VELOCITY MODULATION OF ELECTROMAGNETIC WAVES

by

FREDERIC RICHARD MORGENTHALER

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ABSTRACT

This thesis is concerned with the problem of electromagnetic wave propagation through a dielectric whose propagation constant varies as a function of time. Ferroelectrics give promise of realizing such media since a control voltage is capable of changing the permittivities, and therefore the phase velocity of any electromagnetic wave passing through them. Artificial dielectrics formed with diode junctions as the conducting "dipole" elements also appear to be of some interest. The velocity of propagation of these might be switched between discrete levels. No work has been done on this latter idea.

If the non-linear medium cannot respond to changes of the electric field of the propagating wave then the fields within such media will be linear and related through the corresponding linear Maxwell equations. These are solved for the general case when the permittivity and permeability vary independently with time. When μ and ϵ vary in such a manner so as to keep their ratio constant an exact solution to the wave equation is obtained. When the impedance is not invariant an exact solution is not possible, in general, and a closed form approximation is found. The field solutions are interpreted physically using the simple case of a step change in μ and ϵ to illustrate the fundamentals involved. The field energies and electromagnetic momenta are derived for such a velocity transient and it is seen that, in general, there is an energy change while the momentum remains constant. This energy change is the result of the work required to vary the dielectric parameters when there is an internal field and may be an increase or decrease.

The frequency deviation which results when a monochromatic wave is passed through a section of dielectric with non-constant velocity of propagation is taken up in detail. Approximate solutions are obtained if the electrical length of such a section is small and it is found that essentially linear phase modulation occurs. The solution is also found when the length is arbitrarily long and the permittivity of the medium sinusoidally modulated. The optimum length which gives the greatest frequency deviation is derived and is shown to be impractical when known ferroelectric materials are used.

The present state of knowledge of microwave ferroelectricity is reviewed and the author's attempts to measure the complex dielectric constant of certain barium titanate ceramics are discussed.

Available data are used to predict the performance of a modulator operating with a carrier of 3,000 and 10,000 Mcs. The results of a $PbSnO_3$ - $BaTiO_3$ dielectric modulator operating at 10,000 Mc are also given.

It appears that useful modulators are feasible at least for low power application. The maximum modulating rates to which the ferroelectric materials can respond are unknown.

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F. R. M.

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CHAPTER 1

INTRODUCTION

This thesis considers the problem of modulating the velocity of propagation of a dielectric, and the effects which arise when electromagnetic waves travel through such media.

A time varying velocity of propagation implies time variable permittivity and/or permeability and this immediately suggests that ferroelectrics or ferrites under the influence of external electric and magnetic fields respectively, might be suitable means of obtaining velocity modulated media. Other possibilities include the mechanical substitution of different dielectrics as a function of time and the use of time variable "artificial dielectrics." Kock and others(1) have shown that if small conducting particles such as metal spheres or discs are placed in a supporting foam dielectric the effective permittivity is higher than for the foam dielectric alone. If the physical dimensions of the objects are small compared to the wavelength of the electromagnetic field the artificial dielectric appears homogeneous. It seems possible to utilize diode junctions as the conducting elements. These would be switched "on" and "off" by means of the modulating field. The effective dielectric constant would then be double valued and a chopper modulator would be possible.

The most promising of the schemes listed appears to be the ferroelectric or ferrite since they can be modulated much more rapidly than a mechanical system and are continuously variable unlike the artificial diode dielectrics. Ferroelectrics appear to be better than ferrites because a biasing electric field is easier to provide than a magnetic one. Barium titanate ($BaTiO_3$) was chosen as the dielectric to experiment with since more information is available about its properties than other ferroelectrics.

Since the discovery of ferroelectricity in $BaTiO_3$ over ten years ago, many circuit applications have been proposed to exploit the properties of this non-linear dielectric. Thus "memory cells" for computers and dielectric amplifiers have emerged and the non-linear dielectric constant has been used to frequency modulate a signal by varying the capacitance of a simple tank circuit. All of these applications of ferroelectricity have been for relatively low frequency operation and the dielectric behavior in the microwave region has received rather limited attention.(2) This is presumably because measurement problems are severe and the fact that the higher dielectric loss is somewhat discouraging. Nevertheless, it is quite important that the dielectric parameters be obtained in these frequency ranges so that an estimate of what can and cannot be accomplished may be made.

This work consists of the theoretical solution of the modulation problem and the experimental results of preliminary investigations. Chapter 2 contains the mathematical solution to Maxwell's equations for the general case of independently time varying μ and ϵ. Chapter 3 considers the special case of a velocity step transient by physical reasoning and circuit concepts and evaluates the energy densities of the modulated waves. Chapter 4 derives the frequency variation of monochromatic waves passing through a dielectric slab whose velocity of propagation varies homogeneously as a function of time. Chapter 5 gives the experimental results and methods of the author and others to measure the dielectric parameters of $BaTiO_3$ ceramics in the 3,000 and 11,000 megacycle ranges. The results of an experimental modulator in the X band range are also given.

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CHAPTER 2

SOLUTION OF MAXWELL'S EQUATIONS WHEN THE PERMITTIVITY AND PERMEABILITY ARE TIME VARIABLE

The behavior of an electromagnetic wave passing through a dielectric with time varying velocity of propagation can be predicted if the solution to Maxwell's equations is known when the permittivity and permeability are functions of time. Maxwell's equations are given by

$$\nabla \cdot \overline{D} = \rho \tag{2-1}$$

$$\nabla \cdot \overline{B} = 0 \tag{2-2}$$

$$\nabla \times \overline{E} = -\dot{\overline{B}} \tag{2-3}$$

$$\nabla \times \overline{H} = \overline{J} + \dot{\overline{D}} \tag{2-4}$$

$$D = \epsilon \overline{E} \tag{2-5}$$

$$\overline{B} = \mu \overline{H} \tag{2-6}$$

Assume a charge and current free region where μ and ϵ of the medium are functions of space and time. Then $\rho = 0$ and $J = 0$.

Taking the curl of both sides of Eq. (2-3) and substituting Eq. (2-6) yields

$$\nabla \times \nabla \times \overline{E} = - \frac{\delta}{\delta t} \left[\nabla \times (\mu \overline{H}) \right] \tag{2-7}$$

Making use of the vector identity

$$\nabla \times \nabla \times \bar{A} = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}$$

there results

$$\nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E} = -\frac{\delta}{\delta t}\left[\nabla \times (\mu \bar{H})\right] \tag{2-8}$$

Equations (2-1) and (2-2) may be expanded

$$\nabla \cdot \bar{D} = \nabla \cdot (\epsilon \bar{E}) = \bar{E} \cdot \nabla \epsilon + \epsilon \nabla \cdot \bar{E} = 0 \tag{2-9}$$

$$\nabla \cdot \bar{B} = \nabla \cdot (\mu \bar{H}) = \bar{H} \cdot \nabla \mu + \mu \nabla \cdot \bar{H} = 0 \tag{2-10}$$

If the uniform plane wave solution is sought (T E M) there will exist one component of $\bar{E}$, one orthogonal component of $\bar{H}$ and no others.

Assume $E = E_x(z,t)$ and $H = H_y(z,t)$

If the permeability and permittivity are permitted to vary with respect to distance in the z direction only and to vary with respect to time, then $\mu = \mu(z,t)$, $\epsilon = \epsilon(z,t)$ and

$$\nabla \epsilon = \bar{k}\frac{\delta \epsilon}{\delta z} \quad ; \quad \nabla \mu = \bar{k}\frac{\delta \mu}{\delta z}$$

Since $\bar{E} = \bar{i}E_x$ and $\bar{H} = \bar{j}H_y$ (where $\bar{i}$, $\bar{j}$, and $\bar{k}$ are the orthogonal unit vectors in xyz space) it follows that $\bar{E} \cdot \nabla \epsilon$ and $\bar{H} \cdot \nabla \mu$ are both zero and Eqs. (2-9) and (2-10) reduce to

$$\nabla \cdot \bar{E} = 0 \tag{2-11}$$

$$\nabla \cdot \bar{H} = 0 \tag{2-12}$$

Making use of this last result and the well known identity $\nabla \times (\mu \bar{H}) = \nabla \mu \times \bar{H} + \mu \nabla \times \bar{H}$, Eq. (2-8) becomes

$$\nabla^2 \bar{E} = \frac{\delta}{\delta t}\left[\nabla \mu \times \bar{H} + \mu \nabla \times \bar{H}\right] \tag{2-13}$$

where

$$\nabla \mu \times \bar{H} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & \frac{\delta \mu}{\delta z} \\ 0 & Hy & 0 \end{vmatrix} = -\bar{i}\frac{\delta \mu}{\delta z} Hy \tag{2-14}$$

and $$\nabla^2 \bar{E} = \bar{i}\,\frac{\delta^2 E_x}{\delta z^2} \qquad (2\text{-}15)$$

These results plus Eq. (2-4) transform Eq. (2-13) into

$$\bar{i}\,\frac{\delta^2 E_x}{\delta z^2} = \frac{\delta}{\delta t}\left[-\,\bar{i}\,\frac{\delta\mu}{\delta z}H_y + \bar{i}\,\mu\dot{D}_x\right]; \quad \frac{\delta^2 E_x}{\delta z^2} + \frac{\delta}{\delta t}\left(\frac{\delta\mu}{\delta z}\right)H_y + \frac{\delta\mu}{\delta z}\dot{H}_y$$

$$= \mu\epsilon\ddot{E}_x + (\dot{\mu}\epsilon + 2\mu\dot{\epsilon})E_x + (\mu\ddot{\epsilon} + \dot{\mu}\dot{\epsilon})E_x \qquad (2\text{-}16)$$

By starting with Eq. (2-4) instead of Eq. (2-3) and following the corresponding steps there will result

$$\frac{\delta^2 H_y}{\delta z^2} + \frac{\delta}{\delta t}\left(\frac{\delta\epsilon}{\delta z}\right)E_x + \frac{\delta\epsilon'}{\delta z}E_x$$

$$= \mu\epsilon\ddot{H}_y + (\mu\dot{\epsilon} + 2\ddot{\mu}\epsilon)\dot{H}_y + (\mu\ddot{\epsilon} + \dot{\mu}\dot{\epsilon})H_y \qquad (2\text{-}17)$$

The dot notation is used to indicate partial differentiation with respect to time.

These two partial differential equations describe the electric and magnetic fields within the time varying dielectric. It is obvious that the usual procedure of assuming product solutions will not work because the hoped for separation is rendered impossible by the existence of the middle term on the left hand sides of Eqs. (2-16) and (2-17).

Since the general problem under consideration is one of finding the frequency variation of a monochromatic wave passing through a slab of finite thickness whose velocity of propagation varies homogeneously with time, it will be sufficient to solve the equations for the case where μ and ϵ are not functions of z and then match the boundary conditions. Under the conditions of space invariancy, Eqs. (2-16) and (2-17) become (dropping subscript notation)

$$\frac{\delta^2 E}{\delta z^2} = \mu\epsilon \ddot{E} + (\dot{\mu}\epsilon + 2\mu\dot{\epsilon})\dot{E} + (\mu\ddot{\epsilon} + \dot{\mu}\dot{\epsilon})E \qquad (2\text{-}18)$$

and

$$\frac{\delta^2 H}{\delta z^2} = \mu\epsilon \ddot{H} + (\mu\dot{\epsilon} + 2\dot{\mu}\epsilon)\dot{H} + (\ddot{\mu}\epsilon + \dot{\mu}\dot{\epsilon})H \qquad (2\text{-}19)$$

This pair of equations is separable. Consider the E field and assume

$$E = Z_E(z)\, T_E(t)$$

Then

$$\frac{Z''_E}{Z_E} = \mu\epsilon \frac{T''_E}{T_E} + (\dot{\mu}\epsilon + 2\mu\dot{\epsilon})\frac{T'_E}{T_E} + (\mu\ddot{\epsilon} + \dot{\mu}\dot{\epsilon}) = -\beta^2 \qquad (2\text{-}20)$$

The primes denote differentiation with respect to the arguments and β is the separation constant. Equations (2-21) and (2-22) give the space variation differential equation and the familiar solution respectively. In general, β may take on a series of eigenvalues which are determined by the boundary conditions. The general solution is therefore an infinite series of which Eq. (2-22) is a typical term.

$$Z''_E + \beta^2 Z_E = 0 \qquad (2\text{-}21)$$

$$Z_E = Ae^{+j\beta z} + Be^{-j\beta z} \qquad (2\text{-}22)$$

The time variation is

$$\frac{T_E''}{T_E} + \left(\frac{\dot{\mu}\epsilon + 2\mu\dot{\epsilon}}{\mu\epsilon}\right)\frac{T_E'}{T_E} + \left(\frac{\mu\ddot{\epsilon} + \dot{\mu}\dot{\epsilon} + \beta^2}{\mu\epsilon}\right) = 0 \qquad (2\text{-}23)$$

Equation (2-23) is a standard second order linear differential equation with non-constant coefficients of the form

$$T_E'' + a(t)\,T_E' + b(t)\,T_E = 0 \qquad (2\text{-}24)$$

with

$$a(t) = \frac{\dot{\mu}\epsilon + 2\mu\dot{\epsilon}}{\mu\epsilon} \quad ; \quad b(t) = \frac{\mu\ddot{\epsilon} + \dot{\mu}\dot{\epsilon} + \beta^2}{\mu\epsilon}$$

Any second order equation of this form can be subjected to a transformation which causes the first derivative term to vanish.(1) Let $T_E = W_E\, e^{-\frac{1}{2}\int a(t)dt}$ then

$$W_E + \lambda_E(t)\,W_E = 0 \qquad (2\text{-}25)$$

where

$$\lambda_E(t) = b(t) - \frac{1}{4}a^2 - \frac{1}{2}\dot{a}$$

For Eq. (2-24) $e^{-\frac{1}{2}\int a dt} = \frac{1}{\epsilon\sqrt{\mu}}$ and $\lambda_E(t) = \frac{\beta^2}{\mu\epsilon} + \frac{1}{4}\left(\frac{\dot{\mu}}{\mu}\right)^2 - \frac{1}{2}\left(\frac{\ddot{\mu}}{\mu}\right)$

so that the electric field can be written in the form

$$\left.\begin{aligned} E(z,t) &= \frac{W_E}{\epsilon\sqrt{\mu}}\left(A\,e^{+j\beta z} + B\,e^{-j\beta z}\right) \\ W_E &+ \left[\frac{\beta^2}{\mu\epsilon} + \frac{1}{4}\left(\frac{\dot{\mu}}{\mu}\right)^2 - \frac{1}{2}\left(\frac{\ddot{\mu}}{\mu}\right)\right] W_E = 0 \end{aligned}\right\} \qquad (2\text{-}26)$$

In exactly the same manner the magnetic field is found to be

$$H(z,t) = \frac{W_H}{\mu\sqrt{\epsilon}} (A_e^{+j\beta z} + B_e^{-j\beta z}) \quad (2\text{-}27)$$

$$\ddot{W}_H + \left[\frac{\beta^2}{\mu\epsilon} + \frac{1}{4}\left(\frac{\dot{\epsilon}}{\epsilon}\right)^2 - \frac{1}{2}\left(\frac{\ddot{\epsilon}}{\epsilon}\right)\right] W_H = 0$$

It is reassuring to note that if μ and ϵ are constants Eqs. (2-26) and (2-27) reduce to the familiar wave equations.

Special Case

From the form of Eqs. (2-26) and (2-27) it is obvious that W_E will equal W_H only if μ and ϵ are constants or if their ratio is always constant. If $\frac{\mu(t)}{\epsilon(t)} = \eta^2$ = constant, then it is seen that the ratio of the electric field to the magnetic field is a constant and so the two fields are everywhere in space and time phase.

$$E(z,t) = \eta\, H(z,t) \quad (2\text{-}28)$$

This relation is true only for the very special case when the impedance, η, of the time varying medium is always constant. Under these conditions an exact solution of the fields is possible and is

$$E(z,t) = \eta\, H(z,t) = \frac{A}{\epsilon}\, e^{\pm j\beta z}\, e^{\pm j\frac{\beta}{\eta}\int\frac{dt}{\epsilon}} \quad (2\text{-}29)$$

β again takes on eigenvalues subject to the boundary conditions and a series of terms like Eq. (2-29) is the general solution which can be verified by direct substitution. There will be no reflections as long as the impedance of the dielectric remains constant and strictly progressive waves are possible. Since no physical materials are available whose permeability and permittivity can both be varied simultaneously so as to keep η constant, the result is largely of academic interest but one which sheds a great deal of light upon the general problem of time varying dielectrics.*

It is seen that

$$\frac{1}{\eta} \int \frac{dt}{\epsilon} = \int v dt \tag{2-30}$$

Differentiating

$$v(t) = \frac{1}{\sqrt{\mu(t)\,\epsilon(t)}} = \frac{1}{\eta\epsilon} \tag{2-31}$$

The velocity of propagation is given by the same form as when μ and ϵ are constant.

The total phase of the wave given by Eq. (2-29) is

$$\phi = \frac{\beta}{\eta} \int \frac{dt}{\epsilon}$$

and the instantaneous frequency is given by

$$\omega(t) = \frac{d\phi}{dt} = \frac{\beta}{\sqrt{\mu\epsilon}} = \beta v(t) \tag{2-32}$$

This indicates that the frequency is simply proportional to the velocity of propagation and this is true if it is remembered that the derivation was based upon the assumption that μ and ϵ did not vary with position. This implies that the medium is infinite in extent and, moreover, that any wave now in the dielectric has always been there and has

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* It is interesting in this connection to speculate on a ceramic dielectric made up of a mixture of ferroelectric and ferrite materials under the influence of both electric and magnetic control fields.

been influenced by any variation in velocity that has occurred since the infinite past. It is appropriate to point out here that the separation of the partial differential equation implies that the space variation of the wave is unaffected by any changes in μ and ϵ . Consider in connection with this that at some point in the distant past a wave train of length L and frequency f_1 was started in the medium and that at that time μ and ϵ were stationary with time. This wave train is characterized by the frequency f_1 and some constant velocity of propagation V_1. It therefore has a wavelength $\lambda_1 = \frac{V_1}{f_1}$. Now suppose that the velocity of propagation suddenly changes to some new value V_2. All portions of the original wave train will be acted upon simultaneously, that is, slowed down or speeded up together. The new wave train will therefore still be L meters long and the space waveform will not have changed. This means that the wavelength is still the same value λ_1, but because $V = f\lambda$ it follows that the frequency must have changed to a value $f_2 = \frac{V_2}{\lambda_1}$. Since $f_1 = \frac{V_1}{\lambda_1}$ it must be that $\frac{f_2}{f_1} = \frac{V_2}{V_1}$. Equation (2-32) is merely expressing this fact in general terms. As long as the original wave stays in the medium its frequency will follow the velocity changes of the medium. If a fresh wave enters the dielectric it is, of course, not subject to the past history of the medium. For example, if a new wave train of length L and frequency f_1 (as before) enters the dielectric <u>after</u> the velocity has changed from V_1 to V_2, then its frequency will still be f_1 and its wavelength will change to $\lambda_2 = \frac{V_2}{V_1}\lambda_1$. The total length will no longer be L but $\frac{V_2}{V_1}L$ meters. If now the velocity changes to some new value the frequency will change accordingly and the wavelength remain constant.

The exact solution obtained for the special case of constant impedance is illuminating but not very useful since in practice the impedance will not remain constant. It is desirable to solve Eqs. (2-26) and (2-27) for the general case when μ and ϵ vary independently with time. No exact solution is possible in general and the task remains to find a suitable approximation. A series solution is very difficult to interpret physically, therefore a closed form solution is preferable. Since the equation to be solved is a second order linear differential equation, the Liouville approximation offers hope and turns out to be entirely suitable.

