Reconstruction of Modular Data from $SL_2(Z)$ Representations

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RECONSTRUCTION OF MODULAR DATA FROM $\mathrm{SL}_2(\mathbb{Z})$ REPRESENTATIONS

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Abstract. Modular data is a significant invariant of a modular tensor category. We pursue an approach to the classification of modular data of modular tensor categories by building the modular S and T matrices directly from irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$. We discover and collect many conditions on the $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ representations to identify those that correspond to some modular data. To arrive at concrete matrices from representations, we also develop methods that allow us to select the proper basis of the $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ representations so that they have the form of modular data. We apply this technique to the classification of rank-6 modular tensor categories, obtaining a classification of modular data, up to Galois conjugation and changing spherical structure. Most of the calculations can be automated using a computer algebraic system, which can be employed to classify modular data of higher rank modular tensor categories. Our classification employs a hybrid of automated computational methods and by-hand calculations.

Contents

1. Introduction

Just as conventional symmetries are described by groups, gapped quantum liquid phases of bosonic matter (*i.e* bosonic topological order) seem to be described by non-degenerate higher braided fusion categories. It has been conjectured that topological orders are classified by the collection of projective representations of mapping class groups for various topologies of closed space manifolds [39]. In particular, we believe that a gapped phase of quantum matter in two spacial dimensions is classified by a pair $(\mathcal{B}, c)$, where $\mathcal{B}$ is a unitary modular tensor category (MTC) and c is a rational number equal to the central charge of $\mathcal{B}$ mod 8. Physically, $\mathcal{B}$ models the topological excitations (*i.e.* the anyons) in the gapped phase [19], and c measures the possible stacking of E_8 quantum Hall state, which has central charge $c = 8$. Therefore, a classification of unitary MTCs should give rise to a classification of all gapped quantum phases of bosons without symmetry in two spacial dimensions.

MTCs are defined by very complicated data. The classification of MTCs naturally breaks into two steps: the first step is to classify the modular data (MD), and the second is to classify modular isotopes with a given MD if not unique. The MD (S, T) of an MTC form a projective representation of the mapping class group of the 2-dimensional torus. (In fact, the notion of topological order was first introduced based on modular data (S, T) [39].) We will see that the classification of MDs is much more manageable than the full classification of MTCs.

Modular data (S, T) corresponding to MTCs of rank $r \leqslant 5$ have been completely classified [5, 33, 17]. More recently, such a classification for MTCs of rank 6 containing a pair of non-self-dual simple objects and a partial classification of general MTCs of rank 6 has also been obtained [9]. The strategy employed in those classifications begins with a stratification of the Galois group of the extension of $\mathbb{Q}$ by the entries of the modular S matrix, followed by a case by case analysis on the inferred polynomial constraints. As the Galois group is isomorphic to an abelian subgroup of $\mathfrak{S}_r$, this program is tractable, although somewhat tedious. As a last resort in a few cases, the classification of low-dimensional representations of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ for small n was required as well. The typical outcome is that most Galois groups can be eliminated and one eventually finds a finite list of modular data which can then be realized from known constructions.

In this article we complete the classification of rank 6 MDs using the reverse strategy: we build upon the approach in [12, 5] by constructing the MDs directly from $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ representations of low dimension. Since n is bounded in terms of the rank, expressing irreducible $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ representations as tensor products of prime-power level representations (i.e. $\mathrm{SL}_2(\mathbb{Z}/p^k\mathbb{Z})$ for primes p) allows us to stratify by representation type and level. Thus, up to basis choice, the $\mathrm{SL}_2(\mathbb{Z})$ representations can be presented as pairs (s, t), where s is symmetric and t is diagonal. The construction of *symmetric* representations of $\mathrm{SL}_2(\mathbb{Z})$ is an interesting problem of its own [27, 28]. We note that the number of inequivalent $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ representations is finite at a given dimension, since the dimension and n are bounded in terms of the rank. These facts make our classification possible. We find that up to Galois conjugation and altering spherical structures there are 12 classes (orbits) of modular data, all of which are realized via quantum groups, see Table 2. Only one of these orbits has no pseudo-unitary representative, while two distinct orbits have the same fusion rules.

In the next step of our classification, for each representation (s, t), we conjugate s by an arbitrary (real orthogonal) matrix that commutes with t to reconstruct the potential MD (S, T) with S symmetric and T diagonal. We find several methods that allow us to select a finite number of possible real orthogonal matrices from the uncountable set of real orthogonal matrices, so that the resulting (S, T) include all the MDs. Up to reordering the objects in the category, *i.e.*, the rows/columns of the resulting (S, T), these must satisfy the algebraic and number-theoretic constraints of MDs. Case by case analyses, following a similar pattern, then yield our classification. We remark that this approach was used in a particular case in [14] to construct modular data for the center of the fusion category associated with the extended Haagerup subfactor. At a BIRS workshop in 2014 with the first 3 authors present, Gannon suggested that the classification of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ representations could provide an alternative proof to the rank-finiteness theorem [6] if one could show there are at most finitely many modular data (S, T) associated to any given $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ representation. In fact, we found this to be true for dimension $\leqslant 6$. The difficulty is to find the appropriate basis changes, even if their existence is known. For small ranks, doing this by hand is a serious hurdle, although feasible. For larger ranks, this can be overcome through computer implementation.

The approach to the classification of MDs by building the modular S and T matrices directly from irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is applicable to much more general cases than the rank 6 case in this paper. One version of our approaches that is presented in the Appendix can be automated and almost all of the calculations in this approach can be implemented using the GAP computer algebra system.

The content of the paper is as follows: In sections 2 and 3, we discover and collect many conditions on the $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ representations to help us identifying those that are from some MDs. To arrive at concrete matrices from representations, we also develop methods that allow us to select the proper basis of the $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ representations so that they become the MDs. In sections 4 and 5, we apply this technique to the classification of rank-6 MTCs, obtaining a classification up to MD. Most of the calculations can be automated using a computer algebraic system, which can be employed to classify MDs of higher rank MTCs.

2. Modular tensor categories and modular data

Given a modular tensor category (MTC)* $\mathcal{B}$, the modular data (MD) of $\mathcal{B}$ consists of the unnormalized S- and T- matrices of $\mathcal{B}$, hence the MD of an MTC is independent of any normalizations. Though the MD of an MTC does not determine the MTC uniquely [22], it is still the most useful and important invariant of an MTC. Moreover, the MDs of MTCs have enchanting relations with diverse areas from congruence subgroups to vector-valued modular forms to topological phases of matter.

2.1. Necessary conditions for the modular data of an MTC.

An obvious strategy to classify MDs would be first to find all necessary and sufficient conditions for MDs, and then simply look for solutions. But it seems very hard to find such a complete characterization of MD. Instead we will list some necessary conditions and then appeal to other methods to finish a classification.

The following collection of results on modular data which will be useful in the sequel. Many are well-known and found in, e.g. [2].

Theorem 2.1. *The modular data (S, T) of an MTC satisfies:*

(1) *S, T are symmetric complex matrices, indexed by $i, j = 0, \ldots, r-1$.*†

*We use the terminology of MTC as in its original sense [23], which is equivalent to a semi-simple modular category of [36], i.e. a semi-simple modular category.

†The index also labels the simple objects in the MTC, with $i = 0$ corresponding to the unit object, and r is the **rank** of the modular data and the MTC.

(2) *T is unitary, diagonal, and $T_{00} = 1$.*

(3) *$S_{00} = 1$. Let $d_i = S_{0i}$ and $D = \sqrt{\sum_{i=0}^{r-1} d_i d_i^*}$. Then*

$$SS^\dagger = D^2 \,\mathrm{id}, \tag{2.1}$$

and the $d_i \in \mathbb{R}$.

(4) *S_{ij} are cyclotomic integers in $\mathbb{Q}_{\mathrm{ord}(T)}$*[‡] [26]*. The ratios S_{ij}/S_{0j} are cyclotomic integers for all i, j* [8]*. Also there is a j such that $S_{ij}/S_{0j} \in [1, +\infty)$ for all i* [13].

(5) *Let $\theta_i = T_{ii}$ and $p_\pm = \sum_{i=0}^{r-1} d_i^2 (\theta_i)^{\pm 1}$.*
Then p_+/p_- is a root of unity, and $p_+ = D e^{i2\pi c/8}$ for some rational number c.[§] *Moreover, the modular data (S, T) is associated with a projective $\mathrm{SL}_2(\mathbb{Z})$ representation, since:*

$$(ST)^3 = p_+ S^2, \quad \frac{S^2}{D^2} = C, \quad C^2 = \mathrm{id}, \tag{2.2}$$

where C is a permutation matrix satisfying

$$\mathrm{Tr}(C) > 0. \tag{2.3}$$

(6) *Cauchy Theorem* [6]*: The set of distinct prime factors of* $\mathrm{ord}(T)$ *coincides with the distinct prime factors of* $\mathrm{norm}(D^2)$.[¶]

(7) *Verlinde formula (cf.* [37]*) :*

$$N_k^{ij} = \frac{1}{D^2} \sum_{l=0}^{r-1} \frac{S_{li} S_{lj} S_{lk}^*}{d_l} \in \mathbb{N}, \tag{2.4}$$

where $i, j, k = 0, 1, \ldots, r-1$ and $\mathbb{N}$ is the set of non-negative integers.[‖] *The N_0^{ij} satisfy*

$$N_0^{ij} = C_{ij}, \tag{2.5}$$

which defines a charge conjugation $i \to \bar{i}$ via

$$N_0^{\bar{i}j} = \delta_{ij}. \tag{2.6}$$

(8) *Let $n \in \mathbb{N}_+$. The n^{th} Frobenius-Schur indicator of the i-th simple object*

$$\nu_n(i) = \frac{1}{D^2} \sum_{j,k} N_i^{jk} (d_j \theta_j^n)(d_k \theta_k^n)^* \tag{2.7}$$

is a cyclotomic integer whose conductor divides n and $\mathrm{ord}(T)$ [25, 24]*. The 1st Frobenius-Schur indicator satisfies $\nu_1(i) = \delta_{i,0}$ while the 2nd Frobenius-Schur indicator $\nu_2(i)$ satisfies $\nu_2(i) = 0$ if $i \neq \bar{i}$, and $\nu_2(i) = \pm 1$ if $i = \bar{i}$ (see* [3, 25, 33]*).*

We denote by $\mathrm{Gal}(\mathbb{Q}_n)$ the Galois group of the cyclotomic field $\mathbb{Q}_n$.

Remark 2.2. The above conditions are for modular data of unitary or non-unitary MTCs. In particular, the above conditions are invariant under Galois conjugations in $\mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(T)}/\mathbb{Q})$. Therefore, we can group modular data into Galois orbits.

The mathematical definition of Frobenius-Schur indicators of an object in pivotal fusion category was introduced in [25] and the trichotomy of the 2nd Frobenius-Schur indicator of a simple object was also proved therein. If the underlying *pivotal structure* is not spherical, the d_i in the preceding

[‡]Here $\mathbb{Q}_n$ denotes the field $\mathbb{Q}(\zeta_n)$ for a primitive nth root of unity ζ_n

[§]The **central charge** c of the modular data and of the MTC is only defined modulo 8.

[¶]Here $\mathrm{norm}(x)$ is the product of the distinct Galois conjugates of the algebraic number x.

[‖]The N_k^{ij} are called the fusion coefficients.

TABLE 1. Rank $\leqslant 5$ modular data

N_c	D^2	$d_0, d_1, \cdots$	$s_0, s_1, \cdots$	N_c	D^2	$d_0, d_1, \cdots$	$s_0, s_1, \cdots$
1_1	1	1	0				
2_1	2	$1, 1$	$0, \frac{1}{4}$	2_{-1}	2	$1, 1$	$0, -\frac{1}{4}$
$2_{14/5}$	3.6180	$1, \frac{1+\sqrt{5}}{2}$	$0, \frac{2}{5}$	$2_{-14/5}$	3.6180	$1, \frac{1+\sqrt{5}}{2}$	$0, -\frac{2}{5}$
3_2	3	$1, 1, 1$	$0, \frac{1}{3}, \frac{1}{3}$	3_{-2}	3	$1, 1, 1$	$0, -\frac{1}{3}, -\frac{1}{3}$
$3_{1/2}$	4	$1, 1, \sqrt{2}$	$0, \frac{1}{2}, \frac{1}{16}$	$3_{-1/2}$	4	$1, 1, \sqrt{2}$	$0, \frac{1}{2}, -\frac{1}{16}$
$3_{3/2}$	4	$1, 1, \sqrt{2}$	$0, \frac{1}{2}, \frac{3}{16}$	$3_{-3/2}$	4	$1, 1, \sqrt{2}$	$0, \frac{1}{2}, -\frac{3}{16}$
$3_{5/2}$	4	$1, 1, \sqrt{2}$	$0, \frac{1}{2}, \frac{5}{16}$	$3_{-5/2}$	4	$1, 1, \sqrt{2}$	$0, \frac{1}{2}, -\frac{5}{16}$
$3_{7/2}$	4	$1, 1, \sqrt{2}$	$0, \frac{1}{2}, \frac{7}{16}$	$3_{-7/2}$	4	$1, 1, \sqrt{2}$	$0, \frac{1}{2}, -\frac{7}{16}$
$3_{8/7}$	9.2946	$1, \xi_7^2, \xi_7^3$	$0, -\frac{1}{7}, \frac{2}{7}$	$3_{-8/7}$	9.2946	$1, \xi_7^2, \xi_7^3$	$0, \frac{1}{7}, -\frac{2}{7}$
4_0^a	4	$1, 1, 1, 1$	$0, 0, 0, \frac{1}{2}$	4_0^b	4	$1, 1, 1, 1$	$0, 0, \frac{1}{4}, -\frac{1}{4}$
4_1	4	$1, 1, 1, 1$	$0, \frac{1}{8}, \frac{1}{8}, \frac{1}{2}$	4_{-1}	4	$1, 1, 1, 1$	$0, -\frac{1}{8}, -\frac{1}{8}, \frac{1}{2}$
4_2	4	$1, 1, 1, 1$	$0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	4_{-2}	4	$1, 1, 1, 1$	$0, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{2}$
4_3	4	$1, 1, 1, 1$	$0, \frac{3}{8}, \frac{3}{8}, \frac{1}{2}$	4_{-3}	4	$1, 1, 1, 1$	$0, -\frac{3}{8}, -\frac{3}{8}, \frac{1}{2}$
4_4	4	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$4_{9/5}$	7.2360	$1, 1, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}$	$0, -\frac{1}{4}, \frac{3}{20}, \frac{2}{5}$
$4_{-9/5}$	7.2360	$1, 1, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}$	$0, \frac{1}{4}, -\frac{3}{20}, -\frac{2}{5}$	$4_{19/5}$	7.2360	$1, 1, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}$	$0, \frac{1}{4}, -\frac{7}{20}, \frac{2}{5}$
$4_{-19/5}$	7.2360	$1, 1, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}$	$0, -\frac{1}{4}, \frac{7}{20}, -\frac{2}{5}$	4_0^c	13.090	$1, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} \frac{1+\sqrt{5}}{2}$	$0, \frac{2}{5}, -\frac{2}{5}, 0$
$4_{12/5}$	13.090	$1, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} \frac{1+\sqrt{5}}{2}$	$0, -\frac{2}{5}, -\frac{2}{5}, \frac{1}{5}$	$4_{-12/5}$	13.090	$1, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} \frac{1+\sqrt{5}}{2}$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{1}{5}$
$4_{10/3}$	19.234	$1, \xi_9^2, \xi_9^3, \xi_9^4$	$0, \frac{1}{3}, \frac{2}{9}, -\frac{1}{3}$	$4_{-10/3}$	19.234	$1, \xi_9^2, \xi_9^3, \xi_9^4$	$0, -\frac{1}{3}, -\frac{2}{9}, \frac{1}{3}$
5_0	5	$1, 1, 1, 1, 1$	$0, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}$	5_4	5	$1, 1, 1, 1, 1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}$
5_2^a	12	$1, 1, \xi_6^2, \xi_6^2, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, \frac{1}{3}$	5_2^b	12	$1, 1, \xi_6^2, \xi_6^2, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, \frac{1}{3}$
5_{-2}^b	12	$1, 1, \xi_6^2, \xi_6^2, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, -\frac{1}{3}$	5_{-2}^a	12	$1, 1, \xi_6^2, \xi_6^2, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, -\frac{1}{3}$
$5_{16/11}$	34.645	$1, \xi_{11}^2, \xi_{11}^3, \xi_{11}^4, \xi_{11}^5$	$0, -\frac{2}{11}, \frac{2}{11}, \frac{1}{11}, -\frac{5}{11}$	$5_{-16/11}$	34.645	$1, \xi_{11}^2, \xi_{11}^3, \xi_{11}^4, \xi_{11}^5$	$0, \frac{2}{11}, -\frac{2}{11}, -\frac{1}{11}, \frac{5}{11}$
$5_{18/7}$	35.339	$1, \xi_7^3, \xi_7^3, \xi_{14}^3, \xi_{14}^5$	$0, -\frac{1}{7}, -\frac{1}{7}, \frac{1}{7}, \frac{3}{7}$	$5_{-18/7}$	35.339	$1, \xi_7^3, \xi_7^3, \xi_{14}^3, \xi_{14}^5$	$0, \frac{1}{7}, \frac{1}{7}, -\frac{1}{7}, -\frac{3}{7}$

theorem could be complex. We do not need this for the sequel, but it may lead to an interesting generalization.

2.2. Classification of modular data up to rank=5 and candidate list of rank=6.

2.2.1. *Rank 1-5 MTCs.* The rank$\leqslant 5$ unitary MTCs are classified [5, 33, 17]; Table 1 lists all 45 rank $\leqslant 5$ cases, only the quantum dimensions and twists are displayed. These are labeled by N_c, where N is the rank and c the (additive) central charge. The entries of the table are ordered by the total quantum dimension D^2. Also d_i is the quantum dimension and $s_i = \arg(T_{ii})$ is the **topological spin** of the i^{th} simple object in the MTC. The quantum dimensions are given in terms of $\xi_n^{m,k} = \frac{\sin(m\pi/n)}{\sin(k\pi/n)}$ and $\xi_n^m = \xi_n^{m,1}$. The fusion coefficients N_k^{ij} and the S-matrices of MTCs can be deduced from the given data in these low rank cases, and we do not list them for brevity's sake.

2.2.2. *Known rank-6 MD of MTCs and their Galois Groups.* Among the known rank 6 modular tensor categories there are 11 distinct fusion rules. We can determine their Galois groups $\mathrm{Gal}(\mathbb{Q}(S_{ij})/\mathbb{Q})$ and the representation type (i.e. dimensions of their irreducible subrepresentations) of their $\mathrm{SL}_2(\mathbb{Z})$ representation, displayed in Table 3. Six are realized as product categories, the other 5 by prime categories. Note that there are two types that yield the fusion rules of $SO(5)_2$: $(3, 2, 1)$ is realized by a zesting of $SO(5)_2$, denoted $SO(5)_2'$ in Table 2, see Theorem 4.15.

Table 2. Realizations of known rank 6 modular data, their Galois groups and representation types.

$\mathcal{C}$	$\mathrm{Gal}(\mathcal{C})$	Type
$PSU(2)_3 \boxtimes SU(2)_2$	$\langle(0\ 1)(2\ 3),(0\ 2)(1\ 3)(4\ 5)\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	(6)
$PSU(2)_3 \boxtimes U(3)_1$	$\langle(0\ 1)(2\ 3)(4\ 5),(2\ 4)(3\ 5)\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$(4,2)$
$PSU(2)_3 \boxtimes PSU(2)_5$	$\langle(0\ 1)(2\ 3)(4\ 5),(0\ 2\ 4)(1\ 3\ 5)\rangle \cong \mathbb{Z}_6$	(6)
$U(2)_1 \boxtimes SU(2)_2$	$\langle(0\ 1)(2\ 3)\rangle \cong \mathbb{Z}_2$	(6)
$U(2)_1 \boxtimes U(3)_1$	$\langle(1\ 2)(3\ 4)\rangle \cong \mathbb{Z}_2$	$(4,2)$
$U(2)_1 \boxtimes PSU(2)_5$	$\langle(0\ 1\ 2)(3\ 4\ 5)\rangle \cong \mathbb{Z}_3$	(6)
$SO(5)_2$,$SO(5)_2'$	$\langle(0\ 1)(2\ 3)\rangle \cong \mathbb{Z}_2$	$(3,3)$, $(3,2,1)$
$PSU(2)_{11}$	$\langle(0\ 1\ 2\ 3\ 4\ 5)\rangle \cong \mathbb{Z}_6$	(6)
$G(2)_3$	$\langle(0\ 1),(2\ 3\ 4)\rangle \cong \mathbb{Z}_6$	$(4,2)$
$PSO(8)_3$	$\langle(0\ 1\ 2)\rangle \cong \mathbb{Z}_3$	$(4,1,1)$
$PSO(5)_{\frac{3}{2}}$	$\langle(0\ 1\ 2)(3\ 4\ 5)\rangle \cong \mathbb{Z}_3$	(6)

The example $PSO(5)_{\frac{3}{2}}$ is noteworthy–it is the smallest example of a MTC the fusion rules of which are never realized as those of a *unitary* MTC. We also remark that the fusion rules of $SO(5)_2$ are realized by categories with distinct representation types: namely the *zested* version of $SO(5)_2$, see Theorem 4.15. In particular, the fusion rules do not determine the representation type.

We also did a computer search for all rank-6 unitary modular data with $N_k^{ij} \leqslant 3$. (Ref. [40] computed all rank-6 unitary modular data with $N_k^{ij} \leqslant 2$.) The Tables 3 and 4 list all 50 of the resulting modular data, we include only the quantum dimensions and twists. In the last column, $N_c \boxtimes N'_{c'}$ indicates that the rank-6 MTC is the product of two MTCs labeled by N_c and $N'_{c'}$. The prime MTCs are all non-Abelian roots of MTCs from Kac-Moody algebra. (The notion of non-Abelian roots is introduced in Ref. [21].) In this paper, we will show that the Tables 3 and 4 include all modular data of rank-6 unitary MTCs.

3. Modular data representations of modular tensor categories

While the number theoretical properties of MD allow the classification of MTCs up to rank=4, the deeper properties of the $\mathrm{SL}_2(\mathbb{Z})$ representations of MD (cf. Definition 3.1) lead to a more streamlined approach with the potential to achieve a classification up to rank=10. The classification of rank=5 MTCs is already a mixture of both Galois theory and representation techniques. Instead of working on cases labeled by abelian subgroups of S_r for rank=r as in earlier classification, we introduce the notion *type* of the MD of an MTC–the list of dimensions of irreducible subrepresentations, so that the cases are indexed by Young diagrams with r boxes.

Every MTC $\mathcal{B}$ leads to a $(2+1)$-TQFT, hence there is a corresponding projective matrix representation $\overline{\rho}_{\mathcal{B}}$ of $\mathrm{SL}_2(\mathbb{Z})$—the mapping class group of the torus. We will refer to this representation as the projective $\mathrm{SL}_2(\mathbb{Z})$ representation of the MTC $\mathcal{B}$, and is given by the S-, T- matrices of $\mathcal{B}$. The linearizations of this projective matrix $\mathrm{SL}_2(\mathbb{Z})$ representation $\overline{\rho}_{\mathcal{B}}$, called $\mathrm{SL}_2(\mathbb{Z})$ representations of $\mathcal{B}$, will be elaborated upon in next section.

3.1. $\mathrm{SL}_2(\mathbb{Z})$ representations of MTC or MD.

Since our classification is based on $\mathrm{SL}_2(\mathbb{Z})$ representations, let us first summarize some important facts about them. Let $\mathfrak{s} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\mathfrak{t} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

TABLE 3. Table of rank 6 modular data with $N_k^{ij} \leqslant 3$ and $D^2 \leqslant 18$.

N_c	D^2	$d_0, d_1, \cdots$	$s_0, s_1, \cdots$	comment
6_1	6	$1,1,1,1,1,1$	$0, \frac{1}{12}, \frac{1}{12}, -\frac{1}{4}, \frac{1}{3}, \frac{1}{3}$	$2_{-1} \boxtimes 3_2$
6_{-1}	6	$1,1,1,1,1,1$	$0, -\frac{1}{12}, -\frac{1}{12}, \frac{1}{4}, -\frac{1}{3}, -\frac{1}{3}$	$2_1 \boxtimes 3_{-2}$
6_3	6	$1,1,1,1,1,1$	$0, \frac{1}{4}, \frac{1}{3}, \frac{1}{3}, -\frac{5}{12}, -\frac{5}{12}$	$2_1 \boxtimes 3_2$
6_{-3}	6	$1,1,1,1,1,1$	$0, -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{3}, \frac{5}{12}, \frac{5}{12}$	$2_{-1} \boxtimes 3_{-2}$
$6_{1/2}$	8	$1,1,1,1,\sqrt{2},\sqrt{2}$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, -\frac{1}{16}, \frac{3}{16}$	$2_1 \boxtimes 3_{-1/2}$
$6_{-1/2}$	8	$1,1,1,1,\sqrt{2},\sqrt{2}$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{1}{16}, -\frac{3}{16}$	$2_1 \boxtimes 3_{-3/2}$
$6_{3/2}$	8	$1,1,1,1,\sqrt{2},\sqrt{2}$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{1}{16}, \frac{5}{16}$	$2_1 \boxtimes 3_{1/2}$
$6_{-3/2}$	8	$1,1,1,1,\sqrt{2},\sqrt{2}$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, -\frac{1}{16}, -\frac{5}{16}$	$2_1 \boxtimes 3_{-5/2}$
$6_{5/2}$	8	$1,1,1,1,\sqrt{2},\sqrt{2}$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{3}{16}, \frac{7}{16}$	$2_1 \boxtimes 3_{3/2}$
$6_{-5/2}$	8	$1,1,1,1,\sqrt{2},\sqrt{2}$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, -\frac{3}{16}, -\frac{7}{16}$	$2_1 \boxtimes 3_{-7/2}$
$6_{7/2}$	8	$1,1,1,1,\sqrt{2},\sqrt{2}$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, \frac{5}{16}, -\frac{7}{16}$	$2_1 \boxtimes 3_{5/2}$
$6_{-7/2}$	8	$1,1,1,1,\sqrt{2},\sqrt{2}$	$0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, -\frac{5}{16}, \frac{7}{16}$	$2_1 \boxtimes 3_{7/2}$
$6_{4/5}$	10.854	$1,1,1,\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}$	$0, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{15}, \frac{1}{15}, \frac{2}{5}$	$2_{14/5} \boxtimes 3_{-2}$
$6_{-4/5}$	10.854	$1,1,1,\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{3}, \frac{1}{3}, -\frac{1}{15}, -\frac{1}{15}, -\frac{2}{5}$	$2_{-14/5} \boxtimes 3_2$
$6_{16/5}$	10.854	$1,1,1,\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}$	$0, -\frac{1}{3}, -\frac{1}{3}, \frac{4}{15}, \frac{4}{15}, -\frac{2}{5}$	$2_{-14/5} \boxtimes 3_{-2}$
$6_{-16/5}$	10.854	$1,1,1,\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{3}, \frac{1}{3}, -\frac{4}{15}, -\frac{4}{15}, \frac{2}{5}$	$2_{14/5} \boxtimes 3_2$
$6_{3/10}$	14.472	$1,1,\sqrt{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\sqrt{2}\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{2}, -\frac{5}{16}, -\frac{1}{10}, \frac{2}{5}, \frac{7}{80}$	$2_{14/5} \boxtimes 3_{-5/2}$
$6_{-3/10}$	14.472	$1,1,\sqrt{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\sqrt{2}\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{2}, \frac{5}{16}, \frac{1}{10}, -\frac{2}{5}, -\frac{7}{80}$	$2_{-14/5} \boxtimes 3_{5/2}$
$6_{7/10}$	14.472	$1,1,\sqrt{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\sqrt{2}\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{2}, \frac{7}{16}, \frac{1}{10}, -\frac{2}{5}, \frac{3}{80}$	$2_{-14/5} \boxtimes 3_{7/2}$
$6_{-7/10}$	14.472	$1,1,\sqrt{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\sqrt{2}\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{2}, -\frac{7}{16}, -\frac{1}{10}, \frac{2}{5}, -\frac{3}{80}$	$2_{14/5} \boxtimes 3_{-7/2}$
$6_{13/10}$	14.472	$1,1,\sqrt{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\sqrt{2}\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{2}, -\frac{3}{16}, -\frac{1}{10}, \frac{2}{5}, \frac{17}{80}$	$2_{14/5} \boxtimes 3_{-3/2}$
$6_{-13/10}$	14.472	$1,1,\sqrt{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\sqrt{2}\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{2}, \frac{3}{16}, \frac{1}{10}, -\frac{2}{5}, -\frac{17}{80}$	$2_{-14/5} \boxtimes 3_{3/2}$
$6_{17/10}$	14.472	$1,1,\sqrt{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\sqrt{2}\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{2}, -\frac{7}{16}, \frac{1}{10}, -\frac{2}{5}, \frac{13}{80}$	$2_{-14/5} \boxtimes 3_{-7/2}$
$6_{-17/10}$	14.472	$1,1,\sqrt{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\sqrt{2}\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{2}, \frac{7}{16}, -\frac{1}{10}, \frac{2}{5}, -\frac{13}{80}$	$2_{14/5} \boxtimes 3_{7/2}$
$6_{23/10}$	14.472	$1,1,\sqrt{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\sqrt{2}\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{2}, -\frac{1}{16}, -\frac{1}{10}, \frac{2}{5}, \frac{27}{80}$	$2_{14/5} \boxtimes 3_{-1/2}$
$6_{-23/10}$	14.472	$1,1,\sqrt{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\sqrt{2}\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{2}, \frac{1}{16}, \frac{1}{10}, -\frac{2}{5}, -\frac{27}{80}$	$2_{-14/5} \boxtimes 3_{1/2}$
$6_{27/10}$	14.472	$1,1,\sqrt{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\sqrt{2}\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{2}, -\frac{5}{16}, \frac{1}{10}, -\frac{2}{5}, \frac{23}{80}$	$2_{-14/5} \boxtimes 3_{-5/2}$
$6_{-27/10}$	14.472	$1,1,\sqrt{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\sqrt{2}\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{2}, \frac{5}{16}, -\frac{1}{10}, \frac{2}{5}, -\frac{23}{80}$	$2_{14/5} \boxtimes 3_{5/2}$
$6_{33/10}$	14.472	$1,1,\sqrt{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\sqrt{2}\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{2}, \frac{1}{16}, -\frac{1}{10}, \frac{2}{5}, \frac{37}{80}$	$2_{14/5} \boxtimes 3_{1/2}$
$6_{-33/10}$	14.472	$1,1,\sqrt{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\sqrt{2}\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{2}, -\frac{1}{16}, \frac{1}{10}, -\frac{2}{5}, -\frac{37}{80}$	$2_{-14/5} \boxtimes 3_{-1/2}$
$6_{37/10}$	14.472	$1,1,\sqrt{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\sqrt{2}\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{2}, -\frac{3}{16}, \frac{1}{10}, -\frac{2}{5}, \frac{33}{80}$	$2_{-14/5} \boxtimes 3_{-3/2}$
$6_{-37/10}$	14.472	$1,1,\sqrt{2},\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\sqrt{2}\frac{1+\sqrt{5}}{2}$	$0, \frac{1}{2}, \frac{3}{16}, -\frac{1}{10}, \frac{2}{5}, -\frac{33}{80}$	$2_{14/5} \boxtimes 3_{3/2}$

be the standard generators of $\mathrm{SL}_2(\mathbb{Z})$. This admits the presentation:

$$\mathrm{SL}_2(\mathbb{Z}) = \langle \mathfrak{s}, \mathfrak{t} \mid \mathfrak{s}^4 = \mathrm{id}, (\mathfrak{s}\mathfrak{t})^3 = \mathfrak{s}^2 \rangle .$$

The 1-dimensional representations of $\mathrm{SL}_2(\mathbb{Z})$, denoted $\widehat{\mathrm{SL}_2(\mathbb{Z})}$, form a cyclic group of order 12 under tensor product. We will take $\chi \in \widehat{\mathrm{SL}_2(\mathbb{Z})}$ defined by $\chi(\mathfrak{t}) = \zeta_{12}$ to be the generator, where $\zeta_n^k := \mathrm{e}^{2\pi\mathrm{i}k/n}$. Under this convention, every 1-dimensional representation of $\mathrm{SL}_2(\mathbb{Z})$ is equivalent to χ^α for some integer α, unique modulo 12:

$$\chi^\alpha(\mathfrak{s}) = \bar{\zeta}_4^\alpha, \quad \chi^\alpha(\mathfrak{t}) = \zeta_{12}^\alpha. \tag{3.1}$$

TABLE 4. Table of rank 6 modular data with $N_k^{ij} \leqslant 3$ and $D^2 > 18$.

N_c	D^2	$d_0, d_1, \cdots$	$s_0, s_1, \cdots$	comment
$6_{1/7}$	18.591	$1, 1, \xi_7^2, \xi_7^2, \xi_7^3, \xi_7^3$	$0, -\frac{1}{4}, -\frac{1}{7}, -\frac{11}{28}, \frac{1}{28}, \frac{2}{7}$	$2_{-1} \boxtimes 3_{8/7}$
$6_{-1/7}$	18.591	$1, 1, \xi_7^2, \xi_7^2, \xi_7^3, \xi_7^3$	$0, \frac{1}{4}, \frac{1}{7}, \frac{11}{28}, -\frac{1}{28}, -\frac{2}{7}$	$2_{1} \boxtimes 3_{-8/7}$
$6_{15/7}$	18.591	$1, 1, \xi_7^2, \xi_7^2, \xi_7^3, \xi_7^3$	$0, \frac{1}{4}, \frac{3}{28}, -\frac{1}{7}, \frac{2}{7}, -\frac{13}{28}$	$2_{1} \boxtimes 3_{8/7}$
$6_{-15/7}$	18.591	$1, 1, \xi_7^2, \xi_7^2, \xi_7^3, \xi_7^3$	$0, -\frac{1}{4}, -\frac{3}{28}, \frac{1}{7}, -\frac{2}{7}, \frac{13}{28}$	$2_{-1} \boxtimes 3_{-8/7}$
6_0^a	20	$1, 1, 2, 2, \sqrt{5}, \sqrt{5}$	$0, 0, \frac{1}{5}, -\frac{1}{5}, 0, \frac{1}{2}$	root of $SO(10)_2$
6_0^b	20	$1, 1, 2, 2, \sqrt{5}, \sqrt{5}$	$0, 0, \frac{1}{5}, -\frac{1}{5}, \frac{1}{4}, -\frac{1}{4}$	root of $SO(10)_2$
6_4^a	20	$1, 1, 2, 2, \sqrt{5}, \sqrt{5}$	$0, 0, \frac{2}{5}, -\frac{2}{5}, 0, \frac{1}{2}$	root of $SO(5)_2$
6_4^b	20	$1, 1, 2, 2, \sqrt{5}, \sqrt{5}$	$0, 0, \frac{2}{5}, -\frac{2}{5}, \frac{1}{4}, -\frac{1}{4}$	$SO(5)_2$
$6_{58/35}$	33.632	$1, \frac{1+\sqrt{5}}{2}, \xi_7^2, \xi_7^3, \frac{1+\sqrt{5}}{2}\xi_7^2, \frac{1+\sqrt{5}}{2}\xi_7^3$	$0, \frac{2}{5}, \frac{1}{7}, -\frac{2}{7}, -\frac{16}{35}, \frac{4}{35}$	$2_{14/5} \boxtimes 3_{-8/7}$
$6_{-58/35}$	33.632	$1, \frac{1+\sqrt{5}}{2}, \xi_7^2, \xi_7^3, \frac{1+\sqrt{5}}{2}\xi_7^2, \frac{1+\sqrt{5}}{2}\xi_7^3$	$0, -\frac{2}{5}, -\frac{1}{7}, \frac{2}{7}, \frac{16}{35}, -\frac{4}{35}$	$2_{-14/5} \boxtimes 3_{8/7}$
$6_{138/35}$	33.632	$1, \frac{1+\sqrt{5}}{2}, \xi_7^2, \xi_7^3, \frac{1+\sqrt{5}}{2}\xi_7^2, \frac{1+\sqrt{5}}{2}\xi_7^3$	$0, \frac{2}{5}, -\frac{1}{7}, \frac{2}{7}, \frac{9}{35}, -\frac{11}{35}$	$2_{14/5} \boxtimes 3_{8/7}$
$6_{-138/35}$	33.632	$1, \frac{1+\sqrt{5}}{2}, \xi_7^2, \xi_7^3, \frac{1+\sqrt{5}}{2}\xi_7^2, \frac{1+\sqrt{5}}{2}\xi_7^3$	$0, -\frac{2}{5}, \frac{1}{7}, -\frac{2}{7}, -\frac{9}{35}, \frac{11}{35}$	$2_{-14/5} \boxtimes 3_{-8/7}$
$6_{46/13}$	56.746	$1, \xi_{13}^2, \xi_{13}^3, \xi_{13}^4, \xi_{13}^5, \xi_{13}^6$	$0, \frac{4}{13}, \frac{2}{13}, -\frac{6}{13}, \frac{6}{13}, -\frac{1}{13}$	root of $SU(2)_{11}$
$6_{-46/13}$	56.746	$1, \xi_{13}^2, \xi_{13}^3, \xi_{13}^4, \xi_{13}^5, \xi_{13}^6$	$0, -\frac{4}{13}, -\frac{2}{13}, \frac{6}{13}, -\frac{6}{13}, \frac{1}{13}$	root of $SU(2)_{\overline{11}}$
$6_{8/3}$	74.617	$1, \xi_{18}^3, \xi_{18}^3, \xi_{18}^3, \xi_{18}^5, \xi_{18}^7$	$0, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{3}, -\frac{1}{3}$	root of $SO(8)_{\bar{3}}$
$6_{-8/3}$	74.617	$1, \xi_{18}^3, \xi_{18}^3, \xi_{18}^3, \xi_{18}^5, \xi_{18}^7$	$0, -\frac{1}{9}, -\frac{1}{9}, -\frac{1}{9}, -\frac{1}{3}, \frac{1}{3}$	root of $SO(8)_3$
6_2	100.61	$1, \frac{3+\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2}, \frac{5+\sqrt{21}}{2}, \frac{7+\sqrt{21}}{2}$	$0, -\frac{1}{7}, -\frac{2}{7}, \frac{3}{7}, 0, \frac{1}{3}$	root of $G(2)_{\bar{3}}$
6_{-2}	100.61	$1, \frac{3+\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2}, \frac{5+\sqrt{21}}{2}, \frac{7+\sqrt{21}}{2}$	$0, \frac{1}{7}, \frac{2}{7}, -\frac{3}{7}, 0, -\frac{1}{3}$	root of $G(2)_3$

Given a modular tensor category $\mathcal{B}$ with the modular data (S, T) and central charge c, the assignment

$$\rho_\alpha(\mathfrak{s}) = \bar{\zeta}_4^\alpha S/D, \quad \rho_\alpha(\mathfrak{t}) = \zeta_{12}^\alpha \mathrm{e}^{-2\pi\mathrm{i}\frac{c}{24}} T \quad (\alpha \in \mathbb{Z}_{12}). \tag{3.2}$$

define a (linear) representation of $\mathrm{SL}_2(\mathbb{Z})$, and we call these representations ρ_α the $\mathrm{SL}_2(\mathbb{Z})$ **representations of** $\mathcal{B}$ or the $\mathrm{SL}_2(\mathbb{Z})$ **representations of the modular data** (S, T). For any $\alpha, \alpha' \in \mathbb{Z}_{12}$,

$$\rho_\alpha \cong \chi^{\alpha-\alpha'} \otimes \rho_{\alpha'}$$

as $\mathrm{SL}_2(\mathbb{Z})$ representations. Therefore, the $\mathrm{SL}_2(\mathbb{Z})$ representation $\rho_{\mathcal{B}}$ of $\mathcal{B}$ is unique up to a tensor factor of linear characters of $\widehat{\mathrm{SL}_2(\mathbb{Z})}$.

Note that two modular data (S, T) and (S', T') are regarded as *the same* if they differ only by a permutation of indices:

$$S' = PSP^\top, \qquad T' = PTP^\top, \tag{3.3}$$

where P is a permutation matrix. Throughout this paper, we simply *identify* ρ_α and its conjugations by permutation matrices.

Definition 3.1. A unitary matrix representation ρ of $\mathrm{SL}_2(\mathbb{Z})$ is called an *MD representation* if ρ is an $\mathrm{SL}_2(\mathbb{Z})$ representation of some modular tensor category. It is called a *pseudo-MD (pMD) representation* if $V\rho V$ is an MD representation for some signed diagonal matrix V.

3.2. Type and level of modular data.

Definition 3.2. Given an MTC $\mathcal{B}$ of rank r, an $\mathrm{SL}_2(\mathbb{Z})$ representation $\rho_{\mathcal{B}}$ decomposes into direct sum of irreducible representations of dimensions $\lambda_1, \ldots, \lambda_m$ in non-increasing order. The *type* of the corresponding MD of $\mathcal{B}$ of rank=r is the Young diagram of r boxes $(\lambda_1, \ldots, \lambda_m)$ with $\sum_{i=1}^m \lambda_i = r$. The type of an MTC simply refers to the type of its MD.

The modular representations of the Fibonacci and Ising theories are both irreducible, so they are of types $(2), (3)$, respectively. The modular representation of the toric code has an image isomorphic to $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ and is reducible of type $(2, 1, 1)$.

We note that for any positive integer n, the reduction $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ defines a surjective group homomorphism $\pi_n : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$. Thus, a representation of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is also a representation of $\mathrm{SL}_2(\mathbb{Z})$, which will be called a *congruence* representation of $\mathrm{SL}_2(\mathbb{Z})$ in this paper. It is immediate to see that a representation of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is also a $\mathrm{SL}_2(\mathbb{Z}/mn\mathbb{Z})$ representation for any positive integer m. The smallest positive integer n such that a congruence representation ρ of $\mathrm{SL}_2(\mathbb{Z})$ factors through $\pi_n : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is called the *level* of ρ. It is known that the level $n = \mathrm{ord}(\rho(\mathfrak{t}))$ (cf. [11, Lem. A.1]). Here $\mathrm{ord}(t)$ is the *order* of t, i.e., the smallest positive integer such that

$$t^{\mathrm{ord}(t)} = \mathrm{id}\,. \tag{3.4}$$

There are many more finite-dimensional noncongruence representations of $\mathrm{SL}_2(\mathbb{Z})$ (cf. [20]) but they are not associated with any modular tensor category by [11, Thm. II]. Since we only deal with congruence representations of $\mathrm{SL}_2(\mathbb{Z})$, all the representations of $\mathrm{SL}_2(\mathbb{Z})$ throughout this paper are assumed to be congruence and finite-dimensional over $\mathbb{C}$.

An $\mathrm{SL}_2(\mathbb{Z})$ representation ρ of an MTC is also *symmetric*, which means ρ is a unitary matrix representation with $\rho(\mathfrak{s})$ symmetric and $\rho(\mathfrak{t})$ diagonal. The following theorem proved in [28] provides the theoretic background for the GAP package [27] and our reconstruction process:

Theorem 3.3. *Every finite-dimensional congruence representation of* $\mathrm{SL}_2(\mathbb{Z})$ *is equivalent to a symmetric one.*

Therefore, throughout this paper, we always assume our *general* representations of $\mathrm{SL}_2(\mathbb{Z})$ to be congruence and symmetric.

In Appendix A, we list all the irreducible $\mathrm{SL}_2(\mathbb{Z})$ representations, generated by [27], of prime-power levels and dimensions $\leqslant 6$. These $\mathrm{SL}_2(\mathbb{Z})$ representations are congruence and symmetric. From these representations, we can construct all the inequivalent $\mathrm{SL}_2(\mathbb{Z})$ representations with dimensions $\leqslant 6$. The MD representations of dimensions $\leqslant 6$ can be reconstructed from these symmetric representations with the help of the following theorem.

Theorem 3.4. *Let* $\rho, \rho' : \mathrm{SL}_2(\mathbb{Z}) \to U_n(\mathbb{C})$ *be unitarily equivalent symmetric representations of* $\mathrm{SL}_2(\mathbb{Z})$ *such that* $\rho(\mathfrak{t}) = \rho'(\mathfrak{t}) = t$, *and define* $s = \rho(\mathfrak{s})$ *and* $s' = \rho'(\mathfrak{s})$. *Then there exists a (real) orthogonal matrix* U *such that*

$$s' = UsU^\top \quad \textit{and} \quad Ut = tU.$$

Proof. Let Q be a unitary matrix such that

$$s' = QsQ^\dagger \quad \text{and} \quad Qt = tQ.$$

Since t is diagonal and unitary, $t^\dagger = \bar{t}$. Taking the conjugate transpose of the second equality implies

$$Q^\dagger \bar{t} = \bar{t} Q^\dagger \quad \text{or} \quad \overline{Q} t = t \overline{Q}\,.$$

Let $Q = X_1 + iX_2$ for some real matrices X_1 and X_2. Then we have

$$(X_1 \pm iX_2)t = t(X_1 \pm iX_2)$$

which implies $[X_i, t] = 0$ for $i = 1, 2$. Similarly, $s'Q = Qs$ implies $\overline{s}'Q = Q\overline{s}$ since both s and s' are symmetric. Therefore, we also have $s'\overline{Q} = \overline{Q}s$, which implies

$$X_i s = s' X_i \quad \text{for } i = 1, 2.$$

Since there are only finitely many roots for the equation $\det(X_1 + xX_2) = 0$, one can take $\lambda \in \mathbb{R}$ such that $X = X_1 + \lambda X_2$ is invertible. Then

$$Xs = s'X \quad \text{and} \quad Xt = tX\,.$$

Let $X = UP$ be the polar decomposition of X where U is orthogonal and P is the unique positive definite satisfying $P^2 = X^\top X$. In fact, P is a polynomial of P^2 (cf. [16, Chap.9. Thm 11.]). Since $s^{-1} = \overline{s}$ and $s'^{-1} = \overline{s'}$,

$$P^2 = X^\top X = (s'X\overline{s})^\dagger(s'X\overline{s}) = sX^\top s'^\dagger s'Xs^\top = sP^2\overline{s}$$

and

$$X^\top t = tX^\top\,.$$

Therefore,

$$P^2 s = sP^2 \quad \text{and} \quad P^2 t = tP^2\,.$$

Since P is a polynomial of P^2, we find

$$Ps = sP \quad \text{and} \quad Pt = tP\,.$$

Therefore,

$$Us = UPsP^{-1} = XsP^{-1} = s'XP^{-1} = s'U$$

and

$$Ut = UPtP^{-1} = XtP^{-1} = tXP^{-1} = tU\,. \quad \square$$

Remark 3.5. An $\mathrm{SL}_2(\mathbb{Z})$ representation ρ is said to be *even* (resp. *odd*) if $\rho(\mathfrak{s}^2) = \mathrm{id}$ (resp. $\rho(\mathfrak{s}^2) = -\mathrm{id}$). If ρ is symmetric and irreducible, then $\rho(\mathfrak{s})$ or $i\rho(\mathfrak{s})$ is a real symmetric matrix, depending on whether ρ is even or odd respectively. A direct sum of irreducible representations of opposite parties is neither even nor odd. In particular, if ρ is an $\mathrm{SL}_2(\mathbb{Z})$ representation of a modular tensor category $\mathcal{C}$, then ρ is even or odd if, and only if, $\mathcal{C}$ is self-dual.

3.3. **Useful conditions on $\mathrm{SL}_2(\mathbb{Z})$ representations.** The set of all the roots of unity can be totally ordered as follows: For any roots of unity x, y, we say that $x < y$ if one the following conditions hold:

(i) $\mathrm{ord}(x) < \mathrm{ord}(y)$, or
(ii) $\mathrm{ord}(x) = \mathrm{ord}(y)$ and $\arg(x) < \arg(y)$,

where $\arg(\zeta)$ denotes the unique number $s_\zeta \in [0, 1) \cap \mathbb{Q}$ such that $e^{2i\pi s_\zeta} = \zeta$.

Definition 3.6. For any representation ρ of $\mathrm{SL}_2(\mathbb{Z})$, $\rho(\mathfrak{t})$ has finite order. We denoted by $\mathrm{spec}(\rho(\mathfrak{t}))$ the increasing ordered set of eigenvalues of $\rho(\mathfrak{t})$ with multiplicities. If $\mathrm{spec}(\rho(\mathfrak{t}))$ is multiplicity free ρ is called *non-degenerate*. If ρ' is another representation of $\mathrm{SL}_2(\mathbb{Z})$, $\mathrm{spec}(\rho(\mathfrak{t})) = \{x_1, \dots, x_m\}$ and $\mathrm{spec}(\rho'(\mathfrak{t})) = \{y_1, \dots, y_n\}$ can be compared by the lexicographical order.

Two representations ρ, ρ' of $\mathrm{SL}_2(\mathbb{Z})$ are called *projectively equivalent* if

$$\rho' \cong \chi^\alpha \otimes \rho \text{ for some } \alpha \in \mathbb{Z}/12\mathbb{Z}\,.$$

A representations ρ of $\mathrm{SL}_2(\mathbb{Z})$ is said to have a *minimal* $\mathfrak{t}$*-spectrum* if $\mathrm{spec}(\rho(\mathfrak{t}))$ is minimal among all the representations projectively equivalent to ρ, i.e.,

$$\mathrm{spec}(\rho(\mathfrak{t})) \leqslant \mathrm{spec}((\chi^\alpha \otimes \rho)(\mathfrak{t})) \text{ for all } \alpha \in \mathbb{Z}/12\mathbb{Z}\,.$$

Let t be any matrix over $\mathbb{C}$. The smallest positive integer n such that $t^n = \alpha\,\mathrm{id}$ for some $\alpha \in \mathbb{C}$ is called the *projective order* of t, and denoted by $\mathrm{pord}(t) := n$. If such integer does not exist, we define $\mathrm{pord}(t) := \infty$.

We can organize the irreducible representations of $\mathrm{SL}_2(\mathbb{Z})$ by the level and the dimension of the representations. Due to the Chinese remainder theorem, if the level of a irreducible representation ρ factors as $n = \prod_i p_i^{k_i}$ where p_i are distinct primes, then $\rho \cong \bigotimes_i \rho_i$ where ρ_i are level $p_i^{k_i}$ representations. Thus we can construct all irreducible $\mathrm{SL}_2(\mathbb{Z})$ representations as tensor products of irreducible $\mathrm{SL}_2(\mathbb{Z})$ representations of prime-power levels, which in turn, yields a construction of all $\mathrm{SL}_2(\mathbb{Z})$ representations ρ via direct sums of the irreducible representations.

Define $\mathbb{Q}_n = \mathbb{Q}(\zeta_n)$ to be the cyclotomic field of order n. For any positive integer n, we can construct a faithful representation $D_n : \mathrm{Gal}(\mathbb{Q}_n) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$, which identifies the Galois group $\mathrm{Gal}(\mathbb{Q}_n) \cong \mathbb{Z}_n^\times$ with the diagonal subgroup of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ [11, Remark 4.5]. More generally, for any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}})$, $\sigma(\mathbb{Q}_n) = \mathbb{Q}_n$ and so there exists an integer a (unique modulo n) such that $\sigma(\zeta_n) = \zeta_n^a$ and

$$D_n(\sigma) := \mathfrak{t}^a \mathfrak{s} \mathfrak{t}^b \mathfrak{s} \mathfrak{t}^a \mathfrak{s}^{-1} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})\,, \tag{3.5}$$

where b satisfies $ab \equiv 1 \mod n$. If ρ is a level n representation of $\mathrm{SL}_2(\mathbb{Z})$, the composition

$$D_\rho(\sigma) := \rho \circ D_n(\sigma) \tag{3.6}$$

defines a representation of $\mathrm{Gal}(\bar{\mathbb{Q}})$. We may also write $D_n(\sigma)$ as $D_n(a)$. Such a representation of Galois group captures the Galois conjugation action on $\mathrm{SL}_2(\mathbb{Z})$ representations ρ_{MD} of modular data, and plays a very important role in our classification. Many of the following collection of results on ρ_{MD} were proved in [26, 11].

Theorem 3.7. *Every $\mathrm{SL}_2(\mathbb{Z})$ representation ρ of an MTC $\mathcal{B}$ is a matrix representation with the standard basis $(e_0, \ldots, e_{r-1})$ identified with $\mathrm{irr}(\mathcal{B})$. Assuming $e_0 = \mathbb{1}$, ρ satisfies the following conditions:*

1. *Let $n = \mathrm{ord}(\rho(\mathfrak{t}))$. For any $\mathfrak{g} \in \mathrm{SL}_2(\mathbb{Z})$, $\rho(\mathfrak{g})$ is a matrix over $\mathbb{Q}_n$. In particular, $\rho(\mathfrak{s})_{ij}$ are cyclotomic numbers in $\mathbb{Q}_n$ for all i, j.*
2. *The modular data (S, T) of $\mathcal{B}$ is given by*
$$S = \frac{\rho(\mathfrak{s})}{\rho(\mathfrak{s})_{00}}, \qquad T = \frac{\rho(\mathfrak{t})}{\rho(\mathfrak{t})_{00}}. \tag{3.7}$$
3. *In particular, ρ is symmetric, $\mathrm{ord}(T) = \mathrm{pord}\,\rho(\mathfrak{t}))$ and (cf. Theorem 2.1(4))*
$$\frac{\rho(\mathfrak{s})_{ij}}{\rho(\mathfrak{s})_{0j}} \in \mathbb{Z}[\zeta_{\mathrm{ord}(T)}]\,.$$
4. *The representation ρ is congruence of level n $\mathrm{ord}(T) \mid n \mid 12\,\mathrm{ord}(T)$. Thus, ρ is a symmetric and congruence $\mathrm{SL}_2(\mathbb{Z})$ representation.*
5. *One has $1/\rho(\mathfrak{s})_{i0} \in \mathbb{Z}[\zeta_n]$, and the set of distinct prime factors of $\mathrm{ord}(T)$ coincides with that of the integer $\mathrm{norm}(1/\rho(\mathfrak{s})_{00})$.*
6. *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_n)$ be a Galois automorphism. Then (cf. (3.5))*
$$D_\rho(\sigma)_{ij} = \epsilon_\sigma(i)\delta_{\hat{\sigma}(i),j}, \tag{3.8}$$
where $\epsilon_\sigma(i) \in \{1, -1\}$ and $\hat{\sigma}$ is a permutation on $\{0, \ldots, r-1\}$ determined by
$$\sigma\left(\frac{\rho(\mathfrak{s})_{ij}}{\rho(\mathfrak{s})_{0j}}\right) = \frac{\rho(\mathfrak{s})_{i\hat{\sigma}(j)}}{\rho(\mathfrak{s})_{0\hat{\sigma}(j)}}\,. \tag{3.9}$$
Moreover,
$$\sigma(\rho(\mathfrak{s})) = D_\rho(\sigma)\rho(\mathfrak{s}) = \rho(\mathfrak{s})D_\rho^\top(\sigma) \quad \text{and} \quad \sigma^2(\rho(\mathfrak{t})) = D_\rho(\sigma)\rho(\mathfrak{t})D_\rho^\top(\sigma)\,. \tag{3.10}$$

(7) *The matrix $\rho(\mathfrak{s})$ satisfies the Verlinde formula (cf. [37]):*

$$N_k^{ij} = \sum_{l=0}^{r-1} \frac{\rho(\mathfrak{s})_{li}\rho(\mathfrak{s})_{lj}\rho(\mathfrak{s})^*_{lk}}{\rho(\mathfrak{s})_{l0}}, \quad i,j = 0,1,\ldots,r-1\,. \tag{3.11}$$

(8) *For $m \in \mathbb{N}_+$, the m^{th} Frobenius-Schur indicator of the i-th simple object can also be expressed in terms of $\rho(\mathfrak{s})$ and $\rho(\mathfrak{t})$:*

$$\nu_m(i) = \sum_{j,k} N_i^{jk} \rho(\mathfrak{s})_{j0}\rho(\mathfrak{t})^m_{jj} \cdot (\rho(\mathfrak{s})_{k0}\rho(\mathfrak{t})^m_{kk})^*\,. \tag{3.12}$$

Remark 3.8. It is worth noting that a pMD representation ρ_{pMD} shares arithmetic properties with MD representations as $\rho = V\rho_{\text{pMD}}V$ is an MD representation for some signed diagonal matrix V. Therefore, Theorem 3.7 (1) and (3-6) also hold for any pMD representation. In particular, for $\sigma \in \text{Gal}(\bar{\mathbb{Q}})$, $D_{\rho_{\text{pMD}}}(\sigma) = VD_\rho(\sigma)V$, and so

$$\sigma(\rho_{\text{pMD}}(\mathfrak{s})_{ij}) = \epsilon'_\sigma(i)\rho_{\text{pMD}}(\mathfrak{s})_{\hat{\sigma}(i)j} = \epsilon'_\sigma(j)\rho_{\text{pMD}}(\mathfrak{s})_{i\hat{\sigma}(j)}$$

but the sign function ϵ'_σ is different from ϵ_σ in Theorem 3.7 (6) in general.

