ESSAYS ON EQUILIBRIA IN DYNAMIC ECONOMIES

by

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ABSTRACT

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Submitted to the Department of Economics on May 16, 1985 in partial fulfillment of the requirement for the degree of Doctor of Philosophy.

This thesis is focused on furthuring the study of equibria in dynamic economies. In chapters 1 and 2 we consider a straightforward extention of a classical exchange economy that is modified to accomodate an infinite number of agents and goods. In chapter 3 we consider a more general dynamic model that incorporates incomplete commodity markets, production, and stock markets.

Our analysis naturally begins in chapter I where we establish the general existence of competitive equilibria for dynamic models. We then proceed to establish some general properties of "monetary" equilibria. Specifically, we establish the general existence of debt equilibria (i.e. equilibria in which there are negative quantities of fiat money outstanding), the sufficiency of monetary policy to implement any Pareto optimal allocation scheme (i.e. the second fundamental theorem of welfare economics), and some continuity properties of monetary policies.

In chapter 2 we focus on characterizing the simplest monetary policies that quarentee intertemporal efficiency. Although this problem has been extensivly studied by others, our results are significant since we permit a quite general form of non-stationarity of agents tastes and endowments. In a sence, ours is the only model considered to date that evolves over time (i.e. is genuinely dynamic).

Finally, in chapter 3, we provide a foundation to the study of incomplete markets by establishing the existence of competitive equilibria for a general class of sequencial economies with production and stock markets.

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whatever you do
work at it with all your heart
as working for the Lord

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CHAPTER I

EQUILIBRIA IN DYNAMIC ECONOMIES

1. INTRODUCTION

In the study of the general properties of competitive equilibria, a lot of attention has been focused on just a few areas. Perhaps the most fundamental area one can explore are the conditions under which competitive equilibria are guaranteed to exist. Once it is known that a certain class of economies has equilibria, one can then proceed to investigate their optimality properties. Another, closely related, area that has received extensive examination are the characteristics of optimal governmental policies. For example, one may ask when the second fundamental theorem of welfare economics is applicable (see Arrow [1951] or Debreu [1959]). That is, one determines when any Pareto optimal distribution of goods can be implemented through a system of lump-sum income or commodity transfers.

For many years, the above issues have been studied in the context of static models. (We call these models static because they involve only finite numbers of agents and goods and hence presuppose the existence of some (arbitrary) terminal date.) It is only recently that these issues have been rigorously studied for general dynamic models (i.e. models that involve infinite numbers of agents and goods). This paper is focused on furthering the latter line of research.

The early equilibrium existence theorems of Arrow and Debreu [1954], McKensie [1954], and Nikaido [1956], for static models, have been refined over the years as the assumptions imposed to guarantee the existence of equilibria have been successively weakened. McKensie [1981] provides a brief review of this line of research as well as what appears to be the minimal set of sufficient restrictions known to date. For dynamic models, Balasko, Cass, Shell [1980(b)] and Wilson [1981] have been able to establish existence results under assumptions that are, for the most part, analogous to those imposed in McKensie [1981]. In fact, the only assumption crucial to the existence proofs of Balasko, Cass, Shell and Wilson that is sufficiently stronger then the corresponding assumption of McKensie is there irreducibility conditions. One of the contributions of this paper is a proof of the existence of competitive equilibria that does not require the stronger form of irreducibility imposed by Balasko et al. Hence, we have a general existence theorem for dynamic economies that is a direct analog of, perhaps, the strongest result yet attained for static economies.

It is well-known that, under very general conditions, the competitive equilibria of a static economy are Pareto optimal (see Arrow [1951] or Debreu [1959]). On the other hand, it is also well-known (see Samuelson [1958]) that this first fundamental theorem of welfare economics fails for a general class of dynamic economies. The crucial feature of these models that accounts for this discrepancy is that the set of Pareto optimal allocation schemes is not necessary closed in dynamic economies. In fact, in this paper, we characterize a very large class of dynamic economies for which the Pareto optimal set is not closed. To illustrate the effect of the non-closedness of the Pareto optimal set on the non-existence of optimal competitive equilibria, we establish the general existence of competitive equilibria that are in the closure of the set of Pareto optimal allocations. That is, even if a particular dynamic economy has no optimal competitive equilibria, we know that at least one of its equilibria is arbitrarily close to the collection of optimal allocations. In a sense, these equilibria are "approximately" optimal. The non-closedness of the set of Pareto optimal allocations prevents us from concluding that every economy has optimal competitive equilibria.

The potential non-optimality of competitive equilibria in dynamic economies makes the study of optimal governmental policies of particular importance. That is, since there may be no optimal competitive equilibria in a particular dynamic economy, governmental intervention that restores efficiency is clearly desirable. In contrast, it is ambiguous whether intervention is beneficial in a static economy since the purely competitive outcomes are already Pareto optimal.

In dealing with the issue of the existence of optimal public policies, Balasko, Cass, Shell [1980(a)] establish the second fundamental theorem of welfare economics for a fairly general class of overlapping-generations economies. That is, they show that any Pareto optimal allocation scheme can be supported as an equilibrium after a lump-sum redistribution of agents incomes or endowments. In our paper, we extend this welfare theorem to cover a much more general class of dynamic economies.

Given that any desirable outcome can be attained through lump-sum redistribution of income (which one can think of as a (long-term) monetary policy), we can ask if it is possible to move continuously from one optimal policy to another. That is, given that the government is currently implementing a particular optimal policy, if a new target allocation scheme is chosen, we ask if the government's policy can be smoothly changed to implement the new target while maintaining an optimal policy at each point in time. Formally, we ask if the class of optimal policies is arc-connected. This connectedness problem was first considered by Balasko and Shell [1981] for a class of overlapping generations economies. Unfortunately, their proof presupposes the closedness of the set of Pareto optimal allocations (which we refute in Corollary 7.6). In this paper, we manage to alter their proof and establish the arc-connectedness of optimal governmental policies for our general class of dynamic economies.

In this paper, we examine each of the afore mentioned properties of equilibria in the context of a dynamic economy. In section 2, we introduce our model with its definitions and assumptions. In section 3, we establish the general compactness of equilibrium prices and allocations. In section 4, we exploit our compactness result to establish the general existence of a competitive equilibrium. As previously mentioned, our existence result strengthens the results of

Balasko, Cass, Shell [1980(b)] and Wilson [1981] in that we impose a weaker (and much more natural) irreducibility assumption. In section 5, we discuss which of our assumptions are essential to our existence proof and which can be relaxed or eliminated. In section 6, we return again to our compactness result to establish the general existence of equilibria with outside debt (i.e. equilibria in which there are negative quantities of fiat money outstanding).

In section 7, we establish the general non-closedness of the set of Pareto optimal allocations and discuss the implications on the optimality of equilibria in dynamic models.

In section 8, we further explore the compactness of equilibrium prices and allocations and derive results that we exploit in the following sections. In section 9, we establish the second fundamental theorem of welfare economics for our general class of dynamic economies. Finally, in section 10, we establish the continuity of governmental monetary policy by supplying a proof of the arc-connectedness of the set of Pareto optimal allocations and monetary policies.

2. THE MODEL

Our model is a straightforward extension of a classical pure exchange economy (e.g. see McKensie [1981]) modified to accommodate an infinite number of agents and goods. Let A be the set of agents and I the set of goods. For each agent $a \in A$, let X_a be his consumption set, $(>)_a$ his preference relation over X_a, and w_a his endowment. An economy can then be specified as a collection $(X_a, (>)_a, w_a)_{a \in A}$.

Both the set of agents, A, and goods, I, will be indexed by the natural numbers. All subsets of R^{∞} will be endowed with the product topology (see Bourbaki [1966]) unless otherwise specified. In particular, the convergence of a sequence of points in R^{∞}, $\{ s_k \}$, will be taken to mean pointwise convergence, i.e. $s_k \rightarrow s$ is equivalent to $s_k^i \rightarrow s^i$ for each i.

The following is a list of assumptions that we employ throughout this paper. As previously mentioned, the necessity of these assumptions in guaranteeing the existence of equilibria is discussed in section 5.

Assumption I (Regularity of consumption spaces and the survival assumption)

For each $a \in A$:

(i) $X_a = R_+^{\infty}$,

(ii) $w_a \in X_a$.

Assumption II (Regularity of preferences)

For each $a \in A$; the preferences of agent a, $(>)_a$, are representable by a utility function, $u_a(\bullet)$, that satisfies:

(i) $u_a : X_a \rightarrow R$ is continuous,

(ii) $u_a(\bullet)$ is quasi-concave.

Assumption III (Free disposal)

For all $a \in A$ and $x \in X_a$; if $y \geq x$, then $y \, (\geq)_a \, x$.

We assume that the total endowment of each good is finite and define $w = \sum_{a \in A} w_a$ as the aggregate endowment of the economy.

Assumption IV (Positive aggregate endowment)

$w >> 0$.

We say that an agent $a \in A$ desires good $i \in I$ if, for some bundle $x_a \in X_a$ and scalar $\epsilon > 0$,

$(x_a^1, \ldots, x_a^i+\epsilon, \ldots) \, (>)_a \, (x_a^1, \ldots, x_a^i, \ldots)$.

For simplicity, we assume that each good is desired by at least one agent (this assumption is clearly non-restrictive). For later notational convenience, we redefine our indices, if necessary, so that good i is desired by some agent $a \leq i$.

Assumption V (Finite number of agents for each good)

For each good, there are only a finite number of agents that desire the good.

Note that assumption V is the only assumption imposed thus far that is not analogous to one of the assumptions typically imposed on a static economy. Assumption V should be innocuous for most interpretations of our model. That is, since there can be only a finite number of agents "alive" at any given point in time, when a particular good is available for consumption, the sub-collection of agents that have tastes for a good must also be finite. In section 5, we present an example that demonstrates the necessity of imposing an assumption like V to guarantee the existence of competitive equilibria. (The necessity of assumption V was overlooked in the existence theorem of Wilson [1981].)

An allocation scheme for the economy is a collection $(x_a)_{a \in A}$ such that $x_a \in X_a$ for each a. The set of all such schemes is denoted by $X = \bigtimes_{a \in A} X_a = X_1 \times X_2 \times \ldots$.
The sub-collection of feasible allocation schemes is defined $X^f = \{ x \in X : \sum_{a \in A} x_a = w \}$.

Assumption VI (Irreducibility)

For any partition of A into two non-empty subsets A_0 and A_1 and for any feasible allocation scheme $(x_a) \in X_f$; there is an agent $a \in A_0$ such that $x_a + \sum_{a \in A_1} w_a \, (>)_a \, x_a$.

As in static models, the irreducibility assumption is imposed to insure that each agent has income. (The positivity of income being required to insure that market demands behave continuously (see lemma 3.2).)

A price system for the economy is a vector $p \in R^{\infty}$, where p^i denotes the price of good i. If we interpret our model as being dynamic, then we think of prices expressed in terms of present discounted value. By Assumption III, we can restrict our attention to non-negative price vectors $p \in R_+^{\infty}$. Agent a's market demand correspondence $D_a(p,y)$ (given prices $p \in R_+^{\infty}$ and income $y > 0$) is defined to be the collection of all solutions to

$$\max u_a(x)$$
$$px = y$$
$$x \in X_a.$$

2.1 Definition.

(i) A Market Equilibrium consists of a price system $p \in R_+^{\infty}$ and a feasible allocation scheme $(x_a) \in X_f$ such that for each agent $a \in A$:

$$px_a < \infty$$
$$x_a \in D_a(p,px_a)$$

(ii) A Monetary Equilibrium is a Market Equilibrium where

$$pw_a < \infty \quad \text{for each agent } a \in A.$$

(iii) A Competitive Equilibrium is a Market Equilibrium where

$$px_a = pw_a \quad \text{for each agent } a \in A.$$

The market equilibria of an economy are those equilibria that can be attained through the open market after a lump-sum redistribution of endowments. The monetary equilibria are those that can be implemented by a lump-sum redistribution of income. The difference between the two equilibria arises when the value of some agents endowment is infinite, $pw_a = \infty$, since in this case an income transfer is not well-defined. We employ the familiar notion of competitive equilibria to be those equilibria that can be attained without any governmental intervention.

3. COMPACTNESS OF PRICES AND ALLOCATIONS

In this section, we establish compactness results that are later employed to establish the existence of competitive equilibria (propositions 4.3 and 4.4), the existence of debt equilibria (proposition 6.1), and the existence of competitive equilibria in the closure of the set of optimal allocations (proposition 7.2).

In order to guarantee the upper-hemi continuity of demand, we follow Debreu [1959] and (temporarily) bound consumption sets. Specifically, let us restrict consumption to be no greater than three times the aggregate endowment. Agent a's demand, $D_a'(p,y)$, is then the collection of all solutions to

$$\max u_a(x)$$
$$\text{s.t. } px = y$$
$$x \in X_a'$$

where the agents feasibility space is defined to be

$$X_a' = \{ x \in X_a : x \leqslant 3w \}.$$

The following lemma is an immediate consequence of assumption I(i) and the Tychonoff theorem (see Bourbaki [1966]).

3.1 Lemma. For each $a \in A$, X_a' is compact.

Our next lemma extends the well-known result, for static economies, that demand is upper hemi-continuous as long as income is positive.

3.2 Lemma. For each agent $a \in A$, $D'_a(\cdot,\cdot)$ is upper hemi-continuous at each point $(p,y) \in R^{\infty}_{+} \times R_{++}$.

Proof of Lemma 3.2.

Consider any sequence $\{(p^k,x^k)\}$ in $R^{\infty}_{+} \times X_a$ such that $x^k \in D'_a(p^k,p^kx^k)$ for each k, $(p^k,x^k) \longrightarrow (p,x)$, and $p^kx^k \longrightarrow y > 0$. We establish the desired continuity property by showing $x \in D'_a(p,y)$.

Since the components of p^k and x^k are non-negative for each k, one may verify that $px \leqslant \lim p^kx^k = y$.

Therefore, x is affordable given prices p. If x was not also optimal, i.e. $x \notin D'_a(p,y)$, then there is a $z \in X'_a$ such that $pz \leqslant y$ and

$z \,(>)_a\, x$.

By assumption II(i), there exists a t sufficiently large and a $\lambda < 1$ such that

(a) $\lambda z' \,(>)_a\, x$, where $z' = (z^1, \ldots, z^t)$.

$y > 0$ implies

(b) $p(\lambda z') < y$.

But, for large k, (a) together with assumption II(i) implies

$\lambda z' \,(>)_a\, x^k$

while (b) and the finiteness of z' imply

$p^k(\lambda z') < p^k x^k$.

But, this contradicts $x^k \in D'_a(p^k,p^kx^k)$.

Q.E.D.

The following corollary of lemma 3.2 is useful in establishing both our existence results of sections 4 and 6 and our results concerning the continuity of governmental policies.

3.3 Corollary. Consider any sequence $\{(p^k, x^k)\}$ in $R_+^{\infty} \times X_a$ such that $(p^k, x^k) \longrightarrow (p, x)$. If $x^k \in D_a(p^k, p^k x^k)$ for all k and if $px > 0$, then $\lim p^k x^k$ exists and $px = \lim p^k x^k$.

Proof of Corollary 3.3.

If either $\lim p^k x^k$ does not exist or it does exist and $px \neq \lim p^k x^k$, then, since $px \leqslant \liminf p^k x^k$, we can choose a subsequence such that $\lim p^k x^k$ exists and $px < \lim p^k x^k$ (where in general $\lim p^k x^k \leqslant \infty$).

If $\lim p^k x^k = \infty$, then by assumption VI, there exists a point $y \in X_a'$ such that $y \,(>)_a\, x$. By assumption II(i), there exists a t such that $(y^1, \ldots, y^t, 0, \ldots) \,(>)_a\, x$. But, this contradicts $x^k \in D_a'(p^k, p^k x^k)$, since for large k, $p^k(y^1, \ldots, y^t, 0, \ldots) < p^k x^k$. Therefore, $\lim p^k x^k < \infty$.

By lemma 3.2, $(p^k, p^k x^k) \rightarrow (p, \lim p^k x^k)$ and $\lim p^k x^k \geqslant px > 0$ implies $x^k \in D_a'(p^k, \lim p^k x^k)$. But, by the non-satiation implied by assumption VI, $x^k \in D_a'(p^k, \lim p^k x^k)$ implies $px = \lim p^k x^k$.

Q.E.D.

The following proposition allows us to establish the existence of competitive equilibria for our infinite economy by considering a sequence of equilibria of truncated (finite) sub-economies. Propositions like this are at the heart of virtually all existence proofs for dynamic economies (e.g. see Bewley [1972], Balasko, Cass, Shell [1980(b)], and Wilson [1981]).

3.4 Proposition. Consider any collection of positive scalars

$(s_a) \in R^{\infty}_{++}$ and any sequence of points in $R^{\infty}_{+} \times X$, $\{ (p^k,(x^k_a)) \}$, that satisfy the following for each k = 1, 2, ... and a = 1, ... , k:

(i) $p^k x_1 = 1$

(ii) $p^k x^k_a < \infty$

(iii) $x^k_a \in D'_a(p^k, p^k x^k_a)$

(iv) $(x^k_a)^i = 0$ unless agent a has tastes for good i

(v) $p^k x^k_a \geqslant s_a p^k w_a$

(vi) $\sum_{a=1}^{k} x^k_a \to w$ as $k \to \infty$.

Assuming that the above conditions are met, there is a limit point, $(p,(x_a))$, of the sequence. Furthermore, the limit point is a monetary equilibrium.

Proof of Proposition 3.4.

By Lemma 3.1 and the Cantor diagonalization process, we can restrict our attention to a subsequence such that $(x^k_a) \to (x_a)$. By assumption V and hypothesis (iv) and (vi), $(x_a) \in X_f$. (Note that without assumption V and hypothesis (iv), we could merely conclude $\sum_{a \in A} x_a \leqslant w$ since, in general $\sum_{a \in A} x_a \leqslant \liminf \sum_{a=1}^{k} x^k_a = w$.)

In order to guarantee that there is a further subsequence such that prices converge, we establish

(a) for each $a \in A$, there is a $B_a < \infty$ such that $p^k x^k_a < B_a$.

If (a) did not hold, then there is an agent $a \in A$ and a further subsequence such that $p^k x^k_a \to \infty$.

Let $A_0 = \{ a \in A : p^k x^k_a \to \infty \}$

and

$A_1 = \{ a \in A :$ for some $B_a < \infty, p^k x^k_a < B_a \}$.

By the Cantor diagonalization process, there exists a subsequence such that each $a \in A$ is in either A_0 or A_1. That is, A_0 and A_1 form a partition of A.

By our assumption that (a) does not hold, we know that $A_0 \neq \emptyset$.

By hypothesis (i), $1 \in A_1$ implies $A_1 \neq 0$.

Therefore, since $(x_a) \in X_f$, assumption VI implies that there is an agent $a \in A_0$ such that $x_a + \sum_{a \in A_1} w_a \, (>)_a \, x_a$.

By assumption II(i), there is a $\lambda < 1$ and a finite subset of A_1, A_1', such that

$\lambda x_a + \sum_{a \in A_1'} w_a \, (>)_a \, x_a$.

Again, by II(i), this implies

$\lambda x_a^k + \sum_{a \in A_1'} w_a \, (>)_a \, x_a^k$ for large k.

Clearly, $\lambda x_a^k + \sum_{a \in A_1} w_a \in X_a$ for large k, since $x_a \leqslant w$.

Therefore, hypothesis (iii) implies

$\sum_{a \in A_1} p^k w_a > (1-\lambda) p^k x_a^k$ for large k.

But, by the construction of A_1 and hypothesis (v),

$(1-\lambda) p^k x_a^k < \sum_{a \in A_1'} p^k w_a \leqslant \sum_{a \in A_1'} (1/s_a) p^k x_a^k < \sum_{a \in A_1'} B_a / s_a < \infty$.

But, this contradicts $a \in A_0$ since $p^k x_a^k \longrightarrow \infty$.

The above paragraph establishes (a). To bound prices, we note that $x \in X_f$ implies that for each i, there is an agent a such that $x_a^i > 0$. Hence, for large k,

$(x_a^k)^i > x_a^i/2 > 0$.

By (a), this implies $(p^k)^i x_a^i/2 < (p^k)^i (x_a^k)^i \leqslant p^k x_a^k < B_a$, which yields $(p^k)^i < 2B_a/x_a^i$ for all large k.

Therefore, the price sequence $\{ p^k \}$ is bounded.

Hence, by the Tychonoff theorem, we can further restrict our attention to a subsequence such that $p^k \longrightarrow p$.

To sum up our results thus far, we have established (a),

$(p^k,(x_a^k)) \longrightarrow (p,(x_a))$, and $(x_a) \in X_f$.

To show that the limit point, $(p,(x_a))$, is a monetary equilibrium, we need to establish the following three facts.

(b) $px_a < \infty$ for $a \in A$

(c) $x_a \in D_a(p,px_a)$ for $a \in A$

(d) $pw_a < \infty$ for $a \in A$.

Fix any $a \in A$.

Since each of the components of p^k, w_a, and x_a^k are non-negative, one may verify

$pw_a \leqslant \lim\inf p^k w_a$ and $px_a \leqslant \lim\inf p^k x_a^k$.

(b) now follows from (a) since $px_a \leqslant \lim\inf p^k x_a^k < B_a < \infty$.

Similarly, (d) follows from (a) and the hypothesis (v) since

$pw_a \leqslant \lim\inf p^k w_a \leqslant (1/s_a)\lim\inf p^k x_a^k \leqslant B_a/s_a < \infty$.

In order to verify (c), we employ

(e) $px_a > 0$ for $a \in A$.

We will not provide a proof of (e) here since its derivation would directly parallel our proof of (a). In particular, if (e) did not hold, then we can obtain a contradiction by applying assumption III to the sets $A_0 = \{ a \in A : px_a > 0 \}$ and $A_1 = \{ a \in A : px_a = 0 \}$. Given (e), (c) follows from corollary 3.3 (since $px_a = \lim p^k x_a^k$), (iii), and lemma 3.2.

Q.E.D.

The following corollary is employed to extend our result concerning the existence of competitive equilibria (proposition 4.3). To establish the existence of general debt equilibria (proposition 6.1) and the existence of competitive equilibria in the closure of the set of optimal allocations (proposition 7.2), we construct a sequence of economies in which endowments are purposefully redistributed. By applying our existence result (proposition 4.3) to the k'th perturbed economy, we know that there exists an equilibrium $(p^k,(x_a^k))$. By applying the following corollary, we obtain a limiting equilibrium for our full economy. The sequence of perturbed economies will be constructed so that the limiting equilibrium inherits the desired properties.

One may readily verify that the following corollary is an immediate consequence of lemma 3.2.

3.5 Corollary. For any collection of positive scalars $(s_a) \in R^{\infty}_{++}$, the collection of all monetary equilibrium such that

$px_1 = 1$

$x_a^i = 0$ unless agent a has tastes for good i

$px_a \geqslant s_a pw_a$ for all $a \in A$;

is a compact subset of $R^{\infty}_{+} \times X^f$.

In static models, the above compactness results (proposition 3.4 and corollary 3.5) are trivial consequences of the continuity of demand (lemma 3.2). This is due to the fact that, with only a finite number of goods, prices can be normalized to lie in a compact set (typically the unit simplex). Feasible allocation schemes also lie in a compact set since they are bounded by the aggregate endowment. Hence, to establish corollary 3.5, for example, one need only show that the specified collection of equilibria is closed, since it is already known that the set lies in a compact space. The closedness of the set follows directly from lemma 3.2 and assumption VI (VI insures that all incomes are positive).

4. GENERAL EXISTENCE OF EQUILIBRIA

In this section, we establish the general existence of competitive equilibria for dynamic economies. Following Balasko, Cass, Shell [1980(b)] and Wilson [1981], we find an equilibrium for our full economy as a limit point of equilibria from a sequence of truncated sub-economies. In order to guarantee the existence of equilibria for each of the truncated sub-economies, Balasko et al. require that each such economy is irreducible. In this paper, however, we refine their techniques and, thereby, eliminate the need for the sub-economy irreducibility assumptions. To avoid using these irreducibility assumptions (which are imposed to guarantee that each agent has a positive income), we simply perturb endowments in the truncated economies by adding positive quantities to each agents endowment of each good. Since each agent then has a strictly-positive (net) endowment of all goods, the positivity of income is guaranteed. The existence of equilibria for each truncated economy then follows from classical methods.

The following lemma establishes the existence of equilibria for each truncated sub-economy. In the k'th sub-economy, agent a ($a = 1, \ldots, k$) receives the extra endowment vector $(1/2)^a \epsilon_k$.

4.1 Lemma. For each $k = 1, 2, \ldots$ and $\epsilon_k \in R^{\infty}_{++}$ with $\epsilon_k \leqslant w$; there is a pair $(p^k, (x^k_a)) \in R^{\infty}_{+} \times X$ such that for $a = 1, \ldots, k$:

(i) $p^k x^k_a = p^k w_a + (1/2)^a p^k \epsilon_k < \infty$

(ii) $x^k_a \in D'_a(p^k, p^k x^k_a)$

(iii) For each $i = 1, \ldots, k$; $\sum_{a=1}^{k} ((x_a^k)^i - w_a^i) = (1-(1/2)^k)\varepsilon_k^i$.

Proof of Lemma 4.1.

For any $k = 2, 3, \ldots$, we restrict our attention to the following sub-economy, E^k, which involves only the first k agents and the first k goods.

For $a = 1, \ldots, k$; agent a's preferences are given by

$$u_a(x^1, \ldots, x^k) = u_a(x^1, \ldots, x^k, 3w^{k+1}, \ldots)$$

and his endowments are

$$w_a = (w_a^1, \ldots, w_a^k) + (1/2)^a(\varepsilon_k^1, \ldots, \varepsilon_k^k).$$

Note that, since $\varepsilon_k >> 0$, each of the agents has a positive endowment of all goods.

One may readily verify that assumptions I, II, III, and the non-satiation implied by VI, are sufficient to guarantee that the above sub-economy has a competitive equilibrium (e.g. see Debreu [1959] of Arrow and Hahn [1971]). We conclude our proof by completing this equilibrium into a pair $(p^k,(x_a^k))$ that satisfies (i) - (iii) of our lemma.

Let $(p^1, \ldots, p^k)$ be the equilibrium prices and $(x_a^1, \ldots, x_a^k)$ be the allocation to agent a.

The equilibrium properties of these prices and allocations can be written:

(a) $(p^1, \ldots, p^k) \geqslant 0$

(b) $(x_a^1, \ldots, x_a^k)$ solves

$$\max u_a(x^1, \ldots, x^k, 3w^{k+1}, \ldots)$$
$$\text{s.t.} \sum_{i=1}^{k} p^i x^i = \sum_{i=1}^{k} p^i w_a^i + \sum_{i=1}^{k} p^i \varepsilon_k^i$$
$$0 \leqslant x^i \leqslant 3w^i \text{ for } i = 1, \ldots, k;$$

(c) $\sum_{a=1}^{k} (x_a^i - w_a^i) = (1-(1/2)^k)\, \varepsilon_k^i$ for $i = 1, \ldots, k$.

We define the desired pair $(p^k,(x_a^k))$ by

$p^k = (p^1, \ldots, p^k, 0, \ldots)$

$x_a^k = (x_a^1, \ldots, x_a^k, 3w^{k+1}, \ldots)$ for $a = 1, \ldots, k$

and for completeness

$x_a^k = 0$ for $a = k+1, k+2, \ldots$.

One may readily verify that (a) - (c) imply that $(p^k,(x_a^k))$ satisfies (i) - (iii).

Q. E. D.

We now exploit our general compactness result, proposition 3.4, to find a competitive equilibrium for our full economy as a limit point of the sequence of equilibria, $\{ (p^k,(x_a^k)) \}$, described in lemma 4.1. The only subtlety in our proof lies in showing that we can choose a sequence of perturbations, $\{ \varepsilon_k \}$, that tend to zero fast enough so as not to effect each agents income in the limit, i.e. $p^k \varepsilon_k \to 0$.

Corollary 3.3 gives some insight as to why transfer payments may be required in the general equilibrium described in proposition 4.2 and why these payments are non-negative. Since $p^k \varepsilon_k \to 0$, lemma 4.1(i) implies $\lim p^k x_a^k = \lim p^k w_a$ while corollary 3.3 implies $px_a = \lim p^k x_a^k$. But, if agent a has a positive endowment of an infinite number of goods, then in general $pw_a \leqslant \lim p^k w_a$. That is, income, pw_a, is only a semi-continuous function of prices. We therefore have the inequality, $pw_a \leqslant px_a$.

4.2 Proposition. There exists a monetary equilibrium such that for each $a \in A$:

(i) $px_a \geq pw_a$

(ii) $px_a = pw_a$ if $w_a^i > 0$ for only a finite number of i.

Before we establish the above result, we present sufficient conditions under which competitive equilibria are guaranteed to exist. (Proposition 4.3 below corresponds to Theorem 2 in Wilson [1981].)

4.3 Proposition. If either:

(i) for each $a \in A$, $w_a^i > 0$ for only a finite number of i, or

(ii) there is a finite subset of agents $B \subseteq A$ and an $\epsilon > 0$ such that $\sum_{a \in B} w_a \geq \epsilon w$;

then a competitive equilibrium exists.

Proof of Proposition 4.3 (given Proposition 4.2).

In each case (i) or (ii), we simply show that the monetary equilibrium given in proposition 4.2 satisfies $px_a = pw_a$ for a - A (and is therefore a competitive equilibrium).

If (i) is satisfied, then $px_a = pw_a$ for $a \in A$ follows immediately from part (ii) of proposition 4.2.

Assume that (ii) is satisfied. By definition 2.1(ii), $px_a < \infty$ for $a \in B$. Hence, since B is finite, (ii) implies

(a) $pw \leq (1/\epsilon) \sum_{a \in B} pw_a < \infty$.

But, the market clearing constraint $\sum_{a \in A} x_a = \sum_{a \in A} w_a$, part (i) of proposition 4.2, and (a) imply

$px_a = pw_a$ for $a \in A$.

Q.E.D.

Proof of Proposition 4.2.

Let Q_{++} denote the collection of positive rational numbers. Since Q_{++} is countable, it follows that Q_{++} is also countable. Therefore, we can list its elements as

$Q^{\infty}_{++} = \{ q_1, q_2, \dots \}$.

We recursively define the desired sequence of perturbations $\{ \epsilon_k \}$ to be elements in Q^{∞}_{++}.

Let $\epsilon_0 = w$.

For $k = 1, 2, \dots$;

let $\epsilon_k = \min \{ (1/2)\epsilon_{k-1}, q_k \}$,

where for any vectors $u,v \in R^{\infty}$, $w = \min \{ u, v \}$ denotes the vector defined $w^t = \min \{ u^t, v^t \}$ for $t = 1, 2, \dots$.

For each k, let $(p^k,(x^k_a)) \in R^{\infty}_{+} \times X$ be the pair of equilibria from lemma 4.1 that correspond to ϵ_k. That is, for $a = 1, \dots , k$:

(a) $0 < p^k x^k_a = p^k w_a + (1/2)^a p^k \epsilon_k < \infty$;

(b) $x^k_a \in D'_a(p^k, p^k x^k_a)$;

(c) $\sum_{a=1}^{k} ((x^k_a)^i - w^i_a) = (1-(1/2)^k)\epsilon^i_k$.

We now verify that the sequence $\{ (p^k,(x^k_a)) \}$ satisfies the hypothesies of proposition 3.3. (i) follows by (a) if we suitably normalize prices. (ii) also follows directly from (a). (iii) corresponds to (b). Since for each good $i = 1, \dots , k$ there is some agent $a \leq k$ that has tastes for the good and since we have imposed our free disposal assumption III, we can trivially change allocations of free goods to insure that (iv) is satisfied. (v) follows by (a) since $p^k x^k_a \geq p^k w_a$. Finally, (vi) follows by (c) since $\epsilon_k \leq (1/2)^k \epsilon_1$ implies $\epsilon_k \rightarrow 0$.

Therefore, by proposition 3.3, there exists a monetary equilibrium

$(p,(x_a))$ such that, for some subsequence of our equilibria,
$(p^k,(x_a^k)) \longrightarrow (p,(x_a))$.

Before we can establish that $(p,(x_a))$ satisfies conditions (i) and (ii) of this proposition, we must guarantee

(d) $p^k \varepsilon_k \longrightarrow 0$.

Since $p^k \longrightarrow p$, there exists a $q \in R^\infty$ such that $p^k \leqslant q$ for all k. Clearly, we may assume $q \in Q^\infty_{++}$. Now consider the vector $q_t \in Q^\infty_{++}$ defined by $q_t = (1/q^1, (1/2)(1/q^2), (1/2)^2(1/q^3), \dots)$, where $q = (q^1, q^2, \dots)$.

Obviously, $q \cdot q_t = 2$.

But, by our definition of $\{ \varepsilon_k \}$, for all large $k > t$,
$p^k \varepsilon_k \leqslant q\, \varepsilon_k \leqslant (1/2)^{k-t} q\, \varepsilon_t \leqslant (1/2)^{k-t} q \cdot q_t = (1/2)^{k-t}$.

(d) immediately follows from the above inequality.

Recall that corollary 3.3 implies $px_a = \lim p^k x_a^k$. In contrast, the fact that all allocations and prices are non-negative only guarantees $pw_a \leqslant \lim\inf p^k w_a$. Part (i) of our proposition now follows from (a) and (b) since

$pw_a \leqslant \lim p^k w_a = \lim p^k x_a^k - (1/2)^a \lim \quad p^k \varepsilon_k = px_a$.

Similarly, part (ii) follows since if $w_a^i > 0$ for only a finite number of i, then $pw_a = \lim p^k w_a$.

Q.E.D.

5. EXTENTION OF EXISTENCE RESULTS

In this section, we discuss the directions in which our existence results (propositions 4.2 and 4.3) can be generalized. In particular, for each of the assumptions of section 2, we either provide a weaker alternative restriction or we discuss why it is unlikely that one can find a significantly weaker alternative.

Throughout this section, we compare our assumptions with those of some of the most general and well-known existence theorems. Specifically, we contrast our results with those of McKensie [1981], Balasko, Cass, Shell [1980(b)] and Wilson [1981]. Our comparison reveals that, in the special case when agents are finitely-lived, our assumptions can be weakened to the point that they are direct counterparts to the restrictions imposed by McKensie. In particular, we point out how our results extend those of Balasko, Cass, Shell [1980(b)] (for overlapping-generations economies) as well as those of Wilson [1981] by replacing their irreducibility assumptions with the significantly weaker version imposed by McKensie.

We also provide counter-examples to demonstrate the necessity of the three non-standard assumptions (I(i), V, and the finite endowment restriction of Proposition 4.3) to guarantee the existence of competitive equilibria for our general dynamic model.

If an agent is finitely-lived, then assumption I(i) can be weakened to require that consumption set are closed, convex, and bounded from below. However, if the agent has tastes for an infinite number of goods, then not only must we require that X_a is closed, convex, and bounded from below, but we must impose a "separability" assumption such as the following.

For each $a \in A$, there exists a T_a such that $x \in X_a$ implies $(x^1, \ldots, x^t, 0, \ldots) \in X_a$ for $t \geq T_a$.

A separability assumption like the one above is crucial to our technique of establishing the existence of an equilibrium for our full (infinite) economy by considering the limit point of a sequence of equilibria from truncated (finite) sub-economies. Intuitively, the separability assumption allows us to approximate any allocation $x \in X_a$ by a sequence of finite allocations, $\{ (x^1, \ldots, x^t, 0, \ldots) \}_{t=T_a}^{\infty}$, thus establishing a link between equilibria of the full economy and the equilibria of the truncated economies.

