Double affine galleries

by

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ABSTRACT

We develop a theory of galleries for double affine hyperplane arrangements. A gallery is an infinite sequence of chambers, indexed by an ordered set, which is maximal with respect to a finiteness condition on the multiset of wall-crossings.

We study the possible order types of galleries. We also use galleries to define a double affine Bruhat order which generalizes the one introduced by Braverman, Kazhdan, and Patnaik, and studied by Muthiah and Orr. We prove an analogue of the classical characterization of the Bruhat order in terms of subexpressions of reduced expressions, and we define an analogue of the Demazure product.

We also study tours, which are certain finite sequences of chambers. Using the previous results, we show that tours form a category which behaves similarly to the category of generalized galleries defined in the classical setting. We construct a functor from tours to schemes, whose image consists of double affine analogues of Demazure varieties. We show that the colimit of this functor recovers the double affine flag variety at the level of sets, but we do not think that the colimit of schemes is well-behaved. Instead, we describe a different way of equipping the colimit set with a ringed space structure, and we conjecture that this ringed space is a scheme.

Our main result is that the category of tours (with fixed start and end chambers, and subject to certain constraints) is contractible. We call this result 'homotopical deletion' because it generalizes the Coxeter deletion lemma.

Thesis supervisor: Roman Bezrukavnikov
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I dedicate this work to my father
who drew circles and lines with me
in an old dairy shop.

The limits of the instrument,
which belonged to my grandfather,
are proved by the same subject
where I have found my own.

The shop and its columns are gone.
Innovation and decay
cannot take the pride from his eyes
nor take what he has given to me.

Let fire burn, machines erase
every true beloved place.
Our circles stand untouched by time:
there is no world as good as mine.

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It seems larger because of its dimness, its symmetry, its mirrors, its age, my unfamiliarity with it, and this solitude.

Jorge Luis Borges, *Death and the Compass*

Contents

List of Figures

1. Introduction

1.1. **Classical galleries.** In the study of Kac–Moody groups, *galleries* are combinatorial objects which are useful for answering group-theoretic and topological questions. The goal of this thesis is to develop a theory of galleries for double loop groups.

In this subsection, we review the classical results which we seek to generalize. For simplicity, let us consider an affine-type Kac–Moody group G^{aff}. There is an affine hyperplane arrangement $\mathcal{H}^{\mathsf{aff}}$ defined by the roots of G^{aff}. Its *chambers* correspond to choices of sets of positive roots, hence to (positive) Borel subgroups containing the torus. A *gallery* is a sequence of chambers

$$([n], C) = (C_0, C_1, \dots, C_n),$$

such that C_{i-1}, C_i are adjacent (and not equal) for all $i \in [1, n]$.

Let W^{aff} be the affine Weyl group associated to G^{aff}, and let I be a set of simple reflections. This choice equips W^{aff} with a *length* function $\ell(-)$. The *(strong) Bruhat order* is a partial order on W^{aff}, graded by $\ell(-)$, which admits two equivalent definitions:

(1) It is the transitive closure of the relation defined by $w \prec rw$ whenever r is a reflection and $\ell(w) < \ell(rw)$.

(2) It is defined by $u \preceq w$ for all u, w such that, for some (equiv. any) reduced expression $w = s_1 \cdots s_n$ of simple reflections, we have $u = s_{i_1} \cdots s_{i_m}$ for some $i_1 < \cdots < i_m$.

Both of these definitions can be interpreted using chambers and galleries. Let C_0 be the 'fundamental' chamber corresponding to the set I, and associate each group element $w \in W^{\mathsf{aff}}$ with the chamber wC_0. In (1), we have $w \prec rw$ if and only if the wall of the reflection r does not separate wC_0 from C_0. In (2), the choice of an expression $w = s_1 \cdots s_n$ is equivalent to the choice of a gallery from C_0 to wC_0, and the expression is reduced if and only if the gallery does not double-cross any wall. A subexpression $s_{i_1} \cdots s_{i_m}$ corresponds to a *folding* of the gallery, where one 'fold' occurs for each omitted simple reflection.

Galleries are also useful for studying G^{aff}. A gallery $([n], C)$ specifies a Demazure variety

$$X([n], C) := P_{C_0 \wedge C_1} \overset{B_{C_1}}{\times} P_{C_1 \wedge C_2} \overset{B_{C_2}}{\times} \cdots \overset{B_{C_{n-1}}}{\times} P_{C_{n-1} \wedge C_n} / B_{C_n}$$

where $C_{i-1} \wedge C_i$ is the face of $\mathcal{H}^{\mathsf{aff}}$ along which C_{i-1}, C_i are adjacent, and each face F gives a (positive) parabolic subgroup P_F whose roots are $\{\alpha \mid \alpha(F) \geq 0\}$. (When F is a chamber, this is a Borel subgroup.) If $wC_0 = C_n$, then we get a resolution of the w Schubert variety:

$$\begin{aligned} X([n], C) &\longrightarrow G^{\mathsf{aff}} / B_{C_0} \\ (p_1, \dots, p_n) &\longmapsto p_1 \cdots p_n w \, B_{C_0} \end{aligned}$$

As observed by Contou-Carrère, we can view the points of $X([n], C)$ as galleries in the Tits building of G^{aff} which begin at C_0 and are shaped like $([n], C)$. Under this interpretation, the above map sends each Tits gallery to its end chamber.

Finally, one can define a category of generalized galleries. This is useful for topological questions because a gallery is a combinatorial analogue of a path, so a category of galleries behaves like a path space. A *generalized gallery* is a sequence of chambers $([n], C)$ such that C_{i-1}, C_i are touching (i.e. $C_{i-1} \wedge C_i$ is nonempty). In [TaTr], we defined a category of generalized galleries, denoted $\mathsf{Rig}^{\mathsf{d}}$, whose morphisms encode deletions of chambers and unfolding moves. This category is designed so that $X(-)$ enhances to a functor from $\mathsf{Rig}^{\mathsf{d}}$ to varieties. Finally, the identification

$$\operatorname*{colim}_{([n],C)\in\mathsf{Rig}^{\mathsf{d}}} X([n],C) \simeq G^{\mathsf{aff}}/B_{C_0}$$

is valid in various senses (schemes, constructible sheaves).

This thesis will generalize all of the above constructions and results to double loop groups, except for the last sentence. This means that we were not able to construct the double affine flag variety or its category of constructible sheaves. However, we will show that one version of the double affine flag variety is uniquely determined, see 1.5.

1.2. **Double affine galleries.** We work with the double affine group $G := G^{\mathsf{aff}}(\mathbb{C}(\!(t)\!))$. Let $T^{\mathsf{aff}} \subset G^{\mathsf{aff}}$ be a maximal torus, and consider the action $T^{\mathsf{aff}} \times \mathbb{G}_m \curvearrowright G$ where T^{aff} acts by conjugation and $\mathbb{G}_m$ acts via loop rotation on $\mathbb{C}(\!(t)\!)$. Then the roots of G are given by the linear functions $\alpha + m\pi$ on the real Lie algebra $\mathfrak{h}^{\mathsf{aff}} \oplus \mathbb{R}$, where $\alpha \in \mathfrak{h}^{\mathsf{aff},\vee}$ is any root of $T^{\mathsf{aff}} \curvearrowright G^{\mathsf{aff}}$, and $\pi : \mathfrak{h}^{\mathsf{aff}} \oplus \mathbb{R} \to \mathbb{R}$ is the projection.

It is natural to replace the vector space $\mathfrak{h}^{\mathsf{aff}} \oplus \mathbb{R}$ by the affine subspace $\mathfrak{h}^{\mathsf{aff}} \times \{1\}$, since an analogous step is used to define the affine hyperplane arrangement of G^{aff}. Then $\alpha + m\pi$ restricts to the affine function $\alpha + m$. These affine functions define an arrangement of (affine) hyperplanes in $\mathfrak{h}^{\mathsf{aff}}$, denoted $\mathcal{H}$. In other words, $\mathcal{H}$ is obtained from the Kac–Moody hyperplane arrangement for G^{aff} by taking all integer translates of all hyperplanes. The union of the hyperplanes is dense in $\mathfrak{h}^{\mathsf{aff}}$.

Gaussent and Rousseau [GR] showed that G is governed by $\mathcal{H}$ in the following sense: they created a generalization of the notion of a Tits building, called a *hovel* or *masure*, and G gives rise to a hovel whose apartments look like $\mathcal{H}$.

We will define chambers and galleries in $\mathcal{H}$. A *chamber* is a compatible system of chambers in the locally finite subarrangements of $\mathcal{H}$. (This definition can be extracted from [GR, 2.2]. Note that it equips the set of chambers with an ‘inverse limit’ topology.) A *gallery* is a sequence of chambers $(c_i)_{i\in I}$, indexed by an ordered set I, such that the following hold:

- The index set I is nonempty and bounded, i.e. it has a maximum and a minimum.
- It is non-stuttering, i.e. it is non-constant on each interval of I of size > 1.
- It crosses each wall only a finite number of times, and it crosses only finitely many walls more than once.
- It is maximal under refinement, i.e. it is impossible to insert a new chamber without increasing the multiset of walls which are crossed.

We will prove the following:

- Proposition 3.1.7: a chamber sequence $(c_i)_{i\in I}$ is a gallery if and only if I is complete, c is continuous, and c sends each pair of consecutive elements of I to a pair of adjacent chambers of $\mathcal{H}$.
- 3.4.6: If $(c_i)_{i\in I}$ is a *positive* gallery, then exactly one of the following holds:

– I is isomorphic to an ordered set of the form

$$\underbrace{\mathbb{Z}_{\geq 0} \sqcup \cdots \sqcup \mathbb{Z}_{\geq 0}}_{<\dim A \text{ copies}} \sqcup S \sqcup \underbrace{\mathbb{Z}_{\leq 0} \sqcup \cdots \sqcup \mathbb{Z}_{\leq 0}}_{<\dim A \text{ copies}},$$

where S is finite and nonempty.

– I can be obtained from the Cantor set as follows: replace each gap by an ordered set of the above form. Also, replace the maximal and minimal elements by an ordered set of the above form.

In 3.4.6, the 'positivity' condition is a requirement on the start and end chambers of the gallery (denoted $c_{\hat{0}}, c_{\hat{1}}$) which is slightly weaker than the requirement that $c_{\hat{1}}$ lies at higher level than $c_{\hat{0}}$, when viewed as subsets of $\mathfrak{h}^{\mathsf{aff}}$. (As usual, the *level* is the coordinate on $\mathfrak{h}^{\mathsf{aff}}$ which corresponds to the loop-rotation $\mathbb{G}_m$ in the Kac–Moody group G^{aff}.)

From here on, we implicitly impose positivity. This corresponds to a restriction, which is standard in the literature, of working inside a certain 'positive' subsemigroup $G_+ \subset G$, see [BKP, 1.2.2] and [M1, §1]. As mentioned there, G_+ has a Bruhat decomposition but G does not. For this reason, it is the 'positive' combinatorics which should be the most important for describing the double loop group.

Remark. If G^{aff} is an affine SL_2, then its Cartan algebra is $\mathfrak{h}^{\mathsf{aff}} = \mathbb{R}\alpha^{\mathsf{fin},\vee} \oplus \mathbb{R}d \oplus \mathbb{R}K$, where $\alpha^{\mathsf{fin},\vee}$ is a coroot of the finite-type SL_2. The coefficient of d is the level. The Kac–Moody arrangement consists of 2-dimensional subspaces $\{\mathsf{span}\langle n\alpha^{\mathsf{fin},\vee} + d, K\rangle \,|\, n \in \mathbb{Z}\}$, and $\mathcal{H}$ is obtained by translating this arrangement by all vectors $\mathbb{Z}\alpha^{\mathsf{fin},\vee}$. If we quotient by K, then $\mathcal{H}$ can be visualized as the set of lines in $\mathbb{R}^2$ with integer slope and y-intercept, after a 90-degree rotation. All of our results can be illuminated by visualizing them for this arrangement.

1.3. **Bruhat order.** By definition, the double affine Weyl group W is generated by orthogonal reflections through the walls of $\mathcal{H}$. In contrast to the classical setting, the action of W on the set of chambers of $\mathcal{H}$ is neither free nor transitive. Should we define the Bruhat order on W or on the set of chambers?

In fact, we can attain maximum generality by defining the Bruhat order on the set of *tethered chambers*: after fixing a chamber T, we say that a tethered chamber is a pair (C, w) such that $wT = C$. To define the Bruhat order, we fix another chamber C_0 and declare that $(C, w) \prec (rC, rw)$ for every (C, w) and every reflection $r \in W$ such that the reflection wall does not separate C_0 from C. Compare this with the classical definition 1.1(1).

A special case of this was defined in Appendix B of [BKP]. To recover their definition, take C_0 to be a particular chamber which touches $0 \in \mathfrak{h}^{\mathsf{aff}}$, and take $T = C_0 + nd$ for some integer $n > 0$. The following is known about the Bruhat order in this case:

- It is a partial order, i.e. there are no cycles among the given relations. [M1]
- It is $\mathbb{Z}$-graded by a *length* function, which is related to the *inversion set* of a chamber C, i.e. the (infinite) set of walls which separate C from C_0. [MO]
- Every element has finitely many covers (i.e. minimal elements lying strictly above it) and cocovers (i.e. maximal elements lying strictly below it). [W]

We generalize these results to arbitrary C_0, T as follows:

- Corollary 4.4.8: The first two bullets (above) hold in general.
- Corollary 4.5.8: The third bullet (above) holds when C_0 and T are rational-level.

We also prove the following statements relating the Bruhat order to galleries, cf. 1.1(2):

- Theorem 4.7.4: The characterization of the Bruhat order in terms of subexpressions of an expression generalizes to the current setting.
- Corollary 4.7.5: The characterization of the Demazure product of an expression as the Bruhat-maximal product of a subexpression generalizes to the current setting.

Recall that, in the classical setting, an 'expression' is a sequence of simple reflections. Because a gallery typically crosses infinitely many walls, we are led to consider infinite expressions, which are difficult to work with. It is better to work directly with galleries and foldings, and it turns out that one should restrict to foldings along *finitely* many walls.

Remark. If we try to define an analogous 'Bruhat order' on the set of (non-tethered) chambers, then we do not get a partial order, because there are nontrivial cycles in the relations. The subgroup of W which fixes one chamber (equiv. all chambers) is generated by an element τ_{nK}, corresponding to translation by nK, where $K \in \mathfrak{h}^{\mathsf{aff}}$ is the central vector. For most choices of C_0, T, and (C, w), one can show directly that $(C, w) \prec (C, \tau_{nK} w)$ in the Bruhat order on tethered chambers. This implies that $C \prec C$ in the Bruhat order on (non-tethered) chambers, so it cannot be a partial order.

1.4. **Demazure varieties.** In the double affine setting, is not possible to define an interesting category of galleries, essentially because one cannot change infinitely many chambers using finitely many chamber-deletion moves. Instead, in Section 5, we define a category of *finite tethered jointed tours*, which are triples

$$(c_0, \ldots, c_n), (f_1, \ldots, f_n), w$$

where c_i are chambers, f_i are faces, $f_i \preceq c_i$, and (c_n, w) is a tethered chamber. This category is denoted D and is the correct generalization of $\mathsf{Rig}^{\mathsf{d}}$ from the classical setting.

In Section 6, we construct a functor $X(-)$ sending D to certain quasicompact schemes which we call *sub-Demazure varieties*. This uses a similar formula as before, except the faces f_i are used to specify the relevant parabolic subgroups. (We can no longer use the meets $c_{i-1} \wedge c_i$ because they can be empty.) To construct a *Demazure variety*, we fix a gallery g and take the colimit of $X(-)$ along a sequence of increasingly-fine finite jointed tours which are refined by g, see 6.6. The colimit is filtered and occurs along open embeddings, so the result is a (non-quasicompact) scheme.

Thus, there are many interesting maps between sub-Demazure varieties, but no maps between Demazure varieties.

1.5. **Double affine flag variety.** The question which initially motivated this thesis is whether it is reasonable to define the double affine flag variety by

$$\mathcal{F}\ell := \operatorname*{colim}_{t \in \mathsf{D}} X(t).$$

Since we are focused on answering this question, we do not attempt to study G/B_{C_0} or any other proposed construction of $\mathcal{F}\ell$.

We show that the colimit definition makes sense at the level of sets, i.e. $\mathcal{F}\ell$ decomposes into the expected Schubert cells (Theorem 7.1.2).

Unfortunately, our method for computing the colimit does not give good results at the level of topological spaces or schemes. The issue is that maps between sub-Demazure varieties $X(t_1) \to X(t_2)$ are often non-proper, even if we require t_1, t_2 to be arbitrarily fine. A related issue is that $X(t) \to \mathcal{F}\ell$ can be non-proper, so we cannot control the topology of $\mathcal{F}\ell$ in a straightforward way. (In 7.3, we illustrate this by analyzing a special case where $\mathcal{F}\ell$ is already known: namely, if C_0 and T are touching chambers, then $\mathcal{F}\ell$ is the thick affine flag variety, which is a scheme.) This issue is why we are not able to construct the double affine flag variety or its category of constructible sheaves.

We do, however, give a conjectural construction which determines it uniquely. The idea is inspired by Mathieu's construction of Kac–Moody flag varieties [Ma1], [Ma2].

To explain this, let us temporarily return to the classical setting and describe the construction which we have in mind.

- For each gallery g, let $E(g)$ be the variant of $X(g)$ obtained by not performing the final quotient by B_{C_n}.
- For each $w \in W$, equip the set $\mathcal{F}\ell_{\preceq w}$ with a topology using the following basis of open sets: A subset $U \subseteq \mathcal{F}\ell_{\preceq w}$ belongs to the basis if and only if its preimages under the available maps $E(g) \to \mathcal{F}\ell_{\preceq w}$ are cut out by regular functions on the $E(g)$'s which are compatible under the maps between the $E(g)$'s. In other words, we require that there exists a system of regular functions
$$(a_g)_g \in \lim_{g \in \mathsf{Rig}^{\mathsf{d,op}}_{\preceq w}} \Gamma(E(g), \mathcal{O}_{E(g)})$$
such that, for each gallery g, we have
$$\pi_g^{-1}(U) = D(a_g)$$
as subsets of $E(g)$, where $\pi_g : E(g) \to \mathcal{F}\ell_{\preceq w}$ is an analogue of the Demazure map, and $D(a_g)$ is the locus where a_g is nonzero.
- Choose a reduced gallery g which ends at wC_0. We turn the topological space $\mathcal{F}\ell_{\preceq w}$ into a ringed space by equipping it with the sheaf of rings $\pi_{g,*}\mathcal{O}_{E(g)}$.

This gives the correct scheme structure on $\mathcal{F}\ell_{\preceq w}$.

Properness does not play a role in this construction. In fact, the construction remains correct if, in the last bullet, we replace $E(g)$ by any dense open union of cells which includes all of the cells which lie over the codimension-one strata $\mathcal{F}\ell_u \subset \mathcal{F}\ell_{\preceq w}$, i.e. $\ell(u) = \ell(w) - 1$. This follows from the normality of Schubert varieties, because functions on normal varieties extend across codimension-two subsets.

This, we believe it is reasonable to conjecture that an analogous construction works in the double affine setting, making $\mathcal{F}\ell_{\preceq w}$ a scheme and $\mathcal{F}\ell$ an ind-scheme (7.2). Moreover, we conjecture that the scheme structure on $\mathcal{F}\ell_{\preceq w}$ does not depend on the choice of the tour t (which replaces the gallery g in the third bullet), as long as $E(t)$ satisfies the 'codimension-one' condition from the previous paragraph. This conjecture uniquely determines the ind-scheme structure on $\mathcal{F}\ell$, but we do not know that an ind-scheme structure exists.

In the classical setting, it is also true that $G^{\mathsf{aff}} = \mathrm{colim}_g E(g)$. However, in the double affine setting, we will not discuss the analogous colimit in any serious way or attempt to relate it to any other version of the double loop group such as $G := G^{\mathsf{aff}}(\mathbb{C}(\!(t)\!))$.

1.6. **Homotopical deletion.** We now elaborate on the proof of Theorem 7.1.2, which states that the set

$$\mathcal{F}\ell := \operatorname*{colim}_{t\in \mathsf{D}} X(t)$$

canonically identifies with the disjoint union of Schubert cells which one would expect in the double affine flag variety. In the classical setting, the identification between the colimit and the flag variety also holds at the level of schemes or constructible sheaves, see 1.1.

If t is reduced, then the Demazure map $X(t) \to \mathcal{F}\ell$ should map the big cell of $X(t)$ isomorphically onto a Schubert cell of $\mathcal{F}\ell$. In other words, the expected Schubert cells in the colimit should come from these $X(t)$'s. In order to show that there is no redundancy, we need to know the following:

(1) Any two reduced finite jointed tethered tours $t_1, t_2 \in \mathsf{D}$ are related by a zig-zag of maps which do not change their end chamber.

(These maps correspond to *birational* maps of sub-Demazure varieties.)

The classical analogue of this result is the statement that any two reduced expressions for a Weyl group element are related by a sequence of braid moves.

In addition, we need to show that, when t is *non-reduced*, $X(t)$ does not contribute to the colimit. This requires

(2) If $t \in \mathsf{D}$ is non-reduced, then there are maps

$$t \xrightarrow{\text{zig-zag}} t' \longrightarrow t''$$

where the zig-zag involves only maps which do not change the end chamber, and the last map decreases the total length of the tour.

(The last map gives a dominant *non-birational* map of sub-Demazure varieties.)

The classical analogue of this result is the Coxeter deletion lemma, which states that any non-reduced expression can be modified, by a sequence of braid moves, so that it contains a repeated pair of simple reflections $s_i s_i$, which can then be canceled.

These are purely combinatorial statements, so we are not hampered by the geometric issues described in 1.5. Moreover, we are able to prove stronger versions of (1) and (2) which state that the required diagrams are unique up to homotopy. (This means that the set of choices can be viewed as the objects of a category, which is shown to be contractible.) For technical reasons, we restrict to tours which only involve *rational-level* chambers. (We suspect that this restriction can be removed.)

We also strengthen (1) and (2) in another way. In the double affine setting, we are working with finite tours (rather than galleries), and these can be modified much more flexibly. For instance, we can prove (1) in a silly way by deleting all chambers of t_1 except for the start and end chambers, and then inserting all chambers of t_2. To forbid this, we restrict to tours which satisfy a 'lower bound' in terms of the refinement relation. We show that (1) and (2) continue to hold with this restriction.

The full 'homotopical deletion' statement is Theorem 8.8.1. We view this result as the centerpiece of this thesis, since it is the double affine generalization of the main step of the proof of the identification $\mathcal{F}\ell \simeq \operatorname{colim}_t X(t)$ at the level of constructible sheaves [TaTr]. Unfortunately, due to the aforementioned geometric issues, we do not know any application of the full strength of this result.

1.7. **Notations.** We use the following notations from [TaTr, 1.4]:

- $[n] := \{0, 1, \ldots, n\}$.
- The simplex category is $\mathbf{\Delta}$, and the n-dimensional simplex is Δ^n.
- The category of schemes is Sch.
- A functor is *initial* or *final* if precomposition by it preserves limits or colimits, respectively.
- The *comma category* associated to a pair of functors

$$\mathcal{C}_1 \xrightarrow{F_1} \mathcal{D} \xleftarrow{F_2} \mathcal{C}_2$$

  is denoted $\langle F_1 \downarrow F_2 \rangle$ or $\langle \mathcal{C}_1 \xrightarrow{\mathcal{D}} \mathcal{C}_2 \rangle$. The latter notation is convenient when the functors are obvious.
- An ∞-*category* is an $(\infty, 1)$-category modeled as a weak Kan complex, as in [HTT]. We do not distinguish between a category and its nerve, so we say that every category *is* an ∞-category.
- A *categorical equivalence* of simplicial sets is a weak equivalence in the Joyal model structure, while a *homotopy equivalence* is a weak equivalence in the Kan model structure. (See [HTT, Def. 1.1.5.14].) When working with ∞-categories, we abbreviate 'categorical equivalence' as 'equivalence,' but the phrase 'homotopy equivalence' will never be abbreviated. For us, *contractible* means 'weakly contractible' (i.e. homotopy equivalent to a point).

2. Hyperplane arrangements

2.1. **Double affine hyperplane arrangements.**

2.1.1. Let A be a finite-dimensional affine space over $\mathbb{R}$.

Let A^* be the vector space of affine functions on A, and let $\pi \in A^*$ be the constant function with value 1. Note that specifying A is equivalent to specifying the pair (A^*, π).

A *double affine arrangement* in A consists of the following:

- A *vertical (a.k.a. level) coordinate* $\delta \in A^* \smallsetminus \mathbb{R}\pi$.
- A collection of *roots* $R \subset A^*$ such that $R = -R$ and R is a finite union of subsets of the form $\alpha + \mathbb{Z}\delta + \mathbb{Z}\pi$.

From now on, we fix such an arrangement, denoted $\mathcal{H}$. The following definitions pertaining to $\mathcal{H}$ are standard:

- A *wall* is a subset $H_\alpha := \{\alpha = 0\} \subset A$ for $\alpha \in R \smallsetminus \mathbb{R}\pi$.
- A *flat* is a nonempty intersection of walls.
- A *root half-space* is a subset $\{\alpha \geq 0\}$ or $\{\alpha > 0\}$ for $\alpha \in R \smallsetminus \mathbb{R}\pi$.

A subset $S \subset A$ is *horizontal* if $\delta(S)$ is a single real number.

2.1.2. Call a root α *imaginary* if $\alpha \in \mathbb{R}\delta + \mathbb{R}\pi$ and *real* otherwise. A root is real if and only if it determines a non-horizontal wall.

A wall H is the zero locus of infinitely many roots if and only if H is horizontal and rational-level (i.e. $\delta(H) \in \mathbb{Q}$). If such a wall exists, then there must be imaginary roots. (The converse is not true.)

2.1.3. The *irrelevant space* is the vector space of translations of A which preserve each root. Clearly, taking the quotient of A by these translations does not affect the combinatorics of $\mathcal{H}$. From now on, we assume that the irrelevant space is $\{0\}$, or equivalently $\mathsf{span}\, R = A^*$.

2.1.4. *Local and induced arrangements.* Let $B \subset A$ be an affine subspace.

- The *induced arrangement* $\mathcal{H}|_B$ has ambient space B, with roots restricted from A.
- The *local arrangement* $\mathcal{H}_B$ has ambient space A, with roots $\{\alpha \in R \,|\, \alpha(B) = 0\}$.

These arrangements always belong to one of the four classes defined as follows. (These criteria pertain to a pair (A^*, π) and a subset $R \subset A^*$.)

- *Affine:* R is a finite union of sets $\alpha + \mathbb{Z}\pi$.
- *Irrational-affine:* For some $r \in \mathbb{R} \smallsetminus \mathbb{Q}$, the set R is a finite union of sets $\alpha + (\mathbb{Z} + \mathbb{Z}r)\pi$.
- *Pre-affine:* For some $\delta \in A^* \smallsetminus \mathbb{R}\pi$, the set R is a finite union of sets $\alpha + \mathbb{Z}\delta$.
- *Finite:* R is finite.

The five classes of arrangements which we have defined so far are pairwise disjoint, except for the empty arrangement which belongs to all of them.

Given an affine subspace $B \subset A$, the induced and local arrangements are as follows:

hypothesis	induced arr. $\mathcal{H}\|_B$	local arr. $\mathcal{H}_B$
B is horizontal, $\delta(B)$ is rational	affine	pre-affine
B is horizontal, $\delta(B)$ is irrational	irrational-affine	finite
B is not horizontal	double affine	finite

If B is a maximal horizontal subspace (i.e. $\mathsf{dim}\, B = \mathsf{dim}\, A - 1$), then $\mathcal{H}_B$ has either one wall (which is B) or no wall. If it has one wall, then that wall corresponds to a finite set of roots if $\delta(B)$ is irrational and to an infinite set of roots if $\delta(B)$ is rational. This distinction was pointed out in 2.1.2.

2.2. **Faces.**

2.2.1. Each subset $R' \subseteq R$ determines a *subarrangement* $\mathcal{H}'$ of $\mathcal{H}$ whose ambient space is A. If $\mathcal{H}'$ is locally finite, define $\mathsf{Faces}(\mathcal{H}')$ to be the poset of faces, where $F_1 \preceq F_2$ means $F_1 \subseteq \overline{F_2}$. In this case, we say that F_1 is a *face* of F_2.

For any two locally finite subarrangements $\mathcal{H}'_1 \subseteq \mathcal{H}'_2$, there is an obvious *projection* map $\mathsf{Faces}(\mathcal{H}'_2) \to \mathsf{Faces}(\mathcal{H}'_1)$.

2.2.2. *Main definition.* For any subarrangement $\mathcal{H}'$ of $\mathcal{H}$, consider the limit of posets

$$\mathsf{Faces}(\mathcal{H}') := \lim_{\mathcal{H}''} \mathsf{Faces}(\mathcal{H}''),$$

where $\mathcal{H}''$ runs over the locally finite subarrangements of $\mathcal{H}'$. This limit is filtered. The definition ensures that, if $\mathcal{H}'_1 \subseteq \mathcal{H}'_2$ are any two subarrangements, then there is a projection map $\mathsf{Faces}(\mathcal{H}'_2) \to \mathsf{Faces}(\mathcal{H}'_1)$.

For each $F \in \mathsf{Faces}(\mathcal{H})$, we make the following definitions:

- The *support* $\overline{F}$ is the intersection of closures of projections of F to locally finite subarrangements. This is a nonempty closed subset of A.
- For any subset $S \subset A$, we write $F \subset S$ if the projection of F to some locally finite subarrangement is contained in S. Let us specialize this definition in three ways:
  - F is *horizontal* if $F \subset \delta^{-1}(a)$ for some real number a.
  - $\mathsf{span}\, F$ is the smallest flat which contains F. Define $\mathcal{H}_F := \mathcal{H}_{\mathsf{span}\, F}$.
  - If $F \subset \{\alpha > 0\}$, we write $\alpha(F) > 0$. Define $R^+(F) := \{\alpha \in R \,|\, \alpha(F) > 0\}$, and define $R^0(F), R^-(F)$ similarly.
- F is a *chamber* if $\mathsf{dim}\,\mathsf{span}\, F = \mathsf{dim}\, A$. Equivalently, every projection of F to a locally finite subarrangement is a chamber therein. A chamber is never horizontal.

Remark. If we only allow $\mathcal{H}''$ to run over the finite subarrangements of $\mathcal{H}'$, then the resulting limit $\mathsf{Faces}^{\mathsf{fin}}(\mathcal{H}')$ is strictly larger than $\mathsf{Faces}(\mathcal{H}')$, because it may contain faces which 'escape to infinity.' The 'boundedness' of faces in $\mathsf{Faces}(\mathcal{H}')$ corresponds to (iv) below.

2.2.3. *Concrete description.* For each $F \in \mathsf{Faces}(\mathcal{H})$, the sets $R^+(F), R^0(F), R^-(F)$ define a partition of R with the following properties:

(i) If $\alpha \in R$ is a linear combination of vectors in $R^0(F)$, then $\alpha \in R^0(F)$.

(ii) If $\alpha \in R$ is a positive linear combination of at least one vector in $R^+(F)$ and some vectors in $R^0(F)$, then $\alpha \in R^+(F)$.

(iii) $R^-(F) = -R^+(F)$.

(iv) For each $\alpha \in R$, we have $\alpha + n\pi \in R^+(F)$ for sufficiently large n.

Lemma. *The set* $\mathsf{Faces}(\mathcal{H})$ *is in bijection with the set of such partitions.*

Proof. For any locally finite subarrangement $\mathcal{H}'$, the analogous description of $\mathsf{Faces}(\mathcal{H}')$ is well-known. Now the lemma follows because this description of faces is also well-behaved under limits. (For (iv) to make sense, we should require that the subarrangement $\mathcal{H}'$ has roots R' which satisfy $R' + \mathbb{Z}\pi = R'$. These subarrangements are cofinal with respect to inclusion, so this restriction does not affect the limit.) $\square$

2.2.4. *Faces of pre-affine subarrangements.* Suppose that $B \subset A$ is a horizontal rational-level affine subspace, so that $\mathcal{H}_B$ is pre-affine (2.1.4). Let $\delta_B \in A^*$ be a positive multiple of $\delta - \delta(B)$ such that the roots of $\mathcal{H}_B$ are a union of finitely many $\mathbb{Z}\delta_B$-cosets. If $\mathcal{H}_B$ is nonempty, then each $F \in \mathsf{Faces}(\mathcal{H}_B)$ satisfies exactly one of the following:

- F is *upward* if, for every root α of $\mathcal{H}_B$, we have
$$(\alpha + n\,\delta_B)(F) > 0$$
for all sufficiently large n.
- F is *downward* if the same holds with $<$ in place of $>$.
- F is *liminal* if neither of the above hold.

Note that a face $F \in \mathsf{Faces}(\mathcal{H}_B)$ is liminal if and only if its support $\overline{F}$ is horizontal. This holds if F is horizontal, but the converse need not be true.

These definitions are motivated by the idea that $\mathcal{H}_B$ is a cone over an affine arrangement, namely $\mathcal{H}_B|_{\delta_B^{-1}(1)}$. The upward faces, the downward faces, and the faces of the affine arrangement are in bijection with one another. The liminal faces could be thought of as 'faces at infinity' of the affine arrangement.

If $\mathcal{H}_B$ is empty, then there is exactly one face, which we define to be liminal.

If $B \subset A$ is a horizontal irrational-level affine subspace, then $\mathcal{H}_B$ is finite. In this case, we say that every face is non-liminal, and we do not define 'upward' or 'downward.'

2.3. **Tits product.**

2.3.1. For each $F \in \mathsf{Faces}(\mathcal{H})$, define $\mathsf{Faces}(\mathcal{H})_{\succeq F} \subset \mathsf{Faces}(\mathcal{H})$ to consist of all faces $\succeq F$.

Lemma. *The projection map* $\mathsf{Faces}(\mathcal{H})_{\succeq F} \to \mathsf{Faces}(\mathcal{H}_F)$ *is a bijection.*

Proof. The inverse map sends $G \in \mathsf{Faces}(\mathcal{H}_F)$ to the face $G' \in \mathsf{Faces}(\mathcal{H})_{\succeq F}$ which is characterized by
$$\begin{aligned} R^+(G') &= R^+(F) \sqcup R^+(G) \\ R^0(G') &= R^0(G) \\ R^-(G') &= R^-(F) \sqcup R^-(G) \end{aligned}$$
using the concrete description of Lemma 2.2.3. Note that $R^+(G), R^0(G), R^-(G)$ are subsets of $R^0(F)$, because this is the root set of $\mathcal{H}_F$. $\square$

2.3.2. For fixed F as above, the *Tits product* $G \mapsto FG$ is the composition
$$\mathsf{Faces}(\mathcal{H}) \longrightarrow \mathsf{Faces}(\mathcal{H}_F) \longrightarrow \mathsf{Faces}(\mathcal{H})_{\succeq F}$$
of the projection map and the inverse map from the lemma. We also use $G \mapsto FG$ to denote the first or second map separately; the meaning will always be clear from context.

The Tits product is associative. (The usual proof still works.)

2.4. **Classification of faces.**

2.4.1. *Classification of supports.* Define the following types of subsets of A:

- A *rational support set* is the closure of a face of $\mathcal{H}|_{\delta^{-1}(a)}$ for some $a \in \mathbb{Q}$.
- An *irrational support set* is a point $p \in A$ with $\delta(p) \notin \mathbb{Q}$.

This terminology is justified by the following lemma.

Lemma. *For any $F \in \mathsf{Faces}(\mathcal{H})$, $\overline{F}$ is a rational or irrational support set. In the rational case, exactly one of the following is true:*

- *F is horizontal.*

  $F \subset \overline{F}$, and the projection of F to $\mathcal{H}_{\mathsf{span}\,\overline{F}}$ equals $\mathsf{span}\,\overline{F}$.

- *F is not horizontal. Either $\mathsf{dim}\,\overline{F} < \mathsf{dim}\,A - 1$ or $\mathcal{H}_{\mathsf{span}\,\overline{F}}$ has a horizontal wall.*

  $F \not\subset \overline{F}$, and the projection of F to $\mathcal{H}_{\mathsf{span}\,\overline{F}}$ is non-liminal.

- *F is not horizontal, $\mathsf{dim}\,\overline{F} = \mathsf{dim}\,A - 1$, and $\mathcal{H}_{\mathsf{span}\,\overline{F}}$ has no horizontal wall.*

  F is a chamber, and $\mathcal{H}_{\mathsf{span}\,\overline{F}}$ is empty.

  $F \not\subset \overline{F}$, and the projection of F to $\mathcal{H}_{\mathsf{span}\,\overline{F}}$ is liminal.

Proof. We first claim that $\overline{F}$ is horizontal. If not, then there are points $p, q \in \overline{F}$ at different levels. For any root α, there exists another root $\alpha + n\delta + m\pi$ which is positive on p and negative on q, contradiction.

Now let $a = \delta(\overline{F})$. The definition of supports implies that

($*$) $\overline{F} \subset \delta^{-1}(a)$ is the intersection of closed root half-spaces $\{\alpha \geq 0\}$ or $\{\alpha \leq 0\}$ of $\mathcal{H}|_{\delta^{-1}(a)}$, with every root appearing at least once.

If a is irrational, then $\mathcal{H}|_{\delta^{-1}(a)}$ is irrational-affine. Since we have assumed that the irrelevant space of $\mathcal{H}$ equals $\{0\}$, ($*$) implies that $\overline{F}$ is a point.

Now assume that a is rational, so that $\mathcal{H}|_{\delta^{-1}(a)}$ is affine. Statement ($*$) implies that $\overline{F}$ is the closure of a face of $\mathcal{H}|_{\delta^{-1}(a)}$. This completes the proof of the first sentence.

For the bullets, the idea is to let P be the closure of the projection of F to any sufficiently fine locally finite arrangement, so that $F \subset \mathsf{relint}\,P$ and $P \cap \delta^{-1}(a) = \overline{F}$.

If F is horizontal, then we may choose a locally finite subarrangement so that P is horizontal. This means that $P = P \cap \delta^{-1}(a) = \overline{F}$, so $F \subset \overline{F}$, and the projection of F to $\mathcal{H}_{\mathsf{span}\,\overline{F}}$ equals $\mathsf{span}\,\overline{F}$.

If F is not horizontal, and [$\mathsf{dim}\,\overline{F} < \mathsf{dim}\,A - 1$ or $\mathcal{H}_{\mathsf{span}\,\overline{F}}$ has a horizontal wall], then we may choose a locally finite subarrangement so that $\mathsf{relint}\,P$ is disjoint from $\delta^{-1}(a)$. Now

$F \subset \mathsf{relint}\, P$ implies that $F \not\subset \overline{F}$, and $P \cap \delta^{-1}(a) = \overline{F}$ implies that the projection of F to $\mathcal{H}_{\mathsf{span}\,\overline{F}}$ is non-liminal.

Lastly, assume that F is not horizontal, $\dim \overline{F} = \dim A - 1$, and $\mathcal{H}_{\mathsf{span}\,\overline{F}}$ has no horizontal wall. If F is not a chamber, then we may choose a locally finite subarrangement so that $\dim P < \dim A$. Since P is not horizontal, this implies that

$$\dim \overline{F} = \dim P \cap \delta^{-1}(a) \leq \dim P - 1 < \dim A - 1,$$

contradiction. Therefore F must be a chamber, which implies $F \not\subset \overline{F}$. Since $\dim \overline{F} = \dim A - 1$, the only possible wall in $\mathcal{H}_{\mathsf{span}\,\overline{F}}$ is $\mathsf{span}\,\overline{F}$, which is horizontal, so $\mathcal{H}_{\mathsf{span}\,\overline{F}}$ must be empty. Therefore the projection of F to $\mathcal{H}_{\mathsf{span}\,\overline{F}}$ is liminal. $\square$

For any $F \in \mathsf{Faces}(\mathcal{H})$, define $\delta(F) := \delta(\overline{F})$. If this level is rational, we say that F is *upward*, *downward*, or *liminal* if its projection to $\mathcal{H}_{\mathsf{span}\,\overline{F}}$ has this property.

Intuitively, for a rational-level face F, the lemma says that it is *almost* true that the projection of F to $\mathcal{H}_{\mathsf{span}\,\overline{F}}$ equals $\mathsf{span}\,\overline{F}$ if and only if F is horizontal; otherwise the projection is non-liminal. (Hence, it is *almost* true that F is liminal if and only if it is horizontal.) The only exception is when F is a liminal chamber, i.e. the third bullet holds. Liminal chambers exist if and only if there is no horizontal wall at some rational level.

2.4.2. *Support faces.* If S is a support set, then the corresponding *support face* $\mathring{S} \in \mathsf{Faces}(\mathcal{H})$ is defined by $R^+(\mathring{S}) = \{\alpha \mid \alpha(\mathsf{relint}\, S) > 0\}$ and similarly for $R^0(\mathring{S})$ and $R^-(\mathring{S})$. It is easy to see that

$$(\text{rational support faces}) = (\text{rational horizontal faces}) \sqcup (\text{liminal chambers}).$$

For any $F \in \mathsf{Faces}(\mathcal{H})$, let us write $\tilde{F}$ instead of $\mathring{\overline{F}}$. We have $\tilde{F} \preceq F$. Equality holds if and only if F is a support face. (Remark: This is true if and only if F contains a point of A.)

2.4.3. **Theorem.** *The following map, defined by $(S, G) \mapsto SG$, is a bijection:*

$$\left\{ (S, G) \;\middle|\; \begin{array}{c} S \textit{ is a support face} \\ G \in \mathsf{Faces}(\mathcal{H}_{\mathsf{span}\,S}) \textit{ equals } \mathsf{span}\,S \textit{ or is non-liminal} \end{array} \right\} \to \mathsf{Faces}(\mathcal{H})$$

The inverse map is $F \mapsto (\tilde{F}, \tilde{F}F)$.

Proof. The previous lemma implies that the inverse map is well-defined. In checking that the two maps are indeed inverse, the only nontrivial point is to show that $\overline{SG} = \overline{S}$. This is obvious if $G = \mathsf{span}\,S$, so we instead assume that G is non-liminal. Since $S \preceq SG$, we have $\overline{S} \subseteq \overline{SG}$. In particular, the two have equal levels. If the level is irrational, then they are points, so they are equal. If the level is rational, denote it by a. The non-liminality of G gives an intersection of closed root half-spaces which contains the face SG and whose intersection with $\delta^{-1}(a)$ equals $\mathsf{span}\,S$. This proves that $\overline{SG}$ is no larger than $\overline{S}$, as desired. $\square$

We remind the reader that, when S has irrational level, all faces of $\mathcal{H}_{\mathsf{span}\,S}$ are non-liminal by definition (2.2.4). When S has rational level, each non-liminal face of $\mathcal{H}_{\mathsf{span}\,S}$ is upward or downward (but not both), and $\mathsf{span}\,S$ is liminal *unless* S is a liminal chamber.

Roughly speaking, the theorem says that every double affine face can be specified by choosing a point $p \in A$ and a face of $\mathcal{H}_p$. The issue is that a given double affine face can be specified in multiple ways. To remove the redundancy, we replace the point p by the unique face which contains it (which is necessarily a support face), and we require that the $\mathcal{H}_p$-chamber is non-liminal.

2.5. **Separation and adjacency.**

2.5.1. Two chambers are *separated* by a wall H_α if the root α takes different signs on the two chambers. Two chambers are *adjacent* if they are separated by exactly one wall. Given three chambers C_1, C_2, C_3, we say that C_2 is *between* C_1 and C_3 if there is no wall separating C_2 from C_1 and C_3.

2.5.2. **Lemma.** *If C_1 and C_3 are distinct non-adjacent chambers, then there exists another chamber C_2 which lies between them.*

Proof. Let C_1^{loc} denote the projection of C_1 to $\mathcal{H}_{\overline{C}_1}$, which is non-liminal. For every wall H of C_1^{loc}, there exists a unique chamber C_H of $\mathcal{H}_{\overline{C}_1}$ which is adjacent to it along H. If any H separates C_3 from C_1, then $C_2 = \tilde{C}_1 C_H$ works. From now on, assume that no such H exists. This means that the projection of C_3 to $\mathcal{H}_{\overline{C}_1}$ equals C_1^{loc}, i.e. $\tilde{C}_1 C_3 = C_1$.

If $\overline{C}_1 = \overline{C}_3$, then $C_3 = \tilde{C}_3 C_3 = \tilde{C}_1 C_3 = C_1$, contradiction. Thus, $\overline{C}_1 \neq \overline{C}_3$.

Suppose that $\overline{C}_1, \overline{C}_3$ are disjoint. Then we can find another support set S (disjoint from $\overline{C}_1, \overline{C}_3$) and distinct collinear points, as shown, with p_2 lying between p_1 and p_3.

$$p_1 \in \mathsf{relint}\,\overline{C}_1, \qquad p_2 \in \mathsf{relint}\,S, \qquad p_3 \in \mathsf{relint}\,\overline{C}_3$$

We claim that $C_2 = \mathring{S} C_3$ works. First, the collinearity implies that $\tilde{C}_1 \mathring{S} \tilde{C}_3 = \tilde{C}_1 \tilde{C}_3$, so

$$\tilde{C}_1 C_2 = \tilde{C}_1 \mathring{S} C_3 = \tilde{C}_1 \mathring{S} \tilde{C}_3 C_3 = \tilde{C}_1 \tilde{C}_3 C_3 = \tilde{C}_1 C_3 = \tilde{C}_1 C_1.$$

Next, suppose for sake of contradiction that a root α is positive on C_2 and negative on C_1 and C_3. Then it is nonnegative on p_2 and nonpositive on p_1 and p_3, so the collinearity implies that it is zero on p_1, hence on $\overline{C}_1$. Now $\tilde{C}_1 C_2 = \tilde{C}_1 C_1$ implies that $\alpha(C_2)$ and $\alpha(C_1)$ have the same sign, contradiction.

Lastly, suppose that $\overline{C}_1, \overline{C}_3$ are not disjoint. Then $\delta(C_1) = \delta(C_3) = \text{(rational)}$, and we may assume without loss of generality that $\overline{C}_1 \subsetneq \overline{C}_3$. Since the projection of C_3 to $\mathcal{H}_{\overline{C}_1}$ equals C_1^{loc}, the definition of supports implies that $\overline{C}_3 \subset \mathsf{span}\,\overline{C}_1$, contradiction. $\square$

2.5.3. *Remark.* This proof shows that two chambers C_1 and C_3 are adjacent if and only if $\overline{C}_1 = \overline{C}_3$ and their projections to $\mathcal{H}_{\overline{C}_1}$ are adjacent.

2.6. **Topology.**

2.6.1. Recall that $\mathsf{Faces}(\mathcal{H}) = \lim \mathsf{Faces}(\mathcal{H}')$ where $\mathcal{H}'$ runs over locally finite subarrangements. Then a similar equality holds for the subsets of chambers, which we denote by $\mathsf{Chambers}(-) \subseteq \mathsf{Faces}(-)$. Let us equip $\mathsf{Chambers}(\mathcal{H})$ with the inverse limit topology. This topology is generated by the root half-spaces.

2.6.2. **Lemma.** *Let $\mathcal{H}'$ be a locally finite subarrangement whose chambers are bounded, and let $p : \mathsf{Chambers}(\mathcal{H}) \to \mathsf{Chambers}(\mathcal{H}')$ be the projection map. The fibers of p are homeomorphic to the Cantor set, so $\mathsf{Chambers}(\mathcal{H})$ is homeomorphic to a countable disjoint union of Cantor sets.*

Proof. For each $C \in \mathsf{Chambers}(\mathcal{H}')$, the fiber $p^{-1}(C)$ is obtained by taking the limit as C is increasingly subdivided by the walls of $\mathcal{H}$. Now apply this well-known characterization of the Cantor set: an inverse limit of finite sets $\lim_{i \in \mathbb{Z}_{\geq 0}} S_i$ is homeomorphic to the Cantor set if and only if, for each i and each $s \in S_i$, there exists $j > i$ such that the preimage of s under $S_j \to S_i$ has size ≥ 2. The condition holds because the walls of $\mathcal{H}$ are dense. □

2.6.3. *Neighborhood basis.* Fix a chamber C and write $a = \delta(C)$. A *C-wedge* is any subset $S \subset A$ constructed as follows. Let C^{loc} be the projection of C to $\mathcal{H}_{\overline{C}}$.

- Assume $a \in \mathbb{Q}$. Choose finitely many affine functions f_i on $\mathsf{span}\, \overline{C}$ whose nonnegative half-spaces cut out $\overline{C}$, and arbitrarily extend them to $\hat{f}_i \in A^*$. If C^{loc} is upward, then define
$$S = \overline{C^{\mathsf{loc}}} \cap \left(\bigcap_i \left\{ \hat{f}_i - n\delta + na\pi \geq 0 \right\} \right)$$
for any $n > 0$. If C^{loc} is downward, use $+n$ instead of $-n$.
- Assume $a \notin \mathbb{Q}$. Choose a closed ball B such that $\overline{C} \in \mathsf{int}\, B$, and define
$$S = \overline{C^{\mathsf{loc}}} \cap B.$$

Lemma. *The chamber sets $\{C' \subset S\}$, where S ranges over C-wedges, determine a neighborhood basis of C.*

Proof. An obvious neighborhood basis of C is given as follows: for each finite intersection of root half-spaces containing C, take the set of chambers contained in the intersection. Thus, it suffices to show that every such intersection contains a C-wedge, and vice versa.

Fix a finite intersection of root half-spaces containing C. In the definition of S, the first factor $\overline{C^{\mathsf{loc}}}$ ensures that S is contained in the root half-spaces whose walls contain $\overline{C}$. For all other root half-spaces, the second factor of S can be chosen small enough so that the intersection contains S, as desired.

Now fix a C-wedge S. Since C^{loc} is non-liminal, $\overline{C^{\mathsf{loc}}}$ is the intersection of finitely many root half-spaces. It remains to choose a finite intersection of root half-spaces which contains C and lies in the second factor in the definition of S.

If $a \in \mathbb{Q}$, choose finitely many roots α whose restrictions to $\mathsf{span}\, \overline{C}$ cut out $\overline{C}$. These roots are positive on $\tilde{C}$ and hence C. Fix n such that na is an integer and replace each α by $\alpha \pm (n\delta - na\pi)$. If n is sufficiently large, the intersection of the resulting root half-spaces will lie in S.

Now suppose that $a \notin \mathbb{Q}$. For any root α, integer $N > 0$, and real number $\epsilon > 0$, there are infinitely many roots of the form $\alpha + n\delta + m\pi$ such that $n > N$ and
$$\alpha(\overline{C}) + na + m \in (0, \epsilon).$$

(Proof: $\mathbb{Z}_{>N} \cdot a + \mathbb{Z}$ is dense in $\mathbb{R}$.) The statement remains true if $n > N$ is replaced by $n < -N$. Using this statement, we can find a finite collection of roots which are positive on $\overline{C}$ and hence C, such that the intersection of half-spaces is contained in B. (Proof: Denote the intersection by D. By choosing a set of α's which spans A^*, we ensure that D is bounded. By choosing ϵ to be small, we ensure that $D \cap \delta^{-1}(a)$ lies in B. By choosing N to be large, we ensure that $\delta(D)$ is a small interval, and moreover $D \subset B$.) ☐

2.6.4. **Corollary.** *The map $\delta : \mathsf{Chambers}(\mathcal{H}) \to \mathbb{R}$ is continuous.*

Proof. For each chamber C and each real interval (a, b) containing $\delta(C)$, there exists a C-wedge S whose closure is contained in $\delta^{-1}((a, b))$. Each chamber $C' \subset S$ satisfies $\delta(C') \in (a, b)$, as desired. ☐

3. Galleries

3.1. **Tours and galleries.**

3.1.1. *Ordered sets.* Recall the following standard definitions concerning an ordered[1] set I.

- A *cut* is a downwards-closed subset.
- A *gap* is a pair of elements $i < j$ with no element between them. Equivalently, a gap is a cut J such that J has a maximal element and $I \setminus J$ has a minimal element.
- I is *bounded* if it has a minimal element $\hat{0}$ and a maximal element $\hat{1}$.
- I is *complete* if every subset of I has a supremum in I. This implies the same statement with 'infimum' instead of 'supremum.' It is equivalent to require that every cut equals $[\hat{0}, i)$ or $[\hat{0}, i]$ for some $i \in I$.

A map between bounded ordered sets is *bound-preserving* if it preserves $\hat{0}$ and $\hat{1}$.

3.1.2. A *sequence of chambers* is a pair (I, c) where I is an ordered set and $c : I \to \mathsf{Chambers}(\mathcal{H})$ is a map. We write c_i in place of $c(i)$.

Lemma. *A sequence (I, c) converges if and only if it is eventually bounded and, for every root α, the sequence $\alpha(c_i) \in \{-, +\}$ is eventually constant.*

Proof. Follows from the definition of the topology on chambers. ☐

For each wall H, the number of times the sequence *crosses* H is defined to be $m - 1$, where m is the largest possible size of a finite subset $J \subset I$ such that the sequence of signs $(\alpha(c_j))_{j \in J}$ is alternating. This number lies in $\mathbb{Z}_{\geq 0} \sqcup \{\infty\}$. The sequence is *reduced* if each wall is crossed at most once. Let $\mathsf{Walls}(I, c)$ be the multiset of walls crossed by the sequence.

The sequence (I, c) *stutters* if there exists an interval $[i_1, i_2]$ with $i_1 < i_2$ such that the restriction $([i_1, i_2], c)$ is constant.

[1] For us, an 'ordered set' is a *linearly* or *totally* ordered set. This is in contrast to a *partially* ordered set, which is usually called a poset.

3.1.3. A *tour* is a sequence of chambers (I, c) such that

- I is bounded.
- (I, c) is non-stuttering.
- Each wall is crossed only a finite number of times.
- Only finitely many walls are crossed more than once.

The last two conditions ensure that every tour can be (nonuniquely) expressed as the concatenation of finitely many reduced tours. The last condition also implies that every tour has bounded image.

Define $\mathsf{Tours}(\mathcal{H})$ to be the category whose objects are tours, and a morphism $(I, c) \to (I', c')$ is a bound-preserving weakly increasing *index* map $\varphi : I \to I'$ satisfying $c'\varphi = c$ and $\mathsf{Walls}(I, c) = \mathsf{Walls}(I', c')$. These morphisms are called *refinements*. The non-stuttering condition implies that φ is in fact strictly increasing.

3.1.4. **Lemma.** $\mathsf{Tours}(\mathcal{H})$ *is a poset.*

Proof. Let $(I, c) \to (I', c')$ be any morphism, with index map φ. The definition of morphisms implies that, for any i, we have

$$\mathsf{Walls}([\hat{0}, i], c) = \mathsf{Walls}([\hat{0}, \varphi(i)], c').$$

The non-stuttering condition implies that the multiset $\mathsf{Walls}([\hat{0}, i], c)$ is strictly increasing in i, and similarly for c' in place of c. Thus, the displayed equation determines φ uniquely, i.e. there is at most one morphism between any two objects. $\square$

3.1.5. Fix a chamber C_0. The *weak Bruhat order* centered at C_0 is the partial order on chambers defined as follows: $C_1 \preceq C_2$ means that $\mathsf{Walls}(C_0, C_1) \subseteq \mathsf{Walls}(C_0, C_2)$.[2] Reduced tours starting at C_0 are equivalent to chains in the weak Bruhat order starting at C_0.

Zorn's lemma implies that every reduced tour can be refined to a reduced tour which is maximal under refinement (i.e. a reduced *gallery*, as defined below). Since every tour can be expressed as the concatenation of finitely many reduced tours, we can remove the word 'reduced' from the previous sentence.

3.1.6. **Lemma.** *Let (I, c) be a chamber sequence which is the concatenation of two tours $([\hat{0}, i], c)$ and $([i, \hat{1}], c)$. If there are only finitely many walls which separate c_i from $c_{\hat{0}}$ and $c_{\hat{1}}$, then (I, c) is a tour.*

Proof. The only nontrivial point is to show that there are only finitely many walls which are crossed by both $([\hat{0}, i], c)$ and $([i, \hat{1}], c)$. Since $([\hat{0}, i], c)$ is a tour, the sets of walls crossed by $([\hat{0}, i], c)$ and the two-step tour $(c_{\hat{0}}, c_i)$ differ by only finitely many elements. Similarly for $([i, \hat{1}], c)$ and $(c_i, c_{\hat{1}})$. Thus, it suffices to show that only finitely many walls are crossed by both $(c_{\hat{0}}, c_i)$ and $(c_i, c_{\hat{1}})$. This is equivalent to the hypothesis. $\square$

[2] Here (C_0, C_1) and (C_0, C_2) are two-chamber tours. In this case, $\mathsf{Walls}(-)$ is a set.

3.1.7. A *gallery* is a tour which is maximal under refinement.

Proposition. *A tour (I, c) is a gallery if and only if these conditions hold:*

(1) I is complete.

(2) c is continuous.

(3) c sends each gap to a pair of adjacent chambers.

Proof. First, suppose that (I, c) is a gallery. We will prove (1), (2), (3).

Proof of (1) and (2). Let $J \subset I$ be any cut which has no maximal element. It suffices to show that $J = [\hat{0}, i)$ for some i and that c is continuous from the left at i. (A symmetrical argument shows that c is continuous from the right at i.)

The definition of tours implies that the subsequence (J, c) is bounded and crosses each wall finitely many times. Lemma 3.1.2 implies that the subsequence converges to some chamber C. Define a refinement $(I, c) \to (I', c')$ by inserting a new index j right after J and defining $c'_j = C$. This does not change the number of times any wall is crossed. Since (I, c) is a gallery, (I', c') must fail to be a tour, so it stutters. This implies that $I \smallsetminus J = [i, \hat{1}]$ for some i, and $c'_j = c_i$, because (I, c) is non-stuttering. This proves the claim.

Proof of (3). Suppose for sake of contradiction that $i < j$ is a gap such that c_i and c_j are not adjacent. Use Lemma 2.5.2 to find a chamber C between c_i and c_j, and refine (I, c) by inserting C. This contradicts the assumption that (I, c) is a gallery.

Now suppose that (I, c) is a tour but not a gallery. We assume conditions (1) + (2) and disprove (3).

Since (I, c) is not a gallery, there is a nontrivial refinement $(I, c) \to (I', c')$. We may assume that $I' = I \cup \{j\}$. Define $J = [\hat{0}, j)$ so that $I \smallsetminus J = (j, \hat{1}]$. Applying (1) shows that one of the following descriptions is valid, where $i, i' \in I$:

(A) $J < j < [i, \hat{1}] = I \smallsetminus J$, and J has no maximum.

(B) $J = [\hat{0}, i] < j < I \smallsetminus J$, and $I \smallsetminus J$ has no minimum.

(C) $J = [\hat{0}, i] < j < [i', \hat{1}] = I \smallsetminus J$.

Assume (A). Since (I', c') is non-stuttering, $c'_j \neq c_i$. Let H be any wall which separates these two chambers. By (2), c is continuous, so there exists $k \in J$ such that H does not separate c_k from c_i. Then H is crossed more often by (I', c') than by (I, c) contradiction.

A symmetrical argument rules out (B). Thus, (C) holds. Then $i < i'$ is a gap, but c_i and $c_{i'}$ cannot be adjacent because c'_j lies between them. This disproves (3). □

3.2. **Expressions.**

3.2.1. Fix a 'start' chamber C_0. An *expression* is a sequence of walls (I, h) such that

- The fibers of $h : I \to \mathsf{Walls}(\mathcal{H})$ are finite, and only finitely many have size ≥ 2.

- For each cut $J \subset I$, there exists a (unique) chamber c^h_J such that the following are equivalent, for each wall H:
  - H separates C_0 and c^h_J, i.e. $H \in \mathsf{Walls}(C_0, c^h_J)$.
  - The number of $j \in J$ such that $H = h_j$ is odd.

Thus, each expression gives rise to a chamber sequence $(\mathsf{Cuts}(I), c^h)$.

3.2.2. **Theorem.** *The map* $(I, h) \mapsto (\mathsf{Cuts}(I), c^h)$ *gives a bijection from expressions to galleries starting at* C_0*.*

The rest of this subsection is devoted to proving the theorem.

3.2.3. **Lemma.** *The chamber sequence* $(\mathsf{Cuts}(I), c^h)$ *is a gallery.*

Proof. It is clearly a tour. We will check the three conditions of Proposition 3.1.7. The index set $\mathsf{Cuts}(I)$ is complete because the supremum of a collection of cuts is given by their union.

Next, we show that c^h is continuous from the left at $J \in \mathsf{Cuts}(I)$. (A symmetrical argument shows continuity from the right.) By Lemma 3.1.2, it suffices to check that c^h is bounded and, for every root α, the sign sequence $(\alpha(c^h_{J'}))_{J' \subset J}$ is eventually constant. The boundedness follows from the fact that c^h is a tour. Take $J' \subset J$ to be large enough so that $H_\alpha \notin h(J \smallsetminus J')$. Then the sign sequence is constant after J', because any two chambers which occur after J' are separated only by walls in $h(J \smallsetminus J')$.

Finally, we show that every gap gives a pair of adjacent chambers. In fact, if $J \subset J'$ is a gap, then $J' = J \cup \{i\}$ for some i. By construction, $R^+(c^h_{J'})$ differs from $R^+(c^h_J)$ only for the wall h_i, so the two chambers are adjacent along h_i. □

3.2.4. Let (I, c) be a gallery. Let $\mathsf{Walls}'(I, c)$ be the set of *wall-crossing pairs* (H, n), where n is a positive integer and H is a wall which is crossed at least n times. For each pair (H, n), define a cut of I as follows:

$$\mathsf{cut}(H, n) := \{i \,|\, ([\hat{0}, i], c) \text{ crosses } H \text{ fewer than } n \text{ times}\}.$$

Intuitively, this cut tells us when (I, c) crosses H for the n-th time.

Lemma.

(i) Each $\mathsf{cut}(H, n)$ *is the cut of some gap* $i < j$*, and* c_i *and* c_j *are adjacent along* H*.*

(ii) The map $\mathsf{cut} : \mathsf{Walls}'(I, c) \to \mathsf{Gaps}(I)$ *is a bijection.*

Proof. Proof of (i). Since I is complete, $\mathsf{cut}(H, n)$ and its complement have a supremum and an infimum in I. Since c is continuous, these subsets are closed, so they have a maximum and a minimum. This means that $\mathsf{cut}(H, n)$ is the cut of some gap $i < j$. The definition of $\mathsf{cut}(H, n)$ implies that c_i and c_j are separated by H.

Proof of (ii). The inverse map sends a gap $i < j$ to (H, n), where H is the adjacency wall for c_i and c_j, and n is the number of times $([\hat{0}, j], c)$ crosses H. □

3.2.5. Let I be any ordered set. There is a weakly increasing map

$$\gamma : I \to \mathsf{Cuts}(\mathsf{Gaps}(I))$$

defined by $i \mapsto \{\text{all gaps lying below } i\}$.