Liouville Approximation[2]

It has been shown previously that any second order linear differential equation can be transformed to the form

$$y'' + \lambda(x)y = 0 \qquad (2\text{-}33)$$

If a new variable ϕ is introduced so that

$$\phi = \int \sqrt{\lambda}\, dx \qquad (2\text{-}34)$$

then Eq. (2-33) can be rewritten as

$$\frac{d^2y}{d\phi^2} + \frac{1}{\sqrt{\lambda}}\frac{d}{d\phi}(\sqrt{\lambda})\frac{dy}{d\phi} + y = 0 \qquad (2\text{-}35)$$

This transformation of variable has normalized the coefficient of y to unity. If use is made of Eq. (2-25) it is evident that Eq. (2-35) is equivalent to

$$y = \hat{y}\, e^{-\frac{1}{2} \int \frac{d\sqrt{\lambda}}{\sqrt{\lambda}}} = \frac{\hat{y}}{\sqrt[4]{\lambda}} \tag{2-36}$$

where

$$\frac{d^2\hat{y}}{d\phi^2} + \gamma(\phi)\hat{y} = 0 \tag{2-37}$$

Then $$\gamma(\phi) = 1 + \frac{3}{16}\frac{1}{\lambda^2}\left(\frac{d\lambda}{d\phi}\right)^2 - \frac{1}{4}\frac{1}{\lambda}\frac{d^2\lambda}{d\phi^2}$$

or $$\gamma(x) = 1 + \frac{5}{16}\frac{1}{\lambda}\left(\frac{\lambda'}{\lambda}\right)^2 - \frac{1}{4}\frac{\lambda''}{\lambda^2} \tag{2-38}$$

If the last two terms of Eq. (2-38) are small compared to unity then $\gamma \simeq 1$ and

$$\hat{y} \simeq A e^{\pm j \int \sqrt{\lambda}\, dx} \tag{2-39}$$

or $$y \simeq \frac{A}{\sqrt[4]{\lambda}} e^{\pm j \int \sqrt{\lambda}\, dx} \tag{2-40}$$

The equations of interest, Eqs. (2-26) and (2-27) are repeated for convenience.

$$\ddot{W}_E + \left[\frac{\beta^2}{\mu\epsilon} + \frac{1}{4}\left(\frac{\dot{\mu}}{\mu}\right)^2 - \frac{1}{2}\left(\frac{\ddot{\mu}}{\mu}\right)\right] W_E = 0 \tag{2-26}$$

$$\ddot{W}_H + \left[\frac{\beta^2}{\mu\epsilon} + \frac{1}{4}\left(\frac{\dot{\epsilon}}{\epsilon}\right)^2 - \frac{1}{2}\left(\frac{\ddot{\epsilon}}{\epsilon}\right)\right] W_H = 0 \tag{2-27}$$

If the Liouville approximation is valid, then

$$1 >> \frac{5}{16}\frac{1}{\lambda}\left(\frac{\dot{\lambda}}{\lambda}\right)^2 - \frac{1}{4}\left(\frac{\ddot{\lambda}}{\lambda^2}\right) \tag{2-41}$$

It can be shown that if λ_E and λ_H of Eqs. (2-26) and (2-27) respectively fulfill the condition that $\lambda_E \simeq \lambda_H \simeq \frac{\beta^2}{\mu\epsilon}$ or $W_E \simeq W_H = W$, then Eq. (2-41) is automatically satisfied and

$$\left.\begin{aligned} &\ddot{W} + \frac{\beta^2}{\mu\epsilon} W \simeq 0 \\ &W \simeq B\sqrt[4]{\mu\epsilon}\; e^{\pm j\int \frac{\beta}{\sqrt{\mu\epsilon}}\, dt} \end{aligned}\right\} \tag{2-42}$$

Since $E(z,t) = \frac{W_E}{\epsilon\sqrt{\mu}} Ae^{\pm j\beta z}$ and $H(z,t) = \frac{W_H}{\mu\sqrt{\epsilon}} Ae^{\pm j\beta z}$ the completed approximate solution is

$$E(z,t) \quad \frac{A}{\sqrt[4]{\mu\epsilon^3}} e^{\pm j\beta z} e^{\pm j\beta \int \frac{dt}{\sqrt{\mu\epsilon}}} \tag{2-43}$$

and

$$H(z,t) \quad \frac{A}{\sqrt[4]{\mu^3\epsilon}} e^{\pm j\beta z} e^{\pm j\beta \int \frac{dt}{\sqrt{\mu\epsilon}}} \tag{2-44}$$

For slowly varying μ and ϵ reflections are small and progressive waves are possible.

As before

$$\int v\, dt = \int \frac{dt}{\sqrt{\mu\epsilon}} \tag{2-45}$$

Differentiation yields

$$v(t) = \frac{1}{\sqrt{\mu\epsilon}} \tag{2-46}$$

The total phase is

$$\phi = \beta \int \frac{dt}{\sqrt{\mu\epsilon}} \tag{2-47}$$

and the instantaneous frequency

$$\omega(t) = \frac{d\phi}{dt} = \frac{\beta}{\sqrt{\mu\epsilon}} = \beta v(t) \tag{2-48}$$

As was shown previously for the special case of constant impedance, the instantaneous frequency is proportional to the velocity of propagation. The physical interpretation, utilizing a medium of infinite extent, given before, is applicable in this situation too. The previous remarks concerning eigenvalues of β apply here also so that Eqs. (2-43), (2-44), (2-47) and (2-48) are in general infinite series.

Sinusoidal Variation of the Permittivity

1. Constant Impedance

Let the permittivity be given by

$$\epsilon = K\epsilon_o\,(1 + b\sin\omega_m t) = \frac{1}{\eta^2}\mu \tag{2-49}$$

In this expression ϵ_o is the permittivity of free space; K is the dielectric constant at the operating point; b is the modulating index and ω_m is the modulating frequency. Since ϵ is postulated never to become less than ϵ_o it follows that $b_{max} = \frac{K-1}{K}$ is always less than unity.

Substituting Eq. (2-49) into Eq. (2-29) results in

$$E(z,t) = \eta H(z,t)$$

$$= \frac{A}{1 + b\sin\omega_m t}\, e^{\pm j\beta z}\, e^{\pm j\frac{\beta}{K\epsilon_o\eta}\int\frac{dt}{1 + b\sin\omega_m t}} \tag{2-50}$$

The velocity of propagation is given by

$$v(t) = \frac{1}{\eta K\epsilon_o\,(1 + b\sin\omega_m t)} \tag{2-51}$$

and the instantaneous frequency by

$$\omega(t) = \frac{\beta}{\eta k \epsilon_o (1 + b \sin \omega_m t)} \tag{2-52}$$

2. Approximate Solution when Permeability is Constant

The Liouville approximation is valid if

$$\frac{\beta^2}{\mu\epsilon} >> \frac{1}{4}\left(\frac{\dot{\epsilon}}{\epsilon}\right)^2 - \frac{1}{2}\left(\frac{\ddot{\epsilon}}{\epsilon}\right) \tag{2-53}$$

For the case when $\mu = \mu_o$ and $\epsilon = k\epsilon_o(1 + b \sin \omega_m t)$, Eq. (2-53) requires that

$$\frac{4\beta^2}{\mu_o} >> \frac{b^2 K_o \omega_m^2 (1 + \sin^2 \omega_m t) + 2bK\epsilon_o \omega_m^2 \sin \omega_m t}{(1 + b \sin \omega_m t)} \tag{2-54}$$

The maximum value of the right hand side occurs when $\cos \omega_m t = 0$. The severest requirement is therefore

$$\frac{4\beta^2}{\mu_o} >> 2bk\epsilon_o \omega_m^2 \tag{2-55}$$

The constant β may be evaluated under the conditions of a sinusoidal carrier of frequency ω_c passing through a medium whose operating point velocity is $\frac{1}{\sqrt{k\mu_o\epsilon_o}}$. In this instance the boundary conditions are chosen so that β has only one value.

$$\beta = \frac{\omega_c}{V} = \omega_c \sqrt{k\mu_o \epsilon_o} \tag{2-56}$$

Substituting this value into the inequality Eq. (2-55) gives the result

$$\left(\frac{\omega_c}{\omega_m}\right)^2 >> \frac{b}{2} \tag{2-57}$$

The maximum value of b approaches unity as a limit. If $(\frac{\omega_c}{\omega_m})^2 >> \frac{1}{2}$ the approximation will be valid under all conditions.

Equations (2-43) and (2-44) give as the fields

$$E(z,t) \simeq A(1 + b \sin \omega_m t)^{-\frac{3}{4}} e^{\pm j\omega_c \sqrt{k\mu_o \epsilon_o} z} e^{\pm j\omega_c \int \frac{dt}{\sqrt{1 + b \sin \omega_m t}}} \tag{2-58}$$

and

$$\frac{E}{H} \simeq \eta \frac{1}{\sqrt{1 + b \sin \omega_m t}} \tag{2-59}$$

The velocity of propagation and instantaneous frequency are given by

$$v(t) = \frac{V_o}{\sqrt{1 + b \sin \omega_m t}} \tag{2-60}$$

with $V_o = \frac{1}{\sqrt{k\mu_o \epsilon_o}}$

and

$$\omega(t) = \frac{\omega_c}{\sqrt{1 + b \sin \omega_m t}} \tag{2-61}$$

References

(1) Schelkunoff, S.A., Applied Mathematics for Engineers and Scientists, Chapter II. D. Van Nostrand, 1948.

(2) Schelkunoff, S.A., Applied Mathematics for Engineers and Scientists, p.210. D. Van Nostrand, 1948.

CHAPTER 3

SOLUTIONS BASED ON PHYSICAL REASONING

The fact that the general partial differential Equations (2-18) and (2-19) were separable led to the physical interpretation that the space variation of a wave was invariant after it once entered a time varying dielectric. The transition across the boundary certainly will cause space distortion, but once this has happened no further perturbations of space wave-form will occur until the wave leaves the medium. During the journey through the dielectric all of the individual frequency components of the wave will follow the variations of the velocity of propagation. The physical picture of the phase variations of the electric and magnetic fields is fairly straightforward, namely since the wave length remains constant and the velocity does not, the frequency must change to fulfill the condition $v = f\lambda$. The important point to be realized is that there is nothing sacred about the frequency remaining invariant as a wave passes through a series of different dielectrics. That this is so in the usual case is due only to the fact that the velocity of propagation is not a function of time.

It is desirable to understand physically why the amplitudes of the electric and magnetic fields vary as they do. Consideration of the instantaneous flux and charge offers a convenient method of obtaining this physical picture. The simple step transient depicted in Fig. 1 offers an easily analyzed example which demonstrates all the relevant principles. The transient electromagnetic wave is propagated through a variable dielectric whose constants are given as functions of time. Both μ and ϵ are assumed to step from their initial values μ_1 and ϵ_1 to μ_2 and ϵ_2 respectively. The initial velocity of propagation is $V_1 = \frac{1}{\sqrt{\mu_1 \epsilon_1}}$ and the final value is $V_2 = \frac{1}{\sqrt{\mu_2 \epsilon_2}}$. Since the velocity is constant except at the jump, the standard wave equation must apply except at the discontinuity. From the previous discussions it is clear that the space wave-form but not the amplitude of the transient will be invariant.

At the instant of the jump it is necessary that the total charge Q and the total flux ψ remain constant. An invariant Q and ψ implies that D and B respectively do not change instantaneously.

Before the step $(t < t_o)$

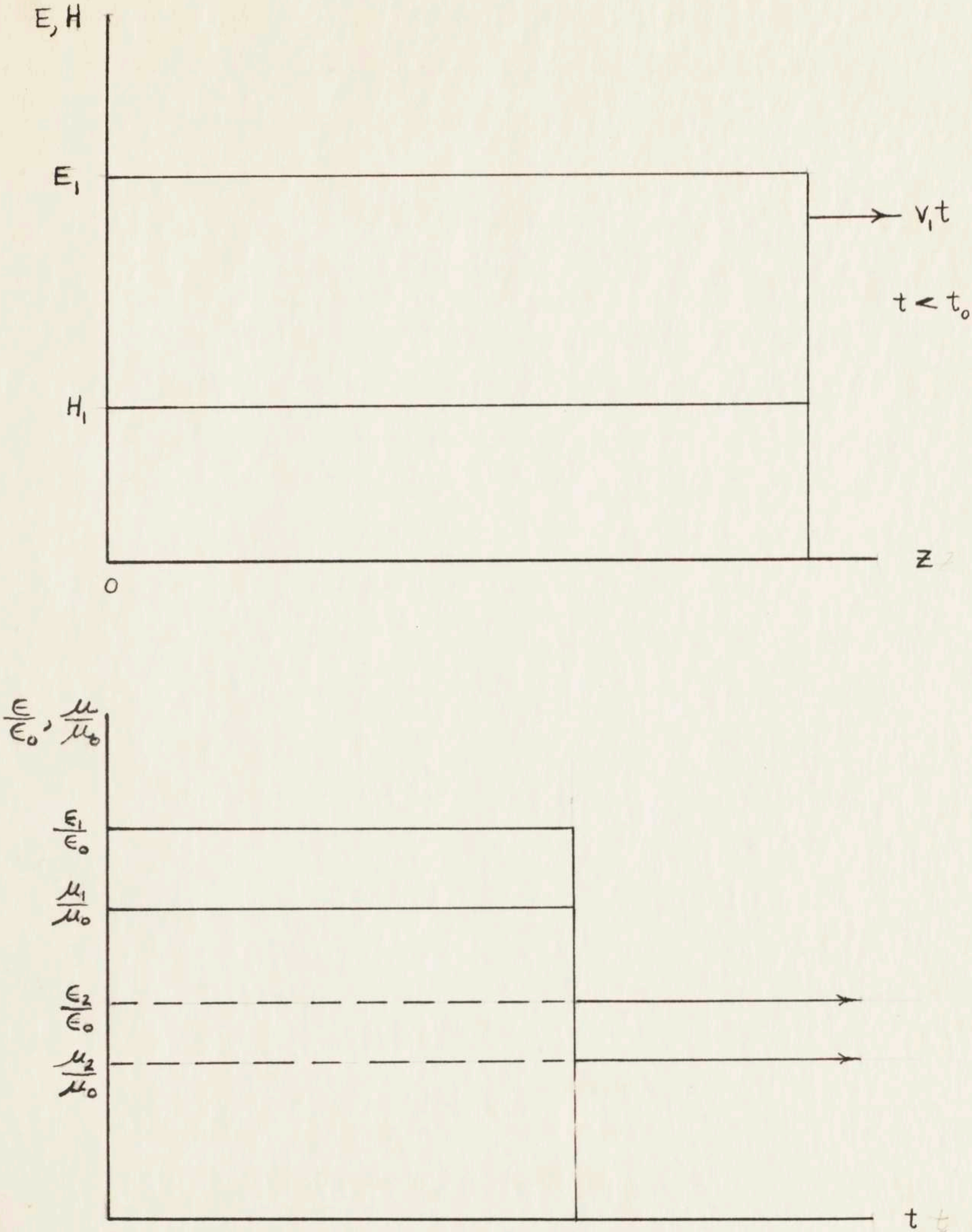
$$B = \mu_1 H_1 \tag{3-1}$$

$$D = \epsilon_1 E_1 \tag{3-2}$$

After the step $(t > t_o)$

$$B = \mu_2 H_2 \tag{3-3}$$

$$D = \epsilon_2 E_2 \tag{3-4}$$



E, H
E_1
H_1
$v_1 t$
$t < t_0$
0
z
$\frac{\epsilon}{\epsilon_0}, \frac{\mu}{\mu_0}$
$\frac{\epsilon_1}{\epsilon_0}$
$\frac{\mu_1}{\mu_0}$
$\frac{\epsilon_2}{\epsilon_0}$
$\frac{\mu_2}{\mu_0}$
0
t_0
t

FIG 1

The most general form of E_2 and H_2 is for both fields to have a backward as well as a forward travelling wave component.

$$H_2 = H_2^+ - H_2^- \tag{3-5}$$

$$E_2 = E_2^+ + E_2^- \tag{3-6}$$

A difference in sign of the backward components is necessary because one of the field components changes phase by 180° upon reflection and the other does not.

The characteristic impedances of the dielectric are

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} = \frac{E_1}{H_1} \quad t < t_o \tag{3-7}$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}} = \frac{E_2}{H_2} \quad t > t_o \tag{3-8}$$

The combination of Eqs. (3-1), (3-2), (3-3), and (3-4) yields

$$B = \mu_1 H_1 = \mu_2 (H_2^+ - H_2^-) \tag{3-9}$$

$$D = \epsilon_1 E_1 = \epsilon_2 (E_2^+ + E_2^-) \tag{3-10}$$

Dividing Eq. (3-1) by Eq. (3-2) and making use of Eq. (3-7) gives

$$B = \eta_1 D \tag{3-11}$$

Substituting Eq. (3-11) into Eq. (3-9) and combining with Eq. (3-10) results in the following set of equations

$$\frac{\eta_1 \eta_2}{\mu_2} D = E_2^+ - E_2^- \tag{3-12}$$

$$\frac{1}{\epsilon_2} D = E_2^+ + E_2^- \tag{3-13}$$

The solution for this pair of equations for E_2^+ and E_2^- gives*

$$E_2^+ = \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}\right)E_1 = \eta_2\,\frac{1}{2}\left(\frac{\mu_1}{\mu_2} + \sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}\right)H_1 = \eta_2 H_2^+ \qquad (3\text{-}14)$$

$$E_2^- = \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} - \sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}\right)E_1 = \eta_2\,\frac{1}{2}\left(\frac{\mu_1}{\mu_2} - \sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}\right)H_1 = \eta_2 H_2^- \qquad (3\text{-}15)$$

For the special case when $\eta_1 = \eta_2 = \eta_o$ or equivalently $\frac{\mu_1}{\mu_2} = \frac{\epsilon_1}{\epsilon_2}$ Eqs. (3-14) and (3-15) reduce to

$$E_2^+ = \frac{\epsilon_1}{\epsilon_2}\,E_1 = \frac{\mu_1}{\mu_2}\,\eta_o\,H_1 = \eta_o\,H_2^+ \qquad (3\text{-}16)$$

$$E_2^- = H_2^- = 0 \qquad (3\text{-}17)$$

Under these conditions of constant impedance there is no reflected wave.

It is worth pointing out that the general solution obtained for this case agrees with Eq. (3-16). From Eq.(2-29) $|E_1| = \frac{A}{\epsilon_1}$ and $|E_2| = \frac{A}{\epsilon_2}$; it follows that $|E_2| = \frac{\epsilon_1}{\epsilon_2}|E_1|$. The amplitude of D is constant as is seen by writing Eq.(2-29) in the alternate form

$$D = \epsilon E = Ae^{\pm j\beta z}\; e^{\pm j\frac{\beta}{\eta}\int\frac{dt}{\epsilon}} \qquad (3\text{-}18)$$

For the case where $\mu_1 = \mu_2$ Eqs. (3-14) and (3-15) become

$$E_2^+ = \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \sqrt{\frac{\epsilon_1}{\epsilon_2}}\right)E_1 = \eta_2 H_2^+ \qquad (3\text{-}19)$$

$$E_2^- = \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} - \sqrt{\frac{\epsilon_1}{\epsilon_2}}\right)E_1 = \eta_2 H_2^- \qquad (3\text{-}20)$$

* A difference equation may be formulated on the basis of an incremental jump in the velocity. In the limit of a differential step this passes to the differential equation.

If ϵ_1 and ϵ_2 do not differ greatly, E_2^- will be small compared to E_2^+ and may be neglected without great loss in accuracy. In that event E_2^+ can be approximated by

$$E_2^+ \simeq \frac{\epsilon_1}{\epsilon_2} E_1 \simeq \eta_2 H_2^+ \tag{3-21}$$

This indicates that the exact solution obtained for the case of constant impedance may be a reasonable approximation when one of the dielectric parameters remains invariant. The Liouville approximation, previously derived, is of slightly different form in that the amplitudes of D and B are not constant. This discrepancy will be discussed in detail in a later section.