Not only is a separability assumption necessary to our proof techniques, but, as the following example demonstrates, one like it is necessary to guarantee the existence of equilibria. Specifically, the economy we present consists of a collection of finitely-lived agents together with one infinitely-lived agent. The finitely-lived agents each have the standard consumption set R_+^{∞} but the infinitely-lived agent does not. Although the particular consumption set of the infinitely-lived agent is closed, convex, and bounded from below and although the economy satisfies all of the requirements of section 2 (except I(i)), we are able to show that the economy has no equilibria of the type described in Proposition 4.2. That is, there are no monetary equilibria where the only monetary transfer is a non-negative one to the infinitely-lived agent.

5.1 Example. Let agent 1 have the consumption set

$X_1 = \{ (x_1^1, x_1^2, \ldots) \in R_+^\infty : x^{i+1} \geqslant 2x^i \text{ for } i = 2, 3, \ldots \}$

and give each of the other agents t = 2, 3, ... the set

$X_t = R_+^\infty$.

Let $w_1^i = 1$ if i = 1 and 0 otherwise and let $u_1(x_1) = x_1^1 + 2x_1^2$. For t = 2, 3, ...; let $w_t^i = 1$ if i = t-1, t and 0 otherwise and let $u_t(x_t) = x_t^{t-1} + (1/3)x_t^t$.

Assume that the above economy has a monetary equilibrium, $(p,(x_a))$, where the only monetary transfer is a non-negative transfer to agent 1, i.e. $px_1 \geqslant pw_1$. By definition, x_1 being in agent 1's consumption set implies $x_1^t \geqslant 2^{t-2}x_1^2$ for all t = 2, 3, ... Since the aggregate endowment of each good is bounded ($w^t = 2$), feasibility implies $x_1^2 = 0$.

Clearly, all prices are positive, $p \gg 0$, since each good is insatiably desired by some agent. In particular, utility maximization by agent 1 together with $x_1^2 = 0$ implies

$x_1^i = 0$ for i = 2, 3, ...

while utility maximization by agents t = 2, 3, ... imply

$x_t^i = 0$ for t = 2, 3, ... and i = t-1, t.

Hence, the agents budget constraints simplify to

(a) $p^{t-1}(x_t^{t-1} - 1) + p^t(x_t^t - 1) = 0$ for t = 2, 3, ...

and the feasibility constraints become

(b) $x_t^t + x_{t+1}^t = 2$ for t = 1, 2, ...

We now obtain a contradiction by pinning down the price system p and then showing that, given p, x_1 is not a utility maximizing choice for agent 1. Since agent 1 consumes only good 1, the fact that agent 1 spends at least the value of his endowment (i.e. receives a non-negative income transfer), implies $x_1^1 = px_1 \geq pw_1 = 1$. Using $x_1^1 \geq 1$ and repeated applications of (a) and (b), one may verify

(c) $1 \leq x_t^t \leq 2$ for $t = 2, 3, \ldots$.

Consider any $t = 2, 3, \ldots$.

If $p^t < (1/3)p^{t-1}$, then utility maximization by agent t implies $x_t^t > 4$, which contradicts (c).

If $p^t > (1/3)p^{t-1}$, then utility maximization implies $x_t^t = 0$, which also contradicts (c). Therefore, $p^t = (1/3)p^{t-1}$ for $t = 2, 3, \ldots$. Hence,

(d) $p^t = 3^{1-t}$ for $t = 2, 3, \ldots$.

We now show that x_1 cannot be a utility maximizing bundle for agent 1. Specifically, instead of the bundle $x_1 = (x_1^1, 0, 0, \ldots)$, the agent could have afforded the bundle $z_1 = (x_1^1-1, 1, 2, 2^2, \ldots)$ since by (d),

$$pz_1 = x_1^1-1 + \sum_{t=2}^{\infty} 3^{1-t}2^{t-2} = x_1^1 = px_1.$$

One may readily verify that z_1 is also feasible, i.e. $z_1 \epsilon X_1$, and is preferable to x_1, i.e. $u_1(z_1) = (x_1^1-1) + 2 > x_1^1 = u_1(x_1)$. Hence, agent 1's alleged choice of x_1 is not consistent with utility maximization. Therefore, the economy has no monetary equilibria as described in proposition 4.2.

As in static models, assumption I(ii), which states that each agent can survive without trade, can be eliminated after a suitable adjustment of our irreducibility assumption (see Moore [1975]). The survival assumption is merely imposed to simplify our techniques.

Specifically, $w_a \geqslant 0$ guarantees that if we perturb agent a's endowment by adding any positive quantity of each good (as in Lemma 4.2), then agent a's net endowment income will be strictly positive at any set of prices. It then follows by assumption II, that the agents market demand correspondence has the necessary continuity properties that allow us to employ finite-dimensional fixed-point theorems to establish the existence of the truncated equilibria of Lemma 4.2.

For agents that have tastes for only a finite number of goods, we claim that assumption II can be weakened to the level of generality considered by McKensie. Specifically, if we define the preferred to set

$P_a(x) = \{ z \in X_a : z (>)_a x \}$ for each $x \in X_a$,

then we can weaken assumption II to the requirement that

(i) $P_a(\circ)$ is open-valued (relative to X_a) and lower hemi-continuous, and

(ii) $x \notin$ convex-hull of $P_a(x)$ for each $x \in X_a$.

On the other hand, if an agent has tastes for an infinite number of goods, then our techniques require a stronger continuity condition of the form

(i)* the graph of $P_a(\cdot) = \{ (x,z) \in X_a \times X_a : z \in P_a(x) \}$ is open (relative to $X_a \times X_a$).

(It is unknown to us, at this time, if the general existence of equilibria can be guaranteed under the weaker condition (i).)

To see that our proof techniques only require (i)* and (ii), instead of assumption II, let us first verify that without loss of generality, one may impose the addition convexity condition

(iii) If $z \in P_a(x)$, then $\lambda z + (1-\lambda)x \in P_a(x)$ for $0 < \lambda \leqslant 1$. This condition entails no loss of generality since if we were given an economy that satisfies all of the assumptions of section 2 (with (i)* and (ii) replacing II), then we could simply expand the preference relations by considering the preferred to sets $P'_a(x) = \{ \lambda z + (1-\lambda)x : z \in P_a(x) \text{ and } 0 < \lambda \leqslant 1 \}$ for each $x \in X_a$. This new economy satisfies all of our assumptions (including (iii)). In addition, since the preference relation has been expanded, any equilibrium of the altered economy is an equilibrium of the original economy.

With the exception of the proof of the existence of the truncated equilibrium in Lemma 4.1, one may verify that (i)* and (iii) were the only restrictions on preferences that we employed in all of the proofs in sections 3 and 4. Finally, note that given (i)* and (iii), lemma 4.1 can be established by employing the existence theorem of McKensie [1981] directly on the truncated economies constructed in the proof of the lemma.

As in static models, the free disposal assumption III is only a convenience that allows us to restrict our attention to systems of non-negative price systems $p \in R^{\infty}_{+}$. In contrast, our assumption IV that the aggregate endowment is positive is necessary to insure that we can restrict our attention to finite prices.

The following example demonstrates the necessity of our assumption

V that each good can be desired by at most a finite number of agents. Specifically, the economy we present satisfies all of the requirements of section 2 except assumption V. Nevertheless, the economy does not satisfy the conclusions of propositions 4.2 or 4.3. That is, we demonstrate that, although each agent has a positive endowment of only a finite number of goods, there are no competitive equilibria.

5.2 Example. For notational convenience, we index the set of goods by $\{0, 1, \ldots\}$. For $t = 1, 2, \ldots$; let $w_t^i = 2^{-t}$ if $i = t-1$, t and $w_t^i = 0$ otherwise and let $u_t(x_t) = x_t^0 + 3^{-t}x_t^t$.

Assume that the above economy has a competitive equilibrium, $(p,(x_a))$. Since each good is insatiably desired by at least one agent, prices must be strictly positive, $p \gg 0$. Let $p^0 = 1$ be our price normalization. Since all prices are positive, utility maximization implies

$x_t^i = 0$ for all $t = 1, 2, \ldots$ and $i = t-1, t$.

Hence, we can write our budget constraints as

(a) $x_t^0 + p^t x_t^t = 2^{-t}(p^{t-1} + p^t)$ for $t = 1, 2, \ldots$

and our feasibility constraints simplifies to

(b) $\sum_{t=1}^{\infty} x_t^0 = w^0 = 2^{-1}$ for good 0

$x_t^t = w^t = 3 \cdot 2^{-t}$ for goods $t = 1, 2, \ldots$.

We can now finish our demonstration by pinning down the price system p and then contradicting (b) by showing that there is an excess supply of good 1. Specifically, we establish

(c) $p^t = 3^{-t}$ for $t = 0, 1, \ldots$,

by induction. Trivially, $p^0 = 1 = 3^{-0}$. We assume the inductive hypothesis $p^t = 3^{-t}$ (for some $t = 0, 1, \ldots$) and prove

$p^{t+1} = 3^{-(t+1)}$.

If $p^{t+1} < 3^{-(t+1)}$, then utility maximization by agent t+1 implies $x^0_{t+1} = 0$. Hence, the agents budget constraint from (a), together with the inductive hypothesis $p^t = 3^{-t}$, implies

$x^{t+1}_{t+1} = 2^{-(t+1)}(1 + p^t/p^{t+1}) > 3 \cdot 2^{-(t+1)}$,

which contradict (b).

If $p^{t+1} > 3^{-(t+1)}$, then utility maximization by agent t+1 implies $x^{t+1}_{t+1} = 0$, which again contradicts (b). Therefore, $p^{t+1} = 3^{-(t+1)}$ and our inductive proof is complete.

We can now determine x^0_t for t = 1, 2, ... By (a), (b), and (c);

$x^0_t = 2^{-t}(p^{t-1} + p^t) - p^t x^t_t = 2^{-t}(3^{1-t} + 3^{-t}) - 3^{-t}(3 \cdot 2^{-t})$

$= 2^{-t} \cdot 3^{-t} = 6^{-t}$.

Therefore, $\sum_{t=1}^{\infty} x^0_t = \sum_{t=1}^{\infty} 6^{-t} = 7^{-1} < 2^{-1}$,

which contradicts (b).

The irreducibility assumption VI is a direct analog of the irreducibility assumption imposed by McKensie. Our single requirement that the entire economy is irreducible replaces the assumptions made by Balasko, Cass, Shell, and Wilson that not only require that the entire economy be irreducible, but that there is a sequence of irreducible sub-economies, $\{ (X_a, (>)_a, w_a)_{a \in A_n} \}$, where for each n, A_n is a finite subset of A, $A_n \subseteq A_{n+1}$, and $\bigcup_n A_n = A$. The critical feature of our proof that allows us to drop these additional irreducibility assumptions is that, in establishing the existence of the equilibria for the finite sub-economies (lemma 4.1), we perturbed each agents endowment to insure that he always has a positive (net) income. In the theorems of Balasko, Cass, Shell and Wilson, the existence of equilibria for the sub-economies required the irreducibility of these economies to guarantee that each agent has a positive income in equilibrium.

To illustrate the greater generality that our techniques allow, we provide a simple example of an economy that satisfies all of our assumptions but does not satisfy the afore mentioned sub-economy irreducibility restrictions. In fact, in this example, we are able to demonstrate that there are no irreducible finite sub-economies that consist of more than one agent.

5.3 Example. Let $w_1^i = 1$ if $i = 1, 2$ and 0 otherwise and let $u_1(x_1) = x_1^1 + \max\{x_1^3, x_1^4\}$. For $t = 2, 3, \ldots$; let $w_t^i = 1$ if $i = t+1$ and 0 otherwise and let $u_t(x_t) = x_t^t + \max\{x_t^{t+2}, x_t^{t+3}\}$.

One may readily verify that this economy satisfies all of our assumptions in section 2. In particular, to see that the economy satisfies our irreducibility assumption VI, simply consider any partitioning of A into non-empty sets A_0 and A_1 and any allocation scheme $(x_a) \in X$.

If A_0 is finite, then let t be the largest index of an agent in A_0 (in particular; $t+1, t+2 \in A_1$). By our construction,

$$u_t(x_t + \sum_{t \in A_1} w_t) \geq u_t(x_t + w_{t+1} + w_{t+2}) = x_t^t + \max\{x_t^{t+2} + 1, x_t^{t+3} + 1\} \geq u_t(x_t) + 1 > u_t(x_t).$$

If A_0 is infinite, then since A_1 is non-empty, there exists a $t \in A_0$ such that $t-1 \in A_1$. Therefore,

$$u_t(x_t + \sum_{t \in A_1} w_t) \geq u_t(x_t + w_{t-1}) = x_t^t + 1 + \max\{x_t^{t+2}, x_t^{t+3}\}$$
$$= u_t(x_t) + 1 > u_t(x_t).$$

Hence, the full economy is irreducible.

Now consider any finite sub-economy $(X_a,(>)_a,w_a)_{a \in A*}$ that consists of more that one agent. That is, the number of agents in A* is greater that 1 but less than infinity. Let t be the highest element of A*. We can show that this sub-economy is reducible by considering the partitioning of A* into $A_1 = \{ t \}$ and $A_0 = A* - \{ t \}$ together with any feasible allocation scheme with the property $x_{t-2}^{t} = x_{t-2}^{t+1}$ and $x_{t-1}^{t+1} = x_{t-1}^{t+2}$. By our construction and the restriction $x_{t-1}^{t+1} = x_{t-1}^{t+2}$, $u_{t-1}(x_{t-1}+w_t) = x_{t-1}^{t-1} + \min \{ x_{t-1}^{t+1}+1, x_{t-1}^{t+2} \} = x_{t-1}^{t-1} + \min \{ x_{t-1}^{t+1}, x_{t-1}^{t+2} \} = u_{t-1}(x_{t-1})$.

Similarly, $x_{t-2}^{t} = x_{t-2}^{t+1}$ implies, $u_{t-2}(x_{t-2}+w_t) = u_{t-2}(x_{t-2})$.

But, none of the other agents a ($a < t-2$) can be made better off by receiving t's endowment since they do not have tastes for any goods dated later than t. Hence, the economy is reducible.

6. EXISTENCE OF MONETARY EQUILIBRIA

In this section, we employ our model to examine the well-studied question of how (or why) fiat money can have value. Most of the previous work in this area is concerned with establishing the existence of equilibria where agents receive positive lump sum transfers of fiat money (e.g. Cass and Yaari [1966], Gale [1973], or Samuelson [1958]). But, as our proposition 6.2 below points out, the existence of equilibria with positive transfers of money may not be very robust. In contrast, in proposition 6.3, we establish the general existence of debt equilibria. That is, equilibria with negative transfers (tax's) of fiat money.

6.1. Definition. A feasible allocation scheme $(x_a) \in X_f$ is Pareto Optimal if there are no other feasible schemes $(y_a) \in X_f$ such that $u_a(y_a) \geqslant u_a(x_a)$ for $a \in A$ with at least one strict inequality.

6.2 Proposition. If the initial endowment sequence (w_a) is Pareto optimal, then there are no monetary equilibria such that $px_a \geqslant pw_a$ for $a \in A$ with at least one strict inequality.

Proof of Proposition 6.2.

Assume that there is such an equilibrium, $(p,(x_a))$.

For each $a \in A$, since $px_a \geqslant pw_a$, utility maximization implies

(a) $u_a(x_a) \geqslant u_a(w_a)$.

By hypothesis, $px_{a_0} > pw_{a_0}$ for some $a_0 \in A$.

By assumption V, preferences are monotonic. Hence, utility maximization implies

(b) $u_{a_0}(x_{a_0}) > u_{a_0}(w_{a_0})$.

But, (a) and (b) imply that (w_a) is not a Pareto optimal allocation scheme since, in particular, it is dominated by (x_a).

Q.E.D.

6.3 Proposition. If each agent has a positive endowment of only a finite number of goods, then given any collection of scalars $(s_a) \in R^{\infty}$ such that $0 < s_a \leqslant 1$ for each $a \in A$; there exists a monetary equilibrium such that $px_a = s_a pw_a$ for each $a \in A$.

Proof of Proposition 6.3.

For k = 1, 2, ... ; we perturb the endowments of our original economy thus forming a new economy E^k. Specifically, the endowments in E^k are given by

$$w_a^k = s_a w_a \quad \text{for } a = 1, \dots, k-1$$
$$w_k^k = w_k + \sum_{a=1}^{k-1} (1-s_a) w_a \quad \text{for } a = k$$
$$w_a^k = w_a \quad \text{for } a = k+1, k+2, \dots$$

By inspection, one may verify that the economy E^k inherits all the properties of our original economy as specified in section 2. In addition, each agent still has a positive endowment of only a finite number of goods. Therefore, proposition 4.3 establishes the existence of a competitive equilibrium $(p^k,(x_a^k))$ for the economy E^k.

By definition 2.1(ii) and (iii), one may verify that $(p^k,(x_a^k))$ constitutes a monetary equilibrium for our original economy. By the definition of the economy E^k,

(a) $p^k x_a^k = s_a p^k w_a$ for $a = 1, \ldots, k-1$.

Technically, we restrict $(x_a^k)^i = 0$ unless agent a has tastes for good i. We also normalize prices so that $p^k x_1^k = 1$ for each k.

We now apply corollary 3.4 to obtain a subsequence of the "k-equilibria" and a (limiting) monetary equilibrium $(p,(x_a))$ such that $(p^k,(x_a^k)) \longrightarrow (p,(x_a))$.

Corollary 3.3 implies $px_a = \lim p^k x_a^k$ while the finiteness property of w_a implies $pw_a = \lim p^k w_a$.

Therefore, by (a), $px_a = s_a pw_a$ for $a \in A$.

Q.E.D.

7. NON-CLOSEDNESS OF THE PARETO OPTIMAL SET AND THE FAILURE OF THE FIRST FUNDAMENTAL WELFARE THEOREM

In this section, we examine the relationship between the non-closedness of the set of pareto optimal allocations and the existence of pareto optimal equilibria. It is well-known that the set of pareto optimal allocations is closed if an economy is static (i.e. finite). Clearly, this closedness is preserved in dynamic economies that are formed by simply concatenating a series of disjoint (and finite) collections of agents. For example, the pareto optimal set is closed in a dynamic economy where agents in a given generation do not interact with agents in any other generation. That is, there tastes and endowments do not intersect. In this section, we characterize a large class of economies that are sufficiently well-connected so as to insure that the pareto optimal set is not closed.

7.1 Lemma. Consider any market equilibrium $(p,(x_a))$. If $pw < \infty$, then the equilibrium allocation scheme, (x_a), is Pareto optimal.

Proof of Lemma 7.1.

Suppose that there was a feasible allocation scheme $(y_a) \in X_f$ such that $u_a(y_a) \geq u_a(x_a)$ for each $a \in A$ and $u_{a_0}(y_{a_0}) > u_{a_0}(x_{a_0})$ for some $a_0 \in A$.

We first show

(a) $py_a \geq px_a$ for $a \in A$.

If (a) did not hold, then since $pw < \infty$, $p(y_a+\lambda w) = px_a$ for some

$\lambda > 0$. But $u_a(y_a) \succsim u_a(x_a)$, together with assumptions II(ii), III, IV, and the non-satiation implied by VI; yields $u_a(y_a+\lambda w) > u_a(x_a)$. But this contradicts x_a being demanded by agent a since $y_a + \lambda w$ is affordable.

The above paragraph establishes (a). By similar arguments, one can conclude

(b) $py_{a_0} > px_{a_0}$.

To obtain a contradiction, we first note that $pw < \infty$, together with the feasibility conditions $\sum_a x_a = \sum_a y_a = w$, implies $\sum_a px_a = \sum_a py_a = pw < \infty$.
But, (a) and (b) imply $\sum_a py_a > \sum_a px_a$.

Q.E.D.

7.2 Proposition. If each agent has a finite endowment, then there is a competitive equilibrium, $(p,(x_a))$, such that the equilibrium allocation scheme, (x_a), is in the closure of the set of Pareto optimal allocation schemes. That is, there is a sequence of Pareto optimal allocation schemes, $\{ (x_a^k) \}_{k=1}^{\infty}$, such that $(x_a^k) \rightarrow (x_a)$ as $k \rightarrow \infty$.

Proof of Proposition 7.2.

As in the proof of proposition 6.3, we find the desired equilibrium as a limit point of equilibria from a sequence of perturbed economies. For our present purposes, we define the economy E^k by changing the endowments in our original economy to

$$w_a^k = w_a \qquad \text{for } a = 1, \ldots, k-1$$
$$w_k^k = w_k + (1/2) \sum_{a=k+1}^{\infty} w_a \qquad \text{for } a = k$$

$w_a^k = (1/2)w_a$ for $a = k+1, k+2, \ldots$.

Again the economy inherits all of the properties of our original economy. Agents 1, ... , k collectively hold more than a fraction (1/2) of all endowments. Hence, proposition 4.3 establishes the existence of a competitive equilibrium $(p^k,(x_a^k))$ for the economy E^k.

We interpret the competitive equilibrium $(p^k,(x_a^k))$ for the economy E^k to be a monetary equilibrium for our original economy.

By the definition of E^k,

(a) $p^k x_a^k = p^k w_a$ for $a = 1, \ldots , k-1$

and

(b) $p^k x_k^k = p^k w_k + (1/2) \sum_{a=k+1}^{\infty} p^k w_a$ for $a = k$.

But, by definition 2.1(ii), $p^k x_a^k < \infty$ for $a \in A$.

In particular, $p^k x_k^k < \infty$. Hence, (b) implies $p^k w < \infty$.

Therefore, lemma 7.1 implies that the allocation scheme (x_a^k) is pareto optimal.

After suitable normalization, as in the proof of proposition 6.3, we can apply corollary 3.4 to obtain a subsequence of the "k-equilibria" and a monetary equilibrium $(p,(x_a))$ such that $(p^k,(x_a^k)) \rightarrow (p,(x_a))$.

Again we argue that $px_a = \lim p^k x_a^k$ and $pw_a = \lim p^k w_a$. Hence, (a) implies $px_a = pw_a$ for $a \in A$. Therefore, $(p,(x_a))$ is a competitive equilibrium.

Q.E.D.

A feasible allocation scheme $(x_a) \in X_f$ is said to be Irreducible if, for any partitioning of A into two non-empty subsets A_0 and A_1, there is an agent $a \in A_0$ such that $x_a + \sum_{a \in A_1} x_a \ (>)_a \ x_a$.

An economy is said to satisfy the Irreducible Allocations assumption if all feasible allocation schemes $(x_a) \in X_f$ such that $u_a(x_a) > u_a(0)$ for $a \in A$, are irreducible.

Consider any feasible allocation scheme $(x_a) \in X_f$. We say that agent a_0 can always benefit at a_1's expense if for any $\varepsilon > 0$, there exists a feasible allocation scheme $(y_a) \in X_f$ such that

$$u_a(y_a) \geqslant u_a(x_a) \text{ for all } a \neq a_1$$
$$u_{a_0}(y_{a_0}) > u_{a_0}(x_{a_0}) \text{ for } a = a_0.$$
$$u_{a_1}(y_{a_1}) > u_{a_1}(x_{a_1}) - \varepsilon \text{ for } a = a_1$$

7.3 Lemma. If the irreducible allocations assumption is satisfied, then given any feasible allocation scheme $(x_a) \in X_f$ such that $u_a(x_a) > u_a(0)$ for $a \in A$:
Each agent a_0 can benefit at each other agents $(a_1 \in A-\{a_0\})$ expense.

Proof of Lemma 7.3.

Consider any allocation scheme $(x_a) \in X_f$ such that $x_a \,(>)_a\, 0$ for $a \in A$. Assume that the implications of this lemma are not satisfied. That is, there are agents a and a_1 $(a \neq a_1)$ such that a cannot benefit at a_1's expence.

We partition agents into the two sets $A_0 = \{ a \in A-\{a_1\}$: agent a cannot benefit at a_1's expense $\}$ and $A_1 = A - A_0$.
By construction, both A_0 and A_1 are non-empty.
Hence, by our irreducibility hypothesis, there exists an agent $a_0 \in A_0$ such that

(a) $x_{a_0} + \sum_{a \in A_1} x_a \,(>)_{a_0}\, x_{a_0}$.

Consider any $\epsilon > 0$. By the definition of A_1, for each $\tilde{a} \in A_1' - \{a_1\}$, there exists a feasible allocation scheme $(y_a^{\tilde{a}}) \in X_f$ such that

$$y_a^{\tilde{a}} (>)_a x_a \quad \text{for all } a \neq a_1$$
$$y_{\tilde{a}}^{\tilde{a}} (>)_{\tilde{a}} x_{\tilde{a}} \quad \text{for } a = a$$
$$u_{a_1} (y_{a_1}^{\tilde{a}}) > u_{a_1}(x_{a_1})) - \epsilon \quad \text{for } a = a_1.$$

Now consider any convex combination (y_a) of these schemes, i.e.

$$(y_a) = \sum_{\tilde{a} \in A_1' - \{a_1\}} \lambda_{\tilde{a}} (y_a^{\tilde{a}}),$$

where the $\lambda_{\tilde{a}}$'s are positive weights that sum to 1.

Clearly, $(y_a) \in X_f$.

Our concavity assumption II(ii) implies,

$$\text{(c)} \quad u_a(y_a) \gtrsim u_a(x_a) \quad \text{for all } a \neq a_1$$
$$u_a(y_a) > u_a(x_a) \quad \text{for } a \in A_1 - \{a_1\}$$
$$u_{a_1}(y_{a_1}) > u_{a_1}(x_{a_1}) - \hat{\epsilon} \quad \text{for } a = a_1.$$

For $\lambda < 1/2$, consider the allocation scheme (z_a) defined by

$$z_{a_0} = (1/2)x_{a_0} + (1/2)y_{a_0} + (1/2 - \lambda) \sum_{a \in A_1'} x_a \quad \text{for } a = a_0$$
$$z_a = (1/2)x_a + (1/2)y_a \quad \text{for } a \neq a_0, a_1 \text{ and } a \notin A_1'$$
$$z_a = \lambda x_a + (1/2)y_a \quad \text{for } a \in A_1' - \{a_1\}$$
$$z_{a_1} = \lambda x_{a_1} + (1/2)y_{a_1} \quad \text{for } a = a_1.$$

Clearly, $(x_a),(y_a) \in X_f$ imply $(z_a) \in X_f$.

Now (c), together with assumption II, implies that for some sufficiently close to 1/2,

$$u_a(z_a) \gtrsim u_a(x_a) \text{ for all } a \neq a_1 \text{ and } a \notin A_1$$
$$u_a(z_a) > u_a(x_a) \text{ for } a \in A_1 - \{a_1\}$$
$$u_{a_1}(z_{a_1}) > u_{a_1}(x_{a_1}) - \epsilon \text{ for } a = a_1.$$

However, again by assumption II(ii), (b) implies

$$u_{a_0}(z_{a_0}) > u_{a_0}(x_{a_0}) \quad \text{for } a = a_0.$$

But, this means that agent a_0 can benefit at the expense of agent a_1 (in contradiction to $a_0 \in A_0$).

Q.E.D.

Let U_1 be the collection of all utility vectors $(u_2, u_3, \ldots) \in R^{\infty}$ such that for some feasible allocation scheme $(x_a) \in X_f$,

$$u_1(x_1) > u_1(0)$$

$$u_a(x_a) \geqslant u_a > u_a(0) \text{ for } a = 2, 3, \ldots .$$

For $u \in U_1$, define $x(u)$ to be the collection of all solutions to

$$\max\ u_1(x_1)$$

$$\text{s.t. } x \in \phi(u),$$

where

$$\phi(u) = \{\ x \in X_f : u_a(x_a) \geqslant u_a \quad \text{for } a = 2, 3, \ldots \}.$$

The range of the correspondence $x(\cdot)$ is defined to be the collection of all x such that $x \in x(u)$ for some $u \in U_1$.

Let P denote the collection of all Pareto optimal allocation schemes (x_a) such that $u_a(x_a) > u_a(0)$ for $a \in A$.

7.4 Lemma. P = the range of $x(u)$.

Proof of Lemma 7.4.

It immediately follows, by inspection, that $P \subseteq$ the range of $x(\cdot)$.

Consider any $x \in x(u)$ for some $u \in U_1$. By construction,

(a) $u_1(x) > u_1(0)$ and $u_a(x_a) \geqslant u_a > u_a(0)$ for $a = 2, 3, \ldots$.

If x is not pareto optimal, then there is a feasible allocation scheme

$(y_a) \in X_f$ such that

(b) $u_a(y_a) \geq u_a(x_a)$ for $a \in A$ with $u_{a_1}(y_{a_1}) > u_{a_1}(x_{a_1})$ for some $a_1 \in A$.

(a) and (b) imply $u_a(y_a) > u_a(0)$ for $a \in A$.

Hence, we can apply lemma 7.3 to agents $a_0 = 1$ and a_1 with $\varepsilon = u_{a_1}(y_{a_1}) - u_{a_1}(x_{a_1}) > 0$ to obtain a feasible allocation $(z_a) \in X_f$ such that

(c) $u_a(z_a) \geq u_a(y_a)$ for all $a \neq a_1$

$u_1(z_1) > u_1(y_1)$ for $a = 1$

$u_{a_1}(z_{a_1}) > u_{a_1}(y_{a_1}) - \varepsilon$ for $a = a_1$.

(a), (b), (c) and the definition of ε then imply

$$u_a(z_a) \geq u_a(x_a) \geq u_a \text{ for } a \in A-\{1\}$$

and

$$u_1(z_1) > u_1(x_1) \text{ for } a = 1.$$

But, this contradicts $x \in x(u)$.

Q.E.D.

7.5 Proposition. If the irreducible allocations assumption is satisfied, then the correspondence $x(\circ)$ is not upper hemi-continuous at any point in its domain.

Proof of Proposition 7.5.

We show that $x(\circ)$ is discontinuous at each $u \in U_1$ by proving that $u_1(x_1(\circ))$ is discontinuous at u. Specifically, we construct a sequence $\{ u^a \}$ in U_1 converging to u, such that

$u_1(x_1(u^a)) = b$ for all a,

where $b = u_1((1/2)x) < u_1(x_1(u))$ for some $x \in x_1(u)$

(The strict inequality follows from the quasi-concavity of $u_1(\bullet)$ and the restriction $u_1(x_1(u)) > u_1(0)$.)

For a = 1, 2, ... ,

let $B_a = \{ \tilde{u}_a \geq u_a : (u_2, \ldots, \tilde{u}_a, \ldots) \in U_1$ and $u_1(x_1(u_2, \ldots, \tilde{u}_a, \ldots)) \geq b \}$.

One may verify that the continuity of preferences implies that B_a is closed. B_a is also non-empty (since $u_a \in B_a$) and is obviously bounded. Therefore, we can define u_a^* to be the maximal element of B_a.

We employ lemma 7.3 to argue that $u_1(x_1(u_2, \ldots, u_a^*, \ldots)) = b$, since otherwise we could allow agent a* to become better off at the expense of agent 1 and therefore generate a utility level, u_a', that is higher than u_a^*. But, if the loss of agent 1 was small enough (maintaining $u_1(\bullet) > b$), then $u_a' \in B_a$, which contradicts u_a^* being maximal.

The desired sequence of utility vectors can then be defined by

$u^a = (u_2, \ldots, u_a^*, \ldots)$ for $a \in A$.

(Clearly, u^a converges to $u = (u_1, \ldots)$ pointwise.)

Q.E.D.

7.6 Corollary. If the irreducible allocations assumption is satisfied, then the set of Pareto optimal allocations is not closed.

Proof of Corollary 7.6.

Consider any point $u \in U_1$. By proposition 7.5, there exists a sequence $\{ (x^k,u^k) \}$ in $X_f \times U_1$ such that $(x^k,u^k) \longrightarrow (x,u)$ and $x^k \in x(u^k)$ for each k but $x \notin x(u)$.

It follows from assumption II(ii) that $\phi(\bullet)$ is a continuous correspondence. In particular, $x \in \phi(u)$. Hence, the only way that we can have $x \notin x(u)$ is that x does not maximize agent 1's utility subject to all other utilities constant. In other words, x is not pareto optimal. But, by lemma 7.4, each of the allocations x^k is optimal.

Q.E.D.

We illustrate the non-closedness of the pareto optimal set and its relation to the non-existence of pareto optimal competitive equilibria by considering a simple overlapping generations economy of the type considered in Samuelson [1958].

7.7 Example.

Let $u_1(x) = x_1^1$ and for a = 2, 3, ... , let $u_a(x) = x_a^{a-1} + x_a^a$. Let $w_1^i = 1$ if i = 1 and 0 otherwise and let $w_a^i = 1$ if i = a-1 or a and 0 otherwise.

One may readily verify that the only competitive equilibrium for the above economy is autarky. That is, the only equilibrium allocation is

$x = (w_1, w_2, \dots)$

$= ((1,0,\dots), (1,1,0,\dots), (0,1,1,0,\dots), (0,0,1,1,\dots), \dots)$. We note that x is not pareto optimal since, in particular, it is dominated by the scheme

$x(0) = ((2,0,\dots), (0,2,0,\dots), (0,0,2,\dots), (0,0,0,2,\dots), \dots)$.

One may readily verify the x(0) is optimal as are each of the schemes { x(k) } given by

$x(1) = ((1,0,\dots), (1,2,0,\dots), (0,0,2,\dots), (0,0,0,2,\dots), \dots)$

$x(2) = ((1,0,\ldots), (1,1,0,\ldots), (0,1,2,\ldots), (0,0,0,2,\ldots), \ldots)$

$x(3) = ((1,0,\ldots), (1,1,0,\ldots), (0,1,1,\ldots), (0,0,1,2,\ldots), \ldots)$

etc. .

Our demonstration is therefore complete since $x(k) \longrightarrow x$ as $k \longrightarrow \infty$.

Q. E. D.

8. FURTHUR COMPACTNESS RESULTS

8.1 Proposition. Consider any collection of utility levels $(u_a) \in R^{\infty}$ such that $u_a > u_a(0)$ for $a \in A$ and any sequence of points in $R_+^{\infty} \times X_f$, $\{ (p^k,(x_a^k)) \}$, that satisfy the following for k = 1, 2, ... and a = 1, ... , k:

$$p^k x_1^k = 1$$

$$u_a(x_a^k) \geqslant u_a$$

$$p^k x_a^k < \infty$$

$$(p^k)^i = 0 \quad \text{for } i = k+1, k+2, \ldots$$

$$x_a^k \text{ solves } \max u_a(x)$$

$$\text{s.t. } p^k x \leqslant p^k x_a^k$$

$$x^i = (x_a^k)^i \text{ for } i = k+1, k+2, \ldots$$

$$x \in X_a$$

$$(x_a^k)^i = 0 \text{ unless agent a has tastes for good i.}$$

If, in addition to the above conditions, the irreducible allocations assumption is also satisfied, then there is a limit point, $(p,(x_a))$, of the sequence $\{ (p^k,(x_a^k)) \}$. Furthermore, the limit point is a market equilibrium.

Proof of Proposition 8.1.

By the Tychonoff theorem, X_f is compact. Hence, there exists a subsequence such that $(x_a^k) \longrightarrow (x_a)$ for some limiting allocation scheme $(x_a) \in X_f$.

We argue that for each $a \in A$, there exists a bound $B_a < \infty$ such that $p^k x_a^k < B_a$ for all k. Otherwise, by the Cantor diagonalization process, we can partition A into two non-empty sets A_0 and A_1 such that for some subsequence

$A_0 = \{ a \in A : p^k x_a^k \to \infty \}$

and

$A_1 = \{ a \in A :$ there is a $B_a < \infty$ such that $p^k x_a^k < B_a \}$.