3.4. **Modular data representations and our classification strategy.** The MD representation introduced in Definition 3.1 plays an important role in our approach. We now explain the strategy of a systematic construction for low rank modular data, implementable on a computer. In Section 4 we provide a largely by-hand approach to the classification of rank 6 MD.

For a given rank, we first construct all the inequivalent $\text{SL}_2(\mathbb{Z})$ representations ρ_{isum} of finite levels, as direct sums of irreducible $\text{SL}_2(\mathbb{Z})$ representations obtained as tensor products of the prime-power level representations listed in Appendix A. Each of these $\text{SL}_2(\mathbb{Z})$ irreducible representations is symmetric, and so is ρ_{isum}.

Although the number of the $\text{SL}_2(\mathbb{Z})$ representations ρ_{isum} is finite, most of these representations are not associated to any MTC. In next section, we introduce and collect conditions on MD representations, to reject as much as possible the $\text{SL}_2(\mathbb{Z})$ representations that are not associated to MTCs.

After we obtain a short list of candidate $\text{SL}_2(\mathbb{Z})$ representations ρ_{isum}, we permute the indices using a permutation matrix P

$$\tilde{\rho} = P\rho_{\text{isum}}P^\top \tag{3.13}$$

such that $\arg(\tilde{\rho}(\mathfrak{t})_{ii})$ is ordered for computer implementation or mathematical deduction.

Suppose $\tilde{\rho}$ is equivalent to an MD representation ρ. Without losing generality, we can further assume $\rho(\mathfrak{t}) = \tilde{\rho}(\mathfrak{t})$. It follows from Theorem 3.4 there exists an orthogonal matrix U such that $\rho(\mathfrak{s}) = U\tilde{\rho}(\mathfrak{s})U^\top$ and $\rho(\mathfrak{t}) = U\tilde{\rho}(\mathfrak{t})U^\top$. In this case, U is a block-diagonal orthogonal matrix. The size of each block U_i is equal to the multiplicity of the eigenvalue $\tilde{\rho}(\mathfrak{t})_{ii}$. We first assume that each of these blocks is of determinant 1. Then

$$\rho_{\text{pMD}} = U\tilde{\rho}U^\top \tag{3.14}$$

is a pseudo-MD representation. Using Theorem 3.7, Remark 3.8 and the conditions established in the next section, the existence of such U could either imply contradiction or be determined for all the rank 6 modular data. In the former case, representation ρ_{isum} will be rejected. Once the matrix U is determined, one can determine the correct signed diagonal matrix by using the Frobenius-Perron dimensions or the Verlinde formula.

The eigenvectors of the diagonal matrix $\tilde{\rho}(\mathfrak{t})$ corresponding to the eigenvalues of multiplicity 1 are of particular importance in the determination of the orthogonal matrix U. We simply called the block of $\tilde{\rho}(\mathfrak{s})$ corresponding to these eigenvectors the *non-degenerate block*, and denoted by $\tilde{\rho}(\mathfrak{s})^{\text{ndeg}}$.

The following proposition provide a convenient sufficient condition for any $\mathrm{SL}_2(\mathbb{Z})$ representation equivalent to an MD representation.

Proposition 3.9. *Let $\tilde{\rho}$ be any (symmetric) $\mathrm{SL}_2(\mathbb{Z})$ representation. If $\tilde{\rho}$ is equivalent to an MD representation, then the entries of $\tilde{\rho}(\mathfrak{s})^{\mathrm{ndeg}}$ are cyclotomic numbers in $\mathbb{Q}_{\mathrm{ord}(\tilde{\rho})}$.*

Proof. The statement is an immediate consequence of Theorem 3.4 and Theorem 3.7(1). □

The proposition can be implemented for computer automation to eliminate many ρ_{isum}. Theorem 3.7 (6) and the property of second Frobenius-Schur indicators are implemented to eliminate ρ_{isum} or solving the matrix U. When the matrix U is determined, the signed diagonal matrix P_{sgn} can be searched by using the nonnegative integral fusion coefficients (cf. Theorem 3.7 (7)). The potential MD representation ρ_{MD} is then given by

$$\rho_{\mathrm{MD}} = P_{\mathrm{sgn}} \rho_{\mathrm{pMD}} P_{\mathrm{sgn}}^{\top}, \tag{3.15}$$

Again, ρ_{isum} will be rejected if no such P_{sgn} is found. From the potential MD representations ρ_{MD} we can then obtain the potential modular data (S, T) via (3.7), and they will be verified if Theorems 2.1 and 3.7 are satisfied. This allows us to get a list of (S, T) pairs that include all the modular data. The computer automation for the endeavor is robust particularly when $\rho_{\mathrm{isum}} = \rho_{\mathrm{isum}}^{\mathrm{ndeg}}$.

By comparing the list of (S, T) pairs to known rank-6 MTCs, we obtain a classification of all modular data via matrix representations of $\mathrm{SL}_2(\mathbb{Z})$.

3.5. **More general properties of $\mathrm{SL}_2(\mathbb{Z})$ representations.** In this subsection, we introduce and collect conditions on $\mathrm{SL}_2(\mathbb{Z})$ representations necessary for them to be MD representations

The decomposition criteria on $\mathfrak{t}$-spectrum [5] of a linear representation of $\mathrm{SL}_2(\mathbb{Z})$ associated with a MTC is one of the major tools.

Theorem 3.10 ($\mathfrak{t}$-spectrum criteria)**.** *Let ρ be an MD representation. If*

$$\rho \cong \rho_1 \oplus \rho_2$$

for some representations ρ_1, ρ_2 of $\mathrm{SL}_2(\mathbb{Z})$, then $\mathrm{spec}(\rho_1(\mathfrak{t})) \cap \mathrm{spec}(\rho_2(\mathfrak{t})) \neq \varnothing$.

Let p be a prime. We denote by G_p the Galois group $\mathrm{Gal}(\mathbb{Q}_p)$. The least dimension of an irreducible representation of $\mathrm{SL}_2(\mathbb{Z})$ of level p is $\frac{p-1}{2}$. Their $\mathfrak{t}$-spectrum is either $G_p^2 \cdot \zeta_p$ or $G_p^2 \cdot \zeta_p^a$ where $x^2 \equiv a \mod p$ has no integer solution. Note that an integer a is called a *nonresidue* modulo p if $x^2 \equiv a \mod p$ has no integral solution. The second least dimension irreducible representation ρ of $\mathrm{SL}_2(\mathbb{Z})$ of level p is $\frac{p+1}{2}$ whose $\mathfrak{t}$-spectrum is either $G_p^2 \cdot e^{2\pi i/p} \cup \{1\}$ or $G_p^2 \cdot e^{2\pi i a/p} \cup \{1\}$ where a is any nonresidue modulo p. In this case, $\rho(\mathfrak{s})^2 = \left(\frac{-1}{p}\right) \mathrm{id}$ (see for example [18]).

Proposition 3.11. *Let $3 < p < q$ be prime such that $pq \equiv 3 \mod 4$. For any modular tensor category $\mathcal{C}$ such that $\mathrm{ord}(T) = pq$, then $rank(\mathcal{C}) \neq \frac{p+q}{2} + 1$. Moreover, if $p > 5$, $rank(\mathcal{C}) > \frac{p+q}{2} + 1$.*

Proof. Let $\mathcal{C}$ be a modular tensor category of rank $r \leqslant \frac{p+q}{2} + 1$ and $\mathrm{ord}(T) = pq$. There exists an $\mathrm{SL}_2(\mathbb{Z})$ representation ρ of $\mathcal{C}$ with level pq [11]. Suppose ρ has an irreducible subrepresentation ρ' of level pq. By the Chinese remainder theorem, the $\rho' \cong \rho_1 \otimes \rho_2$, where ρ_1, ρ_2 are irreducible representations of $\mathrm{SL}_2(\mathbb{Z})$ of levels p and q respectively. Then

$$\frac{p+q}{2} + 1 \geqslant \dim \rho' = (\dim \rho_1)(\dim \rho_2) \geqslant \left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right).$$

The inequality implies $p = 5$ and $q = 7$, and hence $\dim \rho' = 6$. Therefore, the $\mathfrak{t}$-spectrum of ρ' consists of 6 distinct primitive 35-th roots of unity, and $rank(\mathcal{C}) = 6$ or 7. There exists a modular tensor category of rank 6 with $\mathrm{ord}(T) = 35$. However, if $rank(\mathcal{C}) = 7$, then $\rho \cong \rho' \oplus \rho_0$ where ρ_0 is a 1-dimensional representation. The level of ρ_0 is a divisor of 12 but this is not possible by Theorem

3.10. In conclusion, if ρ has an irreducible subrepresentation of level pq, then $p = 5$, $q = 7$ and $rank(\mathcal{C}) = 6$.

Now, we assume ρ has no irreducible subrepresentation of level pq. Then ρ must have irreducible subrepresentations ρ_1, ρ_2 of levels p and q respectively. If $\dim \rho_1 < \frac{p+1}{2}$ or $\dim \rho_2 < \frac{q+1}{2}$, then

$$\rho \cong \rho_1 \oplus \rho_2 \oplus \rho_3$$

where ρ_3 is a subrepresentation of ρ of dimension $\leqslant 2$. If ρ_3 has a 1-dimensional component ρ_4, then $\rho_4(\mathfrak{t})$ must be a 12-th root of unity. Since $3 < p < q$, the only 12-th root which could appear in the $\mathfrak{t}$-spectrum of ρ is 1, or ρ_4 is trivial. However, $\operatorname{spec}(\rho_1(\mathfrak{t}))$ and $\operatorname{spec}(\rho_2(\mathfrak{t}))$ do not contain 1 by the remark preceding this proposition, and this contradicts Theorem 3.10. Note that irreducible $\mathrm{SL}_2(\mathbb{Z})$ representation of dimension 2 at prime levels only appear for the primes 2, 3 and 5. Therefore, if ρ_3 is irreducible of dimension 2, then $p = 5$ and ρ_3 is of level 5, but this contradicts Theorem 3.10 again. Thus, $\dim \rho_1 \geqslant \frac{p+1}{2}$ and $\dim \rho_2 \geqslant \frac{q+1}{2}$. Since $rank(\mathcal{C}) \leqslant \frac{p+q}{2} + 1$, we find $rank(\mathcal{C}) = \frac{p+q}{2} + 1$, $\dim \rho_1 = \frac{p+1}{2}$ and $\dim \rho_2 = \frac{q+1}{2}$. Now, we would like to show that this also impossible.

Without loss of generality, we may assume $\left(\frac{-1}{p}\right) = 1$ and $\left(\frac{-1}{q}\right) = -1$. Then $\rho(\mathfrak{s})^2$ is a signed diagonal matrix and the multiplicities $1, -1$ are respectively $\frac{p+1}{2}$ and $\frac{q+1}{2}$. Thus, $|\operatorname{Tr}(\rho(\mathfrak{s})^2)| = \frac{q-p}{2}$. Since $\operatorname{rank} \mathcal{C} - \frac{q-p}{2} = p + 1$, $\mathcal{C}$ has $\frac{p+1}{2} \geqslant 3$ pairs of simple objects which are not self-dual. Since $\rho(\mathfrak{t})$ has only one eigenvalue of multiplicity 2 and all other eigenvalues are of multiplicity 1, $\mathcal{C}$ has at most 1 pair of simple objects which dual of each other, a contradiction! $\square$

Let ρ be an $\mathrm{SL}_2(\mathbb{Z})$ representation of a modular tensor category $\mathcal{C}$ and let n be the level of ρ. For any $\sigma \in \operatorname{Gal}(\bar{\mathbb{Q}})$, $D_\rho(\sigma)$ defined in (3.6) is a signed permutation matrix of $\hat{\sigma}$ by [11, Theorem II] (or Theorem 3.7 (6)). The permutation $\hat{\sigma}$ on $\operatorname{irr}(\mathcal{C})$ is given by (3.9), and we set

$$\operatorname{Inv}_{\mathcal{C}}(\sigma) := \{i \in \operatorname{irr}(\mathcal{C}) \mid \hat{\sigma}(i) = i\}\,.$$

If γ is complex conjugation, by (3.10),

$$D_\rho(\gamma) = \overline{\rho(\mathfrak{s})}\rho(\mathfrak{s})^{-1} = \rho(\mathfrak{s})^2 = \pm C,$$

where C is the charge conjugation matrix of $\mathcal{C}$. Since $\hat{\gamma}(i) = i^*$ for $i \in \operatorname{irr}(\mathcal{C})$,

$$|\operatorname{Tr}(D_\rho(\gamma))| = |\operatorname{Tr}(\rho(\mathfrak{s}^2))| = \operatorname{Tr}(C) = |\{i \in \operatorname{irr}(\mathcal{C}) \mid i^* = i\} = |\operatorname{Inv}_{\mathcal{C}}(\gamma)|\,.$$

This equality can be generalized to any $\sigma \in \operatorname{Gal}(\bar{\mathbb{Q}})$ as an inequality in the following proposition.

Proposition 3.12. *Let ρ be an $\mathrm{SL}_2(\mathbb{Z})$ representation of a modular tensor category $\mathcal{C}$. For any $\sigma \in \operatorname{Gal}(\bar{\mathbb{Q}})$,*

$$|\operatorname{Tr}(D_\rho(\sigma))| \leqslant |\operatorname{Inv}_{\mathcal{C}}(\sigma)|\,.$$

Let $s := \rho(\mathfrak{s})$, and follow the notation of Theorem 3.7(6). If $s_{ij} \neq 0$ for any $i, j \in \operatorname{Inv}_{\mathcal{C}}(\sigma)$, then $\epsilon_\sigma(i) = \epsilon_\sigma(j)$. If there exists $i \in \operatorname{Inv}_{\mathcal{C}}(\sigma)$ such that $s_{ij} \neq 0$ for all $j \in \operatorname{Inv}_{\mathcal{C}}(\sigma)$, then

$$|\operatorname{Tr}(D_\rho(\sigma))| = |\operatorname{Inv}_{\mathcal{C}}(\sigma)|\,.$$

In particular,

$$\operatorname{Tr}(s^2) = |\{i \in \operatorname{irr}(\mathcal{C}) \mid i^* = i\}| > 0\,.$$

Proof. By Theorem 3.7(6), $D_\rho(\sigma) = \varepsilon_\sigma(i)\delta_{\hat{\sigma}(i),j}$. Therefore,

$$|\operatorname{Tr}(D_\rho(\sigma))| = \left|\sum_{i \in \operatorname{Inv}_{\mathcal{C}}(\sigma)} \varepsilon_\sigma(i)\right| \leqslant |\operatorname{Inv}_{\mathcal{C}}(\sigma)|\,.$$

If $s_{ij} \neq 0$ for any $i, j \in \mathrm{Inv}_{\mathcal{C}}(\sigma)$, then $\sigma(s_{ij}) = \varepsilon_\sigma(i)s_{ij} = \varepsilon_\sigma(j)s_{ij}$, and so $\varepsilon_\sigma(i) = \varepsilon_\sigma(j)$. Thus, if there exists $i \in \mathrm{Inv}_{\mathcal{C}}(\sigma)$ such that $s_{ij} \neq 0$ for all $j \in \mathrm{Inv}_{\mathcal{C}}(\sigma)$, then $\varepsilon_\sigma(i) = \varepsilon_\sigma(j)$ for all j and hence the equality

$$|\,\mathrm{Tr}(D_\rho(\sigma))| = |\,\mathrm{Inv}_{\mathcal{C}}(\sigma)|\,.$$

The last statement was proved in the preceding remark and since $\mathbb{1}^* = \mathbb{1}$ this completes the proof of the proposition. □

According to [11], if ρ is an MD representation of an integral modular tensor category $\mathcal{C}$, then $\rho(\mathfrak{t})_{\mathbb{1},\mathbb{1}} = \zeta$ for some 24-th root of unity ζ under the identification of the standard basis for ρ and $\mathrm{irr}(\mathcal{C})$. The following proposition provides a sufficient condition on the representation type of ρ for $\mathcal{C}$ to be integral.

Proposition 3.13. *Let $\tilde{\rho}$ be any $\mathrm{SL}_2(\mathbb{Z})$ representation. For any $\zeta \in \mathrm{spec}(\tilde{\rho}(\mathfrak{t}))$, denote by $E_\zeta(\tilde{\rho})$ the eigenspace of $\tilde{\rho}(\mathfrak{t})$ for the eigenvalue ζ. Suppose $\tilde{\rho}$ is equivalent to an MD representation ρ of a modular tensor category $\mathcal{C}$. Then*

(1) *$D_{\tilde{\rho}}(\sigma)(E_\zeta(\tilde{\rho})) \subseteq E_\zeta(\tilde{\rho})$ for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}})$ if and only if $\zeta^{24} = 1$.*

(2) *If $\mathbb{1} \in E_\zeta(\rho)$ for some $\zeta \in \mathrm{spec}(\rho(\mathfrak{t}))$, and for each $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}})$, there exists $\epsilon_\sigma = \pm 1$ such that*

$$D_{\tilde{\rho}}(\sigma)|_{E_\zeta(\tilde{\rho})} = \epsilon_\sigma \,\mathrm{id}_{E_\zeta(\tilde{\rho})}\,,$$

then $\mathcal{C}$ is integral. In particular, $\zeta^{24} = 1$.

(3) *If $\mathbb{1} \in \bigoplus_{\gamma \in A} E_\gamma(\rho)$ for some subset $A \subseteq \mathrm{spec}(\rho(\mathfrak{t}))$, and for any $\gamma \in A$, $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}})$, there exists $\epsilon_\sigma(\gamma) = \pm 1$ such that $D_{\tilde{\rho}}(\sigma)|_{E_\gamma(\tilde{\rho})} = \epsilon_\sigma(\gamma)\,\mathrm{id}_{E_\gamma(\tilde{\rho})}$, then A is a set of 24-th roots of unity and $\mathcal{C}$ is integral.*

(4) *If $\mathcal{C}$ is integral and $d_i > 0$ for all i, then for any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}})$, $d_i = d_{\hat{\sigma}(i)}$ for all i, $D_\rho(\sigma)(\mathbb{1}) = \epsilon_\sigma \mathbb{1}$ for some $\epsilon_\sigma = \pm 1$, and $\frac{1}{\epsilon_\sigma} D_\rho(\sigma)$ is the permutation matrix of $\hat{\sigma}$.*

Proof. Assuming the identification of the standard basis for ρ and $\mathrm{irr}(\mathcal{C})$, $E_\zeta(\rho)$ is spanned by the objects $X \in \mathrm{irr}(\mathcal{C})$ such that $\rho(\mathfrak{t})_{X,X} = \zeta$. Let $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}})$. It follows from Theorem 3.7(6) that $D_\rho(\sigma)$ is a signed permutation matrix of a permutation $\hat{\sigma}$ on $\mathrm{irr}(\mathcal{C})$, and that $\sigma^2(\rho(\mathfrak{t})) = D_\rho(\sigma)\rho(\mathfrak{t})D_\rho(\sigma)^{-1}$, or equivalently $\rho(\mathfrak{t})D_\rho(\sigma) = D_\rho(\sigma)\sigma^{-2}(\rho(\mathfrak{t}))$. If $\zeta^{24} = 1$, then $\sigma^2(\zeta) = \zeta$ for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}})$. Thus, for any simple object $X \in E_\zeta(\rho)$,

$$\rho(\mathfrak{t})D_\rho(\sigma)(X) = D_\rho(\sigma)\sigma^{-2}(\rho(\mathfrak{t}))(X) = \sigma^{-2}(\zeta)D_\rho(\sigma)(X) = \zeta D_\rho(\sigma)(X)\,.$$

Therefore, $D_\rho(\sigma)(E_\zeta(\rho)) \subseteq E_\zeta(\rho)$. Let $\phi : \tilde{\rho} \to \rho$ be an isomorphism of $\mathrm{SL}_2(\mathbb{Z})$ representations. Then $\phi(E_\zeta(\tilde{\rho})) = E_\zeta(\rho)$, and $\phi D_{\tilde{\rho}}(\sigma) = D_\rho(\sigma)\phi$ for any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}})$. This implies $D_{\tilde{\rho}}(\sigma)(E_\zeta(\tilde{\rho})) \subseteq E_\zeta(\tilde{\rho})$.

Conversely, if $D_{\tilde{\rho}}(\sigma)(E_\zeta(\tilde{\rho})) \subseteq E_\zeta(\tilde{\rho})$, then $D_\rho(\sigma)(E_\zeta(\rho)) \subseteq E_\zeta(\rho)$ by the same reason. Thus, for any $X \in E_\zeta(\rho)$, $\rho(\mathfrak{t})D_\rho(\sigma)(X) = \zeta D_\rho(\sigma)(X)$. However, we also have

$$\rho(\mathfrak{t})D_\rho(\sigma)(X) = D_\rho(\sigma)\sigma^{-2}(\rho(\mathfrak{t}))(X) = \sigma^{-2}(\zeta)D_\rho(\sigma)(X).$$

Therefore, $\sigma^{-2}(\zeta) = \zeta$ for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}})$. This implies ζ is a 24-th root (cf. [11, Prop. 6.7 and Lem. A.2]). This proves statement (1).

For statement (2), we assume $\mathbb{1} \in E_\zeta(\rho)$, and for each $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}})$ there exists $\epsilon_\sigma = \pm 1$ such that $D_{\tilde{\rho}}(\sigma)|_{E_\zeta(\tilde{\rho})} = \epsilon_\sigma \,\mathrm{id}_{E_\zeta(\tilde{\rho})}$. It follows from (1) that $\zeta^{24} = 1$. Moreover, $D_\rho(\sigma)|_{E_\zeta(\rho)} = \epsilon_\sigma \,\mathrm{id}_{E_\zeta(\rho)}$ and $D_\rho(\sigma)(\mathbb{1}) = \epsilon_\sigma \mathbb{1} = \pm\hat{\sigma}(\mathbb{1})$. Therefore, $\hat{\sigma}(\mathbb{1}) = \mathbb{1}$, and hence $\sigma(\dim(V)) = \dim(V)$ for any $V \in \mathrm{irr}(\mathcal{C})$ by Theorem 3.7(6). Thus, $\dim(V)$ are integers for $V \in \mathrm{irr}(\mathcal{C})$. It follows from [11, Rem. 6.3] that $\mathrm{FPdim}(V) \in \mathbb{Z}$, and hence $\mathcal{C}$ is integral.

Statement (3) follows directly from (2), and this completes the proof of the proposition.

(4) Since $\mathcal{C}$ is integral, $S_{j,0} = d_j \in \mathbb{Z}$ for all j. For any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}})$, $\sigma(S_{j,0}) = S_{j,0}$. Therefore, $\hat{\sigma}(0) = 0$, and so $\sigma(\rho(\mathfrak{s})_{i,0}) = \epsilon_\sigma(0)\rho(\mathfrak{s})_{i,0}$ for all i, where $\epsilon_\sigma(0) = \pm 1$. This is equivalent to that $D_\rho(\sigma)(\mathbb{1}) = \epsilon_\sigma \mathbb{1}$.

Since $\sigma(\rho(\mathfrak{s})_{i,0}) = \epsilon_\sigma(i)\rho(\mathfrak{s})_{\hat{\sigma}(i),0}$ for some $\epsilon_\sigma(i) = \pm 1$, we have $\epsilon_\sigma(i)\rho(\mathfrak{s})_{\hat{\sigma}(i),0} = \epsilon_\sigma \rho(\mathfrak{s})_{i,0}$ or $\epsilon_\sigma(i) d_{\sigma(i)} = \epsilon_\sigma(0) d_i$. This implies $\epsilon_\sigma(0) = \epsilon_\sigma(j)$ and $d_i = d_{\sigma(i)}$ as $d_i, d_{\hat{\sigma}(i)} > 0$.

For any $i, j \in \mathrm{irr}(\mathcal{C})$,

$$\sigma(\rho(\mathfrak{s})_{i,j}) = \epsilon_\sigma(i)\rho(\mathfrak{s})_{\hat{\sigma}(i),j} = \epsilon_\sigma(0)\rho(\mathfrak{s})_{\hat{\sigma}(i),j}$$

which implies $D_\rho(\sigma)\rho(\mathfrak{s}) = \epsilon_\sigma(0)P(\hat{\sigma})\rho(\mathfrak{s})$, where $P(\hat{\sigma})_{ij} = \delta_{\sigma(i),j}$. Thus, $D_\rho(\sigma) = \epsilon_\sigma(0)P(\hat{\sigma})$. □

The following result in [7] is important for determining whether an $\mathrm{SL}_2(\mathbb{Z})$ representation of small level is an MD representation.

Theorem 3.14. *Modular tensor categories with* $\mathrm{ord}(T) = 2, 3, 4, 6$ *are integral.*

Then the case for $\mathrm{ord}(T) = 2$ is completely classified in [38], and the types of these MTCs are given in the following proposition.

Proposition 3.15. *Let $\mathcal{C}$ be a modular tensor category with* $\mathrm{ord}(T) = 2$. *Then* $\mathrm{rank}(\mathcal{C}) = 4^n$ *for some positive integer n, and every* $\mathrm{SL}_2(\mathbb{Z})$ *representation ρ of $\mathcal{C}$ is projectively equivalent to*

$$(\rho_2 \oplus 2\chi_0)^{\otimes n} \cong a_n\rho_2 \oplus b_n\rho_1 \oplus c_n\chi_0\,,$$

where ρ_1, ρ_2 are respectively the level 2 irreducible representations of dimension 1 and 2, and

$$a_n = \frac{4^n - 1}{3}, \quad b_n = \frac{2 \cdot 4^{n-1} + 1}{3} - 2^{n-1}, \quad c_n = \frac{2 \cdot 4^{n-1} + 1}{3} + 2^{n-1}\,.$$

Proof. By [38], $\mathcal{C}$ is a Deligne product of the pointed modular tensor categories $\mathcal{C}(\mathbb{Z}_2^2, q)$ and $\mathcal{C}(\mathbb{Z}_2^2, q')$ with the quadratic forms $q, q' : \mathbb{Z}_2^2 \to \{\pm 1\}$ given by

$$q(x,y) = (-1)^{xy}, \quad q'(x,y) = (-1)^{x^2+xy+y^2}\,.$$

Both modular tensor categories, up to a linear character, have a representation of $\mathrm{SL}_2(\mathbb{Z})$ equivalent to $\rho_2 \oplus 2\chi_0$. Note that $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}) \cong S_3$, symmetric group of degree 3. Thus, $\rho \cong (\rho_2 \oplus 2\chi_0)^{\otimes n} = a_n\rho_2 \oplus b_n\rho_1 \oplus c_n\chi_0$ for some nonnegative integers a_n, b_n, c_n. The fusion matrix of $\rho_2 \oplus 2\chi_0$ relative to the basis $\{\chi_0, \rho_1, \rho_2\}$ is given by

$$F = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix} = P \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1} \quad \text{where } P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{bmatrix}.$$

Thus,

$$F^n = P \begin{bmatrix} 4^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1} = \begin{bmatrix} 2^{n-1} + \frac{2\cdot 4^{n-1}+1}{3} & -2^{n-1} + \frac{2\cdot 4^{n-1}+1}{3} & \frac{4^n-1}{3} \\ -2^{n-1} + \frac{2\cdot 4^{n-1}+1}{3} & 2^{n-1} + \frac{2\cdot 4^{n-1}+1}{3} & \frac{4^n-1}{3} \\ \frac{4^n-1}{3} & \frac{4^n-1}{3} & \frac{2\cdot 4^n+1}{3} = \end{bmatrix}.$$

The result follows from the first column of F^n. □

The following proposition follows immediately from the classification of [4], where strictly weakly integral means $\mathrm{FPdim}(\mathcal{C}) \in \mathbb{Z}$ while $\mathrm{FPdim}(X) \notin \mathbb{Z}$ for some object X.

Proposition 3.16. *Let $\mathcal{C}$ be a modular tensor category of rank 6.*

(1) *If $\mathcal{C}$ is integral, then $\mathcal{C}$ is pointed and hence $\mathcal{C}$ is of type* $(4, 2)$ *and every* $\mathrm{SL}_2(\mathbb{Z})$ *representation of $\mathcal{C}$ has level 24.*

(2) *If $\mathcal{C}$ is strictly weakly integral, then $\mathcal{C}$ is braided equivalent to a Galois conjugate of $U(2)_1 \boxtimes SU(2)_3$, $SO(5)_2$ or its zesting. If ρ is an $\mathrm{SL}_2(\mathbb{Z})$ representations of $\mathcal{C}$ with a minimal* t-*spectrum, then one of the following holds:* (i) *$\mathcal{C}$ is of type* (6) *and ρ has level 16,* (ii) *$\mathcal{C}$ is of type* $(3,3)$ *and ρ has level 20, or* (iii) *$\mathcal{C}$ is of type* $(3,2,1)$ *and ρ has level 10.*

(3) *In particular, if $\mathcal{C}$ is weakly integral, then* $\dim(\mathcal{C}) = 6, 8, 20$.

When a potential modular data is obtained from a representation of $\mathrm{SL}_2(\mathbb{Z})$, one could obtain the $\mathrm{FPdim}(X)$ of each simple object X. Those simple objects X with $\mathrm{FPdim}(X) = 1$ generate a pointed ribbon subcategory. The next proposition, which can be derived from [32] in different notation, describes some relations between the rank of a pointed ribbon category and the orders of the twists.

Proposition 3.17. *Let $\mathcal{C}$ be a pointed ribbon category of rank n. Then $\mathrm{ord}(T) \mid n$ if n is odd, and $\mathrm{ord}(T) \mid 2n$ if n is even. If, in addition, $\mathcal{C}$ is symmetric and $\dim(a) > 0$ for all $a \in \mathrm{irr}(\mathcal{C})$, then either $\mathrm{ord}(T) = 1$ or 2. In the latter case, n must be even and there are exactly $n/2$ simple objects with twist -1.*

Proof. Since $\mathcal{C}$ pointed, the set $G = \mathrm{irr}(\mathcal{C})$ forms an abelian group under the tensor product and the map $q : G \to \mathbb{C}^\times, q(a) = \theta_a$ defines a quadratic form on G. Therefore, $B_q(a,b) = \frac{q(ab)}{q(a)q(b)}$ defines a bicharacter on G. In particular, $B_q(a,b)$ is an n-th root of unity for any $a, b \in G$. Now, for any positive integer m and $a \in G$, we have

$$q(a^m) = q(a)q(a^{m-1})B_q(a, a^{m-1}) = q(a)q(a^{m-1})B_q(a,a)^{m-1}\,.$$

Therefore, by induction, we have

$$q(a^m) = q(a)^m B_q(a,a)^{m(m-1)/2}\,.$$

In particular, $q(a)^n = B_q(a,a)^{-n(n-1)/2}$. If n is odd, $\frac{n-1}{2} \in \mathbb{Z}$ and so $q(a)^n = 1$. If n is even, then $q(a)^{2n} = 1$. This completes the proof of the first statement.

If, in addition, $\mathcal{C}$ is symmetric and $\dim(a) = 1$ for $a \in G$, then

$$1 = S_{a,b} = B_q(a^{-1}, b) = B_q(a,b)^{-1} = \frac{q(a)q(b)}{q(ab)}$$

for any $a, b \in G$. Therefore, q is a character of G. Since $q(a^{-1}) = q(a)$, $q(a)^2 = 1$ for all $a \in G$. If q is of order 1, then $q(a) = 1$ for all $a \in G$ or $T = \mathrm{id}$. However, if q is of order 2, then the image of q is the group $\{\pm 1\}$ which is of order 2. Therefore, $\ker q$ is of index 2 which means there are exactly $n/2$ simple objects in G with twists are 1. Thus, the second statement follows. □

It is worth noting that last statement of the preceding proposition does not hold for super-Tannakian fusion categories which are not pointed. For example, if we take Q to be the quaternion group of order 8 and z the unique central element of order 2, then the super-Tannakian fusion categories $\mathrm{Rep}(Q,z)$ has 4 simple objects a of dimension 1 with $\theta_a = 1$ and a unique simple object b of dimension 2 with $\theta_b = -1$.

For any legitimate fusion rules N_{ij}^k, one could obtain the possible $\theta_k = \mathrm{e}^{2\pi \mathrm{i} s_k}$ by solving a system of linear equations with unknowns s_k. The following proposition provides a condition for legitimate s_k of a potential modular data.

Proposition 3.18. *Let $\mathcal{C}$ be a modular tensor category of rank n and central charge c. If the twists of $\mathcal{C}$ are $\mathrm{e}^{2\pi \mathrm{i} s_1}, \dots, \mathrm{e}^{2\pi \mathrm{i} s_n}$ for some rational numbers $s_1, s_2, \dots, s_n$, then*

$$12 \sum_{k=1}^{n} s_k - nc/2 \in \mathbb{Z}$$

Proof. Note that $\mathrm{e}^{\pi \mathrm{i} c/4} = \frac{1}{D}\sum_{k=1}^{n} d_k^2\, \mathrm{e}^{2\pi \mathrm{i} s_k}$ where d_k denotes the dimension of the simple object k with twist $\mathrm{e}^{2\pi \mathrm{i} s_k}$ and $D = \sqrt{\dim(\mathcal{C})}$. Let (S, T) be the modular data of $\mathcal{C}$. Then

$$\rho(\mathfrak{s}) = \frac{1}{D} S, \quad \rho(\mathfrak{t}) = \mathrm{e}^{-2\pi \mathrm{i} c/24}\, T$$

defines an $\mathrm{SL}_2(\mathbb{Z})$ representation of $\mathcal{C}$. Thus, $\det \circ \rho$ is a 1-dimensional representation of $\mathrm{SL}_2(\mathbb{Z})$. Since the group of linear characters of $\mathrm{SL}_2(\mathbb{Z})$ is a cyclic group of order 12, $\det \rho(\mathfrak{g})^{12} = 1$ for all $\mathfrak{g} \in \mathrm{SL}_2(\mathbb{Z})$. In particular,

$$1 = \det \rho(\mathfrak{t})^{12} = (\mathrm{e}^{2\pi \mathrm{i} s_1} \cdots \mathrm{e}^{2\pi \mathrm{i} s_n} \cdot \mathrm{e}^{-2\pi \mathrm{i} nc/24})^{12} = \mathrm{e}^{2\pi \mathrm{i} \cdot 12 \sum_{k=1}^{n} s_k - nc/2}.$$

This implies $12 \sum_{k=1}^{n} s_k - nc/2 \in \mathbb{Z}$. □

The following proposition is proved in [31] will also be useful later.

Proposition 3.19. *Let ρ an MD linear representation. Then*

$$\rho \not\cong n\rho_0$$

for any integer $n > 1$ and any non-degenerate representation ρ_0 of $\mathrm{SL}_2(\mathbb{Z})$.

3.6. **Modular tensor categories of type** $(d, 1, \ldots, 1)$**.** For a representation ρ_{isum} of $\mathrm{SL}_2(\mathbb{Z})$ of type $(d, 1, \ldots, 1)$ it is generally more difficult to determine whether it is equivalent to an MD representation. However, this type of MTC does exist. It is desirable to deduce some conditions for such MD representations.

Lemma 3.20. *Let ρ be an MD representation. If $\rho \cong \rho_0 \oplus \rho_1 \oplus \cdots \oplus \rho_\ell$ for some 1-dimensional representations $\rho_1, \ldots, \rho_\ell$ of $\mathrm{SL}_2(\mathbb{Z})$, then $\mathrm{spec}(\rho_i(\mathfrak{t})) \subset \mathrm{spec}(\rho_0(\mathfrak{t}))$ for all $i > 0$. In particular, if $\rho_0(\mathfrak{t})$ has exactly one eigenvalue which is a 12-th root of unity, then $\rho_1, \ldots, \rho_\ell$ are all equivalent, and $\rho \cong \rho_0 \oplus \ell\rho_1$.*

Proof. By the $\mathfrak{t}$-spectrum criteria, $\mathrm{spec}(\rho_j(\mathfrak{t})) \subset \mathrm{spec}(\rho_0(\mathfrak{t}))$ for some $j > 0$. Suppose there exists $j > 0$ such that $\mathrm{spec}(\rho_j(\mathfrak{t})) \not\subset \mathrm{spec}(\rho_0(\mathfrak{t}))$. Let $J = \{j \in \{0, \ldots, \ell\} \mid \mathrm{spec}(\rho_j(\mathfrak{t})) \not\subset \mathrm{spec}(\rho_0(\mathfrak{t}))\}$. Then, the decomposition

$$\rho \cong \left(\sum_{j \in J} \rho_j\right) \oplus \left(\sum_{j \notin J} \rho_j\right)$$

does not satisfies the $\mathfrak{t}$-spectrum criteria. Therefore, $\mathrm{spec}(\rho_j(\mathfrak{t})) \subset \mathrm{spec}(\rho_0(\mathfrak{t}))$ for all j.

If, in particular, $\mathrm{spec}(\rho(t))$ contains exactly one 12-th root of unity ζ, then $\mathrm{spec}\, \rho_i(t) = \{\zeta\}$ for all $i > 0$. Hence $\rho_1 \cong \rho_i$ for $i > 1$, and the last assertion follows. □

Corollary 3.21. *Let ρ be an $\mathrm{SL}_2(\mathbb{Z})$ representation of a modular tensor category $\mathcal{C}$. Suppose that $\rho \cong \rho_0 \oplus \rho_1 \oplus \cdots \oplus \rho_\ell$ for some 1-dimensional representations $\rho_1, \ldots, \rho_\ell$ and some non-degenerate irreducible representation ρ_0 of $\mathrm{SL}_2(\mathbb{Z})$ such that $\mathrm{spec}(\rho_0(\mathfrak{t}))$ has a unique 12-th root of unity. Then $\mathcal{C}$ admits an MD representation $\rho' \cong \rho_0' \oplus \ell\chi_0$, where χ_0 is the trivial representation and ρ_0' is projectively equivalent to ρ_0 with $1 \in \mathrm{spec}(\rho_0'(\mathfrak{t}))$.*

If $\ell \notin \{1, 2\dim\rho_0 - 1\}$, then $\mathcal{C}$ is self-dual, and ρ_0' is even. If $\ell \in \{1, 2\dim\rho_0 - 1\}$ and $\mathcal{C}$ is not self-dual, then ρ_0' is odd, and the set of non-self-dual objects is given by $\{i \in \mathrm{irr}(\mathcal{C}) \mid \rho'(\mathfrak{t})_{ii} = 1\}$.

Proof. By Lemma 3.21, $\rho \cong \rho_0 \oplus \ell\rho_1$. Since $\dim\rho_1 = 1$, $\rho' = \rho_1^* \otimes \rho$ is another $\mathrm{SL}_2(\mathbb{Z})$ representation of $\mathcal{C}$. Moreover, $\rho' \cong \rho_0' \oplus \ell\chi_0$, where $\rho_0' = \rho_1^* \otimes \rho_0$ which is also non-degenerate.

Suppose $\rho_0'(\mathfrak{s}^2) = -\mathrm{id}$. By Proposition 3.12, the number of self-dual objects in $\mathrm{irr}(\mathcal{C})$ is given by

$$|\operatorname{Tr}(\rho'(\mathfrak{s}^2))| = |\ell - \dim\rho_0'| > 0$$

since $\mathbb{1}$ is self-dual simple object. If $\ell > \dim\rho_0$, then $|\operatorname{Tr}(\rho'(\mathfrak{s}^2))| = \ell - \dim\rho_0$ and so number of non-self-dual objects in $\operatorname{irr}(\mathcal{C})$ is $2\dim\rho_0$. The non-degeneracy of ρ_0' implies that $\rho'(\mathfrak{t})_{ii} = 1$ for any non-self-dual $i \in \operatorname{irr}(\mathcal{C})$. Therefore, $2\dim\rho_0 = \ell + 1$ or $\ell = 2\dim\rho_0 - 1$.

On the other hand, if $\ell < \dim\rho_0$, then $|\operatorname{Tr}(\rho'(\mathfrak{s}^2))| = \dim\rho_1 - \ell$ and so number of non-self-dual objects in $\operatorname{irr}(\mathcal{C})$ is 2ℓ. Since $\rho'(\mathfrak{t})_{ii} = 1$ for any non-self-dual simple object i, $2\ell = \ell + 1$ or $\ell = 1$.

Thus, if $\ell \neq 1$ or $2\dim\rho_1 - 1$, then $\rho_0'(\mathfrak{s}^2) = \mathrm{id}$ and so $\mathcal{C}$ is self-dual. On the other hand, if $\ell \in \{1, 2\dim\rho_0 - 1\}$ and $\mathcal{C}$ is not self-dual, then $\rho_0'(\mathfrak{s}^2) = -\mathrm{id}$ and the above discussion shows that the non-self-dual objects $i \in \operatorname{irr}(\mathcal{C})$ are exactly those i satisfying $\rho'(\mathfrak{t})_{ii} = 1$. □

Now, we can prove a sufficient condition for any MD representation of prime level $p > 3$ and of type $(\frac{p+1}{2}, 1, \dots, 1)$.

Proposition 3.22. *Let $\mathcal{C}$ be an MTC of type $(d, 1, \dots, 1)$ such that $\operatorname{ord}(T)$ is a prime $p > 3$, where $d = \frac{p+1}{2}$. Then $\mathcal{C}$ is of type $(d, 1)$, and hence $\operatorname{rank}\mathcal{C} = d + 1$. Moreover, $\operatorname{Inv}_{\mathcal{C}}(\sigma) = \varnothing$ for any generator $\sigma \in \operatorname{Gal}(\mathbb{Q}_p/\mathbb{Q})$. Furthermore, if $p \equiv 1 \mod 4$, then $\mathcal{C}$ is self-dual; otherwise $\mathcal{C}$ is not self-dual.*

Proof. By [11], there is an $\mathrm{SL}_2(\mathbb{Z})$ representation ρ of $\mathcal{C}$, which has level p. Then, every subrepresentation of ρ must have a level dividing p. Since $\mathcal{C}$ is of type $(d, 1, \dots, 1)$, ρ has a irreducible subsrepresentation ρ_0 of dimension d and level p. By the classification of irreducible representation $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$, $\rho_0(\mathfrak{s}^2) = \left(\frac{-1}{p}\right)\mathrm{id}$, ρ_0 is non-degenerate and 1 is the unique 12-th root of unity in $\operatorname{spec}(\rho_0(\mathfrak{t}))$. By Corollary 3.21,

$$\rho \cong \rho_0 \oplus \ell\chi_0.$$

Thus, if $p \equiv 1 \mod 4$, then ρ_0 is even and hence $\mathcal{C}$ is self-dual. However, if $p \equiv 3 \mod 4$, then ρ_0 is odd and so $\mathcal{C}$ is not self-dual.

One can derive from [30] that

$$\rho_0(\mathfrak{s}) = \frac{\left(\frac{a}{p}\right)}{\sqrt{p^*}}\left[\begin{array}{c|ccc} 1 & \sqrt{2} & \cdots & \sqrt{2} \\ \hline \sqrt{2} & & & \\ \vdots & & 2\cos\left(\frac{4\pi aij}{p}\right) & \\ \sqrt{2} & & & \end{array}\right], \quad \rho_0(\mathfrak{t}) = \begin{bmatrix} \zeta_p^{a\cdot 0} & & \\ & \ddots & \\ & & \zeta_p^{a(d-1)^2} \end{bmatrix}$$

where $1 \leqslant i, j \leqslant d-1$, $p^* = \left(\frac{-1}{p}\right)p$, and a an integer coprime to p. One may assume $\rho(\mathfrak{t}) = \operatorname{diag}(1, \dots, 1, \zeta_p^a, \dots, \zeta_p^{a(d-1)^2})$. By Theorem 3.4, there exists $W \in \mathrm{O}_{d+\ell}(\mathbb{R})$ such that $\rho = W(\ell\chi_0 \oplus \rho_0)W^\top$. Note that $W = VU$ for some signed diagonal matrix V and

$$U = \left[\begin{array}{c|c} f & 0 \\ \hline 0 & I_{d-1} \end{array}\right], \quad \text{where } f \in \mathrm{SO}_{\ell+1}(\mathbb{R}),$$

and $\rho_{\mathrm{pMD}} = U(\ell\chi_0 \oplus \rho_0)U^\top$ is a pseudo-MD representation, where I_{d-1} denotes the identity matrix of dimension $d-1$.

By direct computation,

$$\rho_{\mathrm{pMD}}(\mathfrak{s}) = U\left[\begin{array}{c|c} I_\ell & 0 \\ \hline 0 & \rho_0(\mathfrak{s}) \end{array}\right]U^\top = \left[\begin{array}{c|c} I_{\ell+1} + f_{*,\ell+1}f_{*,\ell+1}^\top(x-1) & x\sqrt{2}f_{*,\ell+1}r_{d-1} \\ \hline x\sqrt{2}\,r_{d-1}^\top f_{*,\ell+1}^\top & 2x\cos\left(\frac{4\pi aij}{p}\right) \end{array}\right] \tag{3.16}$$

where $f_{*,\ell+1} = [f_{1,\ell+1}, \cdots, f_{\ell+1,\ell+1}]^\top$, $r_{d-1} = [1, \cdots, 1] \in \mathbb{R}^{d-1}$, and $x = \left(\frac{a}{p}\right)/\sqrt{p^*}$.

Let σ be the generator of $\operatorname{Gal}(\mathbb{Q}_p/\mathbb{Q})$. For any $j \in \{1, \dots, d-1\}$, there exists $\tilde{j} \in \{1, \dots, d-1\}$ such that

$$\sigma\left(2\cos(2\pi j/p)\right) = 2\cos(2\pi\tilde{j}/p).$$

Since $\sqrt{p^*} \in \mathbb{Q}_p$, $\sigma(\sqrt{p^*}) = -\sqrt{p^*}$, and so

$$\sigma\left(2x\cos(4\pi aij/p)\right) = -2x\cos(4\pi ai\tilde{j}/p)\,.$$

for any $i, j \in \{1, \ldots, d-1\}$. If one identifies $\mathrm{irr}(\mathcal{C})$ with $\{1, \ldots, d+\ell\}$, then we have $\hat{\sigma}(\ell+1+j) = \ell+1+\tilde{j}$ for each $j \in \{1, \ldots, d-1\}$. In particular, $\hat{\sigma}$ has no fixed point in $\{\ell+2, \ldots, \ell+d\}$.

By (3.16) and Remark 3.8,

$$\sigma(x\sqrt{2}f_{i,\ell+1}) = -x\sqrt{2}f_{i,\ell+1} \quad \text{for all } i \in \{1, \ldots, \ell+1\}\,.$$

Since $x, x\sqrt{2}f_{i,\ell+1} \in \mathbb{Q}_p$ and $\sigma(x) = -x$, $\sqrt{2}f_{i,\ell+1} \in \mathbb{Q}_p$ and $\sigma(\sqrt{2}f_{i,\ell+1}) = \sqrt{2}f_{i,\ell+1}$. Therefore, $\sqrt{2}f_{i,\ell+1} \in \mathbb{Q}$ for all $i \in \{1, \ldots, \ell+1\}$, and hence $f_{i,\ell+1}f_{j,\ell+1} \in \mathbb{Q}$ for all $i, j \in \{1, \ldots, \ell+1\}$.

We claim that $0 < f_{i,\ell+1}^2 < 1$ for all $i \in \{1, \ldots, \ell+1\}$. If $f_{i,\ell+1} = 0$ for some i, then each row of $\rho_{pMD}(\mathfrak{s})$ has a zero entry by (3.16). Therefore, $f_{i,\ell+1} \neq 0$ for all i. Since $f_{*,\ell+1}$ as unit length, if $f_{i,\ell+1}^2 = 1$, then $f_{k,\ell+1} = 0$ for all $k \neq i \leqslant \ell+1$, a contradiction. This proves the claim.

Now we can show that $\mathrm{Inv}_{\mathcal{C}}(\sigma) = \varnothing$. It suffices to show that $\hat{\sigma}$ has no fixed point in $\{1, \ldots, \ell+1\}$. Suppose the i-th column of $s := \rho_{pMD}(\mathfrak{s})$ is fixed by $\hat{\sigma}$ for some $i \in \{1, \ldots, \ell+1\}$. Then $\sigma(s_{ii}) = \epsilon'_\sigma(i)s_{ii}$, where $\epsilon'_\sigma(i) = \pm 1$. Since $s_{ii} = 1 + f_{i,\ell+1}^2(x-1)$, the preceding equality implies

$$\epsilon'_\sigma(i)(1 + f_{i,\ell+1}^2(x-1)) = 1 + f_{i,\ell+1}^2(-x-1)\,.$$

Since $f_{i,\ell+1}^2 \leqslant 1$ is rational, the equation forces $f_{i,\ell+1}^2 = 1, \epsilon'_\sigma(i) = -1$ or $f_{i,\ell+1}^2 = 0, \epsilon'_\sigma(i) = 1$. Both are not possible as $0 < f_{i,\ell+1}^2 < 1$. Therefore, $\hat{\sigma}$ has no fixed point in $\mathrm{irr}(\mathcal{C})$.

Let $\sigma(\zeta_p) = \zeta_p^v$. Then $\mathrm{Tr}(D_{\rho_0}(\sigma)) = \mathrm{Tr}(\rho_0(\mathfrak{t}^v\mathfrak{s}\mathfrak{t}^u\mathfrak{s}\mathfrak{t}^v\mathfrak{s}^{-1})) = -1$ (cf. [18]), where $uv \equiv 1 \mod p$. It follows from Proposition 3.12, $|\,\mathrm{Inv}_{\mathcal{C}}(\sigma)| \geqslant |\,\mathrm{Tr}(D_\rho(\sigma))| = \ell - 1$. Therefore, $\ell = 1$. $\square$

3.7. **MD representations with multiplicities.** In this subsection, we investigate the MD representations $\rho \cong \rho_1 \oplus \rho_2$ such that ρ_1, ρ_2 are non-degenerate, symmetric, and their $\mathfrak{t}$-spectrums have nonempty intersection.