Lemma. *If I is complete, and any two elements are separated by a gap, then γ is a bijection.*

Proof. If any two elements are separated by a gap, then the above map is strictly increasing, hence injective. To show that it is surjective, choose any $(i_a < j_a)_{a\in A} \in \mathsf{Cuts}(\mathsf{Gaps}(I))$. Take $i = \sup_{a\in A} j_a$, which exists because I is complete. We claim that i maps to the chosen cut. If not, then there is some other gap $i' < j'$ which lies above all $i_a < j_a$ and lies below i. But then i' is a strictly smaller upper bound of the j_a, contradiction. □

3.2.6. **Corollary.** *If (I, c) is a gallery, then γ is a bijection.*

Proof. In view of the previous lemma, it suffices to show that any two elements $i < j$ of I are separated by a gap. First, suppose that $c_i \neq c_j$. Let H be any wall which separates these two chambers, and let n be the number of times $([\hat{0}, j], c)$ crosses H. Then $\mathsf{cut}(H, n)$ contains c_i but not c_j. By Lemma 3.2.4(i), this cut corresponds to a gap, and it separates i and j, as desired.

If $c_i = c_j$, then the non-stuttering condition gives an index $i < k < j$ with $c_k \neq c_i$. Now replace $i < j$ by $i < k$ and use the previous argument. □

3.2.7. We now construct the inverse map for Theorem 3.2.2. Given a gallery (I, c), define an expression $(\mathsf{Gaps}(I), h^c)$ as follows: for each gap $i < j$, define $h^c_{i<j}$ to be the wall along which c_i and c_j are adjacent.

Lemma. *This is an expression whose chamber sequence equals (I, c).*

Proof. The previous corollary says that $\gamma : I \to \mathsf{Cuts}(\mathsf{Gaps}(I))$ is a bijection. It remains to prove that, for each $i \in I$, the chamber $c^{h^c}_{\gamma(i)}$ exists and equals c_i. The proof of Lemma 3.2.4(ii) gives a bijection

$$\mathsf{Gaps}([\hat{0}, i]) \to \mathsf{Walls}'([\hat{0}, i], c),$$

so $c^{h^c}_{\gamma(i)}$ is characterized by the requirement that $\mathsf{Walls}(C_0, c^{h^c}_{\gamma(i)})$ equals the set of walls which appear an odd number of times in $\mathsf{Walls}'([\hat{0}, i], c)$. The chamber c_i satisfies this requirement, so $c^{h^c}_{\gamma(i)}$ exists and equals c_i, as desired. □

3.2.8. **Lemma.** *Let (I, h) be an expression. Then $(I, h) = (\mathsf{Gaps}(\mathsf{Cuts}(I)), h^{c^h})$.*

Proof. There is a bijection $I \simeq \mathsf{Gaps}(\mathsf{Cuts}(I))$ sending i to the gap $[\hat{0}, i) \subset [\hat{0}, i]$. It remains to prove that

$$h^{c^h}_{[\hat{0},i)\subset[\hat{0},i]} = h_i.$$

By definition, the left hand side is the wall along which $c^h_{[\hat{0},i)}$ and $c^h_{[\hat{0},i]}$ are adjacent. The definition of these chambers implies that this wall is h_i, as desired. □

We have constructed mutually inverse maps between expressions and galleries starting at C_0. This concludes the proof of Theorem 3.2.2.

3.3. **Positivity.**

3.3.1. *Positive pairs.* Choose a finite subset $R^{\mathsf{fin}} \subset R$ such that $R^{\mathsf{fin}} + \mathbb{Z}\delta + \mathbb{Z}\pi = R$. Define a root $\alpha^{\mathsf{fin}} + n\delta + m\pi$ to be *upward* if $n > 0$ and *downward* if $n \leq 0$. For any pair of chambers (C_1, C_2), define $\mathsf{Roots}(C_1, C_2) := R^+(C_2) \smallsetminus R^+(C_1)$. The pair is *positive* if $\mathsf{Roots}(C_1, C_2)$ contains only finitely many downward roots. This does not depend on the choice of R^{fin}.

Lemma. *If (C_1, C_2) and (C_2, C_3) are positive, then so is (C_1, C_3).*

Proof. Follows from $\mathsf{Roots}(C_1, C_3) \subseteq \mathsf{Roots}(C_1, C_2) \sqcup \mathsf{Roots}(C_2, C_3)$. □

Remark. In the classical setting, every pair of chambers is positive, because the hyperplane arrangement is locally finite, and all of the material in this subsection is trivially true.

3.3.2. A tour (I, c) is *positive* if $(c_{\hat{0}}, c_{\hat{1}})$ is positive.

Lemma. *A concatenation of two positive tours is a positive tour.*

Proof. Denote the concatenation by (I, c), with $([\hat{0}, i], c)$ and $([i, \hat{1}], c)$ being the two tours. Since $(c_{\hat{0}}, c_i)$ and $(c_i, c_{\hat{1}})$ are positive, the previous lemma implies that $(c_{\hat{0}}, c_{\hat{1}})$ is positive. It remains to show that (I, c) is a tour. By Lemma 3.1.6, it suffices to show that only finitely many walls separate c_i from $c_{\hat{0}}$ and $c_{\hat{1}}$. Such a wall gives a downward root in exactly one of $\mathsf{Roots}(c_{\hat{0}}, c_i)$ or $\mathsf{Roots}(c_i, c_{\hat{1}})$. The positivity of these pairs implies that there are only finitely many such walls. □

For any finite chamber sequence $([n], c)$, define its *steps* to be the pairs (c_{i-1}, c_i) for $i \in [1, n]$. The previous lemma implies that, if each step is positive, then $([n], c)$ is a positive tour. The next lemma implies the converse.

3.3.3. **Lemma.** *If (I, c) is a positive tour, then (c_i, c_j) is positive for every $i < j$.*

Proof. We have

$$\mathsf{Roots}(c_{\hat{0}}, c_{\hat{1}}) = \Big(\mathsf{Roots}(c_{\hat{0}}, c_i) \cup \mathsf{Roots}(c_i, c_j) \cup \mathsf{Roots}(c_j, c_{\hat{1}})\Big) \smallsetminus (\text{canceling pairs})$$

where the last term consists of pairs $\{\alpha, -\alpha\}$ for which α and $-\alpha$ come from different factors in the disjoint union. Suppose for sake of contradiction that $\mathsf{Roots}(c_i, c_j)$ contains infinitely many downward roots. Since $\mathsf{Roots}(c_{\hat{0}}, c_{\hat{1}})$ contains only finitely many downward roots, there must be infinitely many canceling pairs. This means that (I, c) double-crosses infinitely many walls, contradiction. □

3.3.4. *Classification of positive pairs.*

Proposition. *A pair (C_1, C_2) is positive if and only if one of the following hold:*

(i) $\delta(C_1) < \delta(C_2)$.

(ii) $\delta(C_1) = \delta(C_2) \notin \mathbb{Q}$ *and* $\overline{C}_1 = \overline{C}_2$.

(iii) $\delta(C_1) = \delta(C_2) \in \mathbb{Q}$, *and one of the following is true:*

- *Both chambers are [downward or liminal] and* $\overline{C}_1 \subseteq \overline{C}_2$.
- *Both chambers are [upward or liminal] and* $\overline{C}_1 \supseteq \overline{C}_2$.

- *C_1 is downward, C_2 is upward, and some face of $\mathcal{H}|_{\delta^{-1}(a)}$ is $\succeq$ to both supports. (Here $a := \delta(C_1)$.)*

In particular, if (C_1, C_2) is positive, then $\delta(C_1) \leq \delta(C_2)$.

Proof. Assume that $\delta(C_1) < \delta(C_2)$. If $\alpha^{\mathsf{fin}} + n\delta + m\pi$ lies in $\mathsf{Roots}(C_1, C_2)$, then

$$\alpha^{\mathsf{fin}}(\overline{C}_1) + n\delta(C_1) + m \leq 0$$
$$\alpha^{\mathsf{fin}}(\overline{C}_2) + n\delta(C_2) + m \geq 0,$$

which implies $n\big(\delta(C_2) - \delta(C_1)\big) \geq \alpha^{\mathsf{fin}}(\overline{C}_1) - \alpha^{\mathsf{fin}}(\overline{C}_2)$. Only finitely many $n \leq 0$ can satisfy this inequality. Then, for a fixed n, only finitely many m can satisfy both displayed inequalities. This shows that $\mathsf{Roots}(C_1, C_2)$ contains finitely many downward roots, so (C_1, C_2) is positive.

Assume that $\delta(C_1) > \delta(C_2)$. For any α^{fin}, we can find infinitely many integers $n \leq 0$ and m such that

$$\alpha^{\mathsf{fin}}(\overline{C}_1) + n\delta(C_1) + m < 0$$
$$\alpha^{\mathsf{fin}}(\overline{C}_2) + n\delta(C_2) + m > 0.$$

This gives infinitely many downward roots in $\mathsf{Roots}(C_1, C_2)$, so (C_1, C_2) is not positive.

From now on, assume that $\delta(C_1) = \delta(C_2)$ and denote this level by a.

Assume that $a \notin \mathbb{Q}$ and that $\overline{C}_1 = \overline{C}_2$. Each root in $\mathsf{Roots}(C_1, C_2)$ is zero on the shared support. Since $a \notin \mathbb{Q}$, there are finitely many such roots, so (C_1, C_2) is positive.

Assume that $a \notin \mathbb{Q}$ and that $\overline{C}_1 \neq \overline{C}_2$. Choose a root α^{fin} which takes different values on the supports. As in the proof of Lemma 2.6.3, there exist infinitely many roots of the form $\alpha^{\mathsf{fin}} + n\delta + m\pi$, with $n \leq 0$, which are negative on $\overline{C}_1$ and positive on $\overline{C}_2$. These are downward roots in $\mathsf{Roots}(C_1, C_2)$, so (C_1, C_2) is not positive.

From now on, assume that $a \in \mathbb{Q}$. There exists an integer $N > 0$, depending only on C_1 and C_2, such that any root $\alpha = \alpha^{\mathsf{fin}} + n\delta + m\pi$ with $n < -N$ satisfies the following:

- If C_1 is downward, then $\alpha(C_1) > 0$ if and only if $\alpha(\tilde{C}_1) \geq 0$.
- If C_1 is upward, then $\alpha(C_1) > 0$ if and only if $\alpha(\tilde{C}_1) > 0$.
- Same statements for C_2.

It is clear that all but finitely many downward roots in $\mathsf{Roots}(C_1, C_2)$ must be of this form, so we may restrict attention to downward roots of this form. Now we split into several cases:

- Suppose that both chambers are [downward or liminal].

  Then the pair (C_1, C_2) is positive if and only if there do not exist any α (of the above form) such that $\alpha(\tilde{C}_1) < 0$ and $\alpha(\tilde{C}_2) \geq 0$. (Proof: If there exists one such root, then there exist infinitely many. Also, a chamber C is liminal if and only if $C = \tilde{C}$, so it is easy to adapt the above bullets to the liminal case.)

  This condition is equivalent to '$\alpha(\tilde{C}_2) \geq 0$ implies $\alpha(\tilde{C}_1) \geq 0$,' which is equivalent to $\overline{C}_1 \subseteq \overline{C}_2$.

- Suppose that both chambers are [upward or liminal].

  Then the pair (C_1, C_2) is positive if and only if there do not exist any α such that $\alpha(\tilde{C}_1) \leq 0$ and $\alpha(\tilde{C}_2) > 0$.

  This condition is equivalent to '$\alpha(\tilde{C}_1) \leq 0$ implies $\alpha(\tilde{C}_2) \leq 0$,' which is equivalent to $\overline{C}_1 \supseteq \overline{C}_2$.

- Suppose that C_1 is downward and C_2 is upward.

  Then the pair (C_1, C_2) is positive if and only if there do not exist any α such that $\alpha(\tilde{C}_1) < 0$ and $\alpha(\tilde{C}_2) > 0$.

  This is equivalent to the condition that some face of $\mathcal{H}|_{\delta^{-1}(a)}$ is $\succeq$ to $\tilde{C}_1$ and $\tilde{C}_2$.

- Suppose that C_1 is upward and C_2 is downward.

  Then the pair (C_1, C_2) is positive if and only if there do not exist any α such that $\alpha(\tilde{C}_1) \leq 0$ and $\alpha(\tilde{C}_2) \geq 0$.

  This is equivalent to '$\alpha(\tilde{C}_2) \geq 0$ implies $\alpha(\tilde{C}_1) > 0$.' This implies that $C_1 = C_2$ and both are liminal chambers, but that contradicts the hypothesis of this case. □

3.4. **Structure of positive galleries.**

3.4.1. Let (I, c) be a positive gallery.

Lemma. *The map $\delta c : I \to \mathbb{R}$ is weakly increasing, and its image equals $[\delta(c_{\hat{0}}), \delta(c_{\hat{1}})]$.*

Proof. Since each pair (c_i, c_j) with $i < j$ is positive, the classification of positive pairs implies that $\delta(c_i)$ is weakly increasing. Thus, the image of δc lies in $[\delta(c_{\hat{0}}), \delta(c_{\hat{1}})]$. Recall that δ and c are continuous (2.6.4, 3.1.7), so δc is continuous. If the image fails to contain some $a \in (\delta(c_{\hat{0}}), \delta(c_{\hat{1}}))$, then continuity implies that $(\delta c)^{-1}([\delta(c_{\hat{0}}), a])$ and $(\delta c)^{-1}([a, \delta(c_{\hat{1}})])$ are disjoint closed subsets of I. Since I is complete, these two subsets have a maximum and minimum element (respectively), and these two elements are a gap $i < j$. Since c_i and c_j are adjacent, Remark 2.5.3 implies that their supports are equal, so $\delta(c_i) = \delta(c_j)$. This contradicts the fact that $\delta(c_i) < a$ and $\delta(c_j) > a$. □

Thus, for any $a \in [\delta(c_{\hat{0}}), \delta(c_{\hat{1}})]$, the fiber $(\delta c)^{-1}(a) \subset I$ is a nonempty bounded interval. Let us fix a and study the subgallery $((\delta c)^{-1}(a), c)$.

3.4.2. **Lemma.** *If $a \notin \mathbb{Q}$, then $(\delta c)^{-1}(a)$ is finite.*

Proof. The classification of positive pairs implies that the supports $(\overline{c}_i)_{i \in (\delta c)^{-1}(a)}$ are constant. Denote this common support by S. Any wall which separates any two of these chambers must lie in $\mathcal{H}_S$. Since $\mathcal{H}_S$ is finite, $(\delta c)^{-1}(a)$ must be finite as well; otherwise some wall in $\mathcal{H}_S$ would be crossed infinitely often. □

3.4.3. Now assume that $a \in \mathbb{Q}$. The classification of positive pairs implies that the sequence $(\overline{c}_i)_{i \in (\delta c)^{-1}(a)}$ is unimodal. More precisely, there is a weakly increasing surjective map $\varphi : (\delta c)^{-1}(a) \to [m]$, a sequence of support sets $([m], s)$, and a 'mode' index $k \in [m]$ such that the following hold:

- For all $i \in (\delta c)^{-1}(a)$, we have $\overline{c}_i = s_{\varphi(i)}$.
- $([0,k], s)$ is strictly increasing, and $([k,m], s)$ is strictly decreasing.
- All chambers in the sequence $(\varphi^{-1}([0,k-1]), c)$ are downward, and all chambers in the sequence $(\varphi^{-1}([k+1,m]), c)$ are upward.

The second bullet point implies that the sizes of $[0,k]$ and $[k,m]$ are at most $\mathsf{dim}\, A$.

Lemma. *If $\hat{0}, \hat{1} \notin \varphi^{-1}(k)$, then $\mathsf{dim}\, s_k = \mathsf{dim}\, A - 1$.*

The hypothesis says that $\varphi^{-1}(k)$ does not occur at the beginning or end of I.

Proof. Suppose for sake of contradiction that $\mathsf{dim}\, s_k < \mathsf{dim}\, A - 1$, so that $\mathcal{H}_{s_k}$ is a pre-affine arrangement with at least one non-horizontal wall (hence infinitely many). This implies that a sequence of downward chambers in $\mathcal{H}_{s_k}$ cannot converge to an upward chamber, and vice versa. Also, a downward chamber in $\mathcal{H}_{s_k}$ cannot be adjacent to an upward chamber.

By construction, every chamber in the gallery (I, c) projects to an upward or downward (i.e. not liminal) chamber of $\mathcal{H}_{s_k}$. Since $\hat{0}, \hat{1} \notin \varphi^{-1}(k)$, the third bullet point above implies that both upward and downward projections occur. Since (I, c) is positive, the downward projections must come before the upward projections. Thus, there exists a cut $J \subset I$ such that (J, c) (resp. $(I \smallsetminus J, c)$) projects to downward (resp. upward) chambers in $\mathcal{H}_{s_k}$, and J and $I \smallsetminus J$ are both nonempty.

Since projection preserves convergence, the first paragraph of this proof implies that J has a maximal element, and similarly $I \smallsetminus J$ has a minimal element. These two elements give a gap of I, denoted $i < j$. Since (I, c) is a gallery, c_i is adjacent to c_j, contradicting the first paragraph. □

Lemma.

(i) $\varphi^{-1}(k)$ is finite. If $\mathsf{dim}\, s_k = \mathsf{dim}\, A - 1$, then $|\varphi^{-1}(k)| \leq 2$.

(ii) If $j \in [0, k-1]$, then $\varphi^{-1}(j) \simeq \mathbb{Z}_{\geq 0}$.

(iii) If $j \in [k+1, m]$, then $\varphi^{-1}(j) \simeq \mathbb{Z}_{\leq 0}$.

Proof. First, consider $j \in [0,m]$ such that $\mathsf{dim}\, s_j = \mathsf{dim}\, A - 1$. This forces $j = k$. To prove that $|\varphi^{-1}(k)| = 1$ or 2, we split into two subcases:

- If $\delta^{-1}(a)$ is a wall, then $\mathcal{H}_{s_k}$ consists of just that wall. There are exactly two chambers C_1, C_2 with support s_k, and C_1 is downward while C_2 is upward. The pair (C_2, C_1) is not positive (Proposition 3.3.4), so $(\varphi^{-1}(k), c)$ can only be (C_1), (C_2), or (C_1, C_2).
- If $\delta^{-1}(a)$ is not a wall, then $\mathcal{H}_{s_k}$ is empty. Then there is exactly one chamber C with support s_k (and C is liminal). The non-stuttering property of the gallery implies that $(\varphi^{-1}(k), c) = (C)$, which has length 1.

Next, consider $j \in [0,m]$ such that $\mathsf{dim}\, s_j < \mathsf{dim}\, A - 1$. The reasoning of the previous proof implies that $(\varphi^{-1}(j), c)$ projects to a gallery in $\mathcal{H}_{s_j}$ consisting of all-upward or all-downward chambers. This implies that the ordered set $\varphi^{-1}(j)$ is finite or isomorphic to one of $\mathbb{Z}_{\geq 0}, \mathbb{Z}_{\leq 0}, \mathbb{Z}$. (If not, then the gallery in $\mathcal{H}_{s_j}$ stutters or crosses some wall infinitely often, contradiction.)

If $\varphi^{-1}(j)$ has no minimal element, then it does not contain its infimum i. Since the gallery (I, c) is continuous, $(\varphi^{-1}(j), c)$ converges (downward) to c_i. This implies that $s_j \subsetneq \overline{c}_i$.

Similarly, if $\varphi^{-1}(j)$ has no maximal element, then it does not contain its supremum i', which implies that $s_j \subsetneq \overline{c}_{i'}$.

Suppose that $j = k$. By definition, s_k is the largest level-a support among the chambers in the gallery. Therefore, it is not possible to have $s_k \subsetneq \overline{c}_i$ or $s_k \subsetneq \overline{c}_{i'}$. We conclude that $\varphi^{-1}(k)$ is finite.

At this point, we know that $\varphi^{-1}(k)$ is finite, regardless of whether $\mathsf{dim}\, s_k = \mathsf{dim}\, A - 1$.

Suppose that $j < k$. Then it is not possible to have $s_j \subsetneq \overline{c}_i$, so $\varphi^{-1}(j)$ must have a minimal element. If it has a maximal element, then there is a gap consisting of that element and the minimal element of $\varphi^{-1}(j+1)$,[3] but the two corresponding chambers have different supports so they cannot be adjacent, contradiction. Thus, $\varphi^{-1}(j)$ does not have a maximal element, so it must be isomorphic to $\mathbb{Z}_{\geq 0}$.

Suppose that $j > k$. A symmetrical argument shows that $\varphi^{-1}(j) \simeq \mathbb{Z}_{\leq 0}$. □

3.4.4. **Corollary.** *The union of supports* $\cup_{i \in I} \overline{c}_i$ *is closed.*

Proof. We fix $p \in A$ which does not lie in any support and construct a neighborhood $p \in U$ which is disjoint from the supports. Define $a = \delta(p)$. If $a \notin [\delta(c_{\hat{0}}), \delta(c_{\hat{1}})]$, then take $U = \delta^{-1}(\mathbb{R} \smallsetminus [\delta(c_{\hat{0}}), \delta(c_{\hat{1}})])$. From now on, assume that $a \in [\delta(c_{\hat{0}}), \delta(c_{\hat{1}})]$.

Decompose $I = I_{<a} \sqcup I_a \sqcup I_{>a}$ based on whether $\delta(c_i)$ is less than a, equal to a, or greater than a. The union of supports decomposes analogously into three parts:

$$\cup_{i \in I} \overline{c}_i = (\cup_{i \in I_{<a}} \overline{c}_i) \sqcup (\cup_{i \in I_a} \overline{c}_i) \sqcup (\cup_{i \in I_{>a}} \overline{c}_i).$$

We claim that the middle part is closed. If $a \notin \mathbb{Q}$, then Lemma 3.4.2 implies that the middle part is a point. If $a \in \mathbb{Q}$, the middle part is a union of closures of faces of the affine arrangement $\mathcal{H}|_{\delta^{-1}(a)}$. This is closed because the affine arrangement is locally finite.

Lemma 3.4.1 says that I_a is nonempty. Therefore, the supremum of $I_{<a}$, denoted i, lies in $I_{<a}$ or I_a. If $i \in I_{<a}$, then there are no chambers at any level between $\delta(c_i)$ and a, which contradicts the fact that the image of δc is a real interval. Thus, $i \in I_a$.

Choose a neighborhood $p \in U$ and a c_i-wedge S such that U is disjoint from the middle part and S. Since $(I_{<a}, c)$ converges to c_i, Lemma 2.6.3 implies that the supports $(I_{<a}, \overline{c})$ are eventually contained in S, say after some index $j \in I_{<a}$. Shrink U so that it contains no points at level $\leq \delta(c_j)$. Then U is disjoint from the first two parts. A symmetrical argument allows one to make U disjoint from all three parts. □

Remark. If all supports were points, the corollary would imply that $\cup_{i \in I} \overline{c}_i$ is a continuous section of δ over $[\delta(c_{\hat{0}}), \delta(c_{\hat{1}})]$, i.e. the graph of a continuous 'function' on this interval. This is not true because the rational-level supports can be larger than points. However, 3.4.3

[3] Why does $\varphi^{-1}(j+1)$ have a minimal element? If $j+1 < k$, this follows from the previous sentence. If $j+1 = k$, this follows from the previous paragraph.

tells us that, when $a \in \mathbb{Q}$, the level-a subset $\cup_{i \in I_a} \overline{c}_i$ equals a face closure of $\mathcal{H}|_{\delta^{-1}(a)}$. Thus, the 'function' may be discontinuous at any $a \in \mathbb{Q}$, but the discontinuities are bounded, and the bound becomes tighter as the denominator of a increases.

3.4.5. Define the map $\delta^{\mathsf{gap}} : \mathsf{Gaps}(I) \to \mathbb{R}$ as follows. If $i < j$ is a gap, c_i and c_j are adjacent, so their supports agree (2.5.3). Define $\delta^{\mathsf{gap}}(i < j) := \delta(c_i) = \delta(c_j)$.

Lemma. *The image of δ^{gap} is an at-most-countable dense subset $[\delta(c_{\hat{0}}), \delta(c_{\hat{1}})]$.*

Proof. Since the multiset of walls crossed by (I, c) is at-most-countable, Lemma 3.2.4(ii) implies that $\mathsf{Gaps}(I)$ is at-most-countable, so the image of δ^{gap} is at-most-countable as well. Lemma 3.4.1 says that δc is weakly increasing with image equal to $[\delta(c_{\hat{0}}), \delta(c_{\hat{1}})]$, and the proof of Corollary 3.2.6 implies that every two elements of I are separated by a gap, so the image of δ^{gap} is dense in this interval. $\square$

3.4.6. To conclude this section, let us summarize what we know about the index set of a positive gallery (I, c) assuming that $\delta(c_{\hat{0}}) < \delta(c_{\hat{1}})$. First, we discuss $\mathsf{Gaps}(I)$. The image of δ^{gap} is isomorphic to one of the ordered sets $\mathbb{Q}, \mathbb{Q}_{\leq 0}, \mathbb{Q}_{\geq 0}, \mathbb{Q} \cap [0, 1]$ by Lemma 3.4.5 and Cantor's Isomorphism Theorem. Each fiber of δ^{gap} is an ordered set of the form

$$\underbrace{\mathbb{Z}_{\geq 0} \sqcup \cdots \sqcup \mathbb{Z}_{\geq 0}}_{<\mathsf{dim}\, A \text{ copies}} \sqcup S \sqcup \underbrace{\mathbb{Z}_{\leq 0} \sqcup \cdots \sqcup \mathbb{Z}_{\leq 0}}_{<\mathsf{dim}\, A \text{ copies}},$$

where S is finite, by 3.4.2 and 3.4.3.

Now we discuss I. It is isomorphic to $\mathsf{Cuts}(\mathsf{Gaps}(I))$ by Corollary 3.2.6. A special case of Lemma 2.6.2 says that $\mathsf{Cuts}(\mathbb{Q})$ is isomorphic to the Cantor set K. Similarly, the $\mathsf{Cuts}(-)$ of $\mathbb{Q}_{\leq 0}, \mathbb{Q}_{\geq 0}, \mathbb{Q} \cap [0, 1]$ are given by $K \sqcup \{*\}$, $\{*\} \sqcup K$, and $\{*\} \sqcup K \sqcup \{*\}$, respectively. The previous paragraph implies that $\mathsf{Cuts}(\mathsf{Gaps}(I))$ can be obtained by starting with one of these four sets and replacing each gap with an ordered set of the form

$$\underbrace{\mathbb{Z}_{\geq 0} \sqcup \cdots \sqcup \mathbb{Z}_{\geq 0}}_{<\mathsf{dim}\, A \text{ copies}} \sqcup S' \sqcup \underbrace{\mathbb{Z}_{\leq 0} \sqcup \cdots \sqcup \mathbb{Z}_{\leq 0}}_{<\mathsf{dim}\, A \text{ copies}},$$

where S' is finite and nonempty.

In fact, 3.4.3 tells us a little bit more. It implies that, when a gap is being replaced by the above ordered set, and one of the following hold, then $|S'| \leq 2$:

- The gap does not involve the element $*$, and there is at least one $\mathbb{Z}_{\geq 0}$ or $\mathbb{Z}_{\leq 0}$.
- $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{\leq 0}$ both appear.
- There are $\mathsf{dim}\, A - 1$ copies of $\mathbb{Z}_{\geq 0}$.
- There are $\mathsf{dim}\, A - 1$ copies of $\mathbb{Z}_{\leq 0}$.

4.1. **Double affine Coxeter arrangement.**

4.1.1. *Affine-type Kac–Moody arrangement.* Let G^{fin} be a simply-connected simple group.

- $\mathfrak{h}^{\mathsf{fin}}$ is the Lie algebra of the torus of G^{fin}.
- R^{fin} is the set of roots.
- $R^{\mathsf{fin},\vee}$ is the set of coroots.
- $\Lambda^{\mathsf{fin},\vee}$ is the coweight lattice.
- W^{fin} is the finite Weyl group.

Next, define

$$\mathfrak{h}^{\mathsf{aff}} = \mathfrak{h}^{\mathsf{fin}} \oplus \mathbb{R}d \oplus \mathbb{R}K$$
$$\mathfrak{h}^{\mathsf{aff},\vee} = (\mathfrak{h}^{\mathsf{fin}})^{\vee} \oplus \mathbb{R}\delta \oplus \mathbb{R}\Lambda_0.$$

Here $\delta(d) = 1$, $\Lambda_0(K) = 1$, and all other nonobvious evaluations are zero. The *affine-type Kac–Moody arrangement* is given by the *real affine* roots

$$R^{\mathsf{aff}} = R^{\mathsf{fin}} + \mathbb{Z}\delta \subset \mathfrak{h}^{\mathsf{aff},\vee}.$$

This is a pre-affine arrangement in $\mathfrak{h}^{\mathsf{aff}}$.

- $\Lambda^{\mathsf{aff},\vee} = \Lambda^{\mathsf{fin},\vee} \oplus \mathbb{Z}d \oplus \mathbb{Z}K$ is the affine coweight lattice.
- $W^{\mathsf{aff}} = W^{\mathsf{fin}} \ltimes \mathbb{Z}R^{\mathsf{fin},\vee}$ is the affine Weyl group.

This arrangement governs the Kac–Moody group ind-scheme G^{aff} and its Lie algebra

$$\mathfrak{g}^{\mathsf{aff}} \simeq \mathfrak{g}^{\mathsf{fin}}[s, s^{-1}] \oplus k\, d \oplus k\, K.$$

The roots of $\mathfrak{g}^{\mathsf{aff}}$ are given by $R^{\mathsf{aff}} \sqcup \mathbb{Z}\delta$. The roots in $\mathbb{Z}\delta$ are called *imaginary affine.*

4.1.2. W^{aff} *is a reflection group.* Define a symmetric bilinear form on $\mathfrak{h}^{\mathsf{aff}}$ as follows:

- On $\mathfrak{h}^{\mathsf{fin}}$, it is the Killing form.
- On $\mathbb{R}d \oplus \mathbb{R}K$, it satisfies $\langle d, d\rangle = \langle K, K\rangle = 0$ and $\langle d, K\rangle = 1$.
- $\mathfrak{h}^{\mathsf{fin}}$ is orthogonal to $\mathbb{R}d \oplus \mathbb{R}K$.

Viewed as a map $\nu : \mathfrak{h}^{\mathsf{aff}} \to \mathfrak{h}^{\mathsf{aff},\vee}$, this form sends $d \mapsto \Lambda_0$ and $K \mapsto \delta$. For each root $\alpha \in R^{\mathsf{aff}}$, the orthogonal reflection through H_α is denoted r_α. (The orthogonal reflection exists because R^{aff} does not include the imaginary roots.) The reflections r_α generate a group isomorphic to W^{aff}. This gives a representation $W^{\mathsf{aff}} \curvearrowright \mathfrak{h}^{\mathsf{aff}}$.

Remark. The preceding material comes from [Kac, §6]. However, we have made the following notational replacements:

Kac	us	Kac	us
$\mathring{\mathfrak{h}}$	$\mathfrak{h}^{\mathsf{fin}}$	$\mathfrak{h}$	$\mathfrak{h}^{\mathsf{aff}}$
$\mathring{\Delta}$	R^{fin}	Δ^{re}	R^{aff}
$\mathring{Q}^\vee$	$\mathbb{Z}R^{\mathsf{fin},\vee}$		
$\mathring{\Lambda}^\vee$	$\Lambda^{\mathsf{fin},\vee}$	$\Lambda^\vee$	$\Lambda^{\mathsf{aff},\vee}$
$\mathring{W}$	W^{fin}	W	W^{aff}

To deduce the above material from [Kac, §6], it is helpful to note that 'nontwisted' means $r = 1$. According to §6.1 in [Kac], $r = 1$ implies $a_0 = 1$, and it is always true that $a_0^\vee = 1$. Also, when $r = 1$, Proposition 6.5 in [Kac] says that $W \simeq \mathring{W} \ltimes \mathring{Q}^\vee$, because in that case (6.5.8) implies that the lattice M identifies with the coroot lattice via ν.

4.1.3. The *double affine Coxeter arrangement* is given by the roots

$$R = R^{\mathsf{aff}} + \mathbb{Z}\pi \subset (\mathfrak{h}^{\mathsf{aff}})^*$$

Here $\mathfrak{h}^{\mathsf{aff}}$ is treated as an affine space, and π is the constant function with value 1. This gives a double affine arrangement in $\mathfrak{h}^{\mathsf{aff}}$.

Since R^{aff} includes only the real affine roots, this arrangement has no horizontal walls.

4.1.4. *Weyl group.* Recall that the action $W^{\mathsf{aff}} \curvearrowright \mathfrak{h}^{\mathsf{aff}}$ leaves the affine coweight lattice $\Lambda^{\mathsf{aff},\vee}$ invariant. Define the *extended double affine Weyl group* to be

$$\tilde{W} := W^{\mathsf{aff}} \ltimes \Lambda^{\mathsf{aff},\vee}.$$

It acts faithfully on $\mathfrak{h}^{\mathsf{aff}}$, with $\Lambda^{\mathsf{aff},\vee}$ acting by translations. The *double affine Weyl group* $W \subset \tilde{W}$ is the subgroup generated by orthogonal reflections across the double affine walls. The subgroup of *central translations* is defined to be $W \cap \mathbb{Z}K$ and is finite index in $\mathbb{Z}K$.

For every face F, the *local Weyl group* $W_F \subset W$ is the subgroup generated by the reflections which fix F, or equivalently fix $\mathsf{span}\, F$ pointwise. Note that W_F does not contain any translations. If F is horizontal, then F is rational-level, the local arrangement $\mathcal{H}_F$ is pre-affine (and nonempty), and W_F is an affine Coxeter group; otherwise, $\mathcal{H}_F$ is finite (possibly empty), and W_F is a finite Coxeter group.

Define the *Weyl groupoid* $\mathcal{W}$ as follows: objects are chambers, a morphism $C_1 \to C_2$ is an element $w \in W$ such that $wC_1 = C_2$, and composition of morphisms is given by group multiplication. A morphism is called a *reflection arrow* if w is a reflection. Since $\overline{wC_1} = w\,\overline{C}_1$, morphisms preserve levels. The set of morphisms between two fixed chambers is a torsor for the group of central translations.

4.1.5. *Definition of* $\mathcal{H}$. The irrelevant space of the aforementioned arrangement is $\mathbb{R}K$. It will be more convenient to work with the quotient

$$\mathfrak{h} = \mathfrak{h}^{\mathsf{aff}}/\mathbb{R}K = \mathfrak{h}^{\mathsf{fin}} \oplus \mathbb{R}d$$

whose irrelevant space is $\{0\}$. From now on, let $\mathcal{H}$ denote the arrangement in the quotient.

This quotient does not affect the combinatorics of the hyperplane arrangement or the definition of local Weyl groups. However, the action $W \curvearrowright \mathfrak{h}$ is not faithful because W contains translations of the form $nK \in \Lambda^{\mathsf{aff},\vee}$.

4.2. **Bruhat category.**

4.2.1. A positive pair (C_1, C_2) is *weak* if $\delta(C_1) = \delta(C_2)$ and *strict* if $\delta(C_1) < \delta(C_2)$.

A wall H is *supported* for this pair if there exists a (positive) gallery from C_1 to C_2 which crosses H. Explicitly, we have

- If the pair is weak, then H is supported if and only if $\overline{C}_1 \subset H$ or $\overline{C}_2 \subset H$.
- If the pair is strict, then every wall is supported.

4.2.2. Fix a 'fundamental' chamber B. A chamber C is called *positive*, *weak*, or *strict* if the pair (B, C) has this property.

A reflection arrow $C \xrightarrow{r} rC$ is oriented *away from* B if its wall does not separate B from C. The arrow is *supported* if C is positive and the reflection wall is supported for (B, C). This implies that rC is also positive (see Lemma 4.4.1, whose proof is self-contained).

- Let $\mathcal{W}^B \subset \mathcal{W}$ be the subcategory whose objects are positive chambers and whose morphisms are generated by supported reflection arrows oriented away from B. This is the *Bruhat category centered at* B.
- Let T be a positive chamber. Then $\langle T \xrightarrow{\mathcal{W}} \mathcal{W}^B\rangle$ is the *Bruhat preorder centered at* B *and tethered to* T.

We will show that this preorder is a partial order (Corollary 4.4.8), generalizing a theorem of Muthiah, see [M1, Rmk. 4.26].

4.2.3. *The classical case.* Suppose that T is a weak chamber such that one of the supports $\overline{B}$ or $\overline{T}$ is contained in the other.

Lemma. *For every chamber C contained in $\langle T \xrightarrow{\mathcal{W}} \mathcal{W}^B\rangle$, one of the supports $\overline{B}$ or $\overline{C}$ is contained in the other.*

Proof. Write $a = \delta(B) = \delta(T)$ and assume that it is rational, which is the harder case. Since $C = wT$ for some $w \in W$, the supports $\overline{C}$ and $\overline{T}$ correspond to faces of the same type in the affine arrangement $\mathcal{H}|_{\delta^{-1}(a)}$. Since (B, C) is positive, $\overline{B}$ and $\overline{C}$ are both contained in the closure of some chamber of $\mathcal{H}|_{\delta^{-1}(a)}$ (Proposition 3.3.4), and similarly for $\overline{B}$ and $\overline{T}$. Now the desired containment for $\overline{B}$ and $\overline{C}$ follows from the analogous statement for $\overline{B}$ and $\overline{T}$, which we have taken as an assumption. $\square$

With this lemma in mind, it is easy to see that $\langle T \xrightarrow{\mathcal{W}} \mathcal{W}^B\rangle$ is isomorphic to one of the following classically-studied partial orders on the Coxeter group $W_{\overline{B}\cap\overline{T}}$ (which is finite or affine by 4.1.4):

- The ordinary Bruhat order.
- A semi-infinite Bruhat order, obtained by moving the fundamental chamber to infinity in some direction. (This only makes sense when the Coxeter group is affine.)
- The opposite of one of the above partial orders.

These partial orders satisfy many good properties (see e.g. [BjBr, Ch. 2]), and the goal of this section is to generalize some of these properties.

4.3. **Length.**

4.3.1. Fix a supported reflection arrow $C \xrightarrow{r} rC$ which is oriented away from B. A wall H *lengthens* this arrow if H separates B from rB and separates C from rC. The arrow's *length*, denoted $\ell_B(-)$, is the number of such walls.

Lemma. *The number of lengthening walls is finite.*

Proof. Assume that C is strict. (The weak case is an exercise.) Then the sets $\overline{C} \cup r_\alpha \overline{C}$ and $\overline{B} \cup r_\alpha \overline{B}$ are bounded and lie at different levels, so there are only finitely many walls which intersect the convex hulls of both sets. $\square$

The length is an odd positive integer because rH lengthens the arrow if and only if H does, and the wall of r always lengthens the arrow. The length of a supported reflection arrow oriented *toward* B is defined to be the negative of the length of the inverse arrow.

Remark. This definition of length comes from [MO]. As explained in [MO, 4.1], in the classical setting we have

$$\ell_B(C \xrightarrow{r_\alpha} r_\alpha C) = \#\,\mathsf{Walls}(B, rC) - \#\,\mathsf{Walls}(B, C).$$

for any reflection arrow oriented away from B. In the double affine setting, both wall sets are infinite, so we can only work with a suitably-defined difference.

4.3.2. *Coplanarity.* Here is the key to understanding the definition of 'lengthening.'

Lemma. *Let $C \xrightarrow{r} rC$ be a supported reflection arrow oriented away from B. There is no wall which separates B and rC from rB and C.*

Proof. We assume that C is strict and leave the weak case as an exercise. Choose points $b \in \mathsf{relint}\,\overline{B}$ and $c \in \mathsf{relint}\,\overline{C}$. If a wall H separates B and rC from rB and rC, then one of its roots α satisfies $\alpha(b), \alpha(rc) \geq 0$ and $\alpha(c), \alpha(rb) \leq 0$. Since b, c, rc, rb are coplanar points and form a (possibly degenerate) convex quadrilateral in that order, we conclude that α is zero on all four points. Hence, H is a wall of the local arrangement $\mathcal{H}_{\mathsf{span}(b,c,rc,rb)}$. We may now project everything to the local arrangement. Since C is strict, we have $\delta(b) < \delta(c)$, so the span of the four points is not horizontal, and the local arrangement is finite. Now choose points b' and c' in the *projections* of B and C, and repeat the first half of this proof. $\square$

There are 8 unordered partitions of $\{B, rB, C, rC\}$ into two parts. The lemma says that the partition $\{B, rC\} \sqcup \{rB, C\}$ cannot be realized by a wall. Therefore, a wall lengthens the arrow if and only if it determines the partition $\{B, C\} \sqcup \{rB, rC\}$.

Remark. By checking all 7 possible partitions, we find that

- If H lengthens the arrow, then $H, rH \in \mathsf{Walls}(B, rC) \smallsetminus \mathsf{Walls}(B, C)$.
- If not, then $H \in \mathsf{Walls}(B, rC) \smallsetminus \mathsf{Walls}(B, C)$ and $rH \in \mathsf{Walls}(B, C) \smallsetminus \mathsf{Walls}(B, rC)$, or vice versa.

In the classical setting, this proves the equation in the previous remark.

4.3.3. *Rational versus irrational levels.* Using the explicit neighborhood basis for chambers (2.6.3), one can show

Lemma. *Fix a reflection r. The map from strict pairs to $\mathbb{Z}$, defined by*

$$(B,C) \mapsto \ell_B\big(C \xrightarrow{r_\alpha} r_\alpha C\big),$$

is locally constant.

Let us call $\langle T \xrightarrow{\mathcal{W}} \mathcal{W}^B \rangle$ a *rational-level* Bruhat order if $\delta(T)$ is rational, and similarly for 'irrational.' The lemma implies that every *finite* subposet of an irrational-level Bruhat order is isomorphic to a subposet of a rational-level Bruhat order, in a way which preserves the lengths of reflection arrows. (Proof: take a sequence of rational levels q_n which converges to the given irrational level. For each chamber C in the given finite subposet, take a sequence of level-q_n chambers which converges to C.)

We believe that the word 'finite' is necessary in this statement. Indeed, we will show that every element of a rational-level Bruhat order has finitely many cocovers, but we also believe that, at least when $G^{\mathsf{fin}} = \mathrm{SL}_2$, some irrational-level Bruhat orders contain elements with infinitely many cocovers (Remark 4.5.7). Thus, the statement would fail if we took an (infinite) subposet of an irrational-level Bruhat order consisting of one minimal element which is joined to infinitely many other elements by length-1 arrows.

4.4. **Foldings.**

4.4.1. *Main definition.* Suppose we are given a positive gallery (I,c) and a finite increasing subsequence $(i_k < j_k)_{k\in[1,m]} \subset \mathsf{Gaps}(I)$. Define a new chamber sequence

$$(I',c') := \big([\hat{0},i_1],c\big) \diamond \cdots \diamond \big([j_{k-1},i_k],(r_1\cdots r_{k-1})c\big) \diamond \cdots \diamond \big([j_m,\hat{1}],(r_1\cdots r_m)c\big),$$

where r_k is the reflection across the adjacency wall h_k for the chambers c_{i_k} and c_{j_k}, and $\diamond$ denotes concatenation. The new index set I' is obtained from I by identifying $i_k \sim j_k$, so $\mathsf{Gaps}(I')$ equals $\mathsf{Gaps}(I)$ minus the chosen gaps. The new sequence (I',c') is the *folding* of (I,c) along the chosen gaps. The h_k are the *folding walls.* The morphism in $\mathcal{W}$ given by

$$(r_1\cdots r_m)c \xrightarrow{r_1\cdots r_m} c_{\hat{1}}$$

is the *discrepancy* of the folding.

Lemma. (I',c') *is a positive gallery.*

Proof. For every $w \in W$, the w-translate of a positive pair is positive. This is true because, under w-translation, only a finite number of roots change from positive to negative or *vice versa* (in the sense of 3.3.1). Thus, each of the 'factors' of (I',c') is positive. This implies that (I',c') is positive, because the concatenation of positive tours is positive (Lemma 3.3.2). Lastly, since (I,c) is maximal under refinement, so is (I',c'). □

Here are two equivalent ways to specify a finite increasing subsequence in $\mathsf{Gaps}(I)$:

(1) Give a surjective bound-preserving map $I \to I'$ with finite fibers such that finitely many fibers have size ≥ 2.

(2) Give a sequence of wall-crossing pairs $(h_k,n_k) \in \mathsf{Walls}'(I,c)$.

For (1), the chosen gaps are the ones which appear in fibers of size ≥ 2. For (2), use Lemma 3.2.4(ii) to get a bijection between $\mathsf{Gaps}(I)$ and $\mathsf{Walls}'(I,c)$. In practice, (2) is the most convenient way. As the notation indicates, the first coordinates of the pairs (h_k, n_k) will be the folding walls.

Define the *folding category* as follows. Its objects are positive galleries. A morphism $(I',c') \hookrightarrow (I,c)$ is an index map $I \to I'$ as in (1) such that (I',c') is the corresponding folding of (I,c). Morphisms compose in the obvious way. Taking the discrepancy gives a functor from the folding category to $\mathcal{W}$.

4.4.2. *Excess.* Fix a tour (I,c). For any wall H, define the H-*excess* $e(H,c)$ as follows:

$$\begin{aligned} \mathsf{cross}(H,c) &:= (\text{number of times } H \text{ is crossed by } (I,c)) \\ \mathsf{sep}(H,c_{\hat{0}},c_{\hat{1}}) &:= (1 \text{ if } H \text{ separates } c_{\hat{0}} \text{ from } c_{\hat{1}} \text{ and } 0 \text{ otherwise}) \\ e(H,c) &:= \mathsf{cross}(H,c) - \mathsf{sep}(H,c_{\hat{0}},c_{\hat{1}}) \end{aligned}$$

This number is nonnegative and even, and it is nonzero for only finitely many H. Thus, we may define the *excess* of (I,c) to be $e(c) := \sum_H e(H,c)$.

Lemma. *For any i, if $(c_{\hat{0}}, c_i, c_{\hat{1}})$ is reduced, then $e(I,c) = e([\hat{0},i],c) + e([i,\hat{1}],c)$.*

Proof. The $\mathsf{cross}(H,-)$ term is always additive. The $\mathsf{sep}(H,-)$ term is additive if $(c_{\hat{0}}, c_i, c_{\hat{1}})$ is reduced. $\square$

4.4.3. *Interaction with lengths.* Intuitively, we want to write

$$(\text{number of wall-crossings of } (I,c)) = e(c) + (\text{number of walls separating } c_{\hat{0}} \text{ from } c_{\hat{1}}),$$

but this does not make sense because both sides are infinite. However, we can get an equality of integers by subtracting two copies of this equation, as the next lemma shows.

Lemma. *Let (I,c) be a positive gallery starting at B. Choose a wall-crossing pair (h,n) which gives a single-wall folding $(I',c') \hookrightarrow (I,c)$. Then*

$$1 = e(c) - e(c') + \ell_B(c'_{\hat{1}} \xrightarrow{r_h} c_{\hat{1}}).$$

Proof. The result is equivalent to

$$1 - \ell_B(r_h c_{\hat{1}} \xrightarrow{r_h} c_{\hat{1}}) = \sum_H \Big(\mathsf{cross}(H,c) - \mathsf{sep}(H,B,c_{\hat{1}}) - \mathsf{cross}(H,c') + \mathsf{sep}(H,B,r_h c_{\hat{1}})\Big),$$

because $c_{\hat{0}} = c'_{\hat{0}} = B$ and $c'_{\hat{1}} = r_h c_{\hat{1}}$. To evaluate the sum, we pair up the terms for H and $r_h H$, to obtain

$$\sum_{\substack{\{H, r_h H\} \\ H \neq h}} \left[\begin{array}{l} \mathsf{cross}(H,c) - \mathsf{cross}(H,c') + \mathsf{cross}(r_h H, c) - \mathsf{cross}(r_h H, c') \\ \quad -\Big(\mathsf{sep}(H,B,c_{\hat{1}}) - \mathsf{sep}(H,B,r_h c_{\hat{1}}) + \mathsf{sep}(r_h H,B,c_{\hat{1}}) - \mathsf{sep}(r_h H,B,r_h c_{\hat{1}})\Big) \end{array} \right]$$
$$+ \mathsf{cross}(h,c) - \mathsf{cross}(h,c') - \Big(\mathsf{sep}(h,B,c_{\hat{1}}) - \mathsf{sep}(h,B,r_h c_{\hat{1}})\Big)$$

This manipulation is safe because all but finitely many summands are zero, as we have already noted. The result will follow from

$$\mathsf{cross}(H,c) - \mathsf{cross}(H,c') + \mathsf{cross}(r_h H, c) - \mathsf{cross}(r_h H, c') = 0$$

$$\mathsf{cross}(h,c) - \mathsf{cross}(h,c') = 1$$

$$\begin{aligned} \mathsf{sep}(H,B,c_{\hat{1}}) - \mathsf{sep}(H,B,r_h c_{\hat{1}}) \\ + \mathsf{sep}(r_h H,B,c_{\hat{1}}) - \mathsf{sep}(r_h H,B,r_h c_{\hat{1}}) \end{aligned} = \begin{cases} \pm 2 & \text{if } H \text{ lengthens } r_h c_{\hat{1}} \xrightarrow{r_h} c_{\hat{1}} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathsf{sep}(h,B,c_{\hat{1}}) - \mathsf{sep}(h,B,r_h c_{\hat{1}}) = \pm 1$$

where the two $\pm$ signs are $+$ if the reflection arrow is oriented away from B, and $-$ otherwise. As before, we assume that $H \neq h$.

The first equation holds because reflecting part of (I,c) across h does not change the combined number of crossings with H and $r_h H$. The second equation holds because the folding removes the n-th crossing with h. For the third equation, rewrite the LHS as

$$\mathsf{sep}(H,B,c_{\hat{1}}) - \mathsf{sep}(H,B,r_h c_{\hat{1}}) + \mathsf{sep}(H,r_h B,r_h c_{\hat{1}}) - \mathsf{sep}(H,r_h B,c_{\hat{1}}).$$

Then the equation follows by checking all 7 partitions in 4.3.2. The fourth equation follows from the definition of the phrase 'oriented away from B' (4.2.2). □

4.4.4. **Corollary.** *Every gallery (I,c) admits a reduced folding with trivial discrepancy.*

Proof. Let us recall the classical 'deletion' algorithm. If (I,c) is nonreduced, then some wall H is crossed at least twice. Use the wall-crossing pairs $(H,1),(H,2)$ to get a folding $(I',c') \hookrightarrow (I,c)$ with trivial discrepancy. Repeat this process until (I',c') is reduced.

In the double affine setting, to prove that this algorithm terminates, it suffices to show that $e(c') = e(c) - 2$, since the excess is a nonnegative integer. To show this, apply the previous lemma twice, once for $(H,1)$ and once for $(H,2)$. (This assumes that (I,c) is a positive gallery starting at B, so that $\ell_B(-)$ of the discrepancy is defined. The general case, which we will not use, follows from adapting the proof of the previous lemma.) □

4.4.5. A *disk* in a groupoid is a sequence of arrows whose composite is an identity arrow.

Theorem. [MO, Thm. 3.7] *Let C_0 be an upward chamber supported at $\{0\}$, and let $w \in \tilde{W}$ be an element such that $\delta(wC_0) > 0$. If $\mathcal{K}$ is a disk of reflection arrows in $\mathcal{W}$ which contains wC_0, then the sum of $\ell_{C_0}(-)$ along $\mathcal{K}$ equals zero.*

Proof. It suffices to prove the result for one choice of C_0, because any other choice is related to it by the W-action. For a suitable choice of C_0, [MO, Thm. 3.7] asserts that

$$\ell_{C_0}(wC_0 \xrightarrow{r} rwC_0) = \ell^{\mathsf{MO}}(rw) - \ell^{\mathsf{MO}}(w),$$

for any reflection r, where $\ell^{\mathsf{MO}} : \{w \,|\, \delta(wC_0) > 0\} \to \mathbb{Z}$ is a function defined in [MO, 3.2] via an explicit formula. Summing along the disk $\mathcal{K}$ gives the result. □

4.4.6. **Corollary.** *If $\mathcal{K}$ is a disk of supported reflection arrows in $\mathcal{W}$, then the sum of ℓ_B along $\mathcal{K}$ equals zero.*

Proof. Let $\mathcal{K} = K_0 \xrightarrow{r_m} K_{m-1} \xrightarrow{r_{m-1}} \cdots \xrightarrow{r_1} K_0$ where $r_m = (r_1 \cdots r_{m-1})^{-1}$. Choose a gallery (I,c) from B to K_0 and a sequence of wall-crossing pairs $(h_1,n_1),\ldots,(h_m,n_m)$ such that the sequence of discrepancy maps of the successive single-wall foldings

$$(I^m, c^m) \xhookrightarrow{(h_m,n_m)} (I^{m-1}, c^{m-1}) \xhookrightarrow{(h_{m-1},n_{m-1})} \cdots \xhookrightarrow{(h_1,n_1)} (I^0, c^0) = (I, c)$$

equals $\mathcal{K}$. (This is possible because the reflection arrows are supported. The idea is to choose (I, c) to be sufficiently wiggly.) Applying Lemma 4.4.3 to each of these foldings gives

$$(\text{sum of lengths along } \mathcal{K}) = m + e(c^m) - e(c).$$

We want to apply the previous theorem. By translating everything and choosing $w \in \tilde{W}$ appropriately, we may assume that (C_0, B, K_0, wC_0) is a reduced positive tour. Choose reduced galleries from C_0 to B, and from K_0 to wC_0, and concatenate them to (I, c) to obtain a larger gallery $(\hat{I}, \hat{c})$. Successively fold $(\hat{I}, \hat{c})$ along the same wall-crossing pairs (h_i, n_i) to obtain galleries $(\hat{I}^i, \hat{c}^i)$, and take discrepancy maps to obtain a disk $\hat{\mathcal{K}}$ which contains wC_0. As before, we have

$$(\text{sum of lengths along } \hat{\mathcal{K}}) = m + e(\hat{c}^m) - e(\hat{c}).$$

The previous theorem says that the left hand side is zero. Additivity of excess under ‘reduced’ concatenation (4.4.2) implies that the right hand side equals $n + e(c^m) - e(c)$. Thus, the sum of lengths along $\mathcal{K}$ is also zero. $\square$

4.4.7. Let $\mathcal{W}^{B,\pm} \subset \mathcal{W}$ be the subgroupoid whose objects are positive chambers and whose morphisms are generated by supported reflection arrows. This differs from $\mathcal{W}^B$ because we do not require that the reflection arrows are oriented away from B.

Corollary. *There is a unique function $\ell_B : \mathsf{Arr}(\mathcal{W}^{B,\pm}) \to \mathbb{Z}$ which is additive under composition and sends every reflection arrow to its length.*

Here $\mathsf{Arr}(-)$ denotes the set of arrows of a (small) category.

Proof. ℓ_B is uniquely determined because the morphisms of $\mathcal{W}^{B,\pm}$ are generated by supported reflection arrows. The previous corollary guarantees that no contradictions arise. $\square$

4.4.8. **Corollary.** *For any positive chamber T, the preorder $\langle T \xrightarrow{\mathcal{W}^{B,\pm}} \mathcal{W}^B \rangle$ is a poset, and ℓ_B gives a strictly increasing function from its set of objects to $\mathbb{Z}$.*

Proof. Each nonidentity arrow in $\mathcal{W}^B$ is a composition of reflection arrows oriented away from B, so ℓ_B is positive on it. $\square$

As a consequence, each Bruhat preorder $\langle T \xrightarrow{\mathcal{W}} \mathcal{W}^B \rangle$ is a poset, because each of its connected components identifies with some $\langle T \xrightarrow{\mathcal{W}^{B,\pm}} \mathcal{W}^B \rangle$, possibly for a different T.

4.5. **Cocovers.**

4.5.1. A *dihedral* subarrangement of $\mathcal{H}$ is one which is generated by reflections through two distinct walls. It may be finite, affine, or pre-affine.

Lemma. *Fix a positive chamber C. Let H_1, H_2, H_3 be three consecutive walls in a dihedral subarrangement $\mathcal{H}'$ such that*

- *Each one of H_1, H_2, H_3 separates C from B.*

- *The projections of B and C to $\mathcal{H}'$ are not adjacent to H_2.*
- *The reflection $r_2C \xrightarrow{r_2} C$ is supported.*

After possibly switching H_1 and H_3, the arrow $r_2C \xrightarrow{r_2} C$ admits a nontrivial factorization in $\mathcal{W}^B$ which includes the arrow $r_1C \xrightarrow{r_1} C$.

Proof. The projections of r_1C and r_2C to $\mathcal{H}'$ are separated by exactly two walls, say H_4, H_5, as shown in Figure 1. After possibly switching H_1 and H_3, we may assume that the two

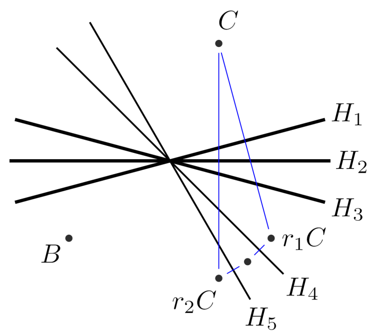

FIGURE 1. The dihedral arrangement generated by H_1, H_2, H_3 which is used for factoring $r_2C \to C$.

reflection arrows

$$r_2C \xrightarrow{r_5} r_5r_2C = r_4r_1C \xrightarrow{r_4} r_1C$$

are oriented away from B. These compose with $r_1C \xrightarrow{r_1} C$ to give $r_2C \xrightarrow{r_2} C$.

If C is weak, then the hypothesis that $r_2C \xrightarrow{r_2} C$ is supported implies that H_2 contains one of $\overline{B}$ or $\overline{C}$. Since the projections of B and C are not adjacent to H_2, every wall of $\mathcal{H}'$ contains one of $\overline{B}$ or $\overline{C}$. Therefore, the reflection arrows discussed above are all supported. $\square$

4.5.2. **Corollary.** *Any non-identity arrow in $\mathcal{W}^B$ factors into length-1 arrows. Thus, each Bruhat poset $\langle T \xrightarrow{\mathcal{W}^{B,\pm}} \mathcal{W}^B \rangle$ is graded by ℓ_B.*

Proof. It suffices to show that every reflection arrow of length > 1 factors nontrivially into reflection arrows oriented away from B. If a reflection arrow $r_\alpha C \xrightarrow{r_\alpha} C$ has length > 1, then it is lengthened by some wall $H \neq H_\alpha$. Let $\mathcal{H}'$ be the dihedral subarrangement generated by H and H_α. Let H_1 and H_2 be the two walls of $\mathcal{H}'$ which are next to H_α. Applying the previous lemma to H_1, H_α, H_2 gives the desired factorization of $r_\alpha C \xrightarrow{r_\alpha} C$. $\square$

4.5.3. Assume that $G^{\mathsf{fin}} = \mathrm{SL}_2$. In this case, the length-1 arrows were explicitly described in [W, Prop. 21]. We review this description here.

The ambient space is $\mathfrak{h} = \mathbb{R} \oplus \mathbb{R}d$, and points inside it will be denoted (x, y). There are only two roots $\pm\alpha^{\mathsf{fin}}$ of SL_2, so the double affine roots are $\pm\alpha^{\mathsf{fin}} + n\delta + m\pi$. The walls are in bijection with the roots of the form $\alpha_{n,m} := \alpha^{\mathsf{fin}} + n\delta + m\pi$, and each wall is a line:

$$H_{\alpha_{n,m}} = \{(x, y) \,|\, x + ny + m = 0\}.$$

We abbreviate $H_{n,m} := H_{\alpha_{n,m}}$ and $r_{n,m} := r_{\alpha_{n,m}}$. For convenience, assume that $\delta(B) > 0$. (This can always be achieved by translation.)

We now fix a strict chamber C and seek to describe length-1 arrows ending at C. Choose points $(x_b, y_b) \in \mathsf{relint}\,\overline{B}$ and $(x_c, y_c) \in \mathsf{relint}\,\overline{C}$. Then $0 < y_b < y_c$. Define the sets

$$\begin{aligned} S &= \{(n,m) \,|\, H_{n,m} \text{ separates } C \text{ from } B\} \\ S^+ &= \{(n,m) \,|\, \alpha_{n,m} \text{ is positive on } C \text{ and negative on } B\} \\ S^- &= \{(n,m) \,|\, \alpha_{n,m} \text{ is negative on } C \text{ and positive on } B\}. \end{aligned}$$

Then we have $S = S^+ \sqcup S^-$ and

$$\mathring{S}^+ := \{\alpha_{n,m}(x_b, y_b) < 0 < \alpha_{n,m}(x_c, y_c)\} \subseteq S^+ \subseteq \{\alpha_{n,m}(x_b, y_b) \le 0 \le \alpha_{n,m}(x_c, y_c)\} =: \overline{S}^+$$

$$\mathring{S}^- := \{\alpha_{n,m}(x_b, y_b) > 0 > \alpha_{n,m}(x_c, y_c)\} \subseteq S^- \subseteq \{\alpha_{n,m}(x_b, y_b) \ge 0 \ge \alpha_{n,m}(x_c, y_c)\} =: \overline{S}^-$$

Explicitly, $\overline{S}^+$ is the set of lattice points in a closed 'sector' bounded by two rays in the directions $(1, -y_b)$ and $(1, -y_c)$, and $\mathring{S}^+$ is the set of lattice points in the interior of that sector. Let R_B and R_C denote the sets of lattice points on the two rays.

Reflection of roots corresponds to reflection of lattice points:

$$r_{n,m}(\alpha_{n',m'}) = -\alpha_{2n-n',2m-m'}.$$

Fix a point $(n,m) \in S$, which gives a positive-length reflection arrow $r_{n,m}C \xrightarrow{r_{n,m}} C$. By 4.3.2, a wall $H_{n',m'}$ lengthens this arrow if and only if $\alpha_{n',m'}$ is positive on $B, r_{n,m}C$ and negative on $r_{n,m}B, C$, or vice versa. Equivalently,

$$\{(n',m'), (2n-n', 2m-m')\} \subset S^+ \quad \text{or} \quad \{(n',m'), (2n-n', 2m-m')\} \subset S^-.$$

If $(n,m) \in S^+$, then only the containment in S^+ is possible. Similarly for S^-. Thus, a point $(n,m) \in S^+$ corresponds to a length-1 arrow if and only if

$$S^+ \cap \big((2n, 2m) - S^+\big) = \{(n,m)\},$$

i.e. the reflection of any other point of S^+ across (n,m) lies outside of S^+. According to [W, Def. 19], these points are called the *corners* of S^+. From now on, we will just consider S^+, since the analysis for S^- is similar.

Fix a point $(n_0, m_0) \in \mathring{S}^+$. Then all corners are contained in $S_B^+ \cup S_C^+$, where

$$\begin{aligned} S_B^+ &:= S^+ \cap \{\alpha_{n,m}(x_b, y_b) \ge \tfrac{1}{2}\alpha_{n_0,m_0}(x_b, y_b)\} \\ S_C^+ &:= S^+ \cap \{\alpha_{n,m}(x_c, y_c) \le \tfrac{1}{2}\alpha_{n_0,m_0}(x_c, y_c)\} \end{aligned}$$

Indeed, if $(n,m) \in S^+$ does not lie in the union of these subsets, then $(2n - n_0, 2m - m_0) \in S^+$, so (n,m) is not a corner. From now on, we will just consider S_C^+, since S_B^+ is similar.