The previous results can be immediately applied to a travelling wave of initial frequency ω_1 and velocity V_1 which undergoes a step to V_2 at time t_o. For $t < t_o$

$$E_1(z,t) = E_1 \cos \omega_1 (t - \frac{z}{V_1}) = \eta_1 H_1 \tag{3-22}$$

$$H_1(z,t) = H_1 \cos \omega_1 (t - \frac{z}{V_1}) \tag{3-23}$$

where $\frac{E_1}{H_1} = \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$ and $V_1 = \frac{1}{\sqrt{\mu_1 \epsilon_1}}$. The point z = 0 is chosen for convenience in determining the amplitudes of the forward and backward waves after the velocity discontinuity occurs. Just prior to the step ($t = t_o^-$)

$$E_1(0,t_o^-) = E_1 \cos \omega_1 t_o \tag{3-24}$$

$$H_1(0,t_o^-) = H_1 \cos \omega_1 t_o \tag{3-25}$$

Immediately after the step ($t = t_o^+$)

$$E_2(0,t_o^+) = (E_2^+ + E_2^-) \cos \omega_2(t_o + \phi) \qquad (3\text{-}26)$$

$$H_2(0,t_o^+) = (H_2^+ - H_2^-) \cos \omega_2(t_o + \phi) \qquad (3\text{-}27)$$

A comparison of the two sets of equations shows that $\cos \omega_1 t_o = \cos \omega_2(t_o + \phi)$. Solving this expression for ϕ and making use of the fact that $\omega_2 = \frac{V_2}{V_1}\omega_1$ gives

$$\phi = (\frac{V_1}{V_2} - 1)t_o \qquad (3\text{-}28)$$

The complete form of E and H after the discontinuity ($t > t_o$) is therefore

$$E_2(z,t) = E_2^+ \cos\left[\frac{V_2}{V_1}\omega_1(t - t_o) + \omega_1 t_o - \beta z\right] + E_2^- \cos\left[\frac{V_2}{V_1}\omega_1(t - t_o) + \omega_1 t_o + \beta z\right] \qquad (3\text{-}29)$$

$$H_2(z,t) = H_2^+ \cos\left[\frac{V_2}{V_1}\omega_1(t - t_o) + \omega_1 t_o - \beta z\right] - H_2^- \cos\left[\frac{V_2}{V_1}\omega_1(t - t_o) + \omega_1 t_o + \beta z\right] \qquad (3\text{-}30)$$

where $\beta = \frac{\omega_1}{V_1}$ and E_2 is given by Eqs. (3-14) and (3-15).

If $\eta_1 = \eta_2$

$$E_2(z,t) = \frac{\epsilon_1}{\epsilon_2} E_1 \cos\left[\frac{\epsilon_1}{\epsilon_2}\omega_1(t - t_o) + \omega_1 t_o - \beta z\right] \qquad (3\text{-}31)$$

and

$$H_2(z,t) = \frac{\mu_1}{\mu_2} H_1 \cos\left[\frac{\mu_1}{\mu_2}\omega_1(t - t_o) + \omega_1 t_o - \beta z\right] \qquad (3\text{-}32)$$

If $\mu_1 = \mu_2$

$$E_2(z,t) = \frac{E_1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \sqrt{\frac{\epsilon_1}{\epsilon_2}}\right) \cos\left[\sqrt{\frac{\epsilon_1}{\epsilon_2}}\,\omega_1(t - t_o) + \omega_1 t_o - \beta z\right]$$
$$+ \frac{E_1}{2}\left(\frac{\epsilon_1}{\epsilon_2} - \sqrt{\frac{\epsilon_1}{\epsilon_2}}\right) \cos\left[\sqrt{\frac{\epsilon_1}{\epsilon_2}}\,\omega_1(t - t_o) + \omega_1 t_o + \beta z\right] \tag{3-33}$$

and

$$H_2(z,t) = \frac{H_1}{2}\left(1 + \sqrt{\frac{\epsilon_1}{\epsilon_2}}\right) \cos\left[\sqrt{\frac{\epsilon_1}{\epsilon_2}}\,\omega_1(t - t_o) + \omega_1 t_o - \beta z\right]$$
$$- \frac{H_1}{2}\left(1 - \sqrt{\frac{\epsilon_1}{\epsilon_2}}\right) \cos\left[\sqrt{\frac{\epsilon_1}{\epsilon_2}}\,\omega_1(t - t_o) + \omega_1 t_o + \beta z\right] \tag{3-34}$$

Energy Density

The uniform step transient of Fig. 1 has a total energy given by

$$U = \frac{1}{2}\int(\epsilon E^2 + \mu H^2)dV \tag{3-35}$$

where the integration extends throughout the entire volume. Prior to t_o the initial energy U_1 is given by

$$U_1 = \frac{1}{2}(\epsilon_1 E_1^2 + \mu H_1^2)V \tag{3-36}$$

The volume energy density is defined as $u = \frac{dU}{dV}$. For the initial wave

$$u_1 = \frac{U_1}{V} = \frac{1}{2}(\epsilon_1 E_1^2 + \mu_1 H_1^2) = \epsilon_1 E_1^2 = \mu_1 H_1^2 \tag{3-37}$$

After the velocity transient is over $(t > t_o)$ the fields are given by $E_2 = E_2^+ + E_2^-$ and $H_2 = H_2^+ - H_2^-$. The energy density is then

$$u_2 = \frac{1}{2}\epsilon_1 E_1^2\left(\frac{\epsilon_1}{\epsilon_2} + \frac{\mu_1}{\mu_2}\right) \qquad (3\text{-}38)$$

The energy gain is defined as $\frac{u_2}{u_1}$ and is evidently given by

$$\frac{u_2}{u_1} = \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \frac{\mu_1}{\mu_2}\right) \qquad (3\text{-}39)$$

This is the energy gain of the electromagnetic wave after the step transient has occurred.

If $\epsilon_1 = \epsilon_2$ and $\mu_1 = \mu_2$ the energy gain is unity as is necessary.

If $\eta_1 = \eta_2$ then

$$\frac{u_2}{u_1} = \frac{\epsilon_1}{\epsilon_2} = \frac{\mu_1}{\mu_2} \qquad (3\text{-}40)$$

and if $\mu_1 = \mu_2$

$$\frac{u_2}{u_1} = \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + 1\right) \qquad (3\text{-}41)$$

The apparent violation of the conservation of energy is reconciled when it is remembered that the difference in energies is needed to do work upon the fields within the dielectric when μ and ϵ are changing. If $\epsilon_1 > \epsilon_2$ work is done on the field within the dielectric, whereas if $\epsilon_2 > \epsilon_1$ the field does work upon the modulating source. It is apparent that here is a mechanism for changing the energy level of an electromagnetic wave. That the frequency changes as well has already been shown.

These results may be obtained by another method which helps to clarify the situation. Consider that the travelling wave of Fig. 1 is propagated between two semi-infinite parallel ground planes* as shown in Fig. 2. If the far end of the transmission line is short circuited and a current source connected across the near terminals then in the steady state there will be a constant current flowing but no voltage. All of the energy stored will be magnetic and given by

$$U_m = \frac{1}{2}\Gamma \psi^2 \tag{3-42}$$

where Γ is the reciprocal inductance of the line, and ψ the flux. For a length ℓ the following relationships hold.

$$\Gamma = \frac{W}{\mu d \ell} \tag{3-43}$$

and

$$\psi = \frac{\mu \ell d}{W} i = Li \tag{3-44}$$

The current is given by Ampere's circuital law

$$i = \int H \cdot dx \tag{3-45}$$

If $\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$ then initially the energy is given by

$$U_{m_1} = \frac{1}{2}\psi_1^2 \frac{W}{\eta_1^2 \epsilon_1 d \ell} = \frac{1}{2}\psi_1^2 \frac{W}{\mu_1 \ell d} \tag{3-46}$$

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* Parallel planes are used for convenience but any geometrically uniform array could be employed.

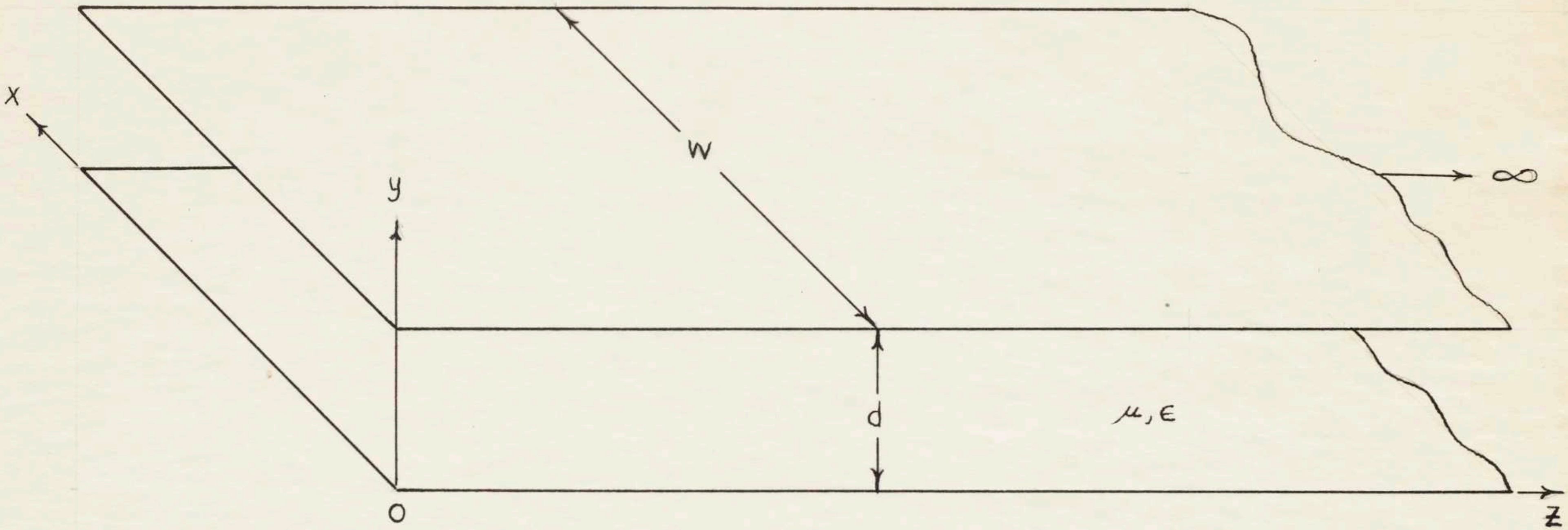


FIG 2

When the medium changes its parameters so that $\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}}$

$$U_{m_2} = \frac{1}{2}\psi_2^2 \frac{W}{\mu_2 \ell d} \qquad (3\text{-}47)$$

Since the flux cannot change instantaneously $\psi_1 = \psi_2$ and

$$U_{m_2} = \frac{1}{2}\psi_1^2 \frac{W}{\ell d \mu_1} \frac{\mu_1}{\mu_2} = U_{m_1} \frac{\mu_1}{\mu_2} \qquad (3\text{-}48)$$

If the far end of the transmission line is now open circuited and a voltage source connected across the near terminals, then in the steady state the current everywhere will be zero and the voltage along the line everywhere a constant. Now all the energy is stored in the electric field and is given by

$$U_e = \frac{1}{2} SQ^2 \qquad (3\text{-}49)$$

where S is the elastance of the line and Q the charge. For a length the following relationships hold.

$$S = \frac{d}{\epsilon W \ell} \qquad (3\text{-}50)$$

and

$$Q = CV \qquad (3\text{-}51)$$

The voltage is given by

$$V = \int E \cdot dy \qquad (3\text{-}52)$$

If $\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$ then initially the energy is

$$U_{e_1} = \frac{1}{2} Q_1^2 \frac{d}{\epsilon_1 W \ell} \qquad (3\text{-}53)$$

When the medium changes its parameters so that $\eta_2 = \frac{\mu_2}{\epsilon_2}$

$$U_{e_2} = \frac{1}{2} Q_2^2 \frac{d}{\epsilon_2 w \ell} \qquad (3\text{-}54)$$

Since the charge cannot change instantaneously $Q_1 = Q_2$ and

$$U_{e_2} = \frac{1}{2} Q_1^2 \frac{d}{\epsilon_1 w \ell} \frac{\epsilon_1}{\epsilon_2} = U_{e_1} \frac{\epsilon_1}{\epsilon_2} \qquad (3\text{-}55)$$

In the original transient problem both electric and magnetic fields exist and both are constants before and after the step in velocity. The principle of superposition is applicable and the total energy is merely the sum of the two energies given in Eqs. (3-42) and (3-49).

$$U_t = U_e + U_m \qquad (3\text{-}56)$$

Originally the energy is split evenly between the electric and magnetic fields $U_{1e} = U_{1m} = \frac{U_{1t}}{2}$ so that the final total energy is given by

$$U_{2t} = U_{e1} \frac{\epsilon_1}{\epsilon_2} + U_{m1} \frac{\mu_1}{\mu_2} = \frac{U_{1t}}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \frac{\mu_1}{\mu_2}\right) \qquad (3\text{-}57)$$

The energy gain is clearly

$$\frac{U_{2t}}{U_{1t}} = \frac{1}{2}\left(\frac{\epsilon_1}{\epsilon_2} + \frac{\mu_1}{\mu_2}\right) \qquad (3\text{-}58)$$

which is in agreement with the previous result, Eq. (3-39).

Approximate Energies

The Liouville approximation is repeated here for convenience.

$$E \simeq A\,\mu^{-\frac{1}{4}}\,\epsilon^{-\frac{3}{4}}\,e^{\pm j\beta z}\,e^{\pm j\beta\int\frac{dt}{\sqrt{\mu\epsilon}}}$$

$$H \simeq A\,\mu^{-\frac{3}{4}}\,\epsilon^{-\frac{1}{4}}\,e^{\pm j\beta z}\,e^{\pm j\beta\int\frac{dt}{\sqrt{\mu\epsilon}}}$$

It is apparent that the amplitudes of D and B are given by

$$|D| \simeq A\,\mu^{-\frac{1}{4}}\,\epsilon^{\frac{1}{4}} \tag{3-59}$$

and

$$|B| \simeq A\,\mu^{\frac{1}{4}}\,\epsilon^{-\frac{1}{4}} \tag{3-60}$$

These field amplitudes are not constant and apparently contradict the statements made regarding the invariancy of the charge and flux. This difficulty is resolved if it is remembered that the approximation is assumed valid only for slowly varying dielectric parameters. In this case the fourth root of this variation is even slower and a constant is approximated by a very slowly varying time function. There is a very definite reason why the approximation takes the form it does. To see this first find the approximate energy density using Eq. (3-35). If the already overworked step transient is considered once more and represented by the Liouville approximation then the final fields will be of the form

$$E_2 \simeq A\,\mu_2^{-\frac{1}{4}}\,\epsilon_2^{-\frac{3}{4}} \tag{3-61}$$

and

$$H_2 \simeq A\,\mu_2^{-\frac{3}{4}}\,\epsilon_2^{-\frac{1}{4}} \tag{3-62}$$

The velocity transient, incidentally, need not be a step and may be as gradual as desired from $\frac{1}{\sqrt{\mu_1 \epsilon_1}}$ to $\frac{1}{\sqrt{\mu_2 \epsilon_2}}$ so that the assumptions under which the approximations were derived are not violated.

The initial fields are

$$E_1 \simeq A \mu_1^{-\frac{1}{4}} \epsilon_1^{-\frac{3}{4}} \tag{3-63}$$

and

$$H_1 \simeq A \mu_1^{-\frac{3}{4}} \epsilon_1^{-\frac{1}{4}} \tag{3-64}$$

The initial energy density is

$$U_1 = \tfrac{1}{2}(\epsilon_1 E_1^2 + \mu_1 H_1^2) = \frac{A^2}{\sqrt{\mu_1 \epsilon_1}} \tag{3-65}$$

The final energy density is

$$U_2 = \tfrac{1}{2}(\epsilon_2 E_2^2 + \mu_2 H_2^2) \simeq \frac{A^2}{\sqrt{\mu_2 \epsilon_2}} \tag{3-66}$$

The energy gain is therefore

$$\frac{U_2}{U_1} \simeq \sqrt{\frac{\mu_1 \epsilon_1}{\mu_2 \epsilon_2}} \tag{3-67}$$

Although Eqs. (3-65) and (3-66) are approximately equal to the exact expressions, their forms are quite different. The pertinent physical interpretation is that Eqs. (3-66) and (3-67) are exact expressions for the energy which is actually propagated by the fields and the energy gain respectively. The energy which is trapped in the standing wave field is neglected by the approximation since it does not admit a reflected component. As long as the reflected wave is small compared to the transmitted one, the approximation is very good. The approximation forces the amplitudes of E and H to vary in order that the approximated power will be exactly equal to the propagated power.

Suppose it is attempted to derive the energy density from power considerations. Poynting's theorem gives for the original fields

$$S_1 = E_1 H_1 = \frac{1}{\eta_1} E_1^2 = \sqrt{\frac{\epsilon_1}{\mu_1}}\, E_1^2 \qquad (3\text{-}68)$$

The total power flowing through an area A is simply

$$P_1 = AS_1 = A\sqrt{\frac{\epsilon_1}{\mu_1}}\, E_1^2 \qquad (3\text{-}69)$$

The energy propagated through a length L of the wave is simply

$$U_1 = \int_0^{\frac{L}{V_1}} P_1 \, dt = \int_0^{\frac{L}{V_1}} A\sqrt{\frac{\epsilon_1}{\mu_1}}\, E_1^2 \, dt = V \epsilon_1 E_1^2 \qquad (3\text{-}70)$$

where V is the volume AL. The initial energy density is therefore

$$u_1 = \frac{U_1}{V} = \epsilon_1 E_1^2 \qquad (3\text{-}71)$$

as was derived previously in a different manner. The propagated energy of the final fields is similarly given by

$$U_2 = A \int_0^{\frac{L}{V_2}} S_2 \, dt = V \frac{S_2}{V_2} \qquad (3\text{-}72)$$

The energy density is therefore

$$u_2 = \sqrt{\mu_2 \epsilon_2}\, S_2 \qquad (3\text{-}73)$$

Poynting's theorem yields

$$S_2 = (E_2^+ + E_2^-)\,(H_2^+ - H_2^-) = E_2^+ H_2^+ - E_2^- H_2^-$$

$$= \frac{1}{\eta_2}\left[(E_2^+)^2 - (E_2^-)^2\right] \qquad (3\text{-}74)$$

Substituting Eqs. (3-14) and (3-15) into Eq. (3-74) results in

$$S_2 = \sqrt{\frac{\epsilon_2}{\mu_2}}\;\frac{E_1^2}{4}\left(\frac{\epsilon_1}{\epsilon_2} + \sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}\right)^2 - \frac{E_1^2}{4}\left(\frac{\epsilon_1}{\epsilon_2} - \sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}\right)^2$$

$$= \epsilon_1 E_1^2\,\frac{\sqrt{\mu_1\epsilon_1}}{\mu_2\epsilon_2} \qquad (3\text{-}75)$$

and the final energy density becomes

$$u_2 = \epsilon_1 E_1^2 \sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}} \qquad (3\text{-}76)$$

Since $u_1 = \epsilon_1 E_1^2$ the energy gain is given by

$$\frac{u_2}{u_1} = \sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}} \qquad (3\text{-}77)$$

Equation (3-74) is the net power passing through a reference boundary. It is the difference between the powers of the forward and backward waves. This is why Eq. (3-76) is the transmitted and not the total energy. The plus and minus waves may be thought of as forming a standing and a travelling wave. The trapped energy is distributed throughout the overlap region of the plus and minus waves and is not taken into account in the above derivation. Equations (3-77) and (3-67) are exactly equal which confirms the previous remarks.