We can obtain a contradiction by employing the irreducible allocations hypothesis to the limiting allocation x (since $u_a(x_a) = \lim u_a(x_a^k) \geq u_a > u_a(0)$ for $a \in A$) to conclude that there is an agent $a \in A$ such that

$$x_a + \sum_{a \in A_1} x_a \; (>)_a \; x_a.$$

By assumption II(ii), we can find a $\lambda < 1$, a finite subset (A_1') of A_1, and a large t such that

$(y^1, \ldots, y^t, 0, \ldots) \; (>)_a \; x_a$,

where $y = \lambda x_a + \sum_{a \in A_1'} x_a$.

Again by assumption II(ii),

$z^k = (y^1, \ldots, y^t, 0, \ldots, (x_a^k)^{k+1}, \ldots) \; (>)_a \; x_a^k$ for all large k.

But, by the definition of A_0, A_1, z^k, and y,

$p^k z^k < p^k x_a^k$ for large k, which contradicts our hypothesis of the (limited) optimality of x_a^k.

The remainder of the proof closely parallels the proof of proposition 3.4 and, therefore, will not be repeated here.

Q.E.D.

8.2 Corollary. For any collection of scalars $(u_a) \in R^\infty$ such that $u_a > u_a(0)$ for $a \in A$, the collection of all market equilibria such that

$px_1 = 1$

$u_a(x_a) \geq u_a$ for $a \in A$;

is a compact subset of $R_+^{\infty} \times X_f$.

9. THE SECOND FUNDAMENTAL WELFARE THEOREM

In this section, we establish the broad scope of governmental tax and transfer policies by deriving the second fundamental welfare theorem for our general class of dynamic economies. Note, in proposition 9.1 below, we can only guarantee that a given outcome is achieved through a transfer of endowments (as opposed to a transfer of income). The sufficiency of a pure monetary (income) policy can only be assured when each agent has a finite number of endowments, since we must guarantee $pw_a < \infty$.

9.1 Proposition (Second fundamental theorem of welfare economics). If the irreducible allocations assumption is satisfied, then any Pareto optimal allocation (x_a), such that $u_a(x_a) > u_a(0)$ for $a \in A$, can be supported as a competitive equilibrium given a suitable redistribution of endowments. Specifically, there is a price system $p \in R_+^\infty$ such that $(p,(x_a))$ is an (autarkic) competitive equilibrium if endowments are given by $w_a = x_a$ for $a \in A$.

Proof of Proposition 9.1.

Consider any such pareto optimal allocation scheme $(x_a) \in X_f$. For each k, we define the following sub-economy consisting of the first k agents and goods. Agent a's tastes are given by

$u_a(x^1, \ldots , x^k) = u_a(x^1, \ldots , x^k, x_a^{k+1}, \ldots)$

and his endowments are

$w_a = (w_a^1, \ldots , w_a^k)$.

Since (x_a) is a pareto optimal allocation for our full economy, the scheme $((x_a^1, \ldots, x_a^k))_{a=1}^{k}$ is pareto optimal for the k'th sub-economy. Therefore, by the classical (finite-dimensional) version of the second welfare theorem (see Arrow [19]), there exists prices $(p^1, \ldots, p^k)$ such that

$$(x_a^1, \ldots, x_a^k) \text{ solves } \max u_a(x)$$
$$\text{s.t. } px \leqslant px_a$$
$$x \in R_+^k.$$

It then follows that if we expand $(p^1, \ldots, p^k)$ into a full price system by defining $p^k = (p^1, \ldots, p^k, 0, \ldots)$ for each k, then the sequence $\{ (p^k,(x_a)) \}$ satisfies the hypothesis of proposition 8.1. Therefore, there exists a market allocation $(p,(y_a))$ such that $(p^k,(x_a)) \rightarrow (p,(y_a))$. Trivially, $(y_a) = (x_a)$. It follows by definition 3.1 that if $(w_a) = (x_a)$, then $(p,(x_a))$ is a competitive equilibrium.

Q.E.D.

10. ARC-CONNECTEDNESS OF THE PARETO OPTIMAL SET AND THE CONTINUITY OF MONETARY POLICY

10.1 Definition. The basis of the From Below (FB) topology, defined on the space $u_1 \subseteq R^{\infty}$, is taken to be all subsets of the form

$$[(a_2,b_2) \times \dots \times (a_n,b_n) \times (-\infty,b_{n+1}) \times (-\infty,b_{n+2}) \quad \dots] \cap U_1$$

for any real numbers $a_2, \dots, a_n, b_2, b_3, \dots$ and $n = 2, 3, \dots$. One may readily verify that the above collection of sets does indeed constitute a basis. Also, the topology is first countable since we could have alternatively defined a countable basis by restricting the coefficients $a_2, \dots, a_n, b_2, b_3, \dots$ to be rational numbers.

10.2 Lemma. A sequence $\{ u^k \}$ in U_1 converges to a point $u \in U_1$ if, and only if,

(i) $u_a^k \rightarrow u_a$ for $a \in A-\{1\}$ (i.e. the sequence converges pointwise) and

(ii) there an a^* such that for large k, $u_a^k < u_a$ for $a > a^*$.

Proof of Lemma 10.2.

(Sufficiency of (i) and (ii)) Consider any such sequence $\{ u^k \}$, point u, and constant N that satisfy (i) and (ii). To show that $u^k \rightarrow u$, we must demonstrate that for any open neighborhood O of u, $u^k \in O$ for large k. Clearly, we can restrict our attention to neighborhoods in the previously described basis of the FB topology, i.e.

$$O = [(a_2,b_2) \times \dots \times (a_n,b_n) \times (-\infty,b_{n+1}) \quad \dots] \cap U_1.$$

For notational convenience (and without loss of generality), assume

$n = a^*$.

For the set O given above, $u \in O$ implies

(a) $u_a \in (a_a, b_a)$ for $a = 1, \ldots, a^*$

and

(b) $u_a \in (-\infty, b_a)$ for $a = a^*+1, a^*+2, \ldots$.

Hypothesis (i), together with (a) implies

(c) there is a K_0 such that $u_a^k \in (a_a, b_a)$ for $a = 1, \ldots, a^*$ and $k > K_0$.

Our second hypothesis (ii), together with (b), implies

(d) there is a K_1 such that $u_a^k \in (-\infty, b_a)$ for $a = a^*+1, \ldots$ and $k > K_1$.

Therefore, by the construction of O, $u^k \in O$ for all $k > K = \max \{ K_0, K_1 \}$.

(Necessity of (i)) The necessity of (i) is immediate since for any a and $\epsilon > 0$, the set

$O = [(u_2-\epsilon, u_2+\epsilon) \times \ldots \times (u_a-\epsilon, u_a+\epsilon) \times (-\infty, u_{a+1}+\epsilon) \times \ldots\ ] \cap U_1$, which contains u, is in the basis of the FB topology. Hence, $u^k \longrightarrow u$ implies $u^k \in O$ for large k. In particular, $u_a^k \in (u_a-\epsilon, u_a+\epsilon)$ for large k. Therefore, $u_a^k \rightarrow u_a$.

We will not demonstrate the necessity of (ii) since this result will not be employed in this paper.

Q.E.D.

10.3 Lemma. U_1 is arc-connected when it is endowed with the FB topology.

Proof of Lemma 10.3.

Since for any two points $v,w \in U_1$, there exists a point u in U_1 such that $u \leqslant v,w$; it is sufficient to show that any $u,v \in U_1$ can be connected by a continuous path in U_1 if $u \leqslant v$.

Fix any increasing sequence of numbers $\{ c_n \}$ such that $c_1 = 0$ and $c_n \rightarrow 1$ as $t \rightarrow \infty$.

Define a path connecting u to v by

$$g(t) = \begin{cases} (v_1,\ldots,\lambda v_n+(1-\lambda)u_n,\ldots), \text{ if } t = \lambda c_{n+1}+(1-\lambda)c_n, \text{ where } 0<\lambda<1 \\ v, \text{ if } t = 1. \end{cases}$$

The range of $g(\bullet)$ is contained in U_1 since for any $t \in [0,1]$; $u \leqslant g(t) \leqslant v$ implies that $g(t) \in U_1$ since $u,v \in U_1$.

We complete the proof by showing that the map $g(\bullet)$ is continuous. Since the FB topology is first countable, the continuity of $g(\bullet)$ can be expressed as follows.

For any sequence $\{ t_k \}$ in $[0,1]$, $t_k \longrightarrow t$ implies $g(t_k) \longrightarrow g(t)$.

Since each $g_a(\bullet)$ is, by construction, piecewise linear (hence continuous), $t_k \longrightarrow t$ implies $g_a(t_k) \longrightarrow g_a(t)$. Hence, the convergence criterion (i) of lemma 10.2 is satisfied. To conclude the proof, we establish criterion (ii) in two cases.

If $t \in [0,1)$, then $t < c_{n+1}$ for some n. Hence, $t_k < c_{n+1}$ for large k. But, by the construction of $g(\bullet)$; $t, t_k < c_{n+1}$ implies $g_a(t_k) = u_a = g_a(t)$ for $a = n+1, n+2, \ldots$. The criterion (ii) of lemma 10.2 now follows.

If $t = 1$, then since $u \leqslant v$, the construction of $g(\bullet)$ implies $g_a(t_k) \leqslant v_a = g_a(t)$ for $a \in A-\{1\}$. Again, (ii) is satisfied.

Q.E.D.

10.4 Proposition. If the irreducible allocations assumption is satisfied and if U_1 is endowed with the FB topology, then $x(\cdot)$: $U_1 \longrightarrow X_f$ is upper hemi-continuous.

Proof of Proposition 10.4.

Since the FB topology is first countable and the range of $x(\cdot)$ is contained in a compact space (X_f), the continuity of $x(\cdot)$ can be characterized as follows.

For any sequence $\{ (u^k,(x_a^k) \}$ in $U_1 \times X_f$ such that $(u^k,(x_a^k) \longrightarrow (u,(x_a)) \in U_1 \times X_f$, $x^k \in x(u^k)$ for all k, implies $x \in x(u)$.

For each k, $x^k \in x(u^k) \subseteq \phi(u^k)$ implies $u_a(x_a^k) \geq u_a^k$ for $a \in A-\{1\}$. By lemma 10.2, $u^k \longrightarrow u$ implies $u_a^k \longrightarrow u_a$ for $a \in A-\{1\}$. Hence, $(x_a^k) \longrightarrow (x_a)$, together with assumption II(ii), implies $u_a(x_a) \geq u_a$ for $a \in A-\{1\}$. In addition, the closedness of X_f implies $(x_a) \in X_f$. Therefore, $(x_a) \in \phi(u)$.

Since $x \in \phi(u)$, if $x \notin x(u)$, then there is some $y \in \phi(u)$ such that (a) $u_1(y_1) > u_1(x_1)$. By repeated applications of Lemma 7.3, all of the agents $a \in A-\{1\}$ can be made better off at the expense of agent 1. Specifically, for any $\epsilon > 0$, there exists a feasible allocation $(z_a) \in X_f$ such that

(b) $u_1(z_1) > u_1(y_1) - \epsilon$ and $u_a(z_a) > u_a(y_a)$ for $a \in A-\{1\}$.

First note that, for ϵ sufficiently small, (a) and (b) imply, $u_1(z_1) > u_1(x_1)$. Hence,

(c) $u_1(z_1) > u_1(x_1^k)$ for large k.

Secondly, $y \in \phi(u)$, together with (b), implies $u_a(z_a) > u_a$ for $a \in A-\{1\}$. This can be written,

$u \in O = [(-\infty,u_2(z_2)) \times (-\infty,u_3(z_3)) \times \ldots] \cap U_1.$

The set C is therefore an open neighborhood of u. Hence, $u^k \longrightarrow u$ implies $u^k \in O$ for large k.

That is, $u_a^k < u_a(z_a)$ for $a \in A-\{1\}$ and large k.

In particular, this implies $z \in \phi(u^k)$ for large k, which, together with (c), contradicts $x^k \in \phi(u^k)$.

Q.E.D.

10.5 Definitions.

(a) A Pareto optimal allocation scheme (x_a) is called Potent if $u_a(x_a) > u_a(0)$ for $a \in A$

(b) A vector $(m_1, m_2, \ldots) \in R^{\infty}$ is called a Potent Gross Monetary Target if there is some price system $p \in R^{\infty}$ and some Potent allocation scheme (x_a) such that $(p,(x_a))$ is a market equilibrium and $m_a = px_a$ for $a \in A$.

(c) A vector $(n_1, n_2, \ldots) \in R^{\infty}$ is called a Potent Net Monetary Target if there is some price system $p \in R^{\infty}$ and some Potent allocation scheme (x_a) such that $(p,(x_a))$ is a monetary equilibrium and $n_a = px_a - pw_a$ for $a \in A$.

The following propositions (10.6 and 10.7) concerning the continuity of governmental intervention can be established for our general economy described in section 2. However, the general proof is notationally quite complex. Therefore, we simplify our analysis by imposing the following constraints.

Assumption VII.

(i) For each $u \in U_1$, $x(u)$ is single-valued.

(ii) Given two market equilibria $(p,(x_a))$ and $(q,(z_a))$, $(x_a) = (z_a)$ implies $p = \lambda q$ for some $\lambda > 0$.

It would be conceptually straightforward to generically express the above assumption in terms of differentiability and strict quasi-concavity assumptions on preferences. However, since assumption VII is only imposed to expedite our exposition, we refrain from doing so.

Our next proposition establishes the continuity of optimal governmental policies in terms of physical redistribution of endowments while proposition 10.7 establishes continuity in terms of monetary targets.

10.6 Proposition. If the irreducible allocations assumption is satisfied, then the collection of all potent allocation schemes, P, is arc-connected.

Proof of Proposition 10.6.

The arc-connectedness of P follows from the arc-connectedness of U_1 (lemma 10.3), the continuity of the function $x(\bullet)$ (proposition 10.4), and the fact that P = the range of $x(\circ)$ (lemma 7.4).

Q.E.D.

10.7 Proposition. If the irreducible allocations assumption is satisfied, then the collection of potent monetary targets is arc-connected. In addition, if each agent has a positive endowment of only a finite number of goods, then the collection of potent net monetary targets is arc-connected.

Proof of Proposition 10.7.

Given any potent allocation $x = (x_a) \in P$, proposition 9.1 guarantees that there is a price system $p \in Q$ such that (p,x) is a market equilibrium. Assumption VII guarantees that the system p is unique. Therefore, we can write $p = q(x)$. Corollary 8.2 implies that $p = q(\bullet)$ is a continuous function of x. That is, $x^k \longrightarrow x$ implies $p^k \longrightarrow p$.

Given any allocation $x = (x_a) \in P$, the corresponding gross monetary target is given by $m(x) = (m_a(x))$, where $m_a(x) = p(x)x_a$. Given the continuity of $p(\circ)$, corollary 3.3 insures that $m(\circ)$ is a continuous function of $x \in P$. That is, $x^k \rightarrow x$ implies $p(x^k) \longrightarrow p(x)$ and, therefore, $p(x^k)x_a^k \longrightarrow p(x)x_a$ by corollary 3.3. Therefore, the arc-connectedness of the set of gross monetary targets follows from the arc-connectedness of the potent allocations (proposition 10.6) and the continuity of the function $m(\circ)$.

The arc-connectedness of the set of net monetary targets follows by a similar argument given that the mapping from P to the corresponding net target is continuous. For $x \in P$, the net target is $n(x) = (n_a(x))$, where $n_a(x) = m_a(x) - p(x)w_a$. Given that $w_a^i > 0$ for only a finite number of i, the continuity of $n(\bullet)$ follows from the continuity of $m_a(\circ)$ and $p(\circ)$.

Q.E.D.

CHAPTER II

OPTIMAL MONETARY EQUILIBRIA IN DYNAMIC ECONOMIES

1. INTRODUCTION

In the study of the classical competitive model, in which a finite number of agents trade in markets for a finite number of goods, two fundamental welfare theorems emerge (e.g., see, Arrow [1951]). The first states that any competitive equilibrium is pareto optimal. In the study of dynamic models, this theorem has been shown to hold in cases where the population of the economy includes a finite number of infinitely-lived agents (e.g., see, Arrow and Hahn [1971]). However, in dynamic models where there are an infinite number of agents which are grouped into a collection of overlapping generations, Samuelson [1958] demonstrates that the first fundamental welfare theorem is no longer valid. As a cure for this non-optimality, Samuelson proposes the introduction of fiat (or outside) money. Samuelson is able to show that, within a certain restricted class of economies, there always exists a pareto optimal equilibrium in which the agents in the initial generation are endowed with non-negative quantities of fiat money. Our paper focuses on the extension of this result to much more general overlapping generations economies. (We will consider the class of economies as defined in Balasko and Shell [1980].)

Traditionally, when economists think of money they consider its two major roles as a medium of exchange and a store of value. Since our analysis is primarily concerned with the use of money to implement intergenerational transfers, we will focus our attention on the store of value aspect of money. In our paper, the scope of monetary policy is limited to the determination of the aggregate level of government debt at any given point in time. In this context, Samuelson's result states that an optimal outcome can always be achieved through a monetary policy which simply maintains a constant level of government debt. In light of the fact that Samuelson only considers economies that are stationary over time, the optimality of maintaining a constant level of debt should be anticipated. However, it may be somewhat surprising that, in this paper, we can essentially extend Samuelson's optimality result to economies that are constantly evolving (i.e. are non-stationary) over time.

Inasmuch as we have focused our attention on only one of the roles of money, the results presented in this paper can be thought of as tentative. It remains to be seen if our results can be meaningfully generalized to accommodate a fully developed model that allows money to play both of its afore mentioned roles. That is, even though the scope of monetary policy now includes the composition of government debt, we want to know under what circumstances can an optimal outcome be achieved through a monetary policy that maintains a constant level of debt.

We will now illustrate Samuelson's basic findings in the following example. We consider a simple overlapping generations economy where there is only one consumption good available in each time period and only one agent in each generation. All agents live for two periods, with the exception of the agent in the initial generation who lives out his life in the first period. Thus, in each time period, there will be one currently "young" agent and one currently "old" agent. We denote by z the consumption of the agent in the initial generation and denote by y and z the levels of consumption of each of the other agents in the economy during their "youth" and "old-age" respectively. Preferences are represented by the utility function $u_0(z) = z$ for the agent in the initial generation and $u(y,z) = y + z$ for each of the other agents. Each agent is endowed with one unit of each good which is available in his lifetime. That is, the agent in the initial generation is endowed with one unit of the first period good and each of the other agents is endowed with one unit of the good which is available during their youth and one unit of the good which is available during their old-age. We will be considering a price system where the price of each good, in terms of present discounted value, is equal to 1. Given these prices, the agent in the initial generation is facing the budget constraint $z \leqslant 1$, while each of the other agents face the constraint $y + z \leqslant 2$. Given our specification of tastes and endowments, it should be clear that these prices support an autarkic equilibrium. In fact, it can be readily verified that the autarkic solution is the only competitive equilibrium for the economy. However, one may also verify that the autarkic allocation scheme is pareto dominated by the scheme where the initial agent receives $z = 2$ units of consumption and where each of the other agents receive $(y,z) = (0,2)$. That is, in the alternative allocation scheme, the agent in the initial generation is better off while each of the other agents is no worse off. Thus, we have an example of an economy for which there is no pareto optimal competitive equilibrium. Finally, we note that this second allocation scheme is pareto optimal and that it can be supported as an equilibrium by our original price system (i.e. with all prices equal to 1) if the agent in the initial generation is given a transfer of one unit of fiat money. That is, the initial agent is now facing the budget constraint $z \leqslant 1 + 1 = 2$, where we normalize the price of money to be 1.

We will see that Samuelson's existence result is also closely related to the second fundamental welfare theorem. This theorem states that any pareto optimal allocation can be supported as a competitive equilibrium after a suitable redistribution of endowments. It is relatively straightforward to establish the second welfare theorem for a quite general overlapping generations economy (e.g., see Balasko and Shell [1980]). Since we wish to interpret the above theorem as a statement about public policy, it seems more natural to think of the transfers as being in terms of fiat money rather than in terms of a redistribution of physical endowments. The net quantity of money transferred to any particular agent would simply be the value of the corresponding transfer of physical endowments that he would have otherwise received. One drawback to the second welfare theorem is that there is no bound imposed on the amount of government intervention that is required to generate any particular allocation. Indeed, in dynamic models, the government might be forced to continually intercede in the economy by imposing an infinite number of monetary transfers. While the second welfare theorem states that there is an appropriate monetary policy that will support any given pareto optimal allocation, Samuelson's result can be viewed as stating that there is always a particular pareto optimal allocation that can be supported by a passive monetary policy. That is, one in which the government intercedes once and for all by giving non-negative endowments of fiat money to the agents in the initial generation.

Since the publication of Samuelson's article in 1958, a lot of work has been done to extend his basic result. Most of these attempts (e.g., see, Cass and Yaari [1966], Diamond [1965], Gale [1973], Shell [1971], and Wallace [1980]) restrict their attention to economies with homogeneous agents and only one good per period. Starrett [1972] allows heterogeneity of agents within each generation but requires that there be only one good per period. Okuno and Zilcha [1980,1983] allow for both heterogeneous agents and many goods per period. All of the papers mentioned above concern models that are stationary with respect to both tastes and endowments. Millan [1981] was able to establish an existence result, which allows only one good per period, where he assumes that the economy is "asymptotically" stationary. That is, the agents tastes and endowments in generation t converge to a limit as t becomes large. In contrast to the above works, we establish optimality results for a class of economies in which there are many heterogeneous agents per generation and many goods per period. Also, we allow the economy to exhibit a quite general form of non-stationarity with respect to both tastes and endowments.

Due to the generality of our model, we cannot establish the exact form of Samuelson's result. In fact, Millan [1981] presents an example of an economy that satisfies all of the assumptions of our paper but does not satisfy the conclusions of Samuelson's theorem. However, we are able to establish general propositions that come arbitrarily close to coinciding with Samuelson's result. The sense of closeness of our results will be defined in the next paragraph. In light of the fact that Millan's example demonstrates that the exact form of Samuelson's result cannot be generalized, it may seem surprising that we can come arbitrarily close. We conclude our paper by exploring a possible cause of this apparent anomaly.

We introduce our model, along with its assumptions, in section 2. In section 3 we establish two existence results. The first concludes that there is a passive monetary policy that is consistent with a pareto optimal allocation that is almost feasible for our economy. The degree of infeasibility of this allocation (as measured by the level of aggregate excess demand for any given good) can be chosen to be less than any given (arbitrarily) small quantity. Our next proposition concludes that there is a feasible and pareto optimal allocation that is consistent with a monetary policy that is almost passive. That is, although the government must intervene in periods other than the first, the extent of intervention (as measured by the magnitude of each agent's monetary transfer as a fraction of the value of his endowments), can be chosen to be less than any (arbitrarily) small quantity.

Generalizations of the above two results are presented in section 4. The first result is significantly strengthened by concluding that the aggregate degree of infeasibility of the pareto optimal allocation scheme (as measured by the summation of the magnitudes of all excess demands) can be chosen to be arbitrarily small. In the strengthening of our second result, we are able to choose an optimal monetary policy where the aggregate interventions in periods other than the first (as measured by the summation of the magnitudes of the transfers to the agents as a fraction of the values of their endowments) can be made arbitrarily small.

In section 5, we explain why our techniques, which are able to establish (arbitrarily) small perturbations of Samuelson's result, cannot be used to extend the exact form of his result to the general class of economies that we consider. The key problem with using our approximate results to obtain an exact result is summed up in Proposition 5.2. This proposition states that, in general, the set of feasible and pareto optimal allocation bundles is not closed. Finally, in section 6, we give a brief summary of our current knowledge of the existence of optimal monetary policies in general overlapping generations models, along with an open question for future research.

2. THE MODEL

We will analyze the overlapping-generations economy specified by Balasko and Shell [1980] with the exception that we allow each generation to consist of many agents. However, we do require that each generation contains the same number of agents. This requirement, along with many of those listed below, insures that we can characterize the set of pareto optimal allocations as in Balasko and Shell [1980]. In each period t ($t = 1, 2, \ldots$), there is a finite (constant) number N of non-producible and non-storable commodities. Consumers are identified by the generation g ($g = 0, 1, \ldots$) in which they belong as well as an index h ($h = 1, \ldots, H$) of their position in that generation. Since the model we consider is, in general, non-stationary, there is no significance attached to an agent's position, h, in a given generation. Agents in generation 0 consume only period 1 goods while agents in generation g ($g = 1, 2, \ldots$) consume goods in periods g and g+1.

Let $x_{g,h}^{t,i}$ be the consumption of commodity i ($i = 1, \ldots, N$) in period t ($t = 1, 2, \ldots$) by consumer (g,h) ($g = t-1, t$ and $h = 1, \ldots, H$). Consumer (g,h)'s preferences are assumed to be representable by a utility function

$$u_{g,h}(x_{g,h}), \quad g = 0, 1, \ldots \quad \text{and} \quad h = 1, \ldots, H,$$

where

$$x_{0,h} = x_{0,h}^{1} = (x_{0,h}^{1,1}, \ldots, x_{0,h}^{1,N}) \in R_{++}^{N} \quad \text{for } g = 0$$

and

$$x_{g,h} = (x_{g,h}^{g}, x_{g,h}^{g+1})$$

$$= (x_{g,h}^{g,1}, \dots, x_{g,h}^{g,N}, x_{g,h}^{g+1,1}, \dots, x_{g,h}^{g+1,N}) \in R_{++}^{2N}$$

for $g = 1, 2, \dots$.

That is, if we define $X_{g,h}$ to be the consumption space for agent (g,h), then $X_{0,h} = R_{++}^{N}$ for $g = 0$ and $X_{g,h} = R_{++}^{2N}$ for $g = 1, 2, \dots$. An allocation scheme for the economy will be denoted by

$$x = (x_{0,1}, \dots, x_{0,H}, x_{1,1}, \dots, x_{1,H}, \dots) \in X,$$

where the set of all such schemes, X, is defined as

$$X = X_{0,1} \times \dots \; X_{0,H} \times X_{1,1} \times \dots \times X_{1,H} \times \dots .$$

For notational convenience, we define the sequences

$$x'_{0,h} = (x_{0,h}^{1}, 0, \dots) \quad \text{for } g = 0$$

and

$$x'_{g,h} = (0, \dots, 0, x_{g,h}^{g}, x_{g,h}^{g+1}, 0, \dots) \quad \text{for } g = 1, 2, \dots .$$

Each consumer has strictly positive endowments of the goods in his lifetime

$$w_{0,h} = w_{0,h}^{1} = (w_{0,h}^{1,1}, \dots, w_{0,h}^{1,N}) \in R_{++}^{N} \quad \text{for } g = 0$$

and

$$w_{g,h} = (w_{g,h}^{g}, w_{g,h}^{g+1})$$

$$= (w_{g,h}^{g,1}, \dots, w_{g,h}^{g,N}, w_{g,h}^{g+1,1}, \dots w_{g,h}^{g+1,N}) \in R_{++}^{2N}$$

for $g = 1, 2, \dots$.

We also find it convenient to define the sequences

$$w'_{0,h} = (w_{0,h}^{1}, 0, \dots) \quad \text{for } g = 0$$

and

$$w'_{g,h} = (0, \dots, 0, w_{g,h}^{g}, w_{g,h}^{g+1}, 0, \dots) \quad \text{for } g = 1, 2, \dots .$$

The aggregate endowment of goods in the economy can then be written

$$w = (w^{1}, w^{2}, \dots) = \sum_{(g,h)} w_{g,h}.$$

The utility functions and the endowment streams are assumed to satisfy the following.

(A.1) $u_{0,h}(\bullet)$ and $u_{g,h}(\bullet,\bullet)$ $(g>0)$ have continuous and strictly positive first-order partial derivatives;

Assumption (A.1) states that preferences are smoothly monotonic.

(A.2) $u_{0,h}(\bullet)$ and $u_{g,h}(\bullet,\bullet)$ $(g>0)$ are strictly quasi-concave.

(A.3) The magnitude of the Gaussian curvature of consumer (g,h)'s indifference surface at any point $x_{g,h}$ such that $0 < x^g_{g,h} \leq w^g$ and $0 < x^{g+1}_{g,h} \leq w^{g+1}$ is uniformly bounded away from 0.

(A.4) There exists constants P and Q (independent of (g,h)) such that

$$0 < P < \frac{D_{g,i}u_{g,h}}{||D_g u_{g,h}||} < Q < \infty$$

and

$$0 < P < \frac{D_{g,i}u_{g,h}}{||(D_g u_{g,h}, D_{g+1}u_{g,h})||} < Q < \infty \quad \text{for } g = 1, 2, \ldots$$

at any point $x_{g,h}$ such that $u_{g,h}(x_{g,h}) \geq u_{g,h}(w_{g,h}/2)$ and $x_{g,h} \leq 2(w^g, w^{g+1})$, where $D_{s,i}u_{g,h}$ $(s = g, g+1)$ is the marginal utility of good i in period s to consumer (g,h) and $D_s u_{g,h} = (D_{s,1}u_{g,h}, \ldots, D_{s,N}u_{g,h})$.

(A.5) For any $x \in R^{2N}_{++}$, the closure of

$$\bigcup_{\substack{(g,h)\\ g>0}} \{ y \in R^{2N}_{++} : u_{g,h}(y) > u_{g,h}(x) \}$$

is contained in R^{2N}_{++} for any $x \in R^{2N}_{++}$.

Assumption (A.5) implies that agents will always demand a strictly positive quantity of all of the goods that are available in their lifetimes.

Assumptions (A.1) and (A.5) insure that we can always find suitable bounds, P and Q, as in (A.4), for each agent (g,h). But, (A.4) restricts these bounds to be the same for every agent.

(A.6) There are vectors w_L, $w_U \in R^{2N}$ such that $0 \ll w_L \ll w_{g,h} \ll w_U$ for all (g,h). For convenience, we scale our measure of quantities so that each component of w_L is greater than 1.

Assumptions (A.4) - (A.6) limit the degree to which our economy can be non-stationary with respect to both preferences and endowments. In later proofs of the existence of optimal monetary equilibria, we will be considering monetary equilibria in which all agents are achieving at least a minimum utility level. That is, for some $x \in R^{2N}_{++}$, $u_{g,h}(x_{g,h}) > u_{g,h}(x)$ for all (g,h). As we shall see later, (A.5) is then employed to deduce that the sequence $\{ x_{g,h} \}_{g>0}$ is bounded from below by a strictly positive vector.

Many of the properties of the overlapping generations model explored in later sections of this paper (such as the compactness of the set of monetary allocations and the non-closure of the set of pareto optimal allocations) hold under much weaker conditions than those listed above. Assumptions (A.1) - (A.6) are employed in the proofs of the existence of optimal monetary policies (Propositions 3.2, 3.3, 4.3 and 4.4). The assumptions allow us to essentially employ Proposition 5.6 of Balasko and Shell [1980] to characterize the set of pareto optimal monetary allocations as in our Lemma 3.1.

Let $p^{t,i}$ denote the price, in terms of present discounted value, of commodity i (i = 1, ... , N) in period t (t = 1, 2, ...). Also let $p^t = (p^{t,1}, \dots, p^{t,N})$ and $p = (p^1, p^2, \dots)$. We choose the normalization $p^{1,1} = 1$ and thus restrict prices to lie in the space $S = \{ p >> 0 : p^{1,1} = 1 \}$. Individual demand functions $f_{g,h}(p,y)$ are determined as the solutions to

(2.1)

maximize $u_{0,h}(x_{0,h})$

subject to $px'_{0,h} \leqslant y$ for $g = 0$

$x_{0,h} \in X_{0,h}$

and

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maximize $u_{g,h}(x_{g,h})$
subject to $px'_{g,h} \leqslant y$ for $g = 1, 2, \ldots$.
$x_{g,h} \in X_{g,h}$

(Given our definition of $x'_{g,h}$, $px'_{0,h} = p^1x^1_{0,h}$ for $g = 0$ and $px'_{g,h} = p^gx^g_{g,h} + p^{g+1}x^{g+1}_{g,h}$ for $g = 1, 2, \ldots$.)

Assumptions (A.1) and (A.3) insure that the functions $f_{g,h}(p,y)$ are well-defined and continuous at any (p,y) such that $p \in S$ and $y > 0$. We let $f^{t,i}_{g,h}(p,y)$ denote the demand by agent (g,h) for good (t,i) ($t = g, g+1$ and $i = 1, \ldots, N$). We also define $f^t_{g,h}(p,y) = (f^{t,1}_{g,h}(p,y), \ldots, f^{t,N}_{g,h}(p,y))$ for $t = g, g+1$.

In this model, monetary policy consists of the government imposing taxes and distributing subsidies of fiat money. A particular policy is a vector $m = (m_{g,h}) \in R^{\infty}$ where $m_{g,h}$ is the net transfer of fiat money to agent (g,h). Throughout this paper, the price of money, when it is valued, will be normalized to be 1. Therefore, given a system of prices p and a monetary policy m, agent (g,h) has a net income of $pw'_{g,h} + m_{g,h}$ and hence demands the consumption bundle $x_{g,h} = f_{g,h}(p, pw'_{g,h} + m_{g,h})$.
Clearly, given prices $p \in S$, we can only consider monetary policies m where each agent has a positive after-tax income, i.e. $pw'_{g,h} + m_{g,h} > 0$ for all (g,h).

We denote by m^t the aggregate stock of fiat money outstanding in the economy at time t. If taxes and subsidies to an agent are executed at the beginning of the agent's life, then $m^t = \sum_{\substack{(g,h) \\ g \leq t}} m_{g,h}$.

The optimality results presented in the next section concern monetary polices where the stock of fiat money is always non-negative, i.e. $m^t \geq 0$ for all t.

2.2 Definition.

(a) A Monetary Allocation consists of a price system p, a monetary policy m, and an allocation scheme x, such that:

$p \in S$,

$pw'_{g,h} + m_{g,h} > 0$ for all (g,h),

$x_{g,h} = f_{g,h}(p, pw'_{g,h} + m_{g,h})$ for all (g,h),

and

$m^t \geq 0$ for all time periods t = 1, 2, ...

(b) A Monetary Equilibrium (p,m,x) is a Monetary Allocation scheme which satisfies the feasibility constraint

$$\sum_{(g,h)} x'_{g,h} = w.$$

(c) A Competitive Equilibrium (p,x) is a Monetary Equilibrium where we consider the associated trivial monetary policy, m, defined by $m_{g,h} = 0$ for all (g,h).

2.3 Definition. The allocation scheme $x \in X$ is Pareto Optimal if there is no $y \in X$ such that

$$\sum_{(g,h)} y'_{g,h} = \sum_{(g,h)} x'_{g,h}$$
and

$u_{g,h}(y_{g,h}) \geq u_{g,h}(x_{g,h})$ for all (g,h), with at least one strict inequality. We will also call a monetary allocation (p,m,x) pareto optimal if the associated allocation scheme, x, is pareto optimal.

3. PRELIMINARY OPTIMAL MONETARY POLICY RESULTS

Before we establish our optimality results, we will find it useful to characterize the set of pareto optimal allocations in terms of the prices that support them. The following lemma is based on Proposition 5.6 of Balasko and Shell [1980]. Their result is extended to allow for generations to consist of many agents.

3.1 Lemma. Given any monetary allocation (p,m,x) such that $pw'_{g,h} + m_{g,h} \geq (1/2)pw'_{g,h}$ for all (g,h), the allocation scheme x is pareto optimal if $\sum_t \frac{1}{||p^t||} = \infty$.