4.5.4. **Lemma.** *If $R_C \cap S^+$ is nonempty, then S_C^+ contains finitely many corners of S^+.*

Proof. Because C projects to a non-liminal chamber of $\mathcal{H}|_{\overline{C}}$, the inclusion $R_C \cap S^+ \subset R_C$ is finite or cofinite. (For a more precise statement, see [W, Prop. 16].) Thus, there exists a point $(n_1, m_1) \in R_C \cap S^+$ such that the set of points in R_C which are strictly lower than (n_1, m_1) is either contained in S^+ or disjoint from it. Reflecting (n_1, m_1) shows that every corner lies in $\{\alpha_{n,m}(x_b, y_b) \ge \frac{1}{2}\alpha_{n_1,m_1}(x_b, y_b)\}$ or in the (finite) part of R_C which is weakly higher than (n_1, m_1). The intersection of the former with S_C^+ is bounded. $\square$

Let us remark that, if $y_c = \delta(C)$ is irrational, then R_C contains at most one lattice point, so the first sentence of the preceding proof is trivial in this case.

4.5.5. **Lemma.** *If $\delta(C)$ is rational, then S_C^+ contains finitely many corners of S^+.*

Proof. (Following [W, Prop. 23].) Since y_c is rational, S_C^+ is covered by finitely many lines of slope $-y_c$. We claim that each line contains finitely many corners. For the outermost line (containing R_C), this follows from the previous lemma. For any other line, let (n_1, m_1) be the highest point in the intersection of this line with S^+. Reflecting (n_1, m_1) shows that no lower point on this line can be a corner. □

4.5.6. **Lemma.** *For any $\epsilon > 0$, there exists N such that, if $(n, m) \in S_C^+$ is a corner of S^+ with $n > N$, then $\alpha_{n,m}(x_c, y_c) < \frac{1+\epsilon}{n}$.*

Proof. Suppose that $(n, m) \in S_C^+$ is a corner of S^+. We will apply Minkowski's Theorem to the parallelogram

$$P := \left\{ (n', m') \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 0 < \alpha_{n',m'}(x_c, y_c) < 2\alpha_{n,m}(x_c, y_c) \\ 0 > \alpha_{n',m'}(x_b, y_b) > 2\alpha_{n,m}(x_b, y_b) \end{array} \right\}$$

which is symmetric about (n, m). Since every lattice point in P lies in $\mathring{S}^+$, and (n, m) is a corner, P cannot contain lattice points other than (n, m). Minkowski's Theorem implies

$$\mathrm{Area}(P) = \frac{4\,\alpha_{n,m}(x_c, y_c)\,\alpha_{n,m}(x_b, y_b)}{y_b - y_c} \leq 4.$$

Note that $\alpha_{n,m}(x_b, y_b)$ and $(y_b - y_c)$ are negative and the area formula is positive.

Next, we bound $\alpha_{n,m}(x_b, y_b)$. Since $(n, m) \in S_C^+$, we have $\alpha_{n,m}(x_c, y_c) \leq \frac{1}{2}\alpha_{n_0,m_0}(x_c, y_c)$. This rewrites as

$$\alpha_{n,m}(x_b, y_b) \leq (y_b - y_c)n + (x_b - x_c) + \tfrac{1}{2}\alpha_{n_0,m_0}(x_c, y_c).$$

For any $\epsilon > 0$, there exists N such that $n > N$ implies that the right hand side is $< \frac{(y_b - y_c)n}{1+\epsilon}$. Combining this with the previous paragraph gives the result. □

4.5.7. *Remark.* In the previous proof, we saw that, if $(n, m) \in S_C^+$ is a corner, then P does not contain any lattice points other than (n, m). The converse holds with $\overline{P}$ in place of P. Now let us set $x_b = y_b = x_c = 0$ and assume that $y_c > 0$ is irrational. Then

$$\overline{P} = \left\{ (n', m') \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} 0 \leq (-m') \leq 2(-m) \\ 0 \leq n' - \frac{1}{y_c}(-m') \leq 2\big(n - \frac{1}{y_c}(-m)\big) \end{array} \right\}.$$

This definition can be interpreted as follows: if we view $\frac{n}{-m}$ as a one-sided Diophantine approximation to $\frac{1}{y_c}$, meaning that $n - \frac{1}{y_c}(-m)$ is small and positive, then a lattice point (n', m') belongs to $\overline{P}$ if and only if $\frac{n'}{-m'}$ is another one-sided Diophantine approximation whose denominator (resp. error) is at most twice as large as the denominator (resp. error) of $\frac{n}{-m}$. In particular, if no such (n', m') exists, then $\frac{n}{-m}$ is a 'best' approximation. These have been studied in [HaTu, §4]. Using those results, it should be possible to show that, for most irrational values of y_c, there are infinitely many corners (n, m).

4.5.8. Now let G^{fin} be arbitrary. Then $\mathcal{H}$ is the disjoint union of finitely many 'rank-2' subarrangements $\mathcal{H}_{\alpha^{\mathsf{fin}}} := \{\pm\alpha^{\mathsf{fin}} + n\delta + m\pi \,|\, (n,m) \in \mathbb{Z}^2\}$, where α^{fin} ranges over roots of G^{fin}. Each length-1 arrow $r_\alpha C \xrightarrow{r_\alpha} C$ in $\mathcal{H}$ projects to a length-1 arrow in the rank-2 subarrangement containing α. Applying the previous lemmas (4.5.4, 4.5.5, 4.5.6) to the finitely many rank-2 subarrangements gives the following statement about $\mathcal{H}$.

Define $\mathsf{slope}(\alpha^{\mathsf{fin}} + n\delta + m\pi) := |n|$.

Corollary. *Let C be a strict chamber. There exist $M, N > 0$ such that, if $r_\alpha C \xrightarrow{r_\alpha} C$ is length-1 with $\mathsf{slope}(\alpha) > N$, then at least one of the following is true:*

- *$\delta(C)$ is irrational, and*
$$(\textit{distance from } \overline{C} \textit{ to } r_\alpha\overline{C}) < \frac{M}{\mathsf{slope}(\alpha)}.$$
- *Same statement with B in place of C.*

In particular, if $\delta(B)$ and $\delta(C)$ are rational, then there are finitely many length-1 arrows ending at C.

If $\delta(C)$ is irrational, then $\overline{C}$ is a point, and the constants M, N can be chosen so as to depend continuously on this point. Similarly for the second bullet.

4.6. **Factorizations of Bruhat arrows.**

4.6.1. For any positive gallery (I, c) starting at B, let $\mathsf{dis}(c) \subset \langle \mathcal{W} \to c_{\hat{1}} \rangle$ be the set of discrepancies of foldings of c.

Lemma. *For any reduced positive gallery (I, c) from B to C, we have $\langle \mathcal{W}^B \to C \rangle \subseteq \mathsf{dis}(c)$.*

Proof. Choose an object $C' \to C$ of $\langle \mathcal{W}^B \to C \rangle$, and factor it as a composite of reflection arrows oriented away from B:
$$C' = C_0 \xrightarrow{r_1} C_1 \xrightarrow{r_2} \cdots \xrightarrow{r_n} C_n = C$$
Let H_i be the wall of r_i. Because each reflection arrow is oriented away from B, it is possible to successively fold c along $H_n, \ldots, H_1$ to obtain a sequence of single-wall foldings
$$c^0 \hookrightarrow c^1 \hookrightarrow \cdots \hookrightarrow c^n = c$$
whose sequence of discrepancies equals the previous sequence. Thus, the chosen object of $\langle \mathcal{W}^B \to C \rangle$ equals the discrepancy of $c^0 \hookrightarrow c$. $\square$

We will eventually show that the inclusion in the lemma is an equality (Theorem 4.7.4). This gives an alternative definition of $\mathcal{W}^B$: it is the subcategory whose morphisms are discrepancies of foldings of reduced galleries.

4.6.2. *Factorizations versus galleries.* Fix a non-identity arrow $C' \to C$ in $\mathcal{W}^B$ and a reduced positive gallery (I, c) from B to C. We have seen that $C' \to C$ factors into length-1 arrows (4.5.2), and the proof of the previous lemma tells us how these factorizations interact with (I, c). Namely, such a factorization is specified by reflections $r_1, \ldots, r_n$, and successively folding along the walls $H_n, \ldots, H_1$ gives galleries $c^n, \ldots, c^0$. Since the reflections are length-1, the folded galleries $c^n, \ldots, c^0$ are reduced. Thus, each reflection wall H_k intersects

c^k exactly once, so we must fold along $(H_k, 1) \in \mathsf{Walls}'(c^k)$ to get c^{k-1}; this means that the sequence of foldings is uniquely determined by the factorization. Let $(h_k, 1) \in \mathsf{Walls}'(c)$ be the pair which corresponds to $(H_k, 1) \in \mathsf{Walls}'(c^k)$ via the folding $c^k \hookrightarrow c$.

Define a total order on $\{h_1, \ldots, h_n\}$ by saying that $h_k \leq h_m$ if c crosses h_k before h_m. (More formally, each h_k gives an element of $\mathsf{Gaps}(I)$, and we use the obvious order on gaps.) For each k, if $h_k < h_{k+1}, \ldots, h_n$, then $h_k = H_k$.

Remark. Let us say that a factorization is *optimal* if $(h_1, \ldots, h_k)$ is increasing. In the classical setting, any non-identity arrow in $\mathcal{W}^B$ has a unique optimal factorization, and this fact can be used to show that the two definitions of the Bruhat order agree.[4] In the double affine setting, the proof of uniqueness still works (see below), but we do not think that an optimal factorization always exists. Instead, we will show that, at irrational levels, there is a sorting algorithm (4.6.4) which 'almost' produces an optimal factorization; this will be enough to prove that the two definitions of the Bruhat order agree (Theorem 4.7.4).

Lemma. *A non-identity arrow in $\mathcal{W}^B$ has at most one optimal factorization.*

Proof. (Following [BjBr, Lem. 2.7.2].) Suppose we are given two optimal factorizations, which correspond to wall sequences $(H_i)_i$ and $(H'_i)_i$. If $H_1 = H'_1$, we may cancel the first reflection and proceed inductively. If $H_1 \neq H'_1$, then assume without loss of generality that c crosses H_1 before H'_1. Since c^0 is obtained by folding the reduced gallery c^1 across H_1, the wall H_1 does not separate $c^0_{\hat{1}}$ from B. Since $c^{0,\prime}$ is reduced and crosses H_1, the wall H_1 separates $c^{0,\prime}_{\hat{1}}$ from B. But $c^0_{\hat{1}} = C' = c^{0,\prime}_{\hat{1}}$, contradiction. □

4.6.3. *Dihedral cases.* Continue to work in the setting of 4.6.2, i.e. $C' \to C$ is a non-identity arrow in $\mathcal{W}^B$, and (I, c) is a reduced gallery from B to C. Suppose we are given a particular factorization of $C' \to C$ into length-1 arrows. Fix an index $k \in [2, n]$ and let $\mathcal{H}'$ be the dihedral subarrangement generated by H_{k-1} and H_k. Then the projections of c^k, c^{k-1}, c^{k-2} to $\mathcal{H}'$ are described by exactly one of the four cases shown in Figure 2. First, we explain the left hand side. The green line is c^k. Let us denote the wall numbered i by $H^{(i)}$. By definition, $H^{(1)}$, $H^{(3)}$, and $H^{(4)}$ are the first, second-to-last, and last walls crossed by c^k, and $H^{(2)}$ is the wall right before $H^{(1)}$. Also, $H^{(4')} := r^{(1)}H^{(4)}$, where $r^{(1)}$ is the reflection along $H^{(1)}$.

Next, we explain the right hand side. An arrow labeled i indicates a single-wall folding along $H^{(i)}$. The four cases are given by the four two-step paths in the diagram, where the second row gives c^{k-1} and the third row gives c^{k-2}. To see that these are the only possible cases, note that H_k must be the first or last wall crossed by c^k (otherwise c^{k-1} would not be reduced), and similarly for $k-1$ in place of k.

We have chosen the number labels to ensure that $h_{k-1} < h_k$ if and only if the number label of H_{k-1} is less than that of H_k. Let us denote the four cases as $21, 4'1, 14, 34$.

To convince the reader that this description is always valid, let us discuss some degenerate situations. First, it is possible that $\mathcal{H}'$ is pre-affine. If the projection of B is liminal, then there is no first wall crossed by c^k, so the only possible case is 34. Similarly, if the

[4] The first definition (which we have chosen) is that the Bruhat order is generated by reflection arrows, and the second definition is that two Weyl group elements satisfy $u \preceq w$ if and only if some (equivalently every) reduced expression for w admits a subexpression with product u, see [BjBr, Thm. 2.2.2].

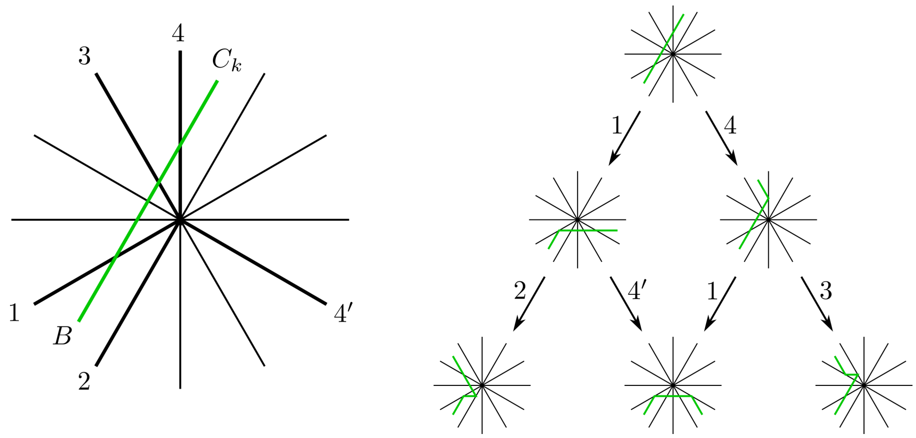

FIGURE 2. All possible cases for two consecutive foldings of a reduced gallery, each of which has a length-1 discrepancy.

projection of C_k is liminal, then the only possible case is 21. From now on, assume that both projections are non-liminal. It is possible that some of the numbered walls coincide, and all such possibilities are summarized below:

$\mathcal{H}'$-walls	c^k-length	coincidences	
>2	2	$1 = 3$	$2 = 4'$
2	2	$1 = 3$	$2 = 4 = 4'$
3	3	$2 = 4$	$3 = 4'$
$n \geq 4$	same n	$2 = 4$	

The 'c^k-length' column shows the number of walls of $\mathcal{H}'$ which are crossed by c^k.

Lemma. *It is never the case that* $H^{(1)} = H^{(4')}$ *or* $H^{(2)} = H^{(3)}$.

Proof. This follows from inspecting the previous table. We can also argue directly. Since c^k crosses at least two walls in $\mathcal{H}'$, we have $H^{(1)} \neq H^{(4)}$ which implies $H^{(1)} \neq H^{(4')}$. Next, we show that $H^{(2)} \neq H^{(3)}$. If c^k does not cross all walls of $\mathcal{H}'$, then c^k does not cross $H^{(2)}$, but it does cross $H^{(3)}$, so these two walls are not equal. If c^k does cross all walls of $\mathcal{H}'$, then one can check directly that $H^{(2)} = H^{(4)}$, which is not equal to $H^{(3)}$ by definition. $\square$

4.6.4. *Sorting algorithm.* Now assume that $\delta(C)$ is irrational.

Let $k \in [2, n]$ be any index such that $h_{k-1} > h_k$. We will describe a *move* associated to (h_{k-1}, h_k), which modifies the factorization by replacing r_{k-1}, r_k by two new reflections r'_{k-1}, r'_k so that the opposite inequality holds for the new walls: $h'_{k-1} < h'_k$. (The other reflections will be unchanged.) Since $h_{k-1} > h_k$, one of the cases 21 or $4'1$ obtains.

(Case 21) Choose the new reflections so that case 34 obtains.

Explicitly, this means that $r_{k-1} = r^{(2)}, r_k = r^{(1)}$, and we define $r'_{k-1} = r^{(3)}, r'_k = r^{(4)}$. Note that, since $\delta(C)$ is irrational, the projection of C is non-liminal, so $H^{(3)}$ and $H^{(4)}$ exist.

(Case $4'1$) Choose the new reflections so that case 14 obtains.

Explicitly, this means that $r_{k-1} = r^{(4')}, r_k = r^{(1)}$, and we define $r'_{k-1} = r^{(1)}, r'_k = r^{(4)}$. Since case $4'1$ obtains, $H^{(1)}$ and $H^{(4)}$ exist, so $H^{(4')}$ exists as well.

The new reflection arrows are oriented away from B, and there is a commutative diagram (below), so the new reflections give a valid factorization into length-1 arrows.

$$\begin{tikzcd} C_{k-2} \arrow[r, "r_{k-1}"] \arrow[d, equal] & C_{k-1} \arrow[dr, "r_k"] & \\ & & C_k \\ C'_{k-2} \arrow[r, "r'_{k-1}"'] & C'_{k-1} \arrow[ur, "r'_k"'] & \end{tikzcd}$$

Since $3 < 4$ and $1 < 4$, the new walls satisfy $h'_{k-1} < h'_k$. More precisely, the new walls are related to the old ones in the following way:

(Case 21) $h_k < h_{k-1} \le h'_{k-1} < h'_k$ (because $1 < 2 < 3 < 4$)

Also, the wall h_m is unchanged if $m > k$, $h_m < h_k$, or $h_m > h'_k$.

(Case $4'1$) Swap h_{k-1}, h_k to get h'_{k-1}, h'_k.

All other walls h_m $(m \ne k, k-1)$ are unchanged.

This follows from inspecting the gallery (green path) in the picture showing the cases.

Here is the algorithm. At each step, choose an index $k \in [2, n]$, ensuring that each index is chosen infinitely often. If $h_{k-1} > h_k$, apply the (h_{k-1}, h_k) move.

4.6.5. The next lemma says that the algorithm almost sorts $(h_1, \ldots, h_n)$, except for some walls which converge toward the level $\delta(C)$. In this lemma, $\delta(h_k)$ denotes the level of the location where c crosses h_k. (More precisely, $(h_k, 1) \in \mathsf{Walls}'(c)$ corresponds to a gap $i < j$ in I, and we define $\delta(h_k) := \delta(c_i) = \delta(c_j)$, where the equality follows from the fact that c_i and c_j are adjacent.) This notation will be used throughout the rest of this subsection.

Lemma. *Assume that C is strict. As the algorithm runs, each index $k \in [1, n]$ satisfies exactly one of the following statements.*

(1) Eventually[5] *$\delta(h_k) = \delta(C)$ and h_k is constant.*

(2) Eventually $\delta(h_k) < \delta(C)$ and $\delta(h_k)$ converges to $\delta(C)$.

(3) Eventually $\delta(h_k) < \delta(C)$ and h_k is constant.

This implies the following:

- *(3)-indices < (2)-indices < (1)-indices.*
- *For the (1) and (3) indices, the eventual value of h_k is increasing with k.*
- *For all (1)-indices, all (3)-indices, and the largest (2)-index, we have $H_k = h_k$.*

[5] I.e., there exists some step in the algorithm after which the statement is always true.

Proof. Apply downward induction on k. The base case $k = n+1$ is vacuously true. Fix $k \in [1, n]$ and assume that each index in $[k+1, n]$ satisfies one of (1), (2), (3). Then the indices in $[k+1, n]$ also satisfy the bullets.

Suppose that eventually $\delta(h_k) = \delta(C)$. Then all indices in $[k+1, n]$ satisfy (1). Since h_{k+1} is eventually constant, the move (h_k, h_{k+1}) eventually does not occur, so h_k is eventually weakly increasing. Since $\delta(C)$ is irrational, c has finitely many wall-crossings at level $\delta(C)$, by Lemma 3.4.2. Thus h_k is eventually constant, so k satisfies (1), as desired. From now on, we may assume that $\delta(h_k) < \delta(C)$ infinitely often.

Suppose that $\delta(h_k) = \delta(C)$ infinitely often. Then there are infinitely many steps at which $\delta(h_k)$ goes from $< \delta(C)$ to $= \delta(C)$ due to a move (h_{k-1}, h_k) or (h_k, h_{k+1}). Each time this happens, the number of indices $m \in [k, n]$ which satisfy $\delta(h_m) = \delta(C)$ increases by at least one, and this number never decreases. This gives a contradiction. From now on, we may assume that eventually $\delta(h_k) < \delta(C)$.

Suppose that h_k decreases infinitely often. Since h_k can decrease only via the move (h_k, h_{k+1}), this move must occur infinitely often. This move is guaranteed to change h_{k+1}, so h_{k+1} does not stabilize. Therefore, the index $k+1$ must satisfy (2). Since the move (h_k, h_{k+1}) only occurs when $h_k > h_{k+1}$, we conclude that $\delta(h_k)$ converges to $\delta(C)$, i.e. the index k satisfies (2), as desired. From now on, we may assume that h_k is weakly increasing.

Suppose for sake of contradiction that the index k does not satisfy (2) or (3), i.e. that $\delta(h_k)$ does not converge to $\delta(C)$, and that h_k does not stabilize. Since h_k is weakly increasing, and there are finitely many walls in $\mathsf{Walls}(c)$ of any fixed slope,[6] the slope of h_k diverges to infinity. Since $\delta(h_k)$ is weakly increasing but does not converge to $\delta(C)$, the indices $m \in [k+1, n]$ which satisfy (1) or (2) eventually also satisfy $h_k < h_m$. This implies that h_k and H_k eventually differ only by reflections r_{h_m} for indices $m \in [k+1, n]$ which satisfy (3). Since these h_m are eventually constant, the slope of H_k diverges to infinity as well.

Define the folding $d^k \hookrightarrow c$ using the walls h_m for all indices $m \in [k+1, n]$ which satisfy (3). Then d^k is eventually constant. We will now give two statements which intuitively mean that c^k 'converges' to d^k as the algorithm progresses. First, the indices in $[k+1, n]$ which are not accounted for in d^k must satisfy (1) or (2), and this implies

(CON1) For any $i \in I$ with $\delta(c_i) < \delta(C)$, the galleries c^k and d^k eventually agree prior to i.

Next, let $m+1 \in [k+1, n]$ be the smallest index which satisfies (1) or (2), and fold $h_{m+1}, \ldots, h_n$ to obtain $c^m \hookrightarrow c$. Since c^m is reduced, Lemma 4.6.6 (below) tells us that the endpoint $\overline{c}^m_{\hat{1}}$ converges to $\overline{c}_{\hat{1}}$. Now fold the indices $[k+1, m]$ (which satisfy (1)) to get

(CON2) The sequence of points $\overline{c}^k_{\hat{1}}$ converges to $\overline{d}^k_{\hat{1}}$.

Finally, we will show that $\delta(h_k)$ converges to $\delta(C)$, which gives the desired contradiction. We will fix $i \in I$ such that $\delta(c_i) < \delta(C)$ and prove that eventually $i < h_k$. Since H_k is the wall of a length-1 arrow ending at $c^k_{\hat{1}}$, and the slope of H_k diverges to infinity, Corollary 4.5.8 says that the *horizontal* distance from H_k to $\overline{c}^k_{\hat{1}}$ converges to zero.[7] Now (CON2) implies that

---

[6]Recall from 4.5.8 that $\mathsf{slope}(\alpha^{\mathsf{fin}} + n\delta + m\pi) := |n|$.

[7]The horizontal distance equals $\frac{1}{2}\|\overline{c}^k_{\hat{1}} - r_{H_k}\overline{c}^k_{\hat{1}}\|$ because orthogonal reflections are level-preserving. Since the point $\overline{c}^k_{\hat{1}}$ moves during the algorithm, we need to know that the constants in Corollary 4.5.8 depend

the horizontal distance from H_k to $\overline{d}^k_{\hat{1}}$ also converges to zero. Since the slope of H_k diverges to infinity, H_k cannot cross $([\hat{0}, i], d^k)$,[8] and this eventually equals $([\hat{0}, i], c^k)$ by (CON1). Since h_k is the c-index where H_k crosses c^k, we conclude that $i < h_k$, as desired. $\square$

4.6.6. **Lemma.** *Let (I, c) be a gallery from B to a strict chamber C at irrational level. For any $\epsilon > 0$, there exists $i \in I$ such that*

- $\delta(c_i) < \delta(C)$.
- *For any folding $c' \hookrightarrow c$ such that c' is reduced and all folding walls come after i, we have $\|\overline{c}'_{\hat{1}} - \overline{c}_{\hat{1}}\| < \epsilon$.*

Proof. Since $\delta(C) \notin \mathbb{Q}$, the proof of Lemma 2.6.3 gives a finite set of roots α such that

(i) $\alpha(B) < 0$ and $\alpha(C) > 0$.

(ii) $\cap\{\alpha \geq 0\}$ contains a neighborhood of $\overline{C}$.

(iii) $(\cap\{\alpha \geq 0\}) \cap \delta^{-1}(\delta(C))$ lies in the ϵ-ball centered at $\overline{C}$.

Let i be the first index such that $c_i \in \cap\{\alpha \geq 0\}$. Statement (ii) implies that $\delta(c_i) < \delta(C)$. Take a folding $c' \hookrightarrow c$ as in the second bullet. Since all folding walls come after i, statement (i) implies that c' crosses each H_α prior to reaching c_i. Since c' is reduced, it cannot cross any H_α for a second time. This implies that $c'_{\hat{1}} \in \cap\{\alpha \geq 0\}$, so (iii) finishes the proof. $\square$

Now we deduce some consequences of the sorting algorithm.

4.6.7. **Lemma.** *Let (B_0, B_1, C) be a reduced positive tour such that $\delta(C)$ is irrational. Then $\langle \mathcal{W}^{B_1} \to C\rangle \subset \langle \mathcal{W}^{B_0} \to C\rangle$*

Proof. We will fix an arrow $C' \to C$ in $\mathcal{W}^{B_1}$ and show that it lies in $\mathcal{W}^{B_0}$. Refine the tour (B_1, C) to a reduced gallery (I, c), and factor the arrow $C' \to C$ into length-1 arrows oriented away from B_1 (4.5.2). Then run the sorting algorithm. It suffices to show that, at some step, each length-1 arrow is also oriented away from B_0. Equivalently, we want to ensure that each wall H_k does not separate B_0 from B_1.

First, suppose that (B_1, C) is weak. Since $\delta(C)$ is irrational, $\overline{B}_1 = \overline{C}$ and $\mathcal{H}_{\overline{C}}$ is a finite arrangement, so the sorting algorithm terminates. Now, since $(h_1, \ldots, h_n)$ is increasing, we have $H_k = h_k$ for all $k \in [1, n]$. Thus H_k crosses c and therefore does not separate B_0 from B_1, as desired.

Next, suppose that (B_1, C) is strict, so Lemma 4.6.5 applies. If $k \in [1, n]$ satisfies (1) or (3), then $H_k = h_k$, so we may argue as in the previous paragraph. From now on, assume that k satisfies (2).

Since each index $m \in [k, n]$ satisfies (1) or (2), Lemma 4.6.6 implies that each distance $\|\overline{c}^{m-1}_{\hat{1}} - \overline{c}_{\hat{1}}\|$ converges to zero. Since reflection across H_k sends $\overline{c}^k_{\hat{1}}$ to $\overline{c}^{k-1}_{\hat{1}}$, the horizontal distance from H_k to $\overline{c}_{\hat{1}} = \overline{C}$ converges to zero. This implies that, during the steps where H_k does not contain $\overline{C}$, the slope of H_k diverges to infinity, so H_k eventually does not separate

continuously on it (which is true), and we need to know that the sequence $\overline{c}^k_{\hat{1}}$ converges (which is (CON2)). Also, note that H_k cannot converge toward the start chamber B because h_k is weakly increasing.

[8] This uses the previous sentence and the assumption on i, which implies that $\delta(d^k_i) < \delta(C)$.

B_0 from B_1. Thus, we are done if we can ensure that there are infinitely many steps such that *none* of the H_k contain $\overline{C}$.

By refining the tour (B_0, B_1) and applying induction, we may assume that one of the following is true:

(i) There is no wall which contains $\overline{C}$ and separates B_0 from B_1.

(ii) There is exactly one such wall, and B_0 and B_1 are adjacent along it.

If (i) holds, then we are done by the previous reasoning.

Now assume that (ii) holds, and let H^{bad} denote the indicated wall. As suggested above, we will modify the algorithm to ensure that there are infinitely many steps such that no H_k equals H^{bad}. Every so often, the new algorithm will enter a second phase whose goal is to eliminate every occurrence of $H_k = H^{\mathsf{bad}}$. The second phase will iterate in increasing order through the set $S^{(2)} \subseteq [1, n]$ of (2)-indices. After step k, we promise that no index $m \leq k$ satisfies $H_m = H^{\mathsf{bad}}$. At the end, we promise that no $m \in [1, n]$ satisfies $H_m = H^{\mathsf{bad}}$.

- Suppose that we are not at the last step, i.e. k is not the largest element of $S^{(2)}$. If $H_k \neq H^{\mathsf{bad}}$, then do nothing. If $H_k = H^{\mathsf{bad}}$, then we apply the dihedral cases (4.6.3) to (h_k, h_{k+1}). Since B_0 and B_1 are adjacent along H^{bad}, we must have $H^{\mathsf{bad}} = H^{(1)}$ or $H^{(2)}$. Split into cases accordingly:

($H^{\mathsf{bad}} = H^{(1)}$) Case 14 obtains. Perform the reverse (h_k, h_{k+1}) move so that case $4'1$ obtains.

By Lemma 4.6.3, $H^{(4')} \neq H^{(1)}$, so $H^{(4')} \neq H^{\mathsf{bad}}$.

($H^{\mathsf{bad}} = H^{(2)}$) Case 21 obtains. Perform the (h_k, h_{k+1}) move so that case 34 obtains.

By Lemma 4.6.3, $H^{(3)} \neq H^{(2)}$, so $H^{(3)} \neq H^{\mathsf{bad}}$.

The new H_k is not equal to H^{bad}, and we have not changed H_m for $m \notin \{k, k+1\}$, so the promise is fulfilled.

- Suppose that we are at the last step, i.e. k is the largest element of $S^{(2)}$. Split into cases depending on $\delta(h_k)$:

($\delta(h_k) < \delta(C)$) Do nothing.

($\delta(h_k) = \delta(C)$) Run the original sorting algorithm on the indices $[k, n]$. Since all of the crossings involved occur at level $\delta(C)$, which is irrational, the algorithm terminates.

Now the sequence $(h_k, \ldots, h_n)$ is increasing. For each $m \in [k, n]$, we have $H_m = h_m$, which implies that H_m is crossed by c, so H_m does not separate B_0 from B_1, so $H_m \neq H^{\mathsf{bad}}$. We have not changed H_m for $m < k$, so the final promise is fulfilled.

If we initiate the second phase infinitely often, then there are infinitely many steps at which no H_k equals H^{bad}, as desired.

Each time the second phase ends, we run the old algorithm for a while. This is necessary for ensuring that the earlier application of Lemma 4.6.6 works, because that relies on the convergence $\delta(h_k) \to \delta(C)$ where k satisfies (1) or (2). After restarting the old algorithm, the old (3)-indices continue to satisfy (3), while some of the (2)-indices may become (1)-indices or (3)-indices, which only makes the situation better. The only way in which the second phase violates the specifications of the old algorithm is that it sometimes performs the reverse (h_k, h_{k+1}) move when k and $k+1$ both satisfy (2), and this only occurs when the case 14 obtains, so the move merely switches h_k and h_{k+1}. This does not affect the convergence $\delta(h_k) \to \delta(C)$ for (2)-indices, provided that we run the old algorithm for long enough between each occurrence of the second phase. □

4.6.8. **Proposition.** *Let (I, c) be a reduced positive gallery from B to a chamber C at irrational level. Then $\langle \mathcal{W}^B \to C \rangle = \mathsf{dis}(c)$.*

Proof. We have already proved the $\subseteq$ direction in Lemma 4.6.1, so we focus on the $\supseteq$ direction. We will fix a folding $c' \hookrightarrow c$ and show that the discrepancy $c'_{\hat{1}} \to C$ lies in $\mathcal{W}^B$.

List the folding walls of $c' \hookrightarrow c$ in increasing order as $h_1, \dots, h_n \in \mathsf{Walls}(I, c)$, and fold them one-by-one in decreasing order to obtain a sequence of single-wall foldings

$$c' = d^0 \xhookrightarrow{h_1} \cdots \xhookrightarrow{h_{n-1}} d^{n-1} \xhookrightarrow{h_n} d^n = c$$

The composition of the discrepancies

$$d^0_{\hat{1}} \xrightarrow{r_1} d^1_{\hat{1}} \xrightarrow{r_2} \cdots \xrightarrow{r_n} d^n_{\hat{1}}$$

equals the discrepancy $c'_{\hat{1}} \to C$. Choose an increasing sequence $\hat{0} = i_0, i_1, \dots, i_n = \hat{1} \in I$ such that i_k lies between h_k and h_{k+1}, where these walls are viewed as elements of $\mathsf{Gaps}(I)$.

We will prove that, for each $k \in [0, n]$, the discrepancy $d^k_{\hat{1}} \to C$ lies in $\mathcal{W}^{c_{i_k}}$. The case $k = 0$ is the desired result. Apply downward induction on k. The base case $k = n$ is obvious because $d^n_{\hat{1}} = c_{\hat{1}} = C$. Assume that $k < n$ and that $d^{k+1}_{\hat{1}} \to C$ lies in $\mathcal{W}^{c_{i_{k+1}}}$. Split into cases depending on whether h_{k+1} separates $d^{k+1}_{\hat{1}}$ from c_{i_k}.

- Assume that h_{k+1} separates $d^{k+1}_{\hat{1}}$ from c_{i_k}, so that $d^k_{\hat{1}} \xrightarrow{r_{k+1}} d^{k+1}_{\hat{1}}$ is oriented away from c_{i_k}. Taking $(B_0, B_1, C) = (c_{i_k}, c_{i_{k+1}}, C)$ in Lemma 4.6.7 allows us to replace $\mathcal{W}^{c_{i_{k+1}}}$ by $\mathcal{W}^{c_{i_k}}$ in the inductive hypothesis. Now
$$d^k_{\hat{1}} \to d^{k+1}_{\hat{1}} \to C$$
is the composite of two maps in $\mathcal{W}^{c_{i_k}}$, so it lies in $\mathcal{W}^{c_{i_k}}$, as desired.
- Assume that h_{k+1} does not separate $d^{k+1}_{\hat{1}}$ from c_{i_k}. This implies that h_{k+1} separates $d^{k+1}_{\hat{1}}$ from $c_{i_{k+1}}$, so that $d^k_{\hat{1}} \xrightarrow{r_{k+1}} d^{k+1}_{\hat{1}}$ is oriented away from $c_{i_{k+1}}$. Now
$$d^k_{\hat{1}} \to d^{k+1}_{\hat{1}} \to C$$
is the composite of two maps in $\mathcal{W}^{c_{i_{k+1}}}$, so it lies in $\mathcal{W}^{c_{i_{k+1}}}$. Finally, applying Lemma 4.6.7 as before allows us to replace $\mathcal{W}^{c_{i_{k+1}}}$ by $\mathcal{W}^{c_{i_k}}$, as desired. □

4.6.9. **Theorem.** *If c_1 and c_2 are two reduced positive galleries with the same start and end chambers, then $\mathsf{dis}(c_1) = \mathsf{dis}(c_2)$.*

Proof. Let the start and end chambers be B and C. Split into three cases.

- Assume that $\delta(C)$ is irrational. Then the previous proposition suffices.

- Assume that $\delta(C)$ is rational and (B, C) is strict. We can find reduced positive galleries as in Figure 3, where unlabeled nodes are irrational-level chambers. Using

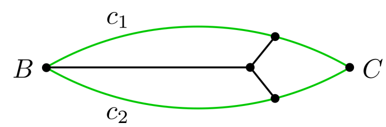

FIGURE 3. Idea for factoring a braid move of galleries with a shared rational-level end chamber into three braid moves involving irrational-level start or end chambers.

the previous case, we can move from c_1 to c_2, passing through two other reduced galleries, without changing $\mathsf{dis}(-)$.

- Assume that $\delta(C)$ is rational and (B, C) is weak. This is the easy case, so we merely sketch an *ad hoc* argument. We claim that, for any reduced positive gallery (I, c), the set $\mathsf{dis}(c)$ can be explicitly described. Split into subcases:
  - Suppose that B and C are both [upward or liminal] or both [downward or liminal]. Then the classification of positive pairs (Proposition 3.3.4) implies that one of $\overline{B}$ or $\overline{C}$ contains the other. In this case, the Bruhat order is classical (4.2.3) and the desired result is known. Specifically, c is equivalent to a gallery in an affine arrangement from one chamber to [another chamber or a 'chamber at infinity'], and $\mathsf{dis}(c)$ is known in either case.
  - The only remaining possibility is that B is downward and C is upward. In this case, the structure of positive galleries (3.4.3) implies that there is a unique index $i \in I$ such that $\mathsf{dim}\, \overline{c}_i = \mathsf{dim}\, \mathfrak{h} - 1$. We think of c as the concatenation $([\hat{0}, i], c) \diamond ([i, \hat{1}], c)$. The discrepancies of these two subgalleries are described by the previous subcase, and $\mathsf{dis}(c)$ is the 'product' of these two discrepancy sets.

  In particular, in the second subcase, $\mathsf{dis}(c)$ depends only on c_i. A further elementary computation shows that it also does not depend on c_i. $\square$

4.7. **Demazure product.**

4.7.1. For any nonreduced positive gallery c, a *greedy move* folds c along the first entry in $\mathsf{Walls}'(c)$ of the form $(H, 2)$, i.e. at the first point where a wall is crossed for the second time.

Lemma. *Every positive gallery becomes reduced after a finite number of greedy moves.*

Proof. Every gallery can be expressed as the concatenation of n reduced galleries, for some n. Induct on n. The base case $n = 1$ is trivial. Assume that $n \geq 2$ and that the claim holds for $n - 1$. Consider a gallery c which is the concatenation of n reduced galleries $c^1, \ldots, c^n$, with start and end chambers as shown:

$$C_0 \xrightarrow{\;c^1\;} C_1 \xrightarrow{\;c^2\;} C_2 \xrightarrow{\;c^3\;} \cdots \xrightarrow{\;c^n\;} C_n$$

The sequence of walls in $\mathsf{Walls}(c)$ which are folded by the greedy moves is increasing. Since c^1 is reduced, the greedy moves do not fold any wall of c^1. We claim that there are finitely many greedy moves which fold walls in c^2. These moves give a sequence of successive foldings $(d^k)_k$ of the concatenation $c^1 \diamond c^2$, hence a sequence of discrepancy maps

$$\cdots \longrightarrow d^1_{\hat{1}} \longrightarrow d^0_{\hat{1}} = C_2$$

Since $c^1 \diamond c^2$ is a concatenation of two reduced galleries, and this property is preserved by greedy moves, these reflection arrows are all oriented toward C_0. Therefore $\ell_{C_0}(d^k_{\hat{1}} \to C_2)$ is negative and strictly increases with k. On the other hand, Lemma 4.4.3 implies that

$$\ell_{C_0}(d^k_{\hat{1}} \to C_2) = k + e(d^k) - e(c^1 \diamond c^2),$$

which is bounded below by $-e(c^1 \diamond c^2)$. This proves the claim.

The output of the aforementioned greedy moves is a gallery which is the concatenation of $(n-1)$ reduced galleries. The inductive hypothesis says that the remaining set of greedy moves is also finite. This completes the inductive step. □

Remark. The lemma remains valid if greedy moves are generalized in the following way: if $(H, 2)$ is the first double-crossing, we have the option to fold $(H, 1)$ or $\{(H, 1), (H, 2)\}$. The same proof works. Indeed, during the inductive step, if we make the former choice infinitely often, then $\ell_{C_0}(d^k_{\hat{1}} \to C_2)$ decreases without bound. If not, then after some point we only make the latter choice, and this produces a reduced gallery in finitely many steps, as observed in Corollary 4.4.4.

For any positive gallery c, let $c^{\mathsf{dem}} \hookrightarrow c$ be the (reduced) folding obtained by performing greedy moves. The *Demazure product* of c is the discrepancy $c^{\mathsf{dem}}_{\hat{1}} \to c_{\hat{1}}$.

4.7.2. Let c be a positive gallery. A *generalized braid move* changes a reduced sub-gallery of c into another reduced gallery with the same start and end chambers. A *deletion move* is a folding along one of two *consecutive* wall-crossings which involve the same wall: $\{(H, n), (H, n+1)\}$. It is easy to see that every greedy move can be expressed as the composite of a generalized braid move, a deletion move, and a generalized braid move.

If c' is obtained from c via one of these two moves, there is an obvious map $\varphi : c'_{\hat{1}} \to c_{\hat{1}}$ specified as follows. For a generalized braid move, note that $c'_{\hat{1}} = c_{\hat{1}}$ and take φ to be the identity. For a deletion move, which is a folding, take φ to be the discrepancy map.

Lemma. *Let c' be obtained from c via a generalized braid move or a deletion move. For every folding $d \hookrightarrow c$, there exists another folding $d' \hookrightarrow c'$ such that the discrepancies fit into a commutative diagram*

$$\begin{tikzcd} d'_{\hat{1}} \arrow[r, equal] \arrow[d] & d_{\hat{1}} \arrow[d] \\ c'_{\hat{1}} \arrow[r, "\varphi"] & c_{\hat{1}} \end{tikzcd}$$

Proof. In the case of a generalized braid move, φ is an identity map. The desired statement follows from the invariance of $\mathsf{dis}(c)$ under generalized braid moves (Theorem 4.6.9). In the case of a deletion move, φ is the discrepancy of the folding $c' \hookrightarrow c$. Given $d \hookrightarrow c$, define $d' \hookrightarrow c'$ as follows:

- If d folds at least one of $\{(H,n),(H,n+1)\}$, then we can view d as a folding of c' and set $d' = d$.
- If d does not fold either of $\{(H,n),(H,n+1)\}$, then define d' by additionally folding both of these wall-crossings.

It is easy to check that this construction of d' works. $\square$

4.7.3. **Corollary.** *Let c be a positive gallery starting at B, and let c' be a reduced gallery obtained from applying a finite sequence of generalized braid and deletion moves to c.*

(i) There is an obvious map $\varphi : c'_{\hat{1}} \to c_{\hat{1}}$ satisfying the conclusion of the previous lemma.

(ii) This map is the unique minimizer of $\ell_B(-)$ on the set $\mathsf{dis}(c)$.

(iii) This map equals the Demazure product $c^{\mathsf{dem}}_{\hat{1}} \to c_{\hat{1}}$.

Proof. Statement (i) follows from the previous lemma applied to each of the chosen generalized braid and deletion moves. For (ii), let $d_{\hat{1}} \to c_{\hat{1}}$ be any other element of $\mathsf{dis}(c)$, which comes from a folding $d \hookrightarrow c$. Statement (i) gives a folding $d' \hookrightarrow c'$ such that

$$\ell_B(d_{\hat{1}} \to c_{\hat{1}}) = \ell_B(d'_{\hat{1}} \to c'_{\hat{1}}) + \ell_B(c'_{\hat{1}} \to c_{\hat{1}}).$$

Since c' is reduced, Lemma 4.4.3 implies that $\ell_B(d'_{\hat{1}} \to c'_{\hat{1}}) \geq 0$ with equality if and only if $d'_{\hat{1}} \to c'_{\hat{1}}$ is the identity. For (iii), note that c^{dem} is also a reduced gallery obtained in the same way as c'. The uniqueness of (ii) implies that the two resulting φ maps are equal. $\square$

Remark. Fix the starting gallery c. The corollary implies that every finite sequence of generalized braid and deletion moves which ends in a reduced gallery has exactly $d(c) := e(c) + \ell_B(c^{\mathsf{dem}}_{\hat{1}} \to c_{\hat{1}})$ deletion moves. In fact, the following improvement is true, although we will not use it: in every (possibly infinite) sequence of such moves, there are at most $d(c)$ deletion moves, and all galleries which come after the $d(c)$-th deletion move are reduced. This follows from a later result (homotopical deletion, 8.8.1) which roughly says that any two sequences of such moves can be 'compared,' meaning that there is a 'homotopy' from one sequence to the other.

4.7.4. **Theorem.** *For any reduced positive gallery (I, c) from B to C, we have $\langle \mathcal{W}^B \to C\rangle = \mathsf{dis}(c)$.*

Proof. We have already proved the $\subseteq$ direction in Lemma 4.6.1, so we focus on the $\supseteq$ direction, i.e. every discrepancy of a reduced positive gallery starting at B lies in $\mathcal{W}^B$. (Let us call these 'reduced discrepancies' for short.) The statement is obvious if the discrepancy has length 1. It suffices to show that every reduced discrepancy of length > 1 factors nontrivially into reduced discrepancies, because the length of a reduced discrepancy must be positive (4.4.3).

Fix a reduced discrepancy, which is a map in $\mathcal{W}$. By definition, it can be realized as $c'_{\hat{1}} \to c_{\hat{1}}$ where c is reduced. Denote the folding walls by $h_1, \ldots, h_n \in \mathsf{Walls}(c)$. Choose a realization which minimizes the (positive) number

$$\min_{m \in [1,n]} \ell_B\big(r_{h_m} c_{\hat{1}} \xrightarrow{r_{h_m}} c_{\hat{1}}\big).$$

Then let m be any index which attains the minimum, and let $c^m \hookrightarrow c$ be the single-wall folding for h_m. Corollary 4.7.3 gives a diagram

$$\begin{tikzcd} d_{\hat{1}} \arrow[r, equal] \arrow[d] & c'_{\hat{1}} \arrow[d] & \\ c_{\hat{1}}^{m,\mathsf{dem}} \arrow[r] & c_{\hat{1}}^{m} \arrow[r] & c_{\hat{1}} \end{tikzcd}$$

for some folding $d \hookrightarrow c^{m,\mathsf{dem}}$. This gives a factorization

$$\left[c'_{\hat{1}} \to c_{\hat{1}}\right] = \left[d_{\hat{1}} \to c_{\hat{1}}^{m,\mathsf{dem}} \to c_{\hat{1}}\right]$$

of the original reduced discrepancy into two reduced discrepancies. If this factorization is nontrivial, then we are done. If it is trivial, then the first factor must be an identity, i.e. the original discrepancy equals $c_{\hat{1}}^{m,\mathsf{dem}} \to c_{\hat{1}}$.

If $c^{m,\mathsf{dem}} = c^m$, then c^m is reduced, so the original discrepancy has length 1, and we are done. If not, then $c^{m,\mathsf{dem}}$ is obtained by performing at least one greedy move on c^m. The first greedy move folds the first double-crossing of c^m, which we denote $(H,2) \in \mathsf{Walls}'(c^m)$. The first two H-crossings of c^m, denoted $(H,1),(H,2) \in \mathsf{Walls}'(c^m)$, correspond to entries $(h,1),(h',1) \in \mathsf{Walls}'(c)$, where $H = h = r_{h_m}h'$. Since $(H,2)$ is the first double-crossing of c^m, the walls h, h_m, h' are consecutive in the dihedral subarrangement which they generate, and applying Lemma 4.5.1 to the chambers B and $c_{\hat{1}}$ shows that

$$\ell_B\big(r_{h_m}c_{\hat{1}} \xrightarrow{r_{h_m}} c_{\hat{1}}\big) > \ell_B\big(r_h c_{\hat{1}} \xrightarrow{r_h} c_{\hat{1}}\big) \text{ or } \ell_B\big(r_{h'}c_{\hat{1}} \xrightarrow{r_{h'}} c_{\hat{1}}\big).$$

Split into cases accordingly.

- Assume that the inequality with h' holds. Since $c^{m,\mathsf{dem}} \hookrightarrow c$ folds h', the above inequality contradicts the minimality assumption.
- Assume that the inequality with h holds. Create $c^{m,\mathsf{dem2}} \hookrightarrow c^m$ by folding at h and then doing greedy moves until the gallery is reduced. Since the folding of c^m at h can be realized as the composite of a generalized braid move, a deletion move, and a generalized braid move (4.7.2), the discrepancy of $c^{m,\mathsf{dem2}} \hookrightarrow c^m$ agrees with that of $c^{m,\mathsf{dem}} \hookrightarrow c^m$ by Corollary 4.7.3. Now $c^{m,\mathsf{dem2}} \hookrightarrow c$ folds h and realizes the original discrepancy, so the above inequality contradicts minimality. $\square$

4.7.5. *Maximality of the Demazure product.*

Corollary. *Let c be a positive gallery starting at B. Then $c_{\hat{1}}^{\mathsf{dem}} \to c_{\hat{1}}$ is the unique maximal element of the full subposet of $\langle \mathcal{W}^B \xrightarrow{\mathcal{W}} c_{\hat{1}}\rangle$ spanned by $\mathsf{dis}(c)$.*

Proof. Every element of $\mathsf{dis}(c)$ is the discrepancy $d_{\hat{1}} \to c_{\hat{1}}$ of some folding $d \hookrightarrow c$. By Corollary 4.7.3(i), this discrepancy factors as $d'_{\hat{1}} \to c_{\hat{1}}^{\mathsf{dem}} \xrightarrow{\varphi} c_{\hat{1}}$ for some folding $d' \hookrightarrow c^{\mathsf{dem}}$. By the previous theorem, $d'_{\hat{1}} \to c_{\hat{1}}^{\mathsf{dem}}$ lies in $\mathcal{W}^B$ because c^{dem} is reduced. $\square$

5. Demazure category

From now on, fix a 'fundamental' chamber as in 4.2.2, but call it C_0 instead of B. All tours are positive and start at C_0 unless otherwise specified. We also allow tours to stutter finitely many times. This will not affect any subsequent application of our earlier results on tours, but note that the category of tours is no longer a poset (Lemma 3.1.4).

In the classical setting, we have constructed, jointly with Roman Travkin, the *Demazure category*, which parameterizes Demazure varieties, see [TaTr, §3]. This uses the framework of *rigidly bistratified categories* in [TaTr, 2.4]. The advantage of using this framework is that it implies that colimits indexed by the Demazure category are easy to compute inductively, as discussed in [TaTr, 3.3.2].

The goal of this section is to construct a double affine analogue of the Demazure category, using the same framework. In fact, all of the hard work has already been done, since we have already developed the notion of tours, the Bruhat order, and the Demazure product. The constructions and proofs which go into the framework are conceptually identical to the classical case. For this reason, we suggest reading 2.4, 3.1, 3.2 in [TaTr] first.

5.1. **Jointed tethered tours.**
We first define the set of objects of the Demazure category.

5.1.1. Fix a positive chamber T. A *tethered chamber* is an object of the Bruhat poset $\langle T \xrightarrow{\mathcal{W}} \mathcal{W}^{C_0}\rangle$. A *tethered tour* $([n], c, w)$ consists of a tour $([n], c)$ and a map $T \xrightarrow{w} c_n$ which makes its end chamber into a tethered chamber. Its *length* is

$$\ell([n], c, w) := \ell_{C_0}\big(T \xrightarrow{w} c_n) + e(c).$$

5.1.2. A *jointed tour* $([n], c, f)$ consists of a tour $([n], c)$ and a sequence of *joint faces* $([1, n], f)$ such that $f_i \preceq c_i$ for all $i \in [1, n]$. We say that the chamber c_i is *unjointed* if $f_i = c_i$. It is sometimes convenient to write a jointed tour as $c_0 \diamond_{f_1} c_1 \diamond_{f_2} \cdots \diamond_{f_n} c_n$. We will omit writing the joint of an unjointed step, i.e. $c_1 \diamond c_2 := c_1 \diamond_{c_2} c_2$.

The i-th *socket chamber* is the Tits product $f_i c_{i-1}$, and these will play a significant role. The *socket tour* is the (unjointed) tour $(c_0, f_1 c_0, c_1, f_2 c_1, c_2, \ldots, c_n)$.

5.1.3. A jointed tour is *threadable* if it satisfies the following conditions:

- *(Opposition.)* For $i \in [1, n]$, if f_i is horizontal, then
  - $f_i = \tilde{c}_i$. (The latter is the support face of c_i, see 2.4.2.)
  - c_i is upward in $\mathcal{H}_{f_i}$.
  - $f_i c_{i-1}$ is downward in $\mathcal{H}_{f_i}$.
- *(Support-matching.)* If c_i and c_j are at the same level, then $\overline{c}_i \subseteq \overline{c}_j$ or $\overline{c}_i \supseteq \overline{c}_j$.

The objects of the Demazure category will be threadable jointed tethered tours. In this section, the threadability condition will merely be a nuisance, but it will become useful in subsequent sections (see the remark below). The word 'threadable' is motivated in 8.4.9.

Note that support-matching implies the following weaker condition:

- *(Support-consistency)* For each level a which has at least one chamber c_i, the intersection of supports of all chambers at level a is nonempty.

5.2. **Rotations.** The first ingredient of [TaTr, 2.4] is a right-cancellable category $\mathsf{Emb}^{\mathsf{c}}$ on the desired set of objects, and a non-full subcategory $\mathsf{Emb}^{\mathsf{d}} \subset \mathsf{Emb}^{\mathsf{c}}$. The latter will provide one class of 'generating morphisms' for the Demazure category. As in the cited paper, an $\mathsf{Emb}^{\mathsf{c}}$-morphism is denoted $\rightarrowtail$, and a $\mathsf{Dom}^{\mathsf{c}}$-morphism is denoted $\hookrightarrow$.

5.2.1. Define the groupoid $\mathsf{Emb}^{\mathsf{c}}$ as follows:

- Its objects are threadable jointed tethered tours.
- A morphism $([n], c, f, w) \to ([n], c', f', w')$ is a tuple $(w_i)_{i \in [1,n]}$, where $w_i \in W_{f_i}$ and
  - For all $i \in [n]$, we have $c'_i = (w_1 \cdots w_i) \cdot c_i$.
  - $w' = (w_1 \cdots w_n) \cdot w$.
- The composite of $([n], c, f, w) \xrightarrow{(w_i)_i} ([n], c', f', w') \xrightarrow{(w'_i)_i} ([n], c'', f'', w'')$ is the tuple $(w''_i)_i$ given by
$$w''_i = \underbrace{(w_1 \cdots w_{i-1})^{-1} w'_i (w_1 \cdots w_{i-1})} \cdot w_i.$$
  The brace term lies in W_{f_i} because $w'_i \in W_{f'_i}$ and $f'_i = (w_1 \cdots w_{i-1}) \cdot f_i$.

Here is the motivation for our definition of morphisms. The element w_n specifies a way of 'rotating' c_n around its joint f_n. Similarly, w_{n-1} specifies a way of 'rotating' c_{n-1} around its joint f_{n-1}, but this 'rotation' should also affect c_n. Continuing in this way yields $([n], c', f')$.

Let $\mathsf{Emb}^{\mathsf{d}} \subset \mathsf{Emb}^{\mathsf{c}}$ be the non-full subcategory consisting of those morphisms $(w_i)_i$ such that, for each $i \in [1, n]$, the arrow $c_i \xrightarrow{w_i} w_i c_i$ lies in the Bruhat category $\mathcal{W}^{f_i c_{i-1}}$.

A morphism in $\mathsf{Emb}^{\mathsf{c}}$ can turn a non-stuttering tour into a stuttering tour. However, this is not true for a morphism in $\mathsf{Emb}^{\mathsf{d}}$.

Remarks.

(1) Requiring that $c_i \xrightarrow{w_i} w_i c_i$ lies in $\mathcal{W}^{f_i c_{i-1}}$ is equivalent to requiring that it lies in $\mathcal{W}^{c_i}$. This is because $w_i \in W_{f_i}$ and the chambers $f_i c_{i-1}, c_i$ project to the same chamber in $\mathcal{H}_{f_i}$.

(2) Threadability implies that $\overline{f_i c_{i-1}} = \overline{f}_i = \overline{c}_i$. (Proof: If f_i is horizontal, this follows from opposition. If f_i is not horizontal, this follows from $f_i \preceq f_i c_{i-1}, c_i$.) Therefore, the valid arrows $c_i \xrightarrow{w_i} w_i c_i$ are governed by a classical Bruhat order (4.2.3).

    Moreover, this classical Bruhat order is either the ordinary Bruhat order on a finite Coxeter group or the opposite of the ordinary Bruhat order on an affine Coxeter group. This more specific statement follows from opposition, which implies that, if f_i is horizontal, then $f_i c_{i-1}$ is downward and c_i is upward.

    In Section 6, the threadability condition will ensure that our Demazure varieties are twisted products of finite type flag varieties and thick affine flag varieties. (In other words, thin affine flag varieties and semi-infinite flag varieties do not arise as factors.) This is essentially forced by the previous paragraph.

In Section 8, the threadability condition will be useful for a completely unrelated reason: it ensures that every tour admits a 'threading path,' which is helpful for showing that a category of tours is contractible.

5.2.2. **Lemma.** *For every morphism* $([n], c, f, w) \xrightarrow{(w_i)_i} ([n], c', f', w')$ *in* $\mathsf{Emb}^{\mathsf{c}}$*, we have*

$$\ell([n], c', f', w') = \ell([n], c, f, w) + \sum_{i=1}^{n} \ell_{f_i c_{i-1}}(c_i \xrightarrow{w_i} w_i c_i).$$

In particular, a non-identity morphism in $\mathsf{Emb}^{\mathsf{d}}$ *is strictly length-increasing.*

The length $\ell(-)$ was defined in 5.1.1.

Proof. By factoring and possibly inverting the arrows $c_i \xrightarrow{w_i} w_i c_i$, we reduce to the case in which one such arrow $c_j \xrightarrow{w_j} w_j c_j$ is a length-1 arrow in $\mathcal{W}^{f_j c_{j-1}}$ and all other arrows are identities. Then w_j is the reflection across some wall H. Refine c' to a gallery g'. The subgallery of g' from c_{i-1} to c_i crosses H exactly once, and folding g' at this point gives a gallery g which refines c. Applying Lemma 4.4.3 to $g \hookrightarrow g'$ shows that

$$1 = e(c') - e(c) + \ell_{C_0}\big(c_n \xrightarrow{r_H} c'_n\big).$$

The definition of morphisms in $\mathsf{Emb}^{\mathsf{c}}$ implies that $w' = r_H w$, so

$$\ell_{C_0}\big(c'_n \xrightarrow{r_H} c_n\big) = \ell_{C_0}(T \xrightarrow{w'} c'_n) - \ell_{C_0}(T \xrightarrow{w} c_n),$$

and the result follows. □

5.3. **Coxeter and Demazure categories.** The second ingredient of [TaTr, 2.4] is a category $\mathsf{Dom}^{\mathsf{c}}$ on the same set of objects. Its morphisms are denoted $\twoheadrightarrow$. We also need $\mathsf{Dom}^{\mathsf{c}}$ to interact with $\mathsf{Emb}^{\mathsf{c}}$ and $\mathsf{Emb}^{\mathsf{d}}$ in a specified way, see below.

5.3.1. Define the category $\mathsf{Dom}^{\mathsf{c}}$ as follows:

- Its objects are threadable jointed tethered tours.
- A morphism $([n], c, f, w) \to ([n'], c', f', w)$ is a bound-preserving weakly-increasing map $\varphi : [n'] \to [n]$ such that
  - For each $j \in [n']$, we have $c'_j = c_{\varphi(j)}$.
  - For each $j \in [1, n']$, the following *Coxeter product conditions* are satisfied:
    - For each $i \in [\varphi(j-1)+1, \varphi(j)]$, we have $f'_j \subset \mathsf{span} f_i$.
    - f'_j lies in every wall which is double-crossed by $([\varphi(j-1), \varphi(j)], c)$.

In other words, $([n'], c', f', w)$ is obtained from $([n], c, f, w)$ by deleting or duplicating chambers and shrinking joint faces. Note that, if f'_j satisfies the Coxeter product conditions, then so does every face smaller than f'_j.

5.3.2. Next, we specify 'commutation relations' between the $\mathsf{Emb}^{\mathsf{c}}$-morphisms and the $\mathsf{Dom}^{\mathsf{c}}$-morphisms, as in [TaTr, 2.4.1]. Once this is done, we will be able to define the *Coxeter category* to be the category whose objects are jointed tethered tours, and whose morphisms are generated by the two aforementioned classes of morphisms, subject to the given commutation relations. The Coxeter category is 'coarser' than the desired Demazure category because it does not know about the Bruhat order or the Demazure product. In fact, the Demazure category will be a non-full subcategory of the Coxeter category.

The commutation relations are given by an *exchange map*, which sends a solid diagram (as below) to a dashed diagram.

$$\begin{tikzcd}
([n^0], c^0, f^0, w^0) \arrow[r, tail, "(w_i)_i"] \arrow[d, two heads, dashed, "\varphi:[n^2]\to[n^0]"] & ([n^0], c^1, f^1, w^1) \arrow[d, two heads, "\varphi:[n^2]\to[n^0]"] \\
([n^2], c^3, f^3, w^0) \arrow[r, tail, dashed, "(w'_j)_j"'] & ([n^2], c^2, f^2, w^1)
\end{tikzcd}$$

In our setting, given the solid diagram, we produce the dashed diagram by requiring that $w'_j = w_{\varphi(j-1)+1} \cdots w_{\varphi(j)}$ for every $j \in [1, n^2]$. Everything else is determined by this.

In [TaTr, 2.4.1], we ask that the exchange map satisfies the following axioms:

- If the following two solid diagrams are equal, then the outer diagrams are equal:

$$\begin{tikzcd}
 & & x_2 \arrow[drr, tail] \arrow[d, two heads, dashed] & & \\
x_1 \arrow[urr, tail] \arrow[d, two heads, dashed] & & \bullet \arrow[drr, tail, dashed] & & x_3 \arrow[d, two heads] \\
\bullet \arrow[urr, tail, dashed] \arrow[rrrr, tail, dashed] & & & & x_4
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x_2 \arrow[dr, tail] & \\
x_1 \arrow[ur, tail] \arrow[rr, tail, dashed] \arrow[d, two heads, dashed] & & x_3 \arrow[d, two heads] \\
\bullet \arrow[rr, tail, dashed] & & x_4
\end{tikzcd}$$

  The squares come from the exchange map, and the triangles come from composition in $\mathsf{Emb}^{\mathsf{c}}$. Also, we impose the analogous axiom for one $\rightarrowtail$ and two $\twoheadrightarrow$'s.

- The exchange map interacts with identity morphisms in the obvious way.

In our setting, the axiom with two $\rightarrowtail$ arrows holds because the formula for w'_j behaves well with respect to composition of rotations. The axiom with two $\twoheadrightarrow$ arrows holds because multiplication in W is associative. The second bullet is obvious.

Since the axioms are satisfied, [TaTr, 2.4.1] produces a category as described above. We call it the *Coxeter category* and denote it by C.

Remark. We do not use the notation $\mathsf{Rig}^{\mathsf{c}}$ from [TaTr], because even in the classical setting our current category is slightly different from the one defined in the cited paper. Indeed, the categories in the cited paper impose the requirement that consecutive chambers touch, i.e. $\overline{c}_{i-1} \cap \overline{c}_i$ is nonempty, see [TaTr, 3.3.3].