Electromagnetic Momentum[1]

The electromagnetic momentum $\overline{g}$ of a field is given by

$$\overline{g} = \frac{1}{V^2}\,\overline{S} \qquad (3\text{-}78)$$

where V is the velocity of propagation and $\overline{S}$ is Poynting's vector. Referring to the problem just discussed, the original fields are described by Eqs. (3-14), (3-15), and (3-68). Substituting the latter equation into Eq. (3-78) yields

$$g_1 = \epsilon_1 \sqrt{\mu_1 \epsilon_1}\; E_1^2 \qquad (3\text{-}79)$$

Since $S_2 = \epsilon_1 \dfrac{\sqrt{\mu_1 \epsilon_1}}{\mu_2 \epsilon_2} E_1^2$ the final value of momentum is given by

$$g_2 = g_1 = \epsilon_1 \sqrt{\mu_1 \epsilon_1}\; E_1^2 \qquad (3\text{-}80)$$

This very important result which has been derived for the special case of the step transient is also true for the general case and states that even though energy may by added or subtracted from the electromagnetic field by varying the velocity of propagation of the medium through which it passes, the electromagnetic momentum of the field is unchanged.

Since the momentum is associated only with the propagated field and not the standing wave field, it is clear that the Liouville approximation should predict conservation of momentum also. The fact that the exact $\overline{S}_2$ is given by the approximation insures that this is indeed true.

Reference

(1) Stratton, J., Electromagnetic Theory, Chapter II, Sec. 2.6, p.103. McGraw-Hill, 1941.

CHAPTER 4

THE DIELECTRIC MODULATOR

The frequency variation of monochromatic electromagnetic waves passing through time varying dielectrics will be considered in detail in this chapter. Fig. 1 indicates the situation to be studied. The variable dielectric extends from z = 0 to z = L and is assumed homogeneous throughout. Since the wave solution has just been found for the interior of such a dielectric and the incoming wave, of frequency ω_c, obeys the standard wave equation outside, it is possible to match boundary conditions and obtain a solution valid over the entire range of z. This procedure will solve the problem but the necessity of matching the boundaries is rather distasteful. An alternate method which circumvents this boundary matching makes use of the transit time concept and will be utilized.

It can be shown that the equation for a wave front propagating with non-constant velocity is[1]

$$E = E_o \sin\left[\omega_c\left(t - \int \frac{dz}{V}\right)\right] \qquad (4\text{-}1)$$

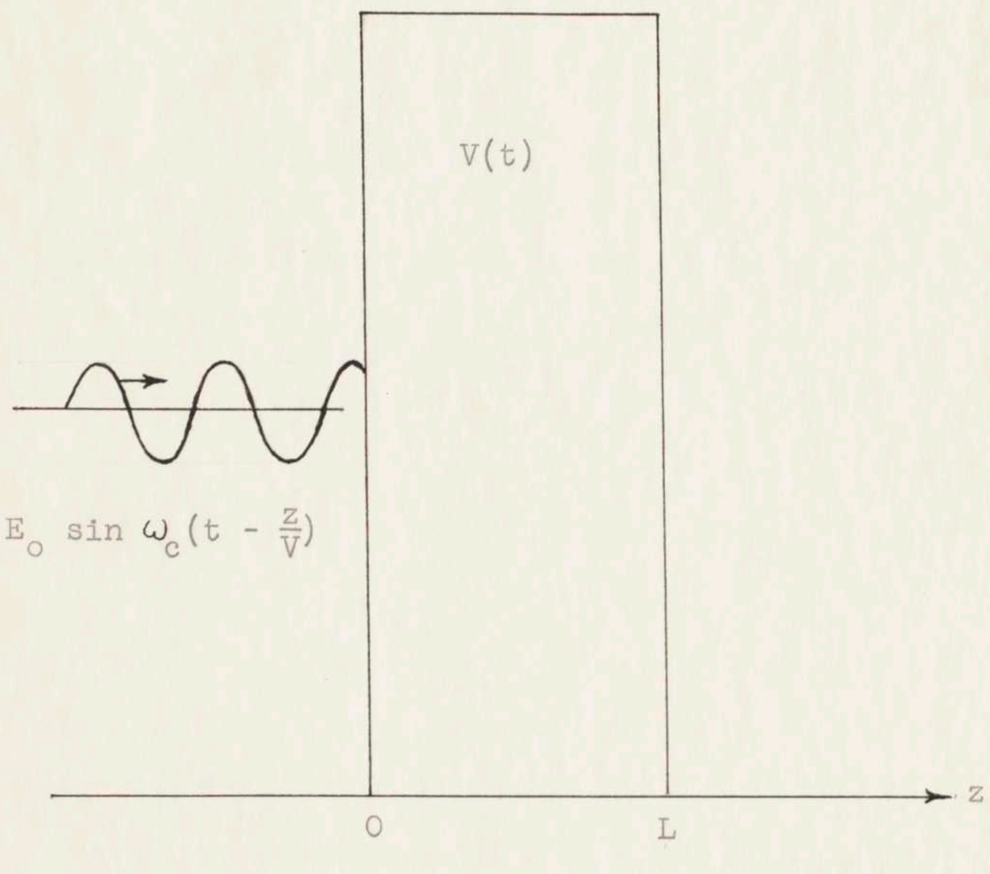


Fig. 1

The transit time is defined as

$$T = \int \frac{dz}{V} \tag{4-2}$$

and is the time required for the wave front to travel a distance z. Since the total phase of the wave is given by

$$\phi = \omega_c \left(t - \int \frac{dz}{V}\right) \tag{4-3}$$

the instantaneous frequency is simply

$$\omega(t) = \frac{d\phi}{dt} = \omega_c \left(1 - \frac{d}{dt} \int \frac{dz}{V}\right) = \omega_c \left(1 - \frac{d}{dt} T\right) \tag{4-4}$$

The change in frequency is thus proportional to the rate of change of transit time. Usually the transit time for any dielectric slab is constant and so the frequency variation is nil.

Thin Sections

If the dielectric slab is very thin the velocity of propagation will not have had time to change appreciably before the wave has passed completely through. The velocity can then be approximated as constant for any given wave front and dependent only upon when, during the modulating cycle, the wave front entered the medium.* Under these conditions

$$T = \int_0^L \frac{dz}{V} \simeq \frac{L}{V(t)} \tag{4-5}$$

---

* This is analagous to the situation found in Klystrons where an electron's velocity is assumed constant while passing through the narrow accelerating gap but dependent on when it entered the gap.

The instantaneous frequency is approximated by

$$\omega(t) \simeq \omega_c \left[1 - L \frac{d}{dt}\left(\frac{1}{V(t)}\right)\right] = \omega_c\left(1 + \frac{L}{V^2}\frac{dV}{dt}\right) \qquad (4\text{-}6)$$

General Solution

In practical situations the approximation Eq. (4-6) is nearly always valid. It is instructive however to consider the general case when the velocity cannot be assumed constant for the transit interval. The transit time for the entire length L is given by

$$T = \int_o^L \frac{dz}{V} = t_1 - t_o \qquad (4\text{-}7)$$

In this equation t_o is the entrance time of a wave front and t_1 is the exit time of the same wave front. The integral may be evaluated as follows

$$\int_{t_o}^{t_1} v(t)\, dt = \int_o^L dz = L \qquad (4\text{-}8)$$

Therefore $f(t_1 t_o) = L$ or $t_o = g(t_1 L)$ and

$$T = t_1 - t_o = t_1 - g(t_1, L) \qquad (4\text{-}9)$$

The case where the velocity of propagation undergoes a step discontinuity as depicted in Fig. 2 is instructive and serves as a check on the previous work.

The evaluation of $\int_{t_o}^{t_1} v\, dt = L$ takes place in three parts:

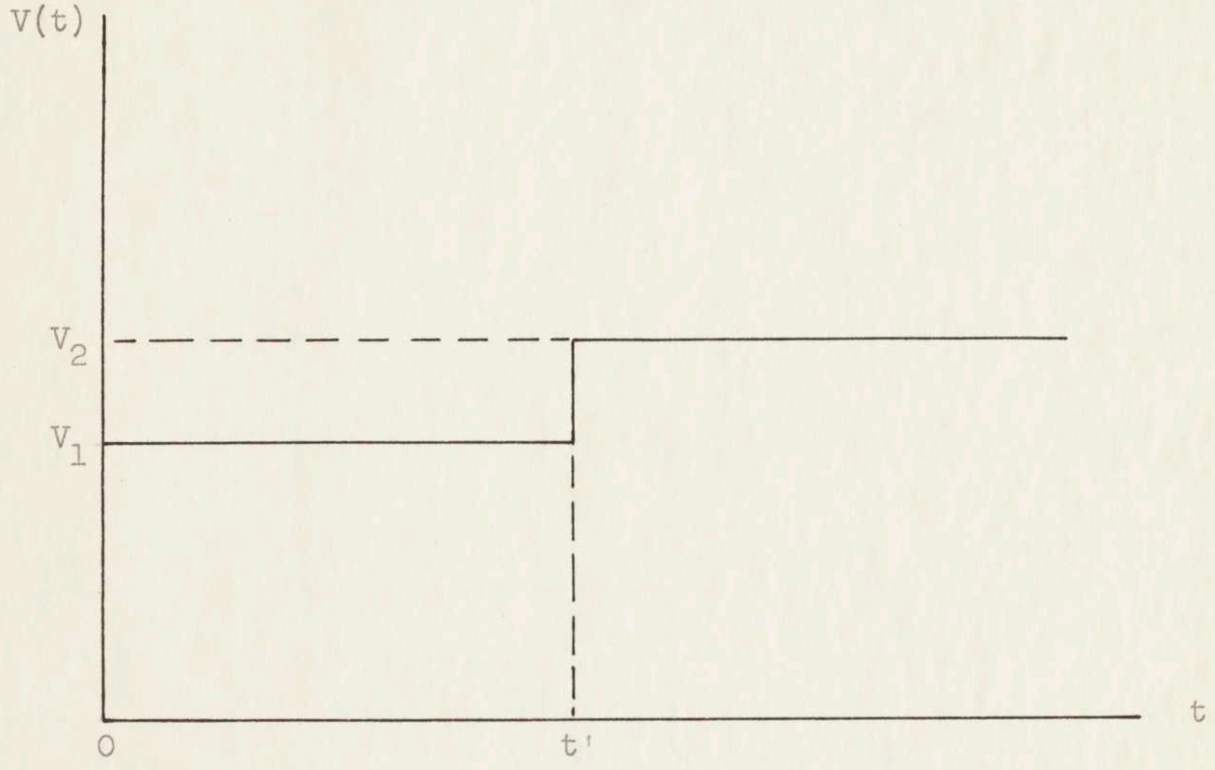


Fig. 2

If $t_1 < t'$, $V(t_1) = V_1$ and

$$T = \frac{L}{V_1} \tag{4-10}$$

If $t_o < t' < t_1$ then $V = V_1 + (V_2 - V_1)\, u_{-1}\,(t - t')$ where $u_{-1}(t)$ is the unit step function and

$$T = \frac{L}{V_1} + \frac{(V_2 - V_1)^{t'}}{V_1} + (1 - \frac{V_2}{V_1})\, t_1 \tag{4-11}$$

Finally if $t_o > t'$, $V = V_2$ and

$$T = \frac{L}{V_2} \tag{4-12}$$

The plot of t_o versus t_1 shown in Fig. 3 gives a graphic representation of the transit time as a function of output time.

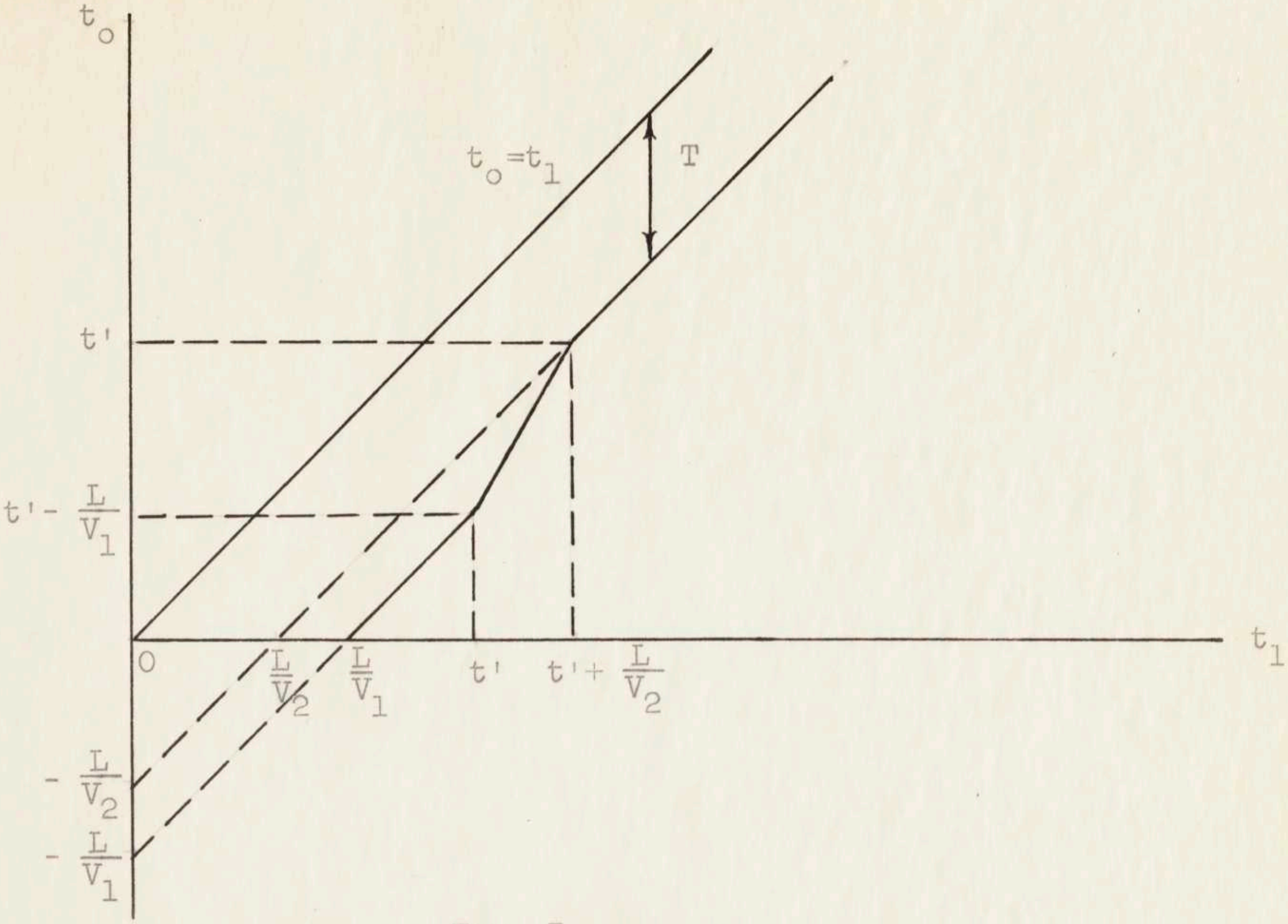
The rate of change of transit time is given by

$$\frac{dT}{dt_1} = \begin{cases} 0 & t_1 < t' \\ 1 - \frac{V_2}{V_1} & t' \leq t_1 \leq \left(t' + \frac{L}{V_2}\right) \\ 0 & t_1 > t' \end{cases} \tag{4-13}$$

Since $\omega(t) = \omega_c(1 - \frac{dT}{dt_1})$ the instantaneous frequency is

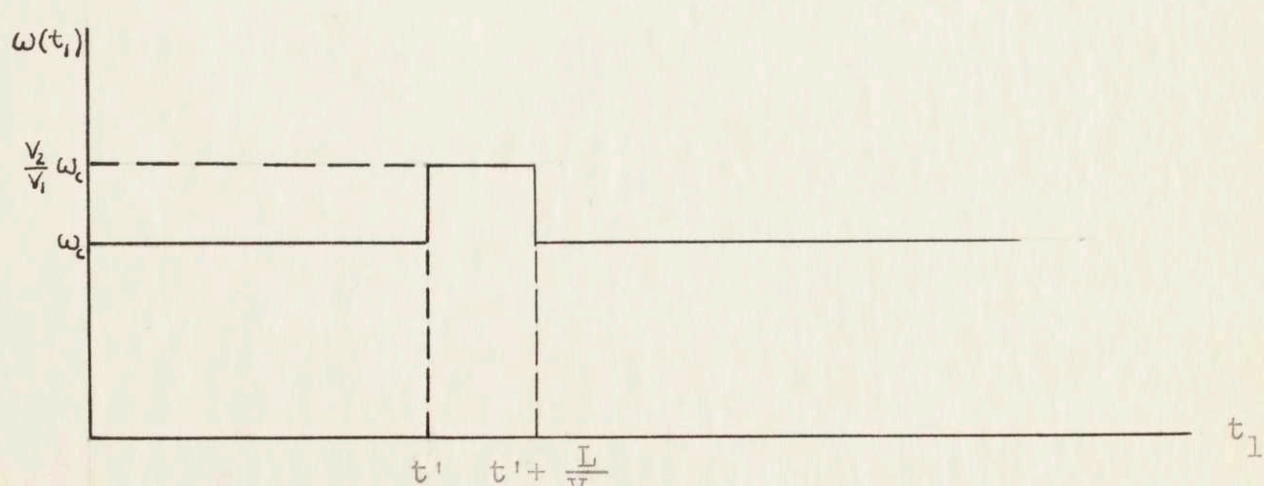
$$\omega(t) = \begin{cases} \omega_c & t_1 < t' \\ \frac{V_2}{V_1}\omega_c & t' \leq t_1 \leq \left(t' + \frac{L}{V_2}\right) \\ \omega_c & t_1 > t' \end{cases} \tag{4-14}$$

as given in Fig. 4.



t_o
t_o=t_1
T
t'
t'- L/V_1
0
L/V_2
L/V_1
t'
t'+ L/V_2
t_1
- L/V_2
- L/V_1

Fig. 3



ω(t_1)
V_2/V_1 ω_c
ω_c
t'
t'+ L/V_2
t_1

Fig. 4

The phase constants involved are

$$\beta = \begin{cases} \dfrac{\omega_c}{V_1} & t_1 < t' + \dfrac{L}{V_2} \\ \dfrac{\omega_c}{V_2} & t_1 > t' + \dfrac{L}{V_2} \end{cases} \qquad (4\text{-}15)$$

This agrees with the previous physical interpretation that the wavelength does not change during the transient of the velocity. If the impedance does not change during the step there will be no reflections assuming the slab is matched to the media on either side. That portion of the wave which undergoes the frequency shift has an energy level gain of $\frac{V_2}{V_1}$.

If the velocity step is replaced by the ramp function shown in Fig. 5, the problem is more meaningful physically since no dielectric can change its velocity of propagation in zero time. The evaluation of the transit time integral proceeds in five parts:

Assume that $\int_{t_a}^{t_b} V dt \leq L$.

