New degrees of freedom in integrable models with q-Hahn weights and their applications to symmetric functions and probability.

by

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Abstract

We present three groups of results about integrable lattice models constructed from orthogonality weights of q-Hahn polynomials. First, we establish that the q-Hahn orthogonality weights appear as matrix coefficients in certain isomorphisms between tensor products of representations of quantum affine $\mathfrak{sl}_2$ algebra. This allows us to find new integrable degrees of freedom in q-Hahn models by constructing an integrable vertex model on a square lattice with weights coming not from an R-matrix, as usually the case, but from our isomorphisms.

Second, we use the partition function of our new vertex model to construct a generalization of $t = 0$ Macdonald symmetric functions, which we call inhomogeneous spin q-Whittaker polynomials. Using integrability we are able to extend several classical properties of symmetric functions to our generalization, in particular, we prove analogues of the Cauchy and dual Cauchy identities. Moreover, we are able to characterize spin q-Whittaker polynomials by vanishing at certain points, which leads to a discovery of interpolation analogues of q-Whittaker and elementary symmetric polynomials.

Finally, we introduce a (colored) stochastic version of our vertex model and prove explicit integral expressions for q-deformed moments of the (colored) height functions of it. Following known techniques our stochastic model can be interpreted as a q-discretization of the Beta polymer model with three families of integrable parameters and we are able to extend the known results about Tracy-Widom large-scale fluctuations to our generalization of this polymer model.

Thesis Supervisor: Alexei Borodin
Title: Professor of Mathematics

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Introduction

This work is devoted to integrability of lattice models constructed from the orthogonality weights of q-Hahn polynomials, that is, from the weights of the following form:

$$(d/b)^m \frac{(d/b;q)_{n-m}(a/d;q)_m}{(a/b;q)_n} \frac{(q;q)_n}{(q;q)_{n-m}(q;q)_m}, \tag{$*$}$$

where $n \geq m \geq 0$ are integers, a, b, d, q are real parameters and $(x;q)_k = \prod_{i=1}^{k}(1-q^{i-1}x)$. Integrable models of such type were first introduced by Povolotsky in [75], where a chipping model with q-Hahn jumping probabilities was shown to be solvable by a coordinate Bethe ansatz. Various integrable stochastic models of significance can be obtained as degenerations of models constructed from q-Hahn weights, notable examples include q-TASEP, q-Boson system, directed Beta polymer and their respective degenerations. In this way q-Hahn models can be used to establish integrability of several other models.

In [20, 36] the q-Hahn TASEP from [75] was reproduced as a particular case of a different model: *fully fused stochastic six-vertex model*. The latter model is a generalization of the stochastic six-vertex model introduced in [46] and these vertex models are directly related to the representation theory of the affine quantum algebra $U_q(\widehat{\mathfrak{sl}}_2)$. Namely, the vertex models are constructed from matrix coefficients of R-matrices $V \otimes W \to W \otimes V$ associated to certain $U_q(\widehat{\mathfrak{sl}}_2)$-modules V, W, and the integrability of the models follows from the corresponding Yang-Baxter equations. In this way integrability of the q-Hahn models can be connected to the algebraic structure behind R-matrices.

In this thesis we introduce a more flexible approach to the integrability of q-Hahn vertex models, generalizing the vertex model construction coming from the six-vertex model. Namely, we show that the q-Hahn weights $(*)$ appear naturally as matrix coefficients of another family of $U_q(\widehat{\mathfrak{sl}}_2)$-module isomorphisms, which is different from the R-matrix. To describe these isomorphisms let $V(s^{-\omega_1})_z$ denote the evaluation module induced from the irreducible $U_q(\mathfrak{sl}_2)$-module with the highest weight s. The isomorphisms in question are

$$V((c/d)^{-\omega_1})_{cd} \otimes V((a/b)^{-\omega_1})_{ab} \cong V((c/b)^{-\omega_1})_{cb} \otimes V((a/d)^{-\omega_1})_{ad}, \tag{$**$}$$

where the two sides differ by a swap of parameters b and d.

One can show that the q-Hahn degeneration of the fully fused six-vertex model from [20, 36] corresponds to using matrix coefficients of $(**)$ in the case $a = c$, when our isomorphism coincides with the R-matrix. However, our findings allow to construct more general versions of the q-Hahn vertex models by using compositions of operators $(**)$ rather than the R-matrices. The resulting models are still integrable and they have additional degrees of freedom, which were not noticed in the preceding research. The current thesis is devoted to describing these new degrees of freedom

and investigating how these new parameters manifest themselves in known applications of q-Hahn models.

The work is divided in three parts. In Chapter 1 we explain the background on quantum affine algebras and identify the q-Hahn weights as the matrix coefficients of certain families isomorphisms of $U_q(\widehat{\mathfrak{sl}_{n+1}})$-modules, which include the isomorphisms $(**)$ above. As a byproduct we also find a new proof of the expression for the fully fused R-matrix of $U_q(\widehat{\mathfrak{sl}_{n+1}})$ from [25]. After that, we introduce the most general formalism of *q-Hahn vertex models*, which are constructed from our isomorphisms and which satisfy Yang-Baxter equations.

In Chapter 2 we apply our new results to study symmetric functions from algebraic combinatorics. Namely, we follow the existing framework where symmetric functions of interest are constructed as partition functions of solvable vertex models and the integrability of the latter is used to study the resulting functions. In particular, in [21], [65] it was understood that the partition functions of q-Hahn vertex models lead to generalizations of *q-Whittaker symmetric functions*, which are $t = 0$ specializations of the celebrated *Macdonald symmetric functions*. We are able to push these generalizations further and in Chapter 2 we introduce *inhomogeneous spin q-Whittaker functions* $\mathbb{F}_\lambda(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B})$, which depend on two infinite sequences of parameters $\mathcal{A}, \mathcal{B}$. We show that several combinatorial properties of the classical q-Whittaker functions extend to our generalization, most importantly, the Cauchy identity which serves the role of a formal orthogonality relation. Moreover, we are able to characterize spin q-Whittaker functions as unique functions vanishing at specific sets of points. The latter result requires our new degrees of freedom in an essential way and it provides connection to the *interpolation symmetric functions*, which are of independent interest. See Section 2.1 for additional background and the references.

Chapter 3 is devoted to the application of our constructions to integrable probability. We introduce the *colored stochastic q-Hahn vertex model*, which generalizes the q-Hahn vertex model from [20, 36] by adding additional degrees of freedom. Using integrability of our model we find contour integral expressions for q-deformed moments of key observables of this model, *colored height functions*. Following known techniques we interpret our generalized q-Hahn model as a q-discretization of the Beta polymer model, which results in a new family of integrable parameters for the latter. To demonstrate how integrability with respect to the new parameters works, we extend the known results about Tracy-Widom large-scale fluctuations of the Beta polymer model. See Section 3.1 for detailed background and descriptions of these two models.

The thesis consists of the material of two works which were written during my time at MIT: [55] (Chapter 3) and [56] (Chapters 1, 2). Sections 2.3.3 and 2.3.4 are based on [18].

General notation

Throughout the work $\mathfrak{S}_n$ denotes the group of permutations of n elements. For a permutation $\pi \in \mathfrak{S}_n$ its length is denoted by $l(\pi)$.

A partition λ is a finite integer sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$, with $l = l(\lambda)$ being called the length of λ. If $\lambda_1 + \cdots + \lambda_{l(\lambda)} = n$ we write $\lambda \vdash n$. The part multiplicities $m_k(\lambda)$ are defined by $m_k(\lambda) = \#\{i \mid \lambda_i = k\}$.

We define the q-Pochhammer symbol by

$$(x;q)_n := \begin{cases} \prod_{i=1}^{n}(1 - xq^{i-1}), & n \geq 0, \\ \prod_{i=1}^{-n}(1 - xq^{-i})^{-1}, & n < 0. \end{cases}$$

Let $(x;q)_\infty := \lim_{n\to\infty}(x;q)_n$, which is well-defined both when q is a number such that $|q| < 1$ and when q is a formal variable.

Chapter 1

Quantum affine algebras and vertex models with q-Hahn weights

In this chapter we show how q-Hahn weights appear as matrix coefficients of intertwining operators between tensor products $V \otimes W$, where V and W are Kirillov-Reshetikhin $U_q'(\widehat{\mathfrak{sl}}_{n+1})$-modules corresponding to the first fundamental weight. This perspective allows us to derive several new identities of Yang-Baxter type, which are key building blocks for main results of this thesis.

Throughout this chapter we use the following q-binomial notation. We treat $q^{\frac{1}{2}}$ either as a fixed transcendental complex number such that $|q^{\frac{1}{2}}| < 1$, or as a formal variable such that $q = (q^{\frac{1}{2}})^2$. Set

$$[n]_q := \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \qquad [n]_q! := \prod_{i=1}^{n} [i]_q, \qquad \begin{bmatrix} n+m \\ m \end{bmatrix}_q := \frac{[n+m]_q!}{[n]_q![m]_q!},$$

Note that we can express the q-binomial coefficients in the following ways

$$\begin{bmatrix} n+m \\ m \end{bmatrix}_q = q^{-\frac{nm}{2}} \frac{(q;q)_{n+m}}{(q;q)_n (q;q)_m} = q^{-\frac{nm}{2}} \frac{(q^{n+1};q)_m}{(q;q)_m} = q^{-\frac{nm}{2}} \frac{(q^{m+1};q)_n}{(q;q)_n},$$

allowing us to extend the definition to the case when either n or m are negative, but not both simultaneously.

Whenever we claim that a statement holds for generic complex parameters $a_1, \ldots, a_n$, we mean that there exists a countable collection of non-constant polynomials $F_i \in \mathbb{C}[x_1, \ldots, x_n]$ such that the statement holds for all $(a_1, \ldots, a_n) \in \mathbb{C}^n$ satisfying $F_i(a_1, \ldots, a_n) \neq 0$ for all i. In particular, the set of such $(a_1, \ldots, a_n)$ is dense in $\mathbb{C}^n$.

1.1 Quantum affine algebra $U_q'(\widehat{\mathfrak{sl}}_{n+1})$ and representations $V(s^{-\omega_1})_z$

In this section we describe the necessary background on the quantum affine algebra $U_q'(\widehat{\mathfrak{sl}}_{n+1})$ and define the representations $V(s^{-\omega_1})_z$. References for the material in this section are [32, Chapter 12], [33], and [66].

1.1.1 Quantum affine algebra $U_q'(\widehat{\mathfrak{sl}}_{n+1})$

Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{sl}_{n+1}$, which we identify with its dual $\mathfrak{h}^\vee$ via the Killing form $(\cdot,\cdot)$. Let α_i denote the simple roots of $\mathfrak{sl}_{n+1}$ with the corresponding Cartan matrix C. We enumerate the nodes $I = \{1,\dots,n\}$ of the Dynkin diagram in a way such that $C_{i,i} = 2$ and $C_{i,i-1} = C_{i-1,i} = -1$, while $C_{i,j} = 0$ for $|i-j| > 1$. We also assume that $(\alpha_i,\alpha_i) = 2$, so the simple coroots $\alpha_i^\vee$ coincide with the roots α_i. We use Q (Q^+ and Q^-) to denote the $\mathbb{Z}$-span ($\mathbb{Z}_{\geq 0}$-span and $\mathbb{Z}_{\leq 0}$-span correspondingly) of the simple roots α_i, while $P := \{\mu \in \mathfrak{h} \mid (\mu,\alpha_i) \in \mathbb{Z}\}$ denotes the lattice of integral weights. Let ω_i denote the fundamental weights corresponding to our choice of the simple roots.

Let $\theta = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ be the maximal positive root and set $\widehat{I} := \{0,1,\dots,n\}$. The extended Cartan matrix $\left(\widehat{C}_{i,j}\right)_{i,j\in\widehat{I}}$ is defined by

$$\widehat{C}_{0,0} = (-\theta,-\theta) = 2, \qquad \widehat{C}_{0,i} = \widehat{C}_{i,0} = -(\alpha_i,\theta),$$

$$\widehat{C}_{i,j} = C_{i,j} = (\alpha_i,\alpha_j) \quad \text{if} \quad i,j \in I.$$

The *quantum affine algebra* $U_q'(\widehat{\mathfrak{sl}}_{n+1})$[1] is the unital associative algebra over $\mathbb{C}$ with generators $\{k_i^{\pm 1}\}_{i\in\widehat{I}}$, $\{x_i^{\pm}\}_{i\in\widehat{I}}$ and relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \qquad k_i k_j = k_j k_i,$$

$$k_i x_j^{\pm} = q^{\pm\frac{1}{2}\widehat{C}_{i,j}} x_j^{\pm} k_i, \qquad [x_i^+, x_j^-] = \delta_{i,j} \frac{k_i - k_i^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \tag{1.1.1}$$

$$\sum_{r=0}^{1-\widehat{C}_{ij}} (-1)^r \begin{bmatrix} 1-\widehat{C}_{ij} \\ r \end{bmatrix}_q (x_i^{\pm})^r x_j^{\pm} (x_i^{\pm})^{1-\widehat{C}_{i,j}-r} = 0 \quad \text{for } i \neq j.$$

The last relation is called the (quantum) *Serre relation*. The algebra $U_q'(\widehat{\mathfrak{sl}}_{n+1})$ has a Hopf algebra structure where coproduct Δ, antipode S and counit ε are given by

$$\Delta k_i = k_i \otimes k_i, \qquad \Delta x_i^+ = x_i^+ \otimes k_i + 1 \otimes x_i^+, \qquad \Delta x_i^- = x_i^- \otimes 1 + k_i^{-1} \otimes x_i^-,$$

$$S(x_i^+) = -x_i^+ k_i^{-1}, \qquad S(x_i^-) = -k_i x_i^-, \qquad S(k_i^{\pm 1}) = k_i^{\mp 1},$$

$$\epsilon(k_i) = 1, \qquad \epsilon(x_i^+) = \epsilon(x_i^-) = 0.$$

We also need an alternative presentation of $U_q'(\widehat{\mathfrak{sl}}_{n+1})$, introduced by Drinfeld in [40]. In this presentation the algebra is generated by $\{k_i^{\pm 1}\}_{i\in I}$, $\{h_{i,m}\}_{i\in I, m\in\mathbb{Z}\setminus\{0\}}$, $\{x_{i,m}^{\pm}\}_{i\in I, m\in\mathbb{Z}}$, $c^{\pm 1}$, with the relations

$$[c^{\pm 1}, U_q'(\widehat{\mathfrak{sl}}_{n+1})] = 0, \qquad k_i k_i^{-1} = k_i^{-1} k_i = 1, \qquad k_i h_{j,m} = h_{j,m} k_i, \qquad k_i x_{j,m}^{\pm} = q^{\pm\frac{1}{2}C_{i,j}} x_{j,m}^{\pm} k_i,$$

$$[h_{i,m}, h_{j,l}] = \delta_{m,-l} \frac{1}{m} [mC_{i,j}]_q \frac{c^m - c^{-m}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \qquad [h_{i,m}, x_{j,l}^{\pm}] = \pm \frac{1}{m} [mC_{i,j}]_q \, c^{-(m\mp|m|)/2} \, x_{j,m+l}^{\pm},$$

[1] The prime in $U_q'(\widehat{\mathfrak{sl}}_{n+1})$ reflects that our definition does not include the additional generator d, which is sometimes used in the context of affine algebras.

$$x^{\pm}_{i,m+1}x^{\pm}_{j,l} - q^{\pm\frac{1}{2}C_{i,j}}x^{\pm}_{i,l}x^{\pm}_{j,m+1} = q^{\pm\frac{1}{2}C_{i,j}}x^{\pm}_{i,m}x^{\pm}_{j,l+1} - x^{\pm}_{i,l+1}x^{\pm}_{j,m}, \tag{1.1.2}$$

$$[x^{+}_{i,m}, x^{-}_{j,l}] = \delta_{i,j}\frac{c^m\phi^{+}_{i,m+l} - c^l\phi^{-}_{i,m+l}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},$$

$$\sum_{\pi\in\mathfrak{S}_{1-C_{i,j}}}\sum_{r=0}^{1-C_{ij}}(-1)^r\begin{bmatrix}1-C_{ij}\\ r\end{bmatrix}_q x^{\pm}_{i,m_{\pi(1)}}\dots x^{\pm}_{i,m_{\pi(r)}}x^{\pm}_{j,l}x^{\pm}_{i,m_{\pi(r+1)}}\dots x^{\pm}_{i,m_{\pi(1-C_{i,j})}} = 0.$$

Here $\phi^{\pm}_{i,m}$ are defined as coefficients in the formal power series

$$\phi^{\pm}_i(u) = \sum_{m\geq 0}\phi^{\pm}_{i,\pm m}u^{\pm m} = k_i^{\pm 1}\exp\left(\pm(q^{\frac{1}{2}} - q^{-\frac{1}{2}})\sum_{m>0}h_{i,\pm m}u^{\pm m}\right)$$

and we set $\phi^{\pm}_{i,\pm m} = 0$ if $m < 0$. The last relation in (1.1.2) is taken for every pair of distinct integers $(i,j) \in I^2$ and for every integer sequence $m_1, \dots, m_{1-C_{i,j}}$, while the first summation is over all permutations π of $1 - C_{i,j}$ elements.

The two presentations of $U'_q(\widehat{\mathfrak{sl}}_{n+1})$ above are related as follows: $\{k_i^{\pm 1}\}_{i\in I}$ denote the same elements in both presentations, $x^{\pm}_{i,0} = x^{\pm}_i$ for $i \in I$ and $c = k_0k_1k_2\dots k_n$. The generators $x^{\pm}_0$ are given in terms of the generators from Drinfeld's presentation by the following expressions, *cf.* [40], [50]:

$$x^{+}_0 = [x^{-}_n, [x^{-}_{n-1}, \dots [x^{-}_2, x^{-}_{1,1}]_{q^{-\frac{1}{2}}}\dots]_{q^{-\frac{1}{2}}}]_{q^{-\frac{1}{2}}}\ ck_1^{-1}k_2^{-1}\dots k_n^{-1}, \tag{1.1.3}$$

$$x^{-}_0 = c^{-1}k_1\dots k_n\ [\dots[[x^{+}_{1,-1}, x^{+}_2]_{q^{\frac{1}{2}}}, x^{+}_3]_{q^{\frac{1}{2}}}, \dots x^{+}_n]_{q^{\frac{1}{2}}}, \tag{1.1.4}$$

where we use the notation $[A,B]_u = AB - uBA$.

The subalgebra of $U'_q(\widehat{\mathfrak{sl}}_{n+1})$ generated by $\{k_i^{\pm 1}\}_{i\in I}$, $\{x^{\pm}_i\}_{i\in I}$ is a Hopf subalgebra isomorphic to the quantized enveloping algebra $U_q(\mathfrak{sl}_{n+1})$, thus any $U'_q(\widehat{\mathfrak{sl}}_{n+1})$-module has also a structure of a $U_q(\mathfrak{sl}_{n+1})$-module. We can define a Q-grading on $U'_q(\widehat{\mathfrak{sl}}_{n+1})$ and $U_q(\mathfrak{sl}_{n+1})$ by setting

$$\deg(x^{\pm}_{i,m}) = \pm\alpha_i, \qquad \deg(x^{\pm}_0) = \mp\theta, \qquad \deg(k_i^{\pm 1}) = \deg(h_{i,n}) = \deg(c) = 0.$$

Let U^+ (resp. U^- and U^0) be the subalgebra of $U_q(\mathfrak{sl}_{n+1})$ generated by $\{x^{+}_i\}_{i\in I}$ (resp. $\{x^{-}_i\}_{i\in I}$ for U^- and $\{k_i^{\pm 1}\}_{i\in I}$ for U^0). Similarly, let $\widehat{U}^+$ (resp. $\widehat{U}^-$,$\widehat{U}^0$) be the subalgebra of $U'_q(\widehat{\mathfrak{sl}}_{n+1})$ generated by $\{x^{+}_{i,m}\}_{i\in I, m\in\mathbb{Z}}$ (resp. $\{x^{-}_{i,m}\}_{i\in I,m\in\mathbb{Z}}$ and $\{k_i^{\pm 1}, h_{i,m}\}_{i\in I, m\in\mathbb{Z}\setminus\{0\}}$). The following factorizations are known [32]:

$$U_q(\mathfrak{sl}_{n+1}) = U^-.U^0.U^+, \qquad U'_q(\widehat{\mathfrak{sl}}_{n+1}) = \widehat{U}^-.\widehat{U}^0.\widehat{U}^+,$$

where $U_1.U_2 := \{u_1u_2 \mid u_1 \in U_1, u_2 \in U_2\}$.

There exists an involution of $U'_q(\widehat{\mathfrak{sl}}_{n+1})$, which is denoted by $\widehat{\omega}$ and is defined by

$$\widehat{\omega}(x^{\pm}_{i,m}) = -x^{\mp}_{i,-m}, \qquad \widehat{\omega}(h_{i,m}) = -h_{i,-m}, \qquad \widehat{\omega}(k_i^{\pm 1}) = k_i^{\mp 1}, \qquad \widehat{\omega}(c^{\pm 1}) = c^{\mp 1}.$$

One can check, using (1.1.3), (1.1.4) that $\widehat{\omega}(x^{\pm}_0) = -q^{\mp\frac{n+1}{2}}x^{\mp}_0$. Moreover, $\widehat{\omega}$ is a coalgebra anti-automorphism:

$$\Delta\circ\widehat{\omega} = (\widehat{\omega}\otimes\widehat{\omega})\circ\Delta'.$$

For a $U_q'(\widehat{\mathfrak{sl}}_{n+1})$-module V we use V^ω to denote the pullback of V through ω.

Remark 1.1.1. Note that we have not listed the expressions for the coproducts of the Drinfeld generators $x_{i,m}^{\pm}, h_{i,m}$. In fact, explicit formulae for $\Delta x_{i,m}^{\pm}$ and $\Delta h_{i,m}$ are not known, but partial expressions are available in the case of the *quantum loop algebra* $U_q(L\mathfrak{sl}_{n+1}) = U_q'(\widehat{\mathfrak{sl}}_{n+1})/(c-1)$, see [31, Proposition 1.2] and references therein.

1.1.2 Highest l-weight modules

To study representations of $U_q(\mathfrak{sl}_{n+1})$ and $U_q'(\widehat{\mathfrak{sl}}_{n+1})$ it is convenient to reintroduce the notions of weights, weight spaces and highest weight representations in the quantum affine setting. For a $U_q(\mathfrak{sl}_{n+1})$-module V we say that $v \in V$ is a weight vector if

$$k_i.v = \rho_i v, \qquad i \in I$$

for an n-tuple $\rho = (\rho_1, \rho_2, \dots, \rho_n) \in (\mathbb{C}^\times)^n$ called the *weight* of v. Treating $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$ as an abelian group, we get an abelian group structure on the set of such weights: $\rho\tau = (\rho_1\tau_1, \dots, \rho_n\tau_n)$. We define an abelian group map $q^{\frac{\bullet}{2}} : P \to (\mathbb{C}^\times)^n$ by setting $q^{\frac{\mu}{2}} := (q^{\frac{1}{2}(\mu,\alpha_1)}, \dots, q^{\frac{1}{2}(\mu,\alpha_n)})$. Since we have assumed that $q^{\frac{1}{2}}$ is transcendental this map is injective.

We call a $U_q(\mathfrak{sl}_{n+1})$-module V a *weight module* if

$$V = \bigoplus_{\rho \in (\mathbb{C}^\times)^n} V_\rho, \qquad V_\rho := \{v \in V \mid k_i.v = \rho_i v\}.$$

The spaces V_ρ are called the *weight spaces* of V, and ρ is called a weight of V if $V_\rho \neq 0$. We say that an $U_q(\mathfrak{sl}_{n+1})$-module V is in the category $\mathcal{O}$ if

- V is a weight module and all its weight spaces are finite-dimensional;
- All weights of V are contained in $\bigcup_{i=1}^m \{\rho_i q^{-\frac{\mu}{2}} \mid \mu \in Q^+\}$ for some weights $\rho_1, \dots, \rho_m \in (\mathbb{C}^\times)^n$.

Similarly to the classical situation, for each weight ρ there exists a unique irreducible $U_q(\mathfrak{sl}_{n+1})$-module $V(\rho) \in \mathcal{O}$ with the highest weight ρ.

To extend the formalism above to $U_q'(\widehat{\mathfrak{sl}}_{n+1})$ we assume for convenience that c always acts by 1; one can check that this assumption is not restrictive as long as we work with simple $U_q'(\widehat{\mathfrak{sl}}_{n+1})$-modules in an appropriate affinization of $\mathcal{O}$, *cf.* [66, Proposition 3.2]. For a $U_q'(\widehat{\mathfrak{sl}}_{n+1})$-module V we call $v \in V$ an *l-weight vector* if

$$\phi_{i,m}^{\pm}.v = \gamma_{i,m}^{\pm} v, \qquad i \in I, m \in \mathbb{Z},$$

where $\gamma = (\gamma_{i,m}^{\pm})_{i \in I, m \in \pm\mathbb{Z}_{\geq 0}}$ is a collection of complex numbers satisfying $\gamma_{i,0}^+ \gamma_{i,0}^- = 1$ for every $i \in I$. Such collections γ are called *l-weights*[2]. We say that an l-weight γ is *rational* if for some rational functions $(f_1(u), \dots, f_n(u))$ the expansions of $f_i(u)$ around 0 and ∞ are

$$f_i(u) = \sum_{m \geq 0} \gamma_{i,m}^+ u^m, \qquad f_i(u) = \sum_{m \geq 0} \gamma_{i,-m}^- u^{-m}.$$

[2] As a warning, l-weights are not as nicely behaved as the usual weights. For example, if v is an l-weight vector $x_i^{\pm}.v$ might fail to be an l-weight vector.

In this situation we the l-weight γ is denoted by $\boldsymbol{f} = (f_1(u), \dots, f_n(u))$. Note that rational functions $f_i(u)$ define a rational l-weight as long as the functions $f_i(u)$ are regular at points $0, \infty$ and satisfy $f_i(0)f_i(\infty) = 1$. For rational l-weights $\boldsymbol{f}, \boldsymbol{g}$ set $\boldsymbol{fg} := (f_1(u)g_1(u), \dots, f_n(u)g_n(u))$.

We call an l-weight vector v *singular* if $x^+_{i,m}.v = 0$ for every i, m. An $U'_q(\widehat{\mathfrak{sl}}_{n+1})$-module V is called *a highest l-weight module* when $V = U'_q(\widehat{\mathfrak{sl}}_{n+1}).v = \widehat{U}^-.v$ for a singular l-weight vector v, in this situation the l-weight of v is called the *highest l-weight* of V. The following statement summarizes the information about irreducible highest l-weight modules with rational highest l-weights:

Proposition 1.1.2 ([66]). *Let $\boldsymbol{f}, \boldsymbol{g}$ be rational l-weights.*

1. *There exists a unique up to isomorphism irreducible representation of $U'_q(\widehat{\mathfrak{sl}}_{n+1})/(c-1)$ with the highest l-weight $\boldsymbol{f}$, which we denote by $L(\boldsymbol{f})$.*
2. *$L(\boldsymbol{f}) \in \mathcal{O}$ as a $U_q(\mathfrak{sl}_{n+1})$-module.*
3. *If $L(\boldsymbol{f}) \otimes L(\boldsymbol{g})$ is an irreducible $U'_q(\widehat{\mathfrak{sl}}_{n+1})$-module then $L(\boldsymbol{f}) \otimes L(\boldsymbol{g}) \cong L(\boldsymbol{fg}) \cong L(\boldsymbol{g}) \otimes L(\boldsymbol{f})$.*

1.1.3 The representations $V(s^{-\omega_1})_z$

For a pair of complex numbers $s, z \in \mathbb{C}^\times$ let $V(s^{-\omega_1})_z$ denote the $U'_q(\widehat{\mathfrak{sl}}_{n+1})$-module $L(\boldsymbol{f})$ with $f_1(u) = s^{-1}\frac{1-szu}{1-s^{-1}zu}$ and $f_r(u) = 1$ for $r > 1$. For $a, b \in \mathbb{C}^\times$ we also set $V^a_b := V((a/b)^{-\omega_1})_{ab}$. These representations can be explicitly described and to do it we use the following notation. By a *composition* $\boldsymbol{I}$ we mean an n-tuple of nonnegative integers $(I_1, \dots, I_n) \in \mathbb{Z}^n_{\geq 0}$ and for any composition $\boldsymbol{I}$ we set $|\boldsymbol{I}| = I_1 + I_2 + \dots + I_n$. Define $\boldsymbol{e}^i = (e^i_1, e^i_2, \dots, e^i_n)$ as the composition with $e^i_j = \delta_{i,j}$, and set $\boldsymbol{0} := (0, \dots, 0)$.

Proposition 1.1.3. *The representation $V(s^{-\omega_1})_z$ is infinite-dimensional and has basis $\{v_{\boldsymbol{I}}\}_{\boldsymbol{I} \in \mathbb{Z}^n_{\geq 0}}$ if $s \neq \pm q^{-\frac{m}{2}}$ for any $m \in \mathbb{Z}_{\geq 0}$, and is finite-dimensional with basis $\{v_{\boldsymbol{I}}\}_{\boldsymbol{I} \in \mathbb{Z}^n_{\geq 0} : |\boldsymbol{I}| \leq m}$ if $s = \pm q^{-\frac{m}{2}}$ for $m \in \mathbb{Z}_{\geq 0}$. The action of $U'_q(\widehat{\mathfrak{sl}}_{n+1})$ is given explicitly by*

$$k_1.v_{\boldsymbol{I}} = s^{-1}q^{\frac{1}{2}(-|\boldsymbol{I}|-I_1)}\, v_{\boldsymbol{I}}, \qquad k_r.v_{\boldsymbol{I}} = q^{\frac{1}{2}(I_{r-1}-I_r)}\, v_{\boldsymbol{I}}, \qquad k_0.v_{\boldsymbol{I}} = sq^{\frac{1}{2}(|\boldsymbol{I}|+I_n)}\, v_{\boldsymbol{I}},$$

$$x^+_1.v_{\boldsymbol{I}} = s^{-1}q^{\frac{1}{2}(-|\boldsymbol{I}|+1)}[I_1]_q\, v_{\boldsymbol{I}-\boldsymbol{e}^1}, \qquad x^+_r.v_{\boldsymbol{I}} = [I_r]_q\, v_{\boldsymbol{I}-\boldsymbol{e}^r+\boldsymbol{e}^{r-1}}, \qquad x^+_0.v_{\boldsymbol{I}} = z\frac{1-s^2q^{|\boldsymbol{I}|}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\, v_{\boldsymbol{I}+\boldsymbol{e}^n},$$

$$x^-_1.v_{\boldsymbol{I}} = \frac{1-s^2q^{|\boldsymbol{I}|}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\, v_{\boldsymbol{I}+\boldsymbol{e}^1}, \qquad x^-_r.v_{\boldsymbol{I}} = [I_{r-1}]_q\, v_{\boldsymbol{I}+\boldsymbol{e}^r-\boldsymbol{e}^{r-1}}, \qquad x^-_0.v_{\boldsymbol{I}} = (zs)^{-1}q^{\frac{1}{2}(-|\boldsymbol{I}|+1)}[I_n]_q\, v_{\boldsymbol{I}-\boldsymbol{e}^n}, \tag{1.1.5}$$

where $r = 2, \dots, n$. Moreover, viewed as a $U_q(\mathfrak{sl}_{n+1})$-module, $V(s^{-\omega_1})_z$ is irreducible with the highest weight $(s^{-1}, 1, \dots, 1)$.

For the finite-dimensional case the claim can be deduced from [61, Proposition 5.1], and the infinite-dimensional situation can be reached using analytic continuation. Below we provide another way to verify Proposition 1.1.3.

Proof. We first check that (1.1.5) gives a well-defined $U'_q(\widehat{\mathfrak{sl}}_{n+1})$-module and then verify that the resulting module is indeed $V(s^{-\omega_1})_z$.

Let V' be a vector space with the basis $\{v_{\boldsymbol{I}}\}_{\boldsymbol{I}\in\mathbb{Z}^n_{\geq 0}}$ and consider V' as a $U'_q(\widehat{\mathfrak{sl}}_{n+1})$-module with the action defined by (1.1.5). To check that this action is well-defined we need to verify the defining relations (1.1.1), which can be done by a direct computation. For instance,

$$[x_1^+,x_1^-].v_{\boldsymbol{I}}=\left(\frac{1-s^2q^{|\boldsymbol{I}|}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}s^{-1}q^{-\frac{1}{2}|\boldsymbol{I}|}\frac{q^{\frac{I_1+1}{2}}-q^{-\frac{I_1+1}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}-s^{-1}q^{\frac{1}{2}(-|\boldsymbol{I}|+1)}\frac{q^{\frac{I_1}{2}}-q^{-\frac{I_1}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\frac{1-s^2q^{|\boldsymbol{I}|-1}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\right)v_{\boldsymbol{I}}$$
$$=\frac{s^{-1}q^{-\frac{|\boldsymbol{I}|}{2}-\frac{I_1}{2}}-sq^{\frac{|\boldsymbol{I}|}{2}+\frac{I_1}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}v_{\boldsymbol{I}}=\frac{k_1-k_1^{-1}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}.v_{\boldsymbol{I}},$$

$$((x_1^+)^2x_2^+-[2]_qx_1^+x_2^+x_1^++x_2^+(x_1^+)^2).v_{\boldsymbol{I}}=s^{-2}q^{-|\boldsymbol{I}|+\frac{3}{2}}[I_2]_q[I_1]_q\left([I_1+1]_q-[2]_q[I_1]_q+[I_1-1]_q\right)v_{\boldsymbol{I}}=0.$$

Other relations (including the Serre relations for $U'_q(\widehat{\mathfrak{sl}}_2)$, when $\widehat{C}_{0,1}=-2$) are either trivial or similar to the two above, so we omit their verification.

Note that when $s=\pm q^{-\frac{m}{2}}$ for $m\in\mathbb{Z}_{\geq 0}$ the module V' defined above has a submodule spanned by $\{v_{\boldsymbol{I}}\}_{\boldsymbol{I}\in\mathbb{Z}^n_{\geq 0}:|\boldsymbol{I}|\leq m}$. Indeed, the only generators $x_i^{\pm}$ which send $v_{\boldsymbol{I}}$ to $v_{\boldsymbol{J}}$ with $|\boldsymbol{J}|>|\boldsymbol{I}|$ are x_0^+ and x_1^-, and we have $x_0^+v_{\boldsymbol{I}}=0, x_1^-v_{\boldsymbol{I}}=0$ when $s=\pm q^{-\frac{m}{2}}$ and $|\boldsymbol{I}|=m$. In this situation, let $V''\subset V'$ denote this submodule spanned by $\{v_{\boldsymbol{I}}\}_{\boldsymbol{I}\in\mathbb{Z}^n_{\geq 0}:|\boldsymbol{I}|\leq m}$, otherwise (when $s\neq\pm q^{-\frac{m}{2}}$ for any $m\in\mathbb{Z}_{\geq 0}$) set $V''=V'$. Looking at the action of $x_i^{\pm}$ for $i=1,\dots,n$, one can see that $V''=U_q(\mathfrak{sl}_{n+1}).v_{\boldsymbol{0}}$ and $v_{\boldsymbol{0}}$ is the only $U_q(\mathfrak{sl}_{n+1})$-singular vector of V'', hence V'' is an irreducible $U_q(\mathfrak{sl}_{n+1})$-module with the highest weight $(s^{-1},1,\dots,1)$.

To finish the proof we only need to show that $V''\cong V(s^{-\omega_1})_z$. To do it we refer to [66, Proposition 5.5], which claims that a $U'_q(\widehat{\mathfrak{sl}}_{n+1})$-module which is irreducible as a $U_q(\mathfrak{sl}_{n+1})$-module and has the highest $U_q(\mathfrak{sl}_{n+1})$-weight $(s^{-1},1,\dots,1)$ is isomorphic to $L(\boldsymbol{g})$ with $g_1(u)=s^{-1}\frac{1-sz'u}{1-s^{-1}z'u}$, $g_2(u)=g_3(u)=\dots=1$ for some $z'\in\mathbb{C}\backslash\{0\}$. So $V''\cong L(\boldsymbol{g})$, and we just need to show that $z=z'$. To do it we can compute the action of $x_{1,1}^-$ on $v_{\boldsymbol{0}}$: Note that for any $A\in U'_q(\widehat{\mathfrak{sl}}_{n+1})$ such that $[A,x_i^+]=0$, $k_iAk_i^{-1}=q^{\frac{1}{2}}A$ we have $[x_i^+,[x_i^-,A]_{q^{-\frac{1}{2}}}k_i^{-1}k_{i-1}^{-1}]_{q^{\frac{1}{2}}}=Ak_{i-1}^{-1}$. Using this relation as an inductive step, we see from (1.1.3) that for any $i=2,\dots,n$

$$[x_i^+,[x_{i+1}^+,\dots[x_n^+,x_0^+]_{q^{\frac{1}{2}}}\dots]_{q^{\frac{1}{2}}}]_{q^{\frac{1}{2}}}=[x_{i-1}^-,\dots[x_2^-,x_{1,1}^-]_{q^{-\frac{1}{2}}}\dots]_{q^{-\frac{1}{2}}}ck_1^{-1}\dots k_{i-1}^{-1}.$$

In particular,

$$x_{1,1}^-.v_{\boldsymbol{0}}=[x_2^+,[x_3^+,\dots[x_n^+,x_0^+]_{q^{\frac{1}{2}}}\dots]_{q^{\frac{1}{2}}}]_{q^{\frac{1}{2}}}k_1.v_{\boldsymbol{0}}=s^{-1}x_2^+\dots x_n^+x_0^+.v_{\boldsymbol{0}}=z\frac{s^{-1}-s}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}v_{\boldsymbol{e}^1},$$

so $\phi_{1,1}^+.v_{\boldsymbol{0}}=(q^{\frac{1}{2}}-q^{-\frac{1}{2}})[x_1^+,x_{1,1}^-].v_{\boldsymbol{0}}=z(s^{-2}-1).v_{\boldsymbol{0}}$. On the other hand, since $g_1(u)=s^{-1}+z'(s^{-2}-1)u+\overline{o}(u)$, we have $\phi_{1,1}^+.v_{\boldsymbol{0}}=z'(s^{-2}-1).v_{\boldsymbol{0}}$. Hence $z=z'$ and $V''\cong V(s^{-\omega_1})_z$. □

Remark 1.1.4. The representation $V(s^{-\omega_1})_z$ is in fact an evaluation module induced from an analytically-continued symmetric tensor power of the standard representation of $U_q(\mathfrak{sl}_{n+1})$. Namely, to obtain $V(s^{-1}\omega_1)_z$ one can consider the extension $U_q(\mathfrak{sl}_{n+1})\subset U_q(\mathfrak{gl}_{n+1})$, define the irreducible $U_q(\mathfrak{gl}_{n+1})$-module $V(s^{-\omega_1})_z$ with the highest weight $(s^{-1},1\dots,1)$ and then pull it along the evaluation map $\mathrm{ev_z}:U'_q(\widehat{\mathfrak{sl}}_{n+1})\to U_q(\mathfrak{gl}_{n+1})$ introduced in [49]. Equivalently, the module $V(s^{-\omega_1})_z$ is a Kirillov-Reshetikhin module corresponding to the fundamental weight ω_1.

The following fact will be crucial in the following section.

Proposition 1.1.5. *For a fixed $L \in \mathbb{Z}_{\geq 1}$ and generic complex parameters $z_1, \dots, z_L, s_1, \dots, s_L$ the $U_q'(\widehat{\mathfrak{sl}}_{n+1})$-module $V(s_1^{-\omega_1})_{z_1} \otimes V(s_2^{-\omega_1})_{z_2} \otimes \cdots \otimes V(s_L^{-\omega_1})_{z_L}$ is irreducible.*

Proof. We refer to a much stronger result [31, Theorem 4] for the case when all $V(s_i^{-\omega_1})_{z_i}$ are finite-dimensional, that is, when $s_i \in q^{-\frac{1}{2}\mathbb{Z}_{\geq 0}}$ for each i.

Theorem 1.1.6 ([31, Theorem 4])**.** *Let $m_1, \dots, m_L \in \mathbb{Z}_{\geq 0}, z_1, \dots, z_L \in \mathbb{C}\backslash\{0\}$. If for any pair i, j such that $i \neq j$ we have $z_i/z_j \notin q^{\frac{1}{2}\mathbb{Z}}$, then $V(q^{\frac{m_1}{2}\omega_1})_{z_1} \otimes V(q^{\frac{m_2}{2}\omega_1})_{z_2} \otimes \cdots \otimes V(q^{\frac{m_L}{2}\omega_1})_{z_L}$ is a highest l-weight module.*

Below we explain how [31, Theorem 4] implies Proposition 1.1.5 but before it we want to clarify how the notation used in [31] match with our notation:

- Our $q^{\frac{1}{2}}$ corresponds to q in [31], to avoid confusion we keep using our $q^{\frac{1}{2}}$ even when describing notation from [31].
- In [31] the irreducible finite-dimensional highest l-weight $U_q'(\widehat{\mathfrak{sl}}_{n+1})$-modules are parametrized by n-tuples $\boldsymbol{\pi} = (\pi_1(u), \dots, \pi_n(u))$ of polynomials $\pi_i(u)$, instead of rational l-weights $\boldsymbol{f}$ like in our work. More precisely, the functions $\pi_i(u)$ corresponding to the representation $V(\boldsymbol{f})$ are given by
$$\pi_i(u) = \begin{cases} \prod_{j\geq 1} \frac{f_i(q^j u)}{f_i(0)}, & |q| < 1, \\ \prod_{j\geq 0} \frac{f_i(0)}{f_i(q^{-j} u)}, & |q| > 1, \end{cases}$$
and it turns out that for the finite-dimensional representation $V(q^{\frac{m}{2}\omega_1})_z$ these functions π_i are the following polynomials (both for $|q| > 1$ and $|q| < 1$):
$$\pi_1(u) = \prod_{i=1}^{m} (1 - q^{\frac{m}{2}-i+1} z u), \qquad \pi_r(u) = 1, \quad r \geq 2.$$
The n-tuple of polynomials $(\pi_1(u), \dots, \pi_n(u))$ above is denoted by $\boldsymbol{\pi}^1_{m, zq^{1/2}}$ in [31], and our representations $V(q^{\frac{m}{2}\omega_1})_z$ correspond to $V(\boldsymbol{\pi}^1_{m, zq^{1/2}})$ in [31].
- In [31] the results are proved over $\mathbb{C}(q^{\frac{1}{2}})$, while here we fix $q^{\frac{1}{2}}$ as a transcendental complex number. Since we work generically and the action of $U_q'(\widehat{\mathfrak{sl}}_{n+1})$ can be expressed in terms of matrices over $\mathbb{Q}(q^{\frac{1}{2}}, z_1, \dots, z_m)$ this difference is irrelevant in view of Lemma 1.1.8 below, we will return to this later when discussing the transition from $s_i = q^{-\frac{m_i}{2}}$ to arbitrary s_i.

To prove Proposition 1.1.5 we first show how Theorem 1.1.6 implies that $V := V(q^{\frac{m_1}{2}\omega_1})_{z_1} \otimes \cdots \otimes V(q^{\frac{m_L}{2}\omega_L})_{z_L}$ is irreducible for arbitrary fixed $m_1, \dots, m_L \in \mathbb{Z}_{\geq 1}$ and generic $z_1, \dots, z_L$. Note that the vector $v := v_{\mathbf{0}} \otimes \dots v_{\mathbf{0}}$ is the unique up to a scalar vector with the maximal $U_q(\mathfrak{sl}_{n+1})$-weight $\rho = (q^{\frac{1}{2}\sum_i m_i}, 1 \dots, 1)$, hence we have $x^+_{i,m}.v = 0$ and v is the highest l-weight vector. So Theorem 1.1.6 states that $U_q'(\widehat{\mathfrak{sl}}_{n+1}).v = V$.

Consider $(V^*)^{\widehat{\omega}}$, that is, the dual representation V^* pulled back along the involution $\widehat{\omega} : U_q'(\widehat{\mathfrak{sl}}_{n+1}) \to U_q'(\widehat{\mathfrak{sl}}_{n+1})$. We use the following fact from [33]:

Lemma 1.1.7 ([33, Proposition 5.1]). *There exists a fixed constant c such that for any m, z we have $((V(q^{\frac{m}{2}\omega_1})_z)^*)^{\widehat{\omega}} \cong V(q^{\frac{m}{2}\omega_1})_{cz^{-1}}$.*

By the lemma above

$$(V^*)^{\widehat{\omega}} = V(q^{\frac{m_1}{2}\omega_1})_{cz_1^{-1}} \otimes \cdots \otimes V(q^{\frac{m_L}{2}\omega_1})_{cz_L^{-1}}$$

and so $(V^*)^{\widehat{\omega}}$ is also a highest l-weight representation for generic $z_1, \ldots, z_L$. By comparing $U_q(\mathfrak{sl}_{n+1})$-weights we moreover can deduce that the highest l-vector v^* of $(V^*)^{\widehat{\omega}}$ is the vector dual to v in the sense that v^* vanishes on all weight spaces of V other than $\mathbb{C}v$.

Let $W \subset V$ be a $U_q'(\widehat{\mathfrak{sl}}_{n+1})$-submodule. W can be decomposed into a sum of $U_q(\mathfrak{sl}_{n+1})$-weight spaces and either $v \in W$ or $v^* \in W^\perp$, where $W^\perp$ is the annihilator of W in V^*. In the first case we have $U_q'(\widehat{\mathfrak{sl}}_{n+1}).v = V \subset W$, while in the second case we get

$$U_q'(\widehat{\mathfrak{sl}}_{n+1}).v^* = \widehat{\omega}\left(U_q'(\widehat{\mathfrak{sl}}_{n+1})\right).v^* = V^* \subset W^\perp,$$

implying $W = 0$. Since W was arbitrary, this implies that $V = V(q^{\frac{m_1}{2}\omega_1})_{z_1} \otimes \cdots \otimes V(q^{\frac{m_L}{2}\omega_1})_{z_L}$ is irreducible for generic $z_1, \ldots, z_L$.

To establish Proposition 1.1.5 we now need to go from $s_i = q^{-\frac{m_i}{2}}$ to generic s_i, which can be done using the following linear algebra fact:

Lemma 1.1.8. *Fix $d_1, d_2 \in \mathbb{Z}_{\geq 1}$ and let $X_{i,j}^{(r)}$ be formal variables enumerated by $i, j, r \in \mathbb{Z}_{\geq 1}$ such that $i \leq d_1, j \leq d_2$. There exist countable families of polynomials $\{F_p\}_{p\in\mathbb{Z}_{\geq 0}}$ and $\{G_t\}_{t\in\mathbb{Z}_{\geq 0}}$ over $\mathbb{Z}$ in variables $X_{i,j}^{(r)}$ with the following property:*

Assume we are given finite-dimensional vector spaces V, W over a field $\mathcal{F}$ with dimensions $\dim(W) = d_1, \dim(V) = d_2$, and a countable family of linear operators $A^{(r)} : W \to V$. Fix bases of V, W and let $A_{i,j}^{(r)}$ be the matrix coefficients of $A^{(r)}$ with respect to those bases. Then $\bigcup_r \operatorname{Im} A^{(r)} = V$ if and only if $F_p(A_{i,j}^{(r)}) \neq 0$ for some p, and $\bigcap_r \operatorname{Ker} A^{(r)} = 0$ if and only if $G_t(A_{i,j}^{(r)}) \neq 0$ for some t. Here $F_p(A_{i,j}^{(r)}), G_t(A_{i,j}^{(r)})$ denote the results of substitution $X_{i,j}^{(r)} = A_{i,j}^{(r)}$ into F_p and G_t. □

We apply Lemma 1.1.8 in the following way. Let, as before, $V = V(s_1^{-\omega_1})_{z_1} \otimes \cdots \otimes V(s_L^{-\omega_1})_{z_L}$, $v = v_{\mathbf{0}} \otimes \cdots \otimes v_{\mathbf{0}}$ and $\rho = (\prod_i s_i^{-1}, 1 \ldots, 1)$. Note that when V is viewed as a $U_q(\mathfrak{sl}_{n+1})$-module, all weights of V are of the form $\rho q^{-\frac{1}{2}\mu}$ for $\mu \in Q^+$ and the weight spaces can be explicitly described: if $\mu = K_1\alpha_1 + \cdots + K_n\alpha_n$ for a composition $\boldsymbol{K} \in \mathbb{Z}_{\geq 0}^n$ then $V_{\rho q^{-\frac{1}{2}\mu}}$ is spanned by $v_{\boldsymbol{I}_1} \otimes \cdots \otimes v_{\boldsymbol{I}_L}$ such that $\boldsymbol{I}_1 + \cdots + \boldsymbol{I}_L = \boldsymbol{K}$ and $v_{\boldsymbol{I}_i} \in V(s_i^{-\omega_1})_{z_i}$. The last condition is only relevant when $s_i = \pm q^{-\frac{m}{2}}, m \in \mathbb{Z}$, when we require $|\boldsymbol{I}_i| \leq m$. Hence, as long as $s_i \notin \{\pm 1, \pm q^{-\frac{1}{2}}, \ldots, \pm q^{-\frac{|\boldsymbol{K}|-1}{2}}\}$ for each i, we can identify the vector space $V_{\rho q^{-\frac{1}{2}\mu}}$ for arbitrary $s_1, \ldots, s_L, z_1, \ldots, z_L$ with the vector space over $\mathbb{C}$ generated by basis vectors $\{|\boldsymbol{I}_1, \ldots, \boldsymbol{I}_L\rangle\}_{\boldsymbol{I}_1 + \cdots + \boldsymbol{I}_L = \boldsymbol{K}}$.

Note that V is irreducible if and only if for any $\mu \in Q^+$ the following holds:

- $V_{\rho q^{-\frac{1}{2}\mu}} \subset U_q'(\widehat{\mathfrak{sl}}_{n+1}).v$;
- $v \in U_q'(\widehat{\mathfrak{sl}}_{n+1}).v'$ for any $v' \in V_{\rho q^{-\frac{1}{2}\mu}} \backslash \{0\}$.

It is enough to show that for a fixed $\mu \in Q^+$ both these conditions hold for generic $s_1, \dots, s_L, z_1, \dots, z_L$. We start with $V_{\rho q^{-\frac{1}{2}\mu}} \subset U_q'(\widehat{\mathfrak{sl}}_{n+1}).v$. Recall that we have a Q-grading on $U_q'(\widehat{\mathfrak{sl}}_{n+1})$, let $\mathcal{G}_\mu$ be the set of words in $k_i^{\pm 1}, x_i^{\pm}, i \in \widehat{I}$ whose total Q-degree is μ. Using $U_q'(\widehat{\mathfrak{sl}}_{n+1})$-action from Proposition 1.1.3, consider each word $w \in \mathcal{G}_{-\mu}$ as an operator $A^{(w)} : \mathbb{C}v \to V_{\rho q^{-\frac{1}{2}\mu}}$. Assuming that $s_i \notin \{\pm 1, \pm q^{-\frac{1}{2}}, \dots, \pm q^{-\frac{K-1}{2}}\}$ for some fixed K depending only on μ, we can identify the spaces $V_{\rho q^{-\frac{1}{2}\mu}}$ with each other and they have a basis of the form $\{|\boldsymbol{I}_1, \dots, \boldsymbol{I}_L\rangle\}_{\boldsymbol{I}_1 + \dots + \boldsymbol{I}_L = \boldsymbol{K}}$. Moreover, all matrix coefficients of $A^{(w)}$ with respect to $v_{\boldsymbol{I}_1} \otimes \dots \otimes v_{\boldsymbol{I}_l}$ are polynomials in $s_1, \dots, s_L, z_1, \dots, z_L$, with coefficients in $\mathbb{Q}(q^{\frac{1}{2}})$. Since the condition $V_{\rho q^{-\frac{1}{2}\mu}} \subset U_q'(\widehat{\mathfrak{sl}}_{n+1}).v$ is equivalent to $\bigcup_w \operatorname{Im} A^{(w)} = V_{\rho q^{-\frac{1}{2}\mu}}$, Lemma 1.1.8 gives a family of polynomials $F_p(s_1, \dots, s_L, z_1, \dots, z_L)$ with coefficients from $\mathbb{Q}(q^{\frac{1}{2}})$, such that $V_{\rho q^{-\frac{1}{2}\mu}} \subset U_q'(\widehat{\mathfrak{sl}}_{n+1}).v$ if and only if $F_p(s_1, \dots, s_L, z_1, \dots, z_L) \neq 0$ for some p, still assuming $s_i \notin \{\pm 1, \pm q^{-\frac{1}{2}}, \dots, \pm q^{-\frac{K-1}{2}}\}$. Since we work generically over $s_1, \dots, s_L, z_1, \dots, z_L$, it is enough to show that $F_p \neq 0$ as a polynomial in s_i, z_i for at least one p. But from the finite-dimensional case we know that $V_{\rho q^{-\frac{1}{2}\mu}} \subset U_q'(\widehat{\mathfrak{sl}}_{n+1}).v$ for generic $z_1, \dots, z_L$ and $s_i \in q^{-\frac{1}{2}\mathbb{Z}_{\geq 0}}$. Hence, for any $m_i \geq K$ and generic z_i, we have $F_p(q^{-\frac{m_1}{2}}, \dots, q^{-\frac{m_L}{2}}, z_1, \dots, z_L) \neq 0$ for some p. Hence $V_{\rho q^{-\frac{1}{2}\mu}} \subset U_q'(\widehat{\mathfrak{sl}}_{n+1}).v$ for generic s_i, z_i. Note in particular, that we have used Lemma 1.1.8 to rewrite the condition $V_{\rho q^{-\frac{1}{2}\mu}} \subset U_q'(\widehat{\mathfrak{sl}}_{n+1}).v$ in terms of vanishing of certain polynomials with coefficients being rational functions in $q^{\frac{1}{2}}$, this readily implies that we can equivalently consider $q^{\frac{1}{2}}$ as a transcendental complex number or as a formal variable, making this difference with [31] irrelevant.

To show that generically $v \in U_q'(\widehat{\mathfrak{sl}}_{n+1}).v'$ for any $v' \in V_{\rho q^{-\frac{1}{2}\mu}} \backslash \{0\}$ we use the other half of Lemma 1.1.8. Namely, considering now the words $w \in \mathcal{G}_\mu$ as operators $A^{(w)} : V_{\rho q^{-\frac{1}{2}\mu}} \to \mathbb{C}v$, we have $v \in U_q'(\widehat{\mathfrak{sl}}_{n+1}).v'$ for any $v' \in V_{\rho q^{-\frac{1}{2}\mu}} \backslash \{0\}$ if and only if $\bigcap_w \operatorname{Ker} A^{(w)} = 0$. By Lemma 1.1.8, there exist polynomials $G_t(s_1, \dots, s_L, z_1, \dots, z_L)$ such that the last condition is equivalent to existence of t such that $G_t(s_1, \dots, s_L, z_1, \dots, z_L) \neq 0$. Irreducibility of V in finite-dimensional situation implies that at least one polynomial G_t is nonzero, hence we have $v \in U_q'(\widehat{\mathfrak{sl}}_{n+1}).v'$ for any $v' \in V_{\rho q^{-\frac{1}{2}\mu}} \backslash \{0\}$ generically. □

1.2 Explicit expressions for isomorphisms between tensor products

In this section we present explicit expressions for isomorphisms between tensor products $V(s_1^{-\omega_1})_{z_1} \otimes V(s_2^{-\omega_1})_{z_2}$. The importance of these expressions is two-fold: on one hand we get explicit expressions for the R-matrix $V(s_1^{-\omega_1})_{z_1} \otimes V(s_2^{-\omega_1})_{z_2} \to V(s_2^{-\omega_1})_{z_2} \otimes V(s_1^{-\omega_1})_{z_1}$, reproducing a result of [25]. On the other hand, using the generic irreducibility of arbitrary tensor products $V(s_1^{-1}\omega_1)_{z_1} \otimes \dots \otimes V(s_L^{-\omega_1})_{z_L}$, we can use the explicit expressions from this section to obtain deformations of the Yang-Baxter equations.

To simplify expressions in this section we use the following notation. Recall that for a pair of parameters $a, b \in \mathbb{C}\backslash\{0\}$ we set $V_b^a := V((a/b)^{-\omega_1})_{ab}$. For compositions $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{Z}_{\geq 0}^n$ and complex parameters a, b define

$$\Phi(\boldsymbol{X}, \boldsymbol{Y}; a, b) := b^{|\boldsymbol{X}|} q^{\sum_{i<j} X_i Y_j} \frac{(a;q)_{|\boldsymbol{X}|} (b;q)_{|\boldsymbol{Y}|}}{(ab;q)_{|\boldsymbol{X}+\boldsymbol{Y}|}} \prod_{i=1}^n \frac{(q;q)_{X_i+Y_i}}{(q;q)_{X_i} (q;q)_{Y_i}}.$$

For parameters $a_1, a_2, a_3, b_1, b_2, b_3$ and compositions $\boldsymbol{I}, \boldsymbol{J}$ we set

$$C_{a_1,b_1;a_2,b_2}(\boldsymbol{I},\boldsymbol{J}) := (b_1/a_1)^{|\boldsymbol{J}|} q^{-\frac{1}{2}\sum_{i<j} J_i I_j}.$$

In view of Proposition 1.1.3 we have a basis of $V(s_1^{-\omega_1})_{z_1} \otimes \cdots \otimes V(s_L^{-\omega_1})_{z_L}$ given by $|v_{\boldsymbol{I}_1} \otimes v_{\boldsymbol{I}_2} \otimes \cdots \otimes v_{\boldsymbol{I}_L}\rangle$, let $\langle v_{\boldsymbol{I}_1} \otimes v_{\boldsymbol{I}_2} \otimes \cdots \otimes v_{\boldsymbol{I}_L}|$ denote the dual vectors in $(V(s_1^{-\omega_1})_{z_1} \otimes \cdots \otimes V(s_L^{-\omega_1})_{z_L})^*$.

1.2.1 Explicit expressions

Let a_1, a_2, b_1, b_2 be generic complex parameters and consider representations $V_{b_1}^{a_1} \otimes V_{b_2}^{a_2}$, $V_{b_1}^{a_2} \otimes V_{b_2}^{a_1}$, $V_{b_2}^{a_1} \otimes V_{b_1}^{a_2}$ and $V_{b_2}^{a_2} \otimes V_{b_1}^{a_1}$. Note that all four representations are irreducible by Proposition 1.1.5, and by Proposition 1.1.2 all of them are isomorphic to $L(\boldsymbol{f})$ where $f_1(u) = \frac{(a_1^{-1}-a_1u)(a_2^{-1}-a_2u)}{(b_1^{-1}-b_1u)(b_2^{-1}-b_2u)}$ and $f_r(u) = 1$ for $r \geq 2$. Our results below give explicit expressions for the isomorphisms between the four representations.

Theorem 1.2.1. *For generic parameters a_1, b_1, a_2, b_2 the representations $V_{b_1}^{a_1} \otimes V_{b_2}^{a_2}$ and $V_{b_1}^{a_2} \otimes V_{b_2}^{a_1}$ are irreducible and isomorphic, with an isomorphism W^a given explicitly by*

$$W^a : V_{b_1}^{a_1} \otimes V_{b_2}^{a_2} \to V_{b_1}^{a_2} \otimes V_{b_2}^{a_1},$$

$$\langle v_{\boldsymbol{K}} \otimes v_{\boldsymbol{L}} | W^a | v_{\boldsymbol{I}} \otimes v_{\boldsymbol{J}} \rangle = \delta_{\boldsymbol{I}+\boldsymbol{J}=\boldsymbol{K}+\boldsymbol{L}} \delta_{\boldsymbol{I} \geq \boldsymbol{K}} \frac{C_{a_2,b_1;a_1,b_2}(\boldsymbol{K},\boldsymbol{L})}{C_{a_1,b_1;a_2,b_2}(\boldsymbol{I},\boldsymbol{J})} \Phi(\boldsymbol{I}-\boldsymbol{K}, \boldsymbol{K}; a_1^2/a_2^2, a_2^2/b_1^2).$$

Similarly, for generic parameters a_1, b_1, a_2, b_2 the representations $V_{b_1}^{a_1} \otimes V_{b_2}^{a_2}$ and $V_{b_2}^{a_1} \otimes V_{b_1}^{a_2}$ are irreducible and isomorphic, with an isomorphism W^b given explicitly by

$$W^b : V_{b_1}^{a_1} \otimes V_{b_2}^{a_2} \to V_{b_2}^{a_1} \otimes V_{b_1}^{a_2}$$

$$\langle v_{\boldsymbol{K}} \otimes v_{\boldsymbol{L}} | W^b | v_{\boldsymbol{I}} \otimes v_{\boldsymbol{J}} \rangle = \delta_{\boldsymbol{I}+\boldsymbol{J}=\boldsymbol{K}+\boldsymbol{L}} \delta_{\boldsymbol{J} \geq \boldsymbol{L}} \frac{C_{a_1,b_2;a_2,b_1}(\boldsymbol{K},\boldsymbol{L})}{C_{a_1,b_1;a_2,b_2}(\boldsymbol{I},\boldsymbol{J})} \Phi(\boldsymbol{L}, \boldsymbol{J}-\boldsymbol{L}; a_2^2/b_1^2, b_1^2/b_2^2).$$

Proof. By the discussion above it is enough to check that the morphisms W^a and W^b presented above commute with the action of $U_q'(\widehat{\mathfrak{sl}}_{n+1})$. We do it by verifying the statement on the generators $\{k_i^{\pm 1}\}$, $\{x_i^{\pm}\}$ of $U_q'(\widehat{\mathfrak{sl}}_{n+1})$. For k_i the check is trivial since W^a and W^b clearly preserve the weight spaces. The checks for $x_i^{\pm}$ are similar to each other and can be readily done by a straightforward but tedious computation. Here we only provide the verification that x_1^+ commutes with the action of W^a.

We need to check that $\langle v_{\boldsymbol{K}} \otimes v_{\boldsymbol{L}} | W^a x_1^+ | v_{\boldsymbol{I}} \otimes v_{\boldsymbol{J}} \rangle = \langle v_{\boldsymbol{K}} \otimes v_{\boldsymbol{L}} | x_1^+ W^a | v_{\boldsymbol{I}} \otimes v_{\boldsymbol{J}} \rangle$ for any compositions $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}, \boldsymbol{L}$. Since W^a preserves the weight and x_1^+ increases it by $q^{\frac{1}{2}\alpha_1}$, both sides are zero unless $\boldsymbol{I}+\boldsymbol{J}-\boldsymbol{e}^1 = \boldsymbol{K}+\boldsymbol{L}$. Assuming $\boldsymbol{I}+\boldsymbol{J}-\boldsymbol{e}^1 = \boldsymbol{K}+\boldsymbol{L}$ from now on, note that

$$x_1^+ | v_{\boldsymbol{I}} \otimes v_{\boldsymbol{J}} \rangle = \frac{b_1 b_2}{a_1 a_2} q^{\frac{1}{2}(-|\boldsymbol{I}|-|\boldsymbol{J}|-J_1+1)} [I_1]_q | v_{\boldsymbol{I}-\boldsymbol{e}^1} \otimes v_{\boldsymbol{J}} \rangle + \frac{b_2}{a_2} q^{\frac{1}{2}(-|\boldsymbol{J}|+1)} [J_1]_q | v_{\boldsymbol{I}} \otimes v_{\boldsymbol{J}-\boldsymbol{e}^1} \rangle,$$

$$\langle v_{\boldsymbol{K}} \otimes v_{\boldsymbol{L}} | x_1^+ = \frac{b_1 b_2}{a_1 a_2} q^{\frac{1}{2}(-|\boldsymbol{K}|-|\boldsymbol{L}|-L_1)} [K_1+1]_q \langle v_{\boldsymbol{K}+\boldsymbol{e}^1} \otimes v_{\boldsymbol{L}} | + \frac{b_2}{a_1} q^{-\frac{1}{2}|\boldsymbol{L}|} [L_1+1]_q \langle v_{\boldsymbol{K}} \otimes v_{\boldsymbol{L}+\boldsymbol{e}^1} |,$$

where the first line describes the left action on $V_{b_1}^{a_1} \otimes V_{b_2}^{a_2}$ with the assumption that $|v_{\boldsymbol{I}'} \otimes v_{\boldsymbol{J}'}\rangle = 0$ when the configurations $\boldsymbol{I}', \boldsymbol{J}'$ are non-positive, while the second line describes the right action on

$(V_{b_1}^{a_2} \otimes V_{b_2}^{a_1})^*$. So we need to verify that

$$\frac{b_1 b_2}{a_1 a_2} q^{\frac{1}{2}(-|\boldsymbol{I}|-|\boldsymbol{J}|-J_1+1)} [I_1]_q \langle v_{\boldsymbol{K}} \otimes v_{\boldsymbol{L}} | W^a | v_{\boldsymbol{I}-\boldsymbol{e}^1} \otimes v_{\boldsymbol{J}} \rangle + \frac{b_2}{a_2} q^{\frac{1}{2}(-|\boldsymbol{J}|+1)} [J_1]_q \langle v_{\boldsymbol{K}} \otimes v_{\boldsymbol{L}} | W^a | v_{\boldsymbol{I}} \otimes v_{\boldsymbol{J}-\boldsymbol{e}^1} \rangle$$
$$= \frac{b_1 b_2}{a_1 a_2} q^{\frac{1}{2}(-|\boldsymbol{K}|-|\boldsymbol{L}|-L_1)} [K_1+1]_q \langle v_{\boldsymbol{K}+\boldsymbol{e}^1} \otimes v_{\boldsymbol{L}} | W^a | v_{\boldsymbol{I}} \otimes v_{\boldsymbol{J}} \rangle + \frac{b_2}{a_1} q^{-\frac{1}{2}|\boldsymbol{L}|} [L_1+1]_q \langle v_{\boldsymbol{K}} \otimes v_{\boldsymbol{L}+\boldsymbol{e}^1} | W^a | v_{\boldsymbol{I}} \otimes v_{\boldsymbol{J}} \rangle.$$

Plugging the expression for W^a, using that $\boldsymbol{I} + \boldsymbol{J} - \boldsymbol{e}^1 = \boldsymbol{K} + \boldsymbol{L}$ and applying relations

$$\frac{C_{a_1,b_1;a_2,b_2}(\boldsymbol{I}-\boldsymbol{e}^1, \boldsymbol{J})}{C_{a_1,b_1;a_2,b_2}(\boldsymbol{I}, \boldsymbol{J})} = 1, \qquad \frac{C_{a_1,b_1;a_2,b_2}(\boldsymbol{I}, \boldsymbol{J}-\boldsymbol{e}^1)}{C_{a_1,b_1;a_2,b_2}(\boldsymbol{I}, \boldsymbol{J})} = (a_1/b_1) q^{\frac{1}{2}(|\boldsymbol{I}|-I_1)},$$

we can rewrite the needed identity as

$$(1-q^{I_1}) \Phi(\boldsymbol{I}-\boldsymbol{K}-\boldsymbol{e}^1, \boldsymbol{K}; a_1^2/a_2^2, a_2^2/b_1^2) + q^{I_1}(1-q^{J_1}) \Phi(\boldsymbol{I}-\boldsymbol{K}, \boldsymbol{K}; a_1^2/a_2^2, a_2^2/b_1^2)$$
$$= (1-q^{K_1+1}) \Phi(\boldsymbol{I}-\boldsymbol{K}-\boldsymbol{e}^1, \boldsymbol{K}+\boldsymbol{e}^1; a_1^2/a_2^2, a_2^2/b_1^2) + q^{K_1}(1-q^{L_1+1}) \Phi(\boldsymbol{I}-\boldsymbol{K}, \boldsymbol{K}; a_1^2/a_2^2, a_2^2/b_1^2).$$

Using

$$\frac{\Phi(\boldsymbol{X}, \boldsymbol{Y}; a, b)}{\Phi(\boldsymbol{X}-\boldsymbol{e}^1, \boldsymbol{Y}; a, b)} = b q^{|\boldsymbol{Y}|-Y_1} \frac{1-aq^{|\boldsymbol{X}|-1}}{1-abq^{|\boldsymbol{X}|+|\boldsymbol{Y}|-1}} \frac{1-q^{X_1+Y_1}}{1-q^{X_1}},$$
$$\frac{\Phi(\boldsymbol{X}, \boldsymbol{Y}+\boldsymbol{e}^1; a, b)}{\Phi(\boldsymbol{X}, \boldsymbol{Y}; a, b)} = \frac{1-bq^{|\boldsymbol{Y}|}}{1-abq^{|\boldsymbol{X}|+|\boldsymbol{Y}|}} \frac{1-q^{X_1+Y_1+1}}{1-q^{Y_1+1}}$$

we finally reduce the verification to

$$\frac{b_1^2}{a_2^2}(1-q^{I_1-K_1}) q^{-|\boldsymbol{K}|+K_1} \frac{1-q^{|\boldsymbol{I}|-1} a_1^2/b_1^2}{1-q^{|\boldsymbol{I}|-|\boldsymbol{K}|-1} a_1^2/a_2^2} + q^{I_1}(1-q^{J_1})$$
$$= \frac{b_1^2}{a_2^2}(1-q^{I_1-K_1}) q^{-|\boldsymbol{K}|+K_1} \frac{1-q^{|\boldsymbol{K}|} a_2^2/b_1^2}{1-q^{|\boldsymbol{I}|-|\boldsymbol{K}|-1} a_1^2/a_2^2} + q^{K_1}(1-q^{L_1+1}).$$

The latter can be verified by standard algebraic manipulations, using $\boldsymbol{I} + \boldsymbol{J} - \boldsymbol{e}^1 = \boldsymbol{K} + \boldsymbol{L}$. □

Proposition 1.2.2. *For generic parameters a_1, b_1, a_2, b_2 there is an isomorphism*

$$R : V_{b_1}^{a_1} \otimes V_{b_2}^{a_2} \to V_{b_2}^{a_2} \otimes V_{b_1}^{a_1},$$

which is given in the following two equivalent ways:

$$\langle v_{\boldsymbol{K}} \otimes v_{\boldsymbol{L}} | R | v_{\boldsymbol{I}} \otimes v_{\boldsymbol{J}} \rangle = \delta_{\boldsymbol{I}+\boldsymbol{J}=\boldsymbol{K}+\boldsymbol{L}} \frac{C_{a_2,b_2;a_1,b_1}(\boldsymbol{K}, \boldsymbol{L})}{C_{a_1,b_1;a_2,b_2}(\boldsymbol{I}, \boldsymbol{J})}$$
$$\times \sum_{\boldsymbol{P}} \Phi(\boldsymbol{L}-\boldsymbol{P}, \boldsymbol{K}; a_1^2/a_2^2, a_2^2/b_2^2) \Phi(\boldsymbol{P}, \boldsymbol{J}-\boldsymbol{P}; a_2^2/b_1^2, b_1^2/b_2^2), \quad (1.2.1)$$

where the sum is over configurations $\boldsymbol{P} \in \mathbb{Z}^n_{\geq 0}$ *such that* $P_i \leq \min(J_i, L_i)$*;*

$$\langle v_{\boldsymbol{K}} \otimes v_{\boldsymbol{L}} | R | v_{\boldsymbol{I}} \otimes v_{\boldsymbol{J}} \rangle = \delta_{\boldsymbol{I}+\boldsymbol{J}=\boldsymbol{K}+\boldsymbol{L}} \frac{C_{a_2,b_2;a_1,b_1}(\boldsymbol{K},\boldsymbol{L})}{C_{a_1,b_1;a_2,b_2}(\boldsymbol{I},\boldsymbol{J})} \times \sum_{\boldsymbol{P}} \Phi(\boldsymbol{L}, \boldsymbol{K}-\boldsymbol{P}; a_1^2/b_1^2, b_1^2/b_2^2) \Phi(\boldsymbol{I}-\boldsymbol{P}, \boldsymbol{P}; a_1^2/a_2^2, a_2^2/b_1^2), \quad (1.2.2)$$

where the sum is over configurations $\boldsymbol{P} \in \mathbb{Z}^n_{\geq 0}$ *such that* $P_i \leq \min(I_i, K_i)$*.*

Proof. For generic a_1, a_2, b_1, b_2 the representations $V^{a_1}_{b_1} \otimes V^{a_2}_{b_2}$ and $V^{a_2}_{b_2} \otimes V^{a_1}_{b_1}$ are irreducible and have one-dimensional highest weight space, so there exists, up to a scalar, at most one isomorphism R between them.

The claim now follows from Proposition 1.2.1, which allows to construct isomorphism R as above in two ways corresponding to (1.2.1) and (1.2.2):

$$W^a \circ W^b : V^{a_1}_{b_1} \otimes V^{a_2}_{b_2} \to V^{a_1}_{b_2} \otimes V^{a_2}_{b_1} \to V^{a_2}_{b_2} \otimes V^{a_1}_{b_1}$$

$$\langle v_{\boldsymbol{K}} \otimes v_{\boldsymbol{L}} | W^a \circ W^b | v_{\boldsymbol{I}} \otimes v_{\boldsymbol{J}} \rangle = \delta_{\boldsymbol{I}+\boldsymbol{J}=\boldsymbol{K}+\boldsymbol{L}} \frac{C_{a_2,b_2;a_1,b_1}(\boldsymbol{K},\boldsymbol{L})}{C_{a_1,b_1;a_2,b_2}(\boldsymbol{I},\boldsymbol{J})} \times \sum_{\substack{\boldsymbol{Q},\boldsymbol{P} \\ \boldsymbol{Q}+\boldsymbol{P}=\boldsymbol{I}+\boldsymbol{J}}} \delta_{\boldsymbol{Q}\geq\boldsymbol{K}} \delta_{\boldsymbol{J}\geq\boldsymbol{P}} \Phi(\boldsymbol{Q}-\boldsymbol{K}, \boldsymbol{K}; a_1^2/a_2^2, a_2^2/b_2^2) \Phi(\boldsymbol{P}, \boldsymbol{J}-\boldsymbol{P}; a_2^2/b_1^2, b_1^2/b_2^2);$$

$$W^b \circ W^a : V^{a_1}_{b_1} \otimes V^{a_2}_{b_2} \to V^{a_2}_{b_1} \otimes V^{a_1}_{b_2} \to V^{a_2}_{b_2} \otimes V^{a_1}_{b_1}$$

$$\langle v_{\boldsymbol{K}} \otimes v_{\boldsymbol{L}} | W^b \circ W^a | v_{\boldsymbol{I}} \otimes v_{\boldsymbol{J}} \rangle = \delta_{\boldsymbol{I}+\boldsymbol{J}=\boldsymbol{K}+\boldsymbol{L}} \frac{C_{a_2,b_2;a_1,b_1}(\boldsymbol{K},\boldsymbol{L})}{C_{a_1,b_1;a_2,b_2}(\boldsymbol{I},\boldsymbol{J})} \times \sum_{\substack{\boldsymbol{P},\boldsymbol{R} \\ \boldsymbol{P}+\boldsymbol{R}=\boldsymbol{I}+\boldsymbol{J}}} \delta_{\boldsymbol{R}\geq\boldsymbol{L}} \delta_{\boldsymbol{I}\geq\boldsymbol{P}} \Phi(\boldsymbol{L}, \boldsymbol{R}-\boldsymbol{L}; a_1^2/b_1^2, b_1^2/b_2^2) \Phi(\boldsymbol{I}-\boldsymbol{P}, \boldsymbol{P}; a_1^2/a_2^2, a_2^2/b_1^2).$$

Note that both morphisms above send $v_{\boldsymbol{0}} \otimes v_{\boldsymbol{0}}$ to $v_{\boldsymbol{0}} \otimes v_{\boldsymbol{0}}$, so these two isomorphisms coincide. □

Remark 1.2.3. Proposition 1.2.2 provides an expression for the action of the R-matrix of $U'_q(\widehat{\mathfrak{sl}}_{n+1})$ on $V(s^{-\omega_1})_z \otimes V(s'^{-\omega_1})_{z'}$. In the case of $U'_q(\widehat{\mathfrak{sl}}_2)$ this expression was obtained from various approaches in the works [52], [62], see also [39], [1] for analogous expressions in the more general elliptic case. For $U_q(\widehat{\mathfrak{sl}}_{n+1})$ the expression of Proposition 1.2.2 was first obtained in [25] using the methods of three-dimensional integrability.

1.2.2 Triple tensor products and deformations of Yang-Baxter equations

By Propositions 1.1.2, 1.1.5 generically we have $V^{a_1}_{b_1} \otimes \cdots \otimes V^{a_m}_{b_m} \cong L(\boldsymbol{f})$ with $f_1(u) = \prod_i \frac{a_i^{-1} - a_i u}{b_i^{-1} - b_i u}$ and $f_r(u) = 1$ for $r > 1$. Hence, for two collections of generic parameters $a_1 \dots a_m, b_1, \dots, b_m$ and $\tilde{a}_1 \dots \tilde{a}_m, \tilde{b}_1, \dots, \tilde{b}_m$ the representations $V^{a_1}_{b_1} \otimes \cdots \otimes V^{a_m}_{b_m}$ and $V^{\tilde{a}_1}_{\tilde{b}_1} \otimes \cdots \otimes V^{\tilde{a}_m}_{\tilde{b}_m}$ are isomorphic if and only if, up to sign changes, $\tilde{a}$ is a permutation of a, $\tilde{b}$ is a permutation of b, and the total number of

sign changes is even. Proposition 1.2.1 allows to explicitly construct all such isomorphisms, since W^a (W^b) is the isomorphism corresponding to a simple transposition of the parameters a_i (respectively b_i), while the sign changes are trivial since $V_b^a = V((a/b)^{-\omega_1})_{ab} = V_{-b}^{-a}$ and we have

$$V_{b_1}^{a_1} \otimes V_{b_2}^{a_2} = V_{b_1}^{a_1} \otimes V_{-b_2}^{-a_2} \cong V_{b_1}^{-a_2} \otimes V_{-b_2}^{a_1} = V_{-b_1}^{a_2} \otimes V_{-b_2}^{a_1} \cong V_{-b_1}^{a_1} \otimes V_{-b_2}^{a_2} = V_{b_1}^{-a_1} \otimes V_{b_2}^{-a_2},$$

where both isomorphisms above are constructed using W^a.