Theorem 3.23. *Let ρ_1, ρ_2 be non-degenerate symmetric representations of $\mathrm{SL}_2(\mathbb{Z})$ such that the intersection of their $\mathfrak{t}$-spectra is of size $l \geqslant 1$. Let $\dim\rho_1 = l+k$ and $\dim\rho_2 = l+m$ and suppose $k, m \geqslant 1$. Let $\rho_1(\mathfrak{s}) = [\psi_{ij}]$, $\rho_1(\mathfrak{t}) = \mathrm{diag}(\alpha_1, \ldots, \alpha_{k+l})$, $\rho_2(\mathfrak{s}) = [\eta_{ij}]$ and $\rho_2(\mathfrak{t}) = \mathrm{diag}(\beta_1, \cdots, \beta_{m+l})$ with $\alpha_i = \beta_i$ for $i = 1, \ldots, l$. Suppose $\rho_1 \oplus \rho_2$ is equivalent to an $\mathrm{SL}_2(\mathbb{Z})$ representation ρ of a modular tensor category $\mathcal{C}$. Then*

(i) *there exists a signed diagonal matrix V and 2×2 orthogonal matrices $U_i = \begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix}$ with $a_i \geqslant 0$ $(i = 1, \ldots, l)$ such that*

$$\rho(\mathfrak{s}) = V\left[\begin{array}{c|c|c} A & B^\top & C^\top \\ \hline B & \psi' & 0 \\ \hline C & 0 & \eta' \end{array}\right]V \text{ and } \rho(\mathfrak{t}) = \mathrm{diag}(\alpha_1 I_2, \ldots, \alpha_l I_2, \alpha_{l+1}, \ldots, \alpha_{l+k}, \beta_{l+1}, \ldots, \beta_{l+m}),$$

where A, B and C are block matrices with

$$A_{ij} = U_i\begin{bmatrix} \psi_{ij} & 0 \\ 0 & \eta_{ij} \end{bmatrix}U_j^\top, \quad B_{i'j} = [\psi_{l+i',j}\ 0]U_j^\top \quad \text{and} \quad C_{i''j} = [0\ \eta_{l+i'',j}]U_j^\top,$$

$1 \leqslant i, j \leqslant l$, $1 \leqslant i' \leqslant k$ and $1 \leqslant i'' \leqslant m$, and ψ', η' are respectively the $k \times k$ and the $m \times m$ bottom diagonal blocks of $\rho_1(\mathfrak{s})$ and $\rho_2(\mathfrak{s})$, i.e.,

$$\rho_1(\mathfrak{s}) = \left[\begin{array}{c|c} * & * \\ \hline * & \psi' \end{array}\right] \quad \text{and} \quad \rho_2(\mathfrak{s}) = \left[\begin{array}{c|c} * & * \\ \hline * & \eta' \end{array}\right].$$

(ii) *Let* $(e_1, \dots, e_{2l+m+k})$ *be the standard basis for* ρ *which is identified with* $\operatorname{irr}(\mathcal{C})$. *Then the unit object* $\mathbb{1}$ *of* $\mathcal{C}$ *is* e_{2u-1} *or* e_{2u} *for some* $u \leqslant l$. *In this case,*

(a) $\psi_{uu} + \eta_{uu} \neq 0$, $\dim(\mathcal{C}) = \frac{4}{|\psi_{uu}+\eta_{uu}|^2}$ *and the modular data of* $\mathcal{C}$ *is given by*

$$S = \frac{2}{\psi_{u,u} + \eta_{u,u}}\rho(\mathfrak{s}) \quad \textit{and} \quad T = \alpha_u^{-1}\rho(\mathfrak{t})\,. \tag{3.17}$$

In particular, the $(2u-1)$*-th, the* $2u$*-th rows of the* S*-matrix have the following form up to signs:*

$$\begin{matrix} \cdots & 1 & \frac{\psi_{uu}-\eta_{uu}}{\psi_{uu}+\eta_{uu}} & \cdots & \frac{\sqrt{2}\psi_{u,l+1}}{\psi_{uu}+\eta_{uu}} & \cdots & \frac{\sqrt{2}\psi_{u,l+k}}{\psi_{uu}+\eta_{uu}} & \frac{\sqrt{2}\eta_{u,l+1}}{\psi_{uu}+\eta_{uu}} & \cdots & \frac{\sqrt{2}\eta_{u,l+m}}{\psi_{uu}+\eta_{uu}} \\ \cdots & \frac{\psi_{uu}-\eta_{uu}}{\psi_{uu}+\eta_{uu}} & 1 & \cdots & \frac{\sqrt{2}\psi_{u,l+1}}{\psi_{uu}+\eta_{uu}} & \cdots & \frac{\sqrt{2}\psi_{u,l+k}}{\psi_{uu}+\eta_{uu}} & \frac{\sqrt{2}\eta_{u,l+1}}{\psi_{uu}+\eta_{uu}} & \cdots & \frac{\sqrt{2}\eta_{u,l+m}}{\psi_{uu}+\eta_{uu}} \end{matrix}\,. \tag{3.18}$$

(b) $\frac{\psi_{uu}-\eta_{uu}}{\psi_{uu}+\eta_{uu}} \in \{\pm\dim(e_{2u-1}), \pm\dim(e_{2u})\}$, *and the dimensions of* $e_{2l+1}, \dots, e_{2l+k+m}$, *up to some signs, are respectively given by*

$$\frac{\sqrt{2}\psi_{u,l+1}}{\psi_{uu}+\eta_{uu}}, \dots, \frac{\sqrt{2}\psi_{u,l+k}}{\psi_{uu}+\eta_{uu}}, \frac{\sqrt{2}\eta_{u,l+1}}{\psi_{uu}+\eta_{uu}}, \dots, \frac{\sqrt{2}\eta_{u,l+m}}{\psi_{uu}+\eta_{uu}}\,.$$

Hence, these numbers are real nonzero cyclotomic integers in $\mathbb{Z}[\zeta_N]$ *where* $N = \operatorname{ord}(T)$. *Moreover,* $\frac{\psi_{uu}-\eta_{uu}}{\psi_{uu}+\eta_{uu}} \in \mathbb{Z}[\zeta_N]$ *is a unit.*

(c) $\frac{\sqrt{2}\psi_{i,l+i'}}{\psi_{u,l+i'}}, \frac{\sqrt{2}\eta_{i,l+i''}}{\eta_{u,l+i''}} \in \mathbb{Z}[\zeta_N]$ *for* $l < i$, $1 \leqslant i' \leqslant k$, $1 \leqslant i'' \leqslant m$.

(iii) *If* ρ_1 *and* ρ_2 *are irreducible, then* ρ_1 *and* ρ_2 *must have the same parity and* $\mathcal{C}$ *is self-dual.*

Proof. We first obtain a representation $\tilde{\rho}$ by conjugating $\rho_1 \oplus \rho_2$ with a permutation matrix so that

$$\tilde{\rho}(\mathfrak{t}) = \operatorname{diag}(\alpha_1 I_2, \dots, \alpha_l I_2, \alpha_{l+1}, \dots, \alpha_{l+k}, \beta_{l+1}, \dots, \beta_{l+k}) \text{ and } \tilde{\rho}(\mathfrak{s}) = \left[\begin{array}{c|c|c} \tilde{A} & \tilde{B}^\top & \tilde{C}^\top \\ \hline \tilde{B} & \psi' & 0 \\ \hline \tilde{C} & 0 & \eta' \end{array}\right]$$

where I_2 is the 2×2 identity matrix, and $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are block matrices given by

$$\tilde{A}_{ij} = \begin{bmatrix} \psi_{ij} & 0 \\ 0 & \eta_{ij} \end{bmatrix}, \quad \tilde{B}_{i'j} = [\psi_{l+i',j}\ 0] \quad \text{and} \quad \tilde{C}_{i''j} = [0\ \eta_{l+i'',j}]$$

with $1 \leqslant i, j \leqslant l$, $1 \leqslant i' \leqslant k$ and $1 \leqslant i'' \leqslant m$, Suppose there exists an MD representation ρ of a modular tensor category $\mathcal{C}$ such that $\rho \cong \rho_1 \oplus \rho_2$. Then $\rho \cong \tilde{\rho}$ and we may assume $\rho(\mathfrak{t}) = \tilde{\rho}(\mathfrak{t})$ by conjugating a permutation matrix to ρ. According to Theorem 3.4, there exists a block diagonal orthogonal matrix U of the form

$$U = \operatorname{diag}(U_1, \dots, U_l, \gamma_{2l+1}, \dots, \gamma_{2l+m+k})$$

such that $\rho(\mathfrak{s}) = U\tilde{\rho}(\mathfrak{s})U^\top$ and $\rho(\mathfrak{t}) = \tilde{\rho}(\mathfrak{t})$, where $\gamma_j = \pm 1$ and U_i is a 2×2 orthogonal matrix for $i = 1, \dots, l$ and $j = 2l+1, \dots, 2l+k+m$. We can always write $U_i = V_i \begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix}$ where $a_i^2 + b_i^2 = 1$, $a_i \geqslant 0$ and V_i a signed diagonal matrix. Now, we set $V = \operatorname{diag}(V_1, \dots, V_l, \gamma_{2l+1}, \dots, \gamma_{2l+k+m})$. Then statement (i) follows.

The standard basis $(e_1, \dots, e_{2l+k+m})$ is now identified with $\operatorname{irr}(\mathcal{C})$. Since only the first $2l$ rows of $\rho(\mathfrak{s})$ may not contain any zero entries, the unit object $\mathbb{1}$ can only be e_x with $1 \leqslant x \leqslant 2l$. Let $u = \lceil x/2 \rceil$, the least integer $\geqslant u/2$. Then,

$$T = \alpha_u^{-1}\operatorname{diag}(\alpha_1 I_2, \dots, \alpha_l I_2, \alpha_{l+1}, \dots, \alpha_{k+l}, \beta_{l+1}, \dots, \beta_{l+m})$$

and the $(2u-1)$-th and $2u$-th rows of $\rho(\mathfrak{s})$ are given by

$$A_{u,i}=\begin{bmatrix} a_ua_j\psi_{uj}+b_ub_j\eta_{uj} & a_ub_j\psi_{uj}-b_ua_j\eta_{uj}\\ b_ua_j\psi_{uj}-a_ub_j\eta_{uj} & b_ub_j\psi_{uj}+a_ua_j\eta_{uj}\end{bmatrix},\quad (B^\top)_{u,i'}=\psi_{u,l+i'}\begin{bmatrix}a_u\\ b_u\end{bmatrix},\quad (C^\top)_{u,i''}=\eta_{u,l+i''}\begin{bmatrix}-b_u\\ a_u\end{bmatrix}.$$

Since $e_x = \mathbb{1}$ and $x \in \{2u-1, 2u\}$, $a_u, b_u, \psi_{u,l+i'}$ and $\eta_{u,l+i''}$ are non-zero for $1 \leqslant i' \leqslant k$ and $1 \leqslant i'' \leqslant m$.

Now, we assume $x = 2u-1$. Then, by [26],

$$\frac{\rho(\mathfrak{s})_{2u,2l+i'}}{\rho(\mathfrak{s})_{2u-1,2l+i'}} = \frac{b_u\psi_{u,l+i'}}{a_u\psi_{u,l+i'}a_u} = \frac{b_u}{a_u} \quad\text{and}\quad \frac{\rho(\mathfrak{s})_{2u,2l+k+i''}}{\rho(\mathfrak{s})_{2u-1,2l+k+i''}} = \frac{-a_u\eta_{u,l+i''}}{b_u\eta_{u,l+i''}} = \frac{-a_u}{b_u} \in \mathbb{Z}[\zeta_N]$$

where $N = \operatorname{ord}(T)$. Therefore, $\frac{a_u}{b_u}$ is a unit in $\mathbb{Z}[\zeta_N]$. According to [29], both $\operatorname{spec}(\rho_1(\mathfrak{t}))$ and $\operatorname{spec}(\rho_2(\mathfrak{t}))$ are closed under the action of σ^2 for any $\sigma \in \operatorname{Gal}(\bar{\mathbb{Q}})$. Therefore, the subsets

$$\{\alpha_{l+1},\dots,\alpha_{l+k}\} \subset \operatorname{spec}(\rho_1(\mathfrak{t})) \quad\text{and}\quad \{\beta_{l+1},\dots,\beta_{l+m}\} \subset \operatorname{spec}(\rho_2(\mathfrak{t}))$$

are closed under σ^2 for all $\sigma \in \operatorname{Gal}(\bar{\mathbb{Q}})$. Thus, $\{2l+1,\dots,2l+k\}$ and $\{2l+k+1,\dots,2l+k+m\}$ are both closed under the action of $\hat{\sigma}$ for $\sigma \in \operatorname{Gal}(\bar{\mathbb{Q}})$. In particular, for $\sigma \in \operatorname{Gal}(\bar{\mathbb{Q}})$, $\hat{\sigma}(2l+1) = 2l+i'$ for some positive integer $i' \leqslant k$. Hence,

$$\sigma\left(\frac{b_u}{a_u}\right) = \sigma\left(\frac{\rho(\mathfrak{s})_{2u,2l+1}}{\rho(\mathfrak{s})_{2u-1,2l+1}}\right) = \frac{\rho(\mathfrak{s})_{2u,\hat{\sigma}(2l+1)}}{\rho(\mathfrak{s})_{2u-1,\hat{\sigma}(2l+1)}} = \frac{\rho(\mathfrak{s})_{2u,2l+i'}}{\rho(\mathfrak{s})_{2u-1,2l+i'}} = \frac{b_u}{a_u}.$$

So, $b_u/a_u \in \mathbb{Q}$ and hence $b_u/a_u = \pm 1 = \epsilon_u$. Since $a_u^2 + b_u^2 = 1$, we have $a_u = \frac{1}{\sqrt{2}}$. This implies that

$$A_{u,u}=\frac{1}{2}\begin{bmatrix}\psi_{uu}+\eta_{uu} & \epsilon_u(\psi_{uu}-\eta_{uu})\\ \epsilon_u(\psi_{uu}-\eta_{uu}) & \psi_{uu}+\eta_{uu}\end{bmatrix},\quad (B^\top)_{u,i'}=\frac{1}{\sqrt{2}}\psi_{u,l+i'}\begin{bmatrix}1\\ \epsilon_u\end{bmatrix},\quad (C^\top)_{u,i''}=\frac{1}{\sqrt{2}}\eta_{u,l+i''}\begin{bmatrix}-\epsilon_u\\ 1\end{bmatrix}.$$

In particular, $\frac{\zeta_4^i}{D} = \frac{\psi_{uu}+\eta_{uu}}{2}$. Therefore, $\psi_{uu}+\eta_{uu} \neq 0$ and so the S-matrix (3.17) of $\mathcal{C}$ is then obtained. In particular, the $(2u-1)$-th and $2u$-th rows of S are displayed in (3.18). Thus, the dimensions of $e_{2u}, e_{2l+1}, \dots, e_{2l+k+m}$, up to some signs, are respectively given by

$$\frac{\psi_{uu}-\eta_{uu}}{\psi_{uu}+\eta_{uu}}, \frac{\sqrt{2}\psi_{u,l+1}}{\psi_{uu}+\eta_{uu}}, \dots, \frac{\sqrt{2}\psi_{u,l+k}}{\psi_{uu}+\eta_{uu}}, \frac{\sqrt{2}\eta_{u,l+1}}{\psi_{uu}+\eta_{uu}}, \dots, \frac{\sqrt{2}\eta_{u,l+m}}{\psi_{uu}+\eta_{uu}},$$

which are non-zero real numbers in $\mathbb{Z}[\zeta_N]$.

Now, the global dimension

$$\dim(\mathcal{C}) = \frac{\pm 4}{(\psi_{uu}+\eta_{uu})^2} \in \mathbb{R}^\times \cap \mathbb{Z}[\zeta_N].$$

It follows from [26] that $\frac{\rho(\mathfrak{s})_{y,z}}{\rho(\mathfrak{s})_{2u-1,z}} \in \mathbb{Z}[\zeta_N]$ for any $y, z = 1, \dots, 2l+k+m$. For $y = z = 2u$, we find

$$\frac{\psi_{uu}+\eta_{uu}}{\psi_{uu}-\eta_{uu}} \in \mathbb{Z}[\zeta_N],$$

and so $\frac{\psi_{uu}-\eta_{uu}}{\psi_{uu}+\eta_{uu}}$ is a real unit in $\mathbb{Z}[\zeta_N]$. For $y, z > 2l$, we find

$$\frac{\sqrt{2}\psi_{i,l+i'}}{\psi_{u,l+i'}} \quad\text{and}\quad \frac{\sqrt{2}\eta_{i,l+i''}}{\eta_{u,l+i''}} \in \mathbb{Z}[\zeta_N].$$

for $i > l$, $1 \leqslant i' \leqslant k$, $1 \leqslant i'' \leqslant m$. This completes the case for $x = 2u-1$.

One can follow the same argument for the case when $x = 2u$. However, the conclusions are identical to the case $x = 2u-1$. Therefore, the proof of statement (ii) is completed.

(iii). Assume the contrary. Then ρ_1, ρ_2 are irreducible representations with opposite parities. Thus, $|\operatorname{Tr}(\rho(\mathfrak{s})^2)| = |k-m|$, which is the number of self-dual objects in $\operatorname{irr}(\mathcal{C})$. Since $\rho(\mathfrak{t})$ has $m+k$

eigenvalues of multiplicity 1, the number of self-dual objects in $\operatorname{irr}(\mathcal{C})$ is at least $m+k$ which is greater than $|k-m|$, a contradiction. The proof of statement (iii) is completed. □

As a consequence of the preceding theorem, two non-degenerate irreducible representations with opposite parities will never satisfy the conditions of the theorem. However, we can solve the modular data if the $\mathfrak{t}$-spectrum of ρ_2 is subset of that of ρ_1.

Theorem 3.24. *Let ρ_1, ρ_2 be non-degenerate symmetric representations of $\mathrm{SL}_2(\mathbb{Z})$such that*

$$\operatorname{spec}(\rho_2(\mathfrak{t})) \subsetneq \operatorname{spec}(\rho_1(\mathfrak{t})).$$

Let $l+k = \dim\rho_1$ and $l = \dim\rho_2$, $\rho_1(\mathfrak{s}) = [\psi_{ij}]$, $\rho_1(\mathfrak{t}) = \operatorname{diag}(\alpha_1, \ldots, \alpha_{k+l})$, $\rho_2(\mathfrak{s}) = [\eta_{ij}]$, $\rho_2(\mathfrak{t}) = \operatorname{diag}(\alpha_1, \ldots, \alpha_l)$. Suppose $\rho_1 \oplus \rho_2$ is equivalent to an $\mathrm{SL}_2(\mathbb{Z})$ representation ρ of a modular tensor category $\mathcal{C}$. Then

(i) *there exists a signed diagonal matrix V and 2×2 orthogonal matrices $U_i = \begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix}$ with $a_i^2 + b_i^2 = 1$ and $a_i \geqslant 0$ $(i = 1, \ldots, l)$ such that*

$$\rho(\mathfrak{s}) = V \left[\begin{array}{c|c} A & B^\top \\ \hline B & \psi' \end{array}\right] V \quad \text{and} \quad \rho(\mathfrak{t}) = \operatorname{diag}(\alpha_1 I_2, \ldots, \alpha_l I_2, \alpha_{l+1}, \ldots, \alpha_{k+l}),$$

where ψ' is the $k\times k$ lower right corner block of $\rho_1(\mathfrak{s})$ and A, B are block matrices given by

$$A_{ij} = U_i \begin{bmatrix} \psi_{ij} & 0 \\ 0 & \eta_{ij} \end{bmatrix} U_j^\top, \quad B_{i'j} = [\psi_{l+i',j}\ 0]U_j^\top,$$

for $1 \leqslant i,j \leqslant l$ and $1 \leqslant j' \leqslant k$.

(ii) *Suppose ρ_1 and ρ_2 have opposite parities. We identify the standard basis $(e_1, \ldots, e_{2l+k})$ of ρ with $\operatorname{irr}(\mathcal{C})$. Then*

(a) *e_{2i-1} and e_{2i} form a dual pair for $i = 1, \ldots, l$.*

(b) *The unit object $\mathbb{1}$ can only be e_{2l+u} with $1 \leqslant u \leqslant k$ such that $\psi_{l+u,l+u} \neq 0$ and*

$$\dim(\mathcal{C}) = |\psi_{l+u,l+u}|^{-2}, \quad \dim(e_{2i-1}) = \dim(e_{2i}) = \frac{\pm\psi_{i,l+u}}{\sqrt{2}\psi_{l+u,l+u}}, \quad \dim(e_j) = \frac{\pm\psi_{j,l+u}}{\psi_{l+u,l+u}}$$

for $i = 1, \ldots, l$ and $j = l+1, \ldots, l+k$. In particular, they are elements of $\mathbb{Z}[\zeta_N] \cap \mathbb{R}^\times$ where N is the order of $T = \alpha_{l+u}^{-1}\rho(\mathfrak{t})$, and the S-matrix of $\mathcal{C}$ is given by

$$S = \psi_{l+u,l+u}^{-1} V' \left[\begin{array}{c|c} A' & B'^\top \\ \hline B' & \psi' \end{array}\right] V' \tag{3.19}$$

for some signed diagonal matrix V' and block matrices A', B' given by

$$A'_{ij} = \begin{bmatrix} \frac{\psi_{i,j}+\epsilon_i\epsilon_j\eta_{i,j}}{2} & \frac{\psi_{i,j}-\epsilon_i\epsilon_j\eta_{i,j}}{2} \\ \frac{\psi_{i,j}-\epsilon_i\epsilon_j\eta_{i,j}}{2} & \frac{\psi_{i,j}+\epsilon_i\epsilon_j\eta_{i,j}}{2} \end{bmatrix} \quad \text{and} \quad B'_{i',j} = \frac{\psi_{l+i',j}}{\sqrt{2}}[1\ 1]$$

where $\epsilon_j = \pm 1$, $1 \leqslant i,j \leqslant l$ and $1 \leqslant i' \leqslant k$.

Proof. By conjugating a permutation matrix to $\rho_1 \oplus \rho_2$, we can obtain an equivalent representation $\tilde\rho$ given by

$$\tilde\rho(\mathfrak{s}) = \left[\begin{array}{c|c} \tilde A & \tilde B^\top \\ \hline \tilde B & \psi' \end{array}\right] \quad \text{and} \quad \tilde\rho(\mathfrak{t}) = \operatorname{diag}(\alpha_1 I_2, \ldots, \alpha_l I_2, \alpha_{l+1}, \ldots, \alpha_{k+l}),$$

where ψ' is the $k \times k$ bottom diagonal block of $\rho_1(\mathfrak{s})$, and $\tilde{A}$, $\tilde{B}$ are block matrices with

$$\tilde{A}_{ij} = \begin{bmatrix} \psi_{ij} & 0 \\ 0 & \eta_{ij} \end{bmatrix}, \quad \tilde{B}_{i',j} = [\psi_{l+i',j}\ 0]$$

for $1 \leqslant i, j \leqslant l$ and $1 \leqslant j' \leqslant k$. By Theorem 3.4, there exists an orthogonal matrix $U = \operatorname{diag}(U_1, \ldots, U_l, \gamma_{2l+1}, \ldots, \gamma_{2l+1})$ such that $\rho(\mathfrak{s}) = U\tilde{\rho}(\mathfrak{s})U^\top$ and $\rho(\mathfrak{t}) = \tilde{\rho}(\mathfrak{t})$ where $\gamma_j = \pm 1$ and U_i is a 2×2 orthogonal matrix for $i = 1, \ldots, l$ and $j = 2l+1, \ldots, 2l+k$. As before, we write $U_i = V_i \begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix}$ where $a_i^2 + b_i^2 = 1$, $a_i \geqslant 0$ and V_i a signed diagonal matrix. Now, we set $V = \operatorname{diag}(V_1, \ldots, V_l, \gamma_{2l+1}, \ldots, \gamma_{2l+k})$, and statement (i) follows immediately.

(ii). Now we assume ρ_1 and ρ_2 are of opposite parities. Then $|\operatorname{Tr}(\rho(\mathfrak{s})^2)| = k$ and so there are exactly k self-dual simple objects in $\operatorname{irr}(\mathcal{C})$ and l dual pairs. Since e_{2i-1} and e_{2i} give rise to the same eigenvalue of $\rho(\mathfrak{t})$ for $i = 1, \ldots, l$, and $\rho(\mathfrak{t})_{2i,2i} \neq \rho(\mathfrak{t})_{j,j}$ for $j \notin \{2i-1, 2i\}$, they must form a dual pair. Since the unit object $\mathbb{1}$ is self-dual, $\mathbb{1} = e_{2l+u}$ for some positive integer $u \leqslant k$, and so $1/\sqrt{\dim(\mathcal{C})}$, up to a 4-th root, is $\rho(\mathfrak{s})_{2l+u,2l+u} = \psi_{l+u,l+u}$. In particular, $\psi_{l+u,l+u} \neq 0$, $\dim(\mathcal{C}) = |\psi_{l+u,l+u}|^{-2}$ and $\psi_{l+u,l+u}^{-2} \in \mathbb{Z}[\zeta_N] \cap \mathbb{R}^\times$, where N is the order of $T = \alpha_{l+u}^{-1}\rho(\mathfrak{t})$. By (i),

$$S = \psi_{l+u,l+u}^{-1} V \left[\begin{array}{c|c} A & B^\top \\ \hline B & \psi' \end{array}\right] V,$$

where A, B are block matrices given by

$$A_{ij} = \begin{bmatrix} \frac{a_i a_j \psi_{i,j} + b_i b_j \eta_{i,j}}{2} & \frac{a_i b_j \psi_{i,j} - a_j b_i \eta_{i,j}}{2} \\ \frac{a_j b_i \psi_{i,j} - a_i b_j \eta_{i,j}}{2} & \frac{b_i b_j \psi_{i,j} + a_i a_j \eta_{i,j}}{2} \end{bmatrix} \quad \text{and} \quad B_{i',j} = \psi_{l+i',j}[a_j\ b_j]\,.$$

Thus, the dimensions of e_{2j-1} and e_{2j} are respectively given by

$$\frac{\psi_{l+u,j} a_j}{\psi_{l+u,l+u}} \quad \text{and} \quad \frac{-\psi_{l+u,j} b_j}{\psi_{l+u,l+u}}$$

which implies $\pm a_j = b_j$. Since $a_j^2 + b_j^2 = 1$ and $a_j \geqslant 0$, we have $a_j = \frac{1}{\sqrt{2}}$ and $b_j = \frac{\epsilon_j}{\sqrt{2}}$ for some $\epsilon_j = \pm 1$ $(j = 1, \ldots, l)$. Therefore, $A_{ij} = \begin{bmatrix} \frac{\psi_{i,j} + \epsilon_i \epsilon_j \eta_{i,j}}{2} & \frac{\epsilon_j \psi_{i,j} - \epsilon_i \eta_{i,j}}{2} \\ \frac{\epsilon_i \psi_{i,j} - \epsilon_j \eta_{i,j}}{2} & \frac{\epsilon_i \epsilon_j \psi_{i,j} + \eta_{i,j}}{2} \end{bmatrix}$ and $B_{i',j} = \frac{\psi_{l+i',j}}{\sqrt{2}}[1\ \epsilon_j]$. Let $E_j = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon_j \end{bmatrix}$ for $j = 1, \ldots, l$. Then

$$A_{ij} = E_i A'_{ij} E_j \quad \text{and} \quad B_{i'j} = B'_{i'j} E_j$$

and the expression (3.19) of the S-matrix follows immediately by setting $V' = VE$ where $E = \operatorname{diag}(E_1, \ldots, E_l, 1, \ldots, 1)$. Moreover, $\dim(e_{2j-1}) = \dim(e_{2j}) = \frac{\pm\psi_{l+u,j}}{\sqrt{2}\psi_{l+u,l+u}}$ for $j = 1, \ldots, l$, and $\dim(e_{2l+i'}) = \frac{\pm\psi_{l+i',l+u}}{\psi_{l+u,l+u}}$ for $1 \leqslant i' \leqslant k$. It follows from [24] that they are elements of $\mathbb{Z}[\zeta_N] \cap \mathbb{R}^\times$. This completes the proof of statement (ii). □

4. Classification of modular data of rank=6: admissible types

In this section, we prove that admissible types of MDs that can be realized by some rank=6 MTCs include $(4,1,1)$, $(4,2)$, $(3,3)$, and $(3,2,1)$.

Definition 4.1. *Let (S, T) be a modular data. Denote by ι the object (label) corresponding to the column of the S-matrix that is a multiple of the column of FP-dimensions.*

4.1. **Classification of modular data of type (4,1,1).** Recall that $SO(8)_3 \cong PSO(8)_3 \boxtimes SO(8)_1$ as modular tensor categories, which defines the notation $PSO(8)_3$. Alternatively, the modular data of $PSO(8)_3$ can be obtained from $SU(3)_6$ via boson condensation [34]. We will prove in this section that the Galois conjugates of the modular data of $PSO(8)_3$ are characterized by the MTCs of type (4,1,1).

Theorem 4.2. *Let $\mathcal{C}$ be a rank 6 modular tensor category of type $(4,1,1)$. Then the modular data of $\mathcal{C}$ is a Galois conjugate of $PSO(8)_3$.*

Let $\mathcal{C}$ be an MTC of type $(4,1,1)$, and ρ an $\mathrm{SL}_2(\mathbb{Z})$ representation of $\mathcal{C}$. Then ρ admits an irreducible decomposition $\rho_0 \oplus \rho_1 \oplus \rho_2$ in which $\dim \rho_0, \dim \rho_1, \dim \rho_2$ respectively 4,1,1. By tensoring a suitable 1-dimensional representation of $\mathrm{SL}_2(\mathbb{Z})$, we will assume ρ_0 has a minimal $\mathfrak{t}$-spectrum.

In particular, all the 4-dimensional irreducible representations of level 6 are even. Now, can prove

Lemma 4.3. *$\mathcal{C}$ is self-dual, ρ_0 must be even of level 9, and $\rho \cong \rho_0 \oplus 2\chi_0$.*

Proof. From Appendix A, 4-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{Z})$ with minimal $\mathfrak{t}$-spectrums appear at the levels 5, 6, 7, 8, 9, 10, 12, 15, 20, 24 and 40. The $\mathfrak{t}$-spectrums of those 4-dimensional irreducible representations of levels 5, 8, 10, 15, 20, 24 and 40 do not contain any 12-th root of unity. It follows from Lemma 3.20 that ρ_0 cannot be of any of these levels.

It remains to show that the level of ρ_0 cannot be 6, 7 or 12. Suppose ρ_0 has level 7. Then $\mathcal{C}$ is of type (4,1,1), which contradicts Proposition 3.22. Therefore, the level of ρ_0 cannot be 7.

Suppose ρ_0 has level 6 or 12. Since there is no 4-dimensional irreducible representation of levels 2, 3 or 4 in the tables of Appendix A, ρ_0 must be projectively equivalent to a tensor product of two 2-dimensional representations, namely $\rho_{2_2^{1,0}} \otimes \rho_{2_3^{1,0}}$ or $\rho_{2_4^{1,0}} \otimes \rho_{2_3^{1,0}}$. However, $\rho_{2_2^{1,0}}$ and $\rho_{2_4^{1,0}}$ are projectively equivalent, hence so are $\rho_{2_2^{1,0}} \otimes \rho_{2_3^{1,0}}$ and $\rho_{2_4^{1,0}} \otimes \rho_{2_3^{1,0}}$. So ρ_0 is projectively equivalent to $\rho_{2_2^{1,0}} \otimes \rho_{2_3^{1,0}}$, which has a minimal $\mathfrak{t}$-spectrums $\{1, -1, \zeta_3, -\zeta_3\}$. Therefore, $\rho_0 \cong \rho_{2_2^{1,0}} \otimes \rho_{2_3^{1,0}}$.

By Lemma 3.20, the levels of ρ_1 and ρ_2 are divisors of 6, and so is the level of ρ. Therefore, $\mathrm{ord}(T)|6$ and hence $\mathcal{C}$ is integral by Theorem 3.14. It follows from Proposition 3.16 that $\mathcal{C}$ is of type (4,2), a contradiction. Therefore, the level of ρ_0 cannot be 6 or 12.

As a consequence, ρ_0 must have level 9, and $\rho \cong \rho_0 \oplus 2\chi_0$ by Lemma 3.20 since 1 is the unique eigenvalue of $\rho(\mathfrak{t})$ with order dividing 12. It follows from Corollary 3.21 that $\rho_0(\mathfrak{s}^2) = \mathrm{id}$ and $\mathcal{C}$ is self-dual. □

4.1.1. *Solving modular data of type (4,1,1).* By Appendix A, there is only one Galois orbit of 4-dimensional irreducible representations of level 9 which is even. This Galois orbit has two projectively equivalent classes given by $\rho_{4_{9,1}^{1,0}}$ and $\rho_{4_{9,1}^{8,0}}$ which are complex conjugate of each other. First, we consider $\rho_0 = \rho_{4_{9,1}^{1,0}}$.

Let $z_1 = c_9^2$, $z_2 = c_9^4$ and $z_3 = c_9^1$ where $c_n^m := \zeta_n^m + \zeta_n^{-m}$. Then

$$\rho_0(\mathfrak{s}) = \frac{1}{3}\begin{bmatrix} 0 & -\sqrt{3} & -\sqrt{3} & -\sqrt{3} \\ -\sqrt{3} & z_1 & z_2 & z_3 \\ -\sqrt{3} & z_2 & z_3 & z_1 \\ -\sqrt{3} & z_3 & z_1 & z_2 \end{bmatrix}, \quad \rho_0(\mathfrak{t}) = \mathrm{diag}(1, \zeta_9, \zeta_9^4, \zeta_9^7)\,.$$

Let $\tilde{\rho} = 2\chi_0 \oplus \rho_0$ and set $s := \rho(\mathfrak{s})$ and $t := \rho(\mathfrak{t})$. By reordering $\mathrm{irr}(\mathcal{C})$, one can assume

$$\rho(\mathfrak{t}) = \tilde{\rho}(\mathfrak{t}) = \mathrm{diag}(1, 1, 1, \zeta_9, \zeta_9^4, \zeta_9^7).$$

By Theorem 3.4, there exists $U \in O_6(\mathbb{R})$ such that $\rho = U\tilde{\rho}U^\top$. Then $U = f \oplus V$ for some signed diagonal matrix $V = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $f \in O_3(\mathbb{R})$ where $f \oplus V$ denotes the block direct sum of f

and V. We may further assume $\varepsilon_3 = 1$, and we get

$$s = U\tilde{\rho}(\mathfrak{s})U^\top = \begin{bmatrix} f_{11}^2 + f_{12}^2 & f_{11}f_{21} + f_{12}f_{22} & f_{11}f_{31} + f_{12}f_{32} & \frac{\varepsilon_1 f_{13}}{-\sqrt{3}} & \frac{f_{13}}{-\sqrt{3}} & \frac{f_{13}}{-\sqrt{3}} \\ f_{11}f_{21} + f_{12}f_{22} & f_{21}^2 + f_{22}^2 & f_{21}f_{31} + f_{22}f_{32} & \frac{\varepsilon_1 f_{23}}{-\sqrt{3}} & \frac{\varepsilon_2 f_{23}}{-\sqrt{3}} & \frac{f_{23}}{-\sqrt{3}} \\ f_{11}f_{31} + f_{12}f_{32} & f_{21}f_{31} + f_{22}f_{32} & f_{31}^2 + f_{32}^2 & \frac{\varepsilon_1 f_{33}}{-\sqrt{3}} & \frac{\varepsilon_2 f_{33}}{-\sqrt{3}} & \frac{f_{33}}{-\sqrt{3}} \\ \frac{\varepsilon_1 f_{13}}{-\sqrt{3}} & \frac{\varepsilon_1 f_{23}}{-\sqrt{3}} & \frac{\varepsilon_1 f_{33}}{-\sqrt{3}} & \frac{z_1}{3} & \frac{\varepsilon_1\varepsilon_2 z_2}{3} & \frac{\varepsilon_1 z_3}{3} \\ \frac{\varepsilon_2 f_{13}}{-\sqrt{3}} & \frac{\varepsilon_2 f_{23}}{-\sqrt{3}} & \frac{\varepsilon_2 f_{33}}{-\sqrt{3}} & \frac{\varepsilon_1\varepsilon_2 z_2}{3} & \frac{z_3}{3} & \frac{\varepsilon_2 z_1}{3} \\ \frac{f_{13}}{-\sqrt{3}} & \frac{f_{23}}{-\sqrt{3}} & \frac{f_{33}}{-\sqrt{3}} & \frac{\varepsilon_1 z_3}{3} & \frac{\varepsilon_2 z_1}{3} & \frac{z_2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 - f_{13}^2 & -f_{13}f_{23} & -f_{13}f_{33} & \frac{\varepsilon_1 f_{13}}{-\sqrt{3}} & \frac{\varepsilon_2 f_{13}}{-\sqrt{3}} & \frac{f_{13}}{-\sqrt{3}} \\ -f_{13}f_{23} & 1 - f_{23}^2 & -f_{23}f_{33} & \frac{\varepsilon_1 f_{23}}{-\sqrt{3}} & \frac{\varepsilon_2 f_{23}}{-\sqrt{3}} & \frac{f_{23}}{-\sqrt{3}} \\ -f_{13}f_{33} & -f_{23}f_{34} & 1 - f_{33}^2 & \frac{\varepsilon_1 f_{33}}{-\sqrt{3}} & \frac{\varepsilon_2 f_{33}}{-\sqrt{3}} & \frac{f_{33}}{-\sqrt{3}} \\ \frac{\varepsilon_1 f_{13}}{-\sqrt{3}} & \frac{\varepsilon_1 f_{23}}{-\sqrt{3}} & \frac{\varepsilon_1 f_{33}}{-\sqrt{3}} & \frac{z_1}{3} & \frac{\varepsilon_1\varepsilon_2 z_2}{3} & \frac{\varepsilon_1 z_3}{3} \\ \frac{\varepsilon_2 f_{13}}{-\sqrt{3}} & \frac{\varepsilon_2 f_{23}}{-\sqrt{3}} & \frac{\varepsilon_2 f_{33}}{-\sqrt{3}} & \frac{\varepsilon_1\varepsilon_2 z_2}{3} & \frac{z_3}{3} & \frac{\varepsilon_2 z_1}{3} \\ \frac{f_{13}}{-\sqrt{3}} & \frac{f_{23}}{-\sqrt{3}} & \frac{f_{33}}{-\sqrt{3}} & \frac{\varepsilon_1 z_3}{3} & \frac{\varepsilon_2 z_1}{3} & \frac{z_2}{3} \end{bmatrix}.$$

We now apply the Galois symmetry [11, Theorem II] of ρ to determine f and $\varepsilon_1, \varepsilon_2$ (cf. Theorem 3.7 (6)). Since $\operatorname{ord}(t) = 9$, then s is a matrix over $\mathbb{Q}_9$. The Galois group $\operatorname{Gal}(\mathbb{Q}_9/\mathbb{Q})$ is generated by σ defined by $\sigma : \zeta_9 \mapsto \zeta_9^2$, and $\hat{\sigma}$ denotes the corresponding permutation on $\operatorname{irr}(\mathcal{C}) = \{1, \ldots, 6\}$. The i-th diagonal entry of t will be denoted by t_i. Under the action of σ^2,

$$t_4 \mapsto t_5, \quad t_5 \mapsto t_6, \quad \text{and} \quad t_6 \mapsto t_4\,.$$

We find $\hat{\sigma}(4) = 5$, $\hat{\sigma}(5) = 6$ and $\hat{\sigma}(6) = 4$. Recall that $\sigma(s_{ij}) = \epsilon_\sigma(i) s_{\hat{\sigma}(i)j}$ where $\epsilon_\sigma(i) = \pm 1$. Applying σ to those s_{ij} with $i, j \in \{4, 5, 6\}$, we have

$$\sigma(z_1) = \epsilon_\sigma(4)\varepsilon_1\varepsilon_2 z_2, \quad \sigma(\varepsilon_1\varepsilon_2 z_2) = \epsilon_\sigma(5)\varepsilon_1 z_3 \quad \text{and} \quad \sigma(\varepsilon_1 z_3) = \epsilon_\sigma(6) z_1\,.$$

Since $\sigma(z_1) = z_2$, $\sigma(z_2) = z_3$ and $\sigma(z_3) = z_1$, we find

$$\epsilon_\sigma(4) = \varepsilon_1\varepsilon_2, \quad \epsilon_\sigma(5) = \varepsilon_2 \quad \text{and} \quad \epsilon_\sigma(6) = \varepsilon_1\,.$$

Now, we apply σ to those s_{ij} with $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$. We have $\sigma(\frac{f_{i3}}{\sqrt{3}}) = \frac{f_{i3}}{\sqrt{3}}$, and hence $\frac{f_{i3}}{\sqrt{3}} \in \mathbb{Q}$ for $i = 1, 2, 3$. This implies that $f_{i3}f_{j3} \in \mathbb{Q}$ for any $i, j \in \{1, 2, 3\}$. Therefore, the first 3 rows of s have rational entries, and hence $\hat{\sigma}$ fixes 1,2,3. Now, we can conclude that $\hat{\sigma} = (4, 5, 6)$.

Since $\mathcal{C}$ is not integral by Proposition 3.16, none of $1, 2$ or 3 cannot be the isomorphism class of the unit object $\mathbb{1}$ or the simple object ι for the Frobenius-Perron dimensions. Therefore, $\dim(\mathcal{C})$ and $\operatorname{FPdim}(\mathcal{C})$ are Galois conjugates, and $\operatorname{FPdim}(\mathcal{C})$ is the largest conjugate of $\dim(\mathcal{C})$. The global dimension $\dim(\mathcal{C})$ can be $9z_1^{-2}$, $9z_3^{-2}$ or $9z_2^{-2}$ depending which of the classes 4,5,6 corresponds $\mathbb{1}$. Since they are conjugates and $-z_2 > z_3 > z_1 > 0$, $\operatorname{FPdim}(\mathcal{C}) = 9z_1^{-2}$.

Let (S, T) be the modular data of $\mathcal{C}$. Note that z_1, z_2, z_3 are units, and they are roots of the irreducible polynomial $x^3 - 3x + 1$. No matter which of 4,5,6 is the isomorphism class of $\mathbb{1}$, for $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$,

$$S_{ij} = \pm\frac{\sqrt{3}f_{i3}}{z_k}$$

for some $k \in \{1, 2, 3\}$. Since S_{ij} is a cyclotomic integer, so is $\sqrt{3}f_{i3}$. Thus, $\sqrt{3}f_{i3}$ is an integer and they satisfy

$$(\sqrt{3}f_{13})^2 + (\sqrt{3}f_{23})^2 + (\sqrt{3}f_{33})^2 = 3\,.$$

Therefore, $\sqrt{3}f_{i3} = \pm 1$ or equivalently $f_{i3} = \pm\frac{1}{\sqrt{3}}$ for $i = 1, 2, 3$. Now, we can compute the modular data for the cases $\mathbb{1} = 4, 5$ or 6:

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(i) Suppose 4 is the isomorphism class of $\mathbb{1}$. Then $D = 3/z_1$ and

$$S = \begin{bmatrix} 3\frac{1-f_{13}^2}{z_1} & 3\frac{f_{13}f_{23}}{-z_1} & 3\frac{f_{13}f_{33}}{-z_1} & \frac{\varepsilon_1\sqrt{3}f_{13}}{-z_1} & \frac{\varepsilon_2\sqrt{3}f_{13}}{-z_1} & \frac{\sqrt{3}f_{13}}{-z_1} \\ 3\frac{f_{13}f_{23}}{-z_1} & 3\frac{1-f_{23}^2}{z_1} & 3\frac{f_{23}f_{33}}{-z_1} & \frac{\varepsilon_1\sqrt{3}f_{23}}{-z_1} & \frac{\varepsilon_2\sqrt{3}f_{23}}{-z_1} & \frac{\sqrt{3}f_{23}}{-z_1} \\ 3\frac{f_{13}f_{33}}{-z_1} & 3\frac{f_{23}f_{33}}{-z_1} & 3\frac{1-f_{33}^2}{z_1} & \frac{\varepsilon_1\sqrt{3}f_{33}}{-z_1} & \frac{\varepsilon_2\sqrt{3}f_{33}}{-z_1} & \frac{\sqrt{3}f_{33}}{-z_1} \\ \frac{\varepsilon_1\sqrt{3}f_{13}}{-z_1} & \frac{\varepsilon_1\sqrt{3}f_{23}}{-z_1} & \frac{\varepsilon_1\sqrt{3}f_{33}}{-z_1} & 1 & \frac{\varepsilon_1\varepsilon_2 z_2}{z_1} & \frac{\varepsilon_1 z_3}{z_1} \\ \frac{\varepsilon_2\sqrt{3}f_{13}}{-z_1} & \frac{\varepsilon_2\sqrt{3}f_{23}}{-z_1} & \frac{\varepsilon_2\sqrt{3}f_{33}}{-z_1} & \frac{\varepsilon_1\varepsilon_2 z_2}{z_1} & \frac{z_3}{z_1} & \varepsilon_2 \\ \frac{\sqrt{3}f_{13}}{-z_1} & \frac{\sqrt{3}f_{23}}{-z_1} & \frac{\sqrt{3}f_{33}}{-z_1} & \frac{\varepsilon_1 z_3}{z_1} & \varepsilon_2 & \frac{z_2}{z_1} \end{bmatrix}.$$

Note that

$$\sum_{i=1}^{6}\left(\frac{S_{i,4}}{S_{4,4}}\right)^2 = \frac{9}{z_1^2} = \mathrm{FPdim}(\mathcal{C})\,.$$

Therefore, 4 is also the isomorphism class of ι (recall Definition 4.1). In particular, $\mathcal{C}$ is pseudounitary and the entries of 4th row of S must be positive. Since $\frac{z_2}{z_1} < 0$ and $\frac{z_3}{z_1} > 0$, we have $\varepsilon_1 = 1$, $\varepsilon_2 = -1$ and $f_{i3} < 0$ for $i = 1, 2, 3$. This implies $\sqrt{3}f_{i3} = -1$ for $i = 1, 2, 3$ and

$$S = \begin{bmatrix} 2z_1^{-1} & -z_1^{-1} & -z_1^{-1} & z_1^{-1} & -z_1^{-1} & z_1^{-1} \\ -z_1^{-1} & 2z_1^{-1} & -z_1^{-1} & z_1^{-1} & -z_1^{-1} & z_1^{-1} \\ -z_1^{-1} & -z_1^{-1} & 2z_1^{-1} & z_1^{-1} & -z_1^{-1} & z_1^{-1} \\ z_1^{-1} & z_1^{-1} & z_1^{-1} & 1 & \frac{-z_2}{z_1} & \frac{z_3}{z_1} \\ -z_1^{-1} & -z_1^{-1} & -z_1^{-1} & \frac{-z_2}{z_1} & \frac{z_3}{z_1} & -1 \\ z_1^{-1} & z_1^{-1} & z_1^{-1} & \frac{z_3}{z_1} & -1 & \frac{z_2}{z_1} \end{bmatrix} \quad \text{and} \quad T = \mathrm{diag}(\zeta_9^8, \zeta_9^8, \zeta_9^8, 1, \zeta_3, \zeta_3^2)\,.$$

(ii) Suppose 5 is the isomorphism class of $\mathbb{1}$. Then $D = 3/z_3$ and hence

$$S = \begin{bmatrix} 3\frac{1-f_{13}^2}{z_3} & 3\frac{f_{13}f_{23}}{-z_3} & 3\frac{f_{13}f_{33}}{-z_3} & \frac{\varepsilon_1\sqrt{3}f_{13}}{-z_3} & \frac{\varepsilon_2\sqrt{3}f_{13}}{-z_3} & \frac{\sqrt{3}f_{13}}{-z_3} \\ 3\frac{f_{13}f_{23}}{-z_3} & 3\frac{1-f_{23}^2}{z_3} & 3\frac{f_{23}f_{33}}{-z_3} & \frac{\varepsilon_1\sqrt{3}f_{23}}{-z_3} & \frac{\varepsilon_2\sqrt{3}f_{23}}{-z_3} & \frac{\sqrt{3}f_{23}}{-z_3} \\ 3\frac{f_{13}f_{33}}{-z_3} & 3\frac{f_{23}f_{33}}{-z_3} & 3\frac{1-f_{33}^2}{z_3} & \frac{\varepsilon_1\sqrt{3}f_{33}}{-z_3} & \frac{\varepsilon_2\sqrt{3}f_{33}}{-z_3} & \frac{\sqrt{3}f_{33}}{-z_3} \\ \frac{\varepsilon_1\sqrt{3}f_{13}}{-z_3} & \frac{\varepsilon_1\sqrt{3}f_{23}}{-z_3} & \frac{\varepsilon_1\sqrt{3}f_{33}}{-z_3} & \frac{z_1}{z_3} & \frac{\varepsilon_1\varepsilon_2 z_2}{z_3} & \varepsilon_1 \\ \frac{\varepsilon_2\sqrt{3}f_{13}}{-z_3} & \frac{\varepsilon_2\sqrt{3}f_{23}}{-z_3} & \frac{\varepsilon_2\sqrt{3}f_{33}}{-z_3} & \frac{\varepsilon_1\varepsilon_2 z_2}{z_3} & 1 & \frac{\varepsilon_2 z_1}{z_3} \\ \frac{\sqrt{3}f_{13}}{-z_3} & \frac{\sqrt{3}f_{23}}{-z_3} & \frac{\sqrt{3}f_{33}}{-z_3} & \varepsilon_1 & \frac{\varepsilon_2 z_1}{z_3} & \frac{z_2}{z_3} \end{bmatrix}.$$

Now, one can check directly that

$$\sum_{i=1}^{6}\left(\frac{S_{i4}}{S_{54}}\right)^2 = \frac{9}{z_2^2} \quad \text{and} \quad \sum_{i=1}^{6}\left(\frac{S_{i6}}{S_{56}}\right)^2 = \frac{9}{z_1^2}\,,$$

which implies 6 is the isomorphism class of ι. Thus, all the entries of the 6th row of S have the same sign. Since $z_2/z_3 < 0$ and $z_1/z_3 > 0$, we obtain that $\varepsilon_1 = \varepsilon_2 = -1$ and $f_{i3} > 0$ for $i = 1, 2, 3$. Therefore,

$$\sqrt{3}f_{13} = \sqrt{3}f_{23} = \sqrt{3}f_{33} = 1$$

and hence

$$S=\begin{bmatrix} 2z_3^{-1} & -z_3^{-1} & -z_3^{-1} & z_3^{-1} & z_3^{-1} & -z_3^{-1} \\ -z_3^{-1} & 2z_3^{-1} & -z_3^{-1} & z_3^{-1} & z_3^{-1} & -z_3^{-1} \\ -z_3^{-1} & -z_3^{-1} & 2z_3^{-1} & z_3^{-1} & z_3^{-1} & -z_3^{-1} \\ z_3^{-1} & z_3^{-1} & z_3^{-1} & \frac{z_1}{z_3} & \frac{z_2}{z_3} & -1 \\ z_3^{-1} & z_3^{-1} & z_3^{-1} & \frac{z_2}{z_3} & 1 & \frac{-z_1}{z_3} \\ -z_3^{-1} & -z_3^{-1} & -z_3^{-1} & -1 & \frac{-z_1}{z_3} & \frac{z_2}{z_3} \end{bmatrix} \quad \text{and} \quad T=\operatorname{diag}(\zeta_9^5,\zeta_9^5,\zeta_9^5,\zeta_3^2,1,\zeta_3)\,.$$

(iii) Suppose 6 is the isomorphism class of $\mathbb{1}$. Then $D=3/z_2$ and

$$S=\begin{bmatrix} 3\frac{1-f_{13}^2}{z_2} & 3\frac{f_{13}f_{23}}{-z_2} & 3\frac{f_{13}f_{33}}{-z_2} & \frac{\varepsilon_1\sqrt{3}f_{13}}{-z_2} & \frac{\varepsilon_2\sqrt{3}f_{13}}{-z_2} & \frac{\sqrt{3}f_{13}}{-z_2} \\ 3\frac{f_{13}f_{23}}{-z_2} & 3\frac{1-f_{23}^2}{z_2} & 3\frac{f_{23}f_{33}}{-z_2} & \frac{\varepsilon_1\sqrt{3}f_{23}}{-z_2} & \frac{\varepsilon_2\sqrt{3}f_{23}}{-z_2} & \frac{\sqrt{3}f_{23}}{-z_2} \\ 3\frac{f_{13}f_{33}}{-z_2} & 3\frac{f_{23}f_{33}}{-z_2} & 3\frac{1-f_{33}^2}{z_2} & \frac{\varepsilon_1\sqrt{3}f_{33}}{-z_2} & \frac{\varepsilon_2\sqrt{3}f_{33}}{-z_2} & \frac{\sqrt{3}f_{33}}{-z_2} \\ \frac{\varepsilon_1\sqrt{3}f_{13}}{-z_2} & \frac{\varepsilon_1\sqrt{3}f_{23}}{-z_2} & \frac{\varepsilon_1\sqrt{3}f_{33}}{-z_2} & \frac{z_1}{z_2} & \varepsilon_1\varepsilon_2 & \frac{\varepsilon_1 z_3}{z_2} \\ \frac{\varepsilon_2\sqrt{3}f_{13}}{-z_2} & \frac{\varepsilon_2\sqrt{3}f_{23}}{-z_2} & \frac{\varepsilon_2\sqrt{3}f_{33}}{-z_2} & \varepsilon_1\varepsilon_2 & \frac{z_3}{z_2} & \frac{\varepsilon_2 z_1}{z_2} \\ \frac{\sqrt{3}f_{13}}{-z_2} & \frac{\sqrt{3}f_{23}}{-z_2} & \frac{\sqrt{3}f_{33}}{-z_2} & \frac{\varepsilon_1 z_3}{z_2} & \frac{\varepsilon_2 z_1}{z_2} & 1 \end{bmatrix}.$$

Now,

$$\sum_{i=1}^{6}\left(\frac{S_{i4}}{S_{64}}\right)^2=\frac{9}{z_3^2} \quad \text{and} \quad \sum_{i=1}^{6}\left(\frac{S_{i5}}{S_{65}}\right)^2=\frac{9}{z_1^2}\,,$$

which implies 5 is the isomorphism class of ι. Thus, all the entries of the 4th row have the same signs. Since $z_3/z_2<0$ and $z_1/z_2<0$, $\varepsilon_1=-1$, $\varepsilon_2=1$ and $f_{i3}>0$ for $i=1,2,3$. Therefore,

$$\sqrt{3}f_{13}=\sqrt{3}f_{23}=\sqrt{3}f_{33}=1$$

and hence

$$S=\begin{bmatrix} 2z_2^{-1} & -z_2^{-1} & -z_2^{-1} & -z_2^{-1} & z_2^{-1} & z_2^{-1} \\ -z_2^{-1} & 2z_2^{-1} & -z_2^{-1} & -z_2^{-1} & z_2^{-1} & z_2^{-1} \\ -z_2^{-1} & -z_2^{-1} & 2z_2^{-1} & -z_2^{-1} & z_2^{-1} & z_2^{-1} \\ -z_2^{-1} & -z_2^{-1} & -z_2^{-1} & \frac{z_1}{z_2} & -1 & \frac{-z_3}{z_2} \\ z_2^{-1} & z_2^{-1} & z_2^{-1} & -1 & \frac{z_3}{z_2} & \frac{z_1}{z_2} \\ z_2^{-1} & z_2^{-1} & z_2^{-1} & \frac{-z_3}{z_2} & \frac{z_1}{z_2} & 1 \end{bmatrix} \quad \text{and} \quad T=\operatorname{diag}(\zeta_9^2,\zeta_9^2,\zeta_9^2,\zeta_3,\zeta_3^2,1)\,.$$

Now, we compute the modular data for $\rho_0=\rho_{4_{9,1}^{8,0}}$, which is the complex conjugate of $\rho_{4_{9,1}^{1,0}}(\mathfrak{s})$. Since $\rho_{4_{9,1}^{1,0}}(\mathfrak{s}))=\rho_{4_{9,1}^{8,0}}(\mathfrak{s})$, modular data are complex conjugations of those obtained for $\rho_0=\rho_{4_{9,1}^{1,0}}$. They are:

(iv)

$$S=\begin{bmatrix} 2z_1^{-1} & -z_1^{-1} & -z_1^{-1} & z_1^{-1} & -z_1^{-1} & z_1^{-1} \\ -z_1^{-1} & 2z_1^{-1} & -z_1^{-1} & z_1^{-1} & -z_1^{-1} & z_1^{-1} \\ -z_1^{-1} & -z_1^{-1} & 2z_1^{-1} & z_1^{-1} & -z_1^{-1} & z_1^{-1} \\ z_1^{-1} & z_1^{-1} & z_1^{-1} & 1 & \frac{-z_2}{z_1} & \frac{z_3}{z_1} \\ -z_1^{-1} & -z_1^{-1} & -z_1^{-1} & \frac{-z_2}{z_1} & \frac{z_3}{z_1} & -1 \\ z_1^{-1} & z_1^{-1} & z_1^{-1} & \frac{z_3}{z_1} & -1 & \frac{z_2}{z_1} \end{bmatrix} \quad \text{and} \quad T=\operatorname{diag}(\zeta_9,\zeta_9,\zeta_9,1,\zeta_3^2,\zeta_3)\,.$$

(v)

$$S = \begin{bmatrix} 2z_3^{-1} & -z_3^{-1} & -z_3^{-1} & z_3^{-1} & z_3^{-1} & -z_3^{-1} \\ -z_3^{-1} & 2z_3^{-1} & -z_3^{-1} & z_3^{-1} & z_3^{-1} & -z_3^{-1} \\ -z_3^{-1} & -z_3^{-1} & 2z_3^{-1} & z_3^{-1} & z_3^{-1} & -z_3^{-1} \\ z_3^{-1} & z_3^{-1} & z_3^{-1} & \frac{z_1}{z_3} & \frac{z_2}{z_3} & -1 \\ z_3^{-1} & z_3^{-1} & z_3^{-1} & \frac{z_2}{z_3} & 1 & \frac{-z_1}{z_3} \\ -z_3^{-1} & -z_3^{-1} & -z_3^{-1} & -1 & \frac{-z_1}{z_3} & \frac{z_2}{z_3} \end{bmatrix} \quad \text{and} \quad T = \operatorname{diag}(\zeta_9^4, \zeta_9^4, \zeta_9^4, \zeta_3, 1, \zeta_3^2)\,.$$

(vi)

$$S = \begin{bmatrix} 2z_2^{-1} & -z_2^{-1} & -z_2^{-1} & -z_2^{-1} & z_2^{-1} & z_2^{-1} \\ -z_2^{-1} & 2z_2^{-1} & -z_2^{-1} & -z_2^{-1} & z_2^{-1} & z_2^{-1} \\ -z_2^{-1} & -z_2^{-1} & 2z_2^{-1} & -z_2^{-1} & z_2^{-1} & z_2^{-1} \\ -z_2^{-1} & -z_2^{-1} & -z_2^{-1} & \frac{z_1}{z_2} & -1 & \frac{-z_3}{z_2} \\ z_2^{-1} & z_2^{-1} & z_2^{-1} & -1 & \frac{z_3}{z_2} & \frac{z_1}{z_2} \\ z_2^{-1} & z_2^{-1} & z_2^{-1} & \frac{-z_3}{z_2} & \frac{z_1}{z_2} & 1 \end{bmatrix} \quad \text{and} \quad T = \operatorname{diag}(\zeta_9^7, \zeta_9^7, \zeta_9^7, \zeta_3^2, \zeta_3, 1)\,.$$

4.1.2. *Proof of Theorem 4.2.* Since modular data of Type (4,1,1) have been completely solved in the last subsection. The modular data of $PSO(8)_3$ coincides with (i) up to a permutation. Let $\sigma \in \operatorname{Gal}(\mathbb{Q}_9)$ be the generator defined by $\sigma : \zeta_9 \mapsto \zeta_9^2$. Applying σ to the modular data (i)-(vi), One can check directly

$$\text{(i)} \xrightarrow{\sigma} \text{(vi)} \xrightarrow{\sigma} \text{(ii)} \xrightarrow{\sigma} \text{(iv)} \xrightarrow{\sigma} \text{(iii)} \xrightarrow{\sigma} \text{(v)} \xrightarrow{\sigma} \text{(i)}$$

up to permutations of the objects. This completes the proof of Theorem 4.2. □

4.2. **Classification of modular data of type (4,2).** In this section, we will complete the classification of modular data of type (4,2) in the following theorem.

