image of (2.3.3.3), then it lies in the kernel of $\mathrm{M}(Z_e^b) \to \mathrm{M}(Z_e^{\mathrm{sch},b})$, hence in the kernel of (2.3.2.1). Otherwise, since we have a factorization $Z_e^{\mathrm{cov},b} \twoheadrightarrow {}^bZ_e^{\mathrm{cov},b} \twoheadrightarrow Z_e^b$ for each $b \in \mathbf{B}_e$, the conclusion follows from (2.3.3.4).

Now, any cocycle of Z_e^b appearing in the canonical basis is pulled back from a cocycle of $Z_e^b/Z(G)$, which is the corresponding stabilizer for the adjoint group $G/Z(G)$. Thus, the commutative square

$$\begin{tikzcd} \mathrm{M}(Z_e/Z(G)) \arrow[r] \arrow[d] & \mathrm{M}(Z_e^b/Z(G)) \arrow[d] \\ \mathrm{M}(Z_e) \arrow[r] & \mathrm{M}(Z_e^b) \end{tikzcd}$$

immediately reduces us to the case where G is semisimple and adjoint (note that a finite cover of $Z_e^b/Z(G)$ pulls back to one of Z_e^b with the corresponding property). In particular, G splits as a product of adjoint simple groups; the cocycles appearing in $\mathbf{B}_e$ clearly all split as products over these simple factors, so we may further assume that G is simple and adjoint. We proceed type by type, using the classification of reductive centralizers of nilpotent elements, and show that (2.3.3.3) is surjective for any subgroup Z_e^b which is a union of connected components, except in the situation of (2.3.3.2), where we obtain a cover satisfying (2.3.3.4).

In type A, the component group of Z_e is always trivial, so there is nothing to check.

In types B, C, and D, recall that Z_e can be written as a semidirect product $Z_e^\circ \rtimes \pi_0(Z_e)$, where Z_e° is the quotient of a product of symplectic groups and special orthogonal groups by $Z(G) \cong \mathbb{Z}/2\mathbb{Z}$, and $\pi_0(Z_e) \cong (\mathbb{Z}/2\mathbb{Z})^n$, where n is the number of orthogonal factors in Z_e°. Lemma 2.1.22 thus gives an exact sequence

$$0 \to \mathrm{H}^1(\pi_0(Z_e), X^*(Z_e^\circ)) \to \ker\left(\mathrm{M}(Z_e) \twoheadrightarrow \mathrm{M}(\pi_0(Z_e))\right) \to \mathrm{M}(Z_e^\circ)^{\pi_0(Z_e)} \to \mathrm{H}^2(\pi_0(Z_e), X^*(Z_e^\circ)). \tag{2.3.3.6}$$

Note that the action of $\pi_0(Z_e)$ on $\mathrm{M}(Z_e^\circ)$ is trivial; moreover, we claim that the final homomorphism is trivial. Indeed, by Lemma 2.1.21 and functoriality of (2.3.3.6), the final homomorphism splits as a product over the factors of Z_e°; thus, we may assume that Z_e° is either a symplectic or special orthogonal group, on which $\pi_0(Z_e)$ acts either trivially, or by projection onto a single factor of $\mathbb{Z}/2\mathbb{Z}$, respectively. In the former case, both $\mathrm{M}(Z_e^\circ)$ and $X^*(Z_e^\circ)$ are trivial as the symplectic groups are simply-connected and simple. In the latter case, if $Z_e^\circ \cong \mathrm{SO}_2 \cong \mathbb{G}_m$, then $\mathrm{M}(Z_e^\circ)$ is again trivial; otherwise, if $Z_e^\circ \cong \mathrm{SO}_m$ for $m > 2$, then we have $X^*(Z_e^\circ) \simeq 0$, so the final term in (2.3.3.6) is trivial.

Now, let $\Gamma \subset \pi_0(Z_e)$ be a finite subgroup. By functoriality of (2.3.3.6), to show that $\mathrm{M}(Z_e) \twoheadrightarrow \mathrm{M}(Z_e^\circ \rtimes \Gamma)$, it suffices to show that

$$\begin{gathered}\mathrm{H}^1(\pi_0(Z_e), X^*(Z_e^\circ)) \twoheadrightarrow \mathrm{H}^1(\Gamma, X^*(Z_e^\circ)),\\ \mathrm{M}(\pi_0(Z_e)) \twoheadrightarrow \mathrm{M}(\Gamma).\end{gathered} \tag{2.3.3.7}$$

The latter is immediate, as any finite subgroup of $(\mathbb{Z}/2\mathbb{Z})^n$ is a direct summand. For the former, we may assume as before that $X^*(Z_e^\circ) \cong \mathbb{Z}$ and that $\pi_0(Z_e)$ acts by projection onto a single factor of $\mathbb{Z}/2\mathbb{Z}$ (via negation). If Γ acts trivially on $\mathbb{Z}$, then $\mathrm{H}^1(\Gamma, \mathbb{Z}) \cong \mathrm{Hom}(\Gamma, \mathbb{Z}) \simeq 0$, and we are done. Otherwise, let K be the index-2 subgroup of $\pi_0(Z_e)$ fixing $\mathbb{Z}$; the inflation-restriction exact sequence then gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \arrow[r] \arrow[d, "\wr"] & \mathrm{H}^1(\pi_0(Z_e), \mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{H}^1(K, \mathbb{Z})^{\mathbb{Z}/2\mathbb{Z}} \arrow[d] \\
0 \arrow[r] & \mathrm{H}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \arrow[r] & \mathrm{H}^1(\Gamma, \mathbb{Z}) \arrow[r] & \mathrm{H}^1(\Gamma \cap K, \mathbb{Z})^{\mathbb{Z}/2\mathbb{Z}}
\end{tikzcd}$$

with exact rows. Since the right-most terms are trivial, the second vertical map is an isomorphism, and we are done.

It remains to treat the exceptional types; we use the tables of centralizers of nilpotent elements for adjoint exceptional groups appearing in [**Ale05**]. When Z_e is connected, the assertion is trivial. We may further disregard all cases in which $\pi_0(Z_e) \cong \mathbb{Z}/2\mathbb{Z}$, and either $\mathrm{M}(Z_e^\circ)$ is trivial or $Z_e \cong Z_e^\circ \times \pi_0(Z_e)$. In all other cases, we have $Z_e \cong Z_e^\circ \rtimes \pi_0(Z_e)$. Moreover, the final map in (2.3.3.6) is again trivial: it is not hard to check that either $\mathrm{M}(Z_e^\circ)$ or $X^*(Z_e^\circ)$ is always trivial. As before, the action of $\pi_0(Z_e)$ on $\mathrm{M}(Z_e^\circ)$ is trivial, except in the case (2.3.3.4). Here, we have $Z_e \simeq \mathrm{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$, and (2.3.3.6) gives $\mathrm{M}(Z_e) \cong 0$, whereas $\mathrm{M}(\mathrm{PGL}_3) \cong \mathbb{Z}/3\mathbb{Z}$. However, Z_e admits a natural 3-fold cover given by $\mathrm{SL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$, which satisfies (2.3.3.4) as $\mathrm{M}(\mathrm{SL}_3) \cong 0$. Thus, we must verify (2.3.3.7) in each of the remaining examples, for all subgroups $\Gamma \subset \pi_0(Z_e)$.

For the former assertion, we need only check the case in which the symmetric group S_3 acts on the weight lattice $\Lambda \cong \mathbb{Z}^3/\mathbb{Z}$ of SL_3 by permutation, which appears in the situation of (2.3.3.1). We may assume that either $\Gamma \cong \langle (123) \rangle$ or $\Gamma \cong \langle (12) \rangle$. In the former case, we have

$$\mathrm{H}^1(\Gamma, \Lambda) \cong \ker((1 + (123) + (132))|_\Lambda)/(1 - (123))\Lambda \cong \Lambda/\langle \Phi \rangle \cong \mathbb{Z}/3\mathbb{Z},$$

with the trivial action of $\mathbb{Z}/2\mathbb{Z}$ (here Φ denotes the root lattice of SL_3). Since $\Lambda^\Gamma \cong 0$, the inflation-restriction exact sequence implies that $\mathrm{H}^1(S_3, \Lambda) \cong \mathrm{H}^1(\Gamma, \Lambda)$, as desired. In the latter case, Λ splits as a Γ-module into a sum of $\mathbb{Z}$ with the trivial Γ-action and $\mathbb{Z}$ with Γ acting by negation; thus, $\mathrm{H}^1(\langle (12) \rangle, \Lambda) \cong \mathbb{Z}/2\mathbb{Z}$, and the restriction map is not surjective. To obtain a cover of $Z_e \cong T \rtimes S_3$ killing this cocycle (here $T \subset \mathrm{SL}_3$ is a maximal torus), consider the S_3-equivariant short exact sequence

$$0 \to \Lambda \xrightarrow{2} \Lambda \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to 0.$$

The endomorphism of $\mathrm{H}^1(\langle(12)\rangle, \Lambda)$ induced by the first map is the zero map, so by the following paragraph, the corresponding surjection

$$1 \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to T \rtimes S_3 \xrightarrow{(-)^2 \rtimes \mathrm{id}} T \rtimes S_3 \to 1$$

suffices.

For the latter assertion, the only nontrivial component groups which appear are the symmetric groups S_2, S_3, S_4, S_5. The assertion is trivial in the first two cases. For the third case, we need only consider the restriction maps $\mathrm{M}(S_4) \to \mathrm{M}(A_4)$, $\mathrm{M}(S_4) \to \mathrm{M}(D_8)$ (the dihedral subgroup of order 8), $\mathrm{M}(S_4) \to \mathrm{M}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ (the normal Klein four-subgroup) and $\mathrm{M}(S_4) \to \mathrm{M}(S_2 \times S_2)$ (the non-normal Klein four-subgroup). Applying Lemma 2.1.22 to $S_4 \cong A_4 \rtimes \mathbb{Z}/2\mathbb{Z}$, and using the well-known identity $\mathrm{M}(A_4) \cong \mathbb{Z}/2\mathbb{Z}$, we see that the first map is an isomorphism. For the second map, applying functoriality of Lemma 2.1.22 reduces us to showing that $\mathrm{M}(A_4) \twoheadrightarrow \mathrm{M}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$; since $A_4 \cong (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}/3\mathbb{Z}$, a further application of Lemma 2.1.22 gives the result. This also shows that $\mathrm{M}(S_4) \twoheadrightarrow \mathrm{M}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$. Finally, applying functoriality of Lemma 2.1.22 to $S_2 \times S_2 \subset \mathbb{Z}/4\mathbb{Z} \rtimes S_2 \cong D_8$ shows that the final map is an isomorphism.

For the fourth case, we need only consider the restriction maps $\mathrm{M}(S_5) \to \mathrm{M}(A_5)$, $\mathrm{M}(S_5) \to \mathrm{M}(S_4)$, and $\mathrm{M}(S_5) \to \mathrm{M}(S_3 \times S_2)$. The first follows as for $A_4 \subset S_4$. For the second, functoriality of Lemma 2.1.22 reduces us to showing that $\mathrm{M}(A_5) \twoheadrightarrow \mathrm{M}(A_4)$, and it is well-known that the restriction of a Schur cover of A_5 to A_4 remains a Schur cover (indeed, the Schur cover of A_n for $n = 4, 5$ and $n \geq 8$ is constructed by pulling back the double cover $\mathrm{Spin}_{n-1} \to \mathrm{SO}_{n-1}$ along the embedding $A_n \hookrightarrow \mathrm{SO}_{n-1}$). For the third, applying functoriality of Lemma 2.1.22 to $S_3 \times S_2 \subset A_5 \rtimes S_2$ shows that the restriction map is in fact trivial. However, consider the "+" and "−" type Schur double covers of S_5; it suffices to show that each of their restrictions to $S_2 \times S_2 \subset S_3 \times S_2$ kills all elements of $M(S_2 \times S_2)$. Indeed, these restrictions are given by the dihedral group and the quaternion group, respectively, both of which are Schur covers of $S_2 \times S_2$. □

2.3.4. We note the following consequence of the proof, with a view towards the discussion following Proposition 2.4.12.

2.3.5. Corollary. *The derived subgroup of $Z_e^{\mathrm{cov},\circ}$ is simply-connected. Moreover, suppose that all simple factors of $[G, G]$ are of classical types. In this case, if Z_e° has simply-connected derived subgroup, then $(\mathcal{C}_b, \alpha_b)$ is trivial for all $b \in \mathbf{B}_e$ (and thus we may take $Z_e^{\mathrm{cov}} = Z_e$).*

Proof. We begin with the first assertion. Following the proof of Proposition 2.3.2, we may check this type-by-type for the adjoint group $G/Z(G)$ by Lemma 2.1.30. In type A, this follows from Lemma 2.1.29. In types B, C, and D, the discussion following (2.3.3.6) gives a short exact sequence

$$0 \to \mathrm{M}(Z_e^\circ) \to \mathrm{M}(Z_e) \to \mathrm{M}(\pi_0(Z_e)) \to 0. \tag{2.3.5.1}$$

The proof of Proposition 2.1.5 then shows that the identity component of $Z_e^{\mathrm{sch}}/X^*(\mathrm{M}(\pi_0(Z_e)))$ is a Schur cover of Z_e°, so the result again follows by Lemma 2.1.29. Finally, in the exceptional types, Lemma 2.1.29 covers the case where Z_e is connected. Likewise, by Proposition 2.1.16 and Lemma 2.1.30, the statement is clear when either $\mathrm{M}(Z_e^\circ)$ is trivial or $Z_e \cong Z_e^\circ \times \pi_0(Z_e)$. Otherwise, we always have $X^*(Z_e^\circ) \cong 0$. When the action of $\pi_0(Z_e)$ on $\mathrm{M}(Z_e^\circ)$ is trivial, the statement follows as for types B, C, and D. In the only remaining case, we have shown that $Z_e^{\mathrm{cov},\circ} \cong \mathrm{SL}_3$, which completes the proof.

For the second assertion, it suffices to show that the image of $[(\mathcal{C}_b, \alpha_b)]$ under $\mathrm{M}(Z_e^b/Z(G)) \to \mathrm{M}(Z_e^b)$ is trivial. We may again reduce to checking this type-by-type. As before, the conclusion is clear whenever $Z_e/Z(G)$ is connected, which proves the claim in type A. In types B, C, and D, note that the universal cover of $G/Z(G)$ is at most a four-fold cover. Thus, if Z_e° has simply-connected derived subgroup, then at most one special orthogonal factor appears in $(Z_e/Z(G))^\circ$. In particular, $\pi_0(Z_e/Z(G))$ is a subgroup of $\mathbb{Z}/2\mathbb{Z}$, and has trivial Schur multiplier. We may therefore assume that $Z_e^b = Z_e$, and the conclusion follows from functoriality of (2.3.5.1). □

2.3.6. In principle, one could verify the latter statement for exceptional types via a finite amount of computation. One must for instance show that the nontrivial cocycles of Z_e in the case (2.3.3.1) do not appear in the canonical basis (as a torus has trivial derived subgroup).

Finally, as for $\mathcal{A}_{S_e}$, we obtain equivariant lifts of $\mathcal{A}_{S_e}^{\mathrm{cov}}$ and its associated sheaves:

2.3.7. COROLLARY. The sheaves $\mathcal{L}_b^{S_e}$ and $\mathcal{E}_b^{S_e}$ admit lifts to $\mathrm{Coh}(\widetilde{S}_e/\widetilde{Z}_e^{\mathrm{cov},b})$, and the objects $\bigoplus_{b'\in\overline{Z}_e\cdot b}\mathcal{L}_{b'}^{S_e}$ and $\bigoplus_{b'\in\overline{Z}_e\cdot b}\mathcal{E}_{b'}^{S_e}$ admit lifts to $\mathrm{Coh}(\widetilde{S}_e/\widetilde{Z}_e^{\mathrm{cov}})$. In particular, we have an equivalence

$$\mathrm{Hom}_{\widetilde{S}_e}(\bigoplus_{b\in\mathbf{B}_e}\mathcal{E}_b^{S_e},-)\colon \mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e^{\mathrm{cov}}) \xrightarrow{\sim} \mathcal{A}_{S_e}^{\mathrm{cov,op}}\text{-mod}^{\widetilde{Z}_e^{\mathrm{cov}}}.$$

PROOF. The first assertion follows from Proposition 2.2.13, (2.1.8.3), and Proposition 2.3.2. For the second assertion, let

$$q_b\colon \widetilde{S}_e/\widetilde{Z}_e^{\mathrm{cov},b} \to \widetilde{S}_e/\widetilde{Z}_e^{\mathrm{cov}}$$

be the projection, and simply note that

$$(q_{b,*}\mathcal{L}_b^{S_e})^{\mathrm{dq}} \simeq \bigoplus_{b'\in\overline{Z}_e\cdot b}\mathcal{L}_{b'}^{S_e},$$
$$(q_{b,*}\mathcal{E}_b^{S_e})^{\mathrm{dq}} \simeq \bigoplus_{b'\in\overline{Z}_e\cdot b}\mathcal{E}_{b'}^{S_e}.$$

The final assertion is now immediate from §2.2.5. □

2.4. Centrally extended sets

2.4.1. In this section, we recall an important theorem of Bezrukavnikov, Ostrik, and Losev, which relates the structure of each block of the asymptotic affine Hecke algebra $\mathcal{J}$ to the canonical basis $\mathbf{B}_e$ and noncommutative Springer resolution $\mathcal{A}_{S_e}$ [**BO04, BL23**]. This further yields an interpretation of each block of $\mathcal{J}$ as the endomorphisms of a certain vector bundle; these "asymptotic coherent Springer sheaves" will be our main object of study in this thesis. Finally, we comment briefly on (and partially resolve) a recent conjecture of Qiu–Xi regarding the structure of $\mathcal{J}$ [**QX22**].

2.4.2. We begin by reviewing the properties of the affine and asymptotic affine Hecke algebras which will be relevant for the present work. References for these facts may be found in the series [**Lus85, Lus87a, Lus87b, Lus89**], though see also [**Bez04**].

Recall that the *affine Hecke algebra* $\mathcal{H}$ is a free $\mathbb{Z}[v^{\pm 1}]$-algebra equipped with a canonical basis $\{C_w\}_{w\in W^{\mathrm{aff}}}$, known as the *Kazhdan–Lusztig basis*, indexed by the extended affine Weyl group W^{aff} of the *Langlands dual* reductive group $\check{G}$. The group W^{aff} decomposes into subsets $\underline{c}\subset W^{\mathrm{aff}}$ known as *two-sided cells*, which are equipped with a standard partial order; the unique minimal two-sided cell consists only of the identity element. Let $W^{\mathrm{aff}}_{\leq\underline{c}} := \bigcup_{\underline{c}'\leq\underline{c}}\underline{c}$, and likewise for $W^{\mathrm{aff}}_{<\underline{c}}$. We set $\mathcal{H}_{\leq\underline{c}}$ and $\mathcal{H}_{<\underline{c}}$ to be the $\mathbb{Z}[v^{\pm 1}]$-submodules of $\mathcal{H}$ spanned by the Kazhdan–Lusztig basis elements $\{C_w\}_{w\in\widetilde{W}_{\leq\underline{c}}}$ and $\{C_w\}_{w\in W^{\mathrm{aff}}_{<\underline{c}}}$, respectively. In fact, these are two-sided ideals of $\mathcal{H}$. We denote the subquotients of the corresponding filtration of $\mathcal{H}$ by $\mathcal{H}_{\underline{c}} := \mathcal{H}_{\leq\underline{c}}/\mathcal{H}_{<\underline{c}}$; the latter is an $\mathcal{H}$-bimodule with $\mathbb{Z}[v^{\pm 1}]$-basis $\{C_w\}_{w\in\underline{c}}$.

Next, recall that the *asymptotic affine Hecke algebra* $\mathcal{J}$ is a free $\mathbb{Z}$-algebra equipped with a canonical basis $\{t_w\}_{w\in W^{\mathrm{aff}}}$. For each two-sided cell $\underline{c}$, we let $\mathbf{D}_{\underline{c}}\subset\underline{c}$ denote the (finite) set of *Duflo involutions* in $\underline{c}$. Then the elements $1_{\mathcal{J}_{\underline{c}}} := \sum_{d\in\mathbf{D}_{\underline{c}}} t_d$ are pairwise-orthogonal idempotents summing to the identity element $1_{\mathcal{J}}$. In particular, setting $\mathcal{J}_{\underline{c}} := 1_{\mathcal{J}_{\underline{c}}}\cdot\mathcal{J}\cdot 1_{\mathcal{J}_{\underline{c}}}$, we obtain a decomposition

$$\mathcal{J}\cong\bigoplus_{\underline{c}}\mathcal{J}_{\underline{c}}$$

into subalgebras, where each $\mathcal{J}_{\underline{c}}$ has $\mathbb{Z}$-basis $\{t_w\}_{w\in\underline{c}}$.

We now recall the construction of Lusztig's homomorphism $\phi\colon\mathcal{H}\to\mathcal{J}[v^{\pm 1}]$, where we have written $\mathcal{J}[v^{\pm 1}] := \mathcal{J}\otimes_{\mathbb{Z}}\mathbb{Z}[v^{\pm 1}]$. Set

$$\mathrm{bas}_{\underline{c}}\colon \mathcal{H}_{\underline{c}} \xrightarrow{\sim} \mathcal{J}_{\underline{c}}[v^{\pm 1}]$$

to be the $\mathbb{Z}[v^{\pm 1}]$-module isomorphism given by $C_w\mapsto t_w$ for $w\in\underline{c}$. We transport the $\mathcal{H}$-bimodule structure on $\mathcal{H}_{\underline{c}}$ to $\mathcal{J}_{\underline{c}}[v^{\pm 1}]$ via $\mathrm{bas}_{\underline{c}}$; one can then show that $\mathcal{J}_{\underline{c}}[v^{\pm 1}]$ carries both $(\mathcal{H},\mathcal{J}_{\underline{c}}[v^{\pm 1}])$- and $(\mathcal{J}_{\underline{c}}[v^{\pm 1}],\mathcal{H})$-bimodule structures (via its natural $\mathcal{J}_{\underline{c}}[v^{\pm 1}]$-bimodule structure). For each two-sided cell $\underline{c}$, we define a map

$$\phi_{\underline{c}}\colon \mathcal{H}\to\mathcal{J}_{\underline{c}}[v^{\pm 1}],$$
$$h\mapsto h\cdot\mathbf{1}_{\mathcal{J}_{\underline{c}}}$$

via the *left* $\mathcal{H}$-module structure on $\mathcal{J}_{\underline{c}}[v^{\pm 1}]$. One can show that this is an algebra homomorphism, and we set $\phi := \bigoplus_{\underline{c}} \phi_{\underline{c}}$. In particular, note that each $\phi_{\underline{c}}$ factors through the quotient algebra $\mathcal{H}/\mathcal{H}_{<\underline{c}}$.

Finally, we record two instances of Langlands duality in these structures. First, Lusztig has established a canonical bijection between the set of nilpotent G-orbits in $\mathfrak{g}$ and the set of two-sided cells in W^{aff} respecting the partial orders on either side. Given a nilpotent element $e \in \mathfrak{g}$, we henceforth misuse notation slightly, and write the subscript "e" in place of the subscript "$\underline{c}$" for the corresponding two-sided cell. For instance, we write $\mathcal{H}_{\leq e}, \mathcal{J}_e, \phi_e$ in place of $\mathcal{H}_{\leq \underline{c}}, \mathcal{J}_{\underline{c}}, \phi_{\underline{c}}$, respectively.

Second, recall that the extended affine Weyl group is given by the semidirect product

$$W^{\mathrm{aff}} := W \ltimes X_*(\check{T}) = W \ltimes X^*(T),$$

where $T \subset G$ is a maximal torus, $\check{T} \subset \check{G}$ is the dual torus, and W denotes the finite Weyl group. Then the *Bernstein isomorphism* identifies the center of $\mathcal{H}$ with the commutative ring

$$\mathbb{Z}[v^{\pm 1}] \otimes_{\mathbb{Z}} \mathbb{Z}[X_*(\check{T})]^W \cong R(\widetilde{T})^W \cong R(\widetilde{G}),$$

such that $\mathbb{Z}[v^{\pm 1}]$ identifies with the subalgebra $R(\mathbb{G}_m)$.

2.4.3. To state Bezrukavnikov–Ostrik–Losev's theorem, we will need the following language from [**BO04**] (see also [**BDD22**, §2.4]):

2.4.4. Definition. Let H be a linear algebraic group.

(1) A *centrally extended H-set* $\mathbf{X}$ is the data of
 (a) an H-set $\underline{\mathbf{X}}$;
 (b) for each $x \in \underline{\mathbf{X}}$, a cocycle $(\mathcal{C}_x, \alpha_x)$ of the stabilizer H^x; and
 (c) for each $h \in H$, an isomorphism $\mathrm{ad}_h^*(\mathcal{C}_x, \alpha_x) \cong (\mathcal{C}_{h\cdot x}, \alpha_{h\cdot x})$, where $\mathrm{ad}_h\colon H^x \to H^{h\cdot x}$ denotes the conjugation map. We require that these isomorphisms satisfy the cocycle condition, and moreover, that they agree with those provided by Lemma 2.1.11 for $h \in H^x$.

(2) Let $\mathbf{X}, \mathbf{Y}$ be two centrally extended H-sets. Their *product* $\mathbf{X} \times \mathbf{Y}$ is the centrally extended H-set with underlying set $\underline{\mathbf{X}} \times \underline{\mathbf{Y}}$, cocycles

$$(\mathcal{C}_{(x,y)}, \alpha_{(x,y)}) := (\mathcal{C}_x, \alpha_x)|_{H^{(x,y)}} \cdot (\mathcal{C}_y, \alpha_y)|_{H^{(x,y)}}, \tag{2.4.4.1}$$

and the induced isomorphisms. Similarly, the *opposite* $\mathbf{X}^{\mathrm{op}}$ is the centrally extended H-set obtained by inverting all cocycles of $\mathbf{X}$.

(3) An *H-equivariant vector bundle on* $\mathbf{X}$ is the data of
 (a) for each $x \in \underline{\mathbf{X}}$, a compact object $\mathcal{F}_x \in \mathrm{Rep}(H^x)^{(\mathcal{C}_x, \alpha_x), \heartsuit}$;
 (b) a compatible projective H-equivariant structure on $\mathcal{F} := \bigoplus_{x \in \mathbf{X}} \mathcal{F}_x$.

 We denote the $(1,1)$-category of such objects by $\mathrm{Vect}^H(\mathbf{X})$.

2.4.5. Remark. After a choice of orbit representatives $\underline{\mathbf{X}}^{\mathrm{orb}}$, the data of (1) is nothing more than the data of a cocycle $(\mathcal{C}_x, \alpha_x)$ of H^x for each $x \in \underline{\mathbf{X}}^{\mathrm{orb}}$. Moreover, $\mathrm{Vect}^H(\mathbf{X})$ is equivalent to the heart of $\bigoplus_{x \in \underline{\mathbf{X}}^{\mathrm{orb}}} \mathrm{Rep}(H^x)^{(\mathcal{C}_x, \alpha_x), c}$. The latter thus gives a non-canonical ∞-categorical version of $\mathrm{Vect}^H(\mathbf{X})$; if we have a covering group $H^{\mathrm{cov}} \twoheadrightarrow H$ trivializing all cocycles appearing in $\mathbf{X}$, we may make it canonical by taking the corresponding full subcategory of $\mathrm{QC}(\underline{\mathbf{X}}/H^{\mathrm{cov}})$. Alternatively, choosing a compact $V_x \in \mathrm{Rep}(H^x)^{(\mathcal{C}_x, \alpha_x), \heartsuit}$ for each $x \in \underline{\mathbf{X}}$, we can take the category of equivariant modules $\bigoplus_{x \in \underline{\mathbf{X}}} \mathrm{End}_k(V_x)\text{-mod}^H$. We will adopt both of these perspectives in the sequel.

For a different ∞-categorical generalization of $\mathrm{Vect}^H(\mathbf{X})$, one can attempt the following construction. Let X be a scheme with an action of H. Recall that the loop space $\mathcal{L}(X/H)$ (see Definition 3.2.2) carries the canonical structure of a relative group scheme over X/H via the loop evaluation map.[19] At the level of k-points, we have

$$\mathcal{L}(X/H)(k) \simeq \{(x,h) \in X(k) \times H(k) : h \cdot x = x\}/H(k),$$

and this group structure is given by multiplication in the second coordinate. Note that Definition 2.1.2 evidently makes sense for relative group schemes; thus, we may speak of *cocycles on* $\mathcal{L}(X/H)$. Now, if X is a finite set, then by Lemma 2.1.11, a cocycle $(\mathcal{C}, \alpha)$ on $\mathcal{L}(X/H)$ is equivalent to the data of a cocycle $(\mathcal{C}_x, \alpha_x)$ on H^x for each $x \in X$, along with descent data for the H-action. If we require the restriction of these descent data to each H^x to agree with the canonical descent datum for $(\mathcal{C}_x, \alpha_x)$, then we recover

[19] In fact, the loop space of any prestack $\mathcal{X}$ has the structure of a group object over $\mathcal{X}$, see [**GKRV22**, Ex. 4.3.3].

the notion of a centrally extended H-set. It is now straightforward to generalize the notions of "product" and "opposite." Moreover, the automorphism group of any $x \in X$ in the classifying stack $\mathrm{B}\mathcal{L}(X/H)$ is identified with $H^x \rtimes H^x$ (where the semidirect product is taken via the adjoint action), and we may replace $\mathrm{Vect}^H(\mathbf{X})$ by the full subcategory of $\mathrm{Rep}(\mathcal{L}(X/H))^{(\mathcal{C},\alpha)}$ spanned by objects such that the representation of $H^x_{(\mathcal{C}_x,\alpha_x)} \rtimes H^x$ at any $x \in X$ restricts to a weight-1 representation of $H^x_{(\mathcal{C}_x,\alpha_x)}$.[20] We shall not need this in the present work, however.

2.4.6. Given a centrally extended H-set $\mathbf{X}$, it is not hard to see that the abelian category $\mathrm{Vect}^H(\mathbf{X}\times\mathbf{X}^{\mathrm{op}})$ carries a monoidal structure under convolution. We now give a sheaf-theoretic realization of the corresponding algebra:

2.4.7. Construction. For each $x \in \underline{\mathbf{X}}$, let $i_x \colon H^x \hookrightarrow H$ denote the (clopen) inclusion, and consider the vector bundle

$$\mathcal{S}^{\mathbf{X}} := \bigoplus_{x\in\underline{\mathbf{X}}} i_{x,*}\mathcal{C}_x^\vee \tag{2.4.7.1}$$

on H. By Definition 2.4.4(1c), this vector bundle canonically descends to the adjoint quotient stack H/H (with the action of H permuting the summands according to the H-action on $\underline{\mathbf{X}}$). Equivalently, letting $\underline{\mathbf{X}}^{\mathrm{orb}}$ be as in Remark 2.4.5 and $i_x^{\mathrm{eq}} \colon H^x/H^x \to H/H$ be the map of adjoint quotient stacks, we have

$$\mathcal{S}^{\mathbf{X}} \simeq \bigoplus_{x\in\underline{\mathbf{X}}^{\mathrm{orb}}} i^{\mathrm{eq}}_{x,*}\mathcal{C}_x^\vee, \tag{2.4.7.2}$$

where $\mathcal{C}_x$ is equipped with its canonical H_x-equivariant structure via Lemma 2.1.11.

2.4.8. Proposition. Suppose H is reductive. Then the dg-algebra of endomorphisms of $\mathcal{S}^{\mathbf{X}}$ is concentrated in degree 0, and

$$K_0\,\mathrm{Vect}^H(\mathbf{X}\times\mathbf{X}^{\mathrm{op}})_k \simeq \mathrm{End}_{\mathrm{Coh}(H/H)}(\mathcal{S}^{\mathbf{X}})^{\mathrm{op}}. \tag{2.4.8.1}$$

Thus, $\mathcal{S}^{\mathbf{X}}$ generates a full embedding

$$K_0\,\mathrm{Vect}^H(\mathbf{X}\times\mathbf{X}^{\mathrm{op}})_k\text{-mod} \simeq \langle\mathcal{S}^{\mathbf{X}}\rangle \subset \mathrm{QC}(H/H). \tag{2.4.8.2}$$

Proof. Let $(x,y) \in \underline{\mathbf{X}}^{\mathrm{orb}}$. We have a base-change diagram

$$\begin{tikzcd} H^{(x,y)}/H^{(x,y)} \arrow[r, "i_2^{\mathrm{eq}}"] \arrow[d, "i_1^{\mathrm{eq}}"] & H^x/H^x \arrow[d, "i_x^{\mathrm{eq}}"] \\ H^y/H^y \arrow[r, "i_y^{\mathrm{eq}}"] & H/H \end{tikzcd}$$

with all maps smooth and proper. Thus,

$$\begin{aligned} \mathrm{Hom}_{H/H}(i^{\mathrm{eq}}_{x,*}\mathcal{C}_x^\vee, i^{\mathrm{eq}}_{y,*}\mathcal{C}_y^\vee) &\simeq \mathrm{Hom}_{H^x/H^x}(\mathcal{C}_x^\vee, i_x^{\mathrm{eq},!}i^{\mathrm{eq}}_{y,*}\mathcal{C}_y^\vee) \\ &\simeq \mathrm{Hom}_{H^x/H^x}(\mathcal{C}_x^\vee, i^{\mathrm{eq}}_{2,*}i_1^{\mathrm{eq},!}\mathcal{C}_y^\vee) \\ &\simeq \mathrm{Hom}_{H^{(x,y)}/H^{(x,y)}}(i_2^{\mathrm{eq},*}\mathcal{C}_x^\vee, i_1^{\mathrm{eq},!}\mathcal{C}_y^\vee) \\ &\simeq \Gamma(H^{(x,y)}/H^{(x,y)}, \mathcal{C}_x|_{H^{(x,y)}} \otimes \mathcal{C}_y^\vee|_{H^{(x,y)}}) \\ &= \Gamma(H^{(x,y)}/H^{(x,y)}, \mathcal{C}_{(x,y)}) \\ &\simeq K_0(\mathrm{Rep}(H^{(x,y)})^{(\mathcal{C}_{(x,y)},\alpha_{(x,y)})})_k, \end{aligned}$$

where the final isomorphism holds by Lemma 2.1.13. The latter identifies with the (x,y) coordinate of $K_0\,\mathrm{Vect}^H(\mathbf{X}\times\mathbf{X}^{\mathrm{op}})_k$ via Remark 2.4.5, so we have an isomorphism (2.4.8.1) of modules over $K_0\,\mathrm{Rep}(H)_k \simeq \mathcal{O}(H/H)$. Finally, it is not hard to check that this isomorphism is independent of the choice of $\underline{\mathbf{X}}^{\mathrm{orb}}$, and respects the algebra structures on either side; we leave this as an exercise. $\square$

[20] Note that given any weight-1 representation V of $H^x_{(\mathcal{C}_x,\alpha_x)}$, the space $V\otimes V^*$ carries a natural $H^x_{(\mathcal{C}_x,\alpha_x)} \rtimes H^x$-representation whose restriction to $H^x_{(\mathcal{C}_x,\alpha_x)}$ is again weight-1, and is irreducible if V is.

2.4.9. In particular, Proposition 2.2.13 shows that $\mathbf{B}_e$ carries the structure of a centrally extended Z_e-set, so we may form the monoidal $(1,1)$-category $\mathrm{Vect}^{Z_e}(\mathbf{B}_e \times \mathbf{B}_e^{\mathrm{op}})$. Let $\mathcal{A}_{\mathfrak{g}} \otimes_{\mathcal{O}(\mathfrak{g})} \mathcal{A}_{\mathfrak{g}}^{\mathrm{op}}\text{-mod}_e^{Z_e,\heartsuit,\mathrm{ss}}$ denote the abelian category of semisimple Z_e-equivariant $\mathcal{A}_{\mathfrak{g}}$-bimodules supported at e; it likewise carries a monoidal structure via the (underived) tensor product. We may finally state the theorem of Bezrukavnikov–Ostrik–Losev:

2.4.10. Theorem ([**BL23**, Prop. 8.25]). *For any $e \in \mathcal{N}$, there are based algebra isomorphisms*

$$\mathcal{J}_e \cong K_0 \,\mathrm{Vect}^{Z_e}(\mathbf{B}_e \times \mathbf{B}_e^{\mathrm{op}}) \cong K_0(\mathcal{A}_{\mathfrak{g}} \otimes_{\mathcal{O}(\mathfrak{g})} \mathcal{A}_{\mathfrak{g}}^{\mathrm{op}}\text{-mod}_e^{Z_e,\heartsuit,\mathrm{ss}}). \tag{2.4.10.1}$$

That is, the basis elements t_w correspond to irreducible vector bundles on $\mathbf{B}_e \times \mathbf{B}_e^{\mathrm{op}}$ and simple objects of $\mathcal{A}_{\mathfrak{g}} \otimes_{\mathcal{O}(\mathfrak{g})} \mathcal{A}_{\mathfrak{g}}^{\mathrm{op}}\text{-mod}_e^{Z_e,\heartsuit}$.

2.4.11. Note that by (2.2.14.1), we may replace $\mathcal{A}_{\mathfrak{g}} \otimes_{\mathcal{O}(\mathfrak{g})} \mathcal{A}_{\mathfrak{g}}^{\mathrm{op}}\text{-mod}_e^{Z_e,\heartsuit,\mathrm{ss}}$ by the semisimple monoidal dg-category $\mathcal{A}_{S_e,0} \otimes_k \mathcal{A}_{S_e,0}^{\mathrm{op}}\text{-mod}^{Z_e}$; the monoidal structures evidently agree. We refer to the sheaf

$$\mathcal{S}^{\mathbf{B}_e} \in \mathrm{Coh}(Z_e/Z_e) \tag{2.4.11.1}$$

as the *asymptotic coherent Springer sheaf* at e. This terminology will be justified in Chapter 5, where we will also reinterpret Proposition 2.4.8 in terms of the “trace formalism” of [**GKRV22**].

Finally, pulling back Theorem 2.4.10 along the surjection $p_{\mathrm{cov}} \colon Z_e^{\mathrm{cov}} \twoheadrightarrow Z_e$ of Proposition 2.3.2 yields:

2.4.12. Proposition. For any $e \in \mathcal{N}$, there is a based injection of algebras

$$\mathcal{J}_e \hookrightarrow K_0 \,\mathrm{Coh}((\underline{\mathbf{B}}_e \times \underline{\mathbf{B}}_e)/Z_e^{\mathrm{cov}}).$$

2.4.13. Remark. This result, along with Corollary 2.3.5, partially resolves the conjecture of Qiu–Xi in [**QX22**, §5.6]. There, the authors furthermore conjecture that $K_0 \,\mathrm{Coh}((\mathbf{B}_e \times \mathbf{B}_e)/Z_e^{\mathrm{cov}})$ is finitely generated over $\mathcal{J}_e$ (as either a left or right module). To this end, it suffices to show that $R(Z_e^{\mathrm{cov}})$ is finitely generated as an $R(Z_e)$-module. The fiber square (2.3.3.5) shows that the kernel of p_{cov} is abelian. Thus, we obtain a $\mathrm{Rep}(Z_e)$-linear decomposition of $\mathrm{Rep}(Z_e^{\mathrm{cov}})$ as in (2.1.25.3), and the claim follows by the same argument as in Lemma 2.1.13. We hope to treat the authors' conjecture regarding the canonical anti-involution of $\mathcal{J}_e$ in a future work.

CHAPTER 3

The trace formalism

3.1. Higher categorical traces

3.1.1. In this section, we recall some facts about the beautiful trace formalism of [**GKRV22**, §3] that will be needed in the sequel. This section may mostly be skipped on a first reading.

3.1.2. Let $\mathbf{dgCat}_k$ denote the $(\infty,2)$-category with objects given by (presentable) cocomplete stable ∞-categories equipped with a Vect_k-module structure (with respect to the Lurie tensor product), 1-morphisms given by continuous (i.e., colimit-preserving) functors, and 2-morphisms given by natural transformations. The Lurie tensor product (over Vect_k) endows $\mathbf{dgCat}_k$ with a symmetric monoidal structure with unit object Vect_k. As in §1.8, We refer to objects of $\mathbf{dgCat}_k$ as "k-linear dg-categories," or simply "dg-categories."

3.1.3. We denote by $\mathrm{Morita}(\mathbf{dgCat}_k)$ the $(\infty,2)$-category whose objects are symbols $\underline{\mathcal{A}\,\textbf{-mod}}$, where $\mathcal{A}$ is a monoidal dg-category, and whose 1-morphisms are given by the $(\infty,1)$-category

$$\mathbf{Map}_{\mathrm{Morita}(\mathbf{dgCat}_k)}(\underline{\mathcal{A}\,\textbf{-mod}},\underline{\mathcal{B}\,\textbf{-mod}}):=\mathcal{B}\otimes\mathcal{A}^{\mathrm{rev}}\,\textbf{-mod},$$

where $\mathcal{A}^{\mathrm{rev}}$ denotes $\mathcal{A}$ with the opposite monoidal structure. The composition law is given by the tensor product of bimodules, and the unit 1-morphism of an object $\underline{\mathcal{A}\,\textbf{-mod}}$ is given by the regular bimodule $\mathcal{A}$.

Moreover, $\mathrm{Morita}(\mathbf{dgCat}_k)$ carries a symmetric monoidal structure via

$$\underline{\mathcal{A}\,\textbf{-mod}}\otimes\underline{\mathcal{B}\,\textbf{-mod}}:=\underline{\mathcal{A}\otimes\mathcal{B}\,\textbf{-mod}},$$

with unit object $\underline{\mathrm{Vect}_k\,\textbf{-mod}}$. Note that we have a canonical identification

$$\mathbf{End}_{\mathrm{Morita}(\mathbf{dgCat}_k)}(\underline{\mathrm{Vect}_k\,\textbf{-mod}})\simeq\mathbf{dgCat}_k. \tag{3.1.3.1}$$

Moreover, any object $\underline{\mathcal{A}\,\textbf{-mod}}$ of $\mathrm{Morita}(\mathbf{dgCat}_k)$ is dualizable, which dual given by $\underline{\mathcal{A}^{\mathrm{rev}}\,\textbf{-mod}}$. Indeed, the unit and counit of this duality are given by

$$\underline{\mathrm{Vect}_k\,\textbf{-mod}}\xrightarrow{\mathcal{A}}\underline{\mathcal{A}\otimes\mathcal{A}^{\mathrm{rev}}\,\textbf{-mod}}\xrightarrow{\mathcal{A}}\underline{\mathrm{Vect}_k\,\textbf{-mod}},$$

respectively. The dual 1-morphism to any $\mathcal{M}\colon\underline{\mathcal{A}\,\textbf{-mod}}\to\underline{\mathcal{B}\,\textbf{-mod}}$ is then given by the same bimodule, i.e., by $\mathcal{M}\colon\underline{\mathcal{B}^{\mathrm{rev}}\,\textbf{-mod}}\to\underline{\mathcal{A}^{\mathrm{rev}}\,\textbf{-mod}}$.

3.1.4. We denote[21] by $L(\mathrm{Morita}(\mathbf{dgCat}_k))_{\mathrm{rgd}}$ the following $(\infty,2)$-category:

(1) An object of $L(\mathrm{Morita}(\mathbf{dgCat}_k))_{\mathrm{rgd}}$ is a pair $(\underline{\mathcal{A}\,\textbf{-mod}},\mathcal{P})$, where
   (a) $\mathcal{A}$ is a monoidal dg-category; and
   (b) $\mathcal{P}$ is an $\mathcal{A}$-bimodule.

(2) Given objects $(\underline{\mathcal{A}\,\textbf{-mod}},\mathcal{P})$ and $(\underline{\mathcal{B}\,\textbf{-mod}},\mathcal{Q})$ of $L(\mathrm{Morita}(\mathbf{dgCat}_k))_{\mathrm{rgd}}$, a 1-morphism

$$(\underline{\mathcal{A}\,\textbf{-mod}},\mathcal{P})\to(\underline{\mathcal{B}\,\textbf{-mod}},\mathcal{Q})$$

in $L(\mathrm{Morita}(\mathbf{dgCat}_k))_{\mathrm{rgd}}$ is a pair $(\mathcal{M},\alpha)$, where
   (a) $\mathcal{M}$ is a right-dualizable $(\mathcal{B},\mathcal{A})$-bimodule $\mathcal{M}$; and
   (b) $\alpha\colon\mathcal{M}\otimes_{\mathcal{A}}\mathcal{P}\to\mathcal{Q}\otimes_{\mathcal{B}}\mathcal{M}$ is a $(\mathcal{B},\mathcal{A})$-linear functor.

The composition law is the evident extension of that for $\mathrm{Morita}(\mathbf{dgCat}_k)$, and the unit 1-morphism is given by the regular bimodule and canonical equivalence.

(3) Given 1-morphisms $(\mathcal{M},\alpha),(\mathcal{N},\beta)\colon(\underline{\mathcal{A}\,\textbf{-mod}},\mathcal{P})\to(\underline{\mathcal{B}\,\textbf{-mod}},\mathcal{Q})$, a 2-morphism

$$(\mathcal{M},\alpha)\Rightarrow(\mathcal{N},\beta)$$

is a pair (γ,θ), where

[21] But do not explain the general meaning of these notations; we refer the reader to *loc. cit.* for the details.

(a) $\gamma\colon \mathcal{M} \to \mathcal{N}$ is a $(\mathcal{B},\mathcal{A})$-linear functor admitting a continuous $(\mathcal{B},\mathcal{A})$-linear right adjoint; and
(b) $\theta\colon (\mathrm{id}_{\mathcal{Q}} \otimes \gamma) \circ \alpha \Rightarrow \beta \circ (\gamma \otimes \mathrm{id}_{\mathcal{P}})$ is a natural transformation of functors $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{P} \to \mathcal{Q} \otimes_{\mathcal{B}} \mathcal{N}$, which we represent diagramatically as

$$\begin{tikzcd}
\mathcal{M} \otimes_{\mathcal{A}} \mathcal{P} \arrow[r, "\alpha"] \arrow[d, "\gamma \otimes \mathrm{id}_{\mathcal{P}}"'] & \mathcal{Q} \otimes_{\mathcal{B}} \mathcal{M} \arrow[d, "\mathrm{id}_{\mathcal{Q}} \otimes \gamma"] \arrow[dl, Rightarrow, "\theta"'] \\
\mathcal{N} \otimes_{\mathcal{A}} \mathcal{P} \arrow[r, "\beta"] & \mathcal{Q} \otimes_{\mathcal{B}} \mathcal{N}.
\end{tikzcd}$$

The composition law is the evident extension of that for Morita($\mathbf{dgCat}_k$), and the unit 2-morphism is given by the identity functor and natural transformation.

Moreover, $L(\mathrm{Morita}(\mathbf{dgCat}_k))_{\mathrm{rgd}}$ carries a symmetric monoidal structure via

$$(\underline{\mathcal{A}\text{-}\mathbf{mod}}, \mathcal{P}) \otimes (\underline{\mathcal{B}\text{-}\mathbf{mod}}, \mathcal{Q}) := (\underline{\mathcal{A} \otimes \mathcal{B}\text{-}\mathbf{mod}}, \mathcal{P} \otimes \mathcal{Q}),$$

with unit object $(\underline{\mathrm{Vect}_k\text{-}\mathbf{mod}}, \mathrm{Vect}_k)$. There is an evident forgetful symmetric monoidal functor

$$L(\mathrm{Morita}(\mathbf{dgCat}_k))_{\mathrm{rgd}} \to \mathrm{Morita}(\mathbf{dgCat}_k).$$

3.1.5. The $(\infty,2)$-category $L(\mathrm{Morita}(\mathbf{dgCat}_k))_{\mathrm{rgd}}$ is equipped with a canonical symmetric monoidal functor of $(\infty,2)$-categories

(3.1.5.1) $$\mathrm{Tr}\colon L(\mathrm{Morita}(\mathbf{dgCat}_k))_{\mathrm{rgd}} \to \mathbf{dgCat}_k,$$

known as the 2-*categorical trace*. We recall (some of) its construction. Given an object $(\underline{\mathcal{A}\text{-}\mathbf{mod}}, \mathcal{P})$ of $L(\mathrm{Morita}(\mathbf{dgCat}_k))_{\mathrm{rgd}}$, we set $\mathrm{Tr}(\underline{\mathcal{A}\text{-}\mathbf{mod}}, \mathcal{P})$ to be the composition

$$\underline{\mathrm{Vect}_k\text{-}\mathbf{mod}} \xrightarrow{\mathcal{A}} \underline{\mathcal{A} \otimes \mathcal{A}^{\mathrm{rev}}\text{-}\mathbf{mod}} \xrightarrow{\mathcal{P} \otimes \mathcal{A}^{\mathrm{rev}}} \underline{\mathcal{A} \otimes \mathcal{A}^{\mathrm{rev}}\text{-}\mathbf{mod}} \xrightarrow{\mathcal{A}} \underline{\mathrm{Vect}_k\text{-}\mathbf{mod}}$$

in Morita($\mathbf{dgCat}_k$), that is,

$$\mathrm{Tr}(\underline{\mathcal{A}\text{-}\mathbf{mod}}, \mathcal{P}) := \mathcal{A} \underset{\mathcal{A} \otimes \mathcal{A}^{\mathrm{rev}}}{\otimes} \mathcal{P} \in \mathbf{dgCat}_k.$$

via (3.1.3.1). Thus, we are simply imitating the usual construction of "trace" in a symmetric monoidal category, one categorical level higher. As in the case of classical algebras, we adopt the shorthand notation $\mathrm{Tr}(\mathcal{A}, \mathcal{P}) := \mathrm{Tr}(\underline{\mathcal{A}\text{-}\mathbf{mod}}, \mathcal{P})$, and refer to it as the *categorical trace* (or *categorical Hochschild homology*) of $\mathcal{P}$.

Moreover, given a 1-morphism

$$(\mathcal{M}, \alpha)\colon (\underline{\mathcal{A}\text{-}\mathbf{mod}}, \mathcal{P}) \to (\underline{\mathcal{B}\text{-}\mathbf{mod}}, \mathcal{Q}),$$

the 1-morphism

$$\mathrm{Tr}(\mathcal{M}, \alpha)\colon\ \mathrm{Tr}(\mathcal{A}, \mathcal{P}) \to \mathrm{Tr}(\mathcal{B}, \mathcal{Q})$$

is given by the composition

(3.1.5.2) $$\begin{tikzcd}
\underline{\mathrm{Vect}_k\text{-}\mathbf{mod}} \arrow[r, "\mathcal{A}"] \arrow[d, "\mathrm{Vect}_k"] & \underline{\mathcal{A} \otimes \mathcal{A}^{\mathrm{rev}}\text{-}\mathbf{mod}} \arrow[r, "\mathcal{P} \otimes \mathcal{A}^{\mathrm{rev}}"] \arrow[d, "\mathcal{M} \otimes \mathcal{M}^R"] \arrow[dl, Rightarrow, "\epsilon_{\mathcal{M}}"'] & \underline{\mathcal{A} \otimes \mathcal{A}^{\mathrm{rev}}\text{-}\mathbf{mod}} \arrow[r, "\mathcal{A}"] \arrow[d, "\mathcal{M} \otimes \mathcal{M}^R"] \arrow[dl, Rightarrow, "\alpha"'] & \underline{\mathrm{Vect}_k\text{-}\mathbf{mod}} \arrow[d, "\mathrm{Vect}_k"] \arrow[dl, Rightarrow, "\eta_{\mathcal{M}}"'] \\
\underline{\mathrm{Vect}_k\text{-}\mathbf{mod}} \arrow[r, "\mathcal{B}"] & \underline{\mathcal{B} \otimes \mathcal{B}^{\mathrm{rev}}\text{-}\mathbf{mod}} \arrow[r, "\mathcal{Q} \otimes \mathcal{B}^{\mathrm{rev}}"] & \underline{\mathcal{B} \otimes \mathcal{B}^{\mathrm{rev}}\text{-}\mathbf{mod}} \arrow[r, "\mathcal{B}"] & \underline{\mathrm{Vect}_k\text{-}\mathbf{mod}},
\end{tikzcd}$$

of 2-morphisms in Morita($\mathbf{dgCat}_k$) (i.e., we obtain a 2-morphism from the outer clockwise circuit to the outer counterclockwise circuit). Here $\mathcal{M}^R$ denotes the right-dual to $\mathcal{M}$, and

(3.1.5.3) $$\eta_{\mathcal{M}}\colon \mathcal{A} \to \mathcal{M}^R \otimes_{\mathcal{B}} \mathcal{M}, \qquad \epsilon_{\mathcal{M}}\colon \mathcal{M} \otimes_{\mathcal{A}} \mathcal{M}^R \to \mathcal{B}$$

are the ($\mathcal{A}$-bilinear) unit and ($\mathcal{B}$-bilinear) counit of this duality, respectively. We refer the reader to [**GKRV22**, §3.9] for the description of the 2-categorical trace of a 2-morphism. Finally, we note that (3.1.5.1) satisfies the usual cyclicity properties for 1- and 2-morphisms; we refer the reader to [**GKRV22**, §3.1.4, §3.2.4] for the details.

3.1.6. Given $(\underline{\mathcal{A}\text{-}\mathbf{mod}}, \mathcal{P}) \in L(\mathrm{Morita}(\mathbf{dgCat}_k))_{\mathrm{rgd}}$, we define the $(\infty,1)$-category

$$(\mathcal{A}, \mathcal{P})\text{-}\mathbf{mod} := \mathbf{Map}_{L(\mathrm{Morita}(\mathbf{dgCat}_k))_{\mathrm{rgd}}}((\underline{\mathrm{Vect}_k\text{-}\mathbf{mod}}, \mathrm{Vect}_k), (\underline{\mathcal{A}\text{-}\mathbf{mod}}, \mathcal{P})).$$

The functor Tr of (3.1.5.1) then restricts to an $(\infty,1)$-functor

(3.1.6.1) $$[-]\colon (\mathcal{A}, \mathcal{P})\text{-}\mathbf{mod} \to \mathrm{Map}_{\mathbf{dgCat}_k}(\mathrm{Vect}_k, \mathrm{Tr}(\mathcal{A}, \mathcal{P})) \simeq \mathrm{Tr}(\mathcal{A}, \mathcal{P}),$$

which we term the 2-*categorical class map*. Let us spell this construction out in more detail.