More importantly, since the representations are irreducible, the isomorphism $V_{b_1}^{a_1} \otimes \cdots \otimes V_{b_m}^{a_m} \cong V_{\tilde{b}_1}^{\tilde{a}_1} \otimes \cdots \otimes V_{\tilde{b}_m}^{\tilde{a}_m}$ is unique up to a scalar. Hence we can obtain nontrivial identities by finding multiple ways of expressing the same isomorphism using W^a, W^b and R. Below we demonstrate this idea for triple tensor products, deriving three equalities which will be used in this work.

For later use we summarize the expressions form Propositions 1.2.1, 1.2.2, setting:

$$\begin{aligned} &W^a_{a_1,a_2,b_1}(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K},\boldsymbol{L}) := \delta_{\boldsymbol{I}+\boldsymbol{J}=\boldsymbol{K}+\boldsymbol{L}}\Phi(\boldsymbol{I}-\boldsymbol{K},\boldsymbol{K};a_1/a_2,a_2/b_1) \\ &= \delta_{\boldsymbol{I}+\boldsymbol{J}=\boldsymbol{K}+\boldsymbol{L}}\delta_{\boldsymbol{I}\geq\boldsymbol{K}}(a_2/b_1)^{|\boldsymbol{I}|-|\boldsymbol{K}|}q^{\sum_{i<j}(I_i-K_i)K_j}\frac{(a_1/a_2;q)_{|\boldsymbol{I}|-|\boldsymbol{K}|}(a_2/b_1;q)_{|\boldsymbol{K}|}}{(a_1/b_1;q)_{|\boldsymbol{I}|}}\prod_{r=1}^{n}\frac{(q;q)_{I_r}}{(q;q)_{I_r-K_r}(q;q)_{K_r}}, \end{aligned} \tag{1.2.3}$$

$$\begin{aligned} &W^b_{a_2,b_1,b_2}(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K},\boldsymbol{L}) := \delta_{\boldsymbol{I}+\boldsymbol{J}=\boldsymbol{K}+\boldsymbol{L}}\Phi(\boldsymbol{L},\boldsymbol{J}-\boldsymbol{L};a_2/b_1,b_1/b_2) \\ &= \delta_{\boldsymbol{I}+\boldsymbol{J}=\boldsymbol{K}+\boldsymbol{L}}\delta_{\boldsymbol{J}\geq\boldsymbol{L}}(b_1/b_2)^{|\boldsymbol{L}|}q^{\sum_{i<j}L_i(J_j-L_j)}\frac{(a_2/b_1;q)_{|\boldsymbol{L}|}(b_1/b_2;q)_{|\boldsymbol{J}|-|\boldsymbol{L}|}}{(a_2/b_2;q)_{|\boldsymbol{J}|}}\prod_{r=1}^{n}\frac{(q;q)_{J_r}}{(q;q)_{L_r}(q;q)_{J_r-L_r}}, \end{aligned} \tag{1.2.4}$$

$$\begin{aligned} R_{a_1,b_1,a_2,b_2}(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K},\boldsymbol{L}) &:= \delta_{\boldsymbol{I}+\boldsymbol{J}=\boldsymbol{K}+\boldsymbol{L}}\sum_{\boldsymbol{P}}\Phi(\boldsymbol{L}-\boldsymbol{P},\boldsymbol{K};a_1/a_2,a_2/b_2)\Phi(\boldsymbol{P},\boldsymbol{J}-\boldsymbol{P};a_2/b_1,b_1/b_2) \\ &= \delta_{\boldsymbol{I}+\boldsymbol{J}=\boldsymbol{K}+\boldsymbol{L}}\sum_{\boldsymbol{P}}\Phi(\boldsymbol{L},\boldsymbol{K}-\boldsymbol{P};a_1/b_1,b_1/b_2)\Phi(\boldsymbol{I}-\boldsymbol{P},\boldsymbol{P};a_1/a_2,a_2/b_1). \end{aligned} \tag{1.2.5}$$

We do not include $C_{a_1,b_1;a_2,b_2}(\boldsymbol{I},\boldsymbol{J})$ from Propositions 1.2.1, 1.2.2 in the expressions above, in this way right-hand sides of (1.2.3)-(1.2.5) are rational functions in q, not just in $q^{\frac{1}{2}}$. Note also that we have replaced a_i^2, b_i^2 from Propositions 1.2.1, 1.2.2 by a_i, b_i.

Remark 1.2.4. For the later use in Sections 2.2 and 2.3 we note that $\Phi(\boldsymbol{X},\boldsymbol{Y},x/y,y/z)$ is a polynomial in y, and $y^{-|\boldsymbol{X}|-|\boldsymbol{Y}|}\Phi(\boldsymbol{X},\boldsymbol{Y},x/y,y/z)$ is a polynomial in y^{-1}. Hence $W^a_{a_1,a_2,b_1}(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K},\boldsymbol{L})$ is a polynomial in a_2, $W^b_{a_2,b_1,b_2}(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K},\boldsymbol{L})$ is a polynomial in b_1 and $a_2^{-|\boldsymbol{I}|}W^a_{a_1,a_2,b_1}(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K},\boldsymbol{L})$ is a polynomial in a_2^{-1}. Moreover, $a_2^{-|\boldsymbol{I}|}R_{a_1,b_1,a_2,b_2}(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K},\boldsymbol{L})$ is a polynomial in b_1 and a_2^{-1}.

Proposition 1.2.5. *The following identity of rational functions in $a_1, a_2, a_3, b_1, b_2, b_3$ holds:*

$$\sum_{\boldsymbol{C}_1,\boldsymbol{C}_2,\boldsymbol{C}_3} W^b_{a_3,b_1,b_2}(\boldsymbol{C}_1,\boldsymbol{C}_2,\boldsymbol{B}_2,\boldsymbol{B}_1) W^b_{a_2,b_1,b_3}(\boldsymbol{A}_1,\boldsymbol{C}_3,\boldsymbol{B}_3,\boldsymbol{C}_1) W^b_{a_3,b_2,b_3}(\boldsymbol{A}_2,\boldsymbol{A}_3,\boldsymbol{C}_3,\boldsymbol{C}_2)$$
$$= \sum_{\boldsymbol{C}_1,\boldsymbol{C}_2,\boldsymbol{C}_3} W^b_{a_2,b_2,b_3}(\boldsymbol{C}_2,\boldsymbol{C}_3,\boldsymbol{B}_3,\boldsymbol{B}_2) W^b_{a_3,b_1,b_3}(\boldsymbol{C}_1,\boldsymbol{A}_3,\boldsymbol{C}_3,\boldsymbol{B}_1) W^b_{a_2,b_1,b_2}(\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{C}_2,\boldsymbol{C}_1).$$

Proof. It is enough to prove that for generic $a_1, a_2, a_3, b_1, b_2, b_3$ we have

$$\sum_{\boldsymbol{C}_1,\boldsymbol{C}_2,\boldsymbol{C}_3} W^b_{a_3^2,b_1^2,b_2^2}(\boldsymbol{C}_1,\boldsymbol{C}_2,\boldsymbol{B}_2,\boldsymbol{B}_1) W^b_{a_2^2,b_1^2,b_3^2}(\boldsymbol{A}_1,\boldsymbol{C}_3,\boldsymbol{B}_3,\boldsymbol{C}_1) W^b_{a_3^2,b_2^2,b_3^2}(\boldsymbol{A}_2,\boldsymbol{A}_3,\boldsymbol{C}_3,\boldsymbol{C}_2)$$
$$= \sum_{\boldsymbol{C}_1,\boldsymbol{C}_2,\boldsymbol{C}_3} W^b_{a_2^2,b_2^2,b_3^2}(\boldsymbol{C}_2,\boldsymbol{C}_3,\boldsymbol{B}_3,\boldsymbol{B}_2) W^b_{a_3^2,b_1^2,b_3^2}(\boldsymbol{C}_1,\boldsymbol{A}_3,\boldsymbol{C}_3,\boldsymbol{B}_1) W^b_{a_2^2,b_1^2,b_2^2}(\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{C}_2,\boldsymbol{C}_1).$$

Consider two isomorphisms $V^{a_1}_{b_1} \otimes V^{a_2}_{b_2} \otimes V^{a_3}_{b_3} \cong V^{a_1}_{b_3} \otimes V^{a_2}_{b_2} \otimes V^{a_3}_{b_1}$:

$$(1\otimes W^b)(W^b\otimes 1)(1\otimes W^b) : V^{a_1}_{b_1}\otimes V^{a_2}_{b_2}\otimes V^{a_3}_{b_3} \to V^{a_1}_{b_1}\otimes V^{a_2}_{b_3}\otimes V^{a_3}_{b_2} \to V^{a_1}_{b_3}\otimes V^{a_2}_{b_1}\otimes V^{a_3}_{b_2} \to V^{a_1}_{b_3}\otimes V^{a_2}_{b_2}\otimes V^{a_3}_{b_1},$$

$$(W^b\otimes 1)(1\otimes W^b)(W^b\otimes 1) : V^{a_1}_{b_1}\otimes V^{a_2}_{b_2}\otimes V^{a_3}_{b_3} \to V^{a_1}_{b_2}\otimes V^{a_2}_{b_1}\otimes V^{a_3}_{b_3} \to V^{a_1}_{b_2}\otimes V^{a_2}_{b_3}\otimes V^{a_3}_{b_1} \to V^{a_1}_{b_3}\otimes V^{a_2}_{b_2}\otimes V^{a_3}_{b_1}.$$

Both isomorphisms send $v_{\boldsymbol{0}} \otimes v_{\boldsymbol{0}} \otimes v_{\boldsymbol{0}}$ to $v_{\boldsymbol{0}} \otimes v_{\boldsymbol{0}} \otimes v_{\boldsymbol{0}}$, so by irreducibility they are equal. The claim now follows by applying Proposition 1.2.1 to

$$\langle v_{\boldsymbol{B}_3} \otimes v_{\boldsymbol{B}_2} \otimes v_{\boldsymbol{B}_1} | (1 \otimes W^b)(W^b \otimes 1)(1 \otimes W^b) | v_{\boldsymbol{A}_1} \otimes v_{\boldsymbol{A}_2} \otimes v_{\boldsymbol{A}_3} \rangle$$
$$= \langle v_{\boldsymbol{B}_3} \otimes v_{\boldsymbol{B}_2} \otimes v_{\boldsymbol{B}_1} | (W^b \otimes 1)(1 \otimes W^b)(W^b \otimes 1) | v_{\boldsymbol{A}_1} \otimes v_{\boldsymbol{A}_2} \otimes v_{\boldsymbol{A}_3} \rangle.$$

Note that all coefficients $C_{a,b;a',b'}(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K},\boldsymbol{L})$ cancel out. □

Proposition 1.2.6. *The following identity of rational functions holds:*

$$\sum_{\boldsymbol{C}_1,\boldsymbol{C}_2,\boldsymbol{C}_3} W^b_{a_2,b_1,b_3}(\boldsymbol{C}_1,\boldsymbol{C}_2,\boldsymbol{B}_2,\boldsymbol{B}_1) R_{a_1,b_1,a_3,b_2}(\boldsymbol{A}_1,\boldsymbol{C}_3,\boldsymbol{B}_3,\boldsymbol{C}_1) W^a_{a_2,a_3,b_2}(\boldsymbol{A}_2,\boldsymbol{A}_3,\boldsymbol{C}_3,\boldsymbol{C}_2)$$
$$= \sum_{\boldsymbol{C}_1,\boldsymbol{C}_2,\boldsymbol{C}_3} W^a_{a_1,a_3,b_2}(\boldsymbol{C}_2,\boldsymbol{C}_3,\boldsymbol{B}_3,\boldsymbol{B}_2) R_{a_2,b_1,a_3,b_3}(\boldsymbol{C}_1,\boldsymbol{A}_3,\boldsymbol{C}_3,\boldsymbol{B}_1) W^b_{a_2,b_1,b_2}(\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{C}_2,\boldsymbol{C}_1)$$

Proof. The claim follows from the same argument as in Proposition 1.2.5, applied to the morphisms

$$(1 \otimes W^b)(R \otimes 1)(1 \otimes W^a) = (W^a \otimes 1)(1 \otimes R)(W^b \otimes 1) : V^{a_1}_{b_1} \otimes V^{a_2}_{b_2} \otimes V^{a_3}_{b_3} \to V^{a_3}_{b_2} \otimes V^{a_1}_{b_3} \otimes V^{a_2}_{b_1},$$

□

Proposition 1.2.7. *The following identity of rational functions holds:*

$$\sum_{\boldsymbol{C}_1,\boldsymbol{C}_2,\boldsymbol{C}_3} R_{a_1,b_1,a_3,b_2}(\boldsymbol{C}_1,\boldsymbol{C}_2,\boldsymbol{B}_2,\boldsymbol{B}_1)R_{a_1,b_1,a_2,b_3}(\boldsymbol{A}_1,\boldsymbol{C}_3,\boldsymbol{B}_3,\boldsymbol{C}_1)W^b_{a_2,b_2,b_3}(\boldsymbol{A}_2,\boldsymbol{A}_3,\boldsymbol{C}_3,\boldsymbol{C}_2)$$
$$= \sum_{\boldsymbol{C}_1,\boldsymbol{C}_2,\boldsymbol{C}_3} W^a_{a_1,a_3,b_2}(\boldsymbol{C}_2,\boldsymbol{C}_3,\boldsymbol{B}_3,\boldsymbol{B}_2)R_{a_2,b_1,a_3,b_3}(\boldsymbol{C}_1,\boldsymbol{A}_3,\boldsymbol{C}_3,\boldsymbol{B}_1)W^b_{a_2,b_1,b_2}(\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{C}_2,\boldsymbol{C}_1)$$

Proof. The claim follows from the same argument as in Proposition 1.2.5, applied to the morphisms

$$(1\otimes R)(R\otimes 1)(1\otimes W^b) = (W^b\otimes 1)(1\otimes R)(R\otimes 1) : V^{a_1}_{b_1}\otimes V^{a_2}_{b_2}\otimes V^{a_3}_{b_3}\to V^{a_2}_{b_3}\otimes V^{a_3}_{b_2}\otimes V^{a_1}_{b_1},$$

□

We can also obtain the celebrated Yang-Baxter equation utilizing the same approach:

Proposition 1.2.8 (Yang-Baxter equation)**.** *The following identity of rational functions holds:*

$$\sum_{\boldsymbol{C}_1,\boldsymbol{C}_2,\boldsymbol{C}_3} R_{a_1,b_1,a_2,b_2}(\boldsymbol{C}_1,\boldsymbol{C}_2,\boldsymbol{B}_2,\boldsymbol{B}_1)R_{a_1,b_1,a_3,b_3}(\boldsymbol{A}_1,\boldsymbol{C}_3,\boldsymbol{B}_3,\boldsymbol{C}_1)R_{a_2,b_2,a_3,b_3}(\boldsymbol{A}_2,\boldsymbol{A}_3,\boldsymbol{C}_3,\boldsymbol{C}_2)$$
$$= \sum_{\boldsymbol{C}_1,\boldsymbol{C}_2,\boldsymbol{C}_3} R_{a_2,b_2,a_3,b_3}(\boldsymbol{C}_2,\boldsymbol{C}_3,\boldsymbol{B}_3,\boldsymbol{B}_2)R_{a_1,b_1,a_3,b_3}(\boldsymbol{C}_1,\boldsymbol{A}_3,\boldsymbol{C}_3,\boldsymbol{B}_1)R_{a_1,b_1,a_2,b_2}(\boldsymbol{A}_1,\boldsymbol{A}_2,\boldsymbol{C}_2,\boldsymbol{C}_1)$$

Proof. The claim follows from the same argument as in Proposition 1.2.5, applied to the morphisms

$$(1\otimes R)(R\otimes 1)(1\otimes R) = (R\otimes 1)(1\otimes R)(R\otimes 1) : V^{a_1}_{b_1}\otimes V^{a_2}_{b_2}\otimes V^{a_3}_{b_3}\to V^{a_3}_{b_3}\otimes V^{a_2}_{b_2}\otimes V^{a_1}_{b_1},$$

□

1.3 Vertex models with q-Hahn weights

In this section we introduce vertex models with q-Hahn weights, which provide a graphical presentation for our results about representations of $U'_q(\widehat{\mathfrak{sl}_{n+1}})$. We also mention how our vertex model degenerates to the higher spin six-vertex model.

1.3.1 General construction

In this work by a *vertex model* we mean the following data:

- A collection of oriented lines in the plane, whose intersections are called *vertices*, while the line segments between vertices are called *edges*. The edges are oriented in the same way as the underlying lines, and each vertex has exactly two incoming edges and two outgoing edges. An edge is *internal* if it connects a pair of distinct vertices, and is *boundary* if it is connected to only one vertex.

- A collection of pairs of *edge parameters* (a, b), which are assigned to the edges and are constrained by the following rule: for each vertex if (a, b) and (a', b') are the parameters of the incoming edges of the vertex, then its outgoing edges have either parameters (a, b) and (a', b'), or parameters (a, b') and (a', b). See Figure 1-1 for the assignments satisfying this constrain.

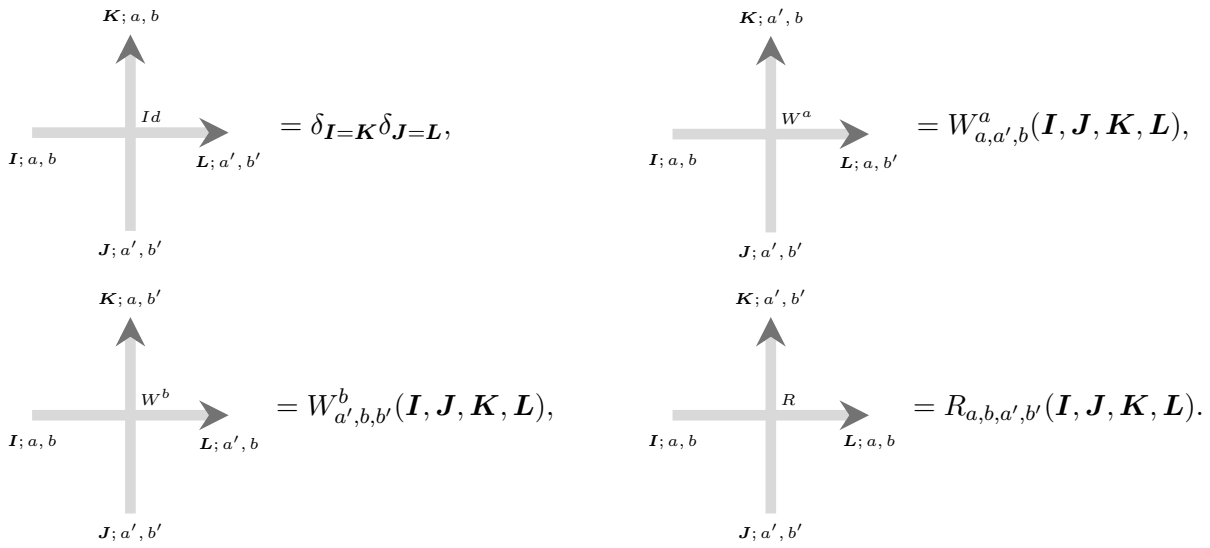


Figure 1-1: Possible vertices in vertex models and their weights.

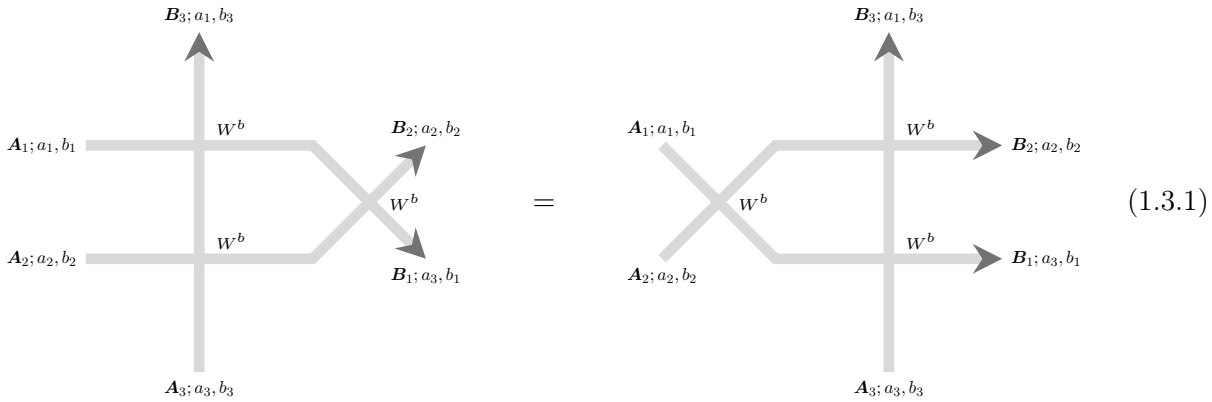
A *configuration* of a vertex model is a labelling of the edges by compositions $\boldsymbol{I} \in \mathbb{Z}_{\geq 0}$. In this text we usually denote these labels by bold capital letters $\boldsymbol{I}, \boldsymbol{J}, \ldots$. Given a configuration around a vertex, that is, a collection of four edge labels $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}, \boldsymbol{L}$ attached to the adjacent edges, we define the corresponding *vertex weight* by tracking the behavior of the edge parameters a, a', b, b' and correspondingly using expressions (1.2.3),(1.2.4) and (1.2.5) in the way demonstrated in Figure 1-1.

In other words, we define four types of vertices Id, W^a, W^b, R, which are determined by the arrangement of the parameters a, b, a', b'. For example, the vertex of type W^a preserves the parameters a, a' but swaps the parameters b, b' between the lines. Note that in each of these situations the vertex weight vanishes unless $\boldsymbol{I} + \boldsymbol{J} = \boldsymbol{K} + \boldsymbol{L}$, we call this fact the *conservation law*.

The *weight* of a configuration of a vertex model is the product of the weights of all vertices. A *boundary condition* for a vertex model is an assignment of labels to the boundary edges. Given a vertex model and a boundary condition we define the corresponding *partition function* as the sum of the weights of all configurations satisfying the boundary condition, that is, configurations whose labels of the boundary edges coincide with the boundary condition.

As an example of this definition, we can rewrite the identity from Proposition 1.2.5 in the

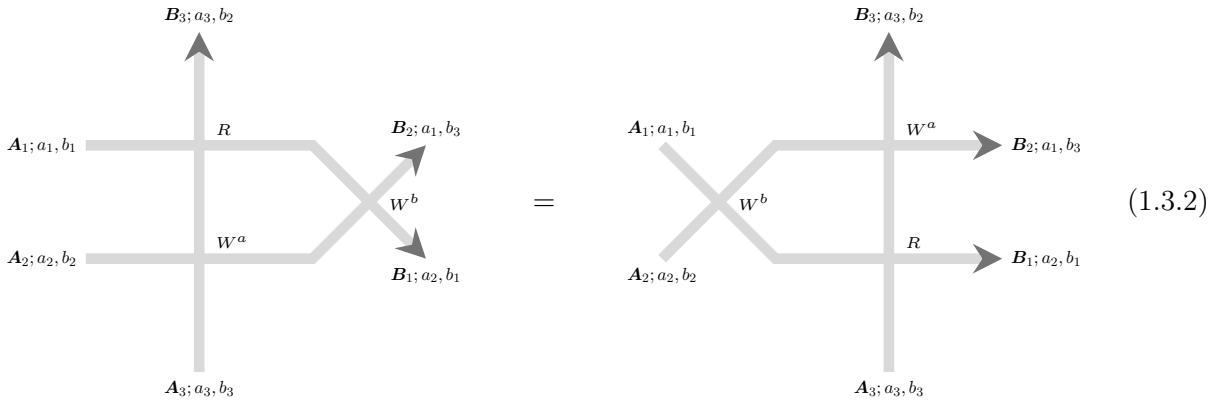
following way:



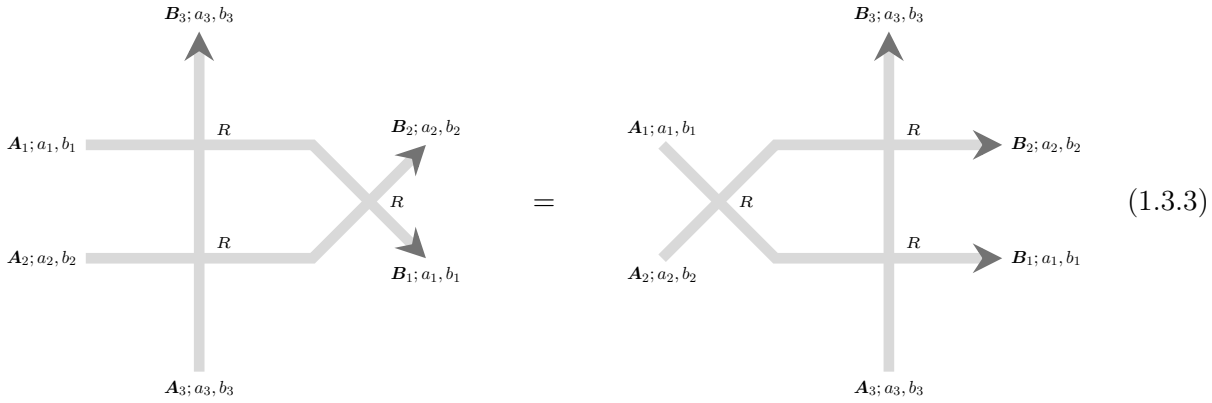
(1.3.1)

Here on each side we have a partition function of a model with three vertices. To each boundary edge we assign a triple $(\boldsymbol{I}; a, b)$, where $\boldsymbol{I}$ is the label of the boundary condition, and (a, b) are the corresponding edge parameters. To lighten the notation we usually only specify the types of the vertices W^a, W^b or R; the labels of the internal edges can be uniquely reconstructed from this information. The summation over $\boldsymbol{C}_i$ from Proposition 1.2.5 correspond to the summation over all possible configurations of internal edges.

In a similar way we can rewrite Propositions 1.2.6, 1.2.8:



(1.3.2)



(1.3.3)

Remark 1.3.1. One can think about the edge with parameters (a, b) as of the representation $V^{\sqrt{a}}_{\sqrt{b}}$, then each vertex corresponds to one of the operators $V^{\sqrt{a_1}}_{\sqrt{b_1}} \otimes V^{\sqrt{a_2}}_{\sqrt{b_2}} \to V^{\sqrt{\tilde{a}_1}}_{\sqrt{\tilde{b}_1}} \otimes V^{\sqrt{\tilde{a}_2}}_{\sqrt{\tilde{b}_2}}$. Note that the whole partition function corresponds to a matrix coefficient of the isomorphism between two tensor products of the form $V^{\sqrt{a_1}}_{\sqrt{b_1}} \otimes \cdots \otimes V^{\sqrt{a_L}}_{\sqrt{b_L}}$, with the parameters a_i and b_i determined by the parameters of the incoming and outgoing edges.

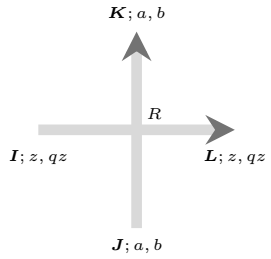
1.3.2 Higher spin six-vertex model

The vertex models introduced above do not pose any restrictions on the labels $\boldsymbol{I}$ of a configuration. However, it is known that the R-weights can be degenerated in a certain way, which prohibits labels $\boldsymbol{I}$ such that $|\boldsymbol{I}| > 1$ for either vertical or horizontal edges. This degeneration is called *higher-spin six vertex model*, and we describe it below.

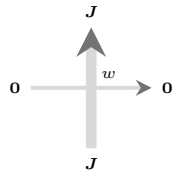
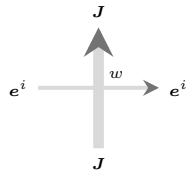
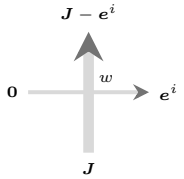
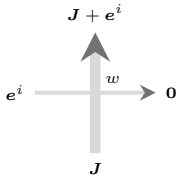
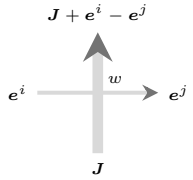
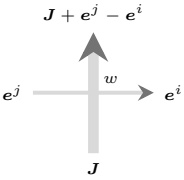
Recall that $\boldsymbol{e}^i \in \mathbb{Z}^n_{\geq 0}$ denotes the standard basis vector with 1 at ith coordinate and 0 elsewhere, set $\boldsymbol{e}^0 = \boldsymbol{0} = (0, \ldots, 0)$. For a composition $\boldsymbol{I} = (I_1, \ldots, I_n)$ and integers $q, p = 1, \ldots, n$ set $I_{[q,p]} = I_q + \cdots + I_p$ with the assumption $I_{[q,p]} = 0$ if $q > p$.

Proposition 1.3.2. *Let a, b, z be generic complex parameters and let $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}, \boldsymbol{L} \in \mathbb{Z}^n_{\geq 0}$ be compositions. Then the following vertex weight is well-defined:*

$$w_{z;a,b}(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}, \boldsymbol{L}) := R_{z,qz,a,b}(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}, \boldsymbol{L}) =$$



Moreover, when $|\boldsymbol{I}| \leq 1$ the weight $w_{z;a,b}(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}, \boldsymbol{L})$ vanishes unless $|\boldsymbol{L}| \leq 1$ and its value is given in the table below, where i, j are arbitrary integers satisfying $i < j$:

 $\frac{1-q^{J_{[1,n]}}z/b}{1-z/b}$	 $\frac{(q^{J_i}a/b-z/b)q^{J_{[i+1,n]}}}{1-z/b}$	 $\frac{(q^{J_i}-1)q^{J_{[i+1,n]}}z/b}{1-z/b}$
 $\frac{1-q^{J_{[1,n]}}a/b}{1-z/b}$	 $\frac{(q^{J_j}-1)q^{J_{[j+1,n]}}z/b}{1-z/b}$	 $\frac{(q^{J_i}-1)q^{J_{[i+1,n]}}a/b}{1-z/b}$

(1.3.4)

Proof. The claim follows readily from (1.2.5). The first line of (1.2.5) gives

$$R_{z,qz,a,b}(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K},\boldsymbol{L}) = \delta_{\boldsymbol{I}+\boldsymbol{J}=\boldsymbol{K}+\boldsymbol{L}} \sum_{\boldsymbol{P}} \Phi(\boldsymbol{L}-\boldsymbol{P},\boldsymbol{K};z/a,a/b)\Phi(\boldsymbol{P},\boldsymbol{J}-\boldsymbol{P};a/(qz),qz/b),$$

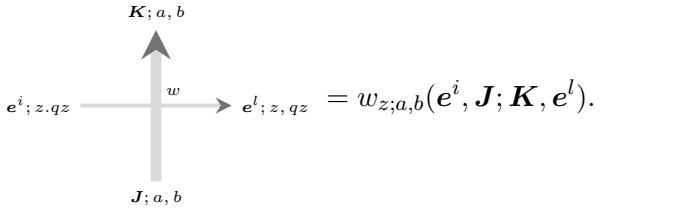
which is well-defined for all compositions $\boldsymbol{I},\boldsymbol{J},\boldsymbol{K},\boldsymbol{L}$.

When $|\boldsymbol{I}| \leq 1$ we can use the second line of (1.2.5) to get

$$R_{z,qz,a,b}(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K},\boldsymbol{L}) = \delta_{\boldsymbol{I}+\boldsymbol{J}=\boldsymbol{K}+\boldsymbol{L}} \sum_{\boldsymbol{P}} \Phi(\boldsymbol{L},\boldsymbol{K}-\boldsymbol{P};1/q,qz/b)\Phi(\boldsymbol{I}-\boldsymbol{P},\boldsymbol{P};z/a,a/(qz)),$$

where the term $\frac{1}{(1/q;q)_{|\boldsymbol{I}|}}$ coming from $\Phi(\boldsymbol{I}-\boldsymbol{P},\boldsymbol{P};z/a,a/(qz))$ is well-defined because $|\boldsymbol{I}| \leq 1$. Note that $\Phi(\boldsymbol{L},\boldsymbol{K}-\boldsymbol{P};1/q,qz/b)$ vanishes unless $|\boldsymbol{L}| \leq 1$, implying the vanishing statement. The exact expressions for $w_{z;a,b}(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K},\boldsymbol{L})$ follow by a direct computation. □

We will use the following notation when working with higher spin six-vertex weights:

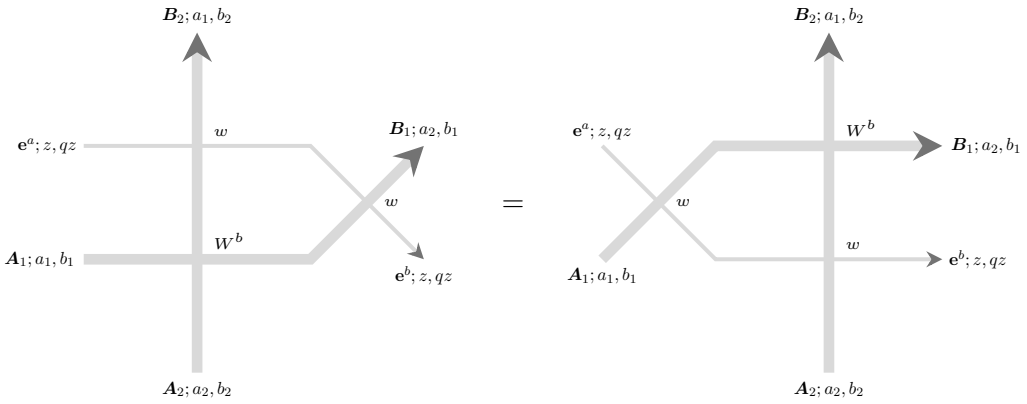


$$= w_{z;a,b}(\boldsymbol{e}^i,\boldsymbol{J};\boldsymbol{K},\boldsymbol{e}^l). \tag{1.3.5}$$

Note that we use a thin line for the horizontal edges in this context, to emphasize the restrictions $|\boldsymbol{I}| \leq 1$ and $|\boldsymbol{L}| \leq 1$ which force $\boldsymbol{I} = \boldsymbol{e}^i, \boldsymbol{L} = \boldsymbol{e}^l$ for some $i,l \in \{0,\ldots,n\}$.

As a particular case of Proposition 1.2.7 we have the following identity involving the higher spin

six-vertex weights, which will be useful later:



(1.3.6)

Remark 1.3.3. We often present configurations of a vertex model as ensembles of oriented colored lattice paths directed along the lines of the model, with colors labeled by integers $1, 2, \ldots, n$. The two descriptions are identified by encoding the colors of paths occupying a single edge by a composition $\boldsymbol{A} = (A_1, \ldots, A_n)$, where A_i is the number of paths with color i.

Chapter 2

Spin q-Whittaker symmetric polynomials

This chapter is devoted to a new family of spin q-Whittaker symmetric polynomials, which are found using q-Hahn models and identities from the previous chapter. In this chapter we only need the constructions of Chapter 1 when $n = 1$, that is, when the quantum algebra in question is $U_q'(\widehat{\mathfrak{sl}}_2)$. In this case all compositions $\boldsymbol{I}$ have length 1 and for this chapter we replace them by nonnegative integers I.

2.1 Overview

Connections between integrable lattice models and the theory of symmetric functions have been long known to be important to both domains; one might, for example, consult the following works (the list is certainly incomplete): Fomin-Kirillov [42], Tsilevich [87], Brubaker-Bump-Friedberg [26], Zinn-Justin [90], Korff [54], Wheeler-Zinn-Justin [88, 89], Motegi-Sakai [64], Motegi [63], Cantini-de Gier-Wheeler [30], Garbali-de Gier-Wheeler [44], Borodin [11, 12], Borodin-Petrov [19, 20], Borodin-Wheeler [21, 23], Garbali-Wheeler [45], Buciumas-Scrimshaw [27], Mucciconi-Petrov [65], Aggarwal-Borodin-Wheeler [2], Aggarwal-Borodin-Petrov-Wheeler [3].

In particular, in [11, 19, 20, 21] families of *spin* Hall-Littlewood and q-Whittaker symmetric functions were introduced. They generalized the classical families of the Hall-Littewood and q-Whittaker symmetric polynomials (both of which are special cases of the Macdonald symmetric functions, see Macdonald [60]), and the additional *spin parameter* was related to the spin in the underlying integrable vertex models. It was shown, in particular, that symmetric functions from those families satisfy explicit branching rules as well as (skew) Cauchy and dual Cauchy identities. A different version of the spin q-Whittaker polynomials was also recently suggested in [65]; it appears to enjoy a few valuable properties that the original one lacked.

In [20] an inhomogeneous version of the spin Hall-Littlewood functions was introduced. The inhomogeneities appear naturally from the point of view of the vertex models involved, and they are useful for some of the probabilistic applications. However, it was not clear how to construct an inhomogeneous version of the spin q-Whittaker polynomials: the approach of [21] did not lead to functions satisfying the dual Cauchy identity involving both q-Whittaker and Hall-Littlewood functions, which is the most natural requirement.

The main goal of this chapter is the construction of the inhomogeneous analogue of spin q-Whittaker polynomials satisfying algebraic properties desired from them, namely the branching rule, the existence of a Cauchy identity involving two spin q-Whittaker polynomials and a dual

Cauchy identity involving spin q-Whittaker and spin Hall-Littlewood functions. It turns out that construction of these polynomials relies on the integrability structure of q-Hahn models described in Chapter 1, which explains why the earlier methods were not sufficient for this task.

Our inhomogeneous generalization reveals a surprising connection of spin q-Whittaker polynomials with another type of symmetric functions, which are *a priori* not related to vertex models: *interpolation symmetric functions*. These are inhomogeneous deformations of classical symmetric functions which can be characterized by vanishing at specific points. The main known classes of interpolation symmetric functions are *factorial Schur functions* and *interpolation Macdonald functions*. The former ones are well-studied and their interpolation properties have appeared in the contexts of Capelli identities, multiplicity free spaces and asymptotic representation theory, see [47, 71, 76]. The interpolation Macdonald functions are more complicated and still somewhat mysterious, however a number of nice combinatorial properties is known about them and their degenerations, see [38, 53, 68, 69, 73, 74, 77].

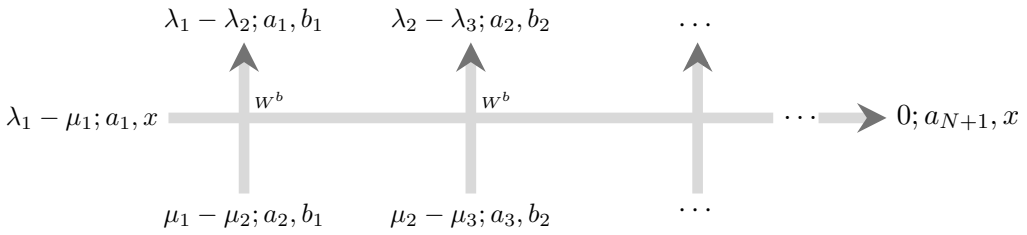
In [70] it was shown that, under a certain constraint, all interpolation symmetric polynomials of interest fall in one of three classes: factorial monomial functions (which were considered trivial), factorial Schur functions and (type BC_n) interpolation Macdonald functions. However, we show that our inhomogeneous spin q-Whittaker polynomials also can be characterized by vanishing at specific points. Moreover, by specializing inhomogeneous spin q-Whittaker polynomials, one can obtain new symmetric functions which should fill the place of *interpolation q-Whittaker* and *interpolation elementary* functions. We are also able to find and prove a parallel version of the classification from [70], which identifies our new interpolation functions as unique functions satisfying a constraint similar to the one from [70]. Overall these findings distinguish spin q-Whittaker polynomials as functions having both a vertex model description and an interpolation characterization, which is rather unique (the only other example are factorial Schur functions, see [29]).

Below we briefly state the results of this chapter.

Inhomogeneous spin q-Whittaker polynomials.

For a pair of sequences of generic complex or formal parameters $\mathcal{A} = (a_0, a_1, a_2, \dots)$ and $\mathcal{B} = (b_0, b_1, b_2, \dots)$ we define functions $\mathbb{F}_{\lambda/\mu}(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B})$, which depend on n variables $x_1, \dots, x_n$ and are labeled by pairs of partitions λ, μ. Here are the main facts we prove about them.

- The functions $\mathbb{F}_{\lambda/\mu}(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B})$ are symmetric polynomials in $x_1, \dots, x_n$.
- For a single variable, $\mathbb{F}_{\lambda/\mu}(x \mid \mathcal{A}, \mathcal{B})$ has a fully factorized expression. To write it down, we first define an auxiliary function $ZW_{\lambda/\mu}(x \mid \mathcal{A}, \mathcal{B})$ as a partition function of the following one-row vertex model with N columns for sufficiently large $N > \max(l(\lambda), l(\mu))$:



  Here we use notation from Chapter 1, where diagrams as above denote a product of vertex weights W^b assigned to each vertex. Note that for the row above there is only one configuration

with non-zero weight, and the partition function is the product of weights

$$(x/b_i)^{\lambda_{i+1}-\mu_{i+1}}\frac{(x/b_i;q)_{\mu_i-\lambda_{i+1}}(a_{i+1}/x;q)_{\lambda_{i+1}-\mu_{i+1}}}{(a_{i+1}/b_i;q)_{\mu_i-\mu_{i+1}}}\frac{(q;q)_{\mu_i-\mu_{i+1}}}{(q;q)_{\mu_i-\lambda_{i+1}}(q;q)_{\lambda_{i+1}-\mu_{i+1}}}.$$

over integers $i\geq 1$. Since these weights are equal to 1 for sufficiently large i and fixed λ,μ, the partition function does not depend on N.

Then the polynomial $\mathbb{F}_{\lambda/\mu}(x\mid\mathcal{A},\mathcal{B})$ is defined by the following expression:

$$\mathbb{F}_{\lambda/\mu}(x\mid\mathcal{A},\mathcal{B}):=(x/b_0)^{\lambda_1-\mu_1}\frac{(a_1/x;q)_{\lambda_1-\mu_1}}{(q;q)_{\lambda_1-\mu_1}}ZW_{\lambda/\mu}(x\mid\mathcal{A},\mathcal{B}).$$

- The following branching rule holds:

$$\mathbb{F}_{\lambda/\nu}(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B})=\sum_{\mu}\mathbb{F}_{\lambda/\mu}(x_1,\dots,x_m\mid\mathcal{A},\mathcal{B})\mathbb{F}_{\mu/\nu}(x_{m+1},\dots,x_n\mid\tau^m\mathcal{A},\mathcal{B}),$$

  where $\tau^m\mathcal{A}$ is the sequence $\mathcal{A}$ shifted to the left by m steps. Together with the partition function expression for the one-variable polynomials, this allows to write the general $\mathbb{F}_{\lambda/\nu}(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B})$ in terms of the partition function for a vertex model with weights W^b.

- The functions $\mathbb{F}_\lambda(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B}):=\mathbb{F}_{\lambda/\varnothing}(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B})$ satisfy the following stability relation:

$$\mathbb{F}_\lambda(x_1,\dots,x_{n-1},a_n\mid\mathcal{A},\mathcal{B})=\mathbb{F}_\lambda(x_1,\dots,x_{n-1}\mid\mathcal{A},\mathcal{B}).$$

- The following Cauchy-type summation identity holds:

$$\sum_{\lambda}\mathbb{F}_\lambda(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B})\,\mathbb{F}^*_\lambda(y_1,\dots,y_m\mid\overline{\mathcal{B}},\overline{\mathcal{A}})=\prod_{i=1}^{n}\prod_{j=1}^{m}\frac{(a_iy_j;q)_\infty(x_i/b_j;q)_\infty}{(x_iy_j;q)_\infty(a_i/b_j;q)_\infty},$$

  Here $\mathbb{F}^*_\lambda$ denotes a certain rescaling of the functions $\mathbb{F}_\lambda$, and we set $\overline{\mathcal{A}}=(a_0^{-1},a_1^{-1},\dots),\overline{\mathcal{B}}=(b_0^{-1},b_1^{-1},\dots)$. These identities could be either viewed as holding in suitable rings of power series or numerically under certain restrictions on the parameters. One can also prove the more general skew version of this identity, see Theorem 2.3.7.

- We have the following dual Cauchy identity:

$$\sum_{\lambda}\widetilde{\mathsf{F}}^*_{\lambda'}(u_1,\dots,u_m\mid\overline{\mathcal{B}},\overline{\mathcal{A}})\mathbb{F}_\lambda(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B})=\prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1-u_jx_i}{1-u_ja_i},$$

  where $\widetilde{\mathsf{F}}^*_{\lambda'}$ denotes a certain stable version of the inhomogeneous spin Hall-Littlewood functions described in Section 2.3.3 below. In this dual Cauchy identity the number of nonzero terms is always finite, thus no convergence conditions are needed.

- The polynomials $\mathbb{F}_\mu$ admit an integral representation of the form

$$\begin{aligned}&\mathbb{F}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})\\ &=\frac{1}{\psi_\lambda(\mathcal{A},\mathcal{B})}\oint_{\mathcal{C}}\frac{dz_1}{2\pi\mathbf{i}z_1}\cdots\oint_{\mathcal{C}}\frac{dz_k}{2\pi\mathbf{i}z_k}\prod_{\alpha<\beta}\frac{z_\alpha-z_\beta}{z_\alpha-qz_\beta}\prod_{\alpha=1}^{k}\left(\frac{-a_0^{-1}}{z_\alpha-b_{\mu'_\alpha}^{-1}}\prod_{j=1}^{\mu'_\alpha-1}\frac{z_\alpha-a_j^{-1}}{z_\alpha-b_j^{-1}}\prod_{i=1}^{n}\frac{1-z_\alpha x_i}{1-z_\alpha a_i}\right)\end{aligned}$$

  where $k=\mu_1$, and the integration contour $\mathcal{C}$ is depicted in Figure 2-3 and described immediately before Theorem 2.3.11.

Comparing with [21], for $a_0=a_1=\cdots=s$ and $b_0=b_1=\cdots=s^{-1}$ our functions $\mathbb{F}_{\lambda/\mu}$ reduce to the spin q-Whittaker from [21]; in particular, our functions degenerate to the usual q-Whittaker polynomials when additionally $s\to 0$ (after suitable renormalization).

Vanishing properties and interpolation characterization

For this group of results the starting point is the following vanishing property of inhomogeneous spin q-Whittaker polynomials $\mathbb{F}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})$:

$$\mathbb{F}_\lambda(a_1q^{\mu_1-\mu_2},a_2q^{\mu_2-\mu_3},\dots,a_nq^{\mu_n} \mid \mathcal{A},\mathcal{B})=0,\qquad \text{unless } \lambda\subseteq\mu.$$

This vanishing property is new, and it leads to an alternative characterization of $\mathbb{F}_\lambda$. Set $\mathbf{x}^n_{\mathcal{A}}(\mu)=(a_1q^{\mu_1-\mu_2},a_2q^{\mu_2-\mu_3},\dots,a_nq^{\mu_n})$ and let $\mathcal{G}^n_0\subset\mathcal{G}^n_1\subset\mathcal{G}^n_2\subset\dots$ denote a certain deformation of the natural filtration of the algebra of symmetric functions in n variables, which is defined before Theorem 2.4.3 in the text.

Theorem (Theorem 2.4.3 in the text)**.** *For each partition $\lambda\in\mathbb{Y}^n$ the spin q-Whittaker polynomial $\mathbb{F}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})$ is uniquely characterized, up to a scalar, by the following properties:*

1. $\mathbb{F}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})\in\mathcal{G}^n_{|\lambda|}$.
2. *For any partition μ such that $|\mu|\le|\lambda|$ and $\mu\ne\lambda$ we have $\mathbb{F}_\lambda(\mathbf{x}^n_{\mathcal{A}}(\mu) \mid \mathcal{A},\mathcal{B})=0$.*
3. $\mathbb{F}_\lambda(\mathbf{x}^n_{\mathcal{A}}(\lambda) \mid \mathcal{A},\mathcal{B})\ne 0$.

Note that in the characterization above the parameters a_i from $\mathcal{A}$ determine the interpolation points, while the dependence on the second family $\mathcal{B}$ is hidden in the subspaces $\mathcal{G}^n_k$.

These vanishing and characterization properties closely resemble the defining properties of interpolation symmetric polynomials, which are families of symmetric polynomials $F_\lambda(x_1,\dots,x_n)$ satisfying

(i) $\deg F_\lambda(x_1,\dots,x_n)\le|\lambda|$;

(ii) $F_\lambda(\mathcal{U}(\mu))=0$ unless $\lambda\subset\mu$;

(iii) $F_\lambda(\mathcal{U}(\lambda))\ne 0$

for some family $\mathcal{U}(\mu)$ of n-dimensional points enumerated by partitions μ. Property (ii), called the *vanishing property*, is the non-trivial one: it overdefines the functions F_λ and makes the existence of these functions for a given family $\mathcal{U}$ exceptional. The factorial Schur functions and the interpolation

Macdonald functions are examples of such functions F_λ for certain choices of $\mathfrak{V}$, in both these examples $\mathfrak{V}$ is of the form

$$\mathfrak{V}(\mu) = (f_1(\mu_1), \dots, f_n(\mu_n))$$

for some functions $f_1, \dots, f_n$.

The spin q-Whittaker functions $\mathbb{F}_\lambda(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B})$ almost satisfy the properties of interpolation symmetric functions F_λ, but in the first property we have the function spaces $\mathcal{G}_k^n$ instead of the natural degree filtration of the algebra of symmetric polynomials. However, when the parameters b_i from $\mathcal{B}$ tend to ∞, the subspaces $\mathcal{G}_k^n$ degenerate to this natural filtration. In this way we can find two degenerations of the spin q-Whittaker functions which satisfy properties (i)-(iii) for certain choices of $\mathfrak{V}$. Namely, when $b_i \to \infty$ we get functions $\widetilde{\mathbb{F}}_\lambda(x_1, \dots, x_n \mid \mathcal{A}, \infty)$, which satisfy the interpolation properties (i)-(iii) with

$$\mathfrak{V}(\mu) = (a_1 q^{\mu_1 - \mu_2}, a_2 q^{\mu_2 - \mu_3}, \dots, a_n q^{\mu_n}).$$

We call these functions *interpolation q-Whittaker polynomials*, as their top homogeneous components coincide with the usual q-Whittaker functions. The other degeneration is obtained by setting $b_i = \infty$ and considering the following limit regime:

$$x_i = e^{\varepsilon r_i}, \quad a_i = e^{\varepsilon c_i}, \quad b_i \to \infty, \quad q = e^{\varepsilon}, \quad \varepsilon \to 0.$$

As a result we obtain symmetric polynomials $\mathbb{F}^{el}(r_1, \dots, r_n \mid \mathcal{C}, \infty)$ in $r_1, \dots, r_n$, depending on a family of parameters $\mathcal{C} = (c_1, c_2, \dots)$ and satisfying an interpolation property with

$$\mathfrak{V}(\mu) = (c_1 + \mu_1 - \mu_2, c_2 + \mu_2 - \mu_3, \dots, c_n + \mu_n).$$

Moreover, the top homogeneous component of $\mathbb{F}^{el}(r_1, \dots, r_n \mid \mathcal{C}, \infty)$ coincides with the elementary symmetric polynomial $e_{\lambda'}(r_1, \dots, r_n)$ for the conjugated partition λ', so we call this second degeneration *elementary interpolation polynomials*.

It turns out that the two degenerations above, interpolation q-Whittaker polynomials and elementary interpolation polynomials, exhaust all interpolation symmetric polynomials for $\mathfrak{V}(\mu)$ of the form

$$\mathfrak{V}(\mu) = (f_1(\mu_1 - \mu_2), f_2(\mu_2 - \mu_3), \dots, f_n(\mu_n)).$$

Note that each part depends on the difference $\mu_i - \mu_{i+1}$, unlike all previously known interpolation symmetric functions where coordinates depend on single parts μ_i. A precise result is stated in Theorem 2.5.2, below we give its brief reformulation:

Theorem (Theorem 2.5.2 in the text)**.** *Assume that $n \geq 3$ and $\mathfrak{V}(\mu)$ is a collection of interpolation points of the form*

$$\mathfrak{V}(\mu) = (f_1(\mu_1 - \mu_2), \dots, f_n(\mu_n)).$$

Then the interpolation polynomials F_λ satisfying the properties (i)-(iii) exist only if either

$$\mathfrak{V}(\mu) = (c + a_1 q^{\mu_1 - \mu_2}, c + a_2 q^{\mu_2 - \mu_3}, \dots, c + a_n q^{\mu_n})$$

for some $c, q, a_1, a_2, \dots, a_n$, or if

$$\mathfrak{V}(\mu) = (c_1 + d(\mu_1 - \mu_2), c_2 + d(\mu_2 - \mu_3), \dots, c_n + d\mu_n)$$

for some $d, c_1, c_2, \dots, c_n$. In these cases they coincide with the interpolation q-Whittaker polynomials and the elementary interpolation polynomials, respectively.

Layout of the chapter.

Section 2.2 is devoted to row operators and exchange relations, which are a convenient technical tool for working with the vertex model description of the spin q-Whittaker polynomials. In Section 2.3 we define spin q-Whittaker polynomials and prove their basic properties, the Cauchy and the dual Cauchy identities. In Section 2.4 we prove th vanishing and characterization properties for the functions $\mathbb{F}_\lambda$ and their degenerations. In Section 2.5 we discuss the general classification for interpolation symmetric polynomials, and prove the version covering interpolation q-Whittaker and elementary functions.

Notation.

In this chapter we actively use the standard notation regarding partitions: A *partition* λ is a monotone infinite sequence $(\lambda_1, \lambda_2, \lambda_3, \dots)$ of nonnegative integers satisfying

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq 0, \qquad |\lambda| := \lambda_1 + \lambda_2 + \lambda_3 + \dots < \infty.$$

The coordinates λ_i are called parts of the partition. For a partition λ, its *length* $l(\lambda)$ is equal to the number of nonzero parts λ_i, while $m_k(\lambda)$ is used to denote the number of parts λ_i equal to k. The *conjugate* λ' of a partition λ is defined by $\lambda'_i = \#\{j \geq 1 : \lambda_j \geq 1\}$. Note that $m_i(\lambda') = \lambda_i - \lambda_{i+1}$.

For a pair of partitions λ, μ, we write $\mu \subset \lambda$ iff for any $i \geq 1$ we have $\mu_i \leq \lambda_i$. Further, we say the partition λ *interlaces* the partition μ, and write $\lambda \succ \mu$, if $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for all $i \geq 1$. Note that in this case we also have $0 \leq l(\lambda) - l(\mu) \leq 1$.

Throughout the chapter we treat q as a formal variable unless stated otherwise, and we fix two infinite sequences of parameters $\mathcal{A} = (a_0, a_1, \dots)$, $\mathcal{B} = (b_0, b_1, \dots)$, which we also treat as formal variables. Let $\Bbbk = \mathbb{Q}(q, \mathcal{A}, \mathcal{B})$ denote the field of rational functions in $q, a_0, a_1, \dots$ and $b_0, b_1, \dots$. For a sequence $\mathcal{X} = (\chi_0, \chi_1, \dots)$ we set

$$\tau^n \mathcal{X} = (\chi_n, \chi_{n+1}, \dots,), \qquad \overline{\mathcal{X}} = (\chi_0^{-1}, \chi_1^{-1}, \dots).$$

2.2 Row operators and exchange relations

In this section we define row operators, which give a convenient algebraic formalism of working with rows of vertices. In particular, using Yang-Baxter equations we can deduce exchange relations between these operators, which is the main result of this section.

2.2.1 Operators $\mathbb{B}$ and $\mathbb{B}^*$.

For $N \in \mathbb{Z}_{\geq 0}$ define $V^{(N)}$ as the vector space over $\Bbbk$ with a basis consisting of the vectors $|A_1, A_2, \dots, A_N\rangle$ enumerated by N-tuples $(A_1, \dots, A_N) \in \mathbb{Z}^N_{\geq 0}$. The same basis can be equivalently enumerated by partitions μ of length at most N in the following way:

$$|\mu\rangle = |\mu_1 - \mu_2, \mu_2 - \mu_3, \dots, \mu_N\rangle;$$

we use both notations interchangeably. For $N < M$ we have an embedding $V^{(N)} \subset V^{(M)}$ which is defined by $|\mu\rangle \mapsto |\mu\rangle$ where $l(\mu) \le N < M$, or, equivalently, $|A_1, \dots, A_N\rangle \mapsto |A_1, \dots, A_N, 0, \dots, 0\rangle$. Let $\{\langle\lambda|\}_{l(\lambda)\le N} \subset {V^{(N)}}^*$ be the vectors dual to the basis $\{|\mu\rangle\}_{l(\mu)\le N}$.

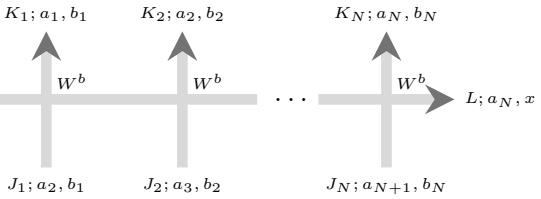
For variables x, y we define row-to-row transfer matrices $\mathbb{T}^{b,N}_{I,L}(x \mid \mathcal{A}, \mathcal{B}), \mathbb{T}^{a,N}_{K,J}(y \mid \mathcal{A}, \mathcal{B}) : V^{(N)} \to V^{(N)}$ by setting

$$\mathbb{T}^{b,N}_{I,L}(x \mid \mathcal{A}, \mathcal{B}) : |J_1, \dots, J_N\rangle \mapsto \sum_{\substack{K_1,\dots,K_N:\\ L_r := I + J_{[1,r]} - K_{[1,r]} \ge 0,\\ L_N = L}} \prod_{r=1}^{N} W^b_{a_{r+1},x,b_r}(L_{r-1}, J_r, K_r, L_r) |K_1, \dots, K_N\rangle \tag{2.2.1}$$

$$\mathbb{T}^{a,N}_{K,J}(y \mid \mathcal{A}, \mathcal{B}) : |I_1, \dots, I_N\rangle \mapsto \sum_{\substack{L_1,\dots,L_N:\\ J_r := K + L_{[1,r]} - I_{[1,r]} \ge 0,\\ J_N = J}} \prod_{r=1}^{N} (y/b_0)^{-I_r} W^a_{a_r,y,b_r}(I_r, J_r, J_{r-1}, L_r) |L_1, \dots, L_N\rangle, \tag{2.2.2}$$

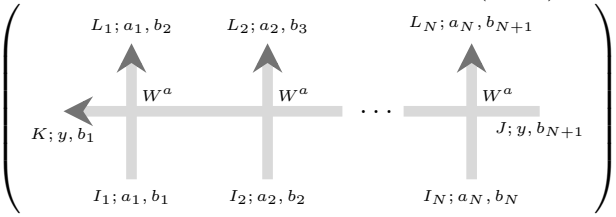
where for a sequence $(A_1, \dots, A_N)$ we set $A_{[1,0]} = 0$, $A_{[1,r]} = A_1 + \dots + A_r$ for $r \ge 1$. The same definitions can be graphically represented using the following partition functions:

$$\langle K_1, \dots, K_N| \, \mathbb{T}^{b,N}_{I,L}(x \mid \mathcal{A}, \mathcal{B}) \, |J_1, \dots, J_N\rangle =$$



(2.2.3)

$$\langle L_1, \dots, L_N| \, \mathbb{T}^{a,N}_{K,J}(y \mid \mathcal{A}, \mathcal{B}) \, |I_1, \dots, I_N\rangle = (y/b_0)^{-I_{[1,N]}}$$



Here we follow the same conventions as in Section 1.3 and in the second partition function we use the vertices of type W^a rotated by 90°. Note that for each partition function above there exists at most one configuration with nonzero weight, and L_r, J_r from (2.2.1), (2.2.2) respectively are the labels of the internal edges in these configurations.

In the following statement we summarize simple facts about the operators $\mathbb{T}^{b,N}_{I,0}$ and $\mathbb{T}^{a,N}_{K,0}$:

Proposition 2.2.1. *Let $N \in \mathbb{Z}_{\ge 0}$ and λ, μ be partitions of length at most N.*

(1) Duality: The operators $\mathbb{T}^{a,N}_{K,0}$ and $\mathbb{T}^{b,N}_{K,0}$ are dual in the following sense:

$$\langle\mu| \, \mathbb{T}^{a,N}_{K,0}(y \mid \mathcal{A}, \mathcal{B}) \, |\lambda\rangle = (a_0/y)^K \frac{(y/b_1; q)_K}{(q; q)_K} \frac{\psi_\lambda(\mathcal{A}, \mathcal{B})}{\psi_\mu(\mathcal{A}, \tau\mathcal{B})} \langle\lambda| \, \mathbb{T}^{b,N}_{K,0}(y^{-1} \mid \overline{\mathcal{B}}, \overline{\mathcal{A}}) \, |\mu\rangle,$$

where $\tau\mathcal{B}$ denotes the shifted sequence and $\psi_\lambda(\mathcal{A},\mathcal{B})$ is defined by

$$\psi_\lambda(\mathcal{A},\mathcal{B}) = \prod_{r\geq 0}(b_r/a_r)^{\lambda_{r+1}} \prod_{r\geq 1}\frac{(q;q)_{\lambda_r-\lambda_{r+1}}}{(a_r/b_r;q)_{\lambda_r-\lambda_{r+1}}}.$$

(2) Polynomiality: $\langle\lambda|\,\mathbb{T}^{b,N}_{I,L}(x\mid\mathcal{A},\mathcal{B})\,|\mu\rangle$ is a polynomial in x, and $\langle\mu|\,\mathbb{T}^{a,N}_{K,J}(y\mid\mathcal{A},\mathcal{B})\,|\lambda\rangle$ is a polynomial in y^{-1}. Moreover, both are rational functions in q regular at $q=0$.

(3) Interlacing: $\langle\lambda|\,\mathbb{T}^{b,N}_{I,0}(x\mid\mathcal{A},\mathcal{B})\,|\mu\rangle = \langle\mu|\,\mathbb{T}^{a,N}_{I,0}(y\mid\mathcal{A},\mathcal{B})\,|\lambda\rangle = 0$ unless $\lambda_1-\mu_1 = I$ and $\mu\prec\lambda$, that is,

$$\lambda_1\geq\mu_1\geq\lambda_2\geq\mu_2\geq\dots.$$

In particular, for any $M>N$ we have $\mathbb{T}^{b,M}_{I,0}(x\mid\mathcal{A},\mathcal{B})\,V^{(N)}\subset V^{(N+1)}$ and $\mathbb{T}^{a,M}_{K,0}(y\mid\mathcal{A},\mathcal{B})\,V^{(N)}\subset V^{(N)}$.

(4) Stability: For any $M>N$ we have

$$\langle\lambda|\,\mathbb{T}^{b,M}_{I,0}(x\mid\mathcal{A},\mathcal{B})\,|\mu\rangle = \langle\lambda|\,\mathbb{T}^{b,N}_{I,0}(x\mid\mathcal{A},\mathcal{B})\,|\mu\rangle,\qquad \langle\mu|\,\mathbb{T}^{a,M}_{K,0}(x\mid\mathcal{A},\mathcal{B})\,|\lambda\rangle = \langle\mu|\,\mathbb{T}^{a,N}_{K,0}(x\mid\mathcal{A},\mathcal{B})\,|\lambda\rangle.$$

Proof. Part (1) follows at once from (2.2.1), (2.2.2) and the relation

$$\begin{aligned}&W^{a}_{a_r,y,b_r}(I_r,J_r,J_{r-1},L_r)\\ &=(a_r/b_r)^{-J_r}(y/b_r)^{L_r}\frac{(q;q)_{I_r}}{(a_r/b_r;q)_{I_r}}\frac{(q;q)_{J_r}}{(y/b_{r+1};q)_{J_r}}\frac{(y/b_r;q)_{J_{r-1}}}{(q;q)_{J_{r-1}}}\frac{(a_r/b_{r+1};q)_{L_r}}{(q;q)_{L_r}}W^{b}_{b^{-1}_{r+1},y^{-1},a_r^{-1}}(J_{r-1},L_r,I_r,J_r),\end{aligned}$$

which is verified using (1.2.3) and (1.2.4). Part (2) is also immediate from (1.2.3), (1.2.4), see Remark 1.2.4.

To prove the interlacing statement from part (3) it is enough to consider $\mathbb{T}^{b,N}_{I,0}(x\mid\mathcal{A},\mathcal{B})$, the claim for $\mathbb{T}^{a,N}_{I,0}(x\mid\mathcal{A},\mathcal{B})$ then follows by part (1). Set $J_r=\mu_r-\mu_{r+1}$, $K_r=\lambda_r-\lambda_{r+1}$ and $L_r = I+J_{[1,r]}-K_{[1,r]}$. Recall from (1.2.4) that the weights $W^{b}_{a_{r+1},x,b_r}(L_{r-1},J_r,K_r,L_r)$ vanish unless $J_r\geq L_r\geq 0$. Hence for $\langle K_1,\dots,K_N|\,\mathbb{T}^{b,N}_{I,0}(x\mid\mathcal{A},\mathcal{B})\,|J_1,\dots,J_N\rangle$ to be nonzero we must have $J_r\geq L_r\geq 0$ for every r and moreover $L_N=0$. Noting that $L_r = I+\mu_1-\mu_{r+1}-\lambda_1+\lambda_{r+1}$ and in particular $L_N = I+\mu_1-\lambda_1$, we get that $\langle\lambda|\,\mathbb{T}^{b,N}_{I,0}(x\mid\mathcal{A},\mathcal{B})\,|\mu\rangle=0$ unless $I=\lambda_1-\mu_1$ and $\mu_r-\mu_{r+1}\geq I+\mu_1-\mu_{r+1}-\lambda_1+\lambda_{r+1}\geq 0$ for every r. The last inequality is equivalent to $\mu_r\geq\lambda_{r+1}\geq\mu_{r+1}$ for all $r=1,\dots,N$, and since $\lambda_1-\mu_1=I\geq 0$ the interlacing $\mu\prec\lambda$ follows.

The second statement of part (3) follows immediately from the interlacing by noticing that $\mu\prec\lambda$ implies $l(\mu)\leq l(\lambda)\leq l(\mu)+1$. Finally, part (4) follows from the fact that $W^{b}_{a,x,b}(0,0,0,0) = W^{a}_{a,y,b}(0,0,0,0)=1$, and so if $N<M$ we have

$$\langle\lambda|\,\mathbb{T}^{b,M}_{I,0}(x\mid\mathcal{A},\mathcal{B})\,|\mu\rangle = W^{b}_{a_{N+1},x,b_N}(0,0,0,0)\langle\lambda|\,\mathbb{T}^{b,M-1}_{I,0}(x\mid\mathcal{A},\mathcal{B})\,|\mu\rangle = \langle\lambda|\,\mathbb{T}^{b,M-1}_{I,0}(x\mid\mathcal{A},\mathcal{B})\,|\mu\rangle$$

and similarly for $\mathbb{T}^{a,M}_{K,0}$. □

Let $V:=\bigcup_N V^{(N)}$ be a vector space over $\Bbbk$ with the basis $\{|\mu\rangle\}_{\mu\in\mathbb{Y}}$ where we have no restrictions on the length of μ. The natural embeddings $V^{(N)}\subset V$ are given by identifying the vectors $|\mu\rangle$ in V and $V^{(N)}$ when $l(\mu)\leq N$. By parts (3) and (4) of Proposition 2.2.1 we can define operators

$\mathbb{T}^b_I(x \mid \mathcal{A},\mathcal{B}), \mathbb{T}^a_K(y \mid \mathcal{A},\mathcal{B}) : V \to V$ by setting for $v \in V^{(N)}$

$$\mathbb{T}^b_I(x \mid \mathcal{A},\mathcal{B})\ v = \mathbb{T}^{b,N+1}_{I,0}(x \mid \mathcal{A},\mathcal{B})\ v, \qquad \mathbb{T}^a_K(y \mid \mathcal{A},\mathcal{B})\ v = \mathbb{T}^{a,N}_{K,0}(y \mid \mathcal{A},\mathcal{B})\ v.$$

Using these operators we can formally define

$$\mathbb{B}(x \mid \mathcal{A},\mathcal{B}) = \sum_{r\geq 0} (x/b_0)^r \frac{(a_1 x^{-1};q)_r}{(q;q)_r} \mathbb{T}^b_r(x \mid \mathcal{A},\mathcal{B}),$$

$$\mathbb{B}^*(y \mid \mathcal{A},\mathcal{B}) = \sum_{r\geq 0} \mathbb{T}^a_r(y \mid \mathcal{A},\mathcal{B}).$$

More precisely, $\mathbb{B}^*(y \mid \mathcal{A},\mathcal{B})$ is an operator $V \to V$ such that

$$\langle\mu|\, \mathbb{B}^*(y \mid \mathcal{A},\mathcal{B})\, |\lambda\rangle = \langle\mu|\, \mathbb{T}^a_{\lambda_1-\mu_1}(y \mid \mathcal{A},\mathcal{B})\, |\lambda\rangle,$$

which is well-defined since $\langle\mu|\, \mathbb{T}^a_{\lambda_1-\mu_1}(y \mid \mathcal{A},\mathcal{B})\, |\lambda\rangle \neq 0$ only if $\mu \prec \lambda$, and there is a finite number of such μ given λ. On the other hand, $\mathbb{B}(x \mid \mathcal{A},\mathcal{B})$ is not a well-defined operator $V \to V$, since for a fixed μ we have $\mathbb{T}^b_r(x \mid \mathcal{A},\mathcal{B})\, |\mu\rangle \neq 0$ for infinitely many r in general. However, we can resolve this issue by considering q and x as formal variables: by Proposition 2.2.1 the matrix coefficients $\langle\lambda|\, \mathbb{T}^b_{\lambda_1-\mu_1}(x \mid \mathcal{A},\mathcal{B})\, |\mu\rangle$ are formal power series from $\Bbbk[[x,q]]$, while the combined degree with respect to x and q of $(x/b_0)^r \frac{(a_1x^{-1};q)_r}{(q;q)_r}$ is at least r. Hence $\mathbb{B}(x \mid \mathcal{A},\mathcal{B})$ is a well-defined operator $V[[x,q]] \to V[[x,q]]$, where $V[[x,q]]$ is the space of formal power series in x, q with coefficients in V.

Proposition 2.2.2. *For any λ, μ we have*

$$\langle\mu|\, \mathbb{B}^*(y \mid \mathcal{A},\mathcal{B})\, |\lambda\rangle = \frac{\psi_\lambda(\mathcal{A},\mathcal{B})}{\psi_\mu(\mathcal{A},\tau\mathcal{B})} \langle\lambda|\, \mathbb{B}(y^{-1} \mid \overline{\mathcal{B}},\overline{\mathcal{A}})\, |\mu\rangle,$$

where $\psi_\lambda(\mathcal{A},\mathcal{B})$ is defined by

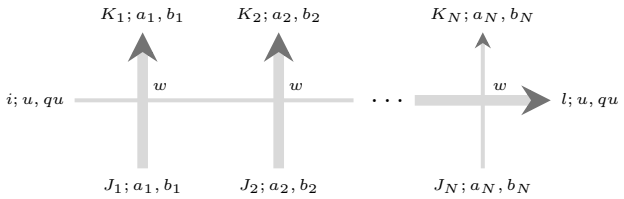
$$\psi_\lambda(\mathcal{A},\mathcal{B}) = \prod_{r\geq 0} (b_r/a_r)^{\lambda_{r+1}} \prod_{r\geq 1} \frac{(q;q)_{\lambda_r-\lambda_{r+1}}}{(a_r/b_r;q)_{\lambda_r-\lambda_{r+1}}}.$$

Proof. Follows from Proposition 2.2.1, part (1). □

2.2.2 Operators B^N and B^*

We can also construct another family of row operators using the higher spin six-vertex weights instead. Fix $N > 0$ and for integers $i, l \in \{0, 1\}$ define operators $T^N_{i,l} : V^{(N)} \to V^{(N)}$ by setting

$$\langle K_1,\dots,K_N|\, T^N_{i,l}(u \mid \mathcal{A},\mathcal{B})\, |J_1,\dots,J_N\rangle =$$



(2.2.4)

We need the following properties of these operators:

Proposition 2.2.3. *Let $N \in \mathbb{Z}_{\geq 0}$ and λ, μ be partitions of length at most N.*

(1) Duality: For any $i \in \{0,1\}$ the operators $T^N_{i,0}$ and $T^N_{1-i,1}$ are dual in the following sense:

$$\langle \mu | \, T^N_{1-i,1}(u \mid \mathcal{A}, \mathcal{B}) \, |\lambda\rangle = (a_0/b_0)^i \left(\prod_{r=1}^{N} \frac{a_r/b_r - u/b_r}{1 - u/b_r} \right) \frac{\psi_\lambda(\mathcal{A}, \mathcal{B})}{\psi_\mu(\mathcal{A}, \mathcal{B})} \langle \lambda | \, T^N_{i,0}(u^{-1} \mid \overline{\mathcal{B}}, \overline{\mathcal{A}}) \, |\mu\rangle,$$

where $\psi_\lambda(\mathcal{A}, \mathcal{B})$ is defined by

$$\psi_\lambda(\mathcal{A}, \mathcal{B}) = \prod_{r \geq 0} (b_r/a_r)^{\lambda_{r+1}} \prod_{r \geq 1} \frac{(q;q)_{\lambda_r - \lambda_{r+1}}}{(a_r/b_r; q)_{\lambda_r - \lambda_{r+1}}}.$$

(2) Interlacing: $\langle \lambda | \, T^N_{i,0}(u \mid \mathcal{A}, \mathcal{B}) \, |\mu\rangle = \langle \mu | \, T^N_{1-i,1}(u \mid \mathcal{A}, \mathcal{B}) \, |\lambda\rangle = 0$ unless $\lambda_1 - \mu_1 = i$ and $\mu' \prec \lambda'$, that is,

$$\lambda'_1 \geq \mu'_1 \geq \lambda'_2 \geq \mu'_2 \geq \dots.$$

In particular, for any $M > N$ we have $T^M_{i,1}(u \mid \mathcal{A}, \mathcal{B}) \, V^{(N)} \subset V^{(N)}$.

(3) Stability: For any $M > N$ we have

$$\langle \lambda | \, T^M_{i,0}(u \mid \mathcal{A}, \mathcal{B}) \, |\mu\rangle = \langle \lambda | \, T^N_{i,0}(x \mid \mathcal{A}, \mathcal{B}) \, |\mu\rangle,$$

$$\left(\prod_{r=1}^{M} \frac{1 - u/b_r}{a_r/b_r - u/b_r} \right) \langle \mu | \, T^M_{i,1}(x \mid \mathcal{A}, \mathcal{B}) \, |\lambda\rangle = \left(\prod_{r=1}^{N} \frac{1 - u/b_r}{a_r/b_r - u/b_r} \right) \langle \mu | \, T^N_{i,1}(x \mid \mathcal{A}, \mathcal{B}) \, |\lambda\rangle.$$

Proof. Note that the partition function on the right-hand side of (2.2.4) has only one configuration with non-zero weight and we have

$$\langle \lambda | \, T^N_{i,l}(u \mid \mathcal{A}, \mathcal{B}) \, |\mu\rangle = \prod_{r=1}^{N} w_{z;a_r,b_r}(i_{r-1}, J_r, K_r, i_r), \tag{2.2.5}$$

where we set $i_r := l + \lambda_{r+1} - \mu_{r+1}$. In particular, to have a non-zero configuration we must have $i = i_0 = \lambda_1 - \mu_1 + l$.

The duality statement follows from (2.2.5) and the following identity

$$w_{z;a,b}(i, J, K, l) = \frac{1 - a/z}{a/b - a/z} (a/b)^l \frac{(q;q)_J}{(q;q)_K} \frac{(a/b;q)_K}{(a/b;q)_J} w_{z^{-1},b^{-1},a^{-1}}(1 - i, K, J, 1 - l)$$

which can be directly verified using (1.3.4).

To show interlacing note that Proposition 1.3.2 forces $i_r \in \{0,1\}$ for every r. Hence for every r we either have $\mu_r = \lambda_r$ or $\mu_r = \lambda_r - 1$. In particular $\mu \subset \lambda$ so $\mu' \subset \lambda'$ and $\lambda'_r \geq \mu'_r$ for every r. On the other hand, if we define a partition $\tilde{\lambda}$ by setting $\tilde{\lambda}_i = \max(0, \lambda_i - 1)$, then $\tilde{\lambda} \subset \mu$ implying $\mu'_r \geq \tilde{\lambda}'_r$. But $\tilde{\lambda}'_r = \lambda'_{r+1}$ and interlacing follows.