Theorem 4.4. *Let $\mathcal{C}$ be a rank 6 modular tensor category of type $(4,2)$. Then the modular data of $\mathcal{C}$ can only be a Galois conjugate of the modular data of the following modular tensor categories:*

1. $\mathcal{C}(\mathbb{Z}_6, q)$ *with* $q(1) = \zeta_{12}$*;*
2. $\mathcal{C}(\mathbb{Z}_3, q) \boxtimes PSU(2)_3$ *with* $q(1) = \zeta_3$*;*
3. $G(2)_3$ *.*

We will use the following level 5 irreducible representations $\rho_{2_5^1}$, $\rho_{4_{5,1}^1}$ and $\rho_{4_{5,2}^1}$ when necessary.

$$\rho_{2_5^1}(\mathfrak{s}) = \frac{1}{s_5^1}\begin{bmatrix} 1 & \varphi \\ \varphi & -1 \end{bmatrix}, \quad \rho_{2_5^1}(\mathfrak{t}) = \operatorname{diag}(\zeta_5, \zeta_5^4)\,. \tag{4.1}$$

Note that $\rho_{2_5^1}$ is defined over $\mathbb{Q}_5$. Let $\sigma \in \operatorname{Gal}(\bar{\mathbb{Q}})$ such that $\sigma(\zeta_5) = \zeta_5^2$. Then $\rho_{2_5^2} := \sigma \circ \rho_{2_5^1}$. $\rho_{2_5^i}$, $i = 1, 2$, form a complete set of inequivalent 2-dimensional representations of level 5. The following irreducible representations also form a complete set of inequivalent 4-dimensional representations of level 5:

$$\rho_{4_{5,1}^1}(\mathfrak{s}) = \frac{s_5^3}{5}\begin{bmatrix} -\varphi^2 & \varphi^{-1} & \sqrt{3}\varphi & \sqrt{3} \\ \varphi^{-1} & \varphi^2 & \sqrt{3} & -\sqrt{3}\varphi \\ \sqrt{3}\varphi & \sqrt{3} & \varphi^{-1} & \varphi^2 \\ \sqrt{3} & -\sqrt{3}\varphi & \varphi^2 & -\varphi^{-1} \end{bmatrix}, \quad \rho_{4_{5,1}^1}(\mathfrak{t}) = \operatorname{diag}(\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4)\,. \tag{4.2}$$

$$\rho_{4^1_{5,2}}(\mathfrak{s}) = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -1 & \varphi^{-1} & \varphi \\ -1 & 1 & \varphi & \varphi^{-1} \\ \varphi^{-1} & \varphi & -1 & 1 \\ \varphi & \varphi^{-1} & 1 & -1 \end{bmatrix}, \quad \rho_{4^1_{5,2}}(\mathfrak{t}) = \mathrm{diag}(\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4)\,. \tag{4.3}$$

We will need to establish a few lemmas to complete the proof of this theorem. Let $\mathcal{C}$ be a modular tensor category of type (4,2) and ρ an $\mathrm{SL}_2(\mathbb{Z})$ representation of $\mathcal{C}$. Then

$$\rho \cong \rho_1 \oplus \rho_2$$

for some irreducible representations ρ_1, ρ_2 of dimensions 4 and 2 respectively. By tensoring with a suitable $\chi^i \in \widehat{\mathrm{SL}_2(\mathbb{Z})}$, we may assume that the $\mathfrak{t}$-spectrum of ρ_1 is minimal. Therefore, ρ_1 has a prime power level or ρ_1 is a tensor product of two 2-dimensional irreducible representations of distinct prime power levels.

According to Appendix A, ρ_1 can only have the prime power levels 5, 7, 8, 9 or the composite levels 6, 10, 15, 24, 40. Note that a 4-dimensional irreducible representation of level 12 is projectively equivalent to an irreducible representation of level 6 as shown in the proof of Lemma 4.3. We will prove that only the levels 7, 15 and 24 are possible.

It follows from Appendix A that the eigenvalues of $\rho_1(\mathfrak{t})$ and $\rho_2(\mathfrak{t})$ are multiplicity free. By the $\mathfrak{t}$-spectrum criteria, $\mathrm{spec}(\rho_1(\mathfrak{t})) \cap \mathrm{spec}(\rho_2(\mathfrak{t})) = \{\tilde{\theta}_0\}$ or $\mathrm{spec}(\rho_2(\mathfrak{t})) \subset \mathrm{spec}(\rho_1(\mathfrak{t})$. These situations have been studied in Theorems 3.23 and 3.24. Now, we can begin to prove the level of ρ_1 cannot 5, 8, or 9.

Lemma 4.5. *The level of ρ_1 cannot be* 5.

Proof. Suppose ρ_1 is of level 5. Since there are exactly two inequivalent irreducible representations of level 5 and dimension 4, which are given by $\rho_{4^1_{5,1}}$ and $\rho_{4^1_{5,2}}$, ρ_1 must be equivalent one of them. In particular, the spectrum of $\rho_1(\mathfrak{t})$ consists of all the primitive 5-th root of unity. By the $\mathfrak{t}$-spectrum criteria, ρ_2 can only be equivalent to $\rho_{2^1_5}$ or $\rho_{2^2_5}$, which are the inequivalent irreducible representations of level 5 and dimension 2. Therefore, ρ is of level 5 and hence $\rho(\mathfrak{s})$ is a matrix over $\mathbb{Q}_5$. Let $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}})$ such that $\sigma(\zeta_5) = \zeta_5^2$. Then $\rho_{2^2_5} = \sigma \circ \rho_{2^1_5}$.

Note that $\tau \circ \rho_{4^1_{5,i}} \cong \rho_{4^1_{5,i}}$ for all $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}})$ and $i = 1, 2$. Thus, if $\rho_1 \oplus \rho_{2^1_5}$ is not equivalent to any MD representation, then so is $\sigma \circ (\rho_1 \oplus \rho_{2^1_5}) \cong \rho_1 \oplus \rho_{2^2_5}$. Therefore, it suffices to show that $\rho_{4^1_{5,1}} \oplus \rho_{2^1_5}$ and $\rho_{4^1_{5,2}} \oplus \rho_{2^1_5}$ are not equivalent to any MD representation.

(i) Suppose $\rho_1 = \rho_{4^1_{5,1}}$ and $\rho_2 = \rho_{2^1_5}$. Using the representations $\rho_{4^1_{5,1}}$ and $\rho_{2^1_5}$ presented in (4.2) and (4.1), we have

$$\begin{aligned} (\rho_1 \oplus \rho_2)(\mathfrak{s}) &= \frac{s_5^3}{5}\begin{bmatrix} -\varphi^2 & \varphi^{-1} & \sqrt{3}\varphi & \sqrt{3} \\ \varphi^{-1} & \varphi^2 & \sqrt{3} & -\sqrt{3}\varphi \\ \sqrt{3}\varphi & \sqrt{3} & \varphi^{-1} & \varphi^2 \\ \sqrt{3} & -\sqrt{3}\varphi & \varphi^2 & -\varphi^{-1} \end{bmatrix} \oplus \frac{1}{s_5^1}\begin{bmatrix} 1 & \varphi \\ \varphi & -1 \end{bmatrix} \\ (\rho_1 \oplus \rho_2)(\mathfrak{t}) &= \mathrm{diag}(\zeta_5, \zeta_5^4, \zeta_5^2, \zeta_5^3, \zeta_5, \zeta_5^4)\,. \end{aligned}$$

By Theorem 3.24 (1), There exists a block diagonal orthogonal matrix

$$U = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \oplus \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \oplus I_2 \quad \text{with } a^2 + b^2 = 1,\ c^2 + d^2 = 1,$$

such that $\rho(\mathfrak{t}) = \mathrm{diag}(\zeta_5, \zeta_5, \zeta_5^4, \zeta_5^4, \zeta_5^2, \zeta_5^3)$ and $\rho(\mathfrak{s})$ is a conjugation of s' by a signed diagonal matrix, where s' is given by

$$s' = \frac{s_5^3}{5}\begin{bmatrix} * & * & * & * & -\sqrt{3}b\varphi & -\sqrt{3}b \\ * & * & * & * & \sqrt{3}a\varphi & \sqrt{3}a \\ * & * & * & * & -\sqrt{3}d & \sqrt{3}d\varphi \\ * & * & * & * & \sqrt{3}c & -\sqrt{3}c\varphi \\ -\sqrt{3}b\varphi & \sqrt{3}a\varphi & -\sqrt{3}d & \sqrt{3}c & \varphi^{-1} & \varphi^2 \\ -\sqrt{3}b & \sqrt{3}a & \sqrt{3}d\varphi & -\sqrt{3}c\varphi & \varphi^2 & -\varphi^{-1} \end{bmatrix}.$$

It follows from the action of σ^2 on $\rho(\mathfrak{t})$, we find $\hat{\sigma}(5) = 6$. Since

$$\sigma(s_5^3/5) = \frac{s_5^1}{5} = -\frac{s_5^3}{5}\varphi \quad \text{and} \quad \sigma(\varphi) = -\varphi^{-1},$$

the action of σ on s'_{55} implies $\epsilon'_\sigma(5) = 1$. Hence, by the action of σ on the 5-th column, we have

$$\sigma(\sqrt{3}x) = \sqrt{3}x \text{ for } x = a, b, c, d.$$

Therefore, $\sqrt{3}a, \sqrt{3}b, \sqrt{3}c, \sqrt{3}d \in \mathbb{Q}$ as $\sigma|_{\mathbb{Q}_5}$ generates $\mathrm{Gal}(\mathbb{Q}_5/\mathbb{Q})$. If 5 (resp. 6) corresponds to the unit object $\mathbb{1}$, then s'/s'_{55} (resp. s'/s'_{66}) is a matrix $\mathbb{Z}[\zeta_5]$. Since φ is a unit in $\mathbb{Z}[\zeta_5]$, $\sqrt{3}a, \sqrt{3}b, \sqrt{3}c, \sqrt{3}d \in \mathbb{Z}[\zeta_5]$ and hence $\sqrt{3}a, \sqrt{3}b, \sqrt{3}c, \sqrt{3}d \in \mathbb{Z}\setminus\{0\}$. However, this contradicts that $(\sqrt{3}a)^2 + (\sqrt{3}b)^2 = 3$. Therefore, 5 and 6 cannot be $\mathbb{1}$.

Suppose 1 is the isomorphism class of $\mathbb{1}$. Then $s'_{i,5}/s'_{1,5} \in \mathbb{Z}[\zeta_5]$ for all i. In particular, $\frac{1}{\sqrt{3}b}$, $a/b \in \mathbb{Z}[\zeta_5]$. So, $\frac{1}{\sqrt{3}b}, a/b \in \mathbb{Z}$. Let $m, n \in \mathbb{Z}$ such that $a = mb$ and $1 = \sqrt{3}bn$. The equality $a^2 + b^2 = 1$ implies $(m^2+1)3b^2 = 3$ and so $m^2 + 1 = 3n^2$. However, $3 \nmid (m^2+1)$ for any integer m. Therefore, 1 cannot the unit object. By the same reason, 2, 3, and 4 are not the isomorphism class of $\mathbb{1}$. This ultimate contradiction implies that $\rho_{4^1_{5,1}} \oplus \rho_{2^1_5}$ is not equivalent any MD representation.

(ii) Now we assume $\rho_1 = \rho_{4^1_{5,2}}$ and $\rho_2 = \rho_{2^1_{5,1}}$. It follows from (4.1) and (4.3) that

$$\begin{aligned}(\rho_1 \oplus \rho_2)(\mathfrak{s}) &= \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -1 & \varphi^{-1} & \varphi \\ -1 & 1 & \varphi & \varphi^{-1} \\ \varphi^{-1} & \varphi & -1 & 1 \\ \varphi & \varphi^{-1} & 1 & -1 \end{bmatrix} \oplus \frac{1}{s_5^1}\begin{bmatrix} 1 & \varphi \\ \varphi & -1 \end{bmatrix} \\ (\rho_1 \oplus \rho_2)(\mathfrak{t}) &= \mathrm{diag}(\zeta_5, \zeta_5^4, \zeta_5^2, \zeta_5^3, \zeta_5, \zeta_5^4).\end{aligned}$$

Note that ρ_1, ρ_2 have opposite parities. We reorder the simple objects as in Theorem 3.24 so that $\rho(\mathfrak{t}) = \mathrm{diag}(\zeta_5, \zeta_5, \zeta_5^4, \zeta_5^4, \zeta_5^2, \zeta_5^3)$. The unit object can only be e_5 or e_6. In either case, we find $\dim(\mathcal{C}) = 5$, and $\dim(e_1) = \dim(e_2) = \frac{\pm\rho_1(\mathfrak{s})_{13}}{\sqrt{2}\rho_1(\mathfrak{s})_{33}} = \frac{\pm\varphi^{-1}}{\sqrt{2}} \notin \mathbb{Q}_5$. This contradicts Theorem 2.1 (4). Therefore, $\rho_{4^1_{5,2}} \oplus \rho_{2^1_{5,1}}$ is not equivalent to any MD representation. This completes the proof of this lemma. □

Lemma 4.6. *The level ρ_1 cannot be 8.*

Proof. Suppose ρ_1 has level 8. Since there is only one projectively equivalent class of irreducible representations of level 8 and dimension 4. One can assume $\rho_1 = \rho_{4_8^{1,0}}$ (cf. Appendix A). In particular, ρ_1 is odd, and $\mathrm{spec}(\rho_1(\mathfrak{t}))$ consists of all the primitive 8-th roots of unity.

By the $\mathfrak{t}$-spectrum criteria, $\mathrm{spec}(\rho_2(\mathfrak{t}))$ must be a set of primitive 8-th roots of unity, and hence ρ_2 has level 8. Therefore, ρ_2 must be projectively equivalent $\rho_{2_8^{1,0}}$, or $\rho_2 \cong \rho_{2_8^{1,\ell}}$, where $\ell = 0, 3, 6, 9$. Note that ρ_1 is equivalent to its complex conjugation while $\{\rho_{2_8^{1,0}}, \rho_{2_8^{1,6}}\}$ and $\{\rho_{2_8^{1,3}}, \rho_{2_8^{1,9}}\}$ are complex conjugation pairs. It suffices to show that ρ_2 is not equivalent to (i) $\rho_{2_8^{1,0}}$ or (ii) $\rho_{2_8^{1,3}}$.

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(i) Suppose $\rho_2 \cong \rho_{2_8^{1,0}}$. Then $\mathrm{spec}(\rho_2(\mathfrak{t})) \subset \mathrm{spec}(\rho_1(\mathfrak{t}))$ and ρ_1, ρ_2 have opposite parities. Their direct sum $\tilde{\rho} = \rho_{4_8^{1,0}} \oplus \rho_{2_8^{1,0}}$ is given by

$$\tilde{\rho}(\mathfrak{s}) = \frac{\mathrm{i}}{\sqrt{8}} \begin{bmatrix} 1 & \sqrt{3} & \sqrt{3} & 1 \\ \sqrt{3} & 1 & -1 & -\sqrt{3} \\ \sqrt{3} & -1 & -1 & \sqrt{3} \\ 1 & -\sqrt{3} & \sqrt{3} & -1 \end{bmatrix} \oplus \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } \tilde{\rho}(\mathfrak{t}) = \mathrm{diag}(\zeta_8, \zeta_8^3, \zeta_8^5, \zeta_8^7, \zeta_8, \zeta_8^3)\,.$$

However,

$$\tilde{\rho}(\mathfrak{s})^{\mathrm{ndeg}} = \frac{\mathrm{i}}{\sqrt{8}} \begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

is not a matrix over $\mathbb{Q}_8$, a contradiction to Proposition 3.9. Therefore, $\rho_2 \not\cong \rho_{2_8^{1,0}}$.

(ii) Now, we assume $\rho_2 \cong \rho_{2_8^{1,3}}$. Then ρ_1, ρ_2 have the same party, and $\tilde{\rho} = \rho_{4_8^{1,0}} \oplus \rho_{2_8^{1,3}}$ is given by

$$\tilde{\rho}(\mathfrak{s}) = \frac{\mathrm{i}}{\sqrt{8}} \begin{bmatrix} 1 & \sqrt{3} & \sqrt{3} & 1 \\ \sqrt{3} & 1 & -1 & -\sqrt{3} \\ \sqrt{3} & -1 & -1 & \sqrt{3} \\ 1 & -\sqrt{3} & \sqrt{3} & -1 \end{bmatrix} \oplus \frac{\mathrm{i}}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } \tilde{\rho}(\mathfrak{t}) = \mathrm{diag}(\zeta_8, \zeta_8^3, \zeta_8^5, \zeta_8^7, \zeta_8^3, \zeta_8^5)\,.$$

However,

$$\tilde{\rho}(\mathfrak{s})^{\mathrm{ndeg}} = \frac{\mathrm{i}}{\sqrt{8}} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$

is not a matrix over $\mathbb{Q}_8$, a contradiction to Proposition 3.9. Therefore, $\rho_2 \not\cong \rho_{2_8^{1,3}}$. □

Lemma 4.7. *The level of ρ_1 cannot be 9.*

Proof. There are 4 projectively inequivalent 4-dimensional irreducible $\mathrm{SL}_2(\mathbb{Z})$ representations of level 9, which are given by $\rho_{4_{9,1}^{1,0}}$, $\rho_{4_{9,1}^{8,0}}$, $\rho_{4_{9,2}^{1,0}}$, and $\rho_{4_{9,2}^{8,0}}$ (cf. Appendix A). $\rho_{4_{9,1}^{1,0}}$, $\rho_{4_{9,1}^{8,0}}$ are complex conjugate of each other and so are $\rho_{4_{9,2}^{1,0}}$, and $\rho_{4_{9,2}^{8,0}}$. Therefore, it suffices to show that ρ_1 cannot be equivalent to (i) $\rho_{4_{9,1}^{1,0}}$ or (ii) $\rho_{4_{9,2}^{1,0}}$.

(i) Suppose $\rho_1 \cong \rho_{4_{9,1}^{1,0}}$, which is odd. By the $\mathfrak{t}$-spectrum criteria, ρ_2 can only be projectively equivalent to $\rho_{2_2^{1,0}}$ or $\rho_{2_3^{1,0}}$, and this implies $\rho_2 \cong \rho_{2_2^{1,0}}$, $\rho_{2_3^{1,0}}$ or $\rho_{2_3^{1,8}}$. In any of these cases, $\mathrm{spec}(\rho_1(\mathfrak{t})) \cap \mathrm{spec}(\rho_2(\mathfrak{t})) = \{1\}$. Therefore, by Theorem 3.23 (iii), ρ_2 is also odd, which means $\rho_2 \not\cong \rho_{2_2^{1,0}}$ as it is even.

Now $\rho_2 \cong \rho_{2_3^{1,0}}$ or $\rho_{2_3^{1,8}}$. Note that

$$\rho_{2_3^{1,0}}(\mathfrak{s}) = \frac{\mathrm{i}}{\sqrt{3}} \begin{bmatrix} -1 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}, \qquad \rho_{2_3^{1,0}}(\mathfrak{t}) = \mathrm{diag}(1, \zeta_3),$$
$$\rho_{2_3^{1,8}}(\mathfrak{s}) = \frac{\mathrm{i}}{\sqrt{3}} \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix}, \qquad \rho_{2_3^{1,8}}(\mathfrak{t}) = \mathrm{diag}(1, \zeta_3^2)\,.$$

By Theorem 3.23 (ii), the unit object $\mathbb{1}$ of $\mathcal{C}$ is an eigenvector of $\rho(\mathfrak{t})$ of eigenvalue 1, and $\dim(\mathcal{C}) = 4/|\rho_1(\mathfrak{s})_{11} + \rho_2(\mathfrak{s})_{11}|^2 = 12$. By the Cauchy Theorem of modular categories, $2 \mid \mathrm{ord}(T) \mid \mathrm{ord}(\rho(\mathfrak{t})) = 9$, a contradiction. Therefore, $\rho_1 \not\cong \rho_{4_{9,1}^{1,0}}$.

(ii) Now, we assume $\rho_1 \cong \rho_{4_{9,2}^{1,0}}$, which is even. Using similar argument as in Case (i), $\rho_2 \cong \rho_{2_2^{1,0}}$ by the $\mathfrak{t}$-spectrum criteria and Theorem 3.23 (iii). In this case, $\mathrm{spec}(\rho_1(\mathfrak{t})) \cap \mathrm{spec}(\rho_2(\mathfrak{t})) = \{1\}$ and ρ has level 18. Theorem 3.23 (ii), the unit object of $\mathcal{C}$ is an eigenvector of $\rho(\mathfrak{t})$ of eigenvalue 1, and

$\dim(\mathcal{C}) = 4/|\rho_1(\mathfrak{s})_{11} + \rho_2(\mathfrak{s})_{11}|^2 = 16$. By the Cauchy Theorem of modular categories, $\mathrm{ord}(T)$ is a 2-power, but this contradicts Theorem 3.7 (4). Therefore, $\rho_1 \not\cong \rho_{4_{9,2}^{1,0}}$. $\square$

Lemma 4.8. *If ρ_1 projectively equivalent to an irreducible representation of prime power level, then the modular data of $\mathcal{C}$ is a Galois conjugate of that of $G(2)_3$.*

Proof. By Lemmas 4.5, 4.6, 4.7 and Appendix A, ρ_1 can only be projective equivalent a level 7 irreducible representation. By the $\mathfrak{t}$-spectrum criteria, $\rho_1 \cong \rho_{4_7^1}$ or its complex conjugate $\rho_{4_7^6}$, which They are defined over $\mathbb{Q}_{56}$.

If there exists some modular data (S, T) whose associated $\mathrm{SL}_2(\mathbb{Z})$ representation $\rho \cong \rho_{4_7^1} \oplus \rho_2$ for some irreducible 2-dimensional representation ρ_2, one can obtain the modular data derived from the MD representation which admits the decomposition $\rho_{4_7^6} \oplus \overline{\rho}_2$ by the complex conjugation of (S, T).

(I) Assume $\rho_1 \cong \rho_{4_7^1}$, which is odd. It follows the $\mathfrak{t}$-spectrum criteria that ρ_2 must be equivalent to a level 2 or level 3 irreducible representation. In any of these cases, $\mathrm{spec}(\rho_1(\mathfrak{t})) \cap \mathrm{spec}(\rho_2(\mathfrak{t})) = \{1\}$. There is only one 2-dimensional irreducible representation of level 2 which is even. By Theorem 3.23 (iii), $\rho_2 \cong \rho_{2_3^{1,0}}$ or $\rho_{2_3^{1,8}}$, which is odd. Since

$$\rho_{2_3^{1,8}} \cong \overline{\rho_{2_3^{1,0}}} = \rho_{2_3^{2,0}}\,.$$

We will solve the modular data for (i) $\rho \cong \rho_{4_7^1} \oplus \rho_{2_3^{1,0}}$ and (ii) $\rho \cong \rho_{4_7^1} \oplus \rho_{2_3^{2,0}}$.

(i) Let $\tilde{\rho} = \rho_{4_7^1} \oplus \rho_{2_3^{1,0}}$. Then $\tilde{\rho}(\mathfrak{t}) = \mathrm{diag}(1, \zeta_7, \zeta_7^2, \zeta_7^4, 1, \zeta_3)$ and

$$\tilde{\rho}(\mathfrak{s}) = \frac{\mathrm{i}}{\sqrt{7}} \begin{bmatrix} -1 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \gamma_1 & \gamma_2 & \gamma_3 \\ \sqrt{2} & \gamma_2 & \gamma_3 & \gamma_1 \\ \sqrt{2} & \gamma_3 & \gamma_1 & \gamma_2 \end{bmatrix} \oplus \frac{\mathrm{i}}{\sqrt{3}} \begin{bmatrix} -1 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$$

where $\gamma_1 = -c_7^2$, $\gamma_2 = -c_7^1$ and $\gamma_3 = -c_7^3$. We reorder $\mathrm{irr}(\mathcal{C})$ so that $\rho(\mathfrak{t}) = \mathrm{diag}(1, 1, \zeta_7, \zeta_7^2, \zeta_7^4, \zeta_3)$, and identify $\mathrm{irr}(\mathcal{C})$ with the standard basis of $\mathbb{C}^6$. By Theorem 3.23,

$$\rho(\mathfrak{s}) = \frac{i}{\sqrt{7}} \begin{bmatrix} \frac{-\sqrt{21}-3}{6} & \frac{(\sqrt{21}-3)\varepsilon_1}{6} & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & -\sqrt{\frac{7}{3}}\varepsilon_1\varepsilon_5 \\ \frac{(\sqrt{21}-3)\varepsilon_1}{6} & \frac{-\sqrt{21}-3}{6} & \varepsilon_1\varepsilon_2 & \varepsilon_1\varepsilon_3 & \varepsilon_1\varepsilon_4 & \sqrt{\frac{7}{3}}\varepsilon_5 \\ \varepsilon_2 & \varepsilon_1\varepsilon_2 & \gamma_1 & \gamma_2\varepsilon_2\varepsilon_3 & \gamma_3\varepsilon_2\varepsilon_4 & 0 \\ \varepsilon_3 & \varepsilon_1\varepsilon_3 & \gamma_2\varepsilon_2\varepsilon_3 & \gamma_3 & \gamma_1\varepsilon_3\varepsilon_4 & 0 \\ \varepsilon_4 & \varepsilon_1\varepsilon_4 & \gamma_3\varepsilon_2\varepsilon_4 & \gamma_1\varepsilon_3\varepsilon_4 & \gamma_2 & 0 \\ -\sqrt{\frac{7}{3}}\varepsilon_1\varepsilon_5 & \sqrt{\frac{7}{3}}\varepsilon_5 & 0 & 0 & 0 & \sqrt{\frac{7}{3}} \end{bmatrix}.$$

for some $\varepsilon_i = \pm 1$, and so $D = 2\left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{7}}\right)^{-1}$ or $\dim(\mathcal{C}) = \frac{21}{2}\left(5 - \sqrt{21}\right)$. Since $\frac{21}{2}\left(5 + \sqrt{21}\right)$ is a Galois conjugate of $\dim(\mathcal{C})$ and

$$\dim(\mathcal{C}) < \frac{21}{2}\left(5 + \sqrt{21}\right) \leqslant \mathrm{FPdim}(\mathcal{C}),$$

the objects $\mathbb{1}$ and ι are distinct. By Theorem 3.23 (ii), e_1, e_2 are the only rows of the S-matrix with no zero entry. Therefore, $\{\mathbb{1}, \iota\} = \{e_1, e_2\}$, and the modular data of $\mathcal{C}$ is given by

$$S = \begin{bmatrix} 1 & -d_1\varepsilon_1 & -d_2\varepsilon_2 & -d_2\varepsilon_3 & -d_2\varepsilon_4 & d_3\varepsilon_1\varepsilon_5 \\ -d_1\varepsilon_1 & 1 & -d_2\varepsilon_1\varepsilon_2 & -d_2\varepsilon_1\varepsilon_3 & -d_2\varepsilon_1\varepsilon_4 & -d_3\varepsilon_5 \\ -d_2\varepsilon_2 & -d_2\varepsilon_1\varepsilon_2 & -d_2\gamma_1 & -d_2\gamma_2\varepsilon_2\varepsilon_3 & -d_2\gamma_3\varepsilon_2\varepsilon_4 & 0 \\ -d_2\varepsilon_3 & -d_2\varepsilon_1\varepsilon_3 & -d_2\gamma_2\varepsilon_2\varepsilon_3 & -d_2\gamma_3 & -d_2\gamma_1\varepsilon_3\varepsilon_4 & 0 \\ -d_2\varepsilon_4 & -d_2\varepsilon_1\varepsilon_4 & -d_2\gamma_3\varepsilon_2\varepsilon_4 & -d_2\gamma_1 & -d_2\gamma_2 & 0 \\ d_3\varepsilon_1\varepsilon_5 & -d_3\varepsilon_5 & 0 & 0 & 0 & -d_3 \end{bmatrix} \quad \text{and} \quad T = \rho(\mathfrak{t})\,,$$

where $d_1 = \frac{1}{2}\left(5-\sqrt{21}\right)$, $d_2 = \frac{1}{2}\left(\sqrt{21}-3\right)$, $d_3 = \frac{1}{2}\left(7-\sqrt{21}\right)$.

If $\mathbb{1} = e_1$, then $\iota = e_2$ and so $S_{2,*} = [d_1, 1, d_2, d_2, d_2, d_3]$. This forces $\varepsilon_1 = \varepsilon_5 = -1, \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$. Thus,

$$S = \begin{bmatrix} 1 & d_1 & -d_2 & -d_2 & -d_2 & d_3 \\ d_1 & 1 & d_2 & d_2 & d_2 & d_3 \\ -d_2 & d_2 & -d_2\gamma_1 & -d_2\gamma_2 & -d_2\gamma_3 & 0 \\ -d_2 & d_2 & -d_2\gamma_2 & -d_2\gamma_3 & -d_2\gamma_1 & 0 \\ -d_2 & d_2 & -d_2\gamma_3 & -d_2\gamma_1 & -d_2\gamma_2 & 0 \\ d_3 & d_3 & 0 & 0 & 0 & -d_3 \end{bmatrix} \quad \text{and} \quad T = \operatorname{diag}(1, 1, \zeta_7, \zeta_7^2, \zeta_7^4, \zeta_3)\,.$$

If $\mathbb{1} = e_2$, then $\iota = e_1$ and so $S_{1,*} = [1, d_1, d_2, d_2, d_2, d_3]$. This forces $\varepsilon_i = -1$ for $i = 1, \ldots, 5$, and so resulting S-matrix is equivalent to the preceding one interchanges the indexes of e_1 and e_2.

(ii) Let $\tilde{\rho} = \rho_{4_7^1} \oplus \rho_{2_3^{2,0}}$. Then $\tilde{\rho}(\mathfrak{t}) = \operatorname{diag}(1, \zeta_7, \zeta_7^2, \zeta_7^4, 1, \zeta_3^2)$ and

$$\tilde{\rho}(\mathfrak{s}) = \frac{\mathrm{i}}{\sqrt{7}} \begin{bmatrix} -1 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \gamma_1 & \gamma_2 & \gamma_3 \\ \sqrt{2} & \gamma_2 & \gamma_3 & \gamma_1 \\ \sqrt{2} & \gamma_3 & \gamma_1 & \gamma_2 \end{bmatrix} \oplus \frac{\mathrm{i}}{\sqrt{3}} \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix}.$$

Note that $\tilde{\rho}$ is defined over $\mathbb{Q}_{168}$. Let $\sigma \in \operatorname{Gal}(\mathbb{Q}_{168}/\mathbb{Q})$ such that $\sigma(\zeta_{168}) = \zeta_{168}^{113}$. Then $\sigma|_{\mathbb{Q}_{56}} = \mathrm{id}$ and $\sigma(\zeta_3) = \zeta_3^2$. One can see easily that

$$\sigma \circ (\rho_{4_7^1} \oplus \rho_{2_3^{1,0}}) = \rho_{4_7^1} \oplus \rho_{2_3^{2,0}}\,.$$

Thus the modular data (S', T') for the MD representation equivalent to $\tilde{\rho}$ is the Galois conjugate by σ of the modular data (S, T) obtained in (i). Therefore,

$$S' = \begin{bmatrix} 1 & d_1' & -d_2' & -d_2' & -d_2' & d_3' \\ d_1' & 1 & d_2' & d_2' & d_2' & d_3' \\ -d_2' & d_2' & -d_2'\gamma_1 & -d_2'\gamma_2 & -d_2'\gamma_3 & 0 \\ -d_2' & d_2' & -d_2'\gamma_2 & -d_2'\gamma_3 & -d_2'\gamma_1 & 0 \\ -d_2' & d_2' & -d_2'\gamma_3 & -d_2'\gamma_1 & -d_2'\gamma_2 & 0 \\ d_3' & d_3' & 0 & 0 & 0 & -d_3' \end{bmatrix} \quad \text{and} \quad T' = \operatorname{diag}(1, 1, \zeta_7, \zeta_7^2, \zeta_7^4, \zeta_3^2)\,,$$

where $d_1' = \sigma(d_1) = \frac{1}{2}\left(5+\sqrt{21}\right)$, $d_2' = \sigma(d_2) = -\frac{1}{2}\left(3+\sqrt{21}\right)$, $d_3' = \sigma(d_3) = \frac{1}{2}\left(7+\sqrt{21}\right)$. Since $S_{1,j}' > 0$, $e_1 = \mathbb{1} = \iota$, and so $\mathcal{C}$ is pseudounitary and $\dim(\mathcal{C}) = \sigma(\frac{21}{2}\left(5-\sqrt{21}\right)) = \frac{21}{2}\left(5+\sqrt{21}\right)$. The modular data of $G(2)_3$ is also (S', T').

(II) Now, we assume $\rho_1 = \rho_{4_7^6}$ and proceed to solve the modular data for (i) $\rho \cong \rho_{4_7^6} \oplus \rho_{2_3^{1,0}}$ and (ii) $\rho \cong \rho_{4_7^6} \oplus \rho_{2_3^{2,0}}$. Note that both of them are defined over $\mathbb{Q}_{168}$.

(i) Let $\tilde{\rho} = \rho_{4_7^6} \oplus \rho_{2_3^{1,0}}$. Then $\overline{\rho_{4_7^1} \oplus \rho_{2_3^{2,0}}} = \tilde{\rho}$. Thus the modular data (S'', T'') of the MD representations equivalent to $\tilde{\rho}$ is $(\overline{S}', \overline{T}')$, which is given by

$$S'' = S' = \begin{bmatrix} 1 & d_1' & -d_2' & -d_2' & -d_2' & d_3' \\ d_1' & 1 & d_2' & d_2' & d_2' & d_3' \\ -d_2' & d_2' & -d_2'\gamma_1 & -d_2'\gamma_2 & -d_2'\gamma_3 & 0 \\ -d_2' & d_2' & -d_2'\gamma_2 & -d_2'\gamma_3 & -d_2'\gamma_1 & 0 \\ -d_2' & d_2' & -d_2'\gamma_3 & -d_2'\gamma_1 & -d_2'\gamma_2 & 0 \\ d_3' & d_3' & 0 & 0 & 0 & -d_3' \end{bmatrix} \quad \text{and} \quad T'' = \overline{T}' = \operatorname{diag}(1, 1, \zeta_7^6, \zeta_7^5, \zeta_7^3, \zeta_3)\,,$$

In particular, the MTC $\mathcal{C}$ is also pseudounitary with $\dim(\mathcal{C}) = \frac{21}{2}\left(5+\sqrt{21}\right)$.

(ii) Finally, we consider $\tilde{\rho} = \rho_{4_7^6} \oplus \rho_{2_3^{2,0}}$ which is the complex conjugate of $\rho_{4_7^1} \oplus \rho_{2_3^{1,0}}$. Thus the modular data (S''', T''') of the MD representations equivalent to $\tilde{\rho}$ is $(\overline{S}, \overline{T})$, which is given by

$$S''' = S = \begin{bmatrix} 1 & d_1 & -d_2 & -d_2 & -d_2 & d_3 \\ d_1 & 1 & d_2 & d_2 & d_2 & d_3 \\ -d_2 & d_2 & -d_2\gamma_1 & -d_2\gamma_2 & -d_2\gamma_3 & 0 \\ -d_2 & d_2 & -d_2\gamma_2 & -d_2\gamma_3 & -d_2\gamma_1 & 0 \\ -d_2 & d_2 & -d_2\gamma_3 & -d_2\gamma_1 & -d_2\gamma_2 & 0 \\ d_3 & d_3 & 0 & 0 & 0 & -d_3 \end{bmatrix} \quad \text{and} \quad T''' = \overline{T} = \operatorname{diag}(1, 1, \zeta_7^6, \zeta_7^5, \zeta_7^3, \zeta_3^2)\,.$$

Therefore, the MTC $\mathcal{C}$ is not pseudounitary and $\dim(\mathcal{C}) = \frac{21}{2}\left(5 - \sqrt{21}\right)$. □

Lemma 4.9. *The level of ρ_1 cannot be 6, 10 or 40.*

Proof. (i) Suppose ρ_1 is of level 6. Then $\rho_1 \cong \psi \otimes \eta$ for some 2-dimensional irreducible representations ψ and η of level 2 and 3 respectively. There is only one 2-dimensional irreducible representation, up to projective equivalence, of levels 2 and 3. Since the $\mathfrak{t}$-spectrum of ρ_1 is minimal, $\rho_1 \cong \rho_{2_2^{1,0}} \otimes \rho_{2_3^{1,0}}$. In particular, $\operatorname{spec}(\rho_1(\mathfrak{t})) = \{1, -1, \zeta_3, -\zeta_3\}$. By the $\mathfrak{t}$-spectrum criteria, ρ_2 can only be equivalent to $\rho_2 \cong \rho_{2_2^{1,i}}$, $i \in \{0, 4, 6, 10\}$, or $\rho_{2_3^{1,j}}$, j even. Therefore, $\operatorname{ord}(\rho_2(\mathfrak{t})) \mid 6$ and so $\operatorname{ord}(\rho(\mathfrak{t})) = 6$. This implies $\operatorname{ord}(T) \mid 6$ and so $\mathcal{C}$ is integral by Theorem 3.14. However, this contradicts Proposition 3.16. Therefore, the level of ρ_1 cannot be 6.

(ii) Suppose ρ_1 is of level 40. Then ρ_1 is projectively equivalent to $\rho_{2_8^{1,0}} \otimes \rho_{2_5^1}$ or $\rho_{2_8^{1,0}} \otimes \rho_{2_5^2}$ (cf. Appendix A). In particular, $\operatorname{spec}(\rho_1(\mathfrak{t}))$ is a set of primitive 40-th roots of unity. However, there does not exist any 2-dimensional representation ρ_2 which satisfies the $\mathfrak{t}$-spectrum criteria. Therefore, the level ρ_1 cannot be 40.

(iii) Suppose ρ_1 is of level 10. Then ρ_1 is projectively equivalent to $\rho_{2_2^{1,0}} \otimes \rho_{2_5^1}$ or $\rho_{2_2^{1,0}} \otimes \rho_{2_5^2}$. Since $\rho_{2_2^{1,0}}$ is equivalent to any of it Galois conjugates, $\rho_{2_2^{1,0}} \otimes \rho_{2_5^1}$ or $\rho_{2_2^{1,0}} \otimes \rho_{2_5^2}$ are Galois conjugate. So, it suffices to show that $\rho_1 \cong \rho_{2_2^{1,0}} \otimes \rho_{2_5^1}$ is not possible.

Assume $\rho_1 \cong \rho_{2_2^{1,0}} \otimes \rho_{2_5^1}$. Then $\operatorname{spec}(\rho_1(\mathfrak{t})) = \{\zeta_5, \zeta_5^4, -\zeta_5, -\zeta_5^4\}$. By the $\mathfrak{t}$-spectrum criteria, $\rho_2 \cong \rho_{2_5^1}$ or $\chi^6 \otimes \rho_{2_5^1}$. Since $\chi^6 \otimes \rho_{2_2^{1,0}} \cong \rho_{2_2^{1,0}}$, $\rho_1 \oplus \rho_{2_5^1}$ and $\rho_1 \oplus \chi^6 \otimes \rho_{2_5^1}$ are projectively equivalent. Therefore, ρ is projectively equivalent to $\tilde{\rho} = (\rho_{2_2^{1,0}} \otimes \rho_{2_5^1}) \oplus \rho_{2_5^1}$ and we can simply assume $\rho \cong \tilde{\rho}$. As in Lemma 4.5, we the use the following equivalent form of $\rho_{2_5^1}$:

$$\rho_{2_5^1}(\mathfrak{s}) = \frac{1}{s_5^1}\begin{bmatrix} 1 & \varphi \\ \varphi & -1 \end{bmatrix}, \quad \rho_{2_5^1}(\mathfrak{t}) = \operatorname{diag}(\zeta_5, \zeta_5^4)\,.$$

Thus, $\tilde{\rho}(\mathfrak{t}) = \operatorname{diag}(\zeta_5, \zeta_5^4, -\zeta_5, -\zeta_5^4, \zeta_5, \zeta_5^4)$ and

$$\tilde{\rho}(\mathfrak{s}) = \frac{1}{2s_5^1}\begin{bmatrix} -1 & -\varphi & -\sqrt{3} & -\sqrt{3}\varphi \\ -\varphi & 1 & -\sqrt{3}\varphi & \sqrt{3} \\ -\sqrt{3} & -\sqrt{3}\varphi & 1 & \varphi \\ -\sqrt{3}\varphi & \sqrt{3} & \varphi & -1 \end{bmatrix} \oplus \frac{1}{s_5^1}\begin{bmatrix} 1 & \varphi \\ \varphi & -1 \end{bmatrix}.$$

By Theorem 3.24 (i), if we reorder $\operatorname{irr}(\mathcal{C})$ so that $\rho(\mathfrak{t}) = \operatorname{diag}(\zeta_5 I_2, \zeta_5^4 I_2, -\zeta_5, -\zeta_5^4)$, then $\rho(\mathfrak{s}) = V s' V$ for some signed diagonal matrix V and

$$s' = \frac{1}{2s_5^1}\begin{bmatrix} * & * & * & * & -\sqrt{3}a & -\sqrt{3}a\varphi \\ * & * & * & * & -\sqrt{3}b & -\sqrt{3}b\varphi \\ * & * & * & * & -\sqrt{3}c\varphi & \sqrt{3}c \\ * & * & * & * & -\sqrt{3}d\varphi & \sqrt{3}d \\ -\sqrt{3}a & -\sqrt{3}b & -\sqrt{3}c\varphi & -\sqrt{3}d\varphi & 1 & \varphi \\ -\sqrt{3}a\varphi & -\sqrt{3}b\varphi & \sqrt{3}c & \sqrt{3}d & \varphi & -1 \end{bmatrix}.$$

where $a, b, c, d \in \mathbb{R}$ satisfying $a^2 + b^2 = 1$ and $c^2 + d^2 = 1$.

Note that φ is a unit in $\mathbb{Z}[\zeta_5]$, and the automorphism σ defined by $\sigma(\zeta_{10}) = \zeta_{10}^7$ generates $\operatorname{Gal}(\mathbb{Q}_{10})$. By the action of σ^2 on $\rho(\mathfrak{t})$, we see $\hat{\sigma}(5) = 6$. Since

$$\sigma(s'_{5,5}) = \frac{1}{2s_5^2} = \frac{\varphi}{2s_5^1} = s'_{56}\,,$$

$\sigma(s'_{i,5}) = s'_{i,6}$ for $i = 1, \dots, 6$. This implies $\sqrt{3}a, \sqrt{3}b, \sqrt{3}c, \sqrt{3}c$ are fixed by σ and so they are rational.

The unit object cannot be e_5, for otherwise $\sqrt{3}a, \sqrt{3}b \in \mathbb{Z}$ and they satisfy the equation $(\sqrt{3}a)^2 + (\sqrt{3}b)^2 = 3$, which is not possible. Similarly, $e_6 \neq \mathbb{1}$. So, the unit object $\mathbb{1} \in \{e_1, e_2, e_3, e_4\}$.

Assume $\mathbb{1} = e_1$. Then $a \neq 0$, $b/a \in \mathbb{Z}$ and $\frac{1}{\sqrt{3}a} \in \mathbb{Z}$. However, this will imply $3 \mid (1 + (b/a)^2)$ which is not possible. Therefore, $\mathbb{1} \neq e_1$. Since φ is a unit in $\mathbb{Z}[\zeta_5]$, if $\mathbb{1} \notin \{e_2, e_3, e_4\}$ for similar reason. Now, we find $1 \notin \{e_1, \dots, e_6\}$, a contradiction. Therefore, the level of ρ_1 cannot be 10. ☐

Lemma 4.10. *If the level of ρ_1 is 24, then $\mathcal{C}$ is equivalent to $\mathcal{C}(\mathbb{Z}_6, q)$ for some non-degenerate quadratic form $q : \mathbb{Z}_6 \to \mathbb{C}^\times$.*

Proof. Since ρ_1 is of level 24, ρ_1 is projectively equivalent to $\rho_{2_3^{1,0}} \otimes \rho_{2_8^{1,0}}$ according to Appendix A. Therefore, we can simply assume $\rho_1 \cong \rho_{2_3^{1,0}} \otimes \rho_{2_8^{1,0}}$ as it has a minimal $\mathfrak{t}$-spectrum. Then, ρ_1 is odd and

$$\rho_1(\mathfrak{s}) = \frac{\mathrm{i}}{\sqrt{6}}\begin{bmatrix} 1 & -1 & -\sqrt{2} & \sqrt{2} \\ -1 & -1 & \sqrt{2} & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} & -1 & 1 \\ \sqrt{2} & \sqrt{2} & 1 & 1 \end{bmatrix}, \quad \rho_1(\mathfrak{t}) = \operatorname{diag}(\zeta_8, \zeta_8^3, \zeta_{24}^{11}, \zeta_{24}^{17})\,.$$

By the $\mathfrak{t}$-spectrum criteria, $\rho_2 \cong \rho_{2_8^{1,j}}$, $j \in \{0, 1, 3, 4, 7, 9\}$, and

$$\rho_{2_8^{1,j}}(\mathfrak{s}) = \frac{(-i)^j}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \rho_{2_8^{1,j}}(\mathfrak{t}) = \operatorname{diag}(\zeta_{24}^{3+2j}, \zeta_{24}^{9+2j})\,.$$

For $j = 1, 3, 7, 9$, $|\operatorname{spec}(\rho_1(\mathfrak{t})) \cap \operatorname{spec}(\rho_2(\mathfrak{t}))| = 1$ and so Theorem 3.23 can be applied.

For $j = 1, 9$, $\operatorname{spec}(\rho_1(\mathfrak{t})) \cap \operatorname{spec}(\rho_2(\mathfrak{t})) = \{\zeta_{24}^{9+2j}\}$, and for $j = 3, 7$, $\operatorname{spec}(\rho_1(\mathfrak{t})) \cap \operatorname{spec}(\rho_2(\mathfrak{t})) = \{\zeta_{24}^{3+2j}\}$. If $\rho \cong \rho_1 \oplus \rho_{2_8^{1,j}}$ is an MD representation of an MTC $\mathcal{C}$, for $j = 1, 3, 7, 9$, then by Theorem 3.23, $\operatorname{ord}(T) = 12$ and

$$D = \sqrt{\dim(\mathcal{C})} = \frac{2}{\frac{1}{\sqrt{2}}(1 \pm \frac{1}{\sqrt{3}})} = \sqrt{6}(\sqrt{3} \mp 1).$$

Note that each row of $\rho_1(\mathfrak{s})$ has an off diagonal entry of the form $\frac{\pm i}{\sqrt{6}}$ and so $\frac{D}{\sqrt{6}}/\sqrt{2}$ is the dimension of an object up to a sign. However,

$$\frac{D}{\sqrt{6}\sqrt{2}} = \frac{\sqrt{3} \pm 1}{\sqrt{2}} \notin \mathbb{Q}_{12}\,.$$

Therefore, $\rho_1 \oplus \rho_{2_8^{1,j}}$ is not equivalent to any MD representation for $j = 1, 3, 7, 9$.

Now, we can conclude that $\rho \cong \rho_1 \oplus \rho_2$ where $\rho_2 \cong \rho_{2_8^{1,j}}$ for some $j = 0, 4$. In particular, ρ_1 and ρ_2 have opposite parties and $\operatorname{spec}(\rho_2(\mathfrak{t})) \subset \operatorname{spec}(\rho_1(\mathfrak{t}))$. By Theorem 3.24 (ii), the unit object $\mathbb{1}$ is an eigenvector of $\rho(\mathfrak{t})$ with eigenvalue $\zeta \in \operatorname{spec}(\rho_1(\mathfrak{t}))\backslash \operatorname{spec}(\rho_2(\mathfrak{t}))$. Let E_j be the subspace of $\mathbb{C}^6$ spanned by the eigenvectors of $\tilde{\rho}_j = \rho_1(\mathfrak{t}) \oplus \rho_{2_8^{1,j}}(\mathfrak{t})$ with eigenvalues in $\operatorname{spec}(\rho_1(\mathfrak{t}))\backslash \operatorname{spec}(\rho_{2_8^{1,j}}(\mathfrak{t}))$ for $j = 0, 4$. One can compute that for $\sigma \in \operatorname{Gal}(\mathbb{Q}_{24}/\mathbb{Q})$, $D_{\tilde{\rho}_j}(\sigma)|_{E_j} = \mathrm{id}$ or $-\mathrm{id}$. By Proposition 3.13, $\mathcal{C}$ is integral. It follows from [4] that $\mathcal{C}$ is a pointed modular tensor category, which is equivalent to to $\mathcal{C}(\mathbb{Z}_6, q)$ for some non-degenerate quadratic form $q : \mathbb{Z}_6 \to \mathbb{C}^\times$. □

Lemma 4.11. *If the level of ρ_1 is 15, then the modular data of $\mathcal{C}$ is a Galois conjugate of that of $\mathcal{C}(\mathbb{Z}_3, q) \boxtimes PSU(2)_3$, where $q : \mathbb{Z}_3 \to \mathbb{C}^\times$ is a quadratic form given by $q(1) = \zeta_3$.*

Proof. Since ρ_1 has a minimal $\mathfrak{t}$-spectrum, it must be equivalent to a tensor product of two 2-dimensional irreducible representations of levels 3 and 5. According to Appendix A, $\rho_1 \cong \rho_{2_3^{1,0}} \otimes \rho_{2_5^i}$, $i = 1, 2$. By the $\mathfrak{t}$-spectrum criteria, $\rho_2 \cong \chi^j \otimes \rho_{2_5^i}$ with $j = 0, 4$. Thus, ρ is equivalent to

$$\tilde{\rho}_{i,j} = (\rho_{2_3^{1,0}} \otimes \rho_{2_5^i}) \oplus (\chi^j \otimes \rho_{2_5^i}), \quad i = 1, 2,\ j = 0, 4\,.$$

Note that $\tilde{\rho}_{i,j}$ is defined over $\mathbb{Q}_{120}$ for i, j. Let $\sigma_a \in \operatorname{Gal}(\mathbb{Q}_{120}/\mathbb{Q})$ such that $\sigma_a(\zeta_{120}) = \zeta_{120}^a$. Then, $\sigma_{97} \circ \tilde{\rho}_{1,j} = \tilde{\rho}_{2,j}$ for $j = 0, 4$.