Case I $(t_1 < t_a)$

$$T = \frac{L}{V_1} \qquad (4\text{-}16)$$

Case II $(t_a < t_1 < t_b)$

$$T = \frac{L}{V_2} - \frac{1}{2}\frac{m}{V_1}(t_1 - t_a)^2 \qquad (4\text{-}17)$$

Case III $(t_o < t_a \;\&\; t_1 > t_b)$

$$T = (1 - \frac{V_2}{V_1})t_1 + (\frac{V_2}{V_1} - 1)t_b + \frac{L}{V_1} - \frac{1}{2}\frac{m}{V_1}(t_b - t_a)^2 \qquad (4\text{-}18)$$

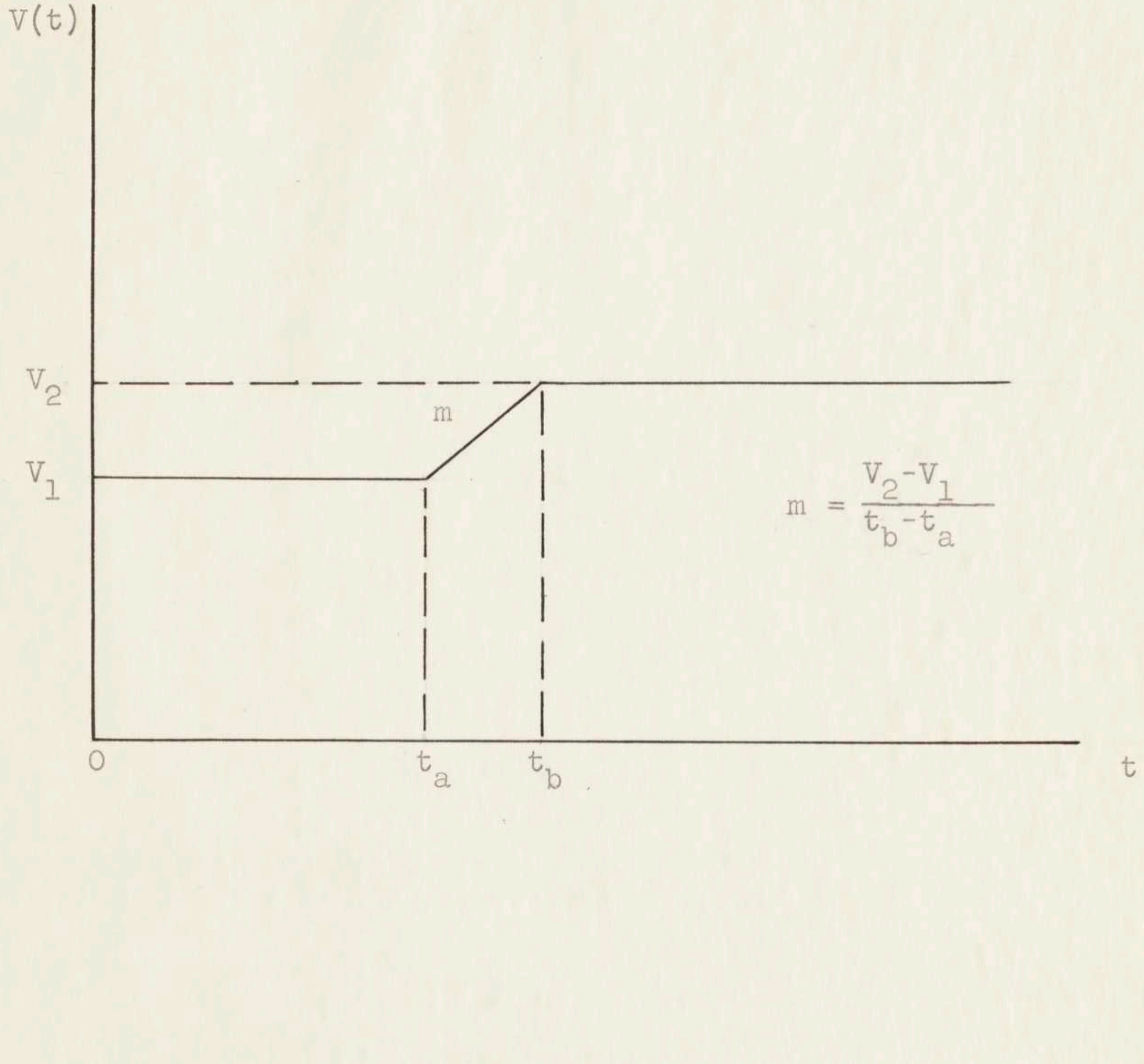


Fig. 5

Case IV $(t_a < t_o < t_b)$

$$T = \frac{V_2}{m} + t_1 - t_b - \sqrt{(\frac{V_2}{m})^2 - \frac{2L}{m} + \frac{2V_2}{m}(t_1 - t_b)} \qquad (4\text{-}19)$$

Case V $(t_o > t_b)$

$$T = \frac{L}{V_2} \qquad (4\text{-}20)$$

The instantaneous frequency is given by $\omega(t) = \omega_c(1 - \frac{dT}{dt_1})$

where

$$\frac{dT}{dt_1} = \begin{cases} 0 & t_1 < t_a \\ -\frac{m}{V_1}(t_1 - t_a) & t_a < t_1 < t_b \\ 1 - \frac{V_2}{V_1} & t_o < t_a \; \& \; t_1 > t_b \\ 1 - \frac{\frac{V_2}{m}}{\sqrt{(\frac{V_2}{m})^2 - \frac{2L}{m} + \frac{2V_2}{m}(t_1 - t_b)}} & t_a < t_o < t_b \\ 0 & t_o > t_b \end{cases} \qquad (4\text{-}21)$$

where $m = \frac{V_2 - V_1}{t_b - t_a}$. It follows that

$$\omega(t_1) = \begin{cases} \omega_c & t_a > t_1 > \left(t_b + \frac{L}{V_2}\right) \\ \frac{V_1(t_b - t_1) + V_2(t_1 - t_a)}{V_1(t_b - t_a)}\omega_c & t_a < t_1 < t_b \\ \frac{V_2}{V_1}\omega_c & t_b < t_1 < \left[t_a + \frac{L}{V_2} + \frac{(V_2 - V_1)(t_b - t_a)}{2V_2}\right] \\ \frac{\frac{V_2}{m}}{\sqrt{(\frac{V_2}{m})^2 - \frac{2L}{m} + \frac{2V_2}{m}(t_1 - t_b)}}\omega_c & \left[t_a + \frac{L}{V_2} + \frac{(V_2 - V_1)(t_b - t_a)}{2V_2}\right] < t_1 < \left(t_b + \frac{L}{V_2}\right) \end{cases} \qquad (4\text{-}22)$$

Equation (4-22) is sketched in Fig. 6 as a function of output time.

If the $\int_{t_a}^{t_b} V dt > L$ the evaluation of Eq. (4-8) proceeds again in five parts.

Case I $(t_1 < t_a)$

$$T = \frac{L}{V_1} \tag{4-23}$$

Case II $(t_o < t_a < t_1)$

$$T = \frac{L}{V_1} - \frac{1}{2}\frac{m}{V_1}(t_1 - t_a)^2 \tag{4-24}$$

Case III $(t_o > t_a \ \& \ t_1 < t_b)$

$$T^2 - 2\left[\frac{V_1}{m} + (t_1 - t_a)\right]T + \frac{2L}{m} = 0 \tag{4-25}$$

Case IV $(t_o < t_b < t_1)$

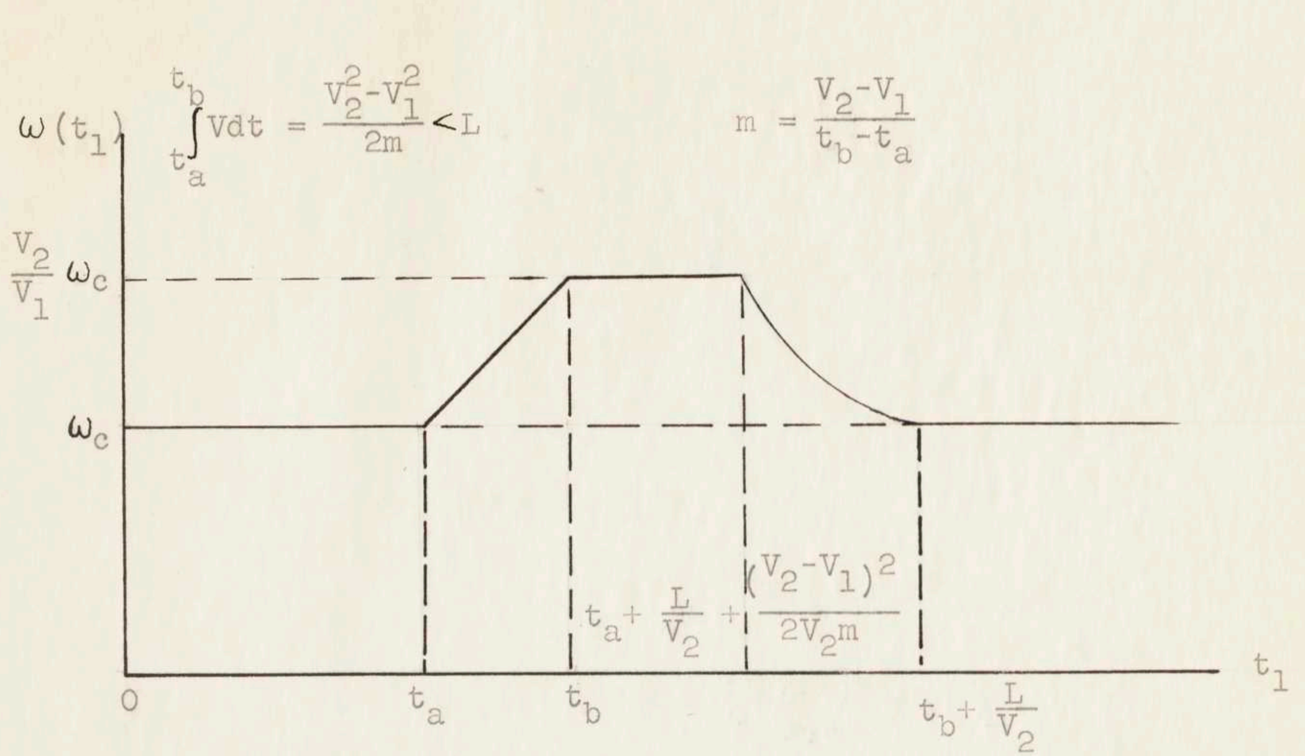
$$T = \frac{V_2}{m} + (t_1 - t_b) - \sqrt{(\frac{V_2}{m})^2 - \frac{2L}{m} + \frac{2V_2}{m}(t_1 - t_b)} \tag{4-26}$$

Case V $(t_o > t_b)$

$$T = \frac{L}{V_2} \tag{4-27}$$

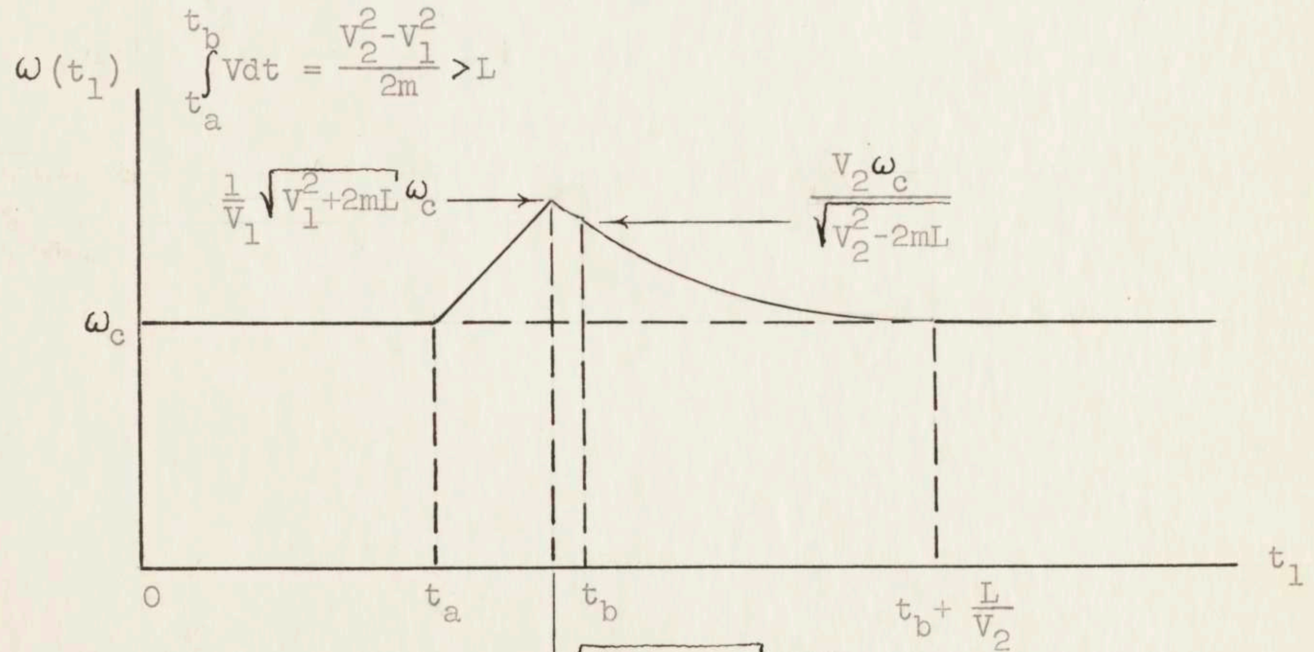
The rate of change of transit time is

$$\frac{dT}{dt_1} = \begin{cases} 0 & (t_b + \frac{L}{V_2}) < t_1 < t_a \\ -\frac{m}{V_1}(t_1 - t_a) & t_o < t_a < t_1 \\ 1 - \dfrac{\frac{V_1}{m} + (t_1 - t_a)}{\sqrt{\left[\frac{V_1}{m} + (t_1 - t_a)\right]^2 - \frac{2L}{m}}} & t_o > t_a \ \& \ t_1 < t_b \\ 1 - \dfrac{\frac{V_2}{m}}{\sqrt{(\frac{V_2}{m})^2 - \frac{2L}{m} + \frac{2V_2}{m}(t_1 - t_b)}} & t_o < t_b < t_1 \end{cases} \tag{4-28}$$



$\omega(t_1)$
$\int_{t_a}^{t_b} V dt = \frac{V_2^2 - V_1^2}{2m} < L$
$m = \frac{V_2 - V_1}{t_b - t_a}$
$\frac{V_2}{V_1}\omega_c$
ω_c
$t_a + \frac{L}{V_2} + \frac{(V_2 - V_1)^2}{2V_2 m}$
t_1
0
t_a
t_b
$t_b + \frac{L}{V_2}$

Fig. 6



$\omega(t_1)$
$\int_{t_a}^{t_b} V dt = \frac{V_2^2 - V_1^2}{2m} > L$
$\frac{1}{V_1}\sqrt{V_1^2 + 2mL}\,\omega_c$
$\frac{V_2 \omega_c}{\sqrt{V_2^2 - 2mL}}$
ω_c
t_1
0
t_a
t_b
$t_b + \frac{L}{V_2}$
$t_a + \sqrt{(\frac{V_1}{m})^2 + \frac{2L}{m}} - \frac{V_1}{m}$

Fig. 7

The frequency is given as

$$\omega(t_1) = \begin{cases} \omega_c & (t_b + \frac{L}{V_2}) < t_1 < t_a \\ \dfrac{V_1(t_b - t_1) + V_2(t_1 - t_a)}{V_1(t_b - t_a)}\,\omega_c & t_a < t_1 < (t_a + \sqrt{(\frac{V_1}{m})^2 + \frac{2L}{m}} - \frac{V_1}{m}) \\ \dfrac{\left[\frac{V_1}{m} + (t_1 - t_a)\right]\omega_c}{\sqrt{\left[\frac{V_1}{m} + (t_1 - t_a)\right]^2 - \frac{2L}{m}}} & (t_a + \sqrt{(\frac{V_1}{m})^2 + \frac{2L}{m}} - \frac{V_1}{m}) < t_1 < t_b \\ \dfrac{\frac{V_2}{m}\,\omega_c}{\sqrt{(\frac{V_2}{m})^2 - \frac{2L}{m} + \frac{2V_2}{m}(t_1 - t_b)}} & t_b < t_1 < (t_b + \frac{L}{V_2}) \end{cases} \qquad (4\text{-}29)$$

Equation (4-29) is sketched in Fig. 7 as a function of output time.

Sinusoidal Modulation

This section will study the case of monochromatic waves passing through a homogeneous dielectric slab whose permittivity is a sinusoidal function of time.

If ϵ is given by

$$\epsilon = K\epsilon_o(1 + b \sin \omega_m t) \tag{4-30}$$

and $\mu = \mu_o$, then the velocity of propagation is given by

$$V = \frac{V_o}{\sqrt{1 + b \sin \omega_m t}} \quad ; \quad V_o = \frac{1}{\sqrt{K\epsilon_o \mu_o}} \tag{4-31}$$

If the dielectric is electrically thin then the approximation (4-6) can be used. Under these circumstances the instantaneous frequency of the wave emerging from the dielectric is

$$\omega(t) \simeq \omega_c \left[1 - \frac{bL}{2V_o} \frac{\cos \omega_m t}{\sqrt{1 + b \sin \omega_m t}} \right] \tag{4-32}$$

and the total variation of frequency

$$\Delta \omega(t) \simeq \frac{-Lb\, \omega_m \omega_c \cos \omega_m t}{2V_o \sqrt{1 + b \sin \omega_m t}} \tag{4-33}$$

If b is very small, Eq. (4-33) becomes

$$\Delta \omega(t) \simeq \frac{-Lb\, \omega_m \omega_c}{2V_o} \cos \omega_m t \tag{4-34}$$

This is the form of linear phase modulation[2] where

$$\omega_d = \frac{b\, \omega_m \omega_c L}{V_o} \tag{4-35}$$

and

$$\theta_d = \frac{\omega_d}{\omega_m} = \frac{b\, \omega_c L}{V_o} \tag{4-36}$$

If L is electrically long so that the velocity cannot be assumed constant over the transit time interval then the integral (4-8) must be evaluated. This becomes

$$L = \int_{t_o}^{t_1} \frac{V_o \, dt}{\sqrt{1 + b \sin \omega_m t}} \qquad (4\text{-}37)$$

This integral can be evaluated approximately if the substitution $\tan\left(\frac{\omega_m t}{2}\right) = X$ is used.* Then $\sin \omega_m t = \frac{2x}{1 + x^2}$ and $\frac{2\,dx}{1 + x^2} = \omega_m \, dt$. Substituting into Eq. (4-37) results in

$$\int \frac{dt}{\sqrt{1 + b \sin \omega_m t}} = \frac{2}{\omega_m} \int \frac{dx}{(1 + x^2)\sqrt{1 + \frac{2bx}{1 + x^2}}}$$

$$= \frac{2}{\omega_m} \int \frac{dx}{\sqrt{x^4 + 2bx^3 + 2x^2 + 2bx + 1}} \qquad (4\text{-}38)$$

Since $(x^2 + bx + 1)^2 = x^4 + 2bx^3 + (2 + b^2)x^2 + 2bx + 1$ it is seen that if $b^2 \ll 2$

$$\int \frac{dt}{\sqrt{1 + b \sin \omega_m t}} \simeq \frac{2}{\omega_m} \int \frac{dx}{x^2 + bx + 1} \qquad (4\text{-}39)$$

where

$$\int \frac{dx}{x^2 + bx + 1} = \frac{2}{\sqrt{4 - b^2}} \tan^{-1}\left[\frac{2x + b}{\sqrt{4 - b^2}}\right] \qquad (4\text{-}40)$$

---

* The transit time integral may be evaluated from tabulated functions if the phase is changed to a cosine function (which obviously makes no difference). Then

$$\int \frac{dt}{\sqrt{1 + b \cos \omega_m t}} = \frac{2}{\omega_m \sqrt{1 + b}} \int \frac{dx}{\sqrt{1 - K \sin^2 x}}$$

(with $\frac{\omega_m t}{2} = x$ and $K = \frac{2b}{1 + b}$) which is an elliptic integral.

