SPHERICAL FUNCTIONS ON GL_n OVER p-ADIC FIELDS

by

Eugene M. Luks

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Signature of Author...Signature redacted......
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Certified by.......Signature redacted...........
Thesis Supervisor

Accepted by....Signature redacted...............
Chairman, Departmental Committee
on Graduate Students

Abstract

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Let k be a p-adic number field, $\mathcal{O}$ its valuation ring, $G = GL(n,k)$, $U = GL(n,\mathcal{O})$. A complex-valued function, constant on the double cosets UgU, $g \in G$, is called a spherical function (s. f.). It is known that the algebra of s. f.'s with compact support can be mapped isomorphically, by a Fourier transform, onto the algebra of symmetric Fourier polynomials in $\{q^{\pm z_j}\}_{1 \leq j \leq n}$ ($q = \#$(residue class field of k)). This paper determines (I) the explicit form of the inverse transformation. This leads to the other main results: (II) the Plancherel measure is computed; (III) the zonal s. f.'s (which identify the maximal ideals of $L(G,U)$) are also given explicitly, they are symmetric rational functions in $\{q^{z_j}\}$; (IV) the bounded zonal s. f.'s are then determined (these correspond to the maximal ideals of the algebra of integrable spherical functions).

Thesis supervisor: Prof. K. Iwasawa, Professor of Mathematics

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Introduction

In the past few years, the theory of spherical functions (i.e. functions constant on the double cosets modulo a maximal compact subgroup) has been extended to some p-adic algebraic groups. Initially, Mautner [4] considered the case of PGL_2. By the method of Fourier transformation, he determined, in particular, (I) the structure of the algebra of spherical functions with compact support, (II) the precise form of the zonal spherical functions, and (III) the Plancherel measure. (I) has since been studied for the case of GL_n by Tamagawa [7], for some other classical groups by Bruhat [2], and for a wider class of reductive algebraic groups by Satake [5]. The authors also investigated the set of zonal spherical functions, showing that it may be identified with a quotient space of the form $(C^*)^n/W$, where W corresponds to the Weyl group of the algebraic group. However, (II) and (III) are still unknown in the general case. In this paper, we determine these for the case GL_n. It is expected that similar calculations can be made for other classical groups.

To be specific, let k be a p-adic number field (that is, a finite extention of Q_p) , $\mathfrak{O}$ its valuation ring, $G = GL(n,k)$, $U = GL(n,\mathfrak{O})$.

Definition. A complex-valued function, f, on G, is called a spherical function if it satisfies

$$(*) \qquad f(ugu_1) = f(g) \qquad \text{for all } g \in G;\ u,u_1 \in U.$$

We shall consider the following sets

1. $L(G,U)$ the algebra of spherical functions with compact support (multiplication by convolution).

2. $L_p(G,U)$, $p = 1,2$. the spaces of spherical functions with Lebesgue integrable (with respect to Haar-measure on G) p-th powers.

3. The set of zonal spherical functions (z. s. f.) i.e. the spherical functions, ω, such that the mapping

$$f \rightarrow \int_G f(g)\,\omega(g^{-1})dg$$

is a homomorphism of $L(G,U)$ onto C. (dg denoting the Haar-measure on G).

It is known (see [5], Chapter III, or section 4, this paper), that $L(G,U)$ is mapped isomorphically by a Fourier transformation to the algebra of functions on C^n which are given by the symmetric polynomials in $q^{\pm z_1},\ldots,q^{\pm z_n}$ (q = the number of elements in the residue class field of k). In section 5, we determine the inverse transformation explicitly. This yields easily the remaining results:

We see that the transformation may be extended to a unitary mapping onto a Hilbert space of functions on a region in R^n and the dual measure, $M_n(y_1,\ldots,y_n)dy_1,\ldots,dy_n$, is given explicitly (Theorem 1 of section 6). It is interesting that the function $M_n(y_1,\ldots,y_n)$, $n > 2$, is, up to a constant multiple, a product of the functions $M_2(y_i,y_j)$, $i < j$.

This result is analogous to that obtained for complex classical groups (see [3]).

In section 7, we discuss the z. s. f., which are also given explicitly (Theorem 2). As expected they correspond to points $(z_1,\ldots,z_n) \in C^n$; they are, indeed, symmetric rational functions in $q^{z_1},\ldots,q^{z_n}$.

As an application of the last result, we determine in section 8 the set of bounded z. s. f. (which corresponds to the set of maximal ideas of the algebra $L_1(G,U)$). The result is analogous to the complex case.

To obtain information about the Fourier transformation, $f \longrightarrow \hat{f}$, we first discuss, as usual, an intermediate mapping $f \longrightarrow \tilde{f}$, $f \in L(G,U)$. (see equation 3.3)). $\tilde{f}$ is a function on the group of diagonal matrices in G. $\tilde{f}$ will be identified with a function on Z^n by letting

$$\tilde{f}(m_1,\ldots,m_n) = \tilde{f}(\mathrm{diag}(\pi^{m_1},\ldots,\pi^{m_n}))$$

where π is a fixed prime element of $\mathcal{O}$. With this convention the mapping $f \longrightarrow \tilde{f}$ is a bijection of L(G,U) to the set of symmetric functions on Z^n which vanish except on a finite set. (Proposition 3.2). Now to know precisely the inverse of $f \longrightarrow \hat{f}$, it is necessary to know the inverse of $f \rightarrow \tilde{f}$. Most of section 4 is devoted to determining the latter. The result, Proposition 3.1, may be considered the key result of this paper.

Section 2 is devoted to the introduction of a certain class of operators on the complex-valued functions on Z^n. By elementary divisors and the same identification as above, f, too, is regarded as a function on Z^n. It is, then, by means of these operators that we are able to express compactly the aforementioned inverse mapping. They also simplify the statements and proofs leading to Proposition 1.

As one further remark, we note that some of the results herein (viz. Lemma 3.1, Proposition 3.2, and Theorem 0) are not really new (see, for example, [5]). The author thus feels some justification should be made for the inclusion of their proofs. In the case of Lemma 3.1, it is felt that the argument helps clarify ideas to be used later in section 4. The proof of Proposition 3.2 (i.e. the part remaining to be proved at that point) requires, essentially, facts from section 2. It seems an interesting reapplication of the latter which were established to prove Proposition 3.1. Finally, the statement of Theorem 0 is necessary for that which follows and the only non-trivial point in the proof is Proposition 3.2.

1. Notations and Preliminaries

k will denote a fixed p-adic number field, $\mathcal{O}$ its valuation ring, $P = (\pi)$ the unique maximal ideal of $\mathcal{O}$ (with generator π). The symbol $|\cdot|$ will be used for the normalized valuation of k. If

$$q = \text{the number of elements in } \mathcal{O}/P < \infty$$

one has

$$|x| = q^{-\mathrm{ord}(x)},$$

where ord(x) is the P-order of x. In particular,

$$|\pi| = q^{-1}.$$

Z will denote the set of rational integers, R the real field, and C the complex field. If $(m) = (m_1,\ldots,m_n) \in Z^n$, we let

$$|(m)| = m_1 + \cdots + m_n.$$

(There will be no confusion with $|x|$ above). Two subsets of Z^n will be of interest:

$$V_n = \{(m) \in Z^n \mid m_1 \geq m_2 \geq \cdots \geq m_n\}$$

$$W_n = \{(m) \in Z^n \mid m_i = 0 \text{ or } 1,\ 1 \leq i \leq n\}.$$

If $(t) = (t_1,\ldots,t_n)$ is an n-tuple (in Z^n, R^n, C^n, or k^n) and $\sigma \in S_n$ the group of permutations of $\{1,\ldots,n\}$, we let

$$(t_\sigma) = (t_{\sigma(1)},\ldots,t_{\sigma(n)}).$$

$G_n = GL(n, k)$, $n \geq 2$, will denote the group of non-singular n×n matrices in k. It is a locally compact group with respect to the p-adic topology. We are concerned with the following subgroups of G_n

1. $U_n = GL(n,\mathcal{O})$, the matrices of G_n with coefficients in $\mathcal{O}$, determinant $\notin P$.

2. A_n the diagonal matrices of G_n.

3. B_n the lower unipotent matrices (i.e. the matrices $b = (x_{ij})$ where $x_{ii} = 1$, $x_{ij} = 0$ for $i<j$). When there is no ambiguity, we write simply G, U, A, B. Note U is an open, compact subgroup of G.

AB = BA is the subgroup of non-singular lower triangular matrices. Then we have easily

(1.1) $\quad G = ABU$

The symbol diag $(a_1,\dots,a_n)$ will represent the diagonal matrix $\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$.

Then, if $(m) = (m_1,\dots,m_n) \in Z^n$, we let $\pi^{(m)} = \operatorname{diag}(\pi^{m_1},\dots,\pi^{m_n})$. by means of elementary divisions, we see, if $g \in G$

$$g = u_1 \pi^{(m)} u_2, \qquad u_1, u_2 \in U$$

where $(m) \in V_n$ is determined uniquely.

Then

(1.2) $$G = \bigcup_{(m)\in V_n} U\pi^{(m)}U, \qquad \text{(disjoint union)}.$$

If f is a spherical function (see introduction), it is constant on the double coset $U\pi^{(m)}U$, $(m) \in Z^n$. f may then be identified with a function on Z^n, by $f((m)) = f(\pi^{(m)})$, and we have at once $f((m)) = f((m_\sigma))$, $\sigma \in S_n$. If, moreover, $f \in L(G,U)$, it is non zero on only a finite number of the double cosets. We conclude that $f((m))$ is a symmetric function of finite support (i.e. it vanishes off a finite set).

<u>Haar measures</u>

Denoting by μ, respectively μ^*, Haar measure on the additive, respectively multiplicative, group of k. Then

$$d\mu(x+c) = d\mu(x)$$
$$d\mu(cx) = |c|\, d\mu(x)$$
$$d\mu^*(cx) = d\mu^*(x).$$

These measures may be normalized so that

(1.3) $$\int_{\mathcal{O}} d\mu(x) = 1$$

(1.4) $$\int_{\mathcal{O}-P} d\mu^*(x) = 1$$

It follows from (1.3) that

(1.5) $$\int_{\operatorname{ord} x \geq r} d\mu(x) = q^{-r} \qquad r \in Z.$$

It is well known that G is a unimodular topological group. ([6], page 389). We normalize its Haar-measure, written dg, so that

(1.6) $$\int_U dg = 1.$$

Now A is topologically isomorphic to $(k^*)^n$, hence a two-sided invariant Haar measure, da, is given by

(1.7) $$da = d\mu^*(a_1)\dots d\mu^*(a_n)$$ where the elements of A are written $a = \operatorname{diag}(a_1,\dots,a_n)$. By (1.4)

(1.8) $$\int_{U\cap A} da = 1.$$

Finally, if $b \in B$ is of the form

$$b = \begin{bmatrix} 1 & & & 0 \\ x_{21} & 1 & & \\ \vdots & & \ddots & \\ x_{n1} & x_{n2} & \cdots & 1 \end{bmatrix}$$

then

(1.9) $\quad db = \prod_{n \geq i > j \geq 1} d\mu(x_{ij})$

is easily seen to be a two-sided invariant measure on B. Also, by (1.3)

(1.10) $\quad \int_{B \cap U} db = 1$

Note that A normalizes B and one has

(1.11) $\quad d(a^{-1}ba) = \gamma(a)db$

where γ is a continuous homomorphism of A to the multiplicative group of the positive reals. One calculates easily, for $(m) \in Z^n$

(1.12) $\quad \gamma(\pi^{(m)}) = q^{\sum_i (2i-n-1)m_i}$.