Since $\sigma_{41} \circ ((\rho_{2_3^{1,0}} \otimes \rho_{2_5^1}) \oplus \rho_{2_5^1}) \cong (\overline{\rho_{2_3^{1,0}}} \otimes \rho_{2_5^1}) \oplus \rho_{2_5^1} \cong (\rho_{2_3^{1,8}} \otimes \rho_{2_5^1}) \oplus \rho_{2_5^1}$, we have $\chi^4 \otimes \sigma_{41} \circ \tilde{\rho}_{1,0} \cong \tilde{\rho}_{1,4}$. Therefore, $\tilde{\rho}_{i,j}$ is projectively equivalent to a Galois conjugate of $\tilde{\rho}_{1,0}$. Hence, it suffices to consider $\tilde{\rho} = \tilde{\rho}_{1,0}$, or equivalently $\rho_1 \cong \rho_{2_3^{1,0}} \otimes \rho_{2_5^1}$ and $\rho_2 \cong \rho_{2_5^1}$

Now, the MD representation ρ of $\mathcal{C}$ is equivalent to $\rho_1 \oplus \rho_2$, where ρ_1 is even, ρ_2 is odd and $\operatorname{spec}(\rho_1(\mathfrak{t})) \subset \operatorname{spec}(\rho_2(\mathfrak{t}))$. Moreover,

$$\tilde{\rho}(\mathfrak{s}) = \frac{\mathrm{i}}{\sqrt{3} s_5^1} \begin{bmatrix} -1 & -\varphi & \sqrt{2} & \sqrt{2}\varphi \\ -\varphi & 1 & \sqrt{2}\varphi & -\sqrt{2} \\ \sqrt{2} & \sqrt{2}\varphi & 1 & \varphi \\ \sqrt{2}\varphi & -\sqrt{2} & \varphi & -1 \end{bmatrix} \oplus \frac{1}{s_5^1} \begin{bmatrix} 1 & \varphi \\ \varphi & -1 \end{bmatrix}, \quad \tilde{\rho}(\mathfrak{t}) = \operatorname{diag}(\zeta_5, \zeta_5^4, \zeta_{15}^8, \zeta_{15}^2, \zeta_5, \zeta_5^4)\,.$$

By Theorem 3.24, $\dim(\mathcal{C}) = 12\sin^2(2\pi/5) = 3(2+\varphi)$. Reorder $\operatorname{irr}(\mathcal{C})$ so that

$$\rho(\mathfrak{t}) = \operatorname{diag}(\zeta_5, \zeta_5^4, \zeta_5, \zeta_5^4, \zeta_{15}^8, \zeta_{15}^2).$$

Again, by Theorem 3.24 (ii), there exist $\gamma_i, \kappa_i, \varepsilon_i \in \{\pm 1\}$ such that

$$\rho(\mathfrak{s}) = \frac{-1}{D} \begin{bmatrix} \frac{1+\mathrm{i}\sqrt{3}}{2} & \frac{(1-\mathrm{i}\sqrt{3})\kappa_1}{2} & \frac{\gamma_3\varphi(1+\mathrm{i}\sqrt{3}\varepsilon_1\varepsilon_2\kappa_1\kappa_2)}{2} & \frac{\gamma_3\varphi(\kappa_2-\mathrm{i}\sqrt{3}\varepsilon_1\varepsilon_2\kappa_1)}{2} & -\gamma_1 & -\gamma_2\varphi \\ \frac{(1-\mathrm{i}\sqrt{3})\kappa_1}{2} & \frac{1+\mathrm{i}\sqrt{3}}{2} & \frac{\gamma_3\varphi(\kappa_1-\mathrm{i}\sqrt{3}\varepsilon_1\varepsilon_2\kappa_2)}{2} & \frac{\gamma_3\varphi(\kappa_1\kappa_2+\mathrm{i}\sqrt{3}\varepsilon_1\varepsilon_2)}{2} & -\gamma_1\kappa_1 & -\gamma_2\kappa_1\varphi \\ \frac{\gamma_3\varphi(1+\mathrm{i}\sqrt{3}\varepsilon_1\varepsilon_2\kappa_1\kappa_2)}{2} & \frac{\gamma_3\varphi(\kappa_1-\mathrm{i}\sqrt{3}\varepsilon_1\varepsilon_2\kappa_2)}{2} & \frac{-(1+\mathrm{i}\sqrt{3})}{2} & \frac{(-1+\mathrm{i}\sqrt{3})\kappa_2}{2} & -\gamma_1\gamma_3\varphi & \gamma_2\gamma_3 \\ \frac{\gamma_3\varphi(\kappa_2-\mathrm{i}\sqrt{3}\varepsilon_1\varepsilon_2\kappa_1)}{2} & \frac{\gamma_3\varphi(\kappa_1\kappa_2+\mathrm{i}\sqrt{3}\varepsilon_1\varepsilon_2)}{2} & \frac{(-1+\mathrm{i}\sqrt{3})\kappa_2}{2} & \frac{-(1+\mathrm{i}\sqrt{3})}{2} & -\gamma_1\gamma_3\kappa_2\varphi & \gamma_2\gamma_3\kappa_2 \\ -\gamma_1 & -\gamma_1\kappa_1 & -\gamma_1\gamma_3\varphi & -\gamma_1\gamma_3\kappa_2\varphi & -1 & -\gamma_1\gamma_2\varphi \\ -\gamma_2\varphi & -\gamma_2\kappa_1\varphi & \gamma_2\gamma_3 & \gamma_2\gamma_3\kappa_2 & -\gamma_1\gamma_2\varphi & 1 \end{bmatrix}.$$

We will use the equalities $\frac{-1+\mathrm{i}\sqrt{3}}{2} = \zeta_3$ and $\frac{1+\mathrm{i}\sqrt{3}}{2} = -\bar{\zeta}_3$ to simplify S-matrix, but we need to determine which of the standard basis elements is the unit object. According to Theorem 3.24 (ii), $\mathbb{1} \in \{e_5, e_6\}$.

(i) Suppose $e_6 = \mathbb{1}$. Then $T = \operatorname{diag}(\zeta_{15}, \zeta_3^2, \zeta_{15}, \zeta_3^2, \zeta_5^2, 1)$. Then $\dim(e_5)^2 = \varphi^2 > 1$ and so $e_6 = \iota$. Thus, all the entries of 6-th rows of $\rho(\mathfrak{s})$ has the same signed, we find $\gamma_2 = \gamma_3 = -1, \gamma_1 = \kappa_1 =$

$\kappa_2 = 1$. Thus,

$$S = \begin{bmatrix} -\bar{\zeta}_3 & -\zeta_3 & \varphi\bar{\zeta}_3 & \varphi\zeta_3 & -1 & \varphi \\ -\zeta_3 & -\bar{\zeta}_3 & \varphi\zeta_3 & \varphi\bar{\zeta}_3 & -1 & \varphi \\ \varphi\bar{\zeta}_3 & \varphi\zeta_3 & \bar{\zeta}_3 & \zeta_3 & \varphi & 1 \\ \varphi\zeta_3 & \varphi\bar{\zeta}_3 & \zeta_3 & \bar{\zeta}_3 & \varphi & 1 \\ -1 & -1 & \varphi & \varphi & -1 & \varphi \\ \varphi & \varphi & 1 & 1 & \varphi & 1 \end{bmatrix}.$$

By reordering $\operatorname{irr}(\mathcal{C})$, we find $T = \operatorname{diag}(1, \zeta_5^2, \zeta_3^2, \zeta_{15}, \zeta_3^2, \zeta_{15}) = T_1 \otimes T_2$ and

$$S = \begin{bmatrix} 1 & \varphi & 1 & \varphi & 1 & \varphi \\ \varphi & -1 & \varphi & -1 & \varphi & -1 \\ 1 & \varphi & \bar{\zeta}_3 & \varphi\bar{\zeta}_3 & \zeta_3 & \varphi\zeta_3 \\ \varphi & -1 & \varphi\bar{\zeta}_3 & -\bar{\zeta}_3 & \varphi\zeta_3 & -\zeta_3 \\ 1 & \varphi & \zeta_3 & \varphi\zeta_3 & \bar{\zeta}_3 & \varphi\bar{\zeta}_3 \\ \varphi & -1 & \varphi\zeta_3 & -\zeta_3 & \varphi\bar{\zeta}_3 & -\bar{\zeta}_3 \end{bmatrix} = S_1 \otimes S_2,$$

where (S_1, T_1) is the modular data of $\mathcal{C}(\mathbb{Z}_3, \bar{q})$ and (S_2, T_2) given by $S_2 = \begin{bmatrix} 1 & \varphi \\ \varphi & -1 \end{bmatrix}$ and $T_2 = \operatorname{diag}(1, \zeta_5^2)$ is the modular data of $PSU(2)_3$. In particular, (S, T) is a Galois conjugate of the modular data of $\mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q_1) \boxtimes PSU(2)_3$.

(ii) Now, we assume $e_5 = \mathbb{1}$. Then $T = \operatorname{diag}(\zeta_3^2, \zeta_{15}^4, \zeta_3^2, \zeta_{15}^4, 1, \zeta_5^3)$ and $\dim(e_6)^2 = \varphi^2 > 1$, and so $e_5 = \iota$. Then $\gamma_1 = \gamma_2 = \gamma_3 = \kappa_1 = \kappa_2 = 1$, and we obtain

$$S = \begin{bmatrix} \bar{\zeta}_3 & \zeta_3 & \bar{\zeta}_3\varphi & \zeta_3\varphi & 1 & \varphi \\ \zeta_3 & \bar{\zeta}_3 & \zeta_3\varphi & \bar{\zeta}_3\varphi & 1 & \varphi \\ \bar{\zeta}_3\varphi & \zeta_3\varphi & -\bar{\zeta}_3 & -\zeta_3 & \varphi & -1 \\ \zeta_3\varphi & \bar{\zeta}_3\varphi & -\zeta_3 & -\bar{\zeta}_3 & \varphi & -1 \\ 1 & 1 & \varphi & \varphi & 1 & \varphi \\ \varphi & \varphi & -1 & -1 & \varphi & -1 \end{bmatrix}.$$

By reordering $\operatorname{irr}(\mathcal{C})$, we find $T = \operatorname{diag}(1, \zeta_5^3, \zeta_3^2, \zeta_{15}^4, \zeta_3^2, \zeta_{15}^4) = T_1 \otimes \overline{T}_2$ and

$$S = \begin{bmatrix} 1 & \varphi & 1 & \varphi & 1 & \varphi \\ \varphi & -1 & \varphi & -1 & \varphi & -1 \\ 1 & \varphi & \bar{\zeta}_3 & \varphi\bar{\zeta}_3 & \zeta_3 & \varphi\zeta_3 \\ \varphi & -1 & \varphi\bar{\zeta}_3 & -\bar{\zeta}_3 & \varphi\zeta_3 & -\zeta_3 \\ 1 & \varphi & \zeta_3 & \varphi\zeta_3 & \bar{\zeta}_3 & \varphi\bar{\zeta}_3 \\ \varphi & -1 & \varphi\zeta_3 & -\zeta_3 & \varphi\bar{\zeta}_3 & -\bar{\zeta}_3 \end{bmatrix} = S_1 \otimes S_2,$$

Since $(S_2, \overline{T}_2)$ is the complex conjugate of modular data of $PSU(2)_3$. Therefore, (S, T) is a Galois conjugate of the modular data of $\mathcal{C}(\mathbb{Z}_3, q) \boxtimes PSU(2)_3$. This completes the proof of statement.

As a consequence, for any i, j, $\tilde{\rho}_{i,j}$ is equivalent to $\mathrm{SL}_2(\mathbb{Z})$ representations of some modular tensor categories Galois conjugate to $\mathcal{C}(\mathbb{Z}_3, q) \boxtimes PSU(2)_3$. □

Proof of Theorem 4.4. The result of Theorem 4.4 is a consequence of Lemmas 4.5 to 4.11. □

4.3. Classification of modular data of type (3,3).

Theorem 4.12. *The modular data of any type* $(3,3)$ *modular tensor category is a Galois conjugate of that of* $SO(5)_2$.

Let $\mathcal{C}$ be a modular tensor category of type (3,3) and ρ an $\mathrm{SL}_2(\mathbb{Z})$ representation of $\mathcal{C}$. Then

$$\rho \cong \rho_1 \oplus \rho_2$$

for some 3-dimensional irreducible representations ρ_1, ρ_2. If ρ_1, ρ_2 have opposite parities, then $\mathrm{Tr}(\rho(\mathfrak{s})) = 0$ which contradicts to Proposition 3.12. Therefore, they have the same parity. We may assume that ρ_1 has a minimal $\mathfrak{t}$-spectrum and show that for ρ_1 cannot be projectively equivalent of any 3-dimensional irreducible representation of levels 3, 7, 8 or 16.

Lemma 4.13. *Neither* ρ_1 *nor* ρ_2 *is projectively equivalent to a 3-dimensional irreducible representation of level* 3, 7, 8 *or* 16.

Proof. Suppose ρ_1 is a 3-dimensional irreducible representation of level 3, 7, 8 or 16 with a minimal $\mathfrak{t}$-spectrum.

(i) ρ_1 **cannot be of level** 7: Suppose ρ_1 is of level 7. Then, by the $\mathfrak{t}$-spectrum criteria and Appendix A, $\rho_2 \cong \rho_1$ but this contradicts Proposition 3.19.

(ii) ρ_1 **cannot be of level** 3: Suppose ρ_1 is the level 3. Then $\rho_1 \cong \rho_{3_3^{1,0}}$. Since $\dim(\rho_2) = 3$ which is a prime number, ρ_2 must be projectively equivalent to a 3-dimensional irreducible representations of prime power level (cf. Appendix A). If ρ_2 is projectively equivalent to $\rho_{3_3^{1,0}}$, then $\rho_2 \cong \rho_{3_3^{1,0}}$ by the $\mathfrak{t}$-spectrum criteria, but this contradicts Proposition 3.19. Therefore, ρ_2 is not projectively equivalent to $\rho_{3_3^{1,0}}$.

It follows from (i) that ρ_2 cannot be projectively equivalent to a level 7 representation. Therefore, by Appendix A, ρ_2 can only be projectively equivalent to a representation of levels 4, 5, 8, 16.

By the $\mathfrak{t}$-spectrum criteria, ρ_2 is not projectively equivalent to any level 16 irreducible representations. If ρ_2 is projectively equivalent a level 8 irreducible representation, then $\rho_2 \cong \chi^j \otimes \psi$ for any level 8 representations in Appendix A. Since ρ_1 is even, $j \equiv 0 \mod 4$, and so ψ must be even. This implies $\psi = \rho_{3_8^{3,3}}, \rho_{3_8^{1,3}}, \rho_{3_8^{3,9}}, \rho_{3_8^{1,9}}$, but none of them satisfies the $\mathfrak{t}$-spectrum criteria. Therefore, ρ_2 can only be projectively equivalent to some ψ of level 5 or 4 in Appendix A. Thus, by the $\mathfrak{t}$-spectrum criteria, $\rho_2 \cong \chi^j \otimes \rho_{3_4^{1,0}}$ or $\chi^j \otimes \rho_{3_5^i}$ for $j = 0, 4, 8$ and $i = 1, 3$. In any of these cases, $|\operatorname{spec}(\rho_1(\mathfrak{t})) \cap \operatorname{spec}(\rho_2(\mathfrak{t}))| = 1$ and $\operatorname{ord}(\rho(\mathfrak{t})) = 12$ or 15. It follows from Theorem 3.23 (ii) (c) that if $\operatorname{spec}(\rho_1(\mathfrak{t})) \cap \operatorname{spec}(\rho_2(\mathfrak{t})) = \{\rho_1(\mathfrak{t})_{u,u}\}$, then $\frac{\sqrt{2}\rho_1(\mathfrak{s})_{jj}}{\rho_1(\mathfrak{s})_{uj}} \in \mathbb{Q}_{12}$ or $\mathbb{Q}_{15}$ for $u \neq j$. However, $\frac{\sqrt{2}\rho_1(\mathfrak{s})_{jj}}{\rho_1(\mathfrak{s})_{uj}} = \frac{-1}{\sqrt{2}} \notin \mathbb{Q}_{12}$ or $\mathbb{Q}_{15}$. Therefore, ρ_2 cannot be projectively equivalent to any irreducible of level 4 or 5. This completes the proof that ρ_1 cannot be of level 3.

(iii) ρ_1 **cannot be of level** 8: Let

$$A = \frac{\mathrm{i}}{2}\begin{bmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{bmatrix}.$$

Then, by Appendix A,

$$\rho_{3_8^{1,0}}(\mathfrak{s}) = A \quad \text{and} \quad \rho_{3_8^{1,0}}(\mathfrak{t}) = \operatorname{diag}(1, \zeta_8, \zeta_8^5)$$

which is odd and has a minimal $\mathfrak{t}$-spectrum. Since all other 4-dimensional level 8 irreducible representations are projectively equivalent to a Galois conjugate of $\rho_{3_8^{1,0}}$, it suffices to show that $\rho_1 \not\cong \rho_{3_8^{1,0}}$.

Assume to the contrary. Then $\rho_1 \cong \rho_{3_8^{1,0}}$, and hence ρ_2 must be odd. It follows from (i) and (ii), ρ_2 cannot be projectively equivalent to any irreducible representation of level 3 or 7. By the $\mathfrak{t}$-spectrum criteria and the parity constraint, ρ_2 cannot be projectively equivalent to any irreducible representations of level 5. Therefore, ρ_2 can only be projectively equivalent to an irreducible representation of level 4, 8 or 16. By the $\mathfrak{t}$-spectrum criteria, ρ_2 is of level 4, 8 or 16.

Suppose ρ_2 has level 4 or 8. Since ρ_2 is odd, $\rho_2 \cong \rho_{3_4^{1,3}}, \rho_{3_4^{1,9}}, \rho_{3_8^{1,0}}, \rho_{3_8^{3,0}}, \rho_{3_8^{1,6}}, \rho_{3_8^{3,6}}$. However, $D_{\rho_1\oplus\rho_2}(\sigma) = \pm\,\mathrm{id}$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}_8/\mathbb{Q})$. By Proposition 3.13, $\mathcal{C}$ is integral which contradicts Proposition 3.16. Therefore, the level of ρ_2 is neither 4 nor 8.

Suppose ρ_2 is an odd irreducible representation of level 16. By the $\mathfrak{t}$-spectrum criteria, $\rho_2 \cong \rho_{3_{16}^{1,0}}, \rho_{3_{16}^{5,6}}, \rho_{3_{16}^{1,6}}, \rho_{3_{16}^{5,0}}$, and they are respectively isomorphic to the following representations:

(1) $\mathfrak{s} \mapsto A, \quad \mathfrak{t} \mapsto \mathrm{diag}(\zeta_8, \zeta_{16}, \zeta_{16}^9)$;
(2) $\mathfrak{s} \mapsto -A, \quad \mathfrak{t} \mapsto \mathrm{diag}(\zeta_8, \zeta_{16}^5, \zeta_{16}^{13})$;
(3) $\mathfrak{s} \mapsto -A, \quad \mathfrak{t} \mapsto \mathrm{diag}(\zeta_8^5, \zeta_{16}, \zeta_{16}^9)$;
(4) $\mathfrak{s} \mapsto A, \quad \mathfrak{t} \mapsto \mathrm{diag}((\zeta_8^5, \zeta_{16}^5, \zeta_{16}^{13})$.

In any of these cases, $\mathrm{spec}(\rho_1(\mathfrak{t})) \cap \mathrm{spec}(\rho_2(\mathfrak{t})) = \{\zeta_8\}$ or $\{\zeta_8^5\}$. It follows from Theorem 3.23 (ii) (a) and (b) that $D = 4$ as $\psi_{uu} = -i/2$ and $\eta_{uu} = 0$. The two nonzero rows of the S-matrix up to some signs are the same:

$$1, 1, 2, \sqrt{2}, 2, 2$$

and one of these rows is ι. Therefore, the Frobenius-Perron dimensions of the simple objects of $\mathcal{C}$ are $1, 1, 2, \sqrt{2}, 2, 2$. In particular, $\mathcal{C}$ is weakly integral, which contradicts Proposition 3.16 (ii). Thus, ρ_2 is not of level 16 either. As a consequence, ρ_1 cannot be of level 8.

(iv) ρ_1 **cannot be of level** 16: Assume contrary. Then $\rho_1 \cong \rho_{3_{16}^{1,0}}$, $\rho_{3_{16}^{3,9}}$, $\rho_{3_{16}^{5,6}}$, $\rho_{3_{16}^{7,3}}$, which are projectively inequivalent and have a minimal $\mathfrak{t}$-spectrum. Moreover,

$$\rho_{3_{16}^{1,0}}(\mathfrak{s}) = A \quad \text{and} \quad \eta(\mathfrak{t}) = \mathrm{diag}(\zeta_8, \zeta_{16}, \zeta_{16}^9)$$

which is odd. Since all the 3-dimensional level 16 irreducible representations are projectively equivalent to a Galois conjugates of $\rho_{3_{16}^{1,0}}$, its suffices consider the case $\rho_1 \cong \rho_{3_{16}^{1,0}}$.

By the $\mathfrak{t}$-spectrum criteria, ρ_2 cannot be projectively equivalent to any irreducible representation of level 4 or 5. By (i), (ii) and (iii), ρ_2 cannot be projectively equivalent to any irreducible representation of level 3, 7, 8. Therefore, ρ_2 can only be projectively equivalent to an irreducible representation of level 16. The $\mathfrak{t}$-spectrum criteria forces ρ_2 to be an irreducible representation of level 16. Since ρ_2 is odd, by Proposition 3.19, $\rho_2 \cong \rho_{3_{16}^{1,6}}$ or $\rho_{3_{16}^{5,6}}$, which are respectively isomorphic to the following irreducible representations:

(1) $\mathfrak{s} \mapsto -A, \quad \mathfrak{t} \mapsto \mathrm{diag}(\zeta_8^5, \zeta_{16}, \zeta_{16}^9)$;
(2) $\mathfrak{s} \mapsto -A, \quad \mathfrak{t} \mapsto \mathrm{diag}(\zeta_8, \zeta_{16}^5, \zeta_{16}^{13})$.

For Case (1), $\mathrm{spec}(\rho_1(\mathfrak{t})) \cap \mathrm{spec}(\rho_2(\mathfrak{t})) = \{\zeta_{16}, \zeta_{16}^9\}$ but

$$\rho_1(\mathfrak{s})_{ii} + \rho_2(\mathfrak{s})_{ii} = A_{ii} - A_{ii} = 0$$

for $i = 2, 3$. Therefore, $\rho \cong \rho_{3_{16}^{1,0}} \oplus \rho_{3_{16}^{1,6}}$ is impossible by Theorem 3.23.

For Case (2), $\mathrm{spec}(\rho_1(\mathfrak{t})) \cap \mathrm{spec}(\rho_2(\mathfrak{t})) = \{\zeta_8\}$ and $\rho_1(\mathfrak{s})_{11} + \rho_2(\mathfrak{s})_{11} = 0$. It follows from Theorem 3.23 that $\rho \cong \rho_{3_{16}^{1,0}} \oplus \rho_{3_{16}^{5,6}}$ is also not possible. ☐

Lemma 4.14. *If ρ_1 is of level 5, then ρ_2 cannot be projectively equivalent to any level 5 irreducible representation.*

Proof. Suppose ρ_2 is projectively equivalent to some level 5 irreducible representation. Then, by the $\mathfrak{t}$-spectrum criteria, ρ_2 is a level 5 irreducible representation. Since there are only two inequivalent level 5 irreducible representation, $\rho_1 \not\cong \rho_2$ by Proposition 3.19. Then $\operatorname{spec}(\rho_1(\mathfrak{t})) \cap \operatorname{spec}(\rho_2(\mathfrak{t})) = \{1\}$. It follows from Appendix A that

$$\rho_1(\mathfrak{s})_{11} + \rho_2(\mathfrak{s})_{11} = 0\,.$$

By Theorem 3.23(i), $\rho_1 \oplus \rho_2$ is not equivalent to any MD representation. Therefore, ρ_2 cannot be projectively equivalent to any level 5 irreducible representation. □

It follows from Lemmas 4.3 and 4.14 that the MD representation ρ of $\mathcal{C}$ of type (3,3) must have the irreducible decomposition $\rho_1 \oplus \rho_2$ where ρ_1 and ρ_2 are 3-dimensional and of levels 5 and 4.

4.3.1. *Solving modular data of type (3,3) level (5,4).* There are only two inequivalent level 5 irreducible representations $\rho_{3_5^1}$ and $\rho_{3_5^3}$. Note that $\sigma \circ \rho_{3_5^1} = \rho_{3_5^3}$ where $\sigma \in \operatorname{Gal}(\bar{\mathbb{Q}})$ such that $\sigma(\zeta_5) = \zeta_5^3$. One may assume $\rho_1 \cong \rho_{3_5^1}$ which is even, and has a minimal $\mathfrak{t}$-spectrum.

By the $\mathfrak{t}$-spectrum criteria and the parity constraint, $\rho_2 \cong \rho_{3_4^{1,0}}$, and so

$$\operatorname{spec}(\rho_1(\mathfrak{t})) \cap \operatorname{spec}(\rho_2(\mathfrak{t})) = \{1\}.$$

By Theorem 3.23, $D = 2/\frac{1}{\sqrt{5}} = 2\sqrt{5}$ or $\dim(\mathcal{C}) = 20$. Moreover, if $\operatorname{irr}(\mathcal{C})$ is reordered so that $\rho(\mathfrak{t}) = \operatorname{diag}(1, 1, \zeta_5, \zeta_5^4, \mathrm{i}, \mathrm{i})$, then

$$\rho(\mathfrak{s}) = \frac{1}{2\sqrt{5}} \begin{bmatrix} 1 & \kappa & -2\gamma_1 & -2\gamma_2 & -\sqrt{5}\gamma_3\kappa & -\sqrt{5}\gamma_4\kappa \\ \kappa & 1 & -2\gamma_1\kappa & -2\gamma_2\kappa & \sqrt{5}\gamma_3 & \sqrt{5}\gamma_4 \\ -2\gamma_1 & -2\gamma_1\kappa & -1-\sqrt{5} & (-1+\sqrt{5})\gamma_1\gamma_2 & 0 & 0 \\ -2\gamma_2 & -2\gamma_2\kappa & (-1+\sqrt{5})\gamma_1\gamma_2 & -1-\sqrt{5} & 0 & 0 \\ -\sqrt{5}\gamma_3\kappa & \sqrt{5}\gamma_3 & 0 & 0 & -\sqrt{5} & \sqrt{5}\gamma_3\gamma_4 \\ -\sqrt{5}\gamma_4\kappa & \sqrt{5}\gamma_4 & 0 & 0 & \sqrt{5}\gamma_3\gamma_4 & -\sqrt{5} \end{bmatrix}$$

for some $\kappa, \gamma_i \in \{\pm 1\}$. One can conclude from S that $\mathcal{C}$ is pseudounitary, and so we can assume $\mathbb{1} = \iota = e_1$. This implies $\kappa = 1$, $\gamma_i = -1$ for $i = 1, \ldots, 4$. Thus, the modular data of $\mathcal{C}$ is given by

$$S = \begin{bmatrix} 1 & 1 & 2 & 2 & \sqrt{5} & \sqrt{5} \\ 1 & 1 & 2 & 2 & -\sqrt{5} & -\sqrt{5} \\ 2 & 2 & -1-\sqrt{5} & -1+\sqrt{5} & 0 & 0 \\ 2 & 2 & -1+\sqrt{5} & -1-\sqrt{5} & 0 & 0 \\ \sqrt{5} & -\sqrt{5} & 0 & 0 & -\sqrt{5} & \sqrt{5} \\ \sqrt{5} & -\sqrt{5} & 0 & 0 & \sqrt{5} & -\sqrt{5} \end{bmatrix} \quad \text{and} \quad T = \operatorname{diag}(1, 1, \zeta_5, \zeta_5^4, \mathrm{i}, -\mathrm{i})\,.$$

However, if $\mathbb{1} \neq \iota$, then one may assume $e_1 = \mathbb{1}$ and $e_2 = \iota$. Then the resulting modular data is (PSP, T) where P is the permutation matrix of the transposition $(1, 2)$. In this sense, the two modular data corresponding to different spherical structures are the *same*.

For $\rho_1 = \rho_{3_5^3}$, the corresponding modular data is $(\sigma(S), \sigma(T))$, where $\sigma \in \operatorname{Gal}(\bar{\mathbb{Q}})$ such that $\sigma(\zeta_5) = \zeta_5^3$ and $\sigma(\mathrm{i}) = \mathrm{i}$. Precisely,

$$\sigma(S) = \begin{bmatrix} 1 & 1 & 2 & 2 & -\sqrt{5} & -\sqrt{5} \\ 1 & 1 & 2 & 2 & \sqrt{5} & \sqrt{5} \\ 2 & 2 & -1+\sqrt{5} & -1-\sqrt{5} & 0 & 0 \\ 2 & 2 & -1-\sqrt{5} & -1+\sqrt{5} & 0 & 0 \\ -\sqrt{5} & \sqrt{5} & 0 & 0 & \sqrt{5} & -\sqrt{5} \\ -\sqrt{5} & \sqrt{5} & 0 & 0 & -\sqrt{5} & \sqrt{5} \end{bmatrix} \quad \text{and} \quad \sigma(T) = \operatorname{diag}(1, 1, \zeta_5^3, \zeta_5^2, \mathrm{i}, -\mathrm{i})\,.$$

In this case, the $e_2 = \iota$. One can use the other spherical structure of $\mathcal{C}$ so that $\mathbb{1} = \iota = e_1$. The resulting modular data is $(P\sigma(S)P, \sigma(T))$, which is the *same* as the modular data $(\sigma(S), \sigma(T))$, and is the modular data of $SO(5)_2$. This completes the proof of Theorem 4.12.

4.4. **Classification of Modular Data of type** $(3, 2, 1)$**.** We now classify modular tensor categories with $\mathrm{SL}_2(\mathbb{Z})$ representations decomposing as a direct sum of irreducible representations of dimension $3, 2$ and 1. The main theorem of this section is:

Theorem 4.15. *The modular data of any type* $(3, 2, 1)$ *modular tensor category is a Galois conjugate of a non-trivial braided zesting of* $SO(5)_2$*.*

The zesting procedure is found in [10]. An alternative approach is to consider the classification of metaplectic modular tensor categories in [1]: this shows that the categories above can be obtained by gauging the particle-hole symmetry (i.e. the $\mathbb{Z}_2$ action $g \leftrightarrow g^{-1}$) on a pointed modular tensor category of the form $\mathcal{C}(\mathbb{Z}_5, q)$. In [15] it is shown that of the 4 modular tensor categories obtained in this way, 2 are $SO(5)_2$ and its (unitary) Galois conjugate and the other two are the non-trivial zesting of $SO(5)_2$ and its (unitary) Galois conjugate.

Let $\rho = \chi_1 \oplus (\rho_2 \otimes \chi_2) \oplus (\rho_3 \otimes \chi_3)$ be the irreducible decomposition a modular representation with ρ_i irreducible of dimension i of prime power level and χ_i a character. This description is possible by the Chinese Remainder Theorem and the fact that 2 and 3 are prime. As before, we may assume $\chi_3 = 1$ and require ρ_3 has a minimal $\mathfrak{t}$-spectrum.

We consider cases in turn, describing the level triples for (ρ_3, ρ_2, χ_1). The $\mathfrak{t}$-spectrum criteria immediately implies that the level of ρ_3 cannot be 7. Similarly the level of ρ_3 cannot be 16: looking at the eigenvalues of the level 16 irreducible 3-dimensional representation we see that $\chi_1(\mathfrak{t}) \notin \mathrm{spec}(\rho_3(\mathfrak{t}))$, and hence $\mathrm{spec}((\rho_2 \otimes \chi_2)(\mathfrak{t})) \cap \mathrm{spec}(\rho_3(\mathfrak{t})) \neq \varnothing$. This implies $\rho_2 \otimes \chi_2$ has level 8 but then $\chi_1(\mathfrak{t}) \notin \mathrm{spec}(\rho_3(\mathfrak{t}) \oplus (\rho_2 \otimes \chi_2)(\mathfrak{t}))$, which contradicts the $\mathfrak{t}$-spectrum criteria.

Suppose the level of ρ_3 is 8. Then $\rho_3 \cong \rho_{2_8^{1,0}}$ or $\rho_{2_8^{3,0}}$, and hence ρ_3 is odd. Note that $\mathrm{spec}(\rho_{2_8^{1,0}}) = \{1, \zeta_8, -\zeta_8\}$ and $\mathrm{spec}(\rho_{2_8^{3,0}}) = \{1, \zeta_8^3, -\zeta_8^3\}$. The level of ρ_2 cannot be 5, by inspection of the corresponding eigenvalues. If the level of ρ_2 is 2 then the $\mathfrak{t}$-spectrum criteria implies that $(\chi_2)^2 = (\chi_1)^2 = 1$. But now $\rho(\mathfrak{s}^2)$ has trace 0, contradicting Proposition 2.1. Thus the level of ρ_2 is either 8 or 3. Applying the $\mathfrak{t}$-spectrum criteria yields the following possible levels in this case: $(8, 8, 1), (8, 3, 3)$ or $(8, 3, 1)$. In particular, if the level of ρ_2 is 8 we cannot have levels $(8, 8, 2)$ or $(8, 8, 4)$ as the $\mathfrak{t}$-spectrum criteria fails in these cases. In all three cases we see that $\rho_2 \otimes \chi_2$ must be odd for otherwise $\mathrm{Tr}(\rho(\mathfrak{s}^2)) = 0$. Hence, the corresponding category would be non-self-dual.

Now suppose that the level of ρ_3 is 5. Then ρ_3 is even. The $\mathfrak{t}$-spectrum criteria implies the level of ρ_2 cannot be 8. Inspecting the remaining possibilities we find the following possible level triples: $(5, 5, 1), (5, 3, 1), (5, 3, 3), (5, 2, 1)$ or $(5, 2, 2)$. The parities imply that the corresponding category would be non-self-dual in the first three cases and self-dual for the last two.

Next if the level of ρ_3 is 4, then $\rho_3 \cong \rho_{3_4^{1,3}}$ which is odd, and has the minimal $\mathfrak{t}$-spectrum $\{1, -1, \mathrm{i}\}$ according Appendix A. The $\mathfrak{t}$-spectrum criteria show that the level of ρ_2 cannot be 8 or 5. If ρ_2 had level 2 then the order of $\rho(\mathfrak{t})$ would be 4, yielding a pointed integral category (by Theorem 3.14) with T-matrix of order 4, which contradicts Proposition 3.16. Thus ρ_2 has level 3 and we find $(4, 3, 1), (4, 3, 2), (4, 3, 3)$ and $(4, 3, 4)$ as possible level triples.

Finally, if the level of ρ_3 were 3 then the $\mathfrak{t}$-spectrum criteria implies that the order of $\rho(\mathfrak{t})$ is a divisor of 6 and hence pointed integral by Theorem 3.14. This contradicts Proposition 3.16.

Below we provide the details of the cases of levels $(4, 3, 2), (5, 2, 2)$ and $(5, 2, 1)$ explicitly. The remaining cases $\{(8, 8, 1), (8, 3, 3), (8, 3, 1), (5, 5, 1), (5, 3, 1), (5, 3, 3), (4, 3, 1), (4, 3, 3), (4, 3, 4)\}$ can be similarly addressed (and indeed are easier). We can eliminate all of these cases computationally as well, see Section B.2.

4.4.1. *Case* $(4,3,2)$. Suppose that the levels of ρ_3, ρ_2 and χ_1 are $4, 3$ and 2, respectively. Without loss of generality we may assume that $\rho_3 \cong \rho_{3_4^{1,3}}$ and $\rho_2 \cong \rho_{2_3^{1,0}}$, which are odd, and respectively have the minimal $\mathfrak{t}$-spectrums $\{1, -1, \mathrm{i}\}$ and $\{1, \zeta_3\}$ according to Appendix A. Let us determine what χ_2 can be. Note that χ_1 is even. Now if χ_2 is odd, then $\mathrm{Tr}(\rho(\mathfrak{s}^2)) = 0$, which is impossible. The $\mathfrak{t}$-spectrum criteria implies that $\chi_2(\mathfrak{t}) \neq \pm\zeta_3$, and a relabeling eliminates ζ_3^{-1}. If $\chi_2(\mathfrak{t}) \in \{-1, -\bar{\zeta}_3\}$, then ρ is projectively equivalent to the complex conjugate the $(4,3,1)$ case. So we will assume that $\chi_2(\mathfrak{t}) = 1, \zeta_3$ or $\rho_2 \otimes \chi_2 \cong \rho_{2_3^{1,0}}$ or $\rho_{2_3^{1,8}}$. In either case, ρ is defined over $\mathbb{Q}_{24}$. If $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}})$ such that $\sigma(\zeta_3) = \zeta_3^2$ and $\sigma(\zeta_8) = \zeta_8$, then we have

$$\rho_3 \oplus \rho_{3_2^{1,8}} \oplus \chi_1 \cong \sigma \circ (\rho_3 \oplus \rho_{3_2^{1,0}} \oplus \chi_1)\,.$$

It suffices to consider $\rho \cong \tilde{\rho} := \rho_3 \oplus \rho_{2_3^{1,0}} \oplus \chi_1$. By Appendix A, we have

$$\tilde{\rho}(\mathfrak{s}) = \frac{\mathrm{i}}{2}\begin{bmatrix} -1 & 1 & \sqrt{2} \\ 1 & -1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{bmatrix} \oplus \frac{\mathrm{i}}{\sqrt{3}}\begin{bmatrix} -1 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \oplus [-1] \quad \text{and} \quad \tilde{\rho}(\mathfrak{t}) = \mathrm{diag}(1, -1, \mathrm{i}, 1, \zeta_3, -1)$$

Reordering $\mathrm{irr}(\mathcal{C})$ so that $\rho(\mathfrak{t}) = \mathrm{diag}(1, 1, -1, -1, \mathrm{i}, \zeta_3)$. By Theorem 3.23, the unit object $\mathbb{1}$ must be an eigenvector $\rho(\mathfrak{t})$ with eigenvalue 1 and so $D = 2/(\frac{1}{2} + \frac{1}{\sqrt{3}}) = 8\sqrt{3} - 12$ or $\dim(\mathcal{C}) = 48(7 - 4\sqrt{3})$. Moreover, $T = \mathrm{diag}(1, 1, -1, -1, \mathrm{i}, \zeta_3)$ and $\rho(\mathfrak{s}) = \frac{-\mathrm{i}(3+2\sqrt{3})}{12} S$, where

$$S = \begin{bmatrix}
1 & -\frac{(2\sqrt{3}-3)\kappa}{2\sqrt{3}+3} & -\frac{3\sqrt{2}a}{2\sqrt{3}+3} & -\frac{3\sqrt{2}b}{2\sqrt{3}+3} & -\frac{6\gamma_1}{2\sqrt{3}+3} & \frac{4\sqrt{3}\gamma_2\kappa}{2\sqrt{3}+3} \\
-\frac{(2\sqrt{3}-3)\kappa}{2\sqrt{3}+3} & 1 & -\frac{3\sqrt{2}a\kappa}{2\sqrt{3}+3} & -\frac{3\sqrt{2}b\kappa}{2\sqrt{3}+3} & -\frac{6\gamma_1\kappa}{2\sqrt{3}+3} & -\frac{4\sqrt{3}\gamma_2}{2\sqrt{3}+3} \\
-\frac{3\sqrt{2}a}{2\sqrt{3}+3} & -\frac{3\sqrt{2}a\kappa}{2\sqrt{3}+3} & -\frac{6(-1+(1+2i)b^2)}{2\sqrt{3}+3} & \frac{(6+12i)ab}{2\sqrt{3}+3} & -\frac{6\sqrt{2}a\gamma_1}{2\sqrt{3}+3} & 0 \\
-\frac{3\sqrt{2}b}{2\sqrt{3}+3} & -\frac{3\sqrt{2}b\kappa}{2\sqrt{3}+3} & \frac{(6+12i)ab}{2\sqrt{3}+3} & \frac{12i(-1+(1-\frac{i}{2})b^2)}{2\sqrt{3}+3} & -\frac{6\sqrt{2}b\gamma_1}{2\sqrt{3}+3} & 0 \\
-\frac{6\gamma_1}{2\sqrt{3}+3} & -\frac{6\gamma_1\kappa}{2\sqrt{3}+3} & -\frac{6\sqrt{2}a\gamma_1}{2\sqrt{3}+3} & -\frac{6\sqrt{2}b\gamma_1}{2\sqrt{3}+3} & 0 & 0 \\
\frac{4\sqrt{3}\gamma_2\kappa}{2\sqrt{3}+3} & -\frac{4\sqrt{3}\gamma_2}{2\sqrt{3}+3} & 0 & 0 & 0 & -\frac{4\sqrt{3}}{2\sqrt{3}+3}
\end{bmatrix}$$

for some $\kappa, \gamma_i \in \{\pm 1\}$ and $a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$. Since $1, \iota \in \{e_1, e_2\}$, $\kappa = -1$, and $a, b \neq 0$. Since $\dim(\iota) = \frac{(2\sqrt{3}-3)}{2\sqrt{3}+3} = 7 - 4\sqrt{3} < 1$, $\mathrm{FPdim}(\mathcal{C}) = 48(7 + 4\sqrt{3})$ and $\iota \neq \mathbb{1}$. We may simply assume $e_1 = \mathbb{1}$ and $e_2 = \iota$. Then $\gamma_1 = 1$, $\gamma_2 = -1$ and $a, b > 0$. Since $\mathrm{Tr}(\rho(\mathfrak{s}^2)) = -4$, there is exactly one dual pair of simple objects, and they can only be e_3, e_4. Therefore, $a = b = \frac{\pm 1}{\sqrt{2}}$ and so $a = b = \frac{1}{\sqrt{2}}$. Thus,

$$S = \begin{bmatrix}
1 & 1-2d & -d & -d & -2d & 2-2d \\
1-2d & 1 & d & d & 2d & 2-2d \\
-d & d & (1-2\mathrm{i})d & (1+2\mathrm{i})d & -2d & 0 \\
-d & d & (1+2\mathrm{i})d & (1-2\mathrm{i})d & -2d & 0 \\
-2d & 2d & -2d & -2d & 0 & 0 \\
2-2d & 2-2d & 0 & 0 & 0 & 2d-2
\end{bmatrix}$$

where $d = 2\sqrt{3} - 3$. Remarkably, the Verlinde formula yields a consistent set of fusion rules. For example the object with twist i has the fusion matrix:

$$N_5 = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 4 & 2 & 2 & 3 & 4 \\
0 & 2 & 2 & 0 & 1 & 2 \\
0 & 2 & 0 & 2 & 1 & 2 \\
1 & 3 & 1 & 1 & 4 & 4 \\
0 & 4 & 2 & 2 & 4 & 4
\end{bmatrix}.$$

However, the second FS-indicator for this object is $\nu_2(e_5) = \frac{1}{\dim(\mathcal{C})}\sum_{j,k} N_{j,k}^5 d_j d_k \left(\frac{\theta_j}{\theta_k}\right)^2 = 2$, a contradiction.

4.4.2. *Case* $(5,2,1)$. Consider the case of levels $(5,2,1)$. Then $\rho \cong \rho_{3_5^1} \oplus \rho_{2_2^{1,0}} \oplus \chi^0$ or $\rho_{3_5^3} \oplus \rho_{2_2^{1,0}} \oplus \chi^0$ according to Appendix A. Since the latter is a Galois conjugate of the former one, it suffices to solve the first case. Let $\tilde{\rho} = \rho_{2_2^{1,0}} \oplus \chi^0 \oplus \rho_{3_5^3}$ in which $\tilde{\rho}(\mathfrak{t}) = \text{diag}\left(1, -1, 1, 1, \zeta_5, \zeta_5^4\right)$. By permuting the first two basis elements, we may assume that $t = \rho(\mathfrak{t}) = \text{diag}(-1, 1, 1, 1, \zeta_5, \zeta_5^4)$. Conjugating by a block diagonal matrix of the form $(r_1) \oplus F \oplus (r_2) \oplus (r_3)$ where $F = \begin{bmatrix} f_{1,1} & f_{1,2} & f_{1,3} \\ f_{2,1} & f_{2,2} & f_{2,3} \\ f_{3,1} & f_{3,2} & f_{3,3} \end{bmatrix}$ is real orthogonal matrix (cf. Prop. 3.4) and $r_i = \pm 1$. One may assume $r_1 = 1$, and we find that $\pm S/D = s = \rho(\mathfrak{s})$ has the form:

$$\begin{bmatrix} \frac{1}{2} & -\frac{f_{1,1}\sqrt{3}}{2} & -\frac{f_{2,1}\sqrt{3}}{2} & -\frac{f_{3,1}\sqrt{3}}{2} & 0 & 0 \\ -\frac{f_{1,1}\sqrt{3}}{2} & & & & \frac{f_{1,3}\sqrt{10}r_2}{5} & \frac{f_{1,3}\sqrt{10}r_3}{5} \\ -\frac{f_{2,1}\sqrt{3}}{2} & & A & & \frac{f_{2,3}\sqrt{10}r_2}{5} & \frac{f_{2,3}\sqrt{10}r_3}{5} \\ -\frac{f_{3,1}\sqrt{3}}{2} & & & & \frac{f_{3,3}\sqrt{10}r_2}{5} & \frac{f_{3,3}\sqrt{10}r_3}{5} \\ 0 & \frac{f_{1,3}\sqrt{10}r_2}{5} & \frac{f_{2,3}\sqrt{10}r_2}{5} & \frac{f_{3,3}\sqrt{10}r_2}{5} & -\frac{\sqrt{5}+5}{10} & \frac{r_2\sqrt{5}(\sqrt{5}-1)r_3}{10} \\ 0 & \frac{f_{1,3}\sqrt{10}r_3}{5} & \frac{f_{2,3}\sqrt{10}r_3}{5} & \frac{f_{3,3}\sqrt{10}r_3}{5} & \frac{r_2\sqrt{5}(\sqrt{5}-1)r_3}{10} & -\frac{\sqrt{5}(\sqrt{5}+1)}{10} \end{bmatrix}$$

where

$$A = \begin{bmatrix} -\frac{f_{1,1}{}^2}{2} + f_{1,2}{}^2 + \frac{f_{1,3}{}^2\sqrt{5}}{5} & * & * \\ -\frac{f_{1,1}f_{2,1}}{2} + f_{1,2}f_{2,2} + \frac{f_{1,3}\sqrt{5}f_{2,3}}{5} & -\frac{f_{2,1}{}^2}{2} + f_{2,2}{}^2 + \frac{f_{2,3}{}^2\sqrt{5}}{5} & * \\ f_{1,2}f_{3,2} + \frac{f_{1,3}\sqrt{5}f_{3,3}}{5} - \frac{f_{1,1}f_{3,1}}{2} & f_{2,2}f_{3,2} + \frac{f_{2,3}\sqrt{5}f_{3,3}}{5} - \frac{f_{2,1}f_{3,1}}{2} & f_{3,2}{}^2 + \frac{f_{3,3}{}^2\sqrt{5}}{5} - \frac{f_{3,1}{}^2}{2} \end{bmatrix}.$$

First we observe that the FP-dimensions and categorical dimensions (which may coincide) must appear as multiples of one of the columns 2, 3 or 4. Moreover, since our category is non-integral by Proposition 3.16, the Galois orbit of the dimension column has size 2. The FP-dimension column of s must have all the same sign, which implies that $r_2 = r_3$.

Let $\sigma \in \text{Gal}(\mathbb{Q}_5/\mathbb{Q})$ be the automorphism defined by $\zeta_5 \to \zeta_5^3$. By Galois symmetry we have: $\hat{\sigma}(1) = 1$, $\hat{\sigma}(5) = 6$. Therefore, $\hat{\sigma}$ has order 2. Reordering the rows of F if necessary (which permutes the corresponding rows/columns of s) we may assume that $\hat{\sigma}(2) = 2$ and $\hat{\sigma}(3) = 4$, so that the FP-dimensions and categorical dimensions correspond to either columns 3 or 4 (or one of each).

We will make frequent use of the fact that $\sigma(s_{ij}) = \epsilon_\sigma(i)s_{\hat{\sigma}(i),j} = \epsilon_\sigma(j)s_{i,\hat{\sigma}(j)}$ where $\epsilon_\sigma(i)$ is a sign.

Now $1/2 = \sigma(s_{1,1}) = \epsilon_\sigma(1)/2$ so that $\epsilon_\sigma(1) = 1$. By a similar computation $\sigma(s_{1,2}) = \epsilon_\sigma(1)s_{1,2} = \epsilon_\sigma(2)s_{1,2}$, so that $\epsilon_\sigma(2) = 1$. From $\sigma(s_{5,5}) = \frac{\sqrt{5}-5}{10} = \epsilon_\sigma(5)s_{5,6}$ we find that $\epsilon_\sigma(5) = -1$. Now we compute two ways: $\sigma(s_{2,5}) = \epsilon_\sigma(2)s_{2,5} = s_{2,5} = \epsilon_\sigma(5)s_{2,6} = -s_{2,6} = -s_{2,5}$, which implies $s_{2,5} = 0$ so that $f_{1,3} = 0$. Now $\sigma(s_{3,5}) = \epsilon_\sigma(3)s_{4,5} = \epsilon_\sigma(5)s_{3,6} = -s_{3,6}$ implies $f_{3,3} = \pm f_{2,3}$ so that $(f_{2,3})^2 = \frac{1}{2}$. Applying a similar calculation we see that $\sigma(s_{1,3}) = s_{1,3} = \epsilon_\sigma(3)s_{1,4}$ implies $f_{2,1} = \pm f_{3,1}$. Setting

$z = f_{1,1}$ and $y = f_{2,1}$ orthogonality yields the following:

$$F = \begin{bmatrix} z & \delta_1\sqrt{2}y & 0 \\ y & \frac{-\delta_1 z}{\sqrt{2}} & \frac{-\delta_2\delta_3}{\sqrt{2}} \\ \delta_2 y & \frac{-\delta_1\delta_2 z}{\sqrt{2}} & \frac{\delta_3}{\sqrt{2}} \end{bmatrix}.$$

One important consequence is that there are only 2 rows of $\rho(\mathfrak{s})$ that have strictly non-zero entries: the 3rd and the 4th.

Next we find that $\sigma(s_{i,1}) = s_{i,1}$ since $\epsilon_\sigma(1) = 1$ and $\hat{\sigma}(1) = 1$. Thus $f_{i,1}\sqrt{3} \in \mathbb{Q}$. Note that $z^2 + 2y^2 = 1$ where $z, y \in \frac{1}{\sqrt{3}}\mathbb{Q}$, and one of $s_{2,1}/s_{3,1} = \pm s_{2,1}/s_{4,1}$ is of the form $S_{X,Y}/d_X$, i.e., an eigenvalue of a fusion matrix. In particular $z/y = \gamma$ is a (rational) algebraic integer, i.e., $\gamma \in \mathbb{Z}$. From this we find that $\gamma^2 + 2 = 1/y^2 \in \mathbb{Z}$ so that $0 < y^2 \leqslant 1/3$, and so $1/3 \leqslant z^2 \leqslant 1$.

Let us compute the values of the submatrix A above. We have:

$$A = \begin{bmatrix} -z^2/2 + 2y^2 & * & * \\ -3yz/2 & \frac{1}{2}(z^2 - y^2 + \frac{1}{\sqrt{5}}) & * \\ -\delta_2 3yz/2 & \frac{\delta_2}{2}(z^2 - y^2 - \frac{1}{\sqrt{5}}) & \frac{1}{2}(z^2 - y^2 + \frac{1}{\sqrt{5}}) \end{bmatrix}.$$

Since the unit object can only correspond to either row 3 or 4 and $s_{32} = \pm s_{42}$, s_{22}/s_{32} is an algebraic integer in $\mathbb{Q}(\sqrt{5})$. Note that

$$\frac{s_{22}}{s_{32}} = \frac{\gamma}{3} - \frac{4}{3\gamma} = \frac{\gamma^2 - 4}{3\gamma} \in \mathbb{Q}.$$

Therefore, $\frac{\gamma^2-4}{3\gamma} \in \mathbb{Z}$ and so $\gamma \mid 4$. Thus, $\gamma^2 = 1, 4$ or 16. However, if $\gamma^2 = 4$ or 16, $y = \frac{\pm 1}{\sqrt{\gamma^2+2}} \notin \frac{1}{\sqrt{3}}\mathbb{Q}$. Thus, $\gamma^2 = 1$ or $z = \pm y$.

This implies that $y = \pm\frac{1}{\sqrt{3}}$, from which we compute: $f_{2,2} = \pm\frac{1}{\sqrt{6}}$, $f_{3,2} = \pm\frac{1}{\sqrt{6}}$, $f_{1,1} = \pm\frac{1}{\sqrt{3}}$, and $f_{1,2} = \pm\frac{2}{\sqrt{6}}$.

Now we may assume $F = \begin{bmatrix} -1/\sqrt{3} & 2x_1/\sqrt{6} & 0 \\ x_2/\sqrt{3} & x_3/\sqrt{6} & x_4/\sqrt{2} \\ x_5/\sqrt{3} & x_6/\sqrt{6} & x_7/\sqrt{2} \end{bmatrix}$ where the $x_i = \pm 1$ after an overall rescaling by ± 1. Orthogonality of F implies several additional conditions on the x_i, so that all are determined by the values of x_2, x_4, x_5 and x_7.

Substituting into s above, rescaling by $\pm D$ and permuting the rows/columns so that the two non-zero rows appear first, we have:

$$S = \begin{bmatrix} 1 & x_4x_7 & \sqrt{5}x_5 & \sqrt{5}x_5 & 2\,x_7r_3 & 2\,x_7r_3 \\ x_4x_7 & 1 & \sqrt{5}x_2 & \sqrt{5}x_2 & 2\,x_4r_3 & 2\,x_4r_3 \\ \sqrt{5}x_5 & \sqrt{5}x_2 & \sqrt{5} & -\sqrt{5} & 0 & 0 \\ \sqrt{5}x_5 & \sqrt{5}x_2 & -\sqrt{5} & \sqrt{5} & 0 & 0 \\ 2\,x_7r_3 & 2\,x_4r_3 & 0 & 0 & -\sqrt{5}-1 & \sqrt{5}-1 \\ 2\,x_7r_3 & 2\,x_4r_3 & 0 & 0 & \sqrt{5}-1 & -\sqrt{5}-1 \end{bmatrix}.$$

Thus we see that the dimensions and FP-dimensions must be, up to sign choices, among $1, 2, \sqrt{5}$. In particular, any such category must be weakly integral, and there is an invertible object of order 2. Therefore, are two spherical structures on $\mathcal{C}$ which make $\mathbb{1} = \iota$ or $\mathbb{1} \neq \iota$. We may assume $\mathbb{1}$ corresponds to the first row. For the first case, we find $x_4x_7 = x_5 = x_7r_3 = 1$ and $x_2 = -1$.

Thus we obtain the following S-matrix:

$$S = \begin{bmatrix} 1 & 1 & \sqrt{5} & \sqrt{5} & 2 & 2 \\ 1 & 1 & -\sqrt{5} & -\sqrt{5} & 2 & 2 \\ \sqrt{5} & \sqrt{5} & \sqrt{5} & -\sqrt{5} & 0 & 0 \\ \sqrt{5} & \sqrt{5} & -\sqrt{5} & \sqrt{5} & 0 & 0 \\ 2 & 2 & 0 & 0 & -\sqrt{5}-1 & \sqrt{5}-1 \\ 2 & 2 & 0 & 0 & \sqrt{5}-1 & -\sqrt{5}-1 \end{bmatrix}$$

For the second case, one can obtain the same S-matrix except the first two rows/columns are interchanged, but the T-matrix is unchanged. Therefore, we have only one modular data for either case.

Applying σ to (S,T), we obtain the modular data for $\rho \cong \rho_{3_5^3} \oplus \rho_{2_2^{1,0}} \oplus \chi^0$ with the T-matrix given by $\sigma(T) = \mathrm{diag}(1,1,1,-1,\zeta_5^3,\zeta_5^2)$. Both of these modular data (S,T) and $(\sigma(S),\sigma(T))$ are modular data of non-trivial braided zesting of MTCs (see [10]) of type (3,3). Notice that the MTCs of type (3,3) have T-matrix of order 20.