In order to fully understand the above lemma, it is helpful to consider the source of the potential non-optimality of monetary allocations. As we saw in the example that was presented in the introduction of this paper, non-optimal allocation schemes can be improved upon by "passing back" consumption from the infinite future to agents in initial generation. That is, all agents, except those in the initial generation, are giving up some consumption during their youth in return for increased consumption in their old-age. The net effect of this redistribution is an increase in the welfare of the agents in the initial generation with no change in the welfare of each of the other agents in the economy.

The factor that prevents all monetary allocations from being improved upon in this manner is that the additional quantities of

goods that agents in generation t require in their old-age, to compensate for their foregone consumption in their youth, becomes unboundedly large for large t. Hence, the pareto-improving redistribution of goods in necessarily infeasible, given our boundedness condition, (A.6), on endowments. We can, therefore, characterize the set of all non-optimal allocations (i.e. the allocations that can be feasibly improved upon) as being those in which, on average, the necessary increase in each agents old-age consumption is small relative to the loss of consumption during their youth. That is, we have a sense in which non-optimal allocation schemes are those in which, at the margin, agents put a large value on old-age consumption relative to consumption during their youth. For any agent (g,h), we can express this condition of willingness to substitute as saying that $||D_{g+1}u_{g,h}(x_{g,h})||/||D_g u_{g,h}(x_{g,h})||$ is large. Since the efficiency condition for individual utility maximazation implies that
$||p^{g+1}||/||p^g|| = ||D_{g+1}u_{g,h}(x_{g,h})||/||D_g u_{g,h}(x_{g,h})||$, we can express our characterization of the non-optimal monetary allocations, (p,m,x), as being those in which, on average, the terms of the sequence $\{ ||p^g|| \}_{g=1}^{\infty}$ grow sufficiently quickly. The precise growth condition of prices that characterizes the monetary allocations that are non-optimal is given by Lemma 3.1 to be $\sum_{g=1}^{\infty} \frac{1}{||p^g||} < \infty$.

Proof of Lemma 3.1. We wish to employ Proposition 5.6 of Balasko and Shell [1980]. However, since the model that they consider restricts each generation to consist of a single agent, we must first transform our economy into one that meets this requirement. For simplicity, we strengthen (A.2) and assume that all utility functions, $u_{g,h}(\bullet)$, are concave.

Fix any $g = 0, 1, \ldots$.

Let $x_g = \sum_{h=1}^{H} x_{g,h}$, $w_g = \sum_{h=1}^{H} w_{g,h}$, $m_g = \sum_{h=1}^{H} m_{g,h}$, and $Z_g = X_{g,1} \times \cdots \times X_{g,H}$.

We first demonstrate that $(x_{g,1}, \ldots, x_{g,H})$ solves

(i)

maximize $u_{g,1}(z_{g,1})$

subject to $p\sum_{h=1}^{H} z'_{g,h} \leqslant pw'_g + m_g$

$u_{g,h}(z_{g,h}) \geqslant u_{g,h}(x_{g,h})$ for $h = 2, \ldots, H$

$(z_{g,1}, \ldots, z_{g,H}) \in Z_g$.

If $(x_{g,1}, \ldots, x_{g,H})$ did not solve the above problem, then there would exist a vector $(z_{g,1}, \ldots, z_{g,H}) \in Z_g$ such that (a) $u_{g,1}(z_{g,1}) > u_{g,1}(x_{g,1})$, (b) $u_{g,h}(z_{g,h}) \geqslant u_{g,h}(x_{g,h})$ for $h = 2, \ldots, H$, and (c) $p\sum_{h=1}^{H} z'_{g,h} \leqslant pw'_g + m_g$. Since each allocation $x_{g,h}$ solves agent (g,h)'s welfare maximazation problem (2.1), (a), together with our monotonicity assumption (A.1), implies $pz'_{g,1} > px'_{g,1}$, while (b), together with (A.1), implies $pz'_{g,h} \geqslant px'_{g,h}$ for $h = 2, \ldots, H$. Hence, $p\sum_{h=1}^{H} z'_{g,h} > px'_g$. But by (A.1), each consumer spends all of his income, i.e. $px'_{g,h} = pw'_{g,h} + m_{g,h}$ for $h = 1, \ldots, H$. Hence, $px'_g = pw'_g + m_g$, which, together with $p\sum_{h=1}^{H} z'_{g,h} > px'_g$, contradicts (c).

From the above paragraph, we conclude that $(x_{g,1}, \ldots, x_{g,H})$ solves the problem (i). Therefore, for any $g = 0, 1, \ldots$, the Kuhn-Tucker Theorem implies that there exist positive scalars $a_{g,1}, \ldots, a_{g,H}$ such that $(x_{g,1}, \ldots, x_{g,H})$ solves

(ii)

$$\text{maximize } \sum_{h=1}^{H} a_{g,h} u_{g,h}(z_{g,h})$$

$$\text{subject to } p\sum_{h=1}^{H} z'_{g,h} \leqslant pw'_g + m_g$$

$$(z_{g,1}, \ldots, z_{g,H}) \in Z_g.$$

Before we proceed further, we must make a technical modification. For the remainder of this proof, we extend agent (g,h)'s preferences to the set $X^*_{g,h}$ = closure $X_{g,h}$. This extension is necessary to insure that the functions $v_g(\circ)$ will be well-defined in (iii) below. For each (g,h), let $u^*_{g,h}$ = the limit of $u_{g,h}(x)$ as $x \rightarrow 0$. (Technically, to insure that the above limit exists we may have to normalize the agents' utility functions to be bounded from below.) By assumption (A.5), the following extension, $u^*_{g,h}(\circ)$, of $u_{g,h}(\circ)$ is continuous on $X^*_{g,h}$. We define,

$$u^*_{g,h}(x) = \begin{cases} u_{g,h}(x) & \text{if } x \in X_{g,h} \\ u^*_{g,h} & \text{if } x \in X_{g,h}, \text{ where } X_{g,h} \text{ denotes the boundary of } X_{g,h}. \end{cases}$$

Clearly, since preferences are quasi-concave, we can replace Z_g with $Z^*_g = X^*_{g,1} \times \ldots \times X^*_{g,h}$ as the constraint set and replace the preferences $u_{g,h}(\circ)$ with their extensions $u^*_{g,h}(\circ)$, in the maximazation problem (ii), and conclude that $(x_{g,1}, \ldots, x_{g,H})$ still solves (ii). Therefore, it follows that x_g solves

(iii)

$$\text{maximize } v_g(z_g)$$

$$\text{subject to } pz'_g \leqslant pw'_g + m_g$$

$$z_g \in X^*_{g,1};$$

where $v_g(\bullet)$ is defined by

(iv) $v_g(z_g) = \max \sum_{h=1}^{H} a_{g,h} u^*_{g,h}(z_{g,h})$ subject to $\sum_{h=1}^{H} z_{g,h} \leqslant z_g$ and $(z_{g,1}, \ldots, z_{g,H}) \in Z^*_g$.

The continuity of the utility functions $u_{g,h}(\circ)$ is sufficient to guarantee that each of the functions $v_g(\circ)$ is well-defined at any point $z_g \in X_{g,1}$.

We now consider an economy E* consisting of N goods per period but only one agent per generation. We give the agent in generation g the consumption set $X_{g,1}$, the endowment w_g, and the utility function $v_g(\bullet)$. By the preceding paragraph, in particular (iii), we can conclude that $(p,(m_g),(x_g))$ is an monetary allocation for the economy E*. It should be obvious that the economy E* inherits all of the properties, (A.1) - (A.6), our original economy. It is also straightforward to verify that (A.1) - (A.6) are sufficient to guarantee that the allocation $(p,(m_g),(x_g))$ for the economy E* satisfies all of the hypothesis of Proposition 5.6 of Balasko and Shell [1980] with the possible exception of the requirement that $\{ x_g \}_{g=1}^{\infty}$ is bounded from below by a strictly positive vector. Once we have established this last requirement, our proof will be completed since if $\sum_t \frac{1}{||p^t||} = \infty$, then by Proposition 5.6 of Balasko and Shell [1980], (x_g) is a pareto optimal allocation scheme for the economy E*. This, in turn, implies that the allocation scheme $(x_{g,h})$ is pareto optimal for our original economy since preferences, $v_g(\circ)$, in the economy E*, as defined in (iv), are monotonic transformations of the preferences of the agents in our original economy.

By our hypothesis $pw'_{g,h} + m_{g,h} \geqslant (1/2)pw'_{g,h}$, the bundle $(1/2)w_{g,h}$ was affordable to agent (g,h). Hence, $u_{g,h}(x_{g,h}) \geqslant u_{g,h}((1/2)w_{g,h})$ for all (g,h), since $x_{g,h}$ was chosen over $(1/2)w_{g,h}$. Therefore, by assumptions (A.1) and (A.6),

(iv) $u_{g,h}(x_{g,h}) \geqslant u_{g,h}((1/2)w_L,(1/2)w_L)$ for all (g,h),

where w_L is the lower bound on endowments as specified in (A.6). By assumption (A.5), (iv) implies that the collection $\{ x_{g,h} : g = 0, 1, \ldots \text{ and } h = 1, 2, \ldots \}$ is uniformly bounded away from 0. This, in turn, implies that $\{ x_g \}_{g=1}^{\infty}$ is also bounded from below by a strictly positive vector.

Q.E.D.

We are now ready to establish our optimality results. The intuition and strategy behind the proofs of these propositions is quite simple and will now be explained. First let us redefine our price set to be

$S = \{ p >> 0 : \sum_i p^{1,i} = 1 \}.$

We will only consider monetary allocation schemes (p,m,x) such that the market for period 1 goods is in equilibrium. Given any such scheme, $p \in S$ and $x_{0,h} = (x^1_{0,h}) \leqslant (\sum_{\substack{(g,h)\\ g=0,1}} x^1_{g,h}) = (w^1)$. Hence,

$p^1 x_{0,h} < p^1 w^1 \leqslant \sum_i w^{1,i},$

where $w^{1,i}$ is the aggregate endowment of good i in period 1. Therefore, $\sum_i w^{1,i}$ serves as an upper bound on the net quantity of money, $m_{0,h}$, that any consumer h in generation 0 can receive. Given this bound, we can construct pareto optimal monetary allocations (p,m,x) by giving consumers in generation 0 monetary transfers which equal the value of (small) quantities of future goods. Prices are kept from exploding (i.e. the terms in the sequence $\{ ||p^t|| \}_{t=0}^{\infty}$ do not diverge to oo as $t \rightarrow \infty$), since otherwise consumers in generation 0 would end up receiving arbitrarily large monetary transfers, which would violate the upper bound $\sum_i w^{1,i}$. By Lemma 3.1, this boundedness property of the price system implies that the allocation scheme, x, is pareto optimal.

3.2 Proposition. For any small $\varepsilon > 0$, there exists a monetary allocation (p,m,x) that satisfies:

(a) $m_{0,h} \geqslant 0$ for g = 0 and h = 1, ... , H;

(b) $m_{g,h} = 0$ for g = 1, 2, ... and h = 1, 2, ... , H;

(c) If we define $E^{t,i}$ to be the aggregate excess demand for good (t,i), $\sum_{\substack{(g,h) \\ g=t-1,t}} (x^{t,i}_{g,h} - w^{t,i}_{g,h})$, then

$E^{1,i} = 0$ for t = 1 and i = 1, ... , N,

$E^{t,i} = \varepsilon$ for t = 2, 3, ... and i = 1, ... , N;

(d) $p^{t,i} < \frac{1}{\varepsilon} \sum_i w^{1,i}$ for t = 2, 3, ... and i = 1, ... , N;

and

(e) x is a pareto optimal allocation scheme.

Proof of Proposition 3.2. Let h* = 1. The desired monetary allocation, i.e. that which satisfies (a) - (e) above, will be found as the competitive equilibrium of an economy E*, which is formed by perturbing our original economy. Specifically, our original economy is altered by giving agent (0,h*) the property rights to an extra endowment of - units of each good i (i = 1, ... , N) in periods t (t = 2, 3, ...). The agent (0,h*) is now interpreted as being an infinitely-lived agent who only has tastes for period 1 goods. Even though the economy E* does not fit into our overlapping generations framework, it does satisfy all of the hypothesis of Theorem 2 in

Wilson [1981]. Wilson's theorem concerns the class of economies which consist of an infinite number of possibly infinitely-lived agents. (In the terminology of Wilson's Theorem 2, the endowment of agent $(0,h^*)$ is a significant fraction of the aggregate endowment in the economy E^*.) Theorem 2 establishes the existence of a competitive equilibrium $(p,(x_{g,h}))$ for the economy E^*. Specifically, $(p,(x_{g,h})$ satisfies:

(i) $p \in S$;

(ii) $x_{0,h^*} = f_{0,h^*}(p,pw'_{0,h^*}+m_{0,h^*})$ for $(g,h)=(0,h^*)$,

where the total value of the extra endowments given to agent $(0,h^*)$ is being denoted by $m_{0,h^*} = \varepsilon \sum_{t=2}^{\infty} \sum_{i=1}^{N} p^{t,i}$;

(iii) $x_{g,h} = f_{g,h}(p,pw'_{g,h})$ for all $(g,h) \neq (0,h^*)$;

and

(iv) $$\sum_{\substack{(g,h)\\ g=t-1,t}} (x^{1,i}_{g,h} - w^{1,i}_{g,h}) = 0 \quad \text{for } t = 1 \text{ and } i = 1, \ldots, N$$

$$\sum_{\substack{(g,h)\\ g=t-1,t}} (x^{t,i}_{g,h} - w^{t,i}_{g,h}) = \varepsilon \quad \text{for } t = 2, 3, \ldots \text{ and } i = 1, \ldots, N.$$

(iv) is the feasibility constraint for $(p,(x_{g,h}))$ to be an equilibrium for the economy E^* since the aggregate endowments of good i ($i = 1, \ldots N$) in period t ($t = 2, \ldots$) is $w^{t,i} + \varepsilon$.

Since only agent $(0,h^*)$ received a monetary transfer, we define the monetary policy associated with $(p,(x_{g,h}))$ as $m_{g,h} = 0$ for all $(g,h) \neq (0,h^*)$ (where m_{0,h^*} has already been defined in (ii)). One may now verify that if we interpret (p,m,x) in terms of our original economy, then (i) - (iv) imply that (p,m,x) is a monetary allocation which satisfies (a) - (c). Property (d) follows immediately since if $p^{t,i} \geqslant \frac{1}{\varepsilon} \sum_i w^{1,i}$, for some $t = 2, 3, \ldots$ and $i = 1, \ldots, N$; then

$$m_{0,h*} = \varepsilon \sum_{\substack{(t,i) \\ t>1}} p^{t,i} > \varepsilon p^{t,i} \geqslant \sum_i w^{1,i},$$

which, as pointed out earlier, is inconsistent with the fact that the market for first period goods is in equilibrium. (e) now follows from (d) by an application of Lemma 3.1.

Q.E.D.

3.3 Proposition. For any small $\varepsilon > 0$, there exists a monetary equilibrium (p,m,x) that satisfies:

(a) $\sum_{(g,h)} m_{g,h} = 0$;

(b) $m_{0,h} \geqslant 0$ for g = 0 and h = 1, ... , H;

(c) $\frac{|m_{g,h}|}{pw'_{g,h}} < \varepsilon$ for g = 1, 2, ... and h = 1, ... , H;

(d) $p^{t,i} < \frac{1}{\varepsilon} \sum_i w^{1,i}$ for t = 1, 2, ... and i = 1, ... , N;

and

(e) x is a pareto optimal allocation scheme.

Condition (c) of this proposition gives us a sense in which the monetary transfers to agents, in all generations other than the first, is small in real terms. Specifically, if the transfers were carried out by physically redistributing endowments, then we are guaranteed that each agent would undergo a transfer that amounts to less than the fraction - of his initial endowment.

Proof of Proposition 3.3. The proof of this proposition is so close to the proof of Proposition 3.2 that we will only highlight the differences.

Let $h^* = 1$. The economy E^* is formed by perturbing the endowments of agents $(0,h^*)$, $(1,h^*)$, ... Agent $(0,h^*)$ receives an extra endowment of – units of each good i ($i = 1, \ldots, N$) in period t ($t = 2, 3, \ldots$). Agents (g,h^*) ($g = 2, 3, \ldots$) lose ϵ units of their endowments of each good i ($i = 1, \ldots, N$) in period g. (By assumption (A.6), this redistribution of endowments is feasible as long as $\epsilon < 1$, which is less then the minimal coordinate of the vector w_L.)

As in the proof of the previous proposition, we apply Theorem 2 of Wilson 1981 to obtain a competitive equilibrium $(p,(x_{g,h}))$ for the economy E^*. Specifically, $(p,(x_{g,h}))$ satisfies:

(i) $p \in S$;

(ii) $x_{g,h^*} = f_{g,h}(p,pw_{g,h^*}+m_{g,h^*})$ for $g = 0, 1, \ldots$ and $h = h^*$, where the net value of the endowment transferred to agent (g,h^*) is denoted by

$m_{0,h^*} = \epsilon \sum_{t=2}^{\infty} \sum_{i=1}^{N} p^{t,i} < \infty$ for $g = 0$ and $h = h^*$,

$m_{1,h^*} = 0$ for $g = 1$ and $h = h^*$,

and $m_{g,h^*} = -\epsilon \sum_{i=1}^{N} p^{g,i}$ for $g = 2, 3, \ldots$ and $h = h^*$;

(iii) $x_{g,h} = f_{g,h}(p,pw'_{g,h})$ for $g = 0, 1, \ldots$ and all $h \neq h^*$,

(iv) $\sum_{(g,h)} x'_{g,h} = w$.

Since the increased endowment of agent $(0,h^*)$ is offset by the decreased endowments of the other agents in the economy, the feasibility constraint for the economy E^*, condition (d^*), corresponds to the feasibility constraint for our original economy.

We complete the specification of the monetary policy associated with $(p,(x_{g,h}))$ by defining $m_{g,h} = 0$ for $g = 0, 1, \ldots$ and all $h \neq h^*$. It now follows that if we interpret (p,m,x) in terms of our original economy, then (i) - (iv) imply that (p,m,x) is a monetary allocation that satisfies (a) - (c).

Q.E.D.

We will call a monetary allocation scheme (p,m,x) passive if $m_{g,h} = 0$ for all (g,h) such that $g > 0$. Proposition 3.2 establishes the existence of a passive and pareto optimal monetary allocation which is "approximately" feasible. Similarly, Proposition 3.3 establishes the existence of a pareto optimal monetary equilibrium which is "approximately" passive. The problem of obtaining monetary allocations which are (exactly) feasible and (exactly) passive is discussed in the next section. For the remainder of this section, we will elaborate further on the implications of Proposition 3.3.

Another interpretation of Proposition 3.3 is that it establishes the existence of a passive and pareto optimal equilibrium where consumers are "approximately" rational. That is, as long as there is some degree of imprecision in the agents measurement of his own income, the taxes $m_{g,h}$ (for $g>0$) in the equilibrium of Proposition 3.3 need not be imposed. Formally, we assume that when facing prices p, agent (g,h) (for $g>0$) might mistakenly calculate his income to be $pw'_{g,h}+m_{g,h}$, instead of $pw'_{g,h}$, which would lead this agent to demand the consumption bundle $x_{g,h}$. Part (c) of Proposition 3.3 guarantees that the necessary imprecision, $m_{g,h}$, in the agents measurement of his income can be made arbitrarily small in percentage terms $(\frac{|m_{g,h}|}{pw'_{g,h}})$. Clearly, if we are going to allow even a small degree of imprecision, as described above, there will be a continuum of monetary equilibria associated with any given monetary policy. Proposition 3.3 then makes the statement that there exists a proper monetary policy such that at least one of its associated monetary equilibria is pareto optimal.

4. OPTIMAL MONETARY POLICY RESULTS

In this section, we strengthen our preliminary existence results. Proposition 3.2 is extended in two ways. First, it is noted that it is only necessary to bound an infinite subset of prices, as opposed to bounding all of the prices, in order to guarantee that a given allocation is pareto optimal, i.e. $\sum_t \frac{1}{||p^t||} = \infty$. Therefore, since the only reason that we allowed the market for a good to be out of equilibrium, in Proposition 3.2, was so that we could bound the price of that good, we can restrict the markets that are out of equilibrium to lie in any specified infinite subset of markets. However, the major strengthening of Proposition 3.2 lies in the bound that we impose on the degree to which our allocation scheme can depart from feasibility. Instead of uniformly bounding each component of the excess demand for goods by any arbitrarily small number, as in Proposition 3.2, we can require that the summation of the magnitudes of the excess demands for all goods be made arbitrarily small. That is, we can bound the aggregate divergence from feasibility by a small number.

Similarly, Proposition 3.3 is extended in two ways. For the same reason that the markets that are out of equilibrium in Proposition 3.2 can be restricted to any infinite set, we can strengthen Proposition 3.3 by requiring that the agents who are taxed can be restricted to any given infinite collection of agents. The major strengthening of Proposition 3.3 lies in the bound that we impose on the degree to which agents in all of the generations, other than the first, are taxed. Instead of uniformly bounding the magnitude of the tax on each agent, measured as a fraction of the agents endowment income, by an arbitrarily small number, as in Proposition 3.3, we can require that the summation of the magnitudes of the taxes can be made arbitrarily small. That is, we can bound the aggregate divergence of our monetary policy from being passive.

The following lemma, about the set of monetary allocations, will be employed in the proofs of our extensions of Propositions 3.2 and 3.3, i.e. Propositions 4.3 and 4.4 respectively.

4.1 Lemma. Consider any point $y \in X$. The set of all monetary allocations (p,m,x) such that

(a) $pw'_{g,h} + m_{g,h} \geqslant (1/2)pw'_{g,h}$ for all (g,h)

and

(b) $x \leqslant y$

is a compact subset of $S \times M \times X$, where $M = R^{\infty}$ is the space of monetary policies and S, M, and X are each endowed with the product topology. Note, in the product topology on $S \times M \times X$, the compactness of a subset of S M X is equivalent to the sequential compactness of a subset.

Proof of Lemma 4.1. Consider any sequence of monetary allocations $\{ (p(k),m(k),x(k)) \}$ each of which satisfies (a) and (b).

Consider any (g,h).

For each k, hypothesis (a) implies that the consumption bundle $(1/2)w_{g,h}$ is in agent (g,h)'s budget set, as defined in (2.1), given prices $p(k)$ and the monetary transfer $m_{g,h}(k)$. Since $x_{g,h}(k)$ is demanded by agent (g,h), when the bundle $(1/2)w_{g,h}$ is available, it must be at least as desirable, i.e. $u_{g,h}(x_{g,h}(k)) \geq u_{g,h}((1/2)w_{g,h})$. Therefore, by hypothesis (b),

$x_{g,h}(k) \in X^*_{g,h}$ for each k,

where

$$X^*_{g,h} = \{ x \in X_{g,h} : u_{g,h}(x) \geq u_{g,h}((1/2)w_{g,h}) \text{ and } x \leq y_{g,h} \}$$

for $g = 0, 1, \ldots,$ and $h = 1, \ldots, H$.

By assumption (A.1), each of the sets $X^*_{g,h}$ is closed and hence compact since, by definition, they are bounded. Therefore, the set $X^* = X^*_{0,1} \times \ldots \times X^*_{0,H} \times X^*_{1,1} \times \ldots \times X^*_{1,H} \times \ldots$, which contains each of the points $x(k)$, is compact for the product topology by Tychonoff's theorem (see, e.g., Bourbaki [1966, I, Sect. 9.5, Theorem 3, p. 88]). Hence we may select a subsequence of $\{ x(k) \}$ such that $x(k) \to x$ for some $x \in X^*$. (Since X is endowed with the product topology, the convergence $x(k) \to x$ means pointwise convergence, i.e. $x_{g,h}(k) \to x_{g,h}$ for each (g,h).) Clearly $X^* \subseteq X$ so $x \in X$.

We will now verify, by induction, that for each $g = 1, 2, \ldots$; $p^g(k) \to p^g$ where $\{ p^g \}$ is defined inductively by

$$p^1 = \frac{D_1 u_{0,1}(x_{0,1})}{D_{1,1} u_{0,1}(x_{0,1})}$$

and

$$\frac{p^{g+1}}{p^{g,1}} = \frac{D_{g+1} u_{g,1}(x_{g,1})}{D_{g,1} u_{g,1}(x_{g,1})} \qquad \text{for } g = 1, 2, \ldots .$$

Note that by assumption (A.1), $p^g \in R^N_{++}$ for all g and that $p^{1,1} = 1$. Hence, $p = (p^1, p^2, \ldots) \in S$.

For each k, the efficiency condition from (2.1) together with the normalization $p^{1,1}(k) = 1$ implies

$$p^1(k) = \frac{D_1 u_{0,1}(x_{0,1}(k))}{D_{1,1} u_{0,1}(x_{0,1}(k))} .$$

Therefore, by assumption (A.1), $x_{0,1}(k) \longrightarrow x_{0,1}$ implies $p^1(k) \longrightarrow p^1$.
Assume the inductive hypothesis $p^g(k) \longrightarrow p^g$.
For each k, the efficiency condition from (2.1) implies

$$\frac{p^{g+1}(k)}{p^{g,1}(k)} = \frac{D_{g+1} u_{g,1}(x_{g,1}(k))}{D_{g,1} u_{g,1}(x_{g,1}(k))} .$$

By assumption (A.1), $x_{g,1}(k) \longrightarrow x_{g,1}$ and $p^{g,1}(k) \longrightarrow p^{g,1}$ imply $p^{g+1}(k) \longrightarrow p^{g+1}$.
We have just shown $p(k) \longrightarrow p$, where $p(k) = (p^1(k), p^2(k), \dots)$.

For each k, (2.1), together with our non-satiation assumption (A.1), implies $m_{g,h}(k) = p(k)x'_{g,h}(k) \in pw'_{g,h}$. Therefore, $x(k) \longrightarrow x$ and $p(k) \longrightarrow p$ imply that $m(k) \longrightarrow m$ where m is defined by $m_{g,h} = px'_{g,h} - pw'_{g,h}$ for all (g,h).

We have just demonstrated that there is a triple $(p,m,x) \in S \times M \times X$ such that $(p(k),m(k),x(k)) \longrightarrow (p,m,x)$. Since $px'_{g,h} = pw'_{g,h} + m_{g,h}$, $p \in S$ and $x \in X$ imply that $pw'_{g,h} + m_{g,h} > 0$. Given this, it is easy to see that (p,m,x) is a monetary allocation and that it satisfies our hypotheses (a) and (b) since each of the triples (p(k),m(k),x(k)) are monetary allocations that satisfy (a) and (b).

Q.E.D.

The following result is an immediate consequence of Proposition 4.1.

4.2 Corollary. The set of all monetary equilibria (p,m,x) such that $pw'_{g,h} + m_{g,h} \geqslant (1/2)pw'_{g,h}$ for all (g,h), is compact in the product topology.

We now present our existence results.

4.3 Proposition. For any infinite collection of goods $I \subseteq \{ (t,i) : t = 2, 3, \ldots \text{ and } i = 1, \ldots, N \}$ and any small $\epsilon > 0$, there exists a monetary allocation (p,m,x) that satisfies:

(a) $m_{0,h} \geqslant 0$ for $g = 0$ and $h = 1, \ldots, H$;

(b) $m_{g,h} = 0$ for $g = 1, 2, \ldots$ and $h = 1, \ldots, H$;

(c) If we define $E^{t,i}$ to be the aggregate excess demand for good (t,i), $\sum_{\substack{(g,h)\\ g=t-1,t}} (x^{t,i}_{g,h} - w^{t,i}_{g,h})$, then

$E^{t,i} = 0$ for each $(t,i) \in I$,

$$\sum_{(t,i)\in I} |E^{t,i}| \leqslant \epsilon;$$

(d) $p^{t,i} \leqslant \frac{1}{\epsilon} \sum_i w^{1,i}$ for each $(t,i) \in I$,

and

(e) x is a pareto optimal allocation scheme.

Our strategy for proving this proposition will be similar to the approach that we used to establish Proposition 3.2. Recall, in the proof of proposition 3.2, our original economy is perturbed by giving agent $(0,h^*)$ the property rights to an extra endowment of - units of all goods in each period other than the first. The market value of this transfer, given prices p, is $m_{0,h^*}(p) = \epsilon \sum_{t=2}^{\infty} \sum_{i=1}^{N} p^{t,i}$. We then considered a competitive equilibrium (p,x) of the perturbed economy. The boundedness property of the equilibrium transfer, $m_{0,h*}(p) \leqslant \sum_i w^{1,i}$, implies that the prices, p, satisfy $p^{t,i} \leqslant \frac{1}{\epsilon}\sum_i w^{1,i}$ for t = 2, 3, ... and i = 1, ... , N. The optimality of the allocation scheme, x, then follows by Lemma 3.1.

In the current proposition, we want to introduce as few extra endowments as possible into our perturbed economy. To accomplish this feat, we are forced to make the quantities of the extra endowments depend on the current market price system. Specifically, we only introduce endowments of goods whose prices are maximal. In this way, we can keep the quantity of extra endowments to a minimum, while still being able to employ the boundedness property on the value of agent (0,h*)'s equilibrium transfer to bound the equilibrium prices. Unfortunately, the price dependency of agent (0,h*)'s endowments put the perturbed economy beyond the reach of all well-known equilibrium existence theorems for dynamic economies. We, therefore, must construct a sequence of approximations { (p(k),m(k),x(k)) } to our desired allocation (p,m,x), i.e. (p(k),m(k),x(k)) $\rightarrow$ (p,m,x) as k $\rightarrow$ oo.

The k'th approximation (p(k),m(k),x(k)) is derived from a competitive equilibrium of a truncated economy in which agent (0,h*) receives an extra endowment of goods which are contained in the given set I, dated no later than k, and have a maximal price. The total distribution of goods is ϵ. Therefore, the equilibrium value of the transfers to agent (0,h*) is

$m_{0,h*}(k) = \epsilon \max \{ p^{t,i}(k) : (t,i) \in I$ and $t = 2, \ldots , k \}$, since the price of each good which makes up the additional endowments must be at the level $\max \{ p^{t,i}(k) : (t,i) \in I$ and $t = 2, \ldots , k \}$. By the afore mentioned boundedness property, $m_{0,h*}(k) < \sum_i w^{1,i}$, we conclude that $p^{t,i}(k) < \frac{1}{\epsilon} \sum_i w^{1,i}$ for each $(t,i) \in I$ such that $2 \leqslant t < k$. Taking the limit of the above inequality as $k \longrightarrow \infty$ yields the desired boundedness property of the limiting prices, p, as specified in condition (d) of the proposition. In turn, this boundedness of prices guarantees that the limiting allocation is pareto optimal.

The bulk of the following proof lies in establishing the existence of each of the approximations $(p(k),m(k),x(k))$. Next, we verify that Lemma 4.1 can be employed to come up with a limit point (p,m,x) of the sequence $\{ (p(k),m(k),x(k)) \}$. It is then a simple matter to verify that the limiting allocation (p,m,x) satisfies the conditions (a) - (e) of the proposition.

Proof of Proposition 4.3. Let $h^* = 1$.

Fix any $k = 1, 2, \ldots$.

To establish the existence of the k'th monetary allocation, $(p(k),m(k),x(k))$, we will consider a finite truncation of our full economy. This truncated economy will consist of all of the agents in generations $0, \ldots , k$. These agents will be allowed to trade in all of the markets for the goods which are available in their lifetimes. The resulting equilibrium for this finite economy will then be (arbitrarily) completed to form the monetary allocation

$(p(k),m(k),x(k))$.

Formally, consider the (truncated) collection of agents

$\{ (g,h) : g = 0, \ldots , k \text{ and } h = 1, \ldots , H \}$,

of goods

$\{ (t,i) : t = 1, \ldots , k+1 \text{ and } i = 1, \ldots N \}$,

of commodity prices

$P_k = \{ (p^1, \ldots , p^{k+1}) \in R^N_{++} \times \ldots \times R^N_{++} \}$,

and the collection of goods from which agent $(0,h^*)$ receives added endowments

$I_k = \{ (t,i) \in I : t = 2, \ldots , k\}$.

We wish to find a monetary allocation $(p(k),m(k),x(k))$ for the truncated economy in which agent $(0,h^*)$ receives the transfer

$m_{0,h^*}(k) = \varepsilon \max \{ p^{t,i}(k) : (t,i) \in I_k \}$.

Let the function

$m_{0,h^*} : S_k \to R_{++}$

be defined by

(a*) $m_{0,h^*}(p) = \varepsilon \max \{ p^{t,i} : (t,i) \in I_k \}$ for each $p \in S_k$.

Given prices $p \in S_k$, agent $(0,h^*)$ will receive the monetary transfer $m_{0,h^*}(p)$. No other agents in the truncated economy will be given any transfers.

The aggregate demand for goods in this economy, which is denoted by $F(p) = (F^1(p), \ldots , F^{k+1}(p)) \in S_k$, is then given by

$$F^1(p) = f^1_{0,h^*}(p,pw'_{0,h^*}+m_{0,h^*}(p)) + \sum_{\substack{(g,h)\neq(0,h^*)\\ g=0,1}} f^1_{g,h}(p,pw'_{g,h}) \quad \text{for } t = 1$$

$$F^t(p) = \sum_{\substack{(g,h)\\ g=t-1,t}} f^t_{g,h}(p,pw'_{g,h}) \quad \text{for } t = 2, \ldots , k$$

and

$$F^{k+1}(p) = \sum_h f^{k+1}_{k+1,h}(p, pw'_{k+1,h}) \qquad \text{for } t = k+1.$$

We denote by $F^{t,i}(p)$ the demand for good (t,i), i.e. the i'th component of $F^t(p)$.

Due to the monetary transfer to agent $(0,h^*)$, the value of aggregate demand exceeds the value of the aggregate endowment by $m_{0,h^*}(p)$. That is, Walras's law is not satisfied. Therefore, in order to establish the existence of an equilibrium, we must first restore Walras's law by introducing further quantities of goods into the economy beyond the existing endowments of the agents. The total value of these additions must offset the transfer $m_{0,h^*}(p)$. We wish to introduce the least amount of extra goods as possible, while still generating the value $m_{0,h^*}(p)$, so that the resulting "equilibrium" allocation, $x(k)$, will be very close to being feasible for our original economy. Therefore, for each $p \in S_k$, we only supply extra quantities of endowments of the goods whose prices are maximal in the set I_k. That is, for any $p \in S_k$, the added endowments are constrained to lie in the set

$$G(p) = \{\, y \in P_k : y^{t,i} = 0 \text{ unless } (t,i) \in I_k(p)$$
$$y^{t,i} \geq 0 \text{ for all } (t,i) \in I_k(p)$$
$$\sum_{(t,i)\in I_k(p)} y^{t,i} = \varepsilon \,\},$$

where the subset of goods whose prices are maximal in I_k is denoted by $I_k(p) = \{\, (t,i) \in I_k : \text{for all } (t',i') \in I_k,\ p^{t,i} \geq p^{t',i'} \,\}$.

One may verify that the correspondence $G(\cdot)$ is upper hemi-continuous on the set of prices S_k.

We now consider the following (modified) excess demand correspondence $H : S_k \longrightarrow R^N \times \ldots \times R^N$, which we define by

$H(p) = \{ F(p) - (w^1, \ldots, w_k^{k+1}) - y : y \in G(p) \}$,

where w_k^{k+1} denotes that aggregate endowment of goods in period k+1 by the agents in generation k, i.e. $w_k^{k+1} = \sum_h w_{k,h}^{k+1}$.