Now $d_1(ab) = dadb$ is a left invariant Haar measure on AB and by (1.1), if ψ is an integrable function on G,

$$\int_G \psi(g)dg = \int_{AB} \int_U \psi(abu)du d_1(ab) .$$

Then, formally,

(1.13) $\quad dg = d_1(ab)du = dadbdu$.

Recall, if ψ and ψ_1 are measurable functions on G, the convolution $\psi * \psi_1$ is defined by

(1.14) $\quad \psi * \psi_1(g_o) = \int_G \psi(g_o g)\psi_1(g^{-1})dg$

when the integral is defined for almost all $g_o \in G$.

Note, in particular, that if ψ, ψ_1 are in L(G,U) (respectively $L_1(G,U)$) then $\psi * \psi_1$ is in $L(G,U)$ (respectively $L_1(G,U)$. Thus $L(G,U)$, respectively $L_1(G,U)$, is an associative algebra with identity equal to the characteristic function of U. (Using 1.6 and 1.14)

As usual, we define, for any function, ψ, on G

$$(1.15) \qquad \psi^*(g) = \overline{\psi(g^{-1})}.$$

The bar $^-$ denotes complex conjugation. Again $\psi \in L(G,U)$ (resp. $L_1(G,U)$), implies $\psi^* \in L(G,U)$(resp. $L_1(G,U)$).

2. The operators λ_i, $D_{ij}(r)$, T_{ij}.

Let F be the set of complex-valued functions on Z^n. To simplify several formulae and calculations in section 3, it is useful to introduce a class of operators on F.

We define first the 2n operators $\lambda_1^{\pm 1},\ldots,\lambda_n^{\pm 1}$ on F by letting

(2.1) $\quad (\lambda_i^{\pm 1}h)((m)) = h(\ldots,m_{i-1},m_i \pm 1,m_{i+1},\ldots)$

for $h \in F$, $(m) \in Z^n$. With products defined by composition, linear combinations as usual, and including I (identity operator) we have a commutative algebra, $C[\lambda_1^{\pm 1},\ldots,\lambda_n^{\pm 1}]$, of operators on F isomorphic to $C[X_1^{\pm 1},\ldots,X_n^{\pm 1}]$.

Of particular interest among these operators is the set $\{D_{ij}(r)\}_{1\le i,j\le n}$, $r \in Z$, given by

(2.2a) $\quad D_{ij}(r) = I - q^r\lambda_i\lambda_j^{-1}.$

And we let

(2.2b) $\quad D_{ij} = D_{ij}(0) = I - \lambda_i\lambda_j^{-1}.$

By direct application of (2.2) one has, for $(s) \in Z^n$,

(2.3) $\quad D_{ij}(r)[q^{(s)\cdot(m)}h((m))] = q^{(s)\cdot(m)}(D_{ij}(r+s_i-s_j))h((m))$

where the bracketed expression is considered as a single function on Z^n. $(s)\cdot(m)$ denotes ordinary dot product, i.e. $\sum_\nu s_\nu m_\nu$. Also, for $(r) \in Z^{n-1}$,

(2.4) $\quad \prod_{i=1}^{n-1} D_{ni}(r_i) = \sum_{(w)\in W_{n-1}} (-1)^{|(w)|} q^{(w)\cdot(r)} \lambda_1^{w_1}\cdots\lambda_{n-1}^{w_{n-1}}\lambda_n^{-|(w)|}$.

Now let F_0 be the collection of members of F which are bounded on each set $Z_r^n = \{(m) \in Z^n \mid |(m)| = r\}$, $r \in Z$. If $h \in F_0$, then $P(\lambda_1,\ldots,\lambda_n)h \in F_0$ for $P(\lambda_1,\ldots,\lambda_n) \in C[\lambda_1^{\pm 1},\ldots,\lambda_n^{\pm 1}]$. $(\lambda_1^{s_1}\cdots\lambda_n^{s_n}h((m)) = h((m'))$, where $(m) \in Z_r^n$ if and only if $(m') \in Z^n_{r+|(s)|}$). F_0 clearly contains the functions with finite support.

We define on F_0, the additional operators $\{T_{ij}\}_{1\le i,j\le n}$ by

(2.5) $\quad T_{ij}h((m)) = \sum_{\nu=0}^{\infty} q^{-\nu}\lambda_i^{\nu}\lambda_j^{-\nu}h((m)),$

for $(m) \in Z^n$, $h \in F_0$.

Since h is bounded on $Z^n_{\langle m\rangle}$ the sum is absolutely convergent. also, if $\sup_{(m)\in Z^n_r}|h((m))| \le M$, then $\sup_{(m)\in Z^n_r}|T_{ij}h((m))| \le M(1+q^{-1})^{-1}$, hence $T_{ij}h \in F_0$. With products again defined by composition, the operators T_{ij} commute with one another (this corresponds to reversing the order of summation) and with the operators $P(\lambda_1,\dots,\lambda_n) \in C[\lambda_1^{\pm1},\dots,\lambda_n^{\pm1}]$. For $h \in F_0$,

$$(I-q^{-1}\lambda_i\lambda_j^{-1})\circ T_{ij}(h)((m)) = \sum_{\nu=0}^{\infty} q^{-\nu}\lambda_i^{\nu}\lambda_j^{-\nu}(h)((m)) - \sum_{\nu=0}^{\infty} q^{-\nu-1}\lambda_i^{\nu+1}\lambda^{-\nu-1}(h)((m))$$

$$= h((m))$$

Thus

$$(2.6)\qquad D_{ij}(-1)\circ T_{ij} = I.$$

Observe also, if $h \in F_0$ is a function such that $h((m)) = 0$ for $m_1 > a$, respectively for $m_n < b$, then if $1\le i<j\le n$, $\lambda_i\lambda_j^{-1}h$ has the respective property, hence so do $D_{ij}(r)h$ and $T_{ij}h$.

We prove here three easy Lemmas involving these operators, which are needed for both Propositions 3.1 and 3.2.

For the first two Lemmas, let $h \in F_0$ be a symmetric function (i.e. $h((m_\sigma)) = h((m))$, $\sigma \in S_n$ and let h_1 be given by

$$(*)\quad h_1((m)) = q^{\sum_i (i-1)m_i} \prod_{n\ge s>r\ge 1} D_{rs}T_{rs}(h)((m)).$$

Lemma 2.1. h, h_1 as above. Suppose $(m) \in Z^n$ satisfies: $m_{j+1} = m_j + 1$ for some $j\le n-1$. Then with $\sigma_0 = (j,j+1) \in S_n$, $h_1((m)) = h_1((m_{\sigma_0}))$.

Proof. By (*), (2.2) and (2.5), for $(\bar m) \in Z^n$

$$h_1((\bar m)) = q^{\sum(i-1)\bar m_i}\sum_{\nu=0}^{\infty} q^{-\nu}\lambda_j^{\nu}\lambda_{j+1}^{-\nu}(I-\lambda_j\lambda_{j+1}^{-1})\, D_{rs}T_{rs}(h)((\bar m)).$$

Define h_0 by

$$h_0((m)) = \prod_{\substack{s>r\\ \langle s,r\rangle \ne \langle j+1,j\rangle}} D_{rs}T_{rs}h((m)).$$

for $(m) \in Z^n$.

Then

$$(**)\quad h_1((m)) = q^{\sum(i-1)m_i}\sum_{\nu=0}^{\infty} q^{-\nu}\lambda_j^{\nu}\lambda_{j+1}^{-\nu}(I-\lambda_j\lambda_{j+1}^{-1})(h_0)((m)).$$

Now since h is a symmetric function and j and j+1 enter the product $\prod_{\substack{s>r \\ \langle s,r\rangle \neq \langle j+1,j\rangle}}$ symmetrically one has for all (m)

(a) $\quad h_o((\bar{m})) = h_o((\bar{m}_{\sigma}))$.

But $m_{j+1} = m_j + 1$ implies

(b) $\quad h_o((m_{\sigma})) = \lambda_j \lambda_{j+1}^{-1} h_o((m))$.

By (a) and (b)

$$(I - \lambda_j \lambda_{j+1}^{-1})(h_o)((m)) = 0 .$$

Thus the summand in (**) is zero for $\nu = 0$, $(\bar{m}) = (m)$. With $(\bar{m}) = (m)$ in (**) we then make the substitution $\nu' = \nu - 1$. $m_{j+1} = m_j + 1$ implies

$$\sum_i (i-1)m_i - \nu = \sum_i (i-1)m_{\sigma(i)} - \nu'$$

and using (a) and (b)

$$\lambda_j^{\nu} \lambda_{j+1}^{-\nu}(I - \lambda_j \lambda_{j+1}^{-1})(h_o)((m)) = \lambda_j^{\nu'} \lambda_{j+1}^{-\nu'}(I - \lambda_j \lambda_{j+1}^{-1})(h_o)((m_{\sigma})).$$

Hence

$$\begin{aligned} h_1((m)) &= q^{\sum(i-1)m_{\sigma(i)}} \sum_{\nu'=0}^{\infty} q^{-\nu'} \lambda_j^{\nu'} \lambda_{j+1}^{-\nu'}(I - \lambda_j \lambda_{j+1}^{-1})(h_o)((m_{\sigma})) \\ &= h_1((m_{\sigma})) . \end{aligned}$$

□

Lemma 2.2. h, h_1 as above. Suppose $h_2 \in F$ is a symmetric function and $h_1 | V_n = h_2 | V_n$ then $h_1 | (V_n + W_n) = h_2 | (V_n + W_n)$. ($V_n + W_n$ denotes vector sum).

Proof. For $(m) \in Z^n$, we define

$$\rho((m)) = \text{the number of i such that } m_i < m_{i+i}.$$

Let $(m) \in V_n + W_n$. Then $m_i = \bar{m}_i + w_i$, $1 \leq i \leq n$ where $(\bar{m}) \in V_n$, $w_i = 0$ or 1. If $\rho((m)) = 0$ then $(m) \in V_n$ and by hypothesis $h_1((m)) = h_2((m))$.