Therefore

$$\frac{V_o\, dt}{\sqrt{1 + b \sin \omega_m t}}$$

$$\simeq \frac{4\, V_o}{\omega_m \sqrt{4 - b^2}} \tan^{-1} \left[\frac{2 \tan(\frac{\omega_m t}{2}) + b}{\sqrt{4 - b^2}} \right] \quad ; \quad (b^2 << 2)$$

(4-41)

The approximate solution of Eq. (4-37) is

$$L = \frac{4\, V_o}{\omega_m \sqrt{4 - b^2}} \tan^{-1} \left[\frac{2 \tan(\frac{\omega_m t}{2}) + b}{\sqrt{4 - b^2}} \right] \Bigg|_{t_o}^{t_1}$$

or

$$t_o \simeq \frac{2}{m} \tan^{-1} \left\{ \frac{\sqrt{4 - b^2}}{2} \tan \left[\tan^{-1} \left\{ \frac{2 \tan(\frac{\omega_m t_1}{2}) + b}{\sqrt{4 - b^2}} \right\} - \frac{\omega_m L \sqrt{4 - b^2}}{4\, V_o} \right] - \frac{b}{2} \right\}$$

(4-42)

Since $\omega(t) = \omega_c(1 - \frac{dT}{dt_1}) = \omega_c \frac{dt_o}{dt_1}$ the instantaneous frequency is

$$\omega(t_1) \simeq \frac{\sec^2(\frac{\omega_m t_1}{2})}{\left[1 + \left\{ \frac{2 \tan(\frac{\omega_m t_1}{2}) + b}{\sqrt{4 - b^2}} \right\}^2 \right]}$$

$$\times \frac{\sec^2 \left[\tan^{-1} \left\{ \frac{2 \tan(\frac{\omega_m t_1}{2}) + b}{\sqrt{4 - b^2}} \right\} - \frac{\omega_m L \sqrt{4 - b^2}}{4\, V_o} \right] \omega_c}{\left[1 + \left\{ \frac{\sqrt{4 - b^2}}{2} \tan \left[\tan^{-1} \left\{ \frac{2 \tan(\frac{\omega_m t_1}{2}) + b}{\sqrt{4 - b^2}} \right\} - \frac{\omega_m L \sqrt{4 - b^2}}{4\, V_o} \right] - \frac{b}{2} \right\}^2 \right]}$$

(4-43)

For $L = 0$, $b = 0$, or $\omega_m = 0$ the instantaneous frequency is simply ω_c as is seen from physical reasoning. Note however that if $\frac{\omega_m L\sqrt{4 - b^2}}{4 V_o} = K\pi$ ($K = 0, 1, 2, 3, \ldots$) the frequency will again be constant and equal to ω_c. This means that if the length L is such as to require an integral number of modulating cycles to elapse before a wave front passes completely through, then surely all wave fronts will have exactly the same transit time. Since the frequency variation is proportional to the rate of change of transit time it is obvious that no frequency variation will take place. These null lengths are given by

$$L = \frac{4K\pi V_o}{\omega_m\sqrt{4 - b^2}} \qquad (K = 0, 1, 2, 3, \ldots) \qquad (4\text{-}44)$$

Since the frequency behavior is periodic there is no advantage to be gained in making L any longer than some value within the first interval. Increased length means greater loss, which in turn means increased modulating power. The optimum length modulator, which will result in the greatest frequency variation, is evidently somewhere within the first interval $0 < L_{opt} < \frac{4\pi V_o}{\omega_m\sqrt{4 - b^2}}$. A guess would place it half-way between the limits. That this is actually correct will be demonstrated in the following section.

Maximization of the Frequency Variation

The frequency given by Eq. (4-43) can be rewritten in the form

$$\omega(t_1) \simeq \frac{A \sec^2(\beta - \alpha)\,\omega_c}{B\left[1 + \left\{C \tan(\beta - \alpha) - \frac{b}{2}\right\}^2\right]} = \frac{N}{D}\,\omega_c \qquad (4\text{-}45)$$

where $A = \sec^2(\frac{\omega_m t_1}{2})$, $B = 1 + \tan^2\beta$, $C = \frac{\sqrt{4 - b^2}}{2}$, $\alpha = \frac{\omega_m L\sqrt{4 - b^2}}{4\,V_o}$, and $\tan\beta = \frac{2\tan(\frac{\omega_m t_1}{2}) + b}{\sqrt{4 - b^2}}$. It is seen that α is the only parameter involving the modulator length L, therefore $\omega(t_1)$ will first be maximized with respect to α.

$$\frac{d\omega}{d\alpha} \simeq \omega_c\,\frac{D\,\frac{dN}{d\alpha} - N\,\frac{dD}{d\alpha}}{D^2} = 0 \qquad (4\text{-}46)$$

or

$$D\,\frac{dN}{d\alpha} = N\,\frac{dD}{d\alpha} \qquad (D^2 \neq 0) \qquad (4\text{-}47)$$

where $N = A\sec^2(\beta - \alpha)$ and $D = B\left[1 + \left\{C\tan(\beta - \alpha) - \frac{b}{2}\right\}^2\right]$.

If a new variable X is introduced such that $X = \beta - \alpha$ then

$$\frac{dN}{d\alpha} = 2A\sec^2 x\,\tan x \qquad (4\text{-}48)$$

and

$$\frac{dD}{d\alpha} = 2BC\sec^2 x\left[C\tan x - \frac{b}{2}\right] \qquad (4\text{-}49)$$

Substituting Eqs. (4-48) and (4.49) into Eq. (4 47) leads to

$$\tan x\left[1 + \left\{C\tan x - \frac{b}{2}\right\}^2\right] = C\sec^2 x\left[C\tan x - \frac{b}{2}\right] \qquad (4\text{-}50)$$

Equation (4-50) can be reduced to

$$\sin^4 x - \sin^2 x + \frac{4 - b^2}{16} = 0 \qquad (4\text{-}51)$$

which has the solution

$$\sin(\beta - \alpha) = \sqrt{\frac{2 \pm b}{4}} \quad ; \quad \beta - \alpha = \sin^{-1}\sqrt{\frac{2 \pm b}{4}} \tag{4-52}$$

If this condition is substituted into Eq. (4-45) the instantaneous frequency becomes

$$\omega(t_1) \simeq K_{\substack{max \\ min}} \frac{\sec^2\left(\frac{\omega_m t_1}{2}\right)}{\left[1 + \left\{\frac{2\tan\left(\frac{\omega_m t_1}{2}\right) + b}{\sqrt{4 - b^2}}\right\}^2\right]} \tag{4-53}$$

K is either a maximum or minimum value so that if Eq. (4-53) is now maximized (minimized) with respect to $\frac{\omega_m t_1}{2}$ the value of β which is required can be found. Substitution of this value of β into Eq. (4-52) yields an equation which can be solved for the maximizing or minimizing values of α. If the substitution $\theta = \frac{\omega_m t_1}{2}$ is made, Eq. (4-53) becomes

$$\omega(t_1) \simeq K_{\substack{max \\ min}} \frac{\sec^2\theta}{\left[1 + \left\{\frac{2\tan\theta + b}{\sqrt{4 - b^2}}\right\}^2\right]} = K_{\substack{max \\ min}} \frac{N(\theta)}{D(\theta)} \tag{4-54}$$

where $N = \sec^2\theta$ and $D = 1 + \left[\frac{2\tan\theta + b}{\sqrt{4 - b^2}}\right]^2$. The maximizing condition is

$$N\frac{dD}{d\theta} - D\frac{dN}{d\theta} = 0 \qquad (D^2 \neq 0) \tag{4-55}$$

The appropriate functions are

$$\frac{dN}{d\theta} = 2\sec^2\theta \tan\theta \tag{4-56}$$

and

$$\frac{dD}{d\theta} = \frac{4}{4 - b^2} \sec^2\theta \left(2 \tan \theta + b \right) \tag{4-57}$$

After substitution and simplification Eq. (4-55) becomes

$$\sin^2\theta = \frac{1}{2} \tag{4-58}$$

The solution is

$$\theta = \frac{\omega_m t_1}{2} = \frac{K\pi}{4} \quad \text{(K odd)} \tag{4-59}$$

The composite conditions for maxima and minima are

$$\left.\begin{array}{l} \beta - \alpha = \sin^{-1} \left(\pm \sqrt{\frac{2 \pm b}{4}}\right) \\ \frac{\omega_m t_1}{2} = \frac{K\pi}{4} \qquad \text{(K odd)} \end{array}\right\} \tag{4-60}$$

The choices of sign and value of K will determine whether a maximum or minimum is found. Utilizing the conditions of Eq. (4-60) the quantities needed for the evaluation of ω_{max} and ω_{min} become

$$\sec^2 \left(\frac{\omega_m t_1}{2}\right) = 2 \tag{4-61}$$

$$\sec^2 (\beta - \alpha) = \frac{4}{2 + (-1)^n b} \tag{4-62}$$

$$\tan \left(\frac{\omega_m t_1}{2}\right) = \pm 1 \tag{4-63}$$

$$\tan (\beta - \alpha) = (-1)^K \sqrt{\frac{2 + (-1)^{n+1} b}{2 + (-1)^n b}} \tag{4-64}$$

n and K are independent and either odd or even to determine the proper signs for the maximums and minimums. Substituting these quantities into Eq. (4-43) gives

$$\omega_{\substack{max\\min}} =$$

$$\frac{(2 \pm b)}{(2\ (-1)^n\ b)\left[2\ (-1)^{n+1}\ b + \frac{b^2}{2}\ (-1)^K\ b\ (-1)^{K+n+1}\ \frac{b^2}{2}\right]} = \frac{N}{D} \tag{4-65}$$

The choice in the numerator comes from the selection of K in Eq. (4-60). Clearly the (-) ($\omega_m t_1 = \frac{\pi}{2}$) is associated with a minimum while the (+) ($\omega_m t_1 = \frac{3\pi}{2}$) is associated with a maximum. The denominator can occur in four distinct combinations.

Case I n odd, K even

$$D = 4 + 2b - b^3 \tag{4-66}$$

Case II n even, K even

$$D = 4 = 2b \tag{4-67}$$

Case III n odd, K odd

$$D = 4 - 2b \tag{4-68}$$

Case IV n even, K odd

$$D = 4 - 2b + b^3 \tag{4-69}$$

The primary maximum value of ω requires the minimum denominator which is clearly Case III since $b < 1$. The primary minimum value of ω requires the maximum value of D which is Case II. Equation (4-65) yields

$$\omega_{max} \simeq \frac{2 + b}{2 - b}\,\omega_c \qquad \text{n odd, K odd, numerator (+)} \qquad (4\text{-}70)$$

and

$$\omega_{min} \simeq \frac{2 - b}{2 + b}\,\omega_c \qquad \text{n even, K even, numerator (-)} \qquad (4\text{-}71)$$

where it is remembered that $b^2 << 2$.

If the maximum conditions are chosen,

$$\beta_{max} - \alpha_{max} = \tan^{-1}\sqrt{\frac{2 + b}{2 - b}} \quad \text{and} \quad \beta_{max} = \tan^{-1}\left(-\sqrt{\frac{2 - b}{2 + b}}\right) .$$

Therefore

$$\alpha_{max} = \tan^{-1}\left(-\sqrt{\frac{2 - b}{2 + b}}\right) - \tan^{-1}\sqrt{\frac{2 + b}{2 - b}} = \frac{\pi}{2} \quad \text{(principal value)} \qquad (4\text{-}72)$$

If minimum conditions are chosen,

$$\beta_{min} - \alpha_{min} = \tan^{-1}\left[-\sqrt{\frac{2 - b}{2 + b}}\right] \text{and} \quad \beta_{min} = \tan^{-1}\sqrt{\frac{2 + b}{2 - b}} .$$

Therefore

$$\alpha_{min} = \tan^{-1}\sqrt{\frac{2 + b}{2 - b}} - \tan^{-1}\left[-\sqrt{\frac{2 - b}{2 + b}}\right] = \frac{\pi}{2} \quad \text{(principal value)} \qquad (4\text{-}73)$$

The transit angle which produces the maximum also produces the minimum.

Since $\alpha = \frac{\omega_m L\sqrt{4 - b^2}}{4\,V_o}$ the optimum length of the modulator becomes

$$L_{opt} = \frac{2\pi V_o}{\omega_m\sqrt{4 - b^2}} = \frac{V_o}{f_m\sqrt{4 - b^2}} = \frac{V_o}{\sqrt{4 - b^2}}\,\tau_m \qquad (4\text{-}74)$$

It was assumed in the derivation that $b^2 << 2$, therefore it is permissible to write

$$L_{opt} \simeq \frac{V_o}{2}\tau_m = \frac{c}{2\sqrt{K}}\tau_m = \frac{c}{2\sqrt{K}}\frac{1}{f_m} \qquad (4-75)$$

where c is the free space velocity of light and K is the dielectric constant at the chosen operating point. τ_m is the modulating period.

The free space velocity of light is approximately $3 \cdot 10^8$ meters/sec. Therefore it is obvious that the optimum length modulator is feasable (if desirable) only when the modulating frequency is very high. A large operating point dielectric constant tends to shorten the length, but since the square root operation is involved the effect is reduced. For low modulating frequencies any physically practical length modulator is well within the assumptions used in deriving Eq. (4-34) and the frequency variation is essentially pure phase modulation.

In conclusion it should be noted that the percent change in frequency is independent of ω_c and therefore the optimum conditions hold for any incoming carrier. In principle, light could be modulated as well as microwaves. The difficulty is that all media have dielectric constants and therefore loss factors which are dependent on the carrier frequency. Known ferroelectrics, such as $BaTiO_3$ ceramics, which give the greatest promise for velocity modulation, do not have variable permittivities much above the microwave bands and are therefore inapplicable for very high frequency operation.

References

1. Slater, J.C., Microwave Electronics, Sec. 11.3, p.268, D. Van Nostrand (1950.

2. Seely, S., Electron-Tube Circuits, Chap.17, Sec.17-5, p.371. McGraw-Hill (1950).

CHAPTER 5

MICROWAVE FERROELECTRICITY

Introduction

The amount of information concerning the properties of ferroelectrics at microwave frequencies is rather limited and it was felt that the measurement of the complex dielectric constant* of certain barium titanate ceramics, at these wavelengths, should precede any attempts to build actual modulators.

Several problems need to be resolved. Powles and Jackson[1] have demonstrated that $BaTiO_3$ ceramics maintain their low frequency dielectric constant, independent of frequency, until some point in the microwave range and then undergo a relaxation phenomenon after which the dielectric constant is reduced perhaps by a factor of ten and the loss is substantially higher. Several questions immediately present themselves. If a low frequency voltage is used to bias the ferroelectric then certainly the low frequency dielectric constant can be controlled, but what of the high frequency constant? Does the same percent change of K' take

* The complex dielectric constant $K^* = K' - jK''$ accounts for dielectric losses and is discussed in detail by A.R. von Hippel in Dielectrics and Waves, Chap. 1, Sec.II, p.3 (Wiley and Sons, 1954).

place or is control lost more or less completely? Another question is what factors affect the frequency of the relaxation? Kittel has calculated that domain resonances are responsible;[2] if so, domain structure is critical and shifts of the relaxation frequency appear possible by suitably tailoring the ferroelectric. The maximum rate of modulation to which the materials can respond is also an open question. The present work is merely a very small step toward solving some of these questions.

Equipment

The 11,000 Mc. range was chosen for preliminary measurements because strip line (50 ohm) was available for this band and could provide a convenient method for applying a DC high voltage bias across a ceramic sample. It was felt that the same apparatus could also be used later on to form the actual modulators. Figure 1 gives a representation of the measuring apparatus. The high voltage terminal is made in the form of a low pass filter so that the microwave energy will not radiate from it. Band pass filters at either end of the sample section of the center conductor were needed to supply DC isolation. This was necessary because the coax to strip line transitions break down at relatively low field strengths. The probe of the slotted line detector could move back and forth to measure the reflection pattern and the tuner could be used to match the sample to the strip line. A standard double detection measuring set utilizing a Kay

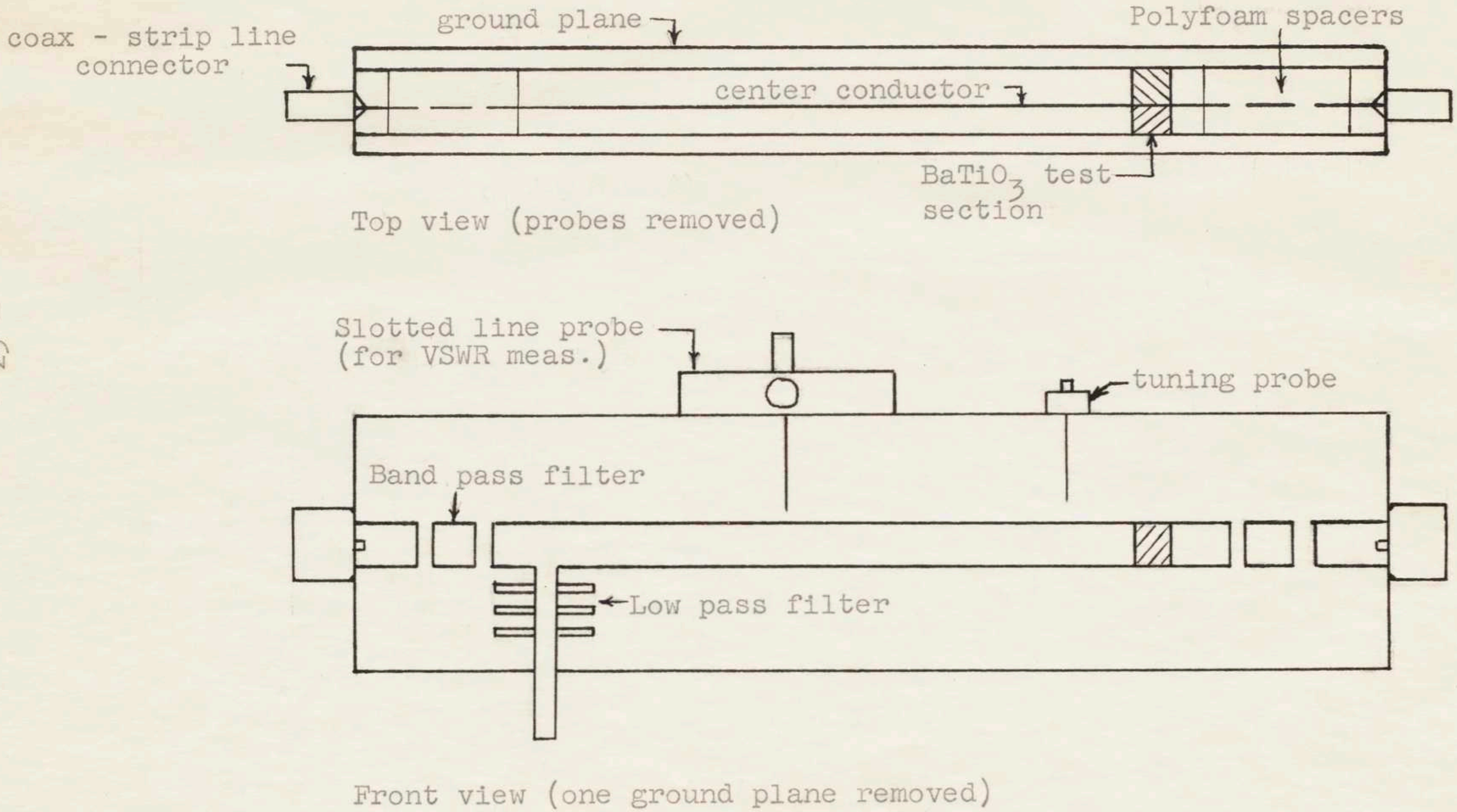

coax - strip line connector
ground plane
center conductor
Polyfoam spacers
$BaTiO_3$ test section
Top view (probes removed)
Slotted line probe (for VSWR meas.)
tuning probe
Band pass filter
Low pass filter
Front view (one ground plane removed)

Fig. 1

Electric meter circuit completed the set up.