5.3.3. Next, [TaTr, 2.4.2] asks that we produce, for each $\mathsf{Dom}^{\mathsf{c}}$-morphism as shown, an $\mathsf{Emb}^{\mathsf{d}}$-morphism as shown. This is called the *turning map*.

$$\begin{tikzcd}
([n], c, f, w) \arrow[r, two heads, "\varphi:[n']\to[n]"] & ([n'], c', f', w) \arrow[r, hook, dashed, "(w_j)_j"] & ([n'], c'', f'', w''),
\end{tikzcd}$$

Given the $\mathsf{Dom}^{\mathsf{c}}$-morphism, we produce the $\mathsf{Emb}^{\mathsf{d}}$ morphism as follows: for each $j \in [1, n']$, choose any gallery g which refines the subtour $([\varphi(j-1), \varphi(j)], c)$, and require that the arrow $c'_j \xrightarrow{w_j} w_j c'_j$ equals the inverse of the discrepancy $g_{\hat{1}}^{\mathsf{dem}} \to g_{\hat{1}}$ defined in 4.7.1. In particular, the new chamber $w_j c'_j$ is obtained via the Demazure product.

Let us show that this construction is valid.

Claim. *For every $j \in [1, n']$, the following are true:*

(i) The inverse-discrepancy $g_{\hat{1}} \to g_{\hat{1}}^{\mathsf{dem}}$ does not depend on the choice of g.

(ii) The inverse-discrepancy can be expressed as $c'_j \xrightarrow{w_j} w_j c'_j$ for some $w_j \in W_{f'_j}$.

(iii) The inverse-discrepancy lies in $\mathcal{W}^{f'_j c'_{j-1}}$ or equivalently $\mathcal{W}^{c'_{j-1}}$.

These properties imply that $(w_i)_i$ gives a morphism in $\mathsf{Emb}^{\mathsf{d}}$.

Proof. Fix $j \in [1, n']$. Statement (i) is true because any two choices of g are related by generalized braid moves (4.7.2), and these do not affect the Demazure product (Corollary 4.7.3).

Next, we prove (ii). Consider the tour

$$c^{\mathsf{alt}} := (c_{\varphi(j-1)}, f'_j c_{\varphi(j-1)}, f'_j c_{\varphi(j-1)+1}, \ldots, f'_j c_{\varphi(j)} = c_{\varphi(j)}).$$

We claim that the tour $([\varphi(j-1), \varphi(j)], c)$ is related to c^{alt} via the following zig-zag of refinements: insert $f'_j c_{\varphi(j)-1}$, delete $c_{\varphi(j)-1}$, insert $f'_j c_{\varphi(j)-2}$, delete $c_{\varphi(j)-2}$, etc. We will explain the first two of these refinements in detail.

- The first step is to insert $f'_j c_{\varphi(j)-1}$ between the last two chambers $c_{\varphi(j)-1}$ and $c_{\varphi(j)}$ of $([\varphi(j-1), \varphi(j)], c)$. To show that this is a refinement, we need to show that the two-step tour $(c_{\varphi(j)-1}, f'_j c_{\varphi(j)-1}, c_{\varphi(j)})$ is reduced, i.e. there does not exist a wall H which separates $f'_j c_{\varphi(j)-1}$ from the other two chambers. Suppose for sake of contradiction that such an H exists. Since $f'_j c_{\varphi(j)-1}$ and $c_{\varphi(j)} = c'_j$ both have f'_j as a face, and H separates them, we must have $f'_j \subset H$. This means that H cannot separate $f'_j c_{\varphi(j)-1}$ from $c_{\varphi(j)-1}$, contradiction.
- The second step is to delete the chamber $c_{\varphi(j)-1}$. To show that this is a refinement, we need to show that the two-step tour $(c_{\varphi(j)-2}, c_{\varphi(j)-1}, f'_j c_{\varphi(j)-1})$ is reduced, i.e. there does not exist a wall H which separates $c_{\varphi(j)-1}$ from the other two chambers. Suppose for sake of contradiction that such an H exists. Since it separates $c_{\varphi(j)-1}$ from $f'_j c_{\varphi(j)-1}$, it does not contain f'_j. Thus, it does not separate $f'_j c_{\varphi(j)-1}$ from $c_{\varphi(j)}$. This implies that it does separate $c_{\varphi(j)-1}$ from $c_{\varphi(j)}$. It also separates $c_{\varphi(j)-1}$ from $c_{\varphi(j)-2}$ by assumption. Thus, it is double-crossed by $([\varphi(j-1), \varphi(j)], c)$. This contradicts the second Coxeter property, because we have already established that H does not contain f'_j.

The rest of the refinements are entirely similar.

Since changing g via generalized braid moves does not affect the Demazure product, we may replace g by a gallery g^{alt} (refining c^{alt}) without changing $g_{\hat{1}} \to g_{\hat{1}}^{\mathsf{dem}}$. Since $c_{\varphi(j-1)}$ and $f'_j c_{\varphi(j-1)}$ are two consecutive chambers of c^{alt}, the subgallery of g^{alt} bounded by these two chambers is reduced. The remaining chambers of g^{alt} all have f'_j as a face. The definition of the Demazure product via greedy moves (4.7.1) implies that the discrepancy $g_{\hat{1}}^{\mathsf{alt,dem}} \to g_{\hat{1}}^{\mathsf{alt}}$ lies in $W_{f'_j}$. This concludes the proof of (ii).

Lastly, we deduce (iii) from the maximality of the Demazure product (Corollary 4.7.5). This says that the discrepancy $g_{\hat{1}}^{\mathsf{dem}} \to g_{\hat{1}}$ is the unique maximal element of the full subposet of $\langle \mathcal{W}^{c'_{j-1}} \xrightarrow{\mathcal{W}} g_{\hat{1}} \rangle$ spanned by $\mathsf{dis}(g)$. Clearly, $g_{\hat{1}} \xrightarrow{\mathsf{id}} g_{\hat{1}}$ is an element of $\mathsf{dis}(g)$, so the inverse discrepancy $g_{\hat{1}} \to g_{\hat{1}}^{\mathsf{dem}}$ lies in $\mathcal{W}^{c'_{j-1}}$, as desired. $\square$

5.3.4. According to [TaTr, 2.4.2], the composite morphisms $\bullet \twoheadrightarrow \bullet \overset{\text{turn}}{\hookrightarrow} \bullet$ produced by the turning map constitute a non-full subcategory $\mathsf{Dom}^{\mathsf{d}} \subset \mathsf{C}$. Then the Demazure category is defined by two generating subcategories $\mathsf{Dom}^{\mathsf{d}}$ and $\mathsf{Emb}^{\mathsf{d}}$, just as the Coxeter category was defined by $\mathsf{Dom}^{\mathsf{c}}$ and $\mathsf{Emb}^{\mathsf{c}}$. The exchange map for $\mathsf{Dom}^{\mathsf{d}}$ and $\mathsf{Emb}^{\mathsf{d}}$ is defined using the exchange map for $\mathsf{Dom}^{\mathsf{c}}$ and $\mathsf{Emb}^{\mathsf{c}}$, which was given in 5.3.2.

For this to work, [TaTr, 2.4.2] says that we must check the following axioms:

- Suppose we are given a solid diagram as below:

$$\begin{tikzcd}
x_1 \arrow[d, two heads] & & \\
x_2 \arrow[r, hook, dashed, "\text{turn}"] \arrow[d, dashed, two heads] & x_3 \arrow[d, two heads] & \\
\bullet \arrow[r, tail, dashed] & x_4 \arrow[r, hook, dashed, "\text{turn}"] & x_5
\end{tikzcd}$$

  Fill in the square using the exchange map. Then we require that the turn of the vertical composite equals the horizontal composite.

- Suppose we are given a solid diagram as below:

$$\begin{tikzcd}
& x_2 \arrow[d, two heads] & \\
x_1 \arrow[ur, hook] \arrow[d, dashed, two heads] & x_3 \arrow[r, hook, dashed, "\text{turn}"] & \bullet \\
\bullet \arrow[ur, tail, dashed] \arrow[r, hook, dashed, "\text{turn}"'] & \bullet \arrow[ur, hook, dotted, "\exists"'] &
\end{tikzcd}$$

  Fill in the square using the exchange map, and create the horizontal $\hookrightarrow$'s using the turning map. We require that there exists a dotted arrow in $\mathsf{Emb}^{\mathsf{d}}$ making the lower square commute. (It is unique by right-cancellability.)

In these axioms, an arrow labeled 'turn' is the result of applying the turning map.

Verification of axioms. For the first axiom, let us denote the relevant index sets, chamber sequences, and index maps as follows. (We omit the joint faces and tethers because they are not important for this argument.)

$$\begin{tikzcd}
([n^1],c^1) \arrow[d, two heads, "\varphi^{2,1}:[n^2]\to[n^1]"] & & \\
([n^2],c^2) \arrow[r, hook, dashed, "\text{turn}"] \arrow[d, two heads, dashed] & ([n^2],c^3) \arrow[d, two heads, "\varphi^{4,2}:[n^4]\to[n^2]"] & \\
([n^4],c^6) \arrow[r, tail, dashed] & ([n^4],c^4) \arrow[r, hook, dashed, "\text{turn}"] & ([n^4],c^5)
\end{tikzcd}$$

Note that some of the index sets are the same because $\mathsf{Emb}^{\mathsf{c}}$-morphisms do not change the index sets. The vertical composition is governed by the composed index map $\varphi^{4,1} := \varphi^{2,1} \circ \varphi^{4,2}$. Let g be any gallery which refines $([n^1], c^1)$ and hence also $([n^2], c^2)$ and $([n^4], c^6)$. The first turn is defined by taking the Demazure products of the subgalleries of g corresponding to $([\varphi^{2,1}(j-1), \varphi^{2,1}(j)], c^1)$ for $j \in [1, n^2]$. Each of these Demazure products can be computed via greedy moves, which collectively define a folding $g' \hookrightarrow g$. By construction, g' is a gallery which refines $([n^2], c^3)$ and hence also $([n^4], c^4)$. Similarly, the second turn is defined by taking the Demazure products of the subgalleries of g' corresponding to $([\varphi^{4,2}(k-1), \varphi^{4,2}(k)], c^3)$ for $k \in [1, n^4]$, and the same procedure gives a folding $g'' \hookrightarrow g'$, where g'' refines $([n^4], c^5)$.

The turn of the vertical composition is defined by taking the Demazure products of the subgalleries of g corresponding to $([\varphi^{4,1}(k-1), \varphi^{4,1}(k)], c^1)$ for $k \in [1, n^4]$. Each such 'big' subgallery is a concatenation of some of the 'small' subgalleries of g which were used to define the first turn map. (Specifically, it is the concatenation of the subgalleries which correspond to intervals $[\varphi^{2,1}(j-1), \varphi^{2,1}(j)]$ for $j \in [1, n^2]$ which lie in the chosen $[\varphi^{4,1}(k-1), \varphi^{4,1}(k)]$.) To show that the turn of the vertical composition equals the horizontal composition, it suffices to show that the Demazure product of the 'big' subgallery can be computed by first folding each 'small' subgallery by a maximal sequence of greedy moves, and then folding the resulting 'modified big' gallery by a maximal sequence of greedy moves. This is true because the Demazure product can be computed using any sequence of moves which terminates in a reduced gallery (Corollary 4.7.3).

We remark that, in the classical setting, the first axiom is verified using the associativity of the Demazure product, see [TaTr, 3.1.4]. The ability to compute the Demazure product by first computing the Demazure product on 'small' subgalleries is a version of the associativity property which makes sense in the double affine setting.

For the second axiom, let us denote the relevant data as follows.

$$\begin{tikzcd}
 & ([n^1],c^2) \arrow[d, two heads, "\varphi^{3,1}:[n^3]\to[n^1]"] & \\
([n^1],c^1) \arrow[ur, tail, "(w_i)_i"] \arrow[d, two heads, dashed, "\varphi^{3,1}:[n^3]\to[n^1]"'] & ([n^3],c^3) \arrow[r, hook, dashed, "\text{turn}"] & ([n^3],c^4) \\
([n^3],c^5) \arrow[ur, tail, dashed] \arrow[r, hook, dashed, "\text{turn}"'] & ([n^3],c^6) \arrow[ur, hook, dotted, "\exists"'] &
\end{tikzcd}$$

The two index maps $\varphi^{3,1}$ are the same because of how the exchange map is defined. Let g be any gallery which refines $([n^1], c^2)$ and hence also $([n^3], c^3)$. Since the upper diagonal map belongs to $\mathsf{Emb}^{\mathsf{d}}$, each 'rotation' map $c_i^2 \xrightarrow{w_i} w_i c_i^2$ lies in $\mathcal{W}^{c^2_{i-1}}$, so Lemma 4.6.1 gives a folding $g' \hookrightarrow g$ such that g' refines $([n^1], c^1)$ and hence also $([n^3], c^5)$. The upper turn map is defined by taking the Demazure products of the subgalleries of g corresponding to $([\varphi^{3,1}(j-1), \varphi^{3,1}(j)], c^2)$ for $j \in [1, n^3]$. Similarly, the lower turn map is defined by taking the Demazure products of the subgalleries of g' corresponding to $([\varphi^{3,1}(j-1), \varphi^{3,1}(j)], c^1)$ for $j \in [1, n^3]$.

Fix an index j as above. Let γ be the corresponding subgallery of g, and let $\gamma' \hookrightarrow \gamma$ be the folding induced by $g' \hookrightarrow g$. In particular, $\gamma_{\hat{0}} = c^1_{\varphi^{3,1}(j-1)} = c^3_{j-1}$ and $\gamma_{\hat{1}} = c^1_{\varphi^{3,1}(j)} = c^3_j$. Unwinding the definitions, we reduce to showing that the following diagram in $\mathcal{W}$ admits a (unique) dashed map which lies in $\mathcal{W}^{c^3_{j-1}}$:

$$\begin{tikzcd}
 & \gamma_{\hat{1}} & \gamma^{\mathsf{dem}}_{\hat{1}} \arrow[l] \\
\gamma'_{\hat{1}} \arrow[ur] & \gamma'^{,\mathsf{dem}}_{\hat{1}} \arrow[l] \arrow[ur, dashed] &
\end{tikzcd}$$

Since $\mathcal{W}$ is a groupoid, the dashed map is uniquely determined. To see that it lies in $\mathcal{W}^{c^3_{j-1}}$, apply the maximality of the Demazure product (Corollary 4.7.5). The composition $\gamma'^{,\mathsf{dem}}_{\hat{1}} \to \gamma'_{\hat{1}} \to \gamma_{\hat{1}}$ lies in $\mathsf{dis}(\gamma)$ because it comes from the folding $\gamma'^{,\mathsf{dem}} \hookrightarrow \gamma' \hookrightarrow \gamma$. (The first folding comes from computing the Demazure product of γ' using greedy moves.) $\square$

Since the axioms are satisfied, [TaTr, 2.4.2] produces a category as described above, which turns out to be a non-full subcategory of C. It is called the *Demazure category* and is denoted by D.

5.3.5. *Rank function.* Lastly, [TaTr, 2.4.2] asks us to define a map from the set of objects to some ordered set J, called the *rank function.* This is needed for the inductive characterization of colimits which was mentioned at the start of this section. Roughly speaking, we want $\mathsf{Emb}^{\mathsf{d}}$-morphisms to be rank-increasing and $\mathsf{Dom}^{\mathsf{d}}$-morphisms to be rank-decreasing. See below for the precise conditions.

Define the subset $J \subset \mathbb{Z} \times \mathsf{Obj}\langle T \to \mathcal{W}\rangle$ to consist of pairs (l, y) such that $l \geq \ell_{C_0}(y)$. Choose an arbitrary well-ordering of $\mathsf{Obj}\langle T \to \mathcal{W}\rangle$, and equip J with the lexicographic order. Note that J is not well-ordered, but its upward intervals $J_{>(l,y)}$ are well-ordered. Define the rank function $\hat{\ell} : \mathsf{Obj}\,\mathsf{D} \to J$ by

$$\hat{\ell}([n], c, f, w) = (\ell([n], c, w), w).$$

Every tethered chamber $y \in \langle T \to \mathcal{W}\rangle$ gives an element $(\ell_{C_0}(y), y) \in J$. If we are comparing elements in J, we sometimes just denote this element by y for convenience.

Claim. *This is 'good' in the sense of* [TaTr, 2.4.2]*, i.e.*

(i) $\hat{\ell}$ *is strictly increasing along* $\mathsf{Emb}^{\mathsf{d}}$*, i.e. a non-identity* $\mathsf{Emb}^{\mathsf{d}}$*-morphism* $x_1 \hookrightarrow x_2$ *implies* $\hat{\ell}(x_1) < \hat{\ell}(x_2)$*.*

(ii) $\hat{\ell}$ *is weakly decreasing along* $\mathsf{Dom}^{\mathsf{d}}$*, i.e. a* $\mathsf{Dom}^{\mathsf{d}}$*-morphism* $x_1 \twoheadrightarrow \bullet \overset{\text{turn}}{\hookrightarrow} x_2$ *implies* $\hat{\ell}(x_1) \geq \hat{\ell}(x_2)$*.*

(iii) Suppose we are given a morphism in $\mathsf{Dom}^{\mathsf{d}}$*, denoted*

$$([n], c, f, w) \xrightarrow{\varphi:[n']\to[n]} ([n'], c', f', w) \xhookrightarrow[\text{turn}]{(w_i)_i} ([n'], c'', f'', w'')$$

The following statements are equivalent:

- $\hat{\ell}([n], c, f, w) = \hat{\ell}([n'], c', f', w)$
- $\hat{\ell}([n], c, f, w) = \hat{\ell}([n'], c'', f'', w'')$
- *For each* $j \in [1, n']$*, the subtour* $([\varphi(j-1), \varphi(j)], c)$ *is reduced.*

Proof. (i) follows from Lemma 5.2.2. Next, fix a morphism in $\mathsf{Dom}^{\mathsf{d}}$ (as shown above) and refine c to a gallery g. Performing greedy moves on each subgallery from $c_{\varphi(i-1)}$ to $c_{\varphi(i)}$ gives a folding $g'' \hookrightarrow g$ which refines c''. If m greedy moves are performed, then the proof of Lemma 5.2.2 implies that

$$\ell([n'], c'', f'', w'') = \ell([n], c, f, w) - m.$$

If $m = 0$, then the turning map is an identity, so $w = w''$. Then equality holds in (ii) and all bullets in (iii) are true. On the other hand, if $m > 0$, then the strict inequality holds in (ii), and all bullets in (iii) are false. □

In the language of [TaTr, 2.4], we have shown that D is ∞-*bistratified* with rank function $\hat{\ell}$. In this situation, it is useful to single out the *basic level morphisms*, which are the D-morphisms $x_1 \to x_2$ such that $\hat{\ell}(x_1) = \hat{\ell}(x_2)$ and there does not exist a factorization $x_1 \to x \to x_2$ with $\hat{\ell}(x) < \hat{\ell}(x_1)$, see [TaTr, 2.2]. As explained in [TaTr, 2.4], the basic level morphisms are precisely the $\mathsf{Dom}^{\mathsf{c}}$-morphisms

$$([n], c, f, w) \xrightarrow{\varphi:[n']\to[n]} ([n'], c', f', w)$$

for which the subtours $([\varphi(j-1), \varphi(j)], c)$ are reduced. These also belong to $\mathsf{Dom}^{\mathsf{d}}$ because their turning maps are identities.

From now on, we rename 'basic level morphisms' to 'birational morphisms' because 'level' already means something else. The motivation is that these morphisms will correspond to birational maps between sub-Demazure varieties.

5.4. **Special classes of maps.**

5.4.1. *Braid maps.* Suppose we are given a map in $\mathsf{Dom}^{\mathsf{c}}$, denoted

$$\phi : ([n], c, f, w) \xrightarrow{\varphi:[n']\to[n]} ([n'], c', f', w).$$

It is a *braid map* if it is birational, $f' = f \circ \varphi$, and the following holds:

- For all $i \in [\varphi(j-1)+1, \varphi(j)-1]$, the chamber c_i is unjointed and $f'_j c_i = f'_j c'_{j-1}$.

In other words, $([n], c, f, w)$ is obtained from $([n'], c', f', w)$ by inserting some unjointed chambers between each pair of chambers $(c'_{j-1}, f'_j c'_{j-1})$.

As a special case of this, it is a *socket-braid map* if, for each $j \in [1, n']$, we have

$$([\varphi(j-1), \varphi(j)], c, f) = c'_{j-1} \diamond f'_j c'_{j-1} \underset{f'_j}{\diamond} c'_j.$$

In other words, ϕ inserts all socket chambers.

5.4.2. *Adherent maps.* A chamber C *adheres* to an affine subspace $A \subset \mathfrak{h}$ if there exists $R \in \mathsf{Chambers}(\mathcal{H}|_A)$ such that $R \preceq C$. In this case, we have $R = C \wedge A$, i.e. for any face F, $F \preceq C$ and $F \subset A$ implies $F \preceq R$.

A jointed tour $([n], c, f)$ is *adherent* if, for each $i \in [1, n]$, c_{i-1} adheres to $\mathsf{span}\, f_i$.

The morphism ϕ is *adherent* if, for each $j \in [1, n']$ and $i \in [\varphi(j-1), \varphi(j)]$, c_i adheres to $\mathsf{span}\, f'_j$. Furthermore, a morphism in D is *adherent* if its $\mathsf{Dom}^{\mathsf{c}}$ component is adherent.

The *adherent subdiagram* is the simplicial subset $\mathsf{aD} \subset \mathsf{D}$ such that a simplex $\sigma : \Delta^n \to \mathsf{D}$ factors through aD if and only if σ sends all vertices to adherent objects and all arrows to adherent morphisms.

5.4.3. *Joint-only and joint-preserving maps.* The morphism ϕ is *joint-only* if $([n], c) = ([n'], c')$ and φ is the identity map. In other words, ϕ merely shrinks the joint faces.

The morphism ϕ is *joint-preserving* if it is adherent and, for all $j \in [1, n']$ and $i \in [\varphi(j-1)+1, \varphi(j)]$, we have $f_i = c_i \wedge \mathsf{span}\, f'_j$. In this case, threadability implies that, for each j such that f'_j is horizontal, we have

$$([\varphi(j-1), \varphi(j)], c, f) = c'_{j-1} \underset{f'_j}{\diamond} c'_j.$$

Furthermore, a morphism in D is *joint-preserving* if its $\mathsf{Dom}^{\mathsf{c}}$ component is joint-preserving.

5.4.4. *Remarks.* We motivate the definition of adherence by making these observations:

(1) If $([n], c, f)$ is a jointed tour, then c_i adheres to $\mathsf{span}\, f_i$, but c_{i-1} might not adhere to $\mathsf{span}\, f_i$. If $([n], c, f)$ is adherent, then c_i and c_{i-1} both adhere to $\mathsf{span}\, f_i$. Intuitively, the joint face f_i appears 'between' the chambers c_{i-1} and c_i, and adherence forces f_i to treat these chambers more symmetrically.

(2) If $([n], c, f)$ is adherent and f_i is horizontal, then $c_{i-1} \succeq f_i$. (Proof: if c_{i-1} adheres to $\mathsf{span}\, f_i$ along any $\mathcal{H}|_{\mathsf{span}\, f_i}$-chamber other than f_i, then the tour is not positive, contradiction.)

(3) Let us say that a *subdiagram* of a category is a simplicial subset obtained by constraining the objects and morphisms, similarly to aD. To construct a map between subdiagrams of categories, it suffices to map the objects, map the arrows, and check compositions of arrows. The higher-dimensional simplices come along for the ride.

(4) For $t \in \mathsf{aD}$, the map $\mathsf{aD}_{t/} \to \mathsf{D}_{t/}$ is fully faithful. In particular, $\mathsf{aD}_{t/}$ is a category.

(5) Suppose we are given an adherent $\mathsf{Dom}^{\mathsf{c}}$-morphism

$$([n^1], c^1, f^1, w) \xrightarrow{\varphi^{31} : [n^3] \to [n^1]} ([n^3], c^3, f^3, w)$$

and a factorization of the index map φ^{31} into

$$[n^3] \xrightarrow{\varphi^{32}} [n^2] \xrightarrow{\varphi^{21}} [n^1].$$

If none of the f^3_k are horizontal, then the $\mathsf{Dom}^{\mathsf{c}}$-morphism factors canonically as

$$([n^1], c^1, f^1, w) \xrightarrow{\varphi^{21}} ([n^2], c^2, f^2, w) \xrightarrow{\varphi^{32}} ([n^3], c^3, f^3, w),$$

where $c^2_j = c^1_{\varphi^{21}(j)}$ and $f^2_j = c^2_j \wedge \mathsf{span}\, f^3_k$ whenever $j \in [\varphi^{32}(k-1)+1, \varphi^{32}(k)]$. Furthermore, the second map is joint-preserving.

(6) In (5), if we take φ^{21} to be the identity, then the first map is joint-only.

There are analogues of (5) and (6) for aD-morphisms, but the notation is more complicated.

In (5) and (6), if some of the f^3 joint faces are horizontal, then the tour $([n^2], c^2, f^2, w)$ may fail to be threadable. This annoyance can be handled in various ways. In 6.5.3, we use the factorization as-is and enlarge aD by dropping the opposition condition. In 8.4.1 and 8.8.4, we delete the problematic chambers from c^2, namely the chambers indexed by $[\varphi^{32}(k-1)+1, \varphi^{32}(k)-1]$ for all $k \in [n^3]$ such that f^3_k is horizontal.

5.5. **Adherence comes for free.**

5.5.1. Let $\mathcal{E}$ be an ∞-category which admits colimits, and let $\mathsf{Fun}^{\mathsf{braid}}(\mathsf{D}, \mathcal{E}) \subset \mathsf{Fun}(\mathsf{D}, \mathcal{E})$ be the full subcategory consisting of functors which send braid maps to isomorphisms.

Theorem. *If $F : \mathsf{aD} \to \mathcal{E}$ sends braid maps to isomorphisms, then $\mathrm{LKE}_{\mathsf{aD} \hookrightarrow \mathsf{D}} F$ has the same property. Furthermore, left Kan extension and restriction give a mutually inverse pair of equivalences*

$$\mathsf{Fun}^{\mathsf{braid}}(\mathsf{aD}, \mathcal{E}) \simeq \mathsf{Fun}^{\mathsf{braid}}(\mathsf{D}, \mathcal{E}).$$

Remark. In fact, the proof of the theorem implies a slightly stronger statement: the ∞-categories obtained from aD and D by inverting the braid maps are equivalent. This will be useful for constructing the Demazure functor in 6.5. We will explicitly construct the functor on aD. Then, since it sends braid maps to isomorphisms, it automatically extends to D. In general, we believe that any 'meaningful' functor on aD or D should send braid maps to isomorphisms. Later, in Section 8, when we study the homotopy types of categories related to D, it will be useful to know that $\mathsf{aD} \hookrightarrow \mathsf{D}$ is a homotopy equivalence. This follows immediately from the theorem.

The rest of the subsection is devoted to proving the theorem.

5.5.2. Let us begin by reformulating the theorem more concretely. Fix $\dot{t} \in \mathsf{D}$, denote its socket-braid map by $\dot{t}^{\mathsf{soc}} \to \dot{t}$, and note that $\dot{t}^{\mathsf{soc}} \in \mathsf{aD}$. We will show that, for any $F : \mathsf{aD} \to \mathcal{E}$ which sends braid maps to isomorphisms, the map

$$F(\dot{t}^{\mathsf{soc}}) \to \big(\mathrm{LKE}_{\mathsf{aD} \hookrightarrow \mathsf{D}} F\big)(\dot{t})$$

is an isomorphism. This statement easily implies the theorem.

In addition, we claim that the left Kan extension is computed in the usual way:

$$\big(\mathrm{LKE}_{\mathsf{aD} \hookrightarrow \mathsf{D}} F\big)(\dot{t}) = \operatorname*{colim}_{\langle \mathsf{aD} \xrightarrow{\mathsf{D}} \dot{t} \rangle} F \circ \mathsf{tail}.$$

This would be immediate if aD were a category. In our setting, it follows from the next lemma, by taking $K = \mathsf{aD}$ and $\mathcal{C} = \mathsf{D}$.

Lemma. *Suppose we are given a diagram $K \xrightarrow{\imath} \mathcal{K} \xrightarrow{f} \mathcal{C}$, where K is a simplicial set, $\mathcal{K}$ and $\mathcal{C}$ are ∞-categories, and $\imath$ is a categorical equivalence. For each $c \in \mathcal{C}$, the map $K_{/c} \xrightarrow{i} \mathcal{K}_{/c}$ is a categorical equivalence.*

Proof. The map $\mathcal{C}_{/c} \to \mathcal{C}$ is a right fibration by [HTT, Cor. 2.1.2.2], hence a cartesian fibration which is fibered in ∞-groupoids. Each vertical map in the following pullback diagram is also a right fibration:

$$\begin{tikzcd} K_{/c} \arrow[r, "i"] \arrow[d] & \mathcal{K}_{/c} \arrow[r] \arrow[d] & \mathcal{C}_{/c} \arrow[d] \\ K \arrow[r, "\imath"] & \mathcal{K} \arrow[r, "f"] & \mathcal{C} \end{tikzcd}$$

The map i is a categorical equivalence because it is the pullback of a categorical equivalence along a cartesian fibration, see [HTT, Prop. 3.3.1.3]. ☐

Let us write $\dot{t} = ([\dot{n}], \dot{c}, \dot{f}, \dot{w})$ and assume that $\dot{n} = 1$, so that $\dot{t}$ looks like $\dot{c}_0 \diamond_{\dot{f}_1} \dot{c}_1$. The general case only requires more bookkeeping, working with each step of $\dot{t}$ separately.

5.5.3. We will only be concerned with objects in $\langle \mathsf{aD} \xrightarrow{\mathsf{D}} \dot{t} \rangle$. Thus, we often denote an object by $t = ([n], c, f)$, omitting the map to $\dot{t}$ for sake of convenience.

For any such object, each chamber of t which has $\dot{f}_1$ as a face comes after each chamber which does not. (Proof: Since the last chamber of t has $\dot{f}_1$ as a face, a counterexample would imply that t double-crosses some wall which does not contain $\dot{f}_1$, contradicting the second Coxeter product condition.) Thus, t has a unique step $c_{i-1} \diamond_{f_i} c_i$ such that c_i is the first chamber which has $\dot{f}_1$ as a face. This is called the *contact step*.

In particular, if $\dot{t}$ satisfies $\dot{f}_1 \preceq \dot{c}_0$, then every chamber of t has $\dot{f}_1$ as a face. This implies that $\mathsf{id}_{\dot{t}} \in \langle \mathsf{aD} \xrightarrow{\mathsf{D}} \dot{t} \rangle$ is a terminal object, and the desired result follows easily. From now on, we assume that $\dot{f}_1 \preceq \dot{c}_0$ does not hold, so the contact step satisfies $i \geq 1$.

5.5.4. Define an exhausting sequence of full subdiagrams as follows:

$$A_0 \subset B_0 \subset A_1 \subset B_1 \subset \cdots \subset \langle \mathsf{aD} \xrightarrow{\mathsf{D}} t \rangle$$

- B_k consists of objects $t \in \langle \mathsf{aD} \xrightarrow{\mathsf{D}} \dot{t} \rangle$ such that the contact step satisfies $i - 1 \leq k$.
- $A_k \subset B_k$ is defined by requiring that the contact step also satisfies $f_i = c_i = \dot{f}_1 c_{i-1}$.

It suffices to show that all of the maps

$$\cdots \to \operatorname*{colim}_{B_{k-1}} F \to \operatorname*{colim}_{A_k} F \to \operatorname*{colim}_{B_k} F \to \cdots$$

are isomorphisms. Indeed, this implies that

$$\operatorname*{colim}_{A_0} F \simeq \operatorname*{colim}_{\langle \mathsf{aD} \xrightarrow{\mathsf{D}} \dot{t} \rangle} F,$$

and the theorem follows because $\dot{t}^{\mathsf{soc}}$ is the terminal object of A_0.

If one partitions B_k into three parts $B_{k-1} \sqcup (A_k \smallsetminus B_{k-1}) \sqcup (B_k \smallsetminus A_k)$, then the maps which go between different parts can only go in the following directions:

$$\begin{tikzcd}
A_k \smallsetminus B_{k-1} \arrow[d, dashed] \arrow[dr, dashed] & \\
B_{k-1} & B_k \smallsetminus A_k \arrow[l, dashed]
\end{tikzcd}$$

5.5.5. **Lemma.** *Suppose we are given maps of simplicial sets $\imath : K \rightleftarrows L : r$ such that $r\imath = \mathsf{id}_K$, and a homotopy $h : L \times \Delta^1 \to L$ sending $\imath r \Rightarrow \mathsf{id}_L$, which respects the fibers of r in the sense that the following diagram commutes:*

$$\begin{tikzcd}
L \times \Delta^1 \arrow[r, "h"] \arrow[dr, "r \circ \mathsf{pr}_1"'] & L \arrow[d, "r"] \\
& K
\end{tikzcd}$$

Let $F : L \to \mathcal{E}$ be a functor such that, for each vertex $l \in L_0$, the arrow $Fh(\{l\} \times \Delta^1)$ is an isomorphism. Then the natural map

$$\operatorname{colim} F\imath \to \operatorname{colim} F$$

is an isomorphism.

Proof. First, we claim that r is final. By Theorem A.3.1, it suffices to show that, for each simplex $\sigma : \Delta^m \to K$, the simplicial set of lifts of σ along r is contractible. This is true because h deformation retracts all lifts onto the lift $\imath\sigma$. (This uses the requirement that h respects the fibers of r.)

Since r is final, we have $\operatorname{colim} F\imath \simeq \operatorname{colim} F\imath r$. By the hypothesis on F, the natural transformation $Fh : F\imath r \Rightarrow F$ is a natural isomorphism, so $\operatorname{colim} F\imath r \simeq \operatorname{colim} F$. □

Remarks.

(1) Under the same hypotheses, $\imath$ is left anodyne and hence initial, by [HTT, Prop. 2.1.2.11] and [HTT, Prop. 4.1.1.3]. Thus, $\lim F\imath \simeq \lim F$ holds without any assumption on F.

(2) The lemma is motivated by the special case when K and L are ∞-categories and $(\imath, r)$ are adjoint functors. In this case, the assumption on F says that $F \Rightarrow \operatorname{LKE}_\imath \operatorname{Res}_\imath F$ is a natural isomorphism, and the lemma follows from left Kan extension to the point category. This was used in [TaTr, Lem. 2.4.4]

5.5.6. We now show that $\operatorname{colim}_{A_k} F \to \operatorname{colim}_{B_k} F$ is an isomorphism. Let $\imath : A_k \hookrightarrow B_k$ be the inclusion, and take the domain of F to be B_k, so that the desired statement reads $\operatorname{colim} F\imath \xrightarrow{\sim} \operatorname{colim} F$. In order to apply the previous lemma, we will define a retraction r of $\imath$ and a homotopy $h : B_k \times \Delta^1 \to B_k$ sending $\imath r \Rightarrow \mathsf{id}_{B_k}$.

For each object $t \in B_k$, define $r(t) \in A_k$ and a B_k-morphism $\eta(t) : r(t) \to t$ as follows. If $t \in A_k$, then $\eta(t)$ is the identity, and otherwise $\eta(t)$ factors the contact step of t into $c_{i-1} \diamond \dot{f}_1 c_{i-1} \diamond_{f_i} c_i$. Next, make r into a functor $B_k \to A_k$ by requiring that, for every morphism $t_1 \xrightarrow{\phi} t_2$ in B_k, the diagram

$$\begin{tikzcd} r(t_1) \arrow[r, "r(\phi)"] \arrow[d, "\eta(t_1)"] & r(t_2) \arrow[d, "\eta(t_2)"] \\ t_1 \arrow[r, "\phi"] & t_2 \end{tikzcd}$$

commutes. Finally, define the homotopy h by requiring that $h(\{t\} \times \Delta^1) = \eta(t)$. The following remarks explain why these constructions work:

- We only need to check that certain compositions of morphisms lie in aD, such as the diagonal map $r(t_1) \to t_2$ in the diagram above. See 5.4.4(3) for a related remark.
- The 'factoring step' of $\eta(t)$ never affects whether a composition of morphisms lies in aD. The reason is that any morphism in $\langle \mathsf{aD} \xrightarrow{\mathsf{D}} \dot{t} \rangle$ which affects the contact step $c_{i-1} \diamond_{f_i} c_i$ will require that c_{i-1} adheres to a certain affine subspace $A \subset \mathfrak{h}$ containing $\dot{f}_1$, and in this case $\dot{f}_1 c_{i-1}$ adheres to A as well.

The construction ensures that h respects the fibers of r. Also, each arrow $F\eta(t)$ is an isomorphism, because $\eta(t)$ is a braid map. (Proof: Since $\dot{f}_1 \preceq f_i$, we have $f_i \dot{f}_1 c_{i-1} = f_i c_{i-1}$.) Thus, Lemma 5.5.5 applies, as desired.

5.5.7. It is well-known that a functor between ∞-categories is final if and only if its under-categories are contractible. We formulate an analogue for arbitrary simplicial sets.

Lemma. *Let $\phi : K \to L$ be a map of simplicial sets, and define*

$$M = \mathrm{Hom}(\Delta^1, L) \times_{\mathrm{Hom}(\{1\}, L)} K.$$

If the map $\pi : M \to L$ defined by evaluation at $\{0\} \in \Delta^1$ is final, then so is ϕ.

Proof. The criterion for final functors between ∞-categories is [HTT, Thm. 4.1.3.1]. Its proof implies that ϕ factors as $K \to M \to L$ where $K \to M$ is final, and the lemma follows. ☐

5.5.8. We now show that $\mathrm{colim}_{B_{k-1}} F \to \mathrm{colim}_{A_k} F$ is an isomorphism, by showing that the embedding $\imath : B_{k-1} \hookrightarrow A_k$ is final. By the previous lemma, it suffices to show that the evaluation map $\pi : M \to A_k$ is final, where M is defined with $K = B_{k-1}$ and $L = A_k$. For this, we apply Theorem A.3.1. Fix a simplex $\sigma : \Delta^m \to A_k$. The simplicial set of lifts of σ along π identifies with the full simplicial subset

$$\mathcal{C} \subset \mathrm{Hom}(\Delta^m, A_k)_{\sigma/}$$

consisting of homotopies $H : \Delta^m \times \Delta^1 \to A_k$ sending $\sigma \Rightarrow \phi$ where $\phi : \Delta^m \to A_k$ is any map which factors through B_{k-1}. In 5.4.4(4), we remarked that each $\mathsf{aD}_{t/}$ is a category, and this implies that $\mathrm{Hom}(\Delta^m, A_k)_{\sigma/}$ and $\mathcal{C}$ are categories. The rest of the proof is devoted to showing that $\mathcal{C}$ is contractible.

Consider the full subcategory $\mathcal{C}_1 \subset \mathcal{C}$ consisting of homotopies such that ϕ is a constant map. The embedding has a left adjoint which sends a homotopy H to its composition with $\phi \circ H_\Delta$, where $H_\Delta : \Delta^m \times \Delta^1 \to \Delta^m$ is the unique homotopy from id_{Δ^m} to the constant map with value $\{m\} \in \Delta^m$. Therefore, the embedding is a homotopy equivalence.

If the image of σ intersects B_{k-1}, then the homotopy $\sigma \circ H_\Delta$ is an initial object of $\mathcal{C}_1$, so $\mathcal{C}_1$ is contractible and we are done. From now on, assume that this is not the case.

An object of $\mathcal{C}_1$ is uniquely determined by the arrow $H(\{m\} \times \Delta^1)$, which we denote $t \to t'$, where $t := \sigma(m)$ and $t' \in B_{k-1}$.[9] Consider the full subcategory $\mathcal{C}_2 \subset \mathcal{C}_1$ for which $t \to t'$ lies in $\mathsf{Dom}^{\mathsf{d}}$. The embedding has a right adjoint, which sends $t \to t'$ to the $\mathsf{Dom}^{\mathsf{d}}$ part of its functorial factorization into $\mathsf{Dom}^{\mathsf{d}}$ and $\mathsf{Emb}^{\mathsf{d}}$ maps. Therefore, the embedding is a homotopy equivalence.

Every map $t \to t'$ in $\mathsf{Dom}^{\mathsf{d}}$ corresponds to a map in $\mathsf{Dom}^{\mathsf{c}}$, which is specified by deleting some chambers and shrinking some joint faces of t. The requirement $t' \in B_{k-1}$ says that this map must delete at least one chamber which does not have $\dot{f}_1$ as a face.

Consider the full subcategory $\mathcal{C}_3 \subset \mathcal{C}_2$ consisting of maps $t \to t'$ which do not delete any chamber $([i, n], c)$ and do not shrink any joint face $([i+1, n], f)$. The embedding has a right adjoint, which modifies t' by restoring the aforementioned chambers and joint faces. The subtlety is that, if c_i was deleted in t', then the newly-restored c_i must be equipped with the joint face $\hat{f}_i := c_i \wedge \operatorname{\mathsf{span}} f'_j$, where $i \in [\varphi(j-1)+1, \varphi(j)]$ and $\varphi : [n'] \to [n]$ is the index map for $t \to t'$. (In other words, the restored c_i gets its joint face from the step of t' which 'contained' the deleted c_i.) Note that $\hat{f}_i$ is not horizontal, so this does not affect threadability. (Proof: if $\hat{f}_i$ is horizontal, then so is f'_j. Then $c_{\varphi(j-1)} = c'_{j-1} \succeq f'_j \succeq \dot{f}_1$, where the middle inequality holds because t' is adherent, see 5.4.4(2). Since $\varphi(j-1) < i$, this contradicts the definition of i.)

Consider the full subcategory $\mathcal{C}_4 \subset \mathcal{C}_3$ consisting of maps $t \to t'$ which delete the chamber c_{i-1}. This embedding has a left adjoint, which modifies the map so that it does delete this chamber. (To show that the modified map still satisfies the Coxeter product conditions, recall from above that, if c_{i-1} adheres to an affine subspace $A \subset \mathfrak{h}$ containing $\dot{f}_1$, then $\dot{f}_1 c_{i-1}$ adheres to A as well. In the present situation, we have assumed that $t \in A_k$, so $f_i = c_i = \dot{f}_1 c_{i-1}$.) Therefore, the embedding is a homotopy equivalence.

Lastly, the poset $\mathcal{C}_4$ has an initial object given by the map $t \to t'$ which deletes only the chamber c_{i-1}. Therefore, $\mathcal{C}_4$ is contractible. This completes the proof of Theorem 5.5.1.

6. Demazure varieties

6.1. **Overview.**

6.1.1. *Main definition.* Given a threadable jointed tour $t = ([n], c, f)$, we will define a *sub-Demazure variety* using the following formula:

$$\begin{aligned} X(t) := \big(I_{c_0} \overset{I_{c_0} \cap I_{f_1 c_0}}{\times} I_{f_1 c_0}\big) \overset{I_{f_1 c_0}}{\times} P_{(f_1 c_0, c_1)} \overset{I_{c_1}}{\times} \cdots \\ \cdots \overset{I_{c_{i-1}}}{\times} \big(I_{c_{i-1}} \overset{I_{c_{i-1}} \cap I_{f_i c_{i-1}}}{\times} I_{f_i c_{i-1}}\big) \overset{I_{f_i c_{i-1}}}{\times} P_{(f_i c_{i-1}, c_i)} \overset{I_{c_i}}{\times} \cdots \\ \cdots \overset{I_{c_{n-1}}}{\times} \big(I_{c_{n-1}} \overset{I_{c_{n-1}} \cap I_{f_n c_{n-1}}}{\times} I_{f_n c_{n-1}}\big) \overset{I_{f_n c_{n-1}}}{\times} P_{(f_n c_{n-1}, c_n)} / I_{c_n}. \end{aligned}$$

[9]We emphasize that the remaining simplex vertices $\sigma|_{\Delta^{m-1}}$ are not redundant. They constrain the maps $t \to t'$ which are allowed in $\mathcal{C}_1$.

The next three subsections are devoted to this. In 6.2 and 6.3, we define the terms in the formula to be certain group ind-schemes or subschemes thereof. In 6.4, we show that $X(t)$ is a quasicompact scheme. Geometrically, it is a twisted product of two kinds of factors:

- $P_{(f_i c_{i-1}, c_i)}/I_{c_i}$ is a Schubert variety in a finite or thick affine flag variety.
- $I_{c_{i-1}}/(I_{c_{i-1}} \cap I_{f_i c_{i-1}})$ is isomorphic to some $\mathbb{A}^m$ or the scheme $\mathbb{A}^\infty$.

When defining the functoriality of $X(t)$ with respect to t, we will make t tethered as well. Finally, *Demazure varieties* are non-quasicompact schemes obtained by taking a colimit of sub-Demazure varieties along a sequence of open embeddings.

6.1.2. *Motivation.* In the classical setting, if we are given a Kac–Moody group G and a chamber sequence $t = ([n], c)$ in the corresponding Coxeter complex, then we can define the Demazure variety

$$X(t) = P_{(c_0,c_1)} \overset{B_{c_1}}{\times} P_{(c_1,c_2)} \overset{B_{c_2}}{\times} \cdots \overset{B_{c_{n-1}}}{\times} P_{(c_{n-1},c_n)}/B_{c_n}.$$

The factors are defined as follows:

- B_{c_i} is the Borel subgroup whose roots are positive on c_i.
- $P_{(c_{i-1},c_i)}$ is the closure of $B_{c_{i-1}} \cdot B_{c_i}$ in G.

Note that $P_{(c_{i-1},c_i)}$ is a union of cells in the Bruhat decomposition of G into double cosets with $B_{c_{i-1}}$ on the left and B_{c_i} on the right. From the viewpoint of Contou–Carrère's thesis, we may interpret this construction as follows:

- B_{c_i} is the stabilizer of c_i in the Tits building.
- $g \in P_{(c_{i-1},c_i)}$ if and only if the relative position (c_{i-1}, gc_i) is at most (c_{i-1}, c_i).
- The k-points of $X(t)$ are in bijection with chamber sequences in the Tits building which are 'bounded above' by t.

If the meet $F = c_{i-1} \wedge c_i$ exists, then $P_{(c_{i-1},c_i)} \subset P_F$, where P_F is the parabolic subgroup associated to the face F. The parabolic subgroup P_F is 'easier' than G because it can be constructed as the semidirect product of its Levi subgroup and its radical. Thus, if we assume that t is a *generalized gallery*, meaning that each meet $c_{i-1} \wedge c_i$ exists, then $X(t)$ can be constructed from groups which are 'easier' than G.

Unfortunately, in the double affine setting, any strictly positive finite tour cannot be a generalized gallery, i.e. at least one meet $c_{i-1} \wedge c_i$ fails to exist. In this case, we cannot construct $P_{(c_{i-1},c_i)}$ as a scheme without already having constructed the double loop group as a scheme. To circumvent this issue, first observe that, in the classical setting, $P_{(c_{i-1},c_i)}$ always contains an open subscheme

$$B_{c_{i-1}} B_{c_i} \simeq B_{c_{i-1}} \overset{B_{c_{i-1}} \cap B_{c_i}}{\times} B_{c_i}.$$

In the double affine setting, if $c_{i-1} \wedge c_i$ does not exist, we simply replace $P_{(c_{i-1},c_i)}$ by the right hand side. With this modification, $X(t)$ becomes an open subscheme of the true Demazure variety, so we call it a *sub-Demazure variety.* As we refine t by adding more chambers, we expect $X(t)$ to approximate the true Demazure variety.

The jointed tours are designed to keep track of two kinds of factors, depending on whether the meet $c_{i-1} \wedge c_i$ exists. Given a jointed tour $t = ([n], c, f)$, we try to construct the Demazure variety of its socket tour (5.1.2). For each step of the form $(f_i c_{i-1}, c_i)$, the meet $f_i c_{i-1} \wedge c_i$ exists, so we can define $P_{(f_i c_{i-1}, c_i)}$. For each step of the form $(c_{i-1}, f_i c_{i-1})$, the meet $c_{i-1} \wedge f_i c_{i-1}$ need not exist, so we instead use $B_{c_{i-1}} \overset{B \cap B}{\times} B_{f_i c_{i-1}}$.

6.2. **Parahoric groups.**

6.2.1. *Lie algebras and imaginary roots.* The affine Lie algebra $\mathfrak{g}^{\mathsf{aff}}$ and the double-affine roots $R = R^{\mathsf{aff}} + \mathbb{Z}\pi$ were defined in 4.1. The double affine Lie algebra is defined as

$$\mathfrak{g} := \mathfrak{g}^{\mathsf{aff}}[t, t^{-1}],$$

and its roots are given by $R^{\mathsf{reim}} := R \sqcup (\mathbb{Z}\delta + \mathbb{Z}\pi)$. The roots in $\mathbb{Z}\delta + \mathbb{Z}\pi$ are called *imaginary*.

Adding the imaginary walls to $\mathcal{H}$ gives a finer hyperplane arrangement $\mathcal{H}^{\mathsf{reim}}$. The projection $p : \mathsf{Faces}(\mathcal{H}^{\mathsf{reim}}) \to \mathsf{Faces}(\mathcal{H})$ is almost a bijection. If $F \in \mathsf{Faces}(\mathcal{H})$ is a top-dimensional support face, then $p^{-1}(F) = \{F^-, F^0, F^+\}$, where F^- is a downward chamber, F^+ is an upward chamber, and they are adjacent along F^0. Otherwise $p^{-1}(F)$ is a singleton. Thus, we can define a section s of p by requiring $s(F) = F^-$ in the first case. Let us redefine $R^+(F) := R^+(s(F)) \subset R^{\mathsf{reim}}$, and define $R^0(-)$ and $R^{+0}(-)$ similarly.

Remark. In brief, we want to identify $\mathsf{Faces}(\mathcal{H})$ with the subset of $\mathsf{Faces}(\mathcal{H}^{\mathsf{reim}})$ obtained by deleting all faces of the form F^0 or F^+. This identification is a little unnatural, but it is motivated by the fact that there is no reflection along the (horizontal) wall $\mathsf{span}\, F^0$, i.e. the deleted faces do not give foldings.

For any *closed* subset $R' \subset R^{\mathsf{reim}}$, meaning that $R' + R' \subseteq R'$, we define $\mathfrak{g}_{R'} \subset \mathfrak{g}$ to be the Lie subalgebra spanned by the root spaces indexed by R'.

6.2.2. Our goal is to construct, for each $F \in \mathsf{Faces}(\mathcal{H})$, a 'parahoric' group ind-scheme P_F. Its Lie algebra will be a completion of $\mathfrak{g}_{R^{+0}(F)}$, and it will be functorial with respect to F. Let us assume that $\delta(F)$ is rational. The irrational-level case is similar and easier.

6.2.3. *The radical U_F when F is not upward.* Assume that F is not upward, i.e. it is downward or horizontal. We will construct a group ind-scheme U_F whose Lie algebra is a completion of $\mathfrak{g}_{R^+(F)}$.

For any point $p \in \mathfrak{h}$ with $\delta(p) < \delta(F)$, consider the closed subset $R^+(F, p) := R^+(F) \cap R^+(p) \subset R^{\mathsf{reim}}$. As p varies, these subsets form a filtered poset. Since F is not upward, the union of these subsets equals $R^+(F)$.

If we are given a point p as above, a point q which lies in the relative interior of the convex hull of $p \cup \overline{F}$, and a positive integer n, then we define $A(q, n)$ to be the subset of roots α such that $\alpha(q) \geq n$. The following statements are easy to check:

(i) $R^+(F, p) + [R^+(F, p) \cap A(q, n)] \subseteq [R^+(F, p) \cap A(q, n)]$.

(ii) $R^+(F, p) \smallsetminus A(q, n)$ is finite.

(iii) For fixed p, as (q, n) varies, the sets $R^+(F, p) \cap A(q, n)$ form a cofiltered poset.

By (i), the quotient $\mathfrak{g}_{R^+(F,p)}/\mathfrak{g}_{R^+(F,p)\cap A(q,n)}$ is a Lie algebra. By (ii), it is finite-dimensional. Since its roots are contained in $R^+(F,p)$, it is nilpotent, so it corresponds to a unipotent algebraic group, which we denote $\overline{U}_{F,p,q,n}$. Next, define

$$U_{F,p} := \lim_{(q,n)} \overline{U}_{F,p,q,n}$$

in the category of schemes. Statement (iii) says that the indexing diagram is cofiltered, and each $\overline{U}_{F,p,q,n}$ is affine, so $U_{F,p}$ exists.

In this paragraph, fix p and p' so that $R^+(F,p) \subset R^+(F,p')$. For any pair (q',n') which is valid for p', there exists a pair (q,n) which is valid for p and satisfies $R^+(F,p)\cap A(q,n) \subset R^+(F,p') \cap A(q',n')$. This containment implies that the obvious map

$$\overline{U}_{F,p,q,n} \to \overline{U}_{F,p',q',n'}$$

of algebraic groups is well-defined. Taking cofiltered limits over (q',n') yields a map

$$U_{F,p} \to U_{F,p'}$$

of group schemes. Comparing root spaces shows that this is a closed embedding. Finally, we take a filtered colimit in the presheaf category:

$$U_F := \operatorname*{colim}_p U_{F,p}.$$

6.2.4. *The group P_F when F is downward.* Suppose that F is downward. Then $\mathfrak{g}_{R^0(F)}$ is the Lie algebra of a reductive group, which we denote L_F. The action of $\mathfrak{g}_{R^0(F)}$ on $\mathfrak{g}_{R^+(F)}$ respects the previously-defined completion,[10] so $\mathfrak{g}_{R^0(F)}$ acts on U_F. Checking integrality of weights shows that this action integrates to $L_F \curvearrowright U_F$. Now define $P_F := L_F \ltimes U_F$.

6.2.5. *The group P_F when F is horizontal.* Suppose that F is horizontal. Define L_F to be the affine-type Kac–Moody group ind-scheme whose roots are $R^0(F)$, and whose positive roots are $R^0(F) \cap R^+(C)$, where C is any downward chamber in $\mathcal{H}_F$. (The group L_F does not depend on the choice of C.) The Lie algebra of L_F is a completion of $\mathfrak{g}_{R^0(F)}$. As before, we will construct an action $L_F \curvearrowright U_F$ and define $P_F := L_F \ltimes U_F$.

Fix C as before, and let E range over the faces of C satisfying $E \succ F$. For each E, the subset $R^0(F) \cap R^{+0}(E) \subset R^0(F)$ determines a parahoric subgroup $L_F^{P_E} \subset L_F$. The semidirect product decomposition

$$\mathfrak{g}_{R^{+0}(E)} = \mathfrak{g}_{R^0(F)\cap R^{+0}(E)} \ltimes \mathfrak{g}_{R^+(F)}$$

is compatible with the previously-defined completions, so it gives a decomposition

$$P_E = L_F^{P_E} \ltimes U_F.$$

In particular, we get an action $L_F^{P_E} \curvearrowright U_F$. Checking Lie algebras shows that these actions are compatible as E varies. Since $L_F = \operatorname{colim}_E L_F^{P_E}$ in the category of group ind-schemes, this gives an action $L_F \curvearrowright U_F$, as desired. Checking Lie algebras again shows that this action does not depend on the choice of C.

[10]To see this, choose the points p and q so that $p, q \in \mathsf{span}\, F$. This restriction makes it easy to show that $\mathfrak{g}_{R^0(F)}$ respects the subalgebras and quotients in the definition of the completion. This restriction defines a cofinal subset of the poset of all choices of p and q, so it does not affect the limit or colimit.

If F is top-dimensional, then $R^0(F) = \{0\}$, so L_F is a torus. In this case, we call P_F an 'Iwahori' group and denote it by I_F.

6.2.6. *The group P_F when F is upward.* Suppose that F is upward. Since $\tilde{F}$ is horizontal, we have already constructed

$$P_{\tilde{F}} = L_{\tilde{F}} \ltimes U_{\tilde{F}}.$$

The subset $R^0(\tilde{F}) \cap R^{+0}(F) \subset R^0(F)$ determines a negative-parahoric subgroup $L_{\tilde{F}}^{P_F} \subset L_{\tilde{F}}$. Define

$$P_F := L_{\tilde{F}}^{P_F} \ltimes U_{\tilde{F}}.$$

(The phrase 'negative-parahoric' is explained in 6.2.8.)

Remark. If we apply this definition when F is downward or horizontal, then the resulting group P_F agrees with the already-defined group P_F. If F is downward, $L_{\tilde{F}}^{P_F} \subset L_{\tilde{F}}$ is a parahoric subgroup. If F is horizontal, then $F = \tilde{F}$ and $L_{\tilde{F}}^{P_F} = L_{\tilde{F}}$.

6.2.7. *Functoriality.* Fix two faces $E \preceq F$. This implies $\tilde{E} \preceq \tilde{F}$. First, suppose that $\tilde{E} = \tilde{F}$, and let S denote both. In the previous remark, we observed that

$$\begin{aligned} P_E &= L_S^{P_E} \ltimes U_S \\ P_F &= L_S^{P_F} \ltimes U_S. \end{aligned}$$

Since $E \preceq F$, we have $L_S^{P_E} \supseteq L_S^{P_F}$, so $P_E \supseteq P_F$, as desired.

Next, suppose that $\tilde{E} \prec \tilde{F}$. Then $E \preceq F$ implies that E is horizontal, i.e. $E = \tilde{E}$. We may factor the original inequality as $E \prec \tilde{F} \preceq F$, where $\tilde{F} \preceq F$ falls under the previous case. Thus, we may replace the original inequality by $E \prec \tilde{F}$ and thereby assume that F is also horizontal. For convenience, assume that L_E is of untwisted type, i.e. $L_E = \hat{\mathfrak{L}}\mathring{H}$, where $\mathring{H}$ is a reductive group and $\hat{\mathfrak{L}}(-)$ denotes the identity component of the Kac–Moody extension of the algebraic loop space. (The twisted case involves more notation but no new ideas.) The roots $R^0(F) \subset R^0(E)$ determine a parabolic subgroup $\mathring{P} \subset \mathring{H}$, and we denote its Levi and radical by $\mathring{L}$ and $\mathring{U}$. Then

$$\begin{aligned} P_E &= L_E \ltimes U_E \\ &\supset \hat{\mathfrak{L}}\mathring{P} \ltimes U_E \\ &= (\hat{\mathfrak{L}}\mathring{L} \ltimes \hat{\mathfrak{L}}\mathring{U}) \ltimes U_E \\ &= \hat{\mathfrak{L}}\mathring{L} \ltimes (\hat{\mathfrak{L}}\mathring{U} \ltimes U_E) \\ &= L_F \ltimes U_F \\ &= P_F, \end{aligned}$$

as desired. Note that the above identifications correspond to the obvious identifications

$$\begin{aligned} \mathfrak{g}_{R^{+0}(E)} &= \mathfrak{g}_{R^0(E)} \ltimes \mathfrak{g}_{R^+(E)} \\ &\supset \mathfrak{g}_{R^0(E) \cap R^{+0}(F)} \ltimes \mathfrak{g}_{R^+(E)} \\ &= \left(\mathfrak{g}_{R^0(F)} \ltimes \mathfrak{g}_{R^0(E) \cap R^+(F)}\right) \ltimes \mathfrak{g}_{R^+(E)} \\ &= \mathfrak{g}_{R^0(F)} \ltimes \left(\mathfrak{g}_{R^0(E) \cap R^+(F)} \ltimes \mathfrak{g}_{R^+(E)}\right) \\ &= \mathfrak{g}_{R^0(F)} \ltimes \mathfrak{g}_{R^+(F)}, \end{aligned}$$

so the key step is to see that $\hat{\mathcal{L}}\mathring{U} \ltimes U_E = U_F$ by checking compatibility with completions.

6.2.8. *Two versions of Kac–Moody groups.* The main difficulty in working with the groups P_F is to keep track of completions and the presence (or absence) of imaginary root subgroups. Let us briefly review how this difficulty plays out in the case of Kac–Moody groups.

Let G be a Kac–Moody group over k defined using Tits's presentation. This means that G has no root subgroups for any imaginary roots, and it is not the set of k-points of any reasonable ind-scheme. Mathieu's construction gives two group ind-schemes $\mathcal{G}^{\mathsf{pmax}}$ and $\mathcal{G}^{\mathsf{nmax}}$ whose groups of k-points are 'completions' of G:

$$\mathcal{G}^{\mathsf{pmax}}(k) \hookleftarrow G \hookrightarrow \mathcal{G}^{\mathsf{nmax}}(k).$$

The group $\mathcal{G}^{\mathsf{pmax}}$ has positive (but not negative) imaginary root subgroups, and vice versa for $\mathcal{G}^{\mathsf{nmax}}$. We always want to work in $\mathcal{G}^{\mathsf{pmax}}$. A choice of positive roots gives a Borel subgroup $\mathcal{B}^{\mathsf{pmax}} \subset \mathcal{G}^{\mathsf{pmax}}$ which is a pro-solvable group scheme. A choice of negative roots gives a Borel subgroup $\mathcal{B}^{-,\mathsf{nmax}} \subset \mathcal{G}^{\mathsf{nmax}}$ in the same way. Taking the preimage of $\mathcal{B}^{-,\mathsf{nmax}}$ in G and then pushing it forward to $\mathcal{G}^{\mathsf{pmax}}(k)$ gives the set of k-points of a group ind-scheme $\mathcal{B}^{-} \subset \mathcal{G}^{\mathsf{pmax}}$ which is of ind-finite-type. We call $\mathcal{B}^{-}$ a *negative-Borel* subgroup of $\mathcal{G}^{\mathsf{pmax}}$, and $\mathcal{B}^{-,\mathsf{nmax}}$ is its *negative-maximal completion.* The geometry of these groups is as follows:

- $\mathcal{G}^{\mathsf{pmax}}/\mathcal{B}^{\mathsf{pmax}}$ is the thin affine flag variety, which is an ind-scheme of ind-finite type. The quotient $\mathcal{B}^{-}/(\text{torus})$ identifies with an open subset.
- $\mathcal{G}^{\mathsf{pmax}}/\mathcal{B}^{-}$ is the thick affine flag variety, which is a scheme of infinite type. The quotient $\mathcal{B}^{\mathsf{pmax}}/(\text{torus})$ identifies with an open subset.

In this discussion, one could replace 'Borel' with 'parabolic.' See [GR, 3.3] for details.

More concrete descriptions are possible if G is of affine extended type. Let G^{fin} be the corresponding reductive group. If we neglect Kac–Moody extensions, then

$$\begin{aligned} G &= G^{\mathsf{fin}}[t, t^{-1}] \\ \mathcal{G}^{\mathsf{pmax}} &= G^{\mathsf{fin}}(\!(t)\!) \\ \mathcal{B}^{\mathsf{pmax}} &= \text{subgroup of } G^{\mathsf{fin}}[\![t]\!] \\ \mathcal{B}^{-} &= \text{subgroup of } G^{\mathsf{fin}}[t^{-1}] \\ \mathcal{G}^{\mathsf{nmax}} &= G^{\mathsf{fin}}(\!(t^{-1})\!) \\ \mathcal{B}^{-,\mathsf{nmax}} &= \text{subgroup of } G^{\mathsf{fin}}[\![t^{-1}]\!] \end{aligned}$$

Of course, $\mathcal{B}^{\mathsf{pmax}}$ is usually called an Iwahori subgroup, and the displayed subgroups are defined by requiring that a t-residue lies in the Borel $B^{\mathsf{fin}} \subset G^{\mathsf{fin}}$ or its opposite.