Finally, stability follows in the same way as in Proposition 2.2.1 by using relations

$$w_{u,a,b}(0,0,0,0) = 1, \qquad \frac{1 - z/b}{a/b - u/b} w_{u,a,b}(1,0,0,1) = 1.$$

□

In view of part (3) of Proposition 2.2.3, we can define an operator $\tilde{T}_{i,1}(u \mid \mathcal{A}, \mathcal{B}) : V \to V$ by setting

$$\langle \mu | \tilde{T}_{i,1}(u \mid \mathcal{A}, \mathcal{B}) | \lambda \rangle = \left(\prod_{r=1}^{N} \frac{1 - u/b_r}{a_r/b_r - u/b_r} \right) \langle \mu | T_{i,1}^N(u \mid \mathcal{A}, \mathcal{B}) | \lambda \rangle$$

for fixed μ, λ and sufficiently large N. Note that we have to insert a normalization factor to make the definition independent of N. Similarly to the operator $\mathbb{T}^a$ before, $\tilde{T}_{i,1}$ is a well-defined operator since for a fixed λ there is only finite number of μ such that $\langle \mu | \tilde{T}_{i,1}(u \mid \mathcal{A}, \mathcal{B}) | \lambda \rangle \neq 0$.

Define operators $B^N(u \mid \mathcal{A}, \mathcal{B}) : V^{(N)} \to V^{(N)}$ and $B^*(u \mid \mathcal{A}, \mathcal{B}) : V \to V$ by setting

$$B^N(u \mid \mathcal{A}, \mathcal{B}) := T_{0,0}^N(u \mid \mathcal{A}, \mathcal{B}) - u b_0^{-1} T_{1,0}^N(u \mid \mathcal{A}, \mathcal{B}),$$

$$B^*(u \mid \mathcal{A}, \mathcal{B}) := T_{1,1}(u \mid \mathcal{A}, \mathcal{B}) - u^{-1} b_0 T_{0,1}(u \mid \mathcal{A}, \mathcal{B}).$$

We have the following duality statement:

Proposition 2.2.4. *For any λ, μ of length at most N we have*

$$\langle \mu | \, B^*(u \mid \mathcal{A}, \mathcal{B}) \, | \lambda \rangle = \frac{\psi_\lambda(\mathcal{A}, \mathcal{B})}{\psi_\mu(\mathcal{A}, \mathcal{B})} \langle \lambda | \, B^N(y^{-1} \mid \overline{\mathcal{B}}, \overline{\mathcal{A}}) \, | \mu \rangle,$$

where $\psi_\lambda(\mathcal{A}, \mathcal{B})$ is defined by

$$\psi_\lambda(\mathcal{A}, \mathcal{B}) = \prod_{r \geq 0} (b_r/a_r)^{\lambda_{r+1}} \prod_{r \geq 1} \frac{(q;q)_{\lambda_r - \lambda_{r+1}}}{(a_r/b_r; q)_{\lambda_r - \lambda_{r+1}}}.$$

Proof. Follows from part (1) of Proposition 2.2.3 by a direct computation. □

Remark 2.2.5. Remark about not defining B.

2.2.3 Exchange relations

Now we consider commutation relations between the operators $\mathbb{B}(x \mid \mathcal{A}, \mathcal{B})$, $\mathbb{B}^*(y \mid \mathcal{A}, \mathcal{B})$ and $B^*(u \mid \mathcal{A}, \mathcal{B})$ defined above. These relations are consequences of Yang-Baxter type equations, and they are proved using the so-called "zipper" argument.

Proposition 2.2.6. *We have*

$$\mathbb{B}^*(y \mid \mathcal{A}, \mathcal{B}) \mathbb{B}(x \mid \mathcal{A}, \mathcal{B}) = \frac{(a_1/y; q)_\infty (x/b_1; q)_\infty}{(a_1/b_1; q)_\infty (x/y; q)_\infty} \, \mathbb{B}(x \mid \mathcal{A}, \tau\mathcal{B}) \mathbb{B}^*(y \mid \tau\mathcal{A}, \mathcal{B}),$$

or, more precisely, the following relation holds:

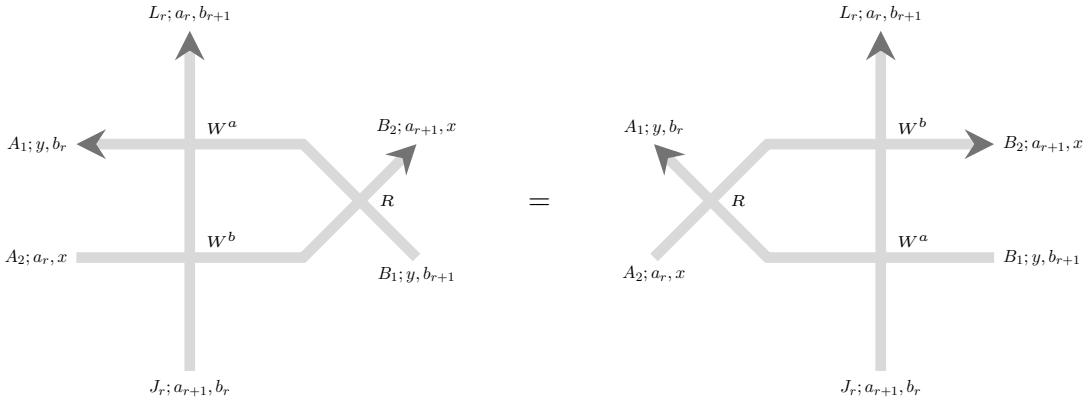
$$\sum_\lambda \langle \nu | \, \mathbb{B}^*(y \mid \mathcal{A}, \mathcal{B}) \, | \lambda \rangle \langle \lambda | \, \mathbb{B}(x \mid \mathcal{A}, \mathcal{B}) \, | \mu \rangle$$

$$= \frac{(a_1/y; q)_\infty (x/b_1; q)_\infty}{(a_1/b_1; q)_\infty (x/y; q)_\infty} \sum_\lambda \langle \nu | \, \mathbb{B}(x \mid \mathcal{A}, \tau\mathcal{B}) \, | \lambda \rangle \langle \lambda | \, \mathbb{B}^*(y \mid \tau\mathcal{A}, \mathcal{B}) \, | \mu \rangle,$$

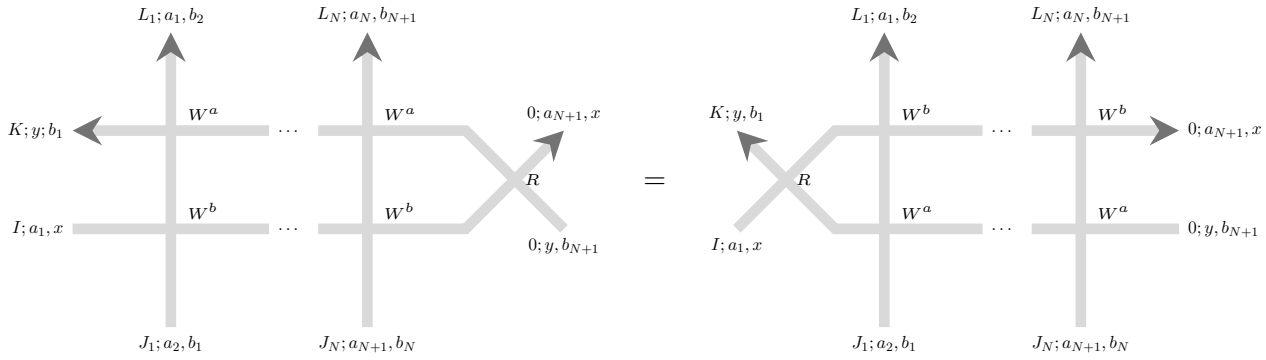
where both sides are viewed as formal power series in x, y^{-1} and q.

Proof. We prove the latter identity, fixing μ, ν from the statement and setting $L_r = \nu_r - \nu_{r+1}$, $J_r = \mu_r - \mu_{r+1}$.

Step 1: We start by consequently applying (1.3.2) to get a relation between the operators $\mathbb{T}^{b,N}_{I,0}$ and $\mathbb{T}^{a,N}_{K,0}$. Using (1.3.2) we have



where A_1, A_2, B_1, B_2 are arbitrary nonnegative integers. Choosing $N \in \mathbb{Z}_{\geq 0}$ such that $N > l(\mu), l(\nu)$ and applying these identities consequently with $r = N, N-1, \ldots, 2, 1$ we get for arbitrary $I, K \in \mathbb{Z}_{\geq 0}$



(2.2.6)

Comparing with (2.2.1), (2.2.2), the identity of partition functions above is equivalent to

$$\sum_{\tilde{I},\tilde{K}} \sum_{\lambda} (y/b_0)^{\lambda_1} \langle \nu | \, \mathbb{T}^{a,N}_{K,\tilde{K}}(y \mid \mathcal{A}, \mathcal{B}) \, |\lambda\rangle \langle \lambda | \, \mathbb{T}^{b,N}_{I,\tilde{I}}(x \mid \mathcal{A}, \mathcal{B}) \, |\mu\rangle \; R_{a_{N+1},x,y,b_{N+1}}(\tilde{I}, 0, \tilde{K}, 0)$$
$$= \sum_{J,L} \sum_{\lambda} R_{a_1,x,y,b_1}(I, J, K, L) \langle \nu | \, \mathbb{T}^{b,N}_{L,0}(x \mid \mathcal{A}, \tau\mathcal{B}) \, |\lambda\rangle (y/b_0)^{\mu_1} \langle \lambda | \, \mathbb{T}^{a,N}_{J,0}(y \mid \tau\mathcal{A}, \mathcal{B}) | \mu \rangle, \quad (2.2.7)$$

where in the left-hand side the sum is over partitions λ such that $l(\lambda) \leq l(\mu) + 1 \leq N$, with the restriction on length coming from Proposition 2.2.1, part (3).

From (2.2.1) we have

$$\langle \lambda | \, \mathbb{T}^{b,N}_{I,\tilde{I}}(x \mid \mathcal{A}, \mathcal{B}) \, |\mu\rangle = \langle \widetilde{\lambda} | \, \mathbb{T}^{b,N-1}_{I,\tilde{I}+\lambda_N-\mu_N}(x \mid \mathcal{A}, \mathcal{B}) \, |\widetilde{\mu}\rangle \; W^b_{a_{N+1},x,b_N}(\tilde{I} + \lambda_N - \mu_N, \mu_N, \lambda_N, \tilde{I}),$$

where $\widetilde{\lambda} := (\lambda_1 - \lambda_N, \dots, \lambda_{N-2} - \lambda_N, \lambda_{N-1} - \lambda_N, 0, \dots)$ and $\widetilde{\mu} := (\mu_1 - \mu_N, \dots, \mu_{N-2} - \mu_N, \mu_{N-1} - \mu_N, 0 \dots)$. But $N > l(\mu)$, so $\mu_N = 0$ and hence

$$\langle\lambda|\, \mathbb{T}^{b,N}_{I,\tilde{I}}(x \mid \mathcal{A}, \mathcal{B})\, |\mu\rangle = \langle\widetilde{\lambda}|\, \mathbb{T}^{b,N-1}_{I,\tilde{I}+\lambda_N}(x \mid \mathcal{A}, \mathcal{B})\, |\mu\rangle\ W^{b}_{a_{N+1},x,b_N}(\tilde{I} + \lambda_N, 0, \lambda_N, \tilde{I}).$$

Recall that $W^{b}_{a,x,b}(I, J, K, L)$ vanishes unless $L \leq J$, so the expression above vanishes unless $\tilde{I} = 0$; hence in the left-hand side of (2.2.7) the nonzero terms have $\tilde{I} = 0$. Since $R_{a_{N+1},x,y,b_{N+1}}(0, 0, \tilde{K}, 0) = \delta_{\tilde{K},0}$, the non-zero summands in the left-hand side of (2.2.7) must also have $\tilde{K} = 0$ and we obtain

$$\begin{aligned}
&\sum_{\lambda} (y/b_0)^{\lambda_1} \langle\nu|\, \mathbb{T}^{a,N}_{K,0}(y \mid \mathcal{A}, \mathcal{B})\, |\lambda\rangle\langle\lambda|\, \mathbb{T}^{b,N}_{I,0}(x \mid \mathcal{A}, \mathcal{B})\, |\mu\rangle \\
&\qquad = \sum_{J,L} \sum_{\lambda} R_{a_1,x,y,b_1}(I, J, K, L) \langle\nu|\, \mathbb{T}^{b,N}_{L,0}(x \mid \mathcal{A}, \tau\mathcal{B})\, |\lambda\rangle (y/b_0)^{\mu_1} \langle\lambda|\, \mathbb{T}^{a,N}_{J,0}(y \mid \tau\mathcal{A}, \mathcal{B}) |\mu\rangle. \quad (2.2.8)
\end{aligned}$$

Step 2: Now we use (2.2.8) to obtain the claim. First, using the definitions of $\mathbb{T}^{b}_{I}$ and $\mathbb{T}^{a}_{K}$, we obtain

$$\begin{aligned}
&\sum_{\lambda} b_0^{-I} \langle\nu|\, \mathbb{T}^{a}_{K}(y \mid \mathcal{A}, \mathcal{B})\, |\lambda\rangle\langle\lambda|\, \mathbb{T}^{b}_{I}(x \mid \mathcal{A}, \mathcal{B})\, |\mu\rangle \\
&\qquad\qquad = \sum_{J,L} y^{-I} R_{a_1,x,y,b_1}(I, J, K, L) \langle\nu|\, \mathbb{T}^{b}_{L}(x \mid \mathcal{A}, \tau\mathcal{B}) \mathbb{T}^{a}_{J}(y \mid \tau\mathcal{A}, \mathcal{B})\, |\mu\rangle,
\end{aligned}$$

where in the left-hand side we have used that $\lambda_1 = \mu_1 + I$ to simplify the term $(y/b_0)^{\lambda_1}$. Both sides are now elements of $\Bbbk[[q, x, y^{-1}]]$, *cf.* Remark 1.2.4 and part (2) of Proposition 2.2.1. So we can multiply both sides by $x^I \frac{(a_1/x;q)_I}{(q;q)_I}$ and take the sum over I, K such that $I + \mu_1 = K + \nu_1$, obtaining an identity in $\Bbbk[[q, x, y^{-1}]]$:

$$\begin{aligned}
&\sum_{\lambda} \langle\nu|\, \mathbb{B}^{*}(y \mid \mathcal{A}, \mathcal{B})\, |\lambda\rangle\langle\lambda|\, \mathbb{B}(x \mid \mathcal{A}, \mathcal{B})\, |\mu\rangle \\
&= \sum_{\substack{I,K,J,L: \\ I-K=L-J=\nu_1-\mu_1}} (x/y)^I \frac{(a_1/x;q)_I}{(q;q)_I} R_{a_1,x,y,b_1}(I, J, K, L) \langle\nu|\, \mathbb{T}^{b}_{L}(x \mid \mathcal{A}, \tau\mathcal{B}) \mathbb{T}^{a}_{J}(y \mid \tau\mathcal{A}, \mathcal{B})\, |\mu\rangle. \quad (2.2.9)
\end{aligned}$$

To prove the claim it is now enough to show that for fixed $J, L \geq 0$

$$\sum_{\substack{I,K\geq 0: \\ I-K=L-J}} (x/y)^I \frac{(a/x;q)_I}{(q;q)_I} R_{a,x,y,b}(I, J, K, L) = \frac{(a/y;q)_\infty (x/b;q)_\infty}{(a/b;q)_\infty (x/y;q)_\infty} (x/b)^L \frac{(a/x;q)_L}{(q;q)_L}. \quad (2.2.10)$$

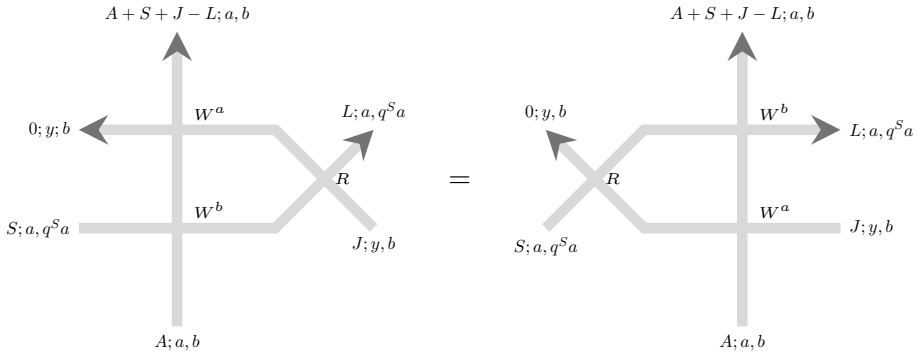
Indeed, if (2.2.10) holds we can rewrite the right-hand side of (2.2.9) as

$$\frac{(a_1/y;q)_\infty (x/b_1;q)_\infty}{(a_1/b_1;q)_\infty (x/y;q)_\infty} \sum_{\substack{J,L:\\ L-J=\nu_1-\mu_1}} (x/b_1)^L \frac{(a_1/x;q)_L}{(q;q)_L} \langle \nu | \, \mathbb{T}_L^b(x \mid \mathcal{A}, \tau\mathcal{B}) \mathbb{T}_J^a(y \mid \tau\mathcal{A}, \mathcal{B}) \, | \mu \rangle$$
$$= \frac{(a_1/y;q)_\infty (x/b_1;q)_\infty}{(a_1/b_1;q)_\infty (x/y;q)_\infty} \langle \nu | \, \mathbb{B}(x \mid \mathcal{A}, \tau\mathcal{B}) \mathbb{B}^*(y \mid \tau\mathcal{A}, \mathcal{B}) \, | \mu \rangle, \quad (2.2.11)$$

which implies the claim.

Step 3: To finish the proof we need to establish (2.2.10). Since both sides are power series in x, y^{-1}, q with coefficients in $\mathbb{C}(a,b)$, it is enough to consider the case when $x = q^S a$ for $S \in \mathbb{Z}_{\geq 0}$: if for a formal series $f(q,x) \in \mathcal{F}[[q,x]]$ we have $f(q,q^S) = 0$ for all $S \in \mathbb{Z}_{\geq 0}$ then $f(q,x) = 0$, *cf.* [80, Lemma 3.2]. So, from now on we set $x = q^S a$.

Let $A \in \mathbb{Z}_{\geq 0}$ be a sufficiently large integer. From (1.3.2) we have



(2.2.12)

Consider the right-hand side of (2.2.12). Note that for $\tilde{J}, \tilde{L} \geq 0$ we have

$$R_{a,q^S a,y,b}(S, \tilde{J}, 0, \tilde{L}) = \delta_{S+\tilde{J}=\tilde{L}} \Phi(\tilde{L}, 0; q^{-S}, q^S a/b) \Phi(S, 0; a/y, q^{-S} y/a)$$
$$= \delta_{S+\tilde{J}=\tilde{L}} (q^S a/b)^{\tilde{L}} \frac{(q^{-S};q)_{\tilde{L}}}{(a/b;q)_{\tilde{L}}} (q^{-S} y/a)^S \frac{(a/y;q)_S}{(q^{-S};q)_S},$$

where for the first equality we have used the second line of (1.2.5), noting that since $K = 0$ the only nonzero summand has $P = 0$. Since $\tilde{L} = \tilde{J} + S \geq S$ and $(q^{-\tilde{L}};q)_S$ vanishes unless $\tilde{L} \leq S$, we get

$$R_{a,q^S a,y,b}(S, \tilde{J}, 0, \tilde{L}) = \begin{cases} (y/b)^S \frac{(a/y;q)_S}{(a/b;q)_S}, & \text{if } \tilde{J} = 0, \tilde{L} = S; \\ 0, & \text{otherwise.} \end{cases}$$

Hence in the right-hand side (2.2.12) the configuration around the vertex of type R is fixed, and

the partition function is equal to

$$(y/b)^S \frac{(a/y;q)_S}{(a/b;q)_S} W^a_{a,y,b}(A,J,0,A+J) W^b_{a,q^S a,b}(S,A+J,A+J+S-L,L)$$
$$= (y/b)^S \frac{(a/y;q)_S}{(a/b;q)_S} (y/b)^A \frac{(a/y;q)_A}{(a/b;q)_A} (q^S a/b)^L \frac{(q^{-S};q)_L (q^S a/b;q)_{A+J-L}}{(a/b;q)_{A+J}} \frac{(q;q)_{A+J}}{(q;q)_L (q;q;)_{A+J-L}}.$$

For the left-hand side of (2.2.12) we can use (1.2.3),(1.2.4) to write it as

$$\sum_{I,K} (q^S a/b)^I \frac{(q^{-S};q)_I (q^S a/b;q)_{A-I}}{(a/b;q)_A} \frac{(q;q)_A}{(q;q)_I (q;q)_{A-I}} (y/b)^{A+S-I} \frac{(a/y;q)_{A+S-I}}{(a/b;q)_{A+S-I}} R_{a,q^S a,y,b}(I,J,K,L).$$

Note that in the summation above we can assume $I \le S$, since $(q^{-S};q)_I$ vanishes otherwise. In particular, the number of terms is bounded by S regardless of the value of A.

To finish the proof we need to consider the dependence on A. Namely, we rewrite (2.2.12) in a way that both sides are rational in q^A: using the identity

$$\frac{(u;q)_{A+X}}{(u;q)_{A+Y}} = (uq^{A+Y};q)_{X-Y}$$

and the expressions above for the both sides we readily obtain

$$\sum_{\substack{I \le S \\ K = I+J-L}} (q^S a/y)^I \frac{(q^{-S};q)_I}{(q;q)_I} \frac{(q^{A-I+1};q)_I}{(q^{A+J-L+1};q)_L} \frac{(aq^A/y;q)_{S-I}}{(aq^{A+J}/b;q)_{S-L}} R_{a,q^S a,y,b}(I,J,K,L)$$
$$= \frac{(a/y;q)_S}{(a/b;q)_S} (q^S a/b)^L \frac{(q^{-S};q)_L}{(q;q)_L}. \quad (2.2.13)$$

Note that the both sides of (2.2.13) are rational functions in q^A, which are equal when A is a sufficiently large integer. Hence (2.2.13) holds for any value of q^A, and in particular we can set $q^A = 0$ getting

$$\sum_{\substack{I \le S \\ K = I+J-L}} (q^S a/y)^I \frac{(q^{-S};q)_I}{(q;q)_I} R_{a,q^S a,y,b}(I,J,K,L) = \frac{(a/y;q)_S}{(a/b;q)_S} (q^S a/b)^L \frac{(q^{-S};q)_L}{(q;q)_L},$$

which is exactly (2.2.10) when $x = q^S a$. □

Remark 2.2.7. One can check that (2.2.10) from the proof above is equivalent to the q-Gauss identity

$$\sum_{k \ge 0} \Big(\frac{c}{ab}\Big)^k \frac{(a;q)_k (b;q)_k}{(c;q)_k (q;q)_k} = \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/(ab);q)_\infty}.$$

For the remaining exchange relations the following statement will be useful:

Lemma 2.2.8. *Let $J \in \mathbb{Z}, L \in \mathbb{Z}_{\ge 0}$ be fixed integers. Then there exists a formal power series $f(q,t) \in \Bbbk[[q,t]]$ such that for sufficiently large A we have $W^b_{a_2,b_1,b_2}(0, A+J, A+J-L, L) = f(q, q^A)$.*

Moreover

$$W^b_{a_2,b_1,b_2}(0,A+J,A+J-L,L)\Big|_{q^A=0} := f(q,0) = (b_1/b_2)^L \frac{(a_2/b_1;q)_L}{(q;q)_L}\frac{(b_1/b_2;q)_\infty}{(a_2/b_1;q)_\infty}.$$

Similarly, for fixed $J\in\mathbb{Z}, l\in\{0,1\}$ the weight $w_{z;a,b}(0,A+J,A+J-l,l)$ depends rationally on q^A and we have

$$w_{z;a,b}(0,A+J,A+J-l,l)\big|_{q^A=0} = \frac{(-z/b)^l}{1-z/b}.$$

Proof. For the first part we can rewrite the weights W^b as

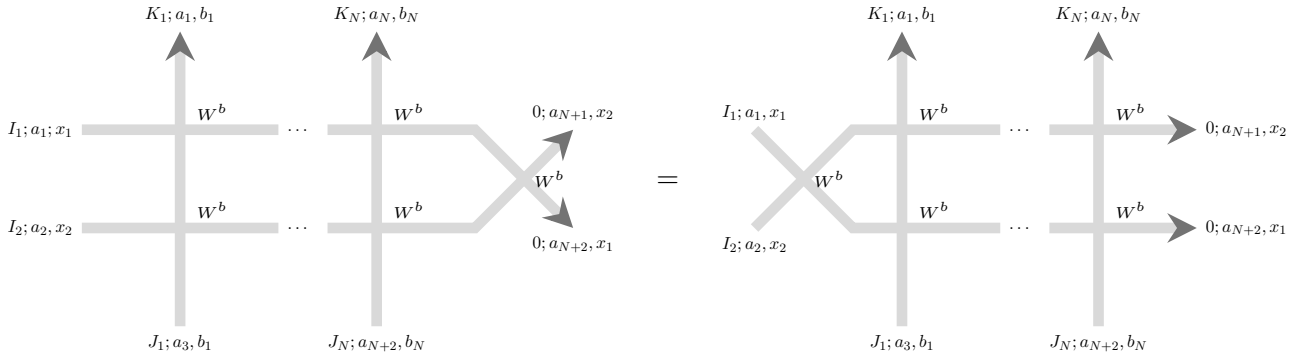
$$W^b_{a_2,b_1,b_2}(0,A+J,A+J-L,L) = (b_1/b_2)^L \frac{(a_2/b_1;q)_L (b_1/b_2;q)_\infty (q^{A+J}a_2/b_1;q)_\infty}{(q^{A+J-L}b_1/b_2;q)_\infty (a_2/b_1;q)_\infty}\frac{(q^{A+J-L+1};q)_L}{(q;q)_L}$$

and by replacing q^A in the expression above with t we get the desired power series. The claim about the weights $w_{z;a,b}$ follows directly from the explicit expressions (1.3.4) □

Proposition 2.2.9. *The following relation holds*

$$\mathbb{B}(x_1 \mid \mathcal{A},\mathcal{B})\mathbb{B}(x_2 \mid \tau\mathcal{A},\mathcal{B}) = \mathbb{B}(x_2 \mid \mathcal{A},\mathcal{B})\mathbb{B}(x_1 \mid \tau\mathcal{A},\mathcal{B})$$

Proof. The proof is similar to Proposition 2.2.6 above. First, we use (1.3.1) to get the relation



(2.2.14)

Note that the right-most vertex in the left-hand side has weight of the form $W^b_{a_{N+2},x_1,x_2}(L_1,L_2,0,0)$, which is non-zero only in the case $W^b_{a_{N+2},x_1,x_2}(0,0,0,0)=1$. Hence we get

$$\mathbb{T}^{b,N}_{I_1,0}(x_1 \mid \mathcal{A},\mathcal{B})\mathbb{T}^{b,N}_{I_2,0}(x_2 \mid \tau\mathcal{A},\mathcal{B}) = \sum_{L_1,L_2} W^b_{a_2,x_1,x_2}(I_1,I_2,L_1,L_2)\mathbb{T}^{b,N}_{L_1,0}(x_2 \mid \mathcal{A},\mathcal{B})\mathbb{T}^{b,N}_{L_2,0}(x_1 \mid \tau\mathcal{A},\mathcal{B}),$$

and by taking arbitrarily large N this implies

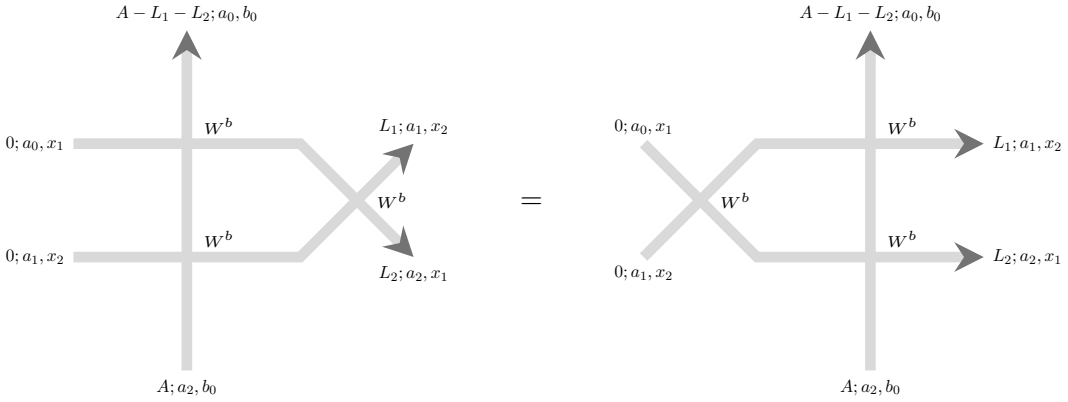
$$\mathbb{T}_{I_1}(x_1 \mid \mathcal{A},\mathcal{B})\mathbb{T}_{I_2}(x_2 \mid \tau\mathcal{A},\mathcal{B}) = \sum_{L_1,L_2} W^b_{a_2,x_1,x_2}(I_1,I_2,L_1,L_2)\mathbb{T}_{L_1}(x_2 \mid \mathcal{A},\mathcal{B})\mathbb{T}_{L_2}(x_1 \mid \tau\mathcal{A},\mathcal{B}).$$

After multiplying both sides by $(x_1/b_0)^{I_1}(x_2/b_0)^{I_2}\frac{(a_1/x_1;q)_{I_1}(a_2/x_2;q)_{I_2}}{(q;q)_{I_1}(q;q)_{I_2}}$ and taking the sum over I_1, I_2,

we see that to finish the proof it is enough to show

$$\sum_{I_1,I_2}(x_1/b_0)^{I_1}(x_2/b_0)^{I_2}\frac{(a_1/x_1;q)_{I_1}(a_2/x_2;q)_{I_2}}{(q;q)_{I_1}(q;q)_{I_2}}W^b_{a_2,x_1,x_2}(I_1,I_2,L_1,L_2)$$
$$=(x_2/b_0)^{L_1}\frac{(a_1/x_2;q)_{L_2}}{(q;q)_{L_1}}(x_1/b_0)^{L_2}\frac{(a_2/x_1;q)_{L_2}}{(q;q)_{L_2}}.$$

To prove the last relation consider the following special case of (1.3.1):



Then the proof is finished by applying Lemma 2.2.8 to both sides and setting $q^A=0$. □

Proposition 2.2.10. *The following relation holds:*

$$B^*(u\mid\mathcal{A},\mathcal{B})\mathbb{B}(x\mid\mathcal{A},\mathcal{B})=\frac{1-x/u}{1-a_1/u}\mathbb{B}(x\mid\mathcal{A},\mathcal{B})B^*(u\mid\tau\mathcal{A},\mathcal{B}).$$

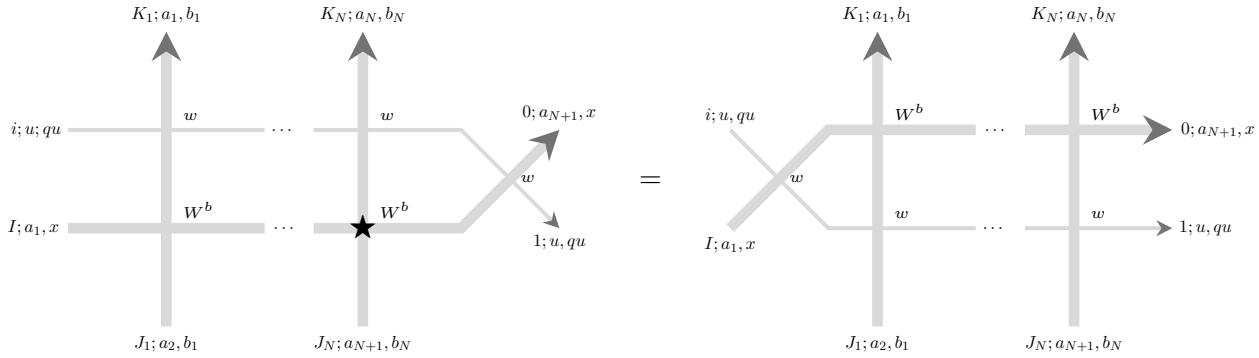
More precisely, for any μ,ν we have

$$\sum_\lambda\langle\nu|B^*(u\mid\mathcal{A},\mathcal{B})|\lambda\rangle\langle\lambda|\mathbb{B}(x\mid\mathcal{A},\mathcal{B})|\mu\rangle=\frac{1-x/u}{1-a_1/u}\sum_\lambda\langle\nu|\mathbb{B}(x\mid\mathcal{A},\mathcal{B})|\lambda\rangle\langle\lambda|B^*(u\mid\tau\mathcal{A},\mathcal{B})|\mu\rangle.$$

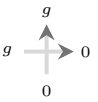
and moreover the sums on both sides have only finite number of non-zero terms.

Proof. We are going to prove the latter identity. Fix μ,ν and let $N>\max(l(\mu),l(\nu)))$. Set

$J_r = \mu_r - \mu_{r+1}$ and $K_r = \nu_r - \nu_{r+1}$. Repeatedly applying (1.3.6) we have



(2.2.15)

Since $N > l(\mu)$ we have $J_N = 0$. Since the weights $W^b_{a_2,b_1,b_2}(I,J,K,L)$ vanish unless $J \geq L$, the $\star$-vertex on the left-hand side of (2.2.6) is forced to have configuration of the form  . Hence, the weight of the right-most vertex on the left-hand side must be $w_{u;a_{N+1},x}(1,0,0,1) = \frac{a_{N+1}/x - u/x}{1-u/x}$ and the left-hand side of (2.2.6) is equal to

$$\frac{a_{N+1}/x - u/x}{1-u/x}\langle\nu|T^N_{i,1}(u \mid \mathcal{A},\mathcal{B})\mathbb{T}^{b,N}_{I,0}(x \mid \mathcal{A},\mathcal{B})|\mu\rangle = \frac{a_{N+1}-u}{x-u}\prod_{r=1}^{N}\frac{a_r - u}{b_r - u}\langle\nu|\tilde{T}_{i,1}(u \mid \mathcal{A},\mathcal{B})\mathbb{T}^b_I(x \mid \mathcal{A},\mathcal{B})|\mu\rangle$$

On the other hand, the right-hand side of (2.2.6) is equal to

$$\sum_{K,l} w_{u,a_1,x}(i,I,K,l)\langle\nu|\mathbb{T}^{b,N}_{K,0}(x \mid \mathcal{A},\mathcal{B})T^N_{l,1}(u \mid \tau\mathcal{A},\mathcal{B})|\mu\rangle,$$

and replacing T by $\tilde{T}$ results in the following expression for the right-hand side:

$$\prod_{r=1}^{N}\frac{a_{r+1}-u}{b_r-u}\sum_{K,l} w_{u,a_1,x}(i,I,K,l)\langle\nu|\mathbb{T}^b_{K,0}(x \mid \mathcal{A},\mathcal{B})\tilde{T}_{l,1}(u \mid \tau\mathcal{A},\mathcal{B})|\mu\rangle.$$

Hence (2.2.6) is equivalent to

$$\frac{a_1-u}{x-u}\langle\nu|\tilde{T}_{i,1}(u \mid \mathcal{A},\mathcal{B})\mathbb{T}^b_I(x \mid \mathcal{A},\mathcal{B})|\mu\rangle = \sum_{K,l} w_{u,a_1,x}(i,I,K,l)\langle\nu|\mathbb{T}^b_{K,0}(x \mid \mathcal{A},\mathcal{B})\tilde{T}_{l,1}(u \mid \tau\mathcal{A},\mathcal{B})|\mu\rangle. \tag{2.2.16}$$

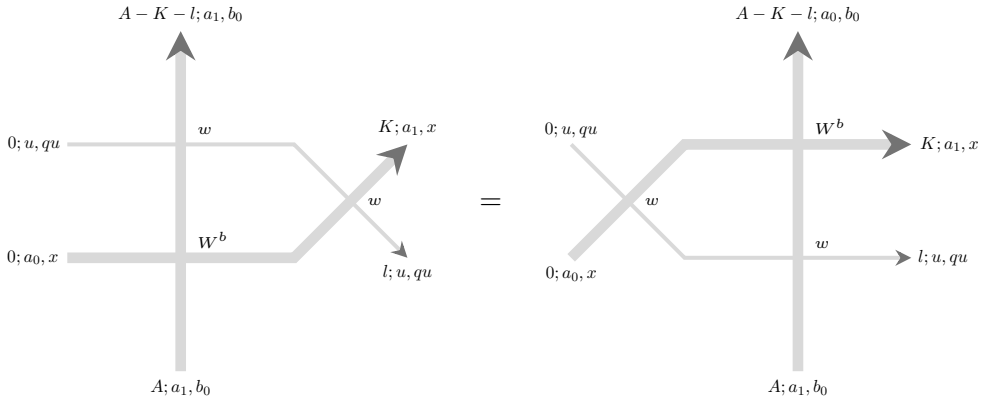
Now we multiply both sides by $(-u/b_0)^{i-1}(x/b_0)^I\frac{(a_1/x;q)_I}{(q;q)_I}$ and take a sum over i, I. Note that this sum has only finite number of non-zero terms: $\langle\nu|\tilde{T}_{i,1}(u \mid \mathcal{A},\mathcal{B})|\lambda\rangle\langle\lambda|\mathbb{T}^b_I(x \mid \mathcal{A},\mathcal{B})|\mu\rangle$ vanishes unless $\mu_1 + I = \lambda_1 = \nu_1 + 1 - i$, hence non-zero summands occur only when $I = \nu_1 - \mu_1$ or $I = \nu_1 - \mu_1 + 1$. After taking the sum, the left-hand side of (2.2.16) becomes

$$\frac{a_1-u}{x-u}\langle\nu|B^*(u \mid \mathcal{A},\mathcal{B})\mathbb{B}(x \mid \mathcal{A},\mathcal{B})|\mu\rangle$$

while the right-hand side becomes

$$\sum_{i,I,K,l} (-u/b_0)^{i-1}(x/b_0)^I \frac{(a_1/x;q)_I}{(q;q)_I} w_{u,a_1,x}(i,I,K,l)\langle\nu|\mathbb{T}^b_{K,0}(x \mid \mathcal{A},\mathcal{B})\tilde{T}_{l,1}(u \mid \tau\mathcal{A},\mathcal{B})|\mu\rangle.$$

To simplify the last expression consider the following specialization of (1.3.6):



Then for sufficiently large A we can apply Lemma 2.2.8 to both sides and by setting $q^A = 0$ we get

$$\sum_{i,I}(-u/b_0)^i(x/b_0)^I \frac{(a_1/x;q)_I}{(q;q)_I} w_{u,a_1,x}(i,I,K,l) = (-u/b_0)^l(x/b_0)^K \frac{(a_1/x;q)_K}{(q;q)_K}.$$

Hence

$$\frac{a_1-u}{x-u}\langle\nu|B^*(u \mid \mathcal{A},\mathcal{B})\mathbb{B}(x \mid \mathcal{A},\mathcal{B})|\mu\rangle = \langle\nu|\mathbb{B}(x \mid \mathcal{A},\mathcal{B})B^*(u \mid \tau\mathcal{A},\mathcal{B})|\mu\rangle, \tag{2.2.17}$$

which is equivalent to the identity from the claim.

□

2.3 Inhomogeneous spin q-Whittaker polynomials

In this section we use the partition function of a vertex model with W^b-weights to construct inhomogeneous spin q-Whittaker polynomials $\mathbb{F}_{\lambda/\mu}$ and to prove several of their properties. We continue to use $\mathcal{A} = (a_0, a_1, \dots)$ and $\mathcal{B} = (b_0, b_1, \dots)$ to denote sequences of parameters and we continue to use the notation $\tau\mathcal{A}, \overline{\mathcal{A}}$ from Section 2.2.

2.3.1 Vertex model construction.

For a pair of partitions λ, μ and a collection of variables $x_1, \dots, x_n$ the *inhomogeneous spin q-Whittaker polynomial* $\mathbb{F}_{\lambda/\mu}(x_1, \dots, x_n \mid \mathcal{A},\mathcal{B})$ is defined in terms of the operators $\mathbb{B}(x \mid \mathcal{A},\mathcal{B})$ from Section 2.2 by

$$\mathbb{F}_{\lambda/\mu}(x_1, \dots, x_n \mid \mathcal{A},\mathcal{B}) = \langle\lambda|\, \mathbb{B}(x_1 \mid \mathcal{A},\mathcal{B})\mathbb{B}(x_2 \mid \tau\mathcal{A},\mathcal{B}) \dots \mathbb{B}(x_n \mid \tau^{n-1}\mathcal{A},\mathcal{B})\, |\mu\rangle. \tag{2.3.1}$$

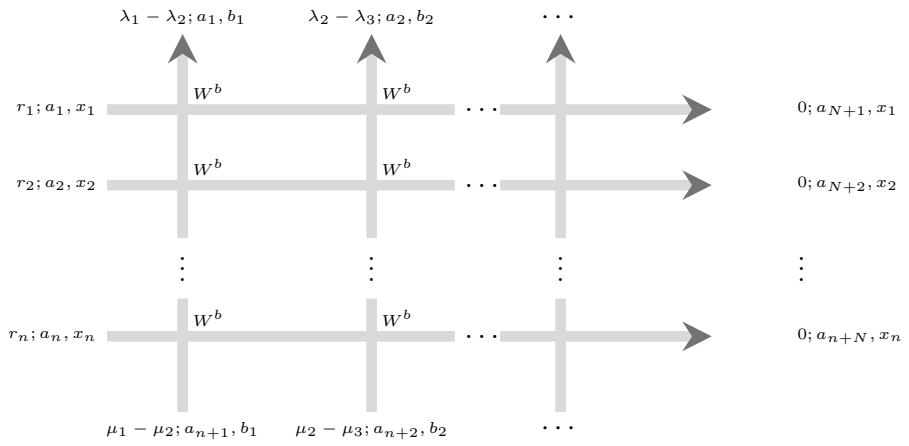


Figure 2-1: The partition function $Z^{r_1,\dots,r_n}_{\lambda/\mu}$ used to compute $\mathbb{F}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})$.

When $\mu=\varnothing$ we write $\mathbb{F}_\lambda$ instead of $\mathbb{F}_{\lambda/\varnothing}$.[1]

Recalling the definitions of the operators $\mathbb{B}(x \mid \mathcal{A},\mathcal{B})$, we can also give a more direct construction in terms of vertex models. For a pair of partitions λ,μ and an integer N with $\max(l(\lambda),l(\mu))<N$ consider a grid of vertices consisting of n rows and N columns. We number the rows from top to bottom, while the columns are numbered from left to right, starting from 1 in both cases. Define the boundary conditions and the edge parameters by setting

- all vertices are of type W^b;
- edge parameters of the ith edge on the left boundary are a_i, x_i, and the corresponding boundary condition is an arbitrary fixed integer $r_i \in \mathbb{Z}_{\geq 0}$;
- the right-most edge in the ith row has edge parameters a_{i+N}, x_i and the boundary condition 0;
- the bottom edge in the jth column has edge parameters a_{n+j}, b_j and the boundary condition $\mu_j-\mu_{j+1}$;
- the top edge in the jth column has edge parameters a_j, b_j and the boundary condition $\lambda_j-\lambda_{j+1}$.

The resulting partition function is denoted by $Z^{(r_1,\dots,r_n)}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})$ and is depicted in Figure 2-1. Using these partition functions, we have the following alternative definition:

$$\mathbb{F}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B}) = \sum_{r_1,\dots,r_n\geq 0}\prod_{i=1}^{n}(x_i/b_0)^{r_i}\frac{(a_i x_i^{-1};q)_{r_i}}{(q;q)_{r_i}} Z^{r_1,\dots,r_n}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B}).$$

Note that this definition does not depend on N since all the vertices to the right of the $l(\lambda)th$ column are forced to have weight 1, see part (4) of Proposition 2.2.1.

[1] Throughout the text we freely use λ instead of $\lambda/\varnothing$ in similar contexts; this should not cause any confusion.

Remark 2.3.1. Spin q-Whittaker polynomials were originally introduced in [21], and later a different but related version was constructed in [65]. The inhomogeneous spin q-Whittaker polynomials $\mathbb{F}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})$ generalize both these versions: when $a_1 = a_2 = \cdots = s$ and $b_1 = b_2 = \cdots = s^{-1}$ the function $\mathbb{F}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})$ is, up to a scalar, the spin q-Whittaker polynomial from [21], while setting $b_1 = b_2 = \cdots = s^{-1}$, $a_{n+l(\mu)} = 0$ and $a_i = s$ for all $i \neq n + l(\mu)$ reduces $\mathbb{F}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})$ to the version from [65].

2.3.2 Main properties

Now we can use the properties of row operators to prove various properties of $\mathbb{F}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})$.

Proposition 2.3.2. *We have the following expression for the single-variable function* $\mathbb{F}_{\lambda/\mu}(x \mid \mathcal{A},\mathcal{B})$*:*

$$\mathbb{F}_{\lambda/\mu}(x \mid \mathcal{A},\mathcal{B}) = \begin{cases} x^{|\lambda|-|\mu|} \prod_{r\geq 1} b_{r-1}^{\mu_r-\lambda_r} \dfrac{(a_r/x;q)_{\lambda_r-\mu_r}(x/b_r;q)_{\mu_r-\lambda_{r+1}}(q;q)_{\mu_r-\mu_{r+1}}}{(q;q)_{\lambda_r-\mu_r}(q;q)_{\mu_r-\lambda_{r+1}}(a_{r+1}/b_r;q)_{\mu_r-\mu_{r+1}}}, & \text{if } \mu \prec \lambda; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The vanishing part follows from Proposition 2.2.1, part (3). For the explicit expression we use (1.2.4), (2.2.1) and the relation

$$\langle\lambda|\, \mathbb{B}(x \mid \mathcal{A},\mathcal{B})\, |\mu\rangle = (x/b_0)^{\lambda_1-\mu_1} \frac{(a_1/x;q)_{\lambda_1-\mu_1}}{(q;q)_{\lambda_1-\mu_1}} \langle\lambda|\, \mathbb{T}^b_{\lambda_1-\mu_1}(x \mid \mathcal{A},\mathcal{B})\, |\mu\rangle.$$

□

Proposition 2.3.3 (Branching rule)**.** *The following identity holds for any* $m < n$*:*

$$\mathbb{F}_{\lambda/\mu}(x_1,x_2,\dots,x_n \mid \mathcal{A},\mathcal{B}) = \sum_{\nu} \mathbb{F}_{\lambda/\nu}(x_1\dots,x_m \mid \mathcal{A},\mathcal{B}) \mathbb{F}_{\nu/\mu}(x_{m+1},\dots,x_n \mid \tau^m\mathcal{A},\mathcal{B}).$$

Proof. Follows from (2.3.1) by an insertion of the summation $\sum_\nu |\nu\rangle\langle\nu| = id$:

$$\begin{aligned} \mathbb{F}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B}) &= \langle\lambda|\mathbb{B}(x_1 \mid \mathcal{A},\mathcal{B})\dots\mathbb{B}(x_n \mid \tau^{n-1}\mathcal{A},\mathcal{B})|\mu\rangle \\ &= \sum_\nu \langle\lambda|\mathbb{B}(x_1 \mid \mathcal{A},\mathcal{B})\dots\mathbb{B}(x_m \mid \tau^{m-1}\mathcal{A},\mathcal{B})|\nu\rangle\langle\nu|\mathbb{B}(x_{m+1} \mid \tau^m\mathcal{A},\mathcal{B})\dots\mathbb{B}(x_n \mid \tau^{n-1}\mathcal{A},\mathcal{B})|\mu\rangle \\ &= \sum_\nu \mathbb{F}_{\lambda/\nu}(x_1 \mid \mathcal{A},\mathcal{B}) \mathbb{F}_{\nu/\mu}(x_2,\dots,x_n \mid \tau\mathcal{A},\mathcal{B}). \end{aligned}$$

□

Corollary 2.3.4. *For a pair of partitions* λ,μ *the function* $\mathbb{F}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})$ *vanishes unless for any* $r \geq 1$ *we have*

$$\mu'_r \leq \lambda'_r \leq \mu'_r + n.$$

In particular,

$$\mathbb{F}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B}) = 0 \qquad \text{unless } \mu \subset \lambda \text{ and } l(\lambda) \leq l(\mu) + n.$$

Proof. We use the following elementary observation: if partitions μ and λ interlace

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \dots,$$

then for any $r \geq 1$ we have

$$\mu'_r \leq \lambda'_r = \#\{i : \lambda_i \geq r\} \leq \#\{i : \mu_i \geq r\} + 1 = \mu'_r + 1.$$

Now the claim follows from Propositions 2.3.2 and 2.3.3, since it implies that the function $\mathbb{F}_{\lambda/\mu}(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B})$ vanishes unless there exists a sequence of interlacing partitions

$$\mu = \nu^{(0)} \prec \nu^{(1)} \prec \dots \prec \nu^{(n)} = \lambda. \qquad \square$$

Theorem 2.3.5. *The functions $\mathbb{F}_{\lambda/\mu}(x_1, x_2, \dots, x_n \mid \mathcal{A}, \mathcal{B})$ are symmetric polynomials in $x_1, \dots, x_n$.*

Proof. $\mathbb{F}_{\lambda/\mu}(x_1, x_2, \dots, x_n \mid \mathcal{A}, \mathcal{B})$ is a polynomial since $\langle\lambda|\, \mathbb{B}(x \mid \mathcal{A}, \mathcal{B})\, |\mu\rangle$ is a polynomial in x for any λ, μ, and $\mathbb{F}_{\lambda/\mu}(x_1, x_2, \dots, x_n \mid \mathcal{A}, \mathcal{B})$ is a finite sum of products of such matrix elements thanks to Proposition 2.3.3. The symmetry in $x_1, \dots, x_n$ follows from the exchange relation in Proposition 2.2.9. $\square$

To formulate the Cauchy identity we need dual functions $\mathbb{F}^*(y_1, \dots, y_m \mid \mathcal{A}, \mathcal{B})$ defined by

$$\mathbb{F}^*_{\lambda/\mu}(y_1, \dots, y_m \mid \mathcal{A}, \mathcal{B}) = \frac{\psi_\lambda(\mathcal{A}, \mathcal{B})}{\psi_\mu(\tau^m \mathcal{A}, \mathcal{B})} \mathbb{F}_{\lambda/\mu}(y_1, \dots, y_m \mid \mathcal{A}, \mathcal{B}),$$

where $\psi_\lambda(\mathcal{A}, \mathcal{B})$ is defined in Proposition 2.2.2:

$$\psi_\lambda(\mathcal{A}, \mathcal{B}) = \prod_{r \geq 0} (b_r/a_r)^{\lambda_{r+1}} \prod_{r \geq 1} \frac{(q;q)_{\lambda_r - \lambda_{r+1}}}{(a_r/b_r; q)_{\lambda_r - \lambda_{r+1}}}.$$

Equivalently, the dual functions can be defined using the operators $\mathbb{B}^*(y \mid \mathcal{A}, \mathcal{B})$ from Section 2.2:

Proposition 2.3.6. *We have*

$$\mathbb{F}^*_{\lambda/\mu}(y_1, \dots, y_m \mid \overline{\mathcal{B}}, \overline{\mathcal{A}}) = \langle\mu|\, \mathbb{B}^*(y_m^{-1} \mid \mathcal{A}, \tau^{m-1}\mathcal{B}) \dots \mathbb{B}^*(y_2^{-1} \mid \mathcal{A}, \tau\mathcal{B}) \mathbb{B}^*(y_1^{-1} \mid \mathcal{A}, \mathcal{B})\, |\lambda\rangle.$$

Proof. Follows immediately from Proposition 2.2.2. Note that $\psi_\lambda(\mathcal{A}, \mathcal{B}) = \psi_\lambda(\overline{\mathcal{B}}, \overline{\mathcal{A}})$. $\square$

Theorem 2.3.7. *For any partitions μ, ν the following equality of formal power series in x_i, y_j, q holds:*

$$\sum_\lambda \mathbb{F}_{\lambda/\mu}(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B}) \mathbb{F}^*_{\lambda/\nu}(y_1, \dots, y_m \mid \overline{\mathcal{B}}, \overline{\mathcal{A}})$$
$$= \prod_{i=1}^n \prod_{j=1}^m \frac{(a_i y_j; q)_\infty (x_i/b_j; q)_\infty}{(x_i y_j; q)_\infty (a_i/b_j; q)_\infty} \sum_\lambda \mathbb{F}^*_{\mu/\lambda}(y_1, \dots, y_m \mid \overline{\mathcal{B}}, \tau^n \overline{\mathcal{A}}) \mathbb{F}_{\nu/\lambda}(x_1, \dots, x_n \mid \mathcal{A}, \tau^m \mathcal{B}).$$

In particular, setting $\mu = \nu = \varnothing$ yields

$$\sum_{\lambda} \mathbb{F}_{\lambda}(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B}) \mathbb{F}^*_{\lambda}(y_1, \dots, y_m \mid \overline{\mathcal{B}}, \overline{\mathcal{A}}) = \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{(a_i y_j; q)_\infty (x_i / b_j; q)_\infty}{(x_i y_j; q)_\infty (a_i / b_j; q)_\infty}. \tag{2.3.2}$$

Proof. This follows from Proposition 2.2.6 in the following way: set

$$\mathbb{B}^{(n)}(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B}) := \mathbb{B}(x_1 \mid \mathcal{A}, \mathcal{B}) \mathbb{B}(x_2 \mid \tau \mathcal{A}, \mathcal{B}) \dots \mathbb{B}(x_n \mid \tau^{n-1} \mathcal{A}, \mathcal{B}).$$

Repeatedly using the exchange relation from Proposition 2.2.6, shifting $\mathcal{A}$ on each step, we have

$$\mathbb{B}^*(y \mid \mathcal{A}, \mathcal{B}) \mathbb{B}^{(n)}(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B}) = \prod_{i=1}^{n} \frac{(a_i / y; q)_\infty (x_i / b_1; q)_\infty}{(x_i / y; q)_\infty (a_i / b_1; q)_\infty} \mathbb{B}^{(n)}(x_1, \dots, x_n \mid \mathcal{A}, \tau \mathcal{B}) \mathbb{B}^*(y \mid \tau^n \mathcal{A}, \mathcal{B}).$$

Iterating the identity above m times, substituting $y = y_j^{-1}$ for $j = 1, \dots, m$, we get

$$\begin{aligned} &\mathbb{B}^*(y_m^{-1} \mid \mathcal{A}, \tau^{m-1} \mathcal{B}) \dots \mathbb{B}^*(y_1^{-1} \mid \mathcal{A}, \mathcal{B}) \mathbb{B}^{(n)}(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B}) \\ &= \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{(a_i y_j; q)_\infty (x_i / b_j; q)_\infty}{(x_i y_j; q)_\infty (a_i / b_j; q)_\infty} \mathbb{B}^{(n)}(x_1, \dots, x_n \mid \mathcal{A}, \tau^m \mathcal{B}) \mathbb{B}^*(y_m^{-1} \mid \tau^n \mathcal{A}, \tau^{m-1} \mathcal{B}) \dots \mathbb{B}^*(y_1^{-1} \mid \tau^n \mathcal{A}, \mathcal{B}). \end{aligned}$$

Evaluating both sides of the last identity at $\langle \nu | \cdot | \mu \rangle$ yields the claim. □

Remark 2.3.8. The convergence from Theorem 2.3.7 also holds numerically if

$$0 < \max_{i,j}(|x_i y_j|) < 1, \qquad 0 < |q| < 1.$$

For the right-hand side of (2.3.2) the convergence is immediate, while for the left-hand side this can be verified as follows: Using Proposition 2.3.2, the one variable functions $\mathbb{F}_{\lambda/\mu}(x_i \mid \mathcal{A}, \mathcal{B})$ can be bounded by $C|x_i|^{|\lambda|-|\mu|} \prod_{i \geq 1} b_{i-1}^{\mu_i - \lambda_i}$, where C is a positive constant not depending on λ, μ (to reach this bound it is enough to bound the finite number of possibly appearing q-Pochhammer symbols $(x, q)_a$ by a uniform constants from above and below). Then, by the branching rule of Proposition 2.3.3, we obtain an upper bound

$$|\mathbb{F}_{\lambda/\mu}(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B})| \leq C|\lambda|^d \max_i (|x_i|)^{|\lambda|} \prod_{i \geq 1} b_{i-1}^{\mu_i - \lambda_i},$$

for some $C, d > 0$ not depending on λ. This bound is sufficient for proving the convergence of the left-hand side of (2.3.2).

2.3.3 Spin Hall-Littlewood functions and dual Cauchy identity.

It is also useful to consider another family of functions that straightforwardly generalizes both inhomogeneous spin Hall-Littlewood functions from [20] and stable Hall-Littlewood functions from [44] and [21].

Let $u_1, \dots, u_n$ be a collection of variables, λ, μ be a pair of partitions and choose an integer $N \geq \max(l(\lambda), l(\mu))$. Define *inhomogeneous stable spin Hall-Littlewood functions* $\widetilde{\mathsf{F}}_{\lambda'/\mu'}(u_1, \dots, u_n \mid$

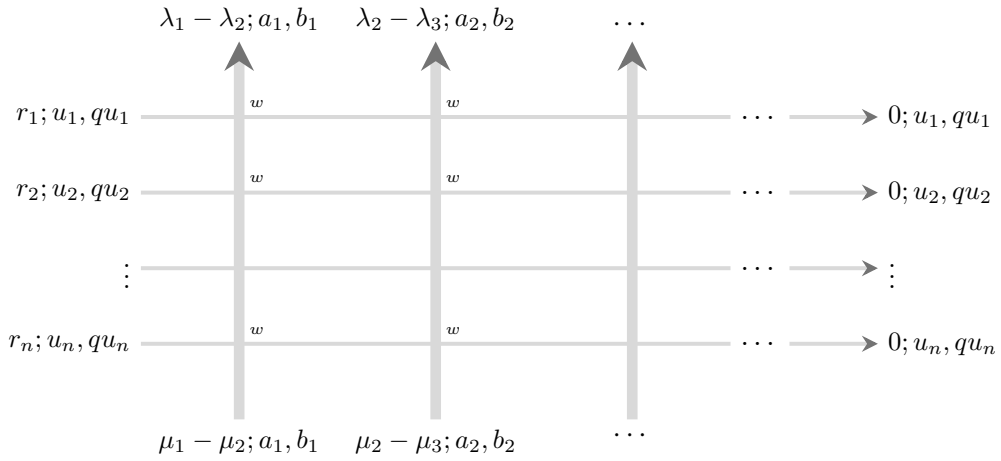


Figure 2-2: Partition function $Zw^{r_1,\dots,r_n}_{\lambda/\mu}$, used to compute $\widetilde{\mathsf{F}}_{\lambda/\mu}(u_1,\dots,u_n \mid \mathcal{A},\mathcal{B})$.

$\mathcal{A},\mathcal{B}$) by

$$\widetilde{\mathsf{F}}_{\lambda'/\mu'}(u_1,\dots,u_n \mid \mathcal{A},\mathcal{B}) = \langle \lambda | B^N(u_1 \mid \mathcal{A},\mathcal{B}) B^N(u_2 \mid \mathcal{A},\mathcal{B}) \dots B^N(u_n \mid \mathcal{A},\mathcal{B}) | \mu \rangle.$$

Note that this definition in fact does not depend on N in view of Proposition 2.2.3, part (3).

Similarly to the construction of the functions $\mathbb{F}_{\lambda/\mu}$, we can give an alternative vertex model description. Consider a grid of vertices consisting of n rows and N columns and define edge parameters and boundary conditions in the following way:

- every vertex has higher spin six-vertex weights w given by Proposition 1.3.2;
- all edge parameters of the ith edge on the left boundary are u_i, qu_i, and the corresponding boundary condition is an arbitrary fixed integer $r_i \in \{0,1\}$;
- the right-most edge in the ith row has edge parameters u_i, qu_i and the boundary condition 0;
- the bottom edge in the jth column has edge parameters a_i, b_i and the boundary condition $\mu_j - \mu_{j+1}$;
- the top edge in the jth column has edge parameters a_i, b_i and the boundary condition $\lambda_j - \lambda_{j+1}$.

The resulting partition function is denoted by $Zw^{r_1,\dots,r_n}_{\lambda/\mu}(u_1,\dots,u_n \mid \mathcal{A},\mathcal{B})$, and its depiction is given in Figure 2-2. Then the functions $\widetilde{\mathsf{F}}_{\lambda'/\mu'}$ are given by

$$\widetilde{\mathsf{F}}_{\lambda'/\mu'}(u_1,\dots,u_n \mid \mathcal{A},\mathcal{B}) := \sum_{r_1,\dots,r_n \in \{0,1\}} \prod_{i=1}^{n} (-ub_0^{-1})^{r_i} Zw^{r_1,\dots,r_n}_{\lambda/\mu}(u_1,\dots,u_n \mid \mathcal{A},\mathcal{B}). \tag{2.3.3}$$

Note that we use conjugated partitions λ' and μ': this is needed for consistency with the usual Hall-Littlewood polynomials. More precisely, when for every i we have $a_i = s = b_i^{-1}$ with s being a fixed complex number, our function $\widetilde{\mathsf{F}}_{\lambda/\mu}(u_1,\dots,u_n \mid \mathcal{A},\mathcal{B})$ degenerate to the stable spin Hall-Littlewood functions used in [21, 44]. The latter functions degenerate to Hall-Littlewood polynomials when $s \to 0$.

Moreover, it turns out that one can interpret the functions $\widetilde{\mathsf{F}}_{\lambda/\mu}(u_1,\dots,u_n \mid \mathcal{A},\mathcal{B})$ as a special case of the *inhomogeneous spin Hall-Littlewood functions* $\mathsf{F}_{\lambda/\mu}$ from [20, Definition 4.4]. Namely, take partitions μ,λ and fix $L > \max(\lambda_1,\mu_1)$. Note that by adding appropriate number of 0 we can view both λ' and μ' as signatures of lengths $L+n$ and L respectively, where by a signature we mean a non-increasing sequence of integers. To reflect this signature structure, define partitions $\widehat{\lambda}^{(L+n)} = (L+n,\lambda_1,\lambda_2,\dots)$ and $\widehat{\mu}^{(L)} = (L,\mu_1,\mu_2,\dots)$ (informally, we add a 0th row which reflects the length of the conjugated partition as a signature). The inhomogeneous spin Hall-Littlewood function $\mathsf{F}_{\lambda'/\mu'}$ from [20] is given in our notation by [2]

$$\mathsf{F}_{\lambda'/\mu'}(u_1,\dots,u_n \mid \mathcal{A},\mathcal{B}) = c\frac{\psi_{\widehat{\lambda}^{(L+n)}}(\widehat{\mathcal{A}},\widehat{\mathcal{B}})}{\psi_{\widehat{\mu}^{(L)}}(\widehat{\mathcal{A}},\widehat{\mathcal{B}})} Zw^{(1,\dots,1)}_{\widehat{\lambda}^{(L+n)}/\widehat{\mu}^{(L)}}(u_1,\dots,u_n \mid \widehat{\mathcal{A}},\widehat{\mathcal{B}}). \tag{2.3.4}$$

Here $\widehat{\mathcal{A}},\widehat{\mathcal{B}}$ denote sequences $(1,a_0,a_1,\dots),(1,b_0,b_1,\dots)$ which are sequences $\mathcal{A},\mathcal{B}$ shifted to the right by one step, so the column parameters in the partition function Zw start from a_0,b_0. The constant c is the normalization constant caused by a difference in vertex weights with [20] and it is given by

$$c = c_{\lambda'/\mu'}(\mathcal{A},\mathcal{B}) = \prod_{i\geq 0}(-\sqrt{a_i/b_i})^{\lambda_{i+1}-\mu_{i+1}}.$$

Using the relation

$$\left(b_0^l \quad \begin{array}{ccc} & k;a_0,b_0 & \\ 1;u,qu & \xrightarrow{\quad w \quad} & l;u,qu \\ & j;a_0,b_0 & \end{array} \right)\Bigg|_{b_0\to\infty} = (-u)^l$$

for the first column of the partition function $Zw^{(1,\dots,1)}_{\widehat{\lambda}^{(L+n)}/\widehat{\mu}^{(L)}}(u_1,\dots,u_n \mid \widehat{\mathcal{A}},\widehat{\mathcal{B}})$ and comparing (2.3.4) with the definition (2.3.3), one can readily get

$$\frac{(q;q)_{L-\mu_1}}{(q;q)_{L+n-\lambda_1}}\left(c^{-1}\mathsf{F}_{\lambda'/\mu'}(u_1,\dots,u_n \mid \mathcal{A},\mathcal{B})\right)\big|_{b_0\to\infty} = b_0^{\lambda_1-\mu_1}\frac{\psi_\lambda(\mathcal{A},\mathcal{B})}{\psi_\mu(\mathcal{A},\mathcal{B})}\widetilde{\mathsf{F}}_{\lambda'/\mu'}(u_1,\dots,u_n \mid \mathcal{A},\mathcal{B}).$$

The observation above allows us to immediately translate certain results from [20] to our setting. For example, we immediately see that the functions $\widetilde{\mathsf{F}}_{\lambda'/\mu'}(u_1,\dots,u_n \mid \mathcal{A},\mathcal{B})$ are symmetric in $u_1,\dots,u_n$, see [20, Proposition 4.5]. We also have an explicit formula in the case $\mu=\varnothing$, which is similar to [20, Theorem 4.14] and [44, Remark 2]:

$$\widetilde{\mathsf{F}}_{\lambda}(u_1,\dots,u_n \mid \mathcal{A},\mathcal{B}) = \frac{1}{(q;q)_{n-l(\lambda)}\psi_{\lambda'}(\mathcal{A},\mathcal{B})}\sum_{\sigma\in\mathfrak{S}_n}\sigma\left(\prod_{1\leq\alpha<\beta\leq n}\frac{u_\alpha-qu_\beta}{u_\alpha-u_\beta}\prod_{i=1}^{n}\widetilde{\varphi}_{\lambda_i}(u_i \mid \mathcal{A},\mathcal{B})\right),$$

where the sum is over permutations of n elements that act on functions in $u_1,\dots,u_n$ by permuting

[2] The paramters ξ_i and s_i used in [20] correspond to $\xi_i = \sqrt{a_i^{-1}b_i^{-1}}$ and $s_i = \sqrt{a_i/b_i}$ in our notation.

variables, and for $k \geq 1$ we set

$$\widetilde{\varphi}_0(u \mid \mathcal{A}, \mathcal{B}) = 1 - q, \qquad \widetilde{\varphi}_k(u \mid \mathcal{A}, \mathcal{B}) = \frac{u(q-1)/b_0}{1 - u/b_k} \prod_{j=1}^{k-1} \frac{a_j/b_j - u/b_j}{1 - u/b_j}. \tag{2.3.5}$$

Define dual stable spin Hall-Littlewood function by

$$\widetilde{\mathsf{F}}^*_{\lambda'/\mu'}(u_1, \dots, u_n \mid \mathcal{A}, \mathcal{B}) := \frac{\psi_\lambda(\mathcal{A}, \mathcal{B})}{\psi_\mu(\mathcal{A}, \mathcal{B})} \widetilde{\mathsf{F}}_{\lambda'/\mu'}(u_1, \dots, u_n \mid \mathcal{A}, \mathcal{B}),$$

In view of Proposition 2.2.4 we have

$$\widetilde{\mathsf{F}}^*_{\lambda'/\mu'}(u_1, \dots, u_n \mid \mathcal{A}, \mathcal{B}) = \langle \mu | B^*(u_n^{-1} \mid \overline{\mathcal{B}}, \overline{\mathcal{A}}) \dots B^*(u_2^{-1} \mid \overline{\mathcal{B}}, \overline{\mathcal{A}}) B^*(u_1^{-1} \mid \overline{\mathcal{B}}, \overline{\mathcal{A}}) | \lambda \rangle. \tag{2.3.6}$$

Theorem 2.3.9. *For any partitions μ, ν the following identity holds*

$$\sum_\lambda \widetilde{\mathsf{F}}^*_{\lambda'/\nu'}(u_1, \dots, u_m \mid \overline{\mathcal{B}}, \overline{\mathcal{A}}) \mathbb{F}_{\lambda/\mu}(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B})$$
$$= \prod_{i=1}^n \prod_{j=1}^m \frac{1 - u_j x_i}{1 - u_j a_i} \sum_\lambda \mathbb{F}_{\nu/\lambda}(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B}) \widetilde{\mathsf{F}}^*_{\mu'/\lambda'}(u_1, \dots, u_m \mid \overline{\mathcal{B}}, \tau^n \overline{\mathcal{A}}),$$

with both sums above having finitely many nonzero terms.

Proof. We rewrite the matrix element

$$\mathcal{E}^\nu_\mu(x_1, \dots, x_n; u_1, \dots, u_m \mid \mathcal{A}, \mathcal{B}) := \langle \nu | B^*(u_m^{-1} \mid \mathcal{A}, \mathcal{B}) \dots B^*(u_1^{-1} \mid \mathcal{A}, \mathcal{B})$$
$$\times \mathbb{B}(x_1 \mid \mathcal{A}, \mathcal{B}) \mathbb{B}(x_2 \mid \tau\mathcal{A}, \mathcal{B}) \dots \mathbb{B}(x_n \mid \tau^n \mathcal{A}, \mathcal{B}) | \mu \rangle$$

in two different ways, corresponding to the two sides of the claim.

For the left-hand side we insert $\sum_\lambda |\lambda\rangle\langle\lambda| = \mathrm{id}$ between the operators $\mathbb{B}$ and B^* to get

$$\mathcal{E}^\nu_\mu(x_1, \dots, x_n; u_1, \dots, u_m \mid \mathcal{A}, \mathcal{B}) = \sum_\lambda \widetilde{\mathsf{F}}^*_{\lambda'/\nu'}(u_1, \dots, u_m \mid \overline{\mathcal{B}}, \overline{\mathcal{A}}) \mathbb{F}_{\lambda/\mu}(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B}).$$

On the other hand, set

$$\mathbb{B}^{(n)}(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B}) := \mathbb{B}(x_1 \mid \mathcal{A}, \mathcal{B}) \mathbb{B}(x_2 \mid \tau\mathcal{A}, \mathcal{B}) \dots \mathbb{B}(x_n \mid \tau^{n-1} \mathcal{A}, \mathcal{B})$$

and iterate the exchange relation from Proposition 2.2.10 to get

$$B^*(u^{-1} \mid \mathcal{A}, \mathcal{B}) \mathbb{B}^{(n)}(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B}) = \prod_{i=1}^n \frac{1 - u x_i}{1 - u a_1} \mathbb{B}^{(n)}(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B}) B^*(u^{-1} \mid \tau\mathcal{A}, \mathcal{B}).$$

Hence,

$$\mathcal{E}^{\nu}_{\mu}(x_1,\dots,x_n;u_1,\dots,u_m \mid \mathcal{A},\mathcal{B}) = \prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1-u_jx_i}{1-u_ja_i} \times \langle \nu|\mathbb{B}^{(n)}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})B^*(u_m^{-1} \mid \tau^n\mathcal{A},\mathcal{B})\dots B^*(u_1^{-1} \mid \tau^n\mathcal{A},\mathcal{B})|\mu\rangle.$$

Now we can again insert $\sum_\lambda |\lambda\rangle\langle\lambda| = \mathrm{id}$ and obtain

$$\mathcal{E}^{\nu}_{\mu}(x_1,\dots,x_n;u_1,\dots,u_m \mid \mathcal{A},\mathcal{B}) = \prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1-u_jx_i}{1-u_ja_i}\sum_{\lambda}\mathbb{F}_{\nu/\lambda}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})\widetilde{\mathsf{F}}^*_{\mu'/\lambda'}(u_1,\dots,u_m \mid \overline{\mathcal{B}},\tau^n\overline{\mathcal{A}}).$$

□

By setting $(\mu,\nu) = (\varnothing,\varnothing)$ and noticing that

$$\mathbb{F}_{\varnothing/\lambda}(\kappa_1,\dots,\kappa_n \mid \mathcal{A},\mathcal{B}) = \widetilde{\mathsf{F}}_{\varnothing/\lambda}(u_1,\dots,u_m \mid \mathcal{A},\mathcal{B}) = \mathbb{1}_{\lambda=\varnothing},$$

Theorem 2.3.9 immediately leads to the following results:

Corollary 2.3.10 (Dual Cauchy identity)**.** *We have*

$$\sum_{\lambda}\widetilde{\mathsf{F}}^*_{\lambda'}(u_1,\dots,u_m \mid \overline{\mathcal{B}},\overline{\mathcal{A}})\mathbb{F}_{\lambda}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B}) = \prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1-u_jx_i}{1-u_ja_i}.$$

2.3.4 Integral representation

We can use the dual Cauchy identity of Corollary 2.3.10 and previously known results about inhomogeneous (stable) spin Hall-Littlewood functions to obtain a multivariate integral representation for the inhomogeneous spin q-Whittaker functions.

For this result we specialize all parameters q, a_i, b_j and variables x_i, u_j to be complex numbers such that $0 < |q| < 1$ and we assume that there exists a positively-oriented simple contour $\mathcal{C}$ on the complex plane satisfying the following conditions:

- all points $\{0\}$ and $\{b_i^{-1}\}_{i=1}^{\infty}$ are inside the contour $\mathcal{C}$;
- all points $\{a_i^{-1}\}_{i=1}^{\infty}$ are outside of the contour $\mathcal{C}$;
- the image $q\mathcal{C}$ of the contour $\mathcal{C}$ under the multiplication by q is inside $\mathcal{C}$.

An example of such a configuration is sketched in Figure 2-3.

Theorem 2.3.11. *Under the above assumptions we have*

$$\mathbb{F}_{\lambda}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B}) = \frac{1}{\psi_\lambda(\mathcal{A},\mathcal{B})}\oint_{\mathcal{C}}\frac{dz_1}{2\pi\mathbf{i}z_1}\cdots\oint_{\mathcal{C}}\frac{dz_k}{2\pi\mathbf{i}z_k}\prod_{\alpha<\beta}\frac{z_\alpha-z_\beta}{z_\alpha-qz_\beta}\prod_{\alpha=1}^{k}\left(\frac{-a_0^{-1}}{z_\alpha-b_{\mu'_\alpha}^{-1}}\prod_{j=1}^{\mu'_\alpha-1}\frac{z_\alpha-a_j^{-1}}{z_\alpha-b_j^{-1}}\prod_{i=1}^{n}\frac{1-z_\alpha x_i}{1-z_\alpha a_i}\right)$$

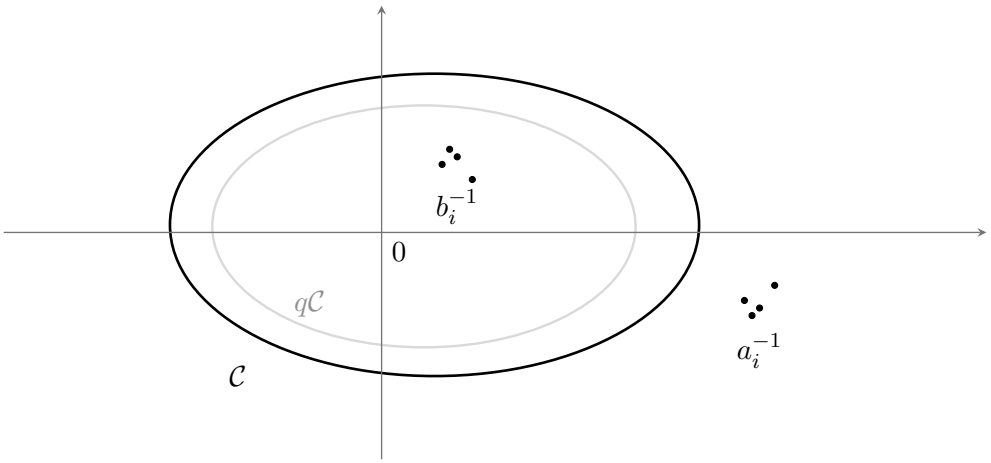


Figure 2-3: A possible of configuration of $\{a_i\}_{i=1}^{\infty}$, $\{b_i\}_{i=1}^{\infty}$ and $\mathcal{C}$ for Theorem 2.3.11.

where $k = \mu_1$.

Proof. We will use the orthogonality of $\widetilde{\mathsf{F}}_\lambda$, *cf.* [20, Theorem 7.4]: for any pair of partitions λ, μ such that $l(\mu) \le L$ we have

$$\frac{(q;q)_{L-l(\mu)}}{(1-q)^L}\prod_{i\ge 0}(b_i/a_i)^{\mu'_{i+1}}\oint_{\mathcal{C}}\frac{dz_1}{2\pi\mathbf{i}z_1}\cdots\oint_{\mathcal{C}}\frac{dz_L}{2\pi\mathbf{i}z_L}\prod_{\alpha<\beta}\frac{z_\alpha-z_\beta}{z_\alpha-qz_\beta}\widetilde{\mathsf{F}}_\lambda(z_1,\dots,z_L\mid\overline{\mathcal{B}},\overline{\mathcal{A}})\prod_{\alpha=1}^{L}\widetilde{\varphi}_{\mu_\alpha}(z_\alpha^{-1}\mid\mathcal{A},\mathcal{B})=\mathbb{1}_{\lambda=\mu}, \tag{2.3.7}$$

where the functions $\widetilde{\varphi}_k(u\mid\mathcal{A},\mathcal{B})$ were defined in (2.3.5).