As in §3.1.4, an object of $(\mathcal{A}, \mathcal{P})$**-mod** is the same as a right-dualizable (left) $\mathcal{A}$-module category $\mathcal{M}$ equipped with an $\mathcal{A}$-module homomorphism $\alpha\colon \mathcal{M} \to \mathcal{P} \otimes_{\mathcal{A}} \mathcal{M}$. We refer to the resulting object $[\mathcal{M}, \alpha] \in \mathrm{Tr}(\mathcal{A}, F_{\mathcal{A}})$ as the 2-*categorical class* of $(\mathcal{M}, \alpha)$. Moreover, given another such pair $(\mathcal{N}, \beta)$, a 1-morphism $(\mathcal{M}, \alpha) \to (\mathcal{N}, \beta)$ in $(\mathcal{A}, \mathcal{P})$**-mod** is the same as an $\mathcal{A}$-linear functor $\gamma\colon \mathcal{M} \to \mathcal{N}$ admitting an $\mathcal{A}$-linear right adjoint, and a natural transformation $\theta\colon (\mathrm{id}_{\mathcal{P}} \otimes \gamma) \circ \alpha \Rightarrow \beta \circ \gamma$ of functors $\mathcal{M} \to \mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}$. We denote the resulting morphism between 2-categorical classes by $[\gamma, \theta]\colon [\mathcal{M}, \alpha] \to [\mathcal{N}, \beta]$.

3.1.7. In particular, for a monoidal category $\mathcal{A}$ equipped with a monoidal endofunctor $F_{\mathcal{A}}$, we set

$$\mathrm{Tr}(\mathcal{A}, F_{\mathcal{A}}) := \mathrm{Tr}(\mathcal{A}, {}_{F_{\mathcal{A}}}\mathcal{A}) := \mathcal{A} \underset{\mathcal{A} \otimes \mathcal{A}^{\mathrm{rev}}}{\otimes} {}_{F_{\mathcal{A}}}\mathcal{A},$$

and refer to it simply as the *categorical trace* of $\mathcal{A}$ (with respect to the endofunctor $F_{\mathcal{A}}$). We will sometimes also write $(\mathcal{A}\text{-}\mathbf{mod}, F_{\mathcal{A}}) := (\mathcal{A}\text{-}\mathbf{mod}, {}_{F_{\mathcal{A}}}\mathcal{A})$. Objects of $(\mathcal{A}, F_{\mathcal{A}})$**-mod** are now pairs $(\mathcal{M}, F_{\mathcal{M}})$ as before; we refer to the $\mathcal{A}$-module homomorphism $F_{\mathcal{M}}\colon \mathcal{M} \to {}_{F_{\mathcal{A}}}\mathcal{M}$ as an $F_{\mathcal{A}}$-*semilinear* endofunctor. When $F_{\mathcal{A}}$ or $F_{\mathcal{M}}$ is the identity, we will often omit it from the notation.

Note that we also have a canonical "universal trace" functor

$$[-]\colon \mathcal{A} \to \mathcal{A} \underset{\mathcal{A} \otimes \mathcal{A}^{\mathrm{rev}}}{\otimes} {}_{F_{\mathcal{A}}}\mathcal{A} \tag{3.1.7.1}$$

given by sending $a \in \mathcal{A}$ to the image of $a \otimes \mathbf{1}_{\mathcal{A}}$, where the latter denotes the monoidal unit of $\mathcal{A}$. This functor factors through (3.1.6.1) as follows: given any $a \in \mathcal{A}$, we may define an $F_{\mathcal{A}}$-semilinear endofunctor $F_{\mathcal{A},a}(-) := F_{\mathcal{A}}(-) \otimes a$ of the regular $\mathcal{A}$-module $\mathcal{A}$. It is then not hard to see that $[a] = [\mathcal{A}, F_{\mathcal{A},a}]$ via (3.1.7.1). In particular, the trace of the monoidal unit agrees with that of the regular representation, that is, $[\mathbf{1}_{\mathcal{A}}] = [\mathcal{A}, F_{\mathcal{A}}]$.

3.1.8. Recall that a monoidal functor $\Psi\colon \mathcal{A} \to \mathcal{B}$ is *rigid* if

(1) the right-adjoint Ψ^R is continuous and $\mathcal{A}$-bilinear; and
(2) the multiplication map $\mathrm{mult}_{\mathcal{B}}\colon \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \to \mathcal{B}$ admits a continuous, $\mathcal{B}$-bilinear right adjoint.

In particular, a monoidal category $\mathcal{A}$ is rigid if the unit functor $\mathrm{unit}_{\mathcal{A}}\colon \mathrm{Vect}_k \to \mathcal{A}$ is rigid. If $\mathcal{A}$ is compactly generated, then by [**GR17a**, Lem. 9.1.5], this is equivalent to requiring

(1) the unit $\mathbf{1}_{\mathcal{A}}$ is compact;
(2) the multiplication $\mathrm{mult}_{\mathcal{A}}\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ preserves compact objects; and
(3) every compact object of $\mathcal{A}$ admits a left and right monoidal dual.

Thus, in this case, we recover a more traditional notion of rigidity. Finally, note that *any* monoidal functor between rigid monoidal categories is itself rigid.[22].

3.1.9. The notion of rigidity also has an interpretation in terms of dualizability of *categories*. Given any monoidal functor $\Psi\colon \mathcal{A} \to \mathcal{B}$, the $(\mathcal{B}, \mathcal{A})$-bimodule $\mathrm{Ind}_{\Psi} := \mathcal{B}_{\Psi}$ is right-dualizable. Specifically, its right-dual is given by the analogous $(\mathcal{A}, \mathcal{B})$-bimodule $\mathrm{Res}_{\Psi} := {}_{\Psi}\mathcal{B}$, with the unit given by

$$\mathcal{A} \xrightarrow{\Psi} {}_{\Psi}\mathcal{B}_{\Psi} \simeq \mathrm{Res}_{\Psi} \otimes_{\mathcal{B}} \mathrm{Ind}_{\Psi} \tag{3.1.9.1}$$

and the counit given by the multiplication map

$$\mathrm{mult}_{\mathcal{B}}\colon \mathrm{Ind}_{\Psi} \otimes_{\mathcal{A}} \mathrm{Res}_{\Psi} \to \mathcal{B}. \tag{3.1.9.2}$$

If Ψ is rigid, then Res_{Ψ} is moreover left-dual to Ind_{Ψ}, via the right-adjoints to (3.1.9.1) and (3.1.9.2).

3.1.10. Given a dualizable dg-category $\mathcal{C}$ with an endofunctor $F_{\mathcal{C}}$, the *Hochschild homology* $\mathrm{HH}(\mathcal{C}, F_{\mathcal{C}})$ is the 2-categorical class $[\mathcal{C}, F_{\mathcal{C}}] \in \mathrm{Tr}(\mathrm{Vect}_k) \in \mathrm{Vect}_k$. Equivalently, by (3.1.5.2), it is the composition

$$\mathrm{Vect}_k \xrightarrow{\eta_{\mathcal{C}}} \mathcal{C} \otimes \mathcal{C}^{\vee} \xrightarrow{F_{\mathcal{C}} \otimes \mathrm{id}_{\mathcal{C}^{\vee}}} \mathcal{C} \otimes \mathcal{C}^{\vee} \xrightarrow{\epsilon_{\mathcal{C}}} \mathrm{Vect}_k, \tag{3.1.10.1}$$

where $\eta_{\mathcal{C}}, \epsilon_{\mathcal{C}}$ are as in (3.1.5.3). Moreover, its functoriality in the pair $(\mathcal{C}, F_{\mathcal{C}})$ is as in §3.1.6. This has the following consequences:

[22]See [**GKRV22**, §3.10.3]. In more detail, this follows from [**GR17a**, Ch. 1, Lem. 9.2.6(b)], and by imitating the proof of [**GR17a**, Ch. 1, Prop. 8.7.2].

(1) For any compact object $c \in \mathcal{C}^c$ equipped with a morphism $\theta\colon c \to F_{\mathcal{C}}(c)$, we obtain a map

$$[c,\theta]\colon k \simeq [\mathrm{Vect}_k, \mathrm{id}_{\mathrm{Vect}_k}] \to [\mathcal{C}, F_{\mathcal{C}}]$$

upon interpreting c as a functor $\mathrm{Vect}_k \to \mathcal{C}$ and θ as a natural transformation. We likewise denote the image of the unit under this map by $[c,\theta] \in \mathrm{HH}(\mathcal{C}, F_{\mathcal{C}})$, and refer to this element as the *(Hochschild) class* of (c,θ) as in (3.1.6.1). In particular, when $F_{\mathcal{C}}$ is the identity, we write $[c] := [c, \mathrm{id}_c]$.

(2) Let $\mathcal{A}$ be a monoidal category equipped with a monoidal endofunctor $F_{\mathcal{A}}$, and suppose that $\mathrm{unit}_{\mathcal{A}}$, $\mathrm{mult}_{\mathcal{A}}$, and $F_{\mathcal{A}}$ all admit continuous right adjoints (for instance, this holds if $\mathcal{A}$ is compactly generated and all preserve $\mathcal{A}^c$). Then the Hochschild homology $\mathrm{HH}(\mathcal{A}, F_{\mathcal{A}})$ inherits an algebra structure with unit $[\mathbf{1}_{\mathcal{A}}]$ (omitting the unit isomorphism of $F_{\mathcal{A}}$). In particular, this holds when $\mathcal{A}$ is rigid, in which case $F_{\mathcal{A}}$ is itself automatically rigid as in §3.1.8.

We now state the main result of [**GKRV22**, §3], and one of our primary technical tools:

3.1.11. Theorem ([**GKRV22**, Thm. 3.8.5]). *Let $\mathcal{A}$ be a rigid monoidal category equipped with a monoidal endofunctor $F_{\mathcal{A}}$. Then there is an equivalence of algebras*[23]

$$\mathrm{HH}(\mathcal{A}, F_{\mathcal{A}}) \simeq \mathrm{End}_{\mathrm{Tr}(\mathcal{A}, F_{\mathcal{A}})}([\mathcal{A}, F_{\mathcal{A}}])^{\mathrm{op}}, \tag{3.1.11.1}$$

which extends to an equivalence of functors

$$\mathrm{HH}(-) \simeq \mathrm{Hom}_{\mathrm{Tr}(\mathcal{A}, F_{\mathcal{A}})}([\mathcal{A}, F_{\mathcal{A}}], [-]) : (\mathcal{A}, F_{\mathcal{A}})\text{-}\mathbf{mod} \to \mathrm{HH}(\mathcal{A}, F_{\mathcal{A}})\text{-mod}. \tag{3.1.11.2}$$

In particular, if $[\mathcal{A}, F_{\mathcal{A}}]$ is compact, then the left adjoint to $\mathrm{Hom}_{\mathrm{Tr}(\mathcal{A})}([\mathcal{A}, F_{\mathcal{A}}], -)$ defines a fully-faithful embedding which preserves compact objects, and whose essential image is the category generated by $[\mathcal{A}, F_{\mathcal{A}}]$:

$$\begin{tikzcd} \mathrm{HH}(\mathcal{A}, F_{\mathcal{A}})\text{-mod} \arrow[rr, hook, shift left, "{[\mathcal{A},F_{\mathcal{A}}]\otimes_{\mathrm{End}([\mathcal{A},F_{\mathcal{A}}])} -}"] \arrow[dr, "\simeq"'] & & \mathrm{Tr}(\mathcal{A}, F_{\mathcal{A}}) \arrow[ll, shift left, "{\mathrm{Hom}([\mathcal{A},F_{\mathcal{A}}],-)}"] \arrow[dl, shift left, "\mathrm{pr}_{[\mathcal{A},F_{\mathcal{A}}]}"] \\ & \langle [\mathcal{A}, F_{\mathcal{A}}] \rangle . \arrow[ur, hook, shift left, "\mathrm{in}_{[\mathcal{A},F_{\mathcal{A}}]}"] & \end{tikzcd} \tag{3.1.11.3}$$

3.1.12. Note that if $\mathcal{A}$ is compactly generated, then by [**GR17a**, Ch. 1, Cor. 8.7.4], the category $\mathrm{Tr}(\mathcal{A}, F_{\mathcal{A}})$ is compactly generated by objects of the form $[a]$, where $a \in \mathcal{A}^c$. In particular, the object $[\mathcal{A}, F_{\mathcal{A}}]$ is compact, and the right adjoint $\mathrm{pr}_{[\mathcal{A},F_{\mathcal{A}}]}$ is continuous.

3.1.13. Finally, we give two methods for constructing adjunctions between monoidal traces, following the approach of [**GKRV22**, §3.10.4].

3.1.14. We begin by discussing the induction-restriction adjunction. Let $(\mathcal{A}, F_{\mathcal{A}})$ and $(\mathcal{B}, F_{\mathcal{B}})$ be as in §3.1.7, and suppose that we are given a rigid monoidal functor $\Psi\colon \mathcal{A} \to \mathcal{B}$, equipped with an isomorphism $\Psi \circ F_{\mathcal{A}} \simeq F_{\mathcal{B}} \circ \Psi$. Then by §3.1.9, we have 1-morphisms

$$(\mathrm{Ind}_{\Psi}, F_{\mathcal{B}} \otimes \mathrm{id}_{\mathcal{A}})\colon (\underline{\mathcal{A}\text{-}\mathbf{mod}}, {}_{F_{\mathcal{A}}}\mathcal{A}) \rightleftarrows (\underline{\mathcal{B}\text{-}\mathbf{mod}}, {}_{F_{\mathcal{B}}}\mathcal{B})\colon (\mathrm{Res}_{\Psi}, \mathrm{id}_{\mathcal{B}}) \tag{3.1.14.1}$$

in $L(\mathrm{Morita}(\mathbf{dgCat}_k))_{\mathrm{rgd}}$. More precisely, these functors are given by the compositions

$$\mathcal{B}_{\Psi} \otimes_{\mathcal{A}} {}_{F_{\mathcal{A}}}\mathcal{A} \xrightarrow{F_{\mathcal{B}} \otimes \mathrm{id}_{\mathcal{A}}} {}_{F_{\mathcal{B}}}\mathcal{B}_{F_{\mathcal{B}} \circ \Psi} \otimes_{\mathcal{A}} {}_{F_{\mathcal{A}}}\mathcal{A} \simeq {}_{F_{\mathcal{B}}}\mathcal{B}_{\Psi \circ F_{\mathcal{A}}} \otimes_{\mathcal{A}} {}_{F_{\mathcal{A}}}\mathcal{A} \to {}_{F_{\mathcal{B}}}\mathcal{B}_{\Psi} \otimes_{\mathcal{A}} \mathcal{A} \simeq {}_{F_{\mathcal{B}}}\mathcal{B}_{\Psi},$$
$${}_{\Psi}\mathcal{B} \otimes_{\mathcal{B}} {}_{F_{\mathcal{B}}}\mathcal{B} \simeq {}_{F_{\mathcal{B}} \circ \Psi}\mathcal{B} \simeq {}_{\Psi \circ F_{\mathcal{A}}}\mathcal{B},$$

[23] We comment on the presence of the opposite algebra, which does not appear in the original formulation of [**GKRV22**]. The algebra isomorphism (3.1.11.1) arises from the adjunction (3.1.14.3) as follows: on the one hand, the monad $\mathrm{Tr}(\mathrm{Res}_{\mathrm{unit}_{\mathcal{A}}} \circ \mathrm{Ind}_{\mathrm{unit}_{\mathcal{A}}}) \in \mathrm{Alg}(\mathrm{Vect}_k)$ identifies with $\mathrm{HH}(\mathcal{A}, F_{\mathcal{A}})$ as in §3.1.10. On the other hand, by functoriality of (3.1.5.1), this monad may equivalently be expressed as $\mathrm{Tr}(\mathrm{Res}_{\mathrm{unit}_{\mathcal{A}}}) \circ \mathrm{Tr}(\mathrm{Ind}_{\mathrm{unit}_{\mathcal{A}}})$. Note that by definition, the functor $\mathrm{Tr}(\mathrm{Ind}_{\mathrm{unit}_{\mathcal{A}}})$ is given by the Vect_k-action on $[\mathcal{A}, F_{\mathcal{A}}]$, and hence its right-adjoint $\mathrm{Tr}(\mathrm{Res}_{\mathrm{unit}_{\mathcal{A}}})$ is given by $\mathrm{Hom}_{\mathrm{Tr}(\mathcal{A}, F_{\mathcal{A}})}([\mathcal{A}, F_{\mathcal{A}}], -)$. It is now straightforward to verify that the algebra structure on

$$\mathrm{Tr}(\mathrm{Res}_{\mathrm{unit}_{\mathcal{A}}}) \circ \mathrm{Tr}(\mathrm{Ind}_{\mathrm{unit}_{\mathcal{A}}}) \simeq \mathrm{Hom}_{\mathrm{Tr}(\mathcal{A}, F_{\mathcal{A}})}([\mathcal{A}, F_{\mathcal{A}}], [\mathcal{A}, F_{\mathcal{A}}])$$

is the opposite one.

respectively. We now aim to endow (3.1.14.1) with the structure of an adjunction in $L(\mathrm{Morita}(\mathbf{dgCat}_k))_{\mathrm{rgd}}$, i.e., to construct unit and counit 2-morphisms

$$\begin{gathered} \mathrm{id}_{(\underline{\mathcal{A}\text{-}\mathbf{mod}}, {}_{F_{\mathcal{A}}}\mathcal{A})} \Rightarrow (\mathrm{Res}_\Psi, \mathrm{id}_{\mathcal{B}}) \circ (\mathrm{Ind}_\Psi, F_{\mathcal{B}} \otimes \mathrm{id}_{\mathcal{A}}), \\ (\mathrm{Ind}_\Psi, F_{\mathcal{B}} \otimes \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{Res}_\Psi, \mathrm{id}_{\mathcal{B}}) \Rightarrow \mathrm{id}_{(\underline{\mathcal{B}\text{-}\mathbf{mod}}, {}_{F_{\mathcal{B}}}\mathcal{B})}, \end{gathered} \tag{3.1.14.2}$$

satisfying the usual identities. By rigidity, the functors (3.1.9.1) and (3.1.9.2) admit continuous bilinear right adjoints. Moreover, we take the natural isomorphisms exhibiting commutativity of the evident diagrams

$$\begin{tikzcd} \mathcal{A} \otimes_{\mathcal{A}} {}_{F_{\mathcal{A}}}\mathcal{A} \arrow[rr, "\sim"] \arrow[d, "\Psi\otimes\mathrm{id}_{\mathcal{A}}"] & & {}_{F_{\mathcal{A}}}\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} \arrow[d, "\mathrm{id}_{\mathcal{A}}\otimes\Psi"] \\ {}_\Psi\mathcal{B}_\Psi \otimes_{\mathcal{A}} {}_{F_{\mathcal{A}}}\mathcal{A} \arrow[r, "F_{\mathcal{B}}\otimes\mathrm{id}_{\mathcal{A}}"] & {}_{F_{\mathcal{B}}\circ\Psi}\mathcal{B}_\Psi \arrow[r, "\sim"] & {}_{F_{\mathcal{A}}}\mathcal{A} \otimes_{\mathcal{A}} {}_\Psi\mathcal{B}_\Psi, \end{tikzcd}$$

$$\begin{tikzcd} (\mathcal{B}_\Psi \otimes_{\mathcal{A}} {}_\Psi\mathcal{B}) \otimes_{\mathcal{B}} {}_{F_{\mathcal{B}}}\mathcal{B} \arrow[r, "\sim"] \arrow[d, "\mathrm{mult}_{\mathcal{B}}\otimes\mathrm{id}_{\mathcal{B}}"] & \mathcal{B}_\Psi \otimes_{\mathcal{A}} {}_{\Psi\circ F_{\mathcal{A}}}\mathcal{B} \arrow[r, "F_{\mathcal{B}}\otimes\mathrm{id}_{\mathcal{B}}"] & {}_{F_{\mathcal{B}}}\mathcal{B} \otimes_{\mathcal{B}} (\mathcal{B}_\Psi \otimes_{\mathcal{A}} {}_\Psi\mathcal{B}) \arrow[d, "\mathrm{id}_{\mathcal{B}}\otimes\mathrm{mult}_{\mathcal{B}}"] \\ \mathcal{B} \otimes_{\mathcal{B}} {}_{F_{\mathcal{B}}}\mathcal{B} \arrow[rr, "\sim"] & & {}_{F_{\mathcal{B}}}\mathcal{B} \otimes_{\mathcal{B}} \mathcal{B}, \end{tikzcd}$$

or equivalently, the identities

$$\begin{gathered} F_{\mathcal{B}}(\Psi(a))\Psi(a') \simeq \Psi(F_{\mathcal{A}}(a)a'), \\ F_{\mathcal{B}}(bb')b'' \simeq F_{\mathcal{B}}(b)(F_{\mathcal{B}}(b')b'') \end{gathered}$$

for $a, a' \in \mathcal{A}$ and $b, b', b'' \in \mathcal{B}$, respectively. It is now straightforward to verify that the resulting 2-morphisms (3.1.14.2) yield an adjunction. Applying the functor (3.1.5.1) then yields an adjunction

$$\mathrm{Tr}(\mathrm{Ind}_\Psi, F_{\mathcal{B}} \otimes \mathrm{id}_{\mathcal{A}})\colon\ \mathrm{Tr}(\mathcal{A}, F_{\mathcal{A}}) \rightleftarrows \mathrm{Tr}(\mathcal{B}, F_{\mathcal{B}})\colon\ \mathrm{Tr}(\mathrm{Res}_\Psi, \mathrm{id}_{\mathcal{B}}), \tag{3.1.14.3}$$

as desired. We henceforth misuse notation slightly and denote the 1-morphisms of (3.1.14.1) simply by Ind_Ψ and Res_Ψ, respectively; the functors of (3.1.14.3) are then denoted by $\mathrm{Tr}(\mathrm{Ind}_\Psi)$ and $\mathrm{Tr}(\mathrm{Res}_\Psi)$ as in [**GKRV22**, (3.30)].

For future reference, we note the following explicit description of $\mathrm{Tr}(\mathrm{Ind}_\Psi)$ (which holds even if Ψ is not necessarily rigid).

3.1.15. Lemma. The functor

$$\mathrm{Tr}(\mathrm{Ind}_\Psi)\colon\ \mathrm{Tr}(\mathcal{A}, F_{\mathcal{A}}) \to \mathrm{Tr}(\mathcal{B}, F_{\mathcal{B}})$$

is given by the composition

$$\mathcal{A} \underset{\mathcal{A}\otimes\mathcal{A}^{\mathrm{rev}}}{\otimes} {}_{F_{\mathcal{A}}}\mathcal{A} \xrightarrow{\Psi\otimes\Psi} {}_\Psi\mathcal{B}_\Psi \underset{\mathcal{A}\otimes\mathcal{A}^{\mathrm{rev}}}{\otimes} {}_{F_{\mathcal{B}}\circ\Psi}\mathcal{B}_\Psi \to \mathcal{B}_\Psi \underset{\mathcal{B}\otimes\mathcal{B}^{\mathrm{rev}}}{\otimes} {}_{F_{\mathcal{B}}}\mathcal{B}.$$

Proof. It follows directly from the definitions (using the duality of §3.1.9) that $\mathrm{Tr}(\mathrm{Ind}_\Psi)$ is given by the composition

$$\begin{aligned} \mathcal{A} \underset{\mathcal{A}\otimes\mathcal{A}^{\mathrm{rev}}}{\otimes} {}_{F_{\mathcal{A}}}\mathcal{A} &\xrightarrow{\Psi\otimes\mathrm{id}_{\mathcal{A}}} {}_\Psi\mathcal{B}_\Psi \underset{\mathcal{A}\otimes\mathcal{A}^{\mathrm{rev}}}{\otimes} {}_{F_{\mathcal{A}}}\mathcal{A} \\ &\xrightarrow{\sim} \mathcal{B} \underset{\mathcal{B}\otimes\mathcal{B}^{\mathrm{rev}}}{\otimes} ({}_\Psi\mathcal{B} \otimes \mathcal{B}_\Psi) \underset{\mathcal{A}\otimes\mathcal{A}^{\mathrm{rev}}}{\otimes} {}_{F_{\mathcal{A}}}\mathcal{A} \\ &\xrightarrow{\sim} \mathcal{B} \underset{\mathcal{B}\otimes\mathcal{B}^{\mathrm{rev}}}{\otimes} (\mathcal{B}_\Psi \underset{\mathcal{A}}{\otimes} {}_{\Psi\circ F_{\mathcal{A}}}\mathcal{B}) \\ &\xrightarrow{\mathrm{id}_{\mathcal{B}}\otimes(F_{\mathcal{B}}\otimes\mathrm{id}_{\mathcal{B}})} \mathcal{B} \underset{\mathcal{B}\otimes\mathcal{B}^{\mathrm{rev}}}{\otimes} ({}_{F_{\mathcal{B}}}\mathcal{B}_{F_{\mathcal{B}}\circ\Psi} \underset{\mathcal{A}}{\otimes} {}_{F_{\mathcal{B}}\circ\Psi}\mathcal{B}) \\ &\xrightarrow{\mathrm{id}_{\mathcal{B}}\otimes\mathrm{mult}_{\mathcal{B}}} \mathcal{B} \underset{\mathcal{B}\otimes\mathcal{B}^{\mathrm{rev}}}{\otimes} {}_{F_{\mathcal{B}}}\mathcal{B}. \end{aligned}$$

Tracing the constructions immediately yields the result. ☐

3.1.16. Next, we discuss adjunctions arising from dualizable objects of the Drinfeld center. Given a monoidal category $\mathcal{B}$, recall that the *Drinfeld center* of $\mathcal{B}$ is the dg-category

$$Z(\mathcal{B}) := \mathrm{Hom}_{\mathcal{B}\otimes\mathcal{B}^{\mathrm{rev}}}(\mathcal{B}, \mathcal{B}),$$

which carries a natural E_2-monoidal structure via composition, and is equipped with a universal central functor

$$Z(\mathcal{B}) \to \mathcal{B} \tag{3.1.16.1}$$

given by evaluation at $\mathbf{1}_{\mathcal{B}}$. In particular, for any $\mathcal{B}$-bimodule $\mathcal{Q}$, there is a natural $Z(\mathcal{B})$-module structure on $\operatorname{Tr}(\mathcal{B}, \mathcal{Q})$. It follows that for any right-dualizable (equivalently, left-dualizable) object $b \in Z(\mathcal{B})$, we have a natural adjunction

$$b \otimes -\colon \operatorname{Tr}(\mathcal{B}, \mathcal{Q}) \rightleftarrows \operatorname{Tr}(\mathcal{B}, \mathcal{Q})\colon b^{\vee,R} \otimes -. \tag{3.1.16.2}$$

In particular, suppose we are given a monoidal functor $\Psi\colon \mathcal{A} \to \mathcal{B}$ which admits a central structure, i.e., a factorization of Ψ through (3.1.16.1). Then for any $a \in \mathcal{A}$, we have a $\mathcal{B}$-bilinear functor $- \otimes \Psi(a)\colon \mathcal{Q} \to \mathcal{Q}$, and hence a 1-morphism

$$(\mathcal{B}, - \otimes \Psi(a))\colon (\underline{\mathcal{B}\text{-}\mathbf{mod}}, \mathcal{Q}) \to (\underline{\mathcal{B}\text{-}\mathbf{mod}}, \mathcal{Q})$$

in $L(\mathrm{Morita}(\mathbf{dgCat}_k))$. Moreover, if a is left-dualizable[24], applying Ψ to the unit and counit maps in $\mathcal{A}$ yields 2-morphisms

$$(\mathcal{B}, - \otimes \Psi(a)) \circ (\mathcal{B}, - \otimes \Psi(a^{\vee,L})) \simeq (\mathcal{B}, - \otimes \Psi(a^{\vee,L} \otimes a)) \Rightarrow (\mathcal{B}, - \otimes \Psi(\mathbf{1}_{\mathcal{A}})) \simeq (\mathcal{B}, \mathrm{id}_{\mathcal{Q}}),$$
$$(\mathcal{B}, \mathrm{id}_{\mathcal{Q}}) \simeq (\mathcal{B}, - \otimes \Psi(\mathbf{1}_{\mathcal{A}})) \Rightarrow (\mathcal{B}, - \otimes \Psi(a \otimes a^{\vee,L})) \simeq (\mathcal{B}, - \otimes \Psi(a^{\vee,L})) \circ (\mathcal{B}, - \otimes \Psi(a))$$

in $L(\mathrm{Morita}(\mathbf{dgCat}_k))$ satisfying the same identities, hence an adjunction

$$(\mathcal{B}, - \otimes \Psi(a))\colon (\underline{\mathcal{B}\text{-}\mathbf{mod}}, \mathcal{Q}) \rightleftarrows (\underline{\mathcal{B}\text{-}\mathbf{mod}}, \mathcal{Q})\colon (\mathcal{B}, - \otimes \Psi(a^{\vee,L})) \tag{3.1.16.3}$$

in $L(\mathrm{Morita}(\mathbf{dgCat}_k))$. Applying the functor (3.1.5.1) then recovers the adjunction (3.1.16.2) (using the E_2-monoidal structure on $Z(\mathcal{B})$). Finally, note that given any $(\mathcal{N}, \beta) \in (\mathcal{B}, \mathcal{Q})\text{-}\mathbf{mod}$, we have

$$(\mathcal{B}, - \otimes \Psi(a)) \circ (\mathcal{N}, \beta) \simeq (\mathcal{N}, \Psi(a) \otimes \beta(-)) \in (\mathcal{B}, \mathcal{Q})\text{-}\mathbf{mod},$$

so we obtain

$$\operatorname{Tr}(\mathcal{B}, - \otimes \Psi(a))([\mathcal{N}, \beta]) \simeq [\mathcal{N}, \Psi(a) \otimes \beta(-)] \tag{3.1.16.4}$$

on 2-categorical classes.

3.2. Traces of categories of quasicoherent sheaves

3.2.1. In this section, we recall some facts about loop spaces of stacks and traces of categories of quasicoherent sheaves that will be needed in the sequel.

3.2.2. Definition. Let $\mathcal{X}$ be a (derived) stack equipped with a self-map $\phi_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$. The *$\phi_{\mathcal{X}}$-twisted loop space* (or *derived $\phi_{\mathcal{X}}$-fixed points*) of $\mathcal{X}$ is given by the fiber product

$$\begin{tikzcd} \mathcal{L}_\phi \mathcal{X} \arrow[r] \arrow[d, "\mathrm{ev}"] & \mathcal{X} \arrow[d, "\Gamma_{\phi_{\mathcal{X}}}"] \\ \mathcal{X} \arrow[r, "\Delta_{\mathcal{X}}"] & \mathcal{X} \times \mathcal{X}, \end{tikzcd}$$

where $\Delta_{\mathcal{X}}$ denotes the diagonal morphism, and $\Gamma_{\phi_{\mathcal{X}}} = (\phi_{\mathcal{X}}, \mathrm{id}_{\mathcal{X}})$ denotes the graph of $\phi_{\mathcal{X}}$. We refer to the morphism ev as the *loop evaluation*. When $\phi_{\mathcal{X}} = \mathrm{id}_{\mathcal{X}}$, we write $\mathcal{L}\mathcal{X} := \mathcal{L}_{\phi_{\mathcal{X}}}\mathcal{X}$, and refer to it simply as the *loop space* of $\mathcal{X}$.

3.2.3. Remark. Alternatively, we have $\mathcal{L}\mathcal{X} \simeq \mathrm{Map}(S^1, \mathcal{X})$, the derived mapping stack from the circle, which thus carries a natural S^1-action. We shall not need this in the present work, however. Note that in this thesis, we reserve the notation $\mathcal{X}^{\phi_{\mathcal{X}}}$ for schemes, in which case it denotes the *classical* fixed points $(\mathcal{L}_{\phi_{\mathcal{X}}}\mathcal{X})^{\mathrm{cl}}$.

3.2.4. The formation of twisted loop spaces is functorial in the pair $(\mathcal{X}, \phi_{\mathcal{X}})$, and commutes with fiber products. More precisely, if $\mathcal{Y}$ is another stack equipped with a self-map $\phi_{\mathcal{Y}}\colon \mathcal{Y} \to \mathcal{Y}$, and $p\colon \mathcal{X} \to \mathcal{Y}$ is a morphism intertwining these self-maps, that is, $p \circ \phi_{\mathcal{X}} \simeq \phi_{\mathcal{Y}} \circ p$, then we have an induced morphism $\mathcal{L}p\colon \mathcal{L}_{\phi_{\mathcal{X}}}\mathcal{X} \to \mathcal{L}_{\phi_{\mathcal{Y}}}\mathcal{Y}$.

[24] A similar construction applies using the left-action of $\Psi(a)$ on $\mathcal{Q}$, in which case it is more natural to take a to be right-dualizable. However, the right-action on $\mathcal{Q}$ agrees more naturally with the left-action on $\mathcal{B}$ in the tensor product describing $\operatorname{Tr}(\mathcal{B}, \mathcal{Q})$. In either case, the two constructions are related by the E_2-monoidal structure on $Z(\mathcal{B})$.

3.2.5. When $\mathcal{X} = X/G$ is a quotient stack, for X a (derived) scheme and G a linear algebraic group, its loop space admits a more explicit description. Suppose that $\phi_{\mathcal{X}}$ commutes with the natural projection $\mathcal{X} \to \mathrm{B}G$, i.e., that $\phi_{\mathcal{X}}$ lifts to an endomorphism $\phi_X\colon X \to X$ commuting with the G-action. Then by [**Che18**, Prop. 3.1.6], we have a Cartesian square

$$\begin{tikzcd} \mathcal{L}_\phi(X/G) \arrow[r, "\mathrm{ev}_G"] \arrow[d, "\mathrm{ev}"] & (X \times G)/G \arrow[d, "(\phi_X \circ \mathrm{act}, \mathrm{pr})"] \\ X/G \arrow[r, "\Delta_X"] & (X \times X)/G, \end{tikzcd} \tag{3.2.5.1}$$

where $\mathrm{act}, \mathrm{pr}\colon X \times G \to X$ denote the action and projection maps, respectively, and G acts diagonally on $X \times X$ and $X \times G$ (via the adjoint action on G). In particular, at the level of k-points, we have

$$\mathcal{L}_\phi(X/G)(k) \cong \{(x,g) \in X(k) \times G(k) : \phi_X(g \cdot x) = x\}/G(k).$$

We now record two lemmas translating properties of stacks into properties of their twisted loop spaces:

3.2.6. Lemma. Suppose we are in the setup of §3.2.4.

(1) If p is a closed immersion (resp. proper), then so is $\mathcal{L}p$.
(2) If p is smooth, then $\mathcal{L}p$ is quasi-smooth.

Proof. (1) Both statements are immediate from the factorization

$$\mathcal{L}_{\phi_{\mathcal{X}}}\mathcal{X} \simeq \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X} \xrightarrow{\Delta_{\mathcal{X}/\mathcal{Y}} \times \mathrm{id}_{\mathcal{X}}} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X} \simeq \mathcal{X} \times_{\mathcal{Y}\times\mathcal{Y}} \mathcal{Y} \xrightarrow{p \times \mathrm{id}_{\mathcal{Y}}} \mathcal{Y} \times_{\mathcal{Y}\times\mathcal{Y}} \mathcal{Y} \simeq \mathcal{L}_{\phi_{\mathcal{Y}}}\mathcal{Y}, \tag{3.2.6.1}$$

of $\mathcal{L}p$, as noted in [**BZN21**, Rem. 4.6] (see also [**BZCHN22**, Lem. 3.10]). Here $\Delta_{\mathcal{X}/\mathcal{Y}}\colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ denotes the relative diagonal, and the latter maps $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ and $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ are given by $\Gamma_{\phi_{\mathcal{X}}}$ and $\Gamma_{\phi_{\mathcal{Y}}}$, respectively.

(2) Suppose that p is smooth, i.e., the cotangent complex $\mathbb{L}_p$ is perfect of Tor-amplitude $[0,1]$. The factorization (3.2.6.1) yields an exact triangle

$$(\Delta_{\mathcal{X}/\mathcal{Y}} \times \mathrm{id}_{\mathcal{X}})^* \mathbb{L}_{p \times \mathrm{id}_{\mathcal{Y}}} \to \mathbb{L}_{\mathcal{L}p} \to \mathbb{L}_{\Delta_{\mathcal{X}/\mathcal{Y}} \times \mathrm{id}_{\mathcal{X}}}.$$

Since $\mathbb{L}_{p \times \mathrm{id}_{\mathcal{Y}}} \simeq \mathrm{pr}_1^* \mathbb{L}_p$ and $\mathbb{L}_{\Delta_{\mathcal{X}/\mathcal{Y}} \times \mathrm{id}_{\mathcal{X}}} \simeq \mathrm{pr}_1^* \mathbb{L}_{\Delta_{\mathcal{X}/\mathcal{Y}}}$ by base-change (where we have let $\mathrm{pr}_1, \mathrm{pr}_2$ denote the respective projections), we reduce to showing that $\mathbb{L}_{\Delta_{\mathcal{X}/\mathcal{Y}}}$ is perfect of Tor-amplitude $[-1,1]$. The exact triangle associated to the composition

$$\mathcal{X} \xrightarrow{\Delta_{\mathcal{X}/\mathcal{Y}}} \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \xrightarrow{\mathrm{pr}_2} \mathcal{X}$$

now yields isomorphisms

$$\mathbb{L}_{\Delta_{\mathcal{X}/\mathcal{Y}}} \simeq \Delta_{\mathcal{X}/\mathcal{Y}}^* \mathbb{L}_{\mathrm{pr}_2}[1] \simeq \Delta_{\mathcal{X}/\mathcal{Y}}^* \mathrm{pr}_1^* \mathbb{L}_p[1] \simeq \mathbb{L}_p[1],$$

and the conclusion follows. □

3.2.7. Lemma. Let X be a derived scheme equipped with an action of a linear algebraic group G and with a self-map $\phi\colon X \to X$ commuting with the G-action. Let $i\colon Z \hookrightarrow X \hookleftarrow U\colon j$ be a complementary closed and open immersion, respectively, and suppose that Z and U are both G-stable and ϕ-stable. Then

$$\mathcal{L}i\colon \mathcal{L}_\phi(Z/G) \to \mathcal{L}_\phi(X/G) \leftarrow \mathcal{L}_\phi(U/G)\colon \mathcal{L}j \tag{3.2.7.1}$$

are a complementary closed and open immersion, respectively.

Proof. The map $\mathcal{L}i$ is a closed immersion by Lemma 3.2.6(1). Alternatively, recall that the property of being a closed immersion depends only on the underlying classical stacks. The diagram (3.2.5.1) implies that $\mathcal{L}_\phi(Z/G)^{\mathrm{cl}}$ is computed by the classical fiber product

$$Z^{\mathrm{cl}}/G \times_{X^{\mathrm{cl}}/G} \mathcal{L}_\phi(X/G)^{\mathrm{cl}}, \tag{3.2.7.2}$$

from which the conclusion is immediate.

Next, consider the commutative cube

$$
\begin{tikzcd}
\mathcal{L}_\phi(U/G) \arrow[rr] \arrow[dr, "\mathcal{L}j"] \arrow[dd] & & (U\times G)/G \arrow[dr, "j\times \mathrm{id}_G"] \arrow[dd, "(\phi\circ\mathrm{act},\mathrm{pr})" near start] & \\
& \mathcal{L}_\phi(X/G) \arrow[rr] \arrow[dd] & & (X\times G)/G \arrow[dd, "(\phi\circ\mathrm{act},\mathrm{pr})"] \\
U/G \arrow[rr, "\Delta_U" near start] \arrow[dr, "j"] & & (U\times U)/G \arrow[dr, "j\times j"] & \\
& X/G \arrow[rr, "\Delta_X"] & & (X\times X)/G.
\end{tikzcd}
$$

Its front and back faces are cartesian; moreover, its bottom face is cartesian as $U \times_X U \simeq U$. Thus, by a standard lemma on cartesian diagrams, the top face is cartesian as well, i.e.,

$$\mathcal{L}_\phi(U/G) \simeq (U\times G)/G \times_{(X\times G)/G} \mathcal{L}_\phi(X/G) \simeq U/G \times_{X/G} \mathcal{L}_\phi(X/G), \tag{3.2.7.3}$$

which implies the result.

Finally, the expressions (3.2.7.2) and (3.2.7.3) imply that the immersions of (3.2.7.1) are complementary, as this condition depends only on the underlying topological spaces. ☐

3.2.8. We now turn to studying the category $\mathrm{QC}(\mathfrak{X})$ of quasi-coherent sheaves on $\mathfrak{X}$ and its trace. Recall that $\mathrm{QC}(\mathfrak{X})$ carries a symmetric monoidal structure via tensor product. For any stack (even prestack) $\mathfrak{X}$, the dualizable objects of $\mathrm{QC}(\mathfrak{X})$ are given by the perfect complexes $\mathrm{Perf}(\mathfrak{X}) \subset \mathrm{QC}(\mathfrak{X})$ (i.e., by sheaves whose pullback to any affine scheme mapping to $\mathfrak{X}$ is perfect). However, the category $\mathrm{QC}(\mathfrak{X})$ is not in general compactly generated, or even rigid. To rectify these problems, we introduce the following conditions, following [**GR17a**, Ch. 3, §3.5] and [**BZFN10**]:

3.2.9. Definition. A stack $\mathfrak{X}$ is *passable* if

(1) its diagonal morphism is quasi-affine;
(2) the structure sheaf $\mathcal{O}_\mathfrak{X} \in \mathrm{QC}(\mathfrak{X})$ is compact; and
(3) the category $\mathrm{QC}(\mathfrak{X})$ is dualizable.

It is *perfect* if its diagonal morphism is furthermore affine, and $\mathrm{QC}(\mathfrak{X})$ is furthermore compactly generated.

3.2.10. By [**GR17a**, Ch. 3, Prop. 3.4.2], the category $\mathrm{QC}(\mathfrak{X})$ is rigid if $\mathfrak{X}$ is passable, and compactly generated by $\mathrm{Perf}(\mathfrak{X})$ if it is moreover perfect. In [**BZFN10**], it is shown that the class of perfect stacks includes

(1) quasi-compact schemes with affine diagonal;
(2) quotient stacks X/G, where G is a linear algebraic group and X is a finite-type scheme endowed with a G-equivariant ample line bundle (for instance, when X is quasi-affine, we may take the structure sheaf); and
(3) fiber products of perfect stacks.

Finally, using [**GR17a**, Ch. 3, Prop. 3.5.3] (though see also [**BZFN10**, Thm. 4.7]), we can compute the categorical trace of $\mathrm{QC}(\mathfrak{X})$ with respect to the endofunctor $\phi_\mathfrak{X}^*$:

3.2.11. Proposition. Suppose that $\mathfrak{X}$ is passable. Then we have a natural identification

$$\mathrm{Tr}(\mathrm{QC}(\mathfrak{X}), \phi_\mathfrak{X}^*) \simeq \mathrm{QC}(\mathcal{L}_{\phi_\mathfrak{X}}\mathfrak{X}), \tag{3.2.11.1}$$

with the universal trace functor (3.1.7.1) is given by the loop evaluation

$$[-] \simeq \mathrm{ev}^* \colon \mathrm{QC}(\mathfrak{X}) \to \mathrm{QC}(\mathcal{L}_{\phi_\mathfrak{X}}\mathfrak{X}).$$

3.2.12. Note that here, Theorem 3.1.11 yields

$$\mathrm{HH}(\mathrm{QC}(\mathfrak{X}), \phi_\mathfrak{X}^*) \simeq \Gamma(\mathcal{O}_{\mathcal{L}_{\phi_\mathfrak{X}}\mathfrak{X}}), \tag{3.2.12.1}$$

which may be checked directly using the duality data for $\mathrm{QC}(\mathfrak{X})$. Finally, we describe the functoriality of (3.2.11.1):

3.2.13. Corollary. Suppose we are in the setup of §3.2.4, and that $\mathcal{X}, \mathcal{Y}$ are both passable, so that

$$p^*: \mathrm{QC}(\mathcal{Y}) \to \mathrm{QC}(\mathcal{X})$$

is a monoidal functor of rigid monoidal categories intertwining the monoidal endofunctors $\phi^*_{\mathcal{Y}}$ and $\phi^*_{\mathcal{X}}$. Then under Proposition 3.2.11, the adjoint functors

$$\mathrm{Tr}(\mathrm{Ind}_{p^*}): \mathrm{Tr}(\mathrm{QC}(\mathcal{Y}), \phi^*_{\mathcal{Y}}) \rightleftarrows \mathrm{Tr}(\mathrm{QC}(\mathcal{X}), \phi^*_{\mathcal{X}}): \mathrm{Tr}(\mathrm{Res}_{p^*})$$

of (3.1.14.3) identify with

$$\mathcal{L}p^*: \mathrm{QC}(\mathcal{L}_{\phi_{\mathcal{Y}}}\mathcal{Y}) \rightleftarrows \mathrm{QC}(\mathcal{L}_{\phi_{\mathcal{X}}}\mathcal{X}): \mathcal{L}p_*.$$

Proof. The identification $\mathrm{Tr}(\mathrm{Ind}_{p^*}) \simeq \mathcal{L}p^*$ is immediate from Lemma 3.1.15 and naturality of [**GR17a**, Ch. 3, Prop. 3.5.3]. The identification $\mathrm{Tr}(\mathrm{Res}_{p^*}) \simeq \mathcal{L}p_*$ then follows by adjunction. □

3.3. Traces of convolution categories

3.3.1. In this section, we recall some general results of Ben-Zvi–Chen–Helm–Nadler [**BZCHN22**] on traces of convolution categories, and their relationships to the traces of categories of quasi-coherent sheaves computed in the previous section. We then specialize these results to the case of the affine Hecke category (and its restriction to a Slodowy slice).

Before proceeding, we will require some technical recollections:

3.3.2. Definition. An algebraic stack is *QCA*[25] if it is quasi-compact, of finite presentation, and has affine finitely presented diagonal.

3.3.3. For instance, the quotient stack of a finitely presented affine scheme by an affine algebraic group is QCA. Moreover, it is not hard to see that fiber products (in particular, loop spaces) of QCA stacks are QCA. As in [**DG13**, Thm. 4.3.1], any QCA stack is passable. Most saliently for our purposes, the category of ind-coherent sheaves on a QCA stack $\mathcal{X}$ is compactly generated by its coherent subcategory, i.e., $\mathrm{QC}^!(\mathcal{X}) \simeq \mathrm{Ind}(\mathrm{Coh}(\mathcal{X}))$.

3.3.4. Next, given any quasi-smooth Artin stack $\mathcal{X}$, we may consider its (classical) stack of singularities

$$\mathrm{Sing}(\mathcal{X}) := \underline{\mathrm{Spec}}_{\mathcal{X}} \, \mathrm{Sym}_{\mathcal{O}_{\mathcal{X}}} \mathrm{H}^1(\mathbb{L}^\vee_{\mathcal{X}})$$

as in [**AG15**], which carries a canonical fiberwise $\mathbb{G}_m$-action. Given any singular support condition Λ, i.e., a conical closed subset $\Lambda \subset \mathrm{Sing}(\mathcal{X})$, we may define a full subcategory $\mathrm{QC}^!_\Lambda(\mathcal{X}) \subset \mathrm{QC}^!(\mathcal{X})$ spanning sheaves whose singular support is contained in Λ. This inclusion then admits a continuous colocalization; we denote this adjoint pair by

$$\iota_\Lambda: \mathrm{QC}^!_\Lambda(\mathcal{X}) \rightleftarrows \mathrm{QC}^!(\mathcal{X}): \Gamma_\Lambda,$$

In particular, letting $\{0\}_{\mathcal{X}} \subset \mathrm{Sing}(\mathcal{X})$ denote the 0-section, we have $\mathrm{QC}^!_{\{0\}_{\mathcal{X}}}(\mathcal{X}) \simeq \mathrm{QC}(\mathcal{X})$. On the other hand, for the vacuous singular support condition, we have $\mathrm{QC}^!_{\mathrm{Sing}(\mathcal{X})}(\mathcal{X}) \simeq \mathrm{QC}^!(\mathcal{X})$. Moreover, for any closed substack $\mathcal{Z} \subset \mathcal{X}$, we have $\mathrm{QC}^!_{\mathcal{Z}\times_{\mathcal{X}}\{0\}_{\mathcal{X}}}(\mathcal{X}) \simeq \mathrm{QC}_{\mathcal{Z}}(\mathcal{X})$ and $\mathrm{QC}^!_{\mathcal{Z}\times_{\mathcal{X}}\mathrm{Sing}(\mathcal{X})}(\mathcal{X}) \simeq \mathrm{QC}^!_{\mathcal{Z}}(\mathcal{X})$, i.e., the full subcategories of sheaves set-theoretically supported on $\mathcal{Z}$.

Finally, given a map of such stacks $p: \mathcal{X} \to \mathcal{Y}$, there is a correspondence

$$\mathrm{Sing}(\mathcal{X}) \xleftarrow{\mathrm{Sing}(p)} \mathrm{Sing}(\mathcal{Y}) \times_{\mathcal{Y}} \mathcal{X} \xrightarrow{p} \mathrm{Sing}(\mathcal{Y}).$$

Note that by Lemmas 2.4.3 and 2.4.4 in *loc. cit.*, the "singular codifferential" $\mathrm{Sing}(p)$ is closed if p is quasi-smooth, and an isomorphism if p is smooth. Thus, given singular support conditions $\Lambda_{\mathcal{X}}$ and $\Lambda_{\mathcal{Y}}$ for $\mathcal{X}$ and $\mathcal{Y}$, we may define singular support conditions

$$p_*\Lambda_{\mathcal{X}} := \overline{p(\mathrm{Sing}(p)^{-1}(\Lambda_{\mathcal{X}}))}, \qquad p^!\Lambda_{\mathcal{Y}} := \overline{\mathrm{Sing}(p)(p^{-1}(\Lambda_{\mathcal{Y}}))}$$

for $\mathcal{Y}$ and $\mathcal{X}$, respectively. We then have functors

$$p_*: \mathrm{QC}^!_{\Lambda_{\mathcal{X}}}(\mathcal{X}) \to \mathrm{QC}^!_{p_*\Lambda_{\mathcal{X}}}(\mathcal{Y}), \qquad p^!: \mathrm{QC}^!_{\Lambda_{\mathcal{Y}}}(\mathcal{Y}) \to \mathrm{QC}^!_{p^!\Lambda_{\mathcal{Y}}}(\mathcal{X}).$$

If p is moreover quasi-smooth (or more generally, Gorenstein), then the same holds for p^* in place of $p^!$ (as [**Gai13**, Prop. 7.3.8] implies that the two differ by a shifted line bundle).

[25] In the sense of [**BZCHN22**], rather than in the more general sense of [**DG13**].

3.3.5. We may now state the description of the categorical trace of a convolution category given in [**BZCHN22**, §3]. Suppose we are in the setup of §3.2.4, and moreover, that

(1) the stacks $\mathcal{X}, \mathcal{Y}$ are smooth and QCA;
(2) the map $p\colon \mathcal{X} \to \mathcal{Y}$ is proper; and
(3) the self-map $\phi_{\mathcal{X}}, \phi_{\mathcal{Y}}$ are automorphisms.

Then the fiber product category $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is QCA and quasi-smooth (as in Lemma 3.2.6(2)), and the category $\mathrm{QC}^!(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})$ is monoidal under the $*$-convolution.[26] Moreover, the automorphism $\phi := \phi_{\mathcal{X}} \times_{\phi_{\mathcal{Y}}} \phi_{\mathcal{X}}$ of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ yields a monoidal endofunctor ϕ^* of $\mathrm{QC}^!(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})$. We then have:

3.3.6. Proposition. The monoidal category $\mathrm{QC}^!(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})$ is rigid. Moreover, we have a natural identification

$$\mathrm{Tr}(\mathrm{QC}^!(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}), \phi^*) \simeq \mathrm{QC}_{\Lambda_{\mathcal{X}/\mathcal{Y},\phi}}(\mathcal{L}_{\phi_{\mathcal{Y}}} \mathcal{Y}),$$

where

$$\Lambda_{\mathcal{X}/\mathcal{Y},\phi} := (p \times \mathrm{id}_{\mathcal{Y}})_* \mathrm{pr}_1^! \operatorname{Sing}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \subset \operatorname{Sing}(\mathcal{L}_{\phi_{\mathcal{Y}}} \mathcal{Y})$$

is the singular support condition obtained via the "trace correspondence"[27]

$$\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \xleftarrow{\mathrm{pr}_1} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X} \simeq \mathcal{X} \times_{\mathcal{Y} \times \mathcal{Y}} \mathcal{Y} \xrightarrow{p \times \mathrm{id}_{\mathcal{Y}}} \mathcal{L}_{\phi_{\mathcal{Y}}} \mathcal{Y}. \tag{3.3.6.1}$$

Moreover, the universal trace functor (3.1.7.1) is given by $(p \times \mathrm{id}_{\mathcal{Y}})_* \mathrm{pr}_1^*$.

3.3.7. Remark. Let us describe the monoidal duality in $\mathrm{QC}^!(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})$ explicitly. Set $\mathcal{Z} := \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, and let $\mathcal{F} \in \mathrm{Coh}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})$. The proof of [**BZCHN22**, Thm. 3.25] shows that the right-dual of $\mathcal{F}$ is given by

$$\mathcal{F}^{\vee,R} \simeq \sigma^*(\omega_{\mathcal{Z}/\mathcal{X}} \otimes \mathbb{D}_{\mathcal{Z}}(\mathcal{F}) \otimes \omega_{\mathcal{Z}}^{\vee}),$$

where $\sigma\colon \mathcal{Z} \to \mathcal{Z}$ is the "swap" map as in §2.2.7, $\omega_{\mathcal{Z}/\mathcal{X}} := \mathrm{pr}_1^! \mathcal{O}_{\mathcal{X}}$ is the relative dualizing sheaf with respect to the first projection, $\mathbb{D}_{\mathcal{Z}}$ denotes Grothendieck–Serre duality on $\mathcal{Z}$, and $\omega_{\mathcal{Z}}$ denotes the dualizing sheaf of $\mathcal{Z}$ (which is a shifted line bundle as $\mathcal{Z}$ is quasi-smooth).