6.2.9. *Levi decompositions.* We will use the previous discussion to obtain a decomposition

$$P_F = L_F \ltimes U_F.$$

If F is downward or horizontal, this follows from the definition of P_F. Assume that F is upward. Then we have defined

$$P_F := L_{\tilde{F}}^{P_F} \ltimes U_{\tilde{F}}$$

where $L_{\tilde F}^{P_F} \subset L_{\tilde F}$ is a negative-parahoric subgroup. Let $L_{\tilde F}$ play the role of $\mathcal{G}^{\mathsf{pmax}}$ in the previous discussion, and let $L_{\tilde F}^{\mathsf{nmax}}$ be the group $\mathcal{G}^{\mathsf{nmax}}$. The negative-maximal completion $L_{\tilde F}^{P_F,\mathsf{nmax}} \subset L_{\tilde F}^{\mathsf{nmax}}$ admits a decomposition

$$L_{\tilde F}^{P_F,\mathsf{nmax}} = L_F \ltimes L_{\tilde F}^{U_F,\mathsf{nmax}}$$
$$\mathfrak{g}_{R^0(\tilde F)\cap R^{+0}(F)} = \mathfrak{g}_{R^0(F)} \ltimes \mathfrak{g}_{R^0(\tilde F)\cap R^{+}(F)}$$

corresponding to the indicated decomposition of Lie algebras. Pulling back to $L_{\tilde F}$ gives a subgroup $L_{\tilde F}^{U_F} \subset L_{\tilde F}$ and a decomposition

$$L_{\tilde F}^{P_F} = L_F \ltimes L_{\tilde F}^{U_F}.$$

The desired decomposition for P_F comes from defining $U_F := L_{\tilde F}^{U_F} \ltimes U_{\tilde F}$.

Remark. More precisely, if $\mathcal{B}^- \subset L_{\tilde F}$ is any negative-Borel subgroup, there is a map $\mathcal{B}^- \hookrightarrow L_{\tilde F}^{\mathsf{nmax}}$, and $L_{\tilde F}^{U_F}$ is defined to be the preimage of $L_{\tilde F}^{U_F,\mathsf{nmax}}$ under this embedding. It does not depend on the choice of $\mathcal{B}^-$. Although $\mathcal{B}^-$ is generated by its root subgroups, and $L_{\tilde F}^{U_F,\mathsf{nmax}}$ is (topologically) generated by its root subgroups, we do not claim that $L_{\tilde F}^{U_F}$ is generated by its root subgroups, which correspond to $R^0(\tilde F) \cap R^+(F)$ with imaginary roots excluded. The use of negative-maximal completions is inspired by 4) in [GR, 3.3].

6.3. **Sub-Demazure varieties.**

6.3.1. Fix a threadable jointed tour t. We will construct the red parts of the following diagram, where hook arrows are closed embeddings:

$$\begin{tikzcd} I_{f_i c_{i-1}} \arrow[r, hook] & P_{(f_i c_{i-1}, c_i)} \arrow[d, hook] & I_{c_i} \arrow[l, hook'] & I_{c_i} \cap I_{f_{i+1} c_i} \arrow[l, hook'] \arrow[r] & I_{f_{i+1} c_i} \\ & P_{f_i} & & & \end{tikzcd}$$

Then define $X(t)$ as in 6.1.1, taking the quotients in the Zariski topology or any finer one.

6.3.2. *The subgroup* $I_{c_i} \cap I_{f_{i+1}c_i} \subset I_{c_i}$. If $F = c_i \wedge f_{i+1}c_i$ exists, then the desired intersection can be taken inside P_F. Assume that F does not exist. Then support-consistency implies that $\delta(c_i) < \delta(f_{i+1}c_i)$. We will directly construct a subgroup of I_{c_i} which plays the role of this intersection. Recall from 6.2.6 that

$$I_{c_i} = L_{\tilde c_i}^{I_{c_i}} \ltimes U_{\tilde c_i}$$
$$P_{\tilde c_i} = L_{\tilde c_i} \ltimes U_{\tilde c_i}.$$

Define two subgroups

$$L_{\tilde c_i}^{I_{f_{i+1}c_i}} \subset L_{\tilde c_i}$$
$$U_{\tilde c_i}^{I_{f_{i+1}c_i}} \subset U_{\tilde c_i}$$

as follows. The inequality of levels implies that $f_{i+1}c_i$ projects to an upward chamber of $\mathcal{H}_{\tilde c_i}$, so $R^0(\tilde c_i) \cap R^{+0}(f_{i+1}c_i) \subset R^0(\tilde c_i)$ specifies a negative-Iwahori subgroup of $L_{\tilde c_i}$, which we define to be $L_{\tilde c_i}^{I_{f_{i+1}c_i}}$. Next, define the sub-ind-scheme $U_{\tilde c_i}^{I_{f_{i+1}c_i}} \subset U_{\tilde c_i}$ using the closed subset of roots $R^+(\tilde c_i) \cap R^{+0}(f_{i+1}c_i) \subset R^+(\tilde c_i)$ and the construction of $U_{\tilde c_i}$.

Now we would like to define

$$P_{\tilde{c}_i}^{I_{f_{i+1}c_i}} := L_{\tilde{c}_i}^{I_{f_{i+1}c_i}} \ltimes U_{\tilde{c}_i}^{I_{f_{i+1}c_i}} \subset P_{\tilde{c}_i}.$$

To show that this is valid, it suffices to show that $U_{\tilde{c}_i}^{I_{f_{i+1}c_i}}$ is invariant under the action of $L_{\tilde{c}_i}^{I_{f_{i+1}c_i}}$. Since we are working with reduced ind-schemes, it suffices to check this at the level of k-points. A well-known characterization of negative-Iwahori subgroups states that $L_{\tilde{c}_i}^{I_{f_{i+1}c_i}}(k)$ is generated by the torus $T(k)$ and the root subgroups indexed by the *real* roots in $R^0(\tilde{c}_i) \cap R^{+0}(f_{i+1}c_i)$, i.e. roots $\alpha = \alpha^{\mathsf{fin}} + n\delta + m\pi$ where $\alpha^{\mathsf{fin}} \neq 0$. Thus, it suffices to show that $U_{\tilde{c}_i}^{I_{f_{i+1}c_i}}(k)$ is invariant under the action of each of these root subgroups. By the Baker–Campbell–Hausdorff formula, this follows from the fact that, for each root β of $U_{\tilde{c}_i}^{I_{f_{i+1}c_i}}(k)$, i.e. $\beta \in R^+(\tilde{c}_i) \cap R^{+0}(f_{i+1}c_i)$, the set $(\beta + \mathbb{Z}_{\geq 0}\alpha) \cap R^+(\tilde{c}_i)$ is finite and contained in $R^+(\tilde{c}_i) \cap R^{+0}(f_{i+1}c_i)$. (The finiteness relies on $\alpha^{\mathsf{fin}} \neq 0$.)

Lastly, define $I_{c_i} \cap I_{f_{i+1}c_i} := I_{c_i} \cap P_{\tilde{c}_i}^{I_{f_{i+1}c_i}} = (L_{\tilde{c}_i}^{I_{c_i}} \cap L_{\tilde{c}_i}^{I_{f_{i+1}c_i}}) \ltimes U_{\tilde{c}_i}^{I_{f_{i+1}c_i}}$.

6.3.3. *The map* $I_{c_i} \cap I_{f_{i+1}c_i} \to I_{f_{i+1}c_i}$. If $F = c_i \wedge f_{i+1}c_i$ exists, then this map is the obvious embedding. Otherwise, it suffices to construct a map

$$P_{\tilde{c}_i}^{I_{f_{i+1}c_i}} \to I_{f_{i+1}c_i}.$$

Consider the subgroups of $I_{f_{i+1}c_i}$ corresponding to the following subsets of $R^{+0}(f_{i+1}c_i)$:

$R^{+0}(f_{i+1}c_i) \cap R^{+0}(\tilde{c}_i)$	$R^{+0}(f_{i+1}c_i) \cap R^{0}(\tilde{c}_i)$	$R^{+0}(f_{i+1}c_i) \cap R^{+}(\tilde{c}_i)$
$I_{f_{i+1}c_i}^{P_{\tilde{c}_i}}$	$I_{f_{i+1}c_i}^{L_{\tilde{c}_i}}$	$I_{f_{i+1}c_i}^{U_{\tilde{c}_i}}$

These are pro-unipotent group schemes, and there is a semidirect product decomposition

$$I_{f_{i+1}c_i}^{P_{\tilde{c}_i}} = I_{f_{i+1}c_i}^{L_{\tilde{c}_i}} \ltimes I_{f_{i+1}c_i}^{U_{\tilde{c}_i}}.$$

Let us first construct the maps

$$L_{\tilde{c}_i}^{I_{f_{i+1}c_i}} \to I_{f_{i+1}c_i}^{L_{\tilde{c}_i}}$$
$$U_{\tilde{c}_i}^{I_{f_{i+1}c_i}} \to I_{f_{i+1}c_i}^{U_{\tilde{c}_i}}$$

The first map is just the negative-maximal completion of the negative-Iwahori subgroup $L_{\tilde{c}_i}^{I_{f_{i+1}c_i}} \subset L_{\tilde{c}_i}$, see 6.2.8. For both maps, the Lie algebra of the target naturally identifies with a completion of the Lie algebra of the domain. This observation suffices for constructing the second map because its source and target are unipotent. Lastly, checking Lie algebras shows that these two maps intertwine the actions which define the semidirect products, so we obtain the desired map

$$P_{\tilde{c}_i}^{I_{f_{i+1}c_i}} = L_{\tilde{c}_i}^{I_{f_{i+1}c_i}} \ltimes U_{\tilde{c}_i}^{I_{f_{i+1}c_i}} \to I_{f_{i+1}c_i}^{L_{\tilde{c}_i}} \ltimes I_{f_{i+1}c_i}^{U_{\tilde{c}_i}} = I_{f_{i+1}c_i}^{P_{\tilde{c}_i}} \subset I_{f_{i+1}c_i}.$$

6.3.4. *The subscheme* $P_{(f_ic_{i-1},c_i)} \subset P_{f_i}$. First, suppose that f_i is horizontal, so L_{f_i} is of affine type. Opposition implies that the projections of f_ic_{i-1} and c_i to $\mathcal{H}_{f_i}$ are downward and upward, respectively. Thus, $R^0(f_i) \cap R^{+0}(f_ic_{i-1}) \subset R^0(f_i)$ specifies an Iwahori subgroup

$L_{f_i}^{I_{f_i c_{i-1}}} \subset L_{f_i}$, and $R^0(f_i) \cap R^{+0}(c_i) \subset R^0(f_i)$ specifies a negative-Iwahori subgroup $L_{f_i}^{I_{c_i}} \subset L_{f_i}$. The desired construction is the fibered product

$$\begin{tikzcd} P_{(f_i c_{i-1}, c_i)} \arrow[r, hook] \arrow[d] & P_{f_i} \arrow[d] \\ \overline{L_{f_i}^{I_{f_i c_{i-1}}} \cdot L_{f_i}^{I_{c_i}}} \arrow[r, hook] & L_{f_i} \end{tikzcd}$$

where the bar denotes Zariski closure in L_{f_i}. In combinatorial terms, this closure is a downward-closed union of cells in the Birkhoff decomposition of L_{f_i} into double cosets with $L_{f_i}^{I_{f_i c_{i-1}}}$ on the left and its opposite negative-Iwahori subgroup on the right.

Next, suppose that f_i is not horizontal, so L_{f_i} is of finite type. One repeats the above construction, noting that $L_{f_i}^{I_{f_i c_{i-1}}}, L_{f_i}^{I_{c_i}} \subset L_{f_i}$ are Borel subgroups instead of Iwahori and negative-Iwahori subgroups.

6.4. **Scheme structure.**

6.4.1. We claim that $X(t)$ is a scheme. Since it is a twisted product of factors $P_{(f_i c_{i-1}, c_i)}/I_{c_i}$ and $I_{c_i}/[I_{c_i} \cap I_{f_{i+1}c_i}]$, it suffices to show that each of these factors is a scheme.

6.4.2. *The factor* $P_{(f_i c_{i-1}, c_i)}/I_{c_i}$. First, assume that f_i is horizontal. Opposition implies that $f_i = \tilde{c}_i$, so 6.2.6 gives decompositions

$$\begin{aligned} P_{f_i} &= L_{f_i} \ltimes U_{f_i} \\ I_{c_i} &= L_{f_i}^{I_{c_i}} \ltimes U_{f_i}. \end{aligned}$$

Thus $P_{f_i}/I_{c_i} \simeq L_{f_i}/L_{f_i}^{I_{c_i}}$. Opposition also implies that c_i is upward, so $L_{f_i}^{I_{c_i}} \subset L_{f_i}$ is a negative-Iwahori subgroup. Thus, the quotient is the thick affine flag variety of L_{f_i}, which is a scheme. Replacing P_{f_i} by $P_{(f_i c_{i-1}, c_i)}$ gives a Schubert variety therein.

If f_i is not horizontal, L_{f_i} is a reductive group, and similar reasoning shows that the desired factor is a Schubert variety in a finite flag variety.

6.4.3. *The factor* $I_{c_i}/[I_{c_i} \cap I_{f_{i+1}c_i}]$. We saw in 6.3.2 that either $c_i \wedge f_{i+1}c_i$ exists or $\delta(c_i) < \delta(f_{i+1}c_i)$. Let us assume the latter, which is the harder case. By definition, we have

$$\begin{aligned} I_{c_i} &= L_{\tilde{c}_i}^{I_{c_i}} \ltimes U_{\tilde{c}_i} \\ I_{c_i} \cap I_{f_{i+1}c_i} &= (L_{\tilde{c}_i}^{I_{c_i}} \cap L_{\tilde{c}_i}^{I_{f_{i+1}c_i}}) \ltimes U_{\tilde{c}_i}^{I_{f_{i+1}c_i}}, \end{aligned}$$

which implies that

$$I_{c_i}/[I_{c_i} \cap I_{f_{i+1}c_i}] \simeq \left[L_{\tilde{c}_i}^{I_{c_i}} / (L_{\tilde{c}_i}^{I_{c_i}} \cap L_{\tilde{c}_i}^{I_{f_{i+1}c_i}}) \right] \times \left[U_{\tilde{c}_i}/U_{\tilde{c}_i}^{I_{f_{i+1}c_i}} \right].$$

Let us show that the second factor is isomorphic to the scheme $\mathbb{A}^\infty$. By the construction of $U_{\tilde{c}_i}$, it suffices to show that $R^+(\tilde{c}_i) \smallsetminus R^{+0}(f_{i+1}c_i) \subset R^+(\tilde{c}_i, p)$ for some point p with $\delta(p) < \delta(\tilde{c}_i)$. The containment holds if we choose p so that $\tilde{c}_i$ intersects the convex hull of $p \cup \overline{f_{i+1}c_i}$.

For the first factor, we have already seen that $L_{\tilde{c}_i}^{I_{f_{i+1}c_i}} \subset L_{\tilde{c}_i}$ is a negative-Iwahori subgroup. Split into two cases:

- If c_i is downward, then $L_{\tilde{c}_i}^{I_{c_i}} \subset L_{\tilde{c}_i}$ is an Iwahori subgroup. The intersection is a finite-dimensional unipotent subgroup, so the quotient is isomorphic to $\mathbb{A}^\infty$.

- If c_i is upward, then $L_{\tilde{c}_i}^{I_{c_i}} \subset L_{\tilde{c}_i}$ is also a negative-Iwahori subgroup. The quotient is described by the next lemma.

Lemma. *Use the notation of 6.2.8, so $\mathcal{G}^{\mathsf{pmax}}$ is a Kac–Moody group ind-scheme. Let $I, J \subset \mathcal{G}^{\mathsf{pmax}}$ be two negative-Borel subgroups. Then $I/[I \cap J]$ is isomorphic to some $\mathbb{A}^m$.*

Proof. Consider the following diagram of 'short exact sequences':

$$\begin{tikzcd} I \cap J \arrow[r, hook] \arrow[d, hook] & I \arrow[r, two heads] \arrow[d, hook] & I/[I \cap J] \arrow[d] \\ I^{\mathsf{nmax}} \cap J^{\mathsf{nmax}} \arrow[r, hook] & I^{\mathsf{nmax}} \arrow[r, two heads] & I^{\mathsf{nmax}}/[I^{\mathsf{nmax}} \cap J^{\mathsf{nmax}}] \end{tikzcd}$$

The left square is defined by viewing I and J as subgroups of $\mathcal{G}^{\mathsf{nmax}}$. We will show that $I^{\mathsf{nmax}}/[I^{\mathsf{nmax}} \cap J^{\mathsf{nmax}}]$ is isomorphic to some $\mathbb{A}^m$. We will also show that the right vertical map is an isomorphism by showing that it induces a bijection on S-points for any test affine scheme S.

The right vertical map induces an injection on S-points because the left square is fibered, i.e. $I \cap J$ is the intersection of $I^{\mathsf{nmax}} \cap J^{\mathsf{nmax}}$ with I.

Let R be the set of roots of I^{nmax} which are not roots of J^{nmax}. This is called a *nilpotent* set of roots; it is finite and closed. Let $K \subset I^{\mathsf{nmax}}$ is the product of root subgroups indexed by R. Then K is isomorphic to some $\mathbb{A}^m$, it is a subgroup, the map

$$K \to I^{\mathsf{nmax}}/[I^{\mathsf{nmax}} \cap J^{\mathsf{nmax}}]$$

is an isomorphism, and $K \subset I$. Thus, $K \hookrightarrow I \twoheadrightarrow I/[I \cap J]$ is a section of the right vertical map. This shows surjectivity on S-points. ☐

6.5. **Demazure functor.**

6.5.1. *Coxeter varieties.* For this subsection only, let us introduce an enlarged version of aC, denoted aC', whose objects are jointed tethered tours which satisfy support-consistency (5.1.3). We have dropped the opposition condition. All of Section 5 applies to aC'.

Our first goal is to define a functor $X^{\mathsf{c}} : \mathsf{aC}' \to \mathsf{PreShv}$ (presheaves on affine schemes). We now define it on objects. For $t \in \mathsf{aC}'$, let $X^{\mathsf{c}}(t)$ be the presheaf obtained by replacing $P_{(f_i c_{i-1}, c_i)}$ by P_{f_i} in the definition of $X(t)$. This is valid because the opposition condition was only used to define $P_{(f_i c_{i-1}, c_i)}$ and to show that $X(t)$ is a scheme.

6.5.2. *Maps in* $\mathsf{Emb}^{\mathsf{c}}$. Consider any map in $\mathsf{Emb}^{\mathsf{c}}$, denoted

$$t' = ([n], c', f', w') \xrightarrow{(w'_i)_i} ([n], c, f, w) = t$$

where $w_i' \in f_i'$. It will sometimes be more convenient to encode such a map using the tuple of its inverse map, denoted $(w_i)_i$. The two tuples are related as follows: $w_1 \cdots w_i = (w_1' \cdots w_i')^{-1}$, or equivalently $w_i' = (w_1 \cdots w_{i-1}) w_i^{-1} (w_1 \cdots w_{i-1})^{-1}$.

We will first define the map $X^{\mathsf{c}}(t') \to X^{\mathsf{c}}(t)$ using language which is familiar in the classical setting, and then we will explain why the terms still make sense in our setting. Choose lifts $\tilde{w}_i$ for w_i and perform the following maps on each factor:

- *(Construction of $I_{c_{i-1}'} \overset{I \cap I}{\times} I_{f_i' c_{i-1}'} \to I_{c_{i-1}} \overset{I \cap I}{\times} I_{f_i c_{i-1}}$.)* The desired map is
$$x \mapsto (\tilde{w}_1 \cdots \tilde{w}_{i-1})^{-1}\, x\, (\tilde{w}_1 \cdots \tilde{w}_{i-1}).$$
- *(Construction of $P_{f_i'} \to P_{f_i}$.)* The desired map is
$$x \mapsto (\tilde{w}_1 \cdots \tilde{w}_{i-1})^{-1}\, x\, (\tilde{w}_1 \cdots \tilde{w}_{i-1})\, \tilde{w}_i.$$

To make sense of this in our setting (4.1), let $N^{\mathsf{aff}} \subset G^{\mathsf{aff}}$ be the normalizer of the torus of the affine Kac–Moody group, and note that it acts on the affine Kac–Moody algebra $\mathfrak{g}^{\mathsf{aff}}$ by conjugation. Therefore, $N^{\mathsf{aff}}[t, t^{-1}]$ acts on the double affine algebra $\mathfrak{g} := \mathfrak{g}^{\mathsf{aff}}[t, t^{-1}]$ (6.2.1). The components of $N^{\mathsf{aff}}[t, t^{-1}]$ are indexed by the extended double affine Weyl group $\tilde{W} = W^{\mathsf{aff}} \ltimes \Lambda^{\mathsf{aff}, \vee}$ (4.1.4). A *lift* of an element of $\tilde{W}$ is an element in the corresponding component of $N^{\mathsf{aff}}[t, t^{-1}]$. Such elements act on $\mathfrak{g}$ by conjugation.

The maps in the bullets are well-defined because the conjugation by $(\tilde{w}_1 \cdots \tilde{w}_{i-1})^{-1}$ sends the $\mathfrak{g}$-subalgebras corresponding to $I_{c_{i-1}'}, I_{f_i' c_{i-1}'}, P_{f_i'}$ to the $\mathfrak{g}$-subalgebras corresponding to $I_{c_{i-1}}, I_{f_i c_{i-1}}$ (use $f_i' = (w_1 \cdots w_{i-1}) \cdot f_i$), the conjugation respects the relevant completions, and $\tilde{w}_i$ can be interpreted as an element of P_{f_i}.

The resulting map $X^{\mathsf{c}}(t') \to X^{\mathsf{c}}(t)$ does not depend on the choice of lifts because we can ensure that any two lifts differ by a torus factor. Passing from one lift to another will change the bullet maps by a torus factor on either side, but these factors cancel due to the diagonal I-quotients in the definition of $X^{\mathsf{c}}(-)$.

6.5.3. *Maps in* $\mathsf{Dom}^{\mathsf{c}}$. Consider any map $t \twoheadrightarrow t'$ in $\mathsf{Dom}^{\mathsf{c}}$ and factor it as
$$t \xrightarrow{\text{joint-only}} t^{\mathsf{joint}} \xrightarrow{\text{joint-preserving}} t'$$
which is possible by 5.4.4(6). Next, create a commutative triangle involving the joint-only map as shown:

$$\begin{tikzcd} t \arrow[d, "\text{joint-only}"'] & \\ & t^{\mathsf{braid}} \arrow[ul, "\text{braid}"'] \arrow[dl] \\ t^{\mathsf{joint}} & \end{tikzcd} \qquad \begin{array}{ll} c_{i-1} \underset{f_i}{\diamond} c_i & (c_{i-1}, f_i c_{i-1}, c_i) \\ c_{i-1} \diamond f_i^{\mathsf{joint}} c_{i-1} \underset{f_i}{\diamond} c_i & (c_{i-1}, f_i^{\mathsf{joint}} c_{i-1}, f_i c_{i-1}, c_i) \\ c_{i-1} \underset{f_i^{\mathsf{joint}}}{\diamond} c_i & (c_{i-1}, f_i^{\mathsf{joint}} c_{i-1}, c_i) \end{array}$$

Corresponding steps of t and t^{joint} are shown in the middle column, and their socket tours are shown in the right column. Define t^{braid} by replacing each step of t by the indicated two-step tour. The braid map deletes the newly added chambers $f_i^{\mathsf{joint}} c_{i-1}$. Although t^{braid} and t^{joint} might not be threadable, we have defined $X^{\mathsf{c}}(-)$ on them.

The isomorphism $X^{\mathsf{c}}(t^{\mathsf{braid}}) \xrightarrow{\sim} X^{\mathsf{c}}(t)$ is defined using the isomorphisms

$$I_{c_{i-1}} \overset{I\cap I}{\times} I_{f_i c_{i-1}} \simeq (I_{c_{i-1}} \overset{I\cap I}{\times} I_{f_i^{\mathsf{joint}} c_{i-1}}) \overset{I}{\times} (I_{f_i^{\mathsf{joint}} c_{i-1}} \overset{I\cap I}{\times} I_{f_i c_{i-1}}),$$

which exist because the roots match up. (Note that $(c_{i-1}, f_i^{\mathsf{joint}} c_{i-1}, f_i c_{i-1})$ is reduced.)

The map $X^{\mathsf{c}}(t^{\mathsf{braid}}) \to X^{\mathsf{c}}(t^{\mathsf{joint}})$ is defined using the map

$$(I_{f_i^{\mathsf{joint}} c_{i-1}} \overset{I\cap I}{\times} I_{f_i c_{i-1}}) \overset{I}{\times} P_{f_i} \to P_{f_i^{\mathsf{joint}}}$$

obtained by embedding the two factors on the left hand side into $P_{f_i^{\mathsf{joint}}}$ and multiplying.

Composing gives a map $X^{\mathsf{c}}(t) \to X^{\mathsf{c}}(t^{\mathsf{joint}})$. It remains to construct $X^{\mathsf{c}}(t^{\mathsf{joint}}) \to X^{\mathsf{c}}(t')$. Let us simplify the notation by replacing $t^{\mathsf{joint}} \to t'$ by any joint-preserving map, denoted $t \to t'$. Furthermore, by factoring this map, we may assume that it involves a single chamber deletion. For notational simplicity, we assume that c_1 is deleted, so the map looks like

$$c_0 \underset{f_1}{\diamond} c_1 \underset{f_2}{\diamond} c_2 \mapsto c_0 \underset{f_2}{\diamond} c_2.$$

The joint-preserving assumption says that $\mathsf{span}\, f_1 = \mathsf{span}\, f_2$. The desired map will be

$$\begin{gathered}
(I_{c_0} \overset{I\cap I}{\times} I_{f_1 c_0}) \overset{I}{\times} P_{f_1} \overset{I}{\times} (I_{c_1} \overset{I\cap I}{\times} I_{f_2 c_1}) \overset{I}{\times} P_{f_2} \\
\big\downarrow \mathsf{id} \times \mathrm{swap} \times \mathsf{id} \\
(I_{c_0} \overset{I\cap I}{\times} I_{f_1 c_0}) \overset{I}{\times} (I_{f_1 c_0} \overset{I\cap I}{\times} I_{f_2 c_0}) \overset{I}{\times} P_{f_2} \overset{I}{\times} P_{f_2} \\
\big\downarrow \mathrm{braid} \times \mathrm{mult} \\
(I_{c_0} \overset{I\cap I}{\times} I_{f_2 c_0}) \overset{I}{\times} P_{f_2}
\end{gathered}$$

The 'braid' map is an isomorphism which is defined in the same way as $X^{\mathsf{c}}(t^{\mathsf{braid}}) \xrightarrow{\sim} X^{\mathsf{c}}(t)$ above. The 'mult' map is multiplication in P_{f_2}.

Now we construct the 'swap' map, which will be an isomorphism

$$P_{f_1} \overset{I_{c_1}}{\times} (I_{c_1} \overset{I_{c_1} \cap I_{f_2 c_1}}{\times} I_{f_2 c_1}) \simeq (I_{f_1 c_0} \overset{I_{f_1 c_0} \cap I_{f_2 c_0}}{\times} I_{f_2 c_0}) \overset{I_{f_2 c_0}}{\times} P_{f_2}.$$

Write $A := \mathsf{span}\, f_1 = \mathsf{span}\, f_2$ and $L_A := L_{f_1} = L_{f_2}$. If A is horizontal, then positivity of the tour implies that $f_1 = f_2$, so both sides equal P_{f_1}. Assume that A is not horizontal. Then the supports of $f_1, c_1, f_1 c_0$ are equal, and the supports of $f_2, f_2 c_0, f_2 c_1$ are equal; denote the support faces by s_1 and s_2, respectively.

Rewrite the left side as follows:

$$\begin{aligned}
P_{f_1} \overset{I_{c_1}}{\times} (I_{c_1} \overset{I_{c_1} \cap I_{f_2 c_1}}{\times} I_{f_2 c_1}) &= P_{f_1} \overset{I_{c_1} \cap I_{f_2 c_1}}{\times} I_{f_2 c_1} \\
&= (L_A \ltimes U_{f_1}) \overset{L_A^{I_{c_1}} \ltimes (L_{s_1}^{U_{f_1}} \cap L_{s_1}^{U_{f_2}}) \ltimes U_{s_1}^{I_{f_2 c_1}}}{\times} (L_A^{I_{f_2 c_1}} \ltimes U_{f_2}) \\
&= L_A \times U_{f_1} \overset{(L_{s_1}^{U_{f_1}} \cap L_{s_1}^{U_{f_2}}) \ltimes U_{s_1}^{I_{f_2 c_1}}}{\times} U_{f_2}.
\end{aligned}$$

For the second identification, use the Levi decomposition of P_{f_1} (6.2.9), use

$$\begin{aligned} I_{f_2c_1} &:= L_{s_2}^{I_{f_2c_1}} \ltimes U_{s_2} \\ &= L_A^{I_{f_2c_1}} \ltimes \underbrace{L_{s_2}^{U_{f_2}} \ltimes U_{s_2}}_{U_{f_2}} \end{aligned}$$

$$\mathfrak{g}_{R^{+0}(f_2c_1)} = \mathfrak{g}_{R^0(A)\cap R^{+0}(f_2c_1)} \ltimes \mathfrak{g}_{R^0(s_2)\cap R^+(f_2)} \ltimes \mathfrak{g}_{R^+(s_2)},$$

and use

$$\begin{aligned} I_{c_1} \cap I_{f_2c_1} &:= (L_{s_1}^{I_{c_1}} \cap L_{s_1}^{I_{f_2c_1}}) \ltimes U_{s_1}^{I_{f_2c_1}} \\ &= L_A^{I_{c_1}} \ltimes (L_{s_1}^{U_{f_1}} \cap L_{s_1}^{U_{f_2}}) \ltimes U_{s_1}^{I_{f_2c_1}} \end{aligned}$$

which follows from similar identifications

$$\begin{aligned} L_{s_1}^{I_{c_1}} &= L_A^{I_{c_1}} \ltimes L_{s_1}^{U_{f_1}} \\ L_{s_1}^{I_{f_2c_1}} &= L_A^{I_{f_2c_1}} \ltimes L_{s_1}^{U_{f_2}} \end{aligned}$$

and the observation that $L_A^{I_{c_1}} = L_A^{I_{f_2c_1}}$ since any root which vanishes on A must take the same sign on c_1 and f_2c_1. For the third identification, cancel $L_A^{I_{c_1}}$ with $L_A^{I_{f_2c_1}}$, and note that the diagonal quotient group now maps into U_{f_1}.

Similarly, rewrite the right side as follows:

$$(I_{f_1c_0} \overset{I_{f_1c_0}\cap I_{f_2c_0}}{\times} I_{f_2c_0}) \overset{I_{f_2c_0}}{\times} P_{f_2} = U_{f_1} \overset{(L_{s_1}^{U_{f_1}}\cap L_{s_1}^{U_{f_2}})\ltimes U_{s_1}^{I_{f_2c_0}}}{\times} U_{f_2} \times L_A.$$

Observe that $U_{s_1}^{I_{f_2c_1}} = U_{s_1}^{I_{f_2c_0}}$ because any root which takes different signs on f_2c_1 and f_2c_0 must be zero on f_2 and hence cannot be positive on s_1. Thus, the desired isomorphism takes the form

$$L_A \times (\text{factor}) \simeq (\text{factor}) \times L_A.$$

The correct isomorphism uses the conjugation action of L_A on (factor), and this action can be defined first on the level of Lie algebras and then integrated. (Since A is not horizontal, L_A is a reductive group.)

6.5.4. *Construction of functors.* We have now defined $X^{\mathsf{c}}(-)$ on $\mathsf{Emb}^{\mathsf{c}}$ and $\mathsf{Dom}^{\mathsf{c}}$. In order to get a functor $X^{\mathsf{c}}(-) : \mathsf{aC}' \to \mathsf{Sch}$, it suffices to check triangles in $\mathsf{Emb}^{\mathsf{c}}$, triangles in $\mathsf{Dom}^{\mathsf{c}}$, and squares coming from the exchange map. This is tedious but straightforward.

Next, restrict $X^{\mathsf{c}}(-)$ to the subdiagram $\mathsf{aD} \subset \mathsf{aC}'$. We claim that $X(-)$ gives a subfunctor of this restricted functor. It suffices to check that, for every map $t \to t'$ in aD, the image of $X(t)$ under $X^{\mathsf{c}}(t) \to X^{\mathsf{c}}(t')$ is contained in $X(t')$. This is also straightforward.[11]

This gives the functor $X(-) : \mathsf{aD} \to \mathsf{Sch}$. It sends braid maps to isomorphisms. Now left Kan extension (Theorem 5.5.1) produces the desired functor $X(-) : \mathsf{D} \to \mathsf{Sch}$, whose values are defined by the same formula as before.

[11] The verification boils down to well-known statements about Kac–Moody groups. If $t \to t'$ belongs to $\mathsf{Emb}^{\mathsf{d}}$, use the fact that the Bruhat order governs closures of Bruhat cells. If $t \to t'$ belongs to $\mathsf{Dom}^{\mathsf{d}}$, use the fact that the Demazure product governs images of Bruhat cells under convolution.

6.5.5. Fix $t \in \mathsf{D}$. Since $X(t)$ is a twisted product of affine spaces and Schubert varieties, the Schubert stratifications of its factors give a stratification of $X(t)$. Each resulting *Schubert cell* of $X(t)$ comes from a map $t' \hookrightarrow t$ in $\mathsf{Emb}^{\mathsf{d}}$ as follows:

$$X(t^{\prime,\mathsf{unjoint}}) \xhookrightarrow{\text{open}} X(t') \xhookrightarrow{\text{closed}} X(t).$$

Here $t^{\prime,\mathsf{unjoint}}$ is obtained from t' by enlarging each joint face f'_i to equal c'_i. The largest Schubert cell corresponds to $t' = t$ and is denoted $V(t)$.

Next, let $t \to t'$ be any birational map in $\mathsf{Dom}^{\mathsf{d}}$ or equivalently $\mathsf{Dom}^{\mathsf{c}}$. We claim that, in the following diagram, each row gives a partition into a closed subscheme and its open complement, the squares are cartesian, and the right vertical map is an isomorphism.

$$\begin{tikzcd}
\operatorname*{colim}\limits_{\substack{s \hookrightarrow t \\ \text{strict}}} X(s) \arrow[r, hook] \arrow[d] & X(t) \arrow[d] & V(t) \arrow[l, hook'] \arrow[d, "\wr"] \\
\operatorname*{colim}\limits_{\substack{s' \hookrightarrow t' \\ \text{strict}}} X(s') \arrow[r, hook] & X(t') & V(t') \arrow[l, hook']
\end{tikzcd}$$

In fact, the claim follows immediately from the partition into Schubert cells discussed above. To see that the right vertical map is an isomorphism, recall from 5.3.5 that $t \to t'$ is a birational map if and only if its ‘deletion’ subtours $([\varphi(j-1), \varphi(j)], c)$ are reduced.

This is not an excision diagram because $X(t) \to X(t')$ need not be proper.

6.6. **Demazure varieties.**

6.6.1. A *refinement* is a span in $\mathsf{Dom}^{\mathsf{c}}$ of the form

$$t \xleftarrow{\text{braid}} t' \xrightarrow{\text{joint-only}} \hat{t},$$

where the left arrow is a braid map, and the right arrow does not change the joint of any chamber which appears in t. We denote the refinement as $t \dashrightarrow \hat{t}$. Refinements can be composed via the usual recipe for composing spans:

$$\begin{tikzcd}
\bullet \arrow[r, "\text{joint-only}"] \arrow[d, "\text{braid}"'] & \hat{t}' \arrow[r, "\text{joint-only}"] \arrow[d, "\text{braid}"] & \hat{\hat{t}} \\
t' \arrow[r, "\text{joint-only}"] \arrow[d, "\text{braid}"'] & \hat{t} & \\
t & &
\end{tikzcd}$$

Note that the square can be filled in uniquely. This gives a category Ref whose objects are threadable jointed tours and whose morphisms are refinements. Since $X(-)$ sends braid maps to isomorphisms, each refinement as above gives a map $X(t) \to X(\hat{t})$ which is easily seen to be an open embedding. This gives a functor $X(-) : \mathsf{Ref} \to \mathsf{Sch}$.

The current definition of ‘refinement’ is compatible with the original definition in 3.1.3. The proof of Lemma 3.1.4 shows that Ref is a poset.

6.6.2. A positive tour (I, c) is called a *generalized gallery* if it satisfies

(1) I is complete.

(2) c is continuous.

(3) For each gap $i < j$ in I, the meet $c_i \wedge c_j$ exists.

Note that every gallery is a generalized gallery (Proposition 3.1.7) and any strictly positive generalized gallery is infinite. To every generalized gallery we attach a canonical sequence of *joint faces*, namely $(I \setminus \{\hat{0}\}, f)$ where $f_j = c_i \wedge c_j$ if j has a predecessor i, and $f_j = c_j$ otherwise.

Fix a threadable generalized gallery g, and let $\mathsf{Ref}_{/g} \subset \mathsf{Ref}$ be the full subposet consisting of threadable jointed tours t which are refined by g, such that the joint faces of t and g agree. Then $\mathsf{Ref}_{/g}$ is a filtered poset, and we define

$$X^{\mathsf{ind}}(g) := \operatorname*{colim}_{\mathsf{Ref}_{/g}} X(t).$$

This is the *Demazure variety* associated to g.

Remarks.

(1) Generalized galleries do not form any interesting category. There is also no 'diagram of Demazure varieties,' because that would have to be a functor X^{ind} from generalized galleries to schemes.

(2) The above colimit is indexed by an uncountable poset $\mathsf{Ref}_{/g}$. It is possible to replace this by a countable poset without changing the colimit. Simply delete c_j from g whenever j has no predecessor. The resulting tour is countable because g crosses countably many walls, and the colimit does not change because $X(-)$ sends braid moves to isomorphisms.

In particular, $X^{\mathsf{ind}}(g)$ decomposes into countably many Schubert cells.

7. Flag variety

7.1. **The underlying set.** In this section, let us modify the Demazure category by requiring that all tours have only rational-level chambers. This assumption is needed in order to apply homotopical deletion (8.8).

7.1.1. Given a scheme, an underline will denote its underlying set (not just its k-points). For each tethered chamber $y \in \langle T \xrightarrow{\mathcal{W}} \mathcal{W}^{C_0} \rangle$, define

$$\underline{\mathcal{F}\ell}_y := \operatorname*{colim}_{t \in \mathsf{D}_y} \underline{V}(t).$$

Here $\mathsf{D}_y := \mathsf{D}_{(\ell_{C_0}(y), y)}$ is the category consisting of reduced tours ending at y and birational morphisms between them, see 5.3.5. Homotopical deletion (8.8.1) implies that D_y is contractible, and the transition maps are isomorphisms, so the colimit is isomorphic to each term $\underline{V}(t)$, hence isomorphic to the scheme $\mathbb{A}^\infty$. Define $\underline{\mathcal{F}\ell} := \sqcup_y \underline{\mathcal{F}\ell}_y$. This is the underlying set of the double affine flag variety, expressed as a disjoint union of Schubert cells.

7.1.2. The next result implies that our Demazure varieties do in fact map to $\underline{\mathcal{F}\ell}$.

Theorem. *There is a unique isomorphism* $\pi : \operatorname{colim}_{t\in\mathsf{D}} \underline{X}(t) \xrightarrow{\sim} \underline{\mathcal{F}\ell}$ *such that, for all reduced tours* t*, the subset* $\underline{V}(t) \subset \underline{X}(t)$ *is identified with* $\underline{\mathcal{F}\ell}_y \subset \underline{\mathcal{F}\ell}$*.*

Proof. First, fix $j \in J$. We start with $\operatorname{colim}_{t\in\mathsf{D}_{\leq j}} X(t)$ as a 'base case' and inductively compute $\operatorname{colim}_{t\in\mathsf{D}} \underline{X}(t)$ using homotopical deletion (8.8.1). The technique applies because, if $F = \underline{X}(-)$ and $\mathcal{E} = \mathsf{Set}$, then each functor $\tilde{F} : \mathsf{Dom}^{\mathsf{c}}_{\leq j'} \to \tilde{\mathcal{E}}$ sends all maps in $\mathsf{Dom}^{\mathsf{c}}_{j'}$ to isomorphisms, thanks to the diagram in 6.5.5. The induction uses the fact that the upward interval $J_{>j}$ is well-ordered. The result is a canonical isomorphism

$$\operatorname*{colim}_{t\in\mathsf{D}} \underline{X}(t) \simeq \operatorname*{colim}_{t\in\mathsf{D}_{\leq j}} \underline{X}(t) \sqcup \left(\bigsqcup_{z} \underline{\mathcal{F}\ell}_z \right),$$

where z ranges over tethered chambers such that $(\ell_{C_0}(z), z) > j$ in J. Each 'relevant' step contributes a component $\underline{\mathcal{F}\ell}_z$, while the 'irrelevant' steps do not contribute anything.

These isomorphisms are compatible as j varies. Taking j to be arbitrarily small yields an injective map

$$\underline{\mathcal{F}\ell} \hookrightarrow \operatorname*{colim}_{t\in\mathsf{D}} \underline{X}(t).$$

To conclude, we will show that this map is surjective. Equivalently, for every t and every point $x \in \underline{X}(t)$, we must find a zig-zag in D from t to some reduced tour t', and a chain of images and preimages of x along this zig-zag, such that the final point lies in $\underline{V}(t')$. Assume without loss of generality that $x \in \underline{V}(t)$ and apply the following algorithm:

- Choose a gallery (I, g) which refines t.
- Consider the first double-crossing $(H, 2) \in \mathsf{Walls}'(g)$. The first crossing $(H, 1)$ gives a step (g_i, g_{i+1}), and the second crossing $(H, 2)$ gives a later step (g_j, g_{j+1}). Let $F = g_j \wedge g_{j+1}$ be the facet of the second crossing. Consider the gallery

  $$\left([\hat{0}, i) \sqcup [i+1, j] \sqcup [j, \hat{1}], g^{\mathsf{braid}}\right) := g|_{[\hat{0},i)} \diamond r_H g|_{[i+1,j]} \diamond g|_{[j,\hat{1}]}$$

  which is obtained from g by performing a generalized braid move on the $[i, j]$-subgallery. Let j' refer to the first copy of j in the new index set. Then g^{braid} crosses H twice, at consecutive steps $(g^{\mathsf{braid}}_{j'}, g^{\mathsf{braid}}_j, g^{\mathsf{braid}}_{j+1})$. Define $g^{\mathsf{del1}}, g^{\mathsf{del2}} \hookrightarrow g^{\mathsf{braid}}$ by folding at one or both of these crossings, respectively. Note that g^{del1} and g^{del2} are the two possible results of applying the move defined in Remark 4.7.1 to g.
- A special case of homotopical deletion says that any two finite reduced jointed tours with the same start and end chambers are related by a zig-zag of birational maps in D. Using this, we can find a zig-zag of birational maps from t to another jointed tour t^{braid} which is refined by g^{braid} and contains

  $$g^{\mathsf{braid}}_{j'} \underset{F}{\diamond} g^{\mathsf{braid}}_j \underset{F}{\diamond} g^{\mathsf{braid}}_{j+1} = g_{j+1} \underset{F}{\diamond} g_j \underset{F}{\diamond} g_{j+1}$$

  as a subtour. Since $x \in \underline{V}(t)$, taking images and preimages along this zig-zag gives a unique point $x^{\mathsf{braid}} \in \underline{V}(t^{\mathsf{braid}})$.

- Define tours $t^{\mathsf{del1}}, t^{\mathsf{del2}}$ and maps in D by modifying the above subtour as shown:

$$\begin{tikzcd}
 & t^{\mathsf{braid}} \arrow[dd, two heads, "\mathsf{Dom}^{\mathsf{d}}"] & & g_{j+1} \underset{F}{\diamond} g_j \underset{F}{\diamond} g_{j+1} \arrow[d, two heads, "\mathsf{Dom}^{\mathsf{c}}"] \\
 & & & g_{j+1} \underset{F}{\diamond} g_{j+1} \arrow[d, hook, "\mathrm{turn}"] \\
t^{\mathsf{del2}} \arrow[r, tail, "\mathsf{Emb}^{\mathsf{d}}"] & t^{\mathsf{del1}} & g_{j+1} \underset{F}{\diamond} g_{j+1} \arrow[r, tail, "\mathsf{Emb}^{\mathsf{d}}"] & g_{j+1} \underset{F}{\diamond} g_j
\end{tikzcd}$$

  As the notation suggests, $t^{\mathsf{del1}}, t^{\mathsf{del2}}$ are refined by $g^{\mathsf{del1}}, g^{\mathsf{del2}}$, respectively. An SL_2 computation shows that

$$(\text{image of } V(t^{\mathsf{braid}})) = V(t^{\mathsf{del1}}) \sqcup (\text{image of } V(t^{\mathsf{del2}}))$$

  as subsets of $X(t^{\mathsf{del1}})$. Let x^{del1} be the image of x^{braid} and split into cases accordingly:

  - If $x^{\mathsf{del1}} \in V(t^{\mathsf{del1}})$, restart the algorithm with $x^{\mathsf{del1}}, t^{\mathsf{del1}}, g^{\mathsf{del1}}$ in place of x, t, g.
  - If x^{del1} lies in the image of $V(t^{\mathsf{del2}})$, then let $x^{\mathsf{del2}} \in V(t^{\mathsf{del2}})$ be the unique preimage. Restart the algorithm with $x^{\mathsf{del2}}, t^{\mathsf{del2}}, g^{\mathsf{del2}}$ in place of x, t, g.

Remark 4.7.1 guarantees that the algorithm terminates with a reduced tour t' and a point in $\underline{V}(t')$, as desired. $\square$

7.1.3. *Convolution flag varieties.* One can easily construct the convolution diagram for double affine flag varieties (at the level of sets) using a similar colimit. In addition to requiring that the end chamber of the tour is tethered, we also require that certain intermediate chambers are tethered. More precisely, suppose we are given a positive tour $(C_0, T_1, \dots, T_m)$ of chambers. Then define a variant of the category D in which an object is a finite jointed tour $([n], c, f)$ equipped with a (necessarily increasing) map $\varphi : [1, m] \to [n]$ and elements $w_1, \dots, w_m$ such that $c_{\varphi(j)} = w_j T_j$ for all $j \in [1, m]$. One obtains a 'convolution' flag variety by taking the colimit of $X(-)$ along this category. Finally, the 'projection' maps come from restricting the tour $([n], c, f)$ to each of the intervals $[\varphi(j-1), \varphi(j)] \subset [n]$ for $j \in [1, m]$, where $\varphi(0) := 0$. This corresponds to viewing $X(-)$ as a twisted product and projecting onto a subset of the factors.

7.2. **The scheme structure (conjecture).** In 1.5, we outlined one construction of Kac–Moody flag varieties, and we conjectured that it generalizes to the double affine setting. We spell out the details of this conjecture here, at the risk of repetition.

7.2.1. *Lifts of sub-Demazure varieties.* For each tour $t \in \mathsf{D}$, let $E(t)$ be the variant of the sub-Demazure variety $X(t)$ obtained by omitting the final I_{c_n}-quotient. All of the previous results apply to $E(t)$ as well as $X(t)$, except $E(t)$ is an ind-scheme rather than a scheme.

For some subgroups $J \subset I_{c_n}$, the quotient $E(t)/J$ is a scheme. We define

$$\Gamma_{E(t)} := \operatorname*{colim}_J \Gamma(E(t)/J, \mathcal{O}_{E(t)/J})$$

as a colimit of sets. Each $a \in \Gamma_{E(t)}$ pulls back to a regular function on each subscheme of the ind-scheme $E(t)$. Thus, it makes sense to define the subset $D(a) \subseteq E(t)$ to be the locus where a does not vanish.

7.2.2. *Construction of the topology.* For each tethered chamber $y \in \langle T \xrightarrow{\mathcal{W}} \mathcal{W}^{C_0}\rangle$, we equip $\underline{\mathcal{F}\ell}_{\preceq y}$ with a topology using the following basis of open sets: A subset $U \subseteq \underline{\mathcal{F}\ell}_{\preceq y}$ belongs to the basis if and only if there exists a system of elements

$$(a_t)_t \in \lim_{t \in \mathsf{D}^{\mathsf{op}}_{\preceq y}} \Gamma_{E(t)}$$

such that, for each tour t, we have

$$\pi_t^{-1}(U) = D(a_t)$$

as subsets of $E(t)$, where π_t is the composition $E(t) \to X(t) \xrightarrow{m} \underline{\mathcal{F}\ell}_{\preceq y}$.

7.2.3. *The conjecture.* Let us say that $t \in \mathsf{D}_y$ is *fine* if, for each length-1 arrow $y' \to y$ in $\mathcal{W}^{C_0}$, there is a (necessarily unique) map $t' \to t$ in $\mathsf{Emb}^{\mathsf{d}}$ such that t' ends at y'. This is equivalent to requiring that $X(t) \to \underline{\mathcal{F}\ell}_{\preceq y}$ is a bijection over the cells of codimension ≤ 1. (Note that $t \in \mathsf{D}_y$ means, by definition, that t is reduced with end chamber y.)

We assumed at the start of this section that C_0 and y are rational-level. This implies that there are finitely many length-1 arrows $y' \to y$ (Corollary 4.5.8), so a fine tour $t \in \mathsf{D}_y$ exists. For any such t, we may turn the topological space $\underline{\mathcal{F}\ell}_{\preceq y}$ into a ringed space by equipping it with the sheaf of rings $m_{t,*}\mathcal{O}_{X(t)}$.

We conjecture that this ringed space is a scheme and does not depend on the choice of t.

We conclude this subsection with some speculative remarks.

7.2.4. *Mathieu's construction.* We have already mentioned that the conjecture is inspired by Mathieu's construction of Kac–Moody flag varieties. We hope that it is possible to develop a double affine analogue of Mathieu's theory, and we believe that this is the best way to approach the conjecture. The basic idea is to look for a line bundle $\mathcal{L}$ on some $X(t)$ such that the resulting map to projective space factors as

$$\begin{tikzcd} X(t) \arrow[d, "m"'] \arrow[dr] & \\ \underline{\mathcal{F}\ell}_{\preceq y} \arrow[r, hook] & \mathbb{P}\big(\Gamma(\mathcal{L})^\vee\big) \end{tikzcd}$$

where the hook arrow is a bijection onto the closure of an I_{C_0}-orbit. This provides one scheme structure on $\underline{\mathcal{F}\ell}_{\preceq y}$, and then the 'correct' one is obtained by normalization. Work along these lines has been done for the semi-infinite affine flag variety, see [KaNS, 4.4].

7.2.5. *Constructible sheaves and Hecke categories.* If the scheme structure on $\underline{\mathcal{F}\ell}_{\preceq y}$ exists, then we believe that the double affine Hecke category should be constructed as follows. Recall that the intersection of a Schubert variety with an opposite Schubert variety is called a Richardson variety. In the double affine setting, the Schubert varieties are the $\mathcal{F}\ell_{\preceq y}$, and the opposite Schubert variety can be defined as the closure (in $\mathcal{F}\ell$) of the attracting locus of a T-invariant point under a certain $\mathbb{G}_m$-action (or infinitesimal deformation thereof).

We conjecture that these closures are ind-proper. Then we can define the double affine Hecke category to be the colimit of constructible sheaf categories on an increasing union of Richardson varieties.

This definition should be compatible with the definition of Kazhdan–Lusztig polynomials in [M2] because R-polynomials count $\mathbb{F}_p$-points in Richardson varieties. We also hope that the increasing union of Richardson varieties will be isomorphic to the union of the transversal slices defined in [BF], and we hope that its set of k-points will be in bijection with the set of chambers of the hovel (a.k.a. masure) defined in [GR].

7.2.6. *New version of the double loop group.* In the classical setting, the Kac–Moody group G^{aff} can be recovered as $\operatorname{colim}_g E(g)$. Analogously, in the double affine setting, we would expect the set

$$\underline{\mathcal{G}}_{C_0,T} := \operatorname*{colim}_{t \in \mathsf{D}} E(t)$$

to be a version of the double loop group.

However, $\underline{\mathcal{G}}_{C_0,T}$ is not actually a group. Instead, there are multiplication maps

$$\underline{\mathcal{G}}_{T_1,T_2} \times \underline{\mathcal{G}}_{T_2,T_3} \to \underline{\mathcal{G}}_{T_1,T_3}.$$

for every positive tour (T_1, T_2, T_3). These maps are constructed in the same way as the convolution flag varieties (7.1.3). In other words, we can define a category whose objects are chambers and whose morphisms from T_1 to T_2 are given by $\underline{\mathcal{G}}_{T_1,T_2}$.

Next, we explain how to get a monoid. The extended double affine Weyl group $\tilde{W}$ acts on everything discussed so far. We will only use the action of the subgroup $\langle d \rangle \subset \tilde{W}$ generated by the translation $d \in \Lambda^{\mathsf{aff},\vee}$. Fix an upward chamber C_0 whose support equals $\{0\}$, and restrict to the components

$$\underline{\mathcal{G}}_{\mathsf{good}} := \bigsqcup_{\substack{n_1,n_2 \in \mathbb{Z} \\ n_1 \le n_2}} \underline{\mathcal{G}}_{C_0+n_1 d, C_0+n_2 d}.$$

Take the quotient by $\langle d \rangle$ to obtain the set

$$M := \bigsqcup_{n \in \mathbb{Z}_{\ge 0}} \underline{\mathcal{G}}_{C_0, C_0+nd}.$$

It is the set of morphisms of a category with one object, so it is a monoid. The Bruhat decomposition of each component $\underline{\mathcal{G}}_{C_0,T}$ gives a decomposition of M.

This monoid is analogous to the 'positive' subsemigroup $G_+ \subset G$ which was mentioned in 1.2. However, it is different because it has a different set of imaginary root subgroups, and it is completed differently. Specifically, M contains I_{C_0} by construction, so it has root subgroups for the imaginary roots $(\mathbb{Z}\delta + \mathbb{Z}_{>0}\pi) \sqcup (\mathbb{Z}_{>0}\delta + 0\pi)$.

We believe that M is the most natural version of the double loop group when seeking to generalize the Kac–Moody theory, because it ensures that the 'positive' Borel subgroup I_{C_0} has as many imaginary root subgroups as possible, while the 'opposite' Borel subgroup has none. Furthermore, we conjecture that M can be made into an ind-scheme (possibly with affine or quasi-affine strata), in analogy with our previous conjecture about $\mathcal{F}\ell$. However, we suspect that different versions of the double loop group will lead to the same $\mathcal{F}\ell$ as long as they have root subgroups for the imaginary roots $\mathbb{Z}_{>0}\delta + \mathbb{Z}_{>0}\pi$.

7.3. **An example of the failure of properness.** As mentioned in 1.5, the Demazure maps $X(t) \to \mathcal{F}\ell$ should be non-proper. It is natural to try to fix this by enlarging $X(t)$ to the Demazure variety $X^{\mathsf{ind}}(g)$ (6.6). Unfortunately, the resulting map $X^{\mathsf{ind}}(g) \to \mathcal{F}\ell$ should still be non-proper. We will consider an example where the correct scheme structure on $\mathcal{F}\ell$ is already known and show that the map is indeed non-proper.

7.3.1. Let us take $G^{\mathsf{fin}} = \mathrm{SL}_2$ and allow only chambers which touch the origin $0 \in \mathfrak{h}$. These chambers are in bijection with the chambers of the local arrangement $\mathcal{H}^{\mathsf{aff}} := \mathcal{H}_{\{0\}}$, which is the affine-type Kac–Moody arrangement. Let us take the fundamental chamber C_0 to be downward, and take the tether chamber to be $C_0^{\mathsf{op}} := -C_0$. Then the Bruhat poset $\langle C_0^{\mathsf{op}} \xrightarrow{\mathcal{W}} \mathcal{W}^{C_0}\rangle$ identifies with W^{aff} equipped with the opposite Bruhat order, i.e. 1 is the maximal element.

With these choices, we have

$$\mathcal{F}\ell = X(C_0 \diamond_{\{0\}} C_0^{\mathsf{op}}) = G^{\mathsf{aff}}/B_{C_0^{\mathsf{op}}}.$$

Indeed, the tour $C_0 \diamond_{\{0\}} C_0^{\mathsf{op}}$ is the terminal object of D (which is defined using our chosen C_0 and T), while $\mathcal{F}\ell$ can be realized as a colimit along D (Theorem 7.1.2), so the first equality follows. In the second equality, $B_{C_0^{\mathsf{op}}} \subset G^{\mathsf{aff}}$ is the 'negative' Borel subgroup, and the equality follows from the definition of $X(-)$. The upshot is that, in this special case, $\mathcal{F}\ell$ is the thick affine flag variety, which has a well-known scheme structure.

Next, denote the two simple reflections by s and t, and let g be the reduced gallery from C_0 to C_0^{op} whose first adjacency (from C_0 to the second chamber) has type s. The Demazure map

$$m : X^{\mathsf{ind}}(g) \to \mathcal{F}\ell$$

is a map of infinite-type schemes. We will construct a finite-type subvariety of $\mathcal{F}\ell$ over which m can be completely described. The description will imply that m is not proper.

The whole discussion is a special case of the infinite-dimensional analogue of [E, §3].

7.3.2. *Brion's resolution.* Consider the non-reduced gallery g^e obtained by concatenating g with the reduced gallery from C_0^{op} to tsC_0^{op}. In Figure 4, the green path is g^e, and g is the subgallery ending at the blue dot. Let $m^e : X^{\mathsf{ind}}(g^e) \to \mathcal{F}\ell$ be the Demazure map, and let $X^{\mathsf{ind}}(g^e)_1$ be its fiber over $1 \in \mathcal{F}\ell$. (This is a point in the big cell of $\mathcal{F}\ell$.)

Lemma. $X^{\mathsf{ind}}(g^e)_1$ *is irreducible and locally smooth of dimension* $\ell(g^e) - \ell(g) = 2$.

Proof. The finite-dimensional analogue is [E, Thm. 3.3], and the same proof works here. Since the I_{C_0}-orbit of $1 \in \mathcal{F}\ell$ is a dense open subset, the three properties (irreducibility, local smoothness, relative dimension) are inherited from the generic fiber. For the first two properties, simply note that $X^{\mathsf{ind}}(g^e)$ is irreducible and locally smooth. To compute the relative dimension, analyze the fibers of deletion maps as in Theorem 7.1.2. ☐

There is also a 'truncation' map $\tau : X^{\mathsf{ind}}(g^e) \to X^{\mathsf{ind}}(g)$ defined by forgetting the last two parahoric factors:

$$\cdots \overset{I}{\times} P \overset{I}{\times} P \overset{I}{\times} P/I \to \cdots \overset{I}{\times} P/I$$
$$(\ldots, p_1, p_2, p_3) \mapsto (\ldots, p_1)$$

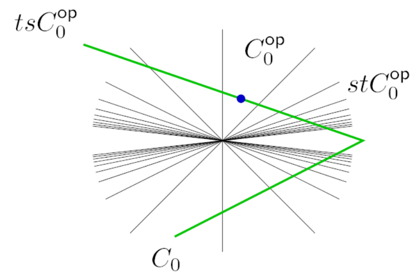

FIGURE 4. The 'extended' gallery g^e in the affine-type Kac-Moody arrangement for SL_2. Of the two walls which are adjacent to C_0, the vertical one is t and the other one is s.

Its restriction to the fiber $X^{\mathsf{ind}}(g^e)_1$ fits into a cartesian diagram

$$\begin{tikzcd} & X^{\mathsf{ind}}(g^e)_1 \arrow[r, hook, "\tau"] \arrow[d, "\mu"] & X^{\mathsf{ind}}(g) \arrow[d, "m"] \\ \mathbb{P}^1 \tilde{\times} \mathbb{P}^1 \arrow[r, "\sim"] & \mathcal{F}\ell^{\succeq st} \arrow[r, hook] & \mathcal{F}\ell \end{tikzcd}$$

where the *opposite Schubert variety* $\mathcal{F}\ell^{\succeq st}$ is isomorphic to a twisted product of two $\mathbb{P}^1$'s. The birational map μ is an analogue of Brion's resolution of Richardson varieties, which is discussed in [E, 3.4].

7.3.3. *Toric description.* We will describe μ using toric geometry. The torus of G^{aff}, which is 3-dimensional, acts on everything in sight. We restrict to the 2-dimensional torus T whose cocharacter lattice is

$$\Lambda_T^\vee := \mathbb{Z}\alpha^{\mathsf{fin},\vee} \oplus \mathbb{Z}d \subset \mathfrak{h},$$

where $\alpha^{\mathsf{fin},\vee}$ is a coroot of SL_2. (In other words, we drop the central $\mathbb{G}_m$.) The action of T on $X^{\mathsf{ind}}(g^e)_1$ has a dense open orbit which is isomorphic to T, so $X^{\mathsf{ind}}(g^e)_1$ is a (non-quasicompact) toric variety. Hence it is governed by an infinite fan in $\Lambda_T^\vee$, which we will describe completely.

The fan is determined by its 2-dimensional 'sectors.' By basic toric geometry, each T-invariant point $p \in X^{\mathsf{ind}}(g^e)_1$ gives a sector

$$\mathcal{S}(p) := \{\mathbb{G}_m \to T \,|\, \mathbb{G}_m \curvearrowright \mathcal{T}_p X^{\mathsf{ind}}(g^e)_1 \text{ has no positive weights}\} \subset \Lambda_T^\vee.$$

Geometrically, $\mathcal{S}(p)$ is the set of cocharacters whose 'limit' in $X^{\mathsf{ind}}(g^e)_1$ exists and equals p.

The T-invariant points $p \in X^{\mathsf{ind}}(g^e)_1$ are in bijection with foldings $g^{e,\prime} \hookrightarrow g^e$. Given such a folding, we make the following definitions.

- Let I be the index set of g^e. For each $(i < j) \in \mathsf{Gaps}(I)$, we know that the chambers g_i^e, g_j^e are adjacent along some wall H (Proposition 3.1.7). Let $\alpha_{i<j} \in R$ be the root which vanishes on H and is negative on g_j^e.

- By the definition of foldings, g_j^e is related to $g_j^{e,\prime}$ by an element of the Weyl group W. Let $\alpha'_{i<j}$ be obtained from $\alpha i < j$ via the same element. Thus $\alpha'_{i<j}$ vanishes on a wall of the chamber $g_j^{e,\prime}$.

The important point is that, even if the chambers $g_i^{e,\prime}$ and $g_j^{e,\prime}$ coincide, we still want to consider the root $\alpha'_{i<j}$, which in this case is negative on *both* chambers.