Fixing a partition μ, conjugating partitions in (2.3.7), setting $L=\mu_1=l(\mu')=k$ and multiplying by $\mathbb{F}_\lambda(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B})$, we get

$$\frac{1}{(1-q)^k\psi_\lambda(\mathcal{A},\mathcal{B})}\prod_{i\ge 0}(b_i/a_i)^{\mu_{i+1}}\oint_{\mathcal{C}}\frac{dz_1}{2\pi\mathbf{i}z_1}\cdots\oint_{\mathcal{C}}\frac{dz_k}{2\pi\mathbf{i}z_k}\prod_{\alpha<\beta}\frac{z_\alpha-z_\beta}{z_\alpha-qz_\beta}$$
$$\times\,\widetilde{\mathsf{F}}^*_{\lambda'}(z_1,\dots,z_k\mid\overline{\mathcal{B}},\overline{\mathcal{A}})\mathbb{F}_\lambda(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B})\prod_{\alpha=1}^{k}\widetilde{\varphi}_{\mu'_\alpha}(z_\alpha^{-1}\mid\mathcal{A},\mathcal{B})=\mathbb{1}_{\lambda=\mu}\cdot\mathbb{F}_\lambda(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B}).$$

Summing over λ and using Corollary 2.3.10, we arrive at the desired integral representation. □

2.4 Vanishing and characterization properties

It turns out that the functions $\mathbb{F}_\lambda(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B})$ satisfy vanishing and characterization properties similar to interpolation symmetric functions. We start with the vanishing property. Recall that $\mathbb{Y}^n$ denotes the set of partitions of length at most n. For a partition $\mu\in\mathbb{Y}^n$ set

$$\mathbf{x}^n_{\mathcal{A}}(\mu) := (a_1q^{\mu_1-\mu_2}, a_2q^{\mu_2-\mu_3},\dots,a_nq^{\mu_n}). \tag{2.4.1}$$

Proposition 2.4.1. *Let $\lambda, \mu \in \mathbb{Y}^n$ be partitions of length at most n. Then we have*

$$\mathbb{F}_\lambda(\mathbf{x}^n_{\mathcal{A}}(\mu) \mid \mathcal{A}, \mathcal{B}) = \mathbb{F}_\lambda(a_1 q^{\mu_1-\mu_2}, a_2 q^{\mu_2-\mu_3}, \dots, a_n q^{\mu_n} \mid \mathcal{A}, \mathcal{B}) = 0, \qquad \text{unless } \lambda \subseteq \mu.$$

Moreover, when $\lambda = \mu$, we have

$$\mathbb{F}_\lambda(\mathbf{x}^n_{\mathcal{A}}(\lambda) \mid \mathcal{A}, \mathcal{B}) = H_\lambda(\mathcal{A}, \mathcal{B}) \neq 0.$$

where

$$H_\lambda(\mathcal{A}, \mathcal{B}) = (-1)^{\lambda_1} q^{\frac{\lambda_1(\lambda_1-1)}{2}} \prod_{i=1}^n (a_i/b_{i-1})^{\lambda_i} \prod_{i,j=1}^n \frac{(q^{\lambda_{i+1}-\lambda_i} a_{j+i}/a_i; q)_{\lambda_{j+i}-\lambda_{j+i+1}}}{(a_{j+i}/b_j; q)_{\lambda_{j+i}-\lambda_{j+i+1}}}.$$

Proof. The proof is by induction on n, with the case $n = 0$ being trivial.

Assume that we have proved the claim for $n-1$. Using the branching rule from Proposition 2.3.3 we get

$$\mathbb{F}_\lambda(\mathbf{x}^n_{\mathcal{A}}(\mu) \mid \mathcal{A}, \mathcal{B}) = \sum_{\nu \prec \lambda} \mathbb{F}_{\lambda/\nu}(a_1 q^{\mu_1-\mu_2} \mid \mathcal{A}, \mathcal{B}) \mathbb{F}_\nu(\mathbf{x}^{n-1}_{\tau\mathcal{A}}(\widetilde{\mu}) \mid \tau\mathcal{A}, \mathcal{B}),$$

where $\widetilde{\mu} = (\mu_2, \mu_3, \dots)$. Note that by induction hypothesis $\mathbb{F}_\nu(\mathbf{x}^{n-1}_{\tau\mathcal{A}}(\widetilde{\mu}) \mid \tau\mathcal{A}, \mathcal{B}) = 0$ unless $\nu \subseteq \widetilde{\mu}$. But $\nu \prec \lambda$ implies $\widetilde{\lambda} := (\lambda_2, \lambda_3, \dots) \subseteq \nu$, so if in the sum above the summand corresponding to ν does not vanish then $\widetilde{\lambda} \subseteq \nu \subseteq \widetilde{\mu}$.

At the same time, $\mathbb{F}_{\lambda/\nu}(a_1 q^{\mu_1-\mu_2} \mid \mathcal{A}, \mathcal{B})$ vanishes unless $\lambda_1 - \nu_1 \leq \mu_1 - \mu_2$, because of the factor $(q^{\mu_2-\mu_1}; q)_{\lambda_1-\nu_1}$ in

$$\begin{aligned}\mathbb{F}_{\lambda/\nu}(a_1 q^{\mu_1-\mu_2} \mid \mathcal{A}, \mathcal{B}) &= \langle \lambda | \, \mathbb{B}(a_1 q^{\mu_1-\mu_2} \mid \mathcal{A}, \mathcal{B}) \, | \mu \rangle \\ &= \left(\frac{a_1 q^{\mu_1-\mu_2}}{b_0}\right)^{\lambda_1-\nu_1} \frac{(q^{\mu_2-\mu_1}; q)_{\lambda_1-\nu_1}}{(q; q)_{\lambda_1-\nu_1}} \langle \lambda | \, \mathbb{T}^b_{\lambda_1-\nu_1, 0}(a_1 q^{\mu_1-\mu_2} \mid \mathcal{A}, \mathcal{B}) \, | \mu \rangle.\end{aligned}$$

Hence, if $\mathbb{F}_\lambda(\mathbf{x}^n_{\mathcal{A}}(\mu) \mid \mathcal{A}, \mathcal{B}) \neq 0$, then for some ν we have $\lambda_1 - \nu_1 \leq \mu_1 - \mu_2$ and $\widetilde{\lambda} \subseteq \nu \subseteq \widetilde{\mu}$. But then

$$\lambda_1 \leq \mu_1 - \mu_2 + \nu_1 \leq \mu_1,$$

so $\lambda \subseteq \mu$ as desired.

To prove the second statement, note that if $\lambda = \mu$ then the only possible ν in the discussion above is $\nu = \widetilde{\lambda} = (\lambda_2, \lambda_3, \dots)$. Hence

$$\mathbb{F}_\lambda(\mathbf{x}^n_{\mathcal{A}}(\lambda) \mid \mathcal{A}, \mathcal{B}) = \mathbb{F}_{\lambda/\widetilde{\lambda}}(a_1 q^{\lambda_1-\lambda_2} \mid \mathcal{A}, \mathcal{B}) \mathbb{F}_{\widetilde{\lambda}}(\mathbf{x}^{n-1}_{\tau\mathcal{A}}(\widetilde{\lambda}) \mid \tau\mathcal{A}, \mathcal{B})$$

and the second statement readily follows from Proposition 2.3.2. □

Remark 2.4.2. We can use the graphical definition of $\mathbb{F}_\lambda(x_1, \dots, x_n \mid \mathcal{A}, \mathcal{B})$ from Remark **??** to sketch an alternative proof of Proposition 2.4.1. Namely, we have

$$\mathbb{F}_\lambda(\mathbf{x}^n_{\mathcal{A}}(\mu) \mid \mathcal{A}, \mathcal{B}) = \sum_{r_1, \dots, r_n \geq 0} \prod_{i=1}^n (a_i q^{\mu_i-\mu_{i+1}}/b_0)^{r_i} \frac{(q^{\mu_{i+1}-\mu_i}; q)_{r_i}}{(q; q)_{r_i}} Z^{r_1, \dots, r_n}_\lambda,$$

where $Z^{r_1, \dots, r_n}_\lambda$ is the partition function from Figure 2-1 with $x_i = a_i q^{\mu_i-\mu_{i+1}}$. Note that the factors $(q^{\mu_{i+1}-\mu_i}; q)_{r_i}$ force $r_i \leq \mu_i - \mu_{i+1}$ in the sum above. On the other hand, since the weights

$W^b(I,J,K,L)$ vanish unless $I+J=K+L$ and $I\le K$, the partition function $Z_\lambda^{r_1,\dots,r_n}$ vanishes unless $r_1+\cdots+r_n=\lambda_1$ and $r_1+\cdots+r_i\le\lambda_1-\lambda_{i+1}$ for any i. These restrictions on r_i imply that non-zero terms in the sum above must satisfy $\lambda_i\le r_i+\cdots+r_n\le\mu_i$, hence $\mathbb{F}_\lambda(\mathbf{x}^n_{\mathcal{A}}(\mu)\mid\mathcal{A},\mathcal{B})$ vanishes unless $\lambda_i\le\mu_i$.

Now we can consider a characterization for $\mathbb{F}_\lambda(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B})$ in terms of the vanishing property. To do so, we need the following notation. For an integer n and a partition $\mu\in\mathbb{Y}^n$ define

$$G^n_\mu(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B})=\prod_{i=1}^n\prod_{r\ge1}\frac{(x_i/b_r;q)_{\mu_r-\mu_{r+1}}}{(a_i/b_r;q)_{\mu_r-\mu_{r+1}}}.$$

Note that G^n_μ are symmetric polynomials in $x_1,\dots,x_n$. Define a filtration $\mathcal{G}^n_0\subset\mathcal{G}^n_1\subset\cdots\subset\Bbbk[x_1,\dots,x_n]^{S_n}$ of the algebra of symmetric polynomials in $x_1,\dots,x_n$ with coefficients in $\Bbbk=\mathbb{Q}(q,\mathcal{A},\mathcal{B})$ by

$$\mathcal{G}^n_m:=\operatorname{span}\{G^n_\mu\}_{\substack{\mu\in\mathbb{Y}^n\\|\mu|\le m}}.$$

Theorem 2.4.3. *For each partition $\lambda\in\mathbb{Y}^n$ the spin q-Whittaker function $\mathbb{F}_\lambda(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B})$ is uniquely characterized by the following properties:*

1. $\mathbb{F}_\lambda(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B})\in\mathcal{G}^n_{|\lambda|}$.
2. *For any partition μ such that $|\mu|\le|\lambda|$ and $\mu\ne\lambda$ we have $\mathbb{F}_\lambda(\mathbf{x}^n_{\mathcal{A}}(\mu)\mid\mathcal{A},\mathcal{B})=0$.*
3. $\mathbb{F}_\lambda(\mathbf{x}^n_{\mathcal{A}}(\lambda)\mid\mathcal{A},\mathcal{B})=H_\lambda(\mathcal{A},\mathcal{B})$, *where $H_\lambda(\mathcal{A},\mathcal{B})$ is defined in Proposition 2.4.1 above.*

Moreover, for each $m\in\mathbb{Z}_{\ge0}$ both $\{\mathbb{F}_\lambda(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B})\}_{\substack{\lambda\in\mathbb{Y}^n\\|\lambda|\le m}}$ and $\{G^n_\mu(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B})\}_{\substack{\mu\in\mathbb{Y}^n\\|\mu|\le m}}$ are bases of $\mathcal{G}^n_m$.

Proof. We start with the last statement. Let $\mu\in\mathbb{Y}^n$ and consider the Cauchy identity (2.3.2) with $m=n$ and $y_i=q^{\mu_i-\mu_{i+1}}/b_i$. We get

$$\sum_\lambda\mathbb{F}_\lambda(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B})\mathbb{F}^*_\lambda(q^{\mu_1-\mu_2}/b_1,\dots,q^{\mu_n}/b_n\mid\overline{\mathcal{B}},\overline{\mathcal{A}})=\prod_{i,j=1}^n\frac{(x_i/b_j;q)_{\mu_j-\mu_{j+1}}}{(a_i/b_j;q)_{\mu_j-\mu_{j+1}}}=G^n_\mu(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B}).\tag{2.4.2}$$

Note that $(q^{\mu_1-\mu_2}/b_1,\dots,q^{\mu_n}/b_n)=\mathbf{x}^n_{\overline{\mathcal{B}}}(\mu)$, so we can apply the vanishing property from Proposition 2.4.1 to $\mathbb{F}^*_\lambda(\mathbf{x}^n_{\overline{\mathcal{B}}}(\mu)\mid\overline{\mathcal{B}},\overline{\mathcal{A}})$, obtaining

$$\mathbb{F}^*_\lambda(\mathbf{x}^n_{\overline{\mathcal{B}}}(\mu)\mid\overline{\mathcal{B}},\overline{\mathcal{A}})=\psi_\lambda(\overline{\mathcal{B}},\overline{\mathcal{A}})\mathbb{F}_\lambda(\mathbf{x}^n_{\overline{\mathcal{B}}}(\mu)\mid\overline{\mathcal{B}},\overline{\mathcal{A}})=0,\qquad\text{unless }\lambda\subseteq\mu,$$

$$\mathbb{F}^*_\mu(\mathbf{x}^n_{\overline{\mathcal{B}}}(\mu)\mid\overline{\mathcal{B}},\overline{\mathcal{A}})=\psi_\mu(\overline{\mathcal{B}},\overline{\mathcal{A}})H_\mu(\overline{\mathcal{B}},\overline{\mathcal{A}})\ne0,$$

where $\psi_\lambda(\mathcal{A},\mathcal{B})$ is defined in Proposition 2.2.2. Hence we can rewrite (2.4.2) as

$$G^n_\mu(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B})=\sum_{\lambda\subseteq\mu}c_{\lambda\mu}\,\mathbb{F}_\lambda(x_1,\dots,x_n\mid\mathcal{A},\mathcal{B})\tag{2.4.3}$$

for some coefficients $c_{\lambda\mu}$ such that $c_{\mu\mu}\ne0$. Thus, the transition matrix expressing G^n_μ in terms of $\mathbb{F}_\lambda$ is upper-triangular with respect to the partial order of inclusion of Young diagrams. Moreover,

this transition matrix has non-zero diagonal entries $c_{\mu\mu}$, so it is invertible and

$$\mathbb{F}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B}) = \sum_{\mu\subseteq\lambda} \overline{c}_{\mu\lambda}\ G^n_\mu(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B}) \tag{2.4.4}$$

for some other coefficients $\overline{c}_{\mu\lambda}$.

Note that (2.4.3),(2.4.4) imply that

$$\operatorname{span}\{\mathbb{F}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})\}_{\substack{\lambda\in\mathbb{Y}^n\\|\lambda|\le m}} = \operatorname{span}\{G^n_\mu(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})\}_{\substack{\mu\in\mathbb{Y}^n\\|\mu|\le m}} = \mathcal{G}^n_m.$$

On the other hand, the vanishing property implies that the functions $\mathbb{F}_\lambda$ are linearly independent: if we have

$$\sum_\lambda \alpha_\lambda \mathbb{F}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B}) = 0,$$

with α_λ being nonzero for some λ, then we get a contradiction by plugging $(x_1,\dots,x_n) = \mathbf{x}^n_{\mathcal{A}}(\mu)$ for a minimal μ such that $\alpha_\mu \neq 0$ and using the vanishing property from Proposition 2.4.1. Hence $\{\mathbb{F}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})\}_{\substack{\lambda\in\mathbb{Y}^n\\|\lambda|\le m}}$ is a basis of $\mathcal{G}^n_m$, and consequently $\{G^n_\mu(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})\}_{\substack{\mu\in\mathbb{Y}^n\\|\mu|\le m}}$ is also a basis of $\mathcal{G}^n_m$ in view of (2.4.3),(2.4.4).

Now we can prove the characterization property. We have already proved that $\mathbb{F}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B}) \in \mathcal{G}^n_m$, and the other properties follow from Proposition 2.4.1, so we only need to prove that $\mathbb{F}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})$ is the unique function satisfying properties (1)-(3) above. Let $f(x_1,\dots,x_n)$ be a function which also satisfies properties (1)-(3), and consider the difference $g(x_1,\dots,x_n) = f(x_1,\dots,x_n) - \mathbb{F}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})$. We have $g(x_1,\dots,x_n) \in \mathcal{G}^n_m$ and $g(\mathbf{x}^n_{\mathcal{A}}(\mu)) = 0$ for any μ such that $|\mu| \le |\lambda|$. Since $\{\mathbb{F}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})\}_{\substack{\lambda\in\mathbb{Y}^n\\|\lambda|\le m}}$ is a basis of $\mathcal{G}^n_{|\lambda|}$, for some coefficients $\alpha_\nu \in \Bbbk$ we have

$$g(x_1,\dots,x_n) = \sum_{\nu: l(\nu)\le n, |\nu|\le|\lambda|} \alpha_\nu \mathbb{F}_\nu(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B}).$$

Assume that $g \neq 0$ and let μ be a minimal partition such that $\alpha_\mu \neq 0$. Then by Proposition 2.4.1

$$g(\mathbf{x}^n_{\mathcal{A}}(\mu)) = \alpha_\mu \mathbb{F}_\mu(\mathbf{x}^n_{\mathcal{A}}(\mu) \mid \mathcal{A},\mathcal{B}) \neq 0,$$

which leads to contradiction. Hence, $g = \mathbb{F}_\lambda - f = 0$, and the uniqueness follows. □

In light of these interpolation properties, two degenerations of $\mathbb{F}_\lambda$ turn out to be of particular interest. For the first degeneration, consider the limit $b_i \to \infty$ of the functions $\mathbb{F}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})$. Recall that $\mathbb{F}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})$ is a rational function in b_i, but Proposition 2.3.2 implies that when we set $b_i^{-1} = 0$ for all $i \ge 1$, the function just vanishes. So, to get a meaningful object we first need to renormalize the function: set

$$\widetilde{\mathbb{F}}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B}) = \prod_{r=1}^n b_{r-1}^{\lambda_r-\mu_r}\ \mathbb{F}(x_1,\dots,x_n \mid \mathcal{A},\mathcal{B})$$

and let $\widetilde{\mathbb{F}}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\infty)$ denote the result of the substitution $b_0^{-1} = b_1^{-1} = \cdots = 0$.

Almost all the properties of $\mathbb{F}_{\lambda/\mu}(x_1,\dots x_n \mid \mathcal{A},\mathcal{B})$ described earlier can be readily modified to obtain properties of $\widetilde{\mathbb{F}}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\infty)$. For instance, we can compute $\widetilde{\mathbb{F}}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\infty)$

using degenerations of the explicit expression from Proposition 2.3.2 and the branching rule from Proposition 2.3.3: we have

$$\widetilde{\mathbb{F}}_{\lambda/\nu}(x_1, x_2, \ldots, x_n \mid \mathcal{A}, \infty) = \sum_{\mu \prec \lambda} \widetilde{\mathbb{F}}_{\lambda/\mu}(x_1 \mid \mathcal{A}, \infty) \widetilde{\mathbb{F}}_{\mu/\nu}(x_2, \ldots, x_n \mid \tau\mathcal{A}, \infty), \tag{2.4.5}$$

$$\widetilde{\mathbb{F}}_{\lambda/\mu}(x \mid \mathcal{A}, \infty) = \begin{cases} x^{|\lambda|-|\mu|} \prod_{r \geq 1} \dfrac{(a_r/x; q)_{\lambda_r - \mu_r} (q; q)_{\mu_r - \mu_{r+1}}}{(q; q)_{\lambda_r - \mu_r} (q; q)_{\mu_r - \lambda_{r+1}}}, & \text{if } \mu \prec \lambda; \\ 0, & \text{otherwise.} \end{cases} \tag{2.4.6}$$

In the same manner we can degenerate the vanishing property from Proposition 2.4.1, since it does not depend on $\mathcal{B}$ in a significant way:

Proposition 2.4.4. *Let λ, μ be partitions of length at most n. Then we have*

$$\widetilde{\mathbb{F}}_{\lambda}(\mathbf{x}^n_{\mathcal{A}}(\mu) \mid \mathcal{A}, \infty) = 0, \qquad \textit{unless } \lambda \subseteq \mu,$$

Moreover, when $\lambda = \mu$, we have

$$\widetilde{\mathbb{F}}_{\lambda}(\mathbf{x}^n_{\mathcal{A}}(\lambda) \mid \mathcal{A}, \infty) = (-1)^{\lambda_1} q^{\frac{\lambda_1^2 - \lambda_1}{2}} \prod_{i=1}^{n} a_i^{\lambda_i} \prod_{i,j=1}^{n} (q^{\lambda_{i+1} - \lambda_i} a_{j+i}/a_i; q)_{\lambda_{j+i} - \lambda_{j+i+1}}.$$

□

The only property proved earlier which cannot be immediately degenerated to $\mathcal{B} = \infty$ is Theorem 2.4.3, since our definition of the filtration $\mathcal{G}^n_m$ makes little sense when $b_i^{-1} \equiv 0$. To state the appropriate characterization, let $\Bbbk[x_1, \ldots, x_n]^{S_n}_{\leq m}$ denote the space of degree $\leq m$ symmetric polynomials with coefficients in $\Bbbk = \mathbb{C}(\mathcal{A}, q)$, and let $P_{\lambda/\mu}(x_1, \ldots, x_n; q, 0)$ denote the q-Whittaker function corresponding to λ/μ, that is, the $t = 0$ specialization of the Macdonald symmetric polynomial $P_{\lambda/\mu}(x_1, \ldots, x_n; q, t)$, *cf.* [60, Chapter VI].

Proposition 2.4.5. *The function $\widetilde{\mathbb{F}}_{\lambda}(x_1, \ldots, x_n \mid \mathcal{A}, \infty)$ can be characterized in the following two equivalent ways: it is the unique function satisfying*

(1) $\widetilde{\mathbb{F}}_{\lambda}(x_1, \ldots, x_n \mid \mathcal{A}, \infty) \in \Bbbk[x_1, \ldots, x_n]^{S_n}_{\leq |\lambda|}$,

(2) $\widetilde{\mathbb{F}}_{\lambda}(\mathbf{x}^n_{\mathcal{A}}(\mu) \mid \mathcal{A}, \infty) = 0$ *for any partition μ such that $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$,*

(3) $\widetilde{\mathbb{F}}_{\lambda}(\mathbf{x}^n_{\mathcal{A}}(\lambda) \mid \mathcal{A}, \infty) = (-1)^{\lambda_1} q^{\frac{\lambda_1^2 - \lambda_1}{2}} \prod_{i=1}^{n} a_i^{\lambda_i} \prod_{i,j=1}^{n} (q^{\lambda_{i+1} - \lambda_i} a_{j+i}/a_i; q)_{\lambda_{j+i} - \lambda_{j+i+1}}$;

and it is also the unique function satisfying

(1') $\widetilde{\mathbb{F}}_{\lambda}(x_1, \ldots, x_n \mid \mathcal{A}, \infty) \in \Bbbk[x_1, \ldots, x_n]^{S_n}_{\leq |\lambda|}$, *identically to (1),*

(2') $\widetilde{\mathbb{F}}_{\lambda}(\mathbf{x}^n_{\mathcal{A}}(\mu) \mid \mathcal{A}, \infty) = 0$ *for any partition μ such that $|\mu| < |\lambda|$,*

(3') *The top degree homogeneous component of $\widetilde{\mathbb{F}}_{\lambda}(x_1, \ldots, x_n \mid \mathcal{A}, \infty)$ has degree $|\lambda|$ and is equal to $P_{\lambda}(x_1, \ldots, x_n; q, 0)$.*

Proof. First we show that $\widetilde{\mathbb{F}}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\infty)$ indeed satisfy the properties from the statement. Properties (2), (2') and (3) follow directly from Proposition 2.4.4. For the remaining properties (1), (1') and (3') note that by (2.4.6) the one-variable function $\widetilde{\mathbb{F}}_{\lambda/\mu}(x \mid \mathcal{A},\infty)$ is a polynomial of degree $|\lambda|-|\mu|$, whose top term coincides with $P_{\lambda/\mu}(x;q,0)$, *cf.* [60, Chapter VI, (7.13')]. Then (2.4.5) implies that $\widetilde{\mathbb{F}}_\lambda(x_1,x_2,\dots,x_n \mid \mathcal{A},\infty)$ has degree at most $|\lambda|$, and since the branching identical to (2.4.5) holds for the functions $P_{\lambda/\mu}$, we deduce that the top-degree component of $\widetilde{\mathbb{F}}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\infty)$ is indeed $P_\lambda(x_1,\dots,x_n;q,0)$.

To show uniqueness for both characterizations it is enough to prove that if $g\in\Bbbk[x_1,\dots,x_n]^{S_n}_{\le m}$ and $g(\mathbf{x}^n_{\mathcal{A}}(\mu))=0$ for all $|\mu|\le m$, then $g=0$. Indeed, for the first chracterization fix λ and assume that $f\in\Bbbk[x_1,\dots,x_n]^{S_n}$ satisfies the conditions (1)-(3). Then $g=f-\widetilde{\mathbb{F}}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\infty)$ is a polynomial in $\Bbbk[x_1,\dots,x_n]^{S_n}_{\le|\lambda|}$ such that $g(\mathbf{x}^n_{\mathcal{A}}(\mu))=0$ for any $\mu\in\mathbb{Y}^n, |\mu|\le|\lambda|$, since the conditions (2) and (3) fix the values of f and $\widetilde{\mathbb{F}}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\infty)$ at $\mathbf{x}^n_{\mathcal{A}}(\mu)$ for $|\mu|\le|\lambda|$. Similarly, if f' is a function satisfying (1')-(3') set $g'=f'-\widetilde{\mathbb{F}}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\infty)$. By (2') the functions f' and $\widetilde{\mathbb{F}}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\infty)$ both vanish at $\mathbf{x}^n_{\mathcal{A}}(\mu)$ when $|\mu|\le|\lambda|-1$, and by (3') the top degree homogeneous components of these functions coincide. Hence $g'\in\Bbbk[x_1,\dots,x_n]^{S_n}_{\le|\lambda|-1}$ and $g'(\mathbf{x}^n_{\mathcal{A}}(\mu))=0$ for $|\mu|\le|\lambda|-1$.

So, it is enough to prove that if $g\in\Bbbk[x_1,\dots,x_n]^{S_n}_{\le m}$ and $g(\mathbf{x}^n_{\mathcal{A}}(\mu))=0$ for all μ such that $|\mu|\le m$, then $g=0$. This follows from Lemma 2.4.6 below, which for later convenience we state in a much greater generality. □

Lemma 2.4.6. *Assume that for a function $\mho:\mathbb{Y}^n\to\Bbbk^n$ there exists a family of polynomials $F_\lambda(x_1,\dots,x_n \mid \mho)$ such that*

- $F_\lambda(x_1,\dots,x_n \mid \mho)\in\Bbbk[x_1,\dots,x_n]^{S_n}_{\le|\lambda|}$;
- $F_\lambda(\mho(\mu) \mid \mho)=0$ *for any partitions* $\lambda,\mu\in\mathbb{Y}^n$ *such that* $|\mu|\le|\lambda|, \lambda\ne\mu$;
- $F_\lambda(\mho(\lambda) \mid \mho)\ne 0$.

Then the following holds

1. *The functions $F_\lambda(x_1,\dots,x_n \mid \mho)$ are uniquely determined up to a scalar;*
2. *Let $m\in\mathbb{Z}_{\ge0}$. If $f\in\Bbbk[x_1,\dots,x_n]^{S_n}_{\le m}$ and $f(\mho(\mu))=0$ for all $\mu\in\mathbb{Y}^n$ such that $|\mu|\le m$, then $f=0$.*
3. *For each $m\in\mathbb{Z}_{\ge0}$ the functions $F_\lambda(x_1,\dots,x_n \mid \mho)$ with $|\lambda|\le m$ form a basis of $\Bbbk[x_1,\dots,x_n]^{S_n}_{\le m}$;*
4. *The degree of $F_\lambda(x_1,\dots,x_n \mid \mho)$ is $|\lambda|$.*

Proof. Fix $m\in\mathbb{Z}_{\ge0}$ and fix a choice of functions $F_\lambda(x_1,\dots,x_n \mid \mho)$ for $|\lambda|\le m$. We claim that these functions are linearly independent: assume that

$$\sum_{\lambda:|\lambda|\le m}\alpha_\lambda F_\lambda(x_1,\dots,x_n \mid \mho)=0$$

with $\alpha_\lambda\ne0$ for some λ. Choose μ such that $\alpha_\mu\ne0$ with minimal possible $|\mu|$. Then for all $\lambda\ne\mu$ either $|\lambda|<|\mu|$ and $\alpha_\lambda=0$, or $|\mu|\le|\lambda|$ and $F_\lambda(\mho(\mu) \mid \mho)=0$ by the definition of $F_\lambda(x_1,\dots,x_n \mid \mho)$.

Hence

$$\sum_{\lambda:|\lambda|\le m} \alpha_\lambda F_\lambda(\mathfrak{V}(\mu) \mid \mathfrak{V}) = \alpha_\mu F_\mu(\mathfrak{V}(\mu) \mid \mathfrak{V}) \neq 0,$$

leading to contradiction.

Note that the $\Bbbk$-dimension of $\Bbbk[x_1,\dots,x_n]^{S_n}_{\le m}$ is equal to the number of partitions $\lambda \in \mathbb{Y}^n$ such that $|\lambda| \le m$, hence the functions $F_\lambda(x_1,\dots,x_n \mid \mathfrak{V})$ with $|\lambda| \le m$ form a basis of $\Bbbk[x_1,\dots,x_n]^{S_n}_{\le m}$, proving (3). In particular, $F_\lambda(x_1,\dots,x_n \mid \mathfrak{V})$ with $|\lambda| \le m-1$ form a basis of $\Bbbk[x_1,\dots,x_n]^{S_n}_{\le m-1}$, so if $|\lambda| = m$ then the degree of $F_\lambda(x_1,\dots,x_n \mid \mathfrak{V})$ is m, proving (4). Finally, (2) implies (1) by considering the difference between two candidates for $F_\lambda(x_1,\dots,x_n \mid \mathfrak{V})$, so we only need to prove the former. Let $f \in \Bbbk[x_1,\dots,x_n]^{S_n}_{\le m}$ be such that $f(\mathfrak{V}(\mu)) = 0$ for all μ satisfying $|\mu| \le m$, and assume that $f \neq 0$. Consider the expansion

$$f = \sum_{\lambda:|\lambda|\le m} \alpha_\lambda F_\lambda(x_1,\dots,x_n \mid \mathfrak{V}).$$

and choose μ such that $\alpha_\mu \neq 0$ with minimal $|\mu|$. Then, in the same way as in the first part of the proof,

$$f(\mathfrak{V}(\mu)) = \sum_{\lambda:|\mu|\le|\lambda|\le m} \alpha_\lambda F_\lambda(\mathfrak{V}(\mu) \mid \mathfrak{V}) = \alpha_\mu F_\mu(\mathfrak{V}(\mu) \mid \mathfrak{V}) \neq 0,$$

leading to contradiction. □

Another degeneration of $\mathbb{F}_\lambda$ is obtained by considering the regime $q, x_i, a_i \to 1$ in $\widetilde{\mathbb{F}}_\lambda(x_1,\dots,x_n \mid \mathcal{A},\infty)$. More precisely, let $\varepsilon, d, r_1,\dots,r_n$ be parameters such that $d \neq 0$, and let $\mathcal{C} = (c_0, c_1, \dots)$ be an infinite sequence of real parameters. Define $\mathbb{F}^{el}_{\lambda/\mu}(r_1,\dots,r_n \mid \mathcal{C},\infty)$ as the limit

$$\mathbb{F}^{el}_{\lambda/\mu}(r_1,\dots,r_n \mid \mathcal{C},\infty) = (-d)^{\lambda_1-\mu_1} \lim_{\varepsilon\to 0} \varepsilon^{|\mu|-\mu_1-|\lambda|+\lambda_1} \widetilde{\mathbb{F}}_{\lambda/\mu}(x_1,\dots,x_n \mid \mathcal{A},\infty)$$

in the following regime:

$$x_i = e^{\varepsilon r_i}, \qquad a_i = e^{\varepsilon c_i}, \qquad q = e^{\varepsilon d}. \tag{2.4.7}$$

From (2.4.6) we get the following expression for the one-variable function

$$\mathbb{F}^{el}_{\lambda/\mu}(r \mid \mathcal{C},\infty) = \begin{cases} \displaystyle\prod_{i\ge 1} \frac{(\mu_i-\mu_{i+1})! \prod_{j=0}^{\lambda_i-\mu_i-1}(r-c_i-jd)}{(\lambda_i-\mu_i)!(\mu_i-\lambda_{i+1})!}, & \text{if } \mu \prec \lambda; \\ 0, & \text{otherwise,} \end{cases} \tag{2.4.8}$$

where we have used the following relation

$$\lim_{\varepsilon\to 0} \varepsilon^{-k} (e^{\varepsilon u}; e^{\varepsilon d})_k = (-1)^k \prod_{j=0}^{k-1} (u+jd).$$

From (2.4.8) we see that $\mathbb{F}^{el}_{\lambda/\mu}(r \mid \mathcal{C},\infty)$ is a polynomial in r of degree $|\lambda|-|\mu|$, with coefficients depending polynomially on d and c_i. Moreover, the top homogeneous degree term is $P_{\lambda/\mu}(r;1,0)$, that is, it is a $q=1, t=0$ specialization of the Macdonald polynomial. The branching rule (2.4.5) implies that $\mathbb{F}^{el}_{\lambda}(r_1,\dots,r_n \mid \mathcal{C},\infty)$ is a symmetric polynomial in $r_1,\dots,r_n$ of degree $|\lambda|$

with coefficients in $\Bbbk^{el} = \mathbb{Q}(d, \mathcal{C})$, and the top homogeneous degree of $\mathbb{F}^{el}_\lambda(r_1, \dots, r_n \mid \mathcal{C}, \infty)$ is $P_\lambda(r_1, \dots, r_n; 1, 0)$. From [60, VI.4] it is known that $P_\lambda(r_1, \dots, r_n; 1, 0) = e_{\lambda'}(r_1, \dots, r_n)$ is the elementary symmetric polynomial, where λ' is the partition conjugate to λ, and the elementary symmetric polynomials are defined by

$$e_\mu(x_1, \dots, x_n) = \prod_{i \geq 1} e_{\mu_i}(x_1, \dots, x_n),$$

$$e_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_n}.$$

We have the following analogues of the vanishing and characterization properties, obtained as limits of Proposition 2.4.4 and Proposition 2.4.5. For a partition $\mu \in \mathbb{Y}^n$ set

$$\mathbf{r}^n_{\mathcal{C}}(\mu) = (c_1 + (\mu_1 - \mu_2)d, c_2 + (\mu_2 - \mu_3)d, \dots, c_n + \mu_n d). \tag{2.4.9}$$

Proposition 2.4.7. *Let λ, μ be partitions of length at most n. Then we have*

$$\mathbb{F}^{el}_\lambda(\mathbf{r}^n_{\mathcal{C}}(\mu) \mid \mathcal{C}, \infty) = 0, \qquad \textit{unless } \lambda \subseteq \mu.$$

Moreover, when $\lambda = \mu$, we have

$$\mathbb{F}^{el}_\lambda(\mathbf{r}^n_{\mathcal{C}}(\lambda) \mid \mathcal{C}, \infty) = (-1)^{|\lambda|}(-d)^{\lambda_1} \prod_{i,j=1}^{n} \prod_{k=1}^{\lambda_{i+j} - \lambda_{i+j+1}} (c_{j+i} - c_i + d(\lambda_{i+1} - \lambda_i + k - 1)).$$

□

Proof. Note that in the regime (2.4.7)

$$\mathbb{F}^{el}_\lambda(\mathbf{r}^n_{\mathcal{C}}(\mu) \mid \mathcal{C}, \infty) = (-d)^{\lambda_1} \lim_{\varepsilon \to 0} \varepsilon^{-|\lambda| + \lambda_1} \widetilde{\mathbb{F}}_\lambda(\mathbf{x}^n_{\mathcal{A}}(\mu) \mid \mathcal{A}, \infty),$$

so we obtain the claim as the limit of Proposition 2.4.4. □

Proposition 2.4.8. *The function $\mathbb{F}^{el}_\lambda(r_1, \dots, r_n \mid \mathcal{C}, \infty)$ is the unique function satisfying*

(1) $\mathbb{F}^{el}_\lambda(r_1, \dots, r_n \mid \mathcal{C}, \infty) \in \Bbbk[r_1, \dots, r_n]^{S_n}_{\leq |\lambda|}$,

(2) $\mathbb{F}^{el}_\lambda(\mathbf{r}^n_{\mathcal{C}}(\mu) \mid \mathcal{C}, \infty) = 0$ *for any partition μ such that $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$,*

(3) $\mathbb{F}^{el}_\lambda(\mathbf{r}^n_{\mathcal{C}}(\lambda) \mid \mathcal{C}, \infty) = (-1)^{|\lambda|}(-d)^{\lambda_1} \prod_{i,j=1}^{n} \prod_{k=1}^{\lambda_{i+j} - \lambda_{i+j+1}} (c_{j+i} - c_i + d(\lambda_{i+1} - \lambda_i + k - 1))$;

and it is also the unique function satisfying

(1') $\mathbb{F}^{el}_\lambda(r_1, \dots, r_n \mid \mathcal{C}, \infty) \in \Bbbk[r_1, \dots, r_n]^{S_n}_{\leq |\lambda|}$, *identically to (1),*

(2') $\mathbb{F}^{el}_\lambda(\mathbf{r}^n_{\mathcal{C}}(\mu) \mid \mathcal{C}, \infty) = 0$ *for any partition μ such that $|\mu| < |\lambda|$,*

(3') *The top degree homogeneous component of $\mathbb{F}^{el}_\lambda(r_1, \dots, r_n \mid \mathcal{C}, \infty)$ has degree $|\lambda|$ and is equal to $e_{\lambda'}(r_1, \dots, r_n)$.*

Proof. By the discussion above and Proposition 2.4.7 $\mathbb{F}^{el}_{\lambda}(r_1,\dots,r_n \mid \mathcal{C},\infty)$ satisfies (1)-(3) and (1')-(3'). The uniqueness follows in the same way as in Theorem 2.4.5, by applying Lemma 2.4.6 with $\mathcal{V}(\mu)=\mathbf{r}^n_{\mathcal{C}}(\mu)$. □

In view of Proposition 2.4.8, we call $\mathbb{F}^{el}_{\lambda}(r_1,\dots,r_n \mid \mathcal{C},\infty)$ interpolation elementary polynomials.

Remark 2.4.9. For the later use we note that instead of considering the fields $\Bbbk=\mathbb{Q}(q,\mathcal{A}),\Bbbk^{cl}=\mathbb{Q}(d,\mathcal{C})$ we can consider q,a_i,d,c_i above as elements of an arbitrary field $\Bbbk$. The induced functions $\widetilde{\mathbb{F}}_{\lambda}(x_1,\dots,x_n \mid \mathcal{A},\infty)\in\Bbbk[x_1,\dots,x_n]^{S_n}$ are well-defined as long as q is not a root of unity, and our proofs of the vanishing property and the characterization for $\widetilde{\mathbb{F}}_{\lambda}(x_1,\dots,x_n \mid \mathcal{A},\infty)$ hold as long as $\widetilde{\mathbb{F}}_{\lambda}(\mathbf{x}^n_{\mathcal{A}}(\lambda) \mid \mathcal{A},\infty)\neq 0$, which is equivalent to $a_i/a_j\notin q^{\mathbb{Z}_{\geq 0}}$ for any i,j such that $i\neq j$. Similarly, the functions $\mathbb{F}^{el}_{\lambda}(r_1,\dots,r_n \mid \mathcal{C},\infty)\in\Bbbk[r_1,\dots,r_n]^{S_n}$ are well-defined as long as the characteristic of $\Bbbk$ is 0, while for the characterization property we also need $c_i-c_j\notin d\mathbb{Z}$ for any $i\neq j$.

2.5 Classification of interpolation symmetric polynomials

Our interest in the vanishing property from Proposition 2.4.4 and the characterization of Proposition 2.4.5 comes from the fact that the symmetric functions with similar properties were actively studied earlier. More precisely, there are roughly three classes of known symmetric polynomials with similar vanishing and characterization properties: factorial monomial polynomials, factorial Schur polynomials and interpolation Macdonald functions. In [70] these three classes and their variations were distinguished as the only solutions to a certain interpolation problem. In this section we describe how our new functions fit into this classification, expanding it.

2.5.1 Interpolation problem

Below we state the general interpolation problem from [70]. Let $\Bbbk$ be a field; for simplicity we assume that $\operatorname{char}\Bbbk=0$. As before, we use $\mathbb{Y}^n$ to denote the set of partitions of length at most n and $\Bbbk[x_1,\dots,x_n]^{S_n}_{\leq m}$ to denote the space of symmetric polynomials in n variables, of degree $\leq m$ and with coefficients in $\Bbbk$.

An *n-grid* is a map $\mathcal{V}:\mathbb{Y}^n\to\Bbbk^n$. An n-grid $\mathcal{V}$ is *non-degenerate* if for every $\lambda\in\mathbb{Y}^n$ there exists a symmetric polynomial $F_{\lambda}(x_1,\dots,x_n \mid \mathcal{V})$ satisfying:

1. $F_{\lambda}(x_1,\dots,x_n \mid \mathcal{V})\in\Bbbk[x_1,\dots,x_n]^{S_n}_{\leq|\lambda|}$;
2. $F_{\lambda}(\mathcal{V}(\mu) \mid \mathcal{V})=0$ for every $\mu\in\mathbb{Y}^n$ such that $|\mu|\leq|\lambda|,\lambda\neq\mu$;
3. $F_{\lambda}(\mathcal{V}(\lambda) \mid \mathcal{V})\neq 0$.

An n-grid $\mathcal{V}$ is called *perfect* if it is non-degenerate and the polynomials F_{λ} additionally satisfy the following vanishing property:

$$F_{\lambda}(\mathcal{V}(\mu) \mid \mathcal{V})=0, \qquad \text{unless } \lambda\subseteq\mu. \tag{2.5.1}$$

The main result of [70] is the following classification of all perfect grids of a certain form.

Theorem 2.5.1 ([70])**.** *Assume that $\mathcal{V}$ is an n-grid of the form*

$$\mathcal{V}(\lambda)=(f_1(\lambda_1),\dots,f_n(\lambda_n)).$$

Then $\mathcal{V}$ is perfect if and only if one of the following cases holds:

($\mathbf{E_1}$) $f_i(j) = c_j$ *where* $c_0, c_1, \ldots$ *are pairwise distinct elements of* $\Bbbk$;

($\mathbf{E_2}$) $f_i(j) = c_{j-i}$ *where* $\ldots, c_{-1}, c_0, c_1, \ldots$ *are pairwise distinct elements of* $\Bbbk$;

($\mathbf{I}$) $f_i(j) = a + bq^j t^i + cq^{-j}t^i$ *where* a, b, c, q, t *are elements of a field extension of* $\Bbbk$;

($\mathbf{II}$) $f_i(j) = \alpha + \beta j + \beta' i + \gamma(\beta j + \beta' i)^2$, *where* $\alpha, \beta, \beta', \gamma \in \Bbbk$;

($\mathbf{III}$) $f_i(j) = \alpha + \epsilon^j \epsilon'^i(\alpha' + \beta j + \beta' i)$ *where* $\epsilon, \epsilon' \in \{\pm 1\}$ *and* $\alpha, \alpha', \beta, \beta' \in \Bbbk$;

($\mathbf{IV}$) *This case only exists when* $n = 2$, *then* $f_1(j) = \alpha + \beta q^j, f_2(j) = \alpha + \beta' q^{-j}$ *where* $\alpha, \beta, \beta', q \in \Bbbk$.

All cases above should additionally satisfy $f_i(j) \neq f_{i'}(j')$ *for all integers* $i \leq i', j > j'$.

The polynomials $F_\lambda(x_1, \ldots, x_n \mid \mathcal{V})$ corresponding to the cases $\mathbf{E_1}, \mathbf{E_2}, \mathbf{I}$ above are called respectively factorial monomial polynomials, factorial Schur polynomials and interpolation Macdonald functions. The polynomial F_λ has degree $|\lambda|$, and in these three cases the top-degree homogeneous component of F_λ is respectively a monomial, Schur or Macdonald symmetric polynomial. Moreover, the functions F_λ for case **II** above include interpolation functions related to Jack polynomials, see [53], [72].

Propositions 2.4.4-2.4.8 for the functions $\widetilde{\mathbb{F}}_\lambda(x_1, \ldots, x_n \mid \mathcal{A}, \infty)$ and $\mathbb{F}^{el}_\lambda(r_1, \ldots, r_n \mid \mathcal{C}, \infty)$ indicate that there is another class of solutions to this interpolation problem, where n-grids have an alternative form $\mathcal{V}(\mu) = (f_1(\mu_1 - \mu_2), \ldots, f_n(\mu_n))$, *cf.* the expressions for $\mathbf{x}^n_{\mathcal{A}}(\mu)$ and $\mathbf{r}^n_{\mathcal{C}}(\mu)$ from (2.4.1),(2.4.9). In this section we show that the function $\widetilde{\mathbb{F}}, \mathbb{F}^{el}$ actually lead to all perfect grids of this alternative form, at least when $n \geq 3$.

Theorem 2.5.2. *Assume that* $n \geq 3$ *and* $\mathcal{V}$ *is an* n*-grid of the form*

$$\mathcal{V}(\lambda) = (f_1(\lambda_1 - \lambda_2), f_2(\lambda_2 - \lambda_3), \ldots, f_n(\lambda_n)).$$

Then $\mathcal{V}$ *is perfect if and only if the functions* f_i *have one of the following two forms:*

1. $f_i(k) = c + a_i q^k$ *for constants* $c, q, a_1, \ldots, a_n \in \Bbbk$ *such that* q *is not a root of unity,* $q \neq 0$ *and* $a_i/a_j \neq q^k$ *for any* $i \neq j, k \in \mathbb{Z}$. *In this case the functions* $F_\lambda(x_1, \ldots, x_n \mid \mathcal{V})$ *are proportional to* $\widetilde{\mathbb{F}}_\lambda(x_1 - c, x_2 - c, \ldots, x_n - c \mid \mathcal{A}, \infty)$, *where* a_i *are identified with elements of* $\mathcal{A}$.

2. $f_i(k) = c_i + kd$ *for constants* $d, c_1, c_2, \ldots, c_n \in \Bbbk$ *such that* $d \neq 0$ *and* $c_i - c_j \neq kd$ *for any* $i \neq j, k \in \mathbb{Z}$. *In this case the functions* $F_\lambda(x_1, \ldots, x_n \mid \mathcal{V})$ *are proportional to* $\mathbb{F}^{el}_\lambda(x_1, \ldots, x_n \mid \mathcal{C}, \infty)$, *where* c_i *are identified with elements of* $\mathcal{C}$.

The remainder of this section is devoted to the proof of Theorem 2.5.2.

Remark 2.5.3. Following the existing terminology, the characterization property from Proposition 2.4.5 allows us to call the functions $\widetilde{\mathbb{F}}$ interpolation q-Whittaker polynomials. Note that, while setting $t = 0$ reduces Macdonald polynomials to q-Whittaker polynomials, setting $t = 0$ in interpolation Macdonald polynomials does not result in any interpolation polynomials, because the characterization property for interpolation Macdonald polynomials does not survive in any form after setting $t = 0$.

Remark 2.5.4. Note that Theorem 2.5.2 does not cover perfect 2-grids of the form $\mho(\mu) = (f_1(\mu_1 - \mu_2), f_2(\mu_2))$. While both types of grids listed in Theorem 2.5.2 are well-defined and perfect when $n = 2$, numerical simulations suggest that there exist more general perfect grids when $n = 2$. However, our proof described below fails to describe the $n = 2$ case.

2.5.2 General properties of n-grids

We start with adapting some arguments from [70] to our setting. From now on we always assume that n-grids $\mho$ have the form

$$\mho(\lambda) = (\mho(1; \lambda_1 - \lambda_2), \mho(2; \lambda_2 - \lambda_3), \dots, \mho(n; \lambda_n)),$$

where $\mho(1; \cdot), \dots, \mho(n; \cdot)$ are functions $\mathbb{Z}_{\geq 0} \to \Bbbk$.

For later convenience, we below restate Lemma 2.4.6 in terms of non-degenerate grids:

Lemma 2.5.5. *Let $\mho$ be a non-degenerate n-grid.*

1. *The functions $F_\lambda(x_1, \dots, x_n \mid \mho)$ are uniquely determined up to a scalar;*
2. *Let $m \in \mathbb{Z}_{\geq 0}$. If $f \in \Bbbk[x_1, \dots, x_n]^{S_n}_{\leq m}$ and $f(\mho(\mu)) = 0$ for all $\mu \in \mathbb{Y}^n$ such that $|\mu| \leq m$, then $f = 0$.*
3. *For each $m \in \mathbb{Z}_{\geq 0}$ the functions $F_\lambda(x_1, \dots, x_n \mid \mho)$ with $|\lambda| \leq m$ form a basis of $\Bbbk[x_1, \dots, x_n]^{S_n}_{\leq m}$;*
4. *The degree of $F_\lambda(x_1, \dots, x_n \mid \mho)$ is $|\lambda|$.*

□

In particular, given a non-degenerate n-grid $\mho$ the functions $F_\lambda(x_1, \dots, x_n \mid \mho)$ are well-defined up to scalars. We will fix a convenient normalization later.

The following operations on grids will be useful. Let $\mho$ be an n-grid.

- For $m \in \mathbb{Z}_{\geq 1}$ such that $m \leq n$ define an m-grid $\mho_m$ by

  $$\mho_m(\lambda) = (\mho(1; \lambda_1 - \lambda_2), \mho(2; \lambda_2 - \lambda_3), \dots, \mho(m; \lambda_m)), \qquad \lambda \in \mathbb{Y}^m.$$

  That is, $\mho_m(i; j) = \mho(i; j)$ for $i = 1, \dots, m$.

- For $l \in \mathbb{Z}_{\geq 0}$ such that $l < n$ define an $(n - l)$-grid ${}_l\mho$ by

  $${}_l\mho(\lambda) = (\mho(l + 1; \lambda_1 - \lambda_2), \mho(l + 2; \lambda_2 - \lambda_3), \dots, \mho(n; \lambda_{n-l})), \qquad \lambda \in \mathbb{Y}^{n-l}.$$

  That is, ${}_l\mho(i; j) = \mho(i + l; j)$ for $i = 1, \dots, n - l$.

- For $k \in \mathbb{Z}$ define an n-grid $\mho^k$ by

  $$\mho^k(\lambda) = \mho(\lambda + k^n) = (\mho(1; \lambda_1 - \lambda_2), \dots, \mho(n - 1; \lambda_{n-1} - \lambda_n), \mho(n; \lambda_n + k)),$$

  where $\lambda + k^n$ denotes the partition with parts $\lambda_i + k$. In other words, $\mho^k(i; j) = \mho(i; j)$ for $i = 1, \dots, n - 1$ and $\mho^k(n, j) = \mho(n; j + k)$.

Our first goal is to show that the three operations above preserve perfect grids. We start with $\mho_m$.

Proposition 2.5.6. *Assume that $\mho$ is a non-degenerate n-grid. Then for any $m \leq n$ the m-grid $\mho_m$ is non-degenerate with*

$$F_\lambda(x_1, \dots, x_m \mid \mho_m) = F_\lambda(x_1, \dots, x_m, \mho(m+1,0), \dots, \mho(n,0) \mid \mho) \tag{2.5.2}$$

for each $\lambda \in \mathbb{Y}^m$. Moreover, if $\mho$ is perfect then $\mho_m$ is perfect as well.

Proof. For the first statement it is enough to show that the functions

$$f_\lambda(x_1, \dots, x_m) := F_\lambda(x_1, \dots, x_m, \mho(m+1,0), \dots, \mho(n,0) \mid \mho), \qquad \lambda \in \mathbb{Y}^m,$$

satisfy the defining properties of $F_\lambda(x_1, \dots, x_m \mid \mho_m)$, which readily follows from noticing that for $\mu \in \mathbb{Y}^m$

$$f_\lambda(\mho_m(\mu)) = F_\lambda(\mho(\mu) \mid \mho).$$

The last statement follows immediately from the vanishing property for $\mho$ and the identity above. □

Proposition 2.5.7. *For any non-degenerate n-grid $\mho$ the following statements hold:*

1. *$\mho(i;j) \neq \mho(n;0)$ for any integer pair $(i,j) \neq (n,0)$;*
2. *$\mho^1$ is non-degenerate and F_λ can be chosen so that*

$$F_\lambda(x_1, \dots, x_n \mid \mho^1) \prod_{i=1}^{n} (x_i - \mho(n;0)) = F_{\lambda+1^n}(x_1, \dots, x_n \mid \mho); \tag{2.5.3}$$

3. *If $\mho$ is perfect, then $\mho^1$ is perfect.*

Proof. Let $\lambda \in \mathbb{Y}^n$ and consider

$$f_\lambda(x_1, \dots, x_{n-1}) := F_{\lambda+1^n}(x_1, \dots, x_{n-1}, \mho(n;0) \mid \mho).$$

Note that $f_\lambda(x_1, \dots, x_{n-1})$ has degree at most $|\lambda| + n$, and for any $\mu \in \mathbb{Y}^{n-1}$ such that $|\mu| \leq |\lambda| + n$ we have

$$f_\lambda(\mho_{n-1}(\mu)) = F_{\lambda+1^n}(\mho(1;\mu_1 - \mu_2), \dots, \mho(n-1;\mu_{n-1}), \mho(n;0) \mid \mho) = F_{\lambda+1^n}(\mho(\mu) \mid \mho) = 0,$$

since $\mu \neq \lambda + 1^n$. Hence, by Lemma 2.5.5 we have $f_\lambda = 0$. In other words, $F_{\lambda+1^n}(x_1, \dots, x_n \mid \mho)\big|_{x_n = \mho(n;0)} = 0$, so $x_n - \mho(n;0)$ divides $F_{\lambda+1^n}(x_1, \dots, x_n \mid \mho)$. By symmetry, $x_i - \mho(n;0)$ for $i = 1, \dots, n$ divide $F_{\lambda+1^n}(x_1, \dots, x_n \mid \mho)$ and so

$$F_{\lambda+1^n}(x_1, \dots, x_n \mid \mho) = g_\lambda(x_1, \dots, x_n) \prod_{i=1}^{n} (x_i - \mho(n;0)) \tag{2.5.4}$$

for some $g_\lambda(x_1, \dots, x_n) \in \Bbbk[x_1, \dots, x_n]^{S_n}_{\leq |\lambda|}$.

To prove (1) take $j \in \mathbb{Z}_{\geq 0}$ and set $\lambda = (nj, (n-1)j, \dots, 2j, j)$. Recall that $F_{\lambda+1^n}(\mho(\lambda+1^n) \mid \mho) \neq 0$, so from (2.5.4) we have

$$F_{\lambda+1^n}(\mho(\lambda+1^n) \mid \mho) = g_\lambda(\mho(\lambda+1^n))(\mho(n;j+1) - \mho(n;0)) \prod_{i=1}^{n-1} (\mho(i;j) - \mho(n;0)) \neq 0.$$

Hence $\mho(i;j) \neq \mho(n;0)$ for any $i < n$ and $\mho(n;j+1) \neq \mho(n;0)$, which implies (1) since $j \in \mathbb{Z}_{\geq 0}$ was arbitrary.

To prove (2) note that from (2.5.4) we have for any $\mu \in \mathbb{Y}^n$

$$g_\lambda(\mho^1(\mu)) \prod_{i=1}^{n} (\mho^1(i; \mu_i - \mu_{i+1}) - \mho(n;0)) = F_{\lambda+1^n}(\mho^1(\mu) \mid \mho) = F_{\lambda+1^n}(\mho(\mu+1^n) \mid \mho). \quad (2.5.5)$$

By (1) we know that $\mho^1(i;j) - \mho(n;0) = \mho(i; j+\delta_{i,n}) - \mho(n;0) \neq 0$ for any $j \geq 0$, hence (2.5.5) implies that $F_{\lambda+1^n}(\mho(\mu+1^n) \mid \mho) = 0$ if and only if $g_\lambda(\mho^1(\mu)) = 0$. In particular, for any $\mu \in \mathbb{Y}^n$ such that $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$ we get $g_\lambda(\mho^1(\mu)) = 0$, while $g_\lambda(\mho^1(\lambda)) \neq 0$. Since the degree of $F_{\lambda+1^n}$ is $|\lambda| + n$, the degree of g_λ is not greater than $|\lambda|$. So, g_λ satisfies the defining properties of $F_\lambda(x_1, \dots, x_n \mid \mho^1)$, which proves (2).

To prove part (3) assume that $\mho$ is perfect. Then $F_{\lambda+1^n}(\mho(\mu+1^n) \mid \mho) = 0$ unless $\lambda \subseteq \mu$, and hence $g_\lambda(\mho^1(\mu)) = F_\lambda(\mho^1(\mu) \mid \mho^1) = 0$ unless $\lambda \subseteq \mu$. So the vanishing property for $\mho$ implies the vanishing property for $\mho^1$. □

Using Proposition 2.5.7 we can inductively get the following results.

Corollary 2.5.8. *If $\mho$ is non-degenerate then $\mho^k$ is non-degenerate for any $k \in \mathbb{Z}_{\geq 0}$. If $\mho$ is perfect then $\mho^k$ is perfect for any $k \in \mathbb{Z}_{\geq 0}$.* □

Corollary 2.5.9. *If $\mho$ is non-degenerate then $\mho(i;j) \neq \mho(i',j')$ for all pairs $(i,j) \neq (i',j')$.*

Proof. Let $\mho$ be a non-degenerate n-grid and fix $(i,j) \neq (i',j')$. Without loss of generality we can assume that $i \leq i'$. By Proposition 2.5.6 $\mho_{i'}$ is also non-degenerate, hence replacing $\mho$ by $\mho_{i'}$ we can assume that $i' = n$. If $i < n$, then $\mho^{j'}$ is non-degenerate, and by Proposition 2.5.7 applied to $\mho^{j'}$

$$\mho(i;j) = \mho^{j'}(i;j) \neq \mho^{j'}(n;0) = \mho(i';j').$$

If $i = i' = n$, then we can additionally assume that $j > j'$ and use Proposition 2.5.7 to get

$$\mho(n;j) = \mho^{j'}(n; j-j') \neq \mho^{j'}(n;0) = \mho(n;j').$$

□

It turns out that the converse to Corollary 2.5.9 is also true, so we can classify all non-degenerate grids:

Proposition 2.5.10. *An n-grid $\mho$ is non-degenerate if and only if $\mho(i;j) \neq \mho(i',j')$ for all $(i,j) \neq (i',j')$.*

Proof. By Corollary 2.5.9 we only need to prove that $\mho$ is non-degenerate if $\mho(i;j) \neq \mho(i',j')$ for all $(i,j) \neq (i',j')$. It is enough to prove the following statement: for any $\mho$ as in the statement, any

$m \in \mathbb{Z}_{\geq 0}$ and any function

$$\phi(\lambda) : \{\lambda \in \mathbb{Y}^n : |\lambda| \leq m\} \to \Bbbk$$

there exists a polynomial $f \in \Bbbk[x_1, \ldots, x_n]^{S_n}_{\leq m}$ such that $f(\mathcal{U}(\lambda)) = \phi(\lambda)$ for any $\lambda \in \mathbb{Y}^n, |\lambda| \leq m$.

We prove the latter claim by induction on n and m, reducing the claim for (n, m) to the claims for (n', m') with either $n' < n$ or $m' < m$. Note that the cases when $n = 1$ or $m = 0$ are trivial.

For the inductive step recall that the monomial symmetric functions $m_\lambda(x_1, \ldots, x_n)$ are defined by

$$m_\lambda(x_1, \ldots, x_n) = \sum_\alpha x_1^{\alpha_1} \ldots x_n^{\alpha_n},$$

where the sum is over all permutations α of the n-tuple $(\lambda_1, \ldots, \lambda_n)$. Let $n \geq 2$ and fix n-grid $\mathcal{U}$ as in the claim. Define a degree-preserving map

$$\mathrm{Sym} : \Bbbk[x_1, \ldots, x_{n-1}]^{S_{n-1}} \to \Bbbk[x_1, \ldots, x_n]^{S_n}$$

by sending $m_\lambda(x_1 - \mathcal{U}(n; 0), \ldots, x_{n-1} - \mathcal{U}(n; 0))$ to $m_\lambda(x_1 - \mathcal{U}(n; 0), \ldots, x_n - \mathcal{U}(n; 0))$. Note that for any $f \in \Bbbk[x_1, \ldots, x_n]^{S_n}$ we have

$$(\mathrm{Sym}\, f)(x_1, \ldots, x_{n-1}, \mathcal{U}(n; 0)) = f(x_1, \ldots, x_{n-1}). \tag{2.5.6}$$

Let $\phi(\lambda)$ be an arbitrary function on the partitions $\lambda \in \mathbb{Y}^n, |\lambda| \leq m$. We will construct the required function f as

$$f = \mathrm{Sym}\, f_1 + f_2 \prod_{i=1}^{n} (x_i - \mathcal{U}(n; 0)).$$

Consider the restriction of ϕ to the partitions from $\mathbb{Y}^{n-1}$. By the induction hypothesis, there exists a function in $n-1$ variables $f_1 \in \Bbbk[x_1, \ldots, x_{n-1}]^{S_{n-1}}_{\leq m}$ such that $f_1(\mathcal{U}_{n-1}(\lambda)) = \phi(\lambda)$ for all partitions $\lambda \in \mathbb{Y}^{n-1}, |\lambda| \leq m$. Since for all $\lambda \in \mathbb{Y}^{n-1}, |\lambda| \leq m$, the substitution $(x_1, \ldots, x_n) = \mathcal{U}(\lambda)$ sets $x_n = \mathcal{U}(n; 0)$, by (2.5.6) we have

$$(\mathrm{Sym}\, f_1)(\mathcal{U}(\lambda)) = f_1(\mathcal{U}_{n-1}(\lambda)) = \phi(\lambda), \qquad \lambda \in \mathbb{Y}^{n-1}, |\lambda| \leq m.$$

If $n < m$ then any partition λ such that $|\lambda| \leq m$ is in $\mathbb{Y}^{n-1}$ and we are done. If $n \geq m$ consider the following function ϕ_2 on $\{\lambda \in \mathbb{Y}^n : |\lambda| \leq m - n\}$:

$$\phi_2(\lambda) = \frac{\phi(\lambda + 1^n) - \mathrm{Sym}\, f_1(\mathcal{U}(\lambda + 1^n))}{(\mathcal{U}(n; 1) - \mathcal{U}(n; 0)) \prod_{i=1}^{n-1} (\mathcal{U}(i; \lambda_i - \lambda_{i+1}) - \mathcal{U}(n; 0))}.$$

By induction in m and since $\mathcal{U}^1$ also satisfies the assumptions on $\mathcal{U}$ from the statement, we can construct a function $f_2 \in \Bbbk[x_1, \ldots, x_n]^{S_n}_{\leq m-n}$ such that $f_2(\mathcal{U}(\lambda + 1^n)) = f_2(\mathcal{U}^1(\lambda)) = \phi_2(\lambda)$. The resulting function

$$f = \mathrm{Sym}\, f_1 + f_2 \prod_{i=1}^{n} (x_i - \mathcal{U}(n; 0))$$

satisfies $f \in \Bbbk[x_1, \ldots, x_n]^{S_n}_{\leq m}$ and $f(\mathcal{U}(\lambda)) = \phi(\lambda)$. □

We can use (2.5.2) and (2.5.3) to introduce a natural normalization for the functions F_λ. For an n-grid $\mathcal{U}$ let $\mathcal{U}^k_m$ denote the m-grid $(\mathcal{U}_m)^k$.

Proposition 2.5.11. *For a non-degenerate n-grid $\mathcal{U}$ there exists a unique choice of functions $F_\lambda(x_1,\dots,x_m \mid \mathcal{U}_m^k)$ for all $m=1,\dots,n$ and $k\in\mathbb{Z}_{\geq 0}$ satisfying*

1. *$F_\varnothing(x_1,\dots,x_m \mid \mathcal{U}_m^k)=1$ for all m,k;*
2. *F_λ are consistent with (2.5.2), that is, for any $\lambda\in\mathbb{Y}^{m-1}$*

$$F_\lambda(x_1,\dots,x_{m-1}\mid \mathcal{U}_{m-1}) = F_\lambda(x_1,\dots,x_{m-1},\mathcal{U}(m;k)\mid \mathcal{U}_m^k).$$

   Note that $(\mathcal{U}_m^k)_{m-1}=\mathcal{U}_{m-1}$.

3. *F_λ are consistent with (2.5.3), that is, for any $\lambda\in\mathbb{Y}^m$*

$$F_{\lambda+1^m}(x_1,\dots,x_m\mid \mathcal{U}_m^k) = F_\lambda(x_1,\dots,x_m\mid \mathcal{U}_m^{k+1})\prod_{i=1}^m (x_i-\mathcal{U}(m;k)).$$

This unique choice of functions $F_\lambda(x_1,\dots,x_m\mid\mathcal{U}_m^k)$ is determined by setting

$$F_\lambda(\mathcal{U}(\lambda)\mid\mathcal{U}) = \prod_{r\geq 1}\prod_{i=1}^{r}\prod_{j=\lambda_{r+1}}^{\lambda_r-1}(\mathcal{U}(i;\lambda_i-\lambda_{i+1})-\mathcal{U}(r;j-\lambda_{r+1})) \tag{2.5.7}$$

for any non-degenerate grid $\mathcal{U}$.

Proof. We use induction on n. When $n=1$ all functions $F_{(\lambda_1)}(x_1\mid\mathcal{U}^k)$ satisfying the conditions (1) and (3) from the statement must be of the form

$$F_{(\lambda_1)}(x_1\mid\mathcal{U}^k)=\prod_{j=0}^{\lambda_1-1}(x_1-\mathcal{U}(1;j+k))=\prod_{j=0}^{\lambda_1-1}(x_1-\mathcal{U}^k(1;j)),$$

and the claim follows.

Now assume that $\mathcal{U}$ is a non-degenerate n-grid and we have proved the claim for $\mathcal{U}_{n-1}$. Note that the functions $F_\lambda(x_1,\dots,x_m\mid\mathcal{U}_m^k)$ are uniquely determined by the numbers $H_\lambda(\mathcal{U}_m^k):=F_\lambda(\mathcal{U}_m^k(\lambda)\mid\mathcal{U}_m^k)$, defined for $\lambda\in\mathbb{Y}^m$. Moreover, if the choice of $F_\lambda(x_1,\dots,x_m\mid\mathcal{U}_m^k)$ satisfies conditions (1)-(3) from the statement, then by the inductive assumption applied to the $(n-1)$-grid $\mathcal{U}_{n-1}$ we have

$$H_\lambda(\mathcal{U}_m^k)=\prod_{r\geq 1}\prod_{i=1}^{r}\prod_{j=\lambda_{r+1}}^{\lambda_r-1}(\mathcal{U}_m^k(i;\lambda_i-\lambda_{i+1})-\mathcal{U}_m^k(r;j-\lambda_{r+1}))$$

for any $k\in\mathbb{Z}_{\geq 0}$, $m<n$ and $\lambda\in\mathbb{Y}^m$. So we only need to prove that $H_\lambda(\mathcal{U}^k)$ are uniquely determined and are given by (2.5.7).

Let $\lambda\in\mathbb{Y}^n$. If $l(\lambda)<n$, then by the consistency with (2.5.2) we have $H_\lambda(\mathcal{U}^k)=H_\lambda(\mathcal{U}_{n-1})$, and we are done. If $l(\lambda)=n$, then we can write $\lambda=\overline{\lambda}+a^n$ for some $\overline{\lambda}\in\mathbb{Y}^{n-1}$ and $a=\lambda_n$. Then, using consistency with (2.5.3), we have

$$H_{\overline{\lambda}+a^n}(\mathcal{U}^k)=H_{\overline{\lambda}}(\mathcal{U}^{k+a})\prod_{i=1}^{n}\prod_{j=0}^{a-1}(\mathcal{U}^{k+a}(i;\overline{\lambda}_i-\overline{\lambda}_{i+1})-\mathcal{U}^k(n;j)).$$

Since $\overline{\lambda} \in \mathbb{Y}^{n-1}$, we already know that $H_{\overline{\lambda}}(\mathcal{U}^{k+a})$ is givn by (2.5.7), and one can readily check that the resulting expression for $H_\lambda(\mathcal{U}^k) = H_{\overline{\lambda}+a^n}(\mathcal{U}^k)$ is also consistent with (2.5.7). $\square$

From now on we will assume that $\{F_\lambda\}_\lambda$ are normalized as in Proposition 2.5.11.

Proposition 2.5.12. *Let $\mathcal{U}$ be a perfect n-grid with $n \geq 2$. Then ${}_1\mathcal{U}$ is perfect and we have*

$$F_\lambda(x_1, \dots, x_n \mid \mathcal{U}) = x_1^{\lambda_1} F_{\widetilde{\lambda}}(x_2, \dots, x_n \mid {}_1\mathcal{U}) + \text{lower } x_1\text{-degree terms},$$

where $\widetilde{\lambda} = (\lambda_2, \lambda_3, \dots)$.

Proof. As before, let $m_\lambda(x_1, \dots, x_n)$ denote monomial symmetric function corresponding to λ, and let $\mu \preceq \lambda$ denote the lexicographical order on the partitions (that is $\mu \preceq \lambda$ if and only if either $\mu = \lambda$ or for some r we have $\lambda_i = \mu_i$ for $i < r$ and $\mu_r < \lambda_r$).

First we want to prove that if $\mathcal{U}$ is a non-degenerate n-grid and $\lambda \in \mathbb{Y}^n$ is a partition such that for $\mu \in \mathbb{Y}^n$

$$F_\lambda(\mathcal{U}(\mu) \mid \mathcal{U}) = 0 \qquad \text{unless } \lambda \subset \mu, \tag{2.5.8}$$

then we have

$$F_\lambda(x_1, \dots, x_n \mid \mathcal{U}) = \sum_{\mu \preceq \lambda} \alpha_{\mu\lambda} m_\mu(x_1, \dots, x_n). \tag{2.5.9}$$

We prove the latter statement by induction on n, with $n = 1$ case being trivial. Fix $\mathcal{U}, \lambda$ as above and let $d = \deg_{x_1} F_\lambda(x_1, \dots, x_n \mid \mathcal{U})$. If $d < \lambda_1$ then we are trivially done with $\alpha_{\lambda\lambda} = 0$, so consider the case when $d \geq \lambda_1$. Define $g(x_2, \dots, x_n)$ by

$$F_\lambda(x_1, \dots, x_n \mid \mathcal{U}) = x_1^d g(x_2, \dots, x_n) + \text{lower } x_1\text{-degree terms}.$$

Clearly, $g \neq 0$. Since the total degree of F_λ is λ, the total degree of g satisfies $\deg g = |\lambda| - d \leq |\lambda| - \lambda_1 = |\widetilde{\lambda}|$, where $\widetilde{\lambda} = (\lambda_2, \lambda_3, \dots)$. Let $\mu \in \mathbb{Y}^{n-1}$ be a partition such that $\widetilde{\lambda} \not\subset \mu$. Then the partition $\mu(k) = (\mu_1 + k, \mu_1, \mu_2, \dots)$ satisfies $\lambda \not\subset \mu(k)$ for any $k \in \mathbb{Z}_{\geq 0}$, so by assumption (2.5.8)

$$F_\lambda(\mathcal{U}(1; k), \mathcal{U}(2; \mu_1 - \mu_2), \dots, \mathcal{U}(n; \mu_{n-1})) = F_\lambda(\mathcal{U}(\mu(k)) \mid \mathcal{U}) = 0$$

for any k. In other words, the polynomial $F_\lambda(x_1, {}_1\mathcal{U}(1; \mu_1 - \mu_2), \dots, {}_1\mathcal{U}(n-1; \mu_{n-1}) \mid \mathcal{U})$ vanishes at $x_1 = \mathcal{U}(1; k)$, and since all these points are distinct by Corollary 2.5.9, this implies

$$F_\lambda(x_1, {}_1\mathcal{U}(1; \mu_1 - \mu_2), \dots, {}_1\mathcal{U}(n-1; \mu_{n-1}) \mid \mathcal{U}) = 0$$

as a polynomial in x_1. In particular, for $\mu \in \mathbb{Y}^{n-1}$ we get

$$g({}_1\mathcal{U}(\mu)) = g(\mathcal{U}(2; \mu_1 - \mu_2), \dots, \mathcal{U}(n; \mu_{n-1})) = 0 \qquad \text{unless } \widetilde{\lambda} \subset \mu. \tag{2.5.10}$$

Recall that the degree of g is at most $|\widetilde{\lambda}|$. By Proposition 2.5.10 the $(n-1)$-grid ${}_1\mathcal{U}$ is non-degenerate, so Lemma 2.5.5 and (2.5.10) imply that $g = c \cdot F_{\widetilde{\lambda}}(x_2, \dots, x_n \mid {}_1\mathcal{U})$ for some $c \in \Bbbk$. Note that $g \neq 0$, hence $c \neq 0$ and the degree of g is exactly $|\widetilde{\lambda}|$, forcing $d = \lambda_1$. Moreover, (2.5.10) implies that the assumption of our statement holds for the pair $({}_1\mathcal{U}, \widetilde{\lambda})$, hence by induction

$$g(x_2, \dots, x_n) = c \cdot F_{\widetilde{\lambda}}(x_2, \dots, x_n \mid {}_1\mathcal{U}) = \sum_{\mu \preceq \widetilde{\lambda}} \alpha'_{\mu\widetilde{\lambda}} m_\mu(x_2, \dots, x_n).$$

This implies (2.5.9):

$$F_\lambda(x_1,\dots,x_n \mid \mho) = \sum_{\mu \preceq \widetilde{\lambda}} \alpha'_{\mu\widetilde{\lambda}} x_1^{\lambda_1} m_\mu(x_2,\dots,x_n) + \text{lower } x_1\text{-degree terms.}$$

Thus, we have proved that if $\mho$ is perfect, then the transition matrix expressing $F_\lambda(x_1,\dots,x_n \mid \mho)$ in terms of $m_\lambda(x_1,\dots,x_n)$ is upper-triangular with respect to the lexicographical order. But both families of functions form bases of the ring of symmetric polynomials, hence the transition matrix must be invertible, and $\alpha_{\lambda\lambda} \neq 0$ in (2.5.9). In particular, in the argument above $d = \lambda_1$ always holds and the case $d < \lambda_1$ never happens, so the construction of $g(x_2,\dots,x_n)$ can be performed for any λ:

$$F_\lambda(x_1,\dots,x_n \mid \mho) = c_\lambda x_1^{\lambda_1} F_{\widetilde{\lambda}}(x_2,\dots,x_n \mid {}_1\mho) + \text{lower } x_1\text{-degree terms}$$

for $c_\lambda \neq 0$. Moreover, (2.5.10) implies that ${}_1\mho$ is perfect.