Now suppose that $\mathcal{X}$ is Calabi–Yau, i.e., has trivial canonical bundle. Then we claim that $\mathcal{F}$ is *pivotal*, that is, its left and right monoidal duals coincide. Indeed, in this case, we have $\omega_{\mathcal{Z}/\mathcal{X}} \simeq \omega_{\mathcal{Z}}[-\dim \mathcal{X}]$, hence $\mathcal{F}^{\vee,R} \simeq \sigma^* \mathbb{D}_{\mathcal{Z}}(\mathcal{F})[-\dim \mathcal{X}]$. Note that the functors σ^* and $\mathbb{D}_{\mathcal{Z}}$ commute. Moreover, σ^* is clearly involutive, and $\mathbb{D}_{\mathcal{Z}}$ is involutive by [**DG13**, §4.4.3]. It follows that

$$(\mathcal{F}^{\vee,R})^{\vee,R} \simeq \sigma^* \mathbb{D}_{\mathcal{Z}}(\sigma^* \mathbb{D}_{\mathcal{Z}}(\mathcal{F})[-\dim \mathcal{X}])[-\dim \mathcal{X}] \simeq \sigma^* \mathbb{D}_{\mathcal{Z}} \sigma^* \mathbb{D}_{\mathcal{Z}}(\mathcal{F}) \simeq \mathcal{F},$$

as desired. In particular, this property holds for the "affine Hecke categories" $\mathcal{H}^{\mathrm{coh}}_{S_e}$ of (2.2.7.2), as

$$\omega_{\widetilde{S}_e/\widetilde{Z}_e} \simeq \mathcal{O}_{\widetilde{S}_e/\widetilde{Z}_e}[\dim \widetilde{S}_e - \dim \widetilde{Z}_e]\langle \dim \widetilde{S}_e \rangle \tag{3.3.7.1}$$

up to a character of Z_e (see for instance [**GR17b**, Ch. 9, Prop. 7.3.4]), and these twists clearly do not disrupt the above argument.

3.3.8. Note that by base-change, we have

$$[\Delta_{\mathcal{X}/\mathcal{Y},*} \mathcal{O}_{\mathcal{X}}] \simeq \mathcal{L} p_* \mathcal{O}_{\mathcal{X}} \in \mathrm{QC}_{\Lambda_{X/Y,\phi}}(\mathcal{L}_{\phi_{\mathcal{Y}}} \mathcal{Y}). \tag{3.3.8.1}$$

Thus, Theorem 3.1.11 yields an algebra isomorphism

$$\mathrm{HH}(\mathrm{QC}^!(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}), \phi^*) \simeq \mathrm{End}_{\mathcal{L}_{\phi_{\mathcal{Y}}} \mathcal{Y}}(\mathcal{L} p_* \mathcal{O}_{\mathcal{X}})^{\mathrm{op}}. \tag{3.3.8.2}$$

In particular, when p is the identity, we recover Proposition 3.2.11 (since the categories of ind-coherent and quasi-coherent sheaves are equivalent for smooth stacks). We will now relate the two situations more generally. Observe the category $\mathrm{QC}(\mathcal{X})$ is equipped with both $(\mathrm{QC}^!(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}), \mathrm{QC}(Y))$- and $(\mathrm{QC}(Y), \mathrm{QC}^!(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}))$-bimodule structures via left and right convolution, respectively; we refer to it in either case as the "regular

[26] Note that our convention differs from that of [**BZCHN22**], where the !-convolution is instead used. The reason for this choice will become clear in §5.3.

[27] Here the middle term is as in (3.2.6.1).

bimodule." Note that the action of the subcategory $\mathrm{Coh}(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X})$ preserves compact objects of $\mathrm{QC}(\mathfrak{X})$ by [**BZNP17b**, Thm. 1.1.3]. Assuming right-dualizability, we obtain a diagram

$$(\mathrm{QC}^!(\mathfrak{Y})\text{-}\mathbf{mod}, \phi_{\mathfrak{Y}}^*) \underset{(\mathrm{QC}(\mathfrak{X}),\phi_{\mathfrak{X}}^*)}{\overset{(\mathrm{QC}(\mathfrak{X}),\phi_{\mathfrak{X}}^*)}{\rightleftarrows}} (\mathrm{QC}^!(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X})\text{-}\mathbf{mod}, \phi^*)$$

in $L(\mathrm{Morita}(\mathbf{dgCat}_k))_{\mathrm{rgd}}$. The following lemma characterizes the induced functors on categorical traces:

3.3.9. Lemma. The regular $(\mathrm{QC}^!(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}), \mathrm{QC}(\mathfrak{Y}))$-bimodule $\mathrm{QC}(\mathfrak{X})$ is both left- and right-dual to the regular $(\mathrm{QC}(\mathfrak{Y}), \mathrm{QC}^!(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}))$-bimodule $\mathrm{QC}(\mathfrak{X})$. Moreover, we have commutative squares

$$\begin{tikzcd}
\mathrm{Tr}(\mathrm{QC}(\mathfrak{Y}), \phi_{\mathfrak{Y}}^*) \arrow[rr, shift left, "\mathrm{Tr}(\mathrm{QC}(\mathfrak{X}),\phi_{\mathfrak{X}}^*)"] \arrow[d, "\wr"] & & \mathrm{Tr}(\mathrm{QC}^!(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}), \phi^*) \arrow[ll, shift left, "\mathrm{Tr}(\mathrm{QC}(\mathfrak{X}),\phi_{\mathfrak{X}}^*)"] \arrow[d, "\wr"] \\
\mathrm{QC}(\mathcal{L}_{\phi_{\mathfrak{Y}}}\mathfrak{Y}) \arrow[r, shift left, "\Gamma_{\{0\}_{p(\mathfrak{X})}}"] & \mathrm{QC}^!_{\{0\}_{p(\mathfrak{X})}}(\mathcal{L}_{\phi_{\mathfrak{Y}}}\mathfrak{Y}) \arrow[l, shift left, "\iota_{\{0\}_{p(\mathfrak{X})}}"] \arrow[r, shift left, "\iota_{\{0\}_{p(\mathfrak{X})}}"] & \mathrm{QC}^!_{\Lambda_{\mathfrak{X}/\mathfrak{Y},\phi}}(\mathcal{L}_{\phi_{\mathfrak{Y}}}\mathfrak{Y}), \arrow[l, shift left, "\Gamma_{\{0\}_{p(\mathfrak{X})}}"]
\end{tikzcd} \tag{3.3.9.1}$$

where the vertical identifications are those of Proposition 3.3.6, and $\{0\}_{p(\mathfrak{X})} := \mathrm{ev}^! \, p(\mathfrak{X}) \subset \{0\}_{\mathcal{L}_{\phi_{\mathfrak{Y}}}\mathfrak{Y}}$ denotes the pullback of the classical support condition $p(\mathfrak{X}) \subset \mathrm{Sing}(\mathfrak{Y}) \simeq \mathfrak{Y}$.

Proof. Right-duality was established in the proof of [**BZCHN22**, Prop. 3.32] (it is not difficult to verify that all results apply in the setting of the $*$-convolution, in addition to the !-convolution). Left-duality now follows immediately using [**Gai12**, Cor. 6.4.1] and rigidity of the monoidal categories $\mathrm{QC}^!(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X})$ and $\mathrm{QC}(\mathfrak{Y})$. More explicitly, the functors

$$\begin{gathered}
\mathrm{QC}(\mathfrak{X}) \otimes_{\mathrm{QC}^!(\mathfrak{X}\times_{\mathfrak{Y}}\mathfrak{X})} \mathrm{QC}(\mathfrak{X}) \simeq \mathrm{QC}^!_{p(\mathfrak{X})}(\mathfrak{Y}) \xhookrightarrow{\iota_{p(\mathfrak{X})}} \mathrm{QC}^!(\mathfrak{Y}) \\
\mathrm{QC}^!(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}) \xrightarrow{\Gamma_{\{0\}_{\mathfrak{X}\times_{\mathfrak{Y}}\mathfrak{X}}}} \mathrm{QC}(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}) \simeq \mathrm{QC}(\mathfrak{X}) \otimes_{\mathrm{QC}(\mathfrak{Y})} \mathrm{QC}(\mathfrak{X})
\end{gathered} \tag{3.3.9.2}$$

exhibit this left-duality. Note that the counit is evidently $\mathrm{QC}^!(\mathfrak{Y})$-bilinear and continuous. Moreover, the unit is the continuous right adjoint to the evident $\mathrm{QC}^!(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X})$-bilinear inclusion, hence $\mathrm{QC}^!(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X})$-bilinear by [**Gai12**, Cor. 6.2.4]. The duality identities now follow immediately from the proof of [**BZCHN22**, Prop. 3.32] by adjunction.

For the latter assertion, commutativity of the clockwise square was established in [**BZCHN22**, Prop. 3.32]. The counter-clockwise square follows by an identical argument, using the functors (3.3.9.2). ☐

3.3.10. We now use this result to give an alternative description of the universal trace functor for the convolution category $\mathrm{QC}^!(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X})$:

3.3.11. Corollary. Given $\mathcal{F} \in \mathrm{QC}^!(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X})$, we have

$$[\mathcal{F}]^{\mathrm{QC}} := \iota_{\{0\}_{p(\mathfrak{X})}} \Gamma_{\{0\}_{p(\mathfrak{X})}}[\mathcal{F}] \simeq [\mathrm{QC}(\mathfrak{X}), \phi_{\mathfrak{X}}^*(-) * \mathcal{F}] \in \mathrm{QC}(\mathcal{L}_{\phi_{\mathfrak{Y}}}\mathfrak{Y}).$$

Moreover, if $\mathcal{F}$ is coherent, then so is $[\mathcal{F}]$, and the latter is connective (resp. coconnective) if and only if $[\mathrm{QC}(\mathfrak{X}), \phi_{\mathfrak{X}}^*(-) * \mathcal{F}]$ is connective (resp. coconnective).

Proof. The first two assertions are immediate from (3.3.9.1) and §3.1.12, respectively. The third assertion then follows from [**Gai13**, Prop. 11.7.5]. ☐

3.3.12. Finally, we explain the functoriality of Proposition 3.3.6 as in Corollary 3.2.13. Let $\mathfrak{Y}'$ be a smooth QCA stack equipped with an automorphism $\phi_{\mathfrak{Y}'}$, and let $f \colon \mathfrak{Y}' \to \mathfrak{Y}$ be a morphism intertwining $\phi_{\mathfrak{Y}'}$ and $\phi_{\mathfrak{Y}}$. Define $p' \colon \mathfrak{X}' := \mathfrak{Y}' \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{Y}'$ and $\phi_{\mathfrak{X}'} := \phi_{\mathfrak{Y}'} \times_{\phi_{\mathfrak{Y}}} \phi_{\mathfrak{X}}$ by pullback along f, and set $\phi' := \phi_{\mathfrak{X}'} \times_{\phi_{\mathfrak{Y}'}} \phi_{\mathfrak{X}'}$ as before. Since f is quasi-smooth, it is locally eventually coconnective by [**AG15**, Cor. 2.2.4], and hence (misusing notation slightly) we have an adjoint pair

$$f^* \colon \mathrm{QC}^!(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}) \rightleftarrows \mathrm{QC}^!(\mathfrak{X}' \times_{\mathfrak{Y}'} \mathfrak{X}') \colon f_*. \tag{3.3.12.1}$$

In particular, f^* is compact object-preserving and monoidal, and it intertwines the monoidal endofunctors ϕ^* and $\phi'^{,*}$.

3.3.13. Corollary. Suppose that f is a smooth relative scheme. Then we have

$$\Lambda_{\mathfrak{X}'/\mathcal{Y}',\phi'} = \mathcal{L}f^!\Lambda_{\mathfrak{X}/\mathcal{Y},\phi}, \qquad \mathcal{L}f_*\Lambda_{\mathfrak{X}'/\mathcal{Y}',\phi'} \subseteq \Lambda_{\mathfrak{X}/\mathcal{Y},\phi},$$

and the adjoint functors

$$\mathrm{Tr}(\mathrm{Ind}_{f^*})\colon \mathrm{Tr}(\mathrm{QC}^!(\mathfrak{X}\times_{\mathcal{Y}}\mathfrak{X}),\phi^*) \rightleftarrows \mathrm{Tr}^!(\mathrm{QC}(\mathfrak{X}'\times_{\mathcal{Y}'}\mathfrak{X}'),\phi'^{,*})\colon \mathrm{Tr}(\mathrm{Res}_{f^*})$$

of (3.1.14.3) identify with

$$\mathcal{L}f^*\colon \mathrm{QC}^!_{\Lambda_{\mathfrak{X}/\mathcal{Y},\phi}}(\mathcal{L}_{\phi_{\mathcal{Y}}}\mathcal{Y}) \rightleftarrows \mathrm{QC}^!_{\Lambda_{\mathfrak{X}'/\mathcal{Y}',\phi'}}(\mathcal{L}_{\phi_{\mathcal{Y}'}}\mathcal{Y}')\colon \mathcal{L}f_* \tag{3.3.13.1}$$

under Proposition 3.3.6.

Proof. The claims about singular supports are easily checked from the definitions and properties listed in §3.3.4 (in particular, the second claim follows from the first). Since $\mathcal{L}f$ is quasi-smooth by Lemma 3.2.6(2), we indeed have an adjoint pair (3.3.13.1) as in (3.3.12.1). It therefore suffices to identify $\mathrm{Tr}(\mathrm{Ind}_{f^*})$ with $\mathcal{L}f^*$ under Proposition 3.3.6. By §3.1.12, we reduce to showing this for images of compact objects under the universal trace functor. Given $\mathcal{F}\in\mathrm{Coh}(\mathfrak{X}\times_{\mathcal{Y}}\mathfrak{X})$, we have $\mathrm{Tr}(\mathrm{Ind}_{f^*})([\mathcal{F}])\simeq[f^*\mathcal{F}]$. The result now follows from (3.3.6.1) by noting that the rightmost square in the commutative diagram

$$\begin{tikzcd}
\mathfrak{X}\times_{\mathcal{Y}}\mathfrak{X} & (\mathfrak{X}\times_{\mathcal{Y}}\mathfrak{X})\times_{\mathfrak{X}\times\mathfrak{X}}\mathfrak{X} \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\sim"] & \mathfrak{X}\times_{\mathcal{Y}\times\mathcal{Y}}\mathcal{Y} \arrow[r, "p\times\mathrm{id}_{\mathcal{Y}}"] & \mathcal{L}_{\phi_{\mathcal{Y}}}\mathcal{Y} \\
\mathfrak{X}'\times_{\mathcal{Y}'}\mathfrak{X}' \arrow[u, "f"] & (\mathfrak{X}'\times_{\mathcal{Y}'}\mathfrak{X}')\times_{\mathfrak{X}'\times\mathfrak{X}'}\mathfrak{X}' \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\sim"] \arrow[u] & \mathfrak{X}'\times_{\mathcal{Y}'\times\mathcal{Y}'}\mathcal{Y}' \arrow[r, "p'\times\mathrm{id}_{\mathcal{Y}'}"] \arrow[u] & \mathcal{L}_{\phi_{\mathcal{Y}'}}\mathcal{Y}' \arrow[u, "\mathcal{L}f"']
\end{tikzcd}$$

is Cartesian; here all vertical arrows are induced by f in the evident manner. ☐

3.3.14. In particular, we obtain $[\Delta_{\mathfrak{X}'/\mathcal{Y}',*}\mathcal{O}_{\mathfrak{X}'}]\simeq\mathcal{L}f^*[\Delta_{\mathfrak{X}/\mathcal{Y},*}\mathcal{O}_{\mathfrak{X}}]$, which agrees with the base-change isomorphism coming from (3.3.8.1).

3.3.15. We now turn to the specific case of the affine Hecke category $\mathcal{H}^{\mathrm{coh}}$ and its restriction $\mathcal{H}^{\mathrm{coh}}_{S_e}$ to a Slodowy slice S_e, as in §2.2.7. Set $S_{e,\mathcal{N}} := S_e\cap\mathcal{N} = \pi_{S_e}(\widetilde{S}_e)$, and let $\widehat{S}_{e,\mathcal{N}}$ denote the formal completion of S_e at $S_{e,\mathcal{N}}$. We set

$$\mathrm{QC}^!(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e)) := \mathrm{QC}^!_{S_{e,\mathcal{N}}}(\mathcal{L}(S_e/\widetilde{Z}_e)),$$

i.e., the subcategory of sheaves set-theoretically supported on $S_{e,\mathcal{N}}$ (via the loop evaluation). In particular, for $e=0$, we write $\widehat{\mathcal{N}} := \widehat{\mathfrak{g}}_{\mathcal{N}}$ following [**BZCHN22**].

3.3.16. We begin by briefly reviewing the "coherent Springer theory" of [**BZCHN22**]. Define the *coherent Springer sheaf*

$$\mathcal{S} := \mathcal{L}\pi_*\mathcal{O}_{\mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G})} \in \mathrm{QC}^!(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})), \tag{3.3.16.1}$$

which identifies with the 2-categorical class of the monoidal unit $\Delta_{\widetilde{\mathcal{N}}/\mathfrak{g},*}\mathcal{O}_{\widetilde{\mathcal{N}}/\widetilde{G}}\in\mathcal{H}^{\mathrm{coh}}$ as in (3.3.8.1). By §2 of *loc. cit.*, the Hochschild homology of $\mathcal{H}^{\mathrm{coh}}$ is concentrated in degree 0 and recovers the affine Hecke algebra of §2.4.2:

$$\mathrm{HH}(\mathcal{H}^{\mathrm{coh}}) \simeq K_0(\mathcal{H}^{\mathrm{coh}})_k \cong \mathcal{H}_k. \tag{3.3.16.2}$$

Here the first isomorphism arises via the natural Chern character from K-theory to Hochschild homology (see §5.2.8), and the latter isomorphism is a generalization (to groups whose derived subgroup is not necessarily simply-connected) of the celebrated results of Kazhdan–Lusztig and Chriss–Ginzburg (see Theorem 1.1.2). These isomorphisms are compatible with the Bernstein isomorphism $R(\widetilde{G})_k\cong Z(\mathcal{H})_k$ in the evident manner. We then have a canonical algebra isomorphism

$$\mathcal{H}_k \simeq \mathrm{End}_{\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})}(\mathcal{S})^{\mathrm{op}},$$

as in (3.3.8.2), realizing a categorical local Langlands correspondence for the unramified principal series of $\check{G}$. These constructions satisfy many other good properties (such as compatibility with parabolic induction), which we will not need in the sequel.

We now identify the traces of these "affine Hecke categories" more precisely:

3.3.17. Corollary. We have a natural identification

$$\mathrm{Tr}(\mathcal{H}^{\mathrm{coh}}_{S_e}) \simeq \mathrm{QC}^!(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e)).$$

Proof. By Proposition 3.3.6, it suffices to show that the singular support condition $\Lambda_{\widetilde{S}_e/S_e}$ is vacuous; the proof follows [**BZCHN22**, Rmk. 4.14]. Identify $\mathfrak{g} \cong \mathfrak{g}^*$ via a non-degenerate form $\langle -,- \rangle$; thus, the cotangent space to any point in S_e may be identified with $\mathfrak{g}^{f,*} \cong \mathfrak{g}/[f,\mathfrak{g}] \cong \mathfrak{g}^e$. The singular locus of $\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e)$ at a k-point $(n,g,q) \in S_e \times \widetilde{Z}_e$ satisfying $gng^{-1} = q^2 n$ is then the set

$$\mathrm{Sing}(\mathcal{L}(\widehat{S}_e/\widetilde{Z}_e))_{(n,g,q)} = \{x \in \mathfrak{g}^e : gxg^{-1} = q^{-2}x, [n,x] = 0, \langle n,x \rangle = 0\}.$$

Moreover, the singular locus of $\widetilde{S}_e \times_{S_e} \widetilde{S}_e$ at a k-point $(n,\mathfrak{b},\mathfrak{b}')$ (where $n \in S_e$ and $\mathfrak{b} \subset \mathfrak{g}$ is a Borel subalgebra containing n) is the set

$$\mathrm{Sing}(\widetilde{S}_e \times_{S_e} \widetilde{S}_e)_{(n,\mathfrak{b},\mathfrak{b}')} = \mathfrak{g}^e \cap \mathfrak{b} \cap \mathfrak{b}'.$$

A calculation then shows that

$$(\Lambda_{\widetilde{S}_e/S_e})_{(n,g,q)} = \{x \in \mathrm{Sing}(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e))_{(n,g,q)} : n,x \in \mathfrak{b} \text{ for some } \mathfrak{b} \in \mathcal{B}\}.$$

Since n and x generate a two-dimensional solvable Lie algebra, they are contained in a Borel subalgebra, and hence $\Lambda_{\widetilde{S}_e/S_e} = \mathrm{Sing}(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e))$, as desired. □

3.3.18. Finally, we identify the trace of the monoidal functor $i^*_{S_e} \colon \mathcal{H}^{\mathrm{coh}} \to \mathcal{H}^{\mathrm{coh}}_{S_e}$. To apply Corollary 3.3.13, we must show:

3.3.19. Lemma. The morphism

$$i_{S_e} \colon S_e/\widetilde{Z}_e \to \mathfrak{g}/\widetilde{G}$$

is a smooth relative scheme.

Proof. The morphism i_{S_e} is clearly schematic; we show that its relative cotangent complex is perfect of Tor-amplitude $[0,1]$. The latter is computed by the complex[28]

$$\mathfrak{g}^* \otimes \mathcal{O}_{S_e/\widetilde{Z}_e} \xrightarrow{(\mathrm{act}^*,\mathrm{in}^*)} (\tilde{\mathfrak{g}}^* \oplus \mathfrak{g}^{f,*}) \otimes \mathcal{O}_{S_e/\widetilde{Z}_e} \xrightarrow{(\mathrm{in}^*,\mathrm{act}^*)} \tilde{\mathfrak{z}}^*_e \otimes \mathcal{O}_{S_e/\widetilde{Z}_e}$$

of (locally) free sheaves, where we have let act, in denote the evident "action" and inclusion maps. Thus, we wish to show that the first map is injective, and its cokernel is locally free. Dualizing and applying Nakayama's lemma reduces us to showing that the dual map is surjective on all fibers. Given $x \in \mathfrak{g}^f$, the fiber of the dual map at $e + x \in S_e$ is computed by

$$([-,e+x], \langle -,e+x \rangle^*, \mathrm{in}) \colon \mathfrak{g} \oplus k \oplus \mathfrak{g}^f \to \mathfrak{g},$$

where $\langle -,- \rangle$ denotes a non-degenerate form on $\mathfrak{g}$ (as in the proof of Corollary 3.3.17). It suffices to show that the composition

$$[\mathfrak{g},f] \hookrightarrow \mathfrak{g} \xrightarrow{[-,e+x]} \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{g}^f \tag{3.3.19.1}$$

is an isomorphism. Consider the basis of $[\mathfrak{g},f]$ given by weight spaces for the $\mathfrak{sl}_2$-action associated to S_e. We then have an associated basis of $\mathfrak{g}/\mathfrak{g}^f$ given by sending each basis vector v to $[v,e]$ (which increases the weight by 2). But for any basis vector v of weight w, the vector $[v,x]$ lies in the span of weight spaces $\leq w$ (as x lies in the span of weight spaces ≤ 0). It follows that the matrix describing (3.3.19.1) with respect to these bases is upper triangular, with 1's along the diagonal, hence an isomorphism. □

3.3.20. Corollary. The adjoint pair

$$\mathrm{Tr}(\mathrm{Ind}_{i^*_{S_e}}) \colon \mathrm{Tr}(\mathcal{H}^{\mathrm{coh}}) \rightleftarrows \mathrm{Tr}(\mathcal{H}^{\mathrm{coh}}_{S_e}) \colon \mathrm{Tr}(\mathrm{Res}_{i^*_{S_e}})$$

identifies with

$$\mathcal{L}i^*_{S_e} \colon \mathrm{QC}^!(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e)) \rightleftarrows \mathrm{QC}^!(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})) \colon \mathcal{L}i_{S_e,*}$$

via Corollary 3.3.17.

[28] Note that here $\tilde{\mathfrak{g}}$ denotes the Lie algebra of $\widetilde{G}$, rather than the Grothendieck simultaneous resolution.

3.4. The Block–Getzler sheaf

3.4.1. In this section, we construct our main technical tool: a complex explicitly computing the 2-categorical class map (3.1.6.1) under certain assumptions on the monoidal category $\mathcal{A}$ (which will suffice for all of our applications). More precisely, we assume that $\mathcal{A}$ is compactly generated and rigid, and that $\mathcal{A}$ admits a central functor from the category of quasi-coherent sheaves on a suitable quotient stack. The "Block–Getzler sheaf" then computes the restriction via (3.1.14.3) of a 2-categorical class to the trace of the latter category, which identifies with the category of quasi-coherent sheaves on a certain (twisted) loop space as in Proposition 3.2.11. Our construction is essentially a straightforward extension of the "Block–Getzler complex" of [**BG94**] (see also [**BZCHN22**, §2.1.2] and [**Che20**, Def. 2.3.4]), which computes Hochschild homology in the equivariant setting.

3.4.2. Let G be a reductive[29] group acting on a scheme X, and suppose that the quotient stack X/G is *perfect* as in Definition 3.2.9 (e.g., X is finite-type and quasi-affine). Let $\phi\colon X \to X$ be a self-map commuting with the G-action, and denote by $\Gamma^G\colon \mathrm{QC}(X/G) \to \mathrm{QC}(\mathrm{B}G) = \mathrm{Rep}(G)$ the functor of equivariant global sections (i.e., the pushforward along the natural projection). Finally, let G act diagonally on $X \times G$ as in §3.2.5.

We begin by constructing a precursor to the Block–Getzler sheaf, which we will soon equip with additional structure in Construction 3.4.7.

3.4.3. Definition. Let $\mathcal{C}$ be a compactly generated dg-category enriched in $\mathrm{QC}(X/G)$, and let $\underline{\mathcal{H}om}_{\mathcal{C}}$ denote the $\mathrm{QC}(X/G)$-internal Hom. Suppose that $\mathcal{C}$ is moreover equipped with a ϕ^*-semilinear endofunctor $F_{\mathcal{C}}$ preserving compact objects.[30] The *pre-Block–Getzler sheaf* ${}^{\mathrm{pre}}\mathsf{BG}_{X/G,\phi}(\mathcal{C}, F_{\mathcal{C}})$ is defined to be the sum totalization of the simplicial quasicoherent sheaf on $(X \times G)/G$ given by

$$ {}^{\mathrm{pre}}\mathsf{BG}^{-n}_{X/G,\phi}(\mathcal{C}, F_{\mathcal{C}}) := \bigoplus_{c_0,\dots,c_n \in \mathcal{C}^c} \big(\Gamma^G \underline{\mathcal{H}om}_{\mathcal{C}}(c_0, c_1) \otimes_k \cdots \otimes_k \Gamma^G \underline{\mathcal{H}om}_{\mathcal{C}}(c_{n-1}, c_n) \otimes_k \underline{\mathcal{H}om}_{\mathcal{C}}(c_n, F_{\mathcal{C}}(c_0))\big) \boxtimes \mathcal{O}_G, $$

where the ($\mathcal{O}_{X\times G}$-linear) face maps $d_i\colon {}^{\mathrm{pre}}\mathsf{BG}^{-n}_{X/G,\phi}(\mathcal{C}, F_{\mathcal{C}}) \to {}^{\mathrm{pre}}\mathsf{BG}^{-(n-1)}_{X/G,\phi}(\mathcal{C}, F_{\mathcal{C}})$ (for $i = 0, \dots, n$) "compose morphisms." More precisely[31], we have

$$ \begin{aligned} d_0(f_0 \otimes \cdots \otimes f_n \boxtimes r) &= f_1 \otimes \cdots \otimes f_{n-1} \otimes \rho(\Gamma^G F_{\mathcal{C}}(f_0)) \circ f_n \boxtimes r, \\ d_i(f_0 \otimes \cdots \otimes f_n \boxtimes r) &= f_0 \otimes \cdots \otimes f_{i+1} \circ f_i \otimes \cdots \otimes f_n \boxtimes r, \text{ for } i = 1, \dots, n, \end{aligned} \tag{3.4.3.1} $$

where for any $V \in \mathrm{Rep}(G)$ we (misusing notation) let $\rho\colon V \to \mathcal{O}(G) \otimes V \simeq V \otimes \mathcal{O}(G)$ denote the *left* coaction map.[32]

3.4.4. We now recall the relationship between the pre-Block–Getzler sheaf and Hochschild homology, following [**BZCHN22**, §2.1.2]. Let $\mathcal{C}$ be a compactly generated $\mathrm{QC}(X/G)$-module category equipped with a ϕ^*-semilinear endofunctor $F_{\mathcal{C}}$ preserving compact objects. Then we may consider the *de-equivariantization*

$$ \mathcal{C}^{\mathrm{dq}} := \mathrm{Vect}_k \otimes_{\mathrm{Rep}(G)} \mathcal{C}, $$

which is a $\mathrm{QC}(X)$-module category, and admits a natural "forgetful" functor $(-)^{\mathrm{dq}}\colon \mathcal{C} \to \mathcal{C}^{\mathrm{dq}}$ preserving compact objects[33]. Since $\mathrm{Vect}_k^{\mathrm{dq}} \simeq \mathrm{QC}(G)$, the category $\mathcal{C}^{\mathrm{dq}}$ carries a natural $\mathrm{QC}(G)$-module structure (where the monoidal structure on $\mathrm{QC}(G)$ is via convolution). In particular, for any $g \in G(k)$, the action of

[29] But as in Chapter 2, not necessarily connected; we need only that the Peter–Weyl theorem holds for G.

[30] I.e., for any $c, c' \in \mathcal{C}$, the map $F_{\mathcal{C}}\colon \underline{\mathrm{Hom}}_{\mathcal{C}}(c, c') \to \underline{\mathrm{Hom}}_{\mathcal{C}}(F(c), F(c'))$ of $\mathcal{O}_{X/G}$-modules is ϕ^*-semilinear, where $\phi^*\colon \mathcal{O}_{X/G} \to \mathcal{O}_{X/G}$ is the induced homomorphism. As in §3.1.6, when $F_{\mathcal{C}}$ is the identity, we will often omit it from the notation.

[31] Note that for any $c, c', c'' \in \mathcal{C}^c$, we have a composition map $\Gamma^G \underline{\mathrm{Hom}}_{\mathcal{C}}(c, c') \otimes_k \underline{\mathrm{Hom}}_{\mathcal{C}}(c', c'') \to \underline{\mathrm{Hom}}_{\mathcal{C}}(c, c'')$ adjoint to Γ^G of the usual composition map $\underline{\mathrm{Hom}}_{\mathcal{C}}(c, c') \to \mathcal{H}om_{X/G}(\underline{\mathrm{Hom}}_{\mathcal{C}}(c', c''), \underline{\mathrm{Hom}}_{\mathcal{C}}(c, c''))$. Taking Γ^G, we also obtain a composition map $\Gamma^G \underline{\mathrm{Hom}}_{\mathcal{C}}(c, c') \otimes_k \Gamma^G \underline{\mathrm{Hom}}_{\mathcal{C}}(c', c'') \to \Gamma^G \underline{\mathrm{Hom}}_{\mathcal{C}}(c, c'')$.

[32] I.e., whose specialization at $g \in G$ is given by the map g^{-1}. Note that this convention is opposite to that of [**BZCHN22**, Def. 2.12], and arises from our different definition of the face maps d_i. We will elaborate more on these conventions in footnote 34.

[33] E.g., by [**GR17a**, Cor. 9.3.3]. Note that the image of $(-)^{\mathrm{dq}}$ also generates $\mathcal{C}^{\mathrm{dq}}$ under colimits.

the skyscraper sheaf at g yields an automorphism $g_*\colon \mathcal{C}^{\mathrm{dq}} \to \mathcal{C}^{\mathrm{dq}}$. We then have a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "(-)^{\mathrm{dq}}"] \arrow[d, "F_{\mathcal{C}}"] & \mathcal{C}^{\mathrm{dq}} \arrow[r, "g_*"] \arrow[d, "F_{\mathcal{C}}^{\mathrm{dq}}"] & \mathcal{C}^{\mathrm{dq}} \arrow[d, "F_{\mathcal{C}}^{\mathrm{dq}}"] \\ \mathcal{C} \arrow[r, "(-)^{\mathrm{dq}}"] & \mathcal{C}^{\mathrm{dq}} \arrow[r, "g_*"] & \mathcal{C}^{\mathrm{dq}}, \end{tikzcd}$$

where the left square is equipped with a canonical commuting structure as $F_{\mathcal{C}}$ is canonically $\mathrm{Rep}(G)$-linear, and the right square is equipped with a canonical commuting structure as $F_{\mathcal{C}}^{\mathrm{dq}}$ acquires a canonical $\mathrm{QC}(G)$-linear structure as before. Note that $g_* \circ (-)^{\mathrm{dq}} \simeq (-)^{\mathrm{dq}}$, as the same is true for $(-)^{\mathrm{dq}}\colon \mathrm{Rep}(G) \to \mathrm{Rep}(G)^{\mathrm{dq}} \simeq \mathrm{Vect}_k$ (here, given $V, V' \in \mathrm{Rep}(G)$, the functor g_* acts according to the canonical G-representation on $\mathrm{Hom}_k(V^{\mathrm{dq}}, V'^{,\mathrm{dq}})$). In particular, writing $F_{\mathcal{C},g}^{\mathrm{dq}} := F_{\mathcal{C}}^{\mathrm{dq}} \circ g_* \simeq g_* \circ F_{\mathcal{C}}$ for the "g-twisted" endofunctor, we obtain a 2-morphism of pairs $\mathrm{dq}_g\colon (\mathcal{C}, F_{\mathcal{C}}) \Rightarrow (\mathcal{C}^{\mathrm{dq}}, F_{\mathcal{C},g}^{\mathrm{dq}})$ in $L(\mathrm{Morita}(\mathbf{dgCat}_k))_{\mathrm{rgd}}$, as in §3.1.10.

3.4.5. Lemma. The pair $(\mathcal{C}, F_{\mathcal{C}})$ admits a natural $\mathrm{QC}(X/G)$-enrichment $(\mathcal{C}^{\mathrm{enr}}, F_{\mathcal{C}}^{\mathrm{enr}})$, such that $F_{\mathcal{C}}^{\mathrm{enr}}$ is a ϕ^*-semilinear endofunctor of $\mathcal{C}^{\mathrm{enr}}$ in the sense of Definition 3.4.3. The global sections of the pre-Block–Getzler sheaf and its fibers over G then compute Hochschild homology, i.e., we have a natural commutative diagram

$$\begin{tikzcd} \Gamma({}^{\mathrm{pre}}\mathsf{BG}_{X/G,\phi}(\mathcal{C}^{\mathrm{enr}}, F_{\mathcal{C}}^{\mathrm{enr}})) \arrow[r] \arrow[d, "\wr"] & \Gamma(i_g^*\, {}^{\mathrm{pre}}\mathsf{BG}_{X/G,\phi}(\mathcal{C}^{\mathrm{enr}}, F_{\mathcal{C}}^{\mathrm{enr}})) \arrow[d, "\wr"] \\ \mathrm{HH}(\mathcal{C}, F_{\mathcal{C}}) \arrow[r, "\mathrm{HH}(\mathrm{dq}_{g^{-1}})"] & \mathrm{HH}(\mathcal{C}^{\mathrm{dq}}, F_{\mathcal{C},g^{-1}}^{\mathrm{dq}}), \end{tikzcd}$$

where $i_g\colon X \times \{g\} \to (X \times G)/G$ denotes the natural map, and the top horizontal arrow is induced by the unit of the adjunction $i_g^* \dashv i_{g,*}$.

Proof. For the first assertion, note that by §3.2.10, the category $\mathrm{QC}(X/G)$ is compactly generated and rigid. Thus, for any $c \in \mathcal{C}$, the functor $\mathrm{act}_c\colon \mathrm{QC}(X/G) \to \mathcal{C}$ given by acting on c has a $\mathrm{QC}(X/G)$-linear right adjoint act_c^R by [**GR17a**, Ch. 1, Lem. 9.3.2] (which is furthermore continuous when c is compact). Given $c, c' \in \mathcal{C}$, we define $\underline{\mathcal{H}om}_{\mathcal{C}^{\mathrm{enr}}}(c, c') := \mathrm{act}_c^R(c')$, and let $F_{\mathcal{C}}^{\mathrm{enr}}$ be adjoint to the composition

$$\mathrm{act}_c^R(c') \otimes F_{\mathcal{C}}(c) \to \phi^* \mathrm{act}_c^R(c') \otimes F_{\mathcal{C}}(c) \simeq F_{\mathcal{C}}(\mathrm{act}_c^R(c') \otimes c) \to F_{\mathcal{C}}(c'), \tag{3.4.5.1}$$

where the first map is the natural ϕ^*-semilinear map, and the final map is obtained by applying $F_{\mathcal{C}}$ to the tautological map $\mathrm{act}_c^R(c') \otimes c \to c'$ obtained by adjunction. We leave it to the reader to verify the requisite axioms. Note that taking global sections is right-adjoint to the unit functor $\mathrm{Vect}_k \to \mathrm{QC}(X/G)$, and hence recovers the Hom-spaces in $\mathcal{C}$ (and the original functor $F_{\mathcal{C}}$). Similarly, taking Γ^G and forgetting the G-equivariance recovers the Hom-spaces in $\mathcal{C}^{\mathrm{dq}}$.

For the latter assertion, we have

$$\begin{aligned} &\Gamma({}^{\mathrm{pre}}\mathsf{BG}^{-n}_{X/G,\phi}(\mathcal{C}^{\mathrm{enr}}, F_{\mathcal{C}}^{\mathrm{enr}})) \\ &\simeq \bigoplus_{c_0,\ldots,c_n \in \mathcal{C}^c} \left(\Gamma^G \underline{\mathcal{H}om}_{\mathcal{C}}(c_0, c_1) \otimes_k \cdots \otimes_k \Gamma^G \underline{\mathcal{H}om}_{\mathcal{C}}(c_{n-1}, c_n) \otimes_k \Gamma^G \underline{\mathcal{H}om}_{\mathcal{C}}(c_n, F_{\mathcal{C}}(c_0)) \otimes_k \mathcal{O}(G)\right)^G, \end{aligned}$$

so we recover (a corrected version of[34]) the Block–Getzler complex of [**BZCHN22**, Def. 2.12]. Likewise, we have

$$\begin{aligned}&\Gamma(i_g^{*\,\mathrm{pre}}\mathsf{BG}^{-n}_{X/G,\phi}(\mathcal{C}^{\mathrm{enr}}, F_{\mathcal{C}}^{\mathrm{enr}}))\\ &\simeq \bigoplus_{c_0,\dots,c_n\in\mathcal{C}^c} \mathrm{Hom}_{\mathcal{C}^{\mathrm{dq}}}(c_0^{\mathrm{dq}}, c_1^{\mathrm{dq}})\otimes_k\cdots\otimes_k \mathrm{Hom}_{\mathcal{C}^{\mathrm{dq}}}(c_{n-1}^{\mathrm{dq}}, c_n^{\mathrm{dq}})\otimes_k \mathrm{Hom}_{\mathcal{C}^{\mathrm{dq}}}(c_n^{\mathrm{dq}}, F_{\mathcal{C}}^{\mathrm{dq}}(c_0^{\mathrm{dq}})),\end{aligned}$$

and specializing the coaction map at g shows that the face map d_0 is computed by $F^{\mathrm{dq}}_{\mathcal{C},g^{-1}}$ in place of $\rho(\Gamma^G F_{\mathcal{C}})$. Thus, we recover (a similarly modified version of) the specialized Block–Getzler complex of *loc. cit.*, and the result follows as in [**BZCHN22**, Prop. 2.13]. □

3.4.6. Remark. Note that, as for the cyclic bar complex, we usually need not consider all of $\mathcal{C}^c$ when computing the pre-Block–Getzler sheaf. Rather, it suffices to restrict $c_0,\dots,c_n$ to any set of compact objects of $\mathcal{C}$ which generate under the action of $\mathrm{Rep}(G)$.

We henceforth omit the notation $(-)^{\mathrm{enr}}$ (and write simply ${}^{\mathrm{pre}}\mathsf{BG}_{X/G,\phi}(\mathcal{C}, F_{\mathcal{C}})$ as in Definition 3.4.3) whenever the $\mathrm{QC}(X/G)$-module structure on $\mathcal{C}$ is implicit.

3.4.7. Construction. We now return to the setting of Definition 3.4.3, and lift ${}^{\mathrm{pre}}\mathsf{BG}_{X/G,\phi}(\mathcal{C}, F_{\mathcal{C}})$ to a quasicoherent sheaf $\mathsf{BG}_{X/G,\phi}(\mathcal{C}, F_{\mathcal{C}})$ on $\mathcal{L}_\phi(X/G)$ satisfying

$$\mathrm{ev}_{G,*}\,\mathsf{BG}_{X/G,\phi}(\mathcal{C}, F_{\mathcal{C}}) \simeq {}^{\mathrm{pre}}\mathsf{BG}_{X/G,\phi}(\mathcal{C}, F_{\mathcal{C}}). \tag{3.4.7.1}$$

By (3.2.5.1), this amounts to giving a homotopy between the two actions

$$\mathcal{O}_X\otimes_k {}^{\mathrm{pre}}\mathsf{BG}_{X/G,\phi}(\mathcal{C}, F_{\mathcal{C}}) \rightrightarrows {}^{\mathrm{pre}}\mathsf{BG}_{X/G,\phi}(\mathcal{C}, F_{\mathcal{C}})$$

coming from $\phi\circ\mathrm{act}$ and pr. For each $i = 0,\dots,n$, we have "degeneracy maps"

$$s_i\colon \mathcal{O}_X\otimes_k {}^{\mathrm{pre}}\mathsf{BG}^{-n}_{X/G,\phi}(\mathcal{C}, F_{\mathcal{C}}) \to {}^{\mathrm{pre}}\mathsf{BG}^{-(n+1)}_{X/G,\phi}(\mathcal{C}, F_{\mathcal{C}})$$

defined on each summand by

$$s_i(f\otimes f_0\otimes\cdots\otimes f_n\boxtimes r) = f_0\otimes\cdots\otimes f_{i-1}\otimes f\cdot \mathrm{id}_{c_i}\otimes f_i\otimes\cdots\otimes f_n\boxtimes r.$$

We claim that the collection of maps

$$s^{-n} := \sum_{i=0}^{n}(-1)^i s_i\colon \mathcal{O}_X\otimes_k {}^{\mathrm{pre}}\mathsf{BG}^{-n}_{X/G,\phi}(\mathcal{C}, F_{\mathcal{C}}) \to {}^{\mathrm{pre}}\mathsf{BG}^{-(n+1)}_{X/G,\phi}(\mathcal{C}, F_{\mathcal{C}}) \tag{3.4.7.2}$$

assemble into the desired homotopy; we leave this as an exercise.[35] We refer to the resulting sheaf

$$\mathsf{BG}_{X/G,\phi}(\mathcal{C}, F_{\mathcal{C}}) \in \mathrm{QC}(\mathcal{L}_\phi(X/G))$$

as the *Block–Getzler sheaf* of the pair $(\mathcal{C}, F_{\mathcal{C}})$.

---

[34] Note that as currently stated, the Block–Getzler complex of [**BZCHN22**, Def. 2.11] does *not* in general constitute a simplicial object, as $d_n s_{n-1}$, for an appropriately defined degeneracy map s_{n-1}, is given by applying $F_{\mathcal{C}}$, rather than the identity. To correct this error, note that the cyclic bar complex of a dg category $\mathcal{C}$ is obtained by computing the (derived) tensor product

$$\mathcal{C}_\Delta \otimes_{\mathcal{C}\otimes\mathcal{C}^{\mathrm{op}}} {}_{F_{\mathcal{C}}}\mathcal{C}_\Delta \tag{3.4.5.2}$$

of $(\mathcal{C},\mathcal{C})$-bimodules, i.e., functors $\mathcal{C}\otimes\mathcal{C}^{\mathrm{op}}\to\mathrm{Vect}_k$ (see for instance [**Kel06**, §3] or [**GHW22**, §2.4, §5.1]). Here the "diagonal bimodule" $\mathcal{C}_\Delta$ is given by the functor $c'\otimes c\mapsto \mathrm{Hom}_{\mathcal{C}}(c,c')$, and ${}_{F_{\mathcal{C}}}\mathcal{C}_\Delta$ denotes its precomposition with $F_{\mathcal{C}}\otimes\mathrm{id}_{\mathcal{C}^{\mathrm{op}}}$. This tensor product may be computed using the usual bar resolution of $\mathcal{C}_\Delta$: an element of the degree-$(-n)$ term of the bar resolution for $c'\otimes c\in\mathcal{C}\otimes\mathcal{C}^{\mathrm{op}}$ is given by

$$f_{-1}\otimes\cdots\otimes f_n \in \mathrm{Hom}_{\mathcal{C}}(c,c_0)\otimes_k \mathrm{Hom}_{\mathcal{C}}(c_0,c_1)\otimes_k\cdots\otimes_k \mathrm{Hom}_{\mathcal{C}}(c_n,c')$$

for some $c_0,\dots,c_n\in\mathcal{C}^c$, and therefore an element of the degree-$(-n)$ term after tensoring with ${}_{F_{\mathcal{C}}}\mathcal{C}_\Delta$ is given by

$$f_0\otimes\cdots\otimes F_{\mathcal{C}}(f_{-1})f_n \in \mathrm{Hom}_{\mathcal{C}}(c_0,c_1)\otimes_k\cdots\otimes_k \mathrm{Hom}_{\mathcal{C}}(c_n, F_{\mathcal{C}}(c_0)).$$

It is then straightforward to verify that the usual face maps yield those in (3.4.3.1).

[35] The only terms of $d\circ s^{-n} + s^{-(n-1)}\circ d$ that survive are

$$(d_0 s_0 - d_{n+1}s_n)(f\otimes f_0\otimes\cdots\otimes f_n\boxtimes r) = (\gamma(\phi^*(f)) - f)\cdot(f_0\otimes\cdots\otimes f_n\boxtimes r),$$

using the ϕ^*-semilinearity of $F_{\mathcal{C}}$.

3.4.8. We now describe the setting in which the Block–Getzler sheaf computes the 2-categorical class map. Let $\mathcal{A}$ be a compactly generated rigid monoidal category, and let

$$\Psi\colon \mathrm{QC}(X/G) \to \mathcal{A} \tag{3.4.8.1}$$

be a monoidal functor admitting a central structure, i.e., a factorization through the Drinfeld center of $\mathcal{A}$ as in §3.1.16. Suppose that we are given a monoidal endofunctor $F_{\mathcal{A}}$ of $\mathcal{A}$ (which automatically preserves compact objects as in §3.1.10), equipped with an isomorphism $F_{\mathcal{A}} \circ \Psi \simeq \Psi \circ \phi^*$. Then as in (3.1.14.3), we have adjoint functors

$$\mathrm{Tr}(\mathrm{Ind}_\Psi)\colon \mathrm{QC}(\mathcal{L}_\phi(X/G)) \simeq \mathrm{Tr}(\mathrm{QC}(X/G), \phi^*) \rightleftarrows \mathrm{Tr}(\mathcal{A}, F_{\mathcal{A}})\colon \mathrm{Tr}(\mathrm{Res}_\Psi), \tag{3.4.8.2}$$

where the first identification follows from Proposition 3.2.11.

Now let $\mathcal{M}$ be a compactly generated right-dualizable $\mathcal{A}$-module category equipped with an $F_{\mathcal{A}}$-semilinear endofunctor $F_{\mathcal{M}}$ which preserves compact objects. As in §3.1.6, its 2-categorical class is an object $[\mathcal{M}, F_{\mathcal{M}}] \in \mathrm{Tr}(\mathcal{A}, F_{\mathcal{A}})$. Our goal is to compute the quasi-coherent sheaf $\mathrm{Tr}(\mathrm{Res}_\Psi)([\mathcal{M}, F_{\mathcal{M}}])$ on $\mathcal{L}_\phi(X/G))$. Note that Ψ gives $\mathrm{QC}(X/G)$-module structures on both $\mathcal{A}$ and $\mathcal{M}$, and that both $F_{\mathcal{A}}$ and $F_{\mathcal{M}}$ are canonically ϕ^*-semilinear with respect to these structures. Thus, the pairs $(\mathcal{A}, F_{\mathcal{A}})$ and $(\mathcal{M}, F_{\mathcal{M}})$ both admit $\mathrm{QC}(X/G)$-enrichments by Lemma 3.4.5, and it is not hard to check that these are compatible with the $\mathcal{A}$-module structure on $\mathcal{M}$.

3.4.9. Remark. Note that, in the terminology of [**GR17a**, Ch. 1, §3.6], the $\mathrm{QC}(X/G)$-enrichments are given by the relative inner Hom with respect to $\mathrm{QC}(X/G)$. In the above setup, both $\mathcal{A}$ and $\mathcal{M}$ also admit relative inner Hom spaces with respect to $\mathcal{A}$, which we denote by $\underline{\mathrm{Hom}}_{\mathcal{A}}$. We may then define the $\mathrm{QC}(X/G)$-enrichments by $\underline{\mathcal{H}om}_{\mathcal{A}^{\mathrm{enr}}}(a, a') := \Psi^R \underline{\mathrm{Hom}}_{\mathcal{A}}(a, a')$ for $a, a' \in \mathcal{A}$, and similarly for $\mathcal{M}$. Likewise, $F_{\mathcal{A}}^{\mathrm{enr}}$ may be described by the composition

$$\Psi^R \underline{\mathrm{Hom}}_{\mathcal{A}}(a, a') \to \phi^* \Psi^R \underline{\mathrm{Hom}}_{\mathcal{A}}(a, a') \to \Psi^R F(\underline{\mathrm{Hom}}_{\mathcal{A}}(a, a')) \simeq \Psi^R \underline{\mathrm{Hom}}_{\mathcal{A}}(F(a), F(a')),$$

where the first map is the natural ϕ^*-semilinear map, and the second map is the usual adjunction base-change map. As usual, we omit the superscripts $(-)^{\mathrm{enr}}$ elsewhere in this document.

3.4.10. We now state the main result of this section:

3.4.11. Proposition. In the setup of §3.4.8, we have a canonical equivalence

$$\mathsf{BG}_{X/G,\phi}(\mathcal{M}, F_{\mathcal{M}}) \simeq \mathrm{Tr}(\mathrm{Res}_\Psi)([\mathcal{M}, F_{\mathcal{M}}]). \tag{3.4.11.1}$$

Proof. As in §3.1.12, it suffices to give a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{L}_\phi(X/G)}([\mathcal{F}], \mathsf{BG}_{X/G,\phi}(\mathcal{M}, F_{\mathcal{M}})) \simeq \mathrm{Hom}_{\mathcal{L}_\phi(X/G)}([\mathcal{F}], \mathrm{Tr}(\mathrm{Res}_\Psi)([\mathcal{M}, F_{\mathcal{M}}])) \tag{3.4.11.2}$$

for each $\mathcal{F} \in \mathrm{Perf}(X/G)$. We begin by unwinding the left-hand side. By Proposition 3.2.11, the universal trace functor is given by

$$[-] \simeq \mathrm{ev}^*\colon \mathrm{QC}(X/G) \to \mathrm{QC}(\mathcal{L}_\phi(X/G)). \tag{3.4.11.3}$$

Thus, adjunction and duality yield

$$\begin{aligned} \mathrm{Hom}_{\mathcal{L}_\phi(X/G)}([\mathcal{F}], \mathsf{BG}_{X/G,\phi}(\mathcal{M}, F_{\mathcal{M}})) &\simeq \mathrm{Hom}_{X/G}(\mathcal{F}, \mathrm{ev}_* \mathsf{BG}_{X/G,\phi}(\mathcal{M}, F_{\mathcal{M}})) \\ &\simeq \Gamma(\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathrm{ev}_* \mathsf{BG}_{X/G,\phi}(\mathcal{M}, F_{\mathcal{M}})) \\ &\simeq \Gamma(\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathrm{pr}_* {}^{\mathrm{pre}}\mathsf{BG}_{X/G,\phi}(\mathcal{M}, F_{\mathcal{M}})) \end{aligned}$$

by (3.2.5.1). Moreover, as in the proof of Lemma 3.4.5, the functor $\underline{\mathrm{Hom}}_{\mathcal{M}}(m, -)$ is $\mathrm{QC}(X/G)$-linear for any $m \in \mathcal{M}$. It follows by construction that

$$\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathrm{pr}_* {}^{\mathrm{pre}}\mathsf{BG}_{X/G,\phi}(\mathcal{M}, F_{\mathcal{M}}) \simeq \mathrm{pr}_* {}^{\mathrm{pre}}\mathsf{BG}_{X/G,\phi}(\mathcal{M}, \Psi(\mathcal{F}^\vee) \otimes F_{\mathcal{M}}(-)),$$

where the latter functor is $F_{\mathcal{A}}$-semilinear by our centrality assumption on Ψ, and preserves compact objects as X/G is perfect. Finally, Lemma 3.4.5 implies that

$$\Gamma(\mathrm{pr}_* {}^{\mathrm{pre}}\mathsf{BG}_{X/G,\phi}(\mathcal{M}, \Psi(\mathcal{F}^\vee) \otimes F_{\mathcal{M}}(-))) \simeq \mathrm{HH}(\mathcal{M}, \Psi(\mathcal{F}^\vee) \otimes F_{\mathcal{M}}(-)). \tag{3.4.11.4}$$

Next, we unwind the right-hand side of (3.4.11.2). By adjunction, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{L}_\phi(X/G)}([\mathcal{F}], \mathrm{Tr}(\mathrm{Res}_\Psi)([\mathcal{M}, F_{\mathcal{M}}])) &\simeq \mathrm{Hom}_{\mathrm{Tr}(\mathcal{A}, F_{\mathcal{A}})}(\mathrm{Tr}(\mathrm{Ind}_\Psi)([\mathcal{F}]), [\mathcal{M}, F_{\mathcal{M}}]) \\ &\simeq \mathrm{Hom}_{\mathrm{Tr}(\mathcal{A}, F_{\mathcal{A}})}([\Psi(\mathcal{F})], [\mathcal{M}, F_{\mathcal{M}}]). \end{aligned} \tag{3.4.11.5}$$

Moreover, the adjunction of (3.1.16.3) and the identity (3.1.16.4) yield

$$\mathrm{Hom}_{\mathrm{Tr}(\mathcal{A},F)}([\Psi(\mathcal{F})],[\mathcal{M},F_{\mathcal{M}}]) \simeq \mathrm{Hom}_{\mathrm{Tr}(\mathcal{A},F_{\mathcal{A}})}([\mathcal{A},F_{\mathcal{A}}],[\mathcal{M},\Psi(\mathcal{F}^\vee)\otimes F_{\mathcal{M}}(-)]). \tag{3.4.11.6}$$

The conclusion now follows from Theorem 3.1.11 and (3.4.11.4). □

3.4.12. Combining this result with (3.1.7) and Corollary 3.2.13, we obtain:

3.4.13. Corollary. Suppose we are in the setup of §3.2.4, and that $\mathcal{X},\mathcal{Y}$ are both perfect. Assume moreover that $\mathcal{Y}\simeq Y/G$ for a scheme Y and reductive group G, and that $\phi_{\mathcal{Y}}$ lifts to a G-equivariant self-map $\phi_Y\colon Y\to Y$. Then:

(1) For any $\mathcal{F}\in\mathrm{Perf}(\mathcal{X})$, we have

$$\mathcal{L}p_*\,\mathrm{ev}^*\,\mathcal{F}\simeq \mathsf{BG}_{Y/G,\phi_Y}(\mathrm{QC}(\mathcal{X}),\phi_{\mathcal{X}}^*(-)\otimes\mathcal{F}).$$

In particular, we have

$$\mathcal{L}p_*\mathcal{O}_{\mathcal{L}_{\phi_{\mathcal{X}}}\mathcal{X}}\simeq \mathsf{BG}_{Y/G,\phi_Y}(\mathrm{QC}(\mathcal{X}),\phi_{\mathcal{X}}^*).$$

(2) Suppose we are moreover in the setting of Corollary 3.3.11. For any $\mathcal{F}\in\mathrm{Coh}(\mathcal{X}\times_{\mathcal{Y}}\mathcal{X})$, we have

$$[\mathcal{F}]^{\mathrm{QC}}\simeq \mathsf{BG}_{Y/G,\phi_Y}(\mathrm{QC}(\mathcal{X}),\phi_{\mathcal{X}}^*(-)*\mathcal{F}).$$

3.4.14. Remark. We will leave the central structure on the functor (3.4.8.1) implicit throughout this thesis. However, it can always be described explicitly; for instance, in the setup of Corollary 3.4.13(1), the Drinfeld center of $\mathrm{QC}(\mathcal{X})$ is computed by the functor $\mathrm{ev}_*\colon \mathrm{QC}(\mathcal{L}\mathcal{X})\to\mathrm{QC}(\mathcal{X})$ as in [**BZFN10**, Cor. 5.2]. Writing $i_{\mathcal{X}}\colon\mathcal{X}\to\mathcal{L}\mathcal{X}$ for the inclusion of the constant loops, we obtain a factorization $\mathcal{L}p^*\simeq \mathrm{ev}_*\circ(i_{\mathcal{X},*}\mathcal{L}p^*)$, as desired.