Suppose $\rho_o = \rho((m)) > 0$ and $h_1((\tilde{m})) = h_2((\tilde{m}))$ for all $(\tilde{m}) \in V_n + W_n$ with $\rho((\tilde{m})) < \rho_o$. Let $j (\geq 1)$ be the smallest integer such that $m_j < m_{j+1}$. Since $\bar{m}_j \geq \bar{m}_{j+1}$, we must have $\bar{m}_j = \bar{m}_{j+1}$, $w_j = 0$, $w_{j+1} = 1$ and so $m_{j+1} = m_j + 1$. Then by Lemma 2.1.

$$h_1(\ldots,m_j,m_{j+1},\ldots) = h_1(\ldots,m_{j+1},m_j,\ldots) .$$

If now $m_{j+1} > m_{j-1}$ we repeat this process, continuing until we get

$$h_1((m)) = h_1(\ldots,m_i,m_{j+1},m_{i+1},\ldots m_j\ldots)$$
$$= h_1((m_\sigma)), \qquad \sigma = (j,j+1,i+1,\ldots,j-1)$$

where $m_i \geq m_{j+1} \geq m_{i+1} \geq \cdots \geq m_j$, $i \geq 0$.

But now $(m_\sigma) = (\overline{m})+(W_\sigma) \in V_n+W_n$ and $\rho(m_\sigma) = \rho_0-1$. By the induction hypothesis

$$h_1((m_\sigma)) = h_2((m_\sigma))$$
$$= h_2((m))$$

since h_2 is a symmetric function. The proof is complete by mathematical induction. □

Finally, we shall use

Lemma 2.3. Let $h \in F$ satisfy

(i) There is an $M \in Z$, such that $h((m)) = 0$ if $m_n \leq M$.

(ii) $h((m)) = h((m_\sigma))$ if $m_{j+1} = m_j + 1$, $\sigma = (j,j+1) \in S_n$ for $1 \leq j \leq n-2$.

(iii) For some $(r) \in Z^{n-1}$

$$[\prod_{i=1}^{n-1} D_{ni}(r_i)]\, h \Big| V_n \equiv 0 .$$

Then $h \Big| V_n \equiv 0$.

Proof. By (i) $h((m)) = 0$ for $m_n \leq M$. Suppose $h((m)) = 0$ for $(m) \in V_n$, $m_n \leq M+j$. Let $(\overline{m}) \in V_n$ be such that $\overline{m}_n = M+j+1$. By (2.4) and (iii)

$$h((\overline{m})) = -\sum_{\substack{(w)\in W_n \\ |(w)|>0}} (-1)^{|(w)|}\, q^{(r)\cdot(w)}\, h(\overline{m}_1+w_1,\ldots,\overline{m}_{n-1}+w_{n-1},\overline{m}_n-|(w)|).$$

Fix a summand on the right. If $m_j + w_j < m_{j+1} + w_{j+1}$, $j \leq n-2$, we may "transpose" these two entrees as in the proof of Lemma 2.2, using (ii). We repeat this until we get, in this summand

$$h(\overline{m}_{\sigma(1)}+w_{\sigma(1)},\ldots,\overline{m}_{\sigma(n-1)}+w_{\sigma(n-1)},\overline{m}_n-(w)).$$

for some $\sigma \in S_{n-1}$ where the argument is in V_n. Since $m_n - |(w)| \leq m_n - 1 = M + j$, the value of the summand is zero. Thus $h((\overline{m})) = 0$. The conclusion follows by induction.

□

3. The mapping $f \to \tilde{f}$.

Throughout this section f will be in L(G,U).

Let δ be the quasi-character of A given by

(3.1) $\delta(a) = \prod_{i=1}^{n} |a_i|^{n-i}$ where $a = \operatorname{diag}(a_1,\ldots,a_n)$.

For $(m) \in Z^n$,

(3.2) $\delta(\pi^{(m)}) = q^{-\sum_{i}^{n}(n-i)m_i}$.

We introduce a function, $\tilde{f}$, on A by letting

(3.3) $\tilde{f}(a) = \delta(a) \int_B f(ba)db$.

Since $\tilde{f}(au) = \tilde{f}(a)$ for $u \in A \cap U$, $\tilde{f}$ is characterized by its values on $\{\pi^{(m)}\}_{(m)\in Z^n}$.

As in the case of f, we shall at times consider $\tilde{f}$ also as a function on Z^n with $\tilde{f}((m)) = \tilde{f}(\pi^{(m)})$. The meaning will be clear in context.

Now, by (1.9) $db = \prod_{i>j} d\mu(y_{ij})$ if $b = \begin{bmatrix} 1 & & & 0 \\ y_{12} & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & 1 \end{bmatrix}$.

Then, if $x_{ij} = y_{ij}\pi^{m_j}$, $\delta(\pi^{(m)})db = \prod_{i>j} d\mu(x_{ij})$. With this, (3.3) yields

(3.4) $$\tilde{f}((m)) = \underbrace{\int_k \cdots \int_k}_{\frac{n(n-1)}{2}} f\left(\begin{bmatrix} \pi^{m_1} & & & & 0 \\ x_{21} & \pi^{m_2} & & & \\ x_{31} & x_{32} & \pi^{m_3} & & \\ \vdots & \vdots & & \ddots & \\ \cdot & \cdot & \cdot & & \cdot \end{bmatrix}\right) \prod_{i>j} d\mu(x_{ij})$$

Lemma 3.1. $\tilde{f}((m))$ is a symmetric function on Z^n with finite support.

Proof. Since f(g) is of compact support, there exist $D, M \in R$ such that $f(g) = 0$ if $\operatorname{ord}(\det(g_{ij})) > D$ or if $\min_{1\le i,j\le n}(\operatorname{ord}(g_{ij})) < M$

where $g = (g_{ij})$. Then the integrand in (3.4) in non-zero only if $|(m)| \leq D$ and $\min_{1 \leq i \leq n} m_i \geq M$. Since there are only a finite number of n-tuples, (m), satisfying these conditions, $\tilde{f}$ is of finite support.

To prove the symmetry, assume first n=2 and $f \in L(G_2, U_2)$. Then with $\sigma = (1,2)$, one has

$$\tilde{f}((m)) - \tilde{f}((m_\sigma)) = \int_k f\left(\begin{bmatrix} \pi^{m1} & 0 \\ x & \pi^{m2} \end{bmatrix}\right) - f\left(\begin{bmatrix} \pi^{m2} & 0 \\ x & \pi^{m1} \end{bmatrix}\right) d\mu(x) .$$

Since the two matrices always have the same elementary divisors, the integrand is identically zero and $\tilde{f}((m)) = \tilde{f}((m_\sigma))$.

Suppose now n is arbitrary. It suffices to show $\tilde{f}((m)) = \tilde{f}((m_\sigma))$, for σ equal to an inversion: $(p,p+1)$, $1 \leq p < n$. Fix $(m_1, \ldots, m_{p-1}, m_{p+2}, \ldots, m_n) \in Z^{n-2}$. We define a function, f_1, on G_2 by

$$(*) \quad f_1(g) = \underbrace{\int_k \cdots \int_k}_{\frac{n(n-1)}{2}-1} f\left(\begin{bmatrix} \ddots & & & & 0 \\ & \ddots & & & \\ & & \ddots & & \\ x_{p1} \cdots\cdots x_{p,p-1} & & \begin{bmatrix} g \end{bmatrix} & & \\ x_{p+1,1} \cdots x_{p+1,p-1} & & & & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix}\right) \prod_{\substack{i>j \\ \langle i,j\rangle \neq \langle p+1,p\rangle}} d\mu(x_{ij})$$

where the integrand is that of (3.4) with 2×2 matrix $\begin{bmatrix} \pi^{m_p} & 0 \\ x_{p+1,p} & \pi^{m_{p+1}} \end{bmatrix}$ replaced by g. Suppose $u \in U_2$. Then

$$f\left(\begin{bmatrix} & \alpha & \\ x_{p,1} \ \ldots \ x_{p,p-1} & \begin{bmatrix} ug \end{bmatrix} & 0 \\ x_{p+1,1} \ldots \ x_{p+1,p-1} & & \\ & \beta & \end{bmatrix}\right) = f\left(\begin{bmatrix} & \alpha & \\ x'_{p,1} \ \ldots \ x'_{p,p-1} & \begin{bmatrix} g \end{bmatrix} & 0 \\ x'_{p+1,1} \ldots \ x'_{p+1,p-1} & & \\ & \beta & \end{bmatrix}\right)$$

where α and β are $p-1 \times n$ and $n-p-1 \times n$ matrices respectively, and $\begin{bmatrix} x_{pi} \\ x_{p+1,i} \end{bmatrix} = u \cdot \begin{bmatrix} x'_{pi} \\ x'_{p+1,i} \end{bmatrix}$, $1 \leq i \leq p-1$. But then $d\mu(x_{pi})d\mu(x_{p+1,i})$

$= d\mu(x'_{pi})d\mu(x'_{p+1,i})$ and so comparing with (✱) $f_1(ug) = f_1(g)$. Similarly, $f_1(gu) = f_1(g)$. f_1 is then a spherical function. As in the proof of the other part $f_1(m_p,m_{p+1})$ is non-zero only on a finite set, hence $f_1(g) \in L(G_2,U_2)$. Finally we note by (✱) and (3.4)

$$\tilde{f}_1(m_p,m_{p+1}) = \tilde{f}(\ldots,m_{p-1},m_p,m_{p+1},m_{p+2},\ldots).$$

Therefore, if $(m) = (\ldots,m_{p-1},m_p,m_{p+1},m_{p+2},\ldots)$

$$\tilde{f}((m)) = \tilde{f}_1(m_p,m_{p+1}) = \tilde{f}_1(m_{p+1},m_p) = \tilde{f}((m_\sigma)). \quad \square$$

Remark. As noted in the introduction this result is not new. The symmetry of $\tilde{f}((m))$ is equivalent to the invariance of $\tilde{f}(a)$ under the operation of the Weyl group of G relative to A. The latter follows from a result of Satake [5, page 22] which is proved by a different method and in a more general setting.

Using now the notation of section 3 the inverse of the mapping $f \rightarrow \tilde{f}$ is given in-

Proposition 3.1. For $(m) \in V_n$, $f \in L(G,U)$

$$(3.5) \qquad f((m)) = q^{\sum_i (i-1)m_i} \prod_{n \geq s > r \geq 1} D_{rs}T_{rs}\, \tilde{f}\,((m))\ .$$

Before this proposition, we shall prove two additional lemmas.