It only remains to show that $c_\lambda = 1$. Note that it is enough to show that for perfect $\mho$ we always have $\alpha_{\lambda\lambda} = 1$ in (2.5.9). Recall that if $\mho$ is perfect then all grids of the form $\mho^k_m$ are also perfect, so we can define a renormalization $\widetilde{F}_\lambda(x_1,\dots,x_m \mid \mho^k_m)$ such that

$$\widetilde{F}_\lambda(x_1,\dots,x_m \mid \mho^k_m) = m_\lambda(x_1,\dots,x_m) + \sum_{\mu \preceq \lambda, \mu \neq \lambda} \alpha_{\mu\lambda} m_\mu(x_1,\dots,x_m)$$

and $\widetilde{F}_\lambda$ is proportional to F_λ. Then the functions $\widetilde{F}_\lambda(x_1,\dots,x_m \mid \mho^k_m)$ satisfy the three conditions of Proposition 2.5.11, hence $\widetilde{F}_\lambda = F_\lambda$, finishing the proof. □

2.5.3 Pieri rule and explicit expressions for some F_λ

Our next goal is to prove Theorem 2.5.2 when $n = 3$. From here on our arguments are completely different from [70], and we will need to compute polynomials F_λ for perfect 2-grids. To do it, we use the following analogue of the Pieri rule:

Lemma 2.5.13. *Let $\mho$ be a perfect 2-grid. Then for any $k \in \mathbb{Z}_{\geq 1}$ we have*

$$\begin{aligned}(x_1 + x_2 - \mho(1;k) - \mho(2;0))F_{(k)}(x_1, x_2 \mid \mho) \\ = F_{(k+1)}(x_1, x_2 \mid \mho) + \kappa_k(\mho)(x_1 - \mho(2;0))(x_2 - \mho(2;0))F_{(k-1)}(x_1, x_2 \mid \mho^1),\end{aligned}$$

where

$$\kappa_k(\mho) = \frac{(\mho(1;k-1) + \mho(2;1) - \mho(1;k) - \mho(2;0))F_{(k)}(\mho(1;k-1), \mho(2;1) \mid \mho)}{(\mho(2;1) - \mho(2;0))(\mho(1;k-1) - \mho(2;0)) \prod_{i=0}^{k-2} (\mho(1;k-1) - \mho(1;i))}.$$

Proof. Fix $k \geq 1$ and set

$$f(x_1, x_2) = (x_1 + x_2 - \mho(1;k) - \mho(2;0))F_{(k)}(x_1, x_2 \mid \mho).$$

Note that $f \in \Bbbk[x_1, x_2]^{S_2}_{\leq k+1}$, and moreover $f(\mho(\mu)) = 0$ for all μ such that $|\mu| \leq k$: for $\mu \neq \lambda$ we have $F_{(k)}(\mho(\mu) \mid \mho) = 0$, while for $\mu = \lambda$ we set $x_1 = \mho(1;k), x_2 = \mho(2;0)$ and $x_1 + x_2 - \mho(1;k) - \mho(2;0)$

vanishes. Hence the expression

$$f(x_1, x_2) - \alpha_k F_{(k+1)}(x_1, x_2 \mid \mho) - \beta_k F_{(k,1)}(x_1, x_2 \mid \mho)$$

with

$$\alpha_k = \frac{f(\mho((k+1)))}{F_{(k+1)}(\mho((k+1)) \mid \mho)}, \qquad \beta_k = \frac{f(\mho((k,1)))}{F_{(k,1)}(\mho((k,1)) \mid \mho)}$$

vanishes for any $(x_1, x_2) = \mho(\mu)$ with $|\mu| \leq k+1$, so by Lemma 2.5.5 we have

$$f(x_1, x_2) = \alpha_k F_{(k+1)}(x_1, x_2 \mid \mho) + \beta_k F_{(k,1)}(x_1, x_2 \mid \mho). \tag{2.5.11}$$

Note that by Lemma 2.5.12 the top x_1-degree term on the sides of (2.5.11) are x_1^{k+1} and $\alpha_k x_1^{k+1}$, hence $\alpha_k = 1$. By (2.5.7), we have

$$F_{(k,1)}(\mho((k,1)) \mid \mho) = (\mho(1; k-1) - \mho(2,0))(\mho(2;1) - \mho(2;0)) \prod_{i=0}^{k-2} (\mho(1; k-1) - \mho(1; i)),$$

hence $\beta_k = \kappa_k(\mho)$ from the statement. Finally, by Proposition 2.5.7 we have

$$F_{(k,1)}(x_1, x_2 \mid \mho) = (x_1 - \mho(2;0))(x_2 - \mho(2;0)) F_{(k-1)}(x_1, x_2 \mid \mho^1),$$

finishing the proof. □

Using Lemma 2.5.13 we can get explicit expressions for $F_{(k)}(x_1, x_2 \mid \mho)$ when $k = 1, 2, 3$. Let $\mho$ be a perfect 2-grid. To make expressions below manageable we use the notation $[i; j] := \mho(i; j)$. By our choice in Proposition 2.5.11, we clearly have $F_{\varnothing}(x_1, x_2 \mid \mho) = 1$. For $F_{(1)}(x_1, x_2 \mid \mho)$ recall that this is a degree 1 symmetric polynomial vanishing at $(x_1, x_2) = \mho(\varnothing)$ and with top term $x_1 + x_2$ by Proposition (2.5.12). Hence

$$F_{(1)}(x_1, x_2 \mid \mho) = x_1 + x_2 - [1;0] - [2;0].$$

From now we can use Lemma 2.5.13. Direct computations give

$$\kappa_1(\mho) = \frac{[1;0] + [2;1] - [1;1] - [2;0]}{[1;0] - [2;0]};$$

$$\begin{aligned} F_{(2)}(x_1, x_2 \mid \mho) = x_1^2 + x_2^2 + \frac{[1;0] + [1;1] - [2;0] - [2;1]}{[1;0] - [2;0]} x_1 x_2 - \frac{[1;0]^2 + [1;0][1;1] - [2;0]^2 - [2;0][2;1]}{[1;0] - [2;0]} (x_1 + x_2) \\ + \frac{[1;0]^2[1;1] + [1;0]^2[2;0] - [1;0][2;0]^2 - [2;0]^2[2;1]}{[1;0] - [2;0]}; \end{aligned} \tag{2.5.12}$$

$$\kappa_2(\mho) = \frac{([1;1] + [2;1] - [1;2] - [2;0])([1;0] + [1;1] - [2;0] - [2;1])}{([1;0] - [2;0])([1;1] - [2;0])};$$

$$F_{(3)}(x_1,x_2 \mid \mathcal{O}) = (x_1+x_2-[1;2]-[2;0])F_{(2)}(x_1,x_2 \mid \mathcal{O})-\kappa_2(\mathcal{O})(x_1-[2;0])(x_2-[2;0])(x_1+x_2-[1;0]-[2;1]). \tag{2.5.13}$$

For Lemma 2.5.16 below we also need an explicit expression for $F_{(2)}(x_1,x_2,x_3 \mid \mathcal{O})$ when $\mathcal{O}$ is a perfect 3-grid such that $\mathcal{O}(3;0) = 0$. To get $F_{(2)}(x_1,x_2,x_3 \mid \mathcal{O})$ we can use the expression for $F_{(2)}(x_1,x_2 \mid \mathcal{O}_2)$ above and follow the construction from Proposition 2.5.10 to get

$$F_{(2)}(x_1,x_2,x_3 \mid \mathcal{O}) = \operatorname{Sym} F_{(2)}(x_1,x_2 \mid \mathcal{O}_2) = x_1^2+x_2^2+x_3^2+\frac{[1;0]+[1;1]-[2;0]-[2;1]}{[1;0]-[2;0]}(x_1x_2+x_1x_3+x_2x_3)$$
$$-\frac{[1;0]^2+[1;0][1;1]-[2;0]^2-[2;0][2;1]}{[1;0]-[2;0]}(x_1+x_2+x_3)+\frac{[1;0]^2[1;1]+[1;0]^2[2;0]-[1;0][2;0]^2-[2;0]^2[2;1]}{[1;0]-[2;0]}. \tag{2.5.14}$$

2.5.4 $n = 3$ case

Now we can prove Theorem 2.5.2 for perfect 3-grids. The main idea is to use vanishings $F_\lambda(\mathcal{O}(\mu) \mid \mathcal{O}) = 0$ for various $\lambda \not\subset \mu$ to obtain enough constraints on $\mathcal{O}(i;j)$ for the desired classification.

Lemma 2.5.14. *Let $\mathcal{O}$ be a perfect n-grid. Then for any $i,j \in \mathbb{Z}_{\geq 1}$ such that $i \leq n-1$ we have*

$$(\mathcal{O}(i;1)-\mathcal{O}(i+1;j-1))\,(\mathcal{O}(i;1)-\mathcal{O}(i+1;j+1)) = (\mathcal{O}(i;0)-\mathcal{O}(i+1;j))(\mathcal{O}(i;2)-\mathcal{O}(i+1;j)). \tag{2.5.15}$$

Proof. Replacing $\mathcal{O}$ by the 2-grid ${}_{i-1}\mathcal{O}_{i+1}$, which is perfect by Propositions 2.5.6, 2.5.12, we can assume that $i = 1$ and $\mathcal{O}$ is a perfect 2-grid. Moreover, replacing $\mathcal{O}$ by $\mathcal{O}^{j-1}$, which is perfect by Corollary 2.5.8, we can assume that $j = 1$.

So, it is enough to prove that for a perfect 2-grid $\mathcal{O}$ we have

$$(\mathcal{O}(1;1)-\mathcal{O}(2;0))\,(\mathcal{O}(1;1)-\mathcal{O}(2;2)) = (\mathcal{O}(1;0)-\mathcal{O}(2;1))(\mathcal{O}(1;2)-\mathcal{O}(2;1)).$$

This follows from the vanishing $F_{(3)}(\mathcal{O}((2,2)) \mid \mathcal{O}) = 0$: by explicit computation using (2.5.13) we have

$$F_{(3)}(\mathcal{O}((2,2)) \mid \mathcal{O}) = F_{(3)}(\mathcal{O}(1;0),\mathcal{O}(2;2) \mid \mathcal{O}) = \frac{(\mathcal{O}(2;0)-\mathcal{O}(2;2))(\mathcal{O}(2;1)-\mathcal{O}(2;2))}{\mathcal{O}(1;1)-\mathcal{O}(2;0)}G,$$

where

$$G = (\mathcal{O}(1;0)-\mathcal{O}(2;1))(\mathcal{O}(1;2)-\mathcal{O}(2;1))-(\mathcal{O}(1;1)-\mathcal{O}(2;0))\,(\mathcal{O}(1;1)-\mathcal{O}(2;2)).$$

Since $\mathcal{O}$ is non-degenerate, $\mathcal{O}(i;j) \neq \mathcal{O}(i';j')$ as long as $(i,j) \neq (i',j')$, hence $F_{(3)}(\mathcal{O}((2,2)) \mid \mathcal{O}) = 0$ implies $G = 0$ and the claim follows. □

From now on, let $\mathcal{O}$ be a perfect 3-grid.

Lemma 2.5.15. *We have*

$$[3;1] = \frac{[1;0][2;1]-[1;1][2;0]-[2;1][3;0]+[1;1][3;0]}{[1;0]-[2;0]}. \tag{2.5.16}$$

Proof. We first note that if a grid $\mathcal{O}$ is perfect then for any constant $c \in k$ the grid $\mathcal{O} + c$, defined by $(\mathcal{O} + c)(i; j) = \mathcal{O}(i; j) + c$, is also perfect with

$$F_\lambda(x_1, x_2, x_3 \mid \mathcal{O} + c) = F_\lambda(x_1 - c, x_2 - c, x_3 - c \mid \mathcal{O}).$$

On the other hand, the desired identity (2.5.16) is equivalent to

$$([2; 1] - [3; 1])([1; 0] - [2; 0]) - ([1; 1] - [2; 1])([2; 0] - [3; 0]) = 0.$$

The left-hand side above clearly stays intact when we change $\mathcal{O}$ to $\mathcal{O} + c$, so it is enough to prove the claim for the perfect 3-grid $\mathcal{O} - [3; 0]$. In other words, it is enough to consider the case $[3; 0] = 0$.

Since $\mathcal{O}$ is perfect, we must have $F_{(2)}(\mathcal{O}((1, 1, 1)) \mid \mathcal{O}) = 0$. Using (2.5.14), we obtain by explicit computation

$$F_{(2)}([1; 0], [2; 0], [3; 1] \mid \mathcal{O}) = \frac{[3; 1]([1; 0][3; 1] - [2; 0][3; 1] + [1; 1][2; 0] - [1; 0][2; 1])}{[1; 0] - [2; 0]}.$$

Since $\mathcal{O}$ is non-degenerate and $[3; 0] = 0$, we have $[3; 1] \neq 0$. Hence $F_{(2)}(\mathcal{O}((1, 1, 1)) \mid \mathcal{O}) = 0$ implies

$$[1; 0][3; 1] - [2; 0][3; 1] + [1; 1][2; 0] - [1; 0][2; 1] = 0,$$

which is equivalent to the claim when $[3; 0] = 0$. □

Lemma 2.5.16. *We have*

$$([2; 1] - [1; 2])([2; 0] - [1; 0]) = ([2; 1] - [1; 1])([2; 0] - [1; 1]).$$

Proof. Similarly to the proof of Lemma 2.5.15, the claim for the grid $\mathcal{O}$ is equivalent to the claim for a grid $\mathcal{O} + c$ for arbitrary c. So, to simplify computations, we can assume that $[3; 0] = 0$ throughout the proof without loss of generality.

To prove the claim we consider two different expressions for $[3; 2]$ in terms of $[1; 0]$, $[1; 1]$, $[1; 2]$, $[2; 0]$, $[2; 1]$, and show that equating them implies the claim. For the first expression we use Lemma 2.5.15 for the grids $\mathcal{O}$ and $\mathcal{O}^1$, obtaining

$$[3; 1] = \frac{[1; 0][2; 1] - [1; 1][2; 0]}{[1; 0] - [2; 0]},$$

$$[3; 2] = \frac{[1; 0][2; 1] - [1; 1][2; 0] - [2; 1][3; 1] + [1; 1][3; 1]}{[1; 0] - [2; 0]}.$$

The other expression comes from Lemma 2.5.14 applied to $(i, j) = (1, 1)$ and $(i, j) = (2, 1)$, leading to

$$[2; 2] = [1; 1] + \frac{([2; 1] - [1; 0])([2; 1] - [1; 2])}{[2; 0] - [1; 1]},$$

$$[3; 2] = [2; 1] - \frac{([3; 1] - [2; 0])([3; 1] - [2; 2])}{[2; 1]}.$$

Subtracting the two expressions for $[3;2]$ from each other we get

$$\frac{[1;0][2;1]-[1;1][2;0]-[2;1][3;1]+[1;1][3;1]}{[1;0]-[2;0]}-[2;1]+\frac{([3;1]-[2;0])([3;1]-[2;2])}{[2;1]}=0,$$

which, after algebraic manipulations, gives

$$\frac{([3;1]-[2;0])([1;1][2;1]-[2;1]^2+[3;1]([1;0]-[2;0])-[2;2]([1;0]-[2;0]))}{[2;1]([1;0]-[2;0])}=0.$$

Since $[3;1]\neq[2;0]$, we get

$$[1;1][2;1]-[2;1]^2+[3;1]([1;0]-[2;0])-[2;2]([1;0]-[2;0])=0.$$

Plugging the values for $[3;1],[2;2]$ above results in

$$-([2;1]-[1;1])([2;1]-[1;0])+\frac{([2;1]-[1;0])([2;1]-[1;2])([1;0]-[2;0])}{[1;1]-[2;0]}=0,$$

and since $[2;0]\neq[1;1]$ and $[2;1]\neq[1;0]$, we finally get

$$-([2;1]-[1;1])([2;0]-[1;1])+([2;1]-[1;2])([2;0]-[1;0])=0.$$

□

The constraints proved above are almost sufficient to prove the classification for $n=3$. Namely, we can now prove the following:

Lemma 2.5.17. *One of the following cases holds for the perfect 3-grid $\mho$:*

1. *For some constants $c,q,a_1,a_2,a_3\in\Bbbk$ we have*

$$[1;0]=c+a_1,\qquad [1;1]=c+a_1q,\qquad [1;2]=c+a_1q^2,$$
$$[2;k]=c+a_2q^k,\quad [3;k]=c+a_3q^k,\qquad k\in\mathbb{Z}_{\geq 0}.$$

2. *For some constants $d,c_1,c_2,c_3\in\Bbbk$ we have*

$$[1;0]=c_1,\qquad [1;1]=c_1+d,\qquad [1;2]=c_1+2d,$$
$$[2;k]=c_2+kd,\quad [3;k]=c_3+kd,\qquad k\in\mathbb{Z}_{\geq 0}.$$

Note that in order to satisfy the non-degeneracy condition from Corollary 2.5.9, we have to assume that $q\neq 0$ and q is not a root of unity in the first case.

Proof. Set

$$q:=\frac{[2;1]-[1;1]}{[2;0]-[1;0]}.$$

We have 2 cases.

Case 1: $q \neq 1$**.** Set

$$a_1 := \frac{[1;0]-[1;1]}{1-q}, \qquad c := [1;0]-a_1, \qquad a_2 := [2;0]-c, \qquad a_3 := [3;0]-c,$$

so we have

$$[1;0] = c + a_1, \qquad [1;1] = c + a_1 q, \qquad [2;0] := c + a_2, \qquad [2;1] = c + a_2 q, \qquad [3;0] = c + a_3.$$

Plugging these values into the constraint from Lemma 2.5.16 we get

$$(c + a_2 q - [1;2])(a_2 - a_1) = (a_2 q - a_1 q)(a_2 - a_1 q) = (a_2 q - a_1 q^2)(a_2 - a_1).$$

Note that since $[1;0] \neq [2;0]$, we have $a_1 \neq a_2$, and hence the equation above implies $[1;2] = c + a_1 q^2$.

To prove that $[2;k] = c + a_2 q^k$ we use induction on k, with cases $k = 0, 1$ already covered. For the inductive step, assume that $[2;k-1] = c + a_2 q^{k-1}, [2;k] = c + a_2 q^k$, and apply Lemma 2.5.14 with $(i,j) = (1,k)$. This gives

$$([2;k+1] - c - a_1 q)(a_2 q^{k-1} - a_1 q) = (a_2 q^k - a_1)(a_2 q^k - a_1 q^2) = (a_2 q^{k+1} - a_1 q)(a_2 q^{k-1} - a_1 q).$$

Since $[2;k-1] \neq [1;1]$ implies $a_2 q^{k-1} - a_1 q \neq 0$, we get $[2;k+1] = c + a_2 q^{k+1}$.

Similarly, we can use induction to prove that $[3;k] = c + a_3 q^k$. We already know the case $k = 0$. Applying Lemma 2.5.15 to the perfect grid $\mathfrak{V}^k$, we get

$$[3;k+1] = \frac{[1;0][2;1] - [1;1][2;0] - [2;1][3;k] + [1;1][3;k]}{[1;0] - [2;0]} = c(1-q) + q[3;k],$$

which implies the inductive step and finishes the proof of this case.

Case 2: $q = 1$**.** The argument is similar to the previous case. Set

$$c_1 := [1;0], \qquad c_2 := [2;0], \qquad c_3 := [3;0], \qquad d := [1;1] - [1;0] = [2;1] - [2;0],$$

where the last equality follows from $q = 1$. Then we have

$$[1;0] = c_1, \qquad [1;1] = c_1 + d, \qquad [2;0] = c_2, \qquad [2;1] = c_2 + d, \qquad [3;0] = c_3.$$

Plugging these values into Lemma 2.5.16 we get

$$(c_2 + d - [1;2])(c_2 - c_1) = (c_2 - c_1)(c_2 - c_1 - d).$$

Since $[1;0] \neq [2;0]$, we have $c_1 \neq c_2$ and hence the equation above implies $[1;2] = c_1 + 2d$.

To prove that $[2;k] = c_2 + kd$ and $[3;k] = c_3 + kd$ we can use Lemma 2.5.14 and Lemma 2.5.15 in exactly the same way as in Case 1. □

At this point, in order to reach the classification from Theorem 2.5.2 for $n = 3$, we only need to determine the values of $[1;k]$ for $k \geq 3$. To do so we focus on the 2-grid $\mathfrak{V}_2$ and, considering each case separately, compute the functions $F_{(k)}(x_1, x_2 \mid \mathfrak{V}_2)$. The result is summarized below.

Lemma 2.5.18. *Let $\widetilde{\mathfrak{V}}$ be a perfect 2-grid.*

1. *Assume that for some $q, c, a_1, a_2 \in \Bbbk$ we have*

$$\widetilde{\mathfrak{V}}(1;0) = c + a_1, \qquad \widetilde{\mathfrak{V}}(1;1) = c + a_1 q, \qquad \widetilde{\mathfrak{V}}(2;k) = c + a_2 q^k, \quad k \in \mathbb{Z}_{\geq 0}.$$

Then we have $\widetilde{\mathfrak{V}}(1;k) = c + a_1 q^k$ for all k and

$$F_{(k)}(x_1 + c, x_2 + c \mid \widetilde{\mathfrak{V}}) = \sum_{i=0}^{k} x_1^i x_2^{k-i} \frac{(a_1/x_1;q)_i (a_2/x_2;q)_{k-i} (q;q)_k}{(q;q)_i (q;q)_{k-i}}, \tag{2.5.17}$$

where we have shifted the variables x_1, x_2 by c.

2. *Assume that for some $d, c_1, c_2 \in \Bbbk$ we have*

$$\widetilde{\mathfrak{V}}(1;0) = c_1, \qquad \widetilde{\mathfrak{V}}(1;1) = c_1 + d, \qquad \widetilde{\mathfrak{V}}(2;k) = c_2 + kd, \quad k \in \mathbb{Z}_{\geq 0}.$$

Then we have $\widetilde{\mathfrak{V}}(1;k) = c_1 + kd$ for all k and

$$F_{(k)}(x_1, x_2 \mid \widetilde{\mathfrak{V}}) = \prod_{i=0}^{k-1} (x_1 + x_2 - c_1 - c_2 - i\, d). \tag{2.5.18}$$

Proof. For both parts we simultaneously prove the expressions for $\widetilde{\mathfrak{V}}(1;k)$ and $F_{(k+1)}(x_1, x_2 \mid \widetilde{\mathfrak{V}})$ by induction on k.

Part 1. Note that without loss of generality we can replace $\widetilde{\mathfrak{V}}$ by $\widetilde{\mathfrak{V}} - c$ and assume that $c = 0$. When $k = 0$ we have $\widetilde{\mathfrak{V}}(1;0) = a_1$ and

$$F_{(1)}(x_1, x_2 \mid \widetilde{\mathfrak{V}}) = x_1 + x_2 - \widetilde{\mathfrak{V}}(1;0) - \widetilde{\mathfrak{V}}(2;0) = x_1 - a_1 + x_2 - a_2.$$

When $k = 1$ we have $\widetilde{\mathfrak{V}}(1;1) = a_1 q$ and from (2.5.12) one can obtain

$$F_{(2)}(x_1, x_2 \mid \widetilde{\mathfrak{V}}) = (x_1 - a_1)(x_1 - a_1 q) + (1 + q)(x_1 - a_1)(x_2 - a_2) + (x_2 - a_2)(x_2 - a_2 q).$$

Now assume that $k \geq 2$ and we have proved the claim for $\widetilde{\mathfrak{V}}(1;i)$ and $F_{(i+1)}(x_1, x_2 \mid \widetilde{\mathfrak{V}})$ with $i < k$. Applying the Pieri rule from Lemma 2.5.13, we have

$$\begin{aligned}(x_1 + x_2 - \widetilde{\mathfrak{V}}(1;k) - a_2) F_{(k)}(x_1, x_2 \mid \widetilde{\mathfrak{V}}) \\ = F_{(k+1)}(x_1, x_2 \mid \widetilde{\mathfrak{V}}) + \kappa_k(\widetilde{\mathfrak{V}})(x_1 - a_2)(x_2 - a_2) F_{(k-1)}(x_1, x_2 \mid \widetilde{\mathfrak{V}}^1),\end{aligned}$$

where

$$\kappa_k(\widetilde{\mathfrak{V}}) = \frac{(a_1 q^{k-1} + a_2 q - \widetilde{\mathfrak{V}}(1;k) - a_2) F_{(k)}(a_1 q^{k-1}, a_2 q \mid \widetilde{\mathfrak{V}})}{(a_2 q - a_2)(a_1 q^{k-1} - a_2) \prod_{i=0}^{k-2} (a_1 q^{k-1} - a_1 q^i)}. \tag{2.5.19}$$

Note that when for any $j = 0, \ldots, k$ we plug $x_1 = a_1 q^{k-j}$ and $x_2 = a_2 q^j$ in the right-hand side of

(2.5.17), there is only one non-zero term in the sum and this term corresponds to $i = j$. So, we have

$$F_{(k)}(\mathfrak{U}((k,j)) \mid \widetilde{\mathfrak{U}}) = F_{(k)}(a_1 q^{k-j}, a_2 q^j \mid \widetilde{\mathfrak{U}}) = a_1^{k-j} a_2^j q^{(k-j)^2+j^2} \frac{(q^{-k+j};q)_{k-j}(q^{-j};q)_j(q;q)_k}{(q;q)_{k-j}(q;q)_j}. \tag{2.5.20}$$

Plugging this expression into (2.5.19), we get

$$\kappa_k(\widetilde{\mathfrak{U}}) = \frac{(a_1 q^{k-1} + a_2 q - \widetilde{\mathfrak{U}}(1;k) - a_2)(1-q^k)}{(a_1 q^{k-1} - a_2)(1-q)},$$

and, consequently,

$$\begin{aligned} F_{(k+1)}(x_1, x_2 \mid \widetilde{\mathfrak{U}}) &= (x_1 + x_2 - \widetilde{\mathfrak{U}}(1;k) - a_2) F_{(k)}(x_1, x_2 \mid \widetilde{\mathfrak{U}}) \\ &\quad - \frac{(a_1 q^{k-1} + a_2 q - \widetilde{\mathfrak{U}}(1;k) - a_2)(1-q^k)}{(a_1 q^{k-1} - a_2)(1-q)} (x_1 - a_2)(x_2 - a_2) F_{(k-1)}(x_1, x_2 \mid \widetilde{\mathfrak{U}}^1). \end{aligned} \tag{2.5.21}$$

Since $\widetilde{\mathfrak{U}}$ is perfect, we must have $F_{(k+1)}(\widetilde{\mathfrak{U}}((k,2)) \mid \widetilde{\mathfrak{U}}) = F_{(k+1)}(a_1 q^{k-2}, a_2 q^2 \mid \widetilde{\mathfrak{U}}) = 0$. Hence the expression above implies

$$\begin{aligned} &(a_1 q^{k-2} + a_2 q^2 - \widetilde{\mathfrak{U}}(1;k) - a_2) F_{(k)}(\widetilde{\mathfrak{U}}((k,2)) \mid \widetilde{\mathfrak{U}}) \\ &\quad = \frac{(a_1 q^{k-1} + a_2 q - \widetilde{\mathfrak{U}}(1;k) - a_2)(1-q^k)}{(a_1 q^{k-1} - a_2)(1-q)} (a_1 q^{k-2} - a_2)(a_2 q^2 - a_2) F_{(k-1)}(\widetilde{\mathfrak{U}}^1((k-1,1)) \mid \widetilde{\mathfrak{U}}^1). \end{aligned}$$

Applying (2.5.20) to $F_{(k)}$ and $F_{(k-1)}$, noting for the latter that $\mathfrak{U}^1$ also satisfies the assumption of the first part with a_2 multiplied by q, we get after elementary manipulations

$$(a_1 q^{k-2} + a_2 q^2 - \widetilde{\mathfrak{U}}(1;k) - a_2)(a_1 q^{k-1} - a_2)(1-q) = (a_1 q^{k-1} + a_2 q - \widetilde{\mathfrak{U}}(1;k) - a_2)(a_1 q^{k-2} - a_2)(1-q^2).$$

This is a linear equation in $\widetilde{\mathfrak{U}}(1;k)$ of the form

$$(q-1)(a_1 q^{k-2} - a_2 q)\mathfrak{U}(1;k) = \beta$$

for some β independent of $\widetilde{\mathfrak{U}}(1;k)$. Since $\widetilde{\mathfrak{U}}$ is non-degenerate, $q \neq 1$ and $a_1 q^{k-2} = \widetilde{\mathfrak{U}}(1;k-2) \neq \widetilde{\mathfrak{U}}(2;1) = a_2 q$, hence the system is non-degenerate and it can be readily checked that $\widetilde{\mathfrak{U}}(1;k) = a_1 q^k$ is its unique solution. So we have proved $\widetilde{\mathfrak{U}}(1;k) = a_1 q^k$.

Returning to (2.5.21), plugging $\widetilde{\mathfrak{U}}(1;k) = a_1 q^k$ we now have

$$\kappa_k(\widetilde{\mathfrak{U}}) = 1 - q^k,$$

$$\begin{aligned} F_{(k+1)}(x_1, x_2 \mid \widetilde{\mathfrak{U}}) &= (x_1 + x_2 - a_1 q^k - a_2) F_{(k)}(x_1, x_2 \mid \widetilde{\mathfrak{U}}) \\ &\quad - (1-q^k)(x_1 - a_2)(x_2 - a_2) F_{(k-1)}(x_1, x_2 \mid \widetilde{\mathfrak{U}}^1). \end{aligned}$$

Plugging expressions for $F_{(k)}$ and $F_{(k-1)}$ and doing elementary manipulations, we get

$$F_{(k+1)}(x_1,x_2 \mid \widetilde{\mho}) = (x_1+x_2-a_1q^k-a_2)\sum_{i=0}^{k} x_1^i x_2^{k-i}\frac{(a_1/x_1;q)_i(a_2/x_2;q)_{k-i}(q;q)_k}{(q;q)_i(q;q)_{k-i}}$$
$$-(x_1-a_2)\sum_{i=0}^{k-1} x_1^i x_2^{k-i}\frac{(a_1/x_1;q)_i(a_2/x_2;q)_{k-i}(q;q)_k}{(q;q)_i(q;q)_{k-i-1}},$$

which can be rewritten as

$$F_{(k+1)}(x_1,x_2 \mid \widetilde{\mho}) = \sum_{i=0}^{k}(x_1q^{k-i}+x_2-a_1q^k-a_2q^{k-i})x_1^i x_2^{k-i}\frac{(a_1/x_1;q)_i(a_2/x_2;q)_{k-i}(q;q)_k}{(q;q)_i(q;q)_{k-i}}$$
$$=\sum_{i=0}^{k} q^{k-i}x_1^{i+1}x_2^{k-i}\frac{(a_1/x_1;q)_{i+1}(a_2/x_2;q)_{k-i}(q;q)_k}{(q;q)_i(q;q)_{k-i}} + x_1^i x_2^{k-i+1}\frac{(a_1/x_1;q)_i(a_2/x_2;q)_{k-i+1}(q;q)_k}{(q;q)_i(q;q)_{k-i}}$$
$$=\sum_{i=0}^{k+1} x_1^i x_2^{k-i}\frac{(a_1/x_1;q)_i(a_2/x_2;q)_{k-i}(q;q)_{k+1}}{(q;q)_i(q;q)_{k-i+1}},$$

finishing the proof of this part.

Part 2. We again proceed by induction on k. When $k=0$ we have $\widetilde{\mho}(1;0)=a_1$ and

$$F_{(1)}(x_1,x_2 \mid \widetilde{\mho}) = x_1+x_2-\widetilde{\mho}(1;0)-\widetilde{\mho}(2;0) = x_1-c_1+x_2-c_2.$$

Similarly, when $k=1$ we have $\widetilde{\mho}(1;0)=a_1+d$ and (2.5.12) implies

$$F_{(2)}(x_1,x_2 \mid \widetilde{\mho}) = (x_1-c_1+x_2-c_2)(x_1-c_1+x_2-c_2-d).$$

Now assume that $k\geq 2$ and we have proved the claim for $\widetilde{\mho}(1;i)$ and $F_{(i+1)}(x_1,x_2 \mid \widetilde{\mho})$ with $i<k$. Applying Lemma 2.5.13 again, we have

$$(x_1+x_2-\widetilde{\mho}(1;k)-c_2)F_{(k)}(x_1,x_2 \mid \widetilde{\mho})$$
$$= F_{(k+1)}(x_1,x_2 \mid \widetilde{\mho}) + \kappa_k(\widetilde{\mho})(x_1-c_2)(x_2-c_2)P_{(k-1)}(x_1,x_2 \mid \widetilde{\mho}^1),$$

where

$$\kappa_k(\widetilde{\mho}) = \frac{(c_1+kd-\widetilde{\mho}(1;k))F_{(k)}(c_1+(k-1)d, c_2+d \mid \widetilde{\mho})}{(c_1+(k-1)d-c_2)d^k(k-1)!}.$$

Plugging the expression for $F_{(k)}(x_1,x_2 \mid \widetilde{\mho})$, we get

$$\kappa_k(\widetilde{\mho}) = \frac{k(c_1+kd-\widetilde{\mho}(1;k))}{(c_1+(k-1)d-c_2)}.$$

$$F_{(k+1)}(x_1, x_2 \mid \widetilde{\mathfrak{O}}) = (x_1 + x_2 - \widetilde{\mathfrak{O}}(1;k) - c_2)F_{(k)}(x_1, x_2 \mid \widetilde{\mathfrak{O}}) \\ - \frac{k(c_1 + kd - \widetilde{\mathfrak{O}}(1;k))}{(c_1 + (k-1)d - c_2)}(x_1 - c_2)(x_2 - c_2)F_{(k-1)}(x_1, x_2 \mid \widetilde{\mathfrak{O}}^1). \quad (2.5.22)$$

Since $F_{(k+1)}(\widetilde{\mathfrak{O}}((k,2)) \mid \widetilde{\mathfrak{O}}) = F_{(k+1)}(c_1 + (k-2)d, c_2 + 2d \mid \widetilde{\mathfrak{O}}) = 0$, we get

$$(x_1 + x_2 - \widetilde{\mathfrak{O}}(1;k) - c_2)F_{(k)}(c_1 + (k-2)d, c_2 + 2d \mid \widetilde{\mathfrak{O}}) \\ = \frac{k(c_1 + kd - \widetilde{\mathfrak{O}}(1;k))}{(c_1 + (k-1)d - c_2)}(x_1 - c_2)(x_2 - c_2)F_{(k-1)}(c_1 + (k-2)d, c_2 + 2d \mid \widetilde{\mathfrak{O}}^1).$$

Using expressions for $F_{(k)}$ and $F_{(k-1)}$, we get after elementary manipulations

$$(c_1 + kd - \widetilde{\mathfrak{O}}(1;k)) = 2(c_1 + (k-2)d - c_2)\frac{(c_1 + kd - \widetilde{\mathfrak{O}}(1;k))}{(c_1 + (k-1)d - c_2)}.$$

which is equivalent to

$$(c_1 + kd - \widetilde{\mathfrak{O}}(1;k))(c_1 + (k-3)d - c_2) = 0.$$

Note that $c_1 + (k-3)t - c_2 = \widetilde{\mathfrak{O}}(1;k-1) - \widetilde{\mathfrak{O}}(2;4) \neq 0$ since $\widetilde{\mathfrak{O}}$ is non-degenerate. Hence the equation above implies $\widetilde{\mathfrak{O}}(1;k) = c_1 + kd$. Plugging this value for $\widetilde{\mathfrak{O}}(1;k)$ into (2.5.22) we get

$$F_{(k+1)}(x_1, x_2 \mid \widetilde{\mathfrak{O}}) = (x_1 + x_2 - c_1 - c_2 - kd)F_{(k)}(x_1, x_2 \mid \widetilde{\mathfrak{O}}),$$

which implies the claim. □

Let us summarize Lemmas 2.5.17 and 2.5.18:

Corollary 2.5.19. *Let $\mathfrak{O}$ be a perfect 3-grid. Then one of the following holds:*

1. *For some constants $c, q, a_1, a_2, a_3 \in \Bbbk$ we have*

$$\mathfrak{O}(1;k) = c + a_1 q^k, \quad \mathfrak{O}(2;k) = c + a_2 q^k, \quad \mathfrak{O}(3;k) = c + a_3 q^k, \qquad k \in \mathbb{Z}_{\geq 0}.$$

2. *For some constants $d, c_1, c_2, c_3 \in \Bbbk$ we have*

$$\mathfrak{O}(1;k) = c_1 + kd, \quad \mathfrak{O}(2;k) = c_2 + kd, \quad \mathfrak{O}(3;k) = c_3 + kd, \qquad k \in \mathbb{Z}_{\geq 0}.$$

□

2.5.5 Proof of Theorem 2.5.2

We first prove that two types of n-grids described in Theorem 2.5.2 are indeed perfect. Propositions 2.4.4, 2.4.5 combined with Remark 2.4.9 imply that the following n-grid $\mathfrak{O}$ is perfect:

$$\mathfrak{O}(i;j) = a_i q^j$$

for $q, a_1, \ldots, a_n \in \Bbbk$ such that $a_i, q \neq 0$, q is not a root of unity, and $a_i/a_j \neq q^k$ for any $i \neq j$ and $k \in \mathbb{Z}$. Hence the grid $\mathfrak{O} + c$ is also perfect for any $c \in \Bbbk$, covering the first type from Theorem 2.5.2.

Similarly, Propositions 2.4.7, 2.4.8 and Remark 2.4.9 imply that the grids of the form $\mho(i;j) = c_i + jd$ are perfect for $c_1, \dots, c_n, d \in \Bbbk$ such $d \neq 0$ and $c_i - c_j \neq kd$ for any $i \neq j$ and $k \in \mathbb{Z}$.

Hence, it is enough to prove that all perfect n-grids for $n \geq 3$ are described by Theorem 2.5.2. We do it by induction on n, where the base case $n = 3$ follows from Corollary 2.5.19. Note that the restrictions $a_i, q \neq 0$, $a_i/a_j \neq q^k$ and $c_i - c_j \neq kd$ follow automatically from Corollary 2.5.9, so its enough to just prove that $\mho$ has the form $\mho(i;j) = c + a_i q^j$ or $\mho(i;j) = c_i + jd$.

For the inductive step assume that we have proved Theorem 2.5.2 for all $(n-1)$-grids $\mho$. Let $\mho$ be a perfect n-grid. Then the $(n-1)$-grid $\mho_{n-1}$ is also perfect, and by inductive assumption we either have $\mho(i;j) = c + a_i q^j$ or $\mho(i;j) = c_i + jd$ for $i \leq n-1$. At the same time, the 3-grid ${}_{n-3}\mho$ is also perfect, and so we either have

$$\mho(n-2;j) = \widetilde{c} + \widetilde{a}_{n-2}\widetilde{q}^j, \qquad \mho(n-1;j) = \widetilde{c} + \widetilde{a}_{n-1}\widetilde{q}^j, \qquad \mho(n;j) = \widetilde{c} + \widetilde{a}_n\widetilde{q}^j$$

for some $\widetilde{c}, \widetilde{q}, \widetilde{a}_{n-2}, \widetilde{a}_{n-1}, \widetilde{a}_n \in \Bbbk$, or

$$\mho(n-2;j) = \widetilde{c}_{n-2} + j\widetilde{d}, \qquad \mho(n-1;j) = \widetilde{c}_{n-1} + j\widetilde{d}, \qquad \mho(n;j) = \widetilde{c}_n + j\widetilde{d}$$

for some $\widetilde{d}, \widetilde{c}_{n-2}, \widetilde{c}_{n-1}, \widetilde{c}_n \in \Bbbk$. But note that for $c, a, q, c', d \in \Bbbk$ we can simultaneously have $\mho(n-1;k) = c + aq^k = c' + kd$ for all $k \in \mathbb{Z}_{\geq 0}$ only if $d = 0$ and all numbers $c + aq^k, c' + kd$ are equal. However, this is impossible since $\mho$ is non-degenerate and the numbers $\mho(n-1;k)$ are distinct. Hence, both grids $\mho_{n-1}$ and ${}_{n-3}\mho$ are of the same type, and moreover $q = \widetilde{q}$ or $d = \widetilde{d}$. Thus, if $\mho(i;j) = c + a_i q^j$ for $i < n$ then $\mho(n;j) = c + a_n q^j$ as well, and similarly for $\mho(i;j) = c_i + jd$. This finishes the proof of Theorem 2.5.2. $\square$

Chapter 3

Applications to integrable probability

3.1 Overview

During the last decades a new conjectural universality law has been actively researched under the name *KPZ universality*, originating from [51]. Establishing this universality in full generality is an extremely challenging task and it is still an open problem. However, for certain integrable models one can directly verify the large-scale properties expected from the KPZ class models by finding explicit expressions describing the precise behavior of natural observables. Recent examples of such large-scale analysis of integrable models can be found in [4, 6, 7, 8, 10, 14, 15, 37, 41, 67, 84, 85, 86] and references therein.

Here we focus on two integrable models, namely, the *q-Hahn model* and the *directed Beta polymer model*. The former model was introduced in [34, 36, 75] and it is of a particular interest for the following twofold reason. On one hand, it is integrable and in the sense above, that is, one can find explicit integral expressions for q-deformed moments of naturally defined *height functions*, *cf* [15, 20, 24, 28]. On the other hand, the q-Hahn model can be degenerated to numerous other integrable models, including TASEP, q-boson particle system, and the mentioned directed Beta polymer model, *cf.* [17, 20, 36]. In particular, one can use q-Hahn models as an explanation of integrability for various other models.

The directed Beta polymer was originally introduced in [7] using two equivalent descriptions: it can either be viewed as a polymer model with Beta distributed weights or as a random walk in a random time-dependent Beta environment. This dual description already distinguished the Beta polymer model among the other integrable discrete polymers, namely, log-Gamma [78], strict-weak [37, 67] and inverse-Beta [81] polymers, which cannot be readily interpreted as random walks in random environments (RWRE). Moreover, there is a whole cluster of integrable models obtainable as degenerations of the Beta polymer, this cluster includes strict-weak polymer, uniform sticky Brownian motions [10, 48, 58], Bernoulli-Exponential directed first passage percolation [7] as well as other percolation models. Overall, the numerous connections to other models give rise to a lot of interesting properties and conjectures, making Beta polymer a popular object in recent research, *cf.* [5, 9, 10, 57].

Here we study new generalizations of these two models, which become accessible due to our integrability results from Chapter 1. Namely, we introduce a new integrable generalization of stochastic q-Hahn vertex model with three families of parameters. The first main result of this chapter is the integral expression for the q-deformed moments of *the colored height functions* of the

q-Hahn vertex. Such explicit expressions encapsulate the behavior of the model in a form suitable for subsequent analysis, in particular, they let us establish the large class of shift-invariance symmetries for our model.

As an immediate degeneration of the q-Hahn model we obtain a new integrable extension of the Beta polymer model, which also contains three families of parameters attached to columns, rows and diagonals. To the best of our knowledge, this is the first integrable example of a directed polymer model inhomogeneous in three directions (there exist several related models with two families of parameters, studied in [8, 35, 82]). We expect that a lot of currently known properties of the Beta polymer can be extended to our inhomogeneous version, and to reinforce this claim, as the second main result of this chapter we demonstrate the Tracy-Widom asymptotics for the inhomogeneous model.

Below we briefly describe the results of this chapter.

3.1.1 Integral expressions for the diagonally inhomogeneous q-Hahn model

Let $q \in (0,1)$ and $(\mu_0, \mu_1, \mu_2, \dots)$, $(\kappa_1, \kappa_2, \dots)$, $(\lambda_1, \lambda_2, \dots)$ be real parameters satisfying

$$\lambda_d < \kappa_j < \mu_i, \qquad i \in \mathbb{Z}_{\geq 0},\ j \in \mathbb{Z}_{>0},\ d \in \mathbb{Z}_{\geq 0}.$$

The *colored diagonally inhomogeneous q-Hahn vertex model* consists of a random ensemble of colored up-right paths in a positive quadrant $\mathbb{Z}^2_{\geq 0}$, with colors labeled by positive integers. The paths are sampled according to the following procedure:

- All paths start from the left boundary of the model $\{0\} \times \mathbb{Z}_{\geq 1}$, with b_j paths of color j starting at $(0, j)$. The numbers b_j are independent and distributed according to

$$\mathbb{P}(b_j = b) = (\kappa_j/\mu_0)^b \frac{(\lambda_j/\kappa_j; q)_b}{(q;q)_b} \frac{(\kappa_j/\mu_0; q)_\infty}{(\lambda_j/\mu_0; q)_\infty}, \qquad b \in \mathbb{Z}_{\geq 0}.$$

- The remaining configuration is sampled sequentially and independently around each integer point (i, j), starting from the bottom and going along the rows. At each integer point all paths coming from the left turn upwards, while the paths coming from below might turn right following the probabilities

$$\mathbb{P}\left(\begin{array}{ccc} & \boldsymbol{K} & \\ \boldsymbol{I} & + & \boldsymbol{L} \\ & \boldsymbol{J} & \end{array} \,\middle|\, \boldsymbol{I}, \boldsymbol{J} \right) = \mathbb{1}_{\boldsymbol{I}+\boldsymbol{J}=\boldsymbol{K}+\boldsymbol{L}}\, (\kappa_j/\mu_i)^{|\boldsymbol{L}|} \frac{(\kappa_j/\mu_i; q)_{|\boldsymbol{J}|-|\boldsymbol{L}|} (\lambda_{j-i}/\kappa_j; q)_{|\boldsymbol{L}|}}{(\lambda_{j-i}/\mu_i; q)_{|\boldsymbol{J}|}} q^{\sum_{i<j} D_i(A_j - D_j)} \times \prod_{i=1}^{n} \frac{(q;q)_{J_i}}{(q;q)_{L_i} (q;q)_{J_i - L_i}},$$

  where $\boldsymbol{I} = (I_1, I_2, \dots)$ is a finite integer sequence with I_c equal to the number of paths of color c entering from the left, $\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{L}$ similarly encode the colors of paths along the bottom, top and the right edges respectively, and we set $|\boldsymbol{J}| = J_1 + J_2 + \dots, |\boldsymbol{L}| = L_1 + L_2 + \dots$.

See Figure 3-1 for a schematic description of the model and its parameters. Note that this model is equivalent to the model from Chapter 1, the only change is the normalization of the weights on

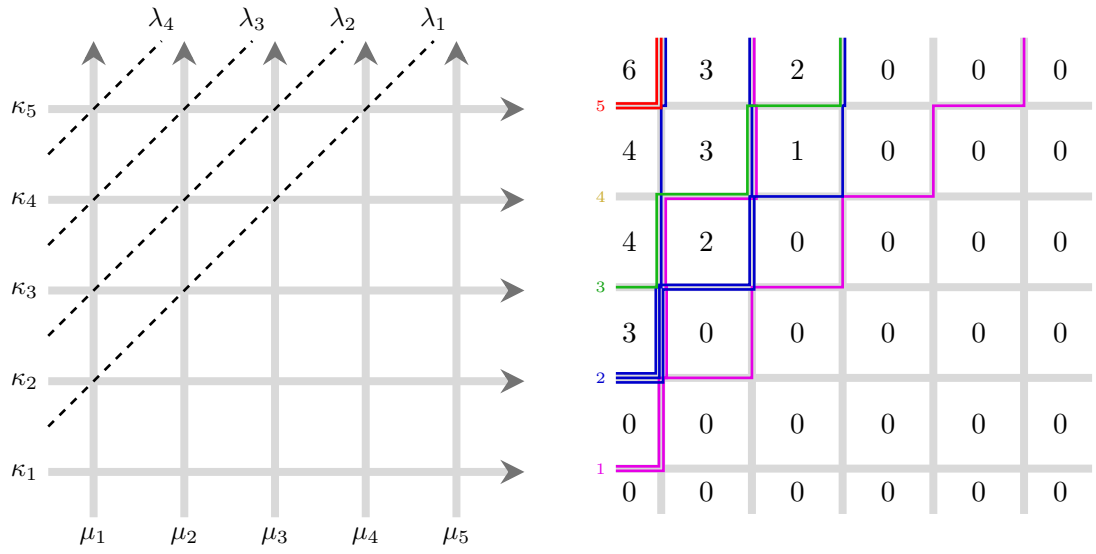


Figure 3-1: Left: parameters of the q-Hahn model. Right: a configuration of the model along with the values of the height function $h_{\geq 2}$ for this configuration.

the left boundary which is needed to make the weights sum to 1.

We describe the configurations of the model above using *colored height functions* $h_{\geq c}^{(x,y)}$, which are assigned to the facets $(x,y) \in \left(\mathbb{Z}+\frac{1}{2}\right)^2$ formed by rows and columns and count the number of paths of color $\geq c$ passing below the facet, as depicted on Figure 3-1. We consider the following observable: for collections

$$x_1 \leq x_2 \leq \cdots \leq x_k, \qquad y_1 \geq y_2 \geq \cdots \geq y_k, \qquad x_a, y_a \in \mathbb{Z}_{\geq 0}+\frac{1}{2};$$

$$c_1 \leq c_2 \leq \cdots \leq c_k, \qquad c_a \in \mathbb{Z}_{\geq 0};$$

and a permutation $\tau \in \mathfrak{S}_k$ set

$$\mathcal{Q}_{\geq \tau.\mathbf{c}}^{(\mathbf{x},\mathbf{y})} := \prod_{a=1}^{k} q^{h_{\geq c_a}^{(x_{\tau(a)},y_{\tau(a)})}}.$$

Theorem 3.1.1 (Theorem 3.4.1 in the text)**.** *With the notation above we have*

$$\mathbb{E}\left[\mathcal{Q}_{\geq \tau.\mathbf{c}}^{\mathbf{x},\mathbf{y}}(\Sigma)\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}-l(\tau)}}{(2\pi\mathbf{i})^k} \oint_{\Gamma_1} \cdots \oint_{\Gamma_k} \prod_{a<b} \frac{w_b-w_a}{w_b-qw_a}\, T_\tau\left(\prod_{a=1}^{k}\prod_{i=1}^{c_a-1}\frac{1-\lambda_i w_a}{1-\kappa_i w_a}\right)$$
$$\times \prod_{a=1}^{k}\left(\prod_{i=0}^{i<x_a}\frac{1}{1-\mu_i w_a}\prod_{j=1}^{j<y_a}(1-\kappa_j w_a)\prod_{d=1}^{d\leq y_a-x_a}\frac{1}{1-\lambda_d w_a}\right)\frac{dw_a}{w_a}, \quad (3.1.1)$$

where the positively oriented contours Γ_b are chosen to enclose μ_i^{-1} and $q\Gamma_a$ for $a<b$, with other singularities being outside of the contours, and T_τ denotes the polynomial representation of the Hecke

algebra generated by the Demazure-Lusztig operators

$$T_i = q + \frac{w_{i+1} - q w_i}{w_{i+1} - w_i}(\mathfrak{s}_i - 1), \qquad \mathfrak{s}_i f(\dots, w_i, w_{i+1}, \dots) = f(\dots, w_{i+1}, w_i, \dots).$$

The proof is based on the idea from [28]: we show that both sides of (3.1.1) satisfy the same discrete recurrence relations, called *local relations*. These relations are consequences of the integrability of the model and we prove them using Yang-Baxter equations from Chapter 1.

As with a lot of inductive verification arguments, our proof of Theorem 3.1.1 does not clearly explain a natural way to arrive to the integral expression. So, let us also mention here is an alternative approach, which is more constructive but does not work completely in full generality hence we do not develop it. The main idea comes from [20], [24]. Note that the average of an observable $\mathcal{O}$ can be written as

$$\mathbb{E}[\mathcal{O}] = \sum_{\lambda} \mathcal{O}(\lambda)\mathbb{P}[\lambda].$$

It turns out that for a special choice of $\mathcal{O}$ one can write both $\mathcal{O}(\lambda)$ and $\mathbb{P}[\lambda]$ as spin q-Whittaker symmetric functions from Chapter 2, and the expectation above becomes one side of the the skew Cauchy summation identity. This lets one to identify $\mathbb{E}[\mathcal{O}]$ with a spin q-Whittaker function and the integral representation for the latter (Theorem 2.3.11) leads to the desired integral expression.

3.1.2 Limit theorem for inhomogeneous Beta polymer

Consider a lattice on the plane formed by the edges $(i,j) \to (i+1, j+1)$ and $(i,j) \to (i, j+1)$ for $(i,j) \in \mathbb{Z}^2_{\geq 0}$ such that $j \geq i$. Let $\widetilde{\sigma}_i, \widetilde{\rho}_j, \widetilde{\omega}_d$ be a family of real parameters attached to columns $i = const$, rows $j = const$ and diagonals $j - i = const$ respectively and satisfying

$$\widetilde{\omega}_d < \widetilde{\rho}_j < \widetilde{\sigma}_i, \qquad i \in \mathbb{Z}_{\geq 0},\ j \in \mathbb{Z}_{>0},\ d \in \mathbb{Z}_{>0}.$$

To each edge we assign a weight in the following way: for each integer point (i,j) we first independently sample a Beta distributed random variable $\eta_{i,j} \sim \mathrm{Beta}(\widetilde{\sigma}_i - \widetilde{\rho}_j, \widetilde{\rho}_j - \widetilde{\omega}_{j-i})$ and then we set

$$w(e) = \begin{cases} \eta_{i,j} & \text{if } e = (i, j-1) \to (i,j), \\ 1 - \eta_{i,j} & \text{if } e = (i-1, j-1) \to (i,j). \end{cases}$$

The *delayed Beta polymer partition function* $\mathfrak{Z}^{(r)}_{x,y}$ is defined as

$$\mathfrak{Z}^{(r)}_{x,y} = \sum_{\pi_0 = (0,r) \to \pi_1 \to \cdots \to \pi_{y-r} = (x,y)} \prod_{i = f(\pi)}^{y-r-1} w(\pi_i \to \pi_{i+1}),$$

where the sum is over all directed lattice paths π from $(0,r)$ to (x,y), and $f(\pi) = \min\{i \mid \pi_{i+1} - \pi_i = (0,1)\}$, so the path starts accumulating its weight with a delay waiting until the first vertical step, see Figure 3-2.

Under a $q \to 1$ limit the diagonally inhomogeneous q-Hahn model converges to the inhomogeneous Beta polymer, and the expression from Theorem 3.1.1 produces an explicit integral expression

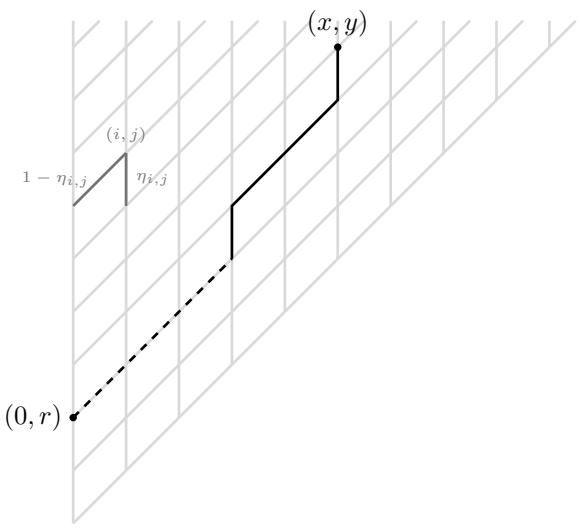


Figure 3-2: Example of a path contributing to the Beta polymer partition function $\mathfrak{Z}^{(r)}_{x,y}$. Due to the delay, the dashed edges do not contribute their weights.

for the moments:

$$\mathbb{E}\left[\left(\mathfrak{Z}^{(r)}_{x,y}\right)^k\right] = \oint_{\mathcal{S}_1}\cdots\oint_{\mathcal{S}_k}\prod_{a<b}\frac{v_b-v_a}{v_b-v_a-1}\prod_{a=1}^{k}\left(\prod_{i=r+1}^{i\le y}(v_a-\widetilde{\rho}_i)\prod_{i=0}^{i\le x}\frac{1}{v_a-\widetilde{\sigma}_i}\prod_{i=r+1}^{i\le y-x}\frac{1}{v_a-\widetilde{\omega}_i}\right)\frac{dv_a}{2\pi\mathbf{i}},$$

where the contour $\mathcal{S}_b$ surrounds $\widetilde{\sigma}_i$ and $\mathcal{S}_a+1$ for $a<b$. Such integral expressions are well-suited for asymptotic analysis and on can follow the standard workflow consisting of first obtaining a Fredholm determinant expression for the Laplace transform of $\mathfrak{Z}^{(r)}_{x,y}$ (which is analytic since $\mathfrak{Z}^{(r)}_{x,y}\in(0,1)$ almost surely) and then performing a steep descent analysis. Below we list our results.

Consider the large scale limit of $\mathfrak{Z}^{(0)}_{\lfloor xt\rfloor,\lfloor yt\rfloor}$ when the parameters $\widetilde{\sigma}_i,\widetilde{\rho}_j,\widetilde{\omega}_d$ have finite number of possible values σ_i,ρ_j,ω_d repeated with frequencies $\alpha_i,\beta_j,\gamma_d$. For example, when $\gamma_1=\gamma_2=\frac{1}{2}$ we assume that half of the diagonals have parameter $\widetilde{\omega}_d=\omega_1$ while the other half has parameter ω_2; see Section 3.8 for a precise description of the model. We conjecture (Conjecture 3.8.2) that the following limit relation holds:

$$\lim_{t\to\infty}\mathbb{P}\left(\frac{\ln\mathfrak{Z}^{(0)}_{\lfloor xt\rfloor,\lfloor yt\rfloor}+It}{ct^{1/3}}\le r\right)=F_{GUE}(r), \tag{3.1.2}$$

where the slope $x/y\in(0,\frac{\sum\beta_i\rho_i-\sum\gamma_i\omega_i}{\sum\alpha_i\sigma_i-\sum\gamma_i\omega_i})$ is fixed,[1] $F_{GUE}(s)$ denotes the GUE Tracy-Widom distribution [83] and the constants I,c are defined in an implicit way involving an auxiliary parameter

[1]The range for the slope looks more natural in the RWRE description: $\frac{\sum\beta_i\rho_i-\sum\gamma_i\omega_i}{\sum\alpha_i\sigma_i-\sum\gamma_i\omega_i}$ corresponds to the average drift of the walk, and we are looking at fluctuations at non-typical velocities. See [7] for more details about the RWRE interpretation of the model.

$\theta \in (\max_i \sigma_i, \infty)$ and polygamma functions Ψ_k:

$$x/y = \frac{\sum_i \beta_i \Psi_1(\theta-\rho_i) - \sum_i \gamma_i \Psi_1(\theta-\omega_i)}{\sum_i \alpha_i \Psi_1(\theta-\sigma_i) - \sum_i \gamma_i \Psi_1(\theta-\omega_i)},$$

$$I = x\sum_i \alpha_i \Psi_0(\theta-\sigma_i) - y\sum_i \beta_i \Psi_0(\theta-\rho_i) + (y-x)\sum_i \gamma_i \Psi_0(\theta-\omega_i),$$

$$c^3 = -\frac{x}{2}\sum_i \alpha_i \Psi_2(\theta-\sigma_i) + \frac{y}{2}\sum_i \beta_i \Psi_2(\theta-\rho_i) - \frac{y-x}{2}\sum_i \gamma_i \Psi_2(\theta-\omega_i).$$

Under assumptions dictated by purely technical reasons we are able to partially verify our conjecture:

Theorem 3.1.2 (Theorem 3.8.4 in the text)**.** *When $\theta \in (0, \frac{1}{2})$, $\sigma_1 = \sigma_2 = \cdots = 0$ and $\rho_1 = \rho_2 = \cdots = -1$ we have*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{\ln \mathfrak{Z}^{(0)}_{\lfloor xt \rfloor, \lfloor yt \rfloor} + It}{ct^{1/3}} \le r \right) = F_{GUE}(r),$$

with the notation specified above.

In other words, we are able to partially verify the conjectural limit statement for the model having only diagonal parameters $\omega_d < 1$, with Beta distributions $\mathrm{Beta}(1, -1-\omega_d)$.

The Beta polymer model and our results about it can be equivalently reformulated in terms of a random walk in a random environment but we omit this description, see [7] for further details.

3.1.3 Layout of the chapter

After preparing the needed algebraic machinery in Sections 3.2, 3.3 we prove the integral formula for q-moments of height functions in Section 3.4. Section 3.5 is a brief overview of the shift-invariance property for the vertex models, which can be extend to the q-Hahn model. Section 3.6 describes the degeneration of our integral formula to Beta polymer model, which is then collected into a Fredholm determinant expression for the Laplace transform of the polymer partition function in Section 3.7. Finally, Section 3.8 is devoted to the large-scale analysis of the inhomogeneous Beta polymer model.

3.1.4 Notation

For a pair of parameters $(\alpha, \beta) \in \mathbb{R}^2_{\ge 0}$ the Beta distribution is defined by the following density function with respect to the Lebesgue measure:

$$f_{\mathrm{Beta}(\alpha,\beta)}(x) = \mathbb{1}_{0<x<1}\, x^{\alpha-1}(1-x)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)},$$

where $\Gamma(z)$ is the Gamma function. We use $\mathrm{Beta}(\alpha, \beta)$ to denote a Beta-distributed random variable with the corresponding parameters.

For a trace-class operator K over $L^2(X)$ we use $\det(I+K)_{L^2(X)}$ to denote its Fredholm determinant. The only operators K considered in this work are integral operators of the form

$$[Kf](x) = \int_X f(y)K(x,y)dy,$$

where $K(x,y)$ is called the kernel of K. In this case the Fredholm determinant is given explicitly by

$$\det(I+K)_{L^2(X)} = \sum_{l\geq 0} \frac{1}{l!} \int_X dx_1 \cdots \int_X dx_l \det\left[K(x_a,x_b)\right]_{a,b=1}^{l}.$$

We often use Fredholm determinants over complex integration contours, so for a contour $\mathcal{C}\subset\mathbb{C}$ we let $L^2(\mathcal{C})$ denote the space of functions on $\mathcal{C}$ with the complex valued integration measure $\frac{dz}{2\pi\mathbf{i}}$.

The Tracy-Widom GUE cumulative distribution function $F_{GUE}(r)$ is defined by

$$F_{GUE}(r) = \det(I - K_{\mathrm{Ai}})_{L^2(r,\infty)},$$

$$K_{\mathrm{Ai}}(\lambda,\mu) = \frac{\mathrm{Ai}(\lambda)\,\mathrm{Ai}'(\mu) - \mathrm{Ai}(\mu)\,\mathrm{Ai}'(\lambda)}{\lambda-\mu} = \int_{e^{-2\pi\mathbf{i}/3}\infty}^{e^{2\pi\mathbf{i}/3}\infty} \frac{dv}{2\pi\mathbf{i}} \int_{e^{-\pi\mathbf{i}/3}\infty}^{e^{\pi\mathbf{i}/3}\infty} \frac{dz}{2\pi\mathbf{i}}\ \frac{1}{z-v} \frac{e^{z^3/3-z\lambda}}{e^{v^3/3-v\mu}},$$

where $\mathrm{Ai}(z)$ is the Airy function and the integration contours in the last expression do not intersect and go to infinity along the prescribed directions.

3.2 Hecke algebras and exchange relations

The *Hecke algebra* $\mathcal{H}_k$ is a $\mathbb{C}$-algebra with basis $\{T_\pi\}_{\pi\in S_k}$ enumerated by partitions $\pi\in S_k$ and satisfying the following defining relations

$$T_\pi T_\tau = T_{\pi\tau} \quad \text{if } \ l(\pi)+l(\tau)=l(\pi\tau), \qquad (T_i-q)(T_i+1)=0,$$

where $T_i = T_{\sigma_i}$ denote the basis elements corresponding to the simple transpositions σ_i exchanging i and $i+1$. Note that the elements T_i generate $\mathcal{H}_k$, and thus we can define a representation of $\mathcal{H}_k$ on the rational functions in $w_1,\ldots,w_k$ by setting

$$T_i = q + \frac{w_{i+1}-qw_i}{w_{i+1}-w_i}(\mathfrak{s}_i - 1). \tag{3.2.1}$$

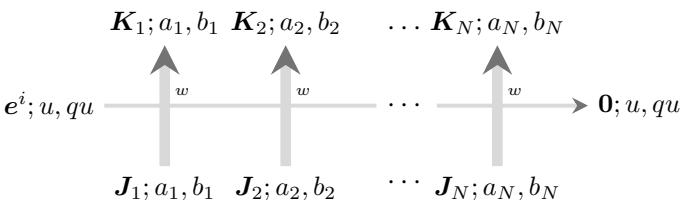
where $\mathfrak{s}_i$ denotes the operator exchanging the variables w_i and w_{i+1}.[2] These operators are called *Demazure-Lusztig operators*. Note that they preserve polynomials in $w_1,\ldots,w_k$, and for any rational function f in $w_1,\ldots,w_k$ the function $T_\pi f$ is well defined on the domain of f, that is, $T_\pi f$ does not have any additional singularities compared to f.

For us the representation above appears as a consequence of exchange relations between higher rank versions of the row operators from Section 2.2. Namely, fix $N>0$ and let $V^{(N)}$ denote an infinite dimensional vector space with basis $\{|\boldsymbol{I}_1,\ldots,\boldsymbol{I}_N\rangle\}_{\boldsymbol{I}_1,\ldots,\boldsymbol{I}_N\in\mathbb{Z}^n_{\geq 0}}$ enumerated by N-tuples of compositions. Let $\langle \boldsymbol{I}_1,\ldots,\boldsymbol{I}_L| \in V_L^*$ denote the elements dual to the basis $\{|\boldsymbol{I}_1,\ldots,\boldsymbol{I}_N\rangle\}_{\boldsymbol{I}_1,\ldots,\boldsymbol{I}_N}$. For a fixed pair of families $\mathcal{A}=(a_1,\ldots,a_N), \mathcal{B}=(b_1,\ldots,b_N)$ the operators $\mathcal{T}_{i,0}(u\mid\mathcal{A},\mathcal{B})$ on $V^{(N)}$

[2] Throughout the text the Hecke algebra appears only via this faithful representation, so identification of the basis elements T_π with the operators T_π should not cause confusion.

are defined by

$$\langle \boldsymbol{K}_1, \ldots, \boldsymbol{K}_N | \, \mathcal{T}_{i,0}(u \mid \mathcal{A}, \mathcal{B}) \, | \boldsymbol{J}_1, \ldots, \boldsymbol{J}_N \rangle :=$$



(3.2.2)

where $1 \le i \le n$ and the partition function on the right-hand side consists of L vertices with the weights $w_{u;a_1,b_1}, w_{u;a_2,s_2}, \ldots, w_{u;a_N,b_N}$. For any $1 \le i < j \le n$ the operators $\mathcal{T}_{i,0}, \mathcal{T}_{j,0}$ satisfy the following commutation relations:

$$\mathcal{T}_{i,0}(v \mid \mathcal{A}, \mathcal{B}) \mathcal{T}_{i,0}(u \mid \mathcal{A}, \mathcal{B}) = \mathcal{T}_{i,0}(u \mid \mathcal{A}, \mathcal{B}) \mathcal{T}_{i,0}(v \mid \mathcal{A}, \mathcal{B}), \tag{3.2.3}$$

$$q\mathcal{T}_{j,0}(v \mid \mathcal{A}, \mathcal{B}) \mathcal{T}_{i,0}(u \mid \mathcal{A}, \mathcal{B}) = \frac{u - qv}{u - v} \mathcal{T}_{i,0}(u \mid \mathcal{A}, \mathcal{B}) \mathcal{T}_{j,0}(v \mid \mathcal{A}, \mathcal{B}) - \frac{(1-q)u}{u - v} \mathcal{T}_{i,0}(v \mid \mathcal{A}, \mathcal{B}) \mathcal{T}_{j,0}(u \mid \mathcal{A}, \mathcal{B}), \tag{3.2.4}$$

$$\mathcal{T}_{i,0}(v \mid \mathcal{A}, \mathcal{B}) \mathcal{T}_{j,0}(u \mid \mathcal{A}, \mathcal{B}) = \frac{u - qv}{u - v} \mathcal{T}_{j,0}(u \mid \mathcal{A}, \mathcal{B}) \mathcal{T}_{i,0}(v \mid \mathcal{A}, \mathcal{B}) - \frac{(1-q)v}{u - v} \mathcal{T}_{j,0}(v \mid \mathcal{A}, \mathcal{B}) \mathcal{T}_{i,0}(u \mid \mathcal{A}, \mathcal{B}). \tag{3.2.5}$$

All these relations can be proved by considering the Yang-Baxter equation (1.3.3) with $(a_1, b_1) = (u, qu), (a_2, b_2) = (v, qv)$ and applying a zipper-like argument similar to Proposition 2.2.6. The exact argument is given in [22, Theorem 3.2.2].[3]

For the purposes of this work we only need the commutation relation (3.2.4), which we rewrite as

$$q\langle \lambda | \mathcal{T}_{j,0}(w_1 \mid \mathcal{A}, \mathcal{B}) \mathcal{T}_{i,0}(w_2 \mid \mathcal{A}, \mathcal{B}) | \mu \rangle = T^{(w_1, w_2)} \langle \lambda | \mathcal{T}_{i,0}(w_1 \mid \mathcal{A}, \mathcal{B}) \mathcal{T}_{j,0}(w_2 \mid \mathcal{A}, \mathcal{B}) | \mu \rangle, \tag{3.2.6}$$

where $T^{(w_1,w_2)}$ is the Demazure-Lisztig operator acting on rational functions in w_1, w_2:

$$T^{(w_1,w_2)} = q + \frac{w_2 - qw_1}{w_2 - w_1} (\mathfrak{s}^{(w_1,w_2)} - 1), \qquad \mathfrak{s}^{(w_1,w_2)} f(w_1, w_2) = f(w_2, w_1).$$

3.3 Local relations

In this section we use the Yang-Baxter equation to derive two recurrence relations: the first relation captures the local behavior of q-moments of the colored height functions, defined in the next section, while the other relation compares rational functions of certain form. As a consequence of one of these relations we establish stochasticity of the vertex weights W^b, and both relations play a central role in the proof of one of the main results of this chapter, namely Theorem 3.4.1.

3.3.1 Ordered permutations.

Both local relations in this section can be described in terms of order-preserving length p subsequences of a generic sequence $\mathbf{c} = (c_1, c_2, \ldots, c_r)$, which can be naturally encoded by subsets

[3] In [22, Theorem 3.2.2] only the *homogeneous* case is considered, when for a complex parameter s we have $a_i = s, b_i = s^{-1}$ for all i. However the proof given in [22, Theorem 3.2.2] works verbatim for general sequences $\mathcal{A}, \mathcal{B}$.

$\mathcal{I} = \{i_1, i_2, \ldots, i_p\} \subset \{1, 2, \ldots, r\}$:

$$\mathbf{c}[\mathcal{I}] = (c_{i_1}, \ldots, c_{i_p}), \quad \text{where } i_1 < i_2 < \cdots < i_p.$$

But for our purposes it is more convenient to encode such subsequences using permutations. Recall that the action of the symmetric group on sequences is defined by

$$\pi.\mathbf{c} = \left(c_{\pi^{-1}(1)}, c_{\pi^{-1}(2)}, \ldots, c_{\pi^{-1}(r)}\right).$$

Then any subsequence of $\mathbf{c}$ can be constructed as

$$\pi.\mathbf{c}[1, p] := (c_{\pi^{-1}(1)}, \ldots, c_{\pi^{-1}(p)})$$

for a permutation π satisfying

$$\pi^{-1}(1) < \pi^{-1}(2) < \cdots < \pi^{-1}(p).$$

This leads us to the following definitions.

A permutation π is called $[a, b]$-ordered if

$$\pi^{-1}(a) < \pi^{-1}(a+1) < \cdots < \pi^{-1}(b).$$

More generally, a permutation π is $[a_1, b_1] \times [a_2, b_2]$-ordered if it is simultaneously $[a_1, b_1]$-ordered and $[a_2, b_2]$-ordered. With this notation the following can be readily verified:

Lemma 3.3.1. *There is a one-to-one correspondence between integer subsets $\{i_1, \ldots, i_p\} \subset [1, n]$ and $[1, p] \times [p+1, r]$-ordered permutations π determined by setting $\pi^{-1}(a) = i_a$ for $1 \le a \le p$, where we have ordered $i_1 < i_2 < \cdots < i_p$. Moreover, for a $[1, p] \times [p+1, r]$-ordered permutation π we have*

$$l(\pi) = \sum_{i=1}^{p} \left(\pi^{-1}(i) - i\right) = \sum_{i=1}^{p} \pi^{-1}(i) - \frac{p(p+1)}{2}.$$

□

So, each order-preserving length p subsequence of $\mathbf{c} = (c_1, \ldots, c_r)$ can be uniquely described as $\pi.\mathbf{c}[1, p]$ for a $[1, p] \times [p+1, r]$-ordered permutation π. We denote the set of such permutations by $\mathfrak{S}^{p|r-p}$.

3.3.2 Local relation: q-moments.

The first local relation describes the local behavior of a certain observable of the weights $W^b_{\lambda,\kappa,\mu}(\boldsymbol{I}, \boldsymbol{J}; \boldsymbol{K}, \boldsymbol{L})$.

Proposition 3.3.2. *For any compositions $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{R}$ we have*

$$\sum_{\boldsymbol{K},\boldsymbol{L}} q^{\sum_{i \le j} R_i L_j} W^b_{\lambda,\kappa,\mu}(\boldsymbol{I}, \boldsymbol{J}; \boldsymbol{K}, \boldsymbol{L})$$
$$= \sum_{\boldsymbol{P} \le \boldsymbol{R}} (\kappa/\mu)^{|\boldsymbol{P}|} \frac{(\kappa/\mu; q)_{|\boldsymbol{R}|-|\boldsymbol{P}|} (\lambda/\kappa; q)_{|\boldsymbol{P}|}}{(\lambda/\mu; q)_{|\boldsymbol{R}|}} q^{\sum_{a<b}(R_a - P_a)P_b} \prod_{a=1}^{n} \frac{(q; q)_{R_a}}{(q; q)_{P_a} (q; q)_{R_a - P_a}} q^{\sum_{a \le b} P_a J_b}, \quad (3.3.1)$$

where the sum in the right-hand side is over compositions $\boldsymbol{P} = (P_1, \dots, P_n)$ *such that* $P_i \le R_i$.

The proof of Proposition 3.3.2 will be given shortly, but first we want to explain the probabilisitic interpretation of (3.3.1). First note that by setting $\boldsymbol{R} = \boldsymbol{0}$ we immediately get the stochasticity of the weights W^b:

Corollary 3.3.3. *For any compositions* $\boldsymbol{I}, \boldsymbol{J}$ *we have* $\sum_{\boldsymbol{K},\boldsymbol{L}} W^b_{\lambda,\kappa,\mu}(\boldsymbol{I}, \boldsymbol{J}; \boldsymbol{K}, \boldsymbol{L}) = 1$.

Hence we can define a (complex-valued for now) probability measure on configurations of the outgoing edges $(\boldsymbol{K}, \boldsymbol{L})$ conditioned on a given fixed incoming configuration $(\boldsymbol{I}, \boldsymbol{J})$. We denote this probability measure as

$$\mathbb{P}_{\lambda,\kappa,\mu}\left[(\boldsymbol{K}, \boldsymbol{L}) \mid (\boldsymbol{I}, \boldsymbol{J})\right] = W_{\lambda,\kappa,\mu}(\boldsymbol{I}, \boldsymbol{J}; \boldsymbol{K}, \boldsymbol{L})$$

and let $\mathbb{E}_{\lambda,\kappa,\mu}$ denote the expectation with respect to this probability measure. For a sequence $\mathbf{c} = (c_1, c_2, \dots, c_r)$ and a composition $\boldsymbol{A} = (A_1, \dots, A_n)$ define

$$\mathcal{Q}_{\ge \mathbf{c}}(\boldsymbol{A}) = \prod_{i=1}^{r} \prod_{j \ge c_i} q^{A_j} = q^{A_{[c_1,n]} + A_{[c_2,n]} + \dots + A_{[c_r,n]}}.$$

Corollary 3.3.4. *For a sequence* $\mathbf{c} = (c_1, \dots, c_r)$ *such that*

$$c_1 \le c_2 \le \dots \le c_r$$

and arbitrary compositions $\boldsymbol{I}, \boldsymbol{J}$ *the following relation holds*

$$\mathbb{E}_{\lambda,\kappa,\mu}\left[\mathcal{Q}_{\ge \mathbf{c}}(\boldsymbol{L}) \;\middle|\; (\boldsymbol{I}, \boldsymbol{J})\right] = \sum_{\mathcal{I}} (\kappa/\mu)^{|\mathcal{I}|} \frac{(\kappa/\mu; q)_{r-|\mathcal{I}|} (\lambda/\kappa; q)_{|\mathcal{I}|}}{(\lambda/\mu; q)_r} q^{-\binom{|\mathcal{I}|+1}{2} + \sum_{i \in \mathcal{I}} i} \; \mathcal{Q}_{\ge \mathbf{c}[\mathcal{I}]}(\boldsymbol{J}), \quad (3.3.2)$$

where the sum is over subsets $\mathcal{I} \subset \{1, 2, \dots, r\}$, $|\mathcal{I}|$ *denotes the number of elements in* $\mathcal{I}$ *and*

$$\mathbf{c}[\mathcal{I}] := (c_{i_1}, \dots, c_{i_{|\mathcal{I}|}}).$$

Equivalently,

$$\mathbb{E}_{\lambda,\kappa,\mu}\left[\mathcal{Q}_{\ge \mathbf{c}}(\boldsymbol{L}) \;\middle|\; (\boldsymbol{I}, \boldsymbol{J})\right] = \sum_{p=0}^{r} \sum_{\tau \in \mathfrak{S}^{p|r-p}} (\kappa/\mu)^p \frac{(\kappa/\mu; q)_{r-p} (\lambda/\kappa; q)_p}{(\lambda/\mu; q)_r} q^{l(\tau)} \mathcal{Q}_{\ge \tau.\mathbf{c}[1,p]}(\boldsymbol{J}), \quad (3.3.3)$$

where the sum is over all $[1, p] \times [p+1, r]$*-ordered permutations and*

$$\tau.\mathbf{c}[1, p] := \left(c_{\tau^{-1}(1)}, \dots, c_{\tau^{-1}(p)}\right).$$

Reduction of Corollary 3.3.4 to Proposition 3.3.2. The second part follows at once from Lemma 3.3.1. So we focus on the first part.