Given a folding (hence a point p), there is a T-equivariant isomorphism of tangent spaces

$$\mathcal{T}_p X^{\mathsf{ind}}(g^e) \simeq \left(\prod_{(i<j)\in \mathsf{Gaps}(I)} k_{\alpha'_{i<j}} \right) \times \left(\prod_{\beta \in R_0^{\mathsf{im},+}(C_0)} k_\beta \right),$$

where k_α is a line with T-weight α, and $R_0^{\mathsf{im},+}(C_0)$ is the set of negative imaginary roots in $\mathcal{H}^{\mathsf{aff}}$, namely $\mathbb{Z}_{<0}\delta$. Clearly, $m^e(p) = 1$ if and only if $g^{e,\prime}$ ends at C_0^{op}. If this is the case, then the map on tangent spaces

$$d_p\, m^e : \mathcal{T}_p X^{\mathsf{ind}}(g^e) \to \mathcal{T}_1 \mathcal{F}\ell$$

identifies with the map

$$\left(\prod_{(i<j)\in \mathsf{Gaps}(I)} k_{\alpha'_{i<j}} \right) \times \left(\prod_{\beta \in R_0^{\mathsf{im},+}(C_0)} k_\beta \right) \to \prod_{\alpha \in R_0^{\mathsf{reim},+}(C_0)} k_\alpha$$

which is defined as follows: each matrix coefficient $k_\alpha \to k_\beta$ is the identity if $\alpha = \beta$ and is zero otherwise. (Here $R_0^{\mathsf{reim},+}(C_0)$ is the set of all roots in $\mathcal{H}^{\mathsf{aff}}$, real or imaginary, which are positive on C_0.) As a consequence, the kernel can be described as

$$\ker d_p\, m^e \simeq \prod_{\text{some } (i<j)\in \mathsf{Gaps}(I)} k_{\alpha'_{i<j}}$$

where the product runs over all gaps $i<j$ such that $\alpha'_{i<j} \in R^-(C_0)$ or $[\alpha'_{i<j} = \alpha'_{i'<j'}$ for some strictly earlier gap $i' < j']$. Of course, the kernel equals $\mathcal{T}_p X^{\mathsf{ind}}(g^e)_1$, so the product must be 2-dimensional. In conclusion, the sector of p is

$$\mathcal{S}(p) = \prod_{\text{some } (i<j)\in \mathsf{Gaps}(I)} \{\alpha \leq 0\}.$$

It is now straightforward to list all foldings of g^e ending at C_0^{op} and compute their sectors. The resulting fan is shown in Figure 5, together with the foldings corresponding to each sector. The precise meaning of this fan is that every finite subfan specifies a (non-proper) toric variety, and $X^{\mathsf{ind}}(g^e)_1$ is the colimit of these toric varieties along open embeddings.

We emphasize that the infinitely small region which opens to the right is not a sector of the fan. This implies that μ (hence m) is not proper, as desired. More precisely, any T-translate of the cocharacter $\alpha^{\mathsf{fin},\vee} : \mathbb{G}_m \to T$ gives a curve in $X^{\mathsf{ind}}(g^e)_1$ which does not have a limit but whose image in $\mathcal{F}\ell^{\succeq st}$ does have a limit.

The four purple rays give the fan of $\mathcal{F}\ell^{\succeq st}$. This confirms that $\mathcal{F}\ell^{\succeq st}$ is a twisted product $\mathbb{P}^1 \tilde{\times} \mathbb{P}^1$. It also shows that μ is an isomorphism away from the point $st \in \mathcal{F}\ell^{\succeq st}$, and its fiber over this point is a disjoint union of two ind-infinite chains of $\mathbb{P}^1$'s, which is ind-proper.

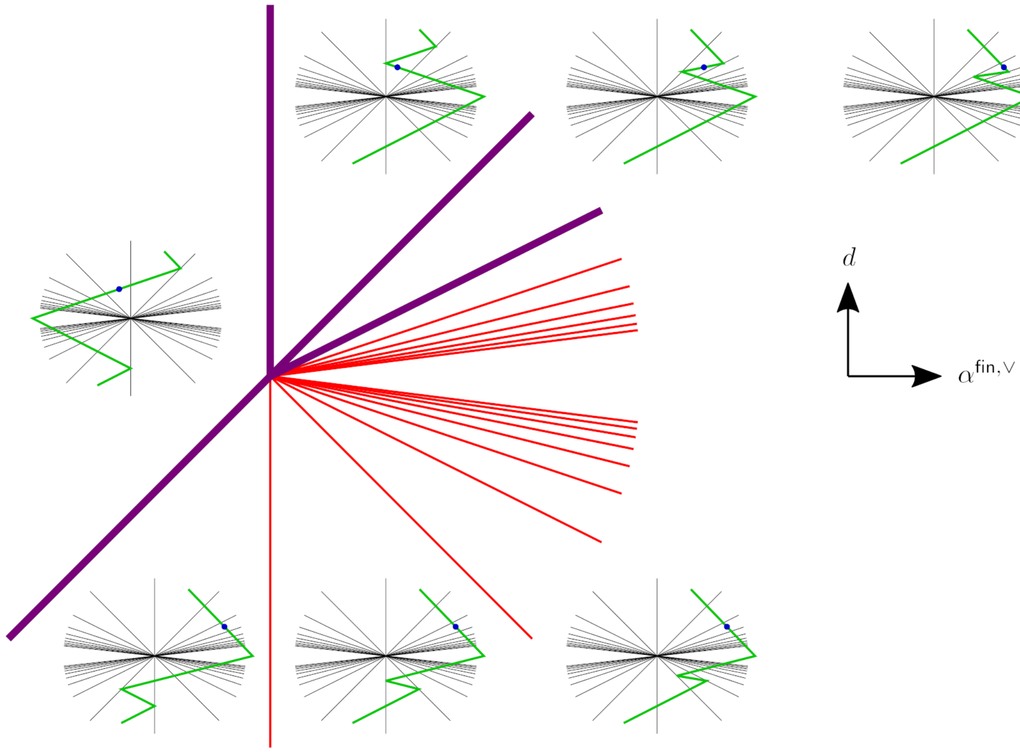

FIGURE 5. The infinite fan which determines $X^{\mathsf{ind}}(g^e)_1$ and the foldings of g^e which correspond to its T-fixed points.

7.3.4. *Pro-Demazure varieties.* There is a 'pro' variant of $X^{\mathsf{ind}}(g)$ which allows us to fill in the missing (infinitely small) sector in the above toric fan. Unfortunately, defining this variant in general requires that we have already constructed the flag variety $\mathcal{F}\ell$. We will only explain how the definition works in the special case considered above.

First, define $\overline{X}(t)$ by modifying $X(t)$ as follows: in 6.1.1, replace each 'bad' factor $(I_{c_{i-1}} \overset{I\cap I}{\times} I_{f_i c_{i-1}})/I_{f_i c_{i-1}}$ by a 'good' factor $P_{(c_{i-1}, f_i c_{i-1})}/I_{f_i c_{i-1}}$, which is a Schubert variety in the thick affine flag variety. Next, let $\mathring{g}^e$ be the chamber sequence obtained from g by deleting the liminal chamber, and define

$$X^{\mathsf{pro}}(\mathring{g}^e) := \lim_{t \in \mathsf{Ref}_{/\mathring{g}^e}} \overline{X}(t),$$

where the limit is 'formal,' i.e. this is a pro-object in the category of schemes. This *pro-Demazure variety* is related to $X^{\mathsf{ind}}(g^e)$ by the following diagram, in which all hook arrows are open embeddings, and $t \dashrightarrow t'$ are any two tours refined by $\mathring{g}^e$:

$$\begin{tikzcd}
X(t) \arrow[r, hook] \arrow[d, hook] & \overline{X}(t) \\
X(t') \arrow[r, hook] \arrow[d, hook] & \overline{X}(t') \arrow[u, two heads, "\text{proper}"'] \\
X^{\mathsf{ind}}(g^e) \arrow[r, hook] & X^{\mathsf{pro}}(\mathring{g}^e) \arrow[u, two heads, "\text{pro-proper}"']
\end{tikzcd}$$

For $X^{\mathsf{ind}}(g^e)$, it does not matter whether we use g^e or $\mathring{g}^e$, because the liminal chamber of g is not adjacent to any walls. On the other hand, for $X^{\mathsf{pro}}(\mathring{g}^e)$, it is important that we use $\mathring{g}^e$. This choice ensures that the labeled arrows (in the above diagram) are (pro-)proper. In addition, each $m : \overline{X}(t) \to \mathcal{F}\ell$ is proper, so $m : X^{\mathsf{pro}}(\mathring{g}^e) \to \mathcal{F}\ell$ is pro-proper.

Let us describe the T-invariant points of $X^{\mathsf{pro}}(\mathring{g}^e)$. First, define a *collapse* of a tour $t = (I, c)$ to be another tour which is obtained from t by repeating the following procedure:

- Choose a cut $J \subset I$ and an element $w \in W$ such that, as $j \in J$ increases and $j' \in I \smallsetminus J$ decreases, it is eventually true that $wc_{j'} \xrightarrow{w} c_{j'}$ lies in $\mathcal{W}^{c_j}$. Output the concatenated tour $(J, c) \diamond (I \smallsetminus J, wc)$.

Note that a folding is equivalent to a collapse in which all of the cuts $J \subset I$ are gaps. Intuitively, a collapse allows us to 'fold' at limit points of I and not just at gaps.

Lemma. *The points $X^{\mathsf{pro}}(\mathring{g}^e)^T$ correspond to collapses of $\mathring{g}^e$.*

Proof. By definition, we have $X^{\mathsf{pro}}(\mathring{g}^e)^T = \lim_t \overline{X}(t)^T$. It is easy to see that the set $\overline{X}(t)^T$ identifies with the set of collapses of t. The result follows because the definition of collapses is well-behaved under the limit along the refinement maps. ☐

The lemma holds if $\mathring{g}^e$ is any chamber sequence. In our case, the collapses of $\mathring{g}^e$ which end at C_0^{op} include the already-discussed foldings and one collapse which is not a folding. (This collapse uses the cut of I which separates the downward chambers of g from the upward ones.) The resulting T-invariant point corresponds to the infinitely small sector in the modified fan, as desired.

Remarks.

(1) $X^{\mathsf{pro}}(-)$ can be defined in general once the scheme structure of the double affine flag variety is known.

The best way to organize this is to enlarge the Demazure category D by allowing joint faces to be empty. Then extend the functor $X : \mathsf{D} \to \mathsf{Sch}$ to this larger category by sending each step $c_{i-1} \underset{\emptyset}{\diamond} c_i$ to the version of $\mathcal{F}\ell$ centered at c_{i-1} and tethered at c_i. The resulting diagram of schemes includes $X(t)$, $\overline{X}(t)$, and mixtures of the two. This gives a canonical way to parameterize the schemes $\overline{X}(t)$, and taking a limit gives the desired $X^{\mathsf{pro}}(-)$.

(2) For the semi-infinite affine flag variety, both versions $X^{\mathsf{ind}}(-)$ and $X^{\mathsf{pro}}(-)$ were studied in [KaNS, 4.3], where they are denoted $\mathbf{Q}_G(\mathbf{i})$ and $\mathbf{Q}_G^{\#}(\mathbf{i})$, respectively.

(3) In our example, what if we use g^e instead of $\mathring{g}^e$? Combinatorially, the gallery g^e has two non-folding collapses which end at C_0^{op}, rather than one. They are shown in Figure 6, where the red line corresponds to $(I \smallsetminus J, wc)$. Geometrically, this means

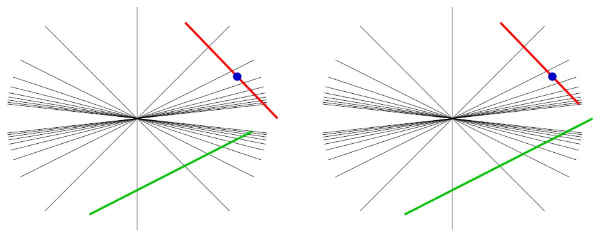

FIGURE 6. The two non-folding collapses of g^e.

that $X^{\mathsf{pro}}(g^e)_1$ should have two new T-invariant points rather than just one. We do not know how to make sense of $X^{\mathsf{pro}}(g^e)$, which is a limit of twisted products of two semi-infinite affine flag varieties with some $\mathbb{P}^1$'s. We also do not know how to describe the map $m : X^{\mathsf{pro}}(g^e)_1 \to \mathcal{F}\ell^{\succeq st}$ in terms of a fan, and we do not think that it is pro-proper.

8. Contractibility

8.1. **Polytopes in locally finite arrangements.** In this subsection, $\mathcal{H}$ is a locally finite arrangement of hyperplanes in an affine space A.

8.1.1. *Poset of chambers.* For any subset $S \subset A$, let $\mathsf{Chambers}(S)$ be the set of chambers which intersect S. If we assign an *orientation* to each wall, then $\mathsf{Chambers}(S)$ acquires a poset structure, in which an arrow $C_1 \to C_2$ exists if and only if (C_1, C_2) does not cross any wall in the negative direction. The orientation is *consistent* if, for every face F, the local arrangement $\mathcal{H}_F$ has a unique maximal chamber.

8.1.2. A *closed polytope* $P \subset A$ is a nonempty bounded intersection of finitely many closed half-spaces, which are not necessarily root half-spaces. *Open polytopes* are defined similarly.

Lemma. *Choose a consistent orientation. If $P \subset A$ is an open polytope, then* $\mathsf{Chambers}(P)$ *is contractible.*

Proof. Let $\mathcal{B}$ be the poset of nonempty intersections of faces with P, ordered by reverse closure, i.e. $F_1 \to F_2$ means that $\overline{F}_1 \supseteq \overline{F}_2$. Since P is contractible, so is $\mathcal{B}$. There is a functor

$$\mathsf{max} : \mathcal{B} \to \mathsf{Chambers}(P)$$

which sends $F \cap P$ to the chamber FC, where C is the unique maximal chamber of $\mathcal{H}_F$. It suffices to show that max is a homotopy equivalence. In fact, we will show that max is initial, i.e. $\langle \mathsf{max} \downarrow C \rangle$ is contractible for all $C \in \mathsf{Chambers}(P)$. Define a smaller open polytope $P^{\preceq C} \subset P$ by intersecting P with all of the negative root open half-spaces containing C. Then $\langle \mathsf{max} \downarrow C \rangle$ is the analogue of $\mathcal{B}$ for the polytope $P^{\preceq C}$, so it is contractible. $\square$

Remark. The lemma may fail if the orientation is inconsistent. For example, there is an inconsistent orientation of the finite root system A_2 for which the poset of chambers is homotopic to a circle.

8.1.3. **Corollary.** *Choose a consistent orientation. For any convex subset $S \subset A$ with nonempty interior, the poset* $\mathsf{Chambers}(S)$ *is contractible.*

Proof. It suffices to show that each finite subposet $\mathcal{P} \subseteq \mathsf{Chambers}(S)$ is contained in a contractible subposet of $\mathsf{Chambers}(S)$. For each such $\mathcal{P}$, one can find an open polytope $P \subset S$ which intersects every chamber in $\mathcal{P}$. Then $\mathcal{P} \subseteq \mathsf{Chambers}(P)$, which is contractible by the previous lemma. □

8.2. **Deformations.** From now on, $(\mathfrak{h}, \mathcal{H})$ is a double affine Coxeter arrangement, and we write $\mathcal{H}^{\mathsf{aff}} := \mathcal{H}_{\{0\}}$.

8.2.1. Consider the ordered $\mathbb{R}$-algebra $E := \mathbb{R}[\epsilon^q \,|\, q \in \mathbb{Q}_{\geq 0}]$ which consists of (finite) linear combinations of symbols ϵ^q. The order on E is obvious. An element of E is *infinitesimal* if its constant term equals zero. A subscript E is shorthand for $E \otimes_{\mathbb{R}} (-)$.

- A *deformed affine function* is a map $f : \mathfrak{h}_E \to E$ of the form $\xi_E - c$, where $\xi : \mathfrak{h} \to \mathbb{R}$ is an $\mathbb{R}$-linear function and $c \in E$. It is *real* if $c \in \mathbb{R}$.
- *Deformed affine subspaces* and *deformed polytopes* are defined using *deformed half-spaces* (e.g. $\{f \geq 0\}$, $\{f > 0\}$) in the obvious way.
- The *specialization* of a subset $S \subset \mathfrak{h}_E$, denoted $\mathsf{sp}\, S \subset \mathfrak{h}$, is the image of S under the map $\mathfrak{h}_E \to \mathfrak{h}$ defined by $\epsilon \mapsto 0$.

We may view $\mathcal{H}$ as an arrangement of deformed hyperplanes in $\mathfrak{h}_E$. For each deformed affine subspace $A \subset \mathfrak{h}_E$, the induced arrangement $\mathcal{H}|_A$ is defined in the obvious way, as an arrangement of deformed hyperplanes in A.

8.2.2. *Remarks.* This construction is motivated by the following observations:

(1) If P is a deformed polytope, then its face poset is isomorphic to that of an ordinary polytope which is constructed as follows. For each $t \in \mathbb{R}_{>0}$, let $P_t \subset \mathfrak{h}$ be the intersection of the half-spaces $\{\xi \geq c_t\}$, where the pair (ξ, c) specifies a deformed half-space defining P, and c_t is the evaluation of c at $\epsilon = t$. There exists $t_1 \in \mathbb{R}_{>0}$ such that the face poset of P_t is constant for $t \in (0, t_1)$. Each such P_t works. The polytopes P_t for different values of t are related by translating the defining half-spaces without changing their slopes. In the literature on polytopes, this is called a *deformation*, which motivates our terminology.

(2) The classification of faces (Theorem 2.4.3) implies that every face of $\mathcal{H}$ contains an element of $\mathfrak{h} + \epsilon\mathfrak{h}$.

(3) For convenience, we have defined everything inside in the ambient space $\mathfrak{h}_E$. Here is a more intrinsic approach. Define a *deformed affine space* to be a pair (V, A), where V is a real vector space and A is a torsor for the additive group V_E. A *deformed affine map* $(V_1, A_1) \to (V_2, A_2)$ is a map $A_1 \to A_2$ which is compatible with the

map $V_{1,E} \to V_{2,E}$ induced by some $\mathbb{R}$-linear map $V_1 \to V_2$. This is called a *deformed affine function* if $(V_2, A_2) = (\mathbb{R}, E)$. We will not need this generality.

8.2.3. A set S of deformed half-spaces is *bounded* if there exists a finite subset of S whose intersection is a bounded subset of $\mathfrak{h}_E$.

Lemma. *Suppose that S is a bounded set of deformed half-spaces such that all but finitely many are real. Then the following are equivalent:*

(i) S has nonempty intersection.

(ii) Every finite subset of S has nonempty intersection.

Proof. It is clear that (i) implies (ii), so we focus on proving the converse. Choose coordinates $x_1, \ldots, x_r$ for $\mathfrak{h}$ and hence $\mathfrak{h}_E$. Viewing S as a system of inequalities, we apply Fourier–Motzkin elimination along these coordinates, obtaining a set T of inequalities between elements of E, which are interpreted as upper and lower bounds for x_r. The procedure guarantees that any given finite subset of S has nonempty intersection if and only if the corresponding finite subset of T is all true. Thus, (ii) implies that T is all true, i.e. the set of upper and lower bounds for x_r is consistent.

We claim that there exists an element $x_r \in E$ satisfying these bounds. The assumption that S is bounded implies that these bounds do not diverge to $\pm\infty$. The assumption that all but finitely many deformed half-spaces in S are real implies that finitely many ϵ^q appear in T. Now the existence of x_r follows from the fact that $\mathbb{R}$ is complete and $\mathbb{Q}_{\geq 0}$ is a dense linear order.

If we fix x_r, then the resulting set of upper and lower bounds for x_{r-1} is consistent. Repeating the previous reasoning yields a point $(x_1, \ldots, x_r)$ in the intersection of S. $\square$

8.2.4. **Corollary.** *For every deformed affine subspace $A \subset \mathfrak{h}_E$, each chamber of the induced arrangement $\mathcal{H}|_A$ contains a point.*

Proof. This follows from the previous lemma because each root is a real affine function, and A is defined by finitely many deformed affine functions. The hypothesis that S is bounded corresponds to the boundedness requirement in the definition of chambers, see 2.2.3(iv). $\square$

8.2.5. **Lemma.** *Suppose we are given a deformed affine subspace $A \subset \mathfrak{h}_E$ and a finite list of deformed affine functions $(f_i)_{i \in [m]}$ such that*

- $\mathsf{sp}\, A$ *is horizontal and rational (i.e. it is defined over $\mathbb{Q}$).*
- *Each $\mathsf{sp}\, f_i$ has rational slope (i.e. ξ is defined over $\mathbb{Q}$).*

Then there exist $u \in \mathfrak{h}^{\mathsf{fin}} + \epsilon\mathfrak{h}$ and $c_i \in \mathbb{R} + \epsilon\mathbb{R}$ such that any bounded set of half-spaces defined by a root or some f_i has nonempty intersection in A if and only if the set of half-spaces obtained by replacing each f_i by $(\mathsf{sp}\, f_i) + c_i$ has nonempty intersection in $(\mathsf{sp}\, A) + u$.

Intuitively, the lemma says that we can replace A and $(f_i)_i$ by 'combinatorially equivalent' objects which involve only ϵ^1 and no other infinitesimals.

Proof. First, we will concretely describe the half-spaces which are defined by a root $\alpha = \alpha^{\mathsf{fin}} + n\delta + m\pi$ or some f_i. Let $r = \dim \mathfrak{h}^{\mathsf{fin}}$, $s = \dim A$, and $l = \delta(\mathsf{sp}\, A)$. The first bullet implies $l \in \mathbb{Q}$. Note that A differs from $\mathsf{sp}\, A$ by an infinitesimal translation $(\varepsilon_j^{(A)})_{j\in[r-s]}$ (where $\varepsilon_j^{(A)} \in E$), and each f_i differs from $\mathsf{sp}\, f_i$ by an infinitesimal constant term $\varepsilon^{(f_i)} \in E$. In particular, if we choose coordinates $x_1, \dots, x_s$ on $\mathsf{sp}\, A$, then we also get coordinates on A by translation. In these coordinates, the root half-space $\{\alpha \geq 0\} \subset A$ looks like

$$a_1x_1 + \cdots + a_sx_s \geq c + m + nl + n\varepsilon_0^{(A)} + \sum_{j=1}^{r-s} b_j\varepsilon_j^{(A)},$$

where $a_j, b_j, c \in \mathbb{Q}$ depend on α^{fin}. Similarly, $\{f_i \geq 0\} \subset A$ looks like

$$a_1x_1 + \cdots + a_sx_s \geq c + \varepsilon^{(f_i)} + \sum_{j=0}^{r-s} b_j\varepsilon_j^{(A)}$$

where $a_j, b_j \in \mathbb{Q}$ and $c \in \mathbb{R}$ depend on f_i. One may replace $\geq$ by $>, \leq, <$.

Next, applying Fourier–Motzkin elimination to any bounded set S of these half-spaces gives a set T of inequalities in E, of the form

$$\frac{n_1}{N} + \frac{n_2}{N}\varepsilon_0^{(A)} + d + \sum_{j=1}^{r-s} d_j\varepsilon_j^{(A)} + \sum_{i=1}^{m} d'_i\varepsilon^{(f_i)} \geq 0,$$

where the positive integer N is fixed, the integers n_1 and n_2 may vary arbitrarily, the numbers $d \in \mathbb{R}$ and $d_j, d'_i \in \mathbb{Q}$ may vary in a fixed finite subset $D \subset \mathbb{R}$, and the inequality sign may be $>$ instead of $\geq$. (The restriction involving the finite subset D comes from the fact that there are finitely many possibilities for α^{fin} and f_i.) The proof of Lemma 8.2.3 implies that S has nonempty intersection in A if and only if T is all true.

Thus, it suffices to find new choices of $\varepsilon_0^{(A)} \in \epsilon\mathbb{R}$ and $(\varepsilon_j^{(A)})_{j\in[1,r-s]}, (\varepsilon^{f_i})_{i\in[m]} \in \mathbb{R} + \epsilon\mathbb{R}$ which do not alter the validity of any inequality in T. (The new choices are used to define u and c_i.) If we treat $\varepsilon_j^{(A)}$ and $\varepsilon^{(f_i)}$ as variables, then the inequalities in T come from as double affine root arrangement (different from $\mathcal{H}$), so the classification of faces in such arrangements implies that good choices exist (see Remark 8.2.2(2)). $\square$

8.3. **Domes.** In this subsection, all deformed half-spaces, deformed affine subspaces, and deformed polytopes are required to have rational specializations.

A vector $v \in \mathfrak{h}$ is *upward* if $\delta(v) > 0$.

8.3.1. Let $A \subset \mathfrak{h}_E$ be a deformed affine subspace. A nonempty bounded subset $D \subset A$ is a *dome* if one of the following holds:

(1) Assume that A is horizontal and $\delta(A) \in \mathbb{Q}$. Then D is a deformed open polytope which intersects exactly one chamber of $\mathcal{H}|_A$.

(2) Assume that A is horizontal and $\delta(A) \notin \mathbb{Q}$. Then D is a deformed open polytope.

(3) Assume that A is not horizontal. Then, for some (equivalently any) upward vector $v \in \mathfrak{h}$, the subset $D \subset A$ equals the intersection of a set of open half-spaces such that all but finitely many are root half-spaces $\{\alpha > 0\}$ such that $\alpha^{\mathsf{aff}}(v) < 0$.

The condition in (3) does not depend on the choice of v because, if v_1 and v_2 are two upward vectors, then the set of roots α for which α^{aff} has different signs on v_1 and v_2 is locally finite. Note that, in all cases, we have $\mathsf{span}\, D = A$.

8.3.2. *Poset of chambers.* Let $D \subset A$ be a dome. Define the subset $\mathsf{Chambers}(D) \subset \mathsf{Chambers}(\mathcal{H}|_A)$ to consist of chambers which intersect D. In case (1), this set is a singleton. In the remaining two cases, choose an upward *orienting vector* $v \in \mathfrak{h}$ as follows:

(2) Using Lemma 8.2.5, we may replace A and D by analogous objects which are defined using only ϵ^0 and ϵ^1, without changing $\mathsf{Chambers}(D)$. After doing so, we can write $\delta(A) = l_1 + \epsilon l_2$, where the hypothesis of (2) implies $l_2 \neq 0$. Next, let $\widetilde{A} \subset \mathfrak{h}$ be the smallest affine subspace which contains A and $\mathsf{sp}\, A$, and let $\widetilde{A}_0$ be its translation to the origin.[12]

Choose $v \in \widetilde{A}_0$ to lie in a chamber of $\mathcal{H}^{\mathsf{aff}}|_{\widetilde{A}_0}$.

(3) Let A_0 be the translation of A to the origin.

Choose $v \in A_0$ to lie in a chamber of $\mathcal{H}^{\mathsf{aff}}|_{A_0}$.

We say that a root α of $\mathcal{H}$ is *v-positive, v-negative, or v-zero* depending on the sign of $\alpha^{\mathsf{aff}}(v)$. Define a partial order on $\mathsf{Chambers}(D)$ as follows: an arrow $C_1 \to C_2$ exists if and only if (C_1, C_2) does not cross into any v-negative root half-space.

Theorem. *If v is chosen as above, then the poset* $\mathsf{Chambers}(D)$ *is contractible.*

The rest of the subsection is devoted to proving the theorem.

8.3.3. Apply induction on $\mathsf{dim}\, A$. The base case $\mathsf{dim}\, A = 0$ is trivial. Assume that $\mathsf{dim}\, A > 0$ and that the theorem holds for all lower dimensions.

For now, assume that case (3) holds. We will handle case (2) at the end (8.3.10), and case (1) is obvious.

8.3.4. *Bottoms.* Perturb v within its $\mathcal{H}^{\mathsf{aff}}|_{A_0}$-chamber so that it is not parallel to any defining hyperplane of D. This does not change the partial order on $\mathsf{Chambers}(D)$.

Let $P \subset A$ be any deformed open polytope such that v is not parallel to its polytopal facets. The *bottom* of P is defined as follows, where f ranges over deformed affine functions.

$$\mathsf{Bot}(P) := \left[\left(\bigcap_{\substack{f(P)>0 \\ f(v)>0}} \{f \geq 0\} \right) \cap \left(\bigcap_{\substack{f(P)>0 \\ f(v)<0}} \{f > 0\} \right) \right] \smallsetminus P.$$

More conceptually, $\mathsf{Bot}(P)$ is the set of points $p \notin P$ such that, when v is rooted at p, it points into P. It is clear that $\mathsf{Bot}(P)$ is a union of polytopal faces of $\overline{P}$ which is open in the boundary of $\overline{P}$.

Define $\mathsf{Bot}(D)$ in the same way. Let P be any deformed open polytope obtained as the intersection of a finite subset S of the defining deformed half-spaces of D. If S includes

[12] Motivation: A is obtained from $\mathsf{sp}\, A$ by translating along an ϵ-multiple of a vector in $\widetilde{A}_0$.

all defining deformed half-spaces which are not v-negative root half-spaces, then $\mathsf{Bot}(D) \subseteq \mathsf{Bot}(P)$, and increasing S can only decrease $\mathsf{Bot}(P)$. For sufficiently large S, every polytopal face of $\mathsf{Bot}(P)$ meets $\mathsf{Bot}(D)$, and increasing S does not change the partition of $\mathsf{Bot}(D)$ induced by the partition of $\mathsf{Bot}(P)$ into polytopal faces. Define a *polytopal face of* $\mathsf{Bot}(D)$ to be a component of the induced partition.

The poset of polytopal faces of $\mathsf{Bot}(D)$, ordered by closure, is contractible because it is isomorphic to the analogous poset for some $\mathsf{Bot}(P)$.

For any $F \in \mathsf{Faces}(\mathcal{H}|_A)$, let F^{max} be the maximal chamber C of $\mathcal{H}|_A$ such that $C \succeq F$, i.e. F^{max} contains $F + cv$ for small $c \in E$. By construction, if F intersects $\mathsf{Bot}(D)$, then F^{max} intersects D.

8.3.5. *Fracture.* We will refine the 'polytopal face' partition of $\mathsf{Bot}(D)$ to obtain another (finite) partition. The new components will be called *fracture faces* and denoted in Fraktur font. Each fracture face will be a dome in its affine span. If a fracture face $\mathfrak{F}$ satisfies (2) or (3), then our construction will also assign to it an upward vector $v_{\mathfrak{F}}$ satisfying 8.3.2.

The construction proceeds in stages, numbered from $\mathsf{dim}\, A - 1$ to 1. At stage m, we refine some of the faces of dimension $\leq m$ as follows. For each m-dimensional face D', split into cases depending on $A' := \mathsf{span}\, D'$:

(1) Assume that A' is horizontal and $\delta(A') \in \mathbb{Q}$. Refine the partition of $\overline{D'}$ by slicing with all walls.

(2) Assume that A' is horizontal and $\delta(A') \notin \mathbb{Q}$. Choose a vector v' by applying 8.3.2(2) to $D' \subset A'$.[13] Refine the partition of $\overline{D'}$ by slicing with the locally finite set of hyperplanes H_α for which α^{aff} takes different signs on v and v'. Then assign v' to each m-dimensional face thus created.

(3) Assume that A' is not horizontal. Proceed as in the previous case, using 8.3.2(3).

This completes the construction. It is easy to see that the resulting partition has the properties claimed above.

Let $\mathsf{Frac}(D)$ be the poset of fracture faces ordered by reverse closure. It is contractible because $\mathsf{Bot}(D)$ is contractible.

8.3.6. For any $F \in \mathsf{Faces}(\mathcal{H}|_A)$ which intersects $\mathsf{Bot}(D)$, define $\mathfrak{p}(F)$ to be the unique fracture face which contains the intersection. Define a poset $\mathcal{B}$ as follows:

- An object is a face in $\mathcal{H}|_A$ which satisfies

  (obj_{fr}) F intersects $\mathsf{Bot}(D)$, and $\mathsf{dim}(F \cap \mathfrak{p}(F)) = \mathsf{dim}\, \mathfrak{p}(F)$.

  (This set of objects is in bijection with the set of chambers of fracture faces.)

- There is a morphism $F_1 \to F_2$ if and only if the following hold:

(mor_{max}) $F_1^{\mathsf{max}} \to F_2^{\mathsf{max}}$ is a morphism in $\mathsf{Chambers}(D)$.

[13]In order to apply 8.3.2(2) to $D' \subset A'$, we need to know that A' and the half-spaces defining D' have rational specializations. This follows from the analogous statements for $D \subset A$, which were assumed at the beginning, see 8.3.

(mor$_{\mathrm{fr}}$) $\mathfrak{p}(F_1) \succeq \mathfrak{p}(F_2)$.

By construction, there are functors

$$\mathsf{Chambers}(D) \xleftarrow{\mathsf{max}} \mathcal{B} \xrightarrow{\mathfrak{p}} \mathsf{Frac}(D)$$

which sends $F \in \mathcal{B}$ to F^{max} and $\mathfrak{p}(F)$, respectively. We will use the next lemma to show that these functors are homotopy equivalences.

8.3.7. **Lemma.** *Suppose we are given a functor $F : \mathcal{C} \to \mathcal{D}$ between two categories. If every fiber of F is contractible and, for every solid diagram as shown, the category of lifts is contractible, then F is final.*

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d, "F"] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

Proof. It suffices to show that $\langle d \downarrow F\rangle$ is contractible, for each $d \in \mathcal{D}$. Since $F^{-1}(d)$ is contractible by the first hypothesis, it suffices to show that $\imath : F^{-1}(d) \hookrightarrow \langle d \downarrow F\rangle$ is a homotopy equivalence. For this, it suffices to show that $\imath$ is initial, i.e. for each $a \in \langle d \downarrow F\rangle$, the category $\langle \imath \downarrow a\rangle$ is contractible. This follows from the second hypothesis, because a specifies a solid diagram, and $\langle \imath \downarrow a\rangle$ is its category of lifts. ☐

Remark. The hypothesis implies that, for every simplex $\sigma : \Delta^n \to \mathcal{D}$, the category of lifts of σ along F is contractible. (Proof: a lift of σ can be constructed by lifting $\{n\} \in \Delta^n$ and then lifting each edge inductively.) Then Theorem A.3.1 implies that F is initial and final. We will not need this extra strength.

8.3.8. **Lemma.** $\mathfrak{p}$ *is a homotopy equivalence, so $\mathcal{B}$ is contractible.*

Proof. We will apply Lemma 8.3.7. First, we must show that the fiber $\mathfrak{p}^{-1}(\mathfrak{F})$ is contractible, for each $\mathfrak{F} \in \mathsf{Frac}(D)$. In fact, $\mathfrak{p}^{-1}(\mathfrak{F}) \simeq \mathsf{Chambers}(\mathfrak{F})$, so the contractibility follows from the inductive hypothesis because $\mathfrak{F}$ is a dome of dimension $< \mathsf{dim}\, A$. The fracture construction ensures that the partial order on $\mathfrak{p}^{-1}(\mathfrak{F})$, which is defined via (mor$_{\max}$) and v, agrees with the partial order on $\mathsf{Chambers}(\mathfrak{F})$, which is defined by $v_{\mathfrak{F}}$.

Next, we must show that the category of lifts for every solid diagram

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d, hook] & \mathcal{B} \arrow[d, "\mathfrak{p}"] & & F_1 \arrow[d, mapsto, "\mathfrak{p}"] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathsf{Frac}(D) & \mathfrak{F}_0 \arrow[r, phantom, "\succeq"] & \mathfrak{F}_1 \end{tikzcd}$$

is contractible. A lift is given by $F_0 \in \mathsf{Chambers}(\mathfrak{F}_0)$ such that $F_0^{\mathsf{max}} \to F_1^{\mathsf{max}}$ is an arrow in $\mathsf{Chambers}(D)$. The category of lifts identifies with a $\mathfrak{p}$-fiber for a smaller dome $D^{\preceq F_1^{\mathsf{max}}}$ obtained by intersecting D with every v-negative root half-space which contains F_1^{max}, so it is contractible by the previous reasoning applied to this smaller dome. Note that $D^{\preceq F_1^{\mathsf{max}}}$ is nonempty because $\mathfrak{F}_0 \succeq \mathfrak{F}_1$ implies that $\mathfrak{F}_0$ contains points arbitrarily close to F_1. ☐

8.3.9. **Lemma.** max *is a homotopy equivalence, so* $\mathsf{Chambers}(D)$ *is contractible.*

Proof. We will show that max is initial. For any $C \in \mathsf{Chambers}(D)$, the overcategory $\langle \mathsf{max} \downarrow C \rangle$ is the poset $\mathcal{B}$ for a smaller dome $D^{\preceq C}$ obtained by intersecting D with every v-negative root half-space which contains C. This is contractible by the previous lemma. $\square$

This completes the inductive step for case (3).

8.3.10. Now suppose that case (2) holds. Replace A and D as described in 8.3.2(2). Assume $l_2 > 0$. (If $l_2 < 0$, then reverse the orientations.) Define the subset $\widetilde{D} \subset \widetilde{A}$ by intersecting with $\{f > 0\}$ for all deformed affine functions f satisfying $f(D) > 0$ and $f(\mathsf{sp}\, D) \geq 0$. Then $\widetilde{D} \subset \widetilde{A}$ is a $(\mathsf{dim}\, A + 1)$-dimensional dome which satisfies case (3), and the old orienting vector v still works. It is easy to see that $\mathsf{Chambers}(D) \simeq \mathsf{Chambers}(\widetilde{D})$, so it suffices to show that the latter is contractible.

Let us apply the previous discussion to $\widetilde{D}$. The fracture faces of $\mathsf{Bot}(\widetilde{D})$ all satisfy (1) or (3) and have dimension $\leq \mathsf{dim}\, A$, so we have already proven that they are contractible. Thus, the previous discussion implies that $\mathsf{Chambers}(\widetilde{D})$ is contractible, which completes the inductive step for case (2). This concludes the proof of Theorem 8.3.2.

8.4. **Precaptive tours.** From now on, all tours in $\mathcal{H}$ are required to have only rational-level chambers. This will allow us to apply Theorem 8.3.2.

8.4.1. Here is a notion which bounds from below the 'fineness' of a jointed tour. Fix a locally finite subarrangement $\mathbf{H}^{\mathsf{pre}}$ of $\mathcal{H}$, and call it the *preclaustral* arrangement.

- A jointed tour $([n], c, f)$ is *precaptive* if, whenever a step $c_{i-1} \diamond_{f_i} c_i$ crosses a preclaustral wall H, we have $f_i \subset H$.

Let us reformulate this in a more illuminating way. A *jointed preclaustral generalized gallery* is a jointed chamber sequence $([n], \mathbf{c}, \mathbf{f})$ for $\mathbf{H}^{\mathsf{pre}}$ which satisfies $\mathbf{f}_i \preceq \mathbf{c}_{i-1}, \mathbf{c}_i$ for all $i \in [1, n]$.

- A jointed tour $([n], c, f)$ is precaptive if and only if projecting c and f to $\mathbf{H}^{\mathsf{pre}}$ gives a jointed preclaustral generalized gallery.

This projection defines a functor from the $\mathsf{Dom}^{\mathsf{c}}$-category of precaptive jointed tours to the category of jointed preclaustral generalized galleries.

8.4.2. Fix a positive pair of chambers (B_0, B_1) and a nonnegative integer e. Define $\mathsf{pre.Dom}^{\mathsf{c}}$ to be the full subcategory of $\mathsf{Dom}^{\mathsf{c}}$ consisting of precaptive threadable jointed tours from B_0 to B_1 with excess $\leq e$. (We can ignore the tether (5.1.1) because maps in $\mathsf{Dom}^{\mathsf{c}}$ do not change the end chamber of a jointed tour.)

Theorem. $\mathsf{pre.Dom}^{\mathsf{c}}$ *is contractible.*

The proof of the theorem begins now and lasts until the end of 8.7.

8.4.3. *Overview of proof.* Let $\mathsf{a.pre.Dom}^{\mathsf{c}} \subset \mathsf{pre.Dom}^{\mathsf{c}}$ be the non-full subdiagram consisting of adherent tours and morphisms. The proof of Theorem 5.5.1 implies that this embedding is a homotopy equivalence. Given any finite diagram $\mathcal{B} \to \mathsf{a.pre.Dom^c}$, we will construct a lax-commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[rr] \arrow[d, "\wr"] \arrow[drr] & \arrow[d, phantom, "\Rightarrow"] & \mathsf{cap.Dom^c} \arrow[dd, "\pi"] \\ \mathbf{\Delta}^{\mathsf{op}}_{/\mathcal{B}} \arrow[d, "\mathsf{ev}_0"', "\wr"] & \arrow[phantom, "\Leftarrow"] & \\ \mathcal{B} \arrow[r] & \mathsf{a.pre.Dom^c} \arrow[r] & \mathsf{pre.Dom^c} \end{tikzcd}$$

where the $\sim$ arrows are homotopy equivalences, and $\mathsf{cap.Dom^c}$ is contractible. This proves the theorem, because it implies that every sphere in $\mathcal{B}$ can be contracted in $\mathsf{pre.Dom^c}$.

The map ev_0 sends a simplex $\beta : \Delta^s \to \mathcal{B}$ to $\beta(0)$. It is a homotopy equivalence by [C, Prop. 7.3.15] and [C, Prop. 7.1.10].

We will construct the following:

- A functor $\mathsf{bel}(-).\mathsf{cap.Dom^c} : \mathbf{\Delta}^{\mathsf{op}}_{/\mathcal{B}} \to \mathsf{Cat}$ whose values are contractible.
- A contractible category $\mathsf{cap.Dom^c}$ and a map $\pi : \mathsf{cap.Dom^c} \to \mathsf{pre.Dom^c}$.
- For each $\beta \in \mathbf{\Delta}^{\mathsf{op}}_{/\mathcal{B}}$, a lax-commutative diagram

$$\begin{tikzcd} \mathsf{bel}(\beta).\mathsf{cap.Dom^c} \arrow[r] \arrow[d] \arrow[dr] & \mathsf{cap.Dom^c} \arrow[d, "\pi"] \arrow[dl, phantom, "\Rightarrow"] \\ \mathsf{pt} \arrow[r, "\beta(0)"'] \arrow[ur, phantom, "\Leftarrow"] & \mathsf{pre.Dom^c} \end{tikzcd}$$

  Furthermore, these diagrams should be lax-functorial with respect to β.

Once this is done, we will be able to construct the desired diagram as follows. Let $\mathcal{E} \to \mathbf{\Delta}^{\mathsf{op}}_{/\mathcal{B}}$ be the cocartesian fibration associated to $\mathsf{bel}(-).\mathsf{cap.Dom^c}$. It is a homotopy equivalence because each fiber is contractible. The remaining maps and natural transformations come from the third bullet point.

8.4.4. *Belayed tours.* In this subsection, we will construct lax-commutative diagrams

$$\begin{tikzcd} \mathsf{bel}(\beta).\mathsf{Dom^c} \arrow[r] \arrow[d] \arrow[dr] & \mathsf{pre.Dom^c} \arrow[d, equal] \arrow[dl, phantom, "\Rightarrow"] \\ \mathsf{pt} \arrow[r, "\beta(0)"'] \arrow[ur, phantom, "\Leftarrow"] & \mathsf{pre.Dom^c} \end{tikzcd}$$

which are lax-functorial with respect to β. We begin by defining $\mathsf{bel}(\beta).\mathsf{Dom^c}$.

Fix $\beta : \Delta^s \to \mathcal{B}$, and abuse notation by viewing it as a map $\Delta^s \to \mathsf{pre.Dom^c}$.

- For each $r \in [s]$, write $\beta(r) = ([n^{\beta(r)}], c^{\beta(r)}, f^{\beta(r)})$.
- For any $r' > r$ in $[s]$, let $\varphi^{r',r} : [n^{\beta(r')}] \to [n^{\beta(r)}]$ be the index map of $\beta(r) \to \beta(r')$.
- For each $r \in [s]$ and $k \in [1, n^{\beta(0)}]$, let $k' \in [1, n^{\beta(r)}]$ be the unique index such that $k \in [\varphi^{r,0}(k'-1)+1, \varphi^{r,0}(k')]$, and define $A^r_k := \mathsf{span}\, f^{\beta(r)}_{k'}$. As r varies, this gives a

flag of affine subspaces of $\mathfrak{h}$:

$$A_k = (A_k^s \subseteq A_k^{s-1} \subseteq \cdots \subseteq A_k^0 \subseteq A_k^{-1} = \mathfrak{h}).$$

Define $\mathsf{mnh}(k) \in [-1, s]$ to be the maximal index r such that A_k^r is non-horizontal.

We say that a chamber *adheres* to A_k if it adheres to each A_k^r.

A *belayed tour* is a precaptive threadable jointed tour $([n], c, f)$ equipped with a *belay map* $\mathsf{bel} : [n^{\beta(0)}] \to [n]$ such that the following *belay conditions* are satisfied:

(B1) For each $k \in [n^{\beta(0)}]$, we have $c_{\mathsf{bel}(k)} = c_k^{\beta(0)}$.

(B2) For each $k \in [1, n^{\beta(0)}]$, we have

- The chambers $([\mathsf{bel}(k-1), \mathsf{bel}(k)], c)$ adhere to A_k.
- The tour $([\mathsf{bel}(k-1), \mathsf{bel}(k)], c \wedge A_k^0)$ in $\mathcal{H}|_{A_k^0}$ is reduced.

(B3) For each $k \in [1, n^{\beta(0)}]$, we have

- If $f_k^{\beta(0)}$ is horizontal, then $([\mathsf{bel}(k-1), \mathsf{bel}(k)], c, f) = c_{k-1}^{\beta(0)} \diamond_{f_k^{\beta(0)}} c_k^{\beta(0)}$.
- If $f_k^{\beta(0)}$ is non-horizontal, then the faces $([\mathsf{bel}(k-1)+1, \mathsf{bel}(k)], f)$ lie in $A_k^{\mathsf{mnh}(k)}$.

A morphism of belayed tours is a morphism of jointed tours which is compatible with the belay maps. Let $\mathsf{bel}(\beta).\mathsf{Dom}^{\mathsf{c}}$ be the category of belayed tours with excess $\leq e$.

8.4.5. Here is the motivation for the above definition. Given any belayed tour $([n], c, f, \mathsf{bel})$, we can construct a diagram in $\mathsf{pre.Dom}^{\mathsf{c}}$

$$\begin{tikzcd}
\beta(0) \arrow[d] & \hat{\sigma}(0) \arrow[l, "\text{joint-preserving}"'] \arrow[r, "\text{joint-only}"] \arrow[d] & \sigma(0) \arrow[d] \\
\beta(1) \arrow[d] & \hat{\sigma}(1) \arrow[l, "\text{joint-preserving}"'] \arrow[r, "\text{joint-only}"] \arrow[d] & \sigma(1) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\beta(s) & \hat{\sigma}(s) \arrow[l, "\text{joint-preserving}"'] \arrow[r, "\text{joint-only}"] & \sigma(s)
\end{tikzcd}$$

as follows. Subject to the 'joint-preserving' and 'joint-only' requirements, the entire diagram is determined by the first row and third column.

- For the first row, $\sigma(0) := ([n], c, f)$, and the index map of $\beta(0) \leftarrow \hat{\sigma}(0)$ is bel.
- For the third column, $\sigma(0) \to \sigma(r)$ is defined as follows: for each $k \in [1, n^{\beta(r)}]$ such that $f_k^{\beta(r)}$ is horizontal, delete the chambers of $\sigma(0)$ indexed by
$$[\mathsf{bel}(\varphi^{r,0}(k-1)) + 1, \mathsf{bel}(\varphi^{r,0}(k)) - 1] \subset [n]$$
and shrink $f_{\mathsf{bel}(\varphi^{r,0}(k))}$ to equal $f_k^{\beta(r)}$.

We leave it as an exercise to check that the belay conditions imply that the diagram can be filled in. Use (B1) to show that $\beta(0) \leftarrow \hat{\sigma}(0)$ is a valid map. Use (B2) to check the second

Coxeter product condition for the maps $\beta(r) \leftarrow \hat{\sigma}(0)$. Use (B3) to show that $\beta(r)$ and $\sigma(r)$ determine a valid jointed tour $\hat{\sigma}(r)$. This diagram is functorial in $([n], c, f, \mathsf{bel})$.

We can now construct the lax-commutative diagram which was claimed in 8.4.4. Let the horizontal map send $([n], c, f, \mathsf{bel})$ to $\sigma(0)$, let the diagonal map send it to $\hat{\sigma}(0)$, and define the two natural transformations using the maps $\beta(0) \leftarrow \hat{\sigma}(0) \rightarrow \sigma(0)$.

8.4.6. *Belay functoriality.* Let us show that $\mathsf{bel}(\beta).\mathsf{Dom}^{\mathsf{c}}$ is functorial with respect to β.

Given a map $(\beta' \Rightarrow \beta) : \Delta^{s'} \rightarrow \Delta^{s} \xrightarrow{\beta} \mathcal{B}$, the desired map

$$\mathsf{bel}(\beta).\mathsf{Dom}^{\mathsf{c}} \longrightarrow \mathsf{bel}(\beta').\mathsf{Dom}^{\mathsf{c}}$$

is defined as follows. Let r be the image of 0 under $\Delta^{s'} \rightarrow \Delta^{s}$. Then send $([n], c, f, \mathsf{bel}) \mapsto (\sigma(r), \mathsf{bel} \circ \varphi^{r,0})$. It is easy to check that the latter satisfies the belay conditions for β'.

It is also easy to equip the diagram in 8.4.4 with lax-functoriality with respect to β.

8.4.7. *Sub-belay tours.* In the diagram of 8.4.5, the third column can be viewed as a sequence of successive 'quotients' of $\sigma(0)$. It is reasonable to define its *associated graded object* to be the collection of tours given by $\sigma(s)$ and $\mathsf{ker}\,\big(\sigma(r-1) \rightarrow \sigma(r)\big)$ for all $r \in [1, s]$, where the 'kernel' is the set of all subtours of $\sigma(r-1)$ which are deleted by $\sigma(r-1) \rightarrow \sigma(r)$. In fact, $\sigma(0)$ can be reconstructed from the associated graded object by successively inserting the kernel subtours back into $\sigma(s)$. Our next goal is to reformulate the notion of a belayed tour in terms of associated graded objects.

Remark. From this point onwards, the diagram of 8.4.5 will not be used in the proof of Theorem 8.4.2. However, it reappears in the proof of Theorem 8.8.1. The discussion there shows that the notion of a belayed tour is more natural than it may seem at first.

For each $r \in [s-1]$ and $k' \in [1, n^{\beta(r+1)}]$ such that $f_{k'}^{\beta(r+1)}$ is horizontal, define the *sub-belay tour* $\gamma(r, k')$ as follows:

- Start with the subtour of $\beta(0)$ indexed by $[\varphi^{r+1,0}(k'-1), \varphi^{r+1,0}(k')] \subset [n^{\beta(0)}]$, and shrink all joint faces to match $\beta(r)$.
- If there exists a (necessarily unique) $k \in [\varphi^{r+1,r}(k'-1)+1, \varphi^{r+1,r}(k')]$ such that $f_k^{\beta(r)}$ is horizontal, then delete all chambers indexed by
$$[\varphi^{r,0}(k-1)+1, \varphi^{r,0}(k)-1] \subset [\varphi^{r+1,0}(k'-1), \varphi^{r+1,0}(k')].$$

Similarly, define $\gamma(s)$ as follows:

- Start with $\beta(0)$, and shrink all joint faces to match $\beta(s)$.
- For each (not necessarily unique) $k \in [1, n^{\beta(s)}]$ such that $f_k^{\beta(s)}$ is horizontal, delete all chambers indexed by
$$[\varphi^{s,0}(k-1)+1, \varphi^{s,0}(k)-1] \subset [n^{\beta(0)}].$$

The chamber-deletion steps ensure that $\gamma(r, k')$ and $\gamma(s)$ are threadable.

Fix a sub-belay tour $\gamma = \gamma(r, k')$ or $\gamma(s)$, and write $\gamma = ([n^\gamma], c^\gamma, f^\gamma)$. If $\gamma = \gamma(s)$, set $r = s$. Since γ is obtained by deleting chambers from $\beta(0)$, it comes with an injection $\iota : [n^\gamma] \hookrightarrow [n^{\beta(0)}]$ such that $c^\gamma = c^{\beta(0)} \circ \iota$.

- For each $k \in [1, n^\gamma]$ such that f^γ_k is non-horizontal, let $\acute{A}_k$ be the subflag of $A_{\iota(k)}$ consisting of the non-horizontal affine subspaces. The construction of γ implies that
$$\acute{A}_k = (\mathsf{span}\, f^\gamma_k = A^r_{\iota(k)} \subseteq \cdots \subseteq A^0_{\iota(k)} \subseteq A^{-1}_{\iota(k)} = \mathfrak{h}).$$
In particular, $r = \mathsf{mnh}(\iota(k))$.

A γ*-belayed tour* is a jointed tour $([n], c, f)$ equipped with a γ*-belay map* $\mathsf{bel} : [n^\gamma] \to [n]$ such that the following analogues of the belay conditions hold:

(B́1) For each $k \in [n^\gamma]$, we have $c_{\mathsf{bel}(k)} = c^\gamma_k$.

(B́2) For each $k \in [1, n^\gamma]$, we have

- The chambers $([\mathsf{bel}(k-1), \mathsf{bel}(k)], c)$ adhere to $\acute{A}_k$.
- The tour $([\mathsf{bel}(k-1), \mathsf{bel}(k)], c \wedge \acute{A}^0_k)$ in $\mathcal{H}|_{\acute{A}^0_k}$ is reduced.

(B́3) For each $k \in [1, n^\gamma]$, we have

- If f^γ_k is horizontal, then $([\mathsf{bel}(k-1), \mathsf{bel}(k)], c, f) = c^\gamma_{k-1} \diamond_{f^\gamma_k} c^\gamma_k$.
- If f^γ_k is non-horizontal, then the faces $([\mathsf{bel}(k-1)+1, \mathsf{bel}(k)], f)$ lie in $\acute{A}^r_k$.

Let $\mathsf{bel}_\beta(\gamma).\mathsf{Dom}^\mathsf{c}$ be the category of γ-belayed tours. Passing to the associated graded object gives a fully faithful embedding
$$\mathsf{bel}(\beta).\mathsf{Dom}^\mathsf{c} \hookrightarrow \prod_\gamma \mathsf{bel}_\beta(\gamma).\mathsf{Dom}^\mathsf{c}$$
onto the subcategory which is defined by the requirement that the total excess is $\leq e$.

8.4.8. *Sub-belay functoriality.* The functoriality of sub-belay tours with respect to β is as follows. Given a map $(\beta' \Rightarrow \beta) : \Delta^{s'} \to \Delta^s \xrightarrow{\beta} \mathcal{B}$, each β'-sub-belay tour γ' is obtained by successively inserting some β-sub-belay tours $\tilde{\gamma}$ into a given β-sub-belay tour γ at horizontal-jointed steps. The insertion of $\tilde{\gamma}$ into γ overwrites the horizontal joint face where the insertion takes place, but it preserves all other joint faces of γ.

8.4.9. *Reformulation of threadability.* Make the following definitions.

- An E-vector $v \in \mathfrak{h}_E$ is *upward* if its leading term $\epsilon^q\, v_q$ satisfies $\delta(v_q) > 0$.
- An *E-path* is a finite sequence $(p_i)_i$ in $\mathfrak{h}_E$ of size ≥ 2. It is *upward* if each $p_i - p_{i-1}$ is upward. Define *paths* similarly, using $\mathfrak{h}$ instead of $\mathfrak{h}_E$.
- An adherent jointed tour $([n], c, f)$ is *threadable* if there exists an upward E-path whose successive vertices are contained in
$$c_{i-1}, c_{i-1} \wedge \mathsf{span}\, f_i, f_i, c_i,$$
cycling through $i \in [1, n]$. This is called a *threading* E-path.

It is an exercise to check that this is equivalent to the old definition of threadability in 5.1.3.

8.4.10. *Slope bound.* The 'upward' condition is hard to work with because it cannot be expressed using a finite set of deformed half-spaces. We now introduce a strictly stronger condition which does have this property.

- A *slope bound* is a nonempty open polytope $P \subset \mathfrak{h}^{\mathsf{fin}}$.

  Define the open cone $P^{\triangleleft} \subset \mathfrak{h}$ to be the $\mathbb{R}_{>0}$-span of $P \times \{d\} \subset \mathfrak{h}$. Define $\overline{P}^{\triangleleft}$ similarly, using $\overline{P} \times \{d\}$, and note that it also does not contain $\{0\}$.
- An E-vector $v \in \mathfrak{h}_E$ is *slope-bounded* by P if $v \in P^{\triangleleft}_E$.
- An E-path $(p_i)_i$ is *slope-bounded* by P if each $p_i - p_{i-1}$ is slope-bounded by P.

Every upward E-path is slope-bounded by some P.

Let us choose a slope bound P such that

- P is defined over $\mathbb{Q}$.
- For every β, each sub-belay tour γ admits a slope-bounded threading E-path.

The second bullet can be achieved because $\mathcal{B}$ is finite.

We will not consider any other threading E-paths. However, we will often use a variant of this notion which pertains to claustral galleries rather than jointed tours. We will also want to ensure that P satisfies a 'general position' condition, which will be achieved by scaling P by a rational number which is close to 1. These points are explained in the next subsection.

8.5. **Claustral galleries.**

8.5.1. Choose a locally finite subarrangement $\mathbf{H} \subset \mathcal{H}$, called the *claustral arrangement*, which contains $\mathbf{H}^{\mathsf{pre}}$ and all walls whose translation to 0 intersects $\overline{P}^{\triangleleft}$. A non-claustral root half-space is *upward* if its translation to 0 is positive on $P^{\triangleleft}$, and *downward* otherwise. The chambers of $\mathbf{H}$ are also denoted in bold font.

We have already defined jointed preclaustral generalized galleries in 8.4.1. Define *jointed claustral generalized galleries* in the same way, using $\mathbf{H}$ instead of $\mathbf{H}^{\mathsf{pre}}$. As usual, these form a $\mathsf{Dom}^{\mathsf{c}}$-style category, in which a morphism $([m], \mathbf{c}, \mathbf{f}) \to ([m'], \mathbf{c}', \mathbf{f}')$ is a weakly increasing bound-preserving index map $\varphi : [m'] \to [m]$ such that

- For all $j \in [m']$, we have $\mathbf{c}'_j = \mathbf{c}_{\varphi(j)}$.
- For all $j \in [1, m']$ and $i \in [\varphi(j-1)+1, \varphi(j)]$, we have $\mathbf{f}_i \succeq \mathbf{f}'_j$.

Let us call these 'claustral galleries' as an abbreviation.

8.5.2. *General position for the slope bound.* Fix an arbitrary point $p_0 \in P$. For any rational number c, define the open polytope $c \cdot P$ by scaling P from p_0. We will replace P by $c \cdot P$ for some $c \geq 1$ which is very close to 1, as promised in 8.4.10.

The following definitions are only interesting when γ contains chambers at different levels. This forces $\gamma = \gamma(s)$.

- Let $(a_k)_{k\in[l]}$ be the strictly increasing sequence of levels of chambers of γ.
- For each $k \in [l]$, let S_k be the intersection of supports of the chambers at level a_k.

An *S-belayed claustral gallery* is a claustral gallery $([m], \mathbf{c}, \mathbf{f})$ equipped with an increasing bound-preserving map $\mathsf{sbel} : [l] \to [m]$ such that, for each $k \in [l]$, we have $S_k \subset \overline{\mathbf{c}}_{\mathsf{sbel}(k)}$.

Fix an S-belayed claustral gallery. A *weak threading path* is an upward path whose successive vertices are contained in

$$S_{k-1}, \overline{\mathbf{f}}_{\mathsf{sbel}(k-1)+1}, \dots, \overline{\mathbf{f}}_{\mathsf{sbel}(k)}, S_k,$$

cycling through $k \in [1, l]$. For any $i \in [1, m]$, *i-partial weak threading paths* are defined similarly, except they end prematurely at $\overline{\mathbf{f}}_i$. The S-belayed claustral gallery determines the following $m + 1$ elements of $\mathbb{R}_{\geq 0} \sqcup \{\infty\}$.

- The infimum of the set of rational numbers c such that there exists a weak threading path which is slope-bounded by $c \cdot P$.
- For any $i \in [1, m]$, the analogous infimum for i-partial weak threading paths.

From now on, we fix an integer st, and restrict attention to claustral galleries which have excess $\leq e$ and at most st stutters. Then there are finitely many possibilities for $([m], \mathbf{c}, \mathbf{f}, \mathsf{sbel})$. Choose $c \geq 1$ such that

- c does not equal any of the aforementioned infima, for any β, γ, and $([m], \mathbf{c}, \mathbf{f}, \mathsf{sbel})$.
- If a wall of $\mathcal{H}^{\mathsf{aff}}$ does not intersect $\overline{P}^{\triangleleft}$, then the same is true with $c \cdot P$ in place of P.

The second bullet is achieved by taking c sufficiently close to 1. Replace P by $c \cdot P$.

Here is what the replacement accomplishes. It is now true that, if $([m], \mathbf{c}, \mathbf{f}, \mathsf{sbel})$ has a (i-partial) weak threading path which is slope-bounded by $\overline{P}$, then it has a (i-partial) weak threading path which is slope-bounded by P.

8.5.3. Fix β and a sub-belay tour γ. A *γ-belayed claustral gallery* is a claustral gallery $([m], \mathbf{c}, \mathbf{f})$ equipped with a weakly-increasing bound-preserving map $\mathsf{bel} : [n^\gamma] \to [m]$ such that the following belay conditions hold:

(**B**1) For each $k \in [n^\gamma]$, the claustral projection of c^γ_k equals $\mathbf{c}_{\mathsf{bel}(k)}$.

(**B**2) For each $k \in [1, n^\gamma]$, we have

- The claustral chambers $([\mathsf{bel}(k-1), \mathsf{bel}(k)], \mathbf{c})$ adhere to $\acute{A}_k$.
- The claustral gallery $([\mathsf{bel}(k-1), \mathsf{bel}(k)], \mathbf{c} \wedge \acute{A}^0_k)$ in $\mathbf{H}|_{\acute{A}^0_k}$ is reduced.

(**B**3) For each $k \in [1, n^\gamma]$, we have

- If f^γ_k is horizontal, then the claustral gallery $([\mathsf{bel}(k-1), \mathsf{bel}(k)], \mathbf{c}, \mathbf{f})$ equals the claustral projection of $c^\gamma_{k-1} \diamond_{f^\gamma_k} c^\gamma_k$.

- If f_k^γ is non-horizontal, then $([\mathsf{bel}(k-1)+1, \mathsf{bel}(k)], \mathbf{f}) \subset \acute{A}_k^r$.

If a γ-belayed jointed tour is precaptive for $\mathbf{H}$, then projecting it to $\mathbf{H}$ gives a γ-belayed claustral gallery.