Lemma 3.2. Let $(m) \in V_n$, $(x) \in k^{n-1}$ and suppose $\operatorname{ord}(x_1) \leq m_n$, then

$$(3.6)\quad \sum_{(w)\in W_{n-1}} (-1)^{|(w)|} f\left(\begin{bmatrix} * & & & & & & 0 \\ & * & & & & & \\ & & * & & & & \\ & & & \pi^{m_j+w_j} & & & \\ & & & & * & & \\ 0 & & & & & * & \\ & & & & & & * \\ *** , & x_j\pi^{-\overline{w}(j)}, & ***, & & x_{n-1}, & & \pi^{m_n+1-|(w)|} \end{bmatrix}\right) = 0$$

where we have let, for convenience, $\overline{w}(j) = \sum_{i=j+1}^{n-1} w_i$. (The entrees are non-zero only on the diagonal and nth row where they are as indicated).

Proof. The proof is by induction on n. If n=2, the sum in (3.6) is

$$f\left(\begin{bmatrix} \pi^{m_1} & 0 \\ x_1 & \pi^{m_2+1} \end{bmatrix}\right) - f\left(\begin{bmatrix} \pi^{m_1+1} & 0 \\ x_1 & \pi^{m_2} \end{bmatrix}\right) .$$

But if $\mathrm{ord}(x_1) \leq m_2 \leq m_1$ both of these matrices are in

$U_2 \cdot \mathrm{diag}((1/x_1)\,\pi^{m_1+m_2+1}, x_1)\cdot U_2$. Hence the above difference is zero.

Now fix n>2, and assume the corresponding version of (3.6) holds for all $f \in L(G_{n-1},U_{n-1})$. Let (m), (x) be as given.

Suppose first $\mathrm{ord}(x_1) > \mathrm{ord}(x_2) > \cdots > \mathrm{ord}(x_{n-1})$. Then for $(w) \in W_{n-1}$

$$m_n+1-|(w)| \geq \mathrm{ord}(x_1)-\overline{w}(1) \geq \mathrm{ord}(x_2)-\overline{w}(2) \geq \cdots \geq \mathrm{ord}(x_{n-1}),$$

and

$$m_{n-1}+w_{n-1} \geq m_n \geq \mathrm{ord}(x_1) \geq \mathrm{ord}(x_{n-1}).$$

We then transform the matrices in (3.6) by subtraction of integral multiples of the (n-1)st column from the others and then one of row n from row(n-1) (these elementary operations correspond of course to multiplication by elementary matrices which are in U) obtaining in the last two rows

$$\begin{bmatrix} \ldots, -\frac{x_j}{x_{n-1}}\pi^{-\bar{w}(j)+w_{n-1}}, \ldots, -\frac{x_{n-2}}{x_{n-1}}, & 0 & , \frac{-\pi^{m_{n-1}+m_n-|(w)|+w_{n-1}}}{x_{n-1}} \\ 0., \quad 0 \quad ,0..0, & 0 & ,x_{n-1}, \quad 0 \end{bmatrix} \begin{matrix} \text{row} \\ n-1 \\ \\ n \end{matrix}$$

Now, for $1 \leq j \leq n-3$

$$-\bar{w}(j) + w_{n-1} = -w_{j+1} - \cdots - w_{n-2}$$

and

$$-|(w)| + w_{n-1} = -w_1 - \cdots - w_{n-2} .$$

It follows that the transformed matrices are independent of "w_{n-1}". But then each summand enters twice, with opposite signs corresponding to $(-1)^{w_{n-1}} = +1$ or -1. Hence, the sum is zero.

One may assume then

$$\operatorname{ord}(x_j) < \operatorname{ord}(x_{j+1}), \quad \text{for some } j, \quad 1 \leq j \leq n-2.$$

For $(w) \in W_{n-1}$

$$\operatorname{ord}(x_j) - \bar{w}(j) \leq \operatorname{ord}(x_{j+1}) - \bar{w}(j+1) .$$

Taking note of this inequality and $m_j \geq m_{j+1}$ we can by elementary operations eliminate the row n - column j+1 entrees in the (3.6) matrices. (This time subtract an appropriate integral multiple of column j from column j+1, then one of row j+1 from row j). With this in mind we define on G_{n-1} the functions

$$f_i(g) = f\left(\begin{bmatrix} g_1 & \begin{matrix}0\\ \vdots \\ 0\end{matrix} & g_2 \\ 0.....0 & \pi^{m(j+1)+i} & 0.....0 \\ g_3 & \begin{matrix}0\\ \vdots \\ 0\end{matrix} & g_4 \end{bmatrix}\right) \quad \text{where } g = \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix} \begin{matrix} \}j \\ \\ \end{matrix}, \quad g_1, g_3 \text{ with } j \text{ columns}$$

$i = 0,1$. Clearly $f_i \in L(G_{n-1}, U_{n-1})$.

Since the last transformation the sum in (3.6) has become

$$\text{(A)} \qquad \sum_{w_{j+1}=0}^{1} (-1)^{w_{j+1}} \sum_{(w')\in W_{n-2}} (-1)^{|(w')|} f_{w_{j+1}}(g_{w'})$$

where $w' = (\ldots,w_j,w_{j+2},\ldots)$ and the matrices $g_{w'}$ are the $n-1 \times n-1$ submatrices of those in (3.6) obtained by the deletion of row- and column- $(j+1)$. For fixed w_{j+1} (=0 or 1) in the inner summation of (A) let $x_i' = x_i \pi^{-w_{j+1}}$, $1 \le i \le j$, $x_i' = x_{i+1}$, $j+1 \le i \le n-1$, and $m_0 = m_n - w_{j+1}$. Then

$$(m') = (m_1,\ldots,m_j,m_{j+2},\ldots,m_{n-1},m_0) \in V_{n-1}$$

and $\mathrm{ord}(x_1') \le m_0$. The inner sum of (A) is now of the type in (3.6) with respect to $(x') \in k^{n-2}$, $(m') \in Z^{n-1}$. By the induction hypothesis, this sum is zero. Therefore the whole sum is zero and by induction the proof is complete. □

For each $f \in L(G,U)$, we define a function f^0 on Z^n by

$$\text{(3.7)} \quad f^0((m)) = \underbrace{\int_k \cdots \int_k}_{n-1} f\left(\begin{bmatrix} * & & & & & 0 \\ & * & & & & \\ 0 & & \pi^{m_j} & & & \\ & & & * & & \\ & & & & * & \\ * * * , & x_j, & * * * , & x_{n-1} & & \pi^{m_n} \end{bmatrix}\right) d\mu(x_1)\ldots d\mu(x_{n-1}).$$

Lemma 3.3. Let $(m) \in V_n$, then

$$\text{(3.8)} \quad \lambda_n \prod_{i=1}^{n-1} D_{in(-i+1)}(f^0)((m)) = q^{-(n-1)(m_n+1)} \lambda_n \prod_{i=1}^{n-1} D_{in(n-i-1)}(f)((m)) .$$

Proof. By (2.4), with $r_i = -(i-1)$, $1 \le i \le n-1$, the left hand side of (3.8) becomes:

$$\sum_{(w)\in W_{n-1}} (-1)^{|(w)|} q^{-(r)\cdot(w)} \int_k \cdots \int_k f\left(\begin{bmatrix} * & & & & 0 \\ & * & & & \\ 0 & & \pi^{m_j+w_j} & & \\ & & & \ddots & \\ * * * & y_j & * * * & & \pi^{m_n+1-|(w)|} \end{bmatrix}\right) \prod_{j=1}^{n-1} d\mu(y_j)$$

In the integrand in each summand, we make the substitutions $y_j = x_j\pi^{\overline{w}(j)}$ ($\overline{w}(j)$ as in Lemma 3.2) and so $d\mu(y_1)\cdots d\mu(y_{n-1})$ $= q^{(r)\cdot(w)}d\mu(x_1)\cdots d\mu(x_{n-1})$.

The last expression becomes

$$(a) \qquad \int_k\cdots\int_k\Big(\sum_{(w)\in W_{n-1}}(-1)^{|(w)|}f([X_{(w)}])\Big)\ d\mu(x_1)\cdots d\mu(x_{n-1})$$

where $[X_{(w)}]$ is the matrix in the corresponding summand in equation (3.6).

By Lemma 3.2 the integral $= 0$ over: $\mathrm{ord}(x_1)\le m_n$. Restricting the integration in (a) to the range: $\mathrm{ord}(x_1) \ge m_n+1$, we note for such (x_1), and $(w)\in W_{n-1}$,

$$\mathrm{ord}(x_1) - \overline{w}(1) \ge m_n + 1 - |(w)| .$$

Then the row n- column 1 entree of $[X_{(w)}]$ is eliminated by suitable subtraction of a column n multiple from column 1.

If $n\ge 3$, we continue and define, for fixed w_1(=0 or 1), the function f_{w_1} on G_{n-1} by

$$f_{w_1}(g) = f\begin{bmatrix}\pi^{m_1+w_1} & 0\cdots 0\\ 0 & \\ \vdots & g\\ 0 & \end{bmatrix}, \quad g\in G_{n-1} .$$

Trivially, $f_{w_1}\in L(G_{n-1},U_{n-1})$.

Now, by the last transformation the integrand in (a) may be rewritten

$$\sum_{w_1=0}^{1}(-1)^{w_1}\sum_{(w')\in W_{n-2}}(-1)^{|(w')|} f_{w_1}[X^1{}_{(w')}] .$$

where $(w') = (w_2,\ldots,w_{n-1})$. $[X^1(w')] \in G_{n-1}$ is the lower right $n-1\times n-1$ submatrix of an $[X_{(w)}]$ and is therefore of a similar form with respect to $(m_2,\ldots,m_{n-1},m_n-w_1) \in V_{n-1}$ and $(x_2,\ldots,x_{n-1}) \in k^{n-2}$. Then again the integral is zero over $\mathrm{ord}(x_2) \leq m_n-w_1$.

Clearly this process may be continued, i.e. at each step for $1\leq j\leq n-2$ and $w_1,\ldots,w_j$ fixed the integrand is zero over

$$\begin{cases}\mathrm{ord}(x_i) \geq m_n+1-w_1-\cdots-w_{i-1} & 1\leq i\leq j\\ \mathrm{ord}(x_{j+1}) \leq m_n-w_1-\cdots-w_j & \end{cases}$$

We are left finally, for each (w), with the integration over

$$\mathrm{ord}(x_i) \geq m_n+1-w_1-\cdots-w_{i-1} \qquad 1\leq i\leq n-1.$$

Furthermore, at this stage we may assume the n^{th} row matrix entrees (except the last) are zero in each summand.