That is, H(p) is the collection of levels of demand which are in excess of both the original endowments, as specified in $(w_{g,h})$, and the artificially added quantities of goods $y \in G(p)$. We wish to verify that $H(\bullet)$ satisfies the following properties (which are typically associated with excess demand correspondences);

(i) $H(\circ)$ is upper hemi-continuous on S_k,

(ii) (Walras's law) For any $p \in P_k$, $pz = 0$ for all $z \in H(p)$,

(iii) $H(\bullet)$ is uniformly bounded from below,

(iv) (Boundary condition) $\inf \{ ||z|| : z \in H(p) \} \longrightarrow \infty$ as p approaches the boundary of S_k,

and

(v) (Convexity) H(p) is a convex subset of $R^N \times \ldots \times R^N$ for each $p \in P_k$.

(i) follows from the continuity of individual demand, $f_{g,h}(\bullet)$, the continuity of the monetary transfer to agent $(0,h^*)$, $m_{0,h^*}(\bullet)$, and from the upper hemi-continuity of $G(\circ)$. By our monotonicity assumption (A.1), every agent spends all of his (after-tax) income. Hence, $pF(p) = p(w^1, \ldots, w^k, w_k^{k+1}) + m_{0,h^*}(p)$. But, by the definition of G(p), $py = m_{0,h^*}(p)$ for every $y \in G(p)$. (ii) now follows from the above two equalities and the definition of H(p). (Recall that $G(\bullet)$ was specifically defined so that (ii) would be satisfied.) For any point $z \in H(p)$, by definition there exists a point $y \in G(p)$ such that $z = F(p) - (w^1, \ldots, w_k^{k+1}) - y$. Hence, since the components of $F(p)$ are strictly positive and the components of y are each no greater than ε, (iii) follows because for all (t,i),

$z^{t,i} = F^{t,i}(p) - w^{t,i} - y^{t,i} \geqslant - w^{t,i} - \varepsilon$

(i.e. the desired lower bound for $H(p)$, b, can be defined by $b^{t,i} = - w^{t,i} - \varepsilon$). By (A.2), aggregate demand, $F(p)$, satisfies the boundary condition (iv). (Technically, the correspondence $F'(p) = \{ F(p) \}$ satisfies (iv).) Hence, since the magnitude of the vectors in $G(p)$ are uniformly bounded, the correspondence $H(p)$ also satisfies (iv). Finally, (v) follows from the convexity of $G(p)$.

Since $H(\bullet)$ satisfies (i) - (v), we can appeal to Lemma 1 of Hildenbrand [1974] to deduce that there are prices $\tilde{p}(k) \in P_k$ such that $0 \in H(\tilde{p}(k))$. By definition, $0 \in H(\tilde{p}(k))$ implies

$F(\tilde{p}(k)) - (w^1, \ldots, w^k) \in G(\tilde{p}(k))$.

Hence, by the definition of $G(\tilde{p}(k))$:

(vi) $F^{t,i}(\tilde{p}(k)) = w^{t,i}$ for $t = 1, \ldots, k$ and $i = 1, \ldots, N$ such that $(t,i) \in I_k$.

$F^{k+1,i}(\tilde{p}(k)) = w_k^{k+1,i}$ for $t = k+1$ and $i = 1, \ldots, N$;

(vii) $F^{t,i}(\tilde{p}(k)) \geqslant w^{t,i}$ for all $(t,i) \in I_k$;

and

(viii) $$\sum_{(t,i)\in I_k} (F^{t,i}(\tilde{p}(k)) - w^{t,i}) = \varepsilon.$$

We can now define the desired monetary allocation $(p(k), m(k), x(k))$ for the full economy by (arbitrarily) extending our equilibrium for the truncated economy. We first normalize the prices $\tilde{p}(k)$ by $\tilde{p}^{1,1}(k) = 1$. Let $q = (1, \ldots, 1) \in R^N$.

We extend our prices $\tilde{p}(k)$ into a full system for our infinite economy by defining

(b*) $p(k) = (\tilde{p}^1(k), \ldots, \tilde{p}^{k+1}(k), q, q, \ldots) \in S$, where $\tilde{p}^t(k)$ is the price of t-period goods ($t = 1, \ldots, k+1$) in the equilibrium price system $\tilde{p}(k)$.

We define our monetary policy m by

(c*) $m_{0,h*}(k) = m_{0,h*}(\tilde{p}(k))$ for $(g,h) = (0,h^*)$ and $m_{g,h}(k) = 0$ for all $(g,h) \neq (0,h^*)$.

allocations are defined by

(d*) $x_{g,h}(k) = f_{g,h}(p(k), p(k)w'_{g,h} + m_{g,h}(k))$ for $g = 0, 1, \ldots$ and $h = 1, \ldots, H$.

By its definition in (b*) - (d*) we conclude that $(p(k), m(k), x(k))$ is a monetary allocation for our full economy. Note that for agents in generations $g = 0, \ldots, k$ and goods in periods $t = 1, \ldots, k+1$, the price, transfers, and allocations defined in (b*) - (d*), $(p(k), m(k), x(k))$, coincide with their respective values in the equilibrium of the truncated economy.

Let $E^{t,i}(k)$ denote the aggregate excess demand for good (t,i), i.e. $E^{t,i}(k) = \sum_{\substack{(g,h) \\ g=t-1,t}} (x^{t,i}_{g,h}(k) - w^{t,i}_{g,h})$.

The following are consequences of (vi) - (viii) respectively;

(e*) $E^{t,i}(k) = 0$ for $t = 1, \ldots, k$ and $i = 1, \ldots, N$ such that $(t,i) \in I_k(k)$,

(f*) $E^{t,i}(k) \geqslant 0$ for each $(t,i) \in I_k$,

and

(g*) $\sum_{(t,i) \in I_k} E^{t,i}(k) = \mathcal{E}$.

We now verify that the hypothesis of Lemma 4.1 are satisfied by the sequence of monetary allocations $\{ (p(k),m(k),x(k)) \}$. Clearly, the first hypothesis is satisfied since $m_{g,h}(k) > 0$ for all agents (g,h) and points k. Define the point $y \in X$ by

$$y^{t,i}_{g,h} = w^{t,i} + \varepsilon + x^{t,i}_{g,h}(g-1) + f^{t,i}_{g,h}((q,q),(q,q)w_{g,h}).$$

Consider any agent (g,h), good (t,i), and iteration k.

If $g < k$, then by (vi) - (viii)

$$x^{t,i}_{g,h}(k) < \sum_{\substack{(g,h)\\ g=t-1,t}} x^{t,i}_{g,h}(k) = F^{t,i}(p(k)) \leqslant w^{t,i} + \varepsilon.$$

If $g = k + 1$, then $x^{t,i}_{g,h}(k) = x^{t,i}_{g,h}(g-1)$.

If $g > k + 1$, then by the definition of $x_{g,h}(k)$ in (d*),

$$x^{t,i}_{g,h}(k) = f^{t,i}_{g,h}((q,q),(q,q)w_{g,h}).$$

By the above arguments, the second hypothesis is also satisfied, i.e. $x(k) \leqslant y$ for all k. Therefore, we may apply Lemma 4.1 to conclude that there is a monetary allocation, (p,m,x), which is a limit point of $\{ (p(k),m(k),x(k)) \}$.

We now verify that (p,m,x) satisfies all of the requirements of our proposition. Properties (a) and (b) follow from the definition of the monetary policy $m(k)$ in (c*) and from the definition of $m_{0,h^*}(\cdot)$ in (a*). The only portion of property (c) that does not obviously follow from (e*) - (g*) is $\sum_{(t,i)\in I} |E^{t,i}| \leqslant \varepsilon$. To prove this, we consider any finite subset, I', of I and show that $\sum_{(t,i)\in I'} |E^{t,i}| \leqslant \varepsilon$. Since $E^{t,i}(k) \geqslant 0$ and $E^{t,i}(k) \to E^{t,i}$ for all $(t,i) \in I$,

$$\sum_{(t,i)\in I'} |E^{t,i}| = \lim \sum_{(t,i)\in I'} E^{t,i}(k) \leqslant \lim \sum_{(t,i)\in I_k} E^{t,i}(k).$$

(c) now follows since by (g*) the limit on the far right is no greater than ε. By previous remarks in section 3, the fact that markets are in equilibrium for all goods in the first period in the allocation

$(p(k), m(k), x(k))$ implies $m_{0,h*}(k) < \sum_i w^{1,i}$. But, for any good $(t,i) \in I$ and $k > t$, the definition of $m_{0,h*}(\circ)$ implies $p^{t,i}(k) < \frac{1}{\varepsilon} m_{0,h*}(p(k))$. Hence, by ($c^*$), $p^{t,i}(k) < \frac{1}{\varepsilon} m_{0,h*}(p(k)) = \frac{1}{\varepsilon} m_{0,h*}(k) < \frac{1}{\varepsilon} \sum_i w^{1,i}$. Therefore, in the limit as $k \to \infty$, $p^{t,i} \leqslant \frac{1}{\varepsilon} \sum_i w^{1,i}$. Finally, since I is infinite, (e) follows from (d) by an application of Lemma 3.1.

Q.E.D.

4.4 Proposition. For any infinite collection of agents $A \subseteq \{ (g,h) : g = 1, 2, \ldots$ and $h = 1, \ldots, H \}$ and any small $\varepsilon > 0$, there exists a monetary equilibrium (p,m,x) that satisfies:

(a) $m_{0,h} \geqslant 0$ for $g = 0$ and $h = 1, \ldots, H$;

(b) $m_{g,h} = 0$ for $g = 1, 2, \ldots$ and $h = 1, \ldots, H$ such that $(g,h) \in A$,

$$\sum_{(g,h) \in A} \frac{|m_{g,h}|}{pw'_{g,h}} < \varepsilon;$$

(c) $p^{g,1} \leqslant \frac{1}{\varepsilon} \sum_i w^{1,i} + 1$ for all g such that $(g,h) \in A$ for some h, and

(d) x is a pareto optimal allocation scheme.

The techniques that we use to establish the above proposition parallel those that were employed in the proof of Proposition 4.3. In particular, the desired equilibrium (p,m,x) will be found as a limit point of allocations, $\{ (p(k), m(k), x(k)) \}$. In turn, each of the allocations are derived from the competitive equilibria of a suitably chosen collection of truncated economies. In the k'th allocation, $(p(k), m(k), x(k))$, agent $(0,h^*)$ receives the monetary transfer $m_{0,h*}(k) = \varepsilon \sum p^{g,1}(k) \min \{1, \max \{0, p^{g,1}(k) - \frac{1}{\varepsilon} \sum_i w^{1,i}\} \}$, where the summation is taken over all $g < k$ such that $(g,h) \in A$ for some h. By the boundedness property of $m_{0,h*}(k)$, we conclude that $p^{g,i}(k) < \frac{1}{\epsilon}\sum_{i} w^{1,i} + 1$ for all such $g < k$. Passing to the limit, the above inequality bounds an infinite number of equilibrium prices, $p^{g,i}$, as specified in condition (c) of the proposition. Hence, the optimality of the limiting allocation scheme, x, is guaranteed.

As with the previous proposition, the bulk of the proof of Proposition 4.4 lies in establishing the existence of each of the allocations (p(k),m(k),x(k)). The remainder of the proof follows along the same lines as the proof of Proposition 4.4 with the single exception that we want to bound the level of monetary transfers instead of the divergence from feasibility.

Proof of Proposition 4.4. Let $h^* = 1$.

Fix any $k = 1, 2, \ldots$.

To establish the existence of the k'th monetary allocation, (p(k),m(k),x(k)), we will consider a finite truncation of our full economy. This truncated economy will consist of all of the agents in generations 0, ... , k. These agents will be allowed to trade in all of the markets for the goods which are available in their lifetimes. The resulting equilibrium for this finite economy will then be (arbitrarily) completed to form the monetary allocation (p(k),m(k),x(k)).

Formally, consider the (truncated) collection of agents

$\{ (g,h) : g = 0, \ldots , k \text{ and } h = 1, \ldots , H \}$,

of goods

$\{ (t,i) : t = 1, \ldots , k+1 \text{ and } i = 1, \ldots , N \}$,

of commodity prices

$P_k = \{ (p^1, \ldots , p^{k+1}) \in R^N_{++} \times \ldots \times R^N_{++} \}$,

and the collection of agents that can be taxed

$A_k = \{ (g,h) \in A : g = 1, \ldots , k \text{ and } h = 1, \ldots , H \}$.

We wish to find a monetary allocation, $(p(k), m(k), x(k))$, for the truncated economy in which agent all agents $(g,h) \in A_k$ pay the tax

$-m_{g,h}(k) = -p^{g,1}(k) \min \{ 1, \max \{ 0, p^{g,1}(k) \in \frac{1}{-} \ _i \ w^{1,i} \} \}$

and the agent $(0,h^*)$ receives the tax revenues, i.e.

$$m_{0,h^*}(k) = \sum_{(g,h)\in A_k} -m_{g,h}(k).$$

For each $(g,h) \in A_k$, let the function

$m_{g,h} : S_k \longrightarrow R$

be defined by

$$(a^*) \ m_{g,h}(p) = - \varepsilon p^{g,1} \min \{ 1, \max \{ 0, p^{g,1} - \frac{1}{\varepsilon} \sum_i w^{1,i} \} \} .$$

We also defined the function

$m_{0,h^*} : S_k \longrightarrow R$

by

$$(b^*) \ m_{0,h^*}(p) = \sum_{(g,h)\in A_k} m_{g,h}(p).$$

The aggregate excess demand for goods in this economy, which is denoted by $H(p) = (H^1(p), \ldots , H^{k+1}(p)) \in S_k$, is then given by

$$H^t(p) = \sum_{\substack{(g,h) \\ g=t-1,t}} (f^t_{g,h}(p, pw'_{g,h}+m_{g,h}(p)) - w^t_{g,h}) \quad \text{for } t = 1, \ldots , k$$

and

$$H^{k+1}(p) = \ _h (f^{k+1}_{k,h}(p, pw_{k,h}+m_{k,h}(p)) - w^{k+1}_{k,h}) \quad \text{for } t = k + 1$$

We now verify the excess demand function, $H(\cdot)$, satisfies the following;

(i) $H(\cdot)$ is continuous on S_k,

(ii) (Walras's law) $pH(p) = 0$ for each $p \in S_k$,

(iii) (Boundary condition) $\|H(p)\| \to 0$ as p approaches the boundary of S_k.

(i) follows from the continuity of demand, $f_{g,h}(\cdot)$, and the continuity of the monetary transfers, $m_{g,h}(\cdot)$. (ii) follows from the fact, by our monotonicity assumption (A.1), that each agent spends all of his after-tax income $pw'_{g,h} + m_{g,h}(p)$ and from the fact that the monetary transfer to agent $(0,h^*)$ is exactly offset by the negative transfers, $m_{g,h}(p)$, to agents (g,h) contained in A_k. (iii) follows immediately from our interiority assumption (A.2).

Since $H(\cdot)$ satisfies (i) - (iii), we can appeal to result in Dierker [1974, Section 8] to deduce that there are prices $\tilde{p}(k) \in P_k$ such that

(iv) $H(\tilde{p}(k)) = 0$.

We can now define the desired monetary allocation $(p(k), m(k), x(k))$ for the full economy by (arbitrarily) extending our equilibrium for the truncated economy. We first normalize the prices $\tilde{p}(k)$ by $\tilde{p}^{1,1}(k) = 1$. Let $q = (1, \ldots, 1) \in R^N$.

We extend our prices $\tilde{p}$ into a full system for our infinite economy by defining

(c*) $p(k) = (\tilde{p}^1(k), \ldots, \tilde{p}^{k+1}(k), q, q, \ldots) \in S$, where $p^t(k)$ is the price of the t-period goods in the equilibrium of our truncated economy.

We define our monetary policy $m(k)$ by

(d*) $m_{g,h}(k) = m_{g,h}(\tilde{p}(k))$ for $(g,h) = (0,h^*)$ and all $(g,h) \in A_k$

and

$m_{g,h}(k) = 0$ for all other (g,h).

allocations are then defined by

$$(e^*)\quad x_{g,h} = f_{g,h}(p(k), p(k)w_{g,h} + m_{g,h}(k)) \quad \text{for } g = 0, 1, \ldots \text{ and } h = 1, \ldots, H.$$

By the definitions in (b^*) – (e^*), we conclude that $(p(k), m(k), x(k))$ is a monetary allocation for our full economy.

Let $E^{t,i}(k)$ denote the aggregate excess demand for good (t,i), i.e. $E^{t,i}(k) = \sum_{\substack{(g,h) \\ g=t-1,t}} (w^{t,i}_{g,h}(k) - w^{t,i}_{g,h})$.

Since the allocations of the agents in generations 0, ... , k+1 in the scheme $x(k)$ coincide with their allocations in the truncated economy, the equilibrium condition (iv) implies

(v) $E^t(k) = 0$ for $t = 1, \ldots, k$.

It can now be shown that the sequence $\{ (p(k), m(k), x(k)) \}$ satisfies the hypothesis of Lemma 4.1. The verification of this fact would be nearly identical to the verification of the corresponding sequence of allocations in the proof of Proposition 4.3 and, therefore, will not be presented here. By Lemma 4.1, there is a monetary allocation, (p,m,x), which is a limit point of $\{ (p(k), m(k), x(k)) \}$. We can employ (v) to deduce that (p,m,x) is a monetary equilibrium for our full economy.

We now verify that (p,m,x) satisfies all of the requirements of our proposition. Property (a) follows from definitions (a^*), (b^*), and (d^*). The only portion of property (b) that does not obviously follow from (b^*) is

$$\sum_{(g,h)\in A} \frac{|m_{g,h}|}{p w'_{g,h}} < \varepsilon$$

We will find it more convenient to verify the stronger statement

$$\sum_{(g,h)\in A} \frac{|m_{g,h}|}{p^{g,1}} < \varepsilon.$$

By definitions (a*) and (d*), $m_{g,h}(k) = 0$ only if $p^{g,1}(k) > \frac{1}{\varepsilon}\sum_i w^{1,i}$.

Hence,

$$\text{(vi)} \quad \sum_{(g,h)\in A_k} \frac{|m_{g,h}(k)|}{p^{g,1}(k)} < \frac{\varepsilon}{\sum_i w^{1,i}} \sum_{(g,h)\in A_k} |m_{g,h}(k)|.$$

By definitions (b*) and (d*),

$$\text{(vii)} \quad m_{0,h^*}(k) = \sum_{(g,h)\in A_k} |m_{g,h}(k)|.$$

Combining (vi), (vii), and our now familiar boundedness property, $m_{0,h^*}(k) < \sum_i w^{1,i}$, yields

$$\text{(viii)} \quad \sum_{(g,h)\in A_k} \frac{|m_{g,h}(k)|}{p^{g,1}(k)} < \varepsilon.$$

Therefore, for any finite $A' \subseteq A$, (viii) implies

$$\sum_{(g,h)\in A'} \frac{|m_{g,h}|}{p^{g,1}} = \lim \sum_{(g,h)\in A'} \frac{|m_{g,h}(k)|}{p^{g,1}(k)}$$

$$\leq \lim \sum_{(g,h)\in A_k} \frac{|m_{g,h}(k)|}{p^{g,1}(k)} < \varepsilon.$$

(b) now follows, since the above inequalities hold for all finite $A' \subseteq A$. By (viii), $\frac{|m_{g,h}(k)|}{p^{g,1}(k)} < \varepsilon$ for all $(g,h) \in A$ and k sufficiently large (i.e. for all $(g,h) \in A_k$). But, by definitions (a*) and (d*), this implies

$p^{g,1}(k) < \frac{1}{\varepsilon}\sum_i w^{1,i} + 1$. (c) now follows if we pass to the limit as k becomes large in the above inequality. Finally, since A is infinite, (d) follows from (c) by an application of Lemma 4.1.

Q.E.D.

5. NON-EXISTENCE OF OPTIMAL AND PROPER MONETARY POLICIES

Upon examination of any one of the Propositions 3.2, 3.3, 4.3, or 4.4, one may wonder why these "approximate" results cannot be used to prove the existence of a pareto optimal and passive monetary equilibrium by letting the tolerance, ε, of the equilibria in these propositions tend to 0. Indeed, if we consider a sequence of monetary allocations $\{ (p(k),m(k),x(k)) \}$, from any of these propositions, which correspond to various choices of $\varepsilon(k)$ with $\varepsilon(k) \rightarrow 0$, then by Lemma 3.2 we may obtain a limit point (p,m,x) that is a passive monetary equilibrium. However, (p,m,x) need not be a pareto optimal equilibrium even though each of the allocations $(p(k),m(k),x(k))$ is optimal. The problem is that the set of feasible and pareto optimal allocation schemes is not closed. Proposition 5.2, of this section, states that this non-closure problem exists for all economies.

We can illustrate the non-closure property by considering the simple overlapping generations economy that was introduced in section 1. Using the notation developed in section 2, preferences are specified by $u_0(x_0) = x_0^1$ for $g = 0$ and $u_g(x_g) = x_g^g + x_g^{g+1}$ for $g = 1, 2, \ldots$. Note, since there is only one good per period and only one agent per generation, all good super-scripts and all agent sub-scripts have been suppressed. In section 1, we considered the allocation scheme $x = ((1), (1,1), \ldots)$ and stated that x was inefficient since it was pareto dominated by the scheme $((2), (0,2), \ldots)$. Consider the sequence of allocation schemes $\{ x(k) \}$ defined by, (5.1) x(0) = ((2), (0,2), ...)

x(1) = ((1), (1,2), (0,2), ...)

x(2) = ((1), (1,1), (1,2), (0,2), ...)

x(3) = ((1), (1,1), (1,1), (1,2), (0,2), ...)

etc. .

Our discontinuity property can now be revealed since x, which is not pareto optimal, is clearly the limit point of the sequence of allocation schemes, { x(k) }, each of which can be shown to be both feasible and pareto optimal.

Obviously, our non-closure result relied on our particular choice of a topology for the space of allocation schemes, X. We have asserted that the set of feasible and pareto optimal allocation schemes is not closed given that X is endowed with the product topology. It may well be the case that the non-closure problem disappears if we endow X with some other well known topology such as that generated by the sup-norm or the L_1-norm. Although endowing X with a topology finer than that of the product topology may establish the general closure of the feasible pareto optimal set, the compactness property of this set could likely fail. Recall, that a crucial step in the argument set forth in the first paragraph of this section was that we could find a limit point of the set of monetary equilibrium, { (p(k),m(k),x(k)) }. Without the compactness of the set of feasible pareto optimal allocations, we could no longer guarantee that such a limit point exists. For example, one may verify that there would be no limit point of the sequence { x(k) }, as defined in 5.1, if X was endowed with the topologies which are generated by either the sup-norm or the L_1-norm. Hence, there would be no limit point of the sequence of monetary equilibria, $\{ (p(k), m(k), x(k)) \}$, where for each k, p(k) and m(k) are the prices and monetary policies which support the allocation x(k) as a monetary equilibrium.

Before we formally state our non-closure result, we make the following definitions.

$X_f = \{ x : x$ is an allocation scheme such that $\sum_{(g,h)} x'_{g,h} \leqslant w \}$ is the set of feasible allocation schemes.

$$U = \{ (u_{0,2}, \ldots, u_{0,h}, \ldots, u_{g,1}, \ldots, u_{g,H}, \ldots) : \text{there is an } x \in X \text{ such that } u_{g,h}(x_{g,h}) \geqslant u_{g,h} \text{ for all } (g,h) \neq (0,1) \}$$

is the set of attainable utility levels of all of the agents except (0, 1).

We also define a function

$x : U \longrightarrow X_f$

by

$$x(u) = (x_{g,h}(u)) = \arg\max\, u_{0,1}(x_{0,1})$$

subject to

$$u_{g,h}(x_{g,h}) \geqslant u_{g,h} \quad \text{for all } (g,h) \neq (0,1)$$

$$x \in X_f.$$

That is, given the utility levels of all of the agents in the economy, except $(0,h^*)$, x(u) is the (unique) pareto optimal allocation that generates these levels. It can be readily verified that for any economy, the closure of the set of pareto optimal allocations is equivalent to the continuity of the function x(u). Therefore, the following proposition demonstrates that, for any economy, the set of feasible pareto optimal allocations is not closed

5.2. Proposition. x(u) is discontinuous at every point in its domain, where both U and X_f are endowed with the product topology.

This proposition, along with some other interesting facts about the nature of the set of pareto optimal allocations in dynamic economies, was established in chapter 1.

6. MILLAN'S COUNTER-EXAMPLE

The purpose of this section is two-fold. First, we specify an economy that satisfies the assumptions of our section 2 but for which there are no passive monetary equilibrium that are optimal. The economy we present is only a specific selection from the class of counter-examples set forth in Millan [1981] but it is provided to pin down Millan's subtle arguments. Secondly, for this same economy, we illustrate our primary existence result (Proposition 4.4) by demonstrating that, even though there are no optimal monetary policies that are passive, there are optimal monetary policies that come arbitrarily close to being passive.

Our counter-example requires only one good per period (N = 1) and two agents (H = 2) in each generation t (t = 1, 2, ...); which we denote (1,t) and (2,t). The economy is stationary with respect to preferences and "asymptotically" stationary with respect to endowments. Specifically, the preferences of agent (1,t) are specified by the utility function

(6.1)
$$u_{1,t}(y,z) = u_1(y,z) = \begin{cases} \dfrac{9^3}{4^8 3^3}\, y^8 z^3 & \text{for} \quad y/z \leq 4/3 \\ (5z-y)^3(y-z) & \text{for} \quad 4/3 \leq y/z \leq 3/2 \\ \dfrac{7^3}{3^3 2^4}\, y^3 z^4 & \text{for} \quad y/z \geq 3/2 \end{cases}$$

where for simplicity y (or $y_{1,t}$) denotes consumption during the agents youth and z (or $z_{1,t}$) denotes old-age consumption. Given these preferences, it is easy to verify that the agents rate of intertemporal substitution is given by

(6.2)

$$\frac{du_1/dz}{du_1/dy}(y/z) = \begin{cases} (3/8)y/z & \text{for} \quad y/z \leqslant 4/3 \\ \dfrac{4y/z - 5}{2 - y/z} & \text{for} \quad 4/3 \leqslant y/z \leqslant 3/2 \\ (4/3)y/z & \text{for} \quad y/z \geqslant 3/2 \end{cases}$$

Fix any e such that $0 < e < 1/2$.

Agent (1,t) is endowed with $5 + 3e^t$ units of period t goods and $1 + 2e^t$ units of period t + 1 goods. Given relative prices $r = p_{t+1}/p_t$ and the monetary transfer (in real terms) $n = m_t/p_t$, agent (1,t)'s market demand is determined by the equations

$$r = \frac{du_1/dz}{du_1/dy}(y_{1,t}/z_{1,t})$$

$$y_{1,t} + rz_{1,t} = (5+3e^t) + (1+2e^t)r + n$$

(Since in our example it will not be necessary to impose a monetary transfer on the other agent in generation t, (2,t), m_t will denote the monetary transfer to agent (1,t).)

One may readily verify that the (unique) solution to these equations is

(6.3)

$$(y_{2,t}, z_{2,t}) = \begin{cases} \dfrac{r + 5 + e^t(2r+3) + n}{11r}(8r,3) & \text{for } 0 < r \leqslant 1/2 \\ \dfrac{r + 5 + e^t(2r+3) + n}{(r+1)(r+5)}(2r+5, r+4) & \text{for } 1/2 \leqslant r \leqslant 2 \\ \dfrac{r + 5 + e^t(2r+3) + n}{7r}(3r,4) & \text{for } r \geqslant 2 \end{cases}$$

Agent (2,t)'s tastes are specified by

(6.4)

$$u_{2,t}(y,z) = u_2(y,z) = \begin{cases} \dfrac{7^3}{2^4 3^3}\, y^4 z^3 & \text{for} \quad y/z \leqslant 2/3 \\ (5y-z)^3(z-y) & \text{for} \quad 2/3 \leqslant y/z \leqslant 3/4 \\ \dfrac{9^3}{3^3 4^8}\, y^3 z^8 & \text{for} \quad y/z \geqslant 3/4 \end{cases}$$

Again one may verify that the agents rate of intertemporal substitution is given by

(6.5)

$$\frac{du_2/dz}{du_2/dy}\,(y/z) = \begin{cases} (3/4)y/z & \text{for } y/z \leqslant 2/3 \\ \dfrac{2y/z - 1}{4-5y/z} & \text{for } 2/3 \leqslant y/z \leqslant 3/4 \\ (8/3)y/z & \text{for } y/z \geqslant 3/4 \end{cases}$$

With endowments of 1 unit of period t goods and 5 units of period t + 1 goods, the agents market demand is defined by the equations

$$r = \frac{du_2/dz}{du_2/dy}\,(y_{1,t}/z_{1,t})$$

$$y_{2,t} + rz_{2,t} = 1 + 5r$$

One may verify that the (unique) solution to these equations is

(6.6)

$$(y_{2,t}(r), z_{2,t}(r)) = \begin{cases} \dfrac{5r+1}{7r}(4r,3) & \text{for } 0 < r \leqslant 1/2 \\ \dfrac{1}{r+1}(4r+1, 5r+2) & \text{for } 1/2 \leqslant r \leqslant 2 \\ \dfrac{5r+1}{11r}(3r,8) & \text{for } r \geqslant 2 \end{cases}$$

The aggregate excess demand of generation t, given relative prices

$r = p_{t+1}/p_t$ and the monetary transfer $n = m_t/p_t$ to agent $(1,t)$ is determined by

$(E_y(r;e^t,n),E_z(r;e^t,n)) = (y_{1,t},z_{1,t}) + (y_{t,2},z_{t,2}) - (6+3e^t,6+2e^t)$.

From (6.3) and (6.6) we obtain

$$(6.7) \qquad (E_y,E_z) = \begin{cases} \frac{138(1-2r)+7(9-16r)e^t}{77r}(-r,1) + \frac{n}{11r}(8r,3) & \text{for } r \leqslant 1/2 \\ \frac{(2-r)e^t}{(r+1)(r+5)}(-r,1) + \frac{n}{(r+1)(r+5)}(2r+5,r+4) & \text{for } 1/2 \leqslant r \leqslant 2 \\ \frac{138(2-r)+66(2-r)e^t}{77r}(-r,1) + \frac{n}{7r}(3r,4) & \text{for } r \geqslant 2 \end{cases}$$

The following properties of these excess demands are immediate consequences of (6.7).

(6.8) (i) $E_y(r;e^t,n) + rE_z(r;e^t,n) = n$

(ii) $E_z(r;e^t,0)$ is decreasing in r

(iii) $E_z(2;e^t,0) = 0$

(iv) $E_z(1/2;e^t,0) = (2/11)e^t$

(v) $E_y(r;e^t,0) \geqslant -\frac{138+63e^t}{77} \geqslant -\frac{138+63e}{77}$

(vi) $(E_y(1;e^t,n),E_z(1;e^t,n)) = e^t/12(-1,1) + n/12(7,5)$

(vii) $E_z(r;e^t,0) \to +\infty$ as $r \to 0$

(viii) $E_y(r;e^t,0) < 0$ for $0 < r < 2$

Given our excess demand functions in (6.7), the requirements of a monetary equilibrium (see (2.2)) can be expressed in terms of prices (p_t) and monetary transfers (m_t) as

(6.9)

(i) $m_0/p_1 + E_y(p_2/p_1;e^1,m_1/p_1) = 0$

(ii) $E_z(p_{t+1}/p_t;e^t,m_t/p_t) + E_y(p_{t+2}/p_{t+1};e^{t+1},m_{t+1}/p_{t+1}) = 0$

for $t = 1, 2, \ldots$

(Formally, we have not yet specified the initial generation of agents. However, since there is only one good available in the first period, as long as there preferences are monotonic, each initial agent will simply spend all of his income on the first period good. Therefore, if we let m_0 denote the net aggregate transfer to agents in the initial generation, then the aggregate excess demand by these agents is m_0/p_1.)

One may readily verify by repeated applications of (6.8)(i) and (6.9), that any monetary equilibrium satisfies

(6.10) $p_{t+1}E_z(p_{t+1}/p_t;e^t,m_t/p_t) = m_0 + \sum_{1\leq k\leq t} m_k$

By inspection, the economy described above satisfies all of the assumptions of section 2. One may also verify that the economy satisfies the additional restriction of Proposition 5.6 of Balasko and Shell [1981], which allows us to strengthen the implications of our Lemma 3.1 to state that a monetary equilibrium for our economy is pareto optimal if, and only if, $\sum_t \frac{1}{p_t} = \infty$.

We will now establish three facts about our economy.

(1) There are no competitive equilibria (i.e. $0 = m_0 = m_1 = \ldots$) that are pareto optimal.

Consider any competitive equilibrium. By (6.10), $E_z(p_{t+1}/p_t;e^t,0) = 0$ for all t. By (6.8)(ii) and (iii), we see that $p_{t+1}/p_t = 2$. Therefore, $\sum_t \frac{1}{p_t} = \frac{2}{p_1} < \infty$. Hence, the equilibrium is not optimal.

(2) There are no passive monetary equilibria (i.e. $m_0 \geqslant 0$ and $0 = m_1 = m_2 = \ldots$) that are pareto optimal.

Consider any passive monetary equilibrium. Let $r_t = p_{t+1}/p_t$. By (6.10),

(a) $E_z(r_t;e^t,0) = m_0/p_{t+1} > 0$ for all t.

Hence, by (6.8)(ii) and (iii),

(b) $r_t < 2$ for all t.

(6.10) then implies

$$\text{(c)}\quad \frac{p_{t+1}E_z(r_t;e^t,0)}{p_{t+2}E_z(r_{t+1};e^{t+1},0)} = \frac{m_0}{m_0} = 1.$$

We now demonstrate

(d) For any t; if $r_t < 1/2$, then $r_{t+1} < 1/2$.

If (d) did not hold, then for some t, $r_t < 1/2$ and $r_{t+1} \geqslant 1/2$.

By (6.8)(ii) and (iv), $E_z(r_t;e^t,0) > E_z(1/2;e^t,0) = (2/11)e^t$

and

$E_z(r_{t+1};e^{t+1},0) \leqslant E_z(1/2;e^{t+1},0) = (2/11)e^{t+1}$. By (a), this implies

$$\frac{E_z(r_t;e^t,0)}{E_z(r_{t+1};e^{t+1},0)} > \frac{e^t}{e^{t+1}} = \frac{1}{e} > 2.$$

But, (c) then implies $r_{t+1} = p_{t+2}/p_{t+1} > 2$, which contradicts (b).

The above paragraph establishes (d). We now use (d) to prove

(e) $r_t \geqslant 1/2$ for each t.

If (e) did not hold, then $r_T < 1/2$ for some T. By (d), this implies $r_t < 1/2$ for all $t > T$. But, (c) then implies $E_z(r_{t+1};e^{t+1},0)/E_z(r_t;e^t,0) = 1/r_{t+1} > 2$ for $t > T - 1$. This, together with $E_z(r_{T-1};e^{T-1},0) > 0$ implies $E_z(r_t;e^t,0) \rightarrow \infty$ as $t \rightarrow \infty$, which contradicts (6.8)(v) and (6.9)(ii) for large t.

By (6.8)(ii), (iv), and (e),
$E_z(r_t;e^t,0) < E_z(1/2;e^t,0) = (2/11)e^t$ for all t.
Therefore, by (6.10),

$$\sum_{t=1}^{\infty} 1/p_t = 1/p_1 + \sum_{t=1}^{\infty} 1/p_{t+1} = 1/p_1 + (1/m_0) \sum_{t=1}^{\infty} E_z(r_t;e^t,0)$$

$$< 1/p_1 + (1/m_0)\sum_{t=1}^{\infty}(2/11)e^t$$

$$= 1/p_1 + \frac{(2/11)e}{m_0(1-e)} < \infty.$$

Therefore, the equilibrium is not optimal. Hence (2) is established.