3.4.15. Finally, we describe functoriality of the Block–Getzler sheaf in the pair $(\mathcal{M},F_{\mathcal{M}})$. Suppose we are given another such pair $(\mathcal{N},F_{\mathcal{N}})$, and a morphism $(\gamma,\theta)\colon(\mathcal{M},F_{\mathcal{M}})\to(\mathcal{N},F_{\mathcal{N}})$ in $(\mathcal{A},F_{\mathcal{A}})$-**mod**, i.e., an $\mathcal{A}$-linear functor $\gamma\colon\mathcal{M}\to\mathcal{N}$ admitting an $\mathcal{A}$-linear right adjoint, and a natural transformation $\theta\colon\gamma\circ F_{\mathcal{M}}\Rightarrow F_{\mathcal{N}}\circ\gamma$ of functors ${}_{F_{\mathcal{A}}}\mathcal{M}\to{}_{F_{\mathcal{A}}}\mathcal{N}$. As in §3.1.6, we obtain a morphism $[\gamma,\theta]\colon[\mathcal{M},F_{\mathcal{M}}]\to[\mathcal{N},F_{\mathcal{N}}]$ between 2-categorical classes in $\mathrm{Tr}(\mathcal{A},F_{\mathcal{A}})$.

On the other hand, we can construct a morphism

$$\mathsf{BG}_{X/G,\phi}(\gamma,\theta)\colon\ \mathsf{BG}_{X/G,\phi}(\mathcal{M},F_{\mathcal{M}})\to\mathsf{BG}_{X/G,\phi}(\mathcal{N},F_{\mathcal{N}}) \tag{3.4.15.1}$$

in $\mathrm{QC}(\mathcal{L}_\phi(X/G))$ associated to the pair (γ,θ). Indeed, for any $m,m'\in\mathcal{M}$, we have a natural map

$$\gamma\colon\underline{\mathcal{H}om}_{\mathcal{M}}(m,m')\to\underline{\mathrm{Hom}}_{\mathcal{N}}(\gamma(m),\gamma(m'))$$

in $\mathrm{QC}(X/G)$ (constructed analogously to (3.4.5.1)), hence a composition

$$\underline{\mathcal{H}om}_{\mathcal{M}}(m,F_{\mathcal{M}}(m'))\xrightarrow{\gamma}\underline{\mathcal{H}om}_{\mathcal{N}}(\gamma(m),\gamma F_{\mathcal{M}}(m'))\xrightarrow{\theta\circ-}\underline{\mathcal{H}om}_{\mathcal{N}}(\gamma(m),F_{\mathcal{N}}\gamma(m')).$$

It is then not hard to check that the maps

$$\begin{aligned}{}^{\mathrm{pre}}\mathsf{BG}^{-n}_{X/G,\phi}(\gamma,\theta)\colon\ {}^{\mathrm{pre}}\mathsf{BG}^{-n}_{X/G,\phi}(\mathcal{M},F_{\mathcal{M}})&\to{}^{\mathrm{pre}}\mathsf{BG}^{-n}_{X/G,\phi}(\mathcal{N},F_{\mathcal{N}})\\ f_0\otimes\cdots\otimes f_{n-1}\otimes f_n\boxtimes r&\mapsto\gamma(f_0)\otimes\cdots\otimes\gamma(f_{n-1})\otimes\theta\circ\gamma(f_n)\boxtimes r\end{aligned}$$

commute with all face maps (3.4.3.1), as well as the homotopy (3.4.7.2), and hence extend to a map as in (3.4.15.1).

These two constructions are compatible under the identification of Proposition 3.4.11:

3.4.16. Proposition. In the above setup, the diagram

$$\begin{tikzcd}
\mathsf{BG}_{X/G,\phi}(\mathcal{M},F_{\mathcal{M}}) \arrow[r, "\mathsf{BG}_{X/G,\phi}(\gamma,\theta)"] \arrow[d, "\wr"] & \mathsf{BG}_{X/G,\phi}(\mathcal{N},F_{\mathcal{N}}) \arrow[d, "\wr"] \\
\mathrm{Tr}(\mathrm{Res}_\Psi)([\mathcal{M},F_{\mathcal{M}}]) \arrow[r, "\mathrm{Tr}(\mathrm{Res}_\Psi)([\gamma,\theta])"] & \mathrm{Tr}(\mathrm{Res}_\Psi)([\mathcal{N},F_{\mathcal{N}}])
\end{tikzcd}$$

commutes (up to a natural isomorphism).

Proof. Let $\mathcal{F} \in \mathrm{Perf}(X/G)$. As in the proof of Proposition 3.4.11, it suffices to show that each of the squares

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{L}_\phi(X/G)}([\mathcal{F}], \mathsf{BG}_{X/G,\phi}(\mathcal{M}, F_{\mathcal{M}})) \arrow[r, "\mathsf{BG}_{X/G,\phi}(\gamma,\theta)\circ -"] \arrow[d, "\wr"] & \mathrm{Hom}_{\mathcal{L}_\phi(X/G)}([\mathcal{F}], \mathsf{BG}_{X/G,\phi}(\mathcal{N}, F_{\mathcal{N}})) \arrow[d, "\wr"] \\
\mathrm{HH}(\mathcal{M}, \Psi(\mathcal{F}^\vee) \otimes F_{\mathcal{M}}(-)) \arrow[r, "\mathrm{HH}(\gamma, \mathrm{id}_{\Psi(\mathcal{F}^\vee)} \otimes \theta)"] & \mathrm{HH}(\mathcal{N}, \Psi(\mathcal{F}^\vee) \otimes F_{\mathcal{N}}(-)) \\
\mathrm{Hom}_{\mathcal{L}_\phi(X/G)}([\mathcal{F}], \mathrm{Tr}(\mathrm{Res}_\Psi)([\mathcal{M}, F_{\mathcal{M}}])) \arrow[r, "\mathrm{Tr}(\mathrm{Res}_\Psi)([\gamma,\theta])\circ -"] \arrow[u, "\wr"] & \mathrm{Hom}_{\mathcal{L}_\phi(X/G)}([\mathcal{F}], \mathrm{Tr}(\mathrm{Res}_\Psi)([\mathcal{N}, F_{\mathcal{N}}])) \arrow[u, "\wr"]
\end{tikzcd}$$

commutes. Commutativity of the lower square is immediate from unwinding the adjunctions (3.4.11.5) and (3.4.11.6). Similarly, as in (3.4.11.4), it suffices to show that the upper square commutes with the top row replaced by the morphism

$$\Gamma({}^{\mathrm{pre}}\mathsf{BG}_{X/G,\phi}(\gamma, \mathrm{id}_{\Psi(\mathcal{F}^\vee)} \otimes \theta))\colon \Gamma({}^{\mathrm{pre}}\mathsf{BG}_{X/G,\phi}(\mathcal{M}, \Psi(\mathcal{F}^\vee) \otimes F_{\mathcal{M}}(-))) \to \Gamma({}^{\mathrm{pre}}\mathsf{BG}_{X/G,\phi}(\mathcal{N}, \Psi(\mathcal{F}^\vee) \otimes F_{\mathcal{N}}(-))).$$

As in the proofs of Lemma 3.4.5 and [**BZCHN22**, Prop. 2.13], this amounts to the analogous functoriality of the cyclic bar complex for a dg-category.[36] □

3.4.17. Thus, the complex $\mathsf{BG}_{X/G,\phi}(-)$ essentially computes the functor

$$\mathrm{Tr}(\mathrm{Res}_\Psi)([-])\colon (\mathcal{A}, F_{\mathcal{A}})\textbf{-mod} \to \mathrm{QC}(\mathcal{L}_\phi(X/G))$$

composed from (3.1.6.1) and (3.4.8.2).

3.4.18. Remark. We expect that, when applicable, the S^1-equivariant structure on the Block–Getzler sheaf may be described by an explicit homotopy, analogous to Connes' boundary operator B on the usual cyclic bar complex. This will not be needed in the sequel, so we leave the details to a future work.

[36]This is well-known, though we could not find a precise reference in the literature (see [**Kel06**, Thm. 5.2(a)] for the case where the endofunctors and natural transformation θ are trivial). Regardless, one can see this directly by examining the functoriality of (3.4.5.2) and (3.1.10.1), using the same general pattern as in Lemma 3.1.15.

CHAPTER 4

Bounding the universal trace functor

4.1. Resolutions of Koszul algebras with multiple simple modules

4.1.1. In this section, we construct a Koszul resolution of the regular bimodule for a Koszul algebra with multiple simple modules. This material is logically independent from the rest of the text, and likely well-known, but we could find no reference in the literature. We will apply this result in §4.3 to the noncommutative Springer resolution, to obtain a bounded complex computing the universal trace functor.

4.1.2. Let A be a non-negatively graded classical k-algebra, and write $A \simeq \bigoplus_{i\in I} E_i$ for the decomposition of the regular right A-module into indecomposable projectives, where I is some finite indexing set. Assume that the $\{E_i\}$ are pairwise nonisomorphic, and let $\{L_i\}$ denote the corresponding simple modules (concentrated in weight 0). As for any k-algebra, the *bar complex* $\mathrm{Bar}_A^\bullet$ is the acyclic complex of A-bimodules

$$\cdots \to A\otimes A\otimes A \xrightarrow{a\otimes b\otimes c\mapsto ab\otimes c - a\otimes bc} A\otimes A \xrightarrow{a\otimes b\mapsto ab} A \to 0 \to 0 \to \cdots$$

concentrated in degrees ≤ 1 (and free in degrees ≤ 0), with differentials given by the usual alternating sum of face maps. Consider the subcomplex $\mathrm{Bar}^\bullet_{\{E_i\}} \subseteq \mathrm{Bar}^\bullet_A$ defined by

$$\mathrm{Bar}^{-n}_{\{E_i\}} := \bigoplus_{i_0,\dots,i_{n+2}\in I} \mathrm{Hom}_{A^{\mathrm{op}}}(E_{i_0},E_{i_1}) \otimes_k \mathrm{Hom}_{A^{\mathrm{op}}}(E_{i_1},E_{i_2}) \otimes_k \cdots \otimes_k \mathrm{Hom}_{A^{\mathrm{op}}}(E_{i_{n+1}},E_{i_{n+2}}) \tag{4.1.2.1}$$

for each $n \geq -1$, which is evidently preserved under each face map $d_1,\dots,d_{n+1}$. It is a complex of A-bimodules (each of which is projective in degrees ≤ 0) via the algebra isomorphism $A \cong \mathrm{End}_{A^{\mathrm{op}}}(\bigoplus_{i\in I} E_i)$, and the restriction of the extra degeneracy from $\mathrm{Bar}^\bullet_A$ exhibits it as acyclic.

Now, for each $n\geq -1$ and $i,j\in I$, consider the summand ${}_i(\mathrm{Bar}^{-n}_{\{E_i\}})_j \subseteq \mathrm{Bar}^{-n}_{\{E_i\}}$ given by setting $i_0 = i$, $i_{n+2} = j$, and letting $i_1,\dots,i_{n+1}\in I$ be arbitrary. Furthermore, let ${}_i(\mathrm{Bar}^{-n}_{\{E_i\}})^{1,\dots,1}_j \subseteq {}_i(\mathrm{Bar}^{-n}_{\{E_i\}})_j$ be the subspace spanned by terms whose $n+2$ tensor factors all lie in weight 1.[37] Set

$${}_i(A^{!,*}_n)_j := \bigcap_{m=1}^{n} \ker\left(d_m \colon {}_i(\mathrm{Bar}^{-n+2}_{\{E_i\}})^{1,\dots,1}_j \to {}_i(\mathrm{Bar}^{-n+3}_{\{E_i\}})_j\right).$$

for each $n\geq 1$; it is clearly a finite-dimensional vector space concentrated in weight n (for instance, we have ${}_i(A^{!,*}_1)_j = \mathrm{Hom}_{A^{\mathrm{op}}}(E_i,E_j)_1 \subseteq A_1$). Consider the subcomplex $\mathrm{Kos}^\bullet_A \subseteq \mathrm{Bar}^\bullet_{\{E_i\}}$ given by

$$\mathrm{Kos}^{-n}_A := \bigoplus_{i_0,i_1,i_{n+1},i_{n+2}\in I} \mathrm{Hom}_{A^{\mathrm{op}}}(E_{i_0},E_{i_1}) \underset{k}{\otimes} {}_{i_1}(A^{!,*}_n)_{i_{n+1}} \underset{k}{\otimes} \mathrm{Hom}_{A^{\mathrm{op}}}(E_{i_{n+1}},E_{i_{n+2}})$$

for each $n\geq 1$, and $\mathrm{Kos}^{-n}_A := \mathrm{Bar}^{-n}_{\{E_i\}}$ otherwise. Equivalently, let E_i^ℓ be the indecomposable projective *left* A-module corresponding to E_i (i.e., generated by the same primitive idempotent), and L_i^ℓ be the corresponding simple module. Setting ${}_i(A^{!,*}_0)_j := \mathrm{Hom}_{A^{\mathrm{op}}}(E_i,E_j)_0$, we may write

$$\mathrm{Kos}^{-n}_A = \bigoplus_{i,j\in I} E_i^\ell \underset{k}{\otimes} {}_i(A^{!,*}_n)_j \underset{k}{\otimes} E_j \tag{4.1.2.2}$$

for each $n\geq 0$, which is a projective A-bimodule. Note that the differential on Kos^{-n}_A is given by the restriction of $d_1 + (-1)^n d_{n+1}$, whose image is clearly contained in Kos^{-n+1}_A.

4.1.3. Recall that A is *Koszul* if for any $i,j\in I$ and $n\geq 0$, the graded vector space $\mathrm{Ext}^n_{A^{\mathrm{op}}}(L_i,L_j)$ is concentrated in weight $-n$. We may now state the main result of this section:

[37] Here we depart slightly from our usual convention of placing weights in subscripts, due to space limitations. This notation will not be used outside of this section.

4.1.4. PROPOSITION. If A is Koszul, then the complex $\mathrm{Kos}_A^\bullet$ is acyclic.

PROOF. It suffices to show that $L_i \otimes_A \mathrm{Kos}_A^\bullet \simeq 0$ for each $i \in I$. Indeed, suppose the complex $\mathrm{Kos}_A^\bullet$ of projective left A-modules has its highest nonzero cohomology in degree d. Then the convergent spectral sequence

$$E_2^{s,t} = \mathrm{H}^s(L_i \otimes_A \mathrm{H}^t(\mathrm{Kos}_A^\bullet)) \Longrightarrow \mathrm{H}^{s+t}(L_i \otimes_A \mathrm{Kos}_A^\bullet) = 0$$

shows that $\mathrm{H}^0(L_i \otimes_A \mathrm{H}^d(\mathrm{Kos}_A^\bullet)) = 0$ for each $i \in I$, and hence $\mathrm{H}^d(\mathrm{Kos}_A^\bullet) = 0$ by the graded version of Nakayama's lemma, a contradiction.

Consider the normalized version $\mathrm{NBar}_{E_i}^\bullet$ of the complex (4.1.2.1) obtained by quotienting by the subspaces generated by the degeneracies of the associated simplicial object. Explicitly, we have

$$\mathrm{NBar}_{\{E_i\}}^{-n} \simeq \bigoplus_{i_1,\dots,i_{n+1} \in I} E_{i_1}^\ell \otimes_k \mathrm{Hom}_{A^{\mathrm{op}}}(E_{i_1}, E_{i_2})_{\geq 1} \otimes_k \cdots \otimes_k \mathrm{Hom}_{A^{\mathrm{op}}}(E_{i_n}, E_{i_{n+1}})_{\geq 1} \otimes_k E_{i_{n+1}} \tag{4.1.4.1}$$

for each $n \geq 0$ (where the subscripts denote the strictly positive components), and the natural inclusion $\mathrm{NBar}_{\{E_i\}}^\bullet \to \mathrm{Bar}_{\{E_i\}}^\bullet \simeq 0$ is a quasi-isomorphism. Thus, it suffices to show that the natural map

$$L_i \otimes_A \mathrm{Kos}_A^\bullet \to L_i \otimes_A \mathrm{NBar}_{\{E_i\}}^\bullet \tag{4.1.4.2}$$

is a quasi-isomorphism. In fact, it suffices to show that it is a quasi-isomorphism after applying $- \otimes_A L_j^\ell$, for any $j \in I$: indeed, the cone of (4.1.4.2) is a complex of projective right A-modules (concentrated in degrees ≤ 0), so we may apply the argument used in our first paragraph.

Let $j \in I$, and observe that

$$L_i \otimes_A \mathrm{Kos}_A^{\leq 0} \otimes_A L_j^\ell \simeq \bigoplus_{n \geq 0} {}_i(A_n^{!,*})_j^*[n],$$

where $\mathrm{Kos}_A^{\leq 0}$ denotes the "naïve" truncation of $\mathrm{Kos}_A^\bullet$. Since each graded component of $\mathrm{NBar}_{\{E_i\}}^\bullet$ is perfect over k (as $\mathrm{NBar}_{\{E_i\}}^{-n}$ lies in weights $\geq n$), and since $\mathrm{Hom}_{A^{\mathrm{op}}}(E_{j'}, L_j) \simeq E_{j'} \otimes_A L_j^\ell$ for any $j' \in I$, we have

$$\mathrm{Hom}_{A^{\mathrm{op}}}(L_i, L_j) \simeq \mathrm{Hom}_{A^{\mathrm{op}}}(L_i \otimes_A \mathrm{NBar}_{\{E_i\}}^{\leq 0}, L_j) \simeq (L_i \otimes_A \mathrm{NBar}_{\{E_i\}}^{\leq 0} \otimes_A L_j^\ell)^*.$$

Thus, Koszulity of A implies that the cohomology of $L_i \otimes_A \mathrm{NBar}_{\{E_i\}}^{\leq 0} \otimes_A L_j^\ell$ lies only in degree $-n$ and weight n for $n \geq 0$. It follows that $\mathrm{H}^{-n}(L_i \otimes_A \mathrm{NBar}_{\{E_i\}}^{\leq 0} \otimes_A L_j^\ell)$ is given by the kernel of the usual differential on

$$\bigoplus_{i_1,\dots,i_{n+1} \in I} \mathrm{Hom}_{A^{\mathrm{op}}}(E_{i_1}, E_{i_2})_1 \otimes_k \cdots \otimes_k \mathrm{Hom}_{A^{\mathrm{op}}}(E_{i_n}, E_{i_{n+1}})_1$$

for each $n \geq 0$. Now, $L_i \otimes_A \mathrm{NBar}_{\{E_i\}}^{-n+1} \otimes_A L_j^\ell$ is graded by $(n-1)$-tuples of weights; since each face map lands in a distinct such tuple, this kernel coincides with ${}_i(A_n^{!,*})_j^*$, as desired. $\square$

4.1.5. In particular, we have shown:

4.1.6. COROLLARY. Suppose A is Koszul. Then for each $n \geq 0$ and $i, j \in I$, we have

$${}_i(A_n^{!,*})_j \simeq \mathrm{Ext}_{A^{\mathrm{op}}}^n(L_i, L_j)^*.$$

4.2. The exotic t-structure

4.2.1. We now give an "intrinsic" reformulation of braid positivity using an exotic t-structure on $\mathcal{H}^{\mathrm{coh}}$ provided by work of Bezrukavnikov–Losev. Intuitively, we would like to identify $\mathcal{H}^{\mathrm{coh}}$ with the category of $\widetilde{G}$-equivariant $\mathcal{A}_{\mathfrak{g}}^{\mathrm{op}}$-bimodules as in (2.2.4.1); however, such an identification would not preserve compact objects (for instance, the regular bimodule is not perfect, just as for the unit sheaf $\Delta_{\widetilde{\mathcal{N}}/\mathfrak{g},*}\mathcal{O}_{\widetilde{\mathcal{N}}}$), so we must "renormalize" the latter as for $\mathcal{H}^{\mathrm{coh}}$. Our presentation follows Preygel's treatment in [**Pre15**, §4].

We begin by recalling the basic properties of t-structures that we shall need:

4.2.2. DEFINITION. A t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ on a dg-category $\mathcal{C}$ is

(1) *accessible* if the subcategory $\mathcal{C}^{\geq 0}$ (or equivalently, $\mathcal{C}^{\leq 0}$) is compactly generated;
(2) *compatible with filtered colimits* if $\mathcal{C}^{\geq 0}$ is closed under filtered colimits in $\mathcal{C}$;
(3) *right-complete* if the inclusion functors induce an equivalence[38] $\mathrm{colim}_n\, \mathcal{C}^{\leq n} \to \mathcal{C}$; and

[38]Equivalently, by [**GR17a**, Ch. 1, Prop. 2.5.7], the truncation functors induce an equivalence $\mathcal{C} \to \lim_n \mathcal{C}^{\leq n}$.

(4) *coherent* if it is compatible with filtered colimits, right-complete, and the compact objects $\mathcal{C}^{\heartsuit,c}$ (in the classical sense) form a generating abelian subcategory of the heart $\mathcal{C}^{\heartsuit}$.

Moreover, suppose that $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is compatible with filtered colimits. We say that an object $X \in \mathcal{C}$ is *coherent* if

(1) X is bounded below, i.e., $X \in \mathcal{C}^{\geq n}$ for some n; and
(2) $\tau^{\geq n}X$ is a compact object of $\mathcal{C}^{\geq n}$ for all n.

We denote the full subcategory of coherent objects of $\mathcal{C}$ by $\mathrm{Coh}(\mathcal{C})$, and refer to the compactly generated category $\mathcal{C}_{\mathrm{ren}} := \mathrm{Ind}(\mathrm{Coh}(\mathcal{C}))$ as the *renormalization*[39] of $\mathcal{C}$. The latter carries a natural t-structure given by

$$\mathcal{C}_{\mathrm{ren}}^{\leq 0} := \mathrm{Ind}(\mathrm{Coh}(\mathcal{C}) \cap \mathcal{C}^{\leq 0}), \qquad \mathcal{C}_{\mathrm{ren}}^{\geq 0} := \mathrm{Ind}(\mathrm{Coh}(\mathcal{C}) \cap \mathcal{C}^{\geq 0}).$$

4.2.3. We now recall some of Preygel's general results regarding these constructions. Let $\mathcal{C}$ be as in the latter part of Definition 4.2.2. If $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is coherent, then the subcategory $\mathrm{Coh}(\mathcal{C})$ consists of cohomologically bounded objects whose cohomologies all lie in $\mathrm{Coh}(\mathcal{C}) \cap \mathcal{C}^{\heartsuit} = \mathcal{C}^{\heartsuit,c}$. Moreover, the natural continuous functor $\mathcal{C}_{\mathrm{ren}} \to \mathcal{C}$ obtained by Ind-extension is t-exact, and induces an equivalence on coconnective (or more generally, bounded below) objects, i.e., $\mathcal{C}_{\mathrm{ren}}^{\geq 0} \to \mathcal{C}^{\geq 0}$. In particular, the t-structure on $\mathcal{C}_{\mathrm{ren}}$ is again coherent, and moreover accessible.[40]

The prototypical example is as follows: if $\mathcal{X}$ is a QCA stack, then the standard t-structure on $\mathrm{QC}(\mathcal{X})$ is coherent and accessible, and $\mathrm{QC}^!(\mathcal{X})$ identifies with the renormalization of $\mathrm{QC}(\mathcal{X})$ (moreover, $\mathrm{QC}(\mathcal{X})$ is the "left-completion" of $\mathrm{QC}^!(\mathcal{X})$, though we shall not need or define this notion).

The following lemma will allow us to compare renormalizations for different t-structures:

4.2.4. Lemma. Let $\mathcal{C}$ be a dg-category equipped with t-structures $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ and $(\mathcal{C}^{\leq' 0}, \mathcal{C}^{\geq' 0})$ which are both compatible with filtered colimits. Suppose that there exist $a, b \in \mathbb{Z}$ such that $\mathcal{C}^{\leq a} \subset \mathcal{C}^{\leq' 0} \subset \mathcal{C}^{\leq b}$ (or equivalently, $\mathcal{C}^{\geq b} \subset \mathcal{C}^{\geq' 0} \subset \mathcal{C}^{\geq a}$). Then the subcategories of coherent objects with respect to each of these t-structures are identical, i.e., $\mathrm{Coh}(\mathcal{C}) = \mathrm{Coh}'(\mathcal{C})$. In particular, the renormalizations $\mathcal{C}_{\mathrm{ren}}$ and $\mathcal{C}_{\mathrm{ren}'}$ are canonically equivalent.

Proof. We show the inclusion $\mathrm{Coh}(\mathcal{C}) \subset \mathrm{Coh}'(\mathcal{C})$; the opposite inclusion follows by symmetry. Suppose that $X \in \mathcal{C}$ is coherent with respect to $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$. Then it is clearly bounded below with respect to $(\mathcal{C}^{\leq' 0}, \mathcal{C}^{\geq' 0})$. Moreover, given $n \in \mathbb{Z}$, we have a diagram

$$\begin{tikzcd} \mathcal{C}^{\geq' n} \arrow[rr, hook, "i"] \arrow[dr, hook, "\iota^{\geq' n}"'] & & \mathcal{C}^{\geq a+n} \arrow[dl, hook', "\iota^{\geq a+n}"] \\ & \mathcal{C} & \end{tikzcd} \tag{4.2.4.1}$$

of fully faithful functors. It suffices to show that i is both continuous and cocontinuous. Indeed, in this case we have $\tau^{\geq' n}X \simeq i^L \tau^{\geq a+n}X$, and the left-adjoint i^L preserves compact objects. Note that the functors $\iota^{\geq' n}, \iota^{\geq a+n}$ are both continuous and cocontinuous by assumption. It is now straightforward to check that i admits left and right adjoints given by $\tau^{\geq' n}\iota^{\geq a+n}$ and $\iota^{\geq' n, R}\iota^{\geq a+n}$, respectively. ☐

4.2.5. We now note some properties of t-structures on module categories for connective algebras:

4.2.6. Lemma. Let $\mathcal{C}$ be a symmetric monoidal dg-category equipped with an accessible t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$. Let $A \in \mathrm{Alg}(\mathcal{C})$ be an algebra object, let $\mathrm{Res}^A_{\mathbf{1}_\mathcal{C}} : A\text{-mod}_\mathcal{C} \to \mathcal{C}$ denote the forgetful functor, and suppose that the functor $A \otimes - : \mathcal{C} \to \mathcal{C}$ is right t-exact. Then the pair

$$A\text{-mod}_\mathcal{C}^{\leq 0} := (\mathrm{Res}^A_{\mathbf{1}_\mathcal{C}})^{-1}(\mathcal{C}^{\leq 0}), \qquad A\text{-mod}_\mathcal{C}^{\geq 0} := (\mathrm{Res}^A_{\mathbf{1}_\mathcal{C}})^{-1}(\mathcal{C}^{\geq 0}), \tag{4.2.6.1}$$

gives a t-structure on $A\text{-mod}_\mathcal{C}$. Moreover, if the t-structure on $\mathcal{C}$ is compatible with filtered colimits (resp. right-complete), then so is that on $A\text{-mod}_\mathcal{C}$.

Finally, suppose that A is a compact object of $\mathcal{C}$, that the tensor product on $\mathcal{C}$ preserves compact objects, and that the t-structure on $\mathcal{C}$ is coherent. Then the t-structure on $A\text{-mod}_\mathcal{C}^{\heartsuit}$ is coherent.

[39]In Preygel's terminology, this is the "regularization" of $\mathcal{C}$; however, we shall not need the notion of regularity in this work.
[40]In fact, accessibility is automatic for compactly generated categories (see for instance [**GR17a**, Ch. 4, Lem. 1.2.4]).

PROOF. The first assertion is a well-known construction (see for instance [**Pol07**, Thm. 2.1.2]). Compatibility with filtered colimits is clear, as $\mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}}$ is continuous. For right-completeness, consider the commutative square

$$\begin{tikzcd} \mathrm{colim}_n\, A\text{-mod}_{\mathcal{C}}^{\le n} \arrow[r, "\mathrm{colim}_n\, \iota^{\le n}"] \arrow[d, "\mathrm{colim}_n\, \mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}}"'] & A\text{-mod}_{\mathcal{C}} \arrow[d, "\mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}}"] \\ \mathrm{colim}_n\, \mathcal{C}^{\le n} \arrow[r, "\sim"] & \mathcal{C}. \end{tikzcd}$$

By our assumption on A, there is a functor $\mathrm{colim}_n\, \mathrm{Ind}^A_{\mathbf{1}_{\mathcal{C}}}$ left-adjoint to $\mathrm{colim}_n\, \mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}}$; moreover, the monad $\mathrm{colim}_n\, \mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}}\, \mathrm{Ind}^A_{\mathbf{1}_{\mathcal{C}}}$ acting on $\mathrm{colim}_n\, \mathcal{C}^{\le n}$ evidently identifies with $\mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}}\, \mathrm{Ind}^A_{\mathbf{1}_{\mathcal{C}}}$ under the lower equivalence. Thus, it suffices to show that the functor $\mathrm{colim}_n\, \mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}}$ is "monadic" (in the sense of [**GR17a**, Ch. 1, Def. 3.7.5]). Indeed, the functors $\mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}}$ and $\mathrm{colim}_n\, \iota^{\le n}$ are both continuous and conservative (as each $\iota^{\le n}$ is), so the same holds for $\mathrm{colim}_n\, \mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}}$. The conclusion now follows from the Barr–Beck–Lurie theorem (see [**Lur17**, Thm. 4.7.0.3]).

For the final assertion, we first claim that $A\text{-mod}_{\mathcal{C}}^{\heartsuit}$ is compactly generated by objects of the form $\tau^{\ge 0}(A \otimes X)$ for $X \in \mathcal{C}^{\heartsuit,c}$. To see that these objects are compact, note that as in (4.2.4.1), there are adjoint pairs

$$\mathcal{C}^{\le 0} \underset{\mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}}}{\overset{\mathrm{Ind}^A_{\mathbf{1}_{\mathcal{C}}}}{\rightleftarrows}} A\text{-mod}_{\mathcal{C}}^{\le 0} \underset{\iota^{\ge 0}}{\overset{\tau^{\ge 0}}{\rightleftarrows}} A\text{-mod}_{\mathcal{C}}^{\heartsuit},$$

where the lower composition is continuous and factors through $\mathcal{C}^{\heartsuit}$. To see that they generate, let $M \in A\text{-mod}_{\mathcal{C}}^{\heartsuit}$. The bar construction expresses M as a simplicial colimit of modules of the form $A^{\otimes i} \otimes \mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}} M$, all of which are connective. Applying $\tau^{\ge 0}$, we obtain an expression for M as a simplicial colimit of modules of the form $\tau^{\ge 0}(A^{\otimes i} \otimes \mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}} M)$, which may be computed in the abelian category $A\text{-mod}_{\mathcal{C}}^{\heartsuit}$. By the Dold–Kan correspondence, M is quasi-isomorphic to the associated "alternating face maps" complex; in particular, we have an exact sequence

$$\tau^{\ge 0}(A \otimes A \otimes \mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}} M) \to \tau^{\ge 0}(A \otimes \mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}} M) \to M \to 0 \tag{4.2.6.2}$$

in $A\text{-mod}_{\mathcal{C}}^{\heartsuit}$. Since $A \otimes -$ preserves colimits and compact objects, it suffices to exhibit each $\mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}} M$ as a filtered colimit of objects of $\mathcal{C}^{\heartsuit,c}$. Such a presentation is immediate from our assumption that $\mathcal{C}^{\heartsuit}$ is compactly generated.

It remains to show that $A\text{-mod}_{\mathcal{C}}^{\heartsuit,c}$ is abelian; we need only establish closure under kernels. So let $f\colon M \to N$ be a morphism in $A\text{-mod}_{\mathcal{C}}^{\heartsuit,c}$. Since A is compact, the functor $\mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}}$ admits a continuous right adjoint which is left t-exact. Moreover, since the t-structure on $\mathcal{C}$ is accessible and compatible with filtered colimits, the truncation $\tau^{\le 0}$ is continuous by [**Che18**, Prop. 2.2.8]. Thus, the restricted functor $\mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}} : A\text{-mod}_{\mathcal{C}}^{\heartsuit} \to \mathcal{C}^{\heartsuit}$ also admits a continuous right adjoint, hence preserves compact objects. In particular, $\mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}} \ker(f) \simeq \ker(\mathrm{Res}^A_{\mathbf{1}_{\mathcal{C}}}(f))$ is compact. Taking $\ker(f)$ in place of M in the exact sequence (4.2.6.2) now immediately exhibits $\ker(f)$ as compact. $\square$

4.2.7. Finally, fix a Slodowy slice S_e, and take $\mathcal{C} := \mathrm{QC}(S_e/\widetilde{Z}_e)$ with its standard t-structure. Then the algebra $\mathcal{A}_{S_e} \otimes_{\mathcal{O}(S_e)} \mathcal{A}_{S_e}^{\mathrm{op}} \in \mathrm{Alg}(\mathrm{QC}(S_e/\widetilde{Z}_e))$ is connective and compact, so by Lemma 4.2.6 and §4.2.3, we have a renormalized category

$$\mathcal{H}_{S_e}^{\mathrm{mod}} := \mathcal{A}_{S_e} \otimes_{\mathcal{O}(S_e)} \mathcal{A}_{S_e}^{\mathrm{op}}\text{-mod}_{\mathrm{ren}}^{\widetilde{Z}_e}. \tag{4.2.7.1}$$

Note that the unrenormalized category $\mathcal{A}_{S_e} \otimes_{\mathcal{O}(S_e)} \mathcal{A}_{S_e}^{\mathrm{op}}\text{-mod}^{\widetilde{Z}_e}$ is monoidal under the tensor product of bimodules (but not rigid!). Since the algebra $\mathcal{A}_{S_e} \otimes_{\mathcal{O}(S_e)} \mathcal{A}_{S_e}^{\mathrm{op}}$ is eventually coconnective, any perfect module is coherent. More precisely, we have:

4.2.8. LEMMA. (1) The category $\mathrm{Coh}(\mathcal{A}_{S_e} \otimes_{\mathcal{O}(S_e)} \mathcal{A}_{S_e}^{\mathrm{op}}\text{-mod}^{\widetilde{Z}_e})$ is given by cohomologically bounded complexes whose cohomology is finitely generated over $\mathrm{H}^0(\mathcal{A}_{S_e} \otimes_{\mathcal{O}(S_e)} \mathcal{A}_{S_e}^{\mathrm{op}})$ (after forgetting the $\widetilde{Z}_e$-equivariance).

(2) The tensor product on $\mathcal{A}_{S_e} \otimes_{\mathcal{O}(S_e)} \mathcal{A}_{S_e}^{\mathrm{op}}\text{-mod}^{\widetilde{Z}_e}$ preserves coherent objects. In particular, it extends to a monoidal structure on $\mathcal{H}_{S_e}^{\mathrm{mod}}$.

Proof. (1) By §4.2.3, it suffices to show that an object $M \in \mathcal{A}_{S_e} \otimes_{\mathcal{O}(S_e)} \mathcal{A}_{S_e}^{\mathrm{op}}\text{-mod}^{\widetilde{Z}_e,\heartsuit}$ is compact if and only if it is finitely generated over $\mathrm{H}^0(\mathcal{A}_{S_e} \otimes_{\mathcal{O}(S_e)} \mathcal{A}_{S_e}^{\mathrm{op}})$. First suppose the latter. Since $\mathrm{H}^0(\mathcal{A}_{S_e} \otimes_{\mathcal{O}(S_e)} \mathcal{A}_{S_e}^{\mathrm{op}})$ is Noetherian (as it is a finite $\mathcal{O}(S_e)$-algebra), finite generation is equivalent to finite presentation. It follows that $\mathrm{H}^0\underline{\mathrm{Hom}}(M,-)$ commutes with (classical) filtered colimits, where we have let $\underline{\mathrm{Hom}}$ denote the $\mathrm{Rep}(\widetilde{Z}_e)$-internal Hom. Thus, $\mathrm{H}^0\,\mathrm{Hom}(M,-) \cong \mathrm{H}^0\underline{\mathrm{Hom}}(M,-)^{\widetilde{Z}_e}$ commutes with filtered colimits, as desired. Conversely, suppose that M is compact. As in the non-equivariant case, we may write M as a direct limit in $\mathcal{A}_{S_e} \otimes_{\mathcal{O}(S_e)} \mathcal{A}_{S_e}^{\mathrm{op}}\text{-mod}^{\widetilde{Z}_e,\heartsuit}$ of its finitely generated submodules (since its underlying vector space decomposes as a direct sum of $\widetilde{Z}_e$-isotypic components). By compactness, the identity map id_M factors through some such submodule, so M is a direct summand of a finitely generated module, hence finitely generated.

(2) Since the algebra $\mathcal{A}_{S_e}$ has finite homological dimension, the tensor product $\otimes_{\mathcal{A}_{S_e}}$ preserves cohomological boundedness. It therefore suffices to show that the tensor product of any finitely generated $\mathrm{H}^0(\mathcal{A}_{S_e} \otimes_{\mathcal{O}(S_e)} \mathcal{A}_{S_e}^{\mathrm{op}})$-modules M, N (in degree 0) has finitely generated cohomology. Since $\mathcal{A}_{S_e}$ is Noetherian, and M, N are in particular finitely generated over $\mathcal{A}_{S_e}$, we may resolve N by a complex of finitely generated free $\mathcal{A}_{S_e}$-modules. It follows that the cohomology of $M \otimes_{\mathcal{A}_{S_e}} N$ is also finitely generated over $\mathcal{A}_{S_e}$, hence over $\mathrm{H}^0(\mathcal{A}_{S_e} \otimes_{\mathcal{O}(S_e)} \mathcal{A}_{S_e}^{\mathrm{op}})$, as desired. $\square$

4.2.9. Altogether, we obtain:

4.2.10. Proposition. Let S_e be a Slodowy slice. Then the functor

$$(-)_{\mathrm{mod}} := \mathrm{Hom}_{\mathcal{H}_{S_e}^{\mathrm{coh}}}(\mathcal{E}^\vee|_{\widetilde{S}_e} \boxtimes \mathcal{E}|_{\widetilde{S}_e}, -)\colon \mathcal{H}_{S_e}^{\mathrm{coh}} \to \mathcal{H}_{S_e}^{\mathrm{mod}} \tag{4.2.10.1}$$

is a left t-exact equivalence of rigid monoidal categories. Moreover, it is compatible with the equivalence (2.2.4.1) on right module categories, and the analogous equivalence on left module categories.

Proof. By [**GR17a**, Ch. 1, Prop. 8.5.4; Ch. 3, Prop. 3.5.3] and (2.2.4.1), we have a commutative diagram

$$\begin{tikzcd}
\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e) \underset{\mathrm{QC}(S_e/\widetilde{Z}_e)}{\otimes} \mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e) \arrow[r, "\mathrm{Hom}(\mathcal{E}^\vee|_{\widetilde{S}_e},-)\otimes\mathrm{Hom}(\mathcal{E}|_{\widetilde{S}_e},-)", "\sim"'] \arrow[d, "\boxtimes", "\wr"'] & \mathcal{A}_{S_e}\text{-mod}^{\widetilde{Z}_e} \underset{\mathrm{QC}(S_e/\widetilde{Z}_e)}{\otimes} \mathcal{A}_{S_e}^{\mathrm{op}}\text{-mod}^{\widetilde{Z}_e} \arrow[d, "\boxtimes", "\wr"'] \\
\mathrm{QC}(\widetilde{S}_e \times_{S_e} \widetilde{S}_e/\widetilde{Z}_e) \arrow[r, "\mathrm{Hom}(\mathcal{E}^\vee|_{\widetilde{S}_e}\boxtimes\mathcal{E}|_{\widetilde{S}_e},-)"] & \mathcal{A}_{S_e} \otimes_{\mathcal{O}(S_e)} \mathcal{A}_{S_e}^{\mathrm{op}}\text{-mod}^{\widetilde{Z}_e},
\end{tikzcd}$$

whose bottom row is therefore an equivalence. Renormalizing the left- and right-hand sides of this equivalence yields $\mathcal{H}_{S_e}^{\mathrm{coh}}$ and $\mathcal{H}_{S_e}^{\mathrm{mod}}$, respectively, so it suffices to show that the standard t-structures on either side satisfy the hypotheses of Lemma 4.2.4.

First let $\mathcal{F} \in \mathcal{H}_{S_e}^{\mathrm{coh},\le 0}$. By [**DG13**, Thm. 1.4.2], there exists a fixed $b \in \mathbb{Z}$ (depending only on $\widetilde{S}_e \times_{S_e} \widetilde{S}_e/\widetilde{Z}_e$, and using the fact that it is QCA) such that the global sections functor has cohomological amplitude $\le b$. Since the sheaf $\mathcal{H}om(\mathcal{E}^\vee|_{\widetilde{S}_e} \boxtimes \mathcal{E}|_{\widetilde{S}_e}, \mathcal{F})$ is connective (as $\mathcal{E}^\vee|_{\widetilde{S}_e} \boxtimes \mathcal{E}|_{\widetilde{S}_e}$ is a vector bundle), its global sections lie in cohomological degrees $\le b$, which gives one inclusion. Now let $\mathcal{F} \in \mathcal{H}_{S_e}^{\mathrm{coh},\ge 0}$. Since the sheaf $\mathcal{H}om(\mathcal{E}^\vee|_{\widetilde{S}_e} \boxtimes \mathcal{E}|_{\widetilde{S}_e}, \mathcal{F})$ is coconnective, its global sections are as well, and we obtain the other inclusion (with $a = 0$). In particular, the functor $(-)_{\mathrm{mod}}$ is left t-exact.

Finally, monoidality and compatibility with the module structures on (2.2.4.1) hold as in [**BL23**, Lem. 4.3] (and are straightforward exercises). $\square$

4.2.11. We refer to the t-structure on $\mathcal{H}_{S_e}^{\mathrm{coh}}$ transported from that on $\mathcal{H}_{S_e}^{\mathrm{mod}}$ as the *exotic t-structure*, as for (2.2.4.1). We may at last state our "intrinsic reformulation" of braid positivity. Namely, right convolution by $\mathcal{F} \in \mathcal{H}_{S_e}^{\mathrm{coh}}$ on $\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e)$ is right t-exact with respect to the exotic t-structure if and only if the image $\mathcal{F}_{\mathrm{mod}} \in \mathcal{H}_{S_e}^{\mathrm{mod}}$ under (4.2.10.1) is connective, i.e., $\mathcal{F}$ is connective with respect to the exotic t-structure on $\mathcal{H}_{S_e}^{\mathrm{coh}}$. Indeed, it suffices to show that the convolution $\mathcal{E}|_{\widetilde{S}_e} * \mathcal{F}$ is connective, and this object is exactly given by the $\mathcal{A}_{S_e}^{\mathrm{op}}$-module structure on $\mathcal{F}_{\mathrm{mod}}$. In particular, for $a \in B_+^{\mathrm{aff}}$, the sheaf $\mathcal{K}_a$ of §2.2.7 is connective for the exotic t-structure.

Finally, we denote by $i^*_{S_e}\colon \mathcal{H}^{\mathrm{mod}} \to \mathcal{H}^{\mathrm{mod}}_{S_e}$ the extension of scalars $-\otimes_{\mathcal{O}(\mathfrak{g})} \mathcal{O}(S_e)$. This functor is evidently intertwined with $i^*_{S_e}\colon \mathcal{H}^{\mathrm{coh}} \to \mathcal{H}^{\mathrm{coh}}_{S_e}$ under Proposition 4.2.10, justifying our duplicate notation. Note that by Lemma 3.3.19, these functors are both t-exact.

4.3. Connectivity and coconnectivity criteria

4.3.1. In this section, we prove the main result of this chapter, i.e., that the coherent Springer sheaf lies in cohomological degree 0. In fact, our result is considerably more general: we use the exotic t-structure of §4.2 to give conditions for the universal trace of a compact object of the affine Hecke category to be either connective or coconnective. These conditions also apply to the traces of the sheaves $\mathcal{K}_a$ giving the affine braid group action on $\mathrm{QC}(\widetilde{\mathcal{N}}/\widetilde{G})$. Our strategy is to use the machinery of Block–Getzler sheaves to compute these traces and their restrictions to any Slodowy slice in terms of the noncommutative Springer resolution. Base-changing to $\mathrm{B}Z_e^{\mathrm{cov}}$ for each nilpotent e, we may then use the Koszul resolution of §4.1 to obtain cohomological bounds.

The precise statement is as follows:

4.3.2. Theorem. *Fix a Slodowy slice S_e for each $e \in \mathcal{N}$.*

(1) For any $e \in \mathcal{N}$, the universal trace functor

$$[-]\colon \mathcal{H}^{\mathrm{coh}}_{S_e} \to \mathrm{QC}^!(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e)) \tag{4.3.2.1}$$

is right t-exact with respect to the exotic t-structure on the former and the standard t-structure on the latter.

(2) Let $\mathcal{F}$ be a compact object of $\mathcal{H}^{\mathrm{coh}}$, and suppose that its right monoidal dual $\mathcal{F}^{\vee,R}$ is connective for the exotic t-structure. Then $[\mathcal{F}]$ is coconnective for the standard t-structure. In particular, for $e=0$, the functor (4.3.2.1) has cohomological amplitude in $[-\dim\mathcal{N}, 0]$.

4.3.3. Note that by Remark 3.3.7, we may equivalently consider the left monoidal dual $\mathcal{F}^{\vee,L}$ in place of $\mathcal{F}^{\vee,R}$. Before commencing the proof, we record some simple consequences. Given $\lambda \in X^*(T)$, define the *λ-twisted coherent Springer sheaf* to be

$$\mathcal{S}^\lambda := \mathcal{L}\pi_* \operatorname{ev}^* \mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda) \simeq [\Delta_{\widetilde{\mathcal{N}}/\mathfrak{g},*}\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)] \in \mathrm{Coh}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})).$$

As in §2.2.7, these are the classes $[\mathcal{K}_a]$ for a contained in the translation subgroup of B^{aff}. The following corollary contains [**BZCHN22**, Conj. 4.15] as a special case:

4.3.4. Corollary. (1) For any $a \in B^{\mathrm{aff}}_+$, the class $[\mathcal{K}_a]$ is connective, and the class $[\mathcal{K}_{a^{-1}}]$ is coconnective.

(2) Let $\lambda \in X^*(T)$. If λ is dominant (resp. anti-dominant), then $\mathcal{S}^\lambda$ is connective (resp. coconnective). In particular, the coherent Springer sheaf $\mathcal{S} = \mathcal{S}^0$ lies in the abelian category $\mathrm{Coh}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))^\heartsuit$.

Proof. (1) The first assertion follows from §4.2.11. For the latter, note that $\mathcal{K}_a$ and $\mathcal{K}_{a^{-1}}$ are monoidal inverses, hence mutually dual.

(2) This follows immediately from the previous statement. $\square$

4.3.5. The proof of our main result will occupy the remainder of this section:

Proof of Theorem 4.3.2. (1) It suffices to establish right t-exactness for compact objects of $\mathcal{H}^{\mathrm{coh}}_{S_e}$ (see for instance [**GR17a**, Ch. 4, Lem. 1.2.4(2)]). So let $\mathcal{F} \in \mathcal{H}^{\mathrm{coh},c}_{S_e}$ be connective for the exotic t-structure. By Corollary 3.3.11, we reduce to showing that

$$[\mathcal{F}]^{\mathrm{QC}} \simeq [\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e), -*\mathcal{F}] \in \mathrm{QC}(\mathcal{L}(S_e/\widetilde{Z}_e)) \tag{4.3.5.1}$$

is connective. By Corollary 3.4.13, we have

$$[\mathcal{F}]^{\mathrm{QC}} \simeq \mathsf{BG}_{S_e/\widetilde{Z}_e}(\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e), -*\mathcal{F}),$$

so by (3.4.7.1), it suffices to show that ${}^{\mathrm{pre}}\mathsf{BG}_{\mathfrak{g}/\widetilde{G}}(\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e), -*\mathcal{F})$ is connective. As in Remark 3.4.6, we may compute this pre-Block–Getzler sheaf using only the compact generator $\mathcal{E}|_{\widetilde{S}_e}$ (i.e., the regular right $\mathcal{A}_{S_e}$-module under the equivalence (2.2.4.1)), which yields a complex of the form

$$\cdots \to \mathcal{A}_{S_e} \otimes_k \mathcal{A}_{S_e} \otimes_k \mathcal{F}_{\mathrm{mod}} \boxtimes \mathcal{O}_{\widetilde{Z}_e} \xrightarrow{d_0-d_1+d_2} \mathcal{A}_{S_e} \otimes_k \mathcal{F}_{\mathrm{mod}} \boxtimes \mathcal{O}_{\widetilde{Z}_e} \xrightarrow{d_0-d_1} \mathcal{F}_{\mathrm{mod}} \boxtimes \mathcal{O}_{\widetilde{Z}_e}. \tag{4.3.5.2}$$

Indeed, we have

$$\mathcal{F}_{\mathrm{mod}} \simeq \mathrm{Hom}_{\widetilde{S}_e}(\mathcal{E}|_{\widetilde{S}_e}, \mathcal{E}|_{\widetilde{S}_e} * \mathcal{F})$$

by definition. The conclusion is now immediate from connectivity of $\mathcal{F}_{\mathrm{mod}}$.

(2) Now let $\mathcal{F} \in \mathcal{H}^{\mathrm{coh},c}$, and suppose that $\mathcal{F}^{\vee,R}$ is connective for the exotic t-structure. As in (4.3.5.1), it suffices to show that $[\mathcal{F}]^{\mathrm{QC}}$ is coconnective. The proof procedes in several steps.

4.3.6. We begin by reducing to a local cohomology calculation on each nilpotent orbit. Define a stratification of $\mathfrak{g}/G$ as follows: choose a total order

$$\{0\} = \mathbb{O}_0 \leq \mathbb{O}_1 \leq \cdots \leq \mathbb{O}_m = \mathbb{O}_{\mathrm{reg}}$$

on the set of nilpotent orbits of $\mathfrak{g}$ extending the usual partial order of §2.2.2. For each $r = 0, \dots, m+1$, set $\mathfrak{g}_r := \mathfrak{g} - \bigcup_{0 \leq r' < r} \mathbb{O}_{r'}$, so that

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \cdots \supseteq \mathfrak{g}_{m+1} = \mathfrak{g} - \mathcal{N},$$

and let

$$\begin{tikzcd} \mathbb{O}_r/\widetilde{G} \arrow[r, hook, "i_r"] & \mathfrak{g}_r/\widetilde{G} \arrow[d, hook, "j_r"] & \mathfrak{g}_{r+1}/\widetilde{G} \arrow[l, hook', "j^{r+1}"'] \\ & \mathfrak{g}/\widetilde{G}. & \end{tikzcd}$$

be the closed and open inclusions, respectively. These remain closed and open after applying the loop space functor by Lemma 3.2.7; moreover, the top row remains complementary. We therefore have a distinguished triangle

$$\Gamma_{\mathcal{L}(\mathbb{O}_r/\widetilde{G})} \to \mathrm{id}_{\mathrm{QC}(\mathcal{L}(\mathfrak{g}_r/\widetilde{G}))} \to \mathcal{L}j^{r+1}_* \mathcal{L}j^{r+1,*}$$

for each r, where the left-most functor is local cohomology with support in $\mathcal{L}(\mathbb{O}_r/\widetilde{G})$.[41] Since $\mathcal{L}j^*_{m+1}[\mathcal{F}]^{\mathrm{QC}} \simeq 0$ and each $\mathcal{L}j^{r+1}_*$ is left t-exact, it suffices to show that

$$\Gamma_{\mathcal{L}(\mathbb{O}_r/\widetilde{G})} \mathcal{L}j_r^*[\mathcal{F}]^{\mathrm{QC}} \in \mathrm{QC}_{\mathcal{L}(\mathbb{O}_r/\widetilde{G})}(\mathcal{L}(\mathfrak{g}_r/\widetilde{G}))^{\geq 0} \tag{4.3.6.1}$$

for each r.