By (1.5) (a) equals

$$q^{-(n-1)(m_n+1)}\sum_{(w)\in W_{n-1}}(-1)^{|(w)|}q^{\sum_i^{n-1}(n-i-1)w_i}\, f(m_1+w_1,\ldots,m_{n-1}+w_{n-1},m_n+1-|(w)|)$$

= r. h. s. of (3.8) by (2.4). □

We are now prepared for the-

Proof of Proposition 3.1. For $f\in L(G,U)$, let f^1, a function on Z^n, be given by

$$(3.9) \qquad f^1((m)) = q^{\sum_i (i-1)m_i} \prod_{n\geq s>r\geq 1} D_{rs}T_{rs}(\tilde{f})((m)).$$

Clearly, for Proposition 3.1, one must show $f^1 \mid V_n = f \mid V_n$.

We note first some immediate consequences of (3.9)

(α) $\tilde{f}, f^1$, respectively, satisfy the conditions on h, h^1 respectively, in Lemmas 2.1, 2.2. (by Lemma 3.1 and (3.9))

(β) f^1 satisfies condition (i) of Lemma 2.3. (since $\tilde{f}$ has that property; see remark after equation (2.6)).

(γ) f^1 satisfies condition (ii) of Lemma 2.3 (by application of Lemma 2.1).

With these facts in mind, the proof is by induction on n. Suppose first $n > 2$ and $\psi \mid V_{n-1} = \psi^1 \mid V_{n-1}$ for $\psi \in L(G_{n-1}, U_{n-1})$.

Applying the product $\prod_{i=1}^{n-1} D_{in}\,(n-i-1)$ to both sides of (3.9) and recalling (2.3) one has

$$\prod_{i=1}^{n-1} D_{in}(n-i-1)(f^1)((m))$$

$$= q^{\sum (i-1)m_i} \prod_{1}^{n-1} D_{in}(-1) \prod_{n\geq s>r} D_{rs}T_{rs}(\tilde{f})((m))$$

$$(3.10) \qquad = q^{\sum (i-1)m_i} \prod_{1}^{n-1} D_{in} \prod_{n-1\geq s>r} D_{rs}T_{rs}(\tilde{f})((m)) \text{ (using (2.6)).}$$

Now, for $p \in Z$, we define a function f_p on G_{n-1}

$$(3.11) \qquad f_p(g) = \underbrace{\int_k \cdots \int_k}_{n-1} f\left(\left[\begin{array}{c|c} g & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline x_1 \cdots x_{n-1} & \pi^p \end{array}\right]\right) d\mu(x_1)\cdots d\mu(x_{n-1}).$$

It is easy to see (as in the case of the function f_1 in the proof of Lemma 3.1) that $f_p \in L(G_{n-1}, U_{n-1})$. Comparing (3.11) with equations (3.4) and (3.7), one has

(3.12a) $f_p(m_1,\ldots,m_{n-1}) = f^{\circ}(m_1,\ldots,m_{n-1},p)$

(3.12b) $\tilde{f}_p(m_1,\ldots,m_{n-1}) = \tilde{f}(m_1,\ldots,m_{n-1},p)$.

By the induction hypothesis, $f_p^1 \mid V_{n-1} \equiv f_p \mid V_{n-1}$.

Then by (α) and Lemma 2.2

(3.13) $f_p^1 \mid (V_{n-1} + W_{n-1}) \equiv f_p \mid (V_{n-1} + W_{n-1})$.

Starting with (3.10) we get, for $(m) \in V_n$

$$\prod_{i=1}^{n-1} D_{in}(n-i-1)(f^1)((m))$$

$$= q^{\sum(i-1)m_i} \sum_{(w)\in W_{n-1}} (-1)^{|(w)|} \prod_{n-1\geq s>r\geq 1} D_{rs}T_{rs}\tilde{f}(m_1+w_1,\ldots,m_{n-1}+w_{n-1},m_n-|(w)|)$$

(by 2.3)

$$= q^{\sum(i-1)m_i} \sum (-1)^{|(w)|} \prod_{n-1\geq s>r\geq 1} D_{rs}T_{rs}\tilde{f}_{m_n-|(w)|}(m_1+w_1,\ldots,m_{n-1}+w_{n-1})$$

(by 3.12b)

$$= q^{\sum(i-1)m_i} \sum (-1)^{|(w)|} q^{-\sum_1^{n-1}(i-1)(m_i+w_i)} \left(f_{m_n-|(w)|}(m_1+w_1,\ldots,m_{n-1}+w_{n-1})\right)$$

(by 3.13 and 3.9)

$$= q^{(n-1)m_n} \sum (-1)^{|(w)|} q^{-\sum_1^{n-1}(i-1)w_i} \left(f^{\circ}(m_1+w_1,\ldots,m_{n-1}+w_{n-1},m_n-|(w)|)\right)$$

(by 3.12a)

$$= q^{(n-1)m_n} \prod_{i=1}^{n-1} D_{in}(-i+1)f^{\circ}((m))$$

(by 2.3)

$$= \prod_{1}^{n-1} D_{in}(n-i-1)(f)((m)),$$

by Lemma 3 applied to $(m_1,\ldots,m_{n-1},m_n-1) \in V_n$. Then by ($\beta$) and ($\gamma$) Lemma 2.3 may be applied to $(f-f^1)$ yielding $(f-f^1) \mid V_n \equiv 0$.

It remains to prove $f|V_2 \equiv f^1|V_2$ for $f \in L(G_2,U_2)$. For such f (3.4) and (3.7) imply $\tilde{f} = f^o$. Let $(m_1,m_2) \in V_2$, then $f^1(m_1,m_2) = q^{m_2}D_{12}f^o(m_1,m_2)$, by (3.9)

$= f(m_1,m_2)$ by Lemma (3.3)

applied to (m_1,m_2-1). This completes the proof.

□

Proposition 3.2. The mapping $f \to \tilde{f}$ is an isomorphism of the vector space L(G,U) onto the space of complex-valued symmetric functions on Z^n with finite support. (in a sense, then, an automorphism of the latter space).

Proof. The mapping is clearly linear by (3.3). Let $\tilde{f}$ be in the latter space, and let f be defined by (3.5) for $(m) \in V_n$ and by $f((m)) = f((m_\sigma))$ otherwise. f is zero for m_1 large or m_n small since $\tilde{f}$ has this property (cf. remark after equation (2.6)) so f vanishes off a finite subset of V_n and is therefore of finite support. Therefore $f(g) \in L(G_n,U_n)$ and there is a well-defined inverse mapping. It remains only to prove that $f \to \tilde{f}$ is surjective or the equivalent that the inverse is one-one.

Thus, we prove: If h is a complex valued symmetric function on Z^n ($\underline{n \geq 1}$), with finite support and

(*) $$q^{\sum(i-1)m_i} \prod_{n \geq s > r \geq 1} D_{rs}T_{rs}h((m)) = 0$$

holds for $(m) \in V_n$, then $h \equiv 0$. The proof is by induction on n.

If $n=1$, $(*)$ reads $h(m_1) = 0$. But $V_1=Z^1$ and so if $(*)$ holds for $m_1 \in V_1$, $h \equiv 0$.

Suppose now $n \geq 2$ and the assertion true for functions on Z^{n-1}. Let h satisfy the hypothesis. Since h is of finite support

(δ) $\quad h((m)) = 0$ if $\min\limits_{1 \leq i \leq n} m_i \leq M$, for some $M \in Z$.

If $(*)$ holds for $(m) \in V_n$, then by Lemma 3.2 (with $h_2 \equiv 0$), $(*)$ holds for $(m) \in V_n + W_n$. Then if $(m) \in V_n$

$$0 = q^{\sum_1^{n-1}(i-1)m_i} \sum_{(w)\in W_{n-1}} (-1)^{|(w)|} q^{-\sum w_i} \Big\{ \prod_{n \geq s > r \geq 1} D_{rs}T_{rs} \cdot h(m_1+w_1,\dots,m_{n-1}+w_{n-1},m_n-|(w)|) \Big\}$$

for the part in the braces is always zero. It follows, for $(m) \in V_n$

$$0 = q^{\sum_1^{n-1}(i-1)m_i} \prod_1^{n-1} D_{in}(-1) \prod_{n \geq s > r \geq 1} D_{rs}T_{rs}h((m)) \qquad \text{(by 2.4)}$$

$$= q^{\sum(i-1)m_i} \prod_1^{n-1} D_{in} \prod_{n-1 \geq s > r} D_{rs}T_{rs}h((m)) \qquad \text{(by 2.6)}$$

$$\text{(iii)} \quad = \prod_1^{n-1} D_{in}(n-i) \Big\{ q^{\sum_1^{n-1}(i-1)m_i} \prod_{n-1 \geq s > r} D_{rs}T_{rs}h((m)) \Big\} \qquad \text{(by 2.3)}$$

Call the function within the braces in (iii), $h_1((m))$. By (δ)

(i) $\quad h_1((m)) = 0$ for $m_n \leq M$.

For fixed m_n, h is a symmetric function on Z^{n-1}, then by Lemma 2.1, h_1 satisfies

(ii) $\quad h((m)) = h((m_\sigma))$ if $m_{j+1} = m_j + 1$, $\sigma = (j,j+1)$, $j \leq n-2$.

By (i), (ii), (iii) and Lemma 2.3.,

(∗∗) $$0 = \prod_{n-1 \geq s > r} D_{rs} T_{rs} h((m))$$

holds for $(m) \in V_n$. (Note that Lemma 2.3 holds trivially for $n-1=1$).