(3) (Proposition 4.4) For any $\epsilon > 0$, there exists a pareto optimal monetary equilibrium such that $\sum_{t=1}^{\infty} |m_t|/p_t < \epsilon$.

Since $0 < e < 1/2$, we can find a date T sufficiently large such that $e^T/(7e+5) < \epsilon$. We claim that the following policy $m_0 = e^T/(7e+5)$; $m_t = 0$ for $t = 1, \ldots, T-1$; and $m_t = -(1-e)/(7e+5)e^t$ for $t = T, T+1, \ldots$; together with prices p_1, p_2, ... satisfying $1 = p_T = p_{T+1} = \ldots$ constitute a monetary equilibrium.

Note, once we have established that the above policy and prices are an equilibrium, then (3) will be established since $\sum_{t=1}^{\infty} 1/p_t \geqslant \sum_{t=T}^{\infty} 1 = \infty$ implies optimality and

$$\sum_{t=1}^{\infty} |m_t|/p_t = \frac{1-e}{7e+5} \sum_{t=T}^{\infty} e^t = e^T/(7e+5) < \varepsilon.$$

Before we define the prices $p_1, \ldots, p_{T-1}$ we demonstrate that the markets for goods in dates T+1, T+2, ... are in equilibrium. Given the above definitions of the m_t, (6.8)(vi) implies $(E_y(1;e^t,m_t), E_z(1;e^t,m_t)) = \frac{e^t}{7e+5}(-1,e)$ for $t > T$. Hence, the equilibrium condition (6.9)(ii) for dates $t > T$ are satisfied since $E_z(1;e^t,m_t) + E_y(1;e^{t+1},m_{t+1}) = \frac{e^t}{7e+5}e - \frac{e^{t+1}}{7e+5} = 0$. (Recall that $r_t = p_t = 1$ for all $t > T$.)

The remaining prices $p_1, \ldots, p_{T-1}$ can be defined recursively by the market clearing conditions (6.8)(ii). Specifically, p_{T-1} is defined by

(6.11)

(i) $E_z(p_T/p_{T-1};e^{T-1},0) = -E_y(1;e^T,m_T) = e^T/(7e+5) > 0$

while p_t ($t = 1, \ldots, T-2$) is defined by

(ii) $E_z(p_{t+1},p_t;e^t,0) = -E_y(p_{t+2}/p_{t+1};e^{t+1},0)$.

(6.8)(iii) and (vii) guarantee that p_{T-1} is well-defined by (6.11)(i) and that $p_T/p_{T-1} < 2$.

Assuming that $p_{t+2}/p_{t+1} < 2$, (6.8)(iii), (vii), and (viii) guarantee that p_t is well-defined in (6.11)(ii) and that $p_{t+1}/p_t < 2$. Hence, each of the $p_{T-1}, p_{T-2}, \ldots, p_1$ are succesivly well-defined.

Finally, if we define $m_0 = p_1 E_y(p_2/p_1;e^1,m_1/p_1)$, then we have a monetary equilibrium. By (6.10),

$$m_0 = p_{T+1} E_z(p_{T+1}/p_T; e^T, m_T) = e^T/(7e+5).$$

7. CONCLUSION

In this paper, we have considered a general class of overlapping generations economies and have characterized some of the properties of their equilibria. We now summarize our current knowledge about these equilibria, along with an open question, in the following table.

	Passivity of the monetary policy	Feasibility of the allocation scheme	Optimality of the allocation scheme
Millan's counter-example: We can not guarentee that there exists a monetary allocation with the following properties.	passive	feasible	pareto optimal
Proposition 4.4: In general, there exists a monetary allocation with the following properties.	approximately passive	feasible	pareto optimal
Proposition 4.3: In general, there exists a monetary allocation with the following properties.	passive	approximately feasible	pareto optimal
Open question: Is it true that, in general, there exists a monetary allocation with the following properties?	passive	feasible	approximately pareto optimal

Of course, before one can answer the above open question, one must first define a sense in which a given allocation is "approximately" pareto optimal. To be precise, one needs a measure of the degree to which an allocation scheme is inefficient. One such measure can be neatly expressed in terms of the function x(u) that was introduced in section 5. The inefficiency of an allocation can be measured by the amount by which agent (0, 1) can be made better off, by a feasible redistribution of goods, while none of the other agents are made worse off. Specifically, given any allocation scheme x, our measure of the inefficiency of x is

$E(x) = u_{0,1}(x_{0,1}(u_{0,2}(x), \ldots, u_{0,H}(x), u_{1,1}(x), \quad)) - u_{0,1}(x).$

Our open question can now be precisely stated as: Is it true that, in general, for any $\epsilon > 0$, there exists a passive monetary equilibrium (p,m,x) such that $E(x) < \epsilon$. Note, even if the answer to the above question is affirmative, the discontinuity of $x(\cdot)$ would prevent us from allowing $\epsilon \rightarrow 0$ and concluding that there is a passive and pareto optimal monetary allocation.

While an affirmative answer to the above question would be a more satisfying statement about the general existence of optimal monetary policies, its proof would likely require different, and probably more subtle, techniques than those employed in this paper.

CHAPTER III

EQUILIBRIA IN DYNAMIC ECONOMIES WITH OR WITHOUT COMPLETE MARKETS

1. INTRODUCTION

In their attempts to characterize rational market behavior in the face of uncertainty, many economists focus their attention on the Arrow-Debreu theory of contingent commodities (see Arrow [1953] or Debreu [1953]). This theory extends the classical Walrasian model by distinguishing commodities not only by their physical characteristics but by the dates and environmental events in which they are available. Implicit in the Arrow-Debreu model is the assumption that all (contingent) commodities are traded in one (complete) market. Over the years, the Arrow-Debreu theory has been criticized (e.g. see Radner [1968, 1970, 1981]) for both the restrictivness of the assumption of a complete market and the inability of the model to explain some basic market behavior such as continual trading in commodity and stock markets.

One obvious way in which one can explain continual trading is to assume that markets for contingent (future) delivery sequentially open and close over time and at no point in time can all future trades be made (i.e. each market is incomplete). Not only does this sequential framework seem more plausible and compatible with continuous trading than the Arrow-Debreu model, but many economists have found it useful to study the role of money in providing insurance (e.g. Bewley [1979] or Foley and Helwig [1975]). In this paper, we provide a firm foundation for the analysis of incomplete markets by establishing the existence of competitive equilibria for a very general class of sequential economies.

The formal specification of our sequential economy is patterned after the work of Radner [1972, 1982]. In this paper, we significantly extend Radner's existence results in many ways. First, since sequential economies are inherently dynamic, we feel that one important refinement of Radner's work is our elimination of his (arbitrary) terminal date. We find that considering an unbounded time horizon permits us to bring out some additional attributes of the sequential model. In particular, the sequential framework allows us to establish a more robust existence result for dynamic economies than is possible in a complete market model. In this paper, we provide counter-examples that demonstrate the inability of complete market models to guarantee the existence of equilibria when there are an infinite number of possibly infinitely-lived agents or when there are infinitely-lived firms. In contrast, our Proposition 3.5 establishes the existence of equilibria for a general class of sequential economies that can incorporate any number of infinitely-lived agents and firms.

Secondly, we significantly strengthen Radner's existence results by relaxing his assumptions to the point that they are direct counterparts to the regularity assumptions typically imposed on classical complete market economies (e.g. see McKensie [1981] for static economies and Balasko, Cass, Shell [1980], Burke [1985], and

Wilson [1981] for dynamic economies). Most importantly, we replace Radner's assumption that each agent has a strictly positive endowment of all goods (an assumption that seems particularly restrictive when we are considering a dynamic model with uncertainty) with an irreducibility assumption of the type first considered in McKensie [1959]. In fact, in Burke [1985], we demonstrate how the general techniques that we employ to establish the existence of equilibria for our incomplete market economy can be employed to generalize the existence results of Balasko, Cass, Shell [1980] and Wilson [1981] for dynamic models with complete markets.

Perhaps one of the most significant features of the incomplete market economy is that it is capable of explaining the existence of active trading in the stock market. Since agents may not be able to trade in enough contingent commodities to diversify all of their risk, there is a need to diversify further by actively buying and selling shares in (heterogeneous) firms. In each of his papers, Radner specifies how production can be incorporated into the sequential framework. However, Radner is no longer able to establish the general existence of competitive equilibria once he considers this extension. In section 6, we exposit our third major strengthening of Radner's work by establishing the general existence of competitive equilibria with production.

In section 2, we specify a skeletal model of an economy that differs from a standard complete market economy only in that trading markets are partitioned. That is, consumers face a sequence of budget constraints, each involving a different subset of goods and each of which must be satisfied for an allocation to be "affordable". In section 3, we derive a general existence result for our skeletal model. In section 4, we discuss the merits of our sequential framework as an alternative to the complete market model. We consider not only its descriptive appeal but also the greater generality that the sequential economy permits.

In section 5, we expand the skeletal framework of section 3 into a fully developed economy with incomplete markets. Finally, in section 6, we incorporate production into our model and derive a general existence result.

2. THE MODEL

We begin with a standard formulation of a pure exchange economy with the aforementioned exception that trading markets are partitioned. Let A be the collection of agents and let I be the set of decision variables for these agents. In classical models with complete markets, these decision variables simply measure various quantities of consumption goods. As we will see in section 6, in general models with incomplete markets, it is more convenient to take these variables to measure holdings of commodity contracts. Nevertheless, in order to ease the difficulty of comprehending the assumptions presented in this section and the derivations of the results of the next section, one should intuitively think of I as if it were the collection of "goods" that agents can "consume".

Given this interpretation, (net) consumption profiles correspond to vectors in $R^{\#I}$, where $R^{\#I}$ denotes the set of all vectors z whose components are super-scripted by the elements of I. That is, z^i denotes the quantity of good $i \in I$ specified in the bundle $z \in R^{\#I}$. Let M denote the partition of I into trading markets. That is, M is a collection of mutually disjoint subsets of goods in I whose union is all of I, i.e. $I = \{ i : i \in m \text{ for some } m \in M \}$. Let $X_a \subseteq R^{\#I}$ be the space of (physically, legally, etc.) feasible consumption bundles for agent $a \in A$ and let $(>)_a$ be his preference relation defined over X_a. Our economy is now fully specified by the collection of agents A, of goods I, of markets M, of consumption sets X_a, and of preference relations $(>)_a$. An Allocation Scheme for the economy is a collection

$(x_a)_{a \in A}$ with $x_a \in X_a$ for $a \in A$.

Throughout the remainder of this paper, we adopt the following conventions. All subsets of $R^{\#I}$ will be endowed with the product topology. In particular, the convergence of a sequence $\{ s(k) \}$ of elements of $R^{\#I}$ to an element s, written $s(k) \rightarrow s$, will mean pointwise convergence i.e. $s^i(k) \rightarrow s^i$ for $i \in I$. For any $x \in R^{\#I}$ and any market $m \in M$ or collection of markets $M' \subseteq M$; let $x|m \in R^{\#I}$ be the element defined by $(x|m)^i = x^i$ if $i \in m$ and 0 otherwise and let $x|M' \in R^{\#I}$ be the element defined by $(x|M')^i = x^i$ if $i \in m$ for some $m \in M$ and 0 otherwise. In particular, given any bundle $x_a \in X_a$ and any market $m \in M_a$, $x_a|m$ denotes the corresponding bundle where agent a trades only in the market m.

We now present our assumptions. In section 3, we establish that assumptions I - V are sufficient to guarantee the existence of competitive equilibria for our sequential model. When our model is fully illustrated as an economy with incomplete markets, we will see that our assumptions are significantly weaker than those employed in Radner [1972, 1982]. In fact, most of the assumptions below correspond to those of McKensie [1981] (for static models) and of Balasko, Cass, Shell [1980], Burke [1985], and Wilson [1981] (for dynamic economies). The necessity of our non-standard assumptions, I(ii) and V(ii), is demonstrated in section 4 when we present examples of economies that, after the relaxation of either of these two restrictions, fail to have equilibria.

Given that we wish to interpret our economy as being dynamic with an unbounded time horizon, we would like to allow for the possibility that any given agent may not be able to participate in all of the markets. Clearly, if an agent is not living at the inception of the economy, it is possible that some markets may already be closed before the agent is born. Conversely, if a given agent is finitely-lived, he will clearly not be able to participate in markets that open at some distant future date. In order to incorporate the possible limited partition of agents into our assumptions, we denote the collection of markets that agent $a \in A$ can participate in by $M_a = \{ m \in M : \text{there is an } x_a \in X_a \text{ and an } i \in m \text{ such that } x_a^i = 0 \}$. That is, M_a is the collection of all markets that contain at least one good that agent $a \in A$ can either buy or sell. We also find it convenient to define the collection of agents that can participate in a given market $m \in M$ by $A_m = \{ a \in A : m \in M_a \}$.

Assumption I (Regularity of feasibility spaces)

For each $a \in A$:

(i) X_a is a closed and convex subset of R^{∞};

(ii) There exists a sequence $\{ M_a(t) \}$ such that each $M_a(t)$ is a finite subset of M_a (that is, $M_a(t)$ contains a finite number of the markets that agent a can trade in), $M_a(t) \subseteq M_a(t+1)$ for all t, $\bigcup_t M_a(t) = M_a$, and such that $x | M_a(t) \in X_a$ for $x \in X_a$ and any t;

(iii) $0 \in X_a$.

As we will see in section 4, assumption I(ii) restricts the degree to which an agent can short futures contracts. However, for our present exposition, it suffices to think of assumption I(ii) as a separability condition on feasibility spaces that allows us to approximate any allocation, $x \in X_a$, with a sequence of allocations, $\{ x|M_a(t) \}$, each of which involves trades in only a finite set of markets. In section 3, this assumption plays a critical role in our technique of finding a competitive equilibrium for our full (infinite) economy by considering a limit point of a sequence of equilibria that come from applying standard existence techniques to truncated (finite) sub-economies.

Assumption II (Positive and finite aggregate endowments)

(i) For each market $m \in M$ and good $i \in m$, there exists a collection of allocations $(x_a)_{a \in A_m}$ that satisfy $x_a \in X_a$ and $x_a = x_a|m$ for all $a \in A_m$, and $\sum_{a \in A_m} x_a^i < 0$;

(ii) $\{ x \in \sum_{a \in A} X_a : x \leqslant 0 \}$ is bounded from below.

If feasibility spaces are of the particular form $X_a = R_+^{\#I} - \{w_a\}$, then assumption II corresponds to the standard requirement of positive but finite aggregate endowments. That is,

$0 < \sum_{a \in A} w_a^i < \infty$ for $i \in I$.

Assumption III (Regularity of preferences)

For each $a \in A$, the preferences of agent a are representable by a utility function $u_a(\cdot): X_a \rightarrow R$ that satisfies:

(i) $u_a(\cdot)$ is continuous;

(ii) $u_a(\cdot)$ is quasi-concave.

Assumption IV (Irreducibility of each market)

Given any market $m \in M$, any partition of A_m into two non-empty subsets B_0 and B_1 (i.e. the sets B_0 and B_1 are disjoint with $A_m = B_0 \cup B_1$) and given any allocation scheme (x_a) such that $\sum_{a \in A_m} x_a^i \leq 0$ for all $i \in m$ (i.e. the market m clears):

There is an agent $b \in B_0$ and allocations $(z_a)_{a \in B_1}$ that satisfy;
$z_a \in X_a$ and $z_a = z_a | m$ for $a \in B_1$, and $x^b - \sum_{a \in B_1} z_a \; (>)_a \; x^b$.

Assumption IV is an extension of the now standard irreducibility condition that first appeared in the complete market model of McKensie [1959]. Recall that, in a complete market model, the irreducibility assumption guarantees that each agent has positive income. By extending this assumption to require that each market taken individually is irreducible, we guarantee that agents have positive income in all of the markets they participate in. In section 3, we see that having positive income in all relevant markets guarantees that agents demands are continuous (which is crucial to any proof of the existence of equilibria).

Note that assumption IV is stated in terms of contracts $x \in R^{\#I}$. Hence, this assumption is making a statement about the availability of contracts as well as a statement about agents tastes and endowments of consumption goods. This point will be discussed further in section 5.

Assumption V (Dimensions of the economy)

(i) Both A and I are countable;

(ii) Each market $m \in M$ is a finite subset of I.

In our general sequential model, we place no restrictions on the duration of each agents life (i.e. we allow agents to be infinitely-lived). However, assumption V(ii) requires that each trading market is finite. The contrast between our finite market assumption and the standard assumption of finite-lifetimes in complete market models (e.g. the overlapping-generations model) is discussed in section 4.

Assumption VI (Simplifying assumption)
For each $a \in A$, X_a is a bounded subset of R^{∞}. That is, there exist positive scalars $(s_a^i)_{i \in I}$ such that $X_a \subseteq \underset{i \in I}{\times} [-s_a^i, s_a^i]$,
i.e. $-s_a^i \leqslant x_a^i \leqslant s_a^i$ for all $x_a \in X_a$ and $i \in I$.

Following Debreu [1959], we can demonstrate that assumption VI is innocuous by finding scalars (s_a^i) sufficiently large such that if we bound consumption sets by them, then the competitive equilibria of the bounded economy coincide with the equilibria of our original economy.

The contrast between sequential economies and standard models of complete markets is brought into focus as we now discuss the agents market behavior. A Price System for the economy will be a vector $p \in R_+^{\# I}$, where p^i denotes the price of good i. The cost of the (net) purchase of the bundle $x \in X_a$ in the market m is the inner product $p(x|m) = \sum_{i \in I} p^i (x|m)^i$, where by assumption V(ii), the inner product is well defined since $(x|m)$ has only a finite number of non-zero components.

Given prices $p \in R_+^{\#I}$, the agents budget set is simply the collection of feasible allocations $x \in X_a$ that are affordable in each market, i.e. $p(x|m) \leq y_m$ for each $m \in M_a$, where y_m denotes the (lump-sum) income transfer to the agent in the market m. Clearly, the agents budget sets will be homogeneous of degree zero in prices and the income transfer in each market. Hence, we can independently normalize prices in each market. We choose the (normalized) price systems to lie in the space $P = \{ p \in R_+^{\#I} : \sum_{i \in m} p^i = 1 \text{ for each } m \in M \}$. We denote the space of allocation schemes by $X = \bigtimes_{a \in A} X_a$.

Given prices $p \in P$ and the income transfer scheme $(y_m) \in R^{\#M_a}$, agent a's market demand correspondence, $D_a(p,(y_m))$, is defined to be the collection of all solutions to

$$\max u_a(x)$$
$$\text{s.t. } p(x|m) \leq y_m \text{ for } m \in M_a$$
$$x \in X_a.$$

N.B. The monotonicity implied by assumption VI insures that the solutions to the above problem do not change if we replace the inequality with $p(x|m) = y_m$. This fact will be exploited later in this paper without being specifically mentioned.

Before we formally define our concept of a competitive equilibrium for our sequential economy, we find it useful to define a weaker notion which we call a competitive allocation. In section 3, we find a competitive equilibrium for our full economy by considering a limit point of an appropriately chosen sequence of competitive allocations (each of which come from applying standard existence techniques to finite sub-economies).

2.1 Definitions.

(i) A Competitive Allocation consists of a price system $p \in P$ and an allocation scheme $(x_a) \in X$ such that for $a \in A$:

$p(x_a|m) \geq 0$ for $m \in M_a$

$x_a \in D_a(p,(y_m))$, where $y_m = p(x_a|m)$ for $m \in M_a$;

(ii) A (Free Disposal) Competitive Equilibrium is a Competitive Allocation that satisfies

$p(x_a|m) = 0$ for $a \in A$ and all $m \in M_a$

$\sum_a x_a \leq 0$.

Competitive allocations are those that can be supported by a system of non-negative income transfers to the various agents. The usefulness of the restriction to non-negative transfers is seen in the proof of Lemma 3.1.

3. GENERAL EXISTENCE OF EQUILIBRIA

In this section, we establish the existence of competitive equilibria for the general class of sequential economies described in section 2. Following Balasko, Cass, Shell [1980], Burke [1985], and Wilson [1981], we find an equilibrium for our full economy by considering a limit point of equilibria of a sequence of truncated sub-economies. Lemma 3.3 establishes the existence of equilibria for each of the truncated economies while Lemmas 3.1 and 3.4 guarantee that any limit point of these equilibria will be a competitive equilibrium for our full economy.

A crucial step in any proof of the existence of competitive equilibria is to show that market demands are appropriately continuous. In order to characterize the conditions under which demand is continuous in a sequential economy, we must first extend the standard notion of income. In classical models, where the market structure is complete and feasibility spaces are of the form $X_a = R_+^{\#I} - \{w_a\}$, an agents income is simply the value of his endowment, pw_a. In our sequential economy, we can generalize this notion by defining the concept of income in a given market as the maximum amount of revenue that can be raised by (feasibly) selling off goods (endowments) in that market only. Specifically, for each $a \in A$, agent a's (potential) income in the market $m \in M_a$, given prices $p \in S$, is

$I_{a,m}(p) = \inf px$ subject to $x = x|m \in X_a$.

If we impose the simplifying assumption VI, as we do throughout the proofs in section 3, then since X_a is compact, we can define $I_{a,m}(p)$ as the minimum income px instead of the infimum of all such incomes.

Lemma 3.1 and Corollary 3.2 below provide useful characterizations of the continuity of demand.

3.1 Lemma.

For $a \in A$, agent a's demand correspondence, $D_a(\bullet,\circ)$, is upper hemi-continuous at each point $(p,(y_m)) \in P \times R_+^{\#M_a}$

Lemma 3.1 and Corollary 3.2 below provide useful characterizations of the continuity of demand.

3.1 Lemma.

For $a \in A$, agent a's demand correspondence, $D_a(\circ,\circ)$, is upper hemi-continuous at each point $(p,(y_m)) \in P \times R_+^{\#M_a}$ such that $I_{a,m}(p) < y_m$ for $m \in M_a$.

Proof of Lemma 3.1.

Consider any such sequence $\{ (p(k),x_a(k)) \}$ and limit point $(p,(x_a))$ in $P \times X_a$ such that $I_{a,m}(p) < p(x_a|m)$ for $m \in M_a$.

Since X_a is closed by assumption I(i), $x_a(k) \in X_a$ for all k and $x_a(k) \longrightarrow x_a$ imply $x_a \in X_a$. Hence, our proof is complete once we have established:

(*) For every $z \in X_a$ such that $z(>)_a x_a$, there is a market $m \in M_a$ such that $p(z|m) > p(x_a|m)$.

Consider any $z \in X_a$ such that $z (>)_a x_a$. Let $\{ M_a(t) \}$ be the sequence of subsets of M_a as defined in assumption II(ii). By assumption II(ii), $z|M_a(t) \in X_a$ for each t and $z|M_a(t) \rightarrow z$ as $t \rightarrow \infty$. Hence, if we fix a sufficiently large t, Assumption III(i) implies $z' (>)_a x_a$, where $z' = z|M_a(t)$. Again, by assumption III(i), $z' (>)_a x_a(k)$ for large k. Hence, since $(p(k), x_a(k))$ is a competitive allocation, there is a $m(k) \in M_a$ such that

(a) $p(k)(z'|m(k)) > p(k)(x_a(k)|m(k))$.

In particular, (a) implies $m(k) \in M_a(t)$. Otherwise, $m(k) \notin M_a(t)$ implies $z'|m(k) = 0$ since $z' = z|M_a(t)$, and hence $p(k)(x_a(k)|m(k)) < 0$ by (a). But, this contradicts our hypothesis that $(p(k),(x_a(k))$ is a competitive allocation. Therefore, for each k, m(k) lies in the (finite) set $M_a(t)$. Hence, there is an $m \in M_a(t)$ such that $m(k) = m$ for an infinite number of k. By the definition of z' and (a), $p(k)(z|m) = p(k)(z'|m) > p(k)(x_a(k)|m)$ for an infinite number of k. Therefore, in the limit as $k \rightarrow \infty$, $p(z|m) \geqslant p(x_a|m)$.

The above paragraph establishes

(b) for every $z \in X_a$ such that $z (>)_a x_a$, there is a market $m \in M_a$ such that $p(z|m) \geqslant p(x_a|m)$.

To verify (*), we need to strengthen (b) by replacing the weak inequality with the strict inequality $p(z|m) > p(x_a|m)$. For the limiting prices p, consider the allocation w_m that defines $I_{a,m}(p)$. That is, $I_{a,m}(p) = pw_m$, where $w_m = w_m|m$ and $w_m \in X_a$.

Consider any $z \in X_a$ such that $z (>)_a x_a$. By assumption I(i) and assumption III(i), there is a collection of sufficiently small (but positive) scalars $(\lambda_m) \in R^{\#M_a}$ such that $\sum_m \lambda_m < 1$ and $z' (>)_a x_a$, where $z' = \lambda z + \sum_m \lambda_m w_m$ and $\lambda = 1 - \sum_m \lambda_m$. By (b), there exists an $m \in M_a$ such that

(c) $p(z'|m) \geqslant p(x_a|m)$.

But,

(d) $p(z'|m) = \lambda p(z|m) + \sum_{m'} \lambda_{m'} p(w_{m'}|m)$

$= \lambda p(z|m) + \lambda_m p w_m$

since $w_{m'}|m = w_m$ if $m' = m$ and 0 otherwise.

By hypothesis,

(e) $pw_m = I_{a,m}(p) < p(z_a|m)$.

The combination of (c), (d), and (e) yields

$p(x_a|m) \lesssim p(z'|m) < \lambda p(z|m) + \lambda_m p(x_a|m)$.

The above inequality implies

$p(x_a|m) \lesssim p(z|m)$, since $0 < 1-\lambda_m < \lambda$ and $p(x_a|m) \geqslant 0$.

(*) is now established.

Q.E.D.

3.2 Corollary. Given any agent $a \in A$ and any system of positive income transfers $(y_m) \in R_{++}^{\#M_a}$, the demand correspondence $D_a(\cdot;(y_m))$ is upper hemi-continuous at each point $p \in P$.

To prove this corollary, simply apply Lemma 3.1 and note that by assumption I(iii), $I_{a,m}(p) < 0$ for all $a \in A$, $m \in M_a$, and $p \in S$.

The next Lemma establishes the existence of equilibria for the afore mentioned sub-economies. By corollary 3.2, we know that, in order to guarantee the continuity of demand (which is crucial to any existence proof), we must insure that each agent has positive income. We do so by adding a strictly positive quantity of each good to each agents endowment. These additional endowments imply that, in the "equilibrium" of the perturbed economy, agents will over-spend, in terms of their original budget constraints. Also, aggregate demand will exceed the original aggregate endowments. However, this excess spending and excess demand will vanish in the limit if we systematically reduce the amount of additional endowments introduced in each successive truncation.

3.3 Lemma. Given any finite collection of markets M', i.e. M' is a finite subset of M; there is a competitive allocation, $(p,(x_a))$, in which all of the agents receive a small income transfer and all of the markets in M' almost clear. Specifically, given any $\epsilon > 0$, there exists a competitive allocation $(p,(x_a))$ such that $0 \leq p(x_a|m) \leq \epsilon$ for $a \in A$ and all $m \in M_a$ and $\sum_{a \in A_m} x_a^i \leq \epsilon$ for $m \in M'$ and all $i \in m$.

Proof of Lemma 3.3.

Since the lemma only restricts aggregate demand in the markets in M', we focus our attention on the sub-collection of goods

$I' = \{ i : i \in m \text{ for some } m \in M' \}$,

the sub-collection of agents that trade for these goods

$A' = \{ a : a \in A_m \text{ for some } m \in M' \}$,

the sub-collection of markets that agent $a \in A'$ trades in

$M_a' = M_a \cap M'$,

the sub-collection of feasible trades that an agent $a \in A'$ can make in the markets in M_a'

$X_a' = \{ x \in R^{\#I'} :$

there is a $z \in X_a$ such that $z^i = x^i$ for all $i \in I' \}$,

and on the sub-collection of prices of the goods in I'

$S' = \{ p \in R_+^{\#I'} : \sum_{i \in m} p^i = 1 \text{ for each } m \in M' \}.$
Since M' is finite, assumption V(ii) and VI(i) imply that there are only a finite number of agents in A' and goods in I'.

For each $i \in I - I'$, let $m(i)$ denote the (unique) market $m \in M$ such that $i \in m$.

Given prices $p' \in P'$, the agents $a \in A'$ will act as if the price of each good $i \in I'$ were $(p')^i$ as if the price of each good $i \in I - I'$ were $\frac{1}{\#m(i)}$. That is, the price system $p' \in P'$, for the goods in I', is (artificially) extended to a full system $p(p') \in P$, for all of the goods in the economy, where $(p(p'))^i = p^{i}$' if $i \in I'$ and $(p(p'))^i = \frac{1}{\#m(i)}$ otherwise. Furthermore, in order to generate positive incomes, each agent $a \in A'$ receives an added endowment of $w_a^i = \frac{\epsilon}{\#A_m}$ units of each good i in each market $m \in M_a$. Since our extended price system always lies in P, the value of the added endowments to agent $a \in A'$, in each market $m \in M_a$, is
$p(p')(w_a|m) = \sum_{i \in m} (p(p'))^i \frac{\epsilon}{\#A_m} = \frac{\epsilon}{\#A_m}$.
Therefore, given prices $p' \in P'$, the value of agent a's demand correspondence is $D_a(p(p');(y_m))$, where $y_m = \frac{\epsilon}{\#A_m}$ for $m \in M_a$. Henceforth, we will suppress the occurrences of (y_m). Since the value of the income transfers is always positive, corollary 4.2 implies (a) the correspondence $D_a(p(p'))$ is upper hemi-continuous at each point $p \in P'$.

In order to satisfy the conclusions of this Lemma, we must only guarantee that the desired allocation scheme is "feasible" (given our added endowments) for all of the goods in the markets in M'. Hence, we can restrict out attention to truncations of our demand correspondences. For $a \in A'$, let $D_a'(\bullet)$ denote the truncation of

$D_a(p(\bullet))$ with respect to the collection of goods I'. That is,

$D'_a(p') = \{ x \in R^{\#I'} :$ there is a $z \in D_a(p(p'))$ such that $z^i = x^i$ for all $i \in I' \}$.

The following regularity properties of demand are immediate consequences of (a) and assumptions I(i) and III(ii).

(b) The correspondence $D'_a(\bullet)$ is upper hemi-continuous at each point $p' \in P'$.

(c) (Walras's law) For any $p' \in P'$, $p'(x-w_a|m) \leq 0$ for $m \in M'_a$ and all $x \in D'_a(p')$.

(d) (Convexity) For any $p' \in P'$, $D'_a(p')$ is a convex subset of $R^{\#I'}$.

We now establish the existence of an equilibrium for our truncated economy by adapting the methods of Debreu [1959] to our sequential framework.

Define the correspondence

$H : R^{\#I'} \longrightarrow P'$

by

$H(z) = \{ p' \in P' :$

$$\text{for any } q' \in P', \sum_{i \in m} (q')^i z^i \leq \sum_{i \in m} (p')^i z^i \text{ for all } m \in M' \}.$$

One may verify that (b) and (d), together with assumption I(i), imply that the correspondence

$(p',(z_a)) \longrightarrow (H(\sum_{a \in A'} (z_a - w_a)), \bigtimes_{a \in A'} D'_a(p')$, that maps $P' \times \bigtimes_{a \in A'} X'_a$ into itself, satisfies the hypotheses of the Kakutani fixed point theorem.

We, therefore, conclude that there is a fixed point $(p',(z_a))$.

That is,

(e) $p' \in H(\sum_{a \in A'} (z_a - w_a))$

(f) $z_a \in D'_a(p')$ for $a \in A'$.

One may verify that (c) and (e) imply

(g) $\sum_{a \in A'} z_a \leq \sum_{a \in A'} w_a$.

We extend the equilibrium $(p',(z_a))$, for the truncated economy, into a competitive allocation $(p,(x_a))$ for our full economy by defining

$p = p(p')$

x_a = the element in $D_a(p,(y_m))$ that corresponds to $z_a \in D'_a(p')$ for $a \in A'$

and

x_a = an (arbitrary) element in $D_a(p,(y_m))$ for $a \in A-A'$.

By (f) and (g), one may verify not only that $(p,(x_a))$ constitutes a competitive allocation, but that it satisfies the additional requirements of Lemma 3.3.

Q.E.D.

3.4 Lemma.

Given any limit point, $(p,(x_a))$, of a convergent sequence of competitive allocations, $\{(p(k),(x_a(k)))\}$, and given any market $m \in M$; if the market m clears at the limiting allocation, i.e. $\sum_{a \in A_m} x_a^i \leq 0$ for all $i \in m$, then each agent $a \in A_m$ has income in the market m at the limiting prices, i.e. $I_{a,m}(p) < 0$.

Proof of Lemma 3.4.

Assume the contrary. That is, assume that there is a sequence of allocations, $\{(p(k),(x_a(k)))\}$, that converge to a limit, $(p,(x_a))$, and assume that there is a market $m \in M$ such that $\sum_{a \in A_m} x_a^i \leq 0$ for all $i \in m$ but $I_m^{a'}(p) \geq 0$ for some $a' \in A_m$. Let

$B_0 = \{ a \in A_m : I_{a,m}(p) < 0 \}$ and $B_1 = \{ a \in A_m : I_{a,m}(p) \geq 0 \}$. By assumption II(i) and the definition of $I_{a,m}(\cdot)$, $\sum_{a\in A_m} I_{a,m}(p) < 0$. Hence, B_0 is non-empty. Since $a' \in B_1$, B_1 is also non-empty. Therefore, given our hypothesis that the market m clears, we can apply assumption IV to the market m, the partition $\{B_0, B_1\}$ of A_m, and to the limiting allocation scheme (x_a), to conclude

(a) there is an agent $b \in B_0$ and allocations $(z_a)_{a\in B_1}$ that satisfy; $z_a \in X_a$ and $z_a = z_a|m$ for each $a \in B_1$, and $x^b - \sum_{a\in B_1} z_a \ (>)_b\ x^b$.

Let w_b define $I_{bm}(p)$, as in the proof of Lemma 4.1. That is, $I_{b,m}(p) = pw_b$, $w_b = w_b|m$, and $w_b \in X^b$. By assumption I(i), III(i), and (a); there is a $\lambda < 1$ such that $\lambda(x^b - \sum_{a\in B_1} z_a) + (1-\lambda)w_b \ (>)_b\ x^b$. By assumption III(i), this implies

(b) $\lambda x^b(k) + y \ (>)^b\ x^b(k)$ for large k,

where $y = (1-\lambda)w_b - \lambda \sum_{a\in B_1} z_a$. Since, by hypothesis, $(p(k),(x_a(k)))$ is a competitive allocation, (b) implies that there is a market $m' \in M^b$ such that $p(k)((\lambda x^b(k)+y)|m') > p(k)(x^b(k)|m')$, i.e.

(c) $p(k)(y|m') > (1-\lambda)p(k)(x^b(k)|m') \geq 0$,

where the last inequality follows from (3.1)(iii). In particular, (c) implies that $m' = m$ since otherwise, by the definition of y, $(y|m') = (1-\lambda)(w_b|m') - \lambda \sum_{a\in B_1} (z_a|m') = 0$. Therefore, again using (c), $p(k)(y|m) > 0$, which by the definition of y, can be written

(d) $(1-\lambda)p(k)w_b > \lambda \sum_{a\in B_1} p(k)z_a$,

since $y = y|m$. Passing to the limit as $k \to \infty$, (d) yields

(e) $(1-\lambda)pw_b \geq \lambda \sum_{a\in B_1} pz_a$.