Several different procedures are possible for measuring K^* (3) and the one which was employed is discussed in the next section.

Measurement of Complex Dielectric Constant

I. Introduction

The dominant mode in strip line is TEM and is sketched in Fig. 2. The fields are concentrated above and below the center conductor and so pass through the ceramic section when it is placed in the line. The geometry of the strip line cancels out when the reflection coefficient is evaluated therefore Fig. 3 represents the situation which is equivalent. The signal flow graph associated with this configuration* has a total reflection coefficient given by

$$\frac{E_r}{E_i} = -r + \frac{rt^2 \, e^{-j2\theta} \, e^{-2\alpha_\theta \theta}}{1 - r^2 \, e^{-j2\theta} \, e^{-2\alpha_\theta \theta}} = \rho \qquad (5\text{-}1)$$

where in general r and t are both complex and related by $|r|^2 + |t|^2 = 1$. The angle θ is equal to $\frac{2\pi L}{\lambda}$ where λ is the wavelength in the dielectric. For ferroelectrics which have large dielectric constants and relatively low loss ($\tan \delta < 1$) the reflection and transmission coefficients r and t can be assumed real and positive with little error.

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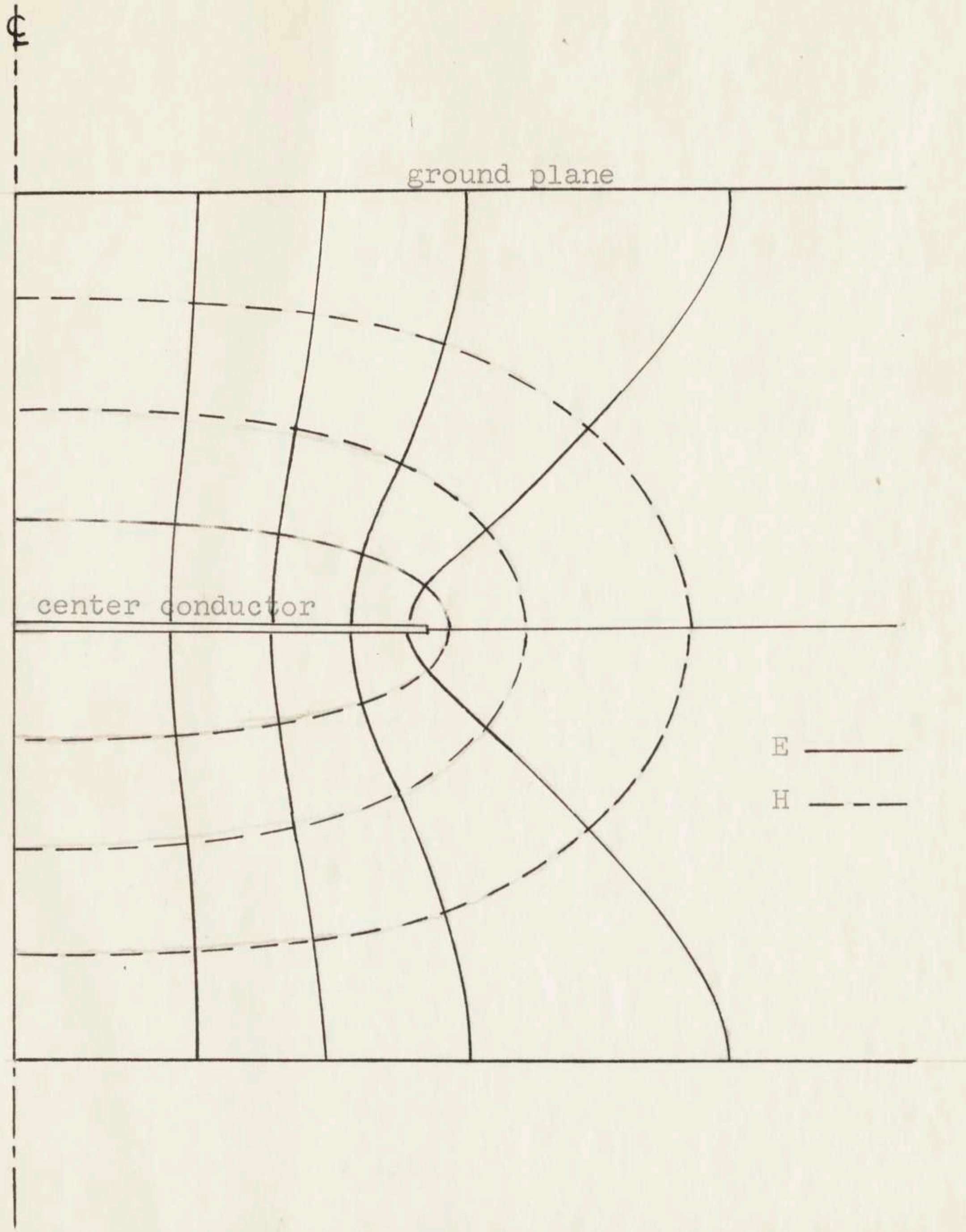
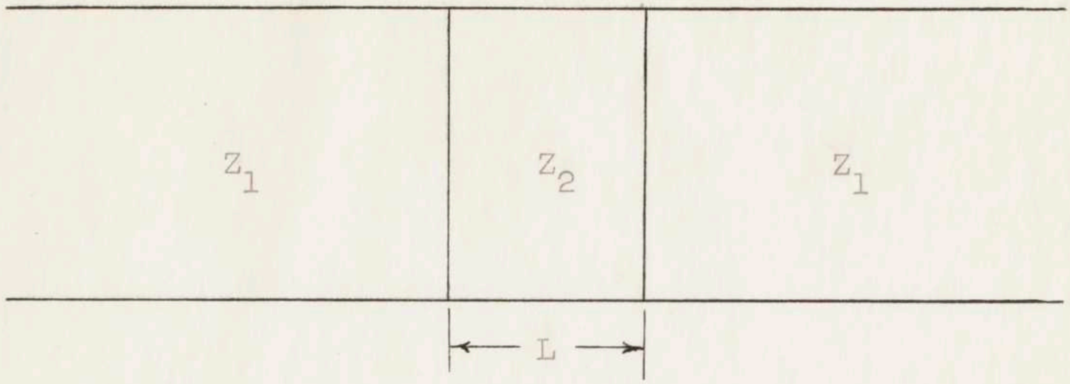
* First shown by S. Mason.

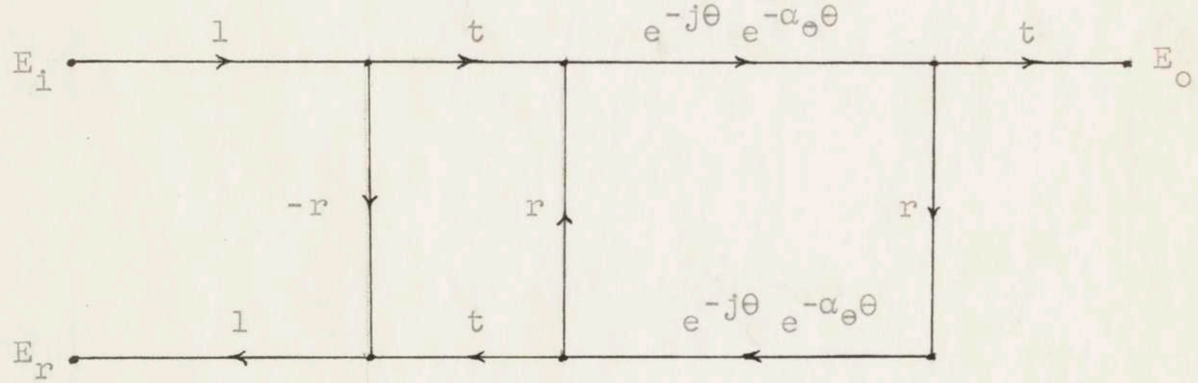


Dominant mode in strip line (from an unpublished memorandum by A. C. Schell). Fields are symmetric with respect to center line.

Fig. 2



$$r = \frac{Z_1 - Z_2}{Z_1 + Z_2} \qquad Z_1 > Z_2 \qquad \theta = \frac{2\pi}{\lambda} L$$



$$|r|^2 + |t|^2 = 1$$

Fig. 3

The magnitude of ρ is then given approximately by

$$|\rho| \simeq r - \frac{r(1 - r^2)\, e^{-2\alpha_\theta \theta}}{\left[1 - 2r^2\, e^{-2\alpha_\theta \theta} \cos 2\theta + r^4\, e^{-4\alpha\theta}\right]^{\frac{1}{2}}} \qquad (5\text{-}2)$$

If $|\rho|$ is known as a function of θ then r and α_θ may be evaluated. This leads to an evaluation of the complex dielectric constant. θ may be varied by changing L or λ and where the dielectric parameters are independent of frequency the latter method is preferable.

II. Lossless Dielectrics

For the lossless case $\alpha_\theta = 0$ and Eq. (5-2) becomes exactly

$$|\rho| = \left|\frac{E_r}{E_i}\right| = r - \frac{r(1 - r^2)}{(1 - 2r^2 \cos 2\theta + r^4)^{\frac{1}{2}}} \qquad (5\text{-}3)$$

This expression is periodic in 2θ so that adjacent maxima or minima are $\Delta\theta = \pi$ radians apart. The change in frequency necessary to produce this $\Delta\theta$ is simply

$$\Delta f = \frac{V}{2L} = \frac{C}{2L\sqrt{K' K_m'}} \qquad (5\text{-}4)$$

where $K' = \frac{\epsilon}{\epsilon_o}$ and $K_m' = \frac{\mu}{\mu_o}$. The ratio of impedances gives

$$\frac{Z_1}{Z_2} = S = \frac{1 + |r|}{1 - |r|} = \sqrt{\frac{K'}{K_m'}} \qquad (5\text{-}5)$$

Solving Eqs. (5-4) and (5-5) together for K' and K_m' yields

$$K_m' = \frac{C}{2LS\,\Delta f} \qquad (5\text{-}6)$$

$$K' = \frac{CS}{2L\,\Delta f} \qquad (5\text{-}7)$$

For non-magnetic dielectrics $K_m' = 1$ and Eq. (5-7) becomes

$$K' = \left(\frac{C}{2L\Delta f}\right)^2 \qquad (5\text{-}8)$$

III. Lossy Dielectrics

If $\alpha_\theta \neq 0$ and the loss is low so that Eq. (5-2) is valid then Eq. (5-8) will give a very good approximation to the real part of K^*. Eq. (5-2) gives $|\rho|$ as a damped oscillatory function of θ and the actual maxima and minima occur near the points where the slope of Eq. (5-2) is zero. These roots are solutions of

$$e^{-2\alpha_\theta \theta} (\sin 2\theta - \alpha_\theta \cos 2\theta) + \frac{\alpha_\theta}{r^2} = 0 \qquad (5\text{-}9)$$

If θ_1 is a solution of Eq. (5-9) then $\theta_1 + \pi$ will also be a solution provided $e^{-2\alpha_\theta \pi} \simeq 1$. The maxima are then spaced π radians apart and occur when $\cos 2\theta \simeq -1$. The envelope which passes through the $|\rho|_{max}$ values is approximately given by

$$|\rho|_{\substack{max \\ envelope}} \simeq r - \frac{r(1 - r^2)\, e^{-2\alpha_\theta \theta}}{1 + r^2\, e^{-2\alpha_\theta \theta}} \qquad (5\text{-}10)$$

The various maxima lie on this surve spaced $\sim\pi$ radians apart. The ratio of two adjacent maxima (ρ_1 and ρ_2 at angles θ_1 and $\pi + \theta_1$ respectively) is given by

$$\frac{\rho_1}{\rho_2} \simeq \frac{(e^{2\alpha_\theta \pi} + r^2\, e^{-2\alpha_\theta \theta_1})(r + 2r^3\, e^{-2\alpha_\theta \theta_1} - r\, e^{-2\alpha_\theta \theta_1})}{(1 + r^2\, e^{-2\alpha_\theta \theta_1})(r\, e^{2\alpha_\theta \pi} + 2r^3\, e^{-2\alpha_\theta \theta_1} - r\, e^{-2\alpha_\theta \theta_1})}$$

(5-11)

Since $e^{2\alpha_\theta \pi} \simeq 1$, Eq. (5-11) can be rewritten by cancelling the second factor on top and bottom and replacing the appropriate numerator term with $e^{2\alpha\pi} \simeq 1 + 2\alpha\pi$. The transformed equation is

$$\frac{\rho_1}{\rho_2} - 1 \simeq \frac{2\pi\alpha_\theta}{1 + r^2 e^{-2\alpha_\theta \theta_1}} \simeq \ln \frac{\rho_1}{\rho_2} \qquad (5\text{-}12)$$

If $\alpha_\theta \theta_1$ is quite small and r is close to unity (as it is for $BaTiO_3$ ceramics) then $\alpha_\theta \simeq \frac{1}{\pi} \ln \frac{\rho_1}{\rho_2}$. If $\alpha_\theta \theta_1$ is very large the exponental term is small compared to unity and $\alpha_\theta \simeq \frac{1}{2\pi} \ln \frac{\rho_1}{\rho_2}$. These two extreme cases differ by a factor of two but in general only the magnitude of α_θ is important so an average can be used

$$\alpha_\theta \simeq \frac{3}{4\pi} \ln \frac{\rho_1}{\rho_2} \qquad (5\text{-}13)$$

If more accuracy is desired the transcendental Eq. (5-12) may be solved by iteration but since this equation is only approximate the additional labor is not justified.

The complex propagation constant is given by

$$\gamma = j\omega\sqrt{\mu^* \epsilon^*} = j\,\frac{2\pi}{\lambda_o} \sqrt{K^* K_m^*} = \alpha + j\beta \qquad (5\text{-}14)$$

For non-magnetic media $K_m^* = 1$ and if K_1 and K_2 are defined so that $K_1 - jK_2 = \sqrt{K^*} = \sqrt{K' - jK''}$ then $\alpha = \frac{2\pi}{\lambda_o} K_2$ and $\beta = \frac{2\pi}{\lambda_o} K_1$. It is readily seen (because $\alpha Z = \alpha_\theta \theta$) that $K_2 = K_1 \alpha_\theta$. The loss tangent is given by

$$\tan \delta = \frac{K''}{K'} = \frac{2K_1K_2}{K_1^2 + K_2^2} \qquad (5\text{-}15)$$

If $\tan \delta \ll 1$ as assumed then $K_2^2 \ll K_1^2$ and Eq. (5-15) becomes

$$\tan \delta \simeq \frac{2K_2}{K_1} = 2\alpha_\theta \simeq \frac{3}{2\pi} \ln \frac{\rho_1}{\rho_2} \qquad (5\text{-}16)$$

This together with $K' \simeq (\frac{C}{2L\Delta f})^2$ characterizes the dielectric. The data required from a plot of $|\rho|$ versus f are the length of the sample (L), the magnitudes of two successive reflection coefficient maxima (ρ_1 and ρ_2) and the difference in frequency between the maxima (Δf).

Materials

Two different types of $BaTiO_3$ ceramics were made available through the efforts of L. Egerton who is in charge of ceramic and glass compositions for Department 1128 at Bell Telephone Laboratories - Murray Hill, New Jersey. These were $CuSnO_3$ - $BaTiO_3$ (10 mole % $CuSnO_3$) and $PbSnO_3$ - $BaTiO_3$ (14 mole % $PbSnO_3$) ceramics. Both have been discussed in the literature at relatively low frequencies[4] but have not received attention at microwave frequencies as far as this author knows. These ceramics were chosen because of their reported good voltage sensitivity and because when made in the proportions listed have Curie points below room temperature. According to von Hippel there is a temperature range above the Curie point in which $BaTiO_3$ ceramics still exhibit ferroelectricity and have lower loss characteristics.[5]

The samples were machined to .320" x .100" x 4 mm. and silver plated on the broad faces. Two of these placed one on either side of the strip line center conductor filled the space above and below it completely. Since practically all of the field is concentrated in this region (for the dominant mode) it was assumed that the absence of the ceramic along the edges of the center conductor was unimportant.

Author's Results

Quantitative results obtained from the reflection pattern data appear to be unreliable and the straightforward theory developed inapplicable. The reasons for this appear to come from several sources. The first is that other modes are present in the strip line which are caused by the least dissymmetry. Normally they are 30 db or more below the main signal but the K' being measured was high so that very large reflections were obtained. When the probe was placed at a voltage minimum the signal level may well have been below that of the undesirable modes which were of arbitrary phase. The second reason may be that the value of K' being high allows other modes to exist within the ceramic section itself. These might be reflected internally many times in various modes before emerging. This could put peaks in the reflection coefficient versus frequency plot not accounted for by the simple theory. The assumption that K^* was invariant over a narrow frequency range may have been wrong but this is unlikely since several different frequency ranges were tried and the per cent deviation of frequency was slight.

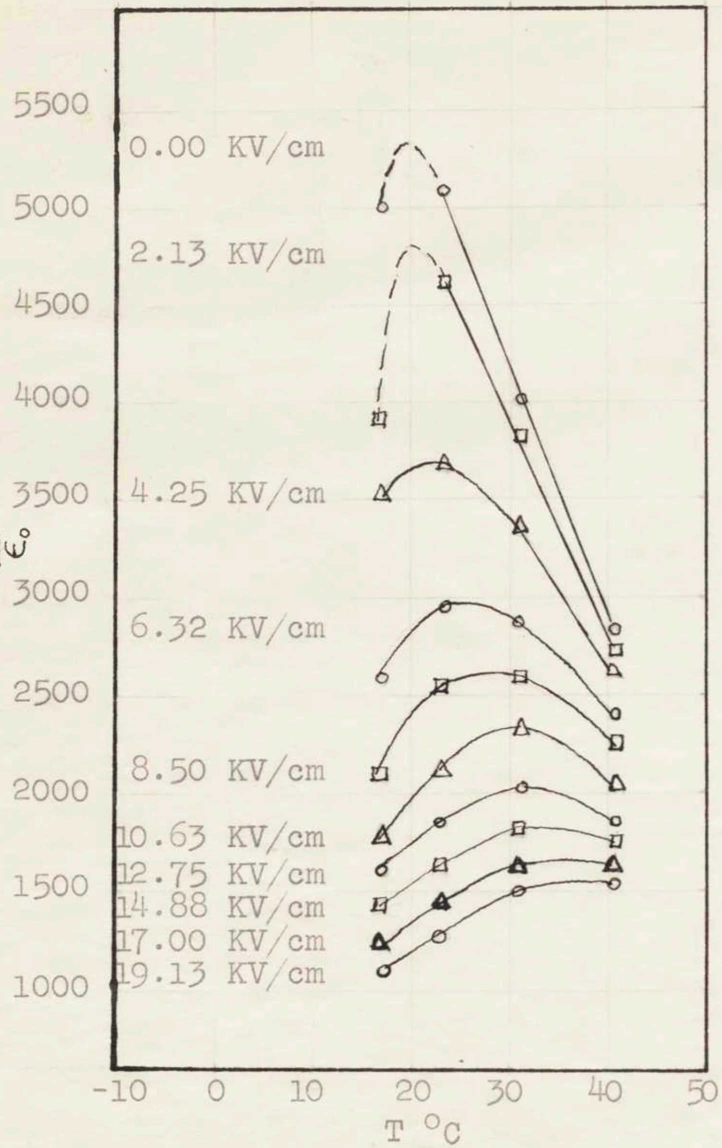
Direct insertion loss measurements were made at 11,000 Mc which indicated about 20 db loss for the 4 mm. samples of both types of ceramics. Any leakage of the field around or between the samples would make the true loss higher but the figure of 5 db/mm may be a fair estimate.