Assume now $h \not\equiv 0$. By (δ) there must exist an M_1 maximal with respect to

(ϵ) $$h((m)) = 0 \text{ if } \min_i m_i \leq M_1.$$

Fix $m_n = M_1 + 1$, and let $(m_1,\ldots,m_{n-1}) \in V_{n-1}$. If $m_{n-1} \geq m_n$, $(m_1,\ldots,m_n) \in V_n$ and (∗∗) holds. If $m_{n-1} \leq m_n - 1 = M_1$, by ($\epsilon$) and the remark after equation (2.6.), $h_1((m))$ is zero and (∗∗) again holds. Hence (∗∗) holds for $m_n = M_1 + 1$ and all $(m_1,\ldots,m_{n-1}) \in V_{n-1}$. By the induction hypothesis

$$h(m_1,\ldots,m_{n-1},M_1 + 1) = 0 \text{ for all } (m_1,\ldots,m_{n-1}) \in Z^{n-1}.$$

Comparing with (ϵ), by the symmetry of h,

$h((m)) = 0$ if $\min_i m_i \leq M_1 + 1$ contradicting the maximality of M_1. Therefore $h \equiv 0$. The proof is complete by induction. □

4. The transform $\hat{f}$.

Let $(z) \in C^n$. We denote by $\alpha_{(z)}$ the quasi-character of A given by

$$(4.1) \qquad \alpha_{(z)}(u\pi^{(m)}) = q^{(m)\cdot(z) + \frac{n-1}{2}\sum_j m_j} \text{ for } u \in U \cap A.$$

$((m)\cdot(z) = m_1 z_1 + \cdots + m_n z_n)$. For $f \in L(G,U)$ we define then a function $\hat{f}$ on C^n by

$$(4.2) \qquad \hat{f}((z)) = \int_A \tilde{f}(a)\, \alpha_{(z)}(a)da.$$

The transformation has the properties

$$(4.3) \qquad \widehat{f * f_1}((z)) = \hat{f}((z))\hat{f}_1((z)),$$

$$(4.4) \qquad \widehat{f^*}((z)) = \hat{f}(-(\bar{z}))$$

(see (1.14), (1.15)),

where $-(z) = (-z_1,\ldots,-z_n)$, $(\bar{z}) = (\bar{z}_1,\ldots,\bar{z}_n)$. Hence,

Proof of (4.3). By (1.14), (3.3), and (1.13)

$$\widetilde{f * f_1}(a_o) = \delta(a_o)\int_A\int_B\int_B f(b_o a_o ab) f_1(b^{-1}a^{-1})\, db_o db da$$

$$= \int_A \tilde{f}(a_o a)\tilde{f}_1(a^{-1})da$$

Hence,

$$\widehat{f * f_1}((z)) = \int_A\int_A \tilde{f}(a_o a)\tilde{f}_1(a^{-1})\alpha_{(z)}(a_o)da da_o$$

$$= \int_A \tilde{f}_1(a^{-1})\alpha_{(z)}(a^{-1})\left\{\int_A \tilde{f}(a_o a)\alpha_{(z)}(a_o a)da_o\right\}da$$

$$= \hat{f}((z))\hat{f}_1((z)) .$$

□

Proof of (4.4). By (1.5), (3.3) and (1.11)

$$\overline{f^*}(a) = \delta(a)\int_B \bar{f}(a^{-1}b^{-1})db$$

$$= \delta^2(a)\ \gamma(a^{-1})\tilde{\bar{f}}(a^{-1}).$$

Then, by (1.12), (3.2) and (4.1)

$$\delta^2(a)\gamma(a^{-1})\alpha_{(z)}(a) = \alpha_{-(z)}(a^{-1}) = \overline{\alpha_{-(\bar{z})}}(a^{-1})$$

$$\widehat{f^*}((z)) = \overline{\int_A \tilde{f}(a^{-1})\alpha_{-(\bar{z})}(a^{-1})da}$$

$$= \hat{f}(-(\bar{z})) \qquad \square$$

Now, using (4.2), (1.7) and (1.8)

$$\hat{f}((z)) = \sum_{(m)\in Z^n}\tilde{f}((m))\alpha_{(z)}(\pi^{(m)})$$

$$(4.5) \qquad = \sum_{(m)\in Z^n}\tilde{f}((m))q^{\frac{n-1}{2}\cdot\sum m_{j_q}(z_1m_1+\cdots+z_nm_n)}$$

By proposition 3.2, $\tilde{f}$ varies over all symmetric functions on Z^n, non-zero only on a finite set. Then (4.5) implies that $\hat{f}((z))$ varies over all symmetric polynomials in $\{q^{\pm z_1},\ldots,q^{\pm z_n}\}$. Furthermore, the mapping $f \to \hat{f}$ is clearly linear and is a homomorphism by (4.3). We have then

Theorem O. (i) The transformation $f \to \hat{f}$ is an isomorphism of the algebra L(G,U) onto the algebra of symmetric polynomials in $\{q^{\pm z_1},\ldots,q^{\pm z_n}\}$.

(ii) The homomorphisms of L(G,U) onto C are then of the form $f \to \hat{f}((c))$, $(c) \in C^n$. Two such homomorphisms, $f \to \hat{f}((c))$, $f \to \hat{f}((c'))$ are identical if and only if

$$c'_i \equiv c_{\sigma(i)} \bmod\left(\frac{2\pi i}{\log q}\right), \quad 1 \leq i \leq n, \text{ for some } \sigma \in S_n.$$

Proof. (i) follows from previous remarks. (ii) is a consequence of (i) and the corresponding fact about the algebra of these polynomials. □

Remarks. Theorem 0 is a known result(cf. introduction)

Note also that (4.5) and Proposition 1 easily provide another known result. (See[5, p.45] and [7, p.395])

Thus -

$$\text{let } (m^\tau) = (\overbrace{1,\ldots,1}^{\tau},0,\ldots,0) \qquad 1 \leq \tau \leq n$$

$$(m^0) = (-1,\ldots\ldots\ldots,-1)$$

and suppose $f_{(m^\tau)}$, $0 \leq \tau \leq n$, is the member of L(G,U) mapping into

$$\hat{f}_{(m^\tau)} = \sum_{\sigma \in S_n} q^{(m^\tau)(z_\sigma)}$$

Note that

$$\hat{f}_{(m^0)}\hat{f}_{(m^n)} = 1.$$

These n+1 symmetric polynomials clearly generate the algebra of all the symmetric polynomials in $\{q^{\pm z_j}\}_{1 \leq j \leq n}$.

Now (4.5) implies

$$(\alpha) \qquad \tilde{f}_{(m^\tau)}((m)) = q^{-\frac{n-1}{2}|(m^\tau)|} \quad \text{if } (m) = (m^\tau_\sigma),$$

Thus for $1 \leq \tau \leq n$

$$(\beta) \qquad \tilde{f}_{(m^\tau)}((m)) = 0 \quad \text{if } m_1 > 1 \text{ or } 0 > m_n.$$

As remarked in section 2, (3.5) implies that (β) holds with respect to $f_{(m^\tau)}((m))$, for $(m) \in V_n$. Moreover, if $f_{(m^\tau)}((m)) \neq 0$, (α) and (3.4) imply $|(m^\tau)| = |(m)|$. It follows at once that,

$$(\text{constant}) \times f_{(m^\tau)} = K_{(m^\tau)}$$

$$= \text{characteristic function of } U\pi^{(m^\tau)}U.$$

We reobtain -

Theorem 0. (iii) L(G,U) is isomorphic to $C[X_1,\ldots,X_{n-1},X_n^{\pm 1}]$, $K_{(m^\tau)} \rightarrow X_\tau$ for $\tau \geq 1$, determining the isomorphism.

5. The inverse transformation

In (4.5) let $z_j = iy_j$ $(i = \sqrt{-1})$, $y_j \in R$, $1 \leq j \leq n$, and let $i(y) = (iy_1, \ldots, iy_n)$. Then

$$\hat{f}(i(y)) = \sum_{(m)\in Z^n} \tilde{f}((m))q^{\frac{n-1}{2}|(m)|}q^{i(y)\cdot(m)}$$

By Fourier series

$$(5.1)\quad \tilde{f}((m)) = q^{-\frac{n-1}{2}|(m)|}\left(\frac{\log q}{2\pi}\right)^n \int_0^{\frac{2\pi}{\log q}}\cdots\int_0^{\frac{2\pi}{\log q}} \hat{f}(i(y))q^{-i(y)\cdot(m)}dy_1\ldots dy_n.$$

Let $Y \subseteq R^n$ be the region

$$(5.2)\quad Y = \left\{(y) \quad 0 \leq y_1 \leq \cdots \leq y_n \leq \frac{2\pi}{\log q}\right\}.$$

Since $\hat{f}(i(y))$ is symmetric in $\{y_1, \ldots, y_n\}$

$$(5.3)\quad \tilde{f}((m)) = q^{-\frac{n-1}{2}|(m)|}(\log q/2\pi)^n\, n! \int\cdots\int_Y \hat{f}(i(y)) \sum_{\sigma\in S_n} q^{-i(y\sigma)\cdot(m)} \prod_j dy_j.$$

With

$$h((m)) = q^{-\frac{n-1}{2}|(m)|}q^{-i(y)\cdot(m)}$$

by (2.2) and (2.5)

$$D_{rs}h((m)) = (1 - q^{i(y_s-y_r)})\, h((m))$$

$$T_{rs}h((m)) = \sum_{\nu=0}^{\infty} q^{-\nu}q^{i\nu(y_s-y_r)}\, h((m))$$

$$= \frac{h((m))}{1-q^{-1-i(y_r-y_s)}}$$

Then from (5.3) and Proposition 3.1, for $(m) \in V_n$

$$(5.4a)\quad f((m)) = \int\cdots\int_Y \hat{f}(i(y)\, H((m),(y))\, dy_1\ldots dy_n$$

with

$$\text{(5.4b)}\quad H((m),(y)) = q^{\sum(j-\frac{n+1}{2})m_j}\left(\frac{\log q}{2\pi}\right)^n n!\left\{\sum_{\sigma\in S_n} q^{-i(y_\sigma)\cdot(m)}\prod_{s>r}\left(\frac{1-q^{i(y_{\sigma(s)}-y_{\sigma(r)})}}{1-q^{-1+i(y_{\sigma(s)}-y_{\sigma(r)})}}\right)\right\}$$

One has easily

$$\text{(5.5)}\quad H((o\ldots o),(y)) = n!\left(\frac{\log q}{2\pi}\right)\prod_{s>r}\left(\frac{q^{-iy_s}-q^{-iy_r}}{\left|1-q^{-1+i(y_s-y_r)}\right|_\infty^2}\right) \times \sum_\sigma \operatorname{sign}(\sigma)\prod_{s>r}\left(q^{iy_{\sigma(s)}}-q^{iy_{\sigma(r)}-1}\right)$$

in which $|\ |_\infty$ denotes ordinary absolute value, and, as usual, sign = +1 if $\sigma \in$ alternating group

= -1 if $\sigma \notin$ alternating group.

We wish to evaluate the sum in (5.5).