4.3.7. Next, we reduce (4.3.6.1) to a computation on a Slodowy slice. Let $e \in \mathbb{O}_r$ and let S_e be a Slodowy slice at e. Let $Z_e^{\mathrm{cov}} \twoheadrightarrow Z_e$ be as in Proposition 2.3.2 (with a view towards applying Corollary 2.3.7). Consider the diagram[42]

$$\begin{tikzcd} e/\widetilde{Z}_e^{\mathrm{cov}} \arrow[r, "i_e"] \arrow[d, "i_{\mathrm{cov}}"] & S_e/\widetilde{Z}_e^{\mathrm{cov}} \arrow[d, "i_{S_e}"] \\ \mathbb{O}_r/\widetilde{G} \arrow[r, "i_r"] & \mathfrak{g}_r/\widetilde{G}. \end{tikzcd}$$

Since S_e and $\mathbb{O}_r$ intersect transversally at e, it is (derived) cartesian, so applying the loop space functor gives a pullback square

$$\begin{tikzcd} \mathcal{L}(e/\widetilde{Z}_e^{\mathrm{cov}}) \arrow[r, "\mathcal{L}i_e"] \arrow[d, "\mathcal{L}i_{\mathrm{cov}}"] & \mathcal{L}(S_e/\widetilde{Z}_e^{\mathrm{cov}}) \arrow[d, "\mathcal{L}i_{S_e}"] \\ \mathcal{L}(\mathbb{O}_r/\widetilde{G}) \arrow[r, "\mathcal{L}i_r"] & \mathcal{L}(\mathfrak{g}_r/\widetilde{G}). \end{tikzcd}$$

In particular, the relative cotangent complexes for the horizontal maps satisfy

$$\mathcal{L}i^*_{\mathrm{cov}} \mathbb{L}_{\mathcal{L}i_r} \simeq \mathbb{L}_{\mathcal{L}i_e}. \tag{4.3.7.1}$$

[41] We comment briefly on our conventions for the local cohomology functor. In [GR14], this functor was only defined for Zariski-closed subsets of derived schemes, rather than for arbitrary closed immersions of stacks. However, we will only be concerned with closed immersions of the form described in the hypotheses of Lemma 3.2.7; in this setup, we write Γ_Z for $\Gamma_{Z^{\mathrm{cl}}}$ (in fact, Z will always be classical in our applications), and the functor $\Gamma_Z \colon \mathrm{QC}(X) \to \mathrm{QC}(X)$ clearly upgrades to a functor $\Gamma_{Z/G} \colon \mathrm{QC}(X/G) \to \mathrm{QC}(X/G)$ using the same distinguished triangle. Regardless, it suffices to check any claim about t-structures after forgetting equivariance, so alternatively, we may simply take local cohomology with respect to the derived scheme underlying each loop space as in (3.2.5.1) (this does not alter any of the arguments).

[42] Our notation conflicts slightly with that of (2.2.2.1), but this should not pose any confusion.

Observe that $\mathbb{L}_{\mathcal{L}i_r}$ is perfect: indeed, $\mathcal{L}(\mathbb{O}_r/\widetilde{G}) \simeq \widetilde{G}^e/\widetilde{G}^e$ is smooth, so it suffices to show that $\mathbb{L}_{\mathcal{L}i_r}$ is coherent; the latter follows from the exact triangle

$$\mathcal{L}i_r^*\mathbb{L}_{\mathcal{L}(\mathfrak{g}_r/\widetilde{G})} \to \mathbb{L}_{\mathcal{L}(\mathbb{O}_r/\widetilde{G})} \to \mathbb{L}_{\mathcal{L}i_r}$$

and Lemma 3.2.6(2). Thus, by [**HL15a**, Lem. 5.2], the sheaf $\Gamma_{\mathcal{L}(\mathbb{O}_r/\widetilde{G})}\mathcal{L}j_r^*[\mathcal{F}]^{\mathrm{QC}}$ has a bounded-below increasing filtration whose associated graded is equivalent to

$$\mathcal{L}i_{r,*}\big(\operatorname{Sym}\mathbb{L}_{\mathcal{L}i_r}^\vee[1] \otimes \mathcal{L}i_r^!\mathcal{O}_{\mathcal{L}(\mathfrak{g}_r/\widetilde{G})} \otimes \mathcal{L}i_r^*\mathcal{L}j_r^*[\mathcal{F}]^{\mathrm{QC}}\big).$$

Since $\mathcal{L}i_{r,*}$ is left t-exact, it suffices to show that

$$\operatorname{Sym}\mathbb{L}_{\mathcal{L}i_r}^\vee[1] \otimes \mathcal{L}i_r^!\mathcal{O}_{\mathcal{L}(\mathfrak{g}_r/\widetilde{G})} \otimes \mathcal{L}i_r^*\mathcal{L}j_r^*[\mathcal{F}]^{\mathrm{QC}} \in \mathrm{QC}(\mathcal{L}(\mathbb{O}_r/\widetilde{G}))^{\geq 0}. \tag{4.3.7.2}$$

We claim that (4.3.7.2) is a direct sum of sheaves lying in $\mathrm{Coh}^-(\mathcal{L}(\mathbb{O}_r/\widetilde{G}))$. Indeed, it suffices to show that $\mathcal{L}i_r^!\mathcal{O}_{\mathcal{L}(\mathfrak{g}_r/\widetilde{G})}$ and $\mathcal{L}i_r^*\mathcal{L}j_r^*[\mathcal{F}]^{\mathrm{QC}}$ both lie in $\mathrm{Coh}^-(\mathcal{L}(\mathbb{O}_r/\widetilde{G}))$. For the former, we have

$$i_r^!\mathcal{O}_{\mathcal{L}(\mathfrak{g}_r/\widetilde{G})} \simeq \det\mathbb{L}_{\mathcal{L}i_r}[\operatorname{rank}\mathbb{L}_{\mathcal{L}i_r}] \tag{4.3.7.3}$$

by [**HL15a**, Lem. 3.8] and Lemma 3.2.6(2). The latter is clear as $[\mathcal{F}]^{\mathrm{QC}}$ is coherent. Thus, the following lemma reduces us to showing that

$$\mathcal{L}i_{\mathrm{cov}}^*\big(\operatorname{Sym}\mathbb{L}_{\mathcal{L}i_r}^\vee[1] \otimes \mathcal{L}i_r^!\mathcal{O}_{\mathcal{L}(\mathfrak{g}_r/\widetilde{G})} \otimes \mathcal{L}i_r^*\mathcal{L}j_r^*[\mathcal{F}]^{\mathrm{QC}}\big) \in \mathrm{QC}(\mathcal{L}(e/\widetilde{Z}_e))^{\geq 0}. \tag{4.3.7.4}$$

4.3.8. Lemma. *Let $\mathcal{G} \in \mathrm{Coh}^-(\widetilde{G}^e/\widetilde{G}^e)$, and suppose that $\mathcal{L}i_{\mathrm{cov}}^*\mathcal{G} \in \mathrm{Coh}^-(\widetilde{Z}_e^{\mathrm{cov}}/\widetilde{Z}_e^{\mathrm{cov}})$ is coconnective. Then $\mathcal{G}$ is coconnective.*

Proof. We have a factorization $\widetilde{Z}_e^{\mathrm{cov}}/\widetilde{Z}_e^{\mathrm{cov}} \to \widetilde{Z}_e/\widetilde{Z}_e \to \widetilde{G}^e/\widetilde{G}^e$ of $\mathcal{L}i_{\mathrm{cov}}$; since the first map is faithfully flat, it suffices to establish the claim with $\mathcal{L}i_{\mathrm{cov}}$ replaced by the latter map.

Recall from §2.2.2 that $\widetilde{G}^e \simeq \mathrm{R}_uG^e \rtimes \widetilde{Z}_e$, where R_uG^e denotes the unipotent radical of G^e, and the factor of $\mathbb{G}_m$ in $\widetilde{Z}_e$ acts on R_uG^e with strictly positive weights. It clearly suffices to verify the claim after forgetting all but the $\mathbb{G}_m$-equivariance; choosing an isomorphism $\mathrm{R}_uG^e/\mathbb{G}_m \simeq \mathbb{A}^n/\mathbb{G}_m$ for some $n \geq 0$, we reduce to the following claim:

(∗) Let $n \geq 0$, and suppose we have an attracting action of $\mathbb{G}_m$ on $\mathbb{A}^n$. Let $\mathcal{G} \in \mathrm{Coh}^-(Z_e \times \mathbb{A}^n/\mathbb{G}_m)$, let $i_0\colon \{0\} \hookrightarrow \mathbb{A}^n$ denote the inclusion, and suppose that $(\mathrm{id}\times i_0)^*\mathcal{G} \in \mathrm{Coh}^-(Z_e \times \mathrm{B}\mathbb{G}_m)$ is coconnective. Then $\mathcal{G}$ is coconnective.

We proceed by induction on n. The claim is trivial for $n = 0$, so suppose $n \geq 1$ and the claim holds for $n-1$. Choose a factorization $\{0\} \overset{i_0'}{\hookrightarrow} \mathbb{A}^{n-1} \overset{i}{\hookrightarrow} \mathbb{A}^n$ of i_0 through some $\mathbb{G}_m$-stable hyperplane. Since $(\mathrm{id}\times i_0)^*\mathcal{G} \simeq (\mathrm{id}\times i_0')^*((\mathrm{id}\times i)^*\mathcal{G})$, the inductive hypothesis gives $(\mathrm{id}\times i)^*\mathcal{G} \in \mathrm{Coh}(Z_e \times \mathbb{A}^{n-1}/\mathbb{G}_m)^{\geq 0}$. The morphism $\mathrm{id}\times i$ has Tor-dimension ≤ 1, so the convergent spectral sequence

$$E_2^{s,t} = \mathrm{H}^s((\mathrm{id}\times i)^*\mathrm{H}^t(\mathcal{G})) \Longrightarrow \mathrm{H}^{s+t}((\mathrm{id}\times i)^*\mathcal{G})$$

degenerates. Thus, it suffices to show that if $\mathrm{H}^t(\mathcal{G})$ is nonzero, then so is $\mathrm{H}^0((\mathrm{id}\times i)^*\mathrm{H}^t(\mathcal{G}))$; this follows from Nakayama's lemma and $\mathbb{G}_m$-equivariance of $\mathrm{H}^t(\mathcal{G})$. □

4.3.9. To compute (4.3.7.4), we first observe that

$$\mathcal{L}i_{\mathrm{cov}}^*\mathcal{L}i_r^!\mathcal{O}_{\mathcal{L}(\mathfrak{g}_r/\widetilde{G})} \simeq \det\mathbb{L}_{\mathcal{L}i_e}[\operatorname{rank}\mathbb{L}_{\mathcal{L}i_e}] \simeq \mathcal{L}i_e^!\mathcal{O}_{\mathcal{L}(S_e/\widetilde{Z}_e^{\mathrm{cov}})}$$

by (4.3.7.1), (4.3.7.3), and the corresponding statement for $\mathcal{L}i_e$. It then follows from (4.3.7.1) and perfectness of the cotangent complexes that

$$\mathcal{L}i_{\mathrm{cov}}^*\big(\operatorname{Sym}\mathbb{L}_{\mathcal{L}i_r}^\vee[1] \otimes \mathcal{L}i_r^!\mathcal{O}_{\mathcal{L}(\mathfrak{g}_r/\widetilde{G})} \otimes \mathcal{L}i_r^*\mathcal{L}j_r^*[\mathcal{F}]^{\mathrm{QC}}\big) \simeq \operatorname{Sym}\mathbb{L}_{\mathcal{L}i_e}^\vee[1] \otimes \mathcal{L}i_e^!\mathcal{O}_{\mathcal{L}(S_e/\widetilde{Z}_e)} \otimes \mathcal{L}i_e^*\mathcal{L}i_{S_e}^*[\mathcal{F}]^{\mathrm{QC}}.$$

Since $\mathcal{L}i_e$ is affine, it is equivalent to show that

$$\mathcal{L}i_{e,*}\big(\operatorname{Sym}\mathbb{L}_{\mathcal{L}i_e}^\vee[1] \otimes \mathcal{L}i_e^!\mathcal{O}_{\mathcal{L}(S_e/\widetilde{Z}_e)} \otimes \mathcal{L}i_e^*\mathcal{L}i_{S_e}^*[\mathcal{F}]^{\mathrm{QC}}\big) \in \mathrm{QC}(\mathcal{L}(S_e/\widetilde{Z}_e))^{\geq 0}.$$

But by a further application of [**HL15a**, Lem. 5.2], this is equivalent to the associated graded of a bounded below increasing filtration on $\Gamma_{\mathcal{L}(e/\widetilde{Z}_e^{\mathrm{cov}})}\mathcal{L}i_{S_e}^*[\mathcal{F}]^{\mathrm{QC}}$, and hence it suffices to show that the latter is coconnective.

We make one further reduction: consider the pullback square

$$\begin{tikzcd}
\mathcal{L}((S_e - \{e\})/\widetilde{Z}_e^{\mathrm{cov}}) \arrow[r, hook, "\mathcal{L}j_e"] \arrow[d, "\mathring{\mathrm{ev}}_{\widetilde{Z}_e^{\mathrm{cov}}}"] & \mathcal{L}(S_e/\widetilde{Z}_e^{\mathrm{cov}}) \arrow[d, "\mathrm{ev}_{\widetilde{Z}_e^{\mathrm{cov}}}"] \\
((S_e - \{e\}) \times \widetilde{Z}_e^{\mathrm{cov}})/\widetilde{Z}_e^{\mathrm{cov}} \arrow[r, hook, "j_e \times \mathrm{id}"] & (S_e \times \widetilde{Z}_e^{\mathrm{cov}})/\widetilde{Z}_e^{\mathrm{cov}}
\end{tikzcd}$$

as in (3.2.7.3). It suffices to show that $\mathrm{ev}_{\widetilde{Z}_e^{\mathrm{cov}},*}\Gamma_{\mathcal{L}(e/\widetilde{Z}_e^{\mathrm{cov}})}\mathcal{L}i_{S_e}^*[\mathcal{F}]^{\mathrm{QC}}$ is coconnective. The exact triangle

$$\mathrm{ev}_{\widetilde{Z}_e^{\mathrm{cov}},*}\Gamma_{\mathcal{L}(e/\widetilde{Z}_e^{\mathrm{cov}})} \to \mathrm{ev}_{\widetilde{Z}_e^{\mathrm{cov}},*} \to \mathrm{ev}_{\widetilde{Z}_e^{\mathrm{cov}},*}\mathcal{L}j_{e,*}\mathcal{L}j_e^*$$

and the equivalences

$$\mathrm{ev}_{\widetilde{Z}_e^{\mathrm{cov}},*}\mathcal{L}j_{e,*}\mathcal{L}j_e^* \simeq (j_e \times \mathrm{id})_*\mathring{\mathrm{ev}}_{\widetilde{Z}_e^{\mathrm{cov}},*}\mathcal{L}j_e^* \simeq (j_e \times \mathrm{id})_*(j_e \times \mathrm{id})^*\,\mathrm{ev}_{\widetilde{Z}_e^{\mathrm{cov}},*}$$

then imply that

$$\mathrm{ev}_{\widetilde{Z}_e^{\mathrm{cov}},*}\Gamma_{\mathcal{L}(e/\widetilde{Z}_e^{\mathrm{cov}})}\mathcal{L}i_{S_e}^*[\mathcal{F}]^{\mathrm{QC}} \simeq \Gamma_{\{e\}\times\widetilde{Z}_e^{\mathrm{cov}}/\widetilde{Z}_e^{\mathrm{cov}}}\,\mathrm{ev}_{\widetilde{Z}_e^{\mathrm{cov}},*}\mathcal{L}i_{S_e}^*[\mathcal{F}]^{\mathrm{QC}}, \tag{4.3.9.1}$$

so we reduce to showing that the latter is coconnective.

4.3.10. We now use our Block–Getzler sheaf to compute (4.3.9.1). By Corollary 3.3.20 (or rather, its obvious analog for Z_e^{cov}) and Proposition 3.4.11, we have

$$\begin{aligned}
\mathrm{ev}_{\widetilde{Z}_e^{\mathrm{cov}},*}\mathcal{L}i_{S_e}^*[\mathcal{F}]^{\mathrm{QC}} &\simeq \mathrm{ev}_{\widetilde{Z}_e^{\mathrm{cov}},*}[\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e^{\mathrm{cov}}), - * i_{S_e}^*\mathcal{F}] \\
&\simeq \mathrm{ev}_{\widetilde{Z}_e^{\mathrm{cov}},*}\mathsf{BG}_{S_e/\widetilde{Z}_e^{\mathrm{cov}}}(\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e^{\mathrm{cov}}), - * i_{S_e}^*\mathcal{F}) \\
&\simeq {}^{\mathrm{pre}}\mathsf{BG}_{S_e/\widetilde{Z}_e^{\mathrm{cov}}}(\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e^{\mathrm{cov}}), - * i_{S_e}^*\mathcal{F}).
\end{aligned} \tag{4.3.10.1}$$

Note that by Corollary 2.3.7 and Remark 3.4.6, we may compute this pre-Block–Getzler sheaf using only the compact generator $\bigoplus_{b\in\mathbf{B}_e}\mathcal{E}_b^{S_e}$ (i.e., the regular right $\mathcal{A}_{S_e}^{\mathrm{cov}}$-module), which yields a complex $C^\bullet$ of the form

$$\cdots \to \mathcal{A}_{S_e}^{\mathrm{cov}}\otimes_k\mathcal{A}_{S_e}^{\mathrm{cov}}\otimes_k i_{S_e}^*\mathcal{F}_{\mathrm{mod}}\boxtimes\mathcal{O}_{\widetilde{Z}_e^{\mathrm{cov}}} \xrightarrow{d_0-d_1+d_2} \mathcal{A}_{S_e}^{\mathrm{cov}}\otimes_k i_{S_e}^*\mathcal{F}_{\mathrm{mod}}\boxtimes\mathcal{O}_{\widetilde{Z}_e^{\mathrm{cov}}} \xrightarrow{d_0-d_1} i_{S_e}^*\mathcal{F}_{\mathrm{mod}}\boxtimes\mathcal{O}_{\widetilde{Z}_e^{\mathrm{cov}}} \tag{4.3.10.2}$$

as in (4.3.5.2). Here we misuse notation slightly by writing $(-)_{\mathrm{mod}}$ for the equivalence given by $\bigoplus_{b\in\mathbf{B}_e}\mathcal{E}_b^{S_e}$ in place of $\mathcal{E}|_{\widetilde{S}_e}$; however, the notation $i_{S_e}^*\mathcal{F}_{\mathrm{mod}}$ is then unambiguous, as $i_{S_e}^*$ and $(-)_{\mathrm{mod}}$ commute. Our goal now is to show that $\Gamma_{\{e\}\times\widetilde{Z}_e^{\mathrm{cov}}/\widetilde{Z}_e^{\mathrm{cov}}}C^\bullet$ is coconnective; we henceforth explicitly forget all $\widetilde{Z}_e^{\mathrm{cov}}$-equivariance (see footnote 41).

4.3.11. We begin by using the Koszul property of $\mathcal{A}_{S_e}^{\mathrm{cov}}$ to replace (4.3.10.2) with a quasi-isomorphic bounded complex. Observe that we may write

$$C^\bullet \simeq \mathcal{A}_{S_e}^{\mathrm{cov}} \underset{\mathcal{A}_{S_e}^{\mathrm{cov}}\underset{k}{\otimes}(\mathcal{A}_{S_e}^{\mathrm{cov}})^{\mathrm{op}}}{\otimes} i_{S_e}^*\mathcal{F}_{\mathrm{mod}}\otimes_k\mathcal{O}(\widetilde{Z}_e^{\mathrm{cov}}),$$

where the $\mathcal{A}_{S_e}^{\mathrm{cov}}$-action on $i_{S_e}^*\mathcal{F}_{\mathrm{mod}}\otimes_k\mathcal{O}(\widetilde{Z}_e^{\mathrm{cov}})$ is given by the left-multiplication on $i_{S_e}^*\mathcal{F}_{\mathrm{mod}}$, and the $(\mathcal{A}_{S_e}^{\mathrm{cov}})^{\mathrm{op}}$-action is given by the algebra homomorphism $\rho\colon \mathcal{A}_{S_e}^{\mathrm{cov}} \to \mathcal{A}_{S_e}^{\mathrm{cov}}\otimes_k\mathcal{O}(\widetilde{Z}_e^{\mathrm{cov}})$ and right-multiplication. Thus, Proposition 4.1.4 and (2.2.5.2) give

$$C^\bullet \simeq \mathrm{Kos}_{\mathcal{A}_{S_e}^{\mathrm{cov}}}^{\leq 0} \underset{\mathcal{A}_{S_e}^{\mathrm{cov}}\underset{k}{\otimes}\mathcal{A}_{S_e}^{\mathrm{cov},\mathrm{op}}}{\otimes} i_{S_e}^*\mathcal{F}_{\mathrm{mod}}\otimes_k\mathcal{O}(\widetilde{Z}_e^{\mathrm{cov}}). \tag{4.3.11.1}$$

Moreover, Corollary 4.1.6 and §2.2.6 show that the projective bimodule resolution $\mathrm{Kos}_{\mathcal{A}_{S_e}^{\mathrm{cov}}}^{\leq 0}$ has length exactly $\dim\widetilde{S}_e$.

Note that $\Gamma_{\{e\}\times\widetilde{Z}_e^{\mathrm{cov}}}$ has cohomological amplitude $[0, \dim S_e]$ (using the standard "local Koszul complex" on a choice of coordinate functions of the affine space S_e). Thus, the spectral sequence of the double complex obtained by applying $\Gamma_{\{e\}\times\widetilde{Z}_e^{\mathrm{cov}}}$ term-by-term to (4.3.11.1) reduces us to showing that

$$\Gamma_{\{e\}\times\widetilde{Z}_e^{\mathrm{cov}}}\big(\mathrm{Kos}_{\mathcal{A}_{S_e}^{\mathrm{cov}}}^{-n} \underset{\mathcal{A}_{S_e}^{\mathrm{cov}}\underset{k}{\otimes}\mathcal{A}_{S_e}^{\mathrm{cov},\mathrm{op}}}{\otimes} i_{S_e}^*\mathcal{F}_{\mathrm{mod}}\otimes_k\mathcal{O}(\widetilde{Z}_e^{\mathrm{cov}})\big) \tag{4.3.11.2}$$

is concentrated in degrees $\geq \dim \widetilde{S}_e$ for each $0 \leq n \leq \dim \widetilde{S}_e$. Furthermore, by (4.1.2.2) and (2.2.6.2), it suffices to show that

$$E_{b'}^{S_e} \otimes_{\mathcal{A}_{S_e}^{\mathrm{cov}}} i_{S_e}^* \mathcal{F}_{\mathrm{mod}} \otimes_{\mathcal{A}_{S_e}^{\mathrm{cov}}} E_b^{S_e,\ell} \simeq \underline{\mathrm{Hom}}_{\widetilde{S}_e}(\mathcal{E}_b^{S_e}, \mathcal{E}_{b'}^{S_e} * i_{S_e}^* \mathcal{F}) \tag{4.3.11.3}$$

has this property after applying Γ_e for each $b, b' \in \mathbf{B}_e$. By Grothendieck local duality (see for instance [**Sta18**, Thm. 0A84]), the $\mathcal{O}(S_e)$-module

$$\Gamma_e \underline{\mathrm{Hom}}_{\widetilde{S}_e}(\mathcal{E}_b^{S_e}, \mathcal{E}_{b'}^{S_e} * i_{S_e}^* \mathcal{F})$$

is Matlis dual to

$$\mathrm{Hom}_{S_e}(\underline{\mathrm{Hom}}_{\widetilde{S}_e}(\mathcal{E}_b^{S_e}, \mathcal{E}_{b'}^{S_e} * i_{S_e}^* \mathcal{F}), \omega_{S_e}), \tag{4.3.11.4}$$

so it suffices to show that the latter is concentrated in cohomological degrees $\leq -\dim \widetilde{S}_e$. Finally, since $\widetilde{S}_e$ is Calabi–Yau as in (3.3.7.1), we have

$$\begin{aligned}
\mathrm{Hom}_{S_e}(\underline{\mathrm{Hom}}_{\widetilde{S}_e}(\mathcal{E}_b^{S_e}, \mathcal{E}_{b'}^{S_e} * i_{S_e}^* \mathcal{F}), \omega_{S_e}) &\simeq \pi_{S_e,*} \mathcal{H}om_{\widetilde{S}_e}(\mathcal{H}om_{\widetilde{S}_e}(\mathcal{E}_b^{S_e}, \mathcal{E}_{b'}^{S_e} * i_{S_e}^* \mathcal{F}), \pi_{S_e}^! \omega_{S_e}) \\
&\simeq \pi_{S_e,*} \mathcal{H}om_{\widetilde{S}_e}(\mathcal{H}om_{\widetilde{S}_e}(\mathcal{E}_b^{S_e}, \mathcal{E}_{b'}^{S_e} * i_{S_e}^* \mathcal{F}), \omega_{\widetilde{S}_e}) \\
&\simeq \underline{\mathrm{Hom}}_{\widetilde{S}_e}(\mathcal{E}_{b'}^{S_e} * i_{S_e}^* \mathcal{F}, \mathcal{E}_b^{S_e})[\dim \widetilde{S}_e] \\
&\simeq \underline{\mathrm{Hom}}_{\widetilde{S}_e}(\mathcal{E}_{b'}^{S_e}, \mathcal{E}_b^{S_e} * i_{S_e}^* \mathcal{F}^{\vee,R})[\dim \widetilde{S}_e].
\end{aligned}$$

The latter is a direct summand of $i_{S_e}^* \mathcal{F}_{\mathrm{mod}}^{\vee,R}[\dim \widetilde{S}_e]$, which lies in cohomological degrees $\leq -\dim \widetilde{S}_e$ by our assumption on $\mathcal{F}^{\vee,R}$ and t-exactness of $i_{S_e}^*$ (see §4.2.11). This proves the first statement of (2). The second statement is immediate from (1) and the claim regarding (4.3.11.2) after noting the inequality $\dim \widetilde{S}_e \leq \dim \widetilde{\mathcal{N}} = \dim \mathcal{N}$.

This concludes the proof of Theorem 4.3.2. $\square$

CHAPTER 5

Asymptotic coherent Springer theory

5.1. Restricting the coherent Springer sheaf

5.1.1. In this short section, we use our Block–Getzler sheaf to deduce the first part of Theorem 1.4.1. That is, we show that a certain restriction of the coherent Springer sheaf to a nilpotent e recovers the corresponding asymptotic coherent Springer sheaf of (2.4.11.1). In the process, we prove a useful lemma computing the 2-categorical class of $\mathrm{Rep}(G)^{(\mathcal{C},\alpha)}$ for reductive G.

5.1.2. Proposition. Let $e \in \mathcal{N}$, and let S_e be a Slodowy slice at e. There is a canonical isomorphism

$$(\mathcal{L}p_{S_e,*}\mathcal{L}i^*_{S_e}\mathcal{S})_0 \simeq \mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m}.$$

Proof. Since $\mathcal{A}_{S_e}$ is concentrated in non-negative weights, the same reasoning as in §4.3.10 implies that $(\mathcal{L}p_{S_e,*}\mathcal{L}i^*_{S_e}\mathcal{S})_0$ is equivalent to the complex

$$\cdots \to \mathcal{A}_{S_e,0} \otimes_k \mathcal{A}_{S_e,0} \otimes_k \mathcal{A}_{S_e,0} \otimes_k \mathcal{O}_{\widetilde{Z}_e} \xrightarrow{d_0-d_1+d_2} \mathcal{A}_{S_e,0} \otimes_k \mathcal{A}_{S_e,0} \otimes_k \mathcal{O}_{\widetilde{Z}_e} \xrightarrow{d_0-d_1} \mathcal{A}_{S_e,0} \otimes_k \mathcal{O}_{\widetilde{Z}_e},$$

which is in turn equivalent[43] to

$$\mathsf{BG}_{*/Z_e}(\mathcal{A}^{\mathrm{op}}_{S_e,0}\text{-mod}^{Z_e}) \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m}.$$

Recall from (2.2.15.1) that

$$\mathcal{A}_{S_e,0} \cong \bigoplus_{b\in\mathbf{B}_e} \mathrm{End}_k(\underline{L}_b^{S_e,*}),$$

with the Z_e-action coming from the $\overline{Z}_e$-action on $\mathbf{B}_e$ and the projective Z_e^b-representation on each $\underline{L}_b^{S_e}$. Thus, by Lemma 2.1.13 and [**BZCHN22**, Lem. 2.11], we have canonical equivalences

$$\mathcal{A}^{\mathrm{op}}_{S_e,0}\text{-mod}^{Z_e} \simeq \bigoplus_{b\in\mathbf{B}_e^{\mathrm{orb}}} \mathrm{End}_k(\underline{L}_b^{S_e,*})^{\mathrm{op}}\text{-mod}^{Z_e^b} \simeq \bigoplus_{b\in\mathbf{B}_e^{\mathrm{orb}}} \mathrm{Rep}(Z_e^b)^{(\mathcal{C}_b,\alpha_b)^\vee}. \tag{5.1.2.1}$$

Proposition 3.4.11 and Corollary 3.2.13 then imply that

$$\mathsf{BG}_{*/Z_e}(\mathcal{A}^{\mathrm{op}}_{S_e,0}\text{-mod}^{Z_e}) \simeq \bigoplus_{b\in\mathbf{B}_e^{\mathrm{orb}}} \mathsf{BG}_{*/Z_e}(\mathrm{Rep}(Z_e^b)^{(\mathcal{C}_b,\alpha_b)^\vee}) \simeq \bigoplus_{b\in\mathbf{B}_e^{\mathrm{orb}}} i^{\mathrm{eq}}_{b,*}\,\mathsf{BG}_{*/Z_e^b}(\mathrm{Rep}(Z_e^b)^{(\mathcal{C}_b,\alpha_b)^\vee}),$$

where $i_b^{\mathrm{eq}} \colon Z_e^b/Z_e^b \to Z_e/Z_e$ is as in (2.4.7.2). Thus, the following lemma completes the proof. ☐

5.1.3. Lemma. Let G be a reductive group, and let $(\mathcal{C},\alpha) \in \mathrm{Coc}(G)$. Then there is a canonical isomorphism

$$\mathsf{BG}_{*/G}(\mathrm{Rep}(G)^{(\mathcal{C},\alpha)}) \simeq \mathcal{C}.$$

Proof. By (2.1.6.1) and Corollary 3.4.13, the sheaf $\mathsf{BG}_{*/G}(\mathrm{Rep}(G)^{(\mathcal{C},\alpha)})$ is a direct summand of

$$\mathsf{BG}_{*/G}(\mathrm{Rep}(\overline{G}_{(\mathcal{C},\alpha)})) \simeq \overline{p}_{(\mathcal{C},\alpha),*}\mathcal{O}_{\overline{G}_{(\mathcal{C},\alpha)}/\overline{G}_{(\mathcal{C},\alpha)}}, \tag{5.1.3.1}$$

where $\overline{p}_{(\mathcal{C},\alpha)} \colon \overline{G}_{(\mathcal{C},\alpha)} \to G$ is as in (2.1.5.2). Observe that the latter is a vector bundle, whose rank is the order of $[(\mathcal{C},\alpha)] \in \mathrm{M}(G)$. It follows that $\mathsf{BG}_{*/G}(\mathrm{Rep}(G)^{(\mathcal{C},\alpha)})$ is also a vector bundle concentrated in cohomological degree 0.

Now let $V \in \mathrm{Rep}(G)^{(\mathcal{C},\alpha),c,\heartsuit}$ be nonzero. By Lemma 2.1.13 and [**BZCHN22**, Lem. 2.11], we have an equivalence

$$\underline{\mathrm{Hom}}_k(V,-) \colon \mathrm{Rep}(G)^{(\mathcal{C},\alpha)} \xrightarrow{\sim} \underline{\mathrm{End}}_k(V)^{\mathrm{op}}\text{-mod}^G, \tag{5.1.3.2}$$

[43] Note that there is no distinction between ${}^{\mathrm{pre}}\mathsf{BG}_{*/G}$ and $\mathsf{BG}_{*/G}$.

and hence

$$\mathsf{BG}_{*/G}(\mathrm{Rep}(G)^{(\mathcal{C},\alpha)}) \simeq \mathsf{BG}_{*/G}(\underline{\mathrm{End}}_k(V)^{\mathrm{op}}\text{-mod}^G). \tag{5.1.3.3}$$

By the previous paragraph, it suffices to compute the cokernel of the final map

$$\begin{aligned} \underline{\mathrm{End}}_k(V) \otimes_k \underline{\mathrm{End}}_k(V) \otimes_k \mathcal{O}_G &\to \underline{\mathrm{End}}_k(V) \otimes_k \mathcal{O}_G, \\ f_0 \otimes f_1 \otimes r &\mapsto (\rho(f_0) \circ f_1 - f_1 \circ f_0) \otimes r, \end{aligned} \tag{5.1.3.4}$$

in the complex defining (5.1.3.3). Regard $\mathcal{O}_G$ and $\mathcal{C}$ as subsheaves of $\mathcal{O}_{G^{\mathrm{sch}}}$, and consider the map

$$\begin{aligned} \underline{\mathrm{End}}_k(V) \otimes_k \mathcal{O}_G &\to \mathcal{C}, \\ f \otimes r &\mapsto \big(g \mapsto \mathrm{tr}(f \circ g|_V) \cdot r(g)\big), \end{aligned} \tag{5.1.3.5}$$

given by the Peter–Weyl theorem for G^{sch} (and extending $\mathcal{O}_G$-linearly). Taking the same map for $V^*, \mathcal{C}^\vee$ and dualizing then yields a natural section[44] of (5.1.3.5), so this map is surjective. Moreover, it is straightforward to verify that (5.1.3.5) factors through the cokernel of (5.1.3.4).[45] In particular, $\mathcal{C}$ is a direct summand of (5.1.3.3), which is therefore a vector bundle of rank ≥ 1. Since this is true for each element of $\langle[(\mathcal{C},\alpha)]\rangle \subset \mathrm{M}(G)$, it follows from (5.1.3.1) and (2.1.6.1) that (5.1.3.3) is a vector bundle of rank precisely 1, hence isomorphic to $\mathcal{C}$. Finally, it is not hard to see that these isomorphisms agree with the canonical decomposition of (5.1.3.1), hence are canonical (i.e., do not depend on the choice of V). $\square$

5.1.4. In particular, applying Theorem 3.1.11, we recover the first isomorphism in (2.1.13.1).

5.2. Homomorphism data

5.2.1. In this section, we introduce a more general notion of "monoidal functor" within the framework of §3.1. Here our express purpose is to identify Lusztig's homomorphism $\phi_{e,k}$ with the homomorphism

$$(\mathcal{L}p_{S_e,*}\mathcal{L}i^*_{S_e})_0\colon \mathrm{End}_{\mathcal{L}(\widehat{N}/\widetilde{G})}(\mathcal{S})^{\mathrm{op}} \to \mathrm{End}_{\widetilde{Z}_e/\widetilde{Z}_e}(\mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m})^{\mathrm{op}} \tag{5.2.1.1}$$

induced by Proposition 5.1.2. We begin by recalling a non-standard perspective on classical algebra homomorphisms, owed to Lusztig:

5.2.2. Proposition. Let A and B be algebras. The datum of a homomorphism $\phi\colon A \to B$ is equivalent to the data of

(1) an (A,B)-bimodule M; and
(2) a right B-module isomorphism $\beta\colon B \xrightarrow{\sim} M$;

where two pairs (M,β) and (M,β') are considered equivalent if there is an (A,B)-module isomorphism $f\colon M \xrightarrow{\sim} M'$ making the diagram

$$\begin{tikzcd} & M \arrow[dd, "f"] \\ B \arrow[ur, "\beta"] \arrow[dr, "\beta'"'] & \\ & M' \end{tikzcd}$$

commute. Moreover, given algebras A, B, and C, and pairs (M,β) and (N,γ) for (A,B) and (B,C), respectively, composition of the corresponding homomorphisms is given by the pair $(M \otimes_B N, (\beta \otimes 1) \circ \gamma)$, where the latter composition is given by

$$C \xrightarrow{\gamma} N \simeq B \otimes_B N \xrightarrow{\beta \otimes \mathrm{id}_N} M \otimes_B N. \tag{5.2.2.1}$$

[44]In formulas, this section is given by

$$\begin{aligned} \mathcal{C} &\to \underline{\mathrm{End}}_k(V) \otimes_k \mathcal{O}_G, \\ r &\mapsto (g \mapsto \tfrac{1}{\dim V} \cdot r(g) \cdot g^{-1}|_V), \end{aligned}$$

as in (5.1.3.5).

[45]Explicitly, their composition sends any $f_0 \otimes f_1 \otimes r$ to the function

$$g \mapsto \mathrm{tr}\left((g^{-1} f_0 g f_1 - f_1 f_0) \circ g|_V\right) \cdot r(g)$$

on G^{sch}, which clearly vanishes.

Proof. We give the constructions in either direction; the rest is left as an exercise. Given a homomorphism $\phi\colon A \to B$, we may take $(M, \beta) = ({}_\phi B, \mathrm{id}_B)$, where ${}_\phi B$ is isomorphic to B as a right B-module, and has left A-module structure given by ϕ. Conversely, given such a pair (M, β), we may take ϕ to be the composition

$$A \to \mathrm{End}_{B^{\mathrm{op}}}(M) \xrightarrow{\mathrm{ad}_\beta} \mathrm{End}_{B^{\mathrm{op}}}(B) \simeq B, \tag{5.2.2.2}$$

where the first map is given by the A-action on M. ☐

5.2.3. We now categorify this notion. In particular, we relax the requirement that β be an equivalence, as our goal is merely to obtain an algebra homomorphism upon decategorifying, i.e., passing to Hochschild homology. Moreover, to obtain a functor on categorical traces, we must *reverse* the A and B actions on M.

5.2.4. Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be rigid monoidal categories. A *homomorphism datum* from $\mathcal{A}$ to $\mathcal{B}$ is a triple $(\mathcal{M}, \beta, \check{\beta})$ consisting of

(1) a right-dualizable $(\mathcal{B}, \mathcal{A})$-bimodule $\mathcal{M}$;
(2) left $\mathcal{B}$-module homomorphisms $\beta\colon \mathcal{B} \to \mathcal{M}$ and $\check{\beta}\colon \mathcal{M} \to \mathcal{B}$, which
   (a) admit continuous right adjoints; and
   (b) induce inverse equivalences on Hochschild homology.

We say that such a homomorphism datum is *unital* if additionally $\check{\beta} \circ \beta \simeq \mathrm{id}_\mathcal{B}$.

5.2.5. Note that the right adjoint $\beta^R, \check{\beta}^R$ are canonically $\mathcal{B}$-linear by [**Gai12**, Cor. 6.2.4]. Moreover, the induced maps

$$\mathrm{HH}(\beta)\colon\ \mathrm{HH}(\mathcal{B}) \rightleftarrows \mathrm{HH}(\mathcal{M})\colon\ \mathrm{HH}(\check{\beta})$$

are automatically $\mathrm{HH}(\mathcal{B})$-module isomorphisms as in Proposition 5.2.2

5.2.6. Given a homomorphism datum $(\mathcal{M}, \beta, \check{\beta})$, we may imitate (5.2.2.2) by forming the composite functor

$$F_{(\mathcal{M},\beta,\check{\beta})}\colon \mathcal{A} \xrightarrow{\mathrm{act}_\mathcal{M}} \mathrm{End}_\mathcal{B}(\mathcal{M})^{\mathrm{rev}} \xrightarrow{\check{\beta}\circ - \circ\beta} \mathrm{End}_\mathcal{B}(\mathcal{B})^{\mathrm{rev}} \simeq \mathcal{B}, \tag{5.2.6.1}$$

where $\mathrm{act}_\mathcal{M}$ denotes the right $\mathcal{A}$-action on $\mathcal{M}$. Explicitly, for $a \in \mathcal{A}$, we have

$$F_{(\mathcal{M},\beta,\check{\beta})}(a) = \check{\beta}(\beta(\mathbf{1}_\mathcal{B}) \otimes a). \tag{5.2.6.2}$$

Thus, unitality of $(\mathcal{M}, \beta, \check{\beta})$ implies that $F_{(\mathcal{M},\beta,\check{\beta})}$ is "unital," i.e., $F_{(\mathcal{M},\beta,\check{\beta})}(\mathbf{1}_\mathcal{A}) \simeq \mathbf{1}_\mathcal{B}$. In general, the functor $F_{(\mathcal{M},\beta,\check{\beta})}$ is *not* monoidal. However, by [**GR17a**, Ch. 1, Lem. 9.3.2] and Definition (5.2.4)(2a), it admits a continuous right adjoint, and therefore induces a map

$$\mathrm{HH}(F_{(\mathcal{M},\beta,\check{\beta})})\colon\ \mathrm{HH}(\mathcal{A}) \to \mathrm{HH}(\mathcal{B}).$$

5.2.7. Our next goal is to give conditions under which $\mathrm{HH}(F_{(\mathcal{M},\beta,\check{\beta})})$ is an algebra homomorphism, and can be realized on categorical traces via Theorem 3.1.11. Consider the composite functor

$$F^{\mathrm{Tr}}_{(\mathcal{M},\beta,\check{\beta})}\colon\ \mathrm{Tr}(\mathcal{A}) \xrightarrow{\mathrm{Tr}(\mathcal{M})} \mathrm{Tr}(\mathcal{B}) \xrightarrow{\mathrm{pr}_{[\mathcal{B}]}} \langle[\mathcal{B}]\rangle,$$

where the functor $\mathrm{Tr}(\mathcal{M})$ is as in (3.1.5.1) and the projection $\mathrm{pr}_{[\mathcal{B}]}$ is as in (3.1.11.3). In particular, we have $\mathrm{Tr}(\mathcal{M})([\mathcal{A}]) \simeq [\mathcal{M}]$, and hence $F^{\mathrm{Tr}}_{(\mathcal{M},\beta,\check{\beta})}([\mathcal{A}]) \simeq [\mathcal{B}]$ by Definition 5.2.4(2b) and (3.1.11.2).

5.2.8. Let $\mathcal{C}$ be a compactly generated dg-category. Recall that the *Chern character* is the natural transformation

$$\mathrm{ch}\colon K(\mathcal{C}^c) \to \mathrm{HH}(\mathcal{C})$$

from the connective K-theory spectrum. We refer the reader to [**BZCHN22**, §2.1.3] for more detailed recollections, and shall only require the following basic facts:

(1) Let $c \in \mathcal{C}^c$. By naturality, the induced map $K_0(\mathcal{C}^c) \to \mathrm{HH}_0(\mathcal{C})$ sends the K-theory class $[c]$ to the Hochschild class $[c]$ (see §3.1.10).
(2) If $\mathcal{C}^c$ is monoidal, then the Chern character is an algebra homomorphism via the lax monoidal structure of K-theory.

We may now state our main criterion; though fairly restrictive, it will suffice for our purposes (where $\mathcal{A}$ is always the mixed affine Hecke category $\mathcal{H}^{\mathrm{coh}}$).

5.2.9. Proposition. Let $(\mathcal{M}, \beta, \check{\beta})$ be a homomorphism datum from $\mathcal{A}$ to $\mathcal{B}$. Suppose that

(1) $\mathcal{A}$ and $\mathcal{M}$ are compactly generated;
(2) $\mathrm{HH}(\mathcal{A})$ is concentrated in cohomological degree 0; and
(3) the k-linearized Chern character $\mathrm{ch}\colon K_0(\mathcal{A})_k \to \mathrm{HH}_0(\mathcal{A})$ is surjective.

Then the induced map

(5.2.9.1) $$F^{\mathrm{Tr}}_{(\mathcal{M},\beta,\check{\beta})}\colon \mathrm{End}_{\mathrm{Tr}(\mathcal{A})}([\mathcal{A}])^{\mathrm{op}} \to \mathrm{End}_{\mathrm{Tr}(\mathcal{A})}([\mathcal{B}])^{\mathrm{op}}$$

identifies with $\mathrm{HH}(F_{(\mathcal{M},\beta,\check{\beta})})$ via (3.1.11.1). In particular, the latter is monoidal.

Proof. Let $\mathrm{act}_{\beta(\mathbf{1}_{\mathcal{B}})}\colon \mathcal{A} \to \mathcal{M}$ denote the functor given by $a \mapsto \beta(\mathbf{1}_{\mathcal{B}}) \otimes a$. As for $F_{(\mathcal{M},\beta,\check{\beta})}$, this functor admits a continuous right adjoint, and we have $F_{(\mathcal{M},\beta,\check{\beta})} \simeq \check{\beta} \circ \mathrm{act}_{\beta(\mathbf{1}_{\mathcal{B}})}$. Thus, it suffices to show that the diagram

$$\begin{tikzcd}
K_0(\mathcal{A})_k \arrow[rr, "K_0(\mathrm{act}_{\beta(\mathbf{1}_{\mathcal{B}})})_k"] \arrow[d, "\mathrm{ch}"] & & K_0(\mathcal{M})_k \arrow[d, "\mathrm{ch}"] & \\
\mathrm{HH}(\mathcal{A}) \arrow[rr, "\mathrm{HH}(\mathrm{act}_{\beta(\mathbf{1}_{\mathcal{B}})})"] \arrow[d, "\wr", "(3.1.11.2)"'] & & \mathrm{HH}(\mathcal{M}) \arrow[r, "\mathrm{HH}(\check{\beta})"] \arrow[d, "\wr", "(3.1.11.2)"'] & \mathrm{HH}(\mathcal{B}) \arrow[d, "\wr", "(3.1.11.2)"'] \\
\mathrm{Hom}_{\mathrm{Tr}(\mathcal{A})}([\mathcal{A}],[\mathcal{A}]) \arrow[r, "\mathrm{Tr}(\mathcal{M})"] & \mathrm{Hom}_{\mathrm{Tr}(\mathcal{B})}([\mathcal{M}],[\mathcal{M}]) \arrow[r, "-\circ[\beta]"] \arrow[rr, bend right, "\mathrm{pr}_{[\mathcal{B}]}"'] & \mathrm{Hom}_{\mathrm{Tr}(\mathcal{B})}([\mathcal{B}],[\mathcal{M}]) \arrow[r, "{[\check{\beta}]\circ-}"] & \mathrm{Hom}_{\mathrm{Tr}(\mathcal{B})}([\mathcal{B}],[\mathcal{B}]),
\end{tikzcd}$$

commutes. All regions of this diagram clearly commute aside from the rectangle involving $\mathrm{Tr}(\mathcal{M})$ and the triangle involving $\mathrm{pr}_{[\mathcal{B}]}$.

For the former, our assumptions reduce us to showing that the outer rectangle involving $K_0(\mathrm{act}_{\beta(\mathbf{1}_{\mathcal{B}})})_k$ commutes. Given $a \in \mathcal{A}^c$, note that (3.1.11.2) factors as the composition

(5.2.9.2)
$$\mathrm{HH}(\mathcal{A}) \simeq \mathrm{Hom}_{\mathrm{Tr}(\mathrm{Vect}_k)}([\mathrm{Vect}_k],[\mathcal{A}]) \xrightarrow{\mathrm{Ind}_{\mathrm{unit}_{\mathcal{A}}}} \mathrm{Hom}_{\mathrm{Tr}(\mathcal{B})}([\mathcal{A}],[\mathcal{A}\otimes\mathcal{A}]) \xrightarrow{[\mathrm{mult}_{\mathcal{A}}]\circ-} \mathrm{Hom}_{\mathrm{Tr}(\mathcal{A})}([\mathcal{A}],[\mathcal{A}])$$

via (3.1.14.3). Thus, the Chern character sends the class of a to $[-\otimes a] \in \mathrm{End}_{\mathrm{Tr}(\mathcal{A})}([\mathcal{A}])$ by functoriality of (3.1.5.1). Likewise, we have

$$\mathrm{Tr}(\mathcal{M})([a]) \circ [\beta] \simeq [(\mathrm{id}_{\mathcal{M}} \otimes a)] \circ [\beta] \simeq [\beta(-) \otimes a] \in \mathrm{Hom}_{\mathrm{Tr}(\mathcal{B})}([\mathcal{B}],[\mathcal{M}]).$$

Finally, the analog of (5.2.9.2) for $\mathrm{Hom}_{\mathrm{Tr}(\mathcal{B})}([\mathcal{B}],[\mathcal{M}])$ shows that the Chern character sends the class of $\beta(\mathbf{1}_{\mathcal{B}}) \otimes a \in \mathcal{M}$ to

$$[-\otimes(\beta(\mathbf{1}_{\mathcal{B}}) \otimes a)] \simeq [\beta(-) \otimes a],$$

as desired.

We now establish commutativity of the triangle involving $\mathrm{pr}_{[\mathcal{B}]}$. Given $\varphi \in \mathrm{End}_{\mathrm{Tr}(\mathcal{B})}([\mathcal{M}])$, the morphism $\mathrm{pr}_{[\mathcal{B}]}(\varphi) \in \mathrm{End}_{\mathrm{Tr}(\mathcal{B})}([\mathcal{B}])$ is unique subject to commutativity of the diagram

$$\begin{tikzcd}
{[\mathcal{B}]} \arrow[r, dashed, "\mathrm{pr}_{[\mathcal{B}]}(\varphi)"] \arrow[d, "{[\beta]}"] & {[\mathcal{B}]} \arrow[d, "{[\beta]}"] \\
{[\mathcal{M}]} \arrow[r, "\varphi"] & {[\mathcal{M}]}.
\end{tikzcd}$$

By Definition 5.2.4(2b) and (3.1.11.2), we have

(5.2.9.3) $$[\check{\beta}] \circ [\beta] \simeq [\check{\beta} \circ \beta] \simeq \mathrm{id}_{[\mathcal{B}]},$$

and therefore $\mathrm{pr}_{[\mathcal{B}]}(\varphi) \simeq [\check{\beta}] \circ \varphi \circ [\beta]$ as desired. □

5.2.10. Note that in particular, $[\mathcal{B}]$ is a direct summand of $[\mathcal{M}]$ via $[\beta], [\check{\beta}]$. Moreover, (3.1.11.3) gives a natural identification

(5.2.10.1) $$F^{\mathrm{Tr}}_{(\mathcal{M},\beta,\check{\beta})}|_{\langle[\mathcal{A}]\rangle} \simeq \mathrm{Ind}^{\mathrm{HH}(\mathcal{B})}_{\mathrm{HH}(\mathcal{A})}.$$

5.2.11. Example. As in the proof of Proposition 5.2.2, any monoidal functor $\Psi\colon \mathcal{A} \to \mathcal{B}$ gives rise to a unital homomorphism datum $(\mathcal{B}_\Psi, \mathrm{id}_\mathcal{B}, \mathrm{id}_\mathcal{B})$, where right-dualizability holds as in (3.1.9). Moreover, if $\mathcal{C}$ is another rigid monoidal category, and $(\mathcal{N}, \gamma, \check{\gamma})$ is a homomorphism datum from $\mathcal{B}$ to $\mathcal{C}$, then we also have a "composed" homomorphism datum

$$(\mathcal{N}, \gamma, \check{\gamma}) \circ \Psi := ({}_\Psi\mathcal{N}, \gamma, \check{\gamma})$$

from $\mathcal{A}$ to $\mathcal{C}$. By (5.2.6.2) and functoriality of (3.1.5.1), we have

$$F_{(\mathcal{N},\gamma,\check{\gamma})\circ\Psi} \simeq F_{(\mathcal{N},\gamma,\check{\gamma})} \circ \Psi, \qquad F^{\mathrm{Tr}}_{(\mathcal{N},\gamma,\check{\gamma})\circ\Psi} \simeq F_{(\mathcal{N},\gamma,\check{\gamma})} \circ \mathrm{Tr}(\mathrm{Ind}_\Psi).$$

Unlike in Proposition 5.2.2, we do not know which conditions allow homomorphism data to be composed in general (i.e., that guarantee that the maps $\mathrm{HH}(\beta), \mathrm{HH}(\check{\beta})$ remain equivalences).

5.3. Recovering Lusztig's homomorphism

5.3.1. As in Proposition 5.1.2, we fix a nilpotent e, and let S_e be a Slodowy slice at e. In this section, we construct a homomorphism datum $\Phi^{\mathrm{mod}}_{S_e}$ categorifying the homomorphism $\phi_{e,k}$, and use Proposition 5.2.9 to identify $\phi_{e,k}$ with the restriction map (5.2.1.1).

5.3.2. We begin by constructing our categorification of $\mathcal{J}_{e,k}$. As suggested by §2.4.11, we define

$$\mathcal{J}^{\mathrm{mod}}_e := \mathcal{A}_{S_e,0} \otimes_k \mathcal{A}^{\mathrm{op}}_{S_e,0}\text{-mod}^{Z_e},$$

which is a monoidal category under the tensor product of bimodules. To verify that it is rigid (as required in Definition 5.2.4), we need the following general lemma:

5.3.3. Lemma. Let $\mathcal{C}$ be a compactly generated, rigid symmetric monoidal category, and let $A \in \mathrm{Alg}(\mathcal{C})$. Suppose that $A \in \mathcal{C}^c$, and the regular A-bimodule lies in $A \otimes A^{\mathrm{op}}\text{-mod}^c_\mathcal{C}$. Then the monoidal category $A \otimes A^{\mathrm{op}}\text{-mod}_\mathcal{C}$ is rigid.

Proof. Tensoring over A preserves compact objects by the first assumption, and the monoidal unit is compact by the latter assumption. Thus, as in §3.1.8, it remains to show that compact objects admit left and right monoidal duals.

We begin with the latter. Given $M \in A \otimes A^{\mathrm{op}}\text{-mod}^c_\mathcal{C}$, we wish to show that $- \otimes_A M$ admits a right adjoint of the form $- \otimes_A M^{\vee,R}$. Since M is a finite colimit of objects of the form $A \otimes X \otimes A$ for $X \in \mathcal{C}^c$, we immediately reduce to this case.[46] Given $L, N \in A \otimes A^{\mathrm{op}}\text{-mod}_\mathcal{C}$, we have

$$\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(L \otimes_A (A \otimes X \otimes A), N) \simeq \mathrm{Hom}_A(L \otimes X, N) \simeq \mathrm{Hom}_A(L, N \otimes X^\vee).$$

Observe that the dual object $A^\vee$ (in $\mathcal{C}$) carries a canonical A-bimodule structure. We claim that

$$\mathrm{Hom}_A(L, N \otimes X^\vee) \simeq \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(L, N \otimes X^\vee \otimes A^\vee). \tag{5.3.3.1}$$

Note that this immediately implies that $(A \otimes X \otimes A)^{\vee,R} \simeq A \otimes X^\vee \otimes A^\vee$. Moreover, this bimodule is compact: writing $A^\vee \simeq A^\vee \otimes_A A$, we may resolve the latter copy of A by bimodules of the form $A \otimes Y \otimes A$ for $Y \in \mathcal{C}^c$, and so $A^\vee \in A^{\mathrm{op}}\text{-mod}^c_\mathcal{C}$ as $A^\vee \in \mathcal{C}^c$. To show (5.3.3.1), note that we may write L as a colimit of bimodules $A \otimes Z \otimes A$ for $Z \in \mathcal{C}^c$, so we may assume L is of this form. We then have

$$\begin{aligned}\mathrm{Hom}_A(A \otimes Z \otimes A, N \otimes X^\vee) &\simeq \mathrm{Hom}_\mathcal{C}(Z \otimes A, N \otimes X^\vee)\\ &\simeq \mathrm{Hom}_\mathcal{C}(Z, N \otimes X^\vee \otimes A^\vee)\\ &\simeq \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A \otimes Z \otimes A, N \otimes X^\vee \otimes A^\vee),\end{aligned}$$

as desired.