Qualitatively it appears that the relaxation spectrum has not set in yet at 12,000 Mc for the $PbSnO_3$ ceramics and that the low frequency dielectric constant, which Coffeen claims is around 4000, is still in effect. The corresponding loss tangent would be on the order of .1. There is a great deal of uncertainty about these conclusions and they need to be rechecked very accurately.

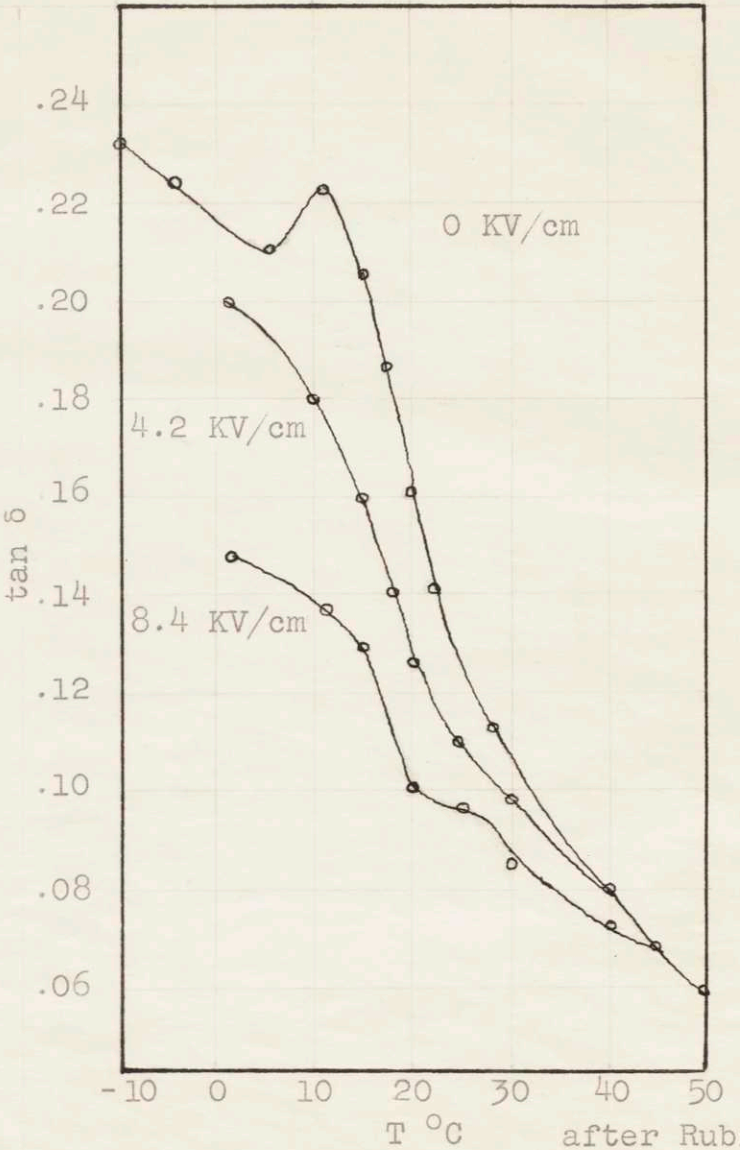
The $CuSnO_3$ ceramics were very lossy at dc. Therefore the $PbSnO_3$ samples, which were not, (less than 3 mw. was dissipated when 3000 V. was applied) were used exclusively in the experiments which attempted to vary the value of K' by applying a bias field. This was possible but the quantitative changes are very much in doubt.

Davis and Rubin's Results

Davis and Rubin have published data on $SrTiO_3$ - $BaTiO_3$ (27% $SrTiO_3$) ceramics at 3000 Mc[6] which is summarized in Fig. 4. Their results show that the relaxation spectrum reported by Powles and Jackson has not been reached and that at room temperature the dielectric constant is on the order of 5000 with a loss tangent of .1 (with no bias field applied). If a field strength of 10 KV/Cm is maintained the loss decreases slightly and the dielectric constant drops to about 2000. If it is postulated that their data also holds at



Temperature and electric field dependence of dielectric constant at 3000 Mc. of 73% $BaTiO_3$ and 27% $SrTiO_3$



after Rubin and Davis.

Temperature and electric field dependence of loss tangent at 3000 Mc. of 73% $BaTiO_3$ and 27% $SrTiO_3$

Fig. 4

11,000 Mc (which is doubtful) then it would appear that the $SrTiO_3$ and $PbSnO_3$ ceramics might be similar in dielectric behavior at least when no bias is applied.

Modulation

I. Theoretical Calculations Based on Experimental Data

On the basis of Rubin and Davis's data it is possible to predict the performance of a $SrTiO_3$ - $BaTiO_3$ modulator operating with a carrier frequency of 3000 Mc. If the ambient temperature is about 25° C then $\tan \delta = .1$ and K' varies between 4000 and 2000 for 0 KV/Cm and 10 KV/Cm respectively. Assuming a linear change of K' with field strength, it is obvious that a dc bias of 5 KV/Cm in series with an ac voltage of magnitude 10 KV/Cm peak to peak will produce a permittivity given by

$$\epsilon = 3000\,\epsilon_o(1 + \tfrac{1}{3}\sin\omega_m t) \qquad (5\text{-}17)$$

Substituting the values of b and K_0 which were found above into Eq. (4-75) one finds that the length of the "optimum modulator" is given by

$$L_{opt} \simeq \frac{3 \times 10^8}{f_m} \quad \text{cm} \qquad (5\text{-}18)$$

The loss per meter can be shown to be

$$\text{Loss} = 8.686\,\frac{2\pi}{\lambda_o}\left[\frac{1}{2} K_0 \left\{\sqrt{1 + \tan^2\delta} - 1\right\}\right]^{\frac{1}{2}} \quad \frac{\text{db}}{\text{meter}} \qquad (5\text{-}19)$$

With a loss tangent of .1 the loss at 3000 Mc. is approximately 15 db/cm. If the maximum allowable loss through the modulator were 15 db then L_{opt} = 1 cm and $f_m = 3 \times 10^8$ cps which presumably is above the maximum value to which the material can respond. Therefore as predicted earlier the optimum modulator is probably not realizable. If a lower value of ω_m is picked and a short L is used so as to keep the loss down, then Eqs. (4-35) and (4 36) apply. The phase deviation at 3000 Mc is

$$\theta_d = \frac{b\omega_c L\sqrt{K_0}}{C} = \frac{11\pi L}{3} \text{ radians} \tag{5-20}$$

where L is in cm. For a $\theta_d = 2.4$ the carrier disappears from the spectrum as is well known. The value of L required is 2 mm and the corresponding loss is 3 db.

If the questionable data concerning the $PbSnO_3$ - $BaTiO_3$ ceramic or the Rubin and Davis data extrapolated to 10 KMc is used then the loss is approximately 5 db/mm. The phase deviation is now

$$\theta_d = \frac{110\pi L}{9} \text{ radians} \tag{5-21}$$

For $\theta_d = 2.4$ as before, $L \simeq \frac{6}{10}$ mm. The corresponding loss is again 3 db. It is readily seen that as long as the dielectric parameters are not functions of frequency the loss through a modulator which gives a $\theta_d = 2.4$ will be 3 db. The db loss is linearly proportional to θ_d.

II. Optimization of Operating Temperature

Above the Curie temperature there exists a range where ferroelectricity is possible and the losses relatively small.

In general the loss decreases with increasing temperature but the modulating index b also decreases. If assumptions are made as to how the values of b and tan δ vary with T in this region then the optimum operating temperature may be computed. When exponential functions are postulated the results are readily obtainable.* Assume therefore that

$$b = b_o \, e^{-\alpha(T - T_c)} \qquad T \geqslant T_c \tag{5-22}$$

and

$$\tan\delta = L_\infty + (L_o - L_\infty)\, e^{-\beta(T - T_c)} \quad T \geqslant T_c \tag{5-23}$$

From Eq. (4-36)

$$L = \frac{\theta_d V_o}{\omega_m b} = \frac{\theta_d V_o}{\omega_m b_o} e^{\alpha(T - T_o)} \tag{5-24}$$

The loss is given by

$$\text{Loss} = 8.686 \frac{2\pi}{\lambda_o} \left[\frac{K_0}{2} \left(\sqrt{1 + \tan^2\delta} - 1 \right) \right]^{\frac{1}{2}} L \text{ db} \tag{5-25}$$

If K_0 is assumed constant with temperature then

$$\text{Loss} = A \left[\sqrt{1 + \tan^2\delta} - 1 \right]^{\frac{1}{2}} \frac{b_o}{b} \tag{5-26}$$

where $A = 8.686 \frac{2\pi}{\lambda_o} \sqrt{\frac{K_0}{2} \frac{\theta_d V_o}{\omega_m b_o}}$. Substituting the exponen-

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* If K'(T), regardless of the field strength, obeyed a Curie-Weiss law exactly, then b(T) would be a constant for all T which is certainly not true. The exponential behavior as postulated allows the ferroelectric effect to gradually die out above the Curie temperature.

tial functions

$$\text{Loss} = A\left[\sqrt{1+\left\{L_{\infty}+(L_o - L_{\infty})e^{-\beta(T-T_c)}\right\}^2} - 1\right]^{\frac{1}{2}} e^{\alpha(T-T_c)}$$

$$T \geq T_c \qquad \text{(5-27)}$$

Minimizing with respect to T requires that $\frac{d\,\text{Loss}}{dT} = 0$ or

$$\frac{d}{dT}\left\{e^{\alpha T}\left[\sqrt{1+\tan^2\delta} - 1\right]^{\frac{1}{2}}\right\} = 0 \qquad \text{(5-28)}$$

Simplifying yields

$$(1+\tan^2\delta) - (1+\tan^2\delta)^{\frac{1}{2}} = \frac{\beta}{2\alpha}(L_o - L_{\infty})e^{\beta(T-T_c)}\tan\delta \qquad \text{(5-29)}$$

For the case where $\tan\delta \ll 1$ the approximation that

$\sqrt{1+\tan^2\delta} \simeq 1 + \frac{1}{2}\tan^2\delta$ is permissible and Eq. (5-13) becomes

$$e^{\beta T} \simeq \frac{(\frac{\beta}{\alpha} - 1)(L_o - L_{\infty})}{L_{\infty}} e^{\beta T_c} \qquad \text{(5-30)}$$

The solution is simply

$$T \simeq T_c + \frac{1}{\beta}\ln\frac{(\beta-\alpha)(L_o - L_{\infty})}{\alpha L_{\infty}} \qquad \text{for } T \geq T_c \qquad \text{(5-31)}$$

therefore $\frac{(\beta-\alpha)(L_o-L_{\infty})}{\alpha L_{\infty}} \geq 1$ or

$$\beta \geq \alpha\frac{L_o}{L_o - L_{\infty}} \qquad \text{(5-32)}$$

Since $L_{\infty} > 0$ it follows that $\frac{L_o}{L_o - L_{\infty}} > 1$ and $\beta \ggg \alpha$ to meet the minimizing condition. In general, the converse is true so that at no value of T in the interval does $\frac{d\,\text{Loss}}{dT} = 0$. The minimum loss occurs therefore at $T = T_c$. This suggests that if the exponential law holds it is advisable to operate just above the Curie temperature. This would automatically fix the value of b and L which in some cases might not be desirable; however, the loss would be the lowest possible. It is apparent that the same temperature minimizes loss for all modulators of the same ceramic regardless of the values of ω_c, or θ_d. The value of this minimum loss is, of course, a function of the latter parameter.

III. Experimental Modulators

An experimental modulator at 11,000 Mc was built and tested using the strip line apparatus previously described. A 800 volt peak to peak 60 cycle power transformer was placed in series with a high voltage dc supply capable of delivering up to 5000 volts. The titanate ceramics ($PbSnO_3$ - $BaTiO_3$) had been coated with ceresin wax after being placed in a drying oven to prevent corona discharge. The transmission signal was observed on a scope after it had been beat down to a 70 Mc IF signal and fed through the meter circuit of a Kay Electric measuring set which passes 60 cps. Fig. 5 shows the waveforms obtained for various values of DC bias. The transmission curves are plots of the modulator output versus time and also output versus the amplitude of the modulating voltage. The S shaped character of these latter curves is explained if it is remembered that the transmission maximum occurs at an electric length of the modulator equal to a multiple of half the wavelength. For

14 Mole % $PbSnO_3$ - $BaTiO_3$ Ceramic used as a Dielectric Modulator at 10140 Mc. Output signal versus DC bias voltage for a fixed 800 volt peak to peak 60 cps modulating voltage.

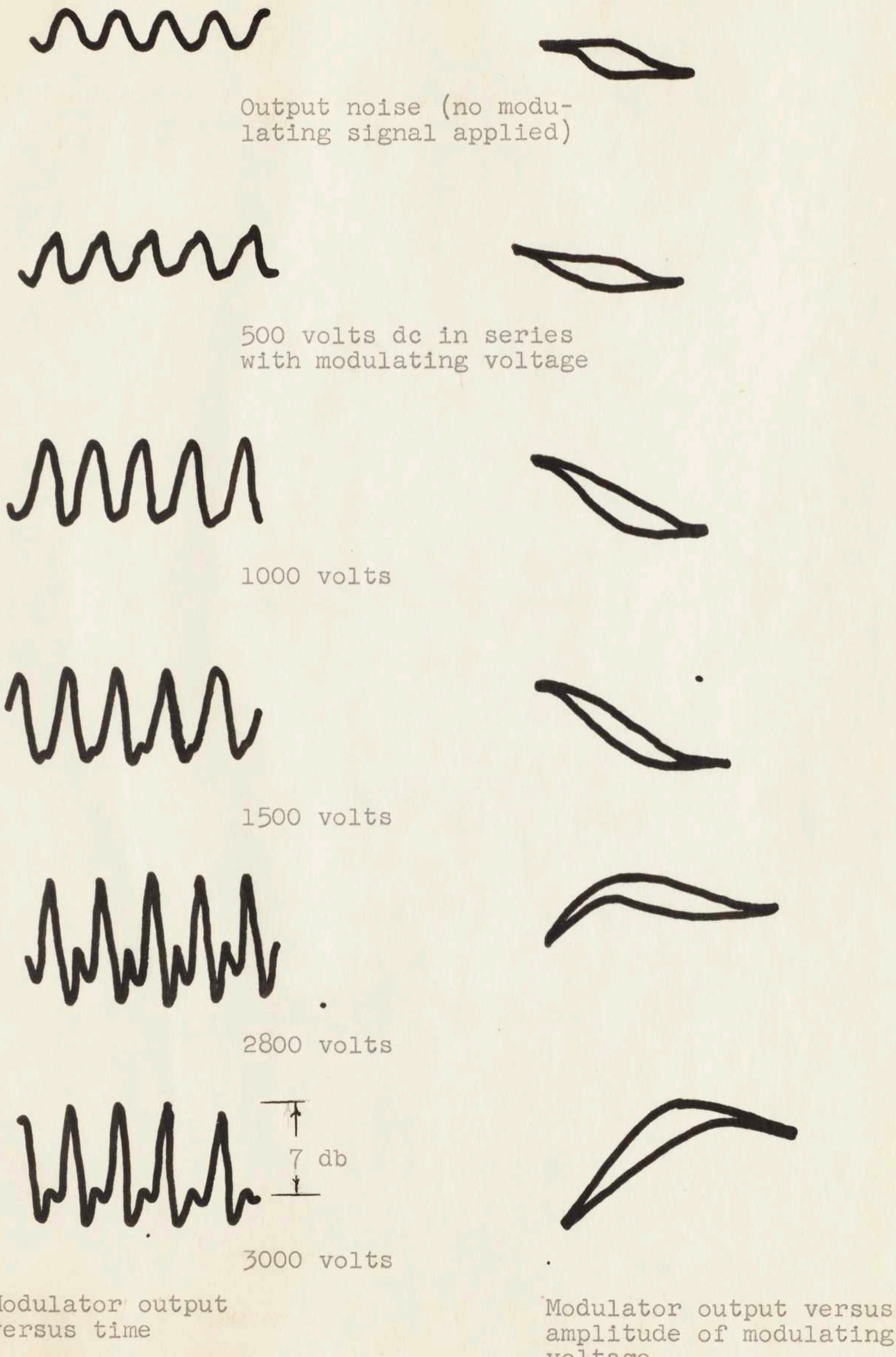

Fig. 5

more or less applied voltage the value of K' hence changes the electric length. No matter whether it is a plus or minus change the transmission is decreased. The hysteresis is due to a noise signal emanating from the meter circuit. These transmission curves show only the amplitude variations of the signal, therefore to detect the phase modulation the signal from the meter circuit was fed into a Hallicrafter SX 62 receiver tuned to 70 Mc. A 60 cycle output from the receiver was observed on a scope and appeared to furnish proof of the PM component. It was observed however that there was either considerable AM to FM conversion or a leak path in the receiver since a standard AM signal generator set on a 70 Mc carrier also produced an output from the receiver when it was tuned to the FM band. An AM signal which was equal in amplitude to the titanate modulator output produced an FM output less than one half that observed from the microwave modulator. Another FM receiver was not available to check the results but it is reasonable to assume that a PM component was actually produced. If the ceramic section had been matched into the strip line the AM component would have been considerably smaller. A higher modulating frequency would have improved the signal to noise ratio since there was a substantial 60 cycle noise source in the Kay Electric meter circuit.

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CHAPTER 6

CONCLUSIONS

The theoretical analysis has indicated the performance which may be expected from a dielectric modulator. In practical situations essentially linear phase modulation may be expected together with the inherent AM component. The electromagnetic momentum of such a wave is unaffected by the modulation process but the energy level will in general be increased. This energy is provided by the modulating source which on the average does work upon the electromagnetic field.

Dielectrics which appear suitable for velocity modulation include "artificial" diode dielectrics (which have not been tried at all so far) and ferroelectric ceramics such as $BaTiO_3$ - $PbSnO_3$ and $BaTiO_3$ - $SrTiO_3$ compositions. There is a temperature range above the Curie Point where these ceramics are still ferroelectric and where the losses are substantially reduced. The Curie temperature can be moved over a wide range by altering the concentrations of the lead and strontium atoms.

One possible application for dielectric modulators appears to be in microwave double detection measuring sets where the difficulty in keeping two frequency sources tuned to the required IF frequency is often a problem and always a nuisance. If the signal from a monochromatic source is split and one half modulated and then recombined with the other (unmodulated) half, an IF component can be detected. The apparatus to be tested would be placed in the path of the unmodulated signal. The IF signal would not be lost when the main source was tuned to a different frequency.

When a ferroelectric is used to frequency modulate at low frequencies by varying the capacitance of a tuned circuit, the operating point value of K' determines the operating point capacitance which in turn determines the carrier frequency. If this value K' changes a little because of temperature variations, etc., then the carrier will drift also. Observe that in the dielectric velocity modulator which has been discussed the carrier frequency is not affected by changes in K' and will always be as stable as the generating source decrees. The phase deviation is, of course, sensitive to changes in the dielectric constant and drift due to temperature changes may be important.

The maximum modulating rate to which the ferroelectric ceramics will respond and the amount of microwave power which can be transmitted through a dielectric modulator are unknown. The extent to which the piezoelectric effect enters the modulation problem is also unknown.

It should be realized that these results assume that the velocity of propagation of the dielectric medium is <u>not</u>

modulated by the electromagnetic field passing through it, but only by the modulating bias. If this is true, the linear analysis which has been derived is valid; if not, the field relations are non-linear and much more difficult to solve. The velocity will not be modulated by the microwave field if the medium cannot respond to microwave frequencies and this is apparently the case. Even if it is not, the results are applicable if the microwave field is not sufficiently strong to make significant changes in the permittivity.

As far as future experimentation is concerned, this author feels that the first and quite formidable task which should be performed is the accurate measurement and tabulation of the dielectric constant and loss tangent of the most promising of the barium titanate ceramics as functions of composition, temperature, and field strength throughout the microwave range.

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