Consider the polynomial in $C[X_1,\ldots,X_n]$ given by,

$$(*)\quad P(X_1,\ldots,X_n) = \sum_\sigma \operatorname{sign}(\sigma)\prod_{n\geq s>r\geq 1}(X_{\sigma(s)} - cX_{\sigma(r)}),\ c\in C.$$

If $x_r = x_s$, $r \neq s$, then if $\tau = (r,s)\in S_n$

$$\prod(x_{\sigma(s)} - cx_{\sigma(r)}) = \prod(x_{\sigma\tau(s)} - cx_{\sigma\tau(r)}) .$$

But these terms enter $(*)$ with opposite signs, and so $P((x)) = 0$. Then $P((X))$ is divisible by $(X_s - X_r)$. We have

$$P(X_1,\ldots,X_n) = p(c)\prod_{s>r}(X_s - X_r) .$$

To calculate $p(c)$ we let $x_j = c^{-j}$, then

$$\prod_{s>r}(c^{-\sigma(s)} - c^{1-\sigma(r)}) = 0 \quad \text{unless } \sigma = (1).$$

So

$$P((x)) = \prod_{s>r}(c^{-s} - c^{1-r}) = p(c)\prod_{s>r}(c^{-s} - c^{-r})$$

which yields

$$p(c) = \prod_{j=1}^{n}\left(\frac{1-c^{j}}{1-c}\right) .$$

Applying this result to(5.5) we obtain

(5.6) $H((o\ldots),(y))$

$$= n!\left(\frac{\log q}{2\pi}\right)^{n}\prod_{j=1}^{n}\left(\frac{1-q^{-j}}{1-q^{-1}}\right)\prod_{s\neq r}\left|\frac{1 - q^{i(y_s-y_r)}}{1 - q^{-1+i(y_s-y_r)}}\right|$$

$$= 2^{n(n-1)}\, n!\left(\frac{\log q}{2\pi}\right)^{n}\prod_{j=1}^{n}\left(\frac{1-q^{-j}}{1-q^{-1}}\right)$$

$$\times\prod_{s>r}\left(\frac{\sin^{2}\left(\frac{y_s-y_r}{2}\log q\right)}{1-2q^{-1}\cos\left(\frac{y_s-y_r}{2}\log q\right)+q^{-2}}\right).$$

With these and (5.4b) in mind, we define the functions $M((z))$ and $\omega((m),(z))$ ($(m)\in Z^n$ fixed) on C^n, by

(5.7) $M((z))$

$$= n!\left(\frac{\log q}{2\pi}\right)^{n}\prod_{j=1}^{n}\left(\frac{1-q^{-j}}{1-q^{-1}}\right)\prod_{s\neq r}\left(\frac{q^{z_s} - q^{z_r}}{q^{z_s} - q^{-1+z_r}}\right)$$

(5.8) $\omega((m),(z))$

$$= q^{\sum(j-\frac{n+1}{2})m_j}\prod_{j=1}^{n}\left(\frac{1-q^{-1}}{1-q^{-j}}\right)\sum_{\sigma\in S_n}q^{-(z\sigma)\cdot(m)}\prod_{s>r}\left(\frac{q^{z_{\sigma(s)}}-q^{-1+z_{\sigma(r)}}}{q^{z_{\sigma(s)}}-q^{z_{\sigma(r)}}}\right) .$$

One has then, for $(y)\in R^n$, $(m)\in V_n$

(5.9a) $M(i(y)) = H((o\ldots o),(y))$

(5.9b) $$\omega((m),i(y)) = \frac{H((m),(y))}{H((o\ldots o),(y))} .$$

Note, by (5.6),

(5.10) $$0 \leq M(i(y)) \leq \underbrace{2^{n(n-1)} n! \left(\frac{\log q}{2\pi}\right)^n \prod_j \left(\frac{1-q^{-j}}{1-q^{-1}}\right) (1-q^{-1})^{-n(n-1)}}_{\text{(constant)}}$$

We see also that $\omega((m),(z))$ is an entire function. For, considering the polynomials in $C[X_1^{\pm 1},\ldots,X_n^{\pm 1}]$:

$$P_{(m)}((X)) = \sum_{\sigma} X_{\sigma(1)}^{m_1} \cdots X_{\sigma(n)}^{m_n} \ \text{sign}\ \sigma \prod_{s>r} (X_{\sigma(s)} - cX_{\sigma(r)}) .$$

Then (cf. proof for P(X)) this polynomial is divisible (in $C[X_1^{1},\ldots,X_n^{1}]$) by $\prod_{s>r}(X_s - X_r)$.

But

$$\frac{P_{(m)}((X))}{\prod(X_s - X_r)} = \sum_{\sigma} X_{\sigma(1)}^{m_1} \cdots X_{\sigma(n)}^{m_n} \prod_{s>r} \frac{(X_{\sigma(s)} - cX_{\sigma(r)})}{(X_{\sigma(s)} - X_{\sigma(r)})}$$

Comparing this with (5.8), we see that for all (m), $\omega((m),(z))$ is actually a polynomial in $\left\{q^{\pm z_j}\right\}_{1 \leq j \leq n}$.

We can make ω a function on G by letting

(5.11) $\omega(\, u\pi^{(m)}u_1 \,,\, (z)\,) = \omega((m),(z))$ for $(m) \in V_n$.

Then (5.4) and (5.9) yield

(5.12) $f(g) = \int \cdots \int_Y \hat{f}(i(y))\ \omega(g,i(y))\ M(i(y))\ dy_1 \dots dy_n$

which is an inverse of the transformation $f \longrightarrow \hat{f}$.

6. The Plancherel measure

By (5.9) and (5.11)

(6.1) $\omega(\,I\,,\,(z)) = \omega((o\ldots o),(z)) = 1$

(I = identity of G). Then (5.12) yields for $f \in L(G,U)$.

(6.2) $$f(I) = \int\cdots\int_Y \hat{f}(i(y))\, M(i(y))dy_1\ldots dy_n$$

Now by (1.14) and (1.15)

$$f{*}f^*(I) = \int_G |f(g)|_\infty^2 dg.$$

But $f{*}f^* \in L(G,U)$ and by (4.3) and (4.4)

$$f{*}f^*(i(y)) = |\hat{f}(i(y))|_\infty^2$$

Then (6.2) applied to $f{*}f^*$ yields

(6.3) $$\int_G |f(g)|_\infty^2\, dg = \int\cdots\int_Y |\hat{f}(i(y)|_\infty^2\, M(i(y))\, dy_1\ldots dy_n$$

It is known that L(G,U) is dense in $L_2(G,U)$ in the square norm. By Theorem O. , as f ranges over L(G,U), $\hat{f}(i(y)$ ranges over the symmetric Fourier polynomials in $\{q^{\pm iy_j}\}_{1\leq j\leq n}$.

But these functions (restricted to Y) are dense in the Hilbert space of functions which are square integrable on Y with respect to the measure $M(i(y))\,dy_1\ldots dy_n$. (cf. 5.10).

Then, with (6.3) we conclude

Theorem 1. The mapping $f \rightarrow \hat{f}$ may be extended uniquely to a unitary mapping of $L_2(G,U)$ onto the Hilbert space of (equivalence classes of) functions which are square $M(i(y))\, dy_1 \dots dy_n$ - integrable over Y.

7. The zonal spherical functions

A spherical function, ω, on G is called a zonal spherical function if the mapping

$$f \longrightarrow \hat{\omega}(f) = \int_G f(g)\ \omega(g^{-1})\ dg$$

is a homomorphism of the algebra L(G,U) onto C. It is known that every homomorphism of L(G,U) onto C is equivlent to $\hat{\omega}$ for a uniquely determined spherical function ω. (see [6, p.366] where this is proved in a more general context).

Consider again the functions $\omega(g,(z)), g \in G$. (section 5). We have noted already that $\omega(g,(z))$ is, for fixed g, a polynomial in $\{q^{\pm z_j}\}_{1 \leq j \leq n}$. By direct observation of (5.8) one has at once that it is actually a symmetric polynomial in these.

Now let τ be the permutation $\begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}$.

For $(m) \in Z^n$, one has

$$(\alpha) \quad \sum_{j=1}^{n} (j - \tfrac{n+1}{2})(-m_{\tau(j)}) = \sum_{j=1}^{n} (j - (\tfrac{n+1}{2}))\ m_j$$

$$(\beta) \quad (z_\sigma)\cdot -(m_\tau) = -(Z_{\sigma\tau})\cdot(m)$$

$$\prod_{s>r}\left(\frac{q^{z_{\sigma(s)}} - q^{-1+z_{\sigma(r)}}}{q^{z_{\sigma(s)}} - q^{z_{\sigma(r)}}}\right) = \prod_{s>r}\left(\frac{q^{z_{\sigma\tau(r)}} - q^{-1+z_{\sigma\tau(s)}}}{q^{z_{\sigma\tau(r)}} - q^{z_{\sigma\tau(r)}}}\right)$$

$$(\gamma) \qquad = \prod_{s>r}\left(\frac{q^{-z_{\sigma\tau(s)}} - q^{-1-z_{\sigma\tau(r)}}}{q^{-z_{\sigma\tau(s)}} - q^{-z_{\sigma\tau(s)}}}\right)$$

Noting also that $\sum_{\sigma\in S_n} = \sum_{\sigma\tau\in S_n}$, and applying ($\alpha$), ($\beta$), ($\gamma$), we see from (5.8)

$$\omega(-(m_\tau),(z)) = \omega((m),-(z)) .$$

Now if $(m)\in V_n$, $-(m_\tau)\in V_n$ so by (5.11)

$$\omega\left(\left(\pi^{(m)}\right)^{-1}, (z)\right) = \omega(-(m_\tau), (z)).$$

Thus

$$(7.1) \qquad \omega(g^{-1}, (z)) = \omega(g, -(z)).$$

It follows from (5.8) that

$$\overline{\omega}(g,(z)) = \omega(g,(\overline{z})).$$

In particular by (7.1)

$$(7.2) \qquad \overline{\omega}(g,i(y)) = \omega(g^{-1},i(y))$$

For $f \in L(G,U)$, (5.12) and (6.3) yield for $(y) \in R^n$.

$$\hat{f}(i(y)) = \int_G f(g)\ \overline{\omega}(g,i(y))\ dg$$

$$(7.3) \qquad = \int_G f(g)\ \omega(g^{-1},i(y))\ dg.$$

Note also, by section 6, (7.3) holds also for $f \in L_2(G,U)$ (up to equivalence in the norm).

Since $\omega(g^{-1},i(y))$, g fixed, is a polynomial in $\{q^{\pm iy_j}\}$, so is the right hand side of (7.3) for $f \in L(G,U)$. Then (7.3) implies

$$(7.4) \qquad \hat{f}((z)) = \int_G f(g)\ \omega(g^{-1},(z))\ dg$$

for all $(z) \in C^n$. Comparing (7.4) with Theorem 0., (ii), one concludes -

Theorem 2. The set of zonal spherical functions on G is exactly the set $\{\omega(g,(z))\}_{(z) \in C^n}$ (see equations (5.8) and (5.11)).