Finally, to establish existence of left duals, we must construct an $M^{\vee,L}$ whose right-dual is M. Observe that writing $A^\vee \simeq A \otimes_A A^\vee \otimes_A A$ exhibits it as a compact bimodule by the same argument as in the previous paragraph. Thus, setting

$$M^{\vee,L} := A^\vee \otimes_A (A^\vee \otimes_A M)^{\vee,R},$$

[46] Recall that, for any compact $A \in \mathrm{Alg}(\mathcal{C})$, the category $A\text{-mod}_\mathcal{C}$ is compactly generated by objects of the form $A \otimes X$ for $X \in \mathcal{C}^c$. Indeed, $\mathrm{Hom}_A(A \otimes X, -) \simeq \mathrm{Hom}_\mathcal{C}(X, -)$, so such objects are evidently compact, and for any $M \in \mathcal{C}$, we may write $M \simeq A \otimes_A M$ and resolve A by its bar complex to obtain a resolution of M by such objects (see also [**Lur17**, Lem. 5.3.2.12]).

we obtain an object of $A \otimes A^{\mathrm{op}}\text{-mod}^c_{\mathcal{C}}$. To show that its right-dual recovers M, we again reduce to the case of $M \simeq A \otimes X \otimes A$, where it is not hard to check that[47]

$$M^{\vee,L} \simeq A^{\vee} \otimes_A (A \otimes X^{\vee} \otimes A) \simeq A^{\vee} \otimes X^{\vee} \otimes A$$

and hence

$$(M^{\vee,L})^{\vee,R} \simeq A \otimes X \otimes A \simeq M$$

as desired. □

5.3.4. Corollary. The monoidal category $\mathcal{J}_e^{\mathrm{mod}}$ is rigid and semisimple.

Proof. Note that $\mathcal{A}_{S_e,0}$ is a finite dimensional algebra, and in particular lies in $\mathrm{Rep}(Z_e)^c$. Thus, by Lemma 5.3.3, it suffices to show that the unit object of $\mathcal{J}_e^{\mathrm{mod}}$ is compact. To this end, note that $\mathrm{Rep}(Z_e)$ is a ($\mathrm{Rep}(Z_e)$-linear) direct summand of $\mathrm{Rep}(Z_e^{\mathrm{cov}})$ as in Remark 2.4.13. It follows that $\mathcal{J}_e^{\mathrm{mod}}$ is a direct summand (and subalgebra) of

$$\mathrm{Rep}(Z_e^{\mathrm{cov}}) \otimes_{\mathrm{Rep}(Z_e)} \mathcal{J}_e^{\mathrm{mod}} \simeq \mathrm{QC}(\underline{\mathbf{B}}_e \times \underline{\mathbf{B}}_e / Z_e^{\mathrm{cov}})$$

as in (5.3.6.2) (which is the categorical analogue of Proposition 2.4.12). It therefore suffices to show that the unit object of the latter is compact, which is clear. Moreover, the latter is evidently semisimple. □

5.3.5. We now compute the categorical trace[48] of $\mathcal{J}_e^{\mathrm{mod}}$ and the universal trace of its unit, with a view towards applying Proposition 5.2.9:

5.3.6. Lemma. We have a natural equivalence

$$\mathrm{Tr}(\mathcal{J}_e^{\mathrm{mod}}) \simeq \mathrm{QC}(Z_e/Z_e), \tag{5.3.6.1}$$

under which there is a canonical isomorphism

$$[\mathcal{J}_e^{\mathrm{mod}}] \simeq \mathcal{S}^{\mathbf{B}_e}.$$

Proof. By [**Lur17**, Rmk. 4.8.5.18], the $\mathrm{Rep}(Z_e)$-modules $\mathcal{A}_{S_e,0}\text{-mod}^{Z_e}$ and $\mathcal{A}^{\mathrm{op}}_{S_e,0}\text{-mod}^{Z_e}$ are dual, so we obtain an adjunction

$$\mathcal{A}^{\mathrm{op}}_{S_e,0}\text{-mod}^{Z_e} \otimes_{\mathrm{Rep}(Z_e)} - \colon \mathrm{Rep}(Z_e)\textbf{-mod} \rightleftarrows \mathrm{Rep}(Z_e)\textbf{-mod} \colon \mathcal{A}_{S_e,0}\text{-mod}^{Z_e} \otimes_{\mathrm{Rep}(Z_e)} - .$$

We claim that the latter functor is conservative. Indeed, by [**GR17a**, Ch. 1, Cor. 8.5.7], (5.1.2.1), and (5.1.3.2), we have equivalences

$$\mathrm{Rep}(Z_e^{\mathrm{cov}}) \otimes_{\mathrm{Rep}(Z_e)} \mathcal{A}_{S_e,0}\text{-mod}^{Z_e} \simeq \mathcal{A}_{S_e,0}\text{-mod}^{Z_e^{\mathrm{cov}}} \simeq \mathcal{A}^{\mathrm{cov}}_{S_e,0}\text{-mod}^{Z_e^{\mathrm{cov}}} \simeq \mathrm{QC}(\underline{\mathbf{B}}_e/Z_e^{\mathrm{cov}}), \tag{5.3.6.2}$$

and the functor $\mathrm{QC}(\underline{\mathbf{B}}_e/Z_e^{\mathrm{cov}}) \otimes_{\mathrm{Rep}(Z_e)} -$ is conservative by the proof of [**BZFN12**, Thm. 1.3]. By [**GR17a**, Ch. 1, Prop. 8.5.4], there is a natural identification of monads

$$\mathcal{A}_{S_e,0}\text{-mod}^{Z_e} \otimes_{\mathrm{Rep}(Z_e)} (\mathcal{A}^{\mathrm{op}}_{S_e,0}\text{-mod}^{Z_e} \otimes_{\mathrm{Rep}(Z_e)} -) \simeq \mathcal{J}_e^{\mathrm{mod}} \otimes_{\mathrm{Rep}(Z_e)} -,$$

so the Barr–Beck–Lurie theorem yields an equivalence

$$\mathcal{A}_{S_e,0}\text{-mod}^{Z_e} \otimes_{\mathrm{Rep}(Z_e)} - \colon \mathrm{Rep}(Z_e)\textbf{-mod} \to \mathcal{J}_e^{\mathrm{mod}}\textbf{-mod}. \tag{5.3.6.3}$$

Taking 2-categorical traces then yields (5.3.6.1). Moreover, under this identification, we have

$$[\mathcal{J}_e^{\mathrm{mod}}] \simeq \mathsf{BG}_{*/Z_e}(\mathcal{A}^{\mathrm{op}}_{S_e,0}\text{-mod}^{Z_e}) \simeq \mathcal{S}^{\mathbf{B}_e}$$

by Proposition 3.4.11 and the proof of Proposition 5.1.2. □

[47] Essentially, we have shown that $M^{\vee,R} \simeq \underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(M, A)$, with the A^{op}-module structure obtained from the A-module structure on M, and the A-module structure obtained from that on A.

[48] We expect such a description to hold more generally, i.e., $\mathrm{Tr}(\mathcal{A} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{A}^{\mathrm{op}}\text{-mod}_{\mathrm{QC}(\mathcal{X})}) \simeq \mathrm{QC}(\mathcal{L}\mathcal{X})$ for a suitable stack $\mathcal{X}$ and faithfully flat $\mathcal{A} \in \mathrm{Alg}(\mathrm{QC}(\mathcal{X}))$. In our case, the group Z_e^{cov} exempts us from such generalities.

5.3.7. In particular, by Theorem 3.1.11 and Proposition 2.4.8, we have natural algebra isomorphisms

$$\mathrm{HH}(\mathcal{J}_e^{\mathrm{mod}}) \simeq \mathrm{End}_{Z_e/Z_e}(\mathcal{S}^{\mathbf{B}_e})^{\mathrm{op}} \simeq \mathcal{J}_{e,k}. \tag{5.3.7.1}$$

Thus, $\mathcal{J}_e^{\mathrm{mod}}$ is indeed a categorification of $\mathcal{J}_{e,k}$ (note that this fact could also have been obtained more directly from Corollary 5.3.4 and Theorem 2.4.10). The tensor product $\mathcal{J}_e^{\mathrm{mod}} \otimes \mathrm{Rep}(\mathbb{G}_m)$ is therefore a categorification of $\mathcal{J}_{e,k}[v^{\pm 1}]$, and Lemma 5.3.6 immediately gives identifications

$$\begin{gathered}\mathrm{Tr}(\mathcal{J}_e^{\mathrm{mod}} \otimes \mathrm{Rep}(\mathbb{G}_m)) \simeq \mathrm{Tr}(\mathcal{J}_e^{\mathrm{mod}}) \otimes \mathrm{Tr}(\mathrm{Rep}(\mathbb{G}_m)) \simeq \mathrm{QC}(\widetilde{Z}_e/\widetilde{Z}_e), \\ [\mathcal{J}_e^{\mathrm{mod}} \otimes \mathrm{Rep}(\mathbb{G}_m)] \simeq [\mathcal{J}_e^{\mathrm{mod}}] \otimes [\mathrm{Rep}(\mathbb{G}_m)] \simeq \mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m}.\end{gathered} \tag{5.3.7.2}$$

We now construct the homomorphism datum $\Phi_{S_e}^{\mathrm{mod}}$. Recall the category $\mathcal{H}_{S_e}^{\mathrm{mod}}$ of (4.2.7.1), which is rigid monoidal by Proposition 4.2.10. Set

$$\mathcal{M}_{S_e}^{\mathrm{mod}} := \mathcal{A}_{S_e,0} \otimes_k \mathcal{A}_{S_e}^{\mathrm{op}}\text{-mod}^{\widetilde{Z}_e},$$

which carries an evident $(\mathcal{J}_e^{\mathrm{mod}} \otimes \mathrm{Rep}(\mathbb{G}_m), \mathcal{H}_{S_e}^{\mathrm{mod}})$-bimodule structure via the tensor products $\otimes_{\mathcal{A}_{S_e,0}}$ and $\otimes_{\mathcal{A}_{S_e}}$, respectively. Consider the functors

$$\mathcal{J}_e^{\mathrm{mod}} \xrightarrow{- \underset{\mathcal{A}_{S_e,0}}{\otimes} \mathcal{A}_{S_e}} \mathcal{M}_{S_e}^{\mathrm{mod}} \xrightarrow{- \underset{\mathcal{A}_{S_e}}{\otimes} \mathcal{A}_{S_e,0}} \mathcal{J}_e^{\mathrm{mod}},$$

where the tensor products are taken with respect to the homomorphisms $\mathrm{in}_0, \mathrm{pr}_0$ of (2.2.15.2). Finally, set[49]

$$\Phi_{S_e}^{\mathrm{mod}} := (\mathcal{M}_{S_e}^{\mathrm{mod}}, - \underset{\mathcal{A}_{S_e,0}}{\otimes} \mathcal{A}_{S_e}, - \underset{\mathcal{A}_{S_e}}{\otimes} \mathcal{A}_{S_e,0}).$$

5.3.8. LEMMA. The triple $\Phi_{S_e}^{\mathrm{mod}}$ is a unital homomorphism datum from $\mathcal{H}_{S_e}^{\mathrm{mod}}$ to $\mathcal{J}_e^{\mathrm{mod}} \otimes \mathrm{Rep}(\mathbb{G}_m)$.

PROOF. The functors $-\otimes_{\mathcal{A}_{S_e,0}} \mathcal{A}_{S_e}$ and $-\otimes_{\mathcal{A}_{S_e,0}} \mathcal{A}_{S_e}$ are evidently $\mathcal{J}_e^{\mathrm{mod}}$-linear, preserve compact objects, and compose to the identity. Moreover, applying Lemma 3.4.5 to the generators $\mathcal{A}_{S_e,0} \otimes_k \mathcal{A}_{S_e,0}$ of $\mathcal{J}_e^{\mathrm{mod}}$ and $\mathcal{A}_{S_e,0} \otimes_k \mathcal{A}_{S_e}$ of $\mathcal{M}_{S_e}^{\mathrm{mod}}$ (using the Koszul grading of §2.2.15) immediately shows that these functors induce inverse equivalences on Hochschild homology.

Thus, it remains to show that the $(\mathcal{J}_e^{\mathrm{mod}} \otimes \mathrm{Rep}(\mathbb{G}_m), \mathcal{H}_{S_e}^{\mathrm{mod}})$-bimodule $\mathcal{M}_{S_e}^{\mathrm{mod}}$ is right-dualizable. Consider the $(\mathcal{H}_{S_e}^{\mathrm{mod}}, \mathcal{J}_e^{\mathrm{mod}} \otimes \mathrm{Rep}(\mathbb{G}_m))$-bimodule

$$\mathcal{M}_{S_e}^{\mathrm{mod},R} := \mathcal{A}_{S_e} \otimes_k \mathcal{A}_{S_e,0}^{\mathrm{op}}\text{-mod}^{\widetilde{Z}_e}.$$

Note that by [GR17a, Ch. 1, Prop. 8.5.4], the external tensor product gives bimodule equivalences

$$\begin{gathered}\mathcal{M}_{S_e}^{\mathrm{mod}} \simeq \mathcal{A}_{S_e,0}\text{-mod}^{\widetilde{Z}_e} \otimes_{\mathrm{Rep}(\widetilde{Z}_e)} \mathcal{A}_{S_e}^{\mathrm{op}}\text{-mod}^{\widetilde{Z}_e}, \\ \mathcal{M}_{S_e}^{\mathrm{mod},R} \simeq \mathcal{A}_{S_e}\text{-mod}^{\widetilde{Z}_e} \otimes_{\mathrm{Rep}(\widetilde{Z}_e)} \mathcal{A}_{S_e,0}^{\mathrm{op}}\text{-mod}^{\widetilde{Z}_e}.\end{gathered} \tag{5.3.8.1}$$

Thus, to show that $\mathcal{M}_{S_e}^{\mathrm{coh}}$ is left-dual to $\mathcal{M}_{S_e}^{\mathrm{coh},R}$, it suffices to show that the $(\mathrm{Rep}(\widetilde{Z}_e), \mathcal{H}_{S_e}^{\mathrm{mod}})$-bimodule $\mathcal{A}_{S_e}^{\mathrm{op}}\text{-mod}^{\widetilde{Z}_e}$ is left-dual to the $(\mathcal{H}_{S_e}^{\mathrm{mod}}, \mathrm{Rep}(\widetilde{Z}_e))$-bimodule $\mathcal{A}_{S_e}\text{-mod}^{\widetilde{Z}_e}$, and the $(\mathcal{J}_e^{\mathrm{mod}} \otimes \mathrm{Rep}(\mathbb{G}_m), \mathrm{Rep}(\widetilde{Z}_e))$-bimodule $\mathcal{A}_{S_e,0}\text{-mod}^{\widetilde{Z}_e}$ is left-dual to the $(\mathrm{Rep}(\widetilde{Z}_e), \mathcal{J}_e^{\mathrm{mod}} \otimes \mathrm{Rep}(\mathbb{G}_m))$-bimodule $\mathcal{A}_{S_e,0}^{\mathrm{op}}\text{-mod}^{\widetilde{Z}_e}$. The former statement holds by Lemma 3.3.9 (using the equivalences (4.2.10.1) and (2.2.4.1)), and the latter statement holds as in [GR17a, Ch. 1, Cor. 8.6.3]. □

5.3.9. As in Example 5.2.11, the monoidal functor $i_{S_e}^* : \mathcal{H}^{\mathrm{mod}} \to \mathcal{H}_{S_e}^{\mathrm{mod}}$ gives rise to a composed homomorphism datum $\Phi_{S_e}^{\mathrm{mod}} \circ i_{S_e}^*$ from $\mathcal{H}^{\mathrm{mod}}$ to $\mathcal{J}_e^{\mathrm{mod}} \otimes \mathrm{Rep}(\mathbb{G}_m)$. We now show that the functor

$$F_{\Phi_{S_e}^{\mathrm{mod}} \circ i_{S_e}^*} : \mathcal{H}^{\mathrm{mod}} \to \mathcal{J}_e^{\mathrm{mod}} \otimes \mathrm{Rep}(\mathbb{G}_m)$$

identifies with $\phi_{e,k}$ on Hochschild homology:

[49] We expect that another homomorphism datum Φ_e^{mod} may be constructed from the bimodule $\mathcal{M}_e^{\mathrm{mod}} := \mathcal{A}_{S_e,0} \otimes_k \mathcal{A}_e^{\mathrm{op}}\text{-mod}^{\widetilde{Z}_e}$, where $\mathcal{A}_e := \mathcal{A}_{S_e} \otimes_{\mathcal{O}(S_e)} k$ is the specialization of $\mathcal{A}_{S_e}$ at e. Our notation $\Phi_{S_e}^{\mathrm{mod}}$ is therefore intended to leave room for this possibility.

5.3.10. Proposition. There is a natural commutative diagram of (classical) algebra homomorphisms

(5.3.10.1)
$$\begin{tikzcd}
\mathcal{H}_k \arrow[r, two heads] \arrow[d, "\wr"] & (\mathcal{H}/\mathcal{H}_{<e})_k \arrow[r, "\phi_{e,k}"] \arrow[d] & \mathcal{J}_{e,k}[v^{\pm 1}] \arrow[d, "\wr"] \\
\mathrm{HH}(\mathscr{H}^{\mathrm{mod}}) \arrow[r, "\mathrm{HH}(i^*_{S_e})"] & \mathrm{HH}(\mathscr{H}^{\mathrm{mod}}_{S_e}) \arrow[r, hook, "\mathrm{HH}(F_{\Phi^{\mathrm{mod}}_{S_e}})"] & \mathrm{HH}(\mathscr{J}^{\mathrm{mod}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)).
\end{tikzcd}$$

Proof. The rightmost identification is immediate from (5.3.7.1), and the leftmost identification is that of [**BZCHN22**, Thm. 1.4]. For the middle vertical map, note that by [**Bez16**, Thm. 55], the Kazhdan–Lusztig basis elements spanning the two-sided ideal $\mathcal{H}_{<e}$ are supported on nilpotent orbits in $\mathcal{N}$ preceding the orbit of e. In particular, their support does not intersect S_e, so the composite map $\mathcal{H}_k \to \mathrm{HH}(\mathscr{H}^{\mathrm{mod}}_{S_e})$ factors through $(\mathcal{H}/\mathcal{H}_{<e})_k$. Moreover, applying Lemma 3.4.5 to the generator $\mathcal{A}_{S_e} \otimes_{\mathcal{O}(S_e)} \mathcal{A}^{\mathrm{op}}_{S_e}$ of $\mathscr{H}^{\mathrm{mod}}_{S_e}$, we immediately obtain an isomorphism (of *vector spaces*)

$$\mathrm{HH}(\mathscr{H}^{\mathrm{mod}}_{S_e}) \simeq \mathrm{HH}(\mathscr{J}^{\mathrm{mod}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)) \tag{5.3.10.2}$$

using the Koszul grading of §2.2.15. Thus, $\mathrm{HH}(\mathscr{H}^{\mathrm{mod}}_{S_e})$ is concentrated in cohomological degree 0.

To show commutativity of the outer rectangle in (5.3.10.1), note that $i^*_{S_e}(-) \simeq - \otimes_{\mathcal{A}_g} \mathcal{A}_{S_e}$, and by (5.2.6.2), we have

$$F_{\Phi^{\mathrm{mod}}_{S_e}}(-) \simeq (\mathcal{A}_{S_e,0} \otimes_{\mathcal{A}_{S_e,0}} \mathcal{A}_{S_e}) \otimes_{\mathcal{A}_{S_e}} - \otimes_{\mathcal{A}_{S_e}} \mathcal{A}_{S_e,0} \simeq {}_{\mathcal{A}_{S_e,0}}(- \otimes_{\mathcal{A}_{S_e}} \mathcal{A}_{S_e,0}).$$

Thus, by §2.4.2 (and monoidality of the leftmost identification in (5.3.10.1)), it suffices to show that $\sum_{d \in \mathbf{D}_e} C_d \in \mathcal{H}_k$ is sent to $[\mathcal{A}_{S_e,0}] \in \mathrm{HH}(\mathscr{H}^{\mathrm{mod}}_{S_e})$, and that we have a commutative diagram

(5.3.10.3)
$$\begin{tikzcd}
(\mathcal{H}/\mathcal{H}_{<e})_k \arrow[d] & \mathcal{H}_{e,k} \arrow[l, hook'] \arrow[r, "\mathrm{bas}_{e,k}", "\sim"'] \arrow[d, "\wr"] & \mathcal{J}_{e,k}[v^{\pm 1}] \arrow[d, "\wr"] \\
\mathrm{HH}(\mathscr{H}^{\mathrm{mod}}_{S_e}) & \mathrm{HH}(\mathscr{H}^{\mathrm{mod}}_{\hat{e}}) \arrow[l, hook'] \arrow[r, "\mathrm{HH}(\mathrm{in}_0^*)"] & \mathrm{HH}(\mathscr{J}^{\mathrm{mod}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)).
\end{tikzcd}$$

Here we have let $\mathscr{H}^{\mathrm{mod}}_{\hat{e}}$ denote the full subcategory of $\mathscr{H}^{\mathrm{mod}}_{S_e}$ consisting of modules supported at e, and in_0^* denote the restriction of scalars along the inclusion $\mathcal{A}_{S_e,0} \otimes_k \mathcal{A}^{\mathrm{op}}_{S_e,0} \hookrightarrow \mathcal{A}_{S_e} \otimes_{\mathcal{O}(S_e)} \mathcal{A}^{\mathrm{op}}_{S_e}$. In fact, the latter assertion implies the former, as $\mathrm{bas}_e(\sum_{d \in \mathbf{D}_e} C_d) = 1_{\mathcal{J}_e}$, and $\mathrm{in}_0^* \mathcal{A}_{S_e,0} \simeq \mathbf{1}_{\mathscr{J}^{\mathrm{mod}}_e}\langle 0 \rangle$. As before, applying Lemma 3.4.5 to the generator $\mathcal{A}_{S_e,0} \otimes \mathcal{A}^{\mathrm{op}}_{S_e,0}$ of $\mathscr{H}^{\mathrm{mod}}_{\hat{e}}$ immediately yields the isomorphism $\mathrm{HH}(\mathrm{in}_0^*)$. Moreover, [**Bez16**, Thm. 54(c)] implies that the Kazhdan–Lusztig basis elements spanning $\mathcal{H}_e$ are sent to (the classes of) shifts of simple objects of $\mathscr{H}^{\mathrm{mod},\heartsuit}_{S_e}$. The identification $\mathcal{H}_{e,k} \simeq \mathrm{HH}(\mathscr{H}^{\mathrm{mod}}_{\hat{e}})$ and commutativity of the leftmost square in (5.3.10.3) are now immediate. Finally, commutativity of the rightmost square follows from the proof of [**BL23**, Prop. 8.25].

Lastly, we show that $\mathrm{HH}(F_{\Phi^{\mathrm{mod}}_{S_e}})$ is injective. By (5.3.10.2) and (2.2.14.3), $\mathrm{HH}(\mathscr{H}^{\mathrm{mod}}_{S_e})$ has an $R(\widetilde{Z}_e)$-basis given by the classes $[\underline{L}_b^{S_e,*} \otimes_k E_b^{S_e} \otimes_{\mathcal{O}(S_e)} E_{b'}^{S_e,\ell} \otimes_k \underline{L}_{b'}^{S_e}]$ for $b, b' \in \mathbf{B}_e$. Likewise, the standard $R(\widetilde{Z}_e)$-basis of $\mathrm{HH}(\mathscr{J}^{\mathrm{mod}}_e \otimes \mathrm{Rep}(\mathbb{G}_m))$ (used in [**BL23**, Prop. 8.25]) is given by the classes $[\underline{L}_b^{S_e,*} \otimes_k L_b^{S_e} \otimes_k L_{b'}^{S_e,\ell} \otimes_k \underline{L}_{b'}^{S_e}]$. Since

$$\begin{aligned}
\mathrm{HH}(F_{\Phi^{\mathrm{mod}}_{S_e}})([\underline{L}_b^{S_e,*} \otimes_k E_b^{S_e} \otimes_{\mathcal{O}(S_e)} E_{b'}^{S_e,\ell} \otimes_k \underline{L}_{b'}^{S_e}]) &= [{}_{\mathcal{A}_{S_e,0}}(\underline{L}_b^{S_e,*} \otimes_k E_b^{S_e} \otimes_{\mathcal{O}(S_e)} k) \otimes_k L_{b'}^{S_e,\ell} \otimes_k \underline{L}_{b'}^{S_e}] \\
&\equiv [\underline{L}_b^{S_e,*} \otimes_k L_b^{S_e} \otimes_k L_{b'}^{S_e,\ell} \otimes_k \underline{L}_{b'}^{S_e}] \bmod \mathcal{J}_{e,k} \otimes_k v \cdot k[v],
\end{aligned}$$

the matrix describing $\mathrm{HH}(F_{\Phi^{\mathrm{mod}}_{S_e}})$ with respect to these bases has nonzero determinant, and is therefore full-rank. $\square$

5.3.11. We now turn to computing the associated functor

$$F^{\mathrm{Tr}}_{\Phi^{\mathrm{mod}}_{S_e} \circ i^*_{S_e}} \colon \mathrm{Tr}(\mathscr{H}^{\mathrm{mod}}) \to \langle [\mathscr{J}^{\mathrm{mod}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)] \rangle.$$

on categorical traces.

5.3.12. Lemma. The functor

$$\mathrm{Tr}(\mathscr{M}^{\mathrm{mod}}_{S_e}) \colon \mathrm{Tr}(\mathscr{H}^{\mathrm{mod}}_{S_e}) \to \mathrm{Tr}(\mathscr{J}^{\mathrm{mod}}_e \otimes \mathrm{Rep}(\mathbb{G}_m))$$

canonically identifies with

$$\mathcal{L}p_{S_e,*}\colon \mathrm{QC}^!(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e)) \to \mathrm{QC}^!(\widetilde{Z}_e/\widetilde{Z}_e)$$

via Corollary 3.3.17 and Lemma 5.3.6.

PROOF. By (5.3.8.1) and (5.3.6.3), we have a commutative diagram

$$\begin{tikzcd}
\mathrm{QC}(S_e/\widetilde{Z}_e)\text{-}\mathbf{mod} \arrow[d, "\mathrm{Res}_{p^*_{S_e}}"] & & \mathcal{H}^{\mathrm{mod}}_{S_e}\text{-}\mathbf{mod} \arrow[ll, "\mathcal{A}^{\mathrm{op}}_{S_e}\text{-}\mathrm{mod}^{\widetilde{Z}_e}"'] \arrow[d, "\mathcal{M}^{\mathrm{mod}}_{S_e}"] \\
\mathrm{QC}(\mathrm{B}\widetilde{Z}_e)\text{-}\mathbf{mod} \arrow[rr, "\mathcal{A}_{S_e,0}\text{-}\mathrm{mod}^{\widetilde{Z}_e}", "\sim"'] & & \mathcal{J}^{\mathrm{mod}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)\text{-}\mathbf{mod}
\end{tikzcd}$$

in $L(\mathrm{Morita}(\mathbf{dgCat}_k))_{\mathrm{rgd}}$ (omitting all identity functors from the notation). Taking categorical traces then yields the commutative diagram

$$\begin{tikzcd}
\mathrm{Tr}(\mathrm{QC}(S_e/\widetilde{Z}_e)) \arrow[d, "\mathrm{Tr}(\mathrm{Res}_{p^*_{S_e}})"] & & \mathrm{Tr}(\mathcal{H}^{\mathrm{mod}}_{S_e}) \arrow[ll, "\mathrm{Tr}(\mathcal{A}^{\mathrm{op}}_{S_e}\text{-}\mathrm{mod}^{\widetilde{Z}_e})"'] \arrow[d, "\mathrm{Tr}(\mathcal{M}^{\mathrm{mod}}_{S_e})"] \\
\mathrm{Tr}(\mathrm{QC}(\mathrm{B}\widetilde{Z}_e)) \arrow[rr, "\mathrm{Tr}(\mathcal{A}_{S_e,0}\text{-}\mathrm{mod}^{\widetilde{Z}_e})", "\sim"'] & & \mathrm{Tr}(\mathcal{J}^{\mathrm{mod}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)).
\end{tikzcd}$$

By Lemma 3.3.9, Proposition 4.2.10, and Corollary 3.2.13, this diagram identifies with

$$\begin{tikzcd}
\mathrm{QC}(\mathcal{L}(S_e/\widetilde{Z}_e)) \arrow[d, "\mathcal{L}p_{S_e,*}"] & & \mathrm{QC}^!(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e)) \arrow[ll, "\iota_{\{0\}_{S_e,\mathcal{N}}}\circ\Gamma_{\{0\}_{S_e,\mathcal{N}}}"'] \arrow[d, "\mathrm{Tr}(\mathcal{M}^{\mathrm{mod}}_{S_e})"] \\
\mathrm{QC}(\mathcal{L}(\mathrm{B}\widetilde{Z}_e)) \arrow[rr, "\mathrm{id}", "\sim"'] & & \mathrm{QC}(\mathcal{L}(\mathrm{B}\widetilde{Z}_e)).
\end{tikzcd} \tag{5.3.12.1}$$

The conclusion is now immediate from [**DG13**, §3.2.12]. □

5.3.13. Finally, we reach the main result of this chapter:

5.3.14. COROLLARY. The induced homomorphism

$$(\mathcal{L}p_{S_e,*}\mathcal{L}i^*_{S_e})_0\colon \mathrm{End}_{\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})}(\mathcal{S})^{\mathrm{op}} \to \mathrm{End}_{\widetilde{Z}_e/\widetilde{Z}_e}(\mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m})^{\mathrm{op}} \tag{5.3.14.1}$$

identifies canonically with $\phi_{e,k}$ via Theorems 1.2.1 and 2.4.10. Moreover, the following squares commute:

$$\begin{tikzcd}
\mathcal{H}_k\text{-mod} \arrow[r, shift left, "\mathrm{Ind}_{\phi_{e,k}}"] \arrow[d, "\wr"] & \mathcal{J}_{e,k}[v^{\pm 1}]\text{-mod} \arrow[l, shift left, "\mathrm{Res}_{\phi_{e,k}}"] \arrow[d, "\wr"] \\
\langle \mathcal{S} \rangle \arrow[r, shift left, "(\mathcal{L}p_{S_e,*}\circ\mathcal{L}i^*_{S_e})_0"] & \langle \mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m} \rangle. \arrow[l, shift left, "\mathrm{pr}_{\mathcal{S}}\circ\mathcal{L}i_{e,*}"]
\end{tikzcd}$$

PROOF. By Proposition 5.2.9, Example 5.2.11, Proposition 5.3.10, and (5.3.7.2), it suffices to identify (5.3.14.1) with the homomorphism induced by

$$F^{\mathrm{Tr}}_{\Phi^{\mathrm{mod}}_{S_e}} \simeq \mathrm{pr}_{[\mathcal{J}^{\mathrm{mod}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)]} \circ \mathrm{Tr}(\mathcal{M}^{\mathrm{mod}}_{S_e}) \circ \mathrm{Tr}(i^*_{S_e}). \tag{5.3.14.2}$$

Indeed, the conditions of Proposition 5.2.9 all hold by Theorem 1.2.1. Such an identification is then immediate from Corollary 3.3.20, Lemma 5.3.12, and Proposition 5.1.2.

The identification of $(\mathcal{L}p_{S_e,*}\mathcal{L}i^*_{S_e})_0|_{\langle\mathcal{S}\rangle}$ with $\mathrm{Ind}_{\phi_{e,k}}$ is as in (5.2.10.1). For the identification with $\mathrm{Res}_{\phi_{e,k}}$, let $\mathrm{Coh}(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e))_{\geq 0}$ denote the full subcategory of $\mathrm{Coh}(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e))$ spanned by complexes $\mathcal{F}$ such that all cohomology sheaves of $\mathcal{L}p_{S_e,*}\mathcal{F}$ lie in non-negative weights, and consider the adjunction

$$(\mathcal{L}i^*_e)_0\colon \mathrm{QC}(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e)) \rightleftarrows \mathrm{QC}(\widetilde{Z}_e/Z_e)\colon \mathcal{L}i_{e,*}\mathrm{in}_0.$$

where we have let in_0 denote the inclusion in weight-0. We claim that this restricts to an adjunction between full subcategories

$$(\mathcal{L}i_e^*)_0 \colon \operatorname{Coh}(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e))_{\geq 0} \rightleftarrows \operatorname{Coh}(\widetilde{Z}_e/Z_e) \colon \mathcal{L}i_{e,*}\mathrm{in}_0. \tag{5.3.14.3}$$

To see that the latter functor factors through $\operatorname{Coh}(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e))_{\geq 0}$, note that $\mathcal{L}i_{e,*}$ is a closed embedding by Lemma 3.2.7, and hence preserves coherent sheaves; moreover, $\mathcal{L}p_{S_e,*}\mathcal{L}i_{e,*}\mathrm{in}_0 \simeq \mathrm{in}_0$ lies in weight-0. To see that the former functor factors through $\operatorname{Coh}(\widetilde{Z}_e/Z_e)$, let $\mathcal{F} \in \operatorname{Coh}(\mathcal{L}(\widehat{S}_e/\widetilde{Z}_e))_{\geq 0}$, and note that we have a canonical identification $(\mathcal{L}i_e^*\mathcal{F})_0 \simeq (\mathcal{L}p_{S_e,*}\mathcal{F})_0$. Indeed, using (3.2.5.1) and the Koszul resolution of the diagonal inside $S_e \times S_e$, we see that $\mathcal{O}_{\mathcal{L}(S_e/\widetilde{Z}_e)}$ admits a semi-free resolution over $\mathcal{O}_{(S_e\times\widetilde{Z}_e)/\widetilde{Z}_e}$ generated by $\mathfrak{g}^{f,*}\langle -2\rangle[1]$. It follows that $\mathcal{O}_{\widetilde{Z}_e/\widetilde{Z}_e}$ admits a semi-free resolution over $\mathcal{O}_{\mathcal{L}(S_e/\widetilde{Z}_e)}$ with all generators in strictly positive weights. Computing $\mathcal{L}i_e^*\mathcal{F}$ via the latter resolution now yields the desired identification. Note that $\mathcal{L}p_{S_e}$ is affine, so $\mathcal{L}p_{S_e,*}$ is t-exact and hence preserves cohomology sheaves. In particular, $(\mathcal{L}p_{S_e,*}\mathcal{F})_0$ has bounded cohomological amplitude, so we reduce to the case where $\mathcal{F}$ lies in cohomological degree 0 and is coherent over $\mathcal{L}(S_e/\widetilde{Z}_e)^{\mathrm{cl}}$. The desired factorization through $\operatorname{Coh}(\widetilde{Z}_e/Z_e)$ is now immediate from the isomorphism $(\mathcal{L}p_{S_e,*}\mathcal{O}_{\mathcal{L}(S_e/\widetilde{Z}_e)^{\mathrm{cl}}})_{\leq 0} \simeq \mathcal{O}_{\widetilde{Z}_e/\widetilde{Z}_e}$.

Ind-completing (5.3.14.3), we obtain an adjunction

$$\operatorname{Ind}((\mathcal{L}i_e^*)_0) \colon \operatorname{Ind}(\operatorname{Coh}(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e)_{\geq 0}) \rightleftarrows \operatorname{QC}^!(\widetilde{Z}_e/Z_e) \colon \operatorname{Ind}(\mathcal{L}i_{e,*}\mathrm{in}_0). \tag{5.3.14.4}$$

Note that $\operatorname{Ind}(\operatorname{Coh}(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e)_{\geq 0})$ is a full subcategory of $\operatorname{QC}^!(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e))$ by [**Lur09**, Prop. 5.3.5.11]. Thus, $\operatorname{Ind}(\mathcal{L}i_{e,*}\mathrm{in}_0)$ identifies with the usual functor $\mathcal{L}i_{e,*}\mathrm{in}_0$, and the previous paragraph identifies $\operatorname{Ind}((\mathcal{L}i_e^*)_0)$ with (the restriction of) $(\mathcal{L}p_{S_e,*})_0$. Finally, observe that the restriction of $\mathcal{L}i_{S_e}^*$ to $\langle \mathcal{S} \rangle$ factors through $\operatorname{Ind}(\operatorname{Coh}(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e)_{\geq 0})$. Indeed, it suffices to show that $\mathcal{L}i_{S_e}^*\mathcal{S}$ lies in $\operatorname{Coh}(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e)_{\geq 0}$. As in (4.3.10.2), this follows from the isomorphism $\mathcal{S} \simeq \mathsf{BG}_{S_e/\widetilde{Z}_e}(\mathcal{A}_{S_e}^{\mathrm{op}}\text{-mod}^{\widetilde{Z}_e}))$ in $\operatorname{QC}(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e))$ and the Koszul grading of $\mathcal{A}_{S_e}$. The adjoint pairs

$$\langle \mathcal{S} \rangle \underset{\mathrm{pr}_{\mathcal{S}}\mathcal{L}i_{S_e,*}}{\overset{\mathcal{L}i_{S_e}^*\mathrm{in}_{\mathcal{S}}}{\rightleftarrows}} \operatorname{Ind}(\operatorname{Coh}(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e)_{\geq 0}) \underset{\mathcal{L}i_{e,*}\mathrm{in}_0}{\overset{(\mathcal{L}p_{S_e,*})_0}{\rightleftarrows}} \operatorname{QC}^!(\widetilde{Z}_e/Z_e) \tag{5.3.14.5}$$

now yield the result. ☐

5.3.15. Remark. When G is rank-1 semisimple, this result may be computed explicitly from the description of $\mathcal{S}$ given in [**BZCHN22**, Ex. 4.21]; we omit the details.

5.3.16. Remark. The proof of Corollary 5.3.14 suggests that we could have simply written $(\mathcal{L}(i_e i_{S_e})^*)_0$ in place of $(\mathcal{L}p_{S_e,*}\mathcal{L}i_{S_e}^*)_0$. Indeed, as in (5.3.14.4), we can make sense of this functor on a full subcategory of $\operatorname{QC}^!(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e))$ containing $\langle \mathcal{S} \rangle$. Alternatively, if we work with categories of *quasi*-coherent sheaves, then it is possible to state an analog of Corollary 5.3.14 with $(\mathcal{L}i_e^*)_0$ in place of $(\mathcal{L}p_{S_e,*}\mathcal{L}i_{S_e}^*)_0$. However, the morphism $\mathcal{L}i_e$ is *not* generally eventually coconnective, so the functor $\mathcal{L}i_e^*$ is not defined on categories of ind-coherent sheaves.[50] In any case, as we have seen, the functor $\mathcal{L}p_{S_e,*}\mathcal{L}i_{S_e}^*$ is the one which arises naturally in the course of our proof.

Still, as always, we do have a functor $\mathcal{L}i_e^!$ on categories of ind-coherent sheaves; we claim that there is a canonical isomorphism

$$(\mathcal{L}(i_e i_{S_e})^!\mathcal{S})_0 \simeq \mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m}$$

just as in Proposition 5.1.2. First note that by Lemma 3.2.6(1), the map $\mathcal{L}i_e$ is a closed immersion; moreover, by Lemma 3.3.19 and [**BZCHN22**, Lem. 3.12], there is a canonical equivalence $\mathcal{L}i_{S_e}^* \simeq \mathcal{L}i_{S_e}^!$. Thus, for any $\mathcal{F} \in \operatorname{QC}^!(\widetilde{Z}_e/Z_e)$, we have a canonical isomorphism

$$\operatorname{Hom}_{\widetilde{Z}_e/Z_e}(\mathcal{F}, (\mathcal{L}(i_e i_{S_e})^!\mathcal{S})_0) \simeq \operatorname{Hom}_{\mathcal{L}(S_e/\widetilde{Z}_e)}(\mathcal{L}i_{e,*}\mathrm{in}_0\mathcal{F}, \mathcal{L}i_{S_e}^*\mathcal{S}).$$

[50] We could of course take the composition

$$\operatorname{QC}^!(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e)) \xrightarrow{\Gamma_{\{0\}}} \operatorname{QC}(\mathcal{L}(\widehat{S}_{e,\mathcal{N}}/\widetilde{Z}_e)) \xrightarrow{\mathcal{L}i_e^*} \operatorname{QC}(\widetilde{Z}_e/\widetilde{Z}_e),$$

noting that the stack $\widetilde{Z}_e/\widetilde{Z}_e$ is smooth, but this will not be left-adjoint to the usual functor $\mathcal{L}i_{e,*}$.

Next, observe that $\mathcal{L}i_{S_e}^*\mathcal{S}$ is Grothendieck–Serre self-dual: indeed, the base-change diagram (2.2.2.1) gives an isomorphism

$$\mathcal{L}i_{S_e}^*\mathcal{S} \simeq \mathcal{L}\pi_{S_e,*}\mathcal{O}_{\mathcal{L}(\widetilde{S}_e/\widetilde{Z}_e)}, \tag{5.3.16.1}$$

and hence

$$\mathbb{D}_{\mathcal{L}(S_e/\widetilde{Z}_e)}(\mathcal{L}i_{S_e}^*\mathcal{S}) \simeq \mathcal{L}\pi_{S_e,*}\mathcal{H}om_{\mathcal{L}(S_e/\widetilde{Z}_e)}(\mathcal{O}_{\mathcal{L}(\widetilde{S}_e/\widetilde{Z}_e)}, \mathcal{L}\pi_{S_e}^!\omega_{\mathcal{L}(S_e/\widetilde{Z}_e)}) \simeq \mathcal{L}\pi_{S_e,*}\omega_{\mathcal{L}(\widetilde{S}_e/\widetilde{Z}_e)} \simeq \mathcal{L}i_{S_e}^*\mathcal{S}.$$

Here we are using the fact that loop spaces of smooth Artin 1-stacks are canonically Calabi–Yau, as shown in *loc. cit.* In the same vein, we have

$$\mathbb{D}_{\mathcal{L}(S_e/\widetilde{Z}_e)}(\mathcal{L}i_{e,*}\mathrm{in}_0\mathcal{F}) \simeq \mathcal{L}i_{e,*}\mathrm{in}_0\mathcal{F}^\vee,$$

and hence

$$\begin{aligned}\mathrm{Hom}_{\mathcal{L}(S_e/\widetilde{Z}_e)}(\mathcal{L}i_{e,*}\mathrm{in}_0\mathcal{F}, \mathcal{L}i_{S_e}^*\mathcal{S}) &\simeq \mathrm{Hom}_{\mathcal{L}(S_e/\widetilde{Z}_e)}(\mathcal{L}i_{S_e}^*\mathcal{S}, \mathcal{L}i_{e,*}\mathrm{in}_0\mathcal{F}^\vee)\\ &\simeq \mathrm{Hom}_{\widetilde{Z}_e/Z_e}((\mathcal{L}p_{S_e,*}\mathcal{L}i_{S_e}^*\mathcal{S})_0, \mathcal{F}^\vee)\\ &\simeq \mathrm{Hom}_{\widetilde{Z}_e/Z_e}(\mathcal{F}, (\mathcal{L}p_{S_e,*}\mathcal{L}i_{S_e}^*\mathcal{S})_0^\vee)\end{aligned}$$

by (5.3.14.5). Altogether, we obtain a canonical isomorphism $(\mathcal{L}(i_e i_{S_e})^!\mathcal{S})_0 \simeq \mathcal{S}^{\mathbf{B}_e,\vee} \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m}$.

Finally, recall that by [**BM13**, §5.4.2(ℵ.iii)], the functor $\widetilde{w_0}\cdot(-)^\vee$ permutes the indecomposable vector bundles $\mathcal{E}_b^{S_e}$; here $w_0 \in W$ denotes the longest element, and $\widetilde{w_0}\cdot -$ denotes the braid group action as in §2.2.7. It is not hard to see that $\widetilde{w_0}\cdot(\mathcal{E}_b^{S_e})^\vee$ is $(\mathcal{C}_b, \alpha_b)^\vee$-equivariant for each $b \in \mathbf{B}_e$; in particular, the corresponding permutation of line bundles yields a canonical isomorphism $\mathcal{S}^{\mathbf{B}_e,\vee} \cong \mathcal{S}^{\mathbf{B}_e}$, as desired. We hope to give an interpretation of the induced homomorphism

$$(\mathcal{L}(i_e i_{S_e})^!)_0\colon \mathrm{End}_{\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})}(\mathcal{S})^{\mathrm{op}} \to \mathrm{End}_{\widetilde{Z}_e/\widetilde{Z}_e}(\mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m})^{\mathrm{op}}, \tag{5.3.16.2}$$

as well as of the anti-involutions of $\mathcal{H}_k$ and $\mathcal{J}_{e,k}$ induced by Grothendieck–Serre duality, in a future work.

CHAPTER 6

Asymptotic Kazhdan–Lusztig theory

6.1. Unstable Θ-stratifications

6.1.1. In this section, we give some brief recollections on Halpern-Leistner's theory of Θ-stratifications, following [**HL21**] (though see also [**HL15b, HL18, HL15a, HLP23, HL22**]). In particular, we construct the "ungraded" inclusion and projection functors $\mathrm{HL}_{\underline{w}}, \check{\mathrm{HL}}_{\underline{w}}$ discussed after (1.5.0.1), for a quasi-compact and quasi-smooth stack equipped with an unstable Θ-stratification satisfying a certain weight-non-positivity condition.

6.1.2. Let $\mathcal{X}$ be a (derived) algebraic stack which is locally almost of finite presentation and has affine diagonal.[51] Set $\Theta := \mathbb{A}^1/\mathbb{G}_m$ with respect to the *repelling* (i.e., weight -1) $\mathbb{G}_m$-action[52], and let

$$\begin{tikzcd} \mathrm{B}\mathbb{G}_m \arrow[r, hook, shift left, "\iota_0"] & \Theta \arrow[l, shift left, "\varpi_0"] & \{1\} \arrow[l, hook', "\iota_1"'] \arrow[ll, bend left, "\varpi_1"] \end{tikzcd}$$

denote the evident inclusions and projections. We therefore have induced maps

$$\begin{tikzcd} \mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{X}) \arrow[r, shift right, "\mathrm{spl}"'] \arrow[rr, bend right, "\mathrm{oblv}_{\mathrm{B}\mathbb{G}_m}"'] & \mathrm{Map}(\Theta, \mathcal{X}) \arrow[l, shift right, "\mathrm{gr}"'] \arrow[r, "\mathrm{oblv}_\Theta"] & \mathcal{X} \end{tikzcd}$$

of derived mapping stacks.[53] By [**HLP23**, Thm. 5.1.1], each of these stacks is representable and satisfies the same hypotheses as $\mathcal{X}$.

6.1.3. We now recall some important multiplicative structures. Regard Θ and $\mathrm{B}\mathbb{G}_m$ as monoid stacks via the usual multiplication maps on $\mathbb{A}^1$ and $\mathbb{G}_m$; we set

$$\mathrm{mult}_\Theta \colon \Theta \times \Theta \to \Theta, \qquad \mathrm{mult}_{\mathrm{B}\mathbb{G}_m} \colon \mathrm{B}\mathbb{G}_m \times \mathrm{B}\mathbb{G}_m \to \mathrm{B}\mathbb{G}_m,$$

to be the resulting products, and note that the unit maps are given by ι_1 and ϖ_1, respectively. The stack $\mathrm{Map}(\Theta, \mathcal{X})$ now carries a tautological "weak Θ-action," i.e., a (unital, associative) monoidal action of Θ in the homotopy category of stacks (see [**HL21**, Def. 1.1.1]). Explicitly, the action map $\Theta \times \mathrm{Map}(\Theta, \mathcal{X}) \to \mathrm{Map}(\Theta, \mathcal{X})$ is classified by the composition

$$\Theta \times \Theta \times \mathrm{Map}(\Theta, \mathcal{X}) \xrightarrow{\mathrm{mult}_\Theta \times \mathrm{id}_{\mathrm{Map}(\Theta,\mathcal{X})}} \Theta \times \mathrm{Map}(\Theta, \mathcal{X}) \xrightarrow{\mathrm{ev}} \mathcal{X}, \tag{6.1.3.1}$$

where ev denotes the tautological evaluation map. Moreover, by Proposition 1.2.4 of *loc. cit.*, this weak Θ-action restricts to any union of connected components of $\mathrm{Map}(\Theta, \mathcal{X})$. Likewise, the stack $\mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{X})$ carries a canonical weak $\mathrm{B}\mathbb{G}_m$-action, which restricts to any union of connected components.

Now, for any stack $\mathcal{S}$ equipped with a weak Θ-action $\mathrm{act}_\mathcal{S} \colon \Theta \times \mathcal{S} \to \mathcal{S}$, the category $\mathrm{QC}(\mathcal{S})$ carries a canonical "baric structure," which we now recall. Given $w \in \mathbb{Z}$, set $\mathrm{QC}(\mathcal{S})_{\geq w}$ (resp. $\mathrm{QC}(\mathcal{S})_{< w}$, $\mathrm{QC}(\mathcal{S})_w$, etc.) to be the full subcategory of $\mathrm{QC}(\mathcal{S})$ spanning sheaves $\mathcal{F}$ such that

$$(\iota_0 \times \mathrm{id}_\mathcal{S})^* \mathrm{act}_\mathcal{S}^* \mathcal{F} \in \mathrm{QC}(\mathrm{B}\mathbb{G}_m \times \mathcal{S}) \simeq \mathrm{QC}(\mathrm{B}\mathbb{G}_m) \otimes \mathrm{QC}(\mathcal{S})$$

---

[51] In general, one can also work over a base stack $\mathcal{B}$, but here we work over $\mathrm{Spec}\, k$ as usual.

[52] Here our convention is opposite to that of [**HL21**]. This ensures that the repelling $\mathbb{G}_m$-action on S_e gives rise to a Θ-stratification of $\widetilde{S}_e$ (see §6.3.2).

[53] Our notation for these maps arises by regarding $\mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{X})$ and $\mathrm{Map}(\Theta, \mathcal{X})$ as the stacks of "graded and filtered objects in $\mathcal{X}$," respectively; we do not elaborate further on this perspective.

is concentrated in weights $\geq w$ (resp. $< w$, exactly w, etc.). Then we have a family of semiorthogonal decompositions $\langle \mathrm{QC}(\mathcal{S})_{>w}, \mathrm{QC}(\mathcal{S})_{\leq w}\rangle$ of $\mathrm{QC}(\mathcal{S})$, which satisfy many desirable properties (see Proposition 1.1.2 in *loc. cit.*). We furthermore set $\mathrm{Coh}(\mathcal{S})_{\geq w} := \mathrm{QC}(\mathcal{S})_{\geq w} \cap \mathrm{Coh}(\mathcal{S})$, and similarly for $\mathrm{Coh}(\mathcal{S})_{<w}$, $\mathrm{Coh}(\mathcal{S})_w$, etc.

Likewise, for any stack $\mathcal{Z}$ equipped with a weak $\mathrm{B}\mathbb{G}_m$-action $\mathrm{act}_{\mathcal{Z}}\colon \mathrm{B}\mathbb{G}_m \times \mathcal{Z} \to \mathcal{Z}$, we have a canonical direct sum decomposition

$$\mathrm{QC}(\mathcal{Z}) \simeq \bigoplus_{w\in\mathbb{Z}} \mathrm{QC}(\mathcal{Z})_w, \tag{6.1.3.2}$$

where $\mathrm{QC}(\mathcal{Z})_w$ denotes the full subcategory of sheaves $\mathcal{F}$ for which $\mathrm{act}_{\mathcal{Z}}^*\mathcal{F}$ is concentrated in weight w. Equivalently, pulling back $\mathrm{act}_{\mathcal{Z}}$ along p_0 gives a weak Θ-action on $\mathcal{Z}$, and the decomposition (6.1.3.2) splits the associated baric structure on $\mathrm{QC}(\mathcal{Z})$ (see Lemma 1.5.3 in *loc. cit.*).

6.1.4. We now recall the definition of a single Θ-stratum in $\mathcal{X}$:

6.1.5. Definition ([**HL21**, Def. 1.2.2, 1.5.1]). A *(derived)* Θ-*stratum* in $\mathcal{X}$ is a union of connected components $\mathcal{S} \subset \mathrm{Map}(\Theta, \mathcal{X})$ such that $\mathrm{oblv}_\Theta|_{\mathcal{S}}\colon \mathcal{S} \to \mathcal{X}$ is a closed immersion. Its *center* is the union of connected components $\mathcal{Z} := \mathrm{spl}^{-1}(\mathcal{S}) \subset \mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{X})$.

6.1.6. Given a Θ-stratum $\mathcal{S}$ with center $\mathcal{Z}$, we set $i_{\mathcal{S}} := \mathrm{oblv}_\Theta|_{\mathcal{S}}\colon \mathcal{S} \hookrightarrow \mathcal{X}$ and $i_{\mathcal{Z}} := \mathrm{oblv}_{\mathrm{B}\mathbb{G}_m}|_{\mathcal{Z}}\colon \mathcal{Z} \to \mathcal{X}$ to be the tautological maps. Note that the latter is often *not* a closed immersion; we shall see an example of this in (6.2.14.1). We furthermore let

$$\mathcal{Z} \xrightarrow{\mathrm{spl}_{\mathcal{Z}}} \mathcal{S} \xrightarrow{\mathrm{gr}_{\mathcal{S}}} \mathcal{Z}$$

denote the evident restrictions of spl and gr, which compose to the identity. As in §6.1.3, we have canonical weak actions

$$\mathrm{act}_{\mathcal{S}}\colon \Theta \times \mathcal{S} \to \mathcal{S}, \qquad \mathrm{act}_{\mathcal{Z}}\colon \mathrm{B}\mathbb{G}_m \times \mathcal{Z} \to \mathcal{Z},$$

and it is not hard to see using (6.1.3.1) that the diagram

$$\begin{tikzcd} \mathrm{B}\mathbb{G}_m \times \mathcal{Z} \arrow[r, "\iota_0 \times \mathrm{spl}_{\mathcal{Z}}"] \arrow[d, "\mathrm{act}_{\mathcal{Z}}"] & \Theta \times \mathcal{S} \arrow[r, "\varpi_0 \times \mathrm{gr}_{\mathcal{S}}"] \arrow[d, "\mathrm{act}_{\mathcal{S}}"] & \mathrm{B}\mathbb{G}_m \times \mathcal{Z} \arrow[d, "\mathrm{act}_{\mathcal{Z}}"] \\ \mathcal{Z} \arrow[r, "\mathrm{spl}_{\mathcal{Z}}"] & \mathcal{S} \arrow[r, "\mathrm{gr}_{\mathcal{S}}"] & \mathcal{Z} \end{tikzcd} \tag{6.1.6.1}$$

commutes. Equivalently, the maps $\mathrm{spl}_{\mathcal{Z}}, \mathrm{gr}_{\mathcal{S}}$ are Θ-equivariant, and hence preserve the baric structures on $\mathrm{QC}(\mathcal{S})$ and $\mathrm{QC}(\mathcal{Z})$.