But, by the definition of w_b and $I_{a,m}(p)$ for each $a \in B_1$, (e) implies

$(1-\lambda)I_{bm}(p) = (1-\lambda)pw_b \geq \lambda \sum_{a\in B_1} pz_a \geq \lambda \sum_{a\in B_1} I_{a,m}(p)$,

which contradicts the definition of B_0 and B_1, since $b \in B_0$ and $0 < \lambda < 1$.

Q.E.D.

Our general existence result will now follow from Lemmas 3.1, 3.3, and 3.4.

3.5 Proposition. There exists a competitive equilibrium.

Proof of Proposition 3.5.

Fix any sequence $\{ M(k) \}$ such that each $M(k)$ is a finite subset of M, $M(k) \subseteq M(k+1)$ for all k, and $\bigcup_k M(k) = M$. Also, fix a sequence of positive scalars $\{ \epsilon(k) \}$ such that $\epsilon(k) \longrightarrow 0$. Let $(p(k),(x_a(k)))$ be the competitive allocation associated with $M(k)$ and $\epsilon(k)$, as specified in Lemma 3.3. That is,

(a) $0 \leqslant p(x_a(k)|m) \leqslant \epsilon(k)$ for $a \in A$ and all $m \in M_a$

(b) $\sum_{a \in A_m} x_a^i(k) = \epsilon(k)$ for $m \in M(k)$ and all $i \in m$.

Since each allocation, $(p(k),(x_a(k)))$, is, by definition, contained in $P \times X$, which is compact by assumption V and the Tychonoff theorem (cf. Bourbaki [1966]), Lemma 3.3 implies that there is a point $(p,(x_a)) \in P \times X$ and a subsequence of allocations such that

(c) $(p(k),(x_a(k))) \longrightarrow (p,(x_a))$.

Since $\epsilon(k) \longrightarrow 0$, (a) and (b) imply

(d) $p(x_a|m) = 0$ for $a \in A$ and all $m \in M_a$

(e) $\sum_{a \in A} x_a \leqslant 0$.

By Lemma 4.4, (c) and (e) imply

(f) $I_{a,m}(p) < 0$ for $a \in A$ and all $m \in M_a$.

By Lemma 4.1, (c), (d), and (f) imply

(g) $(p,(x_a))$ is a competitive allocation.

Therefore, by (g), (d), and (e), $(p,(x_a))$ is a competitive equilibrium.

Q.E.D.

4. ROBUSTNESS OF THE SEQUENCIAL MODEL

In this section, we highlight the robustness of the sequential model by comparing our existence result (proposition 3.5) with those that can be attained for models with a single (complete) market. We find that, with respect to separability requirements on consumption spaces, the two models are comparable in generality. Specifically, our example 4.1 demonstrates the necessity of imposing a condition like I(ii) on a sequential economy while example 4.2 shows that a similar such separability assumption must be imposed on a complete market economy. However, examples 4.3, 4.4, and 4.5 demonstrate that, unlike the sequential model, one can no longer guarantee the existence of competitive equilibria in complete market economies when there are either infinitely-lived agents, infinitely-lived firms, or an infinite number of agents with tastes for the same good.

The following is an example of a sequential economy that satisfies all of the assumptions of section 2 except I(ii). As a result, we are able to demonstrate that the economy has no competitive equilibria. Not only does our example demonstrate the necessity of an assumption like I(ii), but it illustrates how I(ii) can be violated if agents are given too much leverage in shorting futures contracts.

Example 4.1. Let the collection of agents be indexed by $A = \{0, 1, \ldots\}$. There is one good available in each of the time periods $t = 1, 2, \ldots$. Agent 0 is infinitely-lived while agent t ($t = 1, 2, \ldots$) lives during periods t and $t+1$. At time t, there is a market for current goods, $x^{t.5}$, and for futures contracts, x^{t+1}, for period t+1 goods. That is, the set of contingent commodities is indexed by $I = \{1.5,\ 2,\ 2.5,\ 3,\ \dots\}$ with the partitioning of these contracts into markets given by $M = \{\ \{1.5, 2\},\ \{2.5, 3\},\ \dots\ \}$.

For any (contingent) commodity bundle $x \in R^{\#I}$, let agent 0's utility be $u_0(x) = x^{1.5} + v(x^2, x^{2.5}, \dots)$, where $v(\circ)$ is quasi-concave, and strictly-increasing in all of its components, and where $0 < v(\circ) < 1$. For $t = 1, 2, \dots$; let $u_t(x) = x^{t.5} + x^{t+1}$.

Thus far, the economy conforms to the specifications of section 2. We now violate assumption I(ii) by giving agent I(ii) the right to short all of the futures contracts. Suppose that agent 0 has an endowment of $(1/2)^t$ units of goods in period t. Also, suppose that this agent can short up to $2-(1/2)^t$ units of period t goods ($t = 2, 3, \dots$) through the sale of futures contracts at time t–1. Agent 0's space of feasible (net) contract holdings is then given by

$$X_0 = \{\ x \in R^{\#I}\ \mid\ x^{1.5} \geq -(1/2),\ x^2 \geq -2,\ x^2 + x^{2.5} \geq -(1/2)^2,\ \dots,\ x^t \geq -2,\ x^t + x^{t.5} \geq -(1/2)^t,\ \dots\ \}.$$

Suppose that agent t ($t = 1, 2, \dots$) has an endowment of $(1/2)^t$ units of period t goods and $(1/2)^{t+1}$ units of period t+1 goods. Even though agent t is alive when both of the markets $\{t.5, t+1\}$ and $\{(t+1).5, t+2\}$ are open, the agent will only trade in the market $\{t.5, t+1\}$. This is due to the fact that the agent has neither tastes nor endowments for the contract t+2, since it only involves goods in period t+2. Therefore, agent t's space of (net) trades is given by

$$X_t = \{\ x \in R^{\#I}\ \mid\ x^{t.5} \geq -(1/2)^t,\ x^{t+1} \geq -(1/2)^{t+1},\ \text{and } x^i = 0 \text{ for all } i = t.5 \text{ or } t+1\ \}.$$

We now verify that the economy has no equilibria by an indirect argument. Specifically, we pin down any candidate for an equilibrium price system and then show that the corresponding allocation scheme is infeasible.

Suppose that $(p,(x_a))$ was a competitive equilibrium for the above economy. Agent a's budget constraint for trades in the market $m = \{t.5, t+1\} \in M_a$ simplifies to

(a) $p(x_a|m) = p^{t.5}x_a^{t.5} + p^{t+1}x_a^{t+1} \leqslant 0.$

The special form of the feasibility spaces X_1, X_2, ... allows us to simplify the market clearing conditions to

(b) $x_0^{t.5} + x_t^{t.5} \leqslant 0$ and $x_0^{t+1} + x_t^{t+1} \leqslant 0$ for t = 1, 2, ...

Consider any t = 1, 2, ...

If $p^{t.5} > p^{t+1}$, then (a) and utility maximization by agent t implies

$x_t^{t+1} = (p^{t.5}/p^{t+1})(1/2)^t > (1/2)^t.$

By (b), this implies

$-x_0^{t+1} > (1/2)^t.$

Since $x_0 \in X_0$, this implies

$x_0^{(t+1).5} \geqslant - x_0^{t+1} - (1/2)^{t+1} > (1/2)^{t+1}.$

By (b), this implies

$x_{t+1}^{(t+1).5} < - (1/2)^{t+1}.$

But, this contradicts $x_{t+1} \in X_{t+1}$.

The above paragraph establishes

(c) $p^{t.5} \leqslant p^{t+1}$ for t = 1, 2, ...

One may readily verify that

$x = (x^{1.5}, x^2, x^{2.5}, x^3, \ldots) = (2, -2, 2, -2, \ldots) \in R^{\#I}$

is in agent 0's budget set since $x \in X_0$ and, by (c), x satisfies (a) for all markets. Therefore, since x was affordable when x_0 was chosen,

$u_0(x_0) > u_0(x)$.

By the definition of $u_0(\bullet)$, this implies

$x_0^{1.5} + v(x_0^2, x_0^{2.5}, \ldots) > 2 + v(-2, 2, \ldots)$.

This, in turn, implies

$x_0^{1.5} > 1$, since $0 < v(\bullet) < 1$.

But, by (b), this implies

$x_1^{1.5} < -1$, which contradicts $x_1 \in X_1$.

We conclude that the above economy has no competitive equilibria.

All of the examples that follow are of complete market economies, i.e. $M = \{ I \}$. Unless stated otherwise, the collection of agents is indexed by the set

$A = \{0, 1, \ldots \}$

while the collection of goods is indexed by

$I = \{1, 2, \ldots \}$.

For convenience, we now refer to $X_a \subseteq R_+^\infty$ as agent a's space of feasible gross consumption. Unless specified otherwise, assume $X_a = R_+^\infty$. Since we now consider gross consumption, we must explicitly introduce agent a's endowment, which we denote by $w_a \in R_+^\infty$.

The following example demonstrates the necessity of a separability assumption like I(ii) to insure the existence of equilibria for dynamic models with complete markets.

4.2 Example. Let agent 1 have the consumption set

$X_1 = \{ (x_1^1, x_1^2, \ldots) \in R_+^\infty : x^{i+1} \geq 2x^i \text{ for } i = 2, 3, \ldots \}$

and give each of the other agents t = 2, 3, ... the set

$X_t = R_+^\infty$.

Let $w_1^i = 1$ if $i = 1$ and 0 otherwise and let $u_1(x_1) = x_1^1 + 2x_1^2$. For $t = 2, 3, \ldots$; let $w_t^i = 1$ if $i = t-1, t$ and 0 otherwise and let $u_t(x_t) = x_t^{t-1} + (1/3)x_t^t$.

Assume that the above economy has a monetary equilibrium, $(p,(x_a))$, where the only monetary transfer is a non-negative transfer to agent 1, i.e. $px_1 \geqslant pw_1$. By definition, x_1 being in agent 1's consumption set implies $x_1^t \geqslant 2^{t-2}x_1^2$ for all $t = 2, 3, \ldots$. Since the aggregate endowment of each good is bounded ($w^t = 2$), feasibility implies $x_1^2 = 0$.

Clearly, all prices are positive, $p \gg 0$, since each good is insatiably desired by some agent. In particular, utility maximization by agent 1 together with $x_1^2 = 0$ implies

$x_1^i = 0$ for $i = 2, 3, \ldots$

while utility maximization by agents $t = 2, 3, \ldots$ implies

$x_t^i = 0$ for $t = 2, 3, \ldots$ and $i = t-1, t$.

Hence, the agents budget constraints simplify to

(a) $p^{t-1}(x_t^{t-1} - 1) + p^t(x_t^t - 1) = 0$ for $t = 2, 3, \ldots$

and the feasibility constraints become

(b) $x_t^t + x_{t+1}^t = 2$ for $t = 1, 2, \ldots$.

We now obtain a contradiction by pinning down the price system p and then showing that, given p, x_1 is not a utility maximizing choice for agent 1. Since agent 1 consumes only good 1, the fact that agent 1 spends at least the value of his endowment (i.e. receives a non-negative income transfer), implies $x_1^1 = px_1 \geqslant pw_1 = 1$. Using $x_1^1 \geqslant 1$ and repeated applications of (a) and (b), one may verify

(c) $1 \leqslant x_t^t \leqslant 2$ for $t = 2, 3, \ldots$.

Consider any $t = 2, 3, \ldots$.

If $p^t < (1/3)p^{t-1}$, then utility maximization by agent t implies $x_t^t > 4$, which contradicts (c).

If $p^t > (1/3)p^{t-1}$, then utility maximization implies $x_t^t = 0$, which also contradicts (c). Therefore, $p^t = (1/3)p^{t-1}$. Hence,

(d) $p^t = 3^{1-t}$ for $t = 2, 3, \ldots$.

We now show that x_1 cannot be a utility maximizing bundle for agent 1. Specifically, instead of the bundle $x_1 = (x_1^1, 0, 0, \ldots)$, the agent could have afforded the bundle $z_1 = (x_1^1-1, 1, 2, 2^2, \ldots)$ since by (d),

$$pz_1 = x_1^1-1 + \sum_{t=2}^{\infty} 3^{1-t}2^{t-2} = x_1^1 = px_1.$$

One may readily verify that z_1 is also feasible, i.e. $z_1 \in X_1$, and is preferable to x_1, i.e. $u_1(z_1) = x_1^1 + 1 > x_1^1 = u_1(x_1)$. Hence, agent 1's alleged choice of x_1 is not consistent with utility maximization. Therefore, the economy has no monetary equilibria as described in proposition 4.2.

The following example, that demonstrates the potential non-existence of competitive equilibria for complete market economies when there are infinitely-lived agents, is taken from Wilson [1981].

4.3 Example. For $a \geqslant 1$, let $w_a^i = 1$ if $i = a, a+1$ and 0 otherwise and let $u_a(x_a) = x_a^a + 3x_a^{a+1}$. For agent 0, let $w_0^i = (1/2)^i$ for all i and let $u_0(x_0) = x_0^1$. For all a, let $X_a = R_+^{\infty}$.

We will assume that all goods are traded in a single market. That is, agent a faces the single budget constraint $px_a \leqslant pw_a$. Hence, x_a is chosen to maximize $u_a(x_a)$ subject to $px_a \leqslant pw_a$ and $x_a \in R_+^{\infty}$.

We now prove, by contradiction, that there is no competitive equilibrium (see definition (3.5)) for this economy. Assume that p and (x_a) are equilibrium prices and allocations respectively. Clearly, by the monotonicity of preferences, $p \in R^{\infty}_{++}$. Let $I = \{ i > 1 : p^{i+1} < 3p^i \}$. Then for any $i \in I$, utility maximization by agent i implies $x^i_i = 0$. Therefore, since demand equals supply for good i, $x^i_{i-1} = w^i_0 + w^i_{i-1} + w^i_i > 2$. The budget constraint of agent $i-1$ then implies $2p^i < p^i x^i_{i-1} < p^{i-1} w^{i-1}_{i-1} + p^i w^i_{i-1} = p^{i-1} + p^i$. From this it follows that $i \in I$ implies $p^i < p_{i-1}$. In particular, $i \in I$ implies $i-1 \in I$. But the demand of agent 0 will only be well defined if $pw_0 < \infty$. Therefore, I must be an infinite set. It then follows from induction that $i \in I$ for all $i > 1$.

The preceding paragraph then implies that for all i, $p_{i+1} < p^i$ and hence, $p^i < p_1$ for all $i > 1$. Therefore, $pw^0 = \sum_{i=1}^{\infty} p^i 2^{-i} < p_1$, which from the budget constraint of agent 0 implies $x^1_0 < 1$. However, $p_2 < p_1$ also implies that $x^1_1 = 0$ since agent 1 maximizes his utility. But since $x^1_0 + x^1_1 = x^1_0 < 1 < \frac{3}{2} = w^1_0 + w^1_1$, i.e. supply exceeds demand, it follows that a competitive equilibrium cannot exist.

The following is an example of a well-behaved economy in which there is a single trading market and each agent trades for only a finite number of goods in that market. However, we do assume that goods are perfectly storable. As a result of our considering this simple form of production, we are able to demonstrate that the economy has no competitive equilibria. The inability of the complete market model to accommodate even the simplest form of production stands in sharp contrast to the results obtained in section 6 that establish the existence of competitive equilibria for a general class of production processes.

4.4 Example.

For $a > 1$, let $w_a^i = 2$ if $i = a$, $a+1$ and 0 otherwise and let $u_a(x_a) = x_a^a + 2x_a^{a+1}$. For agent 0, let $w_0^i = 1$ if $i = 1$ and 0 otherwise and let $u_0(x_0) = x_0^1$. For all a, let $X_a = R_+^{\infty}$.

As in the previous example, we assume that all goods are traded in a single market. However, in contrast we now consider a simple (perfect storage) production process. Let

$Y = \{ (y_1, y_2, \ldots) : -1 \leqslant y_1 \leqslant 0,\ y^i \geqslant 0 \text{ for } i > 1$

and $\sum_{i>1} y^i \leqslant - y_1 \}$

be the set of net production possibilities.

We now prove that there are no equilibria for this economy. Let p, (x_a), and y be any proposed collection of equilibrium prices, allocations, and net output levels respectively. We first demonstrate, by induction, that $p^{i+1} \geqslant 2p^i$ for all i. Clearly $p^2 \geqslant 2p^1$, since otherwise utility maximization by agents 0 and 1 imply $x_0^1 = 2$ and $x_1^1 = 0$. But these equalities imply, since $y^1 \geqslant -1$, $x_0^1 + x_1^1 = 2 < 3 \leqslant w_0^1 + w_1^1 + y_1$, which contradicts the requirement that equilibrium demand equals supply for all goods. We now assume our inductive hypothesis $p^{i+2} \geqslant 2p^{i+1}$ and demonstrate $p^{i+2} \geqslant 2p^{i+1}$. If $p^{i+2} < 2p^{i+1}$, then utility maximization by agent i+1 implies $x_{i+1}^{i+1} = 0$. Hence, the condition that the demand equals the supply of good i+1 yields

$x_i^{i+1} = x_i^{i+1} + x_{i+1}^{i+1} = w_i^{i+1} + w_{i+1}^{i+1} + y^{i+1} \geqslant 4$

since $y^{i+1} \geqslant 0$. By the budget constraint of agent i, this implies

$$x_i^i \leqslant w_i^i + \frac{p^{i+1}}{p^i}(w_i^{i+1}-x_i^{i+1}) \leqslant 2 - 2\frac{p^{i+1}}{p^i} \leqslant -2$$

since by assumption $p^{i+1} \geqslant 2p^i$. But this contradicts the feasibility of the bundle x_i, i.e. $x_i \in R_+^{\infty}$. It then follows, by induction, that $p^{i+1} \geqslant 2p^i$ for all i.

The above paragraph demonstrates that if we were given an equilibrium, prices would at least double each period. But given such prices, the firm could make any arbitrarily large level of profits by simply storing one unit of period 1 goods until some suitably distant future date. Therefore, there can be no profit maximizing production plan. Hence, the economy has no competitive equilibria.

For completeness, we include our next example that demonstrates the inability of the complete market model to guarantee the existence of equilibria when there are an infinite number of agents with tastes for a single good. We note that the cause of the non-existence of equilibria in examples 4.1-4.4 is due to the lack of continuity of individual demands and supplies. In contrast, the absence of equilibria in the following example stems from the lack of continuity of aggregate demand. That is, even though each agents demand is continuous, the aggregate demand for a good need not be continuous since we are summing up over an infinite number of agents with tastes for that good.

4.5 Example. For notational convenience, we index the set of agents by $A = \{1, 2, \ldots\}$ and the set of goods by $I = \{0, 1, \ldots\}$. For $t = 1, 2, \ldots$; let $w_t^i = 2^{-t}$ if $i = t-1, t$ and $w_t^i = 0$ otherwise and let $u_t(x_t) = x_t^0 + 3^{-t}x_t^t$.

Assume that the above economy has a competitive equilibrium, $(p,(x_a))$. Since each good is insatiably desired by at least one agent, prices must be strictly positive, $p >> 0$. Let $p^0 = 1$ be our price normalization. Since all prices are positive, utility maximization implies

$x_t^i = 0$ for all $t = 1, 2, \ldots$ and $i = t-1, t$.

Hence, we can write our budget constraints as

(a) $x_t^0 + p^t x_t^t = 2^{-t}(p^{t-1} + p^t)$ for $t = 1, 2, \ldots$

and our feasibility constraints simplifies to

(b) $\sum_{t=1}^{\infty} x_t^0 = w^0 = 2^{-1}$ for good 0

$x_t^t = w^t = 3 \cdot 2^{-t}$ for goods $t = 1, 2, \ldots$.

We can now finish our demonstration by pinning down the price system p and then contradicting (b) by showing that there is an excess supply of good 1. Specifically, we establish

(c) $p^t = 3^{-t}$ for $t = 0, 1, \ldots$,

by induction. Trivially, $p^0 = 1 = 3^{-0}$. We assume the inductive hypothesis $p^t = 3^{-t}$ (for some $t = 0, 1, \ldots$) and prove $p^{t+1} = 3^{-(t+1)}$.

If $p^{t+1} < 3^{-(t+1)}$, then utility maximization by agent t+1 implies $x_{t+1}^0 = 0$. Hence, the agents budget constraint from (a), together with the inductive hypothesis $p^t = 3^{-t}$, implies

$x_{t+1}^{t+1} = 2^{-(t+1)}(1 + p^t/p^{t+1}) > 3 \cdot 2^{-(t+1)}$,

which contradict (b).

If $p^{t+1} > 3^{-(t+1)}$, then utility maximization by agent t+1 implies $x_{t+1}^{t+1} = 0$, which again contradicts (b). Therefore, $p^{t+1} = 3^{-(t+1)}$ and our inductive proof is complete.

We can now determine x_t^0 for $t = 1, 2, \ldots$. By (a), (b), and (c);

$x_t^0 = 2^{-t}(p^{t-1} + p^t) - p^t x_t^t = 2^{-t}(3^{1-t} + 3^{-t}) - 3^{-t}(3 \cdot 2^{-t})$
$= 2^{-t} \cdot 3^{-t} = 6^{-t}$.
Therefore, $\sum_{t=1}^{\infty} x_t^0 = \sum_{t=1}^{\infty} 6^{-t} = 7^{-1} < 2^{-1}$,
which contradicts (b).

To further illustrate the generality of the sequential economy and its implicit debt restrictions, we reconsider example 4.3. As it stands, example 4.3 describes a complete market economy in which, due to the presence of an infinitely-lived agent, there are no competitive equilibria. However, if we recast this economy into a sequential framework, then we can easily solve for a competitive equilibrium (the existence of which is guaranteed by proposition 3.5).

We note that the non-existence of equilibria in example 4.3 stems from the fact that, since agent 0 is endowed with an infinite number of goods, his endowment income, pw, can vary discontinuously with prices. This discontinuity of income naturally leads to a discontinuity of demand. When we recast the agents in example 4.3 into a sequential economy, we find that the (implicit) restrictions on the degree to which agents can access their future endowment income sufficiently dampen the above discontinuous income effects to guarantee the continuity of demand.

Example 4.6.

In order to satisfy assumption VI, once the economy is cast into the sequential framework, we assume that agent 0's utility function is $u_0(x_0) = \sum_t (1/4)^t x_0^t$. The remainder of the economy is as specified in example 4.3.

It is straightforward to verify, by the techniques employed in example 4.3, that the above economy has no competitive equilibria.

We now cast the agents into the incomplete market structure described in example 4.1. Specifically, at time t, there is a market for current goods, $x^{t.5}$, and for futures contracts, x^{t+1}, for period t+1 goods. That is, the set of contingent commodities is indexed by $I = \{1.5,\ 2,\ 2.5,\ 3,\ \dots\}$ with the partitioning of these contracts into markets given by $M = \{\ \{1.5,2\},\ \{2.5,3\},\ \dots\ \}$.

For a > 1, agent a's feasibility set is given by

$X_a = \{\ x \in R^{\#I} :\ \ x^i \geq -1 \text{ if } i = a.5 \text{ or } a+1 \text{ and } x^i = 0 \text{ for all other } i\ \}.$

with preferences defined by

$u_a(x) = x^{a.5} + 3x^{a+1}$ for $x \in X_a$.

Agent 0's feasibility set is given by

$X_0 = \{\ x \in R^{\#I} : x^{1.5} \geq -(1/2),\ \dots,\ x^t \geq -(1/2)^t,$
$x^t + x^{t.5} \geq -(1/2)^t,\ \dots\ \}$

with preferences defined by

$$u_0(x) = (1/4)x^{1.5} + \sum_{t=2}^{\infty} (1/4)^t (x^t + x^{t.5}).$$

We now verify that competitive equilibrium prices are given by

$p = (p^{1.5}, p^2, p^{2.5}, p^3, \dots) = (1/4)(1, 3, 1, 3, \dots)$

with equilibrium allocations given by

$x_0 = (x_0^{1.5}, x_0^2, x_0^{2.5}, x_0^3, \dots) = \ (3(1/2)^2, -(1/2)^2, 3(1/2)^3, -(1/2)^3, \dots)$

and

$x_a = (0, \dots, x_a^{a.5}, x_a^{a+1}, 0, \dots) = (0, \dots, -3(1/2)^{a+1}, (1/2)^{a+1}, 0, \dots)$ for $a = 1,\ 2,\ \dots$.

By inspection, the all markets clear in the above allocation scheme and each agent a = 1, 2, ... maximizes his utility.

To verify that agent 0 is maximizing his utility in x_0, by shorting to the limit on every futures contract, we simply note that agent 0's rate of substitution between consumption in periods t+1 vs. t is given by

$$\frac{du_0/dx^{t.5}}{du_0/dx^{t+1}} = 4$$

while, through the sale of futures contracts of period t+1 goods and the purchase of spot contracts for period t goods, the agent can exchange these goods at a rate of 3 units of period t goods for every 1 unit of period t+1 goods.

5. EXISTENCE OF COMPETITIVE EQUILIBRIA FOR DYNAMIC MODELS WITH INCOMPLETE MARKETS

In this section, we describe a general model of a dynamic pure-exchange economy in which, at any given point in time, trading for future commodity contracts takes place in markets that are, in general, incomplete. The model we describe is patterned after the work of Radner [1972, 1982]. Our most significant extensions of Radner's model are that we allow for an unbounded time horizon and that we replace his assumption of each agent having a positive endowment of all goods with a much weaker assumption of irreducibility. In section 6, we further extend Radner's work by establishing the general existence of competitive equilibria when production is considered.

In contrast to the standard Arrow-Debreu economy, since the market structure is incomplete, it may not be possible for a given agent to consummate all of his desired transactions by trading once and for all in a single market. Thus, there will be a demand by agents for a continually reopening of markets over time as new commodity contracts become available. As an extreme example, consider an economy in which, at time t, there are only markets for trade in current goods (i.e. those goods which are consumed at time t). Each time the economy enters a new time period, agents will engage in trade for the now current goods, since these trades could not be made at any previous date. Thus, markets are sequentially opening and closing over time.

After we have described our incomplete market model in detail, we will show how it fits into the general framework developed in section 2. Next, we list restrictions on the incomplete market economy that are sufficient to guarantee that the corresponding assumptions of section 3 are satisfied. It will then follow, by Proposition 3.5, that competitive equilibria exist for our general incomplete market economy.

The economy evolves over dates t = 1, 2, ... Uncertainty about future events is modeled by specifying a set, S, of all possible histories of the economy. Information is modeled as the knowledge that the true history of the economy lies in a specific subset, s, of S. The collection of all alternative collections of information that can be acquired by time t constitutes a partition S_t of S. That is, S_t contains all of the subsets of S, called events, that are observable at time t. If we assume that previously known information is never lost, then the sequence of partitions $\{ S_t \}$ will be monotone non-decreasing in fineness. That is, any element in S_{t+1} is a subset of a element in S_t. For notational simplicity, we assume $S_1 = \{ S \}$.

Agents plan to consume goods at various dates and possibly contingent on (uncertain) events. We denote the set of all such date-event pairs by $D = \{ (t,s) : t = 1, 2, \ldots \text{ and } s \in S_t \}$. We assume that there are a finite number, $H_{t,s}$, of goods that can be consumed at date t in the event $s \in S_t$. We index the collection of all such (contingent) commodities by the set $G = \{ (t,s,h) : (t,s) \in D$ and $h = 1, \ldots, H_{t,s} \}$. For each agent $a \in A$, we denote the collection of all feasible levels of (gross) consumption by $C_a \subseteq R^{\#G}$. That is, given $c \in C_a$, c^g denotes the agents consumption of good $g \in G$. We also let $w_a \in R^{\#G}$ denote the endowment bundle of consumer a. In general, we allow endowments to be state-dependent. For example, at date t, the agents endowments of good 1 in the state $s \in S_t$, $w_{(t,s,1)}$, can differ from the agents endowments of good 1 in the state $s' \in S_t$, $w_{(t,s',1)}$.

At each date-event pair $(t,s) \in D$, consumers can choose to trade among a finite collection of contracts. We index the contacts that are available at (t,s) by $I_{t,s} = \{ (t,s,j) : j = 1, \dots , J_{t,s} \}$. In the terminology of section 3, each of the sets $I_{t,s}$ constitutes a market. We denote the collection of all contracts that are traded on these markets by $I = \bigcup_{(t,s)\in D} I_{t,s}$, while the partitioning of I into trading markets is denoted by $M = \{ I_{t,s} : (t,s) \in D \}$.

One unit of contract $i = (t,s,j) \in I$ entitles the consumer to an allocation bundle $c(i) \in R_+^{\#G}$. Clearly, at any date-event pair $(t,s) \in D$, the consumer can no longer change his consumption plans for any previous date or in any future event that cannot occur given the current state of the economy. Therefore, we must impose the restriction;

For any contract $i \in I_{t,s}$ and any good $g = (t',s',h) \in G$, $(c(i))^g = 0$ unless $t' \geqslant t$ and $s' \subseteq S$.

In general, contracts may be bought and sold in any (divisible) quantity. Holding z^i units of contract i entitles/obligates the agent to a net commodity trade of $z^i c(i) \in R^{\#G}$. A complete description of an agents trade plans are specified by a vector $z \in R^{\#I}$, where z^i denotes the net quantity of contract i that is traded. Given net trades $z \in R^{\#I}$, in terms of contracts, the consumer then incurs a net trade of $\sum_{i \in I} z^i c(i) \in R^{\#G}$ in terms of actual commodities.

To conclude our exposition of the incomplete market economy, we present restrictions that will guarantee that the assumptions of section 2 are satisfied.

As previously noted, in order to satisfy assumption I(ii), we must limit the ability of (infinitely-lived) agents to short futures contracts (i.e. sell so much of a contract that the consumer must buy contracts in latter markets in order to meet his existing obligations). For notational simplicity, we simply eliminate shorting altogether. Before we do so, it is convenient to introduce notation to single out all of the contracts that have been traded up to a given date-event pair $(t,s) \in D$.

We define the relation $\leqslant$ on $I \times D$ by

$(t,s,j) \leqslant (t',s')$ if $t \leqslant t'$ and $s' \subseteq S$.

For later convenience, we define the stronger relation $<$ on $I \times D$ by

$(t,s,j) < (t',s')$ if $t < t'$ and $s' \subseteq s$.

Given an agents space of feasible (gross) consumption profiles $C_a \subseteq R^{\#G}$, we define the agents space of feasible (net) contract holdings $X_a \subseteq R^{\#I}$, as the set of all contract schemes that satisfy the constraint that the agent never shorts a contract. That is,

$$X_a = \{ z \in R^{\#I} \mid w_a + \sum_{i \leqslant d} z^i c(i) \in C_a \text{ for each date-event pair } d \in D \text{ and } z^{(t,s,j)} = 0 \text{ for all } (t,s) \notin D_a \},$$

where $D_a \subseteq D$ denotes the collection of date-event pairs in which agent $a \in A$ is "alive".

In words, a contract scheme is feasible, $z \in X_a$, if at any given date-event pair $d \in D$ the consumer can meet his existing obligations (i.e. fulfill all of the contacts $i \leqslant d$) without trading in any later markets. As in section 2, given X_a, we define the set of markets that agent $a \in A$ participates in by $M_a = \{ m \in M \mid$ there is an $x \in X_a$ and an $i \in m$ such that $x^i = 0 \}$. One may readily verify that Assumption I(ii) is an immediate consequence of the definition of X_a if we consider the sequence $\{ M_a(t) \}$ defined by $M_a(t) = \{ m = I_{t',s} \in M_a : t' \leqslant t$ and $s \in S_{t'} \}$. Assumption I(i) and (iii) will also follow once we impose the following.

Restriction II (Finite endowments).

For each $m \in M$, $\{ c \in \sum_{a \in A} C_a : c \leqslant 0 \}$ is bounded from below.

For convenience, in section 3, we worked solely in terms of trade contracts $x \in R^{\#I}$ (as opposed to consumption profiles $c \in R^{\#G}$). However, since preferences are generically expressed in terms of consumption profiles, we provide the following restriction as an alternative to assumption III.

(R.3) (Regularity of preferences)

For $a \in A$, the preferences of agent $a \in A$ are represented by a utility function $v_a(\cdot)$: $C_a \longrightarrow R$ that satisfies:

(i) $v_a(\cdot)$ is continuous;

(ii) $v_a(\cdot)$ is quasi-concave.

Given these preferences, defined on consumption profiles, we can represent the equivalent preferences over the space of contracts , X_a, by the utility function

$$u(z) = v(\sum_{i \in I} z^i c(i)).$$

One may readily verify that (R.3) implies that assumption III is satisfied by $u(\cdot)$.

In each of the above three cases (except for II(i)), there is a natural equivalence between the generic restrictions in terms of consumption goods and the induced assumptions in terms of contracts. However, the irreducibility assumption IV has no simple generic counterpart. This is due to the fact that the assumption of irreducibility in terms of trade contracts involves not only the degree of irreducibility of preferences and endowments, but it restricts the underlying market structure of the economy as well. To illustrate this point, consider the following simple economy. There are two dates t = 1, 2 (with a trading market open at each date) and two agents a = 1, 2. Each agent has strictly-monotone preferences for all goods and agent 1 has positive endowments of all goods. However, agent 2 only has endowments for goods in period 2 (suppose he has positive endowments for all second period goods). If the first period market is restricted to trading in current goods, then the irreducibility assumption Assumption IV is not satisfied since agent 2 has nothing of value to trade. However, if the first period market also includes trade in some futures contracts, then assumption IV is satisfied since agent 2 can now sell all or part of his second period endowments.

For completeness, we present a restriction on preferences and endowments that is sufficient to guarantee that assumption IV is satisfied. The above example demonstrates that this restriction is not necessary for assumption IV to hold.

(R.4) (Irreducibility at each date-event pair and completeness of current markets)

(i) Given any market $m = I_{t,s} \in M$, any partition of A_m into non-empty subsets B_0 and B_1, and given any consumption scheme $(c_a) \in C$ such that $\sum_{a \in A} c_a^g \leqslant \sum_{a \in A} w_a^g$ for g such that $g = (t,s,h)$ for some h: There is an agent $b \in B_0$ and allocations $(e_a)_{a \in B_1}$ that satisfy; $e_a \in C_a$ and $e_a | (t,s)$ for $a \in B_1$, and $v_b(c_b + \sum_{a \in B_1} (w_a - e_a)) > v_b(c_b)$.

(ii) For each good $g = (t,s,h) \in G$, there exists a contract $i = (t,s,j) \in I$ such that $(c(i))^g > 0$ and $(c(i))^{g'} = 0$ for all goods $g' = g$.

6. EXISTENCE OF COMPETITIVE EQUILIBRIA FOR DYNAMIC MODELS WITH INCOMPLETE MARKETS AND PRODUCTION

In this section, we introduce production into the incomplete market model. In doing so, we closely follow the general scheme set forth in Radner [1972, 1982]. To avoid unnecessary duplication, we abbreviate our basic description of production and the accompanying stock markets. Instead, we focus our attention on our extensions of Radner's work and, in particular, on establishing the general existence of competitive equilibria.

In his commentaries, Radner [1972, 1982] suggests a few areas in which his work should be strengthened. (Of course, the most significant of these is to establish the general existence of equilibria.) Another such area is the incorporation of limited liability of agents that hold shares in the various firms. Having limited liability is essential to our interpretation of agents as "shareholders", as opposed to partners, in firms. We incorporate limited liability by constraining firms to generate only non-negative profits at each date-event pair (thus guaranteeing that agents holding shares in these firms will never suffer a loss of income). Note that these non-negativity constraints do not necessarily prevent firms from undergoing periods of substantial investment. Specifically, firms can invest in excess of revenue from current production provided that they are able to finance at least part of their investments through the sale of contracts for anticipated future output.