4.4.3. *Case* $(5,2,2)$. It suffice to consider the case with $\rho \cong \tilde{\rho} := \rho_{3_5^1} \oplus \rho_{2_2^{1,0}} \oplus \chi^6$. Then

$$\tilde{\rho}(\mathfrak{s}) = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\varphi & \varphi^{-1} \\ -\sqrt{2} & \varphi^{-1} & -\varphi \end{bmatrix} \oplus \frac{1}{2} \begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} \oplus [-1] \quad \text{and} \quad \tilde{\rho}(\mathfrak{t}) = \mathrm{diag}(1,\zeta_5,\zeta_5^4,1,-1,-1)\,.$$

Permute $\mathrm{irr}(\mathcal{C})$ so that $\rho(\mathfrak{t}) = \mathrm{diag}(-1,-1,1,1,\zeta_5,\zeta_5^4)$. By Theorem 3.23, the objects $\mathbb{1}, \iota \in \{e_3, e_4\}$, $D = 2/(\frac{1}{2} - \frac{1}{\sqrt{5}}) = 20 + 8\sqrt{5}$, and

$$s := \rho(\mathfrak{s}) = \begin{bmatrix} \frac{3b^2}{2}-1 & -\frac{1}{2}(3ab) & \frac{1}{2}\sqrt{\frac{3}{2}}b & \frac{1}{2}\sqrt{\frac{3}{2}}b\kappa & 0 & 0 \\ -\frac{1}{2}(3ab) & \frac{1}{2}\left(1-3b^2\right) & -\frac{1}{2}\sqrt{\frac{3}{2}}a & -\frac{1}{2}\sqrt{\frac{3}{2}}a\kappa & 0 & 0 \\ \frac{1}{2}\sqrt{\frac{3}{2}}b & -\frac{1}{2}\sqrt{\frac{3}{2}}a & \frac{1}{20}\left(2\sqrt{5}-5\right) & -\frac{1}{20}\left(2\sqrt{5}+5\right)\kappa & -\frac{\gamma_1\kappa}{\sqrt{5}} & -\frac{\gamma_2\kappa}{\sqrt{5}} \\ \frac{1}{2}\sqrt{\frac{3}{2}}b\kappa & -\frac{1}{2}\sqrt{\frac{3}{2}}a\kappa & -\frac{1}{20}\left(2\sqrt{5}+5\right)\kappa & \frac{1}{20}\left(2\sqrt{5}-5\right) & \frac{\gamma_1}{\sqrt{5}} & \frac{\gamma_2}{\sqrt{5}} \\ 0 & 0 & -\frac{\gamma_1\kappa}{\sqrt{5}} & \frac{\gamma_1}{\sqrt{5}} & \frac{1}{10}\left(-\sqrt{5}-5\right) & \frac{2\gamma_1\gamma_2}{\sqrt{5}+5} \\ 0 & 0 & -\frac{\gamma_2\kappa}{\sqrt{5}} & \frac{\gamma_2}{\sqrt{5}} & \frac{2\gamma_1\gamma_2}{\sqrt{5}+5} & \frac{1}{10}\left(-\sqrt{5}-5\right) \end{bmatrix}$$

for some $\kappa, \gamma_1, \gamma_2 \in \{\pm 1\}$ and $a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$. Since $\frac{5+2\sqrt{5}}{5-2\sqrt{5}} > 1$, $\iota = \mathbb{1}$. We may simply assume $e_4 = \mathbb{1}$. Then $\kappa = 1$, $\gamma_1 = \gamma_2 = -1$ and $a > 0$ and $b < 0$.

By Proposition 3.16, $\mathcal{C}$ is not integral. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_5/\mathbb{Q})$ be a generator. Then $\hat{\sigma}(3) = 4$ and $\epsilon_\sigma(3) = 1$ since $\sigma(s_{3,5}) = s_{4,5}$. Therefore, σ fixes $s_{3,1}, s3,2$, and so $\sqrt{\frac{3}{2}}a, \sqrt{\frac{3}{2}}b \in \mathbb{Q}$. Now, $s_{2,1} = s_{2,1} \in \mathbb{Q}$ since $ab \in \mathbb{Q}$. By Theorem 3.7, $\frac{s_{2,1}}{s4,1}$ and $\frac{s_{1,2}}{s4,2}$ are in $\mathbb{Z}[\zeta_5] \cap \mathbb{Q}$, $\sqrt{6}a, \sqrt{6}b \in \mathbb{Z}$ and $(\sqrt{6}a)^2 + (\sqrt{6}b)^2 = 6$. But the Diophantine equation $X^2 + Y^2 = 6$ has no integral solutions, so we conclude that $\tilde{\rho}$ has no realization.

4.5. **Classification of modular data of type (6).** In this subsection, we discuss the possible rank-6 MDs of type (6) (*i.e.* MDs from dimension-6 irreducible $\mathrm{SL}_2(\mathbb{Z})$ symmetric representations). This part of the classification relies upon computer computations.

Theorem 4.16. *Let $\mathcal{C}$ be a rank 6 modular tensor category of type (6) with $\dim(\mathcal{C}) = D^2 \notin \mathbb{Z}$. Then the modular data of $\mathcal{C}$ can be obtained, up to a choice of (spherical) pivotal structure, as a Galois conjugate of the modular data of the following modular tensor categories:*

(i) $PSU(2)_{11}$ *(entry 10 in Appendix C.2);*
(ii) $PSU(2)_3 \boxtimes PSU(2)_5$ *(entry 20 in Appendix C.2);*
(iii) $SU(2)_1 \boxtimes PSU(2)_5$ *(entry 24 in Appendix C.2);*
(iv) $PSU(2)_3 \boxtimes SU(2)_2$ *(entry 36 in Appendix C.2).*
(v) $PSU(2)_3 \boxtimes E(8)_2$ *(entry 28 in Appendix C.2).*
(vi) $PSO(5)_{3/2}$ *(non-unitary, entry 9 in Appendix C.2);*

It is worth noting that (i), (ii) and (vi) have a unique pivotal structure, up to equivalence (cf. [6]). The categories (i) and (ii) are *transitive* [29], and they are completely determined by their modular data. We note that by [35], any *fusion* category with the same fusion rules as those of (vi) is non-pseudo-unitary.

Recall that a symmetric $\mathrm{SL}_2(\mathbb{Z})$ representation ρ is defined to be an unitary representation which has diagonal $\rho(\mathfrak{t})$ and symmetric $\rho(\mathfrak{s})$. Every finite-dimensional representation of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is equivalent to a symmetric one. Two symmetric $\mathrm{SL}_2(\mathbb{Z})$ representations are equivalent if and only if they are related by a conjugation of a real orthogonal matrix (see Theorem 3.4). There are 70 inequivalent 6-dimensional symmetric irreducible $\mathrm{SL}_2(\mathbb{Z})$ representations of prime-power levels (cf. Appendix A). Up to tensoring one of the 12 1-dimensional representations, other 6-dimensional irreducible representations are tensor products of one of the 11 2-dimensional and one of the 33 3-dimensional irreducible symmetric representations of distinct prime-power levels.

Since there are only a finite number of $\mathrm{SL}_2(\mathbb{Z})$ representations, up to equivalence, for any given dimension, we can examine representatives of each of those symmetric representations by computer and reject those representations that do not satisfy the following necessary conditions (for a symmetric $\mathrm{SL}_2(\mathbb{Z})$ representation equivalent to an MD representation):

(1) If all the eigenvalues of $\rho(\mathfrak{t})$ are distinct (non-degenerate) then $\rho(\mathfrak{s})$ has a row that contains no zero. Note that when $\rho(\mathfrak{t})$ has non-degenerate spectrum, the matrix $\rho(\mathfrak{s})$ differs from that of an MD representation only by a conjugation by signed diagonal matrix. In this case, $\rho(\mathfrak{s})$ must have a row that contains no zero (i.e. the row corresponding to the unit object).
(2) Let $\rho(\mathfrak{s})^{\mathrm{ndeg}}$ (or M^{ndeg}) be the non-degenerate block of $\rho(\mathfrak{s})$ (or M), (i.e., corresponding to the multiplicity 1 eigenvalues of the diagonal matrix $\rho(\mathfrak{t})$, see section 3.4). Then the conductor of $\rho(\mathfrak{s})^{\mathrm{ndeg}}$ divides $\mathrm{ord}(\rho(\mathfrak{t}))$ (cf. Proposition 3.9). If the $\rho(\mathfrak{t})$-spectrum is non-degenerate then we may drop the ndeg superscript.
(3) $\sigma(\rho(\mathfrak{s})^{\mathrm{ndeg}}) = \left(\rho^a(\mathfrak{t})\rho(\mathfrak{s})\rho^b(\mathfrak{t})\rho(\mathfrak{s})\rho^a(\mathfrak{t})\right)^{\mathrm{ndeg}}$ for any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}})$, where $\sigma(\zeta_n) = \zeta_n^a$ for an unique integer a modulo n. Here $n = \mathrm{ord}(\rho(\mathfrak{t}))$ and b satisfies $ab \equiv 1 \mod n$ (cf. Theorem 3.7). Again, this is because $\rho(\mathfrak{s})^{\mathrm{ndeg}}$ can only differ from that of an MD representation by a conjugation of signed diagonal matrix.

Since the weakly integral rank-6 MD of MTCs are classified, we can exclude symmetric $\mathrm{SL}_2(\mathbb{Z})$ representations that must produce such MDs. Thus we also reject the representations that satisfy the following conditions, both of which imply weak integrality:

(1) $\mathrm{pord}(\rho(\mathfrak{t})) \in \{2, 3, 4, 6\}$. In fact, this implies the category is pointed, see Proposition 3.16(i).
(2) The squares of the matrix entries of $\rho(\mathfrak{s})$ in each row containing no zeros are all rational numbers, and ρ is non-degenerate. Indeed, in this case $1/D^2, (d_i/D)^2$ and $(d_i \,\mathrm{FPdim}(X_i)/D)^2$ are rational, where column i is the unique strictly positive (or negative) column. (This condition only rejects one case. See entry 566 in the Supplementary material section of the arXiv version of this paper.)

We remark that there are 6-dimensional irreducible $\mathrm{SL}_2(\mathbb{Z})$ representations where $\rho(\mathfrak{t})$ are degenerate, for example, the representation 6_5^1 in Appendix A. Such a representation is rejected since the conductor of $\rho(\mathfrak{s})^{\mathrm{ndeg}}$ is 40 which does not divides $\mathrm{ord}(\rho(\mathfrak{t})) = 5$ (see also entry 582 in Supplementary material Section of the arXiv version of this paper).

All the passing symmetric $\mathrm{SL}_2(\mathbb{Z})$ representations can be grouped into orbits generated by Galois conjugations and tensoring 1-dimensional representations. There are 7 such orbits. A representative for each orbit is listed in Section B.2, which have (dims; levels) = (6; 9), (6; 13), (6; 15), (6; 16), (6; 35), (6; 56), (6; 80).

Fortunately, we find that all these $\mathrm{SL}_2(\mathbb{Z})$ representations have non-degenerate $\rho(\mathfrak{t})$, so they can only possibly differ from an MD representation by a conjugation of signed diagonal matrix, if they indeed are associated with MDs. We can then search through the finite number of signed diagonal conjugations, and find the (S, T) matrices that satisfy the conditions listed in Theorems 2.1 and 3.7. The results are given in Section C.2, where (S, T) matrices are found from $\mathrm{SL}_2(\mathbb{Z})$ representations that have (dims; levels) = (6; 9), (6; 13), (6; 16), (6; 35), (6; 56), (6; 80). Those computer assisted calculations are described in detail in the Appendix.

5. Classification of modular data of rank=6: non-admissible types

In this section, we complete the classification of rank=6 MDs by eliminating the remaining types.

Theorem 5.1. *There are no rank=6 MTCs of types* $(3,1,1,1), (2,2,2), (2,2,1,1), (2,1,1,1,1), (5,1)$, *or* $(1,1,1,1,1,1)$.

Obviously, type Vec is the only MTC of type (1). However, no MTCs of rank $n > 1$ is of type $(1, \ldots, 1)$, as the associated $\mathrm{SL}_2(\mathbb{Z})$ representations $\rho \cong n\chi^i$ for some integer i by Corollary 3.21. In particular, $\rho(\mathfrak{s})$ has zeros in each row if $n > 1$.

5.1. Nonexistence of type $(3, 1, 1, 1)$.

Proposition 5.2. *There does not exist any modular tensor category of type (3,1,1,1).*

Proof. Assume contrary. Let $\mathcal{C}$ be a modular tensor category of type $(3, 1, 1, 1)$ and ρ an $\mathrm{SL}_2(\mathbb{Z})$ representation of $\mathcal{C}$. Then

$$\rho \cong \rho_0 \oplus \chi_1 \oplus \chi_2 \oplus \chi_3 \, .$$

where ρ_0 is irreducible of dimension 3 and $\chi_i, i = 1, 2, 3$, are 1-dimensional representations. By Lemma 3.20, $\mathrm{spec}(\chi_i(\mathfrak{t})) \subset \mathrm{spec}(\rho_0(\mathfrak{t}))$ for $i = 1, 2, 3$. One may assume ρ_0 has a minimal $\mathfrak{t}$-spectrum. Then ρ_0 must have a prime power level. By Appendix A, the level of ρ_0 can only be 3, 4, 5, 7, 8 or 16. The $\mathfrak{t}$-spectrum of any 3-dimensional irreducible representations of level 7 or 16 does not contain any 12-th root of unity. Therefore, the level of ρ_0 can only be 3, 4, 5, or 8. It suffices to show that none of these levels is possible.

If ρ_0 were of level 3 or 4, then $\mathrm{ord}(\rho(\mathfrak{t})) = 3$ or 4, by Lemma 3.20. This implies $\mathrm{ord}(T) = 2, 3$ or 4 and hence $\mathcal{C}$ is integral by Theorem 3.14. By Proposition 3.16, $\mathcal{C}$ must be of type (4,2), a contradiction. Therefore, ρ_0 can only be of level 5 or 8.

If ρ_0 were of level 5, then $\mathrm{ord}(\rho(\mathfrak{t})) = 5$ by Lemma 3.20. Hence, $\mathrm{ord}(T) = 5$ which is not possible by Proposition 3.22.

If the level of ρ_0 were 8, then $\rho_0 \cong \rho_{3_8^{1,0}}$ or $\rho_{3_8^{3,0}}$ as they are the 3-dimensional irreducible representations of level 8 with a minimal $\mathfrak{t}$-spectrum. In either case, $\mathrm{spec}(\rho_0(\mathfrak{t}))$ has exactly one 12-th root of unity, which is 1, and ρ_0 is odd. Therefore, $\rho \cong \rho_0 \oplus 3\chi^0$ by Corollary 3.21. This implies $\mathrm{Tr}(\rho(\mathfrak{s}^2)) = 0$, which is impossible for any MD representation. $\square$

5.2. **Nonexistence of types (2,2,2), (2,2,1,1) and (2,1,1,1,1).** We will prove the following theorem which leads to the nonexistence of modular tensor categories of these types.

Theorem 5.3. *Let $\mathcal{C}$ a be modular tensor category with $\operatorname{rank}\mathcal{C} > 2$, and ρ an $\mathrm{SL}_2(\mathbb{Z})$ representation of $\mathcal{C}$. If all the irreducible subrepresentations of ρ have dimensions $\leqslant 2$, then $\operatorname{ord}(T) = 1, 2, 3, 4$, or 6 and therefore $\mathcal{C}$ is integral.*

Proof. If every irreducible subrepresentation of ρ is 1-dimensional, then $\mathcal{C}$ is of type $(1, \dots, 1)$ which can only be trivial by the beginning remark of this section. In particular, $\operatorname{ord}(T) = 1$ and $\mathcal{C}$ is integral.

Now, we assume ρ admits a 2-dimensional irreducible subrepresentation ρ_0. By tensoring a 1-dimensional representation to ρ, we may assume the level of ρ_0 to be $2, 3, 5$, or 8.

Suppose ρ_0 is of level 5. Then each irreducible subrepresentations ρ_0' of ρ which is not isomorphic to ρ_0 satisfies $\operatorname{spec}(\rho_0'(\mathfrak{t})) \cap \operatorname{spec}(\rho_0(\mathfrak{t})) = \varnothing$ by Appendix A. This implies $\rho \cong \ell\rho_0$, but this is impossible by Proposition 3.19. Therefore, ρ_0 cannot have level 5.

Assume ρ_0 is of level 8. Note that the $\mathfrak{t}$-spectrum of any 2-dimensional level 8 irreducible representation consists of primitive 8-th roots of unity. By the $\mathfrak{t}$-spectra criterion and Appendix A, all the irreducible subrepresentations of ρ are of dimension 2 and level 8. In particular, $\operatorname{ord}(T) = \operatorname{pord}(\rho(\mathfrak{t})) = 4$.

If ρ_0 is of level 2 or 3, it follows from the preceding discussion that all the 2-dimensional irreducible subrepresentations of ρ are of level 2 or 3. By Lemma 3.20, $\operatorname{ord}(\rho(\mathfrak{t})) = 2, 3$ or 6 and so $\operatorname{ord}(T) = 2, 3$ or 6.

The last assertion follows from Theorem 3.14. □

Corollary 5.4. *There is no modular tensor category of types* $(2, 2, 2)$, $(2, 2, 1, 1)$ *or* $(2, 1, 1, 1, 1)$.

Proof. Suppose there exists a modular tensor category $\mathcal{C}$ of any of these types. By Theorem 5.3, $\mathcal{C}$ is integral, but this contradicts Proposition 3.16 which shows $\mathcal{C}$ is of type $(4, 2)$. □

5.3. **Nonexistence of type** $(5, 1)$**.** Suppose that $\mathcal{C}$ is a modular tensor category of type $(5, 1)$, and ρ an $\mathrm{SL}_2(\mathbb{Z})$ representation of $\mathcal{C}$. Then $\mathcal{C}$ is not integral by Proposition 3.16, and $\rho \cong \rho_0 \oplus \rho_1$ where ρ_0, ρ_1 are irreducible of dimension 5 and 1 respectively. By tensoring a 1-dimensional representation of $\mathrm{SL}_2(\mathbb{Z})$, one may assume ρ_0 is of prime power level. By Appendix A, the level of ρ_0 can only be 11 or 5.

In the former case the $\mathfrak{t}$-spectrum consists primitive 11-th roots of unity. Since $\rho_1(\mathfrak{t})$ is a 12 root of unity, the $\mathfrak{t}$-spectrum criteria shows this is impossible.

Now if ρ_2 has level 5 and $\rho_2 \cong \rho_{5^1_5}$. This implies $\rho_1 \cong \chi^0$. Let $\tilde{\rho} = \chi^0 \oplus \rho_{5^1_5}$. Then $\tilde{\rho}(\mathfrak{t}) = \operatorname{diag}(1, 1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4)$, and

$$\tilde{\rho}(\mathfrak{s}) = [1] \oplus \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{\sqrt{6}}{5} & \frac{\sqrt{6}}{5} & \frac{\sqrt{6}}{5} & \frac{\sqrt{6}}{5} \\ 0 & \frac{\sqrt{6}}{5} & \frac{3-\sqrt{5}}{10} & -\frac{1}{5} - \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} - \frac{1}{5} & \frac{3+\sqrt{5}}{10} \\ 0 & \frac{\sqrt{6}}{5} & -\frac{1}{5} - \frac{1}{\sqrt{5}} & \frac{3+\sqrt{5}}{10} & \frac{3-\sqrt{5}}{10} & \frac{1}{\sqrt{5}} - \frac{1}{5} \\ 0 & \frac{\sqrt{6}}{5} & \frac{1}{\sqrt{5}} - \frac{1}{5} & \frac{3-\sqrt{5}}{10} & \frac{3+\sqrt{5}}{10} & -\frac{1}{5} - \frac{1}{\sqrt{5}} \\ 0 & \frac{\sqrt{6}}{5} & \frac{3+\sqrt{5}}{10} & \frac{1}{\sqrt{5}} - \frac{1}{5} & -\frac{1}{5} - \frac{1}{\sqrt{5}} & \frac{3-\sqrt{5}}{10} \end{bmatrix}.$$

There exists a real orthogonal matrix $U = \operatorname{diag}(f, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ such that $\rho(\mathfrak{s}) = U\tilde{\rho}(\mathfrak{s})U^\top$ and $\rho(\mathfrak{t}) = \tilde{\rho}(\mathfrak{t})$, where $f \in O_2(\mathbb{R})$ and $\varepsilon_i = \pm 1$.

The group $\operatorname{Gal}(\mathbb{Q}_5/\mathbb{Q})$ is generated by σ defined by $\sigma(\zeta_5) = \zeta_5^2$, and

$$D_{\tilde{\rho}}(\sigma) = I_2 \oplus J_4 \quad \text{where } J_4 = [\delta_{i,5-j}]_{1 \leqslant i,j \leqslant 4}\,.$$

So $\hat{\sigma}$ fixes 1 and 2. Since $\mathcal{C}$ is non-integral, the row corresponding to $\mathbb{1}$ must be one of the last 4. Since $\rho(\mathfrak{s})^{\text{ndeg}}$ and $\tilde{\rho}(\mathfrak{s})^{\text{ndeg}}$ are the same up to some signs, $D = \frac{10}{3\pm\sqrt{5}}$ which has norm 25.

Observe that each row of $\tilde{\rho}(\mathfrak{s})^{\text{ndeg}}$ has the entries $-\frac{1}{5} \pm \frac{1}{\sqrt{5}}$. Therefore, $(-\frac{1}{5} \pm \frac{1}{\sqrt{5}})/\frac{3\mp\sqrt{5}}{10} = 1 \pm \sqrt{5}$ are dimensions of some objects up to a sign. However, their norms are -4 which is not a divisor of 25, a contradiction. So, we conclude that such a category cannot exist.

6. Summary and Future Directions

We have developed tools for classifying modular data directly from representations of $\mathrm{SL}_2(\mathbb{Z})$, and have applied them to provide a classification of rank 6 modular data. Sufficiently many of these tools have been implemented as computer algorithms to yield a purely computational approach to the rank 6 classification. A purely "by hand" approach to higher ranks is too involved for the currently theory, but the computational approach can be implemented in higher ranks. It should be noted that in this work we used the classification of weakly integral modular data [4] of rank up to 7 to simplify the computer calculations. For higher ranks this will require further work.

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A. List of $\mathrm{SL}_2(\mathbb{Z})$ irreducible representations of prime-power levels

In this section, we list all the $\mathrm{SL}_2(\mathbb{Z})$ symmetric irreducible representations of dimension $1-6$, whose level ($l = \mathrm{ord}(\rho(\mathfrak{t}))$) is a power of single prime number, which are generated by the GAP program [27]. In the list, $\rho(\mathfrak{t})$ is presented in term of topological spins $(\tilde{s}_1, \tilde{s}_2, \cdots)$ ($\tilde{s}_i = \arg(\rho_a(\mathfrak{t})_{ii})$).

Note that $\rho(\mathfrak{s})$ is symmetric and $\rho(\mathfrak{s})_{ij}$'s are either all real or all imaginary. When $\rho(\mathfrak{s})_{ij}$'s are all real, $\rho(\mathfrak{s})$ is presented as $(\rho_{11}, \rho_{12}, \rho_{13}, \rho_{14}, \cdots;\ \ \rho_{22}, \rho_{23}, \rho_{24}, \cdots)$. In this case, $\rho(\mathfrak{s})^2 = \mathrm{id}$ and the representation ρ is said to be even. When $\rho(\mathfrak{s})_{ij}$'s are all imaginary, $\rho(\mathfrak{s})$ is presented as $\mathrm{i}(-\mathrm{i}\rho_{11}, -\mathrm{i}\rho_{12}, -\mathrm{i}\rho_{13}, -\mathrm{i}\rho_{14}, \cdots; -\mathrm{i}\rho_{22}, -\mathrm{i}\rho_{23}, -\mathrm{i}\rho_{24}, \cdots)$, or as $(s_n^m)^{-1}(s_n^m\rho_{11}, s_n^m\rho_{12}, s_n^m\rho_{13}, s_n^m\rho_{14}, \cdots; s_n^m\rho_{22}, s_n^m\rho_{23}, s_n^m\rho_{24}, \cdots)$, where $s_n^m := \zeta_n^m - \zeta_n^{-m}$. In this case, $\rho(\mathfrak{s})^2 = -\mathrm{id}$ and the representation ρ is said to be odd. In any case, the numbers inside the bracket $(\cdots)$ are all real. We can tell a representation to be even or odd by the absence or the presence of i or $(s_n^m)^{-1}$in front of the bracket $(\cdots)$.

We note that two symmetric representations are equivalent up to a permutation of indices, and a conjugation of signed diagonal matrix. To choose the ordering in indices, we introduce arrays $O_i = [\mathrm{DenominatorOf}(\tilde{s}_i), \tilde{s}_i, \rho_{ii}]$. The order of two arrays is determined by first comparing the lengths of the two arrays. If the lengths are equal, we then compare the first elements of the two arrays. If the first elements are equal, we then compare the second elements of the two arrays, *etc.* To compare two cyclotomic numbers, here we used the ordering of cyclotomic numbers provided by GAP computer algebraic system. We order the indices to make $O_1 \leqslant O_2 \leqslant O_3 \cdots$. The conjugation of signed diagonal matrix is chosen to make $-\rho(\mathfrak{s})_{1j} < \rho(\mathfrak{s})_{1j}$ for $j = 2, 3, \cdots$. If $\rho(\mathfrak{s})_{1j} = 0$, we will try to make $-\rho(\mathfrak{s})_{2j} < \rho(\mathfrak{s})_{2j}$, *etc.*

All the prime-power-level irreducible representations are labeled by index $d_{l,k}^{a,m}$, where d is the dimension and l is the level of the representation. The irreducible representations of a given d, l can be grouped into several orbits, generated by Galois conjugations and tensoring of 1-dimensional representations that do not change the level l: the k in $d^{a,m}_{l,k}$ labels those different orbits. If there is only 1 orbit for a given d, l, the index k will be dropped.

The irreducible representation labeled by $d^{a,m}_{l,k}$ is generated from the irreducible representation labeled by $d^{1,0}_{l,k}$ via the following Galois conjugations and tensoring of 1-dimensional representations

$$\rho_{d^{a,m}_{l,k}}(\mathfrak{t}) = \sigma_a\big(\rho_{d^{1,0}_{l,k}}(\mathfrak{t})\big)\mathrm{e}^{2\pi\mathrm{i}\frac{m}{12}}$$
$$\rho_{d^{a,m}_{l,k}}(\mathfrak{s}) = \sigma_a\big(\rho_{d^{1,0}_{l,k}}(\mathfrak{s})\big)\mathrm{e}^{-2\pi\mathrm{i}\frac{m}{4}} \tag{A.1}$$

where the Galois conjugation σ_a is in $\mathrm{Gal}(\mathbb{Q}_n)$ with n be the least common multiple of $\mathrm{ord}(\rho_{d^{1,0}_{l,k}}(\mathfrak{t}))$ and the conductor of $\rho_{d^{1,0}_{l,k}}(\mathfrak{s})$. The Galois conjugation σ_a is labeled by an integer a, which is given by

$$\sigma_a\big(\mathrm{e}^{2\pi\mathrm{i}/n}\big) = \mathrm{e}^{2\pi\mathrm{i}a/n}. \tag{A.2}$$

Also $m \in \mathbb{Z}_{12}$ is such that $\mathrm{ord}(\rho_{d^{1,0}_{l,k}}(\mathfrak{t})\mathrm{e}^{2\pi\mathrm{i}\frac{m}{12}}) = \mathrm{ord}(\rho_{d^{1,0}_{l,k}}(\mathfrak{t}))$. Due to this condition, when l is not divisible by 2 and 3, m can only be 0. In this case, we will drop m. Here we choose $d^{1,0}_{l,k}$ to be the representation in the orbit with minimal $[\tilde{s}_1, \tilde{s}_2, \cdots]$.

The numbers of distinct irreducible representations with prime-power level (PPL) in each dimension are given by

dim:	1	2	3	4	5	6	7	8	9	10	11	12
# of irreps with PPL	6	11	33	18	3	70	3	10	4	7	3	176
# of irreps	12	54	136	180	36	720	36	456	476	222	36	3214

(A.3)

In the above we also list the numbers of distinct irreducible representations, which are tensor products of the irreducible representations with prime-power levels.

In the following tables, we list all irreducible representations with prime-power levels for rank 2, 3, 4, 5. For rank 6, to save space, we only list all irreducible representations with prime-power levels that have a form $\rho_{d^{1,0}_{l,k}}$. Other irreducible representations, with prime-power levels and the same dimension, can be obtained from those listed ones via Galois conjugations and tensoring 1-dimensional representations. In the Supplementary Material section of the arXiv version of the article we list all distinct irreducible representations of prime-power levels. In the tables $c^m_n := \zeta^m_n + \zeta^{-m}_n$ and $s^m_n := \zeta^m_n - \zeta^{-m}_n$.

$d^{a,m}_{l,k}$	#	$\rho(\mathfrak{t}),\ \ \rho(\mathfrak{s})$
1^1_1	1	(0), (1)
$1^{1,0}_2$	2	$(\frac{1}{2})$, (-1)
$1^{1,0}_3$	3	$(\frac{1}{3})$, (1)
$1^{1,4}_3$	4	$(\frac{2}{3})$, (1)
$1^{1,0}_4$	5	$(\frac{1}{4})$, $\mathrm{i}(1)$
$1^{1,6}_4$	6	$(\frac{3}{4})$, $\mathrm{i}(-1)$

$d^{a,m}_{l,k}$	#	$\rho(\mathfrak{t}),\ \ \rho(\mathfrak{s})$

$2_2^{1,0}$	1	$(0, \frac{1}{2}),\ (-\frac{1}{2},\ -\frac{\sqrt{3}}{2};\ \frac{1}{2})$
$2_3^{1,0}$	2	$(0, \frac{1}{3}),\ (s_3^1)^{-1}(1,\ -\sqrt{2};\ \ -1)$
$2_3^{1,8}$	3	$(0, \frac{2}{3}),\ (s_3^1)^{-1}(-1,\ -\sqrt{2};\ \ 1)$
$2_3^{1,4}$	4	$(\frac{1}{3}, \frac{2}{3}),\ (s_3^1)^{-1}(1,\ -\sqrt{2};\ \ -1)$
$2_4^{1,0}$	5	$(\frac{1}{4}, \frac{3}{4}),\ \mathrm{i}(-\frac{1}{2},\ \frac{\sqrt{3}}{2};\ \ \frac{1}{2})$
2_5^{1}	6	$(\frac{1}{5}, \frac{4}{5}),\ (s_5^1)^{-1}(1,\ -\frac{1+\sqrt{5}}{2};\ \ -1)$
2_5^{2}	7	$(\frac{2}{5}, \frac{3}{5}),\ (s_5^2)^{-1}(1,\ \frac{1-\sqrt{5}}{2};\ \ -1)$
$2_8^{1,0}$	8	$(\frac{1}{8}, \frac{3}{8}),\ (-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \frac{\sqrt{2}}{2})$
$2_8^{1,9}$	9	$(\frac{1}{8}, \frac{7}{8}),\ \mathrm{i}(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \ \frac{\sqrt{2}}{2})$
$2_8^{1,3}$	10	$(\frac{3}{8}, \frac{5}{8}),\ \mathrm{i}(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \ \frac{\sqrt{2}}{2})$
$2_8^{1,6}$	11	$(\frac{5}{8}, \frac{7}{8}),\ (\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ -\frac{\sqrt{2}}{2})$

$d_{l,k}^{a,m}$	#	$\rho(\mathfrak{t}),\ \ \rho(\mathfrak{s})$
$3_3^{1,0}$	1	$(0, \frac{1}{3}, \frac{2}{3}),\ (-\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3};\ -\frac{1}{3},\ \frac{2}{3};\ -\frac{1}{3})$
$3_4^{1,0}$	2	$(0, \frac{1}{4}, \frac{3}{4}),\ (0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ -\frac{1}{2},\ \frac{1}{2};\ -\frac{1}{2})$
$3_4^{1,3}$	3	$(0, \frac{1}{2}, \frac{1}{4}),\ \mathrm{i}(-\frac{1}{2},\ \frac{1}{2},\ \frac{\sqrt{2}}{2};\ \ -\frac{1}{2},\ \frac{\sqrt{2}}{2};\ \ 0)$
$3_4^{1,9}$	4	$(0, \frac{1}{2}, \frac{3}{4}),\ \mathrm{i}(\frac{1}{2},\ \frac{1}{2},\ \frac{\sqrt{2}}{2};\ \ \frac{1}{2},\ -\frac{\sqrt{2}}{2};\ \ 0)$
$3_4^{1,6}$	5	$(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}),\ (0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \frac{1}{2},\ -\frac{1}{2};\ \frac{1}{2})$
3_5^{1}	6	$(0, \frac{1}{5}, \frac{4}{5}),\ (\frac{\sqrt{5}}{5},\ -\frac{\sqrt{10}}{5},\ -\frac{\sqrt{10}}{5};\ -\frac{5+\sqrt{5}}{10},\ \frac{5-\sqrt{5}}{10};\ -\frac{5+\sqrt{5}}{10})$
3_5^{3}	7	$(0, \frac{2}{5}, \frac{3}{5}),\ (-\frac{\sqrt{5}}{5},\ -\frac{\sqrt{10}}{5},\ -\frac{\sqrt{10}}{5};\ -\frac{5-\sqrt{5}}{10},\ \frac{5+\sqrt{5}}{10};\ -\frac{5-\sqrt{5}}{10})$
3_7^{1}	8	$(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}),\ (-\frac{c_{28}^1}{\sqrt{7}},\ -\frac{c_{28}^3}{\sqrt{7}},\ \frac{c_{28}^5}{\sqrt{7}};\ \frac{c_{28}^5}{\sqrt{7}},\ -\frac{c_{28}^1}{\sqrt{7}};\ -\frac{c_{28}^3}{\sqrt{7}})$
3_7^{3}	9	$(\frac{3}{7}, \frac{5}{7}, \frac{6}{7}),\ (-\frac{c_{28}^3}{\sqrt{7}},\ -\frac{c_{28}^1}{\sqrt{7}},\ \frac{c_{28}^5}{\sqrt{7}};\ \frac{c_{28}^5}{\sqrt{7}},\ -\frac{c_{28}^3}{\sqrt{7}};\ -\frac{c_{28}^1}{\sqrt{7}})$
$3_8^{1,0}$	10	$(0, \frac{1}{8}, \frac{5}{8}),\ \mathrm{i}(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \ -\frac{1}{2},\ \frac{1}{2};\ \ -\frac{1}{2})$
$3_8^{3,0}$	11	$(0, \frac{3}{8}, \frac{7}{8}),\ \mathrm{i}(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \ \frac{1}{2},\ -\frac{1}{2};\ \ \frac{1}{2})$
$3_8^{3,3}$	12	$(\frac{1}{4}, \frac{1}{8}, \frac{5}{8}),\ (0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ -\frac{1}{2},\ \frac{1}{2};\ -\frac{1}{2})$
$3_8^{1,3}$	13	$(\frac{1}{4}, \frac{3}{8}, \frac{7}{8}),\ (0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \frac{1}{2},\ -\frac{1}{2};\ \frac{1}{2})$
$3_8^{1,6}$	14	$(\frac{1}{2}, \frac{1}{8}, \frac{5}{8}),\ \mathrm{i}(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \ \frac{1}{2},\ -\frac{1}{2};\ \ \frac{1}{2})$
$3_8^{3,6}$	15	$(\frac{1}{2}, \frac{3}{8}, \frac{7}{8}),\ \mathrm{i}(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \ -\frac{1}{2},\ \frac{1}{2};\ \ -\frac{1}{2})$
$3_8^{3,9}$	16	$(\frac{3}{4}, \frac{1}{8}, \frac{5}{8}),\ (0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \frac{1}{2},\ -\frac{1}{2};\ \frac{1}{2})$

$3_{8}^{1,9}$	17	$(\frac{3}{4}, \frac{3}{8}, \frac{7}{8})$, $(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ -\frac{1}{2},\ \frac{1}{2};\ -\frac{1}{2})$
$3_{16}^{1,0}$	18	$(\frac{1}{8}, \frac{1}{16}, \frac{9}{16})$, $\mathrm{i}(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ -\frac{1}{2},\ \frac{1}{2};\ -\frac{1}{2})$
$3_{16}^{7,3}$	19	$(\frac{1}{8}, \frac{3}{16}, \frac{11}{16})$, $(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ -\frac{1}{2},\ \frac{1}{2};\ -\frac{1}{2})$
$3_{16}^{5,6}$	20	$(\frac{1}{8}, \frac{5}{16}, \frac{13}{16})$, $\mathrm{i}(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \frac{1}{2},\ -\frac{1}{2};\ \frac{1}{2})$
$3_{16}^{3,9}$	21	$(\frac{1}{8}, \frac{7}{16}, \frac{15}{16})$, $(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \frac{1}{2},\ -\frac{1}{2};\ \frac{1}{2})$
$3_{16}^{5,9}$	22	$(\frac{3}{8}, \frac{1}{16}, \frac{9}{16})$, $(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ -\frac{1}{2},\ \frac{1}{2};\ -\frac{1}{2})$
$3_{16}^{3,0}$	23	$(\frac{3}{8}, \frac{3}{16}, \frac{11}{16})$, $\mathrm{i}(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \frac{1}{2},\ -\frac{1}{2};\ \frac{1}{2})$
$3_{16}^{1,3}$	24	$(\frac{3}{8}, \frac{5}{16}, \frac{13}{16})$, $(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \frac{1}{2},\ -\frac{1}{2};\ \frac{1}{2})$
$3_{16}^{7,6}$	25	$(\frac{3}{8}, \frac{7}{16}, \frac{15}{16})$, $\mathrm{i}(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ -\frac{1}{2},\ \frac{1}{2};\ -\frac{1}{2})$
$3_{16}^{1,6}$	26	$(\frac{5}{8}, \frac{1}{16}, \frac{9}{16})$, $\mathrm{i}(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \frac{1}{2},\ -\frac{1}{2};\ \frac{1}{2})$
$3_{16}^{7,9}$	27	$(\frac{5}{8}, \frac{3}{16}, \frac{11}{16})$, $(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \frac{1}{2},\ -\frac{1}{2};\ \frac{1}{2})$
$3_{16}^{5,0}$	28	$(\frac{5}{8}, \frac{5}{16}, \frac{13}{16})$, $\mathrm{i}(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ -\frac{1}{2},\ \frac{1}{2};\ -\frac{1}{2})$
$3_{16}^{3,3}$	29	$(\frac{5}{8}, \frac{7}{16}, \frac{15}{16})$, $(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ -\frac{1}{2},\ \frac{1}{2};\ -\frac{1}{2})$
$3_{16}^{5,3}$	30	$(\frac{7}{8}, \frac{1}{16}, \frac{9}{16})$, $(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \frac{1}{2},\ -\frac{1}{2};\ \frac{1}{2})$
$3_{16}^{3,6}$	31	$(\frac{7}{8}, \frac{3}{16}, \frac{11}{16})$, $\mathrm{i}(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ -\frac{1}{2},\ \frac{1}{2};\ -\frac{1}{2})$
$3_{16}^{1,9}$	32	$(\frac{7}{8}, \frac{5}{16}, \frac{13}{16})$, $(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ -\frac{1}{2},\ \frac{1}{2};\ -\frac{1}{2})$
$3_{16}^{7,0}$	33	$(\frac{7}{8}, \frac{7}{16}, \frac{15}{16})$, $\mathrm{i}(0,\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2};\ \frac{1}{2},\ -\frac{1}{2};\ \frac{1}{2})$

$d_{l,k}^{a,m}$	#	$\rho(\mathfrak{t}),\ \ \rho(\mathfrak{s})$
$4_{5,1}^{1}$	1	$(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$, $(s_5^2)^{-1}(-\frac{5+\sqrt{5}}{10},\ -\frac{\sqrt{15}}{5},\ \frac{3-3\sqrt{5}}{2\sqrt{15}},\ \frac{5-3\sqrt{5}}{10};\ \ -\frac{5-3\sqrt{5}}{10},\ \frac{5+\sqrt{5}}{10},\ -\frac{3-3\sqrt{5}}{2\sqrt{15}};\ \frac{5-3\sqrt{5}}{10},\ -\frac{\sqrt{15}}{5};\ \ \frac{5+\sqrt{5}}{10})$
$4_{5,2}^{1}$	2	$(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$, $(\frac{\sqrt{5}}{5},\ -\frac{5-\sqrt{5}}{10},\ -\frac{5+\sqrt{5}}{10},\ \frac{\sqrt{5}}{5};\ -\frac{\sqrt{5}}{5},\ \frac{\sqrt{5}}{5},\ \frac{5+\sqrt{5}}{10};\ -\frac{\sqrt{5}}{5},\ \frac{5-\sqrt{5}}{10};\ \frac{\sqrt{5}}{5})$
4_{7}^{1}	3	$(0, \frac{1}{7}, \frac{2}{7}, \frac{4}{7})$, $\mathrm{i}(-\frac{\sqrt{7}}{7},\ \frac{\sqrt{14}}{7},\ \frac{\sqrt{14}}{7},\ \frac{\sqrt{14}}{7};\ \ -\frac{c_7^2}{\sqrt{7}},\ -\frac{c_7^1}{\sqrt{7}},\ -\frac{c_7^3}{\sqrt{7}};\ \ -\frac{c_7^3}{\sqrt{7}},\ -\frac{c_7^2}{\sqrt{7}};\ -\frac{c_7^1}{\sqrt{7}})$
4_{7}^{3}	4	$(0, \frac{3}{7}, \frac{5}{7}, \frac{6}{7})$, $\mathrm{i}(\frac{\sqrt{7}}{7},\ \frac{\sqrt{14}}{7},\ \frac{\sqrt{14}}{7},\ \frac{\sqrt{14}}{7};\ \ \frac{c_7^1}{\sqrt{7}},\ \frac{c_7^2}{\sqrt{7}},\ \frac{c_7^3}{\sqrt{7}};\ \ \frac{c_7^3}{\sqrt{7}},\ \frac{c_7^1}{\sqrt{7}};\ \ \frac{c_7^2}{\sqrt{7}})$
$4_{8}^{1,0}$	5	$(\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8})$, $\mathrm{i}(\frac{\sqrt{2}}{4},\ \frac{\sqrt{6}}{4},\ \frac{\sqrt{6}}{4},\ \frac{\sqrt{2}}{4};\ \ \frac{\sqrt{2}}{4},\ -\frac{\sqrt{2}}{4},\ -\frac{\sqrt{6}}{4};\ \ -\frac{\sqrt{2}}{4},\ \frac{\sqrt{6}}{4};\ \ -\frac{\sqrt{2}}{4})$
$4_{8}^{1,3}$	6	$(\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8})$, $(\frac{\sqrt{2}}{4},\ \frac{\sqrt{2}}{4},\ \frac{\sqrt{6}}{4},\ \frac{\sqrt{6}}{4};\ -\frac{\sqrt{2}}{4},\ \frac{\sqrt{6}}{4},\ -\frac{\sqrt{6}}{4};\ -\frac{\sqrt{2}}{4},\ -\frac{\sqrt{2}}{4};\ \frac{\sqrt{2}}{4})$
$4_{9,1}^{1,0}$	7	$(0, \frac{1}{9}, \frac{4}{9}, \frac{7}{9})$, $\mathrm{i}(0,\ \frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3};\ \ -\frac{1}{3}c_{36}^1,\ \frac{1}{3}c_{36}^7,\ \frac{1}{3}c_{36}^5;\ \ \frac{1}{3}c_{36}^5,\ -\frac{1}{3}c_{36}^1;\ \ \frac{1}{3}c_{36}^7)$

$4_{9,1}^{2,0}$	8	$(0, \frac{2}{9}, \frac{5}{9}, \frac{8}{9})$, $\mathrm{i}(0,\ \frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3};\ -\frac{1}{3}c_{36}^{7},\ \frac{1}{3}c_{36}^{1},\ -\frac{1}{3}c_{36}^{5};\ -\frac{1}{3}c_{36}^{5},\ -\frac{1}{3}c_{36}^{7};\ \frac{1}{3}c_{36}^{1})$
$4_{9,1}^{1,4}$	9	$(\frac{1}{3}, \frac{1}{9}, \frac{4}{9}, \frac{7}{9})$, $\mathrm{i}(0,\ \frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3};\ \frac{1}{3}c_{36}^{7},\ \frac{1}{3}c_{36}^{5},\ -\frac{1}{3}c_{36}^{1};\ -\frac{1}{3}c_{36}^{1},\ \frac{1}{3}c_{36}^{7};\ \frac{1}{3}c_{36}^{5})$
$4_{9,1}^{2,4}$	10	$(\frac{1}{3}, \frac{2}{9}, \frac{5}{9}, \frac{8}{9})$, $\mathrm{i}(0,\ \frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3};\ \frac{1}{3}c_{36}^{1},\ -\frac{1}{3}c_{36}^{5},\ -\frac{1}{3}c_{36}^{7};\ -\frac{1}{3}c_{36}^{7},\ \frac{1}{3}c_{36}^{1};\ -\frac{1}{3}c_{36}^{5})$
$4_{9,1}^{1,8}$	11	$(\frac{2}{3}, \frac{1}{9}, \frac{4}{9}, \frac{7}{9})$, $\mathrm{i}(0,\ \frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3};\ \frac{1}{3}c_{36}^{5},\ -\frac{1}{3}c_{36}^{1},\ \frac{1}{3}c_{36}^{7};\ \frac{1}{3}c_{36}^{7},\ \frac{1}{3}c_{36}^{5};\ -\frac{1}{3}c_{36}^{1})$
$4_{9,1}^{2,8}$	12	$(\frac{2}{3}, \frac{2}{9}, \frac{5}{9}, \frac{8}{9})$, $\mathrm{i}(0,\ \frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3};\ -\frac{1}{3}c_{36}^{5},\ -\frac{1}{3}c_{36}^{7},\ \frac{1}{3}c_{36}^{1};\ \frac{1}{3}c_{36}^{1},\ -\frac{1}{3}c_{36}^{5};\ -\frac{1}{3}c_{36}^{7})$
$4_{9,2}^{1,0}$	13	$(0, \frac{1}{9}, \frac{4}{9}, \frac{7}{9})$, $(0,\ -\frac{\sqrt{3}}{3},\ -\frac{\sqrt{3}}{3},\ -\frac{\sqrt{3}}{3};\ \frac{1}{3}c_{9}^{2},\ \frac{1}{3}c_{9}^{4},\ \frac{1}{3}c_{9}^{1};\ \frac{1}{3}c_{9}^{1},\ \frac{1}{3}c_{9}^{2};\ \frac{1}{3}c_{9}^{4})$
$4_{9,2}^{5,0}$	14	$(0, \frac{2}{9}, \frac{5}{9}, \frac{8}{9})$, $(0,\ -\frac{\sqrt{3}}{3},\ -\frac{\sqrt{3}}{3},\ -\frac{\sqrt{3}}{3};\ \frac{1}{3}c_{9}^{4},\ \frac{1}{3}c_{9}^{2},\ \frac{1}{3}c_{9}^{1};\ \frac{1}{3}c_{9}^{1},\ \frac{1}{3}c_{9}^{4};\ \frac{1}{3}c_{9}^{2})$
$4_{9,2}^{1,4}$	15	$(\frac{1}{3}, \frac{1}{9}, \frac{4}{9}, \frac{7}{9})$, $(0,\ -\frac{\sqrt{3}}{3},\ -\frac{\sqrt{3}}{3},\ -\frac{\sqrt{3}}{3};\ \frac{1}{3}c_{9}^{4},\ \frac{1}{3}c_{9}^{1},\ \frac{1}{3}c_{9}^{2};\ \frac{1}{3}c_{9}^{2},\ \frac{1}{3}c_{9}^{4};\ \frac{1}{3}c_{9}^{1})$
$4_{9,2}^{5,4}$	16	$(\frac{1}{3}, \frac{2}{9}, \frac{5}{9}, \frac{8}{9})$, $(0,\ -\frac{\sqrt{3}}{3},\ -\frac{\sqrt{3}}{3},\ -\frac{\sqrt{3}}{3};\ \frac{1}{3}c_{9}^{2},\ \frac{1}{3}c_{9}^{1},\ \frac{1}{3}c_{9}^{4};\ \frac{1}{3}c_{9}^{4},\ \frac{1}{3}c_{9}^{2};\ \frac{1}{3}c_{9}^{1})$
$4_{9,2}^{1,8}$	17	$(\frac{2}{3}, \frac{1}{9}, \frac{4}{9}, \frac{7}{9})$, $(0,\ -\frac{\sqrt{3}}{3},\ -\frac{\sqrt{3}}{3},\ -\frac{\sqrt{3}}{3};\ \frac{1}{3}c_{9}^{1},\ \frac{1}{3}c_{9}^{2},\ \frac{1}{3}c_{9}^{4};\ \frac{1}{3}c_{9}^{4},\ \frac{1}{3}c_{9}^{1};\ \frac{1}{3}c_{9}^{2})$
$4_{9,2}^{5,8}$	18	$(\frac{2}{3}, \frac{2}{9}, \frac{5}{9}, \frac{8}{9})$, $(0,\ -\frac{\sqrt{3}}{3},\ -\frac{\sqrt{3}}{3},\ -\frac{\sqrt{3}}{3};\ \frac{1}{3}c_{9}^{1},\ \frac{1}{3}c_{9}^{4},\ \frac{1}{3}c_{9}^{2};\ \frac{1}{3}c_{9}^{2},\ \frac{1}{3}c_{9}^{1};\ \frac{1}{3}c_{9}^{4})$

$d_{l,k}^{a,m}$	#	$\rho(\mathfrak{t})$, $\rho(\mathfrak{s})$
5_{5}^{1}	1	$(0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$, $(-\frac{1}{5},\ \frac{\sqrt{6}}{5},\ \frac{\sqrt{6}}{5},\ \frac{\sqrt{6}}{5},\ \frac{\sqrt{6}}{5};\ \frac{3-\sqrt{5}}{10},\ -\frac{1+\sqrt{5}}{5},\ -\frac{1-\sqrt{5}}{5},\ \frac{3+\sqrt{5}}{10};\ \frac{3+\sqrt{5}}{10},\ \frac{3-\sqrt{5}}{10},\ -\frac{1-\sqrt{5}}{5};\ \frac{3+\sqrt{5}}{10},\ -\frac{1+\sqrt{5}}{5};\ \frac{3-\sqrt{5}}{10})$
5_{11}^{1}	2	$(\frac{1}{11}, \frac{3}{11}, \frac{4}{11}, \frac{5}{11}, \frac{9}{11})$, $(-\frac{c_{44}^{3}}{\sqrt{11}},\ -\frac{c_{44}^{7}}{\sqrt{11}},\ -\frac{c_{44}^{5}}{\sqrt{11}},\ -\frac{c_{44}^{1}}{\sqrt{11}},\ -\frac{c_{44}^{9}}{\sqrt{11}};\ \frac{c_{44}^{9}}{\sqrt{11}},\ -\frac{c_{44}^{3}}{\sqrt{11}},\ \frac{c_{44}^{5}}{\sqrt{11}},\ \frac{c_{44}^{1}}{\sqrt{11}};\ \frac{c_{44}^{1}}{\sqrt{11}},\ \frac{c_{44}^{9}}{\sqrt{11}},\ \frac{c_{44}^{7}}{\sqrt{11}};\ \frac{c_{44}^{7}}{\sqrt{11}},\ -\frac{c_{44}^{3}}{\sqrt{11}};\ \frac{c_{44}^{5}}{\sqrt{11}})$
5_{11}^{2}	3	$(\frac{2}{11}, \frac{6}{11}, \frac{7}{11}, \frac{8}{11}, \frac{10}{11})$, $(\frac{c_{44}^{5}}{\sqrt{11}},\ \frac{c_{44}^{3}}{\sqrt{11}},\ -\frac{c_{44}^{7}}{\sqrt{11}},\ -\frac{c_{44}^{1}}{\sqrt{11}},\ -\frac{c_{44}^{9}}{\sqrt{11}};\ \frac{c_{44}^{7}}{\sqrt{11}},\ \frac{c_{44}^{9}}{\sqrt{11}},\ \frac{c_{44}^{5}}{\sqrt{11}},\ \frac{c_{44}^{1}}{\sqrt{11}};\ \frac{c_{44}^{1}}{\sqrt{11}},\ -\frac{c_{44}^{3}}{\sqrt{11}},\ \frac{c_{44}^{5}}{\sqrt{11}};\ \frac{c_{44}^{9}}{\sqrt{11}},\ \frac{c_{44}^{7}}{\sqrt{11}};\ -\frac{c_{44}^{3}}{\sqrt{11}})$

$d_{l,k}^{a,m}$	#	$\rho(\mathfrak{t})$, $\rho(\mathfrak{s})$
6_{5}^{1}	1	$(0, 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$, $(s_5^2)^{-1}(\frac{\sqrt{5}}{5},\ -\frac{5-\sqrt{5}}{10},\ -\frac{\sqrt{10}}{5},\ -\frac{5+\sqrt{5}}{10},\ \frac{5-3\sqrt{5}}{10},\ \frac{1-\sqrt{5}}{\sqrt{10}};\ -\frac{\sqrt{5}}{5},\ \frac{-1+\sqrt{5}}{\sqrt{10}},\ \frac{5-3\sqrt{5}}{10},\ \frac{5+\sqrt{5}}{10},\ -\frac{\sqrt{10}}{5};\ \frac{5-\sqrt{5}}{10},\ \frac{1-\sqrt{5}}{\sqrt{10}},\ -\frac{\sqrt{10}}{5},\ -\frac{\sqrt{5}}{5};\ -\frac{\sqrt{5}}{5},\ \frac{5-\sqrt{5}}{10},\ \frac{\sqrt{10}}{5};\ \frac{\sqrt{5}}{5},\ \frac{1-\sqrt{5}}{\sqrt{10}};\ -\frac{5-\sqrt{5}}{10})$