Let $\mathbf{c} = 1^{R_1} 2^{R_2} \dots n^{R_n}$ for a composition $\boldsymbol{R} = (R_1, \dots, R_n)$ with $|\boldsymbol{R}| = r$, that is

$$R_i = \#\{j \in [1; r] \mid c_j = i\}.$$

Then for any composition $\boldsymbol{L}$ we have

$$\mathcal{Q}_{\ge \mathbf{c}}(\boldsymbol{L}) = q^{L_{[c_1,n]} + L_{[c_2,n]} + \dots + L_{[c_r,n]}} = q^{R_1 L_{[1,n]} + R_2 L_{[2,n]} + \dots + R_n L_n} = q^{\sum_{i \le j} R_i L_j}.$$

Similarly, if $\mathbf{c}[\mathcal{I}] = 1^{P_1}2^{P_2}\dots n^{P_n}$ for a composition $\boldsymbol{P}$, then $\mathcal{Q}_{\geq \mathbf{c}[\mathcal{I}]}(\boldsymbol{J}) = q^{\sum_{i\leq j} P_i J_j}$. Thus, (3.3.2) is equivalent to

$$\mathbb{E}_{\lambda,\kappa,\mu}\left[q^{\sum_{i\leq j} R_i L_j} \;\middle|\; (\boldsymbol{I},\boldsymbol{J})\right] = \sum_{\boldsymbol{P}\leq \boldsymbol{R}} \sum_{\substack{\mathcal{I}\subset\{1,\dots,|\boldsymbol{R}|\}\\ \mathbf{c}[\mathcal{I}]=1^{P_1}2^{P_2}\dots}} (\kappa/\mu)^{|\boldsymbol{P}|}\frac{(\kappa/\mu;q)_{|\boldsymbol{R}|-|\boldsymbol{P}|}(\lambda/\kappa;q)_{|\boldsymbol{P}|}}{(\lambda/\mu;q)_{|\boldsymbol{R}|}} q^{-\binom{|\mathcal{I}|+1}{2}+\sum_{i\in\mathcal{I}} i}\, q^{\sum_{i\leq j} P_i J_j}.$$

Note that $P_i = \left|\mathcal{I}\cap\left\{R_{[1,i-1]}+1,\dots,R_{[1,i]}\right\}\right|$, so to reduce Corollary 3.3.4 to Proposition 3.3.2 it is enough to show for any compositions $\boldsymbol{P}\leq\boldsymbol{R}$ that

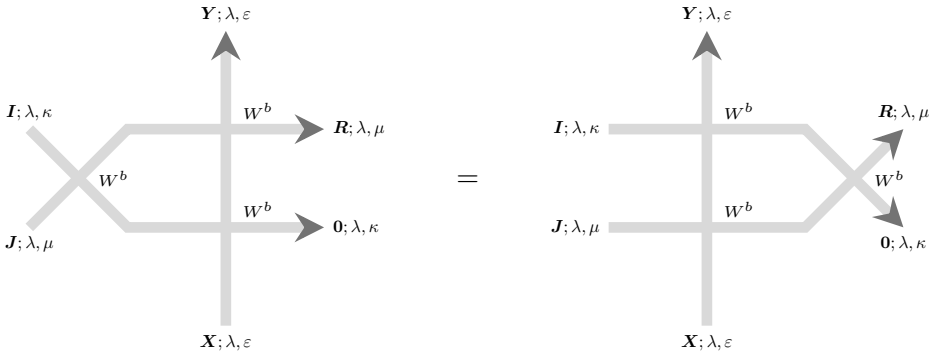
$$\sum_{\substack{\mathcal{I}\subset\{1,\dots,|\boldsymbol{R}|\}:\\ \left|\mathcal{I}\cap\left\{R_{[1,i-1]}+1,\dots,R_{[1,i]}\right\}\right|=P_i}} q^{-\binom{|\mathcal{I}|+1}{2}+\sum_{i\in\mathcal{I}} i} = q^{\sum_{i<j}(R_i-P_i)P_j}\prod_{i=1}^{n}\frac{(q;q)_{R_i}}{(q;q)_{P_i}(q;q)_{R_i-P_i}}.$$

The latter relation readily follows from the standard expression for the q-binomial coefficients:

$$\sum_{\substack{\mathcal{I}_k\subset\{R_{[1,k-1]}+1,\dots,R_{[1,k]}\}\\ |\mathcal{I}_k|=P_k}} q^{-\binom{P_k+1}{2}-R_{[1,k-1]}P_k+\sum_{i\in\mathcal{I}_k} i} = \frac{(q;q)_{R_k}}{(q;q)_{P_k}(q;q)_{R_k-P_k}},$$

applied for $1\leq k\leq n$ to the decomposition $\mathcal{I} = \mathcal{I}_1\sqcup\mathcal{I}_2\sqcup\dots\sqcup\mathcal{I}_n$, with $\mathcal{I}_k := \mathcal{I}\cap\left\{R_{[1,k-1]}+1,\dots,R_{[1,k]}\right\}$. □

Proof of Proposition 3.3.2. The main idea is to see the relation (3.3.1) as a particular instance of the Yang-Baxter equation. We start with the Yang-Baxter equation (1.3.1) with the following parameters and boundary conditions



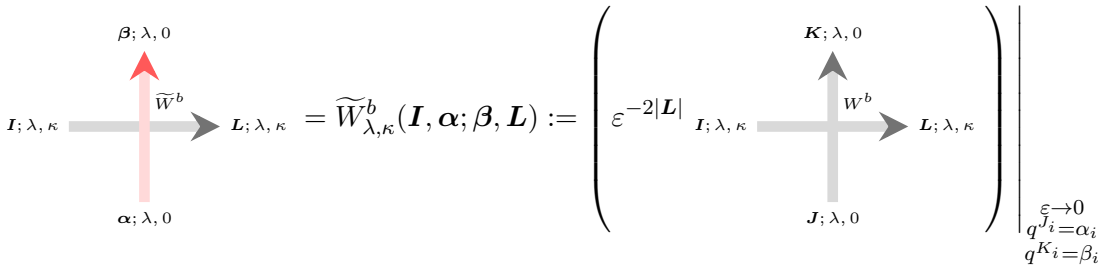
(3.3.4)

where $\lambda,\kappa,\mu,\boldsymbol{I},\boldsymbol{J},\boldsymbol{R}$ are the same as in (3.3.1), while ε is an arbitrary parameter and $\boldsymbol{X},\boldsymbol{Y}$ are arbitrary compositions satisfying

$$\boldsymbol{Y} = \boldsymbol{X}+\boldsymbol{I}+\boldsymbol{J}-\boldsymbol{R}.$$

The next step is to take $\varepsilon\to 0$ and to analytically continue the labels of the vertical edges. Namely,

consider the following vertex weights:



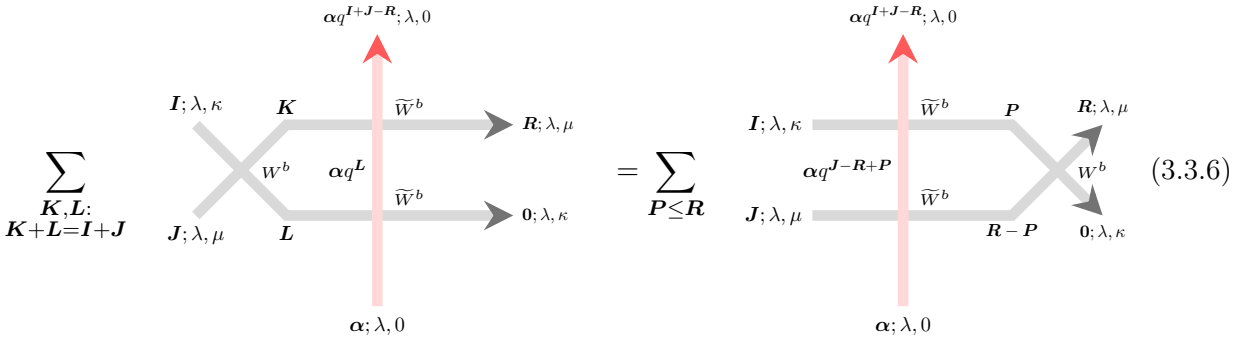
where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$ are analytically continued compositions. Using the notation $\boldsymbol{\alpha} q^{\boldsymbol{I}} = (\alpha_1 q^{I_1}, \ldots, \alpha_n q^{I_n})$ we have

$$\widetilde{W}^b_{\lambda,\kappa}(\boldsymbol{I},\boldsymbol{\alpha};\boldsymbol{\beta},\boldsymbol{L}) = \mathbb{1}_{\boldsymbol{\alpha} q^{\boldsymbol{I}} = \boldsymbol{\beta} q^{\boldsymbol{L}}}\ (\kappa/\lambda)^{|\boldsymbol{L}|}(\lambda/\kappa;q)_{|\boldsymbol{L}|} q^{\sum_{i<j} -L_i L_j} \times \prod_{i=1}^{n} \alpha_i^{L_{[1,i-1]}} \frac{(q^{1-L_i}\alpha_i;q)_{L_i}}{(q;q)_{L_i}}.$$

Note that the weight $\widetilde{W}^b_{\lambda,\kappa}(\boldsymbol{I},\boldsymbol{\alpha};\boldsymbol{\alpha} q^{\boldsymbol{I}-\boldsymbol{L}},\boldsymbol{L})$ is a polynomial in $\alpha_1, \alpha_2, \ldots, \alpha_n$, with the highest combined degree term having an explicit expression

$$\widetilde{W}^b_{\lambda,\kappa}(\boldsymbol{I},\boldsymbol{\alpha};\boldsymbol{\alpha} q^{\boldsymbol{I}-\boldsymbol{L}},\boldsymbol{L}) = (-1)^{|\boldsymbol{L}|}(\kappa/\lambda)^{|\boldsymbol{L}|}(\lambda/\kappa;q)_{|\boldsymbol{L}|} q^{-\binom{|\boldsymbol{L}|}{2}} \prod_{i=1}^{n} \frac{\alpha_i^{L_{[1,i]}}}{(q;q)_{L_i}} + \text{ lower degree terms.} \quad (3.3.5)$$

Multiplying both sides of (3.3.4) by $\varepsilon^{-2|\boldsymbol{R}|}$ and taking $\varepsilon = 0$ we have



(3.3.6)

for any $\boldsymbol{\alpha} = (q^{X_1}, q^{X_2}, \ldots, q^{X_n})$, where $\boldsymbol{X} = (X_1, \ldots, X_n)$ is a composition satisfying $\boldsymbol{X}+\boldsymbol{R}-\boldsymbol{I}-\boldsymbol{J} \geq 0$. Note that we have explicitly indicated the summations over internal edges in both sides of (3.3.6), specifying the internal labels. In particular, both sides are clearly finite sums of polynomial functions in $\alpha_1, \ldots, \alpha_n$ over $\mathbb{C}(q, \lambda, \kappa, \mu)$, hence the equality for $\boldsymbol{\alpha} = q^{\boldsymbol{X}}$ can be continued to any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$.

Having established (3.3.6) for arbitrary $\boldsymbol{\alpha}$, we can now set $\boldsymbol{\alpha} = (\alpha, \alpha, \ldots, \alpha)$, that is, all α_i are equal to a single variable α. Then both sides of (3.3.6) become polynomials in α of degree

$nR_1 + (n-1)R_2 + \cdots + R_n$. Taking the highest degree coefficients using (3.3.5) we get

$$\sum_{\substack{\boldsymbol{K},\boldsymbol{L}:\\ \boldsymbol{K}+\boldsymbol{L}=\boldsymbol{I}+\boldsymbol{J}}} W_{\lambda,\kappa,\mu}(\boldsymbol{I},\boldsymbol{J};\boldsymbol{K},\boldsymbol{L}) \times (-1)^{|\boldsymbol{R}|}(\mu/\lambda)^{|\boldsymbol{R}|}(\lambda/\mu;q)_{|\boldsymbol{R}|} q^{-\binom{|\boldsymbol{R}|}{2}} \prod_{i=1}^{n} \frac{q^{L_i R_{[1,i]}}}{(q;q)_{R_i}}$$

$$= \sum_{\boldsymbol{P}\le\boldsymbol{R}} (-1)^{|\boldsymbol{R}|}(\mu/\lambda)^{|\boldsymbol{R}|-|\boldsymbol{P}|}(\kappa/\lambda)^{|\boldsymbol{P}|}(\kappa/\mu;q)_{|\boldsymbol{R}|-|\boldsymbol{P}|}(\lambda/\kappa;q)_{|\boldsymbol{P}|} q^{-\binom{|\boldsymbol{R}|-|\boldsymbol{P}|}{2}-\binom{|\boldsymbol{P}|}{2}} \prod_{i=1}^{n} \frac{q^{(J_i-R_i+P_i)P_{[1,i]}}}{(q;q)_{P_i}(q;q)_{R_i-P_i}},$$

which is equivalent to (3.3.1) after elementary algebraic manipulations. □

Remark 3.3.5. The local relation exists for the weights $R_{a_1,b_1;a_2,b_2}$ as well and it has the following form:

$$\sum_{\boldsymbol{K},\boldsymbol{L}} q^{\sum_{i\le j} R_i L_j} R_{a_1,b_1;a_2,b_2}(\boldsymbol{I},\boldsymbol{J};\boldsymbol{K},\boldsymbol{L}) = \sum_{\substack{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}\\ \boldsymbol{X}+\boldsymbol{Y}+\boldsymbol{Z}=\boldsymbol{R}}} (a_2/b_2)^{|\boldsymbol{X}|}(b_1/b_2)^{|\boldsymbol{Y}|} \frac{(a_1/a_2;q)_{|\boldsymbol{X}|}(a_2/b_1;q)_{|\boldsymbol{Y}|}(b_1/b_2;q)_{|\boldsymbol{Z}|}}{(a_1/b_2;q)_{|\boldsymbol{R}|}}$$

$$q^{\sum_{i<j} Y_i X_j + Z_i X_j + Z_i Y_j} \prod_{i=1}^{n} \frac{(q;q)_{R_i}}{(q;q)_{X_i}(q;q)_{Y_i}(q;q)_{Z_i}} q^{\sum_{i\le j} Y_i I_j + X_i(I_j+J_j)}. \quad (3.3.7)$$

Since (3.3.7) is not used in the present work we omit its proof, but the idea is as follows: one needs to use the relation $W^a_{\lambda,\kappa,\mu}(\boldsymbol{I},\boldsymbol{J};\boldsymbol{K},\boldsymbol{L}) = W^b_{\lambda,\kappa,\mu}(\boldsymbol{J},\boldsymbol{I};\boldsymbol{K}+\boldsymbol{J},\boldsymbol{L}-\boldsymbol{J})$ to establish an analogue of Proposition 3.3.2 for the weights W^a and then sequentially apply both relations for W^a and W^b to the left-hand side of (3.3.7) using one of the expressions for $R_{a_1,b_1;a_2,b_2}$ from (1.2.5).

3.3.3 Local relation: rational functions

Now we turn to the second relation, which considers certain rational functions in $w_1,\dots,w_r$. Recall that the action of the Hecke algebra $\mathcal{H}_r$ on the rational functions in $w_1,\dots,w_r$ is generated by

$$T_i = q + \frac{w_{i+1}-qw_i}{w_{i+1}-w_i}(\mathfrak{s}_i - 1), \qquad \mathfrak{s}_i f(\dots,w_i,w_{i+1}\dots) = f(\dots,w_{i+1},w_i,\dots).$$

Proposition 3.3.6. *For any parameters t,s,η and any integer $r\in\mathbb{Z}_{\ge 0}$ the following equality of rational functions in $w_1,\dots,w_r$ holds*

$$\prod_{i=1}^{r}\frac{1-\kappa w_i}{1-\lambda w_i} = \sum_{p=0}^{r}\sum_{\tau\in\mathfrak{S}^{p|r-p}} (\kappa/\mu)^p \frac{(\kappa/\mu;q)_{r-p}(\lambda/\kappa;q)_p}{(\lambda/\mu;q)_r}\ T_{\tau^{-1}}\left(\prod_{i=1}^{p}\frac{1-\mu w_i}{1-\lambda w_i}\right),$$

where the sum is taken over $[1,p]\times[p+1,r]$-ordered permutations τ, that is, permutations satisfying

$$\tau^{-1}(1)<\tau^{-1}(2)<\cdots<\tau^{-1}(p), \qquad \tau^{-1}(p+1)<\tau^{-1}(p+2)<\cdots<\tau^{-1}(r).$$

Proof. The main idea of the proof is to take two columns of vertices with weights w, find an explicit expression for their partition function using exchange relations and then apply the equation (1.3.6) to exchange the columns. The resulting identity turns out to be equivalent to the claim. Throughout

the proof we assume that $\lambda, \kappa, \mu, \epsilon$ are fixed generic complex parameters and we fix a composition $\mathbf{1}^r = (1, 1, \dots, 1) \in \mathbb{Z}^r$.

We start with defining the two-column functions mentioned above. For any composition $\boldsymbol{P} \leq \mathbf{1}^r$ define

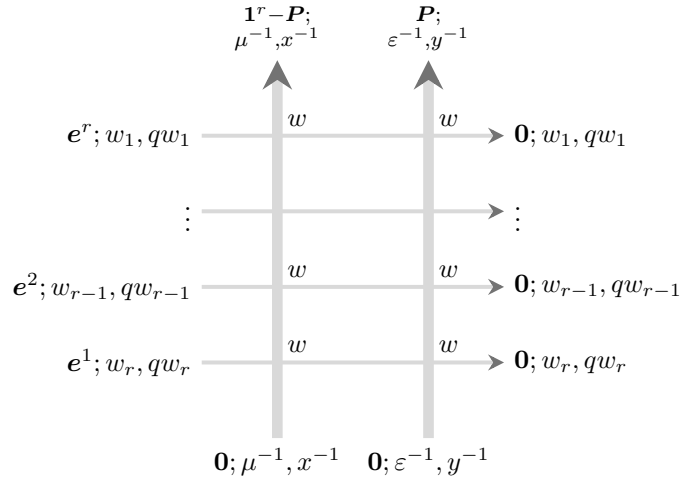
$$f_{\mathbf{1}^r-\boldsymbol{P}|\boldsymbol{P}}(w_1, \dots, w_r \mid x, y) = \langle \mathbf{1}^r - \boldsymbol{P}, \boldsymbol{P}| \ \mathcal{T}_{r,0}(w_1 \mid \mathcal{A}_{x,y}, \mathcal{B}_{x,y}) \dots \mathcal{T}_{1,0}(w_r \mid \mathcal{A}_{x,y}, \mathcal{B}_{x,y}) \ |\mathbf{0}, \mathbf{0}\rangle$$

where we set

$$\mathcal{A}_{x,y} = \left(\mu^{-1}, \varepsilon^{-1}\right), \quad \mathcal{B}_{x,y} = \left(x^{-1}, y^{-1}\right),$$

and $\mathcal{T}_{i,0}$ are the row operators (3.2.2). Alternatively, we can depict these functions as

$f_{\mathbf{1}^r-\boldsymbol{P}|\boldsymbol{P}}(w_1, \dots, w_r \mid x, y) \quad =$



where the diagram consists of $2r$ vertices with the weights w from (1.3.4). The weight of the vertex at the intersection of row i and the first column has parameters $w_{r-i+1}, \mu^{-1}, x^{-1}$, while the parameters of the other vertex in the same row are $w_{r-i+1}, \varepsilon^{-1}, y^{-1}$.

Now we want to use the exchange relations to obtain an explicit expression for $f_{\mathbf{1}^r-\boldsymbol{P}|\boldsymbol{P}}$. Since x, y are fixed in this part of an argument, for now we write $\mathcal{A}, \mathcal{B}$ instead of $\mathcal{A}_{x,y}, \mathcal{B}_{x,y}$. The exchange relation (3.2.6) implies that

$$\begin{aligned}\langle \mathbf{1}^r - \boldsymbol{P}, \boldsymbol{P}| \ \mathcal{T}_{\pi(r),0}(w_1 \mid \mathcal{A}, \mathcal{B}) \dots \mathcal{T}_{\pi(1),0}(w_r \mid \mathcal{A}, \mathcal{B}) \ |\mathbf{0}, \mathbf{0}\rangle \\ = q^{-1} T_{r-i} \ \langle \mathbf{1}^r - \boldsymbol{P}, \boldsymbol{P}| \ \mathcal{T}_{\pi\sigma_i(r),0}(w_1 \mid \mathcal{A}, \mathcal{B}) \dots \mathcal{T}_{\pi\sigma_i(1),0}(w_r \mid \mathcal{A}, \mathcal{B}) \ |\mathbf{0}, \mathbf{0}\rangle\end{aligned}$$

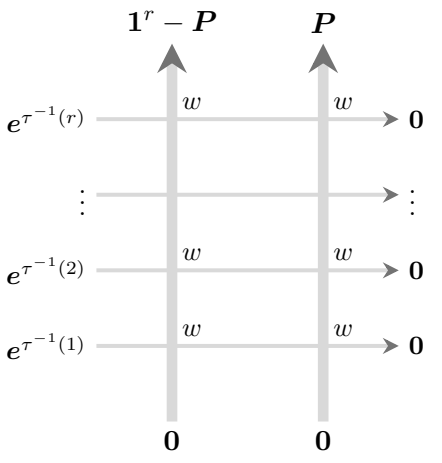
for any permutation π such that $l(\pi) < l(\pi\sigma_i)$, where σ_i is the transposition exchanging i and $i+1$. Iterating the identity above for a reduced expression $\tau = \sigma_{i_1} \dots \sigma_{i_l}$ we obtain

$$\begin{aligned}&\langle \mathbf{1}^r - \boldsymbol{P}, \boldsymbol{P}| \ \mathcal{T}_{r,0}(w_1 \mid \mathcal{A}, \mathcal{B}) \dots \mathcal{T}_{1,0}(w_r \mid \mathcal{A}, \mathcal{B}) \ |\mathbf{0}, \mathbf{0}\rangle \\ &\quad = q^{-l(\tau)} T_{r-i_l} \dots T_{r-i_1} \langle \mathbf{1}^r - \boldsymbol{P}, \boldsymbol{P}| \ \mathcal{T}_{\tau^{-1}(r),0}(w_1 \mid \mathcal{A}, \mathcal{B}) \dots \mathcal{T}_{\tau^{-1}(1),0}(w_r \mid \mathcal{A}, \mathcal{B}) \ |\mathbf{0}, \mathbf{0}\rangle \\ &\quad = q^{-l(\tau)} T_{\widetilde{\tau}^{-1}} \langle \mathbf{1}^r - \boldsymbol{P}, \boldsymbol{P}| \ \mathcal{T}_{\tau^{-1}(r),0}(w_1 \mid \mathcal{A}, \mathcal{B}) \dots \mathcal{T}_{\tau^{-1}(1),0}(w_r \mid \mathcal{A}, \mathcal{B}) \ |\mathbf{0}, \mathbf{0}\rangle, \quad (3.3.8)\end{aligned}$$

where $\widetilde{\tau}$ is the image of τ under the automorphism of $\mathfrak{S}_r$ sending σ_i to σ_{r-i}.

For now we focus on the partition function in the right-hand side of (3.3.8), namely

$$\langle \mathbf{1}^r - \boldsymbol{P}, \boldsymbol{P}|\ \mathcal{T}_{\tau^{-1}(r),0}(w_1 \mid \mathcal{A},\mathcal{B}) \dots \mathcal{T}_{\tau^{-1}(1),0}(w_r \mid \mathcal{A},\mathcal{B})\ |\mathbf{0},\mathbf{0}\rangle =$$



(3.3.9)

It turns out that the partition function above can be explicitly computed for a certain choice of τ. Namely, for a composition $\boldsymbol{P} \le \mathbf{1}^r$ let $\tau_{\boldsymbol{P}} \in \mathfrak{S}_r$ be the unique permutation satisfying

$$\tau_{\boldsymbol{P}}^{-1}(1) < \tau_{\boldsymbol{P}}^{-1}(2) < \dots < \tau_{\boldsymbol{P}}^{-1}(r-|\boldsymbol{P}|), \qquad \tau_{\boldsymbol{P}}^{-1}(r-|\boldsymbol{P}|+1) < \dots < \tau_{\boldsymbol{P}}^{-1}(r),$$

$$P_{\tau_{\boldsymbol{P}}^{-1}(1)} = \dots = P_{\tau_{\boldsymbol{P}}^{-1}(r-|\boldsymbol{P}|)} = 0, \qquad P_{\tau_{\boldsymbol{P}}^{-1}(r-|\boldsymbol{P}|+1)} = \dots = P_{\tau_{\boldsymbol{P}}^{-1}(r)} = 1.$$

In other words, $\tau_{\boldsymbol{P}}$ is the minimal permutation such that $\tau_{\boldsymbol{P}}^{-1}(r-|\boldsymbol{P}|+1), \dots, \tau_{\boldsymbol{P}}^{-1}(r)$ are exactly the colors present in $\boldsymbol{P}$. Then for $\tau = \tau_{\boldsymbol{P}}$ the partition function (3.3.9) has only one configuration with non-vanishing contribution: the path of color $\tau_{\boldsymbol{P}}^{-1}(i)$ for $1 \le i \le r-|\boldsymbol{P}|$ enters at row i and immediately goes upwards, exiting through the first column, while the path of color $\tau_{\boldsymbol{P}}^{-1}(i)$ for $r-|\boldsymbol{P}| < i \le r$ enters at row i, crosses the first column and exits through the second one. Since each horizontal edge can be occupied by at most one path, this is the unique possible configuration. Computing the vertex weights for this configuration gives

$$\begin{aligned}
&\langle \mathbf{1}^r - \boldsymbol{P}, \boldsymbol{P}|\ \mathcal{T}_{\tau_{\boldsymbol{P}}^{-1}(r),0}(w_1 \mid \mathcal{A},\mathcal{B}) \dots \mathcal{T}_{\tau_{\boldsymbol{P}}^{-1}(1),0}(w_r \mid \mathcal{A},\mathcal{B})\ |\mathbf{0},\mathbf{0}\rangle \\
&\quad = \prod_{i=1}^{r-p} \frac{1-x\mu^{-1}q^{i-1}}{1-xw_{r+1-i}} \prod_{i=r-p+1}^{r} \frac{(x\mu^{-1}-xw_{r+1-i})q^{\#\left\{j \le r-p \mid \tau_{\boldsymbol{P}}^{-1}(j) > \tau_{\boldsymbol{P}}^{-1}(i)\right\}}}{1-xw_{r+1-i}} \frac{1-\varepsilon^{-1}yq^{i-r+p-1}}{1-yw_{r+1-i}} \\
&\qquad\qquad = (x/\mu)^p (x/\mu;q)_{r-p} (y/\varepsilon;q)_p\ q^{l(\tau_{\boldsymbol{P}})} \prod_{i=1}^{r} \frac{1}{1-xw_i} \prod_{i=1}^{p} \frac{1-\mu w_i}{1-yw_i}.
\end{aligned}$$

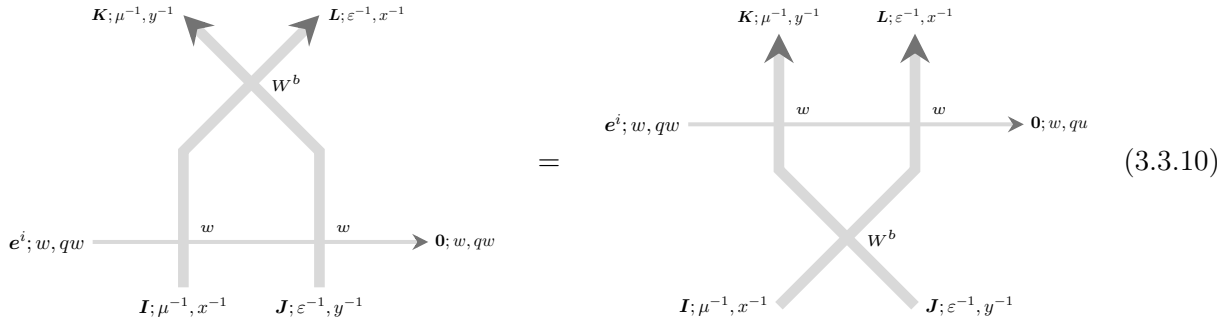
where we set $p := |\boldsymbol{P}|$ and we use the definition of $\tau_{\boldsymbol{P}}$ to deduce

$$\#\left\{i > r-p, j \le r-p \mid \tau_{\boldsymbol{P}}^{-1}(j) > \tau_{\boldsymbol{P}}^{-1}(i)\right\} = l(\tau_{\boldsymbol{P}}).$$

Plugging the expression above into (3.3.8) we get

$$f_{\mathbf{1}^r-\boldsymbol{P}|\boldsymbol{P}}(w_1,\dots,w_r \mid x,y) = (x/\mu)^{|\boldsymbol{P}|}(x/\mu;q)_{r-|\boldsymbol{P}|}(y/\varepsilon;q)_{|\boldsymbol{P}|}\ T_{\widetilde{\tau}_{\boldsymbol{P}}^{-1}} \left(\prod_{i=1}^{r} \frac{1}{1-xw_i} \prod_{i=1}^{|\boldsymbol{P}|} \frac{1-\mu w_i}{1-yw_i} \right).$$

The next step is based on the equation (1.3.6) used in the following form:



(3.3.10)

Here the tilted vertices have the weights $W^b_{\varepsilon^{-1},x^{-1},y^{-1}}$. Using the notation

$$\langle \boldsymbol{K},\boldsymbol{L} \mid W^b \mid \boldsymbol{I},\boldsymbol{J}\rangle = W^b_{\varepsilon^{-1},x^{-1},y^{-1}}(\boldsymbol{I},\boldsymbol{J};\boldsymbol{K},\boldsymbol{L}),$$

we can rewrite (3.3.10) as

$$W^b\ \mathcal{T}_{i,0}(w \mid \mathcal{A}_{x,y},\mathcal{B}_{x,y}) = \mathcal{T}_{i,0}(w \mid \mathcal{A}_{y,x},\mathcal{B}_{y,x})W^b.$$

Using this exchange relation, we obtain

$$f_{\mathbf{1}^r|\mathbf{0}}(w_1,\dots,w_r \mid y,x) = \langle \mathbf{1}^r,\mathbf{0}|\ \mathcal{T}_{r,0}(w_1 \mid \mathcal{A}_{y,x},\mathcal{B}_{y,x})\dots\mathcal{T}_{1,0}(w_r \mid \mathcal{A}_{y,x},\mathcal{B}_{y,x})W^b\ |\mathbf{0},\mathbf{0}\rangle$$
$$= \langle \mathbf{1}^r,\mathbf{0}|\ W^b\mathcal{T}_{r,0}(w_1 \mid \mathcal{A}_{x,y},\mathcal{B}_{x,y})\dots\mathcal{T}_{1,0}(w_r \mid \mathcal{A}_{x,y},\mathcal{B}_{x,y})\ |\mathbf{0},\mathbf{0}\rangle = \sum_{\boldsymbol{P}\le\mathbf{1}^r}\frac{(y/x;q)_{|\boldsymbol{P}|}}{(y/\varepsilon;q)_{|\boldsymbol{P}|}}f_{\mathbf{1}^r-\boldsymbol{P}|\boldsymbol{P}}(w_1,\dots,w_r \mid x,y),$$

where in the last equation we have explicitly evaluated the action of W^b on $\langle \mathbf{1}^r,\mathbf{0}|$. Plugging the earlier expressions for the functions $f_{\mathbf{1}^r-\boldsymbol{P}|\boldsymbol{P}}$ gives

$$(y/\mu;q)_r\prod_{i=1}^r\frac{1}{1-yw_i} = \sum_{\boldsymbol{P}\le\mathbf{1}^r}(x/\mu)^{|\boldsymbol{P}|}(x/\mu;q)_{r-|\boldsymbol{P}|}(y/x;q)_{|\boldsymbol{P}|}\ T_{\widetilde{\tau}_{\boldsymbol{P}}^{-1}}\left(\prod_{i=1}^r\frac{1}{1-xw_i}\prod_{i=1}^{|\boldsymbol{P}|}\frac{1-\mu w_i}{1-yw_i}\right).$$

The remainder of the proof consists of algebraic manipulations bringing the identity above to the desired form. First, we set $x=\kappa$ and $y=\lambda$. Then, recalling the discussion from Section 3.3.1, the summation over $\boldsymbol{P}\le\mathbf{1}^r$ is equivalent to the summation over subsets $\mathcal{I}\subset\{1,2,\dots,r\}$, which is in turn equivalent to the summation over pairs (p,π), where $p\le r$ and $\pi\in\mathfrak{S}^{r-p|p}$. Moreover, tracing back these equivalences, one can readily see that $\boldsymbol{P}$ corresponds to the pair $(|\boldsymbol{P}|,\tau_{\boldsymbol{P}})$, so we get

$$\prod_{i=1}^r\frac{1}{1-\lambda w_i} = \sum_{p=0}^r\sum_{\tau\in\mathfrak{S}^{r-p|p}}(\kappa/\mu)^p\frac{(\kappa/\mu;q)_{r-p}(\lambda/\kappa;q)_p}{(\lambda/\mu;q)_r}\ T_{\widetilde{\tau}^{-1}}\left(\prod_{i=1}^r\frac{1}{1-\kappa w_i}\prod_{i=1}^p\frac{1-\mu w_i}{1-\lambda w_i}\right).$$

The claim follows since the action of the Hecke algebra commutes with the multiplication by $\prod_{i=1}^r(1-\kappa w_i)^{-1}$ and $\tau\in\mathfrak{S}^{r-p|p}$ if and only if $\widetilde{\tau}\in\mathfrak{S}^{p|r-p}$. □

Remark 3.3.7. The functions $f_{\mathbf{1}^r-\boldsymbol{P}|\boldsymbol{P}}$ used throughout the proof of Proposition 3.3.6 are particular

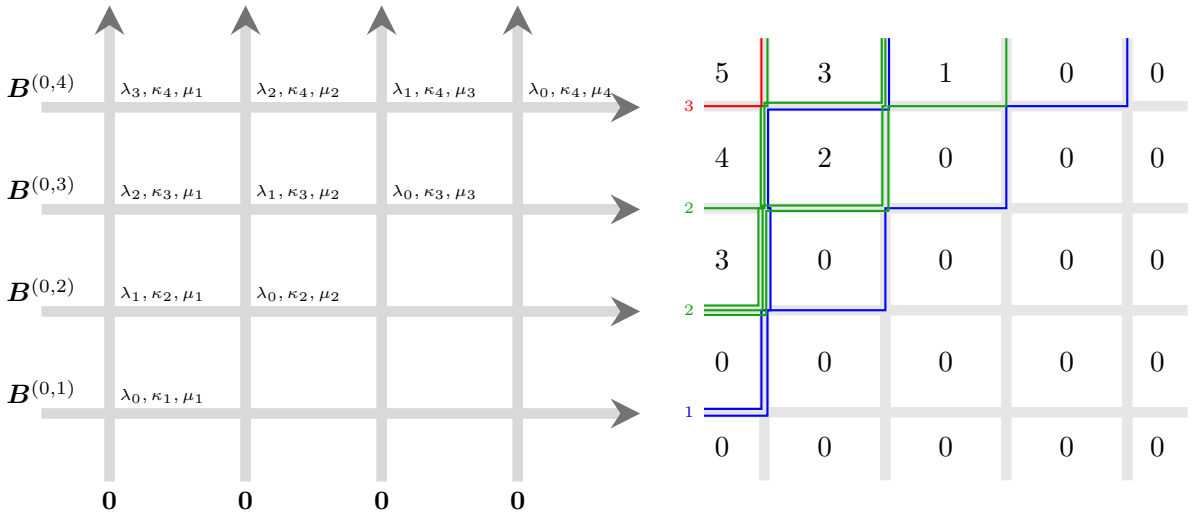


Figure 3-3: Left: parameters of the W^b-weights in the model. Right: a configuration of the q-Hahn vertex model depicted using colored paths, along with the values of the height function $h_{\geq 2}$ for this configuration. Here $\boldsymbol{C} = (1, 2, 1)$.

cases of the *non-symmetric spin Hall-Littlewood functions* introduced in [22]. The first part of the proof partially reproduces the exchange relations from [22, Section 5], while the second part can be interpreted as a new recurrence relation based on the deformed Yang-Baxter relation.

3.4 q-moments of height function

In this section we describe and prove the first main result of this chapter, namely, the integral expression for q-moments of the height functions of the colored diagonally inhomogeneous q-Hahn vertex model.

3.4.1 The model.

As noted before, the weights $W^b_{\lambda,\kappa,\mu}(\boldsymbol{I}, \boldsymbol{J}; \boldsymbol{K}, \boldsymbol{L})$ are stochastic, so they can be used to interpret a single vertex as a random sampling of an outgoing configuration given an incoming one. Now we extend this sampling construction by considering a grid of vertices with weights $W^b_{\lambda,\kappa,\mu}(\boldsymbol{I}, \boldsymbol{J}; \boldsymbol{K}, \boldsymbol{L})$. Then, given an incoming boundary conditions, we can sample a random configuration of the grid, that is, an assignment of a random composition to each lattice edge. Let us describe the construction in more detail.

Fix the number of colors n and consider a square grid of size $N \times N$. We identify the grid with the subset of $\mathbb{Z}^2_{\geq 0}$ formed by intersections of N rows $\mathbb{Z} \times \{j\}$ directed to the right and N columns $\mathbb{Z} \times \{i\}$ directed upwards, where $1 \leq i, j \leq N$. We treat this grid as a vertex model with vertices (i, j) for $i, j \in [1, N]$, vertical edges $(i, j) \to (i, j + 1)$ for $i \in [1, N], j \in [0, N]$ and horizontal edges $(i, j) \to (i + 1, j)$ for $i \in [0, N], j \in [1, N]$. A *configuration* Σ of the model is an assignment of compositions to the edges listed above. We use $\boldsymbol{A}^{(i,j)}$ (resp. $\boldsymbol{B}^{(i,j)}$) to denote the composition of the vertical (resp. horizontal) edge starting at (i, j).

Assume that we are given $q \in (0,1)$, a composition $\boldsymbol{C}$ such that $|\boldsymbol{C}| = N$ and three families of real parameters $(\mu_0, \mu_1, \mu_2, \dots)$, $(\kappa_1, \kappa_2, \dots)$, $(\dots, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \dots)$ satisfying

$$\lambda_d < \kappa_j < \mu_i, \qquad i \in \mathbb{Z}_{\geq 0};\ j \in \mathbb{Z}_{>0};\ d \in \mathbb{Z}. \tag{3.4.1}$$

We construct a random configuration Σ of the model according to the following sampling procedure:

- The compositions $\boldsymbol{A}^{(i,0)}$ of all the bottom incoming vertical edges $(i,0) \to (i,1)$ are set to $\boldsymbol{0}$.
- To construct compositions of the incoming left horizontal edges we first sample a collection of independent random nonnegative integers $(b_1, b_2, \dots, b_N)$ with distributions

$$\mathbb{P}(b_j = b) = (\kappa_j/\mu_0)^b \frac{(\lambda_j/\kappa_j; q)_b}{(q;q)_b} \frac{(\kappa_j/\mu_0; q)_\infty}{(\lambda_j/\mu_0; q)_\infty}, \qquad b \in \mathbb{Z}_{\geq 0}. \tag{3.4.2}$$

  Then the composition $\boldsymbol{B}^{(0,j)}$ of the jth incoming horizontal edge $(0,j) \to (1,j)$ has the form $\boldsymbol{B}^{(0,j)} = (0, \dots, 0, b_j, 0, \dots, 0)$, where all positions but $B_c^{(0,j)} = b_j$ are equal to 0. The non-vanishing color c is determined as the unique color satisfying $C_{[1,c-1]} < j \leq C_{[1,c]}$. In other words, $\boldsymbol{C}$ determines the colors of the left-incoming edges with the first C_1 rows having color 1, the next C_2 rows having color 2 and so on.
- The compositions of the remaining edges are sampled sequentially in N^2 steps, which correspond to the vertices (i,j) of the model and are performed in lexicographical order with respect to (i,j), starting from the vertex $(1,1)$.
- At each step we have a vertex (i,j) with already sampled compositions $\boldsymbol{A}^{(i,j-1)}$ and $\boldsymbol{B}^{(i-1,j)}$ of the incoming edges $(i,j-1) \to (i,j)$ and $(i-1,j) \to (i,j)$, due to the order of the steps. Then we treat the vertex weights $W^b_{\lambda_{j-i},\kappa_j,\mu_i}$ as probabilities for a stochastic sampling algorithm, transforming the incoming compositions into the outgoing ones $\boldsymbol{A}^{(i,j)}, \boldsymbol{B}^{(i,j)}$:

$$\begin{aligned}\mathbb{P}(\boldsymbol{A}^{(i,j)} = \boldsymbol{K}, \boldsymbol{B}^{(i,j)} = \boldsymbol{L} \mid \boldsymbol{A}^{(i,j-1)} = \boldsymbol{J}, \boldsymbol{B}^{(i-1,j)} = \boldsymbol{I}) &= W^b_{\lambda_{j-i},\kappa_j,\mu_i}(\boldsymbol{I}, \boldsymbol{J}; \boldsymbol{K}, \boldsymbol{L}) \\ = \mathbb{1}_{\boldsymbol{I}+\boldsymbol{J}=\boldsymbol{K}+\boldsymbol{L}}\ (\kappa_j/\mu_i)^{|\boldsymbol{L}|} &\frac{(\kappa_j/\mu_i; q)_{|\boldsymbol{J}|-|\boldsymbol{L}|} (\lambda_{j-i}/\kappa_j; q)_{|\boldsymbol{L}|}}{(\lambda_{j-i}/\mu_i; q)_{|\boldsymbol{J}|}} q^{\sum_{a<b} L_a(J_b - L_b)} \prod_{a=1}^{n} \binom{J_a}{L_a}_q. \end{aligned} \tag{3.4.3}$$

Note that all the probabilities are nonnegative for $q \in (0,1)$ and $\mu_i, \kappa_j, \lambda_d$ satisfying (3.4.1). Since the weights at vertex (i,j) depend on σ_i, κ_j and λ_{j-i}, we treat these parameters as attached to column $i = const$, row $j = const$ and diagonal $j - i = const$ respectively. See Figure 3-3 for a depiction of the parameters of the weights in the model.

In the description of the model we have also used that (3.4.2) gives a well-defined probability distribution. This follows from the identity

$$\sum_{k \geq 0} x^k \frac{(y;q)_k}{(q;q)_k} = \frac{(yx;q)_\infty}{(x;q)_\infty}, \tag{3.4.4}$$

which can be proved by setting $y = q^{-n}, n \in \mathbb{Z}_{\geq 0}$ and reducing the claim to the q-binomial theorem.

Finally, note that the vertex sampling is trivial outside of the region $i < j$: indeed, since the weight $W^b_{\lambda,\kappa,\mu}(\boldsymbol{I}, \boldsymbol{J}; \boldsymbol{K}, \boldsymbol{L})$ vanishes unless $\boldsymbol{J} \geq \boldsymbol{L}$, a vertex (i,j) with the incoming bottom

configuration $\boldsymbol{A}^{(i,j-1)} = \mathbf{0}$ is forced to have a deterministic sampling with $\boldsymbol{B}^{(i,j)} = \mathbf{0}$ and $\boldsymbol{A}^{(i,j)} = \boldsymbol{B}^{(i-1,j)}$, that is, the top outgoing configuration is equal to the left incoming one. These rules force all vertices (i,j) with $i > j$ to have edge labels $\mathbf{0}$ around them, while the sampling at the vertices (i,i) is deterministic. In particular, the model does not depend on the parameters $\{\lambda_d\}_{d\le 0}$, thus we can freely omit them from now on.

3.4.2 The colored height functions and the formula for q-moments.

The vertex model just described has a collection of natural observables, called *colored height functions*. We denote them by $h^{(x,y)}_{\ge c}(\Sigma)$, where Σ is a configuration of the model, $c \in [1,n]$ is an integer representing a color and $(x,y) \in \left(\mathbb{Z}_{\ge 0} + \frac{1}{2}\right)^2$ is a point treated as a facet of the grid between columns $x \pm \frac{1}{2}$ and rows $y \pm \frac{1}{2}$.

Informally, the height function $h^{(x,y)}_{\ge c}$ indicates the number of paths of color $\ge c$ passing below (x,y). More precisely, for a given fixed configuration Σ we define the corresponding height functions following two local rules:

- If $\boldsymbol{A}^{(x-\frac{1}{2},y-\frac{1}{2})}$ denotes the label of the vertical edge $(x-\frac{1}{2},y-\frac{1}{2}) \to (x-\frac{1}{2},y+\frac{1}{2})$ then
$$h^{(x-1,y)}_{\ge c}(\Sigma) = h^{(x,y)}_{\ge c}(\Sigma) + A^{(x-\frac{1}{2},y-\frac{1}{2})}_{[c,n]}.$$
- If $\boldsymbol{B}^{(x-\frac{1}{2},y+\frac{1}{2})}$ denotes the label of the horizontal edge $(x-\frac{1}{2},y+\frac{1}{2}) \to (x+\frac{1}{2},y+\frac{1}{2})$ then
$$h^{(x,y+1)}_{\ge c}(\Sigma) = h^{(x,y)}_{\ge c}(\Sigma) + B^{(x-\frac{1}{2},y+\frac{1}{2})}_{[c,n]}.$$

Due to the conservation law around each vertex, these local rules give a well-defined height function $h^{(x,y)}_{\ge c}(\Sigma)$, see Figure 3-3. The resulting height functions are unique up to a global additive shift; we fix the normalization by requiring
$$h^{(\frac{1}{2},\frac{1}{2})}_{\ge c}(\Sigma) = 0$$
for any color $c \in [1,n]$ and configuration Σ.

To simplify expressions in the following sections we also introduce the following notation for the multi-point height function: for a sequence of colors $\mathbf{c} = (c_1,\dots,c_k)$, a pair of sequences of half integers $\mathbf{x} = (x_1,\dots,x_k), \mathbf{y} = (y_1,\dots,y_k)$ and a configuration Σ we set
$$\mathcal{H}^{(\mathbf{x},\mathbf{y})}_{\ge \mathbf{c}}(\Sigma) := \sum_{i=1}^{k} h^{(x_i,y_i)}_{\ge c_i}(\Sigma).$$

3.4.3 The integral expression for q-moments.

The main result of this section expresses a certain observable of the q-Hahn model as a nested contour integral. The observable in question is denoted by $\mathcal{Q}^{(\mathbf{x},\mathbf{y})}_{\ge \tau.\mathbf{c}}(\Sigma)$ and for a configuration of the model Σ, an ordered sequence of colors $\mathbf{c} = (c_1,\dots,c_k)$, a permutation $\tau \in \mathfrak{S}_k$ and a pair of

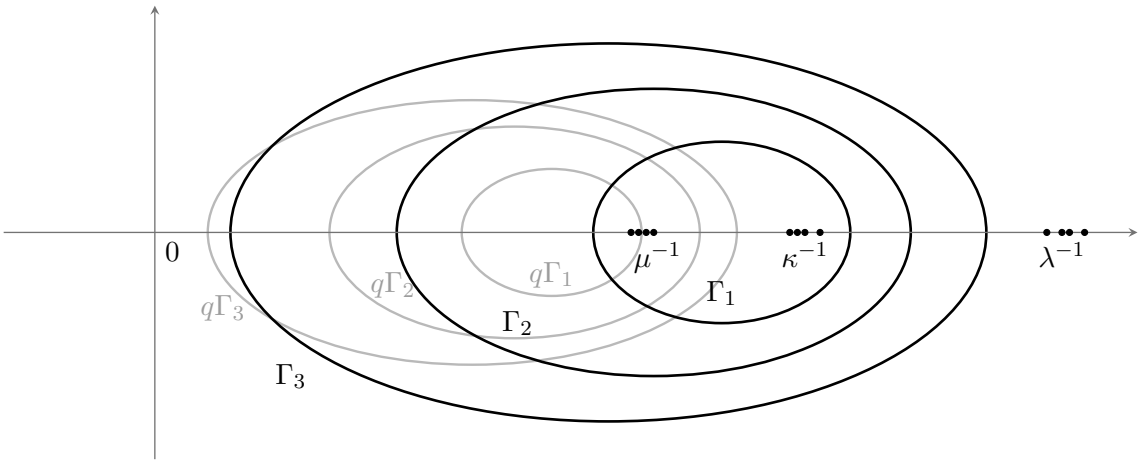


Figure 3-4: A possible of configuration of the contours Γ_i for Theorem 3.4.1.

sequences of half integers $\mathbf{x} = (x_1, \dots, x_k), \mathbf{y} = (y_1, \dots, y_k)$ it is defined by

$$\mathcal{Q}^{(\mathbf{x},\mathbf{y})}_{\geq \tau.\mathbf{c}}(\Sigma) := q^{\mathcal{H}^{(\mathbf{x},\mathbf{y})}_{\geq \tau.\mathbf{c}}(\Sigma)} = \prod_{a=1}^{k} q^{h^{(x_a,y_a)}_{\geq c_{\tau^{-1}(a)}}(\Sigma)},$$

where we follow the notation

$$\tau.\mathbf{c} = \left(c_{\tau^{-1}(1)}, c_{\tau^{-1}(2)}, \dots, c_{\tau^{-1}(k)}\right).$$

To formulate the result we also need to specify integration contours. They are denoted by $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ and are assumed to satisfy the following conditions:

- Every contour Γ_a is a union of simple positively oriented closed contours in $\mathbb{C}$;
- The points from $\{\mu_i^{-1}\}_{i\geq 0}$ and $\{\kappa_j^{-1}\}_{j\geq 1}$ are inside every contour Γ_a, while 0 and the points from $\{\lambda_d^{-1}\}_{d\geq 1}$ are outside of the contours;
- For any $a < b$ the contour Γ_b encircles both contours Γ_a and $q\Gamma_a$. Here $q\Gamma_a$ denotes the contour obtained by multiplying all points of Γ_a by q.

See Figure 3-4 for a possible configuration of the contours. Note that the conditions on the parameters (3.4.1) guarantee the existence of such contours.

Theorem 3.4.1. *Let Σ be a random configuration of the q-Hahn model defined by the data $\{\mu_i\}_{i\geq 0}$, $\{\kappa_j\}_{j\geq 1}$, $\{\lambda_d\}_{d\geq 1}$ and $\boldsymbol{C}$. Then for any sequences $\mathbf{c} = (c_1, \dots, c_k)$, $\mathbf{x} = (x_1, \dots, x_k)$ and $\mathbf{y} = (y_1, \dots, y_k)$ satisfying*

$$x_1 \leq x_2 \leq \dots \leq x_k, \qquad y_1 \geq y_2 \geq \dots \geq y_k, \qquad x_a \leq y_a, \qquad x_a, y_a \in \mathbb{Z}_{\geq 0} + \frac{1}{2};$$

$$c_1 \leq c_2 \leq \dots \leq c_k, \qquad c_a \in \mathbb{Z}_{\geq 0};$$

and for any permutation $\tau \in S_k$ the following holds:

$$\mathbb{E}\left[\mathcal{Q}^{\mathbf{x},\mathbf{y}}_{\geq\tau,\mathbf{c}}(\Sigma)\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}-l(\tau)}}{(2\pi\mathbf{i})^k}\oint_{\Gamma_1}\cdots\oint_{\Gamma_k}\prod_{a<b}\frac{w_b-w_a}{w_b-qw_a}\;T_\tau\left(\prod_{a=1}^{k}\prod_{i=1}^{l_a}\frac{1-\lambda_i w_a}{1-\kappa_i w_a}\right)$$
$$\times\prod_{a=1}^{k}\left(\prod_{i=0}^{i<x_a}\frac{1}{1-\mu_i w_a}\prod_{j=1}^{j<y_a}(1-\kappa_j w_a)\prod_{d=1}^{d\leq y_a-x_a}\frac{1}{1-\lambda_d w_a}\right)\frac{dw_a}{w_a}, \quad (3.4.5)$$

where the integration contours Γ_a are specified before the theorem, T_τ denotes the action of the Hecke algebra defined by (3.2.1) *and the integers l_a are defined by $l_a = C_{[1,c_a-1]}$.*

The proof of the theorem occupies the remainder of the section, but before it we explain how Theorem 3.4.1 simplifies in the one-colored case, when $n = 1$. Then all colors in $\mathbf{c}$ are equal to 1, the colors of the incoming left boundary edges are defined by $\boldsymbol{C} = (N)$ and the operators T_τ act on a constant 1, multiplying it by $q^{l(\tau)}$. Since the only nontrivial height function in the color-blind case is $h_{\geq 1}$, we omit $\mathbf{c}$ and write just $\mathcal{Q}^{\mathbf{x},\mathbf{y}}(\Sigma)$ instead of $\mathcal{Q}^{\mathbf{x},\mathbf{y}}_{\geq(1,\ldots,1)}(\Sigma)$ in the one-colored case:

Corollary 3.4.2. *Let Σ be a random configuration sampled from the one-colored q-Hahn model with parameters $\{\mu_i\}_{i\geq 0}, \{\kappa_j\}_{j\geq 1}, \{\lambda_d\}_{d\geq 1}$. Then for any sequences $\mathbf{x} = (x_1, \ldots, x_k)$ and $\mathbf{y} = (y_1, \ldots, y_k)$ satisfying*

$$x_1 \leq x_2 \leq \cdots \leq x_k, \qquad y_1 \geq y_2 \geq \cdots \geq y_k, \qquad x_a \leq y_a, \qquad x_a, y_b \in \mathbb{Z}_{\geq 0} + \frac{1}{2},$$

the following holds:

$$\mathbb{E}\left[\mathcal{Q}^{\mathbf{x},\mathbf{y}}(\Sigma)\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi\mathbf{i})^k}\oint_{\Gamma_1}\cdots\oint_{\Gamma_k}\prod_{a<b}\frac{w_b-w_a}{w_b-qw_a}$$
$$\times\prod_{a=1}^{k}\left(\prod_{i=0}^{i<x_a}\frac{1}{1-\mu_i w_a}\prod_{j=1}^{j<y_a}(1-\kappa_j w_a)\prod_{d=1}^{d\leq y_a-x_a}\frac{1}{1-\lambda_d w_a}\right)\frac{dw_a}{w_a}, \quad (3.4.6)$$

where the integration contours Γ_a are simple positively oriented curves satisfying

- *The points from $\{\mu_i^{-1}\}_{i\geq 0}$ are inside every contour Γ_a, while 0 and the points from $\{\lambda_d^{-1}\}_{d\geq 1}$ are outside of the contours;*
- *For any $a < b$ the contour Γ_b encircles both contours Γ_a and $q\Gamma_a$.*

Note that, contrary to the colored case, in the one-colored situation the integrand has no singularities at κ_j^{-1}, so we need no restrictions on the positions of κ_j^{-1} with respect to the contours Γ_a.

Remark 3.4.3. When all parameters $\{\lambda_d\}_{d\geq 1}$ are equal and there is no inhomogeneity assigned to diagonals, the q-Hahn model can be obtained from the *colored higher-spin six-vertex model* using *stochastic fusion*. In this case Theorem 3.4.1 can be obtained from analogous integral expressions for the q-moments of the higher-spin model: for one-color case this was done in [20], while the colored situation is covered by [24], [28]. The general case with arbitrary parameters $\{\lambda_d\}_{d\geq 1}$ cannot be reached with fusion.

3.4.4 Explicit integral computations.

The proof of Theorem 3.4.1 consists of two ingredients: explicit computations of contour integrals in certain degenerate cases, and recurrence relations given in Section 3.3. Here we focus on the former.

We fix the contours Γ_a as in the theorem, and let $\mathbf{w} = (w_1, \dots, w_k)$ denote the collection of the integration variables. We need the following fact describing the operators adjoint to T_τ with respect to the scalar product

$$\langle F(\mathbf{w}), G(\mathbf{w}) \rangle_k := \oint_{\Gamma_1} \cdots \oint_{\Gamma_k} \prod_{a<b} \frac{w_b - w_a}{w_b - q w_a} F(\mathbf{w}) G(\mathbf{w}) \prod_{a=1}^{k} \frac{dw_a}{w_a}.$$

Proposition 3.4.4 ([22]). *Assume that $F(w_1, \dots, w_k) = F(\mathbf{w})$ and $G(w_1, \dots, w_k) = G(\mathbf{w})$ are rational functions having only singularities of the form $w_a = \mu_i^{-1}$, $w_a = \kappa_j^{-1}$ or $w_a = \lambda_d^{-1}$. Then for any permutation τ the following relation holds:*

$$\oint_{\Gamma_1} \cdots \oint_{\Gamma_k} \prod_{a<b} \frac{w_b - w_a}{w_b - q w_a} T_\tau(F(\mathbf{w})) G(\mathbf{w}) \prod_{a=1}^{k} \frac{dw_a}{w_a} = \oint_{\Gamma_1} \cdots \oint_{\Gamma_k} \prod_{a<b} \frac{w_b - w_a}{w_b - q w_a} F(\mathbf{w}) T_{\tau^{-1}}(G(\mathbf{w})) \prod_{a=1}^{k} \frac{dw_a}{w_a}, \tag{3.4.7}$$

or, equivalently,

$$\langle T_\tau F(\mathbf{w}), G(\mathbf{w}) \rangle_k = \langle F(\mathbf{w}), T_{\tau^{-1}} G(\mathbf{w}) \rangle_k.$$

Proof. The complete proof is given in [22, Proposition 8.1.3], see also [28, Proposition 4.1]. The main idea is to prove that the generators T_i are self-adjoint, which is enough since T_i generate the Hecke algebra $\mathcal{H}_k$. Each T_i is a sum of a multiplication by a constant and $\frac{w_{i+1} - q w_i}{w_{i+1} - w_i} \mathfrak{s}_i$. The former is clearly self-adjoint, while for the latter note that the multiplication by $\frac{w_{i+1} - q w_i}{w_{i+1} - w_i}$ removes the singularity of the integrand at $w_{i+1} = q w_i$, allowing to exchange the contours Γ_i and Γ_{i+1} without changing the integral. Performing a change of variables swapping w_i and w_{i+1}, one can see that $\frac{w_{i+1} - q w_i}{w_{i+1} - w_i} \mathfrak{s}_i$ is also self-adjoint. □

The next statement describes the situation when one of the integrals in (3.4.5) can be easily computed, reducing the number of integration variables by one.

Proposition 3.4.5. *Assume that $f_1, \dots, f_k$ and $g_1, \dots, g_k$ are rational functions in one variable satisfying*

- *For every $a \in [1, k]$ the function $f_k(w) g_a(w)$ is holomorphic outside of Γ_k;*
- *For every $a \in [1, k]$ the poles of $f_a(w)$ and $g_a(w)$ are inside of $\{\mu_i^{-1}\}_{i \geq 0} \cup \{\kappa_j^{-1}\}_{j \geq 1} \cup \{\lambda_d^{-1}\}_{d \geq 1}$;*
- *For every $a \in [1, k]$ we have $f_a(0) = g_a(0) = 1$.*

Then

$$\frac{(-1)^k q^{\frac{k(k-1)}{2} - l(\tau)}}{(2\pi \mathbf{i})^k} \oint_{\Gamma_1} \cdots \oint_{\Gamma_k} \prod_{a<b} \frac{w_b - w_a}{w_b - q w_a} T_\tau \left(\prod_{a=1}^{k} f_a(w_a) \right) \prod_{a=1}^{k} g_a(w_a) \prod_{a=1}^{k} \frac{dw_a}{w_a}$$

$$= \frac{(-1)^{k-1} q^{\frac{(k-1)(k-2)}{2} - l(\tau')}}{(2\pi \mathbf{i})^{k-1}} \oint_{\Gamma_1} \cdots \oint_{\Gamma_{k-1}} \prod_{a<b \leq k-1} \frac{w_b - w_a}{w_b - q w_a} T_{\tau'} \left(\prod_{a=1}^{k-1} f_a(w_a) \right) \prod_{a=1}^{k-1} g_{\rho(a)}(w_a) \prod_{a=1}^{k-1} \frac{dw_a}{w_a}$$

where $\rho = \sigma_{k-1}\sigma_{k-2}\dots\sigma_{\tau(k)}$ is the cycle sending $\tau(k), \tau(k)+1, \dots, k$ to $\tau(k)+1, \tau(k)+2, \dots, k, \tau(k)$ respectively, while leaving other elements intact, and the permutation τ' is defined by $\tau' = \rho^{-1}\tau$.

Proof. Note that $T_\tau = T_\rho T_{\tau'}$, so by Proposition 3.4.4 the left-hand side can be rewritten as

$$\frac{(-1)^k q^{\frac{k(k-1)}{2}-l(\tau)}}{(2\pi\mathbf{i})^k}\oint_{\Gamma_1}\cdots\oint_{\Gamma_k}\prod_{a<b}\frac{w_b-w_a}{w_b-qw_a}T_{\tau'}\left(\prod_{a=1}^{k}f_a(w_a)\right)\ T_{\rho^{-1}}\left(\prod_{a=1}^{k}g_a(w_a)\right)\prod_{a=1}^{k}\frac{dw_a}{w_a}. \quad (3.4.8)$$

Now we are taking the integral with respect to w_k by computing its residues outside of Γ_k. Note that $\tau'(k) = k$, hence

$$T_{\tau'}\left(\prod_{a=1}^{k}f_a(w_a)\right) = f_k(w_k)T_{\tau'}\left(\prod_{a=1}^{k-1}f_a(w_a)\right).$$

By the conditions of the claim the function

$$f_k(w_k)T_{\rho^{-1}}\left(\prod_{a=1}^{k}g_a(w_a)\right) = \left(\prod_{a=1}^{k-1}f_k(w_a)\right)^{-1}T_{\rho^{-1}}\left(\prod_{a=1}^{k}f_k(w_a)g_a(w_a)\right)$$

is holomorphic with respect to w_k outside of Γ_k, so the only non-vanishing residue of the integrand in (3.4.8) outside of Γ_k is the residue at $w_k = 0$. Computing it, we obtain

$$\frac{(-1)^{k-1}q^{\frac{(k-1)(k-2)}{2}-l(\tau)}}{(2\pi\mathbf{i})^{k-1}}\oint_{\Gamma_1}\cdots\oint_{\Gamma_{k-1}}\prod_{a<b\le k-1}\frac{w_b-w_a}{w_b-qw_a}T_{\tau'}\left(\prod_{a=1}^{k-1}f_a(w_a)\right)\left(T_{\rho^{-1}}\prod_{a=1}^{k}g_a(w_a)\right)\Big|_{w_k=0}\prod_{a=1}^{k-1}\frac{dw_a}{w_a}.$$

To finish the proof, note that for any function $G(\mathbf{w})$ nonsingular at $w_a = 0$ for every a we have

$$\left(T_iG(\mathbf{w})\right)\big|_{w_{i+1}=0} = \left(\frac{(q-1)w_{i+1}}{w_{i+1}-w_i}G(\mathbf{w}) + \frac{w_{i+1}-qw_i}{w_{i+1}-w_i}G(\sigma_i.\mathbf{w})\right)\Big|_{w_{i+1}=0} = qG(\sigma_i.\mathbf{w})\big|_{w_{i+1}=0},$$

where σ_i is a simple transposition acting on the variables $\mathbf{w}$ by permutation. A straightforward induction yields

$$T_{\rho^{-1}}\left(\prod_{a=1}^{k}g_a(w_a)\right)\Big|_{w_k=0} = T_{k-1}T_{k-2}\dots T_{\tau(k)}\left(\prod_{a=1}^{k}g_a(w_a)\right)\Big|_{w_k=0}$$
$$= q^{l(\rho)}\prod_{a=1}^{k}g_a(w_{\rho^{-1}(a)})\Big|_{w_k=0} = q^{l(\rho)}\prod_{a=1}^{k-1}g_{\rho(a)}(w_a).$$

□

Another integral we need to explicitly compute by taking residues is the right-hand side of (3.4.5) when $x_1 = \dots = x_k = \frac{1}{2}$. For convenience, set $p_i := y_i - \frac{1}{2}$ and recall that l_a corresponds to the position of the color c_a on the left boundary.

Proposition 3.4.6. *Fix $N > 0$. For any integer sequences $l_1, l_2, \dots, l_k$ and $p_1, p_2, \dots, p_k$ such that*

$$0 \le l_1 \le l_2 \le \dots \le l_k \le N, \qquad N \ge p_1 \ge p_2 \ge \dots \ge p_k \ge 0,$$

and any permutation τ we have

$$\prod_{j=1}^{N}\frac{(\kappa_j/\mu_0;q)_{r_j}}{(\lambda_j/\mu_0;q)_{r_j}}=\frac{(-1)^k q^{\frac{k(k-1)}{2}-l(\tau)}}{(2\pi\mathbf{i})^k}\oint_{\Gamma_1}\cdots\oint_{\Gamma_k}\prod_{a<b}\frac{w_b-w_a}{w_b-qw_a}\;T_\tau\left(\prod_{a=1}^{k}\prod_{j=1}^{l_a}\frac{1-\lambda_j w_a}{1-\kappa_j w_a}\right)\\ \times\prod_{a=1}^{k}\prod_{j=1}^{p_a}\frac{1-\kappa_j w_a}{1-\lambda_j w_a}\;\prod_{a=1}^{k}\frac{dw_a}{w_a(1-\mu_0 w_a)},\quad(3.4.9)$$

where for every integer $j\in[1,N]$ we set

$$r_j:=\#\{a\in[1,k]\mid l_{\tau^{-1}(a)}<j\le p_a\},$$

that is, r_j is the number of intervals $[l_{\tau^{-1}(a)}+1,p_a]$ containing j.

Proof. The proof is by induction on k and N, where at each step we either reduce k or N by 1, keeping $N,k\ge 0$. The base case $k=0$ is trivial.

Assume that $k>0$ and let $\mathbf{l}=(l_1,\dots,l_k)$ and $\mathbf{p}=(p_1,\dots,p_k)$. Set

$$F^{\mathbf{l}}(\mathbf{w})=\prod_{a=1}^{k}\prod_{j=1}^{l_a}\frac{1-\lambda_j w_a}{1-\kappa_j w_a},\qquad G^{\mathbf{p}}(\mathbf{w})=\prod_{a=1}^{k}\prod_{j=1}^{p_a}\frac{1-\kappa_j w_a}{1-\lambda_j w_a}.$$

We have three cases:

Case 1: $l_k<N$ and $p_1<N$: Then for any a we have $l_a,p_a<N$, so $r_N=0$, $N>0$ and we can freely reduce N by 1 without changing the claim.

Case 2: $l_k=N$: Here we can compute the integral with respect to w_k by applying Proposition 3.4.5 for the functions

$$f_a=\prod_{j=1}^{l_a}\frac{1-\lambda_j w}{1-\kappa_j w},\qquad g_a=\frac{1}{1-\mu_0 w}\prod_{j=1}^{p_a}\frac{1-\kappa_j w}{1-\lambda_j w}.$$

Since $l_k=N\ge p_a$ for any a, the function $f_k(w)g_a(w)$ indeed has no residues outside of Γ_k, so Proposition 3.4.5 implies that

$$\frac{(-1)^k q^{k(k-1)/2-l(\tau)}}{(2\pi\mathbf{i})^k}\oint_{\Gamma_1}\cdots\oint_{\Gamma_k}\prod_{a<b}\frac{w_b-w_a}{w_b-qw_a}T_\tau\left(F^{\mathbf{l}}(\mathbf{w})\right)G^{\mathbf{p}}(\mathbf{w})\prod_{a=1}^{k}\frac{dw_a}{w_a(1-\mu_0 w_a)}$$
$$=\frac{(-1)^{k-1}q^{(k-1)(k-2)/2-l(\tau')}}{(2\pi\mathbf{i})^{k-1}}\oint_{\Gamma_1}\cdots\oint_{\Gamma_{k-1}}\prod_{a<b\le k-1}\frac{w_b-w_a}{w_b-qw_a}T_{\tau'}\left(F^{\mathbf{l}'}(\mathbf{w}')\right)G^{\mathbf{p}'}(\mathbf{w}')\prod_{a=1}^{k-1}\frac{dw_a}{w_a(1-\mu_0 w_a)},$$

where the permutation τ' as defined by $\tau=\rho\tau'$ for the same cycle ρ as in Proposition 3.4.5 and

$$\mathbf{w}'=(w_1,\dots,w_{k-1}),\qquad \mathbf{l}'=(l_1,\dots,l_{k-1}),$$

$$\mathbf{p}'=(p_{\rho(1)},\dots,p_{\rho(k-1)})=(p_1,\dots,p_{\tau(k)-1},p_{\tau(k)+1},\dots,p_k).$$

Hence, the right-hand sides of (3.4.9) for the triples $(\mathbf{l},\mathbf{p},\tau)$ and $(\mathbf{l}',\mathbf{p}',\tau')$ coincide.

At the same time, the families of intervals $\{[l_{\tau^{-1}(a)}+1, p_a]\}_{a\in[1,k]}$ and $\{[l_{\tau'^{-1}(a)}+1, p_{\rho(a)}]\}_{a\in[1,k-1]}$ differ only by the empty interval $[l_k+1, p_{\tau(k)}]$, implying that the left-hand sides of the claim for $(\mathbf{l},\mathbf{p},\tau)$ and $(\mathbf{l}',\mathbf{p}',\tau')$ coincide as well. Thus, we can reduce k by 1 and the induction step hold.

Case 3: $l_k < N$ and $N = p_1 = p_2 = \cdots = p_r > p_{r-1}$: To tackle this case we start computing the integrals with respect to $w_1,\ldots,w_r$, taking residues inside $\Gamma_1,\ldots,\Gamma_r$. Note that possible simple poles at $w_1=\kappa_j^{-1},\ldots,w_r=\kappa_j^{-1}$ coming from $F^{\mathbf{l}}(\mathbf{w})$ are cancelled by $\prod_{a=1}^{r}\prod_{j=1}^{N}(1-\kappa_j w_a)$ from $G^{\mathbf{p}}(\mathbf{w})$, so all non-vanishing residues inside $\Gamma_1,\ldots,\Gamma_r$ come from the singularities of

$$\prod_{a=1}^{k}\frac{1}{w_a(1-\mu_0 w_a)}\prod_{a<b}\frac{w_b-w_a}{w_b-qw_a}.$$

Taking into account the geometry of the contours, the integral with respect to w_1 has only one residue inside Γ_1, which is at $w_1=\mu_0^{-1}$. Computing this residue for the expression above, we obtain

$$\frac{1}{-\mu_0 w_1}\prod_{a=2}^{k}\frac{1}{w_a(1-\mu_0 w_a)}\prod_{a<b}\frac{w_b-w_a}{w_b-qw_a}\Big|_{w_1=\mu_0^{-1}} = -q^{-k+1}\prod_{a=2}^{k}\frac{1}{w_a(1-q^{-1}\mu_0 w_a)}\prod_{2\le a<b}\frac{w_b-w_a}{w_b-qw_a}.$$

Hence, after taking the residue at $w_1=\mu_0^{-1}$, the integral with respect to w_2 again has only one non-vanishing residue inside Γ_2, this time at $w_2=q\mu_0^{-1}$. Continuing this argument, we see that the only non-vanishing contribution in the computation of the right-hand side of (3.4.9) comes from the residue at $w_1=\mu_0^{-1}, w_2=q\mu_0^{-1},\ldots,w_r=q^{r-1}\mu_0^{-1}$. Note that

$$G^{\mathbf{p}}(\mathbf{w})\big|_{\substack{w_a=q^{a-1}\mu_0^{-1}\\ a\in[1,r]}} = \frac{(\kappa_N/\mu_0;q)_r}{(\lambda_N/\mu_0;q)_r}G^{\mathbf{p}'}(\mathbf{w})\Big|_{\substack{w_a=q^{a-1}\mu_0^{-1}\\ a\in[1,r]}},$$

where $\mathbf{p}'=(p_1-1,p_2-1,\ldots,p_r-1,p_{r+1},\ldots,p_k)$. Hence

$$\frac{(-1)^k q^{k(k-1)/2-l(\tau)}}{(2\pi\mathbf{i})^k}\oint_{\Gamma_1}\cdots\oint_{\Gamma_k}\prod_{a<b}\frac{w_b-w_a}{w_b-qw_a}T_\tau\left(F^{\mathbf{l}}(\mathbf{w})\right)G^{\mathbf{p}}(\mathbf{w})\prod_{a=1}^{k}\frac{dw_a}{w_a(1-\mu_0 w_a)}$$
$$=\frac{(\kappa_N/\mu_0;q)_r}{(\lambda_N/\mu_0;q)_r}\frac{(-1)^k q^{k(k-1)/2-l(\tau)}}{(2\pi\mathbf{i})^k}\oint_{\Gamma_1}\cdots\oint_{\Gamma_k}\prod_{a<b}\frac{w_b-w_a}{w_b-qw_a}T_\tau\left(F^{\mathbf{l}}(\mathbf{w})\right)G^{\mathbf{p}'}(\mathbf{w})\prod_{a=1}^{k}\frac{dw_a}{w_a(1-\mu_0 w_a)},$$

At the same time, note that $r_N=r$ for the triple of data $(\mathbf{l},\mathbf{p},\tau)$, while the numbers r_i for $i<N$ are the same for both triples $(\mathbf{l},\mathbf{p},\tau)$ and $(\mathbf{l},\mathbf{p}',\tau)$. So, factoring out the term $\frac{(\kappa_N/\mu_0;q)_r}{(\lambda_N/\mu_0;q)_r}$, we reduce the claim for $(\mathbf{l},\mathbf{p},\tau)$ to the claim for $(\mathbf{l},\mathbf{p}',\tau)$, and the latter one allows to freely reduce N by 1, finishing the step of induction. □

3.4.5 Proof of Theorem 3.4.1.

The main idea of the proof is to use induction, applying the local relations from Section 3.3 to reduce the coordinates $\mathbf{x}$ and $\mathbf{y}$. This process has to eventually reach either the case $x_k=y_k$, when we can use Proposition 3.4.5 to reduce k, or the case $x_1=x_2=\cdots=x_k=\frac{1}{2}$, in which the claim holds by Proposition 3.4.6. Throughout the whole argument the parameters of the model, the contours Γ_a and the sequence $\mathbf{c}$ are fixed.

Step 1 Fix $\mathbf{x}, \mathbf{y}, \tau$ as in the claim and assume that $\frac{1}{2} < x_k < y_k$. In this main step we use the local relations to show that the claim for $(\mathbf{x}, \mathbf{y}, \tau)$ follows from the claims for $(\mathbf{x}', \mathbf{y}', \tau')$ with smaller coordinates $x'_a \leq x_a, y'_b \leq y_b$, $\mathbf{y}' \neq \mathbf{y}$.

Note that for some integers $m \in \mathbb{Z}_{\geq 0}, r \in \mathbb{Z}_{>0}$ we have

$$x_m < x_{m+1} = \cdots = x_{m+r} \leq x_{m+r+1}, \qquad y_m \geq y_{m+1} = \cdots = y_{m+r} > y_{m+r+1},$$

$$y_{m+1} > x_{m+1} > \frac{1}{2}.$$

In other words, all points (x_a, y_a) are either equal to the point (x_{m+1}, y_{m+1}), strictly below it or strictly to the left of it. To show existence of such m, r one can take m equal to the maximal index such that $x_m < x_k$, while r is the maximal index such that $y_{m+r} = y_{m+1}$. For any $a \leq b$ let $\mathbf{e}^{[a,b]} \in \mathbb{Z}^k$ denote the vector with 1 entries at positions $a, a+1, \ldots, b$, and 0s elsewhere:

$$\mathbf{e}^{[a,b]} = \sum_{i=a}^{b} \mathbf{e}^i = (\ldots, 0, 0, 1, 1, \ldots, 1, 0, 0 \ldots).$$

It turns out that the claim for $\mathbf{x}, \mathbf{y}$ and arbitrary τ follows from the claims for $\mathbf{x} - \mathbf{e}^{[m+1,m+p]}, \mathbf{y} - \mathbf{e}^{[m+1,m+r]}$ with $p = 0, \ldots, r$. To show it we proceed in three sub-steps: first we simplify the permutation τ and then apply the local relation to both sides of (3.4.5).

Step 1.1 For now we want to show that it is enough to consider permutations τ which are $[m+1, m+r]$-ordered:

$$\tau^{-1}(m+1) < \tau^{-1}(m+2) < \cdots < \tau^{-1}(m+r).$$

For the left-hand side of (3.4.5) note that for any permutation $\rho \in \mathfrak{S}_{[m+1,m+r]}$ permuting only numbers $m+1, \ldots, m+r$ we have

$$\mathcal{Q}^{\mathbf{x},\mathbf{y}}_{\geq \tau.\mathbf{c}} = \mathcal{Q}^{\rho^{-1}.\mathbf{x},\rho^{-1}.\mathbf{y}}_{\geq \tau.\mathbf{c}} = \mathcal{Q}^{\mathbf{x},\mathbf{y}}_{\geq \rho\tau.\mathbf{c}},$$

thus the left-hand side depends only on the left coset of τ in $\mathfrak{S}_{[m+1,m+r]} \backslash \mathfrak{S}_k$. On the other hand, we can rewrite the right-hand side of (3.4.5) as

$$\frac{(-1)^k q^{\frac{k(k-1)}{2} - l(\tau)}}{(2\pi \mathbf{i})^k} \oint_{\Gamma_1} \cdots \oint_{\Gamma_k} \prod_{a<b} \frac{w_b - w_a}{w_b - q w_a} \, G(\mathbf{w}) \, T_{\tau^{-1}}\left(F^{\mathbf{x},\mathbf{y}}(\mathbf{w})\right) \frac{dw_a}{w_a},$$

where

$$G(\mathbf{w}) = \prod_{a=1}^{k} \prod_{i=1}^{l_a} \frac{1 - \lambda_i w_a}{1 - \kappa_i w_a},$$

$$F^{\mathbf{x},\mathbf{y}}(\mathbf{w}) = \prod_{a=1}^{k} \left(\prod_{i=0}^{i<x_a} \frac{1}{1 - \mu_i w_a} \prod_{j=1}^{j<y_a} (1 - \kappa_j w_a) \prod_{d=1}^{d \leq y_a - x_a} \frac{1}{1 - \lambda_d w_a} \right).$$

Note that $F^{\mathbf{x},\mathbf{y}}(\mathbf{w})$ is symmetric in $w_{m+1}, w_{m+2}, \dots, w_{m+r}$, hence for any $\rho \in \mathfrak{S}_{[m+1,m+r]}$ we have

$$q^{-l(\rho)} T_{\rho^{-1}} F^{\mathbf{x},\mathbf{y}}(\mathbf{w}) = F^{\mathbf{x},\mathbf{y}}(\mathbf{w}), \tag{3.4.10}$$

since $T_i f(w_i, w_{i+1}) = q f(w_i, w_{i+1})$ for any f symmetric in w_i, w_{i+1}. Multiplying (3.4.10) on the left by $T^{-1}_{\tau_{min}}$ for a minimal permutation τ_{min} of a coset in $\mathfrak{S}_{[m+1,m+r]}\backslash\mathfrak{S}_k$, one can readily see that the right-hand side of (3.4.5) also depends only on the left coset of τ, rather than τ itself. So, we can assume that τ is a minimal representative of a left coset in $\mathfrak{S}_{[m+1,m+r]}\backslash\mathfrak{S}_k$, or, equivalently, τ is $[m+1, m+r]$-ordered.

Step 1.2 Now, assuming that τ is $[m+1, m+r]$-ordered, we want to apply an appropriate local relation to the left-hand side of (3.4.5). Let $v = (x_{m+1} - \frac{1}{2}, y_{m+1} - \frac{1}{2})$ denote the vertex immediately to the south-west of the facet $(x, y) := (x_{m+1}, y_{m+1}) = \cdots = (x_{m+r}, y_{m+r})$. It has vertex weights $W^b_{\lambda,\kappa,\mu}$, where

$$\mu := \mu_{x-\frac{1}{2}}, \qquad \kappa := \kappa_{y-\frac{1}{2}}, \qquad \lambda := \lambda_{y-x}.$$

Now we want to change the order in which we perform the sampling of the model: first we sample the outgoing configurations for the vertices (i, j) either below v or to the left of v, obtaining a configuration $\Sigma_{\swarrow v}$, and then we sample the remaining configuration Σ. Since all our samplings are independent, their order does not matter as long as each vertex is sampled after the vertices directly below and to the left of it, so the resulting configuration is distributed in the same way as in the initial model. Note that $\Sigma_{\swarrow v}$ determines the value of the height functions at the facets (x_a, y_a) for $a \le m$, because they are to the left of v, and at the facets (x_a, y_a) for $m + r < a$ because they are below v.