The reason that Radner did not incorporate non-negative profit constraints into his model is that he did not find acceptable conditions that would guarantee the continuity of producers supply correspondences. The possible dis-continuity of supply stems from the fact that, for general classes of production technologies (e.g. in the case of constant returns to scale), producers may not be able to generate positive profits in equilibrium. Thus, firms may be forced to the boundary of their feasibility sets (with respect to the non-negative profit constraints). This problem is similar to the dis-continuities of demand that can arise when some agents have no income.

Fortunately, in this section, we develop techniques that allow us to guarantee the required continuity of producers supply for a quite general class of production technologies. Thus, we are able to incorporate the desired limited liability of shareholders into the sequential model. Quite surprisingly, we find that imposing limited liability is our key to transforming Radner's model into one in which equilibria are guaranteed to exist.

It is well-known that to guarantee the general existence of equilibria (with non-negative prices), one must insure that there is some form of free disposal of goods. In pure exchange economies, this requirement is typically satisfied by assuming that each agent can derive non-negative marginal utilities from consuming each good. This observation directly relates to our incomplete market model in that, in order to establish the existence of equilibria with non-negative commodity and share prices, we must insure that potential investors can derive non-negative benefits from holding additional shares. But, this is implied by the non-negative profit constraints since holding shares can never decrease an agents income.

Before we formally introduce production, we find it necessary to impose further constraints on the underlying structure of commodity contracts. One should note that, at this time, we do not know precisely how necessary most of the assumptions of this section are to insuring the validity of our existence result (proposition 6.9).

Assumption VII.

(i) For each contract $i \in I$, $c(i) = c(i)|d(i)$ for some date-event pair $d(i) \in D$.

(ii) For each contract $i \in I_{t,s}$ and date-event pair $d \in D$;
if $c(i) = c(i)|d$ and $i < d$, then there is an event $s' \in S_{t+1}$ and a contract $i^+ \in I_{t+1,s'}$ such that $i^+ \leqslant d$ and $c(i^+) = c(i)$.

For the remainder of this paper, we adopt the notation defined in assumption VII (i.e. $d(i)$ and i^+). For convenience, we assume that given any contract i, as described in (ii) above, the contract i^+ is unique. Furthermore, given any such contracts i and i^+, we define $j^- = i$ where $j = i^+$. That is, for $i \in I$, i^+ (if it exists) is the contract that corresponds to "reselling" i in the next available market while i^- (if it exists) is the previously available contract that is "resold" as the contract i.

Assumption VII(i) requires that any given commodity contract can involve goods at only one date-event pair. In comparison, our payoff structure is less restrictive than that imposed in Radner [1972, 1982] (who assumes that every contract involves only a single good) but is more restrictive than that considered in Hart [1975] (who allows contracts to involve arbitrary bundles of goods). Assumption VII(ii) requires that futures contracts can always be resold in later markets.

We now specify the market behavior of firms. Let the collection of firms be indexed by F. For notational simplicity, we assume that the number of firms, #F, is finite. It is straightforward to extend the arguments in this section to establish the existence of equilibria if we replace this finiteness assumption with the requirement that there are only a finite number of firms operating at any given point in time.

Let $Y_f \subseteq R^{\#G}$ denote the set of feasible net-output levels of firm $f \in F$. For simplicity, we assume that firms are given access to the same trading markets as consumers. We specify the feasible trades for firms, as we have done for consumers, to be those in which they never short a contract. That is,

$$Z_f = \{ z \in R^{\#I} \mid \sum_{i \leqslant d} z^i c(i) \in Y_f \text{ for every date-event pair } d \in D \}.$$

As in section 3, in order to insure that all market behavior is appropriately continuous, we bound the feasibility set of each agent $a \in A$ and each firm $f \in F$. We choose a $b \in R_+^{\#I}$ and consider the truncated consumption sets

$X'_a = \{ x \in X_a : x \leqslant b \}$ for $a \in A$.

We bound production sets by

$Z'_f = \{ z \in Z_f$: for any maximal string of contracts $i_1, \ldots, i_n$, i.e. $i_{k+1} = i_k^+$ for $k = 1, \ldots, n-1$ and neither i_1^- nor i_n^+ exist: $z^{i_1} \leqslant b^{i_1}, z^{i_2}+z^{i_1} \leqslant b^{i_2}, \ldots, z^{i_1}+\ldots+z^{i_n} \leqslant b^{i_n} \}$.

One may readily verify, by assumption VIII(ii) below, that Z'_f is a bounded set (that is each of the components of elements in Z'_f are bounded). Technically, the reason that we do not simply consider the truncated sets $\{ z \in Z_f : z \leqslant b \}$ is that we want Z'_f to have the property that the purchase of any futures contract can always be postponed to the next available market. That is, for all z, $z = (\ldots, z^i, \ldots, z^{i^+}, \ldots) \in Z'_f$ implies $(\ldots, 0, \ldots, z^{i^+}+z^i, \ldots) - Z'_f$. Following Debreu [1959], we can argue that if we choose b large enough, then the competitive equilibria of the bounded economy correspond to the equilibria of our original economy. We denote the collection of consumer contract schemes by $X = \mathop{\times}_{a \in A} X'_a$ and the collection of producer contract schemes by $Z = \mathop{\times}_{f \in F} Z'_f$.

Given contract prices $p \in R_+^{\# I}$ and net trades $z \in Z'_f$, firm f generates a net revenue of $p(z|m)$ in the market $m \in M$. The stream of all such revenues is denoted $r(p,z) = (r_m(p,z)) \in R^{\# M}$, where $r_m(p,z) = p(z|m)$ for $m \in M$. We assume that firms choose their production profiles to maximize the value of some exogenously given function, $v_f(\bullet)$, of its revenue stream. Formally, given the price system $p \in R_+^{\# I}$, firm f's supply correspondence, $Q_f(p)$, is defined to be the collection of all solutions to

$$\max \ v_f(r(p,z))$$
$$\text{s.t. } r(p,z) \geq 0$$
$$z \in Z_f'.$$

Note that $r(p,z) \geq 0$ restricts net revenue to be non-negative in each market.

We now list restrictions on production technologies and the firms objective functions that, together with assumptions I - VII, guarantee the general existence of competitive equilibria.

Assumption VIII. For each firm $f \in F$:

(i) Y_f is a closed and convex subset of $R^{\#G}$;

(ii) For any $b \in R^{\#G}$, $\{ y \in \sum_f Y_f \mid y \geq b \}$ is bounded;

(iii) $0 \in Y_f$;

(iv) $Y_f = \times_{d \in D} Y_{f,d}$, where $Y_{f,d}$ denotes the set of feasible output levels of goods available at the date-event pair $d \in D$. That is, $y \in Y_f$ if, and only if, $(y(t,s,h))_{h=1}^{H_{t,s}} \in Y_{f,(t,s)}$ for all date-event pairs $(t,s) \in D$.

(v) $v_f(\bullet)$ is continuous, non-decreasing, and strictly quasi-concave.

Assumption VIII(ii) states that it is impossible to produce arbitrarily large quantities of output from a finite quantity of inputs. Assumption VIII(iv) is a separability condition on the firms production technology. There are counter-examples that demonstrates the necessity of an assumption like VIII(iv) (even for finite-horizon economies) but we do not present them here so as to abbreviate our exposition.

Unfortunately, the separability condition VIII(iv) rules out time to build technologies since, in particular, production opportunities at a given date are necessarily independent of any previous production levels. However, time to build technologies can be incorporated if we alternatively specify production to be in terms of contracts, i.e. $Y_f \subseteq R^{\#I}$. For example, if employing one unit of labor and steel today produces one car next year (thus implying a non-separable production set $Y_f \subseteq R^{\#G}$) and if there are futures contracts for cars, then we can alternatively think of the firms technology as being separable if it is expressed in terms of contracts. Specifically, in each period, firms can employ one unit of labor and steel to "produce" one futures contract for a car.

Assumption VIII(v) is analogous to restrictions typically imposed on consumers preferences. In particular, the strict quasi-concavity of $v_f(\bullet)$ implies that a firms revenue stream is uniquely determined given a commodity price system $p \in R_+^{\#I}$. That is, for $f \in F$, we can define firm f's optimal revenue stream by $r_f(p) = (r_m(p))$, where $r_{f,m}(p) = p(z|m)$ for $z \in Q_f(p)$ and $m \in M$.

We now briefly describe the stock market. In each market $m \in M$, agents trade for shares in each firm $f \in F$. We index shares by the set $N = \{ (m,f) : m \in M \text{ and } f \in F \}$. An agents share trading plan is then a point $w \in R^{\#N}$, where w^n denotes the quantity held of share $n \in N$. Share quantities are normalized to be non-negative with the aggregate stock of shares outstanding in each firm taken to be 1. Therefore, the space of feasible share plans for agent $a \in A$ is given by

$W_a = \{ w \in R^{\#N} : 0 \leq w^n \leq 1$ for $n \in N$ and $w^n = 0$ if $n = (m,f)$ and $m \notin M_a \}$.

The collection of all such share plans is denoted $W = \bigtimes_{a \in A} W_a$.

A share price system is a point $q \in R^{\#N}$, where q^n denotes the price of share $n \in N$. As previously mentioned, our non-negativity constraints on producer profit allows us to restrict our attention to non-negative share prices. We jointly normalize commodity and share prices to lie in the space

$$\Pi = \bigtimes_{m \in M} \Pi_m,$$

where $\Pi_m = \{ (p,q) \in R_+^{\#I} \times R_+^N : \sum_{i \in m} p^i + \sum_{f \in F} q^{(m,f)} = 1 \text{ for } m \in M \}$.

Given a (commodity-share) price system $(p,q) \in \Pi$, a portfolio plan $w \in W_a$ implies that agent a's total share of producer revenue in the market $m \in M$ is $\sum_{f \in F} w^{(m,f)} r_{f,m}(p)$. In addition to sharing in producer profit, trading in shares effects income through capital gains or losses. We assume that agents are (exogenously) endowed with shares at the inception of the economy. Let w_a^f denote the endowment of agent $a \in A_m$ of shares in the firm $f \in F$, which he receives before the initial market $m = I_{1,S}$ opens. We normalize share quantities so that, $w_a^f \geq 0$ and $\sum_{a \in A_m} w_a^f = 1$ for $f \in F$.

For completeness, let $w_a^f = 0$ for $a \in A_m$.

For $a \in A_m$, agent a's capital gains in the initial market $m = I_{1,S}$ is

$$\sum_{f \in F} q^{(m,f)} (w_a^{(m,f)} - w_a^f),$$

while his gains in each market $m \in M_a - \{m\}$ are

$$\sum_{f \in F} q^{(m,f)} (w_a^{(m,f)} - w_a^{(m^-,f)}),$$

where m^- denotes the (unique) market immediately preceding m. That is, if $m = (t,s)$, then $m^- = (t-1,s')$ for some $s' \in S_{t-1}$ with $s \subseteq s'$.

We now characterize the agents market behavior. Given a commodity-share price system $(p,q) \in \Pi$ and an income transfer scheme $(y_m) \in R_+^{\#M_a}$, agent a's demand correspondence, $D_a(p,(y_m))$, is defined to be the set of all solutions to

$$\max \ u_a(x)$$

$$\text{s.t. } p(x|m) - \sum_{f\in F} w^{(m,f)} r_{m,f}(p) + \sum_{f\in F} q^{(m,f)}(w_a^{(m,f)} - w_a^f) \leqslant y_m \quad \text{for } m = I_{1,S}$$

$$p(x|m) - \sum_{f\in F} w^{(m,f)} r_{m,f}(p) + \sum_{f\in F} q^{(m,f)}(w_a^{(m,f)} - w_a^{(m^-,f)}) \leqslant y_m \quad \text{for } m \in M-\{m\}$$

$$(x,w) \in X_a' \times W_a.$$

As in our model without production, we find it useful to consider the notion of a competitive allocation as an intermediate step toward obtaining a competitive equilibrium. For purely technical reasons, we introduce $q \in R_+^{\#N}$ as an argument in producers correspondences (i.e. we have $Q_f(p,q)$).

6.1 Definition.

(i) A Competitive Allocation consists of a commodity-share price system $(p,q) \in \Pi$, commodity contract schemes $(x_a) \in X$ and $(z_f) \in Z$, and a distribution of shares $(w_a) \in W$ such that for $a \in A$ and $f \in F$: $(y_{a,m}) \geqslant 0$ for $a \in A$, where $(y_{a,m})$ denotes the income transfer scheme to agent a

$(x_a,w_a) \in D_a(p,(y_{a,m}))$

$z_f \in Q_f(p,q)$.

(ii) A (Free Disposal) Competitive Equilibrium is a Competitive Allocation that satisfies:

$y_{a,m} = 0$ for $a \in A$ and all $m \in M_a$

$\sum_{a \in A} x_a \leqslant \sum_{f \in F} z_f$

$\sum_{a \in A} w_a^n \leqslant 1$ for $n \in N$.

We now establish the general existence of equilibria for our sequential model with production. As previously mentioned, the main difficulty we face is in establishing the continuity of firms supply. Once we do so, the remainder of our proof proceeds as in section 3. Specifically, we find a competitive equilibrium for our full economy as a limit point of equilibria from a sequence of (finite) sub-economies.

In a manner analogous to the income perturbations of section 3, we find it useful to perturb firms trading sets in each of the sub-economies. Specifically, if we "endow" each firm with a positive quantity of all commodities, i.e. change Z_f' to $Z_f' + (\epsilon, \epsilon, \ldots)$, then, by assumption VIII(iii), each firm can generate a positive income in each market. Our first lemma states that this perturbation is sufficient to guarantee the desired continuity of supply.

6.2 Lemma. For each firm $f \in F$; if $Z_f' \times R_{++}^{\#I} = 0$, then $Q_f(p,q)$ is upper hemi-continuous at each point $(p,q) \in \Pi$.

Proof of Lemma 6.2.

Consider a sequence $\{ ((p(k),q(k)),z(k)) \}$ in $\Pi \times Z_f'$ such that

$z(k) \in Q_f((p(k),q(k))$ for each k and $((p(k),q(k)),z(k)) \rightarrow ((p,q),z)$. To establish the desired continuity property we must show $z \in Q_f(p,q)$.

Since each market is finite by assumption V(ii), $r(\circ,\circ)$ is continuous. Hence, $r(p(k),z(k)) \geqslant 0$ for each k implies $r(p,z) \geqslant 0$. Also, assumption VIII implies that Z_f' is closed. Hence, $z(k) \in Z_f'$ for each k implies $z \in Z_f'$.

To conclude our proof, we must demonstrate that z is an optimal trade plan. If z was not optimal, then there would exist an alternative $z_0 \in Z_f'$ such that

$$v_f(r(p,z_0)) > v_f(r(p,z)) \quad \text{and} \quad r(p,z_0) \geqslant 0.$$

By hypothesis, there exists a $z_1 \in Z_f' \cap R_{++}^{\#I}$.

By the continuity of $r(\bullet,\circ)$ and assumption VIII(v)

$$v_f(r(p,z_2)) > v_f(r(p,z)),$$

where $z_2 = \lambda z_0 + (1-\lambda)z_1$ for some $\lambda < 1$.

$z_1 \gg 0$ and $r(p,z_0) > 0$ implies,

(a) $r(p,z_2) \gg 0$.

Let $M_t = \{ I_{k,s} \in M : k < t \}$.

By the definition of Z_f', the continuity of $r(\circ,\circ)$, and assumption VIII(v), there is a t sufficiently large such that

$$z_2|M_t \in Z_f' \text{ and } v_f(r(p,z_2|M_t)) > v_f(r(p,z)).$$

One may now contradict $z(k) \in Q_f(p(k),q(k))$ since, for large k,

$$v_f(r(p(k),z_2|M_t)) > v_f(r(p(k),z(k))$$

and (a) implies

$$r_m(p(k),z_2|M_t) \geqslant 0 \text{ for } m \in M_t.$$

Clearly, $r_m(p(k),z_2|M_t) = 0$ for $m \notin M_t$ since $(z_2|M_t)|m = 0$.

Therefore, z(k) is dominated by $z_2|M_t$.

Q.E.D.

The above lemma establishes the general continuity of supply at all price systems $(p,q) \in \Pi$ provided that each firm can "produce" a positive quantity of all contracts. This result is therefore sufficient to guarantee the existence of equilibria for each of our perturbed economies. However, in order to guarantee that any limit point of these equilibria constitutes an equilibrium for the full economy, we must establish the continuity of producer supply, at the limiting prices, for the original economy (without any added producer endowments).

In proposition 6.9, we verify that all limiting prices satisfy the following "no arbitrage" condition. In lemma 6.5, we guarantee the desired continuity of supply by establishing continuity at all prices that satisfy the no arbitrage requirement.

A commodity price system $p \in R_+^{\#I}$ is said to satisfy the no arbitrage condition if for any market $m \in M$, there is a $\lambda > 0$ such that $p^i = \lambda p^{i^-}$ for all $i \in m$ such that i^- exists. Our interpretation of the above as a requirement of no arbitrage should be apparent. For example, if $p^j = p^{i^-} = 1$ and $p^i = p^{j^-} = 2$ (thus violating the no arbitrage condition); then by buying 1 unit of i^- and j and selling 1 unit of i and j^-, agents generate 1 unit of income in each market m^- and m with no loss of income in any of the other markets.

The following lemma is crucial to establishing the necessary continuity of producer supply. As previously mentioned, it is conceivable that discontinuities in supply can arise when producers are unable to generate positive profits in each market. If we refer to the markets in which the firm cannot generate profits as unproductive, then Lemma 6.3 states that any unproductive market may as well be "eliminated" from the firms opportunity set. Specifically, as pointed our in the corollary to the lemma, a firm can always choose an optimal production trade plan in which it does not trade in unproductive markets. The desired continuity of supply is then established, as in the proof of lemma 6.2, once a firm restricts its trades to productive markets.

Given prices $p \in R_{+}^{\#I}$, let $M_f^+(p)$ denote the collection of markets at which firm f can generate positive profits and let $M_f^0(p)$ denote the remaining (unproductive) markets in which it cannot. That is,

$M_f^+(p)$ = { m ∈ M : there is a $z \in Z_f'$ such that $r(p,z) \geqslant 0$ and r_{m} .$(p,z) > 0$ }

while

$M_f^0(p) = M - M_f^+(p)$.

6.3 Lemma. If the commodity price system $p \in R_{+}^{\#I}$ satisfies the no-arbitrage condition, then for any firm $f \in F$, market $m \in M_f^0(p)$, and trade plan $z \in Z_f'$:

There is an alternative plan $z_* \in Z_f'$ such that $z_*|m = 0$ and $r(p,z_*) = r(p,z)$. Furthermore, z_* can be chosen so that $z_*|m = z|m$ for all markets other than $I_{t,s}$ and those of the form $I_{t+1,s'}$ where $s' \subseteq s$.

Proof of Lemma 6.3.

We define the desired trade plan, z_*, by perturbing the plan z.
For $i \in m$, let $z_*^i = 0$.
For $i \in I$ such that i^- exists and $i^- \in m$, let $z_*^i = z^i + z^{i^-}$.
For all other $i \in I$, let $z_*^i = z^i$.

Basically, we form z_* to differ from z in two ways. First, we delete from z all trades for current goods in the market m. Second, we shift all trades for future goods from the market m to the next available market. Thus, the new trade plan is both feasible, i.e. ($z_* \in Z_f'$), and it satisfies the condition $z|m = 0$. Recall, Z_f' was defined to insure that such a move from z to z_* is feasible.

We conclude our proof by demonstrating that $r(p,z^*) = r(p,z)$. That is, we show that there is no net change in revenue by moving from z to z_*. Clearly, since $m \in M_f^0(p)$, $r_m(p,z) = 0$. Therefore, there is no change in revenue when we delete the trades in m from z. The other markets effected by moving from z to z_* are the markets $I_{t+1,s}$ for $s \in S_{t+1}$. The condition for their to be no revenue change in these markets can be written

(a) $\sum_{i \in m_s} p^{i+} z^i = 0$ for $s \in S_{t+1}$,

where $m_s^s = \{ i \in m : i^+ \text{ exists and } i^+ \in I_{t+1,s} \}$ for $s \in S_{t+1}$.

That is, since each of the contracts effected in the market $I_{t+1,s}$ can be written as i^+ for some $i \in m_s$ and since the total quantity change for the contract i^+ is z^i, the total value of all quantity changes in the market $I_{t+1,s}$ is $\sum_{i \in m_s} p^{i^+} z^i$.

By our hypothesis that the price system p satisfies the no arbitrage condition, we can replace (a) with the equivalent expression

(b) $\sum_{i \in m_s} p^i z^i = 0$ for $s \in S_{t+1}$.

Our proof is therefore complete once we establish (b).

We first verify

(c) $\sum_{i \in m_s} p^i z^i \geqslant 0$ for $s \in S_{t+1}$.

For any $s \in S_{t+1}$, consider the contract scheme z_s defined by

$z_s^i = 0$ if $i \in m_s$ or $i \geqslant (t+1,s)$

and

$z_s^i = z^i$ otherwise.

That is, z_s cuts off future trades that involve goods in the date-event pair $(t+1,s)$ and in its successors.

By examining the definition of Z_f', one may conclude $z_s \in Z_f'$.

By our construction, $z_s|m'$ = either $z|m'$ or 0 for $m' \in M-\{m\}$.

In particular, this implies $r_{m'}(p,z_s) \geqslant 0$ for $m' \in M-\{m\}$. Now since $m \in M_f^s(p)$, we must have

(d) $r_m(p,z_s) \leqslant 0$.

But, by our construction,

$r_m(p,z_s) = \sum_{i \in m} p^i z_s^i = p(z|m) - \sum_{i \in m_s} p^i z^i$.

(c) now follows from (d) since $p(\overset{s}{z}|m) = 0$.

In the above paragraph, we established that the firm must be making a non-negative profit from trading in certain collections (m_s) of futures contracts that are available in the market m. We first showed that the agent always has the freedom of canceling all of his trades in any of these collections (by moving from z to z_s). We then argued that, since these cancellations cannot raise profits at m (or our hypothesis $m \in M_f^s(p)$ would be violated), our non-negative profit condition (c) follows. One may similarly argue that the profit from trading in current goods at m must also be non-negative since these trades can also be deleted. Therefore,

(e) $\sum_{i \in m_c} p^i z^i \geqslant 0,$

where $m_c = m - \bigcup_{s \in S_{t+1}} m_s$ denotes the sub-collection of contracts for current goods.

But, since the aggregate profit at m is zero, i.e. $\sum_{i \in m} p^i z^i = 0$, (b) follows from (c) and (e).

Q.E.D.

The following corollary puts the above result into a form that is useful in establishing the desired continuity of supply. The corollary follows from repeated applications of lemma 6.3.

6.4 Corollary. If prices $p \in R_+^{\#I}$ satisfy the no arbitrage condition, then

(a) there is an optimal trade plan $z \in Q_f(p,q)$ such that $z|m = 0$ for $m \in M_f^0(p)$

(b) there is a trade plan $z \in Z_f^!$ such that $z|m = 0$ for $m \in M_f^0(p)$ and $r_m(p,z) > 0$ for $m \in M_f^+(p)$.

Proof of Corollary 6.4.

(Proof of a). Clearly, assumptions VIII(i) and (v) imply that $Q_f(p,q) = 0$. Let $z^0 \in Q_f(p,q)$.

Order the collection of markets in $Z_f^0(p) = \{ m_1, m_2, \ldots \}$ to be such that $t_1 \leqslant t_2 \leqslant \ldots$, where $m_k = I_{(t_k, s_k)}$.

We recursively define a sequence $\{ z^k \}$ by letting z^{k+1} (k>0) denote the trade plan that results from applying lemma 6.3 to the market m_{k+1} and the trade plan z^k. By induction, one may verify that for each k:

(c) $z_k \in Z'_f$

(d) $z_k' m_t = 0$ for $t = 1, \ldots, k$

and

(e) $r(p, z_k) = r(p, z^0)$.

If $\{ m_1, m_2, \ldots \}$ is finite, then let z_* denote the last plan z_k. If $\{ m_1, m_2, \ldots \}$ is infinite, then the set $\{ z^k \}$ is also infinite. By the Tychonoff theorem, Z'_f is compact. Hence, there is a limit point, call it z_*, of the sequence $\{ z^k \}$.

In either of the above two cases, one may verify that (c) - (e) imply that z_* satisfies (a). (The optimality of z_*, i.e. $z_* \in Q_f(p,q)$, follows from $z^0 \in Q_f(p,q)$, $z_* \in Z'_f$, and $r(p,z_*) = r(p,z^0)$.)

(Proof of b). We can establish (b) by the same inductive process described above. The only difference in the two proofs being the choice of the initial allocation $z^0 \in Z_f$.

For each $m \in M_f^+(p)$, there is, by definition, a trade plan $z_m \in Z'_f$ such that $r(p,z_m) \geqslant 0$ and $r_m(p,z_m) > 0$. Fix any collection of positive scalars (λ_m) such that $\sum_{m \in M_f^+(p)} \lambda_m = 1$.

Let $z_0 = \sum_{m \in M_f^+(p)} \lambda_m z_m$.

Clearly, $r(p,z_0) \geqslant 0$ and $r_m(p,z_0) > 0$ for $m \in M_f^+(p)$.

One may verify that assumption VIII(i) implies $z^0 \in Z'_f$.

If one now follows the procedure used to establish (a), the resulting limiting allocation z_* will satisfy the requirements of part (b) of our corollary.

6.5 Lemma. For each firm $f \in F$, $Q_f(p,q)$ is upper hemi-continuous at each point $(p,q) \in \Pi$ such that p satisfies the no arbitrage condition.

Proof of Lemma 6.5.

Consider a sequence $\{ ((p(k),q(k),(z(k))) \}$ in $\Pi \times Z'_f$ such that $z(k) \in Q_f((p(k),q(k))$ for each k, $((p(k),q(k),z(k)) \longrightarrow ((p,q),z)$, and p satisfies the no arbitrage condition. To establish the desired continuity property, we must show $z \in Q_f(p,q)$.

As in the proof of Lemma 6.2, it immediately follows that $z \in Z'_f$ and $r(p,z) \geqslant 0$. Hence, if $z \in Q_f(p,q)$, then z must not be optimal, i.e.

(a) $v_f(r(p,z)) < v_f(r_f(p))$.

By corollary 6.4, there exists a $z_0 \in Q_f(p,q)$ such that $z|m = 0$ for $m \in M_f^0(p)$ and there is a $z_1 \in Z'_f$ such that $r_m(p,z_1) > 0$ for $m \in M_f^+(p)$ and $z_1|m = 0$ for $m \in M_f^0(p)$.

Since z_0 is optimal, $r_f(p) = r(p,z_0)$.

Hence, (a) implies

(b) $v_f(r(p,z)) < v_f(r(p,z_0))$.

By assumption V(ii), $r(p,\cdot)$ is continuous in z. Hence, (b) together with assumption VIII(v) implies

(c) $v_f(r(p,z)) < v_f(r(p,z_2))$,

where $z_2 = \lambda z_0 + (1-\lambda)z_1$ for some positive $\lambda < 1$.

By assumption VIII(i), $z_2 \in Z'_f$.

Also, by the definitions of z_0 and z_1,

(d) $r_m(p,z_2) > 0$ for $m \in M_f^+(p)$ and $z_2|m = 0$ for $m \in M_f^0(p)$.

Again, by assumption VIII(v),

(e) $v_f(r(p,z)) < v_f(r(p,z_2|M_t))$ for large t,

where $M_t = \{ (t',s) \in M : t' < t \}$.

Fix any such t.

For large k, (d) implies

$r(p(k), z_2|M_t) \gg 0$

while (e) and assumption VIII(v) imply

$v_f(r(p(k), z(k)) < v_f(r(p(k), z_2|M_t))$.

Taken together, the above two inequalities demonstrate that $z_2|M_t$ dominates $z(k)$, in contradiction to our hypothesis

$z(k) \in Q_f(p(k), q(k))$.

Q.E.D.

As previously mentioned, the main difficulty in establishing the general existence of equilibria lies in guaranteeing the above continuity of supply. Since the remainder of our proof is a straightforward extension the techniques employed in section 3, we merely highlight the necessary modifications of our arguments. Specifically, we supply statements (without proofs) of three lemmas, two concerning agents demand and income and one concerning the existence of equilibria for truncated economies. These lemmas, along with our results on the continuity of supply, are then employed to establish of the general existence of competitive equilibria (proposition 6.9).

6.6 Lemma. For $a \in A$, agent a's demand correspondence, $D_a(\circ,\circ)$ is upper hemi-continuous at each point $((p,q),(y_m)) \in \Pi \times R_+^{\#M_a}$ such that $I_{a,m}(p) < y_m$ for $m \in M_a$.

Since considering production enlarges the set of allocation schemes that clear the markets, our irreducibility assumption IV must be strenghthened.

Assumption IV* (Irreducibility of each market).

Given any market $m \in M$, any partition of A_m into two non-empty subsets B_0 and B_1 and given any allocation scheme $(x_a) \in X$ such that for some production scheme $(z_f) \in Z$, $\sum_{a \in A} x_a \leqslant \sum_{f \in F} z_f$ (i.e. the market m clears):

There is an agent $b \in B_0$ and allocations $(z_a)_{a \in B_1}$ that satisfy;
$z_a \in X'_a$ and $z_a = z_a|m$ for $a \in B_1$, and $x_b - \sum_{a \in B_1} z_a \; (>)_b \; x_b$.

6.7 Lemma. Given any limit point, $(p, q, (x_a), (z_f), (w_a))$, of a convergent sequence of competitive allocations, $\{(p(k), q(k), (x_a(k)), (z_f(k)), (w_a(k)))\}$, and given any market $m \in M$; if the market m clears at the limiting allocation, i.e. $\sum_{a \in A} x_a^i \leqslant \sum_{f \in F} z_f^i$ for $i \in m$, then each agent $a \in A_m$ has income in the market m at the limiting prices, i.e. $I_{a,m}(p) < 0$.

In the next lemma, we assert the existence of equilibria for each of our finite sub-economies. Note that in order to guarantee the positivity of income, we give agents additional endowments of shares as well as endowments of commodity contracts. Thus, in each of the "equilibria", share markets, like commodity markets, only come close to clearing.

6.8 Lemma. Given any finite collection of markets M'; there is a competitive allocation, $(p, q, (x_a), (z_f), (w_a))$, in which all of the agents and firms receive a small income transfer and all of the markets in M' almost clear. Specifically, given any $\varepsilon > 0$, if we replace Z'_f with $Z'_f + (\varepsilon, \varepsilon, \ldots)$ for $f \in F$, then there is a competitive allocation $(p,q,(x_a),(z_f),(w_a))$ such that

$0 \leqslant y_{a,m} \leqslant \epsilon$ for $a \in A$ and all $m \in M_a$, where $(y_{a,m})$ denotes the income transfer scheme to agent $a \in A$

$$\sum_{a\in A_m} x_a^i \leqslant \sum_{f\in F} z_f^i + \epsilon \text{ for } m \in M' \text{ and all } i \in m,$$

and

$$\sum_{a\in A_m} w_a^{m,f} = 1 + \epsilon \text{ for } m \in M' \text{ and all } f \in F.$$

6.9 Proposition. Under assumptions I - III, IV*, V, and VII - VIII, there exists a competitive equilibrium.

Proof of Proposition 6.9.

Fix a sequence of collections of markets $\{ M(k) \}$ such that each $M(k)$ is a finite subset of M, $M(k) \subseteq M(k+1)$ for all k, and $\bigcup_k M(k) = M$. Also, fix a sequence of positive scalars $\{ \epsilon(k) \}$ such that $\epsilon(k) \longrightarrow 0$.

Let $(p(k),q(k),(x_a(k)),(z_a(k)),(w_a(k)))$ be the competitive allocation associated with $M(k)$ and $\epsilon(k)$, as specified in lemma 6.8. That is, for the k'th equilibrium

(a) $0 \leqslant y_{a,m}(k) \leqslant \epsilon(k)$ for $a \in A$ and all $m \in M_a$,

where $(y_{a,m}(k))$ denotes the income transfer scheme to agent a

(b) Z_f' is replaced by $Z_f'+(\epsilon(k),\epsilon(k),\ldots)$

(c) $\sum_{a\in A_m} x_a^i \leqslant \sum_{a\in A_m} z_a^i + \epsilon(k)$ for $m \in M(k)$ and all $i \in m$

(d) $\sum_{a\in A_m}^{m} w_a^{m,f} \leqslant 1^m + \epsilon(k)$ for $f \in F$ and all $m \in M(k)$.

Since each allocation $(p(k),q(k),(x_a(k)),(z_f(k)-\epsilon(k)),(w_a(k)))$ is contained in $\Pi \times X \times Z \times W$, which is compact by the Tychonoff theorem, there is a point $(p,q,(x_a),(z_f),(w_a)) \in \Pi \times X \times Z \times W$ and a subsequence of allocations such that

(e) $(p(k),q(k),(x_a(k)),(z_a(k)),(w_a(k))) \rightarrow (p,q,(x_a),(z_f),(w_a))$.

By careful inspection, one may verify that the market clearing conditions (c) imply that prices p(k) satisfy the no arbitrage condition for $\epsilon(k)$ sufficiently small. The no arbitrage condition if clearly preserved in the limit by the price system p.

Since $\epsilon(k) \rightarrow 0$, (a), (c), (d) and (e) imply

(f) $y_{m,a} = 0$ for $a \in A$ and all $m \in M_a$

where $y_{a,m} = \lim y_{a,m}(k)$.

(g) $\sum_{a \in A} x_a \leqslant \sum_{f \in F} z_f$

(h) $\sum_{a \in A_m} w_a^{m,f} \leqslant 1$ for $f \in F$ and all $m \in M$.

By lemma 6.7, (e) and (g) imply

(i) $I_{a,m}(p) < 0$ for $a \in A$ and all $m \in M_a$.

By lemma 6.6, (e) and (i) imply

$(x_a,w_a) \in D_a((p,q),(0))$ for $a \in A$.

By definition 6.1(ii) of competitive equilibria, it only remains to verify

(j) $z_f \in Q_f(p,q)$ for $f \in F$.

Clearly, since $\epsilon(k) \rightarrow 0$, $z_f(k)-(\epsilon(k),\epsilon(k),\ldots) \in Z_f'$ and $r(p(k),z_f(k))) \geqslant 0$ for all k imply $z_f \in Z_f'$ and $r(p,z_f) \geqslant 0$.

Therefore, in order establish (j), we need only show that z_f generates the optimal revenue stream, i.e.

(k) $v_f(r(p,z_f)) \geqslant v_f(r_f(p))$.

For each k, since the feasibility set of firm f is perturbed by adding positive quantities $(\epsilon(k))$ of contracts and since $z_f(k)$ is firm f's optimal trade plan in the k'th equilibria, assumption VIII(v) implies

$v_f(r_f(p(k))) \leqslant v_f(r(p(k),z_f(k)))$.

But, since by lemma 6.5, $Q_f(\bullet)$ is upper hemi-continuous at (p,q), (e) implies $r_f(p) = \lim r_f(p(k))$.

The above two expressions, together with (e) and assumption VIII(v), imply $v_f(r(p,z_f)) = \lim v_f(r(p(k),z_f(k))) \geq \lim v_f(r_f(p(k)))$ $= v_f(r_f(p))$.

Q.E.D.

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