In (7.4), let now $f = f_{(m)}$, the characteristic function of $U\pi^{(m)}U$. We obtain for $(m) \in V_n$

$$(7.5) \qquad \hat{f}_{(m)}((z)) = \text{Meas}(U\pi^{(m)}U)\ \omega((\pi^{(m)})^{-1}(z))$$

$$= \text{Meas}(U\pi^{(m)}U)\ \omega((m), -(z)).$$

We know that

$$\omega((m),-(z)) = \sum_{(m')\in Z^n} K_{(m)}((m'))\, q^{(z)\cdot(m')}.$$

for some $\{K_{(m)}((m'))\} \subset C$. Then letting $(z) = i(y)$, by Fourier series

$$K_{(m)}((m')) = \left(\frac{\log q}{2\pi}\right)^n \int_0^{2\pi/\log q} \cdots \int_0^{2\pi/\log q} \omega((m),-i(y))\, q^{-i(y)\cdot(m')} dy_1 \cdots dy_n$$

$$= \left(\frac{\log q}{2\pi}\right)^n (\mathrm{Meas}(U\pi^{(m)}U))^{-1} \iint_{0\le y_j\le 2\pi/\log q} \hat{f}_{(m)}(i(y))\, q^{-i(y)(m')} dy_1 \cdots dy_n$$

Comparing this with (5.1)

(7.6) $$K_{(m)}((m')) = q^{\frac{n-1}{2}\sum m_j} (\mathrm{Meas}(U\pi^{(m)}U))^{-1}\, \tilde{f}_{(m)}((m')).$$

Remark. The only use we shall make of these last computations will be in the next section where we need the fact that $K_{(m)}((m')) \geq 0$. (This can be proved directly from (5.8) although not so easily). However, it is expected that (7.5), together with the explicit form of ω (5.8), may yield further knowledge of the algebra L(G,U). Note that (iii) of Theorem 0 follows easily from (5.8) and (7.5).

8. The bounded z. s. f.

Since $L(G,U)$ is dense in $L_1(G,U)$, the latter is also a commutative algebra. Now it is easily seen([6], pp. 376-7) that homomorphisms of this normed algebra are of the form

$$f \longrightarrow \hat{\omega}(f) = \int_G f(g)\omega(g^{-1})\, dg$$

where ω is a zonal spherical function, bounded on G. We wish then to determine the set of points, (z) ($\in C^n$) for which $\omega(g,(z))$ is bounded.

By (7.6) the coefficients, $K_{(m)}((m'))$, of the monomials, $q^{-(m')(z)}$, in $\omega((m),(z))$ are all positive. It follows that

(8.1) $$|\omega((m),(z))|_\infty \leq \omega((m),\mathrm{Re}(z))$$

where $\mathrm{Re}(z) = (\mathrm{Re}\, z_1,\ldots,\mathrm{Re}\, z_n) \in R^n$.

Let

$$t_j = \frac{n+1}{2} - j, \quad 1 \leq j \leq n$$

$$(t) = (t_1,\ldots,t_n) .$$

Since $q^{t_{j+1}} = q^{-1+t_j}$, the product

$$\prod_{s>r} \left(q^{t_{\sigma(s)}} - q^{-1+t_{\sigma(r)}}\right), \qquad \sigma \in S_n$$

is non-zero if and only if $\sigma = (1)$. Thus by (5.8)

$$\omega(g,(t)) = \prod_{j=1}^{n} \left(\frac{1-q^{-1}}{1-q^{-j}}\right) \prod_{s>r} \frac{q^{t_s} - q^{-1+t_r}}{q^{t_s} - q^{t_r}}$$

$$= 1$$

Since $\omega(g,(z))$ is symmetric in (z) -

(8.2) $\quad \omega(g,(t_\sigma)) = 1 \qquad$ for $\sigma \in S_n$.

Let H be the convex hull of the set $\{(t_\sigma)\}_{\sigma \in S_n}$.

By a theorem of complex variables ([1], p.109)

$$\max_{Re(z) \in H} |\omega(g,(z))|_\infty = \max_{Re(z) \in \{(t_\sigma)\}} |\omega(g,(z))|_\infty$$

Therefore, with (8.1), one concludes

(8.3) $\quad |\omega(g,(z))|_\infty \leq 1 \qquad$ for $Re(z) \in H$.

H contains the points (x) satisfying the conditions

(α) $\quad x_{\sigma(1)} + x_{\sigma(2)} + \ldots + x_{\sigma(p)} \leq \frac{p(n-p)}{2}$, $1 \leq p \leq n-1$, $\sigma \in S_n$

and

(β) $\quad x_1 + x_2 + \ldots + x_n = 0.$

This is clear since all the points (t_σ) satisfy these.

Assume now $(z) \in C^n$ and $Re(z)$ does not satisfy (α) or (β). We shall see that $\omega(g,(z))$ is then unbounded.

Suppose first (α) is not satisfied by $Re(z)$. Let

$(m) = (M,M,\ldots,M), \quad M \in Z.$

Then by (5.8)

$$\omega((m),(z)) = q^{-M(z_1+\ldots+z_n)} .$$

This is clearly unbounded if $Re(z_1)+\ldots+Re(z_n) \neq 0$.

With $Re(z) = (x)$ suppose now for some $\sigma \in S_n$, $1 \leq p \leq n-1$,

$x_{\sigma(1)} + x_{\sigma(2)} + \ldots + x_{\sigma(p)} > \frac{p(n-p)}{2}$. By the symmetry of

$\omega((m),(z))$ in (z) one may assume

$$x_1 \geq x_2 \geq \dots \geq x_n$$

It follows that

$$(\ast) \qquad x_1 + x_2 + \dots + x_p > \frac{p(n-p)}{2} .$$

We may further suppose

$$x_1 + x_2 + \dots + x_{p-1} \leq \frac{(p-1)(n-p+1)}{2}$$

Then $x_p > \frac{p(n-p)}{2} - \frac{(p-1)(n-p+1)}{2} = \frac{n+1-2p}{2}$.

If now $x_p = x_{p+1} = \dots = x_{p'} > x_{p'+1}$, $(\ast)$ and the last inequality imply

$$x_1 + x_2 + \dots + x_{p'} > \frac{p'(n-p')}{2} .$$

Hence we may assume that $(\ast)$ holds for some p such that

$$(\ast\ast) \qquad x_1 \geq x_2 \geq \dots \geq x_p > x_{p+1} \geq \dots \geq x_n .$$

Let $S_n^1 = \left\{ \sigma \in S_n \,\middle|\, \sigma\{1,2,\dots,p\} = \{1,2,\dots,p\} \right\}$

$S_n^2 = S_n - S_n^1$.

With $(m) = (\overbrace{M\dots M}^{p},0\dots 0)$, we consider the function $\omega((m),-(z))$. Separate the expression (cf.(5.8)) for $\omega((m),-(z))$ into two parts, ω_1, ω_2, corresponding to summation over S_n^1, S_n^2, respectively.

$$\omega_1 = c\, q^{-M\frac{p(n-p)}{2}} \sum_{\sigma \in S_n^1} q^{M(z_1+\dots+z_p)} \prod_{\substack{s>r \\ \{s,r\}\leq p \text{ or } \{s,r\}>p}} \frac{q^{z_{\sigma(r)}} - q^{-1+z_{\sigma(s)}}}{q^{z_{\sigma(r)}} - q^{z_{\sigma(s)}}}$$

where the constant c includes the non-zero and well defined (by **) product

$$\prod_{\substack{s>p \\ r\leq p}} \frac{q^{z_r} - q^{-1+z_s}}{q^{z_r} - q^{z_s}}$$

Then

$$\omega_1 = c\, q^{\frac{-Mp(n-p)}{2}} \sum_{\sigma\in S_n^1} q^{M(z_1+\ldots+z_p)} \prod \{\text{as above}\}$$

$$= c_1\, q^{M(z_1+\ldots+z_p - \frac{p(n-p)}{2})}$$

Note c_1 is again non-zero, since the last sum is (up to a positive constant) a product $\omega'((o\ldots),(z))\ \omega''((o\ldots o),(z\ ))$ $= 1$; ω', ω'' being the z. s. f. defined for G_p, G_{n-p} respectively. (cf. section 5; observe that S_n^1 corresponds naturally to $S_p \times S_{n-p}$).

The final equality implies, by (*), that ω_1 is unbounded as $M \to +\infty$.

It is not difficult to see, by virtue of (**) that

$$\frac{\omega_2}{\omega_1} \to 0 \quad \text{as } M \to +\infty$$

One concludes that $\omega(g,-(z)) = \omega_1 + \omega_2$ is unbounded; so by (7.1) $\omega(g,(z))$ is unbounded.

Let T_H be the complex tube of H, i.e. the set of $(z) \in C^n$, such that $Re(z) \in H$. Then T_H is the set of (z) such that

$$\begin{cases} Re\, z_{j_1} + Re\, z_{j_2} + \ldots + Re\, z_{j_p} \leq \frac{p(n-p)}{2} & \text{for} \\ \qquad 1 \leq j_1 < j_2 < \cdots < j_p \leq n, \quad 1 \leq p \leq n-1 & \\ Re\, z_1 + Re\, z_2 + \ldots + Re\, z_n = 0 & \end{cases}$$

We have

$$|\omega(g,(z))|_\infty \leq 1 \text{ if } (z) \in T_H$$

$$\omega(g,(z)) \text{ is unbounded if } (z) \notin T_H .$$

Thus -

Theorem 3. For $f \in L(G,U)$, f, given by,

$$\hat{f}((z)) = \int_G f(g)\ \omega(g^{-1},(z))\ dg$$

is a symmetric, continuous function on T_H.

The homomorphisms of the Banach algebra $L_1(G,U)$ to the complex field are of the form

$$f \rightarrow \hat{f}((c))$$

for $(c) \in T_H$. Two such homomorphisms $f \rightarrow \hat{f}((c))$, $f \rightarrow \hat{f}((c'))$ are identical if and only if

$$c'_j \equiv c_{\sigma(j)} \mod \left(\frac{2\pi i}{\log q}\right) \quad 1 \leq j \leq n, \text{ for some } \sigma \in S_n.$$

Proof. The last statement follows from properties of $\omega(g_\lambda(z))$. The rest has been proved.

□

Remark. The identification of the homomorphisms of $L_1(G,U)$ to C with the above region T_H appears quite similar to results obtained by Gelfand and Neumark([3], pp. 266-9) for complex groups.

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Index of Special Symbols

$|s|_\infty$ = ordinary absolute of complex number s.

$(u)\cdot(v) = u_1v_1+u_2v_2+ \dots +u_nv_n$ for $(u) = (u_1,u_2,\dots,u_n)$, $(v) = (v_1,v_2,\dots,v_n)$.

$h|V$ function h restricted to V.

Other notation is either standard or clear in context.

Biography

The author was born in Brooklyn, N.Y., in 1940. He received the Bachelor of Science Degree in Mathematics from the City College of New York in June 1960. As an undergraduate, Mr. Luks had represented the school in the Putnam Mathematical Competitions receiving two honorable mentions. He was graduated with honors in Mathematics and was elected to Phi Beta Kappa. The author has since attended the Massachusetts Institute of Technology, having been supported for three of the years by National Science Foundation fellowships.

During some summer vacations, Mr. Luks has been employed by the Air Force Cambridge Research Laboratories, Meteorological Research Laboratory and by the Mitre Corporation of Bedford, Mass. The former organization published one of his research reports entitled, "Transformation of the Equations of Motion of Meteorology Into Orthogonal Coordinates" (AFCRL-62-1109).

The author is employed for the academic year 1965-6 as an instructor at Tufts University.