At the same time, the height functions at (x, y) are determined only by $\Sigma_{\swarrow v}$ and the labels of the edges around v, which are determined by the sampling at v. Let $(\boldsymbol{I}, \boldsymbol{J})$ denote the compositions of the incoming left and bottom edges of v, determined by $\Sigma_{\swarrow v}$, and let $(\boldsymbol{K}, \boldsymbol{L})$ denote the compositions of the outgoing top and right edges of v. Then we have

$$\mathcal{Q}^{\mathbf{x},\mathbf{y}}_{\ge \tau.\mathbf{c}}(\Sigma) = \mathcal{Q}^{\mathbf{x},\mathbf{y}-\mathbf{e}^{[m+1,m+r]}}_{\ge \tau.\mathbf{c}}(\Sigma_{\swarrow v}) \mathcal{Q}_{\ge \tau.\mathbf{c}[m+1,m+r]}(\boldsymbol{L}).$$

where we use the notation $\mathcal{Q}_{\ge \mathbf{c}}(\boldsymbol{L})$ from Section 3.3, we set

$$\tau.\mathbf{c}[m+1, m+r] = (c_{\tau^{-1}(m+1)}, \dots, c_{\tau^{-1}(m+r)})$$

and we use the definition of the height function by local rules to deduce

$$\sum_{a=m+1}^{m+r} h^{(x,y)}_{\ge c_{\tau^{-1}(a)}}(\Sigma) = \sum_{a=m+1}^{m+r} h^{(x,y-1)}_{\ge c_{\tau^{-1}(a)}}(\Sigma_{\swarrow v}) + \sum_{a=m+1}^{m+r} L_{c_{\tau^{-1}(a)}}. \tag{3.4.11}$$

Taking expectations we obtain

$$\mathbb{E}[\mathcal{Q}^{\mathbf{x},\mathbf{y}}_{\ge \tau.\mathbf{c}}(\Sigma)] = \mathbb{E}\left[\mathcal{Q}^{\mathbf{x},\mathbf{y}-\mathbf{e}^{[m+1,m+r]}}_{\ge \tau.\mathbf{c}}(\Sigma_{\swarrow v})\ \mathbb{E}[\mathcal{Q}_{\ge \tau.\mathbf{c}[m+1,m+r]}(\boldsymbol{L}) \mid \Sigma_{\swarrow v}]\right].$$

Now we can apply Corollary 3.3.4 to the conditional expectation inside to get

$$\mathbb{E}[\mathcal{Q}_{\geq\tau.\mathbf{c}[m+1,m+r]}(\boldsymbol{L}) \mid \Sigma_{\swarrow v}] = \sum_{p=0}^{r} \sum_{\rho\in\mathfrak{S}^{p|r-p}_{[m+1,m+r]}} (\kappa/\mu)^p \frac{(\kappa/\mu;q)_{r-p}(\lambda/\kappa)_p}{(\lambda/\mu)_r} q^{l(\rho)} \mathcal{Q}_{\geq\rho\tau.\mathbf{c}[m+1,m+p]}(\boldsymbol{J}),$$

where the second sum is over $[m+1,m+p]\times[m+p+1,m+r]$-ordered permutations of $m+1,\ldots,m+r$. For the same reasons as in Step 1.1, for any such permutation ρ we have $\mathcal{Q}^{\mathbf{x},\mathbf{y}-\mathbf{e}^{[m+1,m+r]}}_{\geq\tau.\mathbf{c}}(\Sigma_{\swarrow v}) = \mathcal{Q}^{\mathbf{x},\mathbf{y}-\mathbf{e}^{[m+1,m+r]}}_{\geq\rho\tau.\mathbf{c}}(\Sigma_{\swarrow v})$ and using the definition of the height function in a way similar to (3.4.11) we have

$$\mathcal{Q}^{\mathbf{x},\mathbf{y}-\mathbf{e}^{[m+1,m+r]}}_{\geq\tau.\mathbf{c}}(\Sigma_{\swarrow v})\mathcal{Q}_{\geq\rho\tau.\mathbf{c}[m+1,m+p]}(\boldsymbol{J}) = \mathcal{Q}^{\mathbf{x}-\mathbf{e}^{[m+1,m+p]},\mathbf{y}-\mathbf{e}^{[m+1,m+r]}}_{\geq\rho\tau.\mathbf{c}}(\Sigma_{\swarrow v}).$$

So the local relation from Corollary 3.3.4 gives the following relation on the left-hand side of (3.4.5):

$$\mathbb{E}[\mathcal{Q}^{\mathbf{x},\mathbf{y}}_{\geq\tau.\mathbf{c}}(\Sigma)] = \sum_{p=0}^{r} \sum_{\rho\in\mathfrak{S}^{p|r-p}_{[m+1,m+r]}} (\kappa/\mu)^p \frac{(\kappa/\mu;q)_{r-p}(\lambda/\kappa;q)_p}{(\lambda/\mu;q)_r} q^{l(\rho)}\ \mathbb{E}\left[\mathcal{Q}^{\mathbf{x}-\mathbf{e}^{[m+1,m+p]},\mathbf{y}-\mathbf{e}^{[m+1,m+r]}}_{\geq\rho\tau.\mathbf{c}}(\Sigma)\right]. \tag{3.4.12}$$

Step 1.3 Continuing with the same setup, we now want to apply the other local relation to the right-hand side of (3.4.5). Using the notation from Step 1.1, we rewrite the right-hand side as

$$\frac{(-1)^k q^{\frac{k(k-1)}{2}-l(\tau)}}{(2\pi\mathbf{i})^k} \oint_{\Gamma_1}\cdots\oint_{\Gamma_k} \prod_{a<b}\frac{w_b-w_a}{w_b-qw_a}\, G(\mathbf{w})\, T_{\tau^{-1}}\left(F^{\mathbf{x},\mathbf{y}}(\mathbf{w})\right)\frac{dw_a}{w_a},$$

where

$$G(\mathbf{w}) = \prod_{a=1}^{k}\prod_{i=1}^{l_a}\frac{1-\lambda_i w_a}{1-\kappa_i w_a},$$

$$F^{\mathbf{x},\mathbf{y}}(\mathbf{w}) = \prod_{a=1}^{k}\left(\prod_{i=0}^{i<x_a}\frac{1}{1-\mu_i w_a}\prod_{j=1}^{j<y_a}(1-\kappa_j w_a)\prod_{d=1}^{d\leq y_a-x_a}\frac{1}{1-\lambda_d w_a}\right).$$

Note that

$$F^{\mathbf{x},\mathbf{y}}(\mathbf{w}) = \left(\prod_{a=m+1}^{m+r}\frac{1-\kappa w_a}{1-\lambda w_a}\right)F^{\mathbf{x},\mathbf{y}-\mathbf{e}^{[m+1,m+r]}}(\mathbf{w}),$$

so we can apply Proposition 3.3.6 to obtain

$$\prod_{a=m+1}^{m+r}\frac{1-\kappa w_a}{1-\lambda w_a} = \sum_{p=0}^{r}\sum_{\rho\in\mathfrak{S}^{p|r-p}_{[m+1,m+r]}}(\kappa/\mu)^p\frac{(\kappa/\mu;q)_{r-p}(\lambda/\kappa;q)_p}{(\lambda/\mu;q)_r}T_{\rho^{-1}}\left(\prod_{a=m+1}^{m+p}\frac{1-\mu w_a}{1-\lambda w_a}\right).$$

Since $F^{\mathbf{x},\mathbf{y}-\mathbf{e}^{[m+1,m+r]}}(\mathbf{w})$ is symmetric in $w_{m+1},\ldots,w_{m+r}$ and

$$F^{\mathbf{x}-\mathbf{e}^{[m+1,m+p]},\mathbf{y}-\mathbf{e}^{[m+1,m+r]}}(\mathbf{w}) = \left(\prod_{a=m+1}^{m+p}\frac{1-\mu w_a}{1-\lambda w_a}\right)F^{\mathbf{x},\mathbf{y}-\mathbf{e}^{[m+1,m+r]}}(\mathbf{w}),$$

we get

$$T_{\tau^{-1}}F^{\mathbf{x},\mathbf{y}}(\mathbf{w})=\sum_{p=0}^{r}\sum_{\rho\in\mathfrak{S}^{p|r-p}_{[m+1,m+r]}}(\kappa/\mu)^p\frac{(\kappa/\mu;q)_{r-p}(\lambda/\kappa;q)_p}{(\lambda/\mu;q)_r}T_{(\rho\tau)^{-1}}F^{\mathbf{x}-\mathbf{e}^{[m+1,m+p]},\mathbf{y}-\mathbf{e}^{[m+1,m+r]}}(\mathbf{w}).$$

Let $\mathcal{I}^{\mathbf{x},\mathbf{y}}_{\tau}$ denote the right-hand side of (3.4.5). Then, the application of the local relation above gives

$$\mathcal{I}^{\mathbf{x},\mathbf{y}}_{\tau}=\sum_{p=0}^{r}\sum_{\rho\in\mathfrak{S}^{p|r-p}_{[m+1,m+r]}}(\kappa/\mu)^p\frac{(\kappa/\mu;q)_{r-p}(\lambda/\kappa;q)_p}{(\lambda/\mu;q)_r}q^{l(\rho)}\mathcal{I}^{\mathbf{x}-\mathbf{e}^{[m+1,m+p]},\mathbf{y}-\mathbf{e}^{[m+1,m+r]}}_{\rho\tau},$$

which is identical to the analogous relation for the left-hand side (3.4.12). Thus, the identity (3.4.5) for $(\mathbf{x},\mathbf{y},\tau)$ follows from $(\mathbf{x}-\mathbf{e}^{[m+1,m+p]},\mathbf{y}-\mathbf{e}^{[m+1,m+r]},\rho\tau)$ for various $p\in[0,r]$ and $\rho\in\mathfrak{S}^{p|r-p}_{[m+1,m+r]}$.

Step 2 Applying Step 1 we can reduce the coordinates $\mathbf{x},\mathbf{y}$ until either $x_1=\cdots=x_k=\frac{1}{2}$ or $x_k=y_k$. Here we deal with the latter case, showing that in this situation we can reduce k by 1, replacing the data $(\mathbf{x},\mathbf{y},\mathbf{c},\tau)$ by

$$\mathbf{x}'=(x_1,\dots,x_{k-1}),\qquad \mathbf{y}'=(y_1,\dots,y_{k-1}),$$

$$\mathbf{c}=(c_{\rho(1)},\dots,c_{\rho(k-1)})=\rho^{-1}.\mathbf{c}[1,k-1],\qquad \tau'=\tau\rho,$$

where ρ is the cycle sending $\tau^{-1}(k),\tau^{-1}(k)+1,\dots,k$ to $\tau^{-1}(k)+1,\dots,k,\tau^{-1}(k)$ respectively.

For the left-hand side of (3.4.5) note that for any color c we have

$$h^{(x_k,y_k)}_{\geq c}(\Sigma)=0,$$

due to the behavior of the model: all edges adjacent to vertices (i,j) for $i>j$ have compositions $\mathbf{0}$. This means that we can freely remove this height function from $\mathcal{Q}^{\mathbf{x},\mathbf{y}}_{\geq\tau.\mathbf{c}}$, obtaining

$$\mathcal{Q}^{\mathbf{x},\mathbf{y}}_{\geq\tau.\mathbf{c}}(\Sigma)=\mathcal{Q}^{\mathbf{x}',\mathbf{y}'}_{\geq\tau'.\mathbf{c}'}(\Sigma).$$

for any possible configuration Σ. Taking expectation we conclude that the left-hand sides for $(\mathbf{x},\mathbf{y},\mathbf{c},\tau)$ and $(\mathbf{x}',\mathbf{y}',\mathbf{c}',\tau')$ are equal.

For the right-hand side of (3.4.5) we are going to apply Proposition 3.4.5. Set

$$f_a(w)=\prod_{i=0}^{i<x_a}\frac{1}{1-\mu_i w}\prod_{j=1}^{j<y_a}(1-\kappa_j w_a)\prod_{d=1}^{d\leq y_a-x_a}\frac{1}{1-\lambda_d w},\qquad g_a(w)=\prod_{j=1}^{l_a}\frac{1-\lambda_j w}{1-\kappa_j w}$$

Since $y_k=x_k$, the function $f_k(w)$ has no singularities outside of Γ_k, as well as the functions $g_a(w)$

for any $a \in [1,k]$. Thus, the conditions of Proposition 3.4.5 hold and we have

$$\frac{(-1)^k q^{\frac{k(k-1)}{2}-l(\tau)}}{(2\pi\mathbf{i})^k}\oint_{\Gamma_1}\cdots\oint_{\Gamma_k}\prod_{a<b}\frac{w_b-w_a}{w_b-qw_a}\prod_{a=1}^{k}g_a(w_a)\,T_{\tau^{-1}}\left(\prod_{a=1}^{k}f_a(w_a)\right)\prod_{a=1}^{k}\frac{dw_a}{w_a}$$
$$=\frac{(-1)^{k-1}q^{\frac{(k-1)(k-2)}{2}-l(\tau')}}{(2\pi\mathbf{i})^{k-1}}\oint_{\Gamma_1}\cdots\oint_{\Gamma_{k-1}}\prod_{a<b\le k-1}\frac{w_b-w_a}{w_b-qw_a}\prod_{a=1}^{k-1}g_{\rho(a)}(w_a)\,T_{\tau'^{-1}}\left(\prod_{a=1}^{k-1}f_a(w_a)\right)\prod_{a=1}^{k-1}\frac{dw_a}{w_a}$$

where ρ and τ' are exactly the permutations defined above. Plugging back the expressions for f_a, g_a, we get the right-hand side of (3.4.5) for $(\mathbf{x}',\mathbf{y}',\tau',\mathbf{c}')$, as desired.

Step 3 Repeating Steps 1 and 2, at some point we either reach $k=0$, when the claim is trivial, or the situation when $x_1=x_2=\cdots=x_k=\frac{1}{2}$. In the latter case Proposition 3.4.6 implies that the right-hand side of (3.4.5) is equal to

$$\prod_{j=1}^{N}\frac{(\kappa_j\mu_0^{-1};q)_{r_j}}{(\lambda_j\mu_0^{-1};q)_{r_j}},\qquad r_j=\#\left\{a\in[1,k]\mid j\in\left[l_{\tau^{-1}(a)}+1,y_a-\frac{1}{2}\right]\right\}.$$

At the same time, recall that the compositions of the incoming edges are mono-colored, and the color of the incoming edge on row j is $\ge c_{\tau^{-1}(a)}$ if only if $j>l_{\tau^{-1}(a)}$. Recall that $b_j=|\boldsymbol{B}^{(0,j)}|$ denote the random integers used to define the compositions at the left boundary and distributed according to (3.4.2), so we get for $\mathbf{x}=\left(\frac{1}{2},\ldots,\frac{1}{2}\right)$

$$\mathbb{E}[\mathcal{Q}_{\tau,\mathbf{c}}^{\mathbf{x};\mathbf{y}}]=\sum_{b_1,b_2,\cdots\ge 0}q^{\sum_{a=1}^{k}\sum_{j=l_a+1}^{j<y_{\tau(a)}}b_j}\prod_{j\ge1}(\kappa_j/\mu_0)^{b_j}\frac{(\lambda_j/\kappa_j;q)_{b_j}}{(q;q)_{b_j}}\frac{(\kappa_j/\mu_0;q)_\infty}{(\lambda_j/\mu_0;q)_\infty}$$
$$=\prod_{i\ge1}\sum_{b\ge0}q^{r_jb}(\kappa_j/\mu_0)^b\frac{(\lambda_j/\kappa_j;q)_b}{(q;q)_b}\frac{(\kappa_j/\mu_0;q)_\infty}{(\lambda_j/\mu_0;q)_\infty}=\prod_{j\ge1}\frac{(q^{r_j}\lambda_j/\mu_0;q)_\infty}{(q^{r_j}\kappa_j/\mu_0;q)_\infty}\frac{(\kappa_j/\mu_0;q)_\infty}{(\lambda_j/\mu_0;q)_\infty},$$

which is equal to the value of the right-hand side, concluding the base case. Note that for the last identity we have used (3.4.4). □

3.5 Shift invariance

In this section we show how the integral expressions from Theorem 3.4.1 lead to certain symmetries of the diagonally inhomogeneous q-Hahn model, which are similar to the shift-invariance symmetries of the stochastic colored six-vertex model from [17] and [43]. Our argument is identical to [28, Section 7].

To state the result we need a better description of the height functions. For simplicity, consider a q-Hahn vertex model with incoming colors along the left boundary defined by $\boldsymbol{C}=(1,1,1,\ldots)$, that is, in the incoming boundary composition at row j only jth entry is nonzero. Then, for each color c we can define a cutoff point $(\frac{1}{2},c-\frac{1}{2})$, which separates the rows with the incoming colors on the left $\ge c$ and $<c$. Since such cutoff points uniquely determine the corresponding colors, each colored height function $h_{\ge c}^{(x,y)}$ can be determined by the points $(\frac{1}{2},c-\frac{1}{2})$ and (x,y).

For a color $c \geq 1$ and a point $(x, y) \in \left(\mathbb{Z}_{\geq 0} + \frac{1}{2}\right)^2$ such that $y - x \geq c$ define

$$\mathrm{Row}_{\geq c}^{(x,y)} = \left[c, y - \frac{1}{2}\right], \quad \mathrm{Col}_{\geq c}^{(x,y)} = \left[0, x - \frac{1}{2}\right], \quad \mathrm{Diag}_{\geq c}^{(x,y)} = [c, y - x].$$

Here $[a, b]$ denotes an integer interval $a, a+1, \ldots, b$. Note that $\mathrm{Row}, \mathrm{Col}, \mathrm{Diag}$ are exactly the rows, columns[4] and diagonals passing between the cutoff point $(\frac{1}{2}, c - \frac{1}{2})$ and (x, y), possibly containing (x, y) but avoiding $(\frac{1}{2}, c - \frac{1}{2})$. The additional constraint $y - x \geq c$ listed above is not restrictive, since $h_{\geq c}^{(x,y)} \equiv 0$ for $y - x < c$.

We start with the informal statement of the shift-invariance symmetry, rigorously described in Proposition 3.5.1. Given a vector of height functions $\left(h_{\geq c_1}^{(x_{\tau(1)}, y_{\tau(1)})}, \ldots, h_{\geq c_k}^{(x_{\tau(k)}, y_{\tau(k)})}\right)$ with x_a, y_a, c_a ordered as before in Theorem 3.4.1, the joint k-dimensional distribution is determined solely by

- The collections of parameters attached to the rows $\mathrm{Row}_{\geq c_a}^{(x_{\tau(a)}, y_{\tau(a)})}$, columns $\mathrm{Col}_{\geq c_a}^{(x_{\tau(a)}, y_{\tau(a)})}$ and diagonals $\mathrm{Diag}_{\geq c_a}^{(x_{\tau(a)}, y_{\tau(a)})}$, corresponding to the height functions,
- The relative order τ of evaluation and cutoff points of the height functions.

The data in the first part corresponds to the one-dimensional distributions of $h_{\geq c}^{(x,y)}$, which are determined by the listed parameters in view of Theorem 3.4.1. Since the one-dimensional distributions are determined by the joint k-dimensional distributions, one can expect this data to be necessary. On the contrary, the second part corresponds to the relative position of the height functions with respect to each other on the plane, and this data is not *a priori* contained in the joint distribution. Moreover, we actually expect it to be redundant, see Remark 3.5.2.

Proposition 3.5.1. *Let $\{\mu_i\}_{i \geq 0}$, $\{\kappa_j\}_{j \geq 1}$, $\{\lambda_d\}_{d \geq 1}$ be parameters of a q-Hahn model, and let $\mathbf{c} = (c_1, \ldots, c_k)$, $\mathbf{x} = (x_1, \ldots, x_k)$ and $\mathbf{y} = (y_1, \ldots, y_k)$ be sequences satisfying*

$$x_1 \leq x_2 \leq \cdots \leq x_k, \qquad y_1 \geq y_2 \geq \cdots \geq y_k, \qquad x_a, y_a \in \mathbb{Z}_{\geq 0} + \frac{1}{2};$$

$$c_1 \leq c_2 \leq \cdots \leq c_k, \qquad c_a \in \mathbb{Z}_{\geq 0}.$$

Additionally, let $\tau \in \mathfrak{S}_k$ be a permutation such that $y_{\tau(a)} - x_{\tau(a)} \geq c_a$. Similarly, let $\{\widetilde{\mu}_i\}_{i \geq 0}$, $\{\widetilde{\kappa}_j\}_{j \geq 1}$, $\{\widetilde{\lambda}_d\}_{d \geq 1}$, $\widetilde{\mathbf{c}} = (\widetilde{c}_1, \ldots, \widetilde{c}_k)$, $\widetilde{\mathbf{x}} = (\widetilde{x}_1, \ldots, \widetilde{x}_k)$, $\widetilde{\mathbf{y}} = (\widetilde{y}_1, \ldots, \widetilde{y}_k)$ and $\widetilde{\tau} = \tau$ be another collection of data satisfying identical constraints, which we treat as another q-Hahn model.

Assume that there exist bijections $\psi_{\mathrm{Row}}, \psi_{\mathrm{Diag}} : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}, \psi_{\mathrm{Col}} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ satisfying

$$\psi_{\mathrm{Row}}\left(\mathrm{Row}_{\geq c_a}^{(x_{\tau(a)}, y_{\tau(a)})}\right) = \mathrm{Row}_{\geq \widetilde{c}_a}^{(\widetilde{x}_{\tau(a)}, \widetilde{y}_{\tau(a)})}, \quad \psi_{\mathrm{Col}}\left(\mathrm{Col}_{\geq c_a}^{(x_{\tau(a)}, y_{\tau(a)})}\right) = \mathrm{Col}_{\geq \widetilde{c}_a}^{(\widetilde{x}_{\tau(a)}, \widetilde{y}_{\tau(a)})},$$

$$\psi_{\mathrm{Diag}}\left(\mathrm{Diag}_{\geq c_a}^{(x_{\tau(a)}, y_{\tau(a)})}\right) = \mathrm{Diag}_{\geq \widetilde{c}_a}^{(\widetilde{x}_{\tau(a)}, \widetilde{y}_{\tau(a)})},$$

for any $a = 1, \ldots, k$ and

$$\widetilde{\kappa}_j = \kappa_{\psi_{\mathrm{Row}}^{-1}(j)}, \quad \widetilde{\mu}_i = \mu_{\psi_{\mathrm{Col}}^{-1}(i)}, \quad \widetilde{\lambda}_d = \lambda_{\psi_{\mathrm{Col}}^{-1}(d)}.$$

[4] With the exception of the artificially added 0th column.

for any $i \geq 0, j \geq 1, d \geq 1$. Then the following random vectors are equal in finite-dimensional distributions:

$$\left(h^{(x_{\tau(1)},y_{\tau(1)})}_{\geq c_1}, \dots, h^{(x_{\tau(k)},y_{\tau(k)})}_{\geq c_k}\right) = \left(\widetilde{h}^{(\widetilde{x}_{\tau(1)},\widetilde{y}_{\tau(1)})}_{\geq \widetilde{c}_1}, \dots, \widetilde{h}^{(\widetilde{x}_{\tau(k)},\widetilde{y}_{\tau(k)})}_{\geq \widetilde{c}_k}\right),$$

where h and $\widetilde{h}$ denote the height functions of the two q-Hahn models.

We are not going to give a complete proof of Proposition 3.5.1, it is identical to the proof of [28, Theorem 7.1]. Below we just outline the main steps of that argument.

Sketch of the proof. The proof is based on the following observation: the joint k-dimensional distribution of a random vector $(h_1, \dots, h_k)$ of nonnegative real numbers is completely determined by the q-moments $\mathbb{E}\left[q^{a_1h_1+\dots+a_kh_k}\right]$ for $a_1, \dots, a_k \in \mathbb{Z}_{\geq 0}$. Repeating in the statement of Proposition 3.5.1 the first height function for a_1 times, the next height function for a_2 times and so on, one readily see that it is enough to prove

$$\mathbb{E}\left[\mathcal{Q}^{(\mathbf{x},\mathbf{y})}_{\geq \tau.\mathbf{c}}(\Sigma)\right] = \widetilde{\mathbb{E}}\left[\mathcal{Q}^{(\widetilde{\mathbf{x}},\widetilde{\mathbf{y}})}_{\geq \tau.\widetilde{\mathbf{c}}}(\Sigma)\right],$$

where $\widetilde{\mathbb{E}}$ denotes expectation in the second model. Applying Theorem 3.4.1 to both sides, the problem is reduced to matching the right-hand sides of the integral expression (3.4.5).

To match integral expressions, we expand the operator T_τ as a linear combination of the permutation operators acting on the variables $\mathbf{w}$:

$$T_\tau = \sum_{\rho \preceq \tau} Z^\rho_\tau(\mathbf{w})\rho, \qquad \rho f(w_1, \dots, w_k) = f(w_{\rho(1)}, \dots, w_{\rho(k)}),$$

where $Z^\rho_\tau(\mathbf{w})$ are rational functions in $\mathbf{w}$ and the inequality $\rho \preceq \tau$ is taken with respect to the strong Bruhat order, see [28, Proposition 3.1]. Then, to match the integral expressions, it is enough to prove for any $\rho \preceq \tau$ that the expression

$$\oint \cdots \oint \prod_{a<b} \frac{w_b - w_a}{w_b - q w_a}\ Z^\rho_\tau(\mathbf{w}) \times \left(\prod_{a=1}^{k} \prod_{i=1}^{c_a - 1} \frac{1 - \lambda_i w_{\rho(a)}}{1 - \kappa_i w_{\rho(a)}} \prod_{i=0}^{i<x_a} \frac{1}{1 - \mu_i w_a} \prod_{j=1}^{j<y_a} (1 - \kappa_j w_a) \prod_{d=1}^{d \leq y_a - x_a} \frac{1}{1 - \lambda_d w_a} \right) \frac{dw_a}{w_a}, \quad (3.5.1)$$

does not change when we replace all $\lambda_i, \mu_i, \kappa_i, c_a, x_a, y_a$ by $\tilde{\lambda}_i, \tilde{\mu}_i, \tilde{\kappa}_i, \tilde{c}_a, \tilde{x}_a, \tilde{y}_a$ When $\rho = \tau$ both sides of (3.5.1) are equal by the assumptions of the claim: for the left-hand side the terms inside the brackets can be rewritten as

$$\prod_{a=1}^{k} \prod_{i \in \mathrm{Col}^{(x_a,y_a)}_{\geq c_{\overline{\rho}(a)}}} \frac{1}{1 - \mu_i w_a} \prod_{j \in \mathrm{Row}^{(x_a,y_a)}_{\geq c_{\overline{\rho}(a)}}} (1 - \kappa_j w_a) \prod_{d \in \mathrm{Diag}^{(x_a,y_a)}_{\geq c_{\overline{\rho}(a)}}} \frac{1}{1 - \lambda_d w_a}, \quad (3.5.2)$$

where $\overline{\rho} = \rho^{-1}$, and the statement of Proposition 3.5.1 essentially just tells that the expression above is equal to the analogous expression for the right-hand side of (3.5.1) when $\rho = \tau$.

For general ρ one can closely follow the proof of [28, Lemma 7.1.3] and show that for any $\rho \preceq \tau$ one of the following situations holds:

- For some a we have $y_{\rho(a)} - x_{\rho(a)} < c_{\rho(a)}$ and $\widetilde{y}_{\rho(a)} - \widetilde{x}_{\rho(a)} < \widetilde{c}_{\rho(a)}$, that is, the corresponding height functions $h^{(x_{\rho(a)}, y_{\rho(a)})}_{\geq c_a}$ and $\widetilde{h}^{(\widetilde{x}_{\rho(a)}, \widetilde{y}_{\rho(a)})}_{\geq \widetilde{c}_a}$ are degenerate and equal to 0. Then both sides of (3.5.1) have no nonzero residues with respect to w_a outside of the integration contours, and hence vanish.

- For every a we have $y_{\rho(a)} - x_{\rho(a)} \geq c_{\rho(a)}$ and $\widetilde{y}_{\rho(a)} - \widetilde{x}_{\rho(a)} \geq \widetilde{c}_{\rho(a)}$. Then one can verify that the conditions of Proposition 3.5.1 still hold when τ is replaced by ρ and, in particular, the expressions of the form (3.5.2) match for ρ as well.

□

Remark 3.5.2. This section is based on [28, Section 7], where the shift-invariance for the *stochastic colored six-vertex model* is derived directly from the integral expressions for the q-moments of the height functions. The shift-invariance for the six-vertex model was initially introduced in [17] and in full generality it was proved in [43], where *flip symmetries* were used as a basic block for all other symmetries. Moreover, it was shown in [43] that the relative position of the height functions, encoded by τ in our notation, is irrelevant and joint distributions of height functions are determined solely by their one-dimensional distributions. Since we don't know how the q-Hahn model with diagonal parameters can be constructed from the six-vertex model, the more general invariance result from [43] cannot be applied to our model, but we suspect it to be true.

3.6 Reduction to Beta polymer

In this section we follow [17] in order to reduce the results from Section 3.4 to the *Beta polymer* model. See also [24] and [28] for similar arguments.

3.6.1 Continuous model.

As before, we treat q-Hahn vertex model from Section 3.4 as a sequence of independent stochastic samplings corresponding to the vertices, depending on the parameters $\mu_i, \kappa_j, \lambda_d$. We slightly simplify the model by assuming that the incoming colors from the left are different and are encoded by the composition $\boldsymbol{C} = (1, 1, 1, \dots)$. We consider the following limit regime:

$$q = e^{-\varepsilon}, \qquad \mu_i = q^{-\sigma_i}, \qquad \kappa_i = q^{-\rho_i}, \qquad \lambda_i = q^{-\omega_i}; \qquad \varepsilon \to 0, \tag{3.6.1}$$

where $\sigma_i, \rho_j, \omega_d$ are new real parameters satisfying

$$\omega_d < \rho_j < \sigma_i, \qquad \text{for any } i, j, d. \tag{3.6.2}$$

It turns out that the limiting model in this regime is described in terms of the Beta distribution. Recall that for a pair of parameters $(\alpha, \beta) \in \mathbb{R}^2_{\geq 0}$ the Beta distribution is defined by the following density function

$$f_{\mathrm{Beta}(\alpha,\beta)}(x) = \mathbb{1}_{0<x<1}\, x^{\alpha-1}(1-x)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}.$$

Proposition 3.6.1 ([17, Proposition 6.16]). *In the regime* (3.6.1) *the random variable* $\exp(-\varepsilon b_j)$ *for* b_j *distributed as in* (3.4.2) *converges weakly as* $\varepsilon \to 0$ *to* $\mathrm{Beta}(\sigma_0 - \rho_j, \rho_j - \omega_j)$.

Proposition 3.6.2 ([17, Corollary 6.21]). *The stochastic sampling rule for* $(\boldsymbol{K}, \boldsymbol{L})$ *given* $(\boldsymbol{I}, \boldsymbol{J})$ *defined by the vertex weights* $W^{b}_{\lambda_{j-i},\kappa_j,\mu_i}(\boldsymbol{I}, \boldsymbol{J}; \boldsymbol{K}, \boldsymbol{L})$ *weakly converges in the limit regime*

$$q = e^{-\varepsilon}, \qquad \mu_i = q^{-\sigma_i}, \qquad \kappa_i = q^{-\rho_i}, \qquad \lambda_i = q^{-\omega_i}; \qquad \varepsilon \to 0,$$

$$\varepsilon \boldsymbol{I} \to \boldsymbol{\alpha}, \qquad \varepsilon \boldsymbol{J} \to \boldsymbol{\beta}, \qquad \varepsilon \boldsymbol{K} \to \boldsymbol{\gamma} \qquad , \varepsilon \boldsymbol{L} \to \boldsymbol{\delta},$$

to the following sampling procedure for the outgoing masses $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$ *and* $\boldsymbol{\delta} = (\delta_1, \dots, \delta_n)$ *given the incoming masses* $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ *and* $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$*: We take a Beta-distributed random variable* $\eta \sim \mathrm{Beta}(\sigma_i - \rho_j, \rho_j - \omega_{j-i})$ *and define* δ_k, $0 \le \delta_k \le \beta_k$, *for all* k *through*

$$\exp(-\delta_{\ge k}) = \exp(-\beta_{\ge k}) + (1 - \exp(-\beta_{\ge k}))\eta, \qquad k = 1, 2, \dots,$$

where we set $\delta_{\ge k} = \sum_{l \ge k} \delta_l$ *and* $\beta_{\ge k} = \sum_{l \ge k} \beta_l$. *The remaining masses* γ_k *are defined by*

$$\gamma_k = \alpha_k + \beta_k - \delta_k, \qquad k = 1, 2, \dots,$$

so the conservation law holds for each color.

The two statements above result in the following model, serving as the limit under (3.6.1) of the inhomogeneous q-Hahn vertex model and depending on the parameters $\sigma_i, \rho_j, \omega_d$:

- Each lattice edge of the positive quadrant $\mathbb{Z}^2_{\ge 1}$ has a random real-valued vector $\boldsymbol{u} = (u_1, u_2, u_3, \dots)$ attached to it. Coordinate $u_i \ge 0$ is interpreted as the *mass* of color i;
- To sample a configuration of the model we first sample independent Beta-distributed variables $\eta_{i,j} \sim \mathrm{Beta}(\sigma_i - \rho_j, \rho_j - \omega_{j-i})$ for all $i \ge 0, j \ge 1$;
- For the edges entering the quadrant from below all masses are set to 0;
- For the edge entering the quadrant at row j only the mass u_j is non-zero and it is equal to $-\ln(\eta_{0,j})$;
- Given the incoming masses of colors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ entering into a vertex (i, j) from left and from bottom, the outgoing masses $\boldsymbol{\gamma}, \boldsymbol{\delta}$ exiting the vertex to the top ant to the right are defined through
$$\exp(-\delta_{\ge k}) = \exp(-\beta_{\ge k}) + (1 - \exp(-\beta_{\ge k}))\eta_{i,j}, \qquad k = 1, 2, \dots,$$
$$\gamma_k = \alpha_k + \beta_k - \delta_k, \qquad k = 1, 2, \dots,$$
where $\delta_{\ge k} = \sum_{l \ge k} \delta_l$ and $\beta_{\ge k} = \sum_{l \ge k} \beta_l$.
- This definition implies that only colors $\le j$ pass through vertex (i, j), guaranteeing that at any vertex we are working only with finite sequences of masses.

We can define the colored height functions $\widetilde{h}^{(x,y)}_{\ge c}$ of the continuous model in the same manner as for the q-Hahn model: we set $\widetilde{h}^{(x,\frac{1}{2})}_{\ge c} = 0$ and

- If the vertical edge $(x-\frac{1}{2}, y-\frac{1}{2}) \to (x-\frac{1}{2}, y+\frac{1}{2})$ has masses $\boldsymbol{u} = (u_1, u_2, \dots)$, then

$$\widetilde{h}_{\geq c}^{(x-1,y)} = \widetilde{h}_{\geq c}^{(x,y)} + u_{\geq c}.$$

- If the horizontal edge $(x-\frac{1}{2}, y+\frac{1}{2}) \to (x+\frac{1}{2}, y+\frac{1}{2})$ has masses $\boldsymbol{u} = (u_1, u_2, \dots)$, then

$$\widetilde{h}_{\geq c}^{(x,y+1)} = \widetilde{h}_{\geq c}^{(x,y)} + u_{\geq c}.$$

The following generalization of [17, Corollary 6.22] straightforwardly follows from the statements above and clarifies in what sense the continuous model is the limit of the q-Hahn model.

Corollary 3.6.3. *Let $h_{\geq c}^{(x,y)}[\varepsilon]$ denote the height function of the q-Hahn model in the regime* (3.6.1). *Then in finite-dimensional distributions*

$$\lim_{\varepsilon \to 0} \varepsilon h_{\geq c}^{(x,y)}[\varepsilon] = \widetilde{h}_{\geq c}^{(x,y)}.$$

3.6.2 Beta polymer.

Fix the same collection of parameters $\{\sigma_i, \rho_j, \omega_d\}$ as before. For $(x,y) \in \mathbb{Z}_{\geq 0}^2$[5] and $r \in \mathbb{Z}_{\geq 0}$ such that $r \leq y - x$ define the *delayed inhomogeneous Beta polymer* partition function $\mathfrak{Z}_{x,y}^{(r)}$ as follows:

- Let $\{\eta_{i,j}\}_{i \geq 0, j \geq 0}$ be a collection of independent Beta distributed variables

$$\eta_{i,j} \sim \mathrm{Beta}(\sigma_i - \rho_j, \rho_j - \omega_{j-i});$$

- For all $t, r \geq 0$ set

$$\mathfrak{Z}_{t,r+t}^{(r)} \equiv 1, \qquad \mathfrak{Z}_{0,r+t}^{(r)} = \eta_{0,r+1}\eta_{0,r+2}\dots\eta_{0,r+t};$$

- For general $(x,y) \in \mathbb{Z}_{\geq 1}^2$ such that $x > y + r$ the partition function $\mathfrak{Z}_{x,y}^{(r)}$ is defined by the recurrence relation

$$\mathfrak{Z}_{x,y}^{(r)} = \eta_{x,y}\mathfrak{Z}_{x,y-1}^{(r)} + (1-\eta_{x,y})\mathfrak{Z}_{x-1,y-1}^{(r)}.$$

The model just described is a generalization of the delayed Beta polymer partition function from [17], which for $r = 0$ coincides with the Beta polymer model from [7]. There is also a graphical description of the model in terms of a point-to-point partition function of a directed polymer, see Section 3.1.

It turns out the the continuous vertex model described earlier can be identified with the delayed inhomogeneous Beta polymer model, in the way identical to the homogeneous situation [17]: repeating the proof of [17, Proposition 7.1] verbatim one can verify that for $(x,y) \in \mathbb{Z}_{\geq 0}^2$ the height functions $\widetilde{h}_{\geq c}^{(x+\frac{1}{2}, y+\frac{1}{2})}$ of the continuous vertex model as a random variables can be defined by the recurrence relation

$$\exp\left(-\widetilde{h}_{\geq c}^{(x+\frac{1}{2}, y+\frac{1}{2})}\right) = \eta_{x,y} \exp\left(-\widetilde{h}_{\geq c}^{(x+\frac{1}{2}, y-\frac{1}{2})}\right) + (1-\eta_{x,y}) \exp\left(-\widetilde{h}_{\geq c}^{(x-\frac{1}{2}, y-\frac{1}{2})}\right)$$

[5] From now on x, y will usually be integers, rather than half-integers as before.

and the initial conditions

$$\widetilde{h}_{\geq c}^{(x+\frac{1}{2},x+\frac{1}{2})}=0,\qquad \widetilde{h}_{\geq c}^{(\frac{1}{2},y+\frac{1}{2})}=\sum_{j=c}^{y}-\ln(\eta_{0,j})$$

for exactly the same choice of $\eta_{i,j}$. Comparing the recurrence relations and initial conditions for $\widetilde{h}_{\geq c}^{(x+\frac{1}{2},y+\frac{1}{2})}$ and $\mathfrak{Z}_{x,y}^{(r)}$ one obtains

Corollary 3.6.4. *The finite dimensional distributions of*

$$\left\{\mathfrak{Z}_{x,y}^{(r)}\right\}_{x,r\geq 0;y\geq x+r}$$

coincide with the finite dimensional distributions of

$$\left\{\exp\left(-\widetilde{h}_{\geq r+1}^{(x+\frac{1}{2},y+\frac{1}{2})}\right)\right\}_{x,r\geq 0;y\geq x+r}.$$

As a result of the discussion in this section, we have realized the Beta polymer partition functions as certain limits of the height functions of the q-Hahn model. Thus, Theorem 3.4.1 can be used to obtain an integral expression for the joint moments of the partitions functions $\{\mathfrak{Z}_{i,j}^{(r)}\}$. To formulate the result, consider a degenerated representation t_τ of the Hecke algebra with $q=1$, acting on functions in $v_1,\dots,v_k$ and defined by the generators

$$t_i=1+\frac{v_{i+1}-v_i-1}{v_{i+1}-v_i}(\mathfrak{s}_i-1).$$

Proposition 3.6.5. *For any sequences $(r_1,\dots,r_k)$, $(x_1,\dots,x_k)$ and $(y_1,\dots,y_k)$ satisfying*

$$x_1\leq x_2\leq\cdots\leq x_k,\qquad y_1\geq y_2\geq\cdots\geq y_k,\qquad x_a\leq y_a,\qquad x_a,y_b\in\mathbb{Z}_{\geq 0};$$

$$r_1\leq r_2\leq\cdots\leq r_k,\qquad r_a\in\mathbb{Z}_{\geq 0};$$

and for any permutation $\tau\in\mathfrak{S}_k$ the following holds:

$$\mathbb{E}\left[\mathfrak{Z}_{x_1,y_1}^{\left(r_{\tau^{-1}(1)}\right)}\mathfrak{Z}_{x_2,y_2}^{\left(r_{\tau^{-1}(2)}\right)}\dots\mathfrak{Z}_{x_k,y_k}^{\left(r_{\tau^{-1}(k)}\right)}\right]=\oint_{\mathcal{S}_1}\cdots\oint_{\mathcal{S}_k}\prod_{a<b}\frac{v_b-v_a}{v_b-v_a-1}\;t_\tau\left(\prod_{a=1}^{k}\prod_{j=1}^{r_a}\frac{v_a-\omega_j}{v_a-\rho_j}\right)$$
$$\times\prod_{a=1}^{k}\left(\prod_{j=1}^{j\leq y_a}(v_a-\rho_j)\prod_{i=0}^{i\leq x_a}\frac{1}{v_a-\sigma_i}\prod_{d=1}^{d\leq y_a-x_a}\frac{1}{v_a-\omega_d}\right)\frac{dv_a}{2\pi\mathbf{i}},\quad(3.6.3)$$

where the integration contours $\mathcal{S}_a$ encircle the points $\{\sigma_i\}_i$ and $\{\rho_j\}_j$ as well as the contours $\mathcal{S}_b,\mathcal{S}_b+1$ for any $b<a$, while the points $\{\omega_d\}_d$ are outside of the contours. Here $\mathcal{S}_b+1$ denotes the contour $\mathcal{S}_b$ shifted to the right by 1.

Proof. We take the limit (3.6.1) in the integral expression from Theorem 3.4.1, applied to the height

functions $h^{(x_a+\frac{1}{2},y_a+\frac{1}{2})}_{\geq r_{\tau^{-1}(a)}+1}$ and assuming $\boldsymbol{C}=(1,1,\dots)$. Set

$$\mathbf{x}+\frac{1}{2}=\left(x_1+\frac{1}{2},\dots,x_k+\frac{1}{2}\right),\quad \mathbf{y}+\frac{1}{2}=\left(y_1+\frac{1}{2},\dots,y_k+\frac{1}{2}\right),\quad \mathbf{c}=(r_1+1,\dots,r_k+1).$$

For the left-hand side of (3.4.5) we can apply Corollaries 3.6.3 and 3.6.4 to obtain

$$\mathbb{E}[\mathcal{Q}^{\mathbf{x}+\frac{1}{2},\mathbf{y}+\frac{1}{2}}_{\geq\tau.\mathbf{c}}]\to\mathbb{E}\left[\mathfrak{Z}^{\left(r_{\tau^{-1}(1)}\right)}_{x_1,y_1}\mathfrak{Z}^{\left(r_{\tau^{-1}(2)}\right)}_{x_2,y_2}\dots\mathfrak{Z}^{\left(r_{\tau^{-1}(k)}\right)}_{x_k,y_k}\right],$$

since the finite-dimensional distributions of $q^{h^{(x+\frac{1}{2},y+\frac{1}{2})}_{\geq r+1}}=\exp(-\varepsilon h^{(x+\frac{1}{2},y+\frac{1}{2})}_{\geq r+1})$ converge to the finite-dimensional distributions of $\exp(-\widetilde{h}^{(x+\frac{1}{2},y+\frac{1}{2})}_{\geq r+1})\sim\mathfrak{Z}^{(r)}_{x,y}$ as $\varepsilon\to 0$.

For the right-hand side of (3.4.5) we perform a change of coordinates $w_a=1-\varepsilon v_a$. One can readily see that for fixed contours $\mathcal{S}_1,\dots,\mathcal{S}_k$ satisfying the conditions of the claim and a sufficiently small ε the contours $1-\varepsilon\mathcal{S}_a$ satisfy the conditions for the contours Γ_a, and so in the right-hand side after a change of contours we have an integral over fixed compact contours, with integrand depending on ε. Point-wise we have

$$\frac{(-1)^k q^{\frac{k(k-1)}{2}-l(\tau)}}{(2\pi\mathbf{i})^k}\prod_{a=1}^k\frac{dw_a}{w_a}=\varepsilon^k\prod_{a=1}^k\frac{dv_a}{2\pi\mathbf{i}}+O(\varepsilon^{k+1}),$$

$$\prod_{a<b}\frac{w_b-w_a}{w_b-qw_a}\to\prod_{a<b}\frac{v_b-v_a}{v_b-v_a-1},$$

$$T_i=q+\frac{w_{i+1}-qw_i}{w_{i+1}-w_i}(\mathfrak{s}_i-1)\to t_i=1+\frac{v_{i+1}-v_i-1}{v_{i+1}-v_i}(\mathfrak{s}_i-1),$$

$$1-\mu_i w_a=\varepsilon(v_a-\sigma_i)+O(\varepsilon^2),\quad 1-\kappa_i w_a=\varepsilon(v_a-\rho_i)+O(\varepsilon^2),\quad 1-\lambda_i w_a=\varepsilon(v_a-\omega_i)+O(\varepsilon^2).$$

One can readily check that the integrand is uniformly bounded for fixed contours $\mathcal{S}_a$ and sufficiently small ε, so by dominated convergence the claim follows. □

As a particular case of Proposition 3.6.5, we have an integral expression for the moments of a single partition function $\mathfrak{Z}^{(r)}_{x,y}$ of the inhomogeneous Beta polymer. Note that, up to a shift of parameters, one dimensional distributions of $\mathfrak{Z}^{(r)}_{x,y}$ depend only on x and $y-r$ (this is the vector between end points of paths inside the polymer), so without loss of generality we can set $r=0$ and consider the moments of $\mathfrak{Z}_{x,y}:=\mathfrak{Z}^{(0)}_{x,y}$:

Corollary 3.6.6. *For any $(x,y)\in\mathbb{Z}^2_{\geq 0}$ such that $y\geq x$ and any $k\in\mathbb{Z}_{\geq 0}$ we have*

$$\mathbb{E}\left[(\mathfrak{Z}_{x,y})^k\right]=\oint_{\mathcal{S}_1}\cdots\oint_{\mathcal{S}_k}\prod_{a<b}\frac{v_b-v_a}{v_b-v_a-1}\prod_{a=1}^k\left(\prod_{i=1}^{i\leq y}(v_a-\rho_i)\prod_{i=0}^{i\leq x}\frac{1}{v_a-\sigma_i}\prod_{i=1}^{i\leq y-x}\frac{1}{v_a-\omega_i}\right)\frac{dv_a}{2\pi\mathbf{i}},\quad(3.6.4)$$

where the integration contours $\mathcal{S}_a$ encircle σ_i, leave ω_d outside and $\mathcal{S}_b$ encircles both $\mathcal{S}_a$ and $\mathcal{S}_a+1$ for $a<b$.

3.7 Laplace transform for inhomogeneous Beta polymer partition function

From now on we focus on the partition function $\mathfrak{Z}_{x,y} = \mathfrak{Z}^{(0)}_{x,y}$ of the inhomogeneous Beta polymer, with parameters $\sigma_i, \rho_j, \omega_d$. Our goal is Theorem 3.8.4, which partially establishes the Tracy-Widom large-scale behavior and which is proved in next section. Our result generalizes the corresponding result from [7], moreover, our proof follows the same outline: in this section we find a suitable expression for the Laplace transform of $\mathfrak{Z}_{x,y}$ in terms of a Fredholm determinant, which is asymptotically analyzed in next section to establish the limit theorem.

Our starting point is the integral expression (3.6.4) for the moments of $\mathfrak{Z}_{x,y}$. Fix x, y and the parameters of the model $\sigma_i, \rho_j, \omega_d$, and assume that

$$\min_{i,j} |\sigma_i - \sigma_j - 1| > 0. \tag{3.7.1}$$

Since our aim is an integral expression for the Laplace transform of $\mathfrak{Z}_{x,y}$ which involves all the moments, it is convenient to replace the integration over shifted contours $\mathcal{S}_1, \dots, \mathcal{S}_k$ in (3.6.4) by an integration over a single contour $\mathcal{C}$. This is done with the help of the following statement:

Proposition 3.7.1. *Suppose that contours $\mathcal{S}_1, \dots, \mathcal{S}_k$ and a rational function $f(v)$ satisfy*

- *The contours S_a are positively-oriented simple closed curves in $\mathbb{C}$;*
- *For any $a < b$ the contour $\mathcal{S}_b$ encircles both contours $\mathcal{S}_a$ and $\mathcal{S}_a + 1$;*
- *The contour $\mathcal{S}_1 + 1$ is completely outside of $\mathcal{S}_1$;*
- *All the poles of the function $f(v)$ are either in the interior of $\mathcal{S}_1$ or in the exterior of $\mathcal{S}_k$.*

Then

$$\begin{aligned}&\oint_{\mathcal{S}_1} \cdots \oint_{\mathcal{S}_k} \prod_{a<b} \frac{v_b - v_a}{v_b - v_a - 1} \prod_{a=1}^{k} f(v_a) \frac{dv_a}{2\pi\mathbf{i}} \\ &\quad = \sum_{\lambda \vdash k} \frac{k!}{m_1! m_2! \dots} \oint_{\mathcal{S}_1} \cdots \oint_{\mathcal{S}_1} \det\left[\frac{1}{v_i + \lambda_i - v_j}\right]_{i,j=1}^{l(\lambda)} \prod_{a=1}^{l(\lambda)} f(v_a) f(v_a + 1) \dots f(v_a + \lambda_a - 1) \frac{dv_a}{2\pi\mathbf{i}},\end{aligned} \tag{3.7.2}$$

where the sum in the right-hand side is over partitions λ of k, $l(\lambda)$ denotes the number of nonzero parts of λ and m_i denotes the number of parts of λ equal to i.

Rough idea of the proof. The statement above follows a type of deduction called *contour shift argument*. The main idea is to shrink the contours $\mathcal{S}_2, \dots, \mathcal{S}_k$ in the left-hand side to the smallest contour $\mathcal{S}_1$. Due to the conditions on the function f and the contours, all non-zero residues along this deformation come from the poles of $\prod_{a<b} \frac{v_b - v_a}{v_b - v_a - 1}$, and a careful bookkeeping produces the right-hand side.

A detailed explanation of the full argument can be found in [13, Proposition 3.2.1] and [16, Proposition 7.4], where more general q-deformed versions of the argument are presented. To adopt those versions to our setting one have to repeat the argument similar to the proof of Proposition 3.6.5, see [13, Proposition 6.2.7] and [7, Proposition 3.6]. □

Let

$$f(z)=\prod_{i=1}^{y}(z-\rho_i)\prod_{i=0}^{x}\frac{1}{z-\sigma_i}\prod_{i=1}^{y-x}\frac{1}{z-\omega_i}$$

and choose $\mathcal{C}=\mathcal{S}_1$ to be a sufficiently small contour around the points σ_i such that the shifted contour $\mathcal{C}+1$ and the points ω_d are completely outside of $\mathcal{C}$. Such contour exists due to the conditions (3.6.2),(3.7.1).

Corollary 3.7.2. *With the notation above, we have*

$$\mathbb{E}\left[(\mathfrak{Z}_{x,y})^k\right]=\sum_{\lambda\vdash r}\frac{k!}{m_1!m_2!\dots}\oint_{\mathcal{C}}\cdots\oint_{\mathcal{C}}\det\left[\frac{1}{v_i+\lambda_i-v_j}\right]_{i,j=1}^{l(\lambda)}\prod_{a=1}^{l(\lambda)}f(v_a)f(v_a+1)\dots f(v_a+\lambda_a-1)\frac{dv_a}{2\pi\mathbf{i}}. \tag{3.7.3}$$

Proof. Direct application of (3.7.2). □

Now we want to take a sum over k in (3.7.3) to obtain the Laplace transform. But to resolve possible convergence issues we need to estimate the integrand for large λ. Set

$$g(z)=\prod_{i=0}^{x}\Gamma(z-\sigma_i)\prod_{j=1}^{y}\frac{1}{\Gamma(z-\rho_j)}\prod_{d=1}^{y-x}\Gamma(z-\omega_d),$$

where $\Gamma(z)$ is the gamma function. Then

$$f(z)f(z+1)f(z+2)\dots f(z+n-1)=\frac{g(z)}{g(z+n)}.$$

Lemma 3.7.3. *For a fixed $\delta>0$ and any $z\in\mathbb{C}$ such that $\arg(z-\sigma)\in[-\pi+\delta;\pi-\delta]$ we have*

$$\ln g(z)=\left(z-\Delta-\frac{1}{2}\right)\ln z-z+\frac{1}{2}\ln(2\pi)+O\left(z^{-1}\right)$$

where $\Delta=\sum_{i=0}^{x}\sigma_i-\sum_{j=1}^{y}\rho_j+\sum_{d=1}^{y-x}\omega_d$.

Proof. We use the following useful asymptotic expansion from [59][(1),(4), pp. 31-32], valid uniformly for $\arg(z)\in[-\pi+\delta/2;\pi-\delta/2]$

$$\ln\Gamma(z)=\left(z-\frac{1}{2}\right)\ln z-z+\frac{1}{2}\ln(2\pi)+O\left(z^{-1}\right), \tag{3.7.4}$$

and, more generally, for a fixed $a\in\mathbb{C}$ and sufficiently large z such that $\arg(z)\in[-\pi+\delta/2;\pi-\delta/2]$

$$\ln\Gamma(z+a)=\left(z+a-\frac{1}{2}\right)\ln z-z+\frac{1}{2}\ln(2\pi)+O\left(z^{-1}\right).$$

Here ln denotes the principal branch of the natural logarithm. Plugging this expression into the

definition of $g(z)$ we get

$$\ln g(z) = \sum_{i=0}^{x} \left(\left(z - \sigma_i - \frac{1}{2} \right) \ln(z) - z + \frac{1}{2} \ln(2\pi) \right) - \sum_{j=1}^{y} \left(\left(z - \rho_j - \frac{1}{2} \right) \ln(z) - z + \frac{1}{2} \ln(2\pi) \right) + \sum_{d=1}^{y-x} \left(\left(z - \omega_d - \frac{1}{2} \right) \ln(z) - z + \frac{1}{2} \ln(2\pi) \right) + O\left(z^{-1}\right),$$

implying the claim. □

Proposition 3.7.4. *For any $u \in \mathbb{C}$ we have*

$$\mathbb{E}\left[e^{u\mathbb{Z}_{x,y}}\right] = \det(I + \widetilde{K_u})_{L^2(\mathbb{Z}_{\geq 1} \times \mathcal{C})} := \sum_{l \geq 0} \frac{1}{l!} \sum_{n_1, \ldots, n_l \geq 1} \oint_{\mathcal{C}} \frac{dv_1}{2\pi \mathbf{i}} \cdots \oint_{\mathcal{C}} \frac{dv_l}{2\pi \mathbf{i}} \det\left[\widetilde{K}_u(v_a, n_a; v_b, n_b) \right]_{a,b=1}^{l}, \tag{3.7.5}$$

where $\det(I + K)$ denotes the Fredholm determinant of K, and $\widetilde{K_u}$ is the integral operator with the kernel

$$\widetilde{K_u}(v, n; v', n') = \frac{f(v) f(v+1) \ldots f(v+n-1) u^n}{v + n - v'} = \frac{g(v)}{g(v+n)} \frac{u^n}{v + n - v'}.$$

Proof. The proof essentially repeats [13, Proposition 3.2.8]. First we rewrite (3.7.3) as

$$\frac{u^k \mathbb{E}\left[(\mathfrak{Z}_{x,y})^k \right]}{k!} = \sum_{\lambda \vdash k} \frac{1}{m_1! m_2! \ldots} \oint_{\mathcal{C}} \frac{dv_1}{2\pi \mathbf{i}} \cdots \oint_{\mathcal{C}} \frac{dv_{l(\lambda)}}{2\pi \mathbf{i}} \det\left[\widetilde{K}_u(v_a, \lambda_a; v_b, \lambda_b) \right]_{a,b=1}^{l(\lambda)}$$

and then change the summation over partitions to a summation over arbitrary sequences, absorbing the multinomial coefficient:

$$\frac{u^k \mathbb{E}\left[(\mathfrak{Z}_{x,y})^k \right]}{k!} = \sum_{l=0}^{k} \frac{1}{l!} \sum_{\substack{n_1, \ldots, n_l \geq 1 \\ n_1 + \cdots + n_l = k}} \oint_{\mathcal{C}} \frac{dv_1}{2\pi \mathbf{i}} \cdots \oint_{\mathcal{C}} \frac{dv_l}{2\pi \mathbf{i}} \det\left[\widetilde{K}_u(v_a, n_a; v_b, n_b) \right]_{a,b=1}^{l}. \tag{3.7.6}$$

Now we want to take the sum over k, but we need to be careful about possible convergence issues. For the left-hand side we have $|\mathfrak{Z}_{x,y}| \leq 1$ almost surely, so the sum converges absolutely for any u and by dominated convergence gives the Laplace transform:

$$\mathbb{E}\left[e^{u\mathfrak{Z}_{x,y}}\right] = \sum_{k \geq 0} \frac{u^k \mathbb{E}\left[\mathfrak{Z}_{x,y}^k\right]}{k!}.$$

For the right-hand side we can use the conditions on the parameters $\sigma_i, \rho_j, \omega_d$ and on the compact contour $\mathcal{C}$ to give a uniform in $(v, v', n) \in \mathcal{C} \times \mathcal{C} \times \mathbb{Z}_{\geq 1}$ upper-bound

$$\left| \frac{g(v)}{v + n - v'} \right| < \Lambda_1.$$

At the same time, by Lemma 3.7.3 we have

$$\frac{1}{g(v+n)} = \exp\left(-\left(v+n-\Delta-\frac{1}{2}\right)\ln(n)+n-\frac{1}{2}\ln(2\pi)+O\left(n^{-1}\right)\right),$$

where $\Delta = \sum_{i=0}^{x}\sigma_i - \sum_{j=1}^{y}\rho_j + \sum_{d=1}^{y-x}\omega_d$ and the constants of $O\left(n^{-1}\right)$ can be chosen independently of $v \in \mathcal{C}$, since $\mathcal{C}$ is bounded. Hence for some constants $\Lambda_2, \varepsilon > 0$ and any $n \in \mathbb{Z}_{\geq 1}$ we have

$$\frac{1}{|g(v+n)|} < \Lambda_2 n^{-\varepsilon n}.$$

Gathering all pieces, we obtain

$$|\widetilde{K}_u(v,n;v',n')| < \Lambda_1\Lambda_2 n^{-\varepsilon n}|u|^n$$

and hence, by Hadamard's inequality,

$$\left|\oint_{\mathcal{C}}\frac{dv_1}{2\pi\mathbf{i}}\cdots\oint_{\mathcal{C}}\frac{dv_l}{2\pi\mathbf{i}}\det\left[\widetilde{K}_u(v_a,n_a;v_b,n_b)\right]_{a,b=1}^{l}\right| \leq \Lambda_1\Lambda_2\xi^l l^{l/2}|u|^{n_1+\ldots n_l}\prod_{i=1}^{l} n_i^{-\varepsilon n_i},$$

where $\xi > 0$ is the length of the contour $\mathbb{C}$. Since

$$\sum_{l\geq 0}\sum_{n_1,\ldots,n_l\geq 1}\frac{\xi^l l^{l/2}}{l!}|u|^{n_1+\ldots n_l}\prod_{i=1}^{l} n_i^{-\varepsilon n_i} < \infty$$

for any $u \in \mathbb{C}$, the right-hand side of (3.7.6) absolutely converges. □

So, we have obtained an expression for the Laplace transform in terms of a Fredholm determinant over $L^2(\mathbb{Z}_{\geq 1}\times\mathcal{C})$. But the summation over $\mathbb{Z}_{\geq 1}$ is inconvenient for the asymptotical analysis, so we replace it by another integration using Mellin-Barnes integral formula. To proceed we add another assumption, namely, we assume that there exists a vertical line $\mathcal{L} = \{z \mid \mathrm{Re}[z] = h\}$ for some h such that the contours $\mathcal{C}$ and $\mathcal{C}+1$ are separated by $\mathcal{L}$. Note that this is possible only if

$$\max_i \sigma_i - \min_i \sigma_i < 1. \tag{3.7.7}$$

Proposition 3.7.5. *For any $u \in \mathbb{C}$ such that $\arg(-u) \in \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ we have*

$$\sum_{n\geq 1}\frac{g(v)}{g(v+n)}\frac{u^n}{v+n-v'} = \frac{1}{2\pi\mathbf{i}}\int_{\mathcal{L}}\frac{-\pi}{\sin(\pi(z-v))}(-u)^{z-v}\frac{g(v)}{g(z)}\frac{dz}{z-v'},$$

where $\mathcal{L}$ is a vertical line $\{z \mid \mathrm{Re}[z] = h\}$ directed upwards, $v, v', n \in \mathcal{C}\times\mathcal{C}\times\mathbb{Z}_{\geq 1}$ and for $(-u)^{z-v} = e^{(z-v)\ln(-u)}$ we use the principal branch of the natural logarithm.

Proof. The statement and the proof are similar to [7, Lemma 3.7], but our restrictions on u are different.

Since the contour $\mathcal{C}$ is compact, there exists $N_0 \in \mathbb{Z}_{>0}$ such that the contour $\mathcal{C}$ is entirely inside the rectangle formed by the lines $\mathcal{L}, \mathcal{L}-1, \mathbb{R}+N_0\mathbf{i}, \mathbb{R}-N_0\mathbf{i}$. For $N > N_0$ let $\mathcal{H}_N$ denote the rectangle

formed by the lines $\mathcal{L}$, $\mathcal{L}+N$, $\mathbb{R}+N\mathbf{i}, \mathbb{R}-N\mathbf{i}$, oriented *clockwise*. Then the integral

$$\frac{1}{2\pi\mathbf{i}}\int_{\mathcal{H}_N}\frac{-\pi}{\sin(\pi(z-v))}(-u)^{z-v}\frac{g(v)}{g(z)}\frac{dz}{z-v'}$$

has nonzero residues only at the poles of $\frac{\pi}{\sin(\pi(z-v))}$, since all the residues of $\frac{g(v)}{g(z)}$ are to the left of the line $\mathcal{L}$. Computing the residues at $z=v+n$ for $n=1,\dots,N$ we get

$$\sum_{n\geq 1}^{N}\frac{g(v)}{g(v+n)}\frac{u^n}{v+n-v'}=\frac{1}{2\pi\mathbf{i}}\int_{\mathcal{H}_N}\frac{-\pi}{\sin(\pi(z-v))}(-u)^{z-v}\frac{g(v)}{g(z)}\frac{dz}{z-v'}.$$

Now we want to take $N\to\infty$ in both sides. For the left-hand side, using the same upper-bound on $K(v,n;v'n')$ as in Proposition 3.7.5, we have

$$\sum_{n\geq 1}\left|\frac{g(v)}{g(v+n)}\frac{u^n}{v+n-v'}\right|<\sum_{n\geq 1}\Lambda_1\Lambda_2 n^{-\varepsilon n}|u|^n<\infty. \tag{3.7.8}$$

so the absolute convergence holds for any $u\in\mathbb{C}$.

For the right-hand side we need to show that the integral along the symmetric difference of $\mathcal{L}$ and $\mathcal{H}_N$ converges to 0. This symmetric difference $\mathcal{D}_N$ consists of the line segments sequentially connecting the points $h-\infty\mathbf{i}, h-N\mathbf{i}, h+N-N\mathbf{i}, h+N+N\mathbf{i}, h+N\mathbf{i}, h+\infty\mathbf{i}$.

Clearly, $\frac{1}{z-v'}$ for $z\in\mathcal{D}_N$ can be bounded uniformly in N, while $\frac{\pi}{\sin(\pi(z-v))}$ decays exponentially in $\operatorname{Im}[z]$:

$$\left|\frac{-\pi}{\sin(\pi(z-v))}\right|<\Lambda_3 e^{-\pi|\operatorname{Im}(z)|},\qquad z\in\mathcal{D}_N,$$

where Λ_3 does not depend on N. To bound $\frac{1}{g(z)}$ we use Lemma 3.7.3 to get

$$\frac{1}{|g(z)|}=\left(1+O(z^{-1})\right)\frac{|e^z z^{-z+\Delta+\frac{1}{2}}|}{\sqrt{2\pi}}<\Lambda_4|z|^{\Delta+\frac{1}{2}}(|z|/e)^{-\operatorname{Re}[z]}e^{\frac{\pi}{2}|\operatorname{Im}[z]|}$$

for a constant Λ_4 and $z\in\mathcal{D}_N$ for sufficiently large N. Finally, due to the condition on u, we have

$$|(-u)^z|<|u|^{\operatorname{Re}[z]}e^{\left(\frac{\pi}{2}-\delta\right)|\operatorname{Im}[z]|}$$

for some $\delta>0$. Combining everything together we see that for some constant $\Lambda_5>0$ the integrand is bounded by

$$\Lambda_5 e^{-\delta|\operatorname{Im}[z]|}|z|^{-\operatorname{Re}[z]}(|u|e)^{\operatorname{Re}[z]}|z|^{\Delta+\frac{1}{2}}$$

which is enough to conclude that the integral over $\mathcal{D}_N$ tends to 0. □

Proposition 3.7.6. *For any $u\in\mathbb{C}$ such that $\arg(-u)\in(-\frac{\pi}{2};\frac{\pi}{2})$ we have*

$$\mathbb{E}\left[e^{u\mathfrak{Z}_{x,y}}\right]=\det(I+K_u)_{L^2(\mathcal{C})}=\sum_{l\geq 0}\frac{1}{l!}\oint_{\mathcal{C}}\frac{dv_1}{2\pi\mathbf{i}}\cdots\oint_{\mathcal{C}}\frac{dv_l}{2\pi\mathbf{i}}\det\left[K_u(v_a;v_b)\right]_{a,b=1}^{l},$$

where the integral operator K_u is defined by the kernel

$$K_u(v;v') = \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{L}} \frac{-\pi}{\sin(\pi(z-v))} (-u)^{z-v} \frac{g(v)}{g(z)} \frac{dz}{z-v'}.$$

Here $\mathcal{C}$ is a sufficiently small contour containing σ_i for all i, and $\mathcal{L}$ is a vertical line separating $\mathcal{C}$ and $\mathcal{C}+1$.

Proof. From Proposition 3.7.4 we have

$$\mathbb{E}\left[e^{u\mathfrak{Z}_{x,y}}\right] = \sum_{l\geq 0} \frac{1}{l!} \sum_{n_1,\ldots,n_l\geq 1} \oint_{\mathcal{C}} \frac{dv_1}{2\pi\mathbf{i}} \cdots \oint_{\mathcal{C}} \frac{dv_l}{2\pi\mathbf{i}} \det\left[\widetilde{K}_u(v_a,n_a;v_b,n_b)\right]_{a,b=1}^{l}.$$

Due to the dominated convergence, which is valid since the contour $\mathcal{C}$ is compact and we have an integrable converging upper-bound (3.7.8), we can exchange the second summation and the integration. Since $\widetilde{K}_u(v,n;v',n')$ does not depend on n', we can use use the same absolute upper-bound (3.7.8) and multi-linearity of the determinant with respect to rows to get

$$\sum_{n_1,\ldots,n_l\geq 1} \det\left[\widetilde{K}_u(v_a,n_a;v_b,n_b)\right]_{a,b=1}^{l} = \det\left[\sum_{n_a\geq 1} \frac{g(v_a)}{g(v_a+n_a)} \frac{u^{n_a}}{v_a+n_a-v'_b}\right]_{a,b=1}^{l}.$$

Then the claim follows from Proposition 3.7.5. □

3.8 Limit theorem

This section is devoted to the Tracy-Widom large-scale limit of the partition function $\mathfrak{Z}_{x,y}$ of the inhomogeneous Beta polymer model, when $x,y\to\infty$ at a constant ratio and the parameters of the model have finite number of values repeated with certain frequencies. Our approach follows [7]: we perform a steepest descent analysis of the Fredholm determinant from Proposition 3.7.6.

3.8.1 Overview

Fix k and consider three families of parameters $\{\sigma_1,\ldots,\sigma_k\}$, $\{\rho_1,\ldots,\rho_k\}$ and $\{\omega_1,\ldots,\omega_k\}$. Let $\{\alpha_1,\ldots,\alpha_k\}$, $\{\beta_1,\ldots,\beta_k\}$, $\{\gamma_1,\ldots,\gamma_k\}$ be collections of corresponding "frequencies" satisfying

$$\sum_{i=1}^{k}\alpha_i = \sum_{j=1}^{k}\beta_j = \sum_{d=1}^{k}\gamma_d = 1, \qquad \alpha_i,\beta_j,\gamma_d\in\mathbb{R}_{\geq 0}.$$

For a fixed slope x/y we consider the Beta polymer partition function $\mathfrak{Z}_{\lfloor xt\rfloor,\lfloor yt\rfloor}$ as $t\to\infty$, where parameters of the model $\{\widetilde{\sigma}_0,\ldots,\widetilde{\sigma}_{\lfloor xt\rfloor}\}$, $\{\widetilde{\rho}_1,\ldots,\widetilde{\rho}_{\lfloor yt\rfloor}\}$, $\{\widetilde{\omega}_1,\ldots,\widetilde{\omega}_{\lfloor yt\rfloor-\lfloor xt\rfloor}\}$[6] are defined in the following way: For the column parameters, exactly $\alpha_i^{[t]}xt$ out of $\lfloor xt\rfloor+1$ parameters $\widetilde{\sigma}_i$ are equal

[6] In earlier sections σ_i,ρ_j,ω_d were used to denote the parameters of individual columns, rows and diagonals. To lighten the expressions in this section it is convenient for us to change the notation by using $\widetilde{\sigma}_i,\widetilde{\rho}_j,\widetilde{\omega}_d$ for individual parameters while σ_i,ρ_j,ω_d denote their possible values; this abuse of notation should not cause confusion.

to σ_i, where $\alpha_i^{[t]}$ is a sequence satisfying

$$\alpha_i^{[t]} - \alpha_i = O(t^{-1}).$$

Similarly, the row parameters ρ_j and the diagonal parameters ω_d are repeated for $\beta_j^{[t]}yt$ and $\gamma_d^{[t]}(y-x)t$ times respectively, where

$$\beta_j^{[t]} - \beta_j = O(t^{-1}), \qquad \gamma_d^{[t]} - \gamma_d = O(t^{-1}).$$

Motivated by the homogeneous setting [7] we are looking for a limit result of the form

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{\ln \mathfrak{Z}_{\lfloor xt\rfloor,\lfloor yt\rfloor} + It}{ct^{1/3}} \le r\right) = F_{GUE}(r), \tag{3.8.1}$$

where $F_{GUE}(p)$ is the GUE Tracy-Widom distribution [83]. A probabilistic interpretation, which one can find more natural, is given by identification of $\mathfrak{Z}_{\lfloor xt\rfloor,\lfloor yt\rfloor}$ with the distribution of a random walk in a random Beta environment. From this point of view, we are looking at large deviations of the random walk, studying fluctuations of a random rate function (depending on the random environment) from the expected limit $I(x/y)$ depending on the slope. See [7] for more details.

To prove a limit relation (3.8.1) it is enough to set $u^{[t]} = -\exp(tI - t^{1/3}cr)$ and show that

$$\lim_{t\to\infty} \mathbb{E}\left[e^{u^{[t]}\mathfrak{Z}_{\lfloor xt\rfloor,\lfloor yt\rfloor}}\right] = F_{GUE}(r),$$

because, by [13, Lemma 4.1.39][7] we have

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{\ln \mathfrak{Z}_{\lfloor xt\rfloor,\lfloor yt\rfloor} + It}{ct^{1/3}} \le r\right) = \lim_{t\to\infty} \mathbb{E}\left[e^{u^{[t]}\mathfrak{Z}_{\lfloor xt\rfloor,\lfloor yt\rfloor}}\right]$$

given that the limit is a continuous distribution function. So, the problem is reduced to studying the Laplace transform, which by Proposition 3.7.6 has an expression in terms of a Fredholm determinant. The latter can be analyzed using *the steepest descent method*: rewrite the kernel $K_{u^{[t]}}$ as

$$K_{u^{[t]}}(v,v') = \frac{1}{2\pi\mathbf{i}}\int_{\mathcal{L}} \frac{-\pi}{\sin(\pi(z-v))} \exp\left(t(h^{[t]}(z) - h^{[t]}(v)) - crt^{1/3}(z-v)\right)\frac{dz}{z-v'},$$

where

$$h^{[t]}(z) = Iz - \sum_{i=1}^{k} x\alpha_i^{[t]} \ln\Gamma(z-\sigma_i) + \sum_{i=1}^{k} y\beta_i^{[t]} \ln\Gamma(z-\rho_i) - \sum_{i=1}^{k}(y-x)\gamma_i^{[t]} \ln\Gamma(z-\omega_i).$$

As $t\to\infty$ the function $h^{[t]}(z)$ converges to the function

$$h(z) = Iz - \sum_{i} x\alpha_i \ln\Gamma(z-\sigma_i) + \sum_{i} y\beta_i \ln\Gamma(z-\rho_i) - \sum_{i}(y-x)\gamma_i \ln\Gamma(z-\omega_i),$$

which, for the right choice of I, has a second order critical point θ depending on the slope y/x in a

[7] Following the notation of [13, Lemma 4.1.39], we need to set $f(x) = e^{-e^{xct^{1/3}}}$.

nontrivial way. However, the dependence of the slope x/y on θ can be easily described, suggesting the following parametrization by $\theta \in (\max_i \sigma_i, \infty)$:

$$x = x(\theta) = \sum_i \beta_i \Psi_1(\theta - \rho_i) - \sum_i \gamma_i \Psi_1(\theta - \omega_i),$$

$$y = y(\theta) = \sum_i \alpha_i \Psi_1(\theta - \sigma_i) - \sum_i \gamma_i \Psi_1(\theta - \omega_i),$$

$$I = I(\theta) = x(\theta) \sum_i \alpha_i \Psi(\theta - \sigma_i) - y(\theta) \sum_i \beta_i \Psi(\theta - \rho_i) + (y(\theta) - x(\theta)) \sum_i \gamma_i \Psi(\theta - \omega_i),$$

$$c^3 = c(\theta)^3 = -\frac{x(\theta)}{2} \sum_i \alpha_i \Psi_2(\theta - \sigma_i) + \frac{y(\theta)}{2} \sum_i \beta_i \Psi_2(\theta - \rho_i) - \frac{y(\theta) - x(\theta)}{2} \sum_i \gamma_i \Psi_2(\theta - \omega_i),$$

where $\Psi(z)$ denotes the digamma function and $\Psi_k(z)$ denotes the kth polygamma function:

$$\Psi(z) = \frac{d}{dz} \ln \Gamma(z), \qquad \Psi_k(z) = \frac{d^{k+1}}{dz^{k+1}} \ln \Gamma(z).$$

Proposition 3.8.1. *1) The function $x(\theta)/y(\theta)$ is increasing in $\theta \in (\max_i \sigma_i, \infty)$, with the limit values*

$$\lim_{\theta \to \max_i \sigma_i} \frac{x(\theta)}{y(\theta)} = 0, \qquad \lim_{\theta \to \infty} \frac{x(\theta)}{y(\theta)} = \frac{\sum_i \beta_i \rho_i - \sum_i \gamma_i \omega_i}{\sum_i \alpha_i \sigma_i - \sum_i \gamma_i \omega_i}.$$

2) $c(\theta)$ can be chosen to be a positive real number.

Proposition 3.8.1 will be proved in Section 3.9.2 later.

With this parametrization we have $h(z) - h(\theta) = \frac{c^3}{3}(z - \theta)^3 + O((z - \theta)^4)$ for z close to θ, and the general philosophy of the steepest descent method suggests that asymptotical behavior of the Fredholm determinant is dominated by its behavior in a small neighborhood of θ, which, after certain manipulation, can be shown to produce $F_{GUE}(r)$.

Conjecture 3.8.2. *For any $\theta \in (\max(\sigma), \infty)$ we have*

$$\lim_{t \to \infty} \mathbb{P}\left(\frac{\ln \mathfrak{Z}_{\lfloor xt \rfloor, \lfloor yt \rfloor} + It}{ct^{1/3}} \le r \right) = F_{GUE}(r),$$

with $x = x(\theta), y = y(\theta), I = I(\theta), c = c(\theta)$ defined above in terms of θ.

Unfortunately, the general case turns out to be technically challenging, and we do not pursue it here. In this section we prove Conjecture 3.8.2 for parameters $\sigma_i, \rho_i, \omega_i, \theta$ satisfying the following restrictions:

Assumption 3.8.3. We assume that

- All σ_i are equal to 0, while all ρ_i are equal to -1:

$$\sigma_1 = \sigma_2 = \cdots = \sigma_k = 0, \qquad \rho_1 = \rho_2 = \cdots = \rho_k = -1.$$

- $\theta \in \left(0, \frac{1}{2}\right)$.