$6^1_{7,1}$	2	$(\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7})$, $\mathrm{i}(\frac{1}{7}c^2_{56} - \frac{1}{7}c^3_{56} + \frac{1}{7}c^{11}_{56},\ \frac{1}{7}c^5_{56} + \frac{1}{7}c^6_{56} + \frac{1}{7}c^9_{56},\ \frac{1}{7}c^3_{112} - \frac{1}{7}c^9_{112} + \frac{1}{7}c^{11}_{112} + \frac{1}{7}c^{23}_{112},\ \frac{2}{7}c^1_{56} - \frac{1}{7}c^3_{56} - \frac{1}{7}c^5_{56} + \frac{1}{7}c^7_{56} + \frac{1}{7}c^9_{56} - \frac{1}{7}c^{10}_{56} - \frac{1}{7}c^{11}_{56},\ \frac{1}{7}c^1_{112} + \frac{1}{7}c^3_{112} - \frac{1}{7}c^5_{112} - \frac{1}{7}c^7_{112} - \frac{1}{7}c^9_{112} - \frac{1}{7}c^{11}_{112} + \frac{2}{7}c^{13}_{112} + \frac{1}{7}c^{15}_{112} + \frac{2}{7}c^{17}_{112} + \frac{1}{7}c^{19}_{112} - \frac{1}{7}c^{21}_{112} - \frac{1}{7}c^{23}_{112},\ \frac{1}{7}c^1_{112} + \frac{1}{7}c^5_{112} - \frac{1}{7}c^{15}_{112} + \frac{1}{7}c^{19}_{112};\ \frac{2}{7}c^1_{56} - \frac{1}{7}c^3_{56} - \frac{1}{7}c^5_{56} + \frac{1}{7}c^7_{56} + \frac{1}{7}c^9_{56} - \frac{1}{7}c^{10}_{56} - \frac{1}{7}c^{11}_{56},\ \frac{1}{7}c^1_{112} + \frac{1}{7}c^5_{112} - \frac{1}{7}c^{15}_{112} + \frac{1}{7}c^{19}_{112},\ \frac{1}{7}c^2_{56} - \frac{1}{7}c^3_{56} + \frac{1}{7}c^{11}_{56},\ -\frac{1}{7}c^3_{112} + \frac{1}{7}c^9_{112} - \frac{1}{7}c^{11}_{112} - \frac{1}{7}c^{23}_{112},\ -\frac{1}{7}c^1_{112} - \frac{1}{7}c^3_{112} + \frac{1}{7}c^5_{112} + \frac{1}{7}c^7_{112} + \frac{1}{7}c^9_{112} + \frac{1}{7}c^{11}_{112} - \frac{2}{7}c^{13}_{112} - \frac{1}{7}c^{15}_{112} - \frac{2}{7}c^{17}_{112} - \frac{1}{7}c^{19}_{112} + \frac{1}{7}c^{21}_{112} + \frac{1}{7}c^{23}_{112};\ -\frac{1}{7}c^5_{56} - \frac{1}{7}c^6_{56} - \frac{1}{7}c^9_{56},\ -\frac{1}{7}c^1_{112} - \frac{1}{7}c^3_{112} + \frac{1}{7}c^5_{112} + \frac{1}{7}c^7_{112} + \frac{1}{7}c^9_{112} + \frac{1}{7}c^{11}_{112} - \frac{2}{7}c^{13}_{112} - \frac{1}{7}c^{15}_{112} - \frac{2}{7}c^{17}_{112} - \frac{1}{7}c^{19}_{112} + \frac{1}{7}c^{21}_{112} + \frac{1}{7}c^{23}_{112},\ \frac{1}{7}c^2_{56} - \frac{1}{7}c^3_{56} + \frac{1}{7}c^{11}_{56},\ -\frac{2}{7}c^1_{56} + \frac{1}{7}c^3_{56} + \frac{1}{7}c^5_{56} - \frac{1}{7}c^7_{56} - \frac{1}{7}c^9_{56} + \frac{1}{7}c^{10}_{56} + \frac{1}{7}c^{11}_{56};\ \frac{1}{7}c^5_{56} + \frac{1}{7}c^6_{56} + \frac{1}{7}c^9_{56},\ -\frac{1}{7}c^1_{112} - \frac{1}{7}c^5_{112} + \frac{1}{7}c^{15}_{112} - \frac{1}{7}c^{19}_{112},\ \frac{1}{7}c^3_{112} - \frac{1}{7}c^9_{112} + \frac{1}{7}c^{11}_{112} + \frac{1}{7}c^{23}_{112};\ -\frac{2}{7}c^1_{56} + \frac{1}{7}c^3_{56} + \frac{1}{7}c^5_{56} - \frac{1}{7}c^7_{56} - \frac{1}{7}c^9_{56} + \frac{1}{7}c^{10}_{56} + \frac{1}{7}c^{11}_{56},\ \frac{1}{7}c^5_{56} + \frac{1}{7}c^6_{56} + \frac{1}{7}c^9_{56};\ -\frac{1}{7}c^2_{56} + \frac{1}{7}c^3_{56} - \frac{1}{7}c^{11}_{56})$
$6^1_{7,2}$	3	$(\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7})$, $(\frac{2}{7} - \frac{1}{7}c^2_7,\ -\frac{2}{7} + \frac{1}{7}c^1_7,\ \frac{1}{7}c^3_{56} + \frac{2}{7}c^5_{56} - \frac{1}{7}c^7_{56} - \frac{2}{7}c^9_{56} + \frac{1}{7}c^{11}_{56},\ -\frac{3}{7} - \frac{1}{7}c^1_7 - \frac{1}{7}c^2_7,\ -\frac{1}{7}c^3_{56} + \frac{1}{7}c^5_{56} - \frac{1}{7}c^9_{56} - \frac{1}{7}c^{11}_{56},\ -\frac{2}{7}c^3_{56} - \frac{1}{7}c^5_{56} + \frac{1}{7}c^7_{56} + \frac{1}{7}c^9_{56} - \frac{2}{7}c^{11}_{56};\ \frac{3}{7} + \frac{1}{7}c^1_7 + \frac{1}{7}c^2_7,\ \frac{2}{7}c^3_{56} + \frac{1}{7}c^5_{56} - \frac{1}{7}c^7_{56} - \frac{1}{7}c^9_{56} + \frac{2}{7}c^{11}_{56},\ \frac{2}{7} - \frac{1}{7}c^2_7,\ \frac{1}{7}c^3_{56} + \frac{2}{7}c^5_{56} - \frac{1}{7}c^7_{56} - \frac{2}{7}c^9_{56} + \frac{1}{7}c^{11}_{56},\ -\frac{1}{7}c^3_{56} + \frac{1}{7}c^5_{56} - \frac{1}{7}c^9_{56} - \frac{1}{7}c^{11}_{56};\ \frac{2}{7} - \frac{1}{7}c^1_7,\ -\frac{1}{7}c^3_{56} + \frac{1}{7}c^5_{56} - \frac{1}{7}c^9_{56} - \frac{1}{7}c^{11}_{56},\ -\frac{2}{7} + \frac{1}{7}c^2_7,\ \frac{3}{7} + \frac{1}{7}c^1_7 + \frac{1}{7}c^2_7;\ \frac{2}{7} - \frac{1}{7}c^1_7,\ -\frac{2}{7}c^3_{56} - \frac{1}{7}c^5_{56} + \frac{1}{7}c^7_{56} + \frac{1}{7}c^9_{56} - \frac{2}{7}c^{11}_{56},\ -\frac{1}{7}c^3_{56} - \frac{2}{7}c^5_{56} + \frac{1}{7}c^7_{56} + \frac{2}{7}c^9_{56} - \frac{1}{7}c^{11}_{56};\ \frac{3}{7} + \frac{1}{7}c^1_7 + \frac{1}{7}c^2_7,\ -\frac{2}{7} + \frac{1}{7}c^1_7;\ \frac{2}{7} - \frac{1}{7}c^2_7)$
$6^{1,0}_{8,1}$	4	$(0, \frac{1}{2}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8})$, $(0,\ 0,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2};\ 0,\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2};\ 0,\ -\frac{1}{2},\ 0,\ \frac{1}{2};\ 0,\ \frac{1}{2},\ 0;\ 0,\ -\frac{1}{2};\ 0)$
$6^{1,0}_{8,2}$	5	$(0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8})$, $\mathrm{i}(-\frac{\sqrt{2}}{4},\ \frac{\sqrt{2}}{4},\ \frac{\sqrt{2}}{4},\ \frac{\sqrt{2}}{4},\ \frac{1}{2},\ \frac{1}{2};\ -\frac{\sqrt{2}}{4},\ -\frac{\sqrt{2}}{4},\ -\frac{\sqrt{2}}{4},\ \frac{1}{2},\ \frac{1}{2};\ \frac{\sqrt{2}}{4},\ \frac{\sqrt{2}}{4},\ -\frac{1}{2},\ \frac{1}{2};\ \frac{\sqrt{2}}{4},\ \frac{1}{2},\ -\frac{1}{2};\ 0,\ 0;\ 0)$
$6^{1,0}_{9,1}$	6	$(\frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9})$, $\mathrm{i}(-\frac{1}{3},\ \frac{1}{3}c^7_{72},\ \frac{1}{3},\ -\frac{1}{3}c^{17}_{72},\ \frac{1}{3},\ -\frac{1}{3}c^5_{72};\ \frac{1}{3},\ \frac{1}{3}c^5_{72},\ -\frac{1}{3},\ -\frac{1}{3}c^{17}_{72},\ \frac{1}{3};\ -\frac{1}{3},\ \frac{1}{3}c^7_{72},\ -\frac{1}{3},\ -\frac{1}{3}c^{17}_{72};\ \frac{1}{3},\ -\frac{1}{3}c^5_{72},\ -\frac{1}{3};\ -\frac{1}{3},\ -\frac{1}{3}c^7_{72};\ \frac{1}{3})$
$6^{1,0}_{9,2}$	7	$(\frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9})$, $(-\frac{1}{3},\ \frac{1}{3}c^1_{36},\ \frac{1}{3},\ \frac{1}{3}c^5_{36},\ \frac{1}{3},\ -\frac{1}{3}c^7_{36};\ -\frac{1}{3},\ \frac{1}{3}c^7_{36},\ \frac{1}{3},\ \frac{1}{3}c^5_{36},\ -\frac{1}{3};\ -\frac{1}{3},\ \frac{1}{3}c^1_{36},\ -\frac{1}{3},\ \frac{1}{3}c^5_{36};\ -\frac{1}{3},\ -\frac{1}{3}c^7_{36},\ \frac{1}{3};\ -\frac{1}{3},\ -\frac{1}{3}c^1_{36};\ -\frac{1}{3})$
$6^{1,0}_{9,3}$	8	$(\frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9})$, $(\frac{1}{3},\ \frac{1}{3}c^2_9,\ \frac{1}{3},\ -\frac{1}{3}c^1_9,\ \frac{1}{3},\ \frac{1}{3}c^4_9;\ \frac{1}{3},\ \frac{1}{3}c^4_9,\ -\frac{1}{3},\ \frac{1}{3}c^1_9,\ \frac{1}{3};\ \frac{1}{3},\ -\frac{1}{3}c^2_9,\ \frac{1}{3},\ \frac{1}{3}c^1_9;\ \frac{1}{3},\ -\frac{1}{3}c^4_9,\ -\frac{1}{3};\ \frac{1}{3},\ \frac{1}{3}c^2_9;\ \frac{1}{3})$
6^1_{11}	9	$(0, \frac{1}{11}, \frac{3}{11}, \frac{4}{11}, \frac{5}{11}, \frac{9}{11})$, $\mathrm{i}(-\frac{\sqrt{11}}{11},\ \frac{\sqrt{22}}{11},\ \frac{\sqrt{22}}{11},\ \frac{\sqrt{22}}{11},\ \frac{\sqrt{22}}{11},\ \frac{\sqrt{22}}{11};\ -\frac{c^2_{11}}{\sqrt{11}},\ -\frac{c^1_{11}}{\sqrt{11}},\ -\frac{c^4_{11}}{\sqrt{11}},\ -\frac{c^3_{11}}{\sqrt{11}},\ -\frac{c^5_{11}}{\sqrt{11}};\ -\frac{c^5_{11}}{\sqrt{11}},\ -\frac{c^2_{11}}{\sqrt{11}},\ -\frac{c^4_{11}}{\sqrt{11}},\ -\frac{c^3_{11}}{\sqrt{11}};\ -\frac{c^3_{11}}{\sqrt{11}},\ -\frac{c^5_{11}}{\sqrt{11}},\ -\frac{c^1_{11}}{\sqrt{11}};\ -\frac{c^1_{11}}{\sqrt{11}},\ -\frac{c^2_{11}}{\sqrt{11}};\ -\frac{c^4_{11}}{\sqrt{11}})$
6^1_{13}	10	$(\frac{1}{13}, \frac{3}{13}, \frac{4}{13}, \frac{9}{13}, \frac{10}{13}, \frac{12}{13})$, $\mathrm{i}(-\frac{c^5_{52}}{\sqrt{13}},\ \frac{c^7_{52}}{\sqrt{13}},\ \frac{c^3_{52}}{\sqrt{13}},\ \frac{c^{11}_{52}}{\sqrt{13}},\ \frac{c^9_{52}}{\sqrt{13}},\ -\frac{c^1_{52}}{\sqrt{13}};\ -\frac{c^{11}_{52}}{\sqrt{13}},\ \frac{c^1_{52}}{\sqrt{13}},\ -\frac{c^5_{52}}{\sqrt{13}},\ \frac{c^3_{52}}{\sqrt{13}},\ \frac{c^9_{52}}{\sqrt{13}};\ \frac{c^7_{52}}{\sqrt{13}},\ \frac{c^9_{52}}{\sqrt{13}},\ -\frac{c^5_{52}}{\sqrt{13}},\ \frac{c^{11}_{52}}{\sqrt{13}};\ -\frac{c^7_{52}}{\sqrt{13}},\ -\frac{c^1_{52}}{\sqrt{13}},\ -\frac{c^3_{52}}{\sqrt{13}};\ \frac{c^{11}_{52}}{\sqrt{13}},\ -\frac{c^7_{52}}{\sqrt{13}};\ \frac{c^5_{52}}{\sqrt{13}})$

$6^{1,0}_{16,1}$	11	$(0, \frac{1}{4}, \frac{1}{16}, \frac{5}{16}, \frac{9}{16}, \frac{13}{16})$, $\mathrm{i}(0,\ 0,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2};\ \ 0,\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2};\ \ -\frac{\sqrt{2}}{4},\ -\frac{\sqrt{2}}{4},\ \frac{\sqrt{2}}{4},\ \frac{\sqrt{2}}{4};\ \ \frac{\sqrt{2}}{4},\ \frac{\sqrt{2}}{4},\ -\frac{\sqrt{2}}{4};\ \ -\frac{\sqrt{2}}{4},\ -\frac{\sqrt{2}}{4};\ \ \frac{\sqrt{2}}{4})$
$6^{1,0}_{16,2}$	12	$(0, \frac{1}{2}, \frac{1}{16}, \frac{3}{16}, \frac{9}{16}, \frac{11}{16})$, $\mathrm{i}(0,\ 0,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2};\ \ 0,\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2};\ \ 0,\ -\frac{1}{2},\ 0,\ \frac{1}{2};\ 0,\ \frac{1}{2},\ 0;\ \ 0,\ -\frac{1}{2};\ \ 0)$
$6^{1,0}_{16,3}$	13	$(0, \frac{1}{2}, \frac{1}{16}, \frac{7}{16}, \frac{9}{16}, \frac{15}{16})$, $(0,\ 0,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2};\ 0,\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2};\ 0,\ -\frac{1}{2},\ 0,\ \frac{1}{2};\ 0,\ \frac{1}{2},\ 0;\ 0,\ -\frac{1}{2};\ 0)$
$6^{1,0}_{16,4}$	14	$(\frac{1}{8}, \frac{5}{8}, \frac{1}{16}, \frac{5}{16}, \frac{9}{16}, \frac{13}{16})$, $(0,\ 0,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2};\ 0,\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2};\ 0,\ -\frac{1}{2},\ 0,\ \frac{1}{2};\ 0,\ \frac{1}{2},\ 0;\ 0,\ -\frac{1}{2};\ 0)$
$6^{1,0}_{32,1}$	15	$(0, \frac{1}{8}, \frac{3}{32}, \frac{11}{32}, \frac{19}{32}, \frac{27}{32})$, $\mathrm{i}(0,\ 0,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2};\ \ 0,\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2};\ \ -\frac{1}{4}c^1_{16},\ -\frac{1}{4}c^3_{16},\ \frac{1}{4}c^1_{16},\ \frac{1}{4}c^3_{16};\ \ \frac{1}{4}c^1_{16},\ \frac{1}{4}c^3_{16},\ -\frac{1}{4}c^1_{16};\ \ -\frac{1}{4}c^1_{16},\ -\frac{1}{4}c^3_{16};\ \ \frac{1}{4}c^1_{16})$
$6^{1,0}_{32,2}$	16	$(0, \frac{1}{8}, \frac{7}{32}, \frac{15}{32}, \frac{23}{32}, \frac{31}{32})$, $(0,\ 0,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2};\ 0,\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2};\ -\frac{1}{4}c^1_{16},\ -\frac{1}{4}c^3_{16},\ \frac{1}{4}c^1_{16},\ \frac{1}{4}c^3_{16};\ \frac{1}{4}c^1_{16},\ \frac{1}{4}c^3_{16},\ -\frac{1}{4}c^1_{16};\ -\frac{1}{4}c^1_{16},\ -\frac{1}{4}c^3_{16};\ \frac{1}{4}c^1_{16})$

B. A list of all candidate $\mathrm{SL}_2(\mathbb{Z})$ representations of MTCs

We will follow the strategy outlined in Section 3.4. We first try to obtain a list that includes all $\mathrm{SL}_2(\mathbb{Z})$ representations associated with MTCs. Certainly, one such list is the list of all $\mathrm{SL}_2(\mathbb{Z})$ representations of finite levels. But such a list is very inefficient since most representations in the list are not associated with MTCs. So in this section we collect the conditions that a representation coming from a MTC must satisfy, to obtain a shorter list.

B.1. **The conditions on $\mathrm{SL}_2(\mathbb{Z})$ representations.** Some of the conditions on $\mathrm{SL}_2(\mathbb{Z})$ representations are obtained from the necessary conditions on modular data Propositions B.1 and 3.7, and others are discussed in the main text of this paper. Let us first translate the conditions on the (S, T) matrices to condition on an MD representations ρ_α:

Proposition B.1. *Given a modular data S, T of rank r, let ρ_α be any one of its 12 MD representations. Then ρ_α has the following properties:*

(1) *ρ_α is an $\mathrm{SL}_2(\mathbb{Z})$ representation of level $\mathrm{ord}(\rho_\alpha(\mathfrak{t}))$, and $\mathrm{ord}(T) \mid \mathrm{ord}(\rho_\alpha(\mathfrak{t})) \mid 12\,\mathrm{ord}(T)$.*
(2) *The conductor of the elements of $\rho_\alpha(\mathfrak{s})$ divides $\mathrm{ord}(\rho_\alpha(\mathfrak{t}))$.*
(3) *If ρ_α is a direct sum of two $\mathrm{SL}_2(\mathbb{Z})$ representations*
$$\rho_\alpha \cong \rho \oplus \rho', \tag{B.1}$$
then the eigenvalues of $\rho(\mathfrak{t})$ and $\rho'(\mathfrak{t})$ must overlap. This implies that if $\rho_\alpha = \rho \oplus \chi_1 \oplus \cdots \oplus \chi_\ell$ for some 1-dimensional representations $\chi_1, \ldots, \chi_\ell$, then $\chi_1, \cdots \chi_\ell$ are the same 1-dimensional representation.
(4) *Suppose that $\rho_\alpha \cong \rho \oplus \ell\chi$ for an irreducible representation ρ with non-degenerate $\rho(\mathfrak{t})$, and an 1-dimensional representation χ. If $\ell \neq 2\dim(\rho) - 1$ or $\ell > 1$, then $(\rho(\mathfrak{s})\chi(\mathfrak{s})^{-1})^2 = \mathrm{id}$.*
(5) *ρ_α satisfies*
$$\rho_\alpha \ncong n\rho \tag{B.2}$$
for any integer $n > 1$ and any representation ρ such that $\rho(\mathfrak{t})$ is non-degenerate.
(6) *If $\rho_\alpha(s)^2 = \pm\,\mathrm{id}$ (i.e. if the modular data or MTC is self dual), $\mathrm{pord}(\rho_\alpha(\mathfrak{t}))$ is a prime and satisfies $\mathrm{pord}(\rho_\alpha(\mathfrak{t})) = 1$ mod 4, then the representation ρ_α cannot be a direct sum of a d-dimensional irreducible $\mathrm{SL}_2(\mathbb{Z})$ representation and two or more 1-dimensional $\mathrm{SL}_2(\mathbb{Z})$ representations with $d = (p+1)/2$.*

(7) *Let* $3 < p < q$ *be prime such that* $pq \equiv 3 \mod 4$ *and* $\text{pord}(\rho_\alpha(\mathfrak{t})) = pq$*, then the rank* $r \neq \frac{p+q}{2} + 1$*. Moreover, if* $p > 5$*, rank* $r > \frac{p+q}{2} + 1$*.*

(8) *The number of self dual objects is greater than 0. Thus*

$$\text{Tr}(\rho_\alpha(\mathfrak{s})^2) \neq 0. \tag{B.3}$$

Since $\text{Tr}(\rho_\alpha(\mathfrak{s})^2) \neq 0$*, let us introduce*

$$C = \frac{\text{Tr}(\rho_\alpha(\mathfrak{s})^2)}{|\text{Tr}(\rho_\alpha(\mathfrak{s})^2)|}\rho_\alpha(\mathfrak{s})^2. \tag{B.4}$$

The above C *is the charge conjugation operator of MTC, i.e.* C *is a permutation matrix of order 2. In particular,* $\text{Tr}(C)$ *is the number of self dual objects. Also, for each eigenvalue* $\tilde{\theta}$ *of* $\rho_\alpha(\mathfrak{t})$*,*

$$\text{Tr}_{\tilde{\theta}}(C) \geqslant 0, \tag{B.5}$$

where $\text{Tr}_{\tilde{\theta}}$ *is the trace in the degenerate subspace of* $\rho_\alpha(\mathfrak{t})$ *with eigenvalue* $\tilde{\theta}$*.*

(9) *For any Galois conjugation* σ *in* $\text{Gal}(\mathbb{Q}_{\text{ord}(\rho_\alpha(\mathfrak{t}))})$*, there is a permutation of the indices,* $i \to \hat{\sigma}(i)$*, and* $\epsilon_\sigma(i) \in \{1, -1\}$*, such that*

$$\sigma\big(\rho_\alpha(\mathfrak{s})_{i,j}\big) = \epsilon_\sigma(i)\rho_\alpha(\mathfrak{s})_{\hat{\sigma}(i),j} = \rho_\alpha(\mathfrak{s})_{i,\hat{\sigma}(j)}\epsilon_\sigma(j) \tag{B.6}$$

$$\sigma^2\big(\rho_\alpha(\mathfrak{t})_{i,i}\big) = \rho_\alpha(\mathfrak{t})_{\hat{\sigma}(i),\hat{\sigma}(i)}, \tag{B.7}$$

for all i, j*.*

(10) *By* [11, Theorem II]*,* $D_{\rho_\alpha}(\sigma)$ *defined in* (3.6) *must be a* ***signed permutation***

$$(D_{\rho_\alpha}(\sigma))_{i,j} = \epsilon_\sigma(i)\delta_{\hat{\sigma}(i),j}.$$

and satisfies

$$\sigma(\rho_\alpha(\mathfrak{s})) = D_{\rho_\alpha}(\sigma)\rho_\alpha(\mathfrak{s}) = \rho_\alpha(\mathfrak{s})D^\top_{\rho_\alpha}(\sigma),$$
$$\sigma^2(\rho_\alpha(\mathfrak{t})) = D_{\rho_\alpha}(\sigma)\rho_\alpha(\mathfrak{t})D^\top_{\rho_\alpha}(\sigma) \tag{B.8}$$

(11) *There exists a* u *such that* $\rho_\alpha(\mathfrak{s})_{uu} \neq 0$ *and*

$$\begin{aligned} &\rho_\alpha(\mathfrak{s})_{ui} \neq 0 \in \mathbb{R}, \quad \frac{\rho_\alpha(\mathfrak{s})_{ij}}{\rho_\alpha(\mathfrak{s})_{uu}}, \ \frac{\rho_\alpha(\mathfrak{s})_{ij}}{\rho_\alpha(\mathfrak{s})_{uj}} \in \mathbb{O}_{\text{ord}(T)}, \quad \frac{\rho_\alpha(\mathfrak{s})_{ij}}{\rho_\alpha(\mathfrak{s})_{i'j'}} \in \mathbb{Q}_{\text{ord}(T)}, \\ &N_k^{ij} = \sum_{l=0}^{r-1} \frac{\rho_\alpha(\mathfrak{s})_{li}\rho_\alpha(\mathfrak{s})_{lj}\rho_\alpha(\mathfrak{s}^{-1})_{lk}}{\rho_\alpha(\mathfrak{s})_{lu}} \in \mathbb{N}. \\ &\forall\ i, j, k = 0, 1, \ldots, r-1. \end{aligned} \tag{B.9}$$

(u corresponds the unit object of MTC).

(12) *Let* $n \in \mathbb{N}_+$*. The* n^{th} *Frobenius-Schur indicator of the i-th simple object*

$$\begin{aligned} \nu_n(i) &= \sum_{j,k=0}^{r-1} N_i^{jk}\rho_\alpha(\mathfrak{s})_{ju}\theta_j^n[\rho_\alpha(\mathfrak{s})_{ku}\theta_k^n]^* = \sum_{j,k=0}^{r-1} N_i^{jk}\rho_\alpha(\mathfrak{t}^n\mathfrak{s})_{ju}\rho_\alpha(\mathfrak{t}^{-n}\mathfrak{s}^{-1})_{ku} \\ &= \sum_{j,k,l=0}^{r-1} \frac{\rho_\alpha(\mathfrak{s})_{lj}\rho_\alpha(\mathfrak{s})_{lk}\rho^*_\alpha(\mathfrak{s})_{li}}{\rho_\alpha(\mathfrak{s})_{lu}}\rho_\alpha(\mathfrak{t}^n\mathfrak{s})_{ju}\rho_\alpha(\mathfrak{t}^{-n}\mathfrak{s}^{-1})_{ku} \\ &= \sum_{l=0}^{r-1} \frac{\rho_\alpha(\mathfrak{s}\mathfrak{t}^n\mathfrak{s})_{lu}\rho_\alpha(\mathfrak{s}\mathfrak{t}^{-n}\mathfrak{s}^{-1})_{lu}\rho_\alpha(\mathfrak{s}^{-1})_{li}}{\rho_\alpha(\mathfrak{s})_{lu}} \end{aligned} \tag{B.10}$$

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is a cyclotomic integer whose conductor divides n *and* $\operatorname{ord}(T)$*. The 1st Frobenius-Schur indicator satisfies* $\nu_1(i) = \delta_{iu}$ *while the 2nd Frobenius-Schur indicator* $\nu_2(i)$ *satisfies* $\nu_2(i) = \pm\rho_\alpha(\mathfrak{s}^2)_{ii}$ *(see* [3, 24, 33]*).*

(13) *If we further assume the modular data or the MTC to be non-integral, then* $\operatorname{pord}(\tilde{\rho}_\alpha(\mathfrak{t})) = \operatorname{ord}(T) \notin \{2,3,4,6\}$*. In particular,* $\operatorname{ord}(\rho_\alpha(\mathfrak{t})) \notin \{2,3,4,6\}$*.*

In Section 3.1 and Appendix A, we have explicitly constructed all irreducible unitary representations of $\mathrm{SL}_2(\mathbb{Z})$ (up to unitary equivalence). However, this only gives the $\mathrm{SL}_2(\mathbb{Z})$ representations in some arbitrary basis, not in the basis yielding MD representations (*i.e.* satisfying (3.7)). We can improve the situation by choosing a basis to make $\rho(\mathfrak{t})$ diagonal and $\rho(\mathfrak{s})$ symmetric. Since we are going to use several types of bases, let us define these choices:

Definition B.2. *An unitary* $\mathrm{SL}_2(\mathbb{Z})$ *representations* $\tilde{\rho}$ *is called a* ***general*** $\mathrm{SL}_2(\mathbb{Z})$ *matrix representations if* $\tilde{\rho}(\mathfrak{t})$ *is diagonal* [**]*. A general* $\mathrm{SL}_2(\mathbb{Z})$ *matrix representation* $\tilde{\rho}$ *is called* ***symmetric*** *if* $\tilde{\rho}(\mathfrak{s})$ *is symmetric. An general* $\mathrm{SL}_2(\mathbb{Z})$ *matrix representation* $\tilde{\rho}$ *is called* ***irrep-sum*** *if* $\tilde{\rho}(\mathfrak{s}), \tilde{\rho}(\mathfrak{t})$ *are matrix-direct sum of irreducible* $\mathrm{SL}_2(\mathbb{Z})$ *representations. An* $\mathrm{SL}_2(\mathbb{Z})$ *matrix representations* $\tilde{\rho}$ *is called an* $\mathrm{SL}_2(\mathbb{Z})$ *representation* ***of modular data*** S, T*, if* $\tilde{\rho}$ *is unitary equivalent to an MD representation of the modular data, i.e.,*

$$\tilde{\rho}(\mathfrak{s}) = \mathrm{e}^{-2\pi \mathrm{i}\frac{\alpha}{4}} \frac{1}{D} U S U^\dagger, \qquad \tilde{\rho}(\mathfrak{t}) = U T U^\dagger \mathrm{e}^{2\pi \mathrm{i}(\frac{-c}{24} + \frac{\alpha}{12})}, \tag{B.11}$$

for some unitary matrix U *and* $\alpha \in \mathbb{Z}_{12}$*, where* c *is the central charge.*[††]

Through our explicit construction, we observe that all irreducible unitary representations of $\mathrm{SL}_2(\mathbb{Z})$ are unitarily equivalent to symmetric matrix representations of $\mathrm{SL}_2(\mathbb{Z})$, at least for dimension equal or less than 12.

We note that different choices of orthogonal basis give rise to different matrix representations of $\mathrm{SL}_2(\mathbb{Z})$. The modular data S, T is obtained from some particular choices of the basis. Some properties on the MD representations of a modular data do not depend on the choices of basis in the eigenspaces of $\tilde{\rho}(\mathfrak{t})$ (induced by the block-diagonal unitary transformation U in (B.11) that leaves $\tilde{\rho}(\mathfrak{t})$ invariant). Those properties remain valid for any general $\mathrm{SL}_2(\mathbb{Z})$ representations $\tilde{\rho}$ of the modular data. In the following, we collect the basis-independent conditions on the $\mathrm{SL}_2(\mathbb{Z})$ matrix representations of modular data. Those conditions have been discussed in the main text.

Proposition B.3. *Let* $\tilde{\rho}$ *be a general* $\mathrm{SL}_2(\mathbb{Z})$ *matrix representations of a modular data or a MTC. Then* $\tilde{\rho}$ *must satisfy the following conditions:*

(1) *If* $\tilde{\rho}$ *is a direct sum of two* $\mathrm{SL}_2(\mathbb{Z})$ *representations*
$$\tilde{\rho} \cong \rho \oplus \rho', \tag{B.12}$$
then the diagonals entries of $\rho(\mathfrak{t})$ *and* $\rho'(\mathfrak{t})$ *must overlap.*

(2) *Suppose that* $\tilde{\rho} \cong \rho \oplus \ell\chi$ *for an irreducible representation* ρ *with* $\rho(\mathfrak{t})$ *non-degenerate, and a character* χ*. If* $\ell \neq 1$ *and* $\ell \neq 2\dim(\rho) - 1$*, then* $(\rho(\mathfrak{s})\chi(\mathfrak{s})^{-1})^2 = \mathrm{id}$*.*

(3) *If* $\tilde{\rho}(\mathfrak{s})^2 = \pm\,\mathrm{id}$*, and* $\operatorname{pord}(\tilde{\rho}(\mathfrak{t})) = 1$ *mod 4 is a prime, then the representation* $\tilde{\rho}$ *cannot be a direct sum of a d-dimensional irreducible* $\mathrm{SL}_2(\mathbb{Z})$ *representation and two or more 1-dimensional* $\mathrm{SL}_2(\mathbb{Z})$ *representations with* $d = (\operatorname{pord}(\tilde{\rho}(\mathfrak{t})) + 1)/2$*.*

(4) $\tilde{\rho}$ *satisfies*
$$\tilde{\rho} \ncong n\rho \tag{B.13}$$
for any integer $n > 1$ *and any representation* ρ *such that* $\rho(\mathfrak{t})$ *is non-degenerate.*

[**]We will consider only $\mathrm{SL}_2(\mathbb{Z})$ matrix representations with diagonal $\tilde{\rho}(\mathfrak{t})$ in this paper.

[††]Note that D^2 is always positive and D in (B.11) is the positive square root of D^2, even for non-unitary cases.

(5) *Let* $3 < p < q$ *be prime such that* $pq \equiv 3 \mod 4$ *and* $\mathrm{pord}(\rho(\mathfrak{t})) = pq$*, then the rank* $r \neq \frac{p+q}{2} + 1$*. Moreover, if* $p > 5$*, rank* $r > \frac{p+q}{2} + 1$*.*

(6) *If we further assume* D^2 *of the modular data or the MTC to be non-integral, then* $\mathrm{pord}(\tilde{\rho}(\mathfrak{t})) = \mathrm{ord}(T) \notin \{2,3,4,6\}$*. This implies that* $\mathrm{ord}(\tilde{\rho}(\mathfrak{t})) \notin \{2,3,4,6\}$*.*

Some properties of an MD representation depend on the choice of basis. To make use of those properties, we can construct some combinations of $\tilde{\rho}(\mathfrak{s})$s that are invariant under the block-diagonal unitary transformation U.

The eigenvalues of $\tilde{\rho}(\mathfrak{t})$ partition the indices of the basis vectors. To construct the invariant combinations of $\tilde{\rho}(\mathfrak{s})$, for any eigenvalue $\tilde{\theta}$ of $\tilde{\rho}(\mathfrak{t})$, let

$$I_{\tilde{\theta}} = \{i \,|\, \tilde{\rho}(\mathfrak{t})_{ii} = \tilde{\theta}\}. \tag{B.14}$$

Let $I = I_{\tilde{\theta}}$, $J = J_{\tilde{\theta}'}$, $K = K_{\tilde{\theta}''}$ for some eigenvalues $\tilde{\theta}$, $\tilde{\theta}'$, $\tilde{\theta}''$ of $\tilde{\rho}(\mathfrak{t})$. We see that the following uniform polynomials of $\tilde{\rho}(\mathfrak{s})$ are invariant

$$\begin{aligned} P_I(\rho(\mathfrak{s})) &= \mathrm{Tr}\,\tilde{\rho}(\mathfrak{s})_{II} \equiv \sum_{i\in I} \tilde{\rho}(\mathfrak{s})_{ii}, \\ P_{IJ}(\rho(\mathfrak{s})) &= \mathrm{Tr}\,\tilde{\rho}(\mathfrak{s})_{IJ}\tilde{\rho}(\mathfrak{s})_{JI} \equiv \sum_{i\in I, j\in J} \tilde{\rho}(\mathfrak{s})_{i,j}\tilde{\rho}(\mathfrak{s})_{ji}, \\ P_{IJK}(\rho(\mathfrak{s})) &= \mathrm{Tr}\,\tilde{\rho}(\mathfrak{s})_{IJ}\tilde{\rho}(\mathfrak{s})_{JK}\tilde{\rho}(\mathfrak{s})_{KI} \equiv \sum_{i\in I, j\in J, k\in K} \tilde{\rho}(\mathfrak{s})_{i,j}\tilde{\rho}(\mathfrak{s})_{j,k}\tilde{\rho}(\mathfrak{s})_{k,i}. \end{aligned} \tag{B.15}$$

Certainly we can construction many other invariant uniform polynomials in the similar way. Using those invariant uniform polynomials, we have the following results

Proposition B.4. *Let* $\tilde{\rho}$ *be a general* $\mathrm{SL}_2(\mathbb{Z})$ *representations of a modular data or a MTC. Then following statements hold:*

(1) $\tilde{\rho}(\mathfrak{s})$ *satisfies*

$$\mathrm{Tr}(\tilde{\rho}(\mathfrak{s})^2) \in \mathbb{Z}\backslash\{0\}. \tag{B.16}$$

Let

$$C = \frac{\mathrm{Tr}(\tilde{\rho}(\mathfrak{s})^2)}{|\,\mathrm{Tr}(\tilde{\rho}(\mathfrak{s})^2)|}\tilde{\rho}(\mathfrak{s})^2. \tag{B.17}$$

For all I*,*

$$P_I(C) \geqslant 0. \tag{B.18}$$

(2) *The conductor of* $P_{\mathrm{odd}}(\tilde{\rho}(\mathfrak{s}))$ *divides* $\mathrm{ord}(\tilde{\rho}(\mathfrak{t}))$ *for all the invariant uniform polynomials* P_{odd} *with odd powers of* $\tilde{\rho}(\mathfrak{s})$ *(such as* P_I *and* P_{IJK} *in* (B.15)*). The conductor of* $P_{\mathrm{even}}(\tilde{\rho}(\mathfrak{s}))$ *divides* $\mathrm{pord}(\tilde{\rho}(\mathfrak{t}))$ *for all the invariant uniform polynomials* P_{even} *with even powers of* $\tilde{\rho}(\mathfrak{s})$ *(such as* P_{IJ} *in* (B.15)*).*

(3) *For any Galois conjugation* $\sigma \in \mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(\rho(\mathfrak{t}))})$*, there is a permutation on the set* $\{I\}$*,* $I \to \hat{\sigma}(I)$*, such that*

$$\begin{aligned} \sigma P_{IJ}(\tilde{\rho}(\mathfrak{s})) &= P_{I\hat{\sigma}(J)}(\tilde{\rho}(\mathfrak{s})) = P_{\hat{\sigma}(I)J}(\tilde{\rho}(\mathfrak{s})) \\ \sigma^2(\tilde{\theta}_I) &= \tilde{\theta}_{\hat{\sigma}(I)}, \end{aligned} \tag{B.19}$$

for all I, J*.*

(4) *For any invariant uniform polynomials* P *(such as those in* (B.15)*)*

$$\sigma P\big(\tilde{\rho}(\mathfrak{s})\big) = P\big(\sigma\tilde{\rho}(\mathfrak{s})\big) = P\big(\tilde{\rho}(\mathfrak{t})^a\tilde{\rho}(\mathfrak{s})\tilde{\rho}(\mathfrak{t})^b\tilde{\rho}(\mathfrak{s})\tilde{\rho}(\mathfrak{t})^a\big) \tag{B.20}$$

where $\sigma \in \mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(\tilde\rho(\mathfrak{t}))})$*, and* a, b *are given by* $\sigma(\mathrm{e}^{\mathrm{i}2\pi/\mathrm{ord}(\tilde\rho(\mathfrak{t}))}) = \mathrm{e}^{a\mathrm{i}2\pi/\mathrm{ord}(\tilde\rho(\mathfrak{t}))}$ *and* $ab \equiv 1$ *mod* $\mathrm{ord}(\tilde\rho(\mathfrak{t}))$*.*

Instead of constructing invariants, there is another way to make use of the properties of an MD representation that depend on the choices of basis. We can choose a more special basis, so that the basis is closer to the basis that leads to the MD representation. For example, we can choose a basis to make $\tilde\rho(\mathfrak{s})$ symmetric (*i.e.* to make $\tilde\rho$ a symmetric representation).

Now consider a symmetric $\mathrm{SL}_2(\mathbb{Z})$ matrix representation $\tilde\rho$ of a modular data or of a MTC. We find that the restriction of the unitary U in (B.11) on the non-degenerate subspace (see Theorem 3.4) must be diagonal with diagonal elements $U_{ii} \in \{1, -1\}$. Therefore, on the non-degenerate subspace, $\tilde\rho(\mathfrak{s})$ of a symmetric representation differs from $\rho(\mathfrak{s})$ of an MD representation only by a diagonal unitary transformation U with diagonal elements ± 1, i.e., a **signed diagonal** matrix. In this case some properties of MD representation apply to the blocks of the symmetric representation within the non-degenerate subspace. This allows us to obtain

Proposition B.5. *Let* $\tilde\rho$ *be a symmetric* $\mathrm{SL}_2(\mathbb{Z})$ *representations equivalent to an MD representation. Let*

$$I_{\mathrm{ndeg}} := \{i \mid \tilde\rho(\mathfrak{t})_{i,i} \text{ is a non-degenerate eigenvalue}\}, \tag{B.21}$$

Then there exists an orthogonal U *such that* $U\tilde\rho U^\top$ *is a pMD representation, and the following statements hold:*

(1) *The conductor of* $(U\tilde\rho(\mathfrak{s})U^\top)_{i,j}$ *divides* $\mathrm{ord}(\tilde\rho(\mathfrak{t}))$ *for all* i, j*. This implies that the conductor of* $(\tilde\rho(\mathfrak{s}))_{i,j}$ *divides* $\mathrm{ord}(\tilde\rho(\mathfrak{t}))$ *for all* $i, j \in I_{\mathrm{ndeg}}$*.*

(2) *For any Galois conjugation* σ *in* $\mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(\tilde\rho(\mathfrak{t}))})$*, there is a permutation* $i \to \hat\sigma(i)$*, such that*

$$\begin{aligned}\sigma\big((U\tilde\rho(\mathfrak{s})U^\top)_{i,j}\big) &= \epsilon_\sigma(i)(U\tilde\rho(\mathfrak{s})U^\top)_{\hat\sigma(i),j} = (U\tilde\rho(\mathfrak{s})U^\top)_{i,\hat\sigma(j)}\epsilon_\sigma(j)\\ \sigma^2\big(\tilde\rho(\mathfrak{t})_{i,i}\big) &= \tilde\rho(\mathfrak{t})_{\hat\sigma(i),\hat\sigma(i)},\end{aligned} \tag{B.22}$$

for all i, j*, where* $\epsilon_\sigma(i) \in \{1, -1\}$*. This implies that*

$$\begin{aligned}\sigma\big(\tilde\rho(\mathfrak{s})_{i,j}\big) &= \tilde\rho(\mathfrak{s})_{\hat\sigma(i),j} \quad\text{or}\quad \sigma\big(\tilde\rho(\mathfrak{s})_{i,j}\big) = -\tilde\rho(\mathfrak{s})_{\hat\sigma(i),j}\\ \sigma\big(\tilde\rho(\mathfrak{s})_{i,j}\big) &= \tilde\rho(\mathfrak{s})_{i,\hat\sigma(j)} \quad\text{or}\quad \sigma\big(\tilde\rho(\mathfrak{s})_{i,j}\big) = -\tilde\rho(\mathfrak{s})_{i,\hat\sigma(j)}\end{aligned} \tag{B.23}$$

for all $i, j \in I_{\mathrm{ndeg}}$*. This also implies that* $D_{\tilde\rho}(\sigma)$ *defined in* (3.6) *is a signed permutation matrix in the* I_{ndeg} *block, i.e.* $(D_{\tilde\rho}(\sigma))_{i,j}$ *for* $i, j \in I_{\mathrm{ndeg}}$ *are matrix elements of a signed permutation matrix.*

(3) *For all* i, j*,*

$$\sigma\big((U\tilde\rho(\mathfrak{s})U^\top)_{i,j}\big) = \big(U\tilde\rho(\mathfrak{t})^a\tilde\rho(\mathfrak{s})\tilde\rho(\mathfrak{t})^b\tilde\rho(\mathfrak{s})\tilde\rho(\mathfrak{t})^aU^\top\big)_{i,j} \tag{B.24}$$

where $\sigma \in \mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(\tilde\rho(\mathfrak{t}))})$*, and* a, b *are given by* $\sigma(\mathrm{e}^{\mathrm{i}2\pi/\mathrm{ord}(\tilde\rho(\mathfrak{t}))}) = \mathrm{e}^{a\mathrm{i}2\pi/\mathrm{ord}(\tilde\rho(\mathfrak{t}))}$ *and* $ab \equiv 1$ *mod* $\mathrm{ord}(\tilde\rho(\mathfrak{t}))$*. This implies that*

$$\sigma\big((\tilde\rho(\mathfrak{s}))_{i,j}\big) = \big(\tilde\rho(\mathfrak{t})^a\tilde\rho(\mathfrak{s})\tilde\rho(\mathfrak{t})^b\tilde\rho(\mathfrak{s})\tilde\rho(\mathfrak{t})^a\big)_{i,j}. \tag{B.25}$$

for all $i, j \in I_{\mathrm{ndeg}}$*.*

(4) *Both* T *and* $\tilde\rho(\mathfrak{t})$ *are diagonal, and without loss of generality, we may assume* $\tilde\rho(\mathfrak{t})$ *is a scalar multiple of* T*. In this case* U *in* (B.11) *is a block diagonal matrix preserving the eigenspaces of* $\tilde\rho(\mathfrak{t})$*. Let* $I_{\mathrm{nonzero}} = \{i\}$ *be a set of indices such that the* i^{th} *row of* $U\tilde\rho(\mathfrak{s})U^\top$ *contains no zeros for some othorgonal* U *satisfying* $U\tilde\rho(\mathfrak{t})U^\top = \tilde\rho(\mathfrak{t})$*. The index for the unit object of MTC must be in* I_{nonzero}*. Thus* I_{nonzero} *must be nonempty:*

$$I_{\mathrm{nonzero}} \neq \varnothing. \tag{B.26}$$

(5) *Let* $I_{\tilde{\theta}}$ *be a set of indices for an eigenspace* $E_{\tilde{\theta}}$ *of* $\tilde{\rho}(\mathfrak{t})$

$$I_{\tilde{\theta}} := \{i \mid \tilde{\rho}(\mathfrak{t})_{i,i} = \tilde{\theta}\}. \tag{B.27}$$

Then there exists a $I_{\tilde{\theta}}$ *such that*

$$I_{\tilde{\theta}} \cap I_{\text{nonzero}} \neq \varnothing \quad \textit{and} \quad \mathrm{Tr}_{E_{\tilde{\theta}}} C > 0, \tag{B.28}$$

where C *is given in* (B.17).

(6) *If we further assume the modular data to be non-integral, then there exists a* $I_{\tilde{\theta}}$ *that has a non-empty overlap with* I_{nonzero}*, such that* $D_{\tilde{\rho}}(\sigma)_{I_{\tilde{\theta}}} \neq \pm\,\mathrm{id}$ *for some* $\sigma \in \mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(\tilde{\rho}(\mathfrak{t}))}/\mathbb{Q})$*. Here* $D_{\tilde{\rho}}(\sigma)$ *is defined in* (3.6)*:*

$$D_{\tilde{\rho}}(\sigma) = \tilde{\rho}(\mathfrak{t})^a \tilde{\rho}(\mathfrak{s}) \tilde{\rho}(\mathfrak{t})^b \tilde{\rho}(\mathfrak{s}) \tilde{\rho}(\mathfrak{t})^a \tilde{\rho}^{-1}(\mathfrak{s}) \tag{B.29}$$

where a, b *are given by* $\sigma(\mathrm{e}^{2\pi\mathrm{i}/\mathrm{ord}(\tilde{\rho}(\mathfrak{t}))}) = \mathrm{e}^{a2\pi\mathrm{i}/\mathrm{ord}(\tilde{\rho}(\mathfrak{t}))}$ *and* $ab \equiv 1 \mod \mathrm{ord}(\tilde{\rho}(\mathfrak{t}))$*. Also* $D_{\tilde{\rho}}(\sigma)_{I_{\tilde{\theta}}}$ *is the block of* $D_{\tilde{\rho}}(\sigma)$ *with indices in* $I_{\tilde{\theta}}$*, i.e. the matrix elements of* $D_{\tilde{\rho}}(\sigma)_{I_{\tilde{\theta}}}$ *are given by* $(D_{\tilde{\rho}}(\sigma))_{i,j},\ i,j \in I_{\tilde{\theta}}$*.*

Proposition B.5(6) is a consequence of Theorem 3.13(3). Using GAP System for Computational Discrete Algebra, we obtain a list of symmetric irrep-sum $\mathrm{SL}_2(\mathbb{Z})$ matrix representations that satisfy the conditions in Propositions B.3, B.4, and B.5. The list is given below for rank $r = 6$ case (see Appendix section B.2).

Some of those symmetric irrep-sum $\mathrm{SL}_2(\mathbb{Z})$ matrix representations are representations of modular data, while others are not. However, the list includes all the symmetric irrep-sum $\mathrm{SL}_2(\mathbb{Z})$ matrix representations of modular data or MTC's which are not weakly integral (and some that are weakly integral).

B.2. **List of symmetric irrep-sum representations.** The following is a list the all rank-6 symmetric irrep-sum representations that satisfy the conditions in Propositions B.3, B.4, and B.5. The list contains all the rank-6 symmetric irrep-sum representations that are unitarily equivalent to rank-6 MD representations, plus some extra ones.

For each symmetric irrep-sum representation, we may generate an orbit by orthogonal transformations

$$\rho_{\text{isum}}(\mathfrak{s}) \to U \rho_{\text{isum}}(\mathfrak{s}) U^\top, \quad \rho_{\text{isum}}(\mathfrak{t}) \to U \rho_{\text{isum}}(\mathfrak{t}) U^\top, \tag{B.30}$$

tensoring 1-dimensional $\mathrm{SL}_2(\mathbb{Z})$ representations χ_α, $\alpha = 1, \dots, 12$:

$$\rho_{\text{isum}}(\mathfrak{s}) \to \chi_\alpha(\mathfrak{s}) \rho_{\text{isum}}(\mathfrak{s}), \quad \rho_{\text{isum}}(\mathfrak{t}) \to \chi_\alpha(\mathfrak{t}) \rho_{\text{isum}}(\mathfrak{t}), \tag{B.31}$$

and applying Galois conjugations σ in $\mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(\rho_{\text{isum}}(\mathfrak{t}))})$:

$$\rho_{\text{isum}}(\mathfrak{s}) \to \sigma(\rho_{\text{isum}}(\mathfrak{s})), \quad \rho_{\text{isum}}(\mathfrak{t}) \to \sigma(\rho_{\text{isum}}(\mathfrak{t})). \tag{B.32}$$

We will call such an orbit a **GT orbit**. The following list includes only one representative for each GT orbit. The list can also be regarded as a list GT orbits.

In the list, a representation ρ_{isum} is expressed as the direct sum of irreducible representations $\rho_{\text{isum}} = \rho_1 \oplus \rho_2 \oplus \cdots$, where $\rho_a(\mathfrak{t})$ is presented as $(\tilde{s}_1, \tilde{s}_2, \cdots)$ with $\tilde{s}_i = \arg(\rho_a(\mathfrak{t})_{ii})$, and $\rho_a(\mathfrak{s})$ is presented as $(\rho_{11}, \rho_{12}, \rho_{13}, \rho_{14}, \cdots;\ \ \rho_{22}, \rho_{23}, \rho_{24}, \cdots)$. The direct sum is also given via an index form, for example, irreps $= 2_2^{1,0} \otimes 2_5^{1,0} \oplus 2_5^{1,0}$. It means that the representation ρ_{isum} is a direct sum of two irreducible representations $2_2^{1,0} \otimes 2_5^{1,0}$ and $2_5^{1,0}$. Here $2_2^{1,0}$, $2_5^{1,0}$ are indices of $\mathrm{SL}_2(\mathbb{Z})$ irreducible representations with prime-power levels. Those prime-power-level $\mathrm{SL}_2(\mathbb{Z})$ irreducible representation are listed in Appendix A, where the meaning of the indices is explained further. $2_2^{1,0} \otimes 2_5^{1,0}$ is the irreducible representation obtained by the tensor product of $2_2^{1,0}$ and $2_5^{1,0}$.

The dimensions of the representations ρ_{isum} are given by dims $= (r_1, r_2, \cdots)$, where r_a is the dimension of the irreducible representation ρ_a, satisfying $r_1 \geqslant r_2 \geqslant \cdots$. The levels of the representations ρ_a are given by levels $= (l_1, l_2, \cdots)$, where $l_a = \text{ord}(\rho_a(\mathfrak{t}))$. We will use (dims;levels) $= (r_1, r_2, \cdots; l_1, l_2, \cdots)$ to label those representations. Now we can explain how the representative of a GT orbit is chosen. The representative for a GT orbit is chosen to be the one with minimal $[[r_1, r_2, \cdots], \text{ord}(\rho_{\text{isum}}(\mathfrak{t})), [l_1, l_2, \cdots]]$. Here the order of two lists is determined by first compare the first elements of the two lists. If the first elements are equal, we then compare the second elements, *etc*. The order of cyclotomic numbers are given by GAP.

To describe the entries of $\rho_a(\mathfrak{s})$, we also introduced the following notations:

$$\zeta_n^m = e^{2\pi i m/n}, \quad c_n^m = \zeta_n^m + \zeta_n^{-m}, \quad s_n^m = \zeta_n^m - \zeta_n^{-m},$$
$$\xi_n^{m,k} = (\zeta_{2n}^m - \zeta_{2n}^{-m})/(\zeta_{2n}^k - \zeta_{2n}^{-k}), \quad \xi_n^m = \xi_n^{m,1}. \tag{B.33}$$

We find that, for rank 6, there are only 25 GT orbits. The GT orbits can be divided into two classes, resolved and unresolved, whose definition will to given in the next section. Below each GT orbit, we indicate whether it is resolved or unresolved. Among 25 GT orbits, 17 are resolved and 8 are unresolved.

For the 17 resolved GT orbits, it is easy to compute all the corresponding pairs of (S, T) matrices that satisfied the conditions in Proposition B.1, which will be done in next section. Below each resolved GT orbit, we indicate the number valid (S, T) pairs obtain with such a computation. Those valid (S, T) pairs will be listed in Appendix C.2. The 8 unresolved GT orbits are difficult to handle by computer, which are discussed in the main text. (The main text also discussed most of the resolved cases.)

1. (dims;levels) $=(3, 2, 1; 5, 5, 1)$, irreps $= 3_5^1 \oplus 2_5^1 \oplus 1_1^1$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 5$,
   $\rho_{\text{isum}}(\mathfrak{t}) = (0, \frac{1}{5}, \frac{4}{5}) \oplus (\frac{1}{5}, \frac{4}{5}) \oplus (0)$,
   $\rho_{\text{isum}}(\mathfrak{s}) = (\sqrt{\frac{1}{5}}, -\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}; -\frac{5+\sqrt{5}}{10}, \frac{5-\sqrt{5}}{10}; -\frac{5+\sqrt{5}}{10}) \oplus i(-\frac{1}{\sqrt{5}}c_{20}^3, \frac{1}{\sqrt{5}}c_{20}^1; \frac{1}{\sqrt{5}}c_{20}^3) \oplus (1)$
   Resolved. Number of valid (S, T) pairs = 0.

2. (dims;levels) $=(3, 2, 1; 8, 8, 1)$, irreps $= 3_8^{1,0} \oplus 2_8^{1,9} \oplus 1_1^1$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 8$,
   $\rho_{\text{isum}}(\mathfrak{t}) = (0, \frac{1}{8}, \frac{5}{8}) \oplus (\frac{1}{8}, \frac{7}{8}) \oplus (0)$,
   $\rho_{\text{isum}}(\mathfrak{s}) = i(0, \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}; -\frac{1}{2}, \frac{1}{2}; -\frac{1}{2}) \oplus i(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}; \sqrt{\frac{1}{2}}) \oplus (1)$
   Resolved. Number of valid (S, T) pairs = 0.

3. (dims;levels) $=(3, 2, 1; 5, 2, 1)$, irreps $= 3_5^1 \oplus 2_2^{1,0} \oplus 1_1^1$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 10$,
   $\rho_{\text{isum}}(\mathfrak{t}) = (0, \frac{1}{5}, \frac{4}{5}) \oplus (0, \frac{1}{2}) \oplus (0)$,
   $\rho_{\text{isum}}(\mathfrak{s}) = (\sqrt{\frac{1}{5}}, -\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}; -\frac{5+\sqrt{5}}{10}, \frac{5-\sqrt{5}}{10}; -\frac{5+\sqrt{5}}{10}) \oplus (-\frac{1}{2}, -\sqrt{\frac{3}{4}}; \frac{1}{2}) \oplus (1)$
   Unresolved.