Moreover, the baric structure on $\mathrm{QC}(\mathcal{S})$ extends to one on $\mathrm{QC}_{\mathcal{S}}(\mathcal{X})$ (see §3.3.4 for this notation). Namely, let $\mathrm{QC}_{\mathcal{S}}(\mathcal{X})_{\geq w} \subset \mathrm{QC}(\mathcal{X})$ denote the smallest full stable subcategory containing the essential image $i_{\mathcal{S},*}(\mathrm{QC}(\mathcal{S})_{\geq w})$ and closed under extensions, filtered colimits, and limits of towers $\cdots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_1$ with $\tau^{\geq n}(\mathcal{F}_i)$ eventually constant for any n. The notations $\mathrm{QC}_{\mathcal{S}}(\mathcal{X})_{<w}$, $\mathrm{QC}_{\mathcal{S}}(\mathcal{X})_w$, $\mathrm{Coh}_{\mathcal{S}}(\mathcal{X})_{\leq w}$, etc., are defined similarly. We refer the reader to [**HL21**, Prop. 1.7.2] for further properties of this construction.

6.1.7. Note that there is a bijection between Θ-strata in $\mathcal{X}$ and *classical* Θ-strata in $\mathcal{X}^{\mathrm{cl}}$, which are defined analogously using the classical mapping stack $\mathrm{Map}(\Theta, \mathcal{X})^{\mathrm{cl}}$ (see Lemma 1.2.3 in *loc. cit.*). This will be our primary tool for constructing Θ-strata. In fact, when $\mathcal{X}$ is smooth, all Θ-strata and centers are equivalent to their classical counterparts:

6.1.8. Lemma. If $\mathcal{X}$ is smooth, then so are the stacks $\mathrm{Map}(\Theta, \mathcal{X})$ and $\mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{X})$ (in particular, they are classical).

Proof. The claim regarding $\mathrm{Map}(\Theta, \mathcal{X})$ is [**HL21**, Cor. 1.3.2.1]. For $\mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{X})$, note that by [**HLP23**, Prop. 5.1.10], its cotangent complex is given by

$$\mathbb{L}_{\mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{X})} \simeq \mathrm{pr}_{2,*}\,\mathrm{ev}^*\,\mathbb{L}_{\mathcal{X}} \simeq (\mathrm{ev}^*\,\mathbb{L}_{\mathcal{X}})_0, \tag{6.1.8.1}$$

where $\mathrm{ev}\colon \mathrm{B}\mathbb{G}_m \times \mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{X}) \to \mathcal{X}$ denotes the tautological evaluation map and pr_2 the second projection. In particular, if $\mathbb{L}_{\mathcal{X}}$ is perfect of Tor-amplitude $[0, 1]$, then so is $\mathbb{L}_{\mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{X})}$. □

6.1.9. We now give an "instrinsic" criterion for recognizing Θ-strata and their centers (in particular, their derived structures) in terms of the relative cotangent complex:

6.1.10. LEMMA. (1) Let $i_{\mathcal{S}}\colon \mathcal{S} \hookrightarrow \mathcal{X}$ be a closed immersion, and suppose $\mathcal{S}$ is equipped with a weak Θ-action $\mathrm{act}_{\mathcal{S}}\colon \Theta \times \mathcal{S} \to \mathcal{S}$. Then $\mathbb{L}_{\mathcal{S}/\mathcal{X}} \in \mathrm{QC}(\mathcal{S})_{<0}$ if and only if the morphism $\mathcal{S} \to \mathrm{Map}(\Theta, \mathcal{X})$ classifying $i_{\mathcal{S}} \circ \mathrm{act}_{\mathcal{S}}$ exhibits $\mathcal{S}$ as a Θ-stratum in $\mathcal{X}$.

(2) Let $i_{\mathcal{Z}}\colon \mathcal{Z} \to \mathcal{X}$ be a morphism, and suppose $\mathcal{Z}$ is equipped with a weak $\mathrm{B}\mathbb{G}_m$-action $\mathrm{act}_{\mathcal{Z}}\colon \mathrm{B}\mathbb{G}_m \times \mathcal{Z} \to \mathcal{Z}$ such that the morphism $\mathcal{Z} \to \mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{X})$ classifying $i_{\mathcal{Z}} \circ \mathrm{act}_{\mathcal{Z}}$ is a closed immersion. Then $\mathbb{L}_{\mathcal{Z}/\mathcal{X}} \in \mathrm{QC}(\mathcal{Z})_{\neq 0}$ if and only if this morphism exhibits $\mathcal{Z}$ as a union of connected components of $\mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{X})$.

PROOF. The first assertion is Lemma 1.3.2 and Proposition 1.4.1 of [**HL21**]. Moreover, by (6.1.8.1), we have $\mathbb{L}_{\mathrm{Map}(\mathrm{B}\mathbb{G}_m,\mathcal{X})} \in \mathrm{QC}(\mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{X}))_0$ and the map

$$d(\mathrm{oblv}_{\mathrm{B}\mathbb{G}_m})\colon \mathrm{oblv}_{\mathrm{B}\mathbb{G}_m}^* \mathbb{L}_{\mathcal{X}} \to \mathbb{L}_{\mathrm{Map}(\mathrm{B}\mathbb{G}_m,\mathcal{X})}$$

is an isomorphism on weight-0 components. This immediately implies the reverse implication of the second assertion.

For the forwards implication, letting $\iota_{\mathcal{X}}\colon \mathcal{Z} \to \mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{X})$ be as above, it suffices to show that the natural map

$$d\iota_{\mathcal{X}}\colon \iota_{\mathcal{X}}^* \mathbb{L}_{\mathrm{Map}(\mathrm{B}\mathbb{G}_m,\mathcal{X})} \to \mathbb{L}_{\mathcal{Z}}$$

is an isomorphism. Since $\iota_{\mathcal{X}}$ is tautologically $\mathrm{B}\mathbb{G}_m$-equivariant, we have $\iota_{\mathcal{X}}^* \mathbb{L}_{\mathrm{Map}(\mathrm{B}\mathbb{G}_m,\mathcal{X})} \in \mathrm{QC}(\mathcal{Z})_0$. Let $\iota_{\mathcal{Z}}\colon \mathcal{Z} \to \mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{Z})$ denote the map classified by $\mathrm{act}_{\mathcal{Z}}$; as before, we have $\iota_{\mathcal{Z}}^* \mathbb{L}_{\mathrm{Map}(\mathrm{B}\mathbb{G}_m,\mathcal{Z})} \in \mathrm{QC}(\mathcal{Z})_0$. Since $\mathrm{oblv}_{\mathrm{B}\mathbb{G}_m} \circ \iota_{\mathcal{Z}} \simeq \mathrm{id}_{\mathcal{Z}}$, the composition

$$\mathbb{L}_{\mathcal{Z}} \simeq \iota_{\mathcal{Z}}^* \mathrm{oblv}_{\mathrm{B}\mathbb{G}_m}^* \mathbb{L}_{\mathcal{Z}} \xrightarrow{\iota_{\mathcal{Z}}^* d(\mathrm{oblv}_{\mathrm{B}\mathbb{G}_m})} \iota_{\mathcal{Z}}^* \mathbb{L}_{\mathrm{Map}(\mathrm{B}\mathbb{G}_m,\mathcal{Z})} \xrightarrow{d\iota_{\mathcal{Z}}} \mathbb{L}_{\mathcal{Z}}$$

is the identity, so we also have $\mathbb{L}_{\mathcal{Z}} \in \mathrm{QC}(\mathcal{Z})_0$. Finally, since $\mathrm{oblv}_{\mathrm{B}\mathbb{G}_m} \circ \iota_{\mathcal{Z}} \simeq i_{\mathcal{Z}}$, the composition

$$i_{\mathcal{Z}}^* \mathbb{L}_{\mathcal{X}} \simeq \iota_{\mathcal{X}}^* \mathrm{oblv}_{\mathrm{B}\mathbb{G}_m}^* \mathbb{L}_{\mathcal{X}} \xrightarrow{\iota_{\mathcal{X}}^* d(\mathrm{oblv}_{\mathrm{B}\mathbb{G}_m})} \iota_{\mathcal{X}}^* \mathbb{L}_{\mathrm{Map}(\mathrm{B}\mathbb{G}_m,\mathcal{X})} \xrightarrow{d\iota_{\mathcal{X}}} \mathbb{L}_{\mathcal{Z}}$$

identifies with $di_{\mathcal{Z}}$. By our assumption that $\mathbb{L}_{\mathcal{Z}/\mathcal{X}} \in \mathrm{QC}(\mathcal{Z})_{\neq 0}$, the map $di_{\mathcal{Z}}$ is an isomorphism on weight-0 components. It follows that $d\iota_{\mathcal{X}}$ is an isomorphism on weight-0 components, hence an isomorphism. □

6.1.11. We now define the main notion of this section:

6.1.12. DEFINITION ([**HL21**, Def. 2.3.1]). A *unstable*[54] *Θ-stratification* of $\mathcal{X}$ indexed by a totally ordered set I is a collection of open substacks $\mathcal{X}_{\leq\alpha} \subset \mathcal{X}$ and Θ-strata $\mathcal{S}_\alpha$ in $\mathcal{X}_{\leq\alpha}$ for each $\alpha \in I$, satisfying

(1) $\mathcal{X}_{\leq\alpha} \subset \mathcal{X}_{\leq\alpha'}$ when $\alpha < \alpha'$;
(2) $\mathcal{X}_{\leq\alpha} \setminus \mathrm{oblv}_\Theta(\mathcal{S}_\alpha) = \bigcup_{\alpha'<\alpha} \mathcal{X}_{\leq\alpha'}$ as topological spaces; and
(3) for every $x \in \mathcal{X}$, there is a minimal $\alpha \in I$ such that $x \in \mathcal{X}_{\leq\alpha}$ (in particular, $\bigcup_{\alpha\in I} X_{\leq\alpha} = X$).

6.1.13. In certain situations, we may "pull back" unstable Θ-stratifications along morphisms of stacks; this will later allow us to make functoriality statements for the corresponding infinite semi-orthogonal decompositions (though we shall not actually need this notion in our applications).

6.1.14. DEFINITION ([**HL21**, Def. 1.2.5]). Let $f\colon \mathcal{X}' \to \mathcal{X}$ be a morphism of stacks, each satisfying the hypotheses of §6.1.2, and let $\mathcal{S}$ be a Θ-stratum in $\mathcal{X}$. We say that $\mathcal{S}$ *induces* a Θ-stratum in $\mathcal{X}'$ if its preimage $\mathcal{S}'$ under the canonical map $\mathrm{Map}(\Theta, \mathcal{X}') \to \mathrm{Map}(\Theta, \mathcal{X})$ is a Θ-stratum, and $\mathrm{oblv}_\Theta(\mathcal{S}') = f^{-1}(\mathrm{oblv}_\Theta(\mathcal{S}))$ as topological spaces.

Moreover, let $\{(\mathcal{X}_{\leq\alpha}, \mathcal{S}_\alpha)\}_{\alpha\in I}$ be an unstable Θ-stratification of $\mathcal{X}$, and set $\mathcal{X}'_{\leq\alpha} := \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}_{\leq\alpha} \subset \mathcal{X}'$ for each $\alpha \in I$. We say that $\{(\mathcal{X}_{\leq\alpha}, \mathcal{S}_\alpha)\}_{\alpha\in I}$ *induces* an unstable Θ-stratification of $\mathcal{X}'$ if $\mathcal{S}_\alpha$ induces a Θ-stratum $\mathcal{S}'_\alpha$ in $\mathcal{X}'_{\leq\alpha}$ for each $\alpha \in I$, and the collection $\{(\mathcal{X}'_{\leq\alpha}, \mathcal{S}'_\alpha)\}_{\alpha\in I}$ is an unstable Θ-stratification of $\mathcal{X}'$.

6.1.15. We let $f_{\mathcal{S}}\colon \mathcal{S}' \to \mathcal{S}$ denote the map of Θ-strata obtained in the above situation. Note that the definition immediately furnishes a canonical map of centers $f_{\mathcal{Z}}\colon \mathcal{Z}' \to \mathcal{Z}$, and we have an evident commutative diagram

(6.1.15.1)
$$\begin{array}{ccccc} \mathcal{Z}' & \xrightarrow{\mathrm{spl}_{\mathcal{Z}}} & \mathcal{S}' & \xrightarrow{\mathrm{gr}_{\mathcal{S}}} & \mathcal{Z}' \\ \downarrow{\scriptstyle f_{\mathcal{Z}}} & & \downarrow{\scriptstyle f_{\mathcal{S}}} & & \downarrow{\scriptstyle f_{\mathcal{Z}}} \\ \mathcal{Z} & \xrightarrow{\mathrm{spl}_{\mathcal{Z}}} & \mathcal{S} & \xrightarrow{\mathrm{gr}_{\mathcal{S}}} & \mathcal{Z}. \end{array}$$

[54]This adjective refers to our requirement that the Θ-strata $\mathcal{S}_\alpha$ cover $\mathcal{X}$; in particular, the "semistable locus" of the Θ-stratification is trivial.

We now reach the main result of this section, known as "derived Kirwan surjectivity" (for bounded coherent sheaves):

6.1.16. THEOREM ([**HL21**, Thm. 2.2.2, 2.3.4, 3.3.1]). *Let $\{(\mathcal{X}_{\le\alpha}, \mathcal{S}_\alpha)\}_{\alpha\in I}$ be an unstable Θ-stratification of $\mathcal{X}$, and let $\underline{w} = \{w_\alpha \in \mathbb{Z}\}_{\alpha\in I}$. Suppose that*

(a) $\mathcal{X}$ is quasi-compact and quasi-smooth; and
(b) $\mathrm{H}^{-1}(i^*_{\mathcal{Z}_\alpha}\mathbb{L}_{\mathcal{X}}) \in \mathrm{QC}(\mathcal{Z}_\alpha)_{\le 0}$ for each $\alpha \in I$.

Then $\mathrm{Coh}(\mathcal{X})$ has an infinite semiorthogonal decomposition[55] *whose pieces are identified with $\mathrm{Coh}(\mathcal{Z}_\alpha)_w$ for each $\alpha \in I$ and $w \in \mathbb{Z}$. More precisely, writing*[56] *$I = \{\alpha_1 < \cdots < \alpha_r\}$, we have*

$$\begin{aligned}
\mathrm{Coh}(\mathcal{X}) = \big\langle &\ldots, \mathrm{Coh}(\mathcal{Z}_{\alpha_r})_{w_{\alpha_r}+2}, \mathrm{Coh}(\mathcal{Z}_{\alpha_r})_{w_{\alpha_r}+1},\\
&\ldots,\\
&\ldots, \mathrm{Coh}(\mathcal{Z}_{\alpha_2})_{w_{\alpha_2}+2}, \mathrm{Coh}(\mathcal{Z}_{\alpha_2})_{w_{\alpha_2}+1},\\
&\ldots, \mathrm{Coh}(\mathcal{Z}_{\alpha_1})_{w_{\alpha_1}+2}, \mathrm{Coh}(\mathcal{Z}_{\alpha_1})_{w_{\alpha_1}+1}, \mathrm{Coh}(\mathcal{Z}_{\alpha_1})_{w_{\alpha_1}}, \mathrm{Coh}(\mathcal{Z}_{\alpha_1})_{w_{\alpha_1}-1}, \ldots,\\
&\qquad \mathrm{Coh}(\mathcal{Z}_{\alpha_2})_{w_{\alpha_2}}, \mathrm{Coh}(\mathcal{Z}_{\alpha_2})_{w_{\alpha_2}-1}, \ldots,\\
&\qquad \ldots,\\
&\qquad \mathrm{Coh}(\mathcal{Z}_{\alpha_r})_{w_{\alpha_r}}, \mathrm{Coh}(\mathcal{Z}_{\alpha_r})_{w_{\alpha_r}-1}, \ldots \big\rangle.
\end{aligned} \tag{6.1.16.1}$$

Furthermore, suppose $\mathcal{X}'$ is another stack satisfying the same hypotheses as $\mathcal{X}$, and let $f\colon \mathcal{X}' \to \mathcal{X}$ be a morphism such that

(c) $f^*\colon \mathrm{QC}(\mathcal{X}) \to \mathrm{QC}(\mathcal{X}')$ preserves the subcategories of coherent sheaves (i.e., f is eventually coconnective);
(d) $\{(\mathcal{X}_{\le\alpha}, \mathcal{S}_\alpha)\}_{\alpha\in I}$ induces an unstable Θ-stratification of $\mathcal{X}'$ satisfying (b);
(e) for each $\alpha \in I$, the canonical map $\mathcal{S}'_\alpha \to \mathcal{S}_\alpha \times_{\mathcal{X}_{\le\alpha}} \mathcal{X}'_{\le\alpha}$ is an isomorphism;

Then $f^*\colon \mathrm{Coh}(\mathcal{X}) \to \mathrm{Coh}(\mathcal{X}')$ preserves all subcategories in the respective semiorthogonal decompositions for $\underline{w}$, and induces the functor $f^*_{\mathcal{Z}_\alpha}\colon \mathrm{Coh}(\mathcal{Z}_\alpha)_w \to \mathrm{Coh}(\mathcal{Z}'_\alpha)_w$ for each $\alpha \in I$ and $w \in \mathbb{Z}$.

PROOF. The only thing to prove is the final sentence. By induction, we reduce to the case of a single Θ-stratum $\mathcal{S}$ in $\mathcal{X}$; as in [**HL21**, Thm. 3.3.1(3)], it suffices to show that the diagram

$$\begin{tikzcd}
\mathrm{Coh}(\mathcal{Z})_w \arrow[r, "i_{\mathcal{S},*}\mathrm{gr}^*_{\mathcal{S}}", "\sim"'] \arrow[d, "f^*_{\mathcal{Z}}"] & \mathrm{Coh}_{\mathcal{S}}(\mathcal{X})_w \arrow[d, "f^*"] \\
\mathrm{Coh}(\mathcal{Z}')_w \arrow[r, "i_{\mathcal{S},*}\mathrm{gr}^*_{\mathcal{S}}", "\sim"'] & \mathrm{Coh}_{\mathcal{S}'}(\mathcal{X}')_w
\end{tikzcd} \tag{6.1.16.2}$$

commutes for any $w \in \mathcal{Z}$. The conclusion is now immediate from (6.1.15.1) and base-change via (e). □

6.1.17. In particular, setting $\mathcal{Z}_I := \coprod_{\alpha\in I} \mathcal{Z}_\alpha$, we obtain inclusion and projection functors

$$\mathrm{Coh}(\mathcal{Z}_I) \xrightarrow{\mathrm{HL}_{\underline{w}}} \mathrm{Coh}(\mathcal{X}) \xrightarrow{\check{\mathrm{HL}}_{\underline{w}}} \mathrm{Coh}(\mathcal{Z}_I)$$

which induce inverse equivalences on Hochschild homology, compose to the identity, and are natural with respect to maps f satisfying the conditions of the theorem. Moreover, the functor $\mathrm{HL}_{\underline{w}}$ satisfies the following orthogonality property, which is immediate from (6.1.16.1):

6.1.18. COROLLARY. Let $\mathcal{F} \in \mathrm{Coh}(\mathcal{Z}_\alpha)_w$ and $\mathcal{F}' \in \mathrm{Coh}(\mathcal{Z}_{\alpha'})_{w'}$. If either $\alpha' < \alpha$ and $w \le w_\alpha$, or $\alpha < \alpha'$ and $w' > w_{\alpha'}$, then

$$\mathrm{Hom}_{\mathcal{X}}(\mathrm{HL}_{\underline{w}}(\mathcal{F}), \mathrm{HL}_{\underline{w}}(\mathcal{F}')) \simeq 0.$$

6.1.19. Finally, we investigate linearity of the functors $\mathrm{HL}_{\underline{w}}, \check{\mathrm{HL}}_{\underline{w}}$:

6.1.20. LEMMA. Suppose we are in the situation of Theorem 6.1.16. Let $\mathcal{B}$ be a stack satisfying the hypotheses of §6.1.2, and let $p\colon \mathcal{X} \to \mathcal{B}$ be a morphism. Suppose that the image of $\mathcal{Z}_I$ under the canonical map $\mathrm{Map}(B\mathbb{G}_m, \mathcal{X}) \to \mathrm{Map}(B\mathbb{G}_m, \mathcal{B})$ lies in a union of connected components on which the tautological weak

[55] In the sense that any object lies in a subcategory generated by finitely many semiorthogonal factors.
[56] Note that since $\mathcal{X}$ is quasi-compact, we may assume the unstable Θ-stratification is finite.

$\mathrm{B}\mathbb{G}_m$-action is trivial. Then the functors $\mathrm{HL}_{\underline{w}}$ and $\check{\mathrm{HL}}_{\underline{w}}$ are canonically $\mathrm{Perf}(\mathcal{B})$-linear (with respect to the actions coming from restriction along p and $i_{\mathcal{Z}_I}\colon \mathcal{Z}_I \to \mathcal{X}$).

Proof. We claim that the $\mathrm{Perf}(\mathcal{B})$-action preserves each subcategory $\mathrm{Coh}(\mathcal{Z}_\alpha)_w$. Indeed, using [**HL21**, Thm. 2.2.2], we may again reduce to the case of a single Θ-stratum $\mathcal{S}$ in $\mathcal{X}$; as in (6.1.16.2), we must show that the subcategory $\mathrm{Coh}_{\mathcal{S}}(\mathcal{X})_w \subset \mathrm{Coh}(\mathcal{X})$ is preserved by the $\mathrm{Perf}(\mathcal{B})$-action. By the definition of this subcategory (see §6.1.6 or Definition 1.7.1 in *loc. cit.*), it suffices to show that the subcategory $i_{\mathcal{S},*}(\mathrm{Coh}(\mathcal{S})_w)$ is preserved by the $\mathrm{Perf}(\mathcal{B})$-action, hence (by the projection formula) that $\mathrm{Coh}(\mathcal{S})_w$ is preserved by the $\mathrm{Perf}(\mathcal{B})$-action given by restriction along $p \circ i_{\mathcal{S}}$. By Proposition 1.1.2(5,6) in *loc. cit.*, it suffices to show that $i_{\mathcal{S}}^* p^* \mathcal{F} \in \mathrm{QC}(\mathcal{S})_0$ for any $\mathcal{F} \in \mathrm{Perf}(\mathcal{B})$. Equivalently, by Lemma 1.5.4 in *loc. cit.*, we must show that $i_{\mathcal{Z}}^* p^* \mathcal{F} \in \mathrm{QC}(\mathcal{Z})_0$. Note that by (6.1.6.1), we have a commutative diagram

(6.1.20.1)
$$\begin{tikzcd}
& \mathrm{B}\mathbb{G}_m \times \mathcal{Z} \arrow[r, "\iota_0 \times \mathrm{spl}_{\mathcal{Z}}"] \arrow[d, "\mathrm{act}_{\mathcal{Z}}"] \arrow[ddl, "\mathrm{id}_{\mathrm{B}\mathbb{G}_m} \times p_{\mathcal{Z}}"'] & \Theta \times \mathcal{S} \arrow[d, "\mathrm{act}_{\mathcal{S}}"] & \\
& \mathcal{Z} \arrow[r, "\mathrm{spl}_{\mathcal{Z}}"] \arrow[d, "p_{\mathcal{Z}}"] \arrow[rr, bend right, "i_{\mathcal{Z}}"'] & \mathcal{S} \arrow[r, "i_{\mathcal{S}}"] & \mathcal{X} \arrow[d, "p"] \\
\mathrm{B}\mathbb{G}_m \times \mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{B}) \arrow[r, "\mathrm{act}"] & \mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{B}) \arrow[rr, "\mathrm{oblv}_{\mathrm{B}\mathbb{G}_m}"] & & \mathcal{B}.
\end{tikzcd}$$

By our assumption, this diagram also commutes with act replaced by the projection onto $\mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{B})$, whence the claim. The same argument shows that the $\mathrm{Perf}(\mathcal{B})$-action on $\mathrm{Coh}(\mathcal{Z})$ preserves $\mathrm{Coh}(\mathcal{Z})_w$. Finally, by the proof of Lemma 3.3.7 in *loc. cit.*, the functor $\mathrm{gr}_{\mathcal{S}}^*\colon \mathrm{Coh}(\mathcal{Z})_w \to \mathrm{Coh}(\mathcal{S})_w$ is an equivalence, with quasi-inverse $\mathrm{spl}_{\mathcal{Z}}^*$; it follows that the functor $i_{\mathcal{S},*}\mathrm{gr}_{\mathcal{S}}^*$ of (6.1.16.2) commutes with the $\mathrm{Perf}(\mathcal{B})$-actions. The remaining assertions are now clear. □

6.2. The graded monad

6.2.1. Suppose now that $\mathcal{X}$ is equipped with a morphism to $\mathrm{B}\mathbb{G}_m$. In general, when the hypothesis of Lemma 6.1.20 is not satisfied, the functors $\mathrm{HL}_{\underline{w}}, \check{\mathrm{HL}}_{\underline{w}}$ are *not* $\mathrm{Perf}(\mathrm{B}\mathbb{G}_m)$-linear. Our goal in this section is to describe a situation in which these functors admit canonical $\mathrm{Perf}(\mathrm{B}\mathbb{G}_m)$-linear modifications satisfying similar properties. In particular, we prove a version of Proposition 1.5.1 in this more general setting; that is, we give a description of the category $\mathrm{QC}^!(\mathcal{X})$ as modules for a graded monad acting on $\mathrm{QC}^!(\mathcal{Z}_I)$.

We begin by recording a simple example in which the functors $\mathrm{HL}_{\underline{w}}, \check{\mathrm{HL}}_{\underline{w}}$ fail to be $\mathrm{Perf}(\mathrm{B}\mathbb{G}_m)$-linear:

6.2.2. Example. Let $\mathcal{X} := \mathbb{P}^1/\mathbb{G}_m$, with $\mathbb{G}_m$ again acting by weight -1 (so that the point $0 \in \mathbb{P}^1$ is repelling). As we shall see in §6.2.14, the Białynicki-Birula decomposition $\mathbb{P}^1/\mathbb{G}_m := \Theta \cup \{\infty\}/\mathbb{G}_m$ is an unstable Θ-stratification, with centers given by $\{0\}/\mathbb{G}_m$ and $\{\infty\}/\mathbb{G}_m$, respectively. Let $\mathcal{O}_0, \mathcal{O}_\infty$ denote the structure sheaves of these centers, and for any $w, w' \in \mathbb{Z}$, let $\mathcal{O}_{\mathbb{P}^1}(w, w')$ denote the $\mathbb{G}_m$-equivariant line bundle on $\mathbb{P}^1$ whose weights at the points $0, \infty$ are given by w, w', respectively (and whose deequivariantization is given by $\mathcal{O}_{\mathbb{P}^1}(w'-w)$, see for instance [**Che18**, Ex. 2.10.6]). Choose weights $w_0, w_\infty \in \mathbb{Z}$. The semiorthogonal decomposition (6.1.16.1) is then given explicitly by

$$\begin{aligned}\mathrm{Coh}(\mathbb{P}^1/\mathbb{G}_m) = \big\langle \ldots, \mathcal{O}_\infty\langle -w_\infty - 2\rangle, \mathcal{O}_\infty\langle -w_\infty - 1\rangle,& \\ \ldots, \mathcal{O}_{\mathbb{P}^1}(w_0+2, w_\infty), \mathcal{O}_{\mathbb{P}^1}(w_0+1, w_\infty), \mathcal{O}_{\mathbb{P}^1}(w_0, w_\infty), \mathcal{O}_{\mathbb{P}^1}(w_0-1, w_\infty), \ldots,& \\ \mathcal{O}_\infty\langle -w_\infty\rangle, \mathcal{O}_\infty\langle -w_\infty + 1\rangle, \ldots \big\rangle,&\end{aligned}$$

and it is straightforward to verify that there are no nonzero maps from right to left. In particular, we have

$$\begin{aligned}\mathrm{HL}_{(w_0,w_\infty)}(\mathcal{O}_0\langle -w_0 - 1\rangle) &\simeq \mathcal{O}_{\mathbb{P}^1}(w_0+1, w_\infty), \\ \mathrm{HL}_{(w_0,w_\infty)}(\mathcal{O}_0\langle -w_0\rangle)\langle -1\rangle &\simeq \mathcal{O}_{\mathbb{P}^1}(w_0+1, w_\infty+1),\end{aligned}$$

which are *not* isomorphic. Likewise, the sheaves

$$\begin{aligned}\check{\mathrm{HL}}_{(w_0,w_\infty)}(\mathcal{O}_{\mathbb{P}^1}(w_0, w_\infty)\langle -1\rangle) &\simeq \mathcal{O}_0\langle -w_0 - 1\rangle \oplus \mathcal{O}_\infty\langle -w_\infty - 1\rangle, \\ \check{\mathrm{HL}}_{(w_0,w_\infty)}(\mathcal{O}_{\mathbb{P}^1}(w_0, w_\infty))\langle -1\rangle &\simeq \mathcal{O}_0\langle -w_0 - 1\rangle,\end{aligned}$$

are not isomorphic. Thus, the functors $\mathrm{HL}_{(w_0,w_\infty)}, \check{\mathrm{HL}}_{(w_0,w_\infty)}$ are not $\mathrm{Perf}(\mathrm{B}\mathbb{G}_m)$-linear.

6.2.3. Now suppose we are in the general situation of Theorem 6.1.16, and let $p\colon \mathcal{X} \to \mathrm{B}\mathbb{G}_m$ be a morphism. We begin by studying the interaction between the functors $\mathrm{HL}_{\underline{w}}, \check{\mathrm{HL}}_{\underline{w}}$ and the $\mathrm{Perf}(\mathrm{B}\mathbb{G}_m)$-actions on $\mathrm{Coh}(\mathcal{Z}_I)$ and $\mathrm{Coh}(\mathcal{X})$. Recall that we have a canonical isomorphism

$$\mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathrm{B}\mathbb{G}_m) \simeq \mathbb{Z} \times \mathrm{B}\mathbb{G}_m,$$

where the connected component $\{n\} \times \mathrm{B}\mathbb{G}_m$ corresponds to the character $(-)^n\colon \mathbb{G}_m \to \mathbb{G}_m$ (more generally, see (6.2.14.1)).

6.2.4. LEMMA. Let $\alpha \in I$ and $n \in \mathbb{Z}$.

(1) Let $\mathcal{F} \in \mathrm{Coh}(\mathcal{Z}_\alpha)$. The sheaf

$$\check{\mathrm{HL}}_{\underline{w}}(\mathrm{HL}_{\underline{w}}(\mathcal{F})\langle n\rangle) \tag{6.2.4.1}$$

is supported on $\mathcal{Z}_{\alpha'}$ with $\alpha \leq \alpha'$, and its component in $\mathrm{Coh}(\mathcal{Z}_\alpha)$ is isomorphic to $\mathcal{F}\langle n\rangle$.

(2) Let $w \in \mathbb{Z}$ and $\mathcal{F} \in \mathrm{Coh}(\mathcal{Z}_\alpha)_w$. Suppose that the image of $\mathcal{Z}_I$ under the canonical map

$$\mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{X}) \to \mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathrm{B}\mathbb{G}_m) \simeq \mathbb{Z} \times \mathrm{B}\mathbb{G}_m \tag{6.2.4.2}$$

lies in $\mathbb{Z}_{>0} \times \mathrm{B}\mathbb{G}_m$. If $n > 0$ (resp. $n < 0$), then the component of (6.2.4.1) in $\mathrm{Coh}(\mathcal{Z}_{\alpha'})_{w'}$ is trivial if this piece lies non-strictly to the left (resp. right) of $\mathrm{Coh}(\mathcal{Z}_\alpha)_w$ in the semiorthogonal decomposition (6.1.16.1).

(3) Suppose we are in the situation of (2). Let $\mathcal{F} \in \mathrm{Coh}(\mathcal{Z}_\alpha)_{w_\alpha}$ and $\mathcal{F}' \in \mathrm{Coh}(\mathcal{Z}_{\alpha'})_{w_{\alpha'}}$. Then the graded inner Hom

$$\underline{\mathrm{Hom}}_{\mathcal{X}}(\mathrm{HL}_{\underline{w}}(\mathcal{F}), \mathrm{HL}_{\underline{w}}(\mathcal{F}'))$$

is concentrated in non-negative weights. Moreover, if $\alpha' < \alpha$, it is concentrated in positive weights, and if $\alpha = \alpha'$, its weight-0 component is isomorphic to $\mathrm{Hom}_{\mathcal{Z}_\alpha}(\mathcal{F}, \mathcal{F}')$.

(4) Suppose that the image of $\mathcal{Z}_I$ under (6.2.4.2) lies in $\{1\} \times \mathrm{B}\mathbb{G}_m$. Set $p_{\mathcal{Z}_I} := p \circ i_{\mathcal{Z}_I}$, and let $\mathcal{Z}_I^{\mathrm{dq}} := \mathcal{Z}_I \times_{\mathrm{B}\mathbb{G}_m} *$. Then we have a canonical decomposition

$$\mathcal{Z}_I \simeq \mathrm{B}\mathbb{G}_m \times \mathcal{Z}_I^{\mathrm{dq}}$$

respecting weak $\mathrm{B}\mathbb{G}_m$-actions and commuting with projections to $\mathrm{B}\mathbb{G}_m$.

PROOF. (1) The first claim is immediate from the proof of [**HL21**, Thm. 2.3.4], as the support of $\mathrm{HL}_{\underline{w}}(\mathcal{F})\langle n\rangle$ is the same as that of $\mathrm{HL}_{\underline{w}}(\mathcal{F})$, and is therefore contained in $\bigcup_{\alpha \leq \alpha'} i_{\mathcal{S}_\alpha}(\mathcal{S}_\alpha)$. Moreover, restricting to $\mathrm{Coh}(\mathcal{X}_{\leq \alpha})$ (i.e., quotienting by the subcategory of sheaves supported on $\bigcup_{\alpha < \alpha'} i_{\mathcal{S}_\alpha}(\mathcal{S}_\alpha)$), the same proof shows that we may assume $i_{\mathcal{S}_\alpha}$ is a closed immersion. In this case, $\mathrm{HL}_{\underline{w}} \simeq i_{\mathcal{S}_\alpha,*}\mathrm{gr}^*_{\mathcal{S}_\alpha}$ as in (6.1.16.2), and the latter is clearly $\mathrm{Perf}(\mathrm{B}\mathbb{G}_m)$-linear as in the proof of Lemma 6.1.20.

(2) Suppose $n > 0$. We first claim that $\mathcal{F}\langle n\rangle \in \mathrm{Coh}(\mathcal{Z}_\alpha)_{<w}$. The diagram (6.1.20.1) with $\mathcal{B} := \mathrm{B}\mathbb{G}_m$ immediately implies that $\mathcal{O}_{\mathcal{Z}_\alpha}\langle n\rangle \in \mathrm{Coh}(\mathcal{Z}_\alpha)_{<0}$ (more precisely, over the component $\{d\} \times \mathrm{B}\mathbb{G}_m \subset \mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathrm{B}\mathbb{G}_m)$, it lies in $\mathrm{Coh}(\mathcal{Z}_\alpha)_{d\cdot(-n)}$), so the claim follows from Lemma 1.5.3 in *loc. cit.* Thus, by (1), it suffices to show that the component of (6.2.4.1) in $\mathrm{Coh}(\mathcal{Z}_{\alpha'})_{>w_{\alpha'}}$ is trivial for each $\alpha' > \alpha$. By the proof of Theorem 2.3.4 in *loc. cit.*, this is equivalent to showing that

$$i^*_{\mathcal{Z}_{\alpha'}}(\mathrm{HL}_{\underline{w}}(\mathcal{F})\langle n\rangle) \in \mathrm{Coh}(\mathcal{Z}_{\alpha'})_{\leq w_{\alpha'}}. \tag{6.2.4.3}$$

Since the same is true with $n = 0$, this is immediate from $\mathrm{Perf}(\mathrm{B}\mathbb{G}_m)$-linearity of $i^*_{\mathcal{Z}_{\alpha'}}$ and the same argument as in the previous claim. For $n > 0$, the proof is analogous, except in place of (6.2.4.3), we must show

$$\mathrm{spl}^*_{\mathcal{Z}_{\alpha'}} i^!_{\mathcal{S}_{\alpha'}}(\mathrm{HL}_{\underline{w}}(\mathcal{F})\langle n\rangle|_{\mathcal{X}_{\leq \alpha'}}) \in \mathrm{Coh}(\mathcal{Z}_{\alpha'})_{>w_{\alpha'}}.$$

This holds by the same argument.

(3) First suppose $\alpha \leq \alpha'$. It suffices to show that

$$\mathrm{Hom}_{\mathcal{X}}(\mathrm{HL}_{\underline{w}}(\mathcal{F}), \mathrm{HL}_{\underline{w}}(\mathcal{F}')\langle n\rangle) \simeq 0 \tag{6.2.4.4}$$

for any $n < 0$. Combining (1) and (2), we see that $\mathrm{HL}_{\underline{w}}(\mathcal{F}')\langle n\rangle$ lies non-strictly to the left of $\mathrm{Coh}(\mathcal{Z}_{\alpha'})_{w_{\alpha'}+1}$ in (6.1.16.1), from which (6.2.4.4) is immediate. Likewise, if $\alpha' < \alpha$, it suffices to show that

$$\mathrm{Hom}_{\mathcal{X}}(\mathrm{HL}_{\underline{w}}(\mathcal{F})\langle n\rangle, \mathrm{HL}_{\underline{w}}(\mathcal{F}')) \simeq 0$$

for any $n \geq 0$, which follows by an analogous argument. For the final statement, simply note that $\mathrm{HL}_{\underline{w}}|_{\mathrm{Coh}(\mathcal{Z}_\alpha)_{w_\alpha}}$ is fully faithful.

(4) Consider the commutative diagram

$$\begin{tikzcd} \mathcal{Z}_I \arrow[r, "p_{\mathcal{Z}_I}"] \arrow[d, "(i\circ p_{\mathcal{Z}_I})\times \mathrm{id}_{\mathcal{Z}_I}"'] & \mathrm{B}\mathbb{G}_m \arrow[r] \arrow[d, "i\times \mathrm{id}_{\mathrm{B}\mathbb{G}_m}"] & * \arrow[d] \\ \mathrm{B}\mathbb{G}_m \times \mathcal{Z}_I \arrow[r, "\mathrm{id}_{\mathrm{B}\mathbb{G}_m}\times p_{\mathcal{Z}_I}"] & \mathrm{B}\mathbb{G}_m \times \mathrm{B}\mathbb{G}_m \arrow[r, "m"] & \mathrm{B}\mathbb{G}_m, \end{tikzcd} \tag{6.2.4.5}$$

where $m\colon \mathrm{B}\mathbb{G}_m \times \mathrm{B}\mathbb{G}_m \to \mathrm{B}\mathbb{G}_m$ denotes the multiplication map and $i\colon \mathrm{B}\mathbb{G}_m \to \mathrm{B}\mathbb{G}_m$ the inversion map. Both squares are clearly cartesian, so the outer rectangle is as well. Now consider the commutative diagram

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Z}_I^{\mathrm{dq}} \arrow[r] \arrow[d] & * \arrow[d] \\ \mathrm{B}\mathbb{G}_m \times \mathcal{Z}_I \arrow[r, "\mathrm{act}_{\mathcal{Z}_I}"] \arrow[d, "(i\circ \mathrm{pr}_1)\times \mathrm{act}_{\mathcal{Z}_I}"'] & \mathcal{Z}_I \arrow[r, "p_{\mathcal{Z}_I}"] \arrow[d, equal] & \mathrm{B}\mathbb{G}_m \\ \mathrm{B}\mathbb{G}_m \times \mathcal{Z}_I \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & \mathcal{Z}_I \arrow[d] & \\ \mathrm{B}\mathbb{G}_m \arrow[r] & *, & \end{tikzcd} \tag{6.2.4.6}$$

where $\mathcal{P}$ is defined by requiring the top-left square to be cartesian. Note that $(i \circ \mathrm{pr}_1) \times \mathrm{act}_{\mathcal{Z}_I}$ is an isomorphism; indeed, it is its own inverse. Thus, all squares in (6.2.4.6) are cartesian; in particular, we have $\mathcal{P} \simeq \mathrm{B}\mathbb{G}_m \times \mathcal{Z}_I^{\mathrm{dq}}$. Moreover, our assumption implies that $p_{\mathcal{Z}_I} \circ \mathrm{act}_{\mathcal{Z}_I} \simeq m \circ (\mathrm{id}_{\mathrm{B}\mathbb{G}_m} \times p_{\mathcal{Z}_I})$, so (6.2.4.5) yields $\mathcal{P} \simeq \mathcal{Z}_I$, as desired.

Finally, note that the weak $\mathrm{B}\mathbb{G}_m$-action on $\mathcal{P}$ coming from its isomorphism with $\mathcal{Z}_I$ is induced by the trivial weak $\mathrm{B}\mathbb{G}_m$-actions on $\mathcal{Z}_I^{\mathrm{dq}}, \mathcal{Z}_I$, as well as the weak $\mathrm{B}\mathbb{G}_m$-action on $\mathrm{B}\mathbb{G}_m \times \mathcal{Z}_I$ given by $m \circ (i \times \mathrm{id}_{\mathrm{B}\mathbb{G}_m})$ on the first factor and $\mathrm{act}_{\mathcal{Z}_I}$ on the second factor. This action is now easily seen to agree with the weak $\mathrm{B}\mathbb{G}_m$-action on $\mathrm{B}\mathbb{G}_m \times \mathcal{Z}_I^{\mathrm{dq}}$ coming from multiplication on the first factor. Moreover, the projection onto the first factor agrees with $p_{\mathcal{Z}_I} \simeq i \circ (i \circ p_{\mathcal{Z}_I})\colon \mathcal{Z}_I \to \mathrm{B}\mathbb{G}_m$, as claimed. $\square$

6.2.5. We now use these results to construct the desired $\mathrm{Perf}(\mathrm{B}\mathbb{G}_m)$-linear modifications of $\mathrm{HL}_{\underline{w}}, \check{\mathrm{HL}}_{\underline{w}}$. Suppose that we are in the situation of Lemma 6.2.4(4). By [**AG15**, Prop. 8.4.14], the external tensor product gives a $\mathrm{Perf}(\mathrm{B}\mathbb{G}_m)$-linear equivalence[57]

$$\boxtimes\colon\ \mathrm{Perf}(\mathrm{B}\mathbb{G}_m) \otimes \mathrm{Coh}(\mathcal{Z}_I^{\mathrm{dq}}) \xrightarrow{\sim} \mathrm{Coh}(\mathcal{Z}_I). \tag{6.2.5.1}$$

Thus, we may define a functor

$$\mathrm{in}_{\underline{w}}\colon\ \mathrm{Coh}(\mathcal{Z}_I^{\mathrm{dq}}) \to \mathrm{Coh}(\mathcal{Z}_I)$$

given by placing each coherent sheaf on $\mathcal{Z}_\alpha^{\mathrm{dq}}$ in weight w_α. We now define $\mathrm{HL}_{\underline{w}}^{\mathrm{gr}}$ to be the composition

$$\begin{tikzcd} \mathrm{Coh}(\mathcal{Z}_I) \arrow[rr, dashed, "\mathrm{HL}_{\underline{w}}^{\mathrm{gr}}"] \arrow[d, "\wr"] & & \mathrm{Coh}(\mathcal{X}) \\ \mathrm{Perf}(\mathrm{B}\mathbb{G}_m) \otimes \mathrm{Coh}(\mathcal{Z}_I^{\mathrm{dq}}) \arrow[r, "\mathrm{id} \otimes \mathrm{in}_{\underline{w}}"] & \mathrm{Perf}(\mathrm{B}\mathbb{G}_m) \otimes \mathrm{Coh}(\mathcal{Z}_I) \arrow[r, "\mathrm{id} \otimes \mathrm{HL}_{\underline{w}}"] & \mathrm{Perf}(\mathrm{B}\mathbb{G}_m) \otimes \mathrm{Coh}(\mathcal{X}), \arrow[u, "p^*(-)\otimes -"'] \end{tikzcd}$$

which is evidently $\mathrm{Perf}(\mathrm{B}\mathbb{G}_m)$-linear. We therefore have an adjunction

$$\mathrm{HL}_{\underline{w}}^{\mathrm{gr}}\colon\ \mathrm{QC}^!(\mathcal{Z}_I) \rightleftarrows \mathrm{QC}^!(\mathcal{X})\colon \mathrm{HL}_{\underline{w}}^{\mathrm{gr},R} \tag{6.2.5.2}$$

with both functors continuous and $\mathrm{QC}(\mathrm{B}\mathbb{G}_m)$-linear (see [**Gai12**, Cor. 6.2.4]). In fact, this adjunction is monadic:

6.2.6. Lemma. The functor $\mathrm{HL}_{\mathrm{gr}}^{\underline{w},R}$ is conservative.

[57] We need only note that the stack $\mathcal{Z}_I^{\mathrm{dq}}$ is quasi-compact and has affine diagonal. Indeed, the latter holds as in §6.1.2. For the former, note that each $\mathcal{S}_\alpha$ is quasi-compact as $\mathcal{X}$ is Noetherian; the surjection $\mathrm{gr}_{\mathcal{S}_\alpha}\colon \mathcal{S}_\alpha \to \mathcal{Z}_\alpha$ now exhibits $\mathcal{Z}_\alpha$, and hence $\mathcal{Z}_\alpha^{\mathrm{dq}}$, as quasi-compact.

Proof. By [**Yan22**, Prop. 2.9], it suffices to show that the image of $\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}$ generates $\mathrm{QC}^!(\mathfrak{X})$ under colimits. In particular, we may work with small categories. Let $\alpha \in I$, $w \in \mathbb{Z}$, and $\mathcal{F} \in \mathrm{Coh}(\mathcal{Z}_\alpha)_w$. By Lemma 6.2.4(1),

$$\check{\mathrm{HL}}_{\underline{w}}(\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}(\mathcal{F})) \simeq \check{\mathrm{HL}}_{\underline{w}}(\mathrm{HL}_{\underline{w}}(\mathcal{F}\langle w - w_\alpha\rangle)\langle -w\rangle) \tag{6.2.6.1}$$

is supported on $\mathcal{Z}_{\alpha'}$ with $\alpha \le \alpha'$, and its component in $\mathrm{Coh}(\mathcal{Z}_\alpha)$ is isomorphic to $\mathcal{F}\langle -w_\alpha\rangle \in \mathrm{Coh}(\mathcal{Z}_\alpha)_{w+w_\alpha}$. The statement is now immediate by induction on α. ☐

6.2.7. Thus, by the Barr–Beck–Lurie theorem (see [**Lur17**, Thm. 4.7.0.3]), the functor $\mathrm{HL}^{\mathrm{gr},R}_{\underline{w}}$ induces an equivalence

$$\mathrm{QC}^!(\mathfrak{X}) \simeq T_{\underline{w}}\text{-mod}_{\mathrm{QC}^!(\mathcal{Z}_I)}, \tag{6.2.7.1}$$

where we have let

$$T_{\underline{w}} := \mathrm{HL}^{\mathrm{gr},R}_{\underline{w}} \circ \mathrm{HL}^{\mathrm{gr}}_{\underline{w}} \in \mathrm{Alg}(\mathrm{End}_{\mathrm{QC}(\mathrm{B}\mathbb{G}_m)}(\mathrm{QC}^!(\mathcal{Z}_I))) \tag{6.2.7.2}$$

denote the monad associated to (6.2.5.2). Moreover, the functors $\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}$ and $\mathrm{HL}^{\mathrm{gr},R}_{\underline{w}}$ are identified with induction and restriction along the unit morphism $\mathrm{id}_{\mathrm{QC}^!(\mathcal{Z}_I)} \to T_{\underline{w}}$, respectively.

Note that the decomposition $\mathrm{QC}^!(\mathcal{Z}_I) \simeq \bigoplus_{\alpha\in I} \mathrm{QC}^!(\mathcal{Z}_\alpha)$ induces a decomposition

$$T_{\underline{w}} \simeq \bigoplus_{\alpha,\alpha'\in I} T^{\alpha,\alpha'}_{\underline{w}} \in \bigoplus_{\alpha,\alpha'\in I} \mathrm{Fun}_{\mathrm{QC}^!(\mathrm{B}\mathbb{G}_m)}(\mathrm{QC}^!(\mathcal{Z}_\alpha), \mathrm{QC}^!(\mathcal{Z}_{\alpha'})).$$

Moreover, by (6.2.5.1), we have

$$\mathrm{End}_{\mathrm{QC}(\mathrm{B}\mathbb{G}_m)}(\mathrm{QC}^!(\mathcal{Z}_I)) \simeq \mathrm{QC}(\mathrm{B}\mathbb{G}_m) \otimes \mathrm{End}(\mathrm{QC}^!(\mathcal{Z}_I^{\mathrm{dq}})), \tag{6.2.7.3}$$

so $T_{\underline{w}}$ also carries a (compatible) grading. We now establish a key structure theorem for $T_{\underline{w}}$ with respect to this decomposition and grading:

6.2.8. Proposition. The monad $T_{\underline{w}}$ is concentrated in non-negative weights with respect to (6.2.7.3). Moreover, if $\alpha < \alpha'$, then $T^{\alpha,\alpha'}_{\underline{w}}$ is concentrated in positive weights, and if $\alpha = \alpha'$, the unit morphism induces an isomorphism $\mathrm{id}_{\mathrm{QC}^!(\mathcal{Z}_\alpha)} \xrightarrow{\sim} T^{\alpha,\alpha}_{\underline{w},0}$ onto the weight-0 component.

Proof. Let $\alpha, \alpha' \in I$, and let $\mathcal{F} \in \mathrm{Coh}(\mathcal{Z}_\alpha)_0$. It suffices to show that for any $\mathcal{F}' \in \mathrm{Coh}(\mathcal{Z}_{\alpha'})_0$, the graded inner Hom

$$\underline{\mathrm{Hom}}_{\mathcal{Z}_{\alpha'}}(\mathcal{F}', T^{\alpha,\alpha'}_{\underline{w}}(\mathcal{F})) \simeq \underline{\mathrm{Hom}}_{\mathcal{Z}_{\alpha'}}(\mathcal{F}', \mathrm{HL}^{\mathrm{gr},R}_{\underline{w}}(\mathrm{HL}_{\underline{w}}(\mathcal{F}\langle -w_\alpha\rangle))) \simeq \underline{\mathrm{Hom}}_{\mathfrak{X}}(\mathrm{HL}_{\underline{w}}(\mathcal{F}'\langle -w_{\alpha'}\rangle), \mathrm{HL}_{\underline{w}}(\mathcal{F}\langle -w_\alpha\rangle))$$

has the claimed properties. These are all immediate from Lemma 6.2.4(3). ☐

6.2.9. The following corollary will allow us to define the graded retraction $\check{\mathrm{HL}}^{\mathrm{gr}}_{\underline{w}}$ of $\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}$:

6.2.10. Corollary. The unit morphism of monads $\mathrm{id}_{\mathrm{QC}^!(\mathcal{Z}_I)} \to T_{\underline{w}}$ admits a canonical $\mathrm{QC}(\mathrm{B}\mathbb{G}_m)$-linear retraction.

Proof. The first statement of Lemma 6.2.8 gives a canonical morphism of monads $T_{\underline{w}} \to T_{\underline{w},0}$, i.e., the projection onto weight-0. The second statement shows that $T_{\underline{w},0}$ is "upper triangular" with respect to I, so we have a canonical morphism $T_{\underline{w},0} \to \bigoplus_{\alpha\in I} T^{\alpha,\alpha}_{\underline{w},0}$ given by projection onto the diagonal. Finally, the third statement identifies the composition of these two maps with a retraction of the unit morphism. ☐

6.2.11. Example. Let us compute this monad explicitly in the situation of Example 6.2.2. By definition, we have

$$\mathrm{HL}^{\mathrm{gr}}_{(w_0,w_\infty)}(\mathcal{O}_0) \simeq \mathcal{O}_{\mathbb{P}^1}(w_0, w_\infty),$$
$$\mathrm{HL}^{\mathrm{gr}}_{(w_0,w_\infty)}(\mathcal{O}_\infty) \simeq \mathcal{O}_\infty\langle -w_\infty\rangle,$$

which clearly generate $\mathrm{Coh}(\mathbb{P}^1/\mathbb{G}_m)$ under the $\mathrm{Perf}(\mathrm{B}\mathbb{G}_m)$-action. We may therefore represent $T_{(w_0,w_\infty)}$ by the 2×2 matrix

$$\underline{\mathrm{End}}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(w_0,w_\infty) \oplus \mathcal{O}_\infty\langle -w_\infty\rangle)^{\mathrm{op}} \simeq \begin{bmatrix} k[0]\langle 0\rangle & k[0]\langle 0\rangle \\ k[-1]\langle -1\rangle & \mathrm{Sym}\, k[-1]\langle -1\rangle \end{bmatrix} \in \mathrm{Alg}(\mathrm{QC}(\{0,\infty\}\times\{0,\infty\}/\mathbb{G}_m)),$$

with the evident algebra structure.[58] In particular, $T_{(w_0,w_\infty)}$ is concentrated in weights $[0,1]$, and in weight-0, it is given by the standard upper triangular matrix algebra (whose diagonal recovers the identity monad).