Theorem 3.8.4. *Under Assumption 3.8.3 we have*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{\ln \mathfrak{Z}_{\lfloor xt\rfloor,\lfloor yt\rfloor} + It}{ct^{1/3}} \leq r\right) = F_{GUE}(r).$$

Remark 3.8.5. Theorem 3.8.4 generalizes [7, Theorem 5.2] by adding a degree of freedom corresponding to the diagonal parameters ω_i, while in [7] they were additionally assumed to be equal to -2. Returning to the Beta polymer model, Theorem 3.8.4 corresponds to the model with Beta distributions $\mathrm{Beta}(1, -\widetilde{\omega} - 1)$ with parameters $\widetilde{\omega} < -1$ constant along the diagonals and having finitely many possible values $\omega_1, \ldots, \omega_k$.

The restrictions from Assumption 3.8.3 come from two technical difficulties. Namely, to justify the steepest descent argument one needs to

- Find contours $\mathcal{C}_\theta, \mathcal{L}_\theta$ passing through θ and satisfying $\mathrm{Re}[h(z) - h(v)] \leq 0$ for $z \in \mathcal{L}_\theta, v \in \mathcal{C}_\theta$, with equality only if $v = z = \theta$;

- Prove that one can deform the initial contours $\mathcal{C}, \mathcal{L}$ to $\mathcal{C}_\theta, \mathcal{L}_\theta$ without changing the limit of the Fredholm determinant.

For the first step a suitable descent contour $\mathcal{L}_\theta$ for $\mathrm{Re}[h(z)]$ valid for any parameters $\sigma_i, \rho_i, \omega_i$ was found in [7] and it is given by a vertical line passing through θ. At the same time, finding a descent contour for $-\mathrm{Re}[h(v)]$ is much more involved for general parameters: we know how to find it only if $\sigma_i - \rho_j = 1$ for any i, j and $\theta - \max_i \sigma_i < 1$, forcing the first part of Assumption 3.8.3[8]. Under these restrictions another contour from [7] turns out to be sufficient, we denote it by $\mathcal{C}_\theta$ and it is equal to a circle around 0 of radius θ. However, numerical computations and related results suggest that a descent contour for $-\mathrm{Re}[h(v)]$ should exist in general: see [10] for an example of a significantly more involved argument used to remove the analogous assumption in a different but related setting.

The second step is complicated by potential singularities of the Fredholm determinant along the deformation of $\mathcal{C}, \mathcal{L}$ to $\mathcal{C}_\theta, \mathcal{L}_\theta$. One can hope to estimate the residues at these singularities and show that their contribution asymptotically vanishes, but the required argument is technically challenging. The second part of Assumption 3.8.3 allows us to avoid this issue, since for $\theta < \frac{1}{2}$ the contours can be deformed without crossing any singularities of the relevant integrands.

The remainder of Section 3.8 is devoted to the proof of Theorem 3.8.4, with technical computations involving polygamma functions moved to Section 3.9. As clarified above, it is enough to prove

$$\lim_{t\to\infty} \mathbb{E}\left[e^{u^{[t]}\mathfrak{Z}_{\lfloor xt\rfloor,\lfloor yt\rfloor}}\right] = F_{GUE}(r), \tag{3.8.2}$$

where $u^{[t]} = -\exp(tI - t^{1/3}cr)$. The proof is structured as follows: first we describe the needed properties of the function $h(z)$, as well as specify estimates on convergence of $h^{(t)}(z)$ to $h(z)$. Then we prove various bounds on the kernel K_u from Proposition 3.7.6. Finally, to prove Theorem 3.8.4 we reduce the computation of the limit to a small neighborhood of θ, perform a change of variables to get rid of lower degree terms and use standard algebraic manipulations to establish (3.8.2).

[8] To simplify the assumptions we have also used the invariance of the model under simultaneous additive shifts of all the parameters $\sigma_i, \rho_i, \omega_i$.

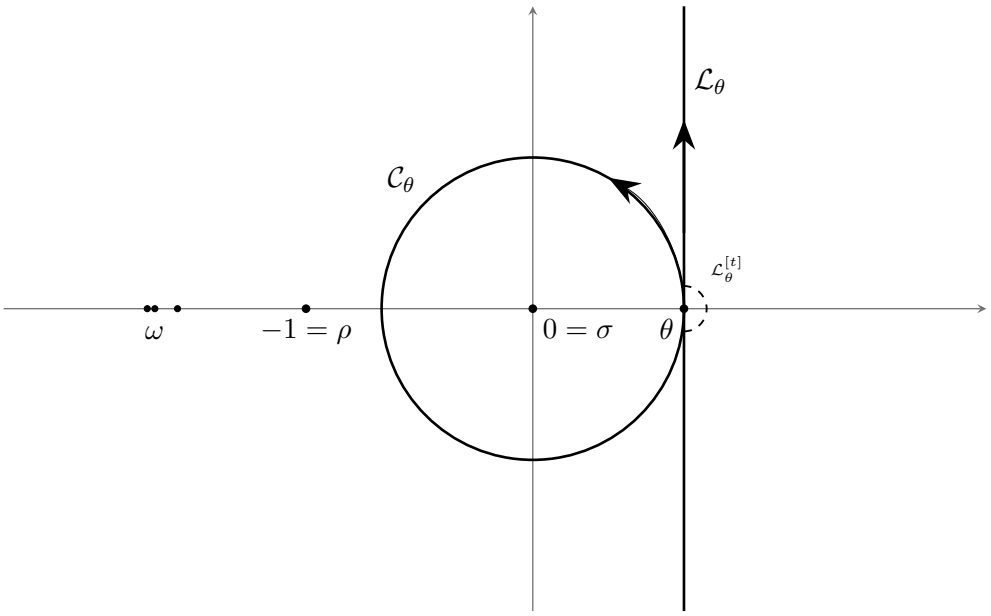


Figure 3-5: Contours $\mathcal{C}_\theta$ and $\mathcal{L}_\theta$, as well as the t-dependent deformation $\mathcal{L}_\theta^{[t]}$ needed to avoid singularity at $\frac{1}{(z-\theta)^2}$

3.8.2 h-functions and descent contours

Here we list the needed properties of the functions $h(z)$ and $h^{[t]}(z)$, which are defined by

$$h^{[t]}(z) = Iz - \sum_{i=1}^{k} x\alpha_i^{[t]} \ln\Gamma(z-\sigma_i) + \sum_{i=1}^{k} y\beta_i^{[t]} \ln\Gamma(z-\rho_i) - \sum_{i=1}^{k} (y-x)\gamma_i^{[t]} \ln\Gamma(z-\omega_i),$$

$$h(z) = Iz - \sum_{i=1}^{k} x\alpha_i \ln\Gamma(z-\sigma_i) + \sum_{i=1}^{k} y\beta_i \ln\Gamma(z-\rho_i) - \sum_{i=1}^{k} (y-x)\gamma_i \ln\Gamma(z-\omega_i),$$

with x, y, I defined in terms of θ as before. For our purposes it is enough to fix a branch of these functions in the half plane $\mathrm{Re}[z] > \max \sigma_i$.

We start with the properties of $h(z)$. Define the contour $\mathcal{L}_\theta$ as a vertical line passing through θ, oriented upwards:

$$\mathcal{L}_\theta = \{\theta + b\mathbf{i} \mid b \in \mathbb{R}\}.$$

Under Assumption 3.8.3 we also define the contour

$$\mathcal{C}_\theta = \{\theta e^{i\phi} \mid \phi \in [-\pi, \pi]\}$$

oriented counter-clockwise. See Figure 3-5 for a depiction of the contours.

The following three statements are proved in Appendix 3.9, since their proofs are similar and require explicit computations of polygamma functions. Note that these statements involve only the real part of $h(z)$, so the branching caused by logarithms is irrelevant here. The first statement does not require Assumption 3.8.3.

Proposition 3.8.6 (3.9.3)**.** *$\mathcal{L}_\theta$ is a steep descent contour for* $\mathrm{Re}[h(z)]$*, that is,* $\mathrm{Re}[h(\theta+bi)]$ *decreases for positive b and increases for negative b.*

Proposition 3.8.7 (3.9.4)**.** *Under Assumption 3.8.3 $\mathcal{C}_\theta$ is a steep descent contour for* $-\mathrm{Re}[h(z)]$*, that is,* $-\mathrm{Re}[h(\theta e^{\phi \mathbf{i}})]$ *decreases for $\phi > 0$ and increases for $\phi < 0$.*

Proposition 3.8.8 (3.9.5)**.** *Suppose that Assumption 3.8.3 is satisfied. Let $\psi(\varepsilon) = \arg(v_\varepsilon - \theta)$, where v_ε is the intersection point of $\mathcal{C}_\theta$ with the circle of radius ε around θ satisfying* $\mathrm{Im}[v_\varepsilon] > 0$*. Then for sufficiently small ε there exists a constant b such that*

$$\mathrm{Re}\left[h(\theta + \varepsilon e^{\pm i\phi}) - h(\theta)\right] > b, \qquad \phi \in [\psi(\varepsilon), 2\pi/3].$$

The next statement reiterates the fact that θ is a critical point of $h(z)$.

Lemma 3.8.9. *For z in a neighborhood of θ we have*

$$h(z) - h(\theta) = \frac{c^3}{3}(z-\theta)^3 + O((z-\theta)^4),$$

where $c > 0$.

Proof. Readily follows by considering the Taylor expansion of $h(z)$, taking derivatives and using the definitions of $x(\theta), y(\theta), I(\theta)$ and $c(\theta)$. The positivity of c follows from Proposition 3.8.1. □

Finally, we need the following statements regarding the convergence of $h^{[t]}(z)$ to $h(z)$ in various situations.

Lemma 3.8.10. *Assume that a sequence of points $z_t \in \mathbb{C}$ converges to θ in a way that $h^{[t]}(z_t)$ is well defined for all $t > 0$. Then*

$$\lim_{t\to\infty} t\left(h^{[t]}(z_t) - h^{[t]}(\theta)\right) = \lim_{t\to\infty} t\left(h(z_t) - h(\theta)\right),$$

assuming that the right-hand side exists.

Proof. Note that

$$th^{[t]}(z) - th(z) = -x\sum_{i=1}^{k} t(\alpha_i^{[t]} - \alpha_i)\ln\Gamma(z-\sigma_i) + y\sum_{i=1}^{k} t(\beta_i^{[t]} - \beta_i)\ln\Gamma(z-\rho_i) - (y-x)\sum_{i=1}^{k} t(\gamma_i^{[t]} - \gamma_i)\ln\Gamma(z-\omega_i)$$

Since $\alpha_i^{[t]} - \alpha_i = O(1/t)$ and $\ln\Gamma(z-\sigma_i)$ is continuous around θ, we have

$$\lim_{t\to\infty} t(\alpha_i^{[t]} - \alpha_i)\left(\ln\Gamma(z_t - \sigma_i) - \ln\Gamma(\theta - \sigma_i)\right) = 0.$$

Taking sum over i and repeating the same argument for $\beta_i, \beta_i^{[t]}, \rho_i$ and $\gamma_i, \gamma_i^{[t]}, \omega_i$ we obtain

$$\lim_{t\to\infty} t\left(h^{[t]}(z_t) - h(z_t)\right) - t\left(h^{[t]}(\theta) - h(\theta)\right) = 0,$$

which implies the claim. □

Lemma 3.8.11. *Let $K \subset \mathbb{C}$ be a compact subset not containing the points $\sigma_i - n, \rho_i - n, \omega_i - n$ for every $n \in \mathbb{Z}_{\geq 0}$. Then for any $z \in K$ and $t > 0$ we have*

$$|\operatorname{Re}[th^{[t]}(z) - th(z)]| < \Lambda$$

for some constant $\Lambda > 0$ depending only on the compact set K but not on t.

Proof. Following the lines of the proof of Lemma 3.8.10, since $\alpha_i^{[t]} - \alpha_i = O(1/t), \beta_i^{[t]} - \beta_i = O(1/t), \gamma_i^{[t]} - \gamma_i = O(1/t)$ there exists a constant $\Lambda_1 > 0$ such that

$$|t(\alpha_i^{[t]} - \alpha_i)|, |t(\beta_i^{[t]} - \beta_i)|, |t(\gamma_i^{[t]} - \gamma_i)| < \Lambda_1, \qquad i = 1, \dots, k.$$

At the same time, since K is bounded away from $\sigma_i, \rho_i, \omega_i$, there exists another constant $\Lambda_2 > 0$ such that

$$|\operatorname{Re}[\ln \Gamma(z - \sigma_i)]|, |\operatorname{Re}[\ln \Gamma(z - \rho_i)]|, |\operatorname{Re}[\ln \Gamma(z - \omega_i)]| < \Lambda_2, \qquad z \in K,\ i = 1, \dots, k.$$

Combining these upper bounds concludes the proof. □

Lemma 3.8.12. *For sufficiently large z such that $\arg(z) \in [-\pi + \delta, \pi - \delta]$ for a fixed δ we have*

$$\exp(th^{[t]}(z)) = \exp(th(z)) \frac{z^{-\Delta^{[t]}}}{\Gamma(z)} (1 + O\left(z^{-1}\right)),$$

where the constants of $O(z^{-1})$ can be chosen independently of t and $\Delta^{[t]}$ is given by

$$\Delta^{[t]} = -\sum_i xt(\alpha_i^{[t]} - \alpha_i)\sigma_i + \sum_i yt(\beta_i^{[t]} - \beta_i)\rho_i - \sum_i t(y - x)(\gamma_i^{[t]} - \gamma_i)\omega_i.$$

Note that, due to our conventions on $\alpha_i^{[t]}, \beta_i^{[t]}$ and $\gamma_i^{[t]}$, we have a uniform in t bound $|\Delta^{[t]}| < \Lambda$ for a constant $\Lambda > 0$ independent of t.

Proof. By taking logarithm the claim can be rephrased as

$$-\sum_i xt(\alpha_i^{[t]} - \alpha_i) \ln \Gamma(z - \sigma_i) + \sum_i yt(\beta_i^{[t]} - \beta_i) \ln \Gamma(z - \rho_i) - \sum_i t(y - x)(\gamma_i^{[t]} - \gamma_i) \ln \Gamma(z - \omega_i)$$
$$= -\Delta^{[t]} \ln(z) - \ln \Gamma(z) + O(z^{-1}).$$

Now we can use the asymptotic expansion for Gamma function, used before in Lemma 3.7.3: with the same restrictions on z as in the claim and any fixed $a \in \mathbb{C}$ we have

$$\ln \Gamma(z - a) = \left(z - a - \frac{1}{2}\right) \ln z - z + \frac{1}{2} \ln(2\pi) + O(z^{-1}).$$

Recall that $\sum_{i=1}^k \alpha_i = \sum_{i=1}^k \beta_i = \sum_{i=1}^k \gamma_i = 1$, while for $\alpha_i^{[t]}, \beta_i^{[t]}, \gamma_i^{[t]}$ we have

$$\sum_{i=1}^k xt\alpha_i^{[t]} = \lfloor xt \rfloor + 1, \quad \sum_{i=1}^k yt\beta_i^{[t]} = \lfloor yt \rfloor, \quad \sum_{i=1}^k (y - x)t\gamma_i^{[t]} = \lfloor yt \rfloor - \lfloor xt \rfloor,$$

with 1 in the first equality coming from having $\lfloor xt \rfloor + 1$ column parameters $\{\widetilde{\sigma}_i\}_{i=0,\dots,\lfloor xt \rfloor}$. Thus

$$-\sum_{i=1}^{k} tx(\alpha_i^{[t]} - \alpha_i) + \sum_{i=1}^{k} ty(\beta_i^{[t]} - \beta_i) - \sum_{i=1}^{k} t(y-x)(\gamma_i^{[t]} - \gamma_i) = -1.$$

Combining the identity above with the asymptotic expansion for the Gamma function, we obtain

$$-\sum_{i} xt(\alpha_i^{[t]} - \alpha_i) \ln \Gamma(z - \sigma_i) + \sum_{i} yt(\beta_i^{[t]} - \beta_i) \ln \Gamma(z - \rho_i) - \sum_{i} t(y-x)(\gamma_i^{[t]} - \gamma_i) \ln \Gamma(z - \omega_i)$$
$$= \left(-z - \Delta^{(t)} + \frac{1}{2}\right) \ln z + z - \frac{1}{2}\ln(2\pi) + 2y\Lambda_1 O(z^{-1}),$$

where Λ_1 is the same constant as in the proof of Lemma 3.8.11. Applying the asymptotic expansion for the Gamma function to the right-hand side implies the claim. □

3.8.3 Kernel estimates

Recall that the substitution of $u = -\exp(tI - t^{1/3}cr)$ into the integral kernel from Proposition 3.7.6 leads to the following kernel:

$$K^{[t]}(v, v') = \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{L}_\theta} \frac{-\pi}{\sin(\pi(z-v))} \exp\left(t(h^{[t]}(z) - h^{[t]}(v)) - crt^{1/3}(z-v)\right) \frac{dz}{z-v'}.$$

For $v, v' = \theta$ we interpret the integral above using its principal value as the contour $\mathcal{L}_\theta$ passes to the right of θ. In other words, we slightly deform the contour $\mathcal{L}_\theta$ to an upwards oriented contour $\mathcal{L}_\theta^{[t]}$ consisting of vertical lines $\{\theta \pm b\mathbf{i} \mid b \in [t^{-1/3}, \infty)\}$ and a semicircle $\{\theta + t^{-1/3}e^{\phi\mathbf{i}} \mid \phi \in [-\pi, \pi]\}$, see Figure 3-5.

The aim of this section is to give asymptotical bounds on behavior of the kernel. We use the following notation. For $\varepsilon > 0$ let $\mathcal{L}_{\theta,\varepsilon}^{[t]}$ and $\mathcal{C}_{\theta,\varepsilon}$ denote the parts of the contours $\mathcal{L}_\theta^{[t]}$ and $\mathcal{C}_\theta$ contained inside the ε-neighborhood of θ. Define the kernel $K_\varepsilon^{[t]}$ by

$$K_\varepsilon^{[t]}(v, v') = \frac{1}{2\pi\mathbf{i}} \int_{\mathcal{L}_{\theta,\varepsilon}^{[t]}} \frac{-\pi}{\sin(\pi(z-v))} \exp\left(t(h^{[t]}(z) - h^{[t]}(v)) - crt^{1/3}(z-v)\right) \frac{dz}{z-v'}.$$

Finally, define $\mathcal{K}(\varepsilon) = \mathcal{D}_\varepsilon \cup \mathcal{A}_\varepsilon \cup (\mathcal{C}_\theta \backslash \mathcal{C}_{\theta,\varepsilon})$ as a contour consisting of the upward-oriented wedge

$$\mathcal{D}_\varepsilon = \{\theta + Re^{\pm 2\pi\mathbf{i}/3} \mid R \in [0, \varepsilon]\},$$

a part of the contour $\mathcal{C}_\theta$ outside of the ε-neighborhood of θ, which we denote as $\mathcal{C}_{\theta,\geq\varepsilon}$, and a couple of upward-oriented arcs

$$\mathcal{A}_\varepsilon = \{\theta + \varepsilon e^{\phi\mathbf{i}} \mid \phi \in [\psi(\varepsilon), 2\pi/3]\},$$

where $\psi(\varepsilon)$ is chosen so that the result is connected. See Figure 3-6 for a depiction of the contour $\mathcal{K}(\varepsilon)$.

For the following statements we fix $\varepsilon > 0$.

Lemma 3.8.13. *Let $v, v' \in \mathcal{K}(\varepsilon)$ be a pair of points such that $\mathrm{Re}[h(v) - h(\theta)] \geq b$ for some fixed*

$b \geq 0$. *Then for* $t > \varepsilon^{-3}$ *we have*

$$|K^{[t]}(v, v') - K_{\varepsilon}^{[t]}(v, v')| < \Lambda e^{-t(a+b)},$$

where the constants $\Lambda, a > 0$ *depend only on* ε *and the parameters of the model.*

Lemma 3.8.14. *Let* $v, v' \in \mathcal{K}(\varepsilon)$ *be a pair of points such that* $\mathrm{Re}[h(v) - h(\theta)] \geq b$ *for a fixed* $b \geq 0$. *Then for* $t > \varepsilon^{-3}$ *we have*

$$|K^{[t]}(v, v')| < \Lambda t^{1/3} e^{-bt},$$

where $\Lambda > 0$ *is a constant depending only on* ε *and* b, *but not on* v, v'. *In particular, if* $b > 0$ *we also have*

$$|K^{[t]}(v, v')| < \Lambda' e^{-bt/2}$$

for some other constant $\Lambda' > 0$ *depending on* ε *and* b.

Proof of Lemma 3.8.13. The assumption $t > \varepsilon^{-3}$ is needed only to ensure that the semicircle of $\mathcal{L}_{\theta}^{[t]}$ is inside the ε neighborhood. Then we need to bound the integral

$$\int_{\mathcal{L}_{\theta} \setminus \{|z| < \varepsilon\}} \frac{-\pi}{\sin(\pi(z-v))} \exp\Big(t(h^{[t]}(z) - h^{[t]}(v)) - crt^{1/3}(z-v)\Big) \frac{dz}{z-v'}$$

taken over two vertical lines $\{\theta \pm b\mathbf{i} | b > \varepsilon\}$. We denote this contour by $\mathcal{L}_{\theta, \geq \varepsilon}$

Note that the contour $\mathcal{K}(\varepsilon)$ is bounded away from $\mathcal{L}_{\theta, \geq \varepsilon} + n$ for any $n \in \mathbb{Z}$, hence we have

$$\left| \frac{-\pi}{\sin \pi(z-v)} \frac{1}{z-v'} \right| < \Lambda_1 e^{-\pi |\operatorname{Im}[z]|}, \qquad z \in \mathcal{L}_{\theta, \geq \varepsilon},$$

where $\Lambda_1 > 0$ is a constant depending only on ε but not on v, v'. So it is enough to give a sufficiently strong upper-bound on $|\exp(t(h^{[t]}(z) - h^{[t]}(v)) - crt^{1/3}(z-v))|$. Due to the steep descent from Proposition 3.8.6, for some constant $a > 0$ we have

$$\mathrm{Re}[h(z) - h(v)] < -2a - b, \qquad z \in \mathcal{L}_{\theta, \geq \varepsilon}$$

But the kernel $K^{[t]}$ is defined in terms of $h^{[t]}(z)$ rather than $h(z)$ and to proceed we have to split the problem in two parts, with $|z|$ large and z compactly supported.

Since $\mathcal{K}(\varepsilon)$ is compact, by Proposition 3.8.11 we have

$$\mathrm{Re}[h(z) - h^{[t]}(v)] < -2(a+b) + \frac{\ln \Lambda_3}{t}, \qquad z \in \mathcal{L}_{\theta}^{[t]} \setminus \mathcal{L}_{\theta\varepsilon}^{[t]},$$

where the constant $\Lambda_3 > 1$ does not depend on v. On the other hand, by Lemma 3.8.12 and the asymptotical expansion for the Gamma function (3.7.4), for sufficiently large $M > \varepsilon$ independent of t, ε we have

$$\left| \exp\Big(t(h^{[t]}(z)\Big) \right| < \left| \exp\left(t(h(z))\right) \frac{2z^{-\Delta^{[t]}}}{\sqrt{2\pi} z^{z-1/2} e^{-z}} \right|, \qquad |z| > M, z \in \mathcal{L}_{\theta}.$$

Recall that $\Delta^{[t]}$ is uniformly bounded, so set $\Delta = \inf_t \Delta^{[t]} > -\infty$. Then, assuming $M > 1$, for some

constant Λ_2' and any $z \in \mathcal{L}_\theta, |z| > M, v \in \mathcal{K}(\varepsilon)$ we obtain

$$|\exp(t(h^{[t]}(z)-h^{[t]}(v))-crt^{1/3}(z-v))| < \frac{2\Lambda_3|z|^{-\Delta}\exp(-2ta-tb-crt^{1/3}\operatorname{Re}[z-v])}{\sqrt{2\pi}|z^{z-1/2}|e^{-\operatorname{Re}[z]}} < \Lambda_2' e^{-(a+b)t}e^{\frac{2\pi}{3}|\operatorname{Im}[z]|},$$

where we have used that $|z^z| > |z|^\theta e^{-\frac{2\pi}{3}|\operatorname{Im}[z]|}$ for $z \in \mathcal{L}_\theta$.

At the same time, having fixed M from the previous part, for $z \in \mathcal{L}_{\theta,\geq\varepsilon}, |z| < M$ we can again apply Lemma 3.8.11 to get

$$\operatorname{Re}[h^{[t]}(z) - h^{[t]}(v)] < -2(a+b) + \frac{\ln \Lambda_4}{t},$$

for $\Lambda_4 > 1$. Hence, for some constant $\Lambda_2'' > 0$ depending only on ε, M we have

$$|\exp(t(h^{[t]}(z) - h^{[t]}(v)) - crt^{1/3}(z-v))| < \Lambda_2'' e^{-(a+b)t}e^{\frac{2\pi}{3}|\operatorname{Im}[z]|}, \qquad z \in \mathcal{L}_{\theta,\geq\varepsilon}, |z| < M.$$

Setting $\Lambda_2 = \max(\Lambda_2', \Lambda_2'')$ and gathering all the pieces together, we get

$$\left|\int_{\mathcal{L}_{\theta,\geq\varepsilon}} \frac{-\pi}{\sin(\pi(z-v))} \exp\left(t(h^{[t]}(z) - h^{[t]}(v)) - crt^{1/3}(z-v)\right)\frac{dz}{z-v'}\right| < \Lambda_1\Lambda_2 e^{-(a+b)t}\int_{\mathcal{L}_{\theta,\geq\varepsilon}} e^{-\frac{\pi}{3}|\operatorname{Im}[z]|}|dz|$$

where Λ_1, Λ_2 depend only on ε, but not on v, v', t. □

Proof of Lemma 3.8.14. By Lemma 3.8.13 it is enough to prove that

$$|K_\varepsilon^{[t]}(v, v')| < \Lambda t^{1/3} e^{-bt}.$$

To bound the integral $K_\varepsilon^{[t]}(v, v')$ we deform the integration contour $\mathcal{L}_\varepsilon^{[t]}$ further, changing it to the contour $\mathcal{P}$ consisting of the line segments $\mathcal{P}_1 = \{\theta + y\mathbf{i} \mid y \in [-\varepsilon, -\delta t^{-\frac{1}{3}}] \cup [\delta t^{-\frac{1}{3}}, \varepsilon]\}$, and the semicircle $\mathcal{P}_2 = \{\theta + \delta t^{-1/3} e^{\phi\mathbf{i}} \mid \phi \in [-\pi/2, \pi/2]\}$, where the parameter $\delta > 0$ possibly depends on ε and is chosen small enough so that $\mathcal{K}(\varepsilon)$ is bounded away from $\mathcal{P} + n$ uniformly in $t > 1$ for $n \neq 0$.[9] Then there exists a constant Λ_1 depending only ε, δ such that

$$\left|\frac{\pi}{\sin \pi(z-v)}\frac{1}{z-v'}\right| < \Lambda_1 \frac{1}{|z-v||z-v'|},$$

which implies, due to the geometry of $K(\varepsilon)$, existence of uniform in v, v', t constants $\Lambda_{1,\mathcal{P}_1}$ and $\Lambda_{1,\mathcal{P}_2}$ such that

$$\left|\frac{\pi}{\sin \pi(z-v)}\frac{1}{z-v'}\right| < \Lambda_{1,\mathcal{P}_1}\frac{1}{|\operatorname{Im}[z]|^2}, \qquad z \in \mathcal{P}_1, \tag{3.8.3}$$

$$\left|\frac{\pi}{\sin \pi(z-v)}\frac{1}{z-v'}\right| < \Lambda_{1,\mathcal{P}_2} t^{2/3}, \qquad z \in \mathcal{P}_2.$$

[9] Basically, we have reduced the semicircle part of $\mathcal{L}_\varepsilon^{[t]}$ to avoid any uneded singularities.

To bound the exponential term in the integrand of $K_\varepsilon^{[t]}$ we use Lemma 3.8.11, obtaining

$$|\exp(t(h^{[t]}(z)-h^{[t]}(v))-crt^{1/3}(z-v))|<\Lambda_2\frac{|\exp(t(h(z)-h(\theta))-crt^{1/3}(z-\theta))|}{|\exp(t(h(v)-h(\theta))-crt^{1/3}(v-\theta))|}$$

for a constant Λ_2 depending only on ε. Since $\mathrm{Re}[v]<\theta$ and $\mathrm{Re}[h(v)-h(\theta)]\geq b$, for any t we have

$$\frac{|\exp(t(h(z)-h(\theta))-crt^{1/3}(z-\theta))|}{|\exp(t(h(v)-h(\theta))-crt^{1/3}(v-\theta))|}\leq e^{-bt}|\exp(t(h(z)-h(\theta))-crt^{1/3}(z-\theta))|.$$

To finish the upper-bounds we have to consider the contours $\mathcal{P}_1$ and $\mathcal{P}_2$ separately, starting with $\mathcal{P}_1$. Since $\mathcal{L}_\theta$ is a steep descent contour for $\mathrm{Re}[h(z)]$ by Proposition 3.8.6, for $z\in\theta+\mathbb{R}\mathbf{i}$ we have

$$\exp(t(h(z)-h(\theta))-rct^{1/3}(z-\theta))|\leq 1.$$

Then the whole integral is bounded as follows:

$$\left|\int_{\mathcal{P}_1}\frac{-\pi}{\sin(\pi(z-v))}\exp\left(t(h^{[t]}(z)-h^{[t]}(v))-cst^{1/3}(z-v)\right)\frac{dz}{z-v'}\right|$$
$$\leq\Lambda_{1,\mathcal{P}_1}\Lambda_2e^{-bt}\int_{y\in[-\varepsilon,-\delta t^{-\frac{1}{3}}]\cup[\delta t^{-\frac{1}{3}},\varepsilon]}\frac{dy}{y^2}\leq\Lambda_3t^{1/3}e^{-bt},$$

for some constant Λ_3 depending only on ε.

To upper-bound the part taken over $\mathcal{P}_2$ we use the Taylor expansion, finding a constant $\Xi>0$ such that

$$\left|h(z)-h(\theta)-\frac{c^3}{3}(z-\theta)^3\right|<\Xi|z-\theta|^4.$$

Plugging $z=\theta+\delta t^{-1/3}e^{i\phi}$ we get

$$\exp(t(h(z)-h(\theta))-t^{1/3}cs(z-\theta))|\leq\exp\left(\frac{c^3\delta^3}{3}\cos(3\phi)+\delta^4\Xi t^{-1/3}\cos(4\phi)-\delta rc\cos(\phi)\right)\leq\Lambda_4$$

where Λ_4 is a constant depending on ε. Gathering the bounds above and using that the length of $\mathcal{P}_2$ is equal to $\pi t^{-1/3}$ we get

$$\left|\int_{\mathcal{P}_2}\frac{-\pi}{\sin(\pi(z-v))}\exp\left(t(h^{[t]}(z)-h^{[t]}(v))-crt^{1/3}(z-v)\right)\frac{dz}{z-v'}\right|<\Lambda_{1,\mathcal{P}_2}\Lambda_4t^{2/3}e^{-bt}\,\pi t^{-1/3}<\Lambda_5t^{1/3}e^{-bt}.$$

The claim follows by combining the bounds for $\mathcal{P}_1$ and $\mathcal{P}_2$, . □

3.8.4 Proof of Theorem 3.8.4

Recall that, in view of Proposition 3.7.6, it is enough to prove

$$\lim_{t\to\infty}\det(I+K^{[t]})_{L^2(\mathcal{C}_\theta)}=F_{GUE}(r)$$

where we use the notation from the previous subsection.

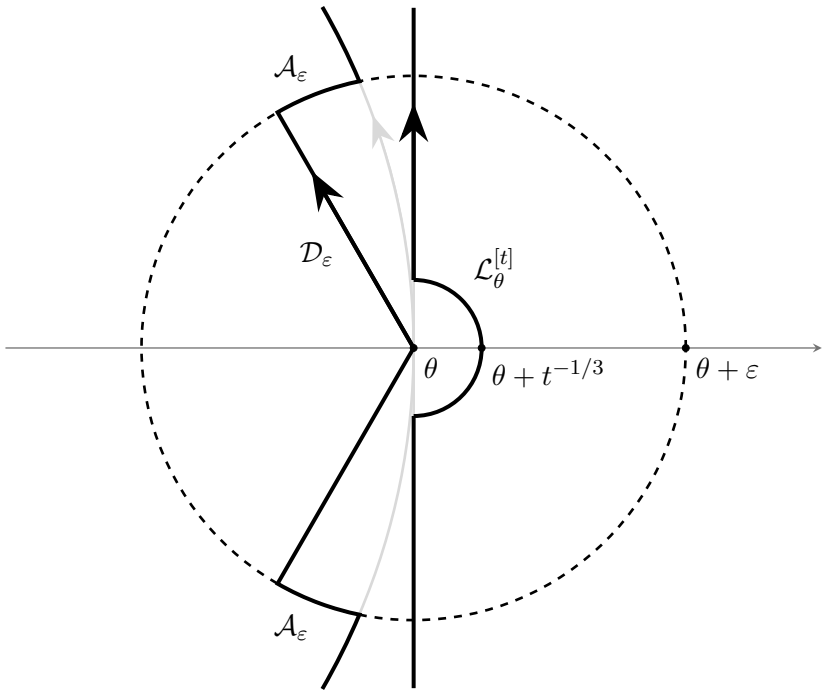


Figure 3-6: The contours $\mathcal{D}_\varepsilon, \mathcal{A}_\varepsilon, \mathcal{K}(\varepsilon)$ and $\mathcal{L}_\varepsilon^{[t]}$ in an ε-neighborhood of θ.

Step 1. First we show that the limit is dominated by the behavior in a neighborhood of θ. Namely, in this step we show that for sufficiently small fixed $\varepsilon > 0$ we have

$$\lim_{t\to\infty} |\det(I + K^{[t]})_{L^2(\mathcal{C}_\theta)} - \det(I + K_\varepsilon^{[t]})_{L^2(\mathcal{D}_\varepsilon)}| = 0,$$

where $\mathcal{D}_\varepsilon = \{\theta + Re^{\pm 2\pi\mathbf{i}/3} \mid R \in [0, \varepsilon]\}$.

It is more convenient to proceed in two steps, reducing first the contour $\mathcal{C}_\theta$ to $\mathcal{D}_\varepsilon$ and then the kernel $K^{[t]}$ to $K_\varepsilon^{[t]}$. Note that we can deform $\mathcal{C}_\theta$ to $\mathcal{K}(\varepsilon) = \mathcal{D}_\varepsilon \cup \mathcal{A}_\varepsilon \cup (\mathcal{C}_\theta \backslash \mathcal{C}_{\theta,\varepsilon})$ without changing the value of the determinant. Since $\mathcal{C}_\theta$ is a steep descent contour for $-\operatorname{Re}[h(z)]$, for each ε there exists a constant $b_1 > 0$ such that

$$\operatorname{Re}[h(v) - h(\theta)] \geq b_1, \qquad v \in \mathcal{C}_\theta \backslash \mathcal{C}_{\theta,\varepsilon}.$$

At the same time, by Proposition 3.8.8, for sufficiently small ε there exists a constant $b_2 > 0$ (depending on ε) such that

$$\operatorname{Re}[h(v) - h(\theta)] \geq b_2, \qquad v = \theta + \varepsilon e^{\pm\phi\mathbf{i}}, \phi \in [\psi(\varepsilon), 2\pi/3].$$

Thus, for sufficiently small fixed ε there exists $b > 0$ such that

$$\operatorname{Re}[h(v) - h(\theta)] \geq b, \qquad v \in \mathcal{A}_\varepsilon \cup (\mathcal{C}_\theta \backslash \mathcal{C}_{\theta,\varepsilon}) = \mathcal{K}(\varepsilon) \backslash \mathcal{D}_\varepsilon.$$

Hence we can apply Lemma 3.8.14, obtaining a constant Λ such that

$$K^{[t]}(v,v') < \Lambda_1 e^{-bt/2}, \qquad v \in \mathcal{K}(\varepsilon)\backslash\mathcal{D}_\varepsilon, v' \in \mathcal{K}(\varepsilon). \tag{3.8.4}$$

Note that the difference of Fredholm determinants $\det(I+K^{[t]})_{L^2(\mathcal{C}_\theta)} - \det(I+K^{[t]})_{L^2(\mathcal{D}_\varepsilon)}$ is equal to $\sum_{l\geq 0}\frac{B_l}{l!}$, where B_l are integrals of $\det[K(v_i,v_j)]_{i,j=1}^l$ taken over the difference $\mathcal{K}(\varepsilon)^l\backslash\mathcal{D}_\varepsilon^l$. Since each point in this difference has at least one of v_i not in $\mathcal{D}_\varepsilon$, by (3.8.4), Lemma 3.8.14 and Hadamard's bound we get

$$\left|\int\limits_{\mathcal{K}(\varepsilon)^l\backslash\mathcal{D}_\varepsilon^l}\cdots\int \det[K^{[t]}(v_i,v_j)]_{i,j=1}^l \frac{dv_1}{2\pi\mathbf{i}}\cdots\frac{dv_l}{2\pi\mathbf{i}}\right| \leq \Lambda_1\Lambda_2^l l^{l/2+1}t^{l/3}e^{-bt/2}.$$

Summing over l, we obtain

$$|\det(I+K^{[t]})_{L^2(\mathcal{C}_\theta)} - \det(I+K^{[t]})_{L^2(\mathcal{D}_\varepsilon)}| \leq \Lambda_1 e^{-bt/2}\sum_l \frac{l^{l/2+1}}{l!}\Lambda_2^l t^{l/3} < \Lambda_1 e^{-bt/2+2\Lambda_2^2 t^{2/3}} \to 0.$$

The argument replacing $K^{[t]}$ by $K_\varepsilon^{[t]}$ is identical and relies on Lemma 3.8.13. We have

$$\begin{aligned}&\left|\det(I+K^{[t]})_{L^2(\mathcal{D}_\varepsilon)} - \det(I+K_\varepsilon^{[t]})_{L^2(\mathcal{D}_\varepsilon)}\right| \\ &\qquad\qquad \leq \sum_{l\geq 0}\frac{1}{l!}\left|\int\limits_{\mathcal{D}_\varepsilon^l}\cdots\int \det[K(v_i,v_j)]_{i,j=1}^l - \det[K_\varepsilon(v_i,v_j)]_{i,j=1}^l \frac{dv_1}{2\pi\mathbf{i}}\cdots\frac{dv_l}{2\pi\mathbf{i}}\right|,\end{aligned}$$

and using multi-linearity of determinants and Lemmas 3.8.13,3.8.14 we have the following upper-bound for some constants $\Lambda_3, a > 0$

$$|\det[K^{[t]}(v_i,v_j)]_{i,j=1}^l - \det[K_\varepsilon^{[t]}(v_i,v_j)]_{i,j=1}^l| < \Lambda_3 e^{-at}{\Lambda_2}^l l^{l/2+1}t^{l/3},$$

which is enough to give an absolutely convergent upper-bound.

So, we have reduced Theorem 3.8.4 to proving

$$\lim_{t\to\infty}\det(I+K_\varepsilon^{[t]})_{L^2(\mathcal{D}_\varepsilon)} = F_{GUE}(r).$$

Step 2. After getting rid of the global parts of the integration contours, in this step we take the needed limit by scaling the neighborhood of θ by $t^{1/3}$.

For now, fix $\varepsilon > 0$ which is sufficiently small so that the argument in the previous step works, but otherwise arbitrary. We would like to perform the following change of coordinates in the Fredholm determinant $\det(I+K_\varepsilon^{[t]})_{L^2(\mathcal{D}_\varepsilon)}$ and the kernel $K_\varepsilon^{[t]}$:

$$z = \theta + t^{-1/3}\widetilde{z}, \qquad v = \theta + t^{-1/3}\widetilde{v}.$$

This leads to

$$\det(I+K_\varepsilon^{[t]})_{L^2(\mathcal{D}_\varepsilon)} = \det(I+\widetilde{K}_\varepsilon^{[t]})_{L^2(\widetilde{\mathcal{D}})},$$

with the contour $\widetilde{\mathcal{D}} = \{|R|e^{\mathrm{sign}(R)2\pi\mathbf{i}/3} \mid R \in [-\infty, \infty]\}$ and the kernel

$$\widetilde{K}_{\varepsilon}^{[t]}(\widetilde{v}, \widetilde{v}') = t^{-1/3}\mathbb{1}_{|\widetilde{v}|,|\widetilde{v}'|<\varepsilon t^{1/3}} K_{\varepsilon}^{[t]}(\theta + t^{-1/3}\widetilde{v}, \theta + t^{-1/3}\widetilde{v}').$$

To take the limit as $t \to \infty$ we start with the point-wise convergence of the kernel $\widetilde{K}_{\varepsilon}^{[t]}(\widetilde{v}, \widetilde{v}')$. Let $\widetilde{\mathcal{L}}_{\varepsilon}^{[t]}$ be the contour consisting of vertical lines $\{\pm Y\mathbf{i} \mid Y \in [1, \varepsilon t^{1/3}]\}$ and a semicircle $\{e^{\phi\mathbf{i}} \mid \phi \in [-\pi, \pi]\}$, oriented so that the imaginary part increases. Also, let $\widetilde{\mathcal{L}}$ denote the limiting contour with the vertical lines $\{\pm Y\mathbf{i} \mid Y \in [1, \infty]\}$ and the same semicircle. Then

$$\widetilde{K}_{\varepsilon}^{[t]}(\widetilde{v}, \widetilde{v}') = \frac{\mathbb{1}_{|\widetilde{v}|,|\widetilde{v}'|<\varepsilon t^{1/3}}}{2\pi\mathbf{i}} \int_{\widetilde{\mathcal{L}}_{\varepsilon}^{[t]}} \frac{-\pi t^{-1/3}}{\sin \pi t^{-1/3}(\widetilde{z} - \widetilde{v})} \frac{e^{t\left(h^{[t]}(\theta + t^{-1/3}\widetilde{z}) - h^{[t]}(\theta)\right) - cr\widetilde{z}}}{e^{t\left(h^{[t]}(\theta + t^{-1/3}\widetilde{v}) - h^{[t]}(\theta)\right) - cr\widetilde{v}}} \frac{d\widetilde{z}}{\widetilde{z} - \widetilde{v}'}.$$

Clearly

$$\frac{-\pi t^{-1/3}}{\sin \pi t^{-1/3}(\widetilde{z} - \widetilde{v})} \to \frac{1}{\widetilde{v} - \widetilde{z}},$$

$$\lim_{t\to\infty} t\left(h^{[t]}(\theta + t^{-1/3}\widetilde{z}) - h^{[t]}(\theta)\right) = \lim_{t\to\infty} t\left(h(\theta + t^{-1/3}\widetilde{z}) - h(\theta)\right) = \frac{c^3}{3}\widetilde{z}^3,$$

where for the second equality we have sequentially used Lemmas 3.8.9 and 3.8.10. So the integrand converges to

$$\frac{1}{2\pi\mathbf{i}} \frac{e^{c^3\widetilde{z}^3/3 - cs\widetilde{z}}}{e^{c^3\widetilde{v}^3/3 - cs\widetilde{v}}} \frac{d\widetilde{z}}{(\widetilde{v} - \widetilde{z})(\widetilde{z} - \widetilde{v}')}.$$

Now we need to give integrable upper-bounds. Note that $\mathrm{Re}\left[t\left(h^{[t]}(\theta + t^{-1/3}\widetilde{z}) - h^{[t]}(\theta)\right)\right]$ is unifromly bounded for $z \in \widetilde{\mathcal{L}}$: by Lemma 3.8.11 it is enough to consider $\mathrm{Re}\left[t\left(h(\theta + t^{-1/3}\widetilde{z}) - h(\theta)\right)\right]$ and for $|\widetilde{z}| > 1$ we can use the steep descent of $\mathcal{L}_\theta$, while for $\widetilde{z}$ on the semicircle this follows from the Taylor expansion of $h(z)$ from Lemma 3.8.9. So for some constant $\Lambda_1 > 0$ we have:

$$\left|\frac{\pi t^{-1/3}}{\sin \pi t^{-1/3}(\widetilde{z} - \widetilde{v})} \frac{e^{t\left(h^{[t]}(\theta + t^{-1/3}\widetilde{z}) - h^{[t]}(\theta)\right) - cs\widetilde{z}}}{\widetilde{z} - \widetilde{v}'}\right| < \frac{\Lambda_1}{|\widetilde{z} - \widetilde{v}|\,|\widetilde{z} - \widetilde{v}'|}, \qquad \widetilde{z} \in \widetilde{\mathcal{L}},\ \widetilde{v}, \widetilde{v}' \in \widetilde{\mathcal{D}},\ |\widetilde{v}| < \varepsilon t^{1/3} \tag{3.8.5}$$

Due to the quadratic decay we can apply dominated convergence, obtaining

$$\lim_{t\to\infty} \widetilde{K}_{\varepsilon}^{[t]}(\widetilde{v}, \widetilde{v}') = \widetilde{K}_{\infty}(\widetilde{v}, \widetilde{v}') := \frac{1}{2\pi\mathbf{i}} \int_{\widetilde{\mathcal{L}}} \frac{e^{c^3\widetilde{z}^3/3 - cr\widetilde{z}}}{e^{c^3\widetilde{v}^3/3 - cr\widetilde{v}}} \frac{d\widetilde{z}}{(\widetilde{v} - \widetilde{z})(\widetilde{z} - \widetilde{v}')}. \tag{3.8.6}$$

To finish we need to justify commutation of the Fredholm determinant and the limit above. To do it, we again use dominated convergence with an absolutely summable upper-bound on the Fredholm determinant. Note that for $\widetilde{v} \in \widetilde{\mathcal{D}}$ such that $|\widetilde{v}| < \varepsilon t^{1/3}$ we have

$$\mathrm{Re}[t(h^{[t]}(\theta + t^{-1/3}\widetilde{v}) - h^{[t]}(\theta)) - cr\widetilde{v}] > \mathrm{Re}[t(h(\theta + t^{-1/3}\widetilde{v}) - h(\theta))] - \ln \Lambda_2$$

for some $\Lambda_2 > 0$. By Taylor expansion, there exists an *independent of* ε constant Ξ such that

$$|t(h(\theta + t^{-1/3}\widetilde{v}) - h(\theta)) - c^3\widetilde{v}^3/3| < \Xi t^{-1/3}|\widetilde{v}|^4, \qquad |\widetilde{v}| < \theta t^{1/3}/2.$$

Then for any $\varepsilon < 1$ we get

$$\left| \frac{1}{e^{t(h^{[t]}(\theta+t^{-1/3}\widetilde{v})-h^{[t]}(\theta))-cs\widetilde{v}}} \right| < \Lambda_2 e^{-(\frac{c^3}{3}-\varepsilon\Xi)\widetilde{v}^3}, \qquad \widetilde{v} \in \widetilde{\mathcal{D}},\ |\widetilde{v}| < \varepsilon t^{1/3}.$$

Recall that for the whole argument ε was an arbitrary sufficiently small number, so we can reduce it even further to get $\frac{c^3}{3} - \varepsilon\Xi = b > 0$. Combining the bound above with (3.8.5) we get

$$|\widetilde{K}_{\varepsilon}^{[t]}(\widetilde{v}, \widetilde{v}')| < \Lambda_3 e^{-b\widetilde{v}^3}$$

for some $\Lambda_3 > 0$, which combined with Hadamard's bound is sufficient for an integrable and absolutely summable upper-bound on the kernel of the Fredholm determinant. Hence

$$\lim_{t\to\infty} \det(I + K_{\varepsilon}^{[t]})_{L^2(\mathcal{D}_\varepsilon)} = \det(I + \widetilde{K}_\infty)_{L^2(\widetilde{\mathcal{D}})},$$

with the kernel $\widetilde{K}_\infty$ defined in (3.8.6).

Step 3. To finish the proof we need to show that the Fredholm determinant in the right-hand side is indeed equal to F_{GUE}, which can be done by standard manipulations. Rescaling $\widetilde{v}, \widetilde{z}$ by c we can assume that

$$\widetilde{K}_\infty(\widetilde{v}, \widetilde{v}') = \frac{1}{2\pi\mathbf{i}} \int_{\widetilde{\mathcal{L}}} \frac{e^{\widetilde{z}^3/3 - r\widetilde{z}}}{e^{\widetilde{v}^3/3 - r\widetilde{v}}} \frac{d\widetilde{z}}{(\widetilde{v} - \widetilde{z})(\widetilde{z} - \widetilde{v}')}$$

Since the integrand has quadratic decay, we can deform the contour $\widetilde{\mathcal{L}}$ to a contour going from $\infty e^{-i\pi/3}$ to $\infty e^{i\pi/3}$ with $\mathrm{Re}[\widetilde{z}] > 0$. Now we can apply the same argument as in [14, Lemma 8.6]: For any $\widetilde{z}$ such that $\mathrm{Re}[\widetilde{z}] > 0$ we use

$$\frac{1}{\widetilde{z} - \widetilde{v}} = \int_{\mathbb{R}_{\geq 0}} e^{-\lambda(\widetilde{z} - \widetilde{v})} d\lambda$$

to write $\widetilde{K}_\infty = -AB$, where A, B are operators with kernels

$$A(v, \lambda) = e^{-\widetilde{v}^3/3 + \widetilde{v}(r+\lambda)}, \qquad B(\lambda, \widetilde{v}') = \frac{1}{2\pi\mathbf{i}} \int_{e^{-\pi\mathbf{i}/3}\infty}^{e^{\pi\mathbf{i}/3}\infty} e^{\widetilde{z}^3/3 - \widetilde{z}(r+\lambda)} \frac{d\widetilde{z}}{\widetilde{z} - \widetilde{v}'}.$$

Then

$$BA(\lambda, \mu) = \int_{e^{-2\pi\mathbf{i}/3}\infty}^{e^{2\pi\mathbf{i}/3}\infty} \frac{d\widetilde{v}}{2\pi\mathbf{i}} \int_{e^{-\pi\mathbf{i}/3}\infty}^{e^{\pi\mathbf{i}/3}\infty} \frac{d\widetilde{z}}{2\pi\mathbf{i}} \frac{1}{\widetilde{z} - \widetilde{v}} \frac{e^{\widetilde{z}^3/3 - \widetilde{z}(r+\lambda)}}{e^{\widetilde{v}^3/3 - \widetilde{v}(r+\mu)}} = K_{\mathrm{Ai}}(r + \lambda, r + \mu),$$

where $K_{\mathrm{Ai}}(\lambda, \mu)$ is the Airy kernel. Since AB and BA are trace-class operators we have $\det(I - AB) = \det(I - BA)$, *cf.* [79], so we finally get

$$\det(I + \widetilde{K}_\infty)_{L^2(\mathcal{D})} = \det(I - K_{\mathrm{Ai}})_{L^2[r,\infty)} = F_{GUE}(r).$$

□

3.9 Computations for Theorem 3.8.4

Here we provide computational arguments for several statements used in the proof of Theorem 3.8.4. Recall that for fixed real parameters $\{\sigma_1,\dots,\sigma_k\}$, $\{\rho_1,\dots,\rho_k\}$, $\{\omega_1,\dots,\omega_k\}$, $\{\alpha_1,\dots,\alpha_k\}$, $\{\beta_1,\dots,\beta_k\}$, $\{\gamma_1,\dots,\gamma_k\}$, θ satisfying

$$\sum_{i=1}^k \alpha_i = \sum_{i=1}^k \beta_i = \sum_{i=1}^k \gamma_i = 1, \qquad \alpha_i,\beta_i,\gamma_i \geq 0,$$

$$\omega_d < \rho_j < \sigma_i < \theta \qquad \text{for any } i,j,d,$$

we have defined

$$x = \sum_i \beta_i \Psi_1(\theta-\rho_i) - \sum_i \gamma_i \Psi_1(\theta-\omega_i), \qquad y = \sum_i \alpha_i \Psi_1(\theta-\sigma_i) - \sum_i \gamma_i \Psi_1(\theta-\omega_i),$$

$$I = x\sum_i \alpha_i \Psi(\theta-\sigma_i) - y\sum_i \beta_i \Psi(\theta-\rho_i) + (y-x)\sum_i \gamma_i \Psi(\theta-\omega_i),$$

$$h(z) = Iz - \sum_i x\alpha_i \ln\Gamma(z-\sigma_i) + \sum_i y\beta_i \ln\Gamma(z-\rho_i) - \sum_i (y-x)\gamma_i \ln\Gamma(z-\omega_i),$$

where

$$\Psi(z) = \frac{d}{dz}\ln\Gamma(z), \qquad \Psi_k(z) = \frac{d^{k+1}}{dz^{k+1}}\ln\Gamma(z).$$

For the computational purposes it is convenient to plug the expressions for x, y, I into $h(z)$ to obtain

$$\begin{aligned} h(z) = &-\left(\sum_i \beta_i \Psi_1(\theta-\rho_i) - \sum_i \gamma_i \Psi_1(\theta-\omega_i)\right)\sum_i \alpha_i \left(\ln\Gamma(z-\sigma_i) - z\Psi(\theta-\sigma_i)\right) \\ &+\left(\sum_i \alpha_i \Psi_1(\theta-\sigma_i) - \sum_i \gamma_i \Psi_1(\theta-\omega_i)\right)\sum_i \beta_i \left(\ln\Gamma(z-\rho_i) - z\Psi(\theta-\rho_i)\right) \\ &-\left(\sum_i \alpha_i \Psi_1(\theta-\sigma_i) - \sum_i \beta_i \Psi_1(\theta-\rho_i)\right)\sum_i \gamma_i \left(\ln\Gamma(z-\omega_i) - z\Psi(\theta-\omega_i)\right). \end{aligned}$$

The last expression can be rewritten as a convex sum of simpler functions, corresponding to $k=1$ case:

$$h(z) = \sum_{i=1}^k \sum_{j=1}^k \sum_{d=1}^k \alpha_i\beta_j\gamma_d h_{\sigma_i,\rho_j,\omega_d}(z), \tag{3.9.1}$$

$$\begin{aligned} h_{\sigma,\rho,\omega}(z) = &-\left(\Psi_1(\theta-\rho) - \Psi_1(\theta-\omega)\right)\left(\ln\Gamma(z-\sigma) - z\Psi(\theta-\sigma)\right) \\ &+\left(\Psi_1(\theta-\sigma) - \Psi_1(\theta-\omega)\right)\left(\ln\Gamma(z-\rho) - z\Psi(\theta-\rho)\right) \\ &-\left(\Psi_1(\theta-\sigma) - \Psi_1(\theta-\rho)\right)\left(\ln\Gamma(z-\omega) - z\Psi(\theta-\omega)\right). \end{aligned}$$

We will mostly work with the latter functions $h_{\sigma,\rho,\omega}(z)$.

In the following computations we frequently use the following series representations:

$$\Psi(z)-\Psi(\theta)=\sum_{n\geq 0}-\frac{1}{n+z}+\frac{1}{n+\theta}=\sum_{n\geq 0}\frac{z-\theta}{(n+\theta)(n+z)}, \tag{3.9.2}$$

$$\Psi_k(z)=\sum_{n\geq 0}\frac{(-1)^{k+1}k!}{(n+z)^{k+1}}. \tag{3.9.3}$$

In particular, the functions $(-1)^k\Psi_k(z)$ are increasing functions from $(0,\infty)$ to $(-\infty,0)$.

3.9.1 Derivatives of $h(z)$

This section is devoted to the signs of $h^{(d)}(\theta)$.

Lemma 3.9.1. *For any $d\geq 3$ we have $(-1)^{d+1}h^{(d)}(\theta)>0$.*

Proof. By (3.9.1) it is enough to prove that $(-1)^{d+1}h^{(d)}_{\sigma,\rho,\omega}(\theta)>0$ for $\omega<\rho<\sigma$. By a slight abuse of notation let h denote $h_{\sigma,\rho,\omega}$ during the proof.

Our approach follows [7, Lemma 5.3]. For $d\geq 3$ we have

$$\begin{aligned}h^{(d)}(\theta)=&\left(\Psi_1(\theta-\rho)-\Psi_1(\theta-\omega)\right)\left(\Psi_{d-1}(\theta-\rho)-\Psi_{d-1}(\theta-\sigma)\right)\\&+\left(\Psi_1(\theta-\sigma)-\Psi_1(\theta-\rho)\right)\left(\Psi_{d-1}(\theta-\rho)-\Psi_{d-1}(\theta-\omega)\right).\end{aligned}$$

Thus, we need to prove

$$\frac{(-1)^{d-1}\Psi_{d-1}(\theta-\rho)-(-1)^{d-1}\Psi_{d-1}(\theta-\omega)}{\Psi_1(\theta-\rho)-\Psi_1(\theta-\omega)}>\frac{(-1)^{d-1}\Psi_{d-1}(\theta-\sigma)-(-1)^{d-1}\Psi_{d-1}(\theta-\rho)}{\Psi_1(\theta-\sigma)-\Psi_1(\theta-\rho)}.$$

By the Cauchy's mean value theorem this inequality is equivalent to

$$\frac{(-1)^{d-1}\Psi_d(\zeta)}{\Psi_2(\zeta)}>\frac{(-1)^{d-1}\Psi_d(\xi)}{\Psi_2(\xi)}$$

for some $\xi\in(\theta-\sigma,\theta-\rho)$ and $\zeta\in(\theta-\rho,\theta-\omega)$. Since $\xi<\zeta$, it is enough to prove that the function $\frac{(-1)^{d-1}\Psi_d(z)}{\Psi_2(z)}$ is increasing on $\mathbb{R}_{>0}$. Taking derivative, it is enough to prove that

$$(-1)^{d-1}\left(\Psi_{d+1}(z)\Psi_2(z)-\Psi_d(z)\Psi_3(z)\right)>0,\qquad z>0.$$

Using series representation (3.9.3), the left hand side can be rewritten as

$$\begin{aligned}(-1)^{d-1}\left(\Psi_{d+1}(z)\Psi_2(z)-\Psi_d(z)\Psi_3(z)\right)&=\sum_{n,m\geq 0}\frac{2(d+1)!}{(n+z)^{d+2}(m+z)^3}-\frac{6\ d!}{(n+z)^{d+1}(m+z)^4}\\&=2d!\sum_{n,m\geq 0}\frac{(d+1)(m+z)^{d-1}-3(n+z)(m+z)^{d-2}}{(n+z)^{d+2}(m+z)^{d+2}}\geq 6d!\sum_{n,m\geq 0}\frac{(m-n)(m+z)^{d-2}}{(n+z)^{d+2}(m+z)^{d+2}},\end{aligned}$$

where we have used $d\geq 3$ to establish inequality. The proof is finished by symmetrizing the last

summation:

$$2\sum_{n,m\geq 0}\frac{(m-n)(m+z)^{d-2}}{(n+z)^{d+2}(m+z)^{d+2}}=\sum_{n,m\geq 0}\frac{(m-n)\left((m+z)^{d-2}-(n+z)^{d-2}\right)}{(n+z)^{d+2}(m+z)^{d+2}}>0,$$

where for the last inequality we have used that the function z^{d-2} is increasing for $d\geq 3$. □

3.9.2 Proof of Proposition 3.8.1

It turns out that the two parts of Proposition 3.8.1 are closely related. In the first part we want to prove that $x(\theta)/y(\theta)$ is increasing as a function of $\theta\in(\max_i\sigma_i,\infty)$, where

$$x=\sum_i\beta_i\Psi_1(\theta-\rho_i)-\sum_i\gamma_i\Psi_1(\theta-\omega_i),\qquad y=\sum_i\alpha_i\Psi_1(\theta-\sigma_i)-\sum_i\gamma_i\Psi_1(\theta-\omega_i).$$

Taking derivative, we need to prove that

$$x'(\theta)y(\theta)-x(\theta)y'(\theta)>0.$$

After algebraic manipulations the left-hand side of the inequality above is equal to

$$-x(\theta)\sum_i\alpha_i\Psi_2(\theta-\sigma_i)+y(\theta)\sum_i\beta_i\Psi_2(\theta-\rho_i)-(y(\theta)-x(\theta))\sum_i\gamma_i\Psi_2(\theta-\omega_i),$$

which is exactly how we have defined $2c(\theta)^3$. So, the monotonicity from the first part boils down to showing that $c(\theta)^3>0$, which would immediately imply that $c(\theta)$ can be chosen to be positive. But from the expression above it is clear that $\frac{c(\theta)^3}{3}=h'''(\theta)>0$, with the inequality following from Lemma 3.9.1.

The limits of $x(\theta)/y(\theta)$ as $\theta\to\max_i\sigma_i,\infty$ are readily obtained from the definitions and the limit behavior of the trigamma function:

$$\lim_{z\to 0}\Psi_1(z)=\infty,\qquad \Psi_1(z)=\frac{1}{z}+\frac{1}{2z^2}+O(z^{-3}).$$

□

3.9.3 Proof of Proposition 3.8.6

Here we give a slightly modified version of the proof of the corresponding fact in [7]. Recall that we want to show that $\mathrm{Re}[h(\theta+b\mathbf{i})]$ decreases for positive b and increases for negative b. Since $h(z)$ is a convex combination of $h_{\sigma,\rho,\omega}(z)$, it is enough to consider the latter functions. By an abuse of notation we again denote the function $h_{\sigma,\rho,\omega}(z)$ simply by $h(z)$. By symmetry, it is also enough to consider the case $\mathrm{Im}(z)>0$, so we want to prove that

$$\mathrm{Re}[\mathbf{i}h'(\theta+b\mathbf{i})]<0,\qquad b>0.$$

We have

$$\begin{aligned}\operatorname{Re}[\mathbf{i}h'(\theta+b\mathbf{i})] = -\operatorname{Im}[&(\Psi_1(\theta-\rho)-\Psi_1(\theta-\omega))(\Psi(\theta+b\mathbf{i}-\rho)-\Psi(\theta+b\mathbf{i}-\sigma))\\ &+(\Psi_1(\theta-\sigma)-\Psi_1(\theta-\rho))(\Psi(\theta+b\mathbf{i}-\rho)-\Psi(\theta+b\mathbf{i}-\omega))].\end{aligned}$$

From (3.9.2) it follows that for any real $A_1, A_2 > 0$

$$\operatorname{Im}[\Psi(A_1+b\mathbf{i})-\Psi(A_2+b\mathbf{i})] = \sum_{n\geq 0}\frac{b}{(n+A_1)^2+b^2}-\frac{b}{(n+A_2)^2+b^2}.$$

Thus, using that $b>0$ and setting

$$\Phi(A) = \sum_{n\geq 0}\frac{1}{(n+A)^2+b^2},$$

the claim is reduced to showing

$$-(\Psi_1(\theta-\rho)-\Psi_1(\theta-\omega))(\Phi(\theta-\rho)-\Phi(\theta-\sigma))-(\Psi_1(\theta-\sigma)-\Psi_1(\theta-\rho))(\Phi(\theta-\rho)-\Phi(\theta-\omega))<0.$$

This inequality can be rewritten in more convenient form

$$\frac{\Psi_1(\theta-\rho)-\Psi_1(\theta-\sigma)}{\Phi(\theta-\rho)-\Phi(\theta-\sigma)} > \frac{\Psi_1(\theta-\omega)-\Psi_1(\theta-\rho)}{\Phi(\theta-\omega)-\Phi(\theta-\rho)},$$

which by Cauchy's mean value theorem is equivalent to

$$\frac{\Psi_2(\xi)}{\Phi'(\xi)} > \frac{\Psi_2(\zeta)}{\Phi'(\zeta)}$$

for some $\xi\in(\theta-\sigma,\theta-\rho)$ and $\zeta\in(\theta-\rho,\theta-\omega)$. Since $0<\xi<\zeta$, it is enough to show that the function $\frac{\Psi_2(z)}{\Phi'(z)}$ is decreasing for $z>0$, and by taking derivative this is equivalent to

$$\Psi_2(z)\Phi''(z)-\Psi_3(z)\Phi'(z)>0, \qquad z>0.$$

Using the definition of $\Phi(x)$ and (3.9.3) we obtain

$$\begin{aligned}\Psi_2(z)\Phi''(z)-\Psi_3(z)\Phi'(z) &= \sum_{n,m\geq 0} -\frac{2}{(n+z)^3}\frac{6(z+m)^2-2b^2}{((m+z)^2+b^2)^3}+\frac{6}{(n+z)^4}\frac{2(m+z)}{((m+z)^2+b^2)^2}\\ &= 12\sum_{n,m\geq 0}\frac{(m+z)^3+b^2(m+z)-(n+z)(m+z)^2+\frac{b^2}{3}(n+z)}{(n+z)^4((m+z)^2+b^2)^3}.\end{aligned}$$

Clearly the terms in the numerator containing b^2 are positive, while for the remaining terms we have

$$\sum_{n,m\geq 0}\frac{(m+z)^3-(n+z)(m+z)^2}{(n+z)^4((m+z)^2+b^2)^3} = \frac{1}{2}\sum_{n,m\geq 0}\frac{(m-n)(m+z)^2}{(n+z)^4((m+z)^2+b^2)^3}+\frac{(n-m)(n+z)^2}{(m+z)^4((n+z)^2+b^2)^3}$$

To finish the proof it is enough to show that each summand in the last expression is nonnegative for any $n, m, z \geq 0$, which is true:

$$\frac{(m-n)(m+z)^2}{(n+z)^4((m+z)^2+b^2)^3} + \frac{(n-m)(n+z)^2}{(m+z)^4((n+z)^2+b^2)^3} \geq 0$$
$$\Leftrightarrow (m-n)\left[\left(\frac{(m+z)^2}{((m+z)^2+b^2)}\right)^3 - \left(\frac{(n+z)^2}{((n+z)^2+b^2)}\right)^3\right] \geq 0$$
$$\Leftrightarrow (m-n)\left[\frac{1}{\left(1+\frac{b^2}{(m+z)^2}\right)^3} - \frac{1}{\left(1+\frac{b^2}{(n+z)^2}\right)^3}\right] \geq 0$$

The last inequality holds since $\left(1+\frac{b^2}{(z+m)^2}\right)^{-3}$ is increasing in m. □

3.9.4 Proof of Proposition 3.8.7

From now on we are working under Assumption 3.8.3, so $\theta \in \left(0, \frac{1}{2}\right)$ and the function $h(z)$ is a convex combination of the functions $h_{0,-1,\omega}(z)$ for $\omega < -1$. The latter functions are simpler, and for further convenience we provide an explicit expression for them:

$$h_{0,-1,\omega}(z)$$
$$= (\Psi_1(\theta+1) - \Psi_1(\theta-\omega))\left(\ln(z) - \frac{z}{\theta}\right) - \frac{1}{\theta^2}\left(\ln\Gamma(z-\omega) - \ln\Gamma(z+1) - z\Psi(\theta-\omega) + z\Psi(\theta+1)\right).$$

Recall that we want to show that $\mathrm{Re}[h(\theta e^{\phi\mathbf{i}})]$ is increasing for $\phi > 0$ and decreasing for $\phi < 0$. Due to assumptions it is enough to consider the functions $h_{0,-1,\omega}(z)$, so by an abuse of notation we let h denote a function $h_{0,-1,\omega}(z)$ for the duration of the proof. We consider only the case $\phi > 0$, the other case follows by symmetry.

We need to prove that

$$\mathrm{Re}[\mathbf{i}\theta e^{\phi\mathbf{i}} h'(\theta e^{\phi\mathbf{i}})] > 0, \qquad \phi \in (0, \pi).$$

Computing the derivative and applying (3.9.2),(3.9.3) we obtain

$$h'(z) = (\Psi_1(\theta+1) - \Psi_1(\theta-\omega))\left(\frac{1}{z} - \frac{1}{\theta}\right) - \frac{1}{\theta^2}(\Psi(z-\omega) - \Psi(z+1) - \Psi(\theta-\omega) + \Psi(\theta+1))$$
$$= \sum_{n\geq 0}\left[\frac{\theta - z}{z\theta}\left(\frac{1}{(n+\theta+1)^2} - \frac{1}{(n+\theta-\omega)^2}\right) - \frac{1}{\theta^2}\left(\frac{z-\theta}{(n+\theta-\omega)(n+z-\omega)} - \frac{z-\theta}{(n+\theta+1)(n+z+1)}\right)\right]$$
$$= \frac{(z-\theta)^2}{z\theta^2}\sum_{n\geq 0}\left[\frac{n+1}{(n+\theta+1)^2(n+z+1)} - \frac{n-\omega}{(n+\theta-\omega)^2(n+z-\omega)}\right].$$

Note that for any real τ

$$\mathrm{Re}\left[\mathbf{i}\theta e^{\phi\mathbf{i}}\frac{(e^{\phi\mathbf{i}}-1)^2}{\theta e^{\phi\mathbf{i}}}\frac{1}{\theta e^{\phi\mathbf{i}}+\tau}\right] = \left|1-e^{\phi\mathbf{i}}\right|^2 \mathrm{Re}\left[-\mathbf{i}\frac{e^{\phi\mathbf{i}}}{\theta e^{\phi\mathbf{i}}+\tau}\right] = \left|1-e^{\phi\mathbf{i}}\right|^2 \frac{\tau\sin\phi}{\tau^2+\theta^2+2\theta\tau\cos\phi},$$

so plugging $z = \theta e^{\phi \mathbf{i}}$ in the expression for $h'(z)$ and factoring out positive terms we need to prove that

$$\sum_{n\geq 0}\left[\frac{(n+1)^2}{((n+1)^2+\theta^2+2\theta(n+1)\cos\phi)(n+1+\theta)^2} - \frac{(n-\omega)^2}{((n-\omega)^2+\theta^2+2\theta(n-\omega)\cos\phi)(n-\omega+\theta)^2}\right] > 0.$$

It is enough to show that each summand above is positive, which can be rewritten as

$$\frac{X_1^2}{(X_1^2+\theta^2+2\theta X_1\cos\phi)(X_1+\theta)^2} > \frac{X_2^2}{(X_2^2+\theta^2+2\theta X_2\cos\phi)(X_2+\theta)^2}$$

for $X_1 = n+1$ and $X_2 = n-\omega$. Since $\omega < -1$ and both sides of the last inequality are clearly positive, we can equivalently show that

$$f(X) = (X^2+\theta^2+2\theta X\cos\phi)\left(1+\frac{\theta}{X}\right)^2$$

is increasing in $X \geq 1$ for any fixed ϕ. Taking derivative and using $\theta \in \left(0, \frac{1}{2}\right)$ we get

$$\begin{aligned} f'(X) = 2(X+\theta\cos\phi)\left(1+\frac{\theta}{X}\right)^2 - \frac{2\theta}{X^2}(X^2+\theta^2+2\theta X\cos\phi)\left(1+\frac{\theta}{X}\right) \\ = 2\left(1-\frac{\theta^2}{X^2}\right)\left(X+\theta(1+\cos\phi)+\frac{\theta^2}{X}\right) > 0. \end{aligned}$$

□

3.9.5 Proof of Proposition 3.8.8

For convenience, we repeat the statement:

Proposition. *Suppose that Assumption 3.8.3 is satisfied. Let $\psi(\varepsilon) = \arg(v_\varepsilon - \theta)$, where v_ε is the intersection point of $\mathcal{C}_\theta$ with the circle of radius $\varepsilon > 0$ around θ satisfying $\operatorname{Im}[v_\varepsilon] > 0$. Then, for sufficiently small fixed $\varepsilon > 0$ there exists a depending on ε constant $b > 0$ such that*

$$\operatorname{Re}\left[h(\theta+\varepsilon e^{\pm\phi\mathbf{i}}) - h(\theta)\right] > b, \qquad \phi \in [\psi(\varepsilon), 2\pi/3].$$

Proof. Recall that $\theta \in \left(0, \frac{1}{2}\right)$. Due to the symmetry, it is enough to consider $\operatorname{Re}[h(\theta+\varepsilon e^{\phi\mathbf{i}}) - h(\theta)]$. Since h is a convex sum of finitely many $h_{0,-1,\omega}$, it is enough to consider $h(z) \equiv h_{0,-1,\omega}(z)$.

We consider only $\varepsilon < \theta/2$. By Taylor expansion there exists a constant $\Xi > 0$ such that

$$\left|h(\theta+\varepsilon e^{\phi\mathbf{i}}) - h(\theta) - \frac{h'''(\theta)}{6}\varepsilon^3 e^{3\phi\mathbf{i}} - \frac{h^{(4)}(\theta)}{24}\varepsilon^4 e^{4\phi\mathbf{i}}\right| < \Xi\varepsilon^5$$

Taking real part, we get

$$\operatorname{Re}\left[h(\theta+\varepsilon e^{\phi\mathbf{i}}) - h(\theta)\right] \geq \frac{h'''(\theta)}{6}\varepsilon^3\sin(3(\phi-\pi/2)) + \frac{h^{(4)}(\theta)}{24}\varepsilon^4\cos(4(\phi-\pi/2)) - \Xi\varepsilon^5$$

Since $h'''(\theta) > 0$ by Lemma 3.9.1, for any $\phi \in [5\pi/8, 2\pi/3]$ we clearly have

$$\mathrm{Re}\left[h(\theta + \varepsilon e^{\phi \mathbf{i}}) - h(\theta)\right] \geq \frac{h'''(\theta)}{6}\varepsilon^3 \sin(3\pi/8) - \Lambda_1 \varepsilon^4 > \Lambda_2 \varepsilon^3$$

for some constants $\Lambda_1, \Lambda_2 > 0$. So it is enough to consider the claim for $\phi \in [\psi(\varepsilon), 5\pi/8]$. Moreover, note that for $\phi \in [\pi/2, 5\pi/8]$ and fixed ε the function

$$\frac{h'''(\theta)}{6}\varepsilon^3 \sin(3(\phi - \pi/2)) + \frac{h^{(4)}(\theta)}{24}\varepsilon^4 \cos(4(\phi - \pi/2)) - \Xi\varepsilon^5$$

is increasing, where we have again used Lemma 3.9.1 to get $h^{(4)}(\theta) < 0$. Hence the claim reduces to showing that the function above is positive at $\phi = \psi(\varepsilon)$.

Recall that $\psi(\varepsilon)$ was defined using v_ε, which can be explicitly described:

$$v_\varepsilon = \frac{2\theta^2 - \varepsilon^2}{2\theta} + \varepsilon\sqrt{1 - \frac{\varepsilon^2}{4\theta^2}}\mathbf{i}.$$

On the other hand, $v_\varepsilon = \theta + \varepsilon e^{\psi(\varepsilon)\mathbf{i}}$, so we obtain

$$\frac{v_\varepsilon - \theta}{\varepsilon} = -\frac{\varepsilon}{2\theta} + \sqrt{1 - \frac{\varepsilon^2}{4\theta^2}}\mathbf{i} = \cos(\psi(\varepsilon)) + \sin(\psi(\varepsilon))\mathbf{i}.$$

This implies that $\psi(\varepsilon) = \frac{\pi}{2} + \frac{\varepsilon}{2\theta} + O(\varepsilon^2)$. Plugging it into the studied function we get

$$\frac{h'''(\theta)}{6}\varepsilon^3 \sin(3(\psi(\varepsilon) - \pi/2)) + \frac{h^{(4)}(\theta)}{24}\varepsilon^4 \cos(4(\psi(\varepsilon) - \pi/2)) - \Xi\varepsilon^5 > \varepsilon^4\left(\frac{h'''(\theta)}{4\theta} + \frac{h^{(4)}(\theta)}{24}\right) - \Lambda_3\varepsilon^5$$

for some constant $\Lambda_3 > 0$. So the claim follows as long as

$$\frac{h'''(\theta)}{\theta} + \frac{h^{(4)}(\theta)}{6} > 0.$$

We prove the last inequality by direct computation. Recall that

$$h(z) = (\Psi_1(\theta+1) - \Psi_1(\theta-\omega))\left(\ln(z) - \frac{z}{\theta}\right) - \frac{1}{\theta^2}(\ln\Gamma(z-\omega) - \ln\Gamma(z+1) - z\Psi(\theta-\omega) + z\Psi(\theta+1)),$$

where $\omega < -1$. The derivatives are given by

$$h'''(\theta) = \frac{2}{\theta^3}(\Psi_1(\theta+1) - \Psi_1(\theta-\omega)) - \frac{1}{\theta^2}(\Psi_2(\theta-\omega) - \Psi_2(\theta+1)),$$

$$h^{(4)}(\theta) = -\frac{6}{\theta^4}(\Psi_1(\theta+1) - \Psi_1(\theta-\omega)) - \frac{1}{\theta^2}(\Psi_3(\theta-\omega) - \Psi_3(\theta+1)).$$

Rearranging terms we obtain

$$\frac{h'''(\theta)}{\theta} + \frac{h^{(4)}(\theta)}{6} = f(-\omega) - f(1)$$

where

$$f(\tau) = -\frac{1}{\theta^4}\Psi_1(\theta+\tau) - \frac{1}{\theta^3}\Psi_2(\theta+\tau) - \frac{1}{6\theta^2}\Psi_3(\theta+\tau).$$

So, it is enough to prove that $f(\tau)$ is increasing in $\tau \geq 1$. Using series representations (3.9.3) write

$$f(\tau) = \frac{1}{\theta^4}\sum_{n\geq 0} -\frac{1}{(n+\theta+\tau)^2} + \frac{2\theta}{(n+\theta+\tau)^3} - \frac{\theta^2}{(n+\theta+\tau)^4} = \sum_{n\geq 0}\frac{-(n+\tau)^2}{\theta^4(n+\theta+\tau)^4}$$

The claim follows since $-\frac{X^2}{(X+\theta)^4}$ is increasing in $X \geq 1$:

$$\left(-\frac{X^2}{(X+\theta)^4}\right)' = \frac{-2X(X+\theta)+4X^2}{(X+\theta)^5} = \frac{2X(X-\theta)}{(X+\theta)^5} > 0.$$

□

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