We say that a γ-belayed claustral gallery is *slope-bounded* if it admits a *slope-bounded threading E-path*, i.e. a slope-bounded E-path whose successive vertices are contained in

$$c_{k-1}^\gamma, \mathbf{f}_{\mathsf{bel}(k-1)+1}, \ldots, \mathbf{f}_{\mathsf{bel}(k)}, c_k^\gamma,$$

cycling through $k \in [1, n^\gamma]$. For any $i \in [1, m]$, an *i-partial E-path* is a version of a slope-bounded threading E-path which ends prematurely at $\mathbf{f}_i$.

Let $\mathsf{bel}_\beta(\gamma).\mathsf{clausGal}$ be the category of slope-bounded γ-belayed claustral galleries with excess $\leq e$ and at most st stutters. (Of course, the morphisms are required to be compatible with the bel maps.) Next, define the subcategory

$$\mathsf{bel}(\beta).\mathsf{clausGal} \subset \prod_\gamma \mathsf{bel}_\beta(\gamma).\mathsf{clausGal}$$

via the requirements that the total excess is $\leq e$ and the total number of stutters is $\leq \mathsf{st}$. The hypothesis on the slope bound P from 8.4.10 implies that $\mathsf{bel}(\beta).\mathsf{clausGal}$ is nonempty.

Remark. If we are given a slope-bounded threading E-path in some γ-belayed claustral gallery, then specializing $\epsilon \mapsto 0$ and deleting repeated vertices gives a weak threading path in an S-belayed claustral gallery. There is a map from the former claustral gallery to the latter, but they are not always equal. Similarly, an i-partial E-path gives a j-partial weak threading path, where j depends on i.

8.5.4. *Functoriality.* Let us show that $\mathsf{bel}(\beta).\mathsf{clausGal}$ is functorial with respect to β.

The functoriality of γ with respect to a map $\beta' \Rightarrow \beta$ was explained in 8.4.8. Namely, each γ' is obtained by successively inserting some γ into a given one. It is clear that a similar insertion procedure creates a γ'-belayed claustral gallery from a collection of γ-belayed claustral galleries. Thus, the only nontrivial point is to show that the insertion procedure preserves slope-boundedness.

Lemma. *Suppose that $\tilde{\gamma}$ and γ are two sub-belay tours for β, where $\tilde{\gamma}$ is to be inserted into a horizontal-jointed step of γ. If $\tilde{\mathbf{g}}$ (resp. $\mathbf{g}$) is a slope-bounded $\tilde{\gamma}$-belayed (resp. γ-belayed) claustral gallery, then the insertion of $\tilde{\mathbf{g}}$ into $\mathbf{g}$ is also slope-bounded.*

Proof. Let $(p_i)_i$ and $(q_j)_j$ be slope-bounded threading E-paths for $\tilde{\mathbf{g}}$ and $\mathbf{g}$, respectively. Let j_0 be the index such that the vertex q_{j_0} lies in the horizontal joint (claustral) face of $\mathbf{g}$ at which the insertion takes place. Fix an integer $N > 0$, and consider the new E-path obtained by replacing the single vertex q_{j_0} by the 'weighted average' sequence

$$\left(\epsilon^N \cdot p_i + (1 - \epsilon^N) \cdot q_{j_0}\right)_i.$$

For sufficiently large N, this new E-path is slope-bounded, because the slope bound P is open. Also, its vertices lie in the correct chambers and claustral faces, so it is threading. $\square$

Remark. The whole purpose of introducing sub-belay tours is to make the above proof work. The more naive approach would have been to fix β and define a *β-belayed claustral gallery* to be a claustral gallery $([m], \mathbf{c}, \mathbf{f})$ equipped with a belay map $\mathsf{bel} : [n^{\beta(0)}] \to [m]$ satisfying similar 'belay' conditions. Most things still work. The slope-bounded condition can be defined as before. Given a β-belayed claustral gallery $\mathbf{g}$ and a map $\beta' \Rightarrow \beta$, one can still obtain a β'-belayed claustral gallery $\mathbf{g}'$, as in Lemma 8.4.6. The issue is that slope-boundedness for $\mathbf{g}$ does not imply slope-boundedness for $\mathbf{g}'$. Indeed, to go from a slope-bounded threading E-path for $\mathbf{g}$ to one for $\mathbf{g}'$, one must delete some vertices and then *add* one vertex for each horizontal joint of $\beta'(0)$, and these vertices cannot always be added while keeping the E-path slope-bounded.

8.5.5. **Proposition.** *The category* $\mathsf{bel}(\beta).\mathsf{clausGal}$ *is contractible.*

The rest of this subsection is devoted to proving the proposition. In fact, we will fix γ and prove that $\mathsf{bel}_\beta(\gamma).\mathsf{clausGal}$ is contractible. The contraction procedure does not increase the excess or number of stutters, so it implies that $\mathsf{bel}(\beta).\mathsf{clausGal}$ is contractible.

8.5.6. *Creation of anchors.* Fix the following elements of $\mathbb{Q}_{\geq 0}$, which will be used as exponents of $\epsilon \in E$.

$$0 = q_0^{\mathsf{hor}} < q_0^{\mathsf{vert}} < \cdots < q_{\dim \mathfrak{h}-1}^{\mathsf{hor}} < q_{\dim \mathfrak{h}-1}^{\mathsf{vert}}$$

For any chamber C, we will construct its *anchor* $\breve{C} \subset C$, which is a deformed open polytope.

- Since $\mathfrak{h} = \mathfrak{h}^{\mathsf{fin}} \times \mathbb{R}d$, we may write $\overline{C} = S \times \{\delta(C)\}$. Let $r := \dim \overline{C}$.
  - If $r > 0$, define $\breve{S} \subset \mathfrak{h}^{\mathsf{fin}}$ by shifting the facets of $\mathsf{relint}\, S$ inwards by $\epsilon^{q_r^{\mathsf{hor}}}$.
  - If $r = 0$, define $\breve{S} := S$.
- Let C^{loc} be the projection of C to $\mathcal{H}_{\overline{C}}$, and define

$$\breve{C} := C^{\mathsf{loc}} \cap \Big(\breve{S} \times \big(\delta(C) - \epsilon^{q_r^{\mathsf{vert}}}, \delta(C) + \epsilon^{q_r^{\mathsf{vert}}}\big)\Big).$$

We say that a threading E-path for a γ-belayed claustral gallery is *anchored* if each vertex which is constrained to lie in a chamber c_k^γ moreover lies in $\breve{c}_k^\gamma$.

Lemma. *If a γ-belayed claustral gallery has a slope-bounded threading E-path, then it has an anchored slope-bounded threading E-path.*

Proof. Let $(p_i)_{i \in I}$ be the given E-path. First, we fix a level a and handle the vertices which satisfy $\delta(\mathsf{sp}\, p_i) = a$. They are indexed by an interval $I_a \subseteq I$. For each possible support-dimension $r \in [\dim \mathfrak{h} - 1]$, we make the following definitions.

- Define $S_{a,r} := \overline{c}_k^\gamma$ for any chamber c_k^γ at level a such that $\dim \overline{c}_k^\gamma = r$.

  It is possible that no such c_k^γ exists, in which case we remove the chosen index r from consideration. Since c^γ is a threadable tour, support-matching implies that the definition of $S_{a,r}$ does not depend on the choice of c_k^γ.
- The subset $I_{a,r} \subseteq I_a$ consists of i such that $p_i \in c_k^\gamma$ for some k such that $\overline{c}_k^\gamma = S_{a,r}$.

Also, $S_a := \cap_r S_{a,r}$. For each $i \in I_a$, define the following elements of $\mathbb{Q}_{\geq 0}$.

- $\nu_a^{\mathsf{vert}}(p_i)$ is the ϵ-exponent of the leading term of $\delta(p_i) - a$.

- For each r, let $\nu^{\mathsf{hor}}_{a,r}(p_i)$ be the ϵ-exponent of the leading term of the distance from $\mathsf{pr}_a\, p_i$ to the boundary of $S_{a,r}$. Here pr_a is the projection of $\mathfrak{h} = \mathfrak{h}^{\mathsf{fin}} \times \mathbb{R}d$ onto the horizontal slice $\mathfrak{h}^{\mathsf{fin}} \times \{a\}$.

The significance of these exponents is that, if $p_i \in c^\gamma_k$ with $\overline{c}^\gamma_k = S_{a,r}$, and $\nu^{\mathsf{hor}}_{a,r}(p_i) < q^{\mathsf{hor}}_r$ and $\nu^{\mathsf{vert}}_a(p_i) > q^{\mathsf{vert}}_r$, then $p_i \in \check{c}^\gamma_k$.

We claim that these exponents are related by the following inequalities.

(i) Fix r and $i \in I_{a,r}$. Then $\nu^{\mathsf{hor}}_{a,r}(p_i) < \nu^{\mathsf{vert}}_a(p_i)$.

(ii) Fix r and distinct $i, j \in I_{a,r}$. Then $\nu^{\mathsf{hor}}_{a,r}(p_i) < \nu^{\mathsf{vert}}_a(p_j)$.

(iii) Fix $r < r'$, $i \in I_{a,r}$, and $i' \in I_{a,r'}$. Then $\nu^{\mathsf{vert}}_a(p_i) \leq \nu^{\mathsf{hor}}_{a,r'}(p_{i'})$.

Indeed, if (i) fails, then p_i can only lie in a chamber whose support has dimension $< r$, contradicting $i \in I_{a,r}$. If (ii) fails, then (i) implies that

$$\nu^{\mathsf{hor}}_{a,r}(p_j) < \nu^{\mathsf{vert}}_a(p_j) \leq \nu^{\mathsf{hor}}_{a,r}(p_i) < \nu^{\mathsf{vert}}_a(p_i),$$

so the horizontal component of $p_j - p_i$ is larger than any real multiple of the vertical component, contradicting the slope-boundedness of $(p_i)_i$. If (iii) fails, then we have

$$\nu^{\mathsf{hor}}_{a,r'}(p_{i'}) < \nu^{\mathsf{vert}}_a(p_i), \nu^{\mathsf{vert}}_a(p_{i'}),$$

where the second inequality follows from (i). Since p_i lies in a chamber with support $S_{a,r}$, which lies in the boundary of $S_{a,r'}$, we must also have $\nu^{\mathsf{vert}}_a(p_i) \leq \nu^{\mathsf{hor}}_{a,r'}(p_i)$. This implies that the horizontal component of $p_{i'} - p_i$ is larger than any real multiple of the vertical component, contradicting the slope-boundedness of $(p_i)_i$.

Consider the finite sets $Q^{\mathsf{vert}}_{a,r} := \{\nu^{\mathsf{vert}}_a(p_i) \,|\, i \in I_{a,r}\}$ and $Q^{\mathsf{hor}}_{a,r} := \{\nu^{\mathsf{hor}}_{a,r}(p_i) \,|\, i \in I_{a,r}\}$. The above inequalities imply the top row in the following diagram:

$$\begin{array}{ccccccccccc}
\cdots & \leq & Q^{\mathsf{hor}}_{a,r} & < & Q^{\mathsf{vert}}_{a,r} & \leq & Q^{\mathsf{hor}}_{a,r+1} & < & Q^{\mathsf{vert}}_{a,r+1} & \leq & \cdots \\
 & & \wedge & & \vee & & \wedge & & \vee & & \\
\cdots & < & q^{\mathsf{hor}}_r & < & q^{\mathsf{vert}}_r & < & q^{\mathsf{hor}}_{r+1} & < & q^{\mathsf{vert}}_r & < & \cdots
\end{array}$$

The middle row of 'vertical' inequalities can be achieved by modifying all ϵ-exponents of $(p_i)_{i\in I_a}$ using a single order-preserving automorphism of $\mathbb{Q}_{\geq 0}$. These inequalities imply that, if $p_i \in c^\gamma_k$, then $p_i \in \check{c}^\gamma_k$. This modification does not affect the slope-boundedness of the sub-E-path $(p_i)_{i\in I_a}$.

Let $(p'_i)_{i\in I}$ denote the E-path obtained from $(p_i)_{i\in I}$ by modifying each sub-E-path $(p_i)_{i\in I_a}$ in this way. If $\gamma = \gamma(s)$, then there may be multiple levels a, and each level requires us to modify the ϵ-exponents using a different automorphism of $\mathbb{Q}_{\geq 0}$. Thus, $(p'_i)_{i\in I}$ may have some steps $p'_i - p'_{i-1}$ which are level-increasing (i.e. $\delta(\mathsf{sp}\, p'_i) > \delta(\mathsf{sp}\, p'_{i-1})$) and not slope-bounded by P. We will fix this by modifying the E-path again:

- Identifying repeated vertices in the path $(\mathsf{sp}\, p'_i)_{i\in I} = (\mathsf{sp}\, p_i)_{i\in I}$ gives a path $(x_j)_{j\in J}$ and a surjective map $\sigma : I \to J$ which tracks the identifications. In fact, the fibers of σ are the subsets $I_a \subseteq I$. As noted in Remark 8.5.3, $(x_j)_{j\in J}$ is a weak threading path for a (possibly different) S-belayed claustral gallery.

- Since the E-path $(p_i)_{i\in I}$ is slope-bounded by P, the path $(x_j)_{j\in J}$ is slope-bounded by $\overline{P}$. The last paragraph of 8.5.2 gives a different weak threading path $(y_j)_{j\in J}$ which is slope-bounded by P.
- Define a new E-path $(p''_i)_{i\in I}$ as follows: $p''_i := p'_i + \frac{1}{2}(y_{\sigma(i)} - x_{\sigma(i)})$. In other words, we modify the ϵ^0-part of p'_i by moving it halfway towards the corresponding y_j.

We now show that $(p''_i)_i$ is an anchored slope-bounded threading E-path.

- Suppose that $p'_i \in \check{c}^\gamma_k$ where $\overline{c}^\gamma_k = S_{a,r}$. Then $x_{\sigma(i)}, y_{\sigma(i)} \in S_a \subseteq S_{a,r}$. Now the construction of anchors implies that $p''_i \in \check{c}^\gamma_k$.
- If $p'_i - p'_{i-1}$ is level-increasing, then we set $j := \sigma(i)$ and note that $j-1 = \sigma(i-1)$. The vector

  $$\mathsf{sp}(p''_i - p''_{i-1}) = \tfrac{1}{2}\big((x_j - x_{j-1}) + (y_j - y_{j-1})\big)$$

  lies in $P^\triangleleft$ because it is the average of a vector in $\overline{P}^\triangleleft$ with a vector in $P^\triangleleft$. This implies that $p''_i - p''_{i-1}$ is slope-bounded by P, since P is open.

  Otherwise, we have $p''_i - p''_{i-1} = p'_i - p'_{i-1}$, and this is slope-bounded by P.
- Suppose that $p'_i \in \mathbf{f}_\imath$. Then $x_{\sigma(i)}, y_{\sigma(i)} \in \overline{\mathbf{f}}_\imath$, which implies $p''_i \in \mathbf{f}_\imath$. □

Remark. The whole purpose of introducing S-belayed claustral galleries and weak threading paths and using them to impose a general position constraint on P (8.5.2) is to make the last step in the above proof work.

8.5.7. *Enlargement of anchors.* We want to simplify the notion of slope-bounded γ-belayed claustral galleries by replacing E-paths by ordinary paths.

For each $k \in [n^\gamma]$, let $\mathbf{c}^\gamma_k$ be the claustral chamber containing c^γ_k. For any choice of open polytopes $\hat{c}^\gamma_k \subseteq \mathbf{c}^\gamma_k$, consider the following condition on γ-belayed claustral galleries:

(S) There exists a slope-bounded path whose successive vertices are contained in

$$\hat{c}^\gamma_{k-1}, \mathbf{f}_{\mathsf{bel}(k-1)+1}, \ldots, \mathbf{f}_{\mathsf{bel}(k)}, \hat{c}^\gamma_k,$$

cycling through $k \in [1, n^\gamma]$.

Lemma. *For some choice of $\hat{c}^\gamma_k$, condition* (S) *is equivalent to slope-boundedness for all γ-belayed claustral galleries with excess $\le e$ and at most* st *stutters.*

Proof. Given a γ-belayed claustral gallery $([m], \mathbf{c}, \mathbf{f}, \mathsf{bel})$, let $L \subset \mathfrak{h}^{[n^\gamma]\sqcup[1,n]}$ be the set of slope-bounded paths whose successive vertices are contained in

$$\mathbf{c}^\gamma_{k-1} = \mathbf{c}_{\mathsf{bel}(k-1)}, \mathbf{f}_{\mathsf{bel}(k-1)+1}, \ldots, \mathbf{f}_{\mathsf{bel}(k)}, \mathbf{c}_{\mathsf{bel}(k)} = \mathbf{c}^\gamma_k,$$

cycling through $k \in [1, n^\gamma]$. Consider the projection $\pi : \mathfrak{h}^{[n^\gamma]\sqcup[1,n]} \to \mathfrak{h}^{[n^\gamma]}$ which remembers only the vertices which are constrained to lie in $\mathbf{c}^\gamma_k$ for some k. Since L is the intersection of finitely many (closed or open) half-spaces, $\pi(L)$ has the same property. The set L_E, obtained from L by extension of scalars, identifies with the set of slope-bounded E-paths whose successive vertices satisfy the same constraint.

It suffices to choose the $\hat{c}^\gamma_k$ so that, as $([m], \mathbf{c}, \mathbf{f}, \mathsf{bel})$ varies among the belayed claustral galleries with excess $\le e$ and at most st stutters, the intersection $\pi(L_E) \cap \prod_{k\in[n^\gamma]} \check{c}^\gamma_k$ is nonempty if and only if $\pi(L) \cap \prod_{k\in[n^\gamma]} \hat{c}_k^\gamma$ is nonempty. (We have reformulated the usual slope-bounded condition to include anchors, which is allowed by Lemma 8.5.6.)

Fix $t \in \mathbb{R}_{>0}$ and define each $\hat{c}_k^\gamma$ to be the evaluation of $\check{c}_k^\gamma$ at $\epsilon = t$, see Remark 8.2.2(1). For a fixed γ-belayed claustral gallery, the intersections in the previous paragraph are defined by finitely many half-spaces, so the 'if and only if' statement holds when t is sufficiently small. The same is true when we consider all γ-belayed claustral galleries with excess $\leq e$ and at most $\mathtt{st}$ stutters, because there are finitely many of them. $\square$

8.5.8. Next, we remove the adherence condition in (**B**2). The idea is to work in a different arrangement whose chambers are in bijection with the chambers of $\mathbf{H}$ which adhere to $\acute{A}_k$.

For each $k \in [1, n^\gamma]$ such that f_k^γ is non-horizontal, define the *k-graded arrangement*

$$\mathbb{H}_k := \mathbf{H}|_{\acute{A}_k^r} \times \left(\prod_{t=1}^{r} \left((\mathbf{H}_{\acute{A}_k^t})|_{\acute{A}_k^{t-1}} \right) / \acute{A}_k^t \right) \times (\mathbf{H}_{\acute{A}_k^0}) / \acute{A}_k^0.$$

The notation $(-)/\acute{A}_k^t$ indicates a quotient by the subspace $\acute{A}_k^t \subset \acute{A}_k^{t-1}$, which is irrelevant for the given arrangement. The ambient space of this arrangement is

$$\mathbb{h}_k := \underbrace{\acute{A}_k^r \oplus \left(\bigoplus_{t=1}^{r} \acute{A}_k^{t-1} / \acute{A}_k^t \right)}_{\acute{\mathbb{A}}_k^0} \oplus \mathfrak{h}/\acute{A}_k^0,$$

i.e. the 'associated graded' space obtained by viewing $\acute{A}_k$ as a filtration of $\mathfrak{h}$. The chambers of this arrangement, which are called *k-graded chambers* and denoted in blackboard font, are in bijection with the chambers of $\mathbf{H}$ which adhere to $\acute{A}_k$. Let $\mathbb{c}_{k-1}^{\gamma,(k)}$ and $\mathbb{c}_k^{\gamma,(k)}$ be the k-graded chambers corresponding to $\mathbf{c}_{k-1}^\gamma$ and $\mathbf{c}_k^\gamma$.

Remark. Note that $\acute{A}_k^r$ is a subspace of $\mathfrak{h}$ and of $\mathbb{h}_k$. Thus, anything in $\mathfrak{h}$ which lies in $\acute{A}_k^r$ may be 'transferred' into $\mathbb{h}_k$. This is our motivation for including, in the belay conditions, the requirement that joint faces belong to $\acute{A}_k^r$.

8.5.9. *Slope-bounded total graded galleries.* Let k be as above. A *k-graded gallery* is a jointed generalized gallery $([m], \mathbb{c}, \mathbb{f})$ in $\mathbb{H}_k$ which satisfies the following belay conditions:

($\mathbb{B}$1) The gallery $([m], \mathbb{c})$ goes from $\mathbb{c}_{k-1}^{\gamma,(k)}$ to $\mathbb{c}_k^{\gamma,(k)}$.

($\mathbb{B}$2) The gallery $([m], \mathbb{c} \wedge \acute{\mathbb{A}}_k^0)$ in $(\mathbb{H}_k)|_{\acute{\mathbb{A}}_k^0}$ is reduced.

($\mathbb{B}$3) Each joint face $\mathbb{f}_j$ lies in $\acute{A}_k^r$.

A *total graded gallery* is a choice of k-graded gallery for each k as above.

A total graded gallery is *slope-bounded* if there is a slope-bounded path in $\mathfrak{h}$ consisting of the following parts, interlaced in the obvious way:

(i) For each $k \in [1, n^\gamma]$ such that f_k^γ is non-horizontal, a path whose vertices are in

$$\hat{c}_{k-1}^\gamma, \mathbb{f}_1, \ldots, \mathbb{f}_m, \hat{c}_k^\gamma,$$

where $\mathbb{f}$ indexes the joint faces of the k-graded gallery.

This is a path in $\mathfrak{h}$, where the joint faces $\mathbb{f}_j$ are interpreted as lying in $\mathfrak{h}$ using the above remark. However, the sub-path from $\mathbb{f}_1$ to $\mathbb{f}_m$ lies in $\acute{A}^r_k$ and thus could be interpreted as lying in $\mathbb{h}_k$.

(ii) For each $k \in [1, n^\gamma]$ such that f^γ_k is horizontal, a path whose vertices are in $\hat{c}^\gamma_{k-1}, \mathbf{f}^\gamma_k, \hat{c}^\gamma_k$.

Let $\mathsf{gradeGal}$ be the category of slope-bounded total graded galleries. By construction, it is isomorphic to $\mathsf{bel}_\beta(\gamma).\mathsf{clausGal}$.

8.5.10. A *superpath* is specified as follows:

(i′) For each $k \in [1, n^\gamma]$ such that f^γ_k is non-horizontal, let $\mathsf{pr}_k : \mathbb{h}_k \to \acute{A}^r_k$ be the projection onto the first factor. Choose a path in $\mathbb{h}_k$ such that

- The first vertex $p_{\hat{0}}$ lies in $\mathbb{c}^{\gamma,(k)}_{k-1}$, and the last vertex $p_{\hat{1}}$ lies in $\mathbb{c}^{\gamma,(k)}_k$.
- The path is transverse to $\mathbb{H}_k$.
- The pr_k-image of the path is a slope-bounded path in $\acute{A}^r_k$.

Choose slope-bounded segments in $\mathfrak{h}$ from $\hat{c}^\gamma_{k-1}$ to $\mathsf{pr}_k(p_{\hat{0}})$, and from $\mathsf{pr}_k(p_{\hat{1}})$ to $\hat{c}^\gamma_k$.

(ii′) For each $k \in [1, n^\gamma]$ such that f^γ_k is horizontal, choose a slope-bounded path in $\mathfrak{h}$ whose vertices are in $\hat{c}^\gamma_{k-1}, \mathbf{f}^\gamma_k, \hat{c}^\gamma_k$.

Of course, we require that these paths are compatible in the sense that, for each k, the two vertices which are constrained to lie in $\hat{c}^\gamma_k$ must be equal. We say that a superpath *threads* a total graded gallery if the following holds:

- For each $k \in [1, n^\gamma]$ such that f^γ_k is non-horizontal, denote the k-graded gallery by $([m], \mathbb{c}, \mathbb{f})$, and require that the vertices of the corresponding (i′)-path lie in

$$\mathbb{c}_{j-1}\ ,\ \overline{\mathbb{c}}_{j-1} \cap (\mathsf{pr}_k^{-1}\, \mathbb{f}_j) \cap \overline{\mathbb{c}}_j\ ,\ \mathbb{c}_j,$$

cycling through $j \in [1, m]$.

In other words, each (i′)-path is required to thread the corresponding k-graded gallery in the usual sense, where the ‘joint faces’ are taken to be $\overline{\mathbb{c}}_{j-1} \cap (\mathsf{pr}_k^{-1}\, \mathbb{f}_j) \cap \overline{\mathbb{c}}_j$.

Each superpath threads exactly one total graded gallery, which is slope-bounded.[14] Conversely, each slope-bounded total graded gallery is threaded by a contractible space of superpaths. Thanks to these facts, it is possible to show that $\mathsf{gradeGal}$ is contractible by applying the method of prefix-straightening, which is explained in 3.3.4 to 3.3.7 in [TaTr], to the space of superpaths. Here are two caveats:

- A superpath consists of several paths which live in different vector spaces $\mathbb{h}_k$, so it does not make sense to say that a superpath *is* a straight line segment. Instead, apply prefix-straightening to each path separately.

[14] This would not be true if we had written $\mathbb{f}_j$ instead of $\mathsf{pr}_k^{-1}\, \mathbb{f}_j$ in the bullet point. In that case, there would be some superpaths which do not thread any total graded gallery, because we required that each joint face of a k-graded gallery must lie in $\acute{A}^r_k$ ($\mathbb{B}$3).

- For the 'galleries to paths' direction [TaTr, 3.3.6], we must choose the path vertices p_I^x so that the resulting paths have slope-bounded pr_k-images. This is possible because every slope-bounded total graded gallery is threaded by at least one superpath. Also, note that the linear interpolation step preserves slope-boundedness because it involves a weighted average and the slope bound P is convex.

Since $\mathsf{gradeGal}$ is contractible, so is $\mathsf{bel}_\beta(\gamma).\mathsf{clausGal}$.

As remarked at the start of the proof, prefix-straightening does not increase the excess or the number of stutters, so applying our argument to all sub-belay tours γ gives a contraction of $\mathsf{bel}(\beta).\mathsf{Dom}^{\mathsf{c}}$. This concludes the proof of Proposition 8.5.5.

8.6. **Marked claustral galleries.**

8.6.1. From now on, let ν range over all downward non-claustral roots. A pair of faces (F_1, F_2) is *nc-vertical* if $\nu(F_2) \geq 0$ implies $\nu(F_1) > 0$. This is a transitive relation. A sequence of faces $([n], f)$ is *nc-vertical* if every pair (f_{i-1}, f_i) is nc-vertical. Observe the following:

- (F_1, F_2) is nc-vertical if and only if every pair of chambers (C_1, C_2) satisfying $C_1 \succeq F_1$ and $C_2 \succeq F_2$ is nc-vertical.
- If there exists a slope-bounded E-path from F_1 to F_2, then (F_1, F_2) is nc-vertical.
- If a tour $([n], c)$ is nc-vertical, then it crosses each non-claustral wall at most once.
- (F, F) is nc-vertical if and only if F lies outside of all non-claustral walls.

  In particular, nc-verticality is not a partial order.

Lemma. *Let $([m], \mathbf{c}, \mathbf{f})$ be a slope-bounded claustral gallery. For each $i \in [1, m]$ and each face $F \subset \mathbf{f}_i$, there exists a unique chamber F^{max} such that $F \preceq F^{\mathsf{max}} \subset \mathbf{c}_i$ and (F, F^{max}) is nc-vertical.*

Proof. Choose a slope-bounded threading E-path from $([m], \mathbf{c}, \mathbf{f})$. This gives an E-point $p \in \mathbf{f}_i$ and a slope-bounded E-vector $v \in \mathfrak{h}_E$ which points into $\overline{\mathbf{c}}_i$ when rooted at p. Since the slope bound P is open, we can perturb v to point into $\mathbf{c}_i$. If v is rooted at any other E-point $p' \in \mathbf{f}_i$, then it still points into $\mathbf{c}_i$. If $p' \in F$, then v determines a chamber C of $\mathcal{H}_F$. The choice $F^{\mathsf{max}} := FC$ works. Since v is slope-bounded, (F, F^{max}) is nc-vertical. $\square$

8.6.2. *Fracture.* We will refine the 'claustral face' partition of $\mathfrak{h}$ by emulating 8.3.5. The resulting partition will still be locally finite, and its components will be called *fracture faces.* If a fracture face $\mathfrak{F}$ is not a claustral chamber and is not horizontal, then our construction will also assign to it an orienting vector $v_{\mathfrak{F}}$ satisfying 8.3.2(3) for $A = \mathsf{span}\,\mathfrak{F}$.

The construction proceeds in stages, numbered from $\dim \mathfrak{h} - 1$ to 1. At stage m, we refine some of the faces of dimension $\leq m$ as follows. For each m-dimensional face D', split into cases depending on $A' := \mathsf{span}\, D'$:

- Assume that A' is horizontal. Refine the partition of $\overline{D'}$ by slicing with all walls.

- Assume that A' is not horizontal. Choose a vector v' which satisfies 8.3.2(3) for A'. Refine the partition of $\overline{D'}$ by slicing with the locally finite set of non-claustral walls H_α for which α^{aff} takes different signs on $P^{\triangleleft}$ and v'. Then assign v' to each m-dimensional face thus created.

This differs from 8.3.5 in two ways. First, it is impossible for A' to be horizontal with $\delta(A') \notin \mathbb{Q}$, since walls are real (i.e. non-deformed). Second, the non-horizontal case only uses non-claustral walls, which is why $P^{\triangleleft}$ can play the role of the vertical vector.

For any face f, let $\mathfrak{p}(f)$ be the unique fracture face which contains it.

8.6.3. Fix β and a sub-belay tour γ. A *marked claustral gallery* is a slope-bounded γ-belayed claustral gallery $([m], \mathbf{c}, \mathbf{f}, \mathsf{bel})$ equipped with a sequence of faces $([1, m], f)$ which satisfies $f_j \subset \mathbf{f}_j$ as well as the following:

($\mathrm{obj}_{\mathrm{vert}}$) The sequence of faces defined by

$$\breve{c}^{\gamma}_{k-1}, f_{\mathsf{bel}(k-1)+1}, \ldots, f_{\mathsf{bel}(k)}, \breve{c}^{\gamma}_{k},$$

cycling through $k \in [1, n^{\gamma}]$, is nc-vertical.

($\mathrm{obj}_{\mathrm{fr}}$) For each $i \in [1, m]$, we have

- There exists a i-partial E-path ending at f_i.
- $\mathsf{dim}\, f_i = \mathsf{dim}\, \mathfrak{p}(f_i)$.

Define the category $\mathsf{bel}_\beta(\gamma).\mathsf{mark.clausGal}$ by modifying $\mathsf{bel}_\beta(\gamma).\mathsf{clausGal}$ as follows. Each object is now marked, and a morphism

$$([m], \mathbf{c}, \mathbf{f}, \mathsf{bel}, f) \xrightarrow{\varphi:[m']\to[m]} ([m'], \mathbf{c}', \mathbf{f}', \mathsf{bel}', f')$$

must satisfy the following for every $j \in [m']$ and $i \in [\varphi(j-1)+1, \varphi(j)]$:

($\mathrm{mor}_{\mathrm{max}}$) $\nu(f'_j) > 0$ implies $\nu(f_i) > 0$.

($\mathrm{mor}_{\mathrm{fr}}$) $\mathfrak{p}(f_i) \succeq \mathfrak{p}(f'_j)$. (This implies $\mathsf{span}\, f_i \supset f'_j$.)

Note that ($\mathrm{mor}_{\mathrm{max}}$) is equivalent to the nc-verticality of $(f_i^{\mathsf{max}}, f_j'^{,\mathsf{max}})$ or $(f_i, f_j'^{,\mathsf{max}})$. However, it is weaker than the nc-verticality of (f_i, f'_j).

Define the full subcategory

$$\mathsf{bel}(\beta).\mathsf{mark.clausGal} \subset \prod_{\gamma} \mathsf{bel}_\beta(\gamma).\mathsf{mark.clausGal}$$

via the requirement that the total excess is $\leq e$ and the total number of stutters is $\leq \mathsf{st}$.

8.6.4. *Functoriality.* The functoriality of $\mathsf{bel}(\beta).\mathsf{mark.clausGal}$ with respect to β is defined by insertion, as was the case for $\mathsf{bel}(\beta).\mathsf{clausGal}$. The part of ($\mathrm{obj}_{\mathrm{fr}}$) which says that "there exists a j-partial E-path ending at f_i" can be handled using the averaging trick of Lemma 8.5.4.

8.6.5. **Proposition.** *The category* $\mathsf{bel}(\beta).\mathsf{mark.clausGal}$ *is contractible.*

The rest of this subsection is devoted to proving the proposition.

8.6.6. *Non-belayed analogues.* For clarity of exposition, we will only prove a non-belayed analogue of the desired result. To get the belayed result, it suffices to work in the graded arrangements which were defined in the proof of Proposition 8.5.5. This adds a layer of notational complexity but does not use any new insights, so we have omitted it.

We will now define the non-belayed categories. Let $\mathsf{clausGal}$ be the category obtained by taking $\gamma = B_0 \diamond B_1$ in the definition of $\mathsf{bel}_\beta(\gamma).\mathsf{clausGal}$ and omitting the second part of ($\mathbf{B}2$) which says that the claustral gallery $\mathbf{c} \wedge \acute{A}^0_k$ is reduced.[15] The proof of Proposition 8.5.5 implies that $\mathsf{clausGal}$ is contractible.[16]

Define the category $\mathsf{mark.clausGal}$ by modifying $\mathsf{bel}_\beta(\gamma).\mathsf{mark.clausGal}$ in a similar way. The new condition ($\mathrm{obj}_{\mathrm{vert}}$) says that the sequence of faces $B_0, f_1, \dots, f_m, B_1$ is nc-vertical.

There is a functor

$$\mathsf{mark.clausGal} \xrightarrow{\mathsf{mark.oblv}} \mathsf{clausGal}$$

which forgets the mark faces. Since the target is contractible by Proposition 8.5.5, it suffices to show that this functor is a homotopy equivalence. In fact, we will show that it is final.

8.6.7. *Threadable regions.* For any fixed $([m], \mathbf{c}, \mathbf{f})$, the i-partial E-path requirement of ($\mathrm{obj}_{\mathrm{fr}}$) is equivalent to the following:

- Let $L_i \subset \mathbf{f}_i$ be the set of endpoints of i-partial E-paths which start in the anchor $\breve{B}_0 \subset B_0$. We require that f_i intersects L_i.

To see this, it suffices to show that if there exists a i-partial E-path which ends at f_i, then there exists a i-partial E-path which starts in $\breve{B}_0$ and ends at f_i. This follows from the proof of Lemma 8.5.6. (The only reason why the general position condition on P (8.5.2) includes i-partial weak threading paths is to make this proof work.)

This reformulation is convenient because L_i is nice. Since P is an open polytope and $\breve{B}_0$ is a deformed open polytope, L_i is a deformed open polytope in $\mathbf{f}_i$. Since P is defined over $\mathbb{Q}$, and B_0 is rational-level, L_i is defined by deformed half-spaces with rational specializations.

8.6.8. We will use Lemma 8.3.7 to show that $\mathsf{mark.clausGal}$ is contractible. First, we fix $([m], \mathbf{c}, \mathbf{f}) \in \mathsf{clausGal}$ and show that the fiber $\mathsf{mark.oblv}^{-1}([m], \mathbf{c}, \mathbf{f})$ is contractible. Concretely, an object of this fiber is a sequence of mark faces $([1, m], f)$.

The idea is to build the sequence of mark faces inductively, starting with f_m, and to show that there is a contractible space of choices at each step. To realize this idea, we will define posets of partially-built face sequences. Fix $i \in [m]$ and a 'postfix' face sequence $([i+1, m], g)$ which is assumed to satisfy $g_j \subset \mathbf{f}_j$ and suitably truncated versions of ($\mathrm{obj}_{\mathrm{vert}}$) and ($\mathrm{obj}_{\mathrm{fr}}$). Let $\mathcal{P}_{i,g}$ be the poset whose objects are face sequences $([1, i], f)$ such that the concatenation $f \diamond g$ gives an object of $\mathsf{mark.oblv}^{-1}([m], \mathbf{c}, \mathbf{f})$, and whose maps are defined using ($\mathrm{mor}_{\mathrm{max}}$) and ($\mathrm{mor}_{\mathrm{fr}}$). In the extreme case $i = m$, the postfix sequence g is vacuous, and the resulting poset $\mathcal{P}_{m,\emptyset}$ equals the fiber $\mathsf{mark.oblv}^{-1}([m], \mathbf{c}, \mathbf{f})$.

---

[15]This is equivalent to dropping ($\mathbf{B}2$) and ($\mathbf{B}3$) entirely, because the jointed tour $\gamma = B_0 \diamond B_1$ has only one joint face B_1, which is full-dimensional, so the flag $\acute{A}_k$ is just the whole space $\mathfrak{h}$. Thus, the conditions of adhering to $\acute{A}_k$ or lying in $\acute{A}^r_k$ are automatically satisfied.

[16]Use 8.5.6, 8.5.7, and prefix-straightening, but do not use the graded arrangements.

We use induction on i to prove that $\mathcal{P}_{i,g}$ is contractible. The base case $i = 0$ is trivial. Fix $i \geq 1$ and assume that the result holds for all smaller i.

8.6.9. Given a postfix sequence g, define the poset $\mathcal{B}$ to consist of faces $F \subset \mathbf{f}_i$ such that the concatenation $\{F\} \diamond g$ is a valid postfix sequence for the index $i-1$. Morphisms are defined using ($\mathrm{mor}_{\mathrm{max}}$) and ($\mathrm{mor}_{\mathrm{fr}}$) as before.

Let us reformulate the definition of $\mathcal{B}$ more concretely. Define $G = B_1$ if $i = m$, and $G = g_{i+1}$ otherwise. Let $D \subset \mathfrak{h}$ be the intersection of $\{\nu > 0\}$ for every downward non-claustral root ν satisfying $\nu(G) \geq 0$. Here is an equivalent definition of $\mathcal{B}$:

- An object of $\mathcal{B}$ is a face $F \subset \mathbf{f}_i$ which satisfies

($\mathrm{obj}_{\mathrm{fr}}$) F intersects $L_i \cap D$, and $\mathsf{dim}\, F = \mathsf{dim}\, \mathfrak{p}(F)$.

- There is a map $F_1 \to F_2$ if and only if the following hold:

($\mathrm{mor}_{\mathrm{max}}$) $(F_1^{\mathsf{max}}, F_2^{\mathsf{max}})$ is nc-vertical.

($\mathrm{mor}_{\mathrm{fr}}$) $\mathfrak{p}(F_1) \succeq \mathfrak{p}(F_2)$.

The equivalent definition shows that this $\mathcal{B}$ is analogous to the poset $\mathcal{B}$ defined in 8.3.6. To further substantiate the analogy, we prove

Lemma. *$L_i \cap D \subset \mathbf{f}_i$ is a dome.*

Proof. It is enough to show that $L_i \cap D$ is nonempty and full-dimensional in $\mathbf{f}_i$. Since g is a valid postfix sequence, the intersection $g_{i+1} \cap L_{i+1}$ is nonempty, so we may choose an E-point p lying inside it. Let $Q \subset \mathbf{f}_i$ be the set of $\mathbf{f}_i$-vertices of $(i+1)$-partial E-paths ending at p. Truncating any $(i+1)$-partial E-path at $\mathbf{f}_i$ gives a i-partial E-path, so $Q \subseteq L_i$. Since every $(i+1)$-partial E-path is slope-bounded (by definition) and reaches G after passing through $\mathbf{f}_i$, we have $Q \subset D$. Since the slope-bound P is an open polytope, Q is an open deformed polytope in $\mathbf{f}_i$, so $\mathsf{dim}\, Q = \mathsf{dim}\, \mathbf{f}_i$ and hence $\mathsf{dim}\, D = \mathsf{dim}\, \mathbf{f}_i$. □

Let $\mathsf{Frac}(L_i \cap D)$ be the poset whose objects are nonempty intersections of fracture faces with $L_i \cap D$, ordered by reverse closure. Since $L_i \cap D$ is nonempty and convex, $\mathsf{Frac}(L_i \cap D)$ is contractible. Define the functors

$$\mathcal{P}_{i,g} \xrightarrow{\pi} \mathcal{B} \xrightarrow{\mathfrak{p}} \mathsf{Frac}(L_i \cap D)$$

via $([1,i], f) \mapsto f_i$ and $F \mapsto \mathfrak{p}(F) \cap L_i \cap D$, respectively.

8.6.10. Here are two important classes of $\mathcal{B}$-morphisms. Let $F_1 \to F_2$ denote a $\mathcal{B}$-morphism.

- It is a *glide* morphism if $\mathsf{span}\, F_1 = \mathsf{span}\, F_2$. (This implies $\mathfrak{p}(F_1) = \mathfrak{p}(F_2)$.)
- It is a *specialization* morphism if $F_1 \succeq F_2$.

The Tits product gives a functorial factorization of any $\mathcal{B}$-morphism into a glide morphism followed by a specialization morphism: $F_1 \to F_2F_1 \to F_2$.

Lemma. *The functor π satisfies the following properties:*

- *Each glide map $F_1 \to F_2$ admits a cocartesian lift to any object in $\pi^{-1}(F_1)$.*

- *Each specialization map $F_1 \to F_2$ admits a cartesian lift to any object in $\pi^{-1}(F_1)$.*
- *Each fiber of π is contractible.*

Proof. Suppose we are given a diagram

$$\begin{tikzcd} f \arrow[d, maps to, "\pi"] & \\ f_i \arrow[r, "\mathsf{glide}"] & F \end{tikzcd}$$

The corresponding cocartesian arrow is $f \to f'$, where f' is obtained from f by replacing f_i with F. The only nontrivial point is to show that f' satisfies ($\mathrm{obj}_{\mathrm{vert}}$). Since f satisfies ($\mathrm{obj}_{\mathrm{vert}}$), it suffices to show that $\nu(F) \geq 0$ implies $\nu(f_i) \geq 0$. If $\nu(F) > 0$, then $\nu(f_i) > 0$, because $f_i \to F$ satisfies ($\mathrm{mor}_{\max}$). If $\nu(F) = 0$, then $\nu(f_i) = 0$ because the two faces have equal spans, by the definition of glide maps.

Suppose we are given a diagram

$$\begin{tikzcd} & f \arrow[d, maps to, "\pi"] \\ F \arrow[r, "\mathsf{sp}"] & f_i \end{tikzcd}$$

The corresponding cocartesian arrow is $f' \to f$, where f' is defined as before. Again, we need to show that f' satisfies ($\mathrm{obj}_{\mathrm{vert}}$), and it suffices to show that $\nu(F) \geq 0$ implies $\nu(f_i) \geq 0$. This follows from $F \succeq f_i$, which is true by the definition of specialization maps.

The inductive hypothesis says that the fibers $\pi^{-1}(F) = \mathcal{P}_{i-1,\{F\}\diamond g}$ are contractible. □

8.6.11. The next lemma completes the inductive step.

Lemma. *Let $\pi : \mathcal{E} \to \mathcal{B}$ be any functor which satisfies the conclusion of the previous lemma. Then $\mathcal{E}$ is contractible.*

Proof. Since we already know that $\mathsf{Frac}(L_i \cap D)$ is contractible, it suffices to prove that $\mathfrak{p} \circ \pi$ is a homotopy equivalence. Apply Lemma 8.3.7.

First, we fix an object $\mathfrak{F} \in \mathsf{Frac}(L_i \cap D)$ and show that $(\mathfrak{p} \circ \pi)^{-1}(\mathfrak{F})$ is contractible. Restricting π gives a functor

$$\pi' : (\mathfrak{p} \circ \pi)^{-1}(\mathfrak{F}) \to \mathfrak{p}^{-1}(\mathfrak{F}).$$

Every map in $\mathfrak{p}^{-1}(\mathfrak{F})$ is a glide map, so π' is cocartesian. Since its fibers are contractible, π' is a homotopy equivalence. Thus, it suffices to show that $\mathfrak{p}^{-1}(\mathfrak{F})$ is contractible. In fact, $\mathfrak{p}^{-1}(\mathfrak{F}) \simeq \mathsf{Chambers}(\mathfrak{F})$, so the contractibility follows from Theorem 8.3.2 because $\mathfrak{F}$ is a dome. The fracture construction ensures that the partial order on $\mathfrak{p}^{-1}(\mathfrak{F})$, which is defined via ($\mathrm{mor}_{\max}$), agrees with the partial order on $\mathsf{Chambers}(\mathfrak{F})$, which is defined via $v_{\mathfrak{F}}$.

Next, we fix a solid diagram (as below) and show that its category of lifts, denoted Lifts, is contractible.

$$
\begin{tikzcd}
\{1\} \arrow[r] \arrow[dd, hook] & \mathcal{E} \arrow[d, "\pi"] & & E_1 \arrow[d, mapsto, "\pi"] \\
& \mathcal{B} \arrow[d, "\mathfrak{p}"] & & F_1 \arrow[d, mapsto, "\mathfrak{p}"] \\
\Delta^1 \arrow[r] \arrow[uur, dashed] & \mathsf{Frac}(L_i \cap D) & \mathfrak{F}_0 \arrow[r, phantom, "\succeq"] & \mathfrak{F}_1
\end{tikzcd}
$$

As the notation indicates, a solid diagram corresponds to a choice of $E_1 \in \mathcal{E}$ and $\mathfrak{F}_0 \in \mathsf{Frac}(L_i \cap D)$ such that $\mathfrak{F}_0 \succeq \mathfrak{p}(\pi(E_1))$. A lift is specified by an object $E_0 \in (\mathfrak{p} \circ \pi)^{-1}(\mathfrak{F}_0)$ and a map $\phi : E_0 \to E_1$.

Define the full subposet $\mathfrak{p}^{-1}(\mathfrak{F}_0)_{\succeq F_1} \subset \mathfrak{p}^{-1}(\mathfrak{F}_0)$ via the requirement that the face is $\succeq F_1$. Consider the functor

$$s : \mathfrak{p}^{-1}(\mathfrak{F}_0)_{\succeq F_1} \to \mathsf{Lifts}$$

which sends a face F to the cartesian lift

$$
\begin{tikzcd}
\mathsf{sp}^* E_1 \arrow[r] \arrow[d, mapsto, "\pi"] & E_1 \arrow[d, mapsto, "\pi"] \\
F \arrow[r, "\mathsf{sp}"] & F_1
\end{tikzcd}
$$

It has a left adjoint $\mathsf{Lifts} \to \mathfrak{p}^{-1}(\mathfrak{F}_0)_{\succeq F_1}$, which sends a lift (E_0, ϕ) to the face $F_1 F_0$, where $F_0 := \pi(E_0)$. The unit map comes from the universal property of cartesian arrows:

$$
\begin{tikzcd}
E_0 \arrow[dd, mapsto, "\pi"] \arrow[dr, dashed, "\epsilon"'] \arrow[drr, "\phi"] & & \\
& \mathsf{sp}^* E_1 \arrow[r] \arrow[d, mapsto, "\pi"] & E_1 \arrow[d, mapsto, "\pi"] \\
F_0 \arrow[r, "\mathsf{glide}"] & F_1 F_0 \arrow[r, "\mathsf{sp}"] & F_1
\end{tikzcd}
$$

Thus, s is a homotopy equivalence, so it suffices to show that $\mathfrak{p}^{-1}(\mathfrak{F}_0)_{\succeq F_1}$ is contractible.

Use the fact that F_1 lies in the boundary of $\mathfrak{F}_0$ to show that

$$\mathfrak{p}^{-1}(\mathfrak{F}_0)_{\succeq F_1} \simeq \mathsf{Chambers}_{\mathcal{H}_{F_1}|_{\mathsf{span}\,\mathfrak{F}_0}}(\mathfrak{F}_0),$$

where the right hand side consists of chambers in $\mathcal{H}_{F_1}|_{\mathsf{span}\,\mathfrak{F}_0}$ which intersect $\mathfrak{F}_0$. The contractibility follows from Theorem 8.3.2 as before. Although the theorem was stated for double affine arrangements, the same proof works for the arrangement $\mathcal{H}_{F_1}|_{\mathsf{span}\,\mathfrak{F}_0}$, which is pre-affine or finite. $\square$

8.6.12. Next, fix a solid diagram as shown. We will show that the category of lifts, denoted Lifts, is contractible.

$$
\begin{tikzcd}
\{1\} \arrow[r] \arrow[d, hook] & \mathsf{mark.clausGal} \arrow[d, "\mathsf{mark.oblv}"] & & ([\dot{m}], \dot{\mathbf{c}}, \dot{\mathbf{f}}, \dot{f}) \arrow[d, mapsto, "\mathsf{mark.oblv}"] \\
\Delta^1 \arrow[r] \arrow[ur, dashed] & \mathsf{clausGal} & ([m], \mathbf{c}, \mathbf{f}) \arrow[r, "\varphi : [\dot{m}] \to [m]"] & ([\dot{m}], \dot{\mathbf{c}}, \dot{\mathbf{f}})
\end{tikzcd}
$$

Concretely, a lift is a sequence $([1,m],f)$ of mark faces for $([m],\mathbf{c},\mathbf{f})$ such that φ gives a map in $\mathsf{mark.clausGal}$. This description shows that Lifts is a poset.

Let L_i and $\dot{L}_j$ denote the threadable regions of the two claustral galleries, respectively.

8.6.13. The condition $(\mathrm{mor}_{\mathrm{fr}})$ for φ says that $\mathfrak{p}(f_i) \succeq \mathfrak{p}(\dot{f}_j)$ for all $j \in [\dot{m}]$ and $i \in [\varphi(j-1)+1, \varphi(j)]$. This condition is hard to work with because the union of all fracture faces $\mathfrak{F} \subset \mathbf{f}_i$ which dominate $\mathfrak{p}(\dot{f}_j)$ can fail to be convex. Our first goal is to replace this condition with a stronger one which is easier to work with.

Define the full subposet $\mathsf{tight.Lifts} \subset \mathsf{Lifts}$ via the requirement that $f_i \succeq \dot{f}_j$ for all i and j as above. The next lemma reduces us to showing that $\mathsf{tight.Lifts}$ is contractible.

Lemma. *The embedding* $\mathsf{tight.Lifts} \subset \mathsf{Lifts}$ *is a right adjoint.*

Proof. We claim that the left adjoint is given by $f \mapsto f^{\mathsf{new}}$, where $f_i^{\mathsf{new}} := \dot{f}_j f_i$. Let us show that f^{new} is a valid sequence, by showing the following:

$(\mathrm{obj}^{\mathrm{new}}_{\mathrm{vert}})$ The following pairs are nc-vertical:

- $(B_0, \dot{f}_1 f_1)$.
- $(\dot{f}_{\dot{m}} f_m, B_1)$.
- $(\dot{f}_j f_i, \dot{f}_{j'} f_{i+1})$, where $i \in [1, m-1]$ and j' satisfies $i+1 \in [\varphi(j'-1)+1, \varphi(j')]$.

$(\mathrm{obj}^{\mathrm{new}}_{\mathrm{fr}})$ $\dot{f}_j f_i$ intersects L_i.

$(\mathrm{mor}^{\mathrm{new}}_{\mathrm{max}})$ $\nu(\dot{f}_j) > 0$ implies $\nu(\dot{f}_j f_i) > 0$.

We have omitted the conditions involving fracture faces because they are easy to check.

Proof of $(\mathrm{obj}^{\mathrm{new}}_{\mathrm{vert}})$. For the first two bullets, use $\dot{f}_1 f_1 \succeq \dot{f}_1$ and $\dot{f}_{\dot{m}} f_m \succeq f_m$ and $(\mathrm{obj}_{\mathrm{vert}})$ for $\dot{f}$. For the third bullet, split into two cases.

- Assume $j' > j$. Then $(\dot{f}_j, \dot{f}_{j'})$ is nc-vertical by $(\mathrm{obj}_{\mathrm{vert}})$ for $\dot{f}$. If $\nu(\dot{f}_{j'} f_{i+1}) \geq 0$, then $\nu(\dot{f}_{j'}) \geq 0$, so $\nu(\dot{f}_j) > 0$ by nc-verticality. This implies $\nu(\dot{f}_j f_i) > 0$, as desired.
- Assume $j' = j$. If $\nu(\dot{f}_j) > 0$, then ν is positive on both Tits products. If $\nu(\dot{f}_j) = 0$, then $\nu(\dot{f}_j f_{i+1}) = \nu(f_{i+1})$ and $\nu(\dot{f}_j f_i) = \nu(f_i)$. Now apply $(\mathrm{obj}_{\mathrm{vert}})$ for f.

Also, $(\mathrm{mor}^{\mathrm{new}}_{\mathrm{max}})$ follows from $\dot{f}_j f_i \succeq \dot{f}_j$.

Proof of $(\mathrm{obj}^{\mathrm{new}}_{\mathrm{fr}})$. By $(\mathrm{obj}_{\mathrm{fr}})$ for f, we know that f_i intersects L_i. Thus, there exists an i-partial E-path for $([m],\mathbf{c},\mathbf{f})$, with vertices $(b_0) \sqcup (p_\imath)_{\imath \in [1,i]}$, where $b_0 \in \breve{B}_0$ and $p_\imath \in \mathbf{f}_\imath$ and $p_i \in f_i$. By the same reasoning, since $\dot{f}_j$ intersects $\dot{L}_j$, there exists a j-partial E-path for $([\dot{m}], \dot{\mathbf{c}}, \dot{\mathbf{f}})$, with vertices $(\dot{b}_0) \sqcup (\dot{p}_\jmath)_{\jmath \in [1,j]}$, where $\dot{p}_j \in \dot{f}_j$. For any integer $N > 0$, consider the weighted average E-path with vertices $(b_0^{\mathsf{new}}) \sqcup (p_\imath^{\mathsf{new}})_{\imath \in [1,i]}$ defined by

$$
\begin{aligned}
b_0^{\mathsf{new}} &:= (1-\epsilon^N)\, \dot{b}_0 + \epsilon^N\, b_0 \\
p_\imath^{\mathsf{new}} &:= (1-\epsilon^N)\, \dot{p}_\jmath + \epsilon^N\, p_\imath,
\end{aligned}
$$

where the $\jmath$ in the second line is determined by the requirement $\imath \in [\varphi(\jmath-1)+1, \varphi(\jmath)]$. This is an i-partial E-path because $\mathbf{f}_\imath \succeq \dot{\mathbf{f}}_\jmath$ implies that $p_\imath^{\mathsf{new}} \in \mathbf{f}_\imath$, and the convexity of the slope bound P implies that the E-path is slope-bounded. Hence $p_i^{\mathsf{new}} \in L_i$. The geometric definition of the Tits product implies that, if N is sufficiently large, then $p_i^{\mathsf{new}} \in f_i^{\mathsf{new}}$, so f_i^{new} intersects L_i, as desired. ☐

Remark. The whole purpose of the nc-verticality condition ($\mathrm{obj}_{\mathrm{vert}}$) in the definition of marked claustral galleries is to make the first bullet in the "Proof of ($\mathrm{obj}^{\mathrm{new}}_{\mathrm{vert}}$)" work. Naively, it would have been more natural to instead impose a weaker condition on the 'mark' faces, that $\nu(f_{i+1}) > 0$ implies $\nu(f_i) > 0$. Everything else would still work. In particular, 8.6.10 and 8.6.11 would become much simpler because all maps in $\mathcal{B}$ would be cocartesian, so there would be no need to handle cartesian maps separately. But it is not true that, if the weaker condition holds for (f_{i+1}, f_i) and $(\dot{f}_{j'}, \dot{f}_j)$, then it holds for $(\dot{f}_{j'} f_{i+1}, \dot{f}_j f_i)$. (It is easy to give a counterexample for SL_2.) The issue is that the weaker condition is not closed under generization, and by insisting on this we arrive at nc-verticality.

8.6.14. From now on, let $([m], f)$ be an object of $\mathsf{tight.Lifts}$. Denote the projection of each f_i to the local arrangement $\mathcal{H}_{\dot{f}_j}$ by f_i^{loc}. We will reformulate everything in terms of f^{loc}.

First, for each fracture face $\mathfrak{F} \subset \mathbf{f}_i$ which dominates $\mathfrak{p}(\dot{f}_j)$ (or equivalently $\dot{f}_j$), define the corresponding *local fracture face* $\mathfrak{F}^{\mathsf{loc}}$ to be the projection of $\mathfrak{F}$ to $\mathcal{H}_{\dot{f}_j}$. For every $F \in \mathsf{Faces}(\mathcal{H}_{\dot{f}_j})$, let $\mathfrak{p}^{\mathsf{loc}}(F)$ be the unique local fracture face which contains F.

Recall that the objects of Lifts and hence $\mathsf{tight.Lifts}$ are governed by four conditions: ($\mathrm{obj}_{\mathrm{vert}}$) and ($\mathrm{obj}_{\mathrm{vert}}$) applied to f, and ($\mathrm{mor}_{\mathrm{max}}$) and ($\mathrm{mor}_{\mathrm{fr}}$) applied to φ. We claim that these conditions can be reformulated as follows:

($\mathrm{obj}^{\mathrm{loc}}_{\mathrm{vert}}$) Suppose that $i \in [1, m-1]$ satisfies $i+1 \in [\varphi(j-1)+1, \varphi(j)]$. (Equivalently, $j' = j$.) Then $[\nu(\dot{f}_j) = 0$ and $\nu(f^{\mathsf{loc}}_{i+1}) \geq 0]$ implies $\nu(f_i^{\mathsf{loc}}) > 0$.

($\mathrm{obj}^{\mathrm{loc}}_{\mathrm{fr}}$) f_i^{loc} intersects L_i, and $\dim f_i^{\mathsf{loc}} = \dim \mathfrak{p}^{\mathsf{loc}}(f_i^{\mathsf{loc}})$.

($\mathrm{mor}^{\mathrm{loc}}_{\mathrm{max}}$) Always true.

($\mathrm{mor}^{\mathrm{loc}}_{\mathrm{fr}}$) Always true.

Proof of equivalence. Equivalence for ($\mathrm{obj}^{\mathrm{loc}}_{\mathrm{vert}}$). The proof of Lemma 8.6.13 implies that, if $f_i \succeq \dot{f}_j$, and ($\mathrm{obj}_{\mathrm{vert}}$) holds for $\dot{f}$, then the following pairs are nc-vertical:

- (B_0, f_1).
- (f_m, B_1).
- (f_i, f_{i+1}) if $i+1 \notin [\varphi(j-1)+1, \varphi(j)]$. (Equivalently, if $j' > j$.)

Since we have assumed that $f_i \succeq \dot{f}_j$, the only part of ($\mathrm{obj}_{\mathrm{vert}}$) which is not guaranteed to hold is the nc-verticality of (f_i, f_{i+1}) when $j' = j$. Thus, it suffices to fix an index $i \in [1, m-1]$ such that $j' = j$ and show that the following are equivalent:

- $\nu(f_{i+1}) \geq 0$ implies $\nu(f_i) > 0$.
- $[\nu(\dot{f}_j) = 0$ and $\nu(f_{i+1}) \geq 0]$ implies $\nu(f_i) > 0$.

The equivalence follows from the observation that, if $\nu(\dot{f}_j) > 0$, then $\nu(f_{i+1}) > 0$ and $\nu(f_i) > 0$, because $f_{i+1}, f_i \succeq \dot{f}_j$.

Equivalence for $(\mathrm{obj}^{\mathrm{loc}}_{\mathrm{fr}})$. The averaging trick of Lemma 8.6.13 shows that, if f_i^{loc} intersects L_i, then f_i intersects L_i. The equivalence for the second part is obvious.

It is easy to show that $f_i \succeq \dot{f}_j$ implies $(\mathrm{mor}_{\max})$ and $(\mathrm{mor}_{\mathrm{fr}})$ for φ. $\square$

8.6.15. We will use the inductive strategy of 8.6.8 to show that $\mathsf{tight.Lifts}$ is contractible. Fix $i \in [m]$ and a 'postfix' sequence of (local) faces $([i+1, m], g)$ which satisfies the following:

- For each $\imath \in [i+1, m]$, define $\jmath$ via $\imath \in [\varphi(\jmath - 1) + 1, \varphi(\jmath)]$. We require that $g_\imath$ is a face of $\mathcal{H}_{\dot{f}_\jmath}$ and lies in the projection of $\mathbf{f}_\imath$ to this local arrangement.
- Suitably truncated versions of $(\mathrm{obj}^{\mathrm{loc}}_{\mathrm{vert}})$ and $(\mathrm{obj}^{\mathrm{loc}}_{\mathrm{fr}})$.

Define the poset $\mathcal{P}^{\mathsf{loc}}_{i,g}$ to consist of sequences of (local) faces $([1, i], f^{\mathsf{loc}})$ which fit with g. Apply induction on i to prove that $\mathcal{P}^{\mathsf{loc}}_{i,g}$ is contractible, and fix $i \geq 1$.

The analogous poset $\mathcal{B}^{\mathsf{loc}}$ consists of certain faces of the local arrangement $\mathcal{H}_{\dot{f}_j}$, where j is defined via $i \in [\varphi(j-1) + 1, \varphi(j)]$. If i satisfies the hypothesis of $(\mathrm{obj}^{\mathrm{loc}}_{\mathrm{vert}})$, i.e. $j' = j$, then define $D \subset \mathfrak{h}$ to be the intersection of $\{\nu > 0\}$ for every downward non-claustral root ν satisfying $\nu(\dot{f}_j) = 0$ and $\nu(g_{i+1}) \geq 0$. Otherwise, define $D := \mathfrak{h}$. Define analogous functors

$$\mathcal{P}^{\mathsf{loc}}_{i,g} \longrightarrow \mathcal{B}^{\mathsf{loc}} \xrightarrow{\mathfrak{p}^{\mathsf{loc}}} \mathsf{Frac}^{\mathsf{loc}}(L_i \cap D),$$

where $\mathsf{Frac}^{\mathsf{loc}}(L_i \cap D)$ consists of intersections of local fracture faces with $L_i \cap D$. As before, $L_i \cap D$ is nonempty and convex, so $\mathsf{Frac}^{\mathsf{loc}}(L_i \cap D)$ is contractible.

It is possible to reuse all of the earlier material to show that $\mathcal{P}^{\mathsf{loc}}_{i,g}$ is contractible, which completes the inductive step. This concludes the proof of Proposition 8.7.4.

8.7. **Captive tours.**

8.7.1. A *γ-belayed captive tour* consists of

- $([n], c, f, \mathsf{bel}) \in \mathsf{bel}_\beta(\gamma).\mathsf{Dom}^{\mathsf{c}}$.
- $([m], \mathbf{c}, \mathbf{f}, \mathsf{beg}, \dot{f}) \in \mathsf{bel}_\beta(\gamma).\mathsf{mark.clausGal}$.
- A weakly-increasing surjective map $\mathsf{cap} : [n] \to [m]$.

such that the following conditions are satisfied:

- $([n], c)$ is nc-vertical.
- For each $i \in [n]$, we have $c_i \subset \mathbf{c}_{\mathsf{cap}(i)}$.
- $\mathsf{beg} = \mathsf{cap} \circ \mathsf{bel}$.
- For each $j \in [1, m]$, write $\mathsf{cap}^{-1}(j) = [\mathsf{cal}(j), \mathsf{car}(j)]$.
  - $\dot{f} = f \circ \mathsf{cal}$.