6.2.12. Using Corollary 6.2.10, we obtain a commutative diagram of continuous, $\mathrm{QC}(\mathrm{B}\mathbb{G}_m)$-linear adjoint pairs

$$\begin{tikzcd}
\mathrm{id}_{\mathrm{QC}^!(\mathcal{Z}_I)}\text{-mod}_{\mathrm{QC}^!(\mathcal{Z}_I)} \arrow[r, shift left, "\mathrm{Ind}_{\mathrm{id}}^{T_{\underline{w}}}"] \arrow[d, equal] & T_{\underline{w}}\text{-mod}_{\mathrm{QC}^!(\mathcal{Z}_I)} \arrow[l, shift left, "\mathrm{Res}_{\mathrm{id}}^{T_{\underline{w}}}"] \arrow[r, shift left, "\mathrm{Ind}^{\mathrm{id}}_{T_{\underline{w}}}"] & \mathrm{id}_{\mathrm{QC}^!(\mathcal{Z}_I)}\text{-mod}_{\mathrm{QC}^!(\mathcal{Z}_I)} \arrow[l, shift left, "\mathrm{Res}^{\mathrm{id}}_{T_{\underline{w}}}"] \arrow[d, equal] \\
\mathrm{QC}^!(\mathcal{Z}_I) \arrow[r, shift left, "\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}"] & \mathrm{QC}^!(\mathcal{X}) \arrow[l, shift left, "\mathrm{HL}^{\mathrm{gr},R}_{\underline{w}}"] \arrow[u, "\wr"', "(6.2.7.1)"] \arrow[r, shift left, dashed, "\check{\mathrm{HL}}^{\mathrm{gr}}_{\underline{w}}"] & \mathrm{QC}^!(\mathcal{Z}_I), \arrow[l, shift left, dashed, "\check{\mathrm{HL}}^{\mathrm{gr},R}_{\underline{w}}"]
\end{tikzcd}$$

where each horizontal composition identifies with the identity. In particular, we have constructed the $\mathrm{QC}(\mathrm{B}\mathbb{G}_m)$-linear functor $\check{\mathrm{HL}}^{\mathrm{gr}}_{\underline{w}}$, which preserves coherent sheaves and satisfies $\check{\mathrm{HL}}^{\mathrm{gr}}_{\underline{w}} \circ \mathrm{HL}^{\mathrm{gr}}_{\underline{w}} \simeq \mathrm{id}_{\mathrm{QC}^!(\mathcal{Z}_I)}$.

We now verify that the desirable properties of the functors $\mathrm{HL}_{\underline{w}}, \check{\mathrm{HL}}_{\underline{w}}$ carry over to their graded counterparts $\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}, \check{\mathrm{HL}}^{\mathrm{gr}}_{\underline{w}}$:

6.2.13. Lemma. (1) The functors $\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}$, $\check{\mathrm{HL}}^{\mathrm{gr}}_{\underline{w}}$ induce inverse equivalences on Hochschild homology.

(2) Let $p' \colon \mathcal{X}' \to \mathrm{B}\mathbb{G}_m$ satisfy the same hypotheses as p, and let $f \colon \mathcal{X}' \to \mathcal{X}$ be as in Theorem 6.1.16. Suppose that $p \simeq f \circ p'$. Then the lefthand square in the diagram

(6.2.13.1)
$$\begin{tikzcd}
\mathrm{Coh}(\mathcal{Z}_I) \arrow[r, "\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}"] \arrow[d, "f^*_{\mathcal{Z}_I}"] & \mathrm{Coh}(\mathcal{X}) \arrow[r, "\check{\mathrm{HL}}^{\mathrm{gr}}_{\underline{w}}"] \arrow[d, "f^*"] & \mathrm{Coh}(\mathcal{Z}_I) \arrow[d, "f^*_{\mathcal{Z}_I}"] \\
\mathrm{Coh}(\mathcal{Z}'_I) \arrow[r, "\mathrm{HL}'^{,\mathrm{gr}}_{\underline{w}}"] & \mathrm{Coh}(\mathcal{X}') \arrow[r, "\check{\mathrm{HL}}'^{,\mathrm{gr}}_{\underline{w}}"] & \mathrm{Coh}(\mathcal{Z}'_I)
\end{tikzcd}$$

commutes. If moreover $f_{\mathcal{Z}_I}$ is an isomorphism, then the righthand square commutes as well.

(3) Let $q \colon \mathcal{X} \to \mathcal{B}$ be as in Lemma 6.1.20. Then $\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}$ and $\check{\mathrm{HL}}^{\mathrm{gr}}_{\underline{w}}$ are $\mathrm{Perf}(\mathcal{B})$-linear.

Proof. (1) It suffices to show that $\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}$ induces an equivalence on Hochschild homology, which is immediate from (6.2.6.1) and a standard upper-triangular matrix argument.

(2) Lemma 6.2.4(4) and §6.1.17 immediately imply commutativity of the lefthand square. For the latter assertion, we implicitly identify $\mathrm{Coh}(\mathcal{Z}_I)$ and $\mathrm{Coh}(\mathcal{Z}'_I)$ henceforth via $f^*_{\mathcal{Z}_I}$. Let $T_{\underline{w}}, T'_{\underline{w}}$ denote the respective monads for $\mathcal{X}, \mathcal{X}'$. Then we have a natural morphism of $\mathrm{QC}(\mathrm{B}\mathbb{G}_m)$-linear monads

$$T_{\underline{w}} = \mathrm{HL}^{\mathrm{gr},R}_{\underline{w}} \circ \mathrm{HL}^{\mathrm{gr}}_{\underline{w}} \to \mathrm{HL}^{\mathrm{gr},R}_{\underline{w}} \circ f_* \circ f^* \circ \mathrm{HL}^{\mathrm{gr}}_{\underline{w}} \simeq \mathrm{HL}'^{,\mathrm{gr},R}_{\underline{w}} \circ \mathrm{HL}'^{,\mathrm{gr}}_{\underline{w}} = T'_{\underline{w}}. \tag{6.2.13.2}$$

It is not hard to see that this morphism commutes with the respective retractions of the unit morphisms, so it suffices to show that the diagram

$$\begin{tikzcd}
\mathrm{QC}^!(\mathcal{X}) \arrow[r, "(6.2.7.1)", "\sim"'] \arrow[d, "f^*"] & T_{\underline{w}}\text{-mod}_{\mathrm{QC}^!(\mathcal{Z}_I)} \arrow[d, "\mathrm{Ind}_{T_{\underline{w}}}^{T'_{\underline{w}}}"] \\
\mathrm{QC}^!(\mathcal{X}') \arrow[r, "(6.2.7.1)", "\sim"'] & T'_{\underline{w}}\text{-mod}_{\mathrm{QC}^!(\mathcal{Z}_I)}
\end{tikzcd}$$

commutes. Passing to right adjoints, we must show that $\mathrm{Res}_{T_{\underline{w}}}^{T'_{\underline{w}}}$ identifies with f_*. Recall that the horizontal equivalences are enhanced versions of $\mathrm{HL}^{\mathrm{gr},R}_{\underline{w}}$ and $\mathrm{HL}'^{,\mathrm{gr},R}_{\underline{w}}$, respectively; more precisely, we must show that the diagram

$$\begin{tikzcd}
(\mathrm{HL}^{\mathrm{gr},R}_{\underline{w}} \circ \mathrm{HL}^{\mathrm{gr}}_{\underline{w}}) \circ (\mathrm{HL}^{\mathrm{gr},R}_{\underline{w}} \circ f_*) \arrow[r, "\mathrm{id}_{\mathrm{HL}^{\mathrm{gr},R}_{\underline{w}}} \circ \epsilon \circ \mathrm{id}_{f_*}"] \arrow[d, "(6.2.13.2)"] & \mathrm{HL}^{\mathrm{gr},R}_{\underline{w}} \circ f_* \arrow[d, "\wr"] \\
(\mathrm{HL}'^{,\mathrm{gr},R}_{\underline{w}} \circ \mathrm{HL}'^{,\mathrm{gr}}_{\underline{w}}) \circ \mathrm{HL}'^{,\mathrm{gr},R}_{\underline{w}} \arrow[r, "\mathrm{id}_{\mathrm{HL}'^{,\mathrm{gr},R}_{\underline{w}}} \circ \epsilon'"] & \mathrm{HL}'^{,\mathrm{gr},R}_{\underline{w}}
\end{tikzcd}$$

commutes, where we have let ϵ, ϵ' denote the respective counit maps. We leave this as an exercise.

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[58] Note that as in Proposition 1.5.1, we have identified $\mathrm{End}_{\mathrm{QC}(\mathrm{B}\mathbb{G}_m)}(\{0,\infty\}/\mathbb{G}_m)$ with the latter convolution category via the usual integral transform, see for instance [**BZNP17b**].

(3) We claim that $q_{\mathcal{Z}_I} := q \circ i_{\mathcal{Z}_I} : \mathcal{Z}_I \to \mathcal{B}$ factors through $\mathcal{Z}_I^{\mathrm{dq}}$. By assumption, it factors through a union of connected components $\mathcal{Z}_{\mathcal{B}} \subset \mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{B})$ on which the canonical weak $\mathrm{B}\mathbb{G}_m$-action is trivial. Misusing notation slightly, we write $q_{\mathcal{Z}_I} : \mathcal{Z}_I \to \mathcal{Z}_{\mathcal{B}}$ for this map as well. We then have a commutative diagram

$$\begin{tikzcd}
\mathrm{B}\mathbb{G}_m \times \mathcal{Z}_I^{\mathrm{dq}} \arrow[rr, "\mathrm{pr}_2"] \arrow[d, "\mathrm{id}_{\mathrm{B}\mathbb{G}_m} \times \varpi_1 \times \mathrm{id}_{\mathcal{Z}_I^{\mathrm{dq}}}"] \arrow[dd, bend right=60, "\mathrm{id}"'] & & \mathcal{Z}_I^{\mathrm{dq}} \arrow[dd, "\varpi_1 \times \mathrm{id}_{\mathcal{Z}_I^{\mathrm{dq}}}"] \\
\mathrm{B}\mathbb{G}_m \times \mathrm{B}\mathbb{G}_m \times \mathcal{Z}_I^{\mathrm{dq}} \arrow[r, "\mathrm{id}_{\mathrm{B}\mathbb{G}_m} \times q_{\mathcal{Z}_I}"] \arrow[d, "m \times \mathrm{id}_{\mathcal{Z}_I^{\mathrm{dq}}}"] & \mathrm{B}\mathbb{G}_m \times \mathcal{Z}_{\mathcal{B}} \arrow[d, "\mathrm{pr}_2"] & \\
\mathrm{B}\mathbb{G}_m \times \mathcal{Z}_I^{\mathrm{dq}} \arrow[r, "q_{\mathcal{Z}_I}"] & \mathcal{Z}_{\mathcal{B}} & \mathrm{B}\mathbb{G}_m \times \mathcal{Z}_I^{\mathrm{dq}}. \arrow[l, "q_{\mathcal{Z}_I}"']
\end{tikzcd}$$

Commutativity of the outer square now gives the desired factorization.

The functor $\mathrm{HL}_{\underline{w}}^{\mathrm{gr}}$ is then clearly $\mathrm{Perf}(\mathcal{B})$-linear, so by [**BZFN10**, Prop. 3.6] and [**Gai12**, Cor. 6.2.4], its right adjoint $\mathrm{HL}_{\underline{w}}^{\mathrm{gr},R}$ and the monad $T_{\underline{w}}$ are as well. Moreover, the $\mathrm{Perf}(\mathcal{B})$-action preserves the decomposition (6.2.7.2) and the grading (6.2.7.3), so the rest of the arguments go through as before. $\square$

6.2.14. We conclude this section by specializing our constructions to the case of quotient stacks. Let X be a finite-type (derived) scheme with an action of a reductive group G, and set $\mathcal{X} := X/G$. Given a cocharacter $\check{\lambda} : \mathbb{G}_m \to G$, let

(1) $P_{\check{\lambda}}$ denote the parabolic subgroup of G associated to the subspace $\mathfrak{p}_{\check{\lambda}} \subset \mathfrak{g}$ spanning non-positive weight-spaces for the adjoint action of $\mathbb{G}_m$;
(2) $L_{\check{\lambda}}$ denote the Levi subgroup of $P_{\check{\lambda}}$, i.e., the subgroup associated to the subspace $\mathfrak{l}_{\check{\lambda}} \subset \mathfrak{g}$ spanning trivial weight-spaces; and
(3) $\mathrm{BB}_{\mathcal{X}}^{\check{\lambda}}$ denote the set of $L_{\check{\lambda}}$-orbits of connected components of the classical fixed points $X^{\mathbb{G}_m}$, where $\mathbb{G}_m$ acts via $\check{\lambda}$.

Finally, for any $\alpha \in \mathrm{BB}_{\mathcal{X}}^{\check{\lambda}}$, let $X_\alpha^{\check{\lambda},0} \subset X_\alpha^{\check{\lambda},-} \subset X^{\mathrm{cl}}$ denote the corresponding component of $X^{\mathbb{G}_m}$ and its repelling locus (i.e., the classical Białynicki-Birula stratum), respectively. Note that $\mathrm{BB}_{\mathcal{X}}^{\check{\lambda}}$ carries a natural partial order via closures, i.e., $\alpha < \alpha'$ if $X_{\alpha'}^{\check{\lambda},-} \subset \overline{X_\alpha^{\check{\lambda},-}}$.

Recall from [**HL18**, Ex. 2.3] that the *classical* mapping stacks studied in §6.1.2 admit decompositions

$$\mathrm{Map}(\Theta, \mathcal{X})^{\mathrm{cl}} = \coprod_{\substack{\check{\lambda} \in X_*(G)/G \\ \alpha \in \mathrm{BB}_{\mathcal{X}}^{\check{\lambda}}}} X_\alpha^{\check{\lambda},-}/P_{\check{\lambda}}, \qquad \mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{X})^{\mathrm{cl}} = \coprod_{\substack{\check{\lambda} \in X_*(G)/G \\ \alpha \in \mathrm{BB}_{\mathcal{X}}^{\check{\lambda}}}} X_\alpha^{\check{\lambda},0}/L_{\check{\lambda}}, \tag{6.2.14.1}$$

where G acts on $X_*(G)$ by conjugation. As in §6.1.7, for each such $\check{\lambda}, \alpha$, there are connected components $\mathcal{X}_\alpha^{\check{\lambda},-} \subset \mathrm{Map}(\Theta, \mathcal{X})$ and $\mathcal{X}_\alpha^{\check{\lambda},0} \subset \mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathcal{X})$ such that $\mathcal{X}_\alpha^{\check{\lambda},-,\mathrm{cl}} \simeq X_\alpha^{\check{\lambda},-}/P_{\check{\lambda}}$ and $\mathcal{X}_\alpha^{\check{\lambda},0,\mathrm{cl}} \simeq X_\alpha^{\check{\lambda},0}/L_{\check{\lambda}}$. Moreover, when X is smooth, these stacks have trivial derived structure by Lemma 6.1.8.

Let us now specialize further to the case of a quasi-smooth quotient stack $\mathcal{X} := X/\widetilde{G}$, for G reductive. Moreover, fix $\check{\lambda} : \mathbb{G}_m \to \widetilde{G}$ to be the tautological cocharacter, and omit it henceforth from the notation. Suppose that the Białynicki-Birula strata $\{X_\alpha^-\}_{\alpha \in \mathrm{BB}_{\mathcal{X}}}$ cover X. Then $\{(\bigcup_{\alpha' \leq \alpha} \mathcal{X}_{\alpha'}^-, \mathcal{X}_\alpha^-)\}_{\alpha \in \mathrm{BB}_{\mathcal{X}}}$ is an unstable Θ-stratification of $\mathcal{X}$ (which is equivalent to $\{(\bigcup_{\alpha' \leq \alpha} X_{\alpha'}^-/\widetilde{G}, X_\alpha^-/\widetilde{G})\}_{\alpha \in \mathrm{BB}_{\mathcal{X}}}$ when X is smooth). In particular, as in §6.2.12, we obtain $\mathrm{Perf}(\mathrm{B}\mathbb{G}_m)$-linear functors

$$\mathrm{Coh}(\mathcal{Z}_{\mathrm{BB}_{\mathcal{X}}}) \xrightarrow{\mathrm{HL}_{\mathrm{gr}}^{\underline{w}}} \mathrm{Coh}(\mathcal{X}) \xrightarrow{\check{\mathrm{HL}}_{\mathrm{gr}}^{\underline{w}}} \mathrm{Coh}(\mathcal{Z}_{\mathrm{BB}_{\mathcal{X}}})$$

for any choice of $\underline{w} = \{w_\alpha\}_{\alpha \in \mathrm{BB}_{\mathcal{X}}}$. Moreover, taking $\mathcal{B} := \mathrm{B}G$ and $p : X/\widetilde{G} \to \mathrm{B}G$ to be the projection, Lemma 6.2.13(3) shows that these functors are naturally $\mathrm{Perf}(\mathrm{B}\widetilde{G})$-linear (since all fixed loci map to the connected component of $\mathrm{Map}(\mathrm{B}\mathbb{G}_m, \mathrm{B}G)$ corresponding to the trivial character).

6.2.15. Example. In the final paragraph of §6.2.14, the strata $\mathcal{X}_\alpha^-$ and centers $\mathcal{X}_\alpha^0$ may not be classical, even if X is itself classical. For instance, consider the stack $\mathcal{X} := X/\mathbb{G}_m$, where $X = \mathrm{Spec}\, k[x,y]/xy$ with

x, y in weights $1, -1$, respectively. Since $X^{\mathbb{G}_m} = \{0\}$, we omit α from the notation, so that $X^0 = \{0\}$ and $X^- = \operatorname{Spec} k[x]$. Observe that $\mathbb{L}_{\mathcal{X}}$ is computed by the complex

$$\mathcal{O}_X \cdot (y\,dx + x\,dy) \to \mathcal{O}_X \cdot dx \oplus \mathcal{O}_X \cdot dy \xrightarrow{(dx,dy)\mapsto(-x,y)} \mathcal{O}_X$$

in degrees $[-1, 1]$. It follows from Lemma 6.1.10 that $\mathbb{L}_{\mathcal{X}^0}$ and $\mathbb{L}_{\mathcal{X}^-}$ are computed by restricting this complex to X^0 and X^-, followed by projecting to the subcategories $\mathrm{QC}(X^0/\mathbb{G}_m)_0$ and $\mathrm{QC}(X^-/\mathbb{G}_m)_{\geq 0}$, respectively.[59] Thus, we have

$$\mathbb{L}_{\mathcal{X}^0} \simeq \big(k \cdot (y\,dx + x\,dy) \to 0 \to k\big),$$
$$\mathbb{L}_{\mathcal{X}^-} \simeq \big(k[x] \cdot (y\,dx + x\,dy) \xrightarrow{0} k[x] \cdot dx \xrightarrow{dx\mapsto -x} k[x]\big),$$

and hence

$$\mathcal{X}^0 \simeq \operatorname{Spec}(\operatorname{Sym}_{\mathcal{O}_{X^0}} \mathcal{O}_{X^0}[1])/\mathbb{G}_m,$$
$$\mathcal{X}^- \simeq \operatorname{Spec}(\operatorname{Sym}_{\mathcal{O}_{X^-}} \mathcal{O}_{X^-}[1])/\mathbb{G}_m,$$

both of which have nontrivial derived structure. For a more general discussion of this phenomenon, see [**HL21**, §1.6].

6.3. The coherent homomorphism datum

6.3.1. We now return to the setting of §5.3, with a fixed nilpotent e and Slodowy slice S_e. In this section, we use the results of §6.2 to construct the homomorphism datum $\Phi^{\mathrm{coh}}_{S_e,\underline{w}}$ of (1.5.2.1) and prove Theorem 1.5.2. In particular, we prove a k-linearized version of Qiu–Xi's conjecture (1.3.0.2). We conclude by explaining how (suitably equivariant) full exceptional collections in $\mathrm{Coh}(\mathcal{B}_e^{\mathbb{G}_m})$ can be used to compute the asymptotic coherent Springer sheaf, and give an illustrative example for $G = \mathrm{Sp}_6$.

6.3.2. We begin by further specializing §6.2.14 to the case of $\mathcal{X} = \widetilde{S}_e/\widetilde{Z}_e$. Let $\underline{w} \in \mathrm{BB}_{\widetilde{S}_e/\widetilde{Z}_e}$ be a weight-vector. Since the $\mathbb{G}_m$-action on S_e is repelling, $\widetilde{S}_e/\widetilde{Z}_e$ is covered by its Białynicki-Birula strata. Using §6.2.14 and Lemma 6.2.13(1), we obtain $\mathrm{Perf}(\mathrm{B}\widetilde{Z}_e)$-linear functors

$$\mathrm{Coh}(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e) \xrightarrow{\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}} \mathrm{Coh}(\widetilde{S}_e/\widetilde{Z}_e) \xrightarrow{\check{\mathrm{HL}}^{\mathrm{gr}}_{\underline{w}}} \mathrm{Coh}(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e).$$

We now use these functors to construct the homomorphism datum $\Phi^{\mathrm{coh}}_{S_e,\underline{w}}$. Set

$$\mathcal{J}^{\mathrm{coh}}_e := \mathrm{QC}(\mathcal{B}_e^{\mathbb{G}_m} \times \mathcal{B}_e^{\mathbb{G}_m}/Z_e).$$

By Proposition 3.3.6, it is a rigid monoidal category. Now let

$$\mathcal{M}^{\mathrm{coh}}_{S_e} := \mathrm{QC}(\mathcal{B}_e^{\mathbb{G}_m} \times \widetilde{S}_e/\widetilde{Z}_e),$$

which carries an evident $(\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m), \mathcal{H}^{\mathrm{coh}}_{S_e})$-bimodule structure via the respective convolutions. By [**GR17a**, Ch. 3, Prop. 3.5.3], the external tensor product gives bimodule equivalences

$$\begin{aligned} \mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m) &\simeq \mathrm{QC}(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e) \otimes_{\mathrm{QC}(\mathrm{B}\widetilde{Z}_e)} \mathrm{QC}(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e), \\ \mathcal{M}^{\mathrm{coh}}_{S_e} &\simeq \mathrm{QC}(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e) \otimes_{\mathrm{QC}(\mathrm{B}\widetilde{Z}_e)} \mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e), \end{aligned} \tag{6.3.2.1}$$

so by §6.3.2, we have functors

$$\mathcal{J}^{\mathrm{coh}}_e \xrightarrow{\mathrm{id}_{\mathrm{QC}(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e)} \otimes \mathrm{HL}^{\mathrm{gr}}_{\underline{w}}} \mathcal{M}^{\mathrm{coh}}_{S_e} \xrightarrow{\mathrm{id}_{\mathrm{QC}(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e)} \otimes \check{\mathrm{HL}}^{\mathrm{gr}}_{\underline{w}}} \mathcal{J}^{\mathrm{coh}}_e.$$

Finally, set

$$\Phi^{\mathrm{coh}}_{S_e,\underline{w}} := (\mathcal{M}^{\mathrm{coh}}_{S_e}, \mathrm{id}_{\mathrm{QC}(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e)} \otimes \mathrm{HL}^{\mathrm{gr}}_{\underline{w}}, \mathrm{id}_{\mathrm{QC}(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e)} \otimes \check{\mathrm{HL}}^{\mathrm{gr}}_{\underline{w}}).$$

6.3.3. Lemma. The triple $\Phi^{\mathrm{coh}}_{S_e,\underline{w}}$ is a unital homomorphism datum from $\mathcal{H}^{\mathrm{coh}}_{S_e}$ to $\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)$.

[59]For the latter, we use the localization functor provided by the semiorthogonal decomposition

$$\mathrm{QC}(X^-/\mathbb{G}_m) = \langle \mathrm{QC}(X^-/\mathbb{G}_m)_{>1}, \mathrm{QC}(X^-/\mathbb{G}_m)_{\leq 1} \rangle$$

as in §6.1.3.

Proof. The same argument as for Lemma 5.3.8 shows that $\mathcal{M}^{\mathrm{coh}}_{S_e}$ is left-dual to the $(\mathcal{H}^{\mathrm{coh}}_{S_e}, \mathcal{J}^{\mathrm{coh}}_e)$-bimodule $\mathcal{M}^{\mathrm{coh},R}_{S_e} := \mathrm{QC}(\widetilde{S}_e \times \mathcal{B}^{\mathbb{G}_m}_e/\widetilde{Z}_e)$ (using Lemma 3.3.9, (6.3.2.1), and the analogous equivalence for $\mathcal{M}^{\mathrm{coh},R}_{S_e}$). Moreover, the functors $\mathrm{id}\otimes\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}$ and $\mathrm{id}\otimes\check{\mathrm{HL}}^{\mathrm{gr}}_{\underline{w}}$ clearly commute with the $\mathcal{J}^{\mathrm{coh}}_e$-actions, preserve compact objects, and satisfy the unitality condition. It remains to show that they induce inverse equivalences on Hochschild homology. Rather than appeal to Lemma 6.2.13(1), we give a different argument.

First, as in §5.2.10 and (5.3.7.2), the maps

$$(6.3.3.1)\qquad [\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)] \xrightarrow{[\mathrm{id}\otimes\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}]} [\mathcal{M}^{\mathrm{coh}}_e] \xrightarrow{[\mathrm{id}\otimes\check{\mathrm{HL}}^{\mathrm{gr}}_{\underline{w}}]} [\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)]$$

in the category

$$\mathrm{Tr}(\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)) \simeq \mathrm{Tr}(\mathcal{J}^{\mathrm{coh}}_e) \otimes \mathrm{QC}(\mathbb{G}_m/\mathbb{G}_m)$$

compose to the identity. Moreover, under this equivalence, we have

$$[\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)] \simeq [\mathcal{J}^{\mathrm{coh}}_e] \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m},$$

which lies in weight-0. Thus, by Theorem 3.1.11, it suffices to show that the split map

$$(6.3.3.2)\qquad [\mathrm{id}\otimes\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}]\colon [\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)] \to [\mathcal{M}^{\mathrm{coh}}_e]_0$$

is an isomorphism.

Next, by [**BZFN12**, Thm. 1.3], we have an equivalence

$$(6.3.3.3)\qquad \mathrm{QC}(\mathcal{B}^{\mathbb{G}_m}_e/Z_e) \otimes_{\mathrm{QC}(\mathrm{B}Z_e)} -\colon\ \mathrm{QC}(\mathrm{B}Z_e)\textbf{-mod} \xrightarrow{\sim} \mathcal{J}^{\mathrm{coh}}_e\textbf{-mod}.$$

This induces equivalences

$$(6.3.3.4)\qquad \mathrm{Tr}(\mathcal{J}^{\mathrm{coh}}_e) \simeq \mathrm{Tr}(\mathrm{QC}(\mathrm{B}Z_e)) \simeq \mathrm{QC}(Z_e/Z_e),$$

and an identification of (6.3.3.1) with the maps

$$(6.3.3.5)\qquad [\mathrm{QC}(\mathcal{B}^{\mathbb{G}_m}_e/\widetilde{Z}_e)] \xrightarrow{[\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}]} [\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e)] \xrightarrow{[\check{\mathrm{HL}}^{\mathrm{gr}}_{\underline{w}}]} [\mathrm{QC}(\mathcal{B}^{\mathbb{G}_m}_e/\widetilde{Z}_e)]$$

in the category $\mathrm{QC}(\widetilde{Z}_e/\widetilde{Z}_e)$. In particular, by Corollary 3.2.13, (5.3.16.1), and Proposition 5.1.2, we have

$$(6.3.3.6)\qquad [\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e)]_0 \simeq (\mathcal{L}p_{S_e,*}\mathcal{L}i^*_{S_e}\mathcal{S})_0 \simeq \mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m}.$$

Thus, $[\mathrm{QC}(\mathcal{B}^{\mathbb{G}_m}_e/\widetilde{Z}_e)]$ is a direct summand of a (finite-dimensional) vector bundle, hence itself a vector bundle. We may therefore reduce to showing that the dimensions of the fibers of $[\mathrm{QC}(\mathcal{B}^{\mathbb{G}_m}_e/Z_e)]$ and $\mathcal{S}^{\mathbf{B}_e}$ at any point $s \in Z_e$ agree.

Finally, by Lemma 3.4.5, Proposition 3.4.11, (3.2.12.1), and [**Che18**, Cor. 1.0.2], we have isomorphisms

$$(6.3.3.7)\qquad [\mathrm{QC}(\mathcal{B}^{\mathbb{G}_m}_e/Z_e)]_s \simeq \mathrm{HH}(\mathrm{QC}(\mathcal{B}^{\mathbb{G}_m}_e), s^{-1}_*) \simeq \mathrm{HH}(\mathrm{QC}(\mathcal{B}^{\mathbb{G}_m}_e), s^*) \simeq \mathcal{O}(\mathcal{L}_s(\mathcal{B}^{\mathbb{G}_m}_e)) \simeq \mathcal{O}(\mathcal{L}(\mathcal{B}^{\mathbb{G}_m,s}_e)).$$

Likewise, for any $q \in \mathbb{G}_m$, we have

$$[\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e)]_{(s,q)} \simeq \mathrm{HH}(\mathrm{QC}(\widetilde{S}_e), (s,q)^*) \simeq \mathcal{O}(\mathcal{L}_{(s,q)}(\widetilde{S}_e)) \simeq \mathcal{O}(\mathcal{L}(\widetilde{S}^{(s,q)}_e)).$$

But for q suitably generic, we have $\widetilde{S}^{(s,q)}_e \simeq \mathcal{B}^{\mathbb{G}_m,s}_e$, hence by (6.3.3.6), we have

$$\dim_k \mathcal{S}^{\mathbf{B}_e}_s = [\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e)]_{0,(s,q)} \le \dim_k[\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e)]_{(s,q)} = \dim_k[\mathrm{QC}(\mathcal{B}^{\mathbb{G}_m}_e/Z_e)]_s.$$

Since the latter is a direct summand of the former, the opposite inequality is clear, and we are done. ☐

6.3.4. In particular, combining (6.3.3.2), (6.3.3.6), and (5.3.7.2), we have constructed canonical isomorphisms

$$(6.3.4.1)\qquad [\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)] \simeq \mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m} \simeq [\mathcal{J}^{\mathrm{mod}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)]$$

for each weight-vector $\underline{w}$ (under the identification (6.3.3.4)). Theorem 3.1.11 then gives corresponding algebra isomorphisms

$$(6.3.4.2)\qquad \mathrm{HH}(\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)) \simeq \mathrm{End}_{\widetilde{Z}_e/\widetilde{Z}_e}([\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)]) \simeq \mathcal{J}_{e,k}[v^{\pm1}].$$

Note that for any two weight-vectors $\underline{w}, \underline{w}'$, the corresponding isomorphisms differ by a canonical inner automorphism of $\mathcal{J}_{e,k}[v^{\pm1}]$. Indeed, the composite isomorphism

$$\mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m} \xrightarrow[\sim]{[\mathrm{HL}_{\underline{w}'}]^{-1}} [\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)] \xrightarrow[\sim]{[\mathrm{HL}_{\underline{w}}]} \mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m}$$

gives an element of $\mathrm{Aut}_{\widetilde{Z}_e/\widetilde{Z}_e}(\mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m})^{\mathrm{op}}$, i.e., a unit in $\mathcal{J}_{e,k}[v^{\pm 1}]$. Conjugating the isomorphism (6.3.4.2) for $\underline{w}$ by this unit then recovers the isomorphism for $\underline{w}'$.

Combining this discussion with the results of Chapter 5 now yields our main theorem:

6.3.5. THEOREM. *For each* $\underline{w} \in \mathrm{BB}_{\widetilde{S}_e/\widetilde{Z}_e}$, *we have a commutative diagram of classical algebra homomorphisms*

$$\begin{tikzcd}
\mathcal{H}_k \arrow[rr, "\phi_{e,k}"] & & \mathcal{J}_{e,k}[v^{\pm 1}] \\
\mathrm{HH}(\mathcal{H}^{\mathrm{coh}}) \arrow[u, "(3.3.16.2)", "\wr"'] \arrow[rr, "\mathrm{HH}(F_{\Phi^{\mathrm{coh}}_{S_e,\underline{w}}} \circ i^*_{S_e})"] & & \mathrm{HH}(\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)) \arrow[u, "\wr", "(6.3.4.2)"'] \\
K_0(\mathcal{H}^{\mathrm{coh}})_k \arrow[u, "\mathrm{ch}", "\wr"'] \arrow[rr, "K_0(F_{\Phi^{\mathrm{coh}}_{S_e,\underline{w}}} \circ i^*_{S_e})_k"] & & K_0(\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m))_k. \arrow[u, "\wr", "\mathrm{ch}"']
\end{tikzcd}$$

Moreover, for any two weight-vectors $\underline{w}, \underline{w}'$, *the homomorphisms* $K_0(F_{\Phi^{\mathrm{coh}}_{S_e,\underline{w}}} \circ i^*_{S_e})_k$ *and* $K_0(F_{\Phi^{\mathrm{coh}}_{S_e,\underline{w}'}} \circ i^*_{S_e})_k$ *differ by a canonical inner automorphism in* $K_0(\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m))_k$.

PROOF. First note that as in Lemma 5.3.12, we have a canonical identification $\mathrm{Tr}(\mathcal{M}^{\mathrm{coh}}_{S_e}) \simeq \mathcal{L}p_{S_e,*}$. Indeed, the argument is identical; we need only replace all categories with their coherent counterparts (i.e., $\mathcal{A}^{\mathrm{op}}_{S_e}\text{-mod}^{\widetilde{Z}_e}$ by $\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e)$, $\mathcal{A}_{S_e,0}\text{-mod}^{\widetilde{Z}_e}$ by $\mathrm{QC}(\mathcal{B}^{\mathbb{G}_m}_e/\widetilde{Z}_e)$, etc.), and the equivalences (5.3.6.3) and (5.3.8.1) with (6.3.3.3) and (6.3.2.1), respectively. The isomorphisms (6.3.4.1) and (5.3.14.2) now identify the functor

$$F^{\mathrm{Tr}}_{\Phi^{\mathrm{coh}}_{S_e,\underline{w}} \circ i^*_{S_e}} \simeq \mathrm{pr}_{[\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)]} \circ \mathrm{Tr}(\mathcal{M}^{\mathrm{coh}}_{S_e}) \circ \mathrm{Tr}(i^*_{S_e})$$

with $F^{\mathrm{Tr}}_{\Phi^{\mathrm{mod}}_{S_e} \circ i^*_{S_e}}$, so commutativity of the upper square follows from Corollary 5.3.14 and Proposition 5.2.9.

It remains to show that the Chern character for $\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)$ is an isomorphism (note that it is monoidal by §5.2.8). First, as in §6.1.17 and §6.3.2, we have a functor

$$\mathrm{HL}_{\underline{w}} \colon \mathrm{QC}(\mathcal{B}^{\mathbb{G}_m}_e \times \mathcal{B}^{\mathbb{G}_m}_e/\widetilde{Z}_e) \to \mathrm{QC}(\widetilde{S}_e \times \widetilde{S}_e/\widetilde{Z}_e)$$

associated to the Białynicki-Birula stratification of $\widetilde{S}_e \times \widetilde{S}_e$ obtained from the diagonal $\mathbb{G}_m$-action. This yields a commutative diagram

$$\begin{tikzcd}
K_0(\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m))_k \arrow[r, "\mathrm{ch}"] \arrow[d, "\wr"', "K_0(\mathrm{HL}_{\underline{w}})_k"] & \mathrm{HH}(\mathcal{J}^{\mathrm{coh}}_e \otimes \mathrm{Rep}(\mathbb{G}_m)) \arrow[d, "\wr"', "\mathrm{HH}(\mathrm{HL}_{\underline{w}})"] \\
K_0(\mathrm{QC}(\widetilde{S}_e \times \widetilde{S}_e/\widetilde{Z}_e))_k \arrow[r, "\mathrm{ch}"] & \mathrm{HH}(\mathrm{QC}(\widetilde{S}_e \times \widetilde{S}_e/\widetilde{Z}_e))
\end{tikzcd} \tag{6.3.5.1}$$

where both vertical maps are isomorphisms. It therefore suffices to show that the Chern character is an isomorphism for $\mathrm{QC}(\widetilde{S}_e \times \widetilde{S}_e/\widetilde{Z}_e)$.

Next, as in (2.2.4.1), the noncommutative Springer resolution gives an equivalence

$$\mathrm{Hom}(\mathcal{E}|_{\widetilde{S}_e} \boxtimes \mathcal{E}|_{\widetilde{S}_e}, -) \colon \mathrm{QC}(\widetilde{S}_e \times \widetilde{S}_e/\widetilde{Z}_e) \xrightarrow{\sim} \mathcal{A}^{\mathrm{op}}_{S_e} \otimes_k \mathcal{A}^{\mathrm{op}}_{S_e}\text{-mod}^{\widetilde{Z}_e}.$$

Note that the functor

$$- \underset{\mathcal{A}_{S_e,0} \underset{k}{\otimes} \mathcal{A}_{S_e,0}}{\otimes} \mathcal{A}_{S_e} \underset{k}{\otimes} \mathcal{A}_{S_e} \colon \mathcal{A}^{\mathrm{op}}_{S_e,0} \otimes_k \mathcal{A}^{\mathrm{op}}_{S_e,0}\text{-mod}^{\widetilde{Z}_e} \to \mathcal{A}^{\mathrm{op}}_{S_e} \otimes_k \mathcal{A}^{\mathrm{op}}_{S_e}\text{-mod}^{\widetilde{Z}_e}$$

gives a bijection between isomorphism classes of indecomposable finitely generated projective modules as in (2.2.14.3), hence induces an equivalence on K_0. Moreover, this functor induces an equivalence on Hochschild homology as in the proof of Lemma 5.3.8. Thus, as in (6.3.5.1), we reduce to showing that the Chern character is an isomorphism for $\mathcal{A}^{\mathrm{op}}_{S_e,0} \otimes_k \mathcal{A}^{\mathrm{op}}_{S_e,0}\text{-mod}^{\widetilde{Z}_e}$. Since the latter is a semisimple category as in Corollary 5.3.4, this is clear. □

6.3.6. Finally, specializing the discussion of §6.3.4 and the Chern character isomorphism (6.3.5.1) to any point of $\mathbb{G}_m$ yields a linearized version of Qiu–Xi's conjecture (1.3.0.2):

6.3.7. Corollary. For each $\underline{w} \in \mathrm{BB}_{\widetilde{S}_e/\widetilde{Z}_e}$ and $q \in \mathbb{G}_m$, there is an isomorphism

$$[\mathcal{J}_e^{\mathrm{coh}}] \simeq \mathcal{S}^{\mathbf{B}_e}. \tag{6.3.7.1}$$

This gives classical algebra isomorphisms

$$\mathcal{J}_{e,k} \simeq \mathrm{HH}(\mathcal{B}_e^{\mathbb{G}_m} \times \mathcal{B}_e^{\mathbb{G}_m}/Z_e) \simeq K_0(\mathcal{B}_e^{\mathbb{G}_m} \times \mathcal{B}_e^{\mathbb{G}_m}/Z_e)_k,$$

where the former depends on the pair $(\underline{w}, q)$, and the latter is given by the Chern character. Moreover, the isomorphisms obtained from two pairs $(\underline{w}, q)$ and $(\underline{w}', q')$ differ by a canonical inner automorphism.

6.3.8. We now make a series of remarks on this corollary:

6.3.9. Remark. Letting $\pi_e^{\mathbb{G}_m} : \mathcal{B}_e^{\mathbb{G}_m}/Z_e \to \mathrm{B}Z_e$ denote the projection, (3.3.8.1) yields a (non-canonical) expression

$$\mathcal{L}\pi_{e,*}^{\mathbb{G}_m}\mathcal{O}_{\mathcal{L}(\mathcal{B}_e^{\mathbb{G}_m}/Z_e)} \simeq \mathcal{S}^{\mathbf{B}_e}$$

for the asymptotic coherent Springer sheaf. This exactly parallels the definition (3.3.16.1) of the coherent Springer sheaf. Philosophically, we may regard the stack $\bigsqcup_e \mathcal{L}(\mathcal{B}_e^{\mathbb{G}_m}/Z_e)$ as an "asymptotic" analogue of $\mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G})$, just as $\bigsqcup_e Z_e/Z_e$ is an "asymptotic" analogue of the stack $\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})$ of Deligne–Langlands parameters.

6.3.10. Remark. Let us explain in more detail why the isomorphism (6.3.7.1) depends on the choice of $q \in \mathbb{G}_m$. By Proposition 3.4.16 and (6.3.3.5), we have a commutative diagram

$$\begin{tikzcd}
\mathsf{BG}_{*/\widetilde{Z}_e}(\mathrm{QC}(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e)) \arrow[r, "\mathsf{BG}_{*/\widetilde{Z}_e}(\mathrm{HL}^{\mathrm{gr}}_{\underline{w}})"] \arrow[d, "\wr"] & \mathsf{BG}_{*/\widetilde{Z}_e}(\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e))_0 \arrow[d, "\wr"] \\
{[\mathrm{QC}(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e)]} \arrow[r, "{[\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}]}"] \arrow[d, "\wr"] & {[\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e)]_0} \arrow[d, "\wr"] \\
{[\mathcal{J}_e^{\mathrm{coh}}] \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m}} \arrow[r, "(6.3.4.1)"] & \mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m}.
\end{tikzcd} \tag{6.3.10.1}$$

We wish to understand where in this diagram the "dependence on the choice of specialization" arises.

On the one hand, by Remark 3.4.6 and the proof of Lemma 6.2.6, we may compute each of these Block–Getzler sheaves using only the compact objects of $\mathrm{QC}(\mathcal{B}_e^{\mathbb{G}_m}/\widetilde{Z}_e)_0$ and their images under $\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}$, respectively. By Lemma 6.2.8, all $\mathrm{Rep}(\widetilde{Z}_e)$-enriched Hom-spaces between these images are concentrated in non-negative weights, so we may compute $\mathsf{BG}_{*/\widetilde{Z}_e}(\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e))_0$ using only the weight-0 components of these Hom-spaces. Note that the coaction map for $\mathbb{G}_m$ is trivial on weight-0 representations; thus, the sheaf $\mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m}$ canonically splits off as a factor of each of the Block–Getzler sheaves in (6.3.10.1), and the map $\mathrm{id}_{\mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m}}$ canonically splits off as a factor of $\mathsf{BG}_{*/\widetilde{Z}_e}(\mathrm{HL}^{\mathrm{gr}}_{\underline{w}})$.

On the other hand, we may compute the Block–Getzler sheaf $\mathsf{BG}_{*/\widetilde{Z}_e}(\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e))$ using only the Bezrukavnikov–Mirković tilting bundle $\mathcal{E}|_{\widetilde{S}_e}$. As we have seen, the resulting complex satisfies a similar non-negativity property, so the sheaf $\mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m}$ again canonically splits off as a factor of its weight-0 component. In general, however, these two decompositions of $\mathsf{BG}_{*/\widetilde{Z}_e}(\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e))_0$ do not agree! Intuitively, this is because we have chosen two distinct generating subcategories of $\mathrm{QC}(\widetilde{S}_e/\widetilde{Z}_e)$. In particular, the decomposition coming from $\mathrm{HL}^{\mathrm{gr}}_{\underline{w}}$ agrees with that of $[\mathcal{J}_e^{\mathrm{coh}}] \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m}$, and the decomposition coming from $\mathcal{E}|_{\widetilde{S}_e}$ agrees with that of $\mathcal{S}^{\mathbf{B}_e} \boxtimes \mathcal{O}_{\mathbb{G}_m/\mathbb{G}_m}$.

6.3.11. Remark. When e is subregular, the isomorphism (6.3.7.1) (and its dependence on q) may be computed explicitly by combining the derived McKay correspondence of [**BKR01**] with the description of $\mathcal{A}_{S_e}$ attached to the Kleinian singularity as in [**Bez06**, Ex. 2.9]. We omit the details.

6.3.12. Remark. When $e = 0$, the stack $\widetilde{S}_e/\widetilde{Z}_e = \widetilde{\mathcal{N}}/\widetilde{G}$ is covered by a single Θ-stratum. It follows that the functors $\mathrm{HL}^{\mathrm{gr}}_w, \check{\mathrm{HL}}^{\mathrm{gr}}_w$ of §6.3.2 are given by $p^*_{\widetilde{\mathcal{N}}}\langle -w\rangle, i^*_{\mathcal{B}}\langle w\rangle$, respectively, where $w \in \mathbb{Z}$ corresponds to the unique connected component of $\mathcal{B}$, and the maps $p_{\widetilde{\mathcal{N}}}, i_{\mathcal{B}}$ are as in (1.3.0.1). In particular, the functor $F_{\Phi^{\mathrm{coh}}_{\mathfrak{g},w}} : \mathcal{H}^{\mathrm{coh}} \to \mathcal{J}_e^{\mathrm{coh}}$ does not depend on the choice of w. Moreover, a straightforward computation shows that $F_{\Phi^{\mathrm{coh}}_{\mathfrak{g},w}}$ recovers Chriss–Ginzburg's functor $(\mathrm{id}_{\mathcal{B}} \times i_{\mathcal{B}})^*(p_{\widetilde{\mathcal{N}}} \times \mathrm{id}_{\widetilde{\mathcal{N}}})_*$, discussed in §1.3. Note that this differs from the functor $(i_{\mathcal{B}} \times \mathrm{id}_{\mathcal{B}})^*(\mathrm{id}_{\widetilde{\mathcal{N}}} \times p_{\widetilde{\mathcal{N}}})_*$ appearing in [**Xi16**]; it follows that the two are related by an inner automorphism in $K_0(\mathcal{B} \times \mathcal{B}/\widetilde{G})$ (in fact, for $G = \mathrm{SL}_2$, the two homomorphisms on K_0 agree by the proof of [**CG10**, Thm 7.5.12] and [**Daw21**, Lem. 7]).

6.3.13. We now explain how Corollary 6.3.7 can be used to compute the sheaf $\mathcal{S}^{\mathbf{B}_e}$ from the geometry of $\mathcal{B}_e^{\mathbb{G}_m}$. We begin with a general lemma computing the Block–Getzler sheaf for a category equipped with an "equivariant" full exceptional collection:

6.3.14. Proposition. Let H be a reductive group, and let $\mathcal{C}$ be a $\mathrm{Rep}(H)$-module category. Fix a totally ordered H-set I, and let $\langle c_x \rangle_{x \in I}$ be a full exceptional collection in the de-equivariantization $\mathcal{C}^{\mathrm{dq},c}$. Suppose that

1. the total order on I descends to the set-theoretic quotient I/H;
2. for any $h \in H$, there exists an isomorphism $h_* c_x \simeq c_{h \cdot x}$ (see §3.4.4 for this notation); and
3. each c_x admits a $(\mathcal{C}_x, \alpha_x)$-equivariant structure for some $(\mathcal{C}_x, \alpha_x) \in \mathrm{Coc}(H^x)$, i.e., a lift to the twist $(\mathrm{Rep}(H^x) \otimes_{\mathrm{Rep}(H)} \mathcal{C})^{(\mathcal{C}_x, \alpha_x)}$.

Let $I^{\mathrm{orb}} \subset I$ denote the set of minimal orbit representatives for the H-action. Then there is a canonical isomorphism

$$\mathsf{BG}_{*/H}(\mathcal{C}) \simeq \bigoplus_{x \in I^{\mathrm{orb}}} i^{\mathrm{eq}}_{x,*} \mathcal{C}_x^\vee,$$

where $i^{\mathrm{eq}}_x \colon H^x/H^x \to H/H$ denotes the natural map of adjoint quotient stacks.

Proof. Choose a nonzero $V_x \in \mathrm{Rep}(H^x)^{(\mathcal{C}_x, \alpha_x), c}$ for each $x \in I^{\mathrm{orb}}$; our third assumption then yields compact objects

$$V_x^* \otimes c_x \in \mathrm{Rep}(H^x) \otimes_{\mathrm{Rep}(H)} \mathcal{C}.$$

Consider the natural $\mathrm{Rep}(H)$-linear, compact object-preserving adjoint pair

$$\mathrm{Res}^H_{H^x} \colon \mathcal{C} \rightleftarrows \mathrm{Rep}(H^x) \otimes_{\mathrm{Rep}(H)} \mathcal{C} \colon \mathrm{Ind}^H_{H^x}.$$

We claim that the objects $\mathrm{Ind}^H_{H^x}(V_x^* \otimes c_x)$ generate $\mathcal{C}$ under the $\mathrm{Rep}(H)$-action. Indeed, it suffices to show that the objects $\mathrm{Ind}^H_{H^x}(V_x^* \otimes c_x)^{\mathrm{dq}}$ generate $\mathcal{C}^{\mathrm{dq}}$. A standard base-change argument then shows that

$$\mathrm{Ind}^H_{H^x}(V_x^* \otimes c_x)^{\mathrm{dq}} \simeq \bigoplus_{[h] \in H/H^x} V_x^{*,\mathrm{dq}} \otimes h_* c_x \tag{6.3.14.1}$$

for a choice of coset representatives of H/H^x, so the claim follows from our second assumption. Now, recall from the proof of Lemma 3.4.5 that the $\mathrm{Rep}(H)$-internal Hom for $\mathcal{C}$ identifies with the Hom in $\mathcal{C}^{\mathrm{dq}}$. Using (6.3.14.1) and our first assumption, we immediately obtain a canonical decomposition

$$\mathsf{BG}_{*/H}(\mathcal{C}) \simeq \bigoplus_{x \in I^{\mathrm{orb}}} \mathsf{BG}_{*/H}(\langle \mathrm{Ind}^H_{H^x}(V_x^* \otimes c_x) \rangle) \simeq \bigoplus_{x \in I^{\mathrm{orb}}} \mathsf{BG}_{*/H}\left(\mathrm{End}_{\mathcal{C}^{\mathrm{dq}}}(\mathrm{Ind}^H_{H^x}(V_x^* \otimes c_x)^{\mathrm{dq}})^{\mathrm{op}}\text{-mod}^H \right).$$

Next, suppose $[h], [h'] \in H/H^x$ are distinct coset representatives. Then

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{dq}}}(h'_* c_x, h_* c_x) \simeq \mathrm{Hom}_{\mathcal{C}^{\mathrm{dq}}}(h_*^{-1} h'_* c_x, c_x) \simeq \mathrm{Hom}_{\mathcal{C}^{\mathrm{dq}}}(c_{h^{-1}h' \cdot x}, c_x) \simeq 0$$

by our second assumption and minimality of x. It follows from (6.3.14.1) that

$$\mathrm{End}_{\mathcal{C}^{\mathrm{dq}}}(\mathrm{Ind}^H_{H^x}(V_x^* \otimes c_x)^{\mathrm{dq}})^{\mathrm{op}}\text{-mod}^H \simeq \mathrm{End}_k(V_x^*)^{\oplus H/H^x, \mathrm{op}}\text{-mod}^H \simeq \mathrm{End}_k(V_x^*)^{\mathrm{op}}\text{-mod}^{H^x} \simeq \mathrm{Rep}(H^x)^{(\mathcal{C}_x, \alpha_x)^\vee}$$

as in the proof of Lemma 5.1.3. Moreover, Proposition 3.4.11 and Corollary 3.2.13 now give

$$\mathsf{BG}_{*/H}(\mathrm{Rep}(H^x)^{(\mathcal{C}_x, \alpha_x)^\vee}) \simeq i^{\mathrm{eq}}_{x,*} \mathsf{BG}_{*/H_x}(\mathrm{Rep}(H^x)^{(\mathcal{C}_x, \alpha_x)^\vee}) \simeq \bigoplus_{x \in I^{\mathrm{orb}}} i^{\mathrm{eq}}_{x,*} \mathcal{C}_x^\vee,$$

hence the conclusion. □

6.3.15. Example. Let $G = \mathrm{Sp}_6$, and take e be the nilpotent corresponding to the partition $2+2+2$. We will use Proposition 6.3.14 and Corollary 6.3.7 to compute the sheaf $\mathcal{S}^{\mathbf{B}_e}$, recovering the direct combinatorial computation of [**BDD22**, Thm. 2]. One computes that

$$Z_e \cong \mathrm{O}_3 \cong \mathrm{PGL}_2 \times \{\pm 1\}.$$

In particular, Z_e is connected modulo the center of G, so its action on $\mathbf{B}_e$ is trivial, and its action on $\mathcal{B}_e$ factors through $\mathrm{SO}_3 \cong \mathrm{PGL}_2$. Moreover, a computation shows that the variety $\mathcal{B}_e^{\mathbb{G}_m}$ has three connected components (see Proposition 1 of *loc. cit.* for a description of the entire Springer fiber $\mathcal{B}_e$):

(1) a component isomorphic to the flag variety $\mathcal{B}_{\mathrm{SL}_3}$ of SL_3, on which Z_e acts via the embedding $\mathrm{SO}_3 \hookrightarrow \mathrm{SL}_3$;
(2) a component isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, on which Z_e acts via the diagonal action of PGL_2; and
(3) a component isomorphic to $\mathbb{P}^1$, on which Z_e acts via the usual action of PGL_2.

Note that by Proposition 2.1.16 and Lemma 2.1.21, the Schur multiplier of Z_e has order 2; let us write $(\mathcal{C}, \alpha)$ for the nontrivial cocycle (corresponding to the "odd" representations of SL_2). We now describe full exceptional collections on each of the three components of $\mathcal{B}_e^{\mathbb{G}_m}$:

(1) The $\mathbb{P}^1$-fibration $\mathcal{B}_{\mathrm{SL}_3} \to \mathbb{P}^2$ yields a full exceptional collection on $\mathcal{B}_{\mathrm{SL}_3}$ consisting of 6 line bundles. Each of these line bundles is SL_3-equivariant, hence in particular Z_e-equivariant.
(2) The variety $\mathbb{P}^1 \times \mathbb{P}^1$ admits a full exceptional collection given by
$$\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1).$$
The first and last line bundles are Z_e-equivariant, whereas the second and third are $(\mathcal{C}, \alpha)$-equivariant.
(3) The variety $\mathbb{P}^1$ admits a full exceptional collection given by $\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)$. The former is Z_e-equivariant, whereas the latter is $(\mathcal{C}, \alpha)$-equivariant.

Altogether, we obtain a full exceptional collection on $\mathcal{B}_e^{\mathbb{G}_m}$ consisting of 9 sheaves, 6 of which are Z_e-equivariant, and 3 of which are $(\mathcal{C}, \alpha)$-equivariant. It follows as in (6.3.10.1) that
$$\mathcal{S}^{\mathbf{B}_e} \simeq \mathsf{BG}_{*/Z_e}(\mathrm{QC}(\mathcal{B}_e^{\mathbb{G}_m}/Z_e)) \simeq \mathcal{O}_{Z_e/Z_e}^{\oplus 6} \oplus \mathcal{C}^{\oplus 3},$$
agreeing with the numerics computed in Theorem 2 of *loc. cit.* In particular, this example shows that nontrivial cocycles may appear in $\mathbf{B}_e$ even when the group G is simply connected.

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