– If i is not in the image of cal, then $f_i = c_i$.

Note the following consequences:

- $([n], c, f)$ is precaptive for $\mathbf{H}$.
- $([n], c)$ and $([m], \mathbf{c})$ have the same excess.
- Everything is determined by $([n], c, f, \mathsf{bel})$ and $\mathsf{cap} : [n] \to [m]$.
- Let $([n], \underline{c}, \underline{f}, \mathsf{bel})$ be the claustral projection of $([n], c, f, \mathsf{bel})$. Then cap defines a map
$$([m], \mathbf{c}, \mathbf{f}, \mathsf{beg}) \xrightarrow{\mathsf{cap}:[n]\to[m]} ([n], \underline{c}, \underline{f}, \mathsf{bel})$$
in $\mathsf{bel}_\beta(\gamma).\mathsf{clausGal}$. The only effect of this map is to create some unjointed stutters.

Define the category $\mathsf{bel}_\beta(\gamma).\mathsf{cap.Dom}^{\mathsf{c}}$ by declaring that an object is a γ-belayed captive tour, and a morphism is a pair of maps

$$([n], c, f, \mathsf{bel}) \longrightarrow ([n'], c', f', \mathsf{bel}')$$
$$([m], \mathbf{c}, \mathbf{f}, \mathsf{beg}, \dot{f}) \longrightarrow ([m'], \mathbf{c}', \mathbf{f}', \mathsf{beg}', \dot{f}')$$

in $\mathsf{bel}_\beta(\gamma).\mathsf{Dom}^{\mathsf{c}}$ and $\mathsf{bel}_\beta(\gamma).\mathsf{mark.clausGal}$, respectively, which are compatible with the maps cap and cap'. Define the full subcategory

$$\mathsf{bel}(\beta).\mathsf{cap.Dom}^{\mathsf{c}} \subset \prod_\gamma \mathsf{bel}_\beta(\gamma).\mathsf{cap.Dom}^{\mathsf{c}}$$

via the requirement that the total excess is $\leq e$ and the total number of stutters in the claustral galleries is $\leq \mathsf{st}$.

8.7.2. *Functoriality.* The functoriality of $\mathsf{bel}(\beta).\mathsf{cap.Dom}^{\mathsf{c}}$ with respect to β is deduced from the functoriality of $\mathsf{bel}(\beta).\mathsf{Dom}^{\mathsf{c}}$ and $\mathsf{bel}(\beta).\mathsf{mark.clausGal}$, see 8.4.6 and 8.6.4 respectively.

8.7.3. *Main construction.* We can now do the constructions which were promised in 8.4.3.

We have already defined $\mathsf{bel}(-).\mathsf{cap.Dom}^{\mathsf{c}}$. Let $\mathsf{cap.Dom}^{\mathsf{c}}$ be its non-belayed analogue in the sense of 8.6.6. We construct the lax-commutative diagram in the third bullet point of 8.4.3 by constructing a (strongly) commutative diagram

$$\begin{tikzcd}
\mathsf{bel}(\beta).\mathsf{cap.Dom}^{\mathsf{c}} \arrow[r] \arrow[d] & \mathsf{cap.Dom}^{\mathsf{c}} \arrow[d, "\pi"] \\
\mathsf{bel}(\beta).\mathsf{Dom}^{\mathsf{c}} \arrow[r, "\mathsf{oblv}"] & \mathsf{pre.Dom}^{\mathsf{c}}
\end{tikzcd}$$

which is lax-functorial with respect to β, and stacking it on top of the diagram in 8.4.4.

- oblv forgets the belay map.
- π forgets everything except for $([n], c, f)$.
- The upper horizontal map is the composition
$$\mathsf{bel}(\beta).\mathsf{cap.Dom}^{\mathsf{c}} \longrightarrow \mathsf{bel}(\beta(0)).\mathsf{cap.Dom}^{\mathsf{c}} \xrightarrow{\mathsf{oblv}} \mathsf{cap.Dom}^{\mathsf{c}},$$

where the first arrow comes from the map of simplices $\{0\} \to \Delta^s \xrightarrow{\beta} \mathcal{B}$, and the second arrow forgets the belay map. (The first arrow assembles a tuple of γ-belayed captive tours into one $\beta(0)$-belayed captive tour.)

- The left vertical map forgets everything except for the γ-belayed tours $([n], c, f)$ for each γ. (Recall from 8.4.7 that $\mathsf{bel}(\beta).\mathsf{Dom}^{\mathsf{c}}$ can be characterized in terms of tuples of γ-belayed tours.)

This completes the construction. The next result gives the required contractibility.

8.7.4. **Proposition.**

(i) The category $\mathsf{bel}(\beta).\mathsf{cap}.\mathsf{Dom}^{\mathsf{c}}$ *is contractible.*

(ii) The category $\mathsf{cap}.\mathsf{Dom}^{\mathsf{c}}$ *is contractible.*

The rest of this subsection is devoted to proving the proposition. In fact, we will only prove (ii), for the same reasons as in 8.6.6.

8.7.5. There is a functor

$$\mathsf{cap}.\mathsf{Dom}^{\mathsf{c}} \xrightarrow{\mathsf{c.oblv}} \mathsf{mark}.\mathsf{clausGal}$$

which sends $\big(([n], c, f), ([m], \mathbf{c}, \mathbf{f}, f \circ \mathsf{cal}), \mathsf{cap}\big)$ to $([m], \mathbf{c}, \mathbf{f}, f \circ \mathsf{cal})$. Since the target is contractible by the non-belayed analogue of Proposition 8.6.5, it suffices to show that this functor is a homotopy equivalence. In fact, we will show that it is initial.

8.7.6. It suffices to fix $([\dot m], \dot{\mathbf{c}}, \dot{\mathbf{f}}, \dot f) \in \mathsf{mark}.\mathsf{clausGal}$ and show that the overcategory

$$\langle \mathsf{cap}.\mathsf{Dom}^{\mathsf{c}} \xrightarrow{\mathsf{mark}.\mathsf{clausGal}} ([\dot m], \dot{\mathbf{c}}, \dot{\mathbf{f}}, \dot f)\rangle$$

is contractible. Explicitly, an object of the overcategory is specified by an object of $\mathsf{cap}.\mathsf{Dom}^{\mathsf{c}}$, denoted $\big(([n], c, f), ([m], \mathbf{c}, \mathbf{f}, f \circ \mathsf{cal}), \mathsf{cap}\big)$, and a map

$$([m], \mathbf{c}, \mathbf{f}, f \circ \mathsf{cal}) \xrightarrow{\varphi : [\dot m] \to [m]} ([\dot m], \dot{\mathbf{c}}, \dot{\mathbf{f}}, \dot f).$$

Define a sequence of full subcategories as follows (cf. 5.5.4):

$$\mathcal{B}_0 \subset \mathcal{A}_1 \subset \mathcal{B}_1 \subset \cdots \subset \mathcal{A}_{\dot m} \subset \mathcal{B}_{\dot m} \subset \mathcal{A}_{\dot m + 1} := \langle \mathsf{cap}.\mathsf{Dom}^{\mathsf{c}} \xrightarrow{\mathsf{mark}.\mathsf{clausGal}} ([\dot m], \dot{\mathbf{c}}, \dot{\mathbf{f}}, \dot f)\rangle$$

- $\mathcal{B}_0$ consists of one object, which is characterized by

$$([n], c, f) = B_0 \underset{\dot f_1}{\diamond} \dot f_1^{\mathsf{max}} \underset{\dot f_2}{\diamond} \cdots \underset{\dot f_{\dot m}}{\diamond} \dot f_{\dot m}^{\mathsf{max}} \diamond B_1,$$

  so that $n - 1 = \dot m$, as well as $\mathsf{cap}(-) = \min(-, \dot m)$ and $\varphi = \mathsf{id}_{[\dot m]}$.

- For each $k \in [1, \dot m]$, an object belongs to $\mathcal{B}_k$ if the following conditions are satisfied.
  - The pair $(c_{\mathsf{car}(\varphi(k))}, \dot f_k^{\mathsf{max}})$ is nc-vertical.
  - The maps $[\mathsf{car}(\varphi(k)) + 1, n - 1] \xrightarrow{\mathsf{cap}} [\varphi(k) + 1, m] \xleftarrow{\varphi} [k + 1, \dot m]$ are bijections. The composition induces isomorphisms of sequences

$$\big([\mathsf{car}(\varphi(k)) + 1, n - 1], c\big) = ([k + 1, \dot m], \dot f^{\mathsf{max}})$$
$$\big([\mathsf{car}(\varphi(k)) + 1, n - 1], f\big) = ([k + 1, \dot m], \dot f).$$

- The full subcategory $\mathcal{A}_k \subset \mathcal{B}_k$ is characterized by the condition $c_{\mathsf{car}(\varphi(k))} = \dot{f}_k^{\mathsf{max}}$.

Since $\mathcal{B}_0$ is contractible, it suffices to show that these embeddings are homotopy equivalences.

8.7.7. We claim that $\mathcal{B}_{k-1} \hookrightarrow \mathcal{A}_k$ is a right adjoint, hence a homotopy equivalence. For any object of $\mathcal{A}_k$ denoted as above, the unit map is given by a map of belayed captive tours

$$
\begin{aligned}
([n], c, f) &\xrightarrow{\quad\quad \eta_1 \quad\quad} ([n'], c', f') \\
([m], \mathbf{c}, \mathbf{f}, f \circ \mathsf{cal}) &\xrightarrow{\eta_2} ([m'], \mathbf{c}', \mathbf{f}', f' \circ \mathsf{cal}')
\end{aligned}
$$

which is defined as follows. Let $i \in [\mathsf{car}(\varphi(k-1))]$ be the last index such that $(c_i, \dot{f}_{k-1}^{\mathsf{max}})$ is nc-vertical. In fact, $(\mathrm{mor}_{\mathrm{max}})$ for φ implies that $i \geq \mathsf{cal}(\varphi(k-1))$. The map η_1 deletes the chambers $\big([i+1, \mathsf{car}(\varphi(k)) - 1], c\big)$ and shrinks $f_{\mathsf{car}(\varphi(k))}$ to equal $\dot{f}_k$. The map η_2 deletes the claustral chambers $\big([\varphi(k-1)+1, \varphi(k)-1], \mathbf{c}\big)$ and shrinks $\mathbf{f}_{\varphi(k)}$ to equal $\dot{\mathbf{f}}_k$. Let us check that these maps are valid:

- The map η_1 must satisfy the Coxeter product conditions. The first condition follows from the span containment in $(\mathrm{mor}_{\mathrm{fr}})$ for φ, and the second condition follows from the fact that $([n], c)$ and $([m], \mathbf{c})$ have the same excess.
- For each $j \in [1, m']$, there must exist a j-partial E-path for $([m'], \mathbf{c}', \mathbf{f}')$ ending at $f'_{\mathsf{cal}'(j)}$. If $j \leq \varphi(k-1)$, then this follows from $(\mathrm{obj}_{\mathrm{fr}})$ for $([m], \mathbf{c}, \mathbf{f}, f \circ \mathsf{cal})$. Otherwise, $f'_{\mathsf{cal}'(j)} = \dot{f}_{k'}$ for some k' for some $k' \geq k$. By $(\mathrm{obj}_{\mathrm{fr}})$ for $([\dot{m}], \dot{\mathbf{c}}, \dot{\mathbf{f}}, \dot{f})$, there exists a k'-partial E-path for $([\dot{m}], \dot{\mathbf{c}}, \dot{\mathbf{f}})$ ending at $\dot{f}_{k'}$. Use the averaging trick of Lemma 8.5.4 to generize this E-path so that it works for $([m'], \mathbf{c}', \mathbf{f}')$.
- For each $j \in [1, m'-1]$, the pair $(f'_{\mathsf{cal}'(j)}, f'_{\mathsf{cal}'(j+1)})$ must be nc-vertical. If $j < \varphi(k-1)$, then this follows from $(\mathrm{obj}_{\mathrm{vert}})$ for $([m], \mathbf{c}, \mathbf{f}, f \circ \mathsf{cal})$. If $j > \varphi(k-1)$, then this follows from $(\mathrm{obj}_{\mathrm{vert}})$ for $([\dot{m}], \dot{\mathbf{c}}, \dot{\mathbf{f}}, \dot{f})$. In the remaining case $j = \varphi(k-1)$, combine the following facts:
  - $(f'_{\mathsf{cal}'(j)}, f'_{\mathsf{cal}'(j+1)}) = (f_{\mathsf{cal}(\varphi(k-1))}, \dot{f}_k)$.
  - $(f_{\mathsf{cal}(\varphi(k-1))}, \dot{f}_{k-1}^{\mathsf{max}})$ is nc-vertical, because φ satisfies $(\mathrm{mor}_{\mathrm{max}})$.
  - $(\dot{f}_{k-1}^{\mathsf{max}}, \dot{f}_k)$ is nc-vertical, because $(\dot{f}_{k-1}, \dot{f}_k)$ is nc-vertical.

The remaining details are left to the reader.

8.7.8. We claim that $\mathcal{A}_k \hookrightarrow \mathcal{B}_k$ is a left adjoint, hence a homotopy equivalence. For any object of $\mathcal{B}_k$ denoted as above, the counit map is given by a map of belayed captive tours

$$
\begin{aligned}
([n'], c', f') &\xrightarrow{\quad\quad \epsilon_1 \quad\quad} ([n], c, f) \\
([m], \mathbf{c}, \mathbf{f}, f \circ \mathsf{cal}) &\xrightarrow{\epsilon_2 = \mathsf{id}} ([m], \mathbf{c}, \mathbf{f}, f \circ \mathsf{cal})
\end{aligned}
$$

which is defined as follows. If $c_{\mathsf{car}(\varphi(k))} = \dot{f}_k^{\mathsf{max}}$, then the map ϵ_1 is an identity; otherwise it inserts $\dot{f}_k^{\mathsf{max}}$ into $([n], c, f)$ as an unjointed chamber:

$$\begin{array}{c}
\cdots \underset{f_{\mathsf{car}(\varphi(k))}}{\diamond} c_{\mathsf{car}(\varphi(k))} \diamond \dot{f}_k^{\mathsf{max}} \underset{\dot{f}_{k+1}}{\diamond} \dot{f}_{k+1}^{\mathsf{max}} \underset{\dot{f}_{k+2}}{\diamond} \cdots \\
\big\downarrow{\scriptstyle \epsilon_1} \\
\cdots \underset{f_{\mathsf{car}(\varphi(k))}}{\diamond} c_{\mathsf{car}(\varphi(k))} \qquad\quad \underset{\dot{f}_{k+1}}{\diamond} \dot{f}_{k+1}^{\mathsf{max}} \underset{\dot{f}_{k+2}}{\diamond} \cdots
\end{array}$$

It is easy to check that this works. This concludes the proof of Proposition 8.7.4 and hence also the proof of Theorem 8.4.2.

8.8. **Homotopical deletion.** We remind the reader that, starting in 8.4, all tours were required to have only rational-level chambers. Thus, the results of this subsection apply to a variant of the Demazure category which incorporates this requirement.

8.8.1. Fix an element $j = (l, w) \in J$ (5.3.5). Let $\mathcal{E}$ be an ∞-category which admits colimits, and fix a functor $F : \mathsf{D} \to \mathcal{E}$. This gives a functor $\tilde{F} : \mathsf{Dom}^{\mathsf{c}}_{\leq j} \to \tilde{\mathcal{E}}$ via [TaTr, 2.4.5].[17] Finally, fix a preclaustral arrangement (8.4.1).

Theorem. *Assume that $\tilde{F}$ sends braid maps, birational joint-preserving maps, and birational maps between precaptive tours to isomorphisms. Then, for any precaptive reduced $t \in \mathsf{Dom}^{\mathsf{c}}_{\leq j}$ which ends at (T, w), we have $\tilde{F}(t) \simeq \operatorname{colim} \tilde{F}$.*

Let us motivate the theorem, which is the main result of this thesis. Although we do not know any application of its full strength, we hope that such applications will exist.

The situation we have in mind is that $F(t)$ is some category of constructible sheaves on $X(t)$. Since $X(-)$ sends braid maps to isomorphisms, so do F and $\tilde{F}$. Since $X(-)$ sends joint-preserving maps to proper maps of schemes, the usual excision argument shows that $\tilde{F}$ sends birational joint-preserving maps to isomorphisms. This is analogous to the classical setting which was studied in [TaTr]. Thus, the first two hypotheses are reasonable.

Unfortunately, we do not know any version of constructible sheaves on $X(t)$ for which the third hypothesis would be satisfied. The issue is that $X(-)$ does not send all birational maps to proper maps of schemes. Imposing a precaptivity condition on t forces $X(t)$ to 'see' more Bruhat cells, but the example in 7.3 shows that $X(-)$ can fail to give a proper map even if it 'sees' infinitely many Bruhat cells.

To find an interesting application of this theorem, it may be necessary to take a different approach to the double affine Hecke category. For example, we can ask whether there exist 'tamer' geometries which are governed by locally finite subarrangements $\mathcal{H}' \subset \mathcal{H}$. One might try to apply the theorem to these geometries and then take the limit as $\mathcal{H}'$ increases.

Lastly, let us recall some material from [TaTr]. Roughly speaking, the functor $\tilde{F}$ is given by F modulo the behavior of F on the strictly lower strata $\mathsf{Dom}^{\mathsf{c}}_{<j}$. In other words, it is an 'associated graded' piece of F for the 'filtration' defined by the rank function $r : \mathsf{Dom}^{\mathsf{c}} \to J$. This construction guarantees that $\tilde{F}|_{\mathsf{Dom}^{\mathsf{c}}_{<j}}$ is constant with value $\hat{0}$ (the initial object of

[17] In the cited paper, we denoted this functor by $\tilde{F}^{\mathsf{c}}$. For notational simplicity, we now write $\tilde{F}$, because this will not overlap with any other notation.

$\tilde{\mathcal{E}}$), so the hypothesis of the theorem is nontrivial only on the highest stratum $\mathsf{Dom}^{\mathsf{c}}_j$. The conclusion of the theorem is useful because it implies that colimits along D can be computed inductively as follows:

(i) If $l = \ell(w)$, then, for any precaptive reduced $t \in \mathsf{Dom}^{\mathsf{c}}_{\leq j}$ which ends at (T, w),[18] the following is a pushout square in $\mathcal{E}$:

$$\begin{tikzcd} \operatorname*{colim}_{\langle \mathsf{Emb}^{\mathsf{d}}_{<j} \xrightarrow{\mathsf{Emb}^{\mathsf{d}}} t\rangle} F \circ \mathsf{tail} \arrow[r] \arrow[d] & F(t) \arrow[d] \\ \operatorname*{colim}_{\mathsf{D}_{<j}} F \arrow[r] & \operatorname*{colim}_{\mathsf{D}_{\leq j}} F \end{tikzcd}$$

(ii) If $l > \ell(w)$, then $\operatorname{colim}_{\mathsf{D}_{<j}} F \simeq \operatorname{colim}_{\mathsf{D}_{\leq j}} F$.

This is analogous to [TaTr, 3.3.2]. For a proof, see [TaTr, 2.3.4].

The proof of the theorem occupies the rest of this section.

8.8.2. Let $\mathsf{Dom}^{\mathsf{c}} \subset \mathsf{Dom}^{\mathsf{c}}_{\leq j}$ be the full subcategory consisting of threadable jointed tethered tours which end at (T, w). Then $\mathsf{Dom}^{\mathsf{c}}$ is isomorphic to the category of threadable jointed (untethered) tours from C_0 to wT, with excess $\leq e := l - \ell_{C_0}(T \xrightarrow{w} wT)$, so this notation agrees with 8.4.2.

Since maps in $\mathsf{Dom}^{\mathsf{c}}_{\leq j}$ do not change the end chamber of a jointed tour, the aforementioned subcategory is a connected component. Since its complement goes to $\hat{0}$ under $\tilde{F}$, we have

$$\operatorname*{colim}_{\mathsf{Dom}^{\mathsf{c}}_{\leq j}} \tilde{F} \simeq \operatorname*{colim}_{\mathsf{Dom}^{\mathsf{c}}} \tilde{F}.$$

Next, let $\mathsf{pre.Dom}^{\mathsf{c}} \subset \mathsf{Dom}^{\mathsf{c}}$ be the full subcategory consisting of precaptive threadable jointed tours. It is contractible by Theorem 8.4.2. Since $\tilde{F}$ sends birational maps between precaptive tours to isomorphisms, [TaTr, Prop. 2.3.3] implies that

$$\operatorname*{colim}_{\mathsf{pre.Dom}^{\mathsf{c}}} \tilde{F} \simeq \tilde{F}(t)$$

for any precaptive reduced t. It remains to show that the $\mathsf{Dom}^{\mathsf{c}}$-colimit agrees with the $\mathsf{pre.Dom}^{\mathsf{c}}$-colimit.

8.8.3. We will compute the $\mathsf{Dom}^{\mathsf{c}}$-colimit using the following technical lemma.

For any simplicial set K, define the simplicial set $\mathsf{lax}.\mathbf{\Delta}^{\mathsf{op}}_{/K}$ as follows.

- A map $\sigma : \Delta^n \to \mathbf{\Delta}^{\mathsf{op}}$ is equivalent to a diagram of functors

$$\sigma(0) \leftarrow \sigma(1) \leftarrow \cdots \leftarrow \sigma(n),$$

where each $\sigma(i)$ is a simplex. Let $\mathcal{M}(\sigma) \to \Delta^n$ be the corresponding cartesian fibration. There is a canonical section $\Delta^n \to \mathcal{M}(\sigma)$ sending $i \in \Delta^n$ to the vertex of $\mathcal{M}(\sigma)$ corresponding to the initial vertex $0 \in \sigma(i)$.

[18] Note that this implies $r(t) = j$.

- By definition, a map $\sigma : \Delta^n \to \mathsf{lax}.\mathbf{\Delta}^{\mathsf{op}}_{/K}$ consists of a map $\sigma^1 : \Delta^n \to \mathbf{\Delta}^{\mathsf{op}}$ and a map $\sigma^2 : \mathcal{M}(\sigma^1) \to K$.

There is a monomorphism

$$\mathbf{\Delta}^{\mathsf{op}}_{/K} \hookrightarrow \mathsf{lax}.\mathbf{\Delta}^{\mathsf{op}}_{/K}$$

whose image is characterized by the requirement that the map $\mathcal{M}(\sigma^1) \xrightarrow{\sigma^2} K$ factors through the projection map $\mathcal{M}(\sigma^1) \to \sigma^1(0)$. There is also a map

$$\mathsf{lax}.\mathbf{\Delta}^{\mathsf{op}}_{/K} \xrightarrow{\mathsf{ev}_0} K$$

which sends $\sigma = (\sigma^1, \sigma^2)$ to the composition $\Delta^n \to \mathcal{M}(\sigma^1) \xrightarrow{\sigma^2} K$, where the first map is the canonical section defined in the first bullet.

Next, given a map of simplicial sets $K \to L$, we define $\mathsf{rel.lax}.\mathbf{\Delta}^{\mathsf{op}}_{/K}$ using the following diagram, in which the left square is a fibered product of simplicial sets:

$$\begin{tikzcd}
\mathsf{rel.lax}.\mathbf{\Delta}^{\mathsf{op}}_{/K} \arrow[r, hook] \arrow[d] & \mathsf{lax}.\mathbf{\Delta}^{\mathsf{op}}_{/K} \arrow[r, "\mathsf{ev}_0"] \arrow[d] & K \arrow[d] \\
\mathbf{\Delta}^{\mathsf{op}}_{/L} \arrow[r, hook] & \mathsf{lax}.\mathbf{\Delta}^{\mathsf{op}}_{/L} \arrow[r, "\mathsf{ev}_0"] & L
\end{tikzcd}$$

Lemma. *The map* $\mathsf{rel.lax}.\mathbf{\Delta}^{\mathsf{op}}_{/K} \to K$ *is final.*

Proof. By Theorem A.3.1, it suffices to show that, for each $\eta : \Delta^n \to K$, the simplicial set of lifts (as shown below) is contractible.

$$\begin{tikzcd}
 & \Delta^n \arrow[dl, dashed, "\sigma"'] \arrow[d, "\eta"] \\
\mathsf{rel.lax}.\mathbf{\Delta}^{\mathsf{op}}_{/K} \arrow[r] & K
\end{tikzcd}$$

Denote the simplicial set of lifts by Lifts. We will show that there is a vertex $x \in \mathsf{Lifts}$ and a homotopy $\mathsf{Lifts} \times \Delta^1 \to \mathsf{Lifts}$ from the identity map to the constant map with value x.

Define the lift $x = (x^1, x^2) : \Delta^n \to \mathsf{rel.lax}.\mathbf{\Delta}^{\mathsf{op}}_{/K}$ using the map $x^1 : \Delta^n \to \mathbf{\Delta}^{\mathsf{op}}$ given by

$$\Delta^{[0,n]} \leftarrow \Delta^{[1,n]} \leftarrow \cdots \leftarrow \Delta^{[n,n]},$$

where the arrows are defined in the obvious way, and the map $x^2 : \mathcal{M}(x^1) \to K$ given by

$$\mathcal{M}(x^1) \to x^1(0) = \Delta^{[0,n]} = \Delta^n \xrightarrow{\eta} K,$$

where the first map is the projection onto the first fiber of the cartesian fibration. For any other lift σ, define a map of lifts $\sigma \xrightarrow{h} x$, i.e. a map $\Delta^n \times \Delta^1 \to \mathsf{rel.lax}.\mathbf{\Delta}^{\mathsf{op}}_{/K}$, using a map $h^1 : \Delta^n \times \Delta^1 \to \mathbf{\Delta}^{\mathsf{op}}$ of the form

$$\begin{tikzcd}
\sigma^1(0) & \sigma^1(1) \arrow[l] & \cdots \arrow[l] & \sigma^1(n) \arrow[l] \\
\Delta^{[0,n]} \arrow[u] & \Delta^{[1,n]} \arrow[l] \arrow[u] & \cdots \arrow[l] & \Delta^{[n,n]} \arrow[l] \arrow[u]
\end{tikzcd}$$

and a map $h^2 : \mathcal{M}(h^1) \to K$ which are defined as follows:

- To define h^1, let the vertical map $\Delta^{[i,n]} \to \sigma^1(i)$ send $j \in \Delta^{[i,n]}$ to the image of $0 \in \sigma^1(j)$ under the horizontal map $\sigma^1(j) \to \sigma^1(i)$.

- The map h^2 is the composition $\mathcal{M}(h^1) \to \mathcal{M}(\sigma^1) \xrightarrow{\sigma^2} K$, where the first map sends $j \in \Delta^{[i,n]}$ to the vertex $0 \in \sigma^1(j)$, viewed as a vertex of $\mathcal{M}(\sigma^1)$.

This specifies the homotopy on vertices of Lifts. It is easy to generalize the above construction to accommodate a simplex of lifts $\sigma_0 \to \cdots \to \sigma_m$, so the desired homotopy exists. $\square$

8.8.4. Consider the category of spans in $\mathsf{Dom}^{\mathsf{c}}$ of the form

$$t^1 \xleftarrow{\text{joint-preserving}} t^2 \xrightarrow{\text{joint-only}} t^3,$$

such that $t^3 \in \mathsf{pre.Dom}^{\mathsf{c}}$. Let $\mathcal{S}$ be the subdiagram (5.4.4(3)) whose objects satisfy that t^1 is adherent, and whose morphisms

$$\begin{tikzcd} t^1 \arrow[d] & t^2 \arrow[l, "\text{joint-preserving}"'] \arrow[r, "\text{joint-only}"] \arrow[d] & t^3 \arrow[d] \\ t^{1,\prime} & t^{2,\prime} \arrow[l, "\text{joint-preserving}"'] \arrow[r, "\text{joint-only}"] & t^{3,\prime} \end{tikzcd}$$

satisfy that the diagonal map $t^2 \to t^{1,\prime}$ is adherent. (This implies that the left square is adherent. Note that joint-preserving maps are adherent by definition.) There are maps

$$p_1, p_2, p_3 : \mathcal{S} \to \mathsf{Dom}^{\mathsf{c}},$$

sending a span to t^1, t^2, t^3, respectively, and p_1 lands in $\mathsf{a.Dom}^{\mathsf{c}}$ while t^3 lands in $\mathsf{pre.Dom}^{\mathsf{c}}$.

Lemma. *The composite map*

$$\operatorname*{colim}_{\mathcal{S}} \tilde{F} p_2 \to \operatorname*{colim}_{\mathcal{S}} \tilde{F} p_1 \to \operatorname*{colim}_{\mathsf{a.Dom}^{\mathsf{c}}} \tilde{F}$$

induced by the natural transformation $p_2 \Rightarrow p_1$ is an isomorphism.

Proof. According to 8.8.3, the map p_1 gives a diagram

$$\begin{tikzcd} \mathsf{rel.lax.}\mathbf{\Delta}^{\mathsf{op}}_{/\mathcal{S}} \arrow[r, "\mathsf{ev}_0'"] \arrow[d, "p_1'"] & \mathcal{S} \arrow[d, "p_1"] \\ \mathbf{\Delta}^{\mathsf{op}}_{/\mathsf{a.Dom}^{\mathsf{c}}} \arrow[r, "\mathsf{ev}_0"] & \mathsf{a.Dom}^{\mathsf{c}} \end{tikzcd}$$

in which ev_0' is final. Also, ev_0 is final by the references in 8.4.3. Thus, the map in question can be rewritten as

$$\begin{aligned} \operatorname*{colim}_{\mathsf{rel.lax.}\mathbf{\Delta}^{\mathsf{op}}_{/\mathcal{S}}} \tilde{F} \circ p_2 \circ \mathsf{ev}_0' \to & \operatorname*{colim}_{\mathsf{rel.lax.}\mathbf{\Delta}^{\mathsf{op}}_{/\mathcal{S}}} \tilde{F} \circ p_1 \circ \mathsf{ev}_0' \\ & = \operatorname*{colim}_{\mathsf{rel.lax.}\mathbf{\Delta}^{\mathsf{op}}_{/\mathcal{S}}} \tilde{F} \circ \mathsf{ev}_0 \circ p_1' \\ & \to \operatorname*{colim}_{\mathbf{\Delta}^{\mathsf{op}}_{/\mathsf{a.Dom}^{\mathsf{c}}}} \tilde{F} \circ \mathsf{ev}_0. \end{aligned}$$

We will show that the natural transformation

$$\mathrm{LKE}_{p_1'} \tilde{F} \circ p_2 \circ \mathsf{ev}_0' \Rightarrow \tilde{F} \circ \mathsf{ev}_0$$

is a natural isomorphism. Then left Kan extension to a point gives the result.

The construction of $\mathcal{S}$ ensures that $\mathsf{rel.lax.}\boldsymbol{\Delta}^{\mathsf{op}}_{/\mathcal{S}}$ is a category. Thus p'_1 is a functor between categories, and it is easy to check that it is cocartesian. This implies that the value of the left Kan extension at any $\beta \in \boldsymbol{\Delta}^{\mathsf{op}}_{/\mathsf{a.Dom}^{\mathsf{c}}}$ can be computed by taking the colimit along the fiber $(p'_1)^{-1}(\beta)$. Thus, our task reduces to showing that

$$\operatorname*{colim}_{(p'_1)^{-1}(\beta)} \tilde{F} \circ p_2 \circ \mathsf{ev}'_0 \to \tilde{F}(\beta(0))$$

is an isomorphism.

To proceed, we describe the fiber $(p'_1)^{-1}(\beta)$ in greater detail. Its objects are certain simplices $\tilde{\sigma} : \Delta^s \to \mathcal{S}$ which correspond to diagrams

$$\begin{tikzcd}
\beta(0) \arrow[d] & \hat{\sigma}(0) \arrow[l, "\text{joint-preserving}"'] \arrow[r, "\text{joint-only}"] \arrow[d] & \sigma(0) \arrow[d] \\
\beta(1) \arrow[d] & \hat{\sigma}(1) \arrow[l, "\text{joint-preserving}"'] \arrow[r, "\text{joint-only}"] \arrow[d] & \sigma(1) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\beta(s) & \hat{\sigma}(s) \arrow[l, "\text{joint-preserving}"'] \arrow[r, "\text{joint-only}"] & \sigma(s)
\end{tikzcd}$$

This diagram matches up with the one in 8.4.5, so it makes sense to use the notation of 8.4.4. We write $\sigma(0) = ([n], c, f)$ and let $\mathsf{bel} : [n^{\beta(0)}] \to [n]$ be the index map of $\beta(0) \leftarrow \hat{\sigma}(0)$.

Let $\iota^{\mathsf{pre}} : (p'_1)^{-1}(\beta)^{\mathsf{pre}} \hookrightarrow (p'_1)^{-1}(\beta)$ be the full subcategory characterized as follows:

- For each $r \in [s]$, the map $\sigma(0) \to \sigma(r)$ must equal the map defined as follows. For each $k \in [1, n^{\beta(r)}]$, shrink the following faces of $\sigma(0)$ by applying $(-) \wedge \mathsf{span}\, f_k^{\beta(r)}$:
$$\big(\big[\mathsf{bel}(\varphi^{r,0}(k-1)) + 1, \mathsf{bel}(\varphi^{r,0}(k))\big], f\big).$$
Furthermore, if $f_k^{\beta(r)}$ is horizontal, delete the chambers of $\sigma(0)$ indexed by
$$\big[\mathsf{bel}(\varphi^{r,0}(k-1)) + 1, \mathsf{bel}(\varphi^{r,0}(k)) - 1\big].$$

The operation $(-) \wedge \mathsf{span} f_k^{\beta(r)}$ is well-defined because the constraint on morphisms in $\mathcal{S}$ forces each map $\hat{\sigma}(0) \to \beta(r)$ to be adherent. Also, note that an object of $(p'_1)^{-1}(\beta)^{\mathsf{pre}}$ is completely determined by $\sigma(0)$ and $\mathsf{bel} : [n^{\beta(0)}] \to [n]$.

Conceptually, the above bullet says that $\sigma(r)$ is the closest possible approximation to $\sigma(0)$ for which there exist joint-preserving and joint-only maps $\beta(r) \leftarrow \hat{\sigma}(r) \to \sigma(r)$.

The embedding ι^{pre} has a right adjoint which replaces $\sigma(1), \ldots, \sigma(r)$ by the tours defined in the above bullet. Composing the counit of this adjunction with the functor $\tilde{F} \circ p_2 \circ \mathsf{ev}'_0$ gives a natural isomorphism because the latter functor only cares about $\hat{\sigma}(0)$. Thus, Remark 5.5.5(2) implies that

$$\operatorname*{colim}_{(p'_1)^{-1}(\beta)^{\mathsf{pre}}} \tilde{F} \circ p_2 \circ \mathsf{ev}'_0 \circ \iota^{\mathsf{pre}} \simeq \operatorname*{colim}_{(p'_1)^{-1}(\beta)} \tilde{F} \circ p_2 \circ \mathsf{ev}'_0.$$

Let $\iota^{\mathsf{post}} : (p'_1)^{-1}(\beta)^{\mathsf{post}} \hookrightarrow (p'_1)^{-1}(\beta)^{\mathsf{pre}}$ be the full subcategory characterized as follows:

- For each $k \in [1, n^{\beta(0)}]$, we must have
  - If $f_k^{\beta(0)}$ is horizontal, then $([\mathsf{bel}(k-1), \mathsf{bel}(k)], c, f) = c_{k-1}^{\beta(0)} \diamond_{f_k^{\beta(0)}} c_k^{\beta(0)}$.
  - If $f_k^{\beta(0)}$ is non-horizontal, then the faces $([\mathsf{bel}(k-1)+1, \mathsf{bel}(k)], f)$ lie in $A_k^{\mathsf{mnh}(k)}$.

The embedding ι^{post} has a left adjoint which changes $\sigma(0)$ by deleting chambers and shrinking joints so that the aforementioned requirement is satisfied.[19] Since ι^{post} is final, we have

$$\operatorname*{colim}_{(p_1')^{-1}(\beta)^{\mathsf{post}}} \tilde{F} \circ p_2 \circ \mathsf{ev}_0' \circ \iota^{\mathsf{pre}} \circ \iota^{\mathsf{post}} \simeq \operatorname*{colim}_{(p_1')^{-1}(\beta)^{\mathsf{pre}}} \tilde{F} \circ p_2 \circ \mathsf{ev}_0' \circ \iota^{\mathsf{pre}}.$$

We now observe that $(p_1')^{-1}(\beta)^{\mathsf{post}}$ identifies with the category of *β-belayed precaptive tours* $\mathsf{bel}(\beta).\mathsf{Dom}^{\mathsf{c}}$ defined in 8.4.4. Indeed, an object in this category can be viewed as a pair $(\sigma(0), \mathsf{bel})$, where $\sigma(0)$ is precaptive. Condition (B1) holds by construction. The first bullet in (B2) follows from the adherence constraint on morphisms in $\mathcal{S}$. The second bullet in (B2) follows from the Coxeter product condition on the map $\hat{\sigma}(0) \to \beta(0)$. Condition (B3) is equivalent to the constraint which defines $(p_1')^{-1}(\beta)^{\mathsf{post}}$. Conversely, a β-belayed precaptive tour gives a valid object of $(p_1')^{-1}(\beta)^{\mathsf{post}}$ due to the discussion in 8.4.5.

We claim that $(p_1')^{-1}(\beta)^{\mathsf{post}} = \mathsf{bel}(\beta).\mathsf{Dom}^{\mathsf{c}}$ is contractible. This follows from modifying the proof of Theorem 8.4.2, which states that the category of (unbelayed) precaptive tours is contractible. The modification requires defining *twice-belayed captive tours*, which depend on the current β as well as a finer belay simplex which plays the role of 'β' in the proof of Theorem 8.4.2. There are no new ideas, so we have omitted the details.

Lastly, we analyze the functor $\tilde{F} \circ p_2 \circ \mathsf{ev}_0' \circ \iota^{\mathsf{pre}} \circ \iota^{\mathsf{post}}$ which is defined on $(p_1')^{-1}(\beta)^{\mathsf{post}}$. For any map $\tilde{\sigma} \to \tilde{\sigma}'$ in $(p_1')^{-1}(\beta)$, the diagram

$$\begin{tikzcd} \beta(0) \arrow[d, equal] & \hat{\sigma}(0) \arrow[l, "\text{joint-preserving}"'] \arrow[d] \\ \beta(0) & \hat{\sigma}'(0) \arrow[l, "\text{joint-preserving}"'] \end{tikzcd}$$

shows that the right vertical map is joint-preserving. If it is also birational, then $\tilde{F}$ sends it to an isomorphism, by the hypothesis of the current theorem. Thus, the aforementioned functor sends every birational map in $(p_1')^{-1}(\beta)^{\mathsf{post}}$ to an isomorphism. Since $(p_1')^{-1}(\beta)^{\mathsf{post}}$ is contractible and its subcategory of tours with excess $< e$ is also contractible (or empty) by the same argument, [TaTr, Prop. 2.3.3] implies that the colimit of this functor is equivalent to its value on any object $(\sigma(0), \mathsf{bel})$ such that the map

$$\beta(0) \xleftarrow{\text{joint-preserving}} \hat{\sigma}(0)$$

is birational. This value is $\tilde{F}(\hat{\sigma}(0))$, which is isomorphic to $\tilde{F}(\beta(0))$ since $\tilde{F}$ sends birational joint-preserving maps to isomorphisms. □

[19] Shrink each joint by applying $(-) \wedge A_k^{\mathsf{mnh}(k)}$ for the appropriate choice of k. The resulting jointed tour may not be threadable, but deleting chambers fixes this issue.

8.8.5. Next, we want to identify $\operatorname{colim}_{\mathcal{S}} \tilde{F}p_2$ with $\operatorname{colim}_{\mathsf{a.pre.Dom}^{\mathsf{c}}} \tilde{F}$. This is much easier.

Consider the subdiagram of $\mathcal{S}$ consisting of spans (and maps between spans) which are contained in $\mathsf{a.Dom}^{\mathsf{c}}$. In other words, the objects are required to satisfy that t^1, t^2, t^3 are adherent, and the morphisms are required to satisfy that every map in the corresponding diagram of two spans (8.8.4) is adherent. The proof of Theorem 5.5.1 implies that we can replace $\mathcal{S}$ by this subdiagram without changing $\operatorname{colim}_{\mathcal{S}} \tilde{F}p_2$. Let us perform this replacement and then denote the subdiagram by $\mathcal{S}$ for notational convenience.

Define the full subdiagram $\imath : \mathcal{S}_{1=2} \hookrightarrow \mathcal{S}$ via the requirement that $t^1 \leftarrow t^2$ is an isomorphism. We will use Lemma 5.5.5 to show that

$$\operatorname*{colim}_{\mathcal{S}_{1=2}} \tilde{F} \circ p_2 \circ \imath \simeq \operatorname*{colim}_{\mathcal{S}} \tilde{F} \circ p_2.$$

The retraction $r : \mathcal{S} \to \mathcal{S}_{1=2}$ sends a general span to

$$t^2 = t^2 \longrightarrow t^3,$$

and the behavior of r on morphisms is obvious. The homotopy h sending $\imath r \Rightarrow \mathsf{id}_{\mathcal{S}}$ is defined using the map of spans

$$\begin{tikzcd} t^2 \arrow[r, equal] \arrow[d] & t^2 \arrow[r] \arrow[d, equal] & t^3 \arrow[d, equal] \\ t^1 & t^2 \arrow[l] \arrow[r] & t^3 \end{tikzcd}$$

The functor $\tilde{F} \circ p_2$ is constant along this map because t^2 does not change. Thus, the lemma applies and proves the claim.

Define the full subdiagram $\jmath : \mathcal{S}_{1=2=3} \hookrightarrow \mathcal{S}_{1=2}$ via the requirement that both maps in the span are isomorphisms. We will use an opposite version of the $\lim F\imath \simeq \lim F$ statement in Remark 5.5.5(1) to show that

$$\operatorname*{colim}_{\mathcal{S}_{1=2=3}} \tilde{F} \circ p_2 \circ \imath \circ \jmath \simeq \operatorname*{colim}_{\mathcal{S}_{1=2}} \tilde{F} \circ p_2 \circ \imath.$$

The retraction $r : \mathcal{S}_{1=2} \to \mathcal{S}_{1=2=3}$ sends a general span to $t^3 = t^3 = t^3$. The homotopy h sending $\mathsf{id}_{\mathcal{S}_{1=2}} \Rightarrow \jmath r$ is defined using the map of spans

$$\begin{tikzcd} t^2 \arrow[r, equal] \arrow[d] & t^2 \arrow[r] \arrow[d] & t^3 \arrow[d, equal] \\ t^3 \arrow[r, equal] & t^3 \arrow[r, equal] & t^3 \end{tikzcd}$$

The aforementioned statement in Remark 5.5.5(1) does not impose any hypothesis on the functor, so it applies and proves the claim. Since $\mathcal{S}_{1=2=3} = \mathsf{a.pre.Dom}^{\mathsf{c}}$, we conclude that

$$\operatorname*{colim}_{\mathcal{S}} \tilde{F}p_2 \simeq \operatorname*{colim}_{\mathsf{a.pre.Dom}^{\mathsf{c}}} \tilde{F}.$$

8.8.6. From Lemma 8.8.4 and 8.8.5, we know that

$$\operatorname*{colim}_{\mathsf{a.Dom}^{\mathsf{c}}} \tilde{F} \simeq \operatorname*{colim}_{\mathcal{S}} \tilde{F}p_2 \simeq \operatorname*{colim}_{\mathsf{a.pre.Dom}^{\mathsf{c}}} \tilde{F}.$$

The proof of Theorem 5.5.1 implies that we can drop the adherence requirements in $\mathsf{a.Dom}^{\mathsf{c}}$ and $\mathsf{a.pre.Dom}^{\mathsf{c}}$ without changing the colimits. Thus

$$\operatorname*{colim}_{\mathsf{Dom}^{\mathsf{c}}} \tilde{F} \simeq \operatorname*{colim}_{\mathsf{pre.Dom}^{\mathsf{c}}} \tilde{F}.$$

This is what we wanted in 8.8.2, so the proof of Theorem 8.8.1 is complete.

Appendix A. Universal homotopy equivalences of simplicial sets

If a map $f : X \to Y$ of topological spaces is nice enough, and the fibers of f are contractible, then f is a homotopy equivalence, see e.g. [Sm]. Our goal is to prove a statement like this when f is a map of simplicial sets (Theorem A.3.1). It is not enough to require that each fiber of f is contractible. We also need to consider 'generalized fibers' which correspond to fibers of $|f|$ over barycenters of simplices in Y.

A.1. **Naive criterion.**

A.1.1. Given a map of simplicial sets $f : K \to L$ and a simplex $\sigma : \Delta^n \to L$, we define the *generalized fiber* of f over σ to be the category of diagrams

$$\begin{tikzcd} \Delta^m \arrow[r, dashed] \arrow[d, dashed, "\text{surj.}"'] & K \arrow[d, "f"] \\ \Delta^n \arrow[r, "\sigma"] & L \end{tikzcd}$$

where Δ^m is an arbitrary simplex, and the indicated map must be surjective. A morphism between two such objects, with simplices Δ^{m_1} and Δ^{m_2}, respectively, is a map $\Delta^{m_1} \to \Delta^{m_2}$ which is compatible with the dashed maps.

A.1.2. **Proposition.** *Let $f : K \to L$ be a map of simplicial sets. If every generalized fiber of f is contractible, then f is initial and final.*

Proof. We will prove that f is initial. Applying this to f^{op} shows that f is also final.

Consider the diagram

$$\begin{tikzcd} \mathbf{\Delta}_{/K} \arrow[r] \arrow[d, "\dot{f}"] & K \arrow[d, "f"] \\ \mathbf{\Delta}_{/L} \arrow[r] & L \end{tikzcd}$$

where the horizontal arrows are given by evaluation at the final vertex of a simplex. The horizontal arrows are initial (and final) by the references in 8.4.3. Thus, it suffices to show that the left vertical arrow is initial, or equivalently that $\langle \dot{f} \downarrow \sigma \rangle$ is contractible for every simplex $\sigma : \Delta^n \to L$. Consider the functor

$$\langle \dot{f} \downarrow \sigma \rangle \xrightarrow{\pi} \mathsf{Faces}(\Delta^n)$$

which sends a diagram

$$\begin{tikzcd} \Delta^m \arrow[r] \arrow[d, "f'"] & K \arrow[d, "f"] \\ \Delta^n \arrow[r, "\sigma"] & L \end{tikzcd}$$

to the image of the left horizontal map. The functor π is cartesian, so it suffices to show that each fiber $\pi^{-1}(\Delta^S)$ is contractible, where $S \subseteq [n]$. Moreover, replacing σ by the simplex $\Delta^S \hookrightarrow \Delta^n \xrightarrow{\sigma} L$ reduces us to showing that $\pi^{-1}(\Delta^n)$ is contractible.

But $\pi^{-1}(\Delta^n)$ is the generalized fiber of f over σ, so it is contractible by hypothesis. $\square$

A.2. **Bar construction for multi-simplicial sets.**

A.2.1. *Multi-simplicial sets.* Fix a simplex Δ^n and let $\mathbf{\Delta}^{\mathsf{surj}}_{/\Delta^n}$ be the category of simplices equipped with surjective maps to Δ^n. For any map of simplicial sets $f : K \to L$ and simplex $\sigma : \Delta^n \to L$, the corresponding generalized fiber (A.1.1) is the unstraightening of the functor $F : \mathbf{\Delta}^{\mathsf{surj,op}}_{/\Delta^n} \to \mathsf{Set}$ which sends an object $\Delta^m \xrightarrow{\varphi} \Delta^n$ to the set of ways to fill in the diagram

$$\begin{tikzcd} \Delta^m \arrow[r, dashed] \arrow[d, "\varphi"] & K \arrow[d, "f"] \\ \Delta^n \arrow[r, "\sigma"] & L \end{tikzcd}$$

There is an obvious equivalence

$$\mathbf{\Delta}^{\mathsf{surj}}_{/\Delta^n} \simeq \mathbf{\Delta}^{\times(n+1)}$$

which sends $\Delta^m \xrightarrow{\varphi} \Delta^n$ to the tuple of fibers $(\varphi^{-1}(0), \dots, \varphi^{-1}(n))$. Thus, the generalized fiber may be viewed as a multi-simplicial set.

A.2.2. *Subdivision and extension.* It would be more convenient to reformulate the criterion of Proposition A.1.2 in terms of simplicial sets rather than multi-simplicial sets. To this end, we define a pair of adjoint functors which relate simplicial and multi-simplicial sets:

$$\mathsf{SSet} \underset{\mathsf{Ex}}{\overset{\mathsf{Sd}}{\rightleftarrows}} \mathsf{S}^{(n+1)}\mathsf{Set} := \mathsf{Fun}(\mathbf{\Delta}^{\times(n+1),\mathsf{op}}, \mathsf{Set})$$

The 'subdivision' functor Sd sends a simplicial set X to the generalized fiber of the projection map $\mathsf{pr}_1 : \Delta^n \times X \to \Delta^n$ over the identity simplex $\mathsf{id} : \Delta^n \to \Delta^n$.

The 'extension' functor Ex is the right adjoint of Sd. Explicitly, it sends a multi-simplicial set Y to the simplicial set

$$\Delta^m \mapsto \mathrm{Hom}_{\mathsf{S}^{(n+1)}\mathsf{Set}}(\mathsf{Sd}\,\Delta^m, Y).$$

A.2.3. *Remarks.* We call Sd a 'subdivision' because it can be geometrically interpreted as follows: if the domain and target of the geometric realization

$$|\mathsf{pr}_1| : |\Delta^n \times X| \to |\Delta^n|$$

are equipped with their canonical CW-complex structures, then the fiber of $|\mathsf{pr}_1|$ over any internal point of $|\Delta^n|$ is a CW-complex whose cells identify with $(n+1)$-fold products of simplices (of varying dimensions). This CW-complex structure refines the original one on $|X|$, and it identifies with the geometric realization of $\mathsf{Sd}\,X$.

We call the right adjoint Ex in analogy with Kan's Ex functor, which is right adjoint to the functor of barycentric subdivision.

These functors are well-studied in simplicial homotopy theory, albeit under different names. Let $+ : \mathbf{\Delta}^{\times(n+1)} \to \mathbf{\Delta}$ be the $(n+1)$-fold ordinal sum, i.e.

$$(\Delta^{m_0}, \dots, \Delta^{m_n}) \mapsto \Delta^{m_0 + \cdots m_n}.$$

Then Sd corresponds to precomposition by $+$, and it is called the *total décalage* functor. The right adjoint Ex is called the *total simplicial set* functor, the *Artin–Mazur codiagonal*, or the *bar construction*.[20]

We also remind the reader that the diagonal functor

$$\mathsf{diag} : \mathsf{S}^{(n+1)}\mathsf{Set} \to \mathsf{SSet}$$

is a more common way to go from a multi-simplicial set to a simplicial set. It is defined by precomposition with the diagonal map $\delta : \mathbf{\Delta} \to \mathbf{\Delta}^{\times(n+1)}$.

Although our notation Sd is non-standard, the composite functor $\mathsf{diag} \circ \mathsf{Sd}$ has been called *edgewise subdivision*, for example in [BöHM, §1] and [EdGr]. The latter paper includes some three-dimensional pictures of this subdivision.

A.2.4. *Comparison map.* For every multi-simplicial set X, there is a canonical monomorphism of simplicial sets

$$\mathsf{diag}\, X \hookrightarrow \mathsf{Ex}\, X$$

defined as follows. Given a map $\Delta^m \to \mathsf{diag}\, X$, i.e. a map $(\Delta^m, \ldots, \Delta^m) \to X$, we need to produce a map $\Delta^m \to \mathsf{Ex}\, X$, i.e. a map $\mathsf{Sd}\, \Delta^m \to X$. The desired map is the composition

$$\mathsf{Sd}\, \Delta^m \to (\Delta^m, \ldots, \Delta^m) \to X,$$

where the second map was given to us, and the first map takes a multi-simplex

$$(\Delta^{k_0}, \ldots, \Delta^{k_n}) \to \mathsf{Sd}\, \Delta^m,$$

i.e. a map $\Delta^{k_0+\cdots+k_n} \to \Delta^m$, and sends it to the multi-simplex

$$(\Delta^{k_0}, \ldots, \Delta^{k_n}) \to (\Delta^m, \ldots, \Delta^m)$$

which is a product of compositions

$$\Delta^{k_i} \xrightarrow{\text{insertion}} \Delta^{k_0+\cdots+k_n} \xrightarrow{\qquad} \Delta^m$$

A.2.5. We believe that the above comparison map is a homotopy equivalence. This is true for bisimplicial sets ($n = 1$), see [CeRe]. We were not able to find a reference for the general case, so we will give *ad hoc* proof for the weaker result which we need:

Lemma. *For any multi-simplicial set X, the simplicial set $\mathsf{Ex}\, X$ is contractible if and only if $\mathsf{diag}\, X$ is contractible.*

The rest of this subsection is devoted to proving this lemma.

The idea is to relate the homology groups and fundamental groups and then apply the Hurewicz Theorem. We learned this standard method from the discussion of Kan's Ex functor in [GoJ, Cor. 4.4], and our notations are inspired by that exposition.

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[20] We learned this terminology from `https://ncatlab.org/nlab/show/bisimplicial+set` which has many useful references.

A.2.6. For any map $f : K \to \Delta^n$ of simplicial sets, we have the *relative internal Hom* $\underline{\mathrm{Hom}}_{\Delta^n}(\Delta^n, K)$, which is a simplicial set whose m-simplices are commutative diagrams

$$\begin{tikzcd} \Delta^n \times \Delta^m \arrow[rr, dashed] \arrow[dr, "\mathsf{pr}_1"'] & & K \arrow[dl, "f"] \\ & \Delta^n & \end{tikzcd}$$

Lemma. *There is a canonical isomorphism of simplicial sets*

$$\underline{\mathrm{Hom}}_{\Delta^n}(\Delta^n, K) \simeq \mathsf{Ex}(\text{generalized fiber of } f \text{ over } \mathsf{id}_{\Delta^n}).$$

Proof. The following data are equivalent:

- A map $\Delta^m \to \underline{\mathrm{Hom}}_{\Delta^n}(\Delta^n, K)$.
- A commutative diagram as above.
- A map from the generalized fiber of $\Delta^n \times \Delta^m$ to that of K (as multi-simplicial sets).
- A map from $\mathsf{Sd}\,\Delta^m$ to the generalized fiber of K.
- A map $\Delta^m \to \mathsf{Ex}(\text{generalized fiber of } K)$.

All of these equivalences are tautologies except for the one between the second and third bullets. For that step, it suffices to note that $\Delta^n \times \Delta^m$ can be expressed as the colimit of a diagram of simplices which map surjectively to Δ^n. □

A.2.7. **Corollary.** *Each simplicial set* $\mathsf{Ex}\,\mathsf{Sd}\,\Delta^m$ *is contractible.*

Proof. Lemma A.2.6 implies that this simplicial set is isomorphic to $\underline{\mathrm{Hom}}_{\Delta^n}(\Delta^n, \Delta^n \times \Delta^m)$, defined using the projection map $\mathsf{pr}_1 : \Delta^n \times \Delta^m \to \Delta^n$. This relative internal Hom is a category with an initial object, corresponding to the map

$$\Delta^n = \Delta^n \times \{0\} \hookrightarrow \Delta^n \times \Delta^m,$$

so it is contractible. □

It is easy to see that $\mathsf{Sd}\,\Delta^m$ is also contractible. This follows from its geometric interpretation as a subdivision of $|\Delta^m|$ (A.2.3) or from a combinatorial argument.

A.2.8. **Lemma.** *The map* $\mathsf{diag}\,X \hookrightarrow \mathsf{Ex}\,X$ *induces an equivalence of fundamental groupoids.*

Proof. We take it for granted that the fundamental groupoid of a simplicial set is generated by 0, 1, and 2-dimensional simplices in the obvious way. The comparison map induces a bijection of 0-dimensional simplices. We will show that every generating arrow in the fundamental groupoid of $\mathsf{Ex}\,X$ equals an arrow in the fundamental groupoid of $\mathsf{diag}\,X$, and similarly for every generating relation.

A generating arrow in $\pi_1(\mathsf{Ex}\,X)$ comes from a map $s : \Delta^1 \to \mathsf{Ex}\,X$. This map corresponds to a map $s' : \mathsf{Sd}\,\Delta^1 \to X$ by adjunction, so we obtain the following commutative diagram:

$$\begin{tikzcd} & \mathsf{diag}\,\mathsf{Sd}\,\Delta^1 \arrow[r, "\mathsf{diag}\,s'"] \arrow[d, hook] & \mathsf{diag}\,X \arrow[d, hook] \\ \Delta^1 \arrow[r, "\text{unit}"] & \mathsf{Ex}\,\mathsf{Sd}\,\Delta^1 \arrow[r, "\mathsf{Ex}\,s'"] & \mathsf{Ex}\,X \end{tikzcd}$$

The tautological arrow $\alpha \in \pi_1(\Delta^1)$ maps to an arrow in $\pi_1(\mathsf{Ex}\,\mathsf{Sd}\,\Delta^1)$. This lifts to an arrow in $\pi_1(\mathsf{diag}\,\mathsf{Sd}\,\Delta^1)$ because $\mathsf{diag}\,\mathsf{Sd}\,\Delta^1$ and $\mathsf{Ex}\,\mathsf{Sd}\,\Delta^1$ are contractible and because the vertical map induces a bijection on 0-dimensional simplices. Thus, the arrow in $\pi_1(\mathsf{Ex}\,X)$ coming from α lies in the image of $\pi_1(\mathsf{diag}\,X)$, as desired.

A generating relation in $\pi_1(\mathsf{Ex}\,X)$ comes from a map $r : \Delta^2 \to \mathsf{Ex}\,X$, and a similar argument shows that every such relation already holds in $\pi_1(\mathsf{diag}\,X)$. (This argument relies on the observation that the above diagram is functorial in the simplex Δ^1.) ☐

A.2.9. **Lemma.** *The map* $\mathsf{diag}\,X \hookrightarrow \mathsf{Ex}\,X$ *induces an isomorphism on integral homology.*

Proof. From Lemma A.2.8, we already know that the map induces an isomorphism on $H_0(-)$. We would like to use the acyclic models theorem to construct the inverse map $H.(\mathsf{Ex}\,X) \to H.(\mathsf{diag}\,X)$.

Let us consider the two functors

$$\mathsf{S}^{(n+1)}\mathsf{Set} \overset{F}{\underset{G}{\rightrightarrows}} \mathsf{Ch}(\mathbb{Z}\text{-}\mathsf{mod})$$

defined by $F(X) := C.(\mathsf{Ex}\,X)$ and $G(X) := C.(\mathsf{diag}\,X)$. Here $C.(-)$ stands for the Moore complex of a simplicial set. Take the models to be $\mathsf{Sd}\,\Delta^m \in \mathsf{S}^{(n+1)}\mathsf{Set}$, equipped with the distinguished cycle

$$[\Delta^m] \in C_m(\Delta^m) \to C_m(\mathsf{Ex}\,\mathsf{Sd}\,\Delta^m) = F_m(\mathsf{Sd}\,\Delta^m)$$

where the map $\Delta^m \to \mathsf{Ex}\,\mathsf{Sd}\,\Delta^m$ is the unit of the adjunction.

By construction, the abelian group $F_m(X) := C_m(\mathsf{Ex}\,X)$ is freely generated by simplices $s : \Delta^m \to \mathsf{Ex}\,X$. To show that F is free, we must show that each generator is the image of the distinguished cycle under the map

$$C_m(\mathsf{Ex}\,\mathsf{Sd}\,\Delta^m) \to C_m(\mathsf{Ex}\,X)$$

induced by some map $\mathsf{Sd}\,\Delta^m \to X$. To see this, define the map $s' : \mathsf{Sd}\,\Delta^m \to X$ by adjunction from s, and observe (again) that s canonically factors as

$$\Delta^m \xrightarrow{\text{unit}} \mathsf{Ex}\,\mathsf{Sd}\,\Delta^m \xrightarrow{\mathsf{Ex}\,s'} \mathsf{Ex}\,X.$$

To show that G is acyclic, we need to check that $H.(\mathsf{diag}\,\mathsf{Sd}\,\Delta^m)$ vanishes in positive degrees. This follows from the contractibility of $\mathsf{Sd}\,\Delta^m$ which we have already observed (A.2.7).

Now the acyclic models theorem gives the desired map $F \to G$ which lifts the given isomorphism on $H_0(-)$.

To show that the induced map on homology is indeed inverse to the map $G \to F$ coming from the comparison map $\mathsf{diag}\,X \hookrightarrow \mathsf{Ex}\,X$, we need to check the following:

- The map $F \to G \to F$ is homotopic to $\mathsf{id}\, F$.
- The map $G \to F \to G$ is homotopic to $\mathsf{id}\, G$.

Of course, we use the acyclic models theorem to construct these homotopies. For the first bullet, to check that F is acyclic, we need to know that $\mathsf{Ex}\,\mathsf{Sd}\,\Delta^m$ is contractible (Corollary A.2.7). For the second bullet, we instead use the models $(\Delta^m, \dots, \Delta^m)$. $\square$

At this point, we know that $\mathsf{diag}\, X \hookrightarrow \mathsf{Ex}\, X$ induces an equivalence of fundamental groupoids (Lemma A.2.8) and on integral homology (Lemma A.2.9). If one of them is contractible, then the other is also contractible by the Hurewicz Theorem. This concludes the proof of Lemma A.2.5.

A.3. **Improved criterion.**

A.3.1. We can now rephrase Proposition A.1.2 as follows:

Theorem. *If $f : K \to L$ is a map of simplicial sets such that, for every simplex $\sigma : \Delta^n \to L$, the relative internal Hom*

$$\underline{\mathrm{Hom}}_{\Delta^n}\left(\Delta^n, \Delta^n \underset{L}{\times} K\right)$$

is contractible, then f is initial and final.

Proof. Since we have already proved Proposition A.1.2, we only need to show that, for every simplex σ, the generalized fiber of f over σ is homotopy equivalent to the relative internal Hom. This follows from a chain of homotopy equivalences between

- The generalized fiber of f over σ.
- The generalized fiber of $\Delta^n \times_L K \to \Delta^n$ over id_{Δ^n}.
- Ex(generalized fiber of $\Delta^n \times_L K \to \Delta^n$ over id_{Δ^n}).
- $\underline{\mathrm{Hom}}_{\Delta^n}(\Delta^n, \Delta^n \times_L K)$.

The first equivalence is tautological, the second uses Lemma A.2.5, and the third uses Lemma A.2.6 (where the relative internal Hom was defined). $\square$

Remark. Let us say that a map of simplicial sets $f : K \to L$ is a *universal homotopy equivalence* if its base-change along any map $L' \to L$ is a homotopy equivalence to L'. One can show (using the theorem) that f is such a map if and only if it satisfies the condition in the theorem or, equivalently, the condition in Proposition A.1.2. In particular, such a map is always initial and final.

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