4. (dims;levels) $=(3, 2, 1; 5, 2, 2)$, irreps $= 3_5^1 \oplus 2_2^{1,0} \oplus 1_2^{1,0}$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 10$,
   $\rho_{\text{isum}}(\mathfrak{t}) = (0, \frac{1}{5}, \frac{4}{5}) \oplus (0, \frac{1}{2}) \oplus (\frac{1}{2})$,
   $\rho_{\text{isum}}(\mathfrak{s}) = (\sqrt{\frac{1}{5}}, -\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}; -\frac{5+\sqrt{5}}{10}, \frac{5-\sqrt{5}}{10}; -\frac{5+\sqrt{5}}{10}) \oplus (-\frac{1}{2}, -\sqrt{\frac{3}{4}}; \frac{1}{2}) \oplus (-1)$
   Unresolved.

5. (dims;levels) $=(3, 2, 1; 4, 3, 2)$, irreps $= 3_4^{1,3} \oplus 2_3^{1,0} \oplus 1_2^{1,0}$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 12$,
   $\rho_{\text{isum}}(\mathfrak{t}) = (0, \frac{1}{2}, \frac{1}{4}) \oplus (0, \frac{1}{3}) \oplus (\frac{1}{2})$,

$\rho_{\text{isum}}(\mathfrak{s}) = \mathrm{i}(-\frac{1}{2},\ \frac{1}{2},\ \sqrt{\frac{1}{2}};\ \ -\frac{1}{2},\ \sqrt{\frac{1}{2}};\ \ 0) \oplus \mathrm{i}(-\sqrt{\frac{1}{3}},\ \sqrt{\frac{2}{3}};\ \ \sqrt{\frac{1}{3}}) \oplus (-1)$
Resolved. Number of valid (S, T) pairs = 0.

6. (dims;levels) =$(3, 2, 1; 4, 3, 4)$, irreps = $3_4^{1,3} \oplus 2_3^{1,0} \oplus 1_4^{1,0}$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 12$,
$\rho_{\text{isum}}(\mathfrak{t}) = (0, \frac{1}{2}, \frac{1}{4}) \oplus (0, \frac{1}{3}) \oplus (\frac{1}{4})$,
$\rho_{\text{isum}}(\mathfrak{s}) = \mathrm{i}(-\frac{1}{2},\ \frac{1}{2},\ \sqrt{\frac{1}{2}};\ \ -\frac{1}{2},\ \sqrt{\frac{1}{2}};\ \ 0) \oplus \mathrm{i}(-\sqrt{\frac{1}{3}},\ \sqrt{\frac{2}{3}};\ \ \sqrt{\frac{1}{3}}) \oplus \mathrm{i}(1)$
Unresolved.

7. (dims;levels) =$(3, 2, 1; 8, 3, 1)$, irreps = $3_8^{1,0} \oplus 2_3^{1,0} \oplus 1_1^{1}$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 24$,
$\rho_{\text{isum}}(\mathfrak{t}) = (0, \frac{1}{8}, \frac{5}{8}) \oplus (0, \frac{1}{3}) \oplus (0)$,
$\rho_{\text{isum}}(\mathfrak{s}) = \mathrm{i}(0,\ \sqrt{\frac{1}{2}},\ \sqrt{\frac{1}{2}};\ \ -\frac{1}{2},\ \frac{1}{2};\ \ -\frac{1}{2}) \oplus \mathrm{i}(-\sqrt{\frac{1}{3}},\ \sqrt{\frac{2}{3}};\ \ \sqrt{\frac{1}{3}}) \oplus (1)$
Resolved. Number of valid (S, T) pairs = 0.

8. (dims;levels) =$(3, 2, 1; 8, 3, 3)$, irreps = $3_8^{1,0} \oplus 2_3^{1,0} \oplus 1_3^{1,0}$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 24$,
$\rho_{\text{isum}}(\mathfrak{t}) = (0, \frac{1}{8}, \frac{5}{8}) \oplus (0, \frac{1}{3}) \oplus (\frac{1}{3})$,
$\rho_{\text{isum}}(\mathfrak{s}) = \mathrm{i}(0,\ \sqrt{\frac{1}{2}},\ \sqrt{\frac{1}{2}};\ \ -\frac{1}{2},\ \frac{1}{2};\ \ -\frac{1}{2}) \oplus \mathrm{i}(-\sqrt{\frac{1}{3}},\ \sqrt{\frac{2}{3}};\ \ \sqrt{\frac{1}{3}}) \oplus (1)$
Resolved. Number of valid (S, T) pairs = 0.

9. (dims;levels) =$(3, 3; 5, 3)$, irreps = $3_5^{1} \oplus 3_3^{1,0}$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 15$,
$\rho_{\text{isum}}(\mathfrak{t}) = (0, \frac{1}{5}, \frac{4}{5}) \oplus (0, \frac{1}{3}, \frac{2}{3})$,
$\rho_{\text{isum}}(\mathfrak{s}) = (\sqrt{\frac{1}{5}},\ -\sqrt{\frac{2}{5}},\ -\sqrt{\frac{2}{5}};\ -\frac{5+\sqrt{5}}{10},\ \frac{5-\sqrt{5}}{10};\ -\frac{5+\sqrt{5}}{10}) \oplus (-\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3};\ -\frac{1}{3},\ \frac{2}{3};\ -\frac{1}{3})$
Resolved. Number of valid (S, T) pairs = 0.

10. (dims;levels) =$(3, 3; 16, 16)$, irreps = $3_{16}^{1,0} \oplus 3_{16}^{1,6}$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 16$,
$\rho_{\text{isum}}(\mathfrak{t}) = (\frac{1}{8}, \frac{1}{16}, \frac{9}{16}) \oplus (\frac{5}{8}, \frac{1}{16}, \frac{9}{16})$,
$\rho_{\text{isum}}(\mathfrak{s}) = \mathrm{i}(0,\ \sqrt{\frac{1}{2}},\ \sqrt{\frac{1}{2}};\ \ -\frac{1}{2},\ \frac{1}{2};\ \ -\frac{1}{2}) \oplus \mathrm{i}(0,\ \sqrt{\frac{1}{2}},\ \sqrt{\frac{1}{2}};\ \ \frac{1}{2},\ -\frac{1}{2};\ \ \frac{1}{2})$
Unresolved.

11. (dims;levels) =$(3, 3; 5, 4)$, irreps = $3_5^{1} \oplus 3_4^{1,0}$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 20$,
$\rho_{\text{isum}}(\mathfrak{t}) = (0, \frac{1}{5}, \frac{4}{5}) \oplus (0, \frac{1}{4}, \frac{3}{4})$,
$\rho_{\text{isum}}(\mathfrak{s}) = (\sqrt{\frac{1}{5}},\ -\sqrt{\frac{2}{5}},\ -\sqrt{\frac{2}{5}};\ -\frac{5+\sqrt{5}}{10},\ \frac{5-\sqrt{5}}{10};\ -\frac{5+\sqrt{5}}{10}) \oplus (0,\ \sqrt{\frac{1}{2}},\ \sqrt{\frac{1}{2}};\ -\frac{1}{2},\ \frac{1}{2};\ -\frac{1}{2})$
Resolved. Number of valid (S, T) pairs = 2.

12. (dims;levels) =$(4, 1, 1; 9, 1, 1)$, irreps = $4_{9,2}^{1,0} \oplus 1_1^{1} \oplus 1_1^{1}$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 9$,
$\rho_{\text{isum}}(\mathfrak{t}) = (0, \frac{1}{9}, \frac{4}{9}, \frac{7}{9}) \oplus (0) \oplus (0)$,
$\rho_{\text{isum}}(\mathfrak{s}) = (0,\ -\sqrt{\frac{1}{3}},\ -\sqrt{\frac{1}{3}},\ -\sqrt{\frac{1}{3}};\ \frac{1}{3}c_9^2,\ \frac{1}{3}c_9^4,\ \frac{1}{3}c_9^1;\ \frac{1}{3}c_9^1,\ \frac{1}{3}c_9^2;\ \frac{1}{3}c_9^4) \oplus (1) \oplus (1)$
Unresolved.

13. (dims;levels) =$(4, 2; 5, 5)$, irreps = $4_{5,1}^{1} \oplus 2_5^{1}$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 5$,
$\rho_{\text{isum}}(\mathfrak{t}) = (\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}) \oplus (\frac{1}{5}, \frac{4}{5})$,

$\rho_{\text{isum}}(\mathfrak{s}) = \mathrm{i}(\frac{1}{5}c_{20}^1 + \frac{1}{5}c_{20}^3,\ \frac{2}{5}c_{15}^2 + \frac{1}{5}c_{15}^3,\ -\frac{1}{5} + \frac{2}{5}c_{15}^1 - \frac{1}{5}c_{15}^3,\ \frac{1}{5}c_{20}^1 - \frac{1}{5}c_{20}^3;\ -\frac{1}{5}c_{20}^1 + \frac{1}{5}c_{20}^3,\ -\frac{1}{5}c_{20}^1 - \frac{1}{5}c_{20}^3,\ \frac{1}{5} - \frac{2}{5}c_{15}^1 + \frac{1}{5}c_{15}^3;\ \frac{1}{5}c_{20}^1 - \frac{1}{5}c_{20}^3,\ \frac{2}{5}c_{15}^2 + \frac{1}{5}c_{15}^3;\ -\frac{1}{5}c_{20}^1 - \frac{1}{5}c_{20}^3) \oplus \mathrm{i}(-\frac{1}{\sqrt{5}}c_{20}^3,\ \frac{1}{\sqrt{5}}c_{20}^1;\ \frac{1}{\sqrt{5}}c_{20}^3)$
Unresolved.

14. (dims;levels) $=(4,2;5,5;a)$, irreps $= 4_{5,2}^1 \oplus 2_5^1$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 5$,
$\rho_{\text{isum}}(\mathfrak{t}) = (\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}) \oplus (\frac{1}{5}, \frac{4}{5})$,
$\rho_{\text{isum}}(\mathfrak{s}) = (\sqrt{\frac{1}{5}},\ \frac{-5+\sqrt{5}}{10},\ -\frac{5+\sqrt{5}}{10},\ \sqrt{\frac{1}{5}};\ -\sqrt{\frac{1}{5}},\ \sqrt{\frac{1}{5}},\ \frac{5+\sqrt{5}}{10};\ -\sqrt{\frac{1}{5}},\ \frac{5-\sqrt{5}}{10};\ \sqrt{\frac{1}{5}}) \oplus \mathrm{i}(-\frac{1}{\sqrt{5}}c_{20}^3,\ \frac{1}{\sqrt{5}}c_{20}^1;\ \frac{1}{\sqrt{5}}c_{20}^3)$
Resolved. Number of valid (S,T) pairs = 0.

15. (dims;levels) $=(4,2;10,5)$, irreps $= 2_5^1 \otimes 2_2^{1,0} \oplus 2_5^1$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 10$,
$\rho_{\text{isum}}(\mathfrak{t}) = (\frac{1}{5}, \frac{4}{5}, \frac{3}{10}, \frac{7}{10}) \oplus (\frac{1}{5}, \frac{4}{5})$,
$\rho_{\text{isum}}(\mathfrak{s}) = \mathrm{i}(\frac{1}{2\sqrt{5}}c_{20}^3,\ \frac{1}{2\sqrt{5}}c_{20}^1,\ \frac{3}{2\sqrt{15}}c_{20}^1,\ \frac{3}{2\sqrt{15}}c_{20}^3;\ -\frac{1}{2\sqrt{5}}c_{20}^3,\ -\frac{3}{2\sqrt{15}}c_{20}^3,\ \frac{3}{2\sqrt{15}}c_{20}^1;\ \frac{1}{2\sqrt{5}}c_{20}^3,\ -\frac{1}{2\sqrt{5}}c_{20}^1;\ -\frac{1}{2\sqrt{5}}c_{20}^3) \oplus \mathrm{i}(-\frac{1}{\sqrt{5}}c_{20}^3,\ \frac{1}{\sqrt{5}}c_{20}^1;\ \frac{1}{\sqrt{5}}c_{20}^3)$
Unresolved.

16. (dims;levels) $=(4,2;15,5)$, irreps $= 2_5^1 \otimes 2_3^{1,0} \oplus 2_5^1$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 15$,
$\rho_{\text{isum}}(\mathfrak{t}) = (\frac{1}{5}, \frac{4}{5}, \frac{2}{15}, \frac{8}{15}) \oplus (\frac{1}{5}, \frac{4}{5})$,
$\rho_{\text{isum}}(\mathfrak{s}) = (-\frac{1}{\sqrt{15}}c_{20}^3,\ \frac{1}{\sqrt{15}}c_{20}^1,\ \frac{2}{\sqrt{30}}c_{20}^1,\ -\frac{2}{\sqrt{30}}c_{20}^3;\ \frac{1}{\sqrt{15}}c_{20}^3,\ \frac{2}{\sqrt{30}}c_{20}^3,\ \frac{2}{\sqrt{30}}c_{20}^1;\ -\frac{1}{\sqrt{15}}c_{20}^3,\ -\frac{1}{\sqrt{15}}c_{20}^1;\ \frac{1}{\sqrt{15}}c_{20}^3) \oplus \mathrm{i}(-\frac{1}{\sqrt{5}}c_{20}^3,\ \frac{1}{\sqrt{5}}c_{20}^1;\ \frac{1}{\sqrt{5}}c_{20}^3)$
Resolved. Number of valid (S,T) pairs = 1.

17. (dims;levels) $=(4,2;7,3)$, irreps $= 4_7^1 \oplus 2_3^{1,0}$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 21$,
$\rho_{\text{isum}}(\mathfrak{t}) = (0, \frac{1}{7}, \frac{2}{7}, \frac{4}{7}) \oplus (0, \frac{1}{3})$,
$\rho_{\text{isum}}(\mathfrak{s}) = \mathrm{i}(-\sqrt{\frac{1}{7}},\ \sqrt{\frac{2}{7}},\ \sqrt{\frac{2}{7}},\ \sqrt{\frac{2}{7}};\ -\frac{1}{\sqrt{7}}c_7^2,\ -\frac{1}{\sqrt{7}}c_7^1,\ \frac{1}{\sqrt{7}\mathrm{i}}s_{28}^5;\ \frac{1}{\sqrt{7}\mathrm{i}}s_{28}^5,\ -\frac{1}{\sqrt{7}}c_7^2;\ -\frac{1}{\sqrt{7}}c_7^1) \oplus \mathrm{i}(-\sqrt{\frac{1}{3}},\ \sqrt{\frac{2}{3}};\ \sqrt{\frac{1}{3}})$
Resolved. Number of valid (S,T) pairs = 1.

18. (dims;levels) $=(5,1;5,1)$, irreps $= 5_5^1 \oplus 1_1^1$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 5$,
$\rho_{\text{isum}}(\mathfrak{t}) = (0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}) \oplus (0)$,
$\rho_{\text{isum}}(\mathfrak{s}) = (-\frac{1}{5},\ \sqrt{\frac{6}{25}},\ \sqrt{\frac{6}{25}},\ \sqrt{\frac{6}{25}},\ \sqrt{\frac{6}{25}};\ \frac{3-\sqrt{5}}{10},\ -\frac{1+\sqrt{5}}{5},\ \frac{-1+\sqrt{5}}{5},\ \frac{3+\sqrt{5}}{10};\ \frac{3+\sqrt{5}}{10},\ \frac{3-\sqrt{5}}{10},\ \frac{-1+\sqrt{5}}{5};\ \frac{3+\sqrt{5}}{10},\ -\frac{1+\sqrt{5}}{5};\ \frac{3-\sqrt{5}}{10}) \oplus (1)$
Unresolved.

19. (dims;levels) $=(6;9)$, irreps $= 6_{9,3}^{1,0}$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 9$,
$\rho_{\text{isum}}(\mathfrak{t}) = (\frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9})$,
$\rho_{\text{isum}}(\mathfrak{s}) = (\frac{1}{3},\ \frac{1}{3}c_9^2,\ \frac{1}{3},\ -\frac{1}{3}c_9^1,\ \frac{1}{3},\ \frac{1}{3}c_9^4;\ \frac{1}{3},\ \frac{1}{3}c_9^4,\ -\frac{1}{3},\ \frac{1}{3}c_9^1,\ \frac{1}{3};\ \frac{1}{3},\ -\frac{1}{3}c_9^2,\ \frac{1}{3},\ \frac{1}{3}c_9^1;\ \frac{1}{3},\ -\frac{1}{3}c_9^4,\ -\frac{1}{3};\ \frac{1}{3},\ \frac{1}{3}c_9^2;\ \frac{1}{3})$
Resolved. Number of valid (S,T) pairs = 1.

20. (dims;levels) $=(6;13)$, irreps $= 6_{13}^1$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 13$,
$\rho_{\text{isum}}(\mathfrak{t}) = (\frac{1}{13}, \frac{3}{13}, \frac{4}{13}, \frac{9}{13}, \frac{10}{13}, \frac{12}{13})$,

$\rho_{\text{isum}}(\mathfrak{s}) = \mathrm{i}(-\frac{1}{\sqrt{13}}c_{52}^{5},\ \frac{1}{\sqrt{13}}c_{52}^{7},\ \frac{1}{\sqrt{13}}c_{52}^{3},\ \frac{1}{\sqrt{13}}c_{52}^{11},\ \frac{1}{\sqrt{13}}c_{52}^{9},\ -\frac{1}{\sqrt{13}}c_{52}^{1};\ -\frac{1}{\sqrt{13}}c_{52}^{11},\ \frac{1}{\sqrt{13}}c_{52}^{1},\ -\frac{1}{\sqrt{13}}c_{52}^{5},\ \frac{1}{\sqrt{13}}c_{52}^{3},\ \frac{1}{\sqrt{13}}c_{52}^{9};\ \frac{1}{\sqrt{13}}c_{52}^{7},\ \frac{1}{\sqrt{13}}c_{52}^{9},\ -\frac{1}{\sqrt{13}}c_{52}^{5},\ \frac{1}{\sqrt{13}}c_{52}^{11};\ -\frac{1}{\sqrt{13}}c_{52}^{7},\ -\frac{1}{\sqrt{13}}c_{52}^{1},\ -\frac{1}{\sqrt{13}}c_{52}^{3};\ \frac{1}{\sqrt{13}}c_{52}^{11},\ -\frac{1}{\sqrt{13}}c_{52}^{7};\ \frac{1}{\sqrt{13}}c_{52}^{5})$

Resolved. Number of valid (S,T) pairs = 1.

21. (dims;levels) $=(6;15)$, irreps $= 3_3^{1,0}\otimes 2_5^1$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 15$,

$\rho_{\text{isum}}(\mathfrak{t}) = (\frac{1}{5}, \frac{4}{5}, \frac{2}{15}, \frac{7}{15}, \frac{8}{15}, \frac{13}{15})$,

$\rho_{\text{isum}}(\mathfrak{s}) = \mathrm{i}(\frac{1}{3\sqrt{5}}c_{20}^{3},\ \frac{1}{3\sqrt{5}}c_{20}^{1},\ \frac{2}{3\sqrt{5}}c_{20}^{1},\ \frac{2}{3\sqrt{5}}c_{20}^{1},\ \frac{2}{3\sqrt{5}}c_{20}^{3},\ \frac{2}{3\sqrt{5}}c_{20}^{3};\ -\frac{1}{3\sqrt{5}}c_{20}^{3},\ -\frac{2}{3\sqrt{5}}c_{20}^{3},\ -\frac{2}{3\sqrt{5}}c_{20}^{3},\ \frac{2}{3\sqrt{5}}c_{20}^{1},\ \frac{2}{3\sqrt{5}}c_{20}^{1};\ -\frac{1}{3\sqrt{5}}c_{20}^{3},\ \frac{2}{3\sqrt{5}}c_{20}^{3},\ \frac{1}{3\sqrt{5}}c_{20}^{1},\ -\frac{2}{3\sqrt{5}}c_{20}^{1};\ -\frac{1}{3\sqrt{5}}c_{20}^{3},\ -\frac{2}{3\sqrt{5}}c_{20}^{1},\ \frac{1}{3\sqrt{5}}c_{20}^{1};\ \frac{1}{3\sqrt{5}}c_{20}^{3},\ -\frac{2}{3\sqrt{5}}c_{20}^{3};\ \frac{1}{3\sqrt{5}}c_{20}^{3})$

Resolved. Number of valid (S,T) pairs = 0.

22. (dims;levels) $=(6;16)$, irreps $= 6_{16,1}^{1,0}$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 16$,

$\rho_{\text{isum}}(\mathfrak{t}) = (0, \frac{1}{4}, \frac{1}{16}, \frac{5}{16}, \frac{9}{16}, \frac{13}{16})$,

$\rho_{\text{isum}}(\mathfrak{s}) = \mathrm{i}(0,\ 0,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2};\ 0,\ \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2};\ -\sqrt{\frac{1}{8}},\ -\sqrt{\frac{1}{8}},\ \sqrt{\frac{1}{8}},\ \sqrt{\frac{1}{8}};\ \sqrt{\frac{1}{8}},\ \sqrt{\frac{1}{8}},\ -\sqrt{\frac{1}{8}};\ -\sqrt{\frac{1}{8}},\ -\sqrt{\frac{1}{8}};\ \sqrt{\frac{1}{8}})$

Resolved. Number of valid (S,T) pairs = 4.

23. (dims;levels) $=(6;35)$, irreps $= 3_7^3\otimes 2_5^2$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 35$,

$\rho_{\text{isum}}(\mathfrak{t}) = (\frac{1}{35}, \frac{4}{35}, \frac{9}{35}, \frac{11}{35}, \frac{16}{35}, \frac{29}{35})$,

$\rho_{\text{isum}}(\mathfrak{s}) = \mathrm{i}(-\frac{4}{35}c_{140}^{1}-\frac{3}{35}c_{140}^{3}-\frac{1}{7}c_{140}^{5}+\frac{1}{35}c_{140}^{7}+\frac{1}{35}c_{140}^{9}+\frac{4}{35}c_{140}^{13}+\frac{2}{35}c_{140}^{15}-\frac{3}{35}c_{140}^{17}+\frac{9}{35}c_{140}^{19}-\frac{4}{35}c_{140}^{21}-\frac{2}{7}c_{140}^{23},\ -\frac{1}{\sqrt{35}}c_{35}^{4}+\frac{1}{\sqrt{35}}c_{35}^{11},\ \frac{1}{\sqrt{35}}c_{35}^{1}-\frac{1}{\sqrt{35}}c_{35}^{6},\ \frac{2}{\sqrt{35}}c_{35}^{3}+\frac{1}{\sqrt{35}}c_{35}^{4}+\frac{1}{\sqrt{35}}c_{35}^{10}+\frac{1}{\sqrt{35}}c_{35}^{11},\ -\frac{1}{\sqrt{35\mathrm{i}}}s_{140}^{3}-\frac{1}{\sqrt{35\mathrm{i}}}s_{140}^{17},\ \frac{2}{35}c_{140}^{1}-\frac{1}{35}c_{140}^{3}-\frac{1}{7}c_{140}^{5}-\frac{3}{35}c_{140}^{7}+\frac{1}{5}c_{140}^{9}-\frac{2}{35}c_{140}^{13}-\frac{1}{35}c_{140}^{15}-\frac{1}{35}c_{140}^{17}+\frac{3}{35}c_{140}^{19}+\frac{2}{35}c_{140}^{21}-\frac{2}{7}c_{140}^{23};\ -\frac{1}{\sqrt{35\mathrm{i}}}s_{140}^{3}-\frac{1}{\sqrt{35\mathrm{i}}}s_{140}^{17},\ \frac{4}{35}c_{140}^{1}+\frac{3}{35}c_{140}^{3}+\frac{1}{7}c_{140}^{5}-\frac{1}{35}c_{140}^{7}-\frac{1}{35}c_{140}^{9}-\frac{4}{35}c_{140}^{13}-\frac{2}{35}c_{140}^{15}+\frac{3}{35}c_{140}^{17}-\frac{9}{35}c_{140}^{19}+\frac{4}{35}c_{140}^{21}+\frac{2}{7}c_{140}^{23},\ -\frac{1}{\sqrt{35}}c_{35}^{1}+\frac{1}{\sqrt{35}}c_{35}^{6},\ -\frac{2}{35}c_{140}^{1}+\frac{1}{35}c_{140}^{3}+\frac{1}{7}c_{140}^{5}+\frac{3}{35}c_{140}^{7}-\frac{1}{5}c_{140}^{9}+\frac{2}{35}c_{140}^{13}+\frac{1}{35}c_{140}^{15}+\frac{1}{35}c_{140}^{17}-\frac{3}{35}c_{140}^{19}-\frac{2}{35}c_{140}^{21}+\frac{2}{7}c_{140}^{23},\ \frac{2}{\sqrt{35}}c_{35}^{3}+\frac{1}{\sqrt{35}}c_{35}^{4}+\frac{1}{\sqrt{35}}c_{35}^{10}+\frac{1}{\sqrt{35}}c_{35}^{11};\ \frac{2}{\sqrt{35}}c_{35}^{3}+\frac{1}{\sqrt{35}}c_{35}^{4}+\frac{1}{\sqrt{35}}c_{35}^{10}+\frac{1}{\sqrt{35}}c_{35}^{11},\ -\frac{2}{35}c_{140}^{1}+\frac{1}{35}c_{140}^{3}+\frac{1}{7}c_{140}^{5}+\frac{3}{35}c_{140}^{7}-\frac{1}{5}c_{140}^{9}+\frac{2}{35}c_{140}^{13}+\frac{1}{35}c_{140}^{15}+\frac{1}{35}c_{140}^{17}-\frac{3}{35}c_{140}^{19}-\frac{2}{35}c_{140}^{21}+\frac{2}{7}c_{140}^{23},\ \frac{1}{\sqrt{35}}c_{35}^{4}-\frac{1}{\sqrt{35}}c_{35}^{11},\ -\frac{1}{\sqrt{35\mathrm{i}}}s_{140}^{3}-\frac{1}{\sqrt{35\mathrm{i}}}s_{140}^{17};\ \frac{1}{\sqrt{35\mathrm{i}}}s_{140}^{3}+\frac{1}{\sqrt{35\mathrm{i}}}s_{140}^{17},\ -\frac{4}{35}c_{140}^{1}-\frac{3}{35}c_{140}^{3}-\frac{1}{7}c_{140}^{5}+\frac{1}{35}c_{140}^{7}+\frac{1}{35}c_{140}^{9}+\frac{4}{35}c_{140}^{13}+\frac{2}{35}c_{140}^{15}-\frac{3}{35}c_{140}^{17}+\frac{9}{35}c_{140}^{19}-\frac{4}{35}c_{140}^{21}-\frac{2}{7}c_{140}^{23},\ \frac{1}{\sqrt{35}}c_{35}^{4}-\frac{1}{\sqrt{35}}c_{35}^{11};\ -\frac{2}{\sqrt{35}}c_{35}^{3}-\frac{1}{\sqrt{35}}c_{35}^{4}-\frac{1}{\sqrt{35}}c_{35}^{10}-\frac{1}{\sqrt{35}}c_{35}^{11},\ -\frac{1}{\sqrt{35}}c_{35}^{1}+\frac{1}{\sqrt{35}}c_{35}^{6};\ \frac{4}{35}c_{140}^{1}+\frac{3}{35}c_{140}^{3}+\frac{1}{7}c_{140}^{5}-\frac{1}{35}c_{140}^{7}-\frac{1}{35}c_{140}^{9}-\frac{4}{35}c_{140}^{13}-\frac{2}{35}c_{140}^{15}+\frac{3}{35}c_{140}^{17}-\frac{9}{35}c_{140}^{19}+\frac{4}{35}c_{140}^{21}+\frac{2}{7}c_{140}^{23})$

Resolved. Number of valid (S,T) pairs = 1.

24. (dims;levels) $=(6;56)$, irreps $= 3_7^1\otimes 2_8^{1,6}$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 28$,

$\rho_{\text{isum}}(\mathfrak{t}) = (\frac{1}{56}, \frac{9}{56}, \frac{11}{56}, \frac{25}{56}, \frac{43}{56}, \frac{51}{56})$,

$\rho_{\text{isum}}(\mathfrak{s}) = (\frac{1}{\sqrt{14}}c_{28}^{1},\ \frac{1}{\sqrt{14}}c_{28}^{3},\ -\frac{1}{\sqrt{14}}c_{28}^{5},\ -\frac{1}{\sqrt{14}}c_{28}^{5},\ \frac{1}{\sqrt{14}}c_{28}^{1},\ \frac{1}{\sqrt{14}}c_{28}^{3};\ -\frac{1}{\sqrt{14}}c_{28}^{5},\ \frac{1}{\sqrt{14}}c_{28}^{1},\ \frac{1}{\sqrt{14}}c_{28}^{1},\ \frac{1}{\sqrt{14}}c_{28}^{3},\ -\frac{1}{\sqrt{14}}c_{28}^{5};\ -\frac{1}{\sqrt{14}}c_{28}^{3},\ \frac{1}{\sqrt{14}}c_{28}^{3},\ \frac{1}{\sqrt{14}}c_{28}^{5},\ -\frac{1}{\sqrt{14}}c_{28}^{1};\ \frac{1}{\sqrt{14}}c_{28}^{3},\ -\frac{1}{\sqrt{14}}c_{28}^{5},\ \frac{1}{\sqrt{14}}c_{28}^{1};\ -\frac{1}{\sqrt{14}}c_{28}^{1},\ -\frac{1}{\sqrt{14}}c_{28}^{3};\ \frac{1}{\sqrt{14}}c_{28}^{5})$

Resolved. Number of valid (S,T) pairs = 2.

25. (dims;levels) $=(6;80)$, irreps $= 3_{16}^{3,3}\otimes 2_5^2$, pord$(\rho_{\text{isum}}(\mathfrak{t})) = 80$,

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$\rho_{\text{isum}}(\mathfrak{t}) = (\frac{1}{40}, \frac{9}{40}, \frac{3}{80}, \frac{27}{80}, \frac{43}{80}, \frac{67}{80}),$

$\rho_{\text{isum}}(\mathfrak{s}) = \mathrm{i}(0,\ 0,\ \frac{1}{\sqrt{10}}c_{20}^{3},\ \frac{1}{\sqrt{10}}c_{20}^{1},\ \frac{1}{\sqrt{10}}c_{20}^{3},\ \frac{1}{\sqrt{10}}c_{20}^{1};\ 0,\ \frac{1}{\sqrt{10}}c_{20}^{1},\ -\frac{1}{\sqrt{10}}c_{20}^{3},\ \frac{1}{\sqrt{10}}c_{20}^{1},\ -\frac{1}{\sqrt{10}}c_{20}^{3};\ -\frac{1}{2\sqrt{5}}c_{20}^{1},$
$-\frac{1}{2\sqrt{5}}c_{20}^{3},\ \frac{1}{2\sqrt{5}}c_{20}^{1},\ \frac{1}{2\sqrt{5}}c_{20}^{3};\ \frac{1}{2\sqrt{5}}c_{20}^{1},\ \frac{1}{2\sqrt{5}}c_{20}^{3},\ -\frac{1}{2\sqrt{5}}c_{20}^{1};\ -\frac{1}{2\sqrt{5}}c_{20}^{1},\ -\frac{1}{2\sqrt{5}}c_{20}^{3};\ \frac{1}{2\sqrt{5}}c_{20}^{1})$

Resolved. Number of valid (S, T) pairs $= 2$.

C. A list of candidate modular data from resolved $\mathrm{SL}_2(\mathbb{Z})$ representations

C.1. **The notion of resolved $\mathrm{SL}_2(\mathbb{Z})$ matrix representations.** In the above, we have chosen a special basis in the eigenspaces of an $\mathrm{SL}_2(\mathbb{Z})$ matrix representation $\tilde{\rho}$ to make $\tilde{\rho}(\mathfrak{s})$ symmetric. But such a special basis is still not special enough to make $\tilde{\rho}$ to be an MD representation ρ.

We can choose a more special basis to make $\tilde{\rho}(\mathfrak{s}^2)$ a signed permutation matrix, and $\tilde{\rho}(\mathfrak{s})$ symmetric. We know that, for an MD representation ρ, $\rho(\mathfrak{s}^2)$ is a signed permutation matrix. So the new special basis makes $\tilde{\rho}$ closer to the MD representation ρ.

We can choose an even more special basis in the eigenspaces of $\tilde{\rho}(\mathfrak{t})$ to make $\tilde{\rho}$ even closer to the MD representation ρ, by using the matrix $D_{\tilde{\rho}}(\sigma)$ in (B.29). For an MD representation ρ, $D_\rho(\sigma)$ is suppose to be signed permutations. So we will try to choose a basis to transform each $D_{\tilde{\rho}}(\sigma)$ into signed permutations. We like to point out that, since both $\tilde{\rho}$ and ρ are symmetric $\mathrm{SL}_2(\mathbb{Z})$ matrix representations that are related by an unitary transformation, according to Theorem 3.4, they can be related by an orthogonal transformation.

Let us consider a simple case to demonstrate our approach. If $\tilde{\rho}(\mathfrak{t})$ is non-degenerate, then $D_{\tilde{\rho}}(\sigma)$ will automatically be a signed permutation matrix. Using signed diagonal matrices V_{sd}, we can transform $\tilde{\rho}$ to many other symmetric representations, ρ's:

$$\rho = V_{\text{sd}} \tilde{\rho} V_{\text{sd}}, \tag{C.1}$$

where $D_\rho(\sigma)$ remains a signed permutation. In fact the signed diagonal matrices V_{sd} are the most general orthogonal matrices that fix $\tilde{\rho}(\mathfrak{t})$ and transform all $D_{\tilde{\rho}}(\sigma)$'s into (potentially different) signed permutations. Thus the resulting symmetric representations, ρ's, include all the symmetric representations where $D_\rho(\sigma)$'s are signed permutations. From those ρ's, we can then construct many pairs of S, T matrices via (3.7), and check which one satisfies the conditions in Proposition B.1. Those S, T matrices that satisfy those conditions may very likely correspond to modular data (or MTC's). If none of the S, T matrices satisfy the conditions, then the representation $\tilde{\rho}$ will not be an $\mathrm{SL}_2(\mathbb{Z})$ representation of any modular data.

When some eigenspaces of $\tilde{\rho}(\mathfrak{t})$ are more than 1-dimensional, then the $D_{\tilde{\rho}}(\sigma)$ may not be signed permutations. There may be infinite many orthogonal matrices that can transform $D_{\tilde{\rho}}(\sigma)$ into signed permutations, which make the subsequent selection difficult. In the following, we will generalize the above notion of non-degenerate representation, to include some cases where some eigenspaces of $\tilde{\rho}(\mathfrak{t})$ are 2-dimensional or more. We will show that, for those special representations, there is only a finite number of orthogonal matrices that can transform $D_{\tilde{\rho}}(\sigma)$ into signed permutations.

To carry through this program, let us concentrate on an eigenspace $E_{\tilde{\theta}}$ of $\tilde{\rho}(\mathfrak{t})$ corresponding to an eigenvalue $\tilde{\theta}$, and let

$$\Omega_{\tilde{\rho}}(\tilde{\theta}) = \{\sigma \in \mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(\tilde{\rho}(\mathfrak{t}))}) \mid \sigma^2(\tilde{\theta}) = \tilde{\theta}\}\,. \tag{C.2}$$

Then $\Omega_{\tilde{\rho}}(\tilde{\theta})$ is a subgroup of $\mathrm{Gal}(\mathbb{Q}_{\mathrm{ord}(\tilde{\rho}(\mathfrak{t}))})$. By definition, $D_{\tilde{\rho}}(\sigma)$ stabilizes the $\tilde{\theta}$-eigenspace $E_{\tilde{\theta}}$ for $\sigma \in \Omega_{\tilde{\rho}}(\tilde{\theta})$, and commute with each other. In particular, $D_{\tilde{\rho}}|_{E_{\tilde{\theta}}}$ (restricted on $E_{\tilde{\theta}}$) defines a representation of $\Omega_{\tilde{\rho}}(\tilde{\theta})$ on $E_{\tilde{\theta}}$.

We can diagonalize $\{D_{\tilde{\rho}}(\sigma)|_{E_{\tilde{\theta}}} \mid \sigma \in \Omega_{\tilde{\rho}}(\tilde{\theta})\}$ simultaneously within $E_{\tilde{\theta}}$. The degeneracy of the $\tilde{\theta}$-eigenspace $E_{\tilde{\theta}}$ is fully resolved by these $D_{\tilde{\rho}}(\sigma)$'s, if the common eigenspace of these $D_{\tilde{\rho}}(\sigma)|_{E_{\tilde{\theta}}}$'s are all 1-dimensional. In terms of the characters of $\Omega_{\tilde{\rho}}(\tilde{\theta})$, the degeneracy of $E_{\tilde{\theta}}$ can be fully resolved if each irreducible character of $\Omega_{\tilde{\rho}}(\tilde{\theta})$ has multiplicity at most 1 in the character decomposition of $E_{\tilde{\theta}}$ as a representation of $\Omega_{\tilde{\rho}}(\tilde{\theta})$. Now we can introduce the notion of resolved representation:

Definition C.1. *A general* $\mathrm{SL}_2(\mathbb{Z})$ *matrix representation* $\tilde{\rho}$ *is called* ***resolved*** *if the degeneracy of each of eigenspace of* $\tilde{\rho}(\mathfrak{t})$ *is fully resolved by* $D_{\tilde{\rho}}(\sigma)$, $\sigma \in \Omega_{\tilde{\rho}}(\tilde{\theta})$, *as described above.*

Given a symmetric irrep-sum matrix representation (denoted as ρ_{isum}), we can use unitary matrices, U's, to transform it into a symmetric representation ρ via

$$\rho(\mathfrak{t}) = U\rho_{\text{isum}}(\mathfrak{t})U^\dagger, \qquad \rho(\mathfrak{s}) = U\rho_{\text{isum}}(\mathfrak{s})U^\dagger. \tag{C.3}$$

where $D_\rho(\sigma)|_{E_{\tilde{\theta}}}$, for all $\sigma \in \Omega_{\tilde{\rho}}(\tilde{\theta})$, are signed permutations within the $\tilde{\theta}$-eigenspace. If ρ_{isum} is resolved, then there is only a finite number of such representations. We then can check which of those representations satisfy Proposition B.1. This is how we compute the potential modular data S, T's from resolved ρ_{isum}'s.

To show a resolved ρ_{isum} is unitarily equivalent to only a finite number representations whose $D_\rho(\sigma)|_{E_{\tilde{\theta}}}$ are signed permutations, we note that both ρ and ρ_{isum} are symmetric, and according to Theorem 3.4, ρ and ρ_{isum} are in fact orthogonally equivalent, *i.e.* the above U can be chosen to satisfy $U = U^*$ and $UU^\top = \text{id}$. If the number of most general orthogonal matrices U that transform ρ_{isum} to ρ is finite, then the number of representations ρ are finite.

Since the orthogonal U acts within the eigenspace of $\rho_{\text{isum}}(\mathfrak{t})$, to show the number of possible U's are finite, we can concentrate on a single $\tilde{\theta}$-eigenspace $E_{\tilde{\theta}}$, and denote $\sigma \in \Omega_{\tilde{\rho}}(\tilde{\theta})$ as σ_{inv}. In the following, we will consider the cases where $E_{\tilde{\theta}}$ is 1-dimensional, 2-dimensional, *etc.*. For each case, we will show the number of possible U's are finite, and give the possible choices of U's.

C.1.1. *Within an 1-dimensional eigenspace of* $\rho_{\text{isum}}(\mathfrak{t})$. $D_{\rho_{\text{isum}}}(\sigma_{\text{inv}})|_{E_{\tilde{\theta}}} = \pm 1$ are already signed permutations. In this case the orthogonal matrix U (within the 1-dimensional eigenspace) has only two choices

$$U = \pm 1, \tag{C.4}$$

which is finite.

C.1.2. *Within a 2-dimensional eigenspace of* $\rho_{\text{isum}}(\mathfrak{t})$. In this case, the matrix groups MG generated by 2-by-2 matrices, $D_{\rho_{\text{isum}}}(\sigma_{\text{inv}})|_{E_{\tilde{\theta}}}$, can have several different forms, for those passing representations. By examine the computer results, we find that, for unresolved cases, matrix groups MG can be

$$\begin{aligned} MG &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, && \text{for } \dim(\rho_{\text{isum}}) \geqslant 5; \\ MG &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, && \text{for } \dim(\rho_{\text{isum}}) \geqslant 6. \end{aligned} \tag{C.5}$$

For resolved cases, we have

$$MG = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \qquad \text{for } \dim(\rho_{\text{isum}}) \geqslant 4;$$

$$MG = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \qquad \text{for } \dim(\rho_{\text{isum}}) \geqslant 6. \tag{C.6}$$

In those two cases,

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{or} \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{or} \quad U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{C.7}$$

will transform all $D_{\rho_{\text{isum}}}(\sigma_{\text{inv}})|_{E_{\tilde{\theta}}}$'s into signed permutations. In general we have

Theorem C.2. *Let*

$$MG_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

$$MG_4 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}. \tag{C.8}$$

The most general orthogonal matrices that transform all matrices in MG_2 or MG_4 into signed permutations must have one of the following forms

$$U = \frac{PV_{\text{sd}}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \text{ or } U = \frac{PV_{\text{sd}}}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ or } U = PV_{\text{sd}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{C.9}$$

where V_{sd} are signed diagonal matrices, and P are permutation matrices. The number of the orthogonal transformations U is finite.

Proof of Theorem C.2. We only needs to consider the first matrix group MG_2, where the matrix group is isomorphic to the $\mathbb{Z}_2$ group. There are only four matrix groups formed by 2-dimensional signed permutations matrices, that are isomorphic $\mathbb{Z}_2$. The four matrix groups are generated by the following four generators respectively:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{C.10}$$

An orthogonal transformation U that transforms MG to one of the above matrix groups must have a from $U = VU_0$, where V transforms MG_2 into itself, and U_0 is a fixed orthogonal transformation that transforms MG_2 to one of the above matrix groups. We can choose U_0 to have the following form

$$U_0 = \frac{P}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \text{ or } U_0 = \frac{P}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ or } U_0 = P \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{C.11}$$

To keep MG unchanged V must satisfy

$$V \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} V. \tag{C.12}$$

We find that V must be diagonal. Thus V, as an orthogonal matrix, must be signed diagonal. This gives us the result (C.9). □

If $\dim(\rho_{\text{isum}}) \geqslant 8$, it is possible that the matrix group of $D_{\rho_{\text{isum}}}(\sigma_{\text{inv}})|_{E_{\tilde{\theta}}}$'s is generated by the following non-diagonal matrix

$$\pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{C.13}$$

This is because the direct sum decomposition of ρ_{isum} contains a dimension-6 irreducible representation $6_1^{0,1}$ in Appendix A, whose $\rho(\mathfrak{t})$ has a 2-dimensional eigenspace. The representation $6_1^{0,1}$ can give rise to such form of $D_{\rho_{\text{isum}}}(\sigma_{\text{inv}})|_{E_{\tilde{\theta}}}$'s.

The eigenvalues of the matrices are $(\mathrm{i}, -\mathrm{i})$. The most general orthogonal matrices that transform all $D_{\rho_{\text{isum}}}(\sigma_{\text{inv}})|_{E_{\tilde{\theta}}}$'s into signed permutations must have a form

$$U = PV_{\text{sd}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{C.14}$$

If $\dim(\rho_{\text{isum}}) \geqslant 8$, it is also possible that $D_{\rho_{\text{isum}}}(\sigma_{\text{inv}})|_{E_{\tilde{\theta}}}$'s form the following matrix group:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{4}} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \sqrt{\frac{3}{4}} \\ -\sqrt{\frac{3}{4}} & -\frac{1}{2} \end{pmatrix} \tag{C.15}$$

This is because the direct sum decomposition of ρ_{isum} contains a dimension-8 irreducible representation whose $\rho(\mathfrak{t})$ has a 2-dimensional eigenspace, which gives rise to the such form of $D_{\rho_{\text{isum}}}(\sigma_{\text{inv}})|_{E_{\tilde{\theta}}}$'s .

The eigenvalues of the later two matrices are $\pm(\mathrm{e}^{\mathrm{i}2\pi/3}, \mathrm{e}^{-\mathrm{i}2\pi/3})$. A permutation of two elements can only have orders 1 or 2. The corresponding 2×2 signed permutation matrix can only have eigenvalues 1, -1 or $\pm\mathrm{i}$. Any other eigenvalue is not possible. Thus, there is no orthogonal matrix that can transform the above two matrices into signed permutation. Such ρ_{isum} is not a representation of any modular data.

C.1.3. *Within a 3-dimensional eigenspace of $\rho_{\text{isum}}(\mathfrak{t})$ for rank $\leqslant 6$.* There is only one such case for rank $\leqslant 6$. The 3×3 matrix group MG generated by $D_{\rho_{\text{isum}}}(\sigma_{\text{inv}})|_{E_{\tilde{\theta}}}$'s is given by

$$MG = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad \text{for } \dim(\rho_{\text{isum}}) = 6. \tag{C.16}$$

which is resolved. To find the most general orthogonal matrices that transform the above 3×3 matrices in MG into signed permutation matrices, we first show

Theorem C.3. *If P is a permutation matrix with $P^2 = \text{id}$, then P is a direct sum of 2×2 and 1×1 matrices. If P_{sgn} is a signed permutation matrix with $P_{\text{sgn}}^2 = \text{id}$, then P_{sgn} is a direct sum of 2×2 and 1×1 matrices.*

Proof of Theorem C.3. If P is a permutation matrix with $P^2 = \text{id}$, then P must be a pair-wise permutation, and thus P is a direct sum of 2×2 and 1×1 matrices. The reduction from signed permutation matrix to permutation matrix by ignoring the signs is homomorphism of the matrix product. If P_{sgn} is a signed permutation matrix with $P_{\text{sgn}}^2 = \text{id}$, then its reduction gives rise to a permutation matrix P with $P^2 = \text{id}$. Since P is a direct sum of 2×2 and 1×1 matrices, P_{sgn} is also a direct sum of 2×2 and 1×1 matrices. □

Using the above result, similarly, we can show that the most general orthogonal matrices that transform all $D_{\rho_{\text{isum}}}(\sigma_{\text{inv}})|_{E_{\tilde{\theta}}}$'s into signed permutations must have a form

$$U = \frac{PV_{\text{sd}}}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ or } U = \frac{PV_{\text{sd}}}{\sqrt{2}} \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{or } U = \frac{PV_{\text{sd}}}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \text{ or } U = \frac{PV_{\text{sd}}}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$$\text{or } U = \frac{PV_{\text{sd}}}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \text{ or } U = \frac{PV_{\text{sd}}}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\text{or } U = \frac{PV_{\text{sd}}}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{C.17}$$

where V_{sd} are signed diagonal matrices, and P are permutation matrices. We note that the non-trivial part of U is a 2×2 block for index $(1, 2)$, $(1, 3)$, and $(2, 3)$. The 2×2 block has three possibilities given in (C.9). Such U's transform the diagonal matrices in MG into a direct sum of a 2×2 and an 1×1 matrices. This is a general pattern that apply for all resolved diagonal matrix group MG generated by $D_{\rho_{\text{isum}}}(\sigma_{\text{inv}})|_{E_{\tilde{\theta}}}$.

The above are all the possibilities that can appear in resolved dimension-6 representations. In the following, we will consider more possibilities, that appear only for resolved representations with dimension larger than 6.

C.2. **List of S, T matrices from resolved representations.** We have constructed a list of irrep-sum symmetric representations (see Appendix B.2) that include all the representations of modular data. Among them, we can select a sublist of resolved symmetric representations, denoted as $\{\rho_{\text{res}}\}$. We then use the orthogonal matrix U constructed above (see (C.4), (C.9) and (C.17)) to transform the resolved symmetric representations ρ_{res} to representations, ρ's:

$$\rho(\mathfrak{t}) = U\rho_{\text{res}}(\mathfrak{t})U^\top, \quad \rho(\mathfrak{s}) = U\rho_{\text{res}}(\mathfrak{s})U^\top. \tag{C.18}$$

such that the corresponding $D_\rho(\sigma)$ are either zero or signed permutation in each eigenspace of $\rho(\mathfrak{t})$. Since the number of such representations is finite, we can examine all resulting representations one by one.

For each U, the resulting representation ρ should satisfy Proposition B.1. In particular, we examine all possible choices of index u that may correspond to the unit object, to see if ρ satisfy the condition (B.9). If no choices of u can satisfy (B.9), then the representations ρ is rejected. If some u's satisfy (B.9), then for each u, we can construct S, T matrices via (3.7). We then check if the resulting S, T matrices satisfy the conditions of modular data summarized in Proposition B.1

In the following, we list all the pairs of S, T matrices that satisfy the conditions in Proposition B.1, and come from the dimension-6 resolved $\text{SL}_2(\mathbb{Z})$ representations listed in Appendix B.2. The list includes all the modular data with $D^2 \notin \mathbb{Z}$ from resolved $\text{SL}_2(\mathbb{Z})$ representations (and the list also includes some modular data with $D^2 \in \mathbb{Z}$). In the list, the S, T matrices are grouped into orbits generated by Galois conjugations, which are called Galois orbits. To save space, we only list one representative for each orbit. If possible, the representative is chosen to have all-positive quantum dimensions.

Each pair of S, T matrices is indexed by $(r_1, r_2, \cdots; l_1, l_2, \cdots)^a_k$, such as $(3,3;5,4)^1_2$. The first part of index, $(3,3;5,4) =$ (dims;levels), is the index of GT orbit listed in Appendix B.2, indicating that the S, T matrices arise from a particular $\mathrm{SL}_2(\mathbb{Z})$ representation in the GT orbit. The subscript k labels the different Galois orbits. The a-index labels the Galois conjugation $\sigma_a : \mathrm{e}^{\mathrm{i}2\pi/\mathrm{ord}(T)} \to \mathrm{e}^{a\mathrm{i}2\pi/\mathrm{ord}(T)}$. Those a-indexed S, T matrices form a Galois orbit.

Some Galois orbits contain no unitary S, T matrices, but some of those S, T matrices are pseudo-unitary, *i.e.* those S, T matrices can be obtained from unitary S, T matrices via a change of spherical structure. In this case those Galois orbits can be obtained from Galois orbits that contain Galois orbits. To save space further, we also drop those Galois orbits that contain pseudo-unitary S, T matrices. There is only one orbit which contains no unitary and no pseudo-unitary S, T matrices. The numbering in the following list includes gaps as we maintain the numbering from the arXiv version for consistency.

In the list, T is presented in terms of topological spin $(s_1, s_2, \cdots)$ with $s_i = \arg(T_{ii})$. S is presented as $(S_{00}, S_{01}, S_{02}, S_{03}, \cdots;\ S_{11}, S_{12}, S_{13}, \cdots)$. $d_i = S_{0i}$ are the quantum dimensions.

Our calculation actually produces 174 pairs of S, T matrices, which are given in Supplementary Material Section in the arXiv version. All those 174 pairs of S, T matrices can be obtained from the pairs of S, T matrices in the following list, via Galois conjugations and change of the spherical structures.

1. ind $= (3,3;5,4)^1_1$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.236,\ 2.236)$
   $D^2 = 20.0 = 20$
   $T = (0, 0, \frac{1}{5}, \frac{4}{5}, \frac{1}{4}, \frac{3}{4})$,
   $S = (1,\ 1,\ 2,\ 2,\ \sqrt{5},\ \sqrt{5};\ \ 1,\ 2,\ 2,\ -\sqrt{5},\ -\sqrt{5};\ \ -1-\sqrt{5},\ -1+\sqrt{5},\ 0,\ 0;\ \ -1-\sqrt{5},\ 0,\ 0;\ \ -\sqrt{5},\ \sqrt{5};\ \ -\sqrt{5})$

2. ind $= (3,3;5,4)^1_2$: $d_i = (1.0,\ 1.0,\ 2.0,\ 2.0,\ 2.236,\ 2.236)$
   $D^2 = 20.0 = 20$
   $T = (0, 0, \frac{2}{5}, \frac{3}{5}, \frac{1}{4}, \frac{3}{4})$,
   $S = (1,\ 1,\ 2,\ 2,\ \sqrt{5},\ \sqrt{5};\ \ 1,\ 2,\ 2,\ -\sqrt{5},\ -\sqrt{5};\ \ -1+\sqrt{5},\ -1-\sqrt{5},\ 0,\ 0;\ \ -1+\sqrt{5},\ 0,\ 0;\ \ \sqrt{5},\ -\sqrt{5};\ \ \sqrt{5})$

3. ind $= (4,2;15,5)^1_1$: $d_i = (1.0,\ 1.0,\ 1.0,\ 1.618,\ 1.618,\ 1.618)$
   $D^2 = 10.854 = \frac{15+3\sqrt{5}}{2}$
   $T = (0, \frac{1}{3}, \frac{1}{3}, \frac{2}{5}, \frac{11}{15}, \frac{11}{15})$,
   $S = (1,\ 1,\ 1,\ \frac{1+\sqrt{5}}{2},\ \frac{1+\sqrt{5}}{2},\ \frac{1+\sqrt{5}}{2};\ \ \zeta_3^1,\ -\zeta_6^1,\ \frac{1+\sqrt{5}}{2},\ \frac{1+\sqrt{5}}{2}\zeta_3^1,\ -\frac{1+\sqrt{5}}{2}\zeta_6^1;\ \ \zeta_3^1,\ \frac{1+\sqrt{5}}{2},\ -\frac{1+\sqrt{5}}{2}\zeta_6^1,\ \frac{1+\sqrt{5}}{2}\zeta_3^1;\ \ -1,\ -1,\ -1;\ \ -\zeta_3^1,\ \zeta_6^1;\ \ -\zeta_3^1)$

7. ind $= (4,2;7,3)^1_1$: $d_i = (1.0,\ 3.791,\ 3.791,\ 3.791,\ 4.791,\ 5.791)$
   $D^2 = 100.617 = \frac{105+21\sqrt{21}}{2}$
   $T = (0, \frac{1}{7}, \frac{2}{7}, \frac{4}{7}, 0, \frac{2}{3})$,

$S = (1,\ \frac{3+\sqrt{21}}{2},\ \frac{3+\sqrt{21}}{2},\ \frac{3+\sqrt{21}}{2},\ \frac{5+\sqrt{21}}{2},\ \frac{7+\sqrt{21}}{2};\ 2-c_{21}^1-2c_{21}^2+3c_{21}^3+2c_{21}^4-2c_{21}^5,\ -c_{21}^2-2c_{21}^3-c_{21}^4+c_{21}^5,\ -1+2c_{21}^1+3c_{21}^2-c_{21}^3+2c_{21}^5,\ -\frac{3+\sqrt{21}}{2},\ 0;\ \ -1+2c_{21}^1+3c_{21}^2-c_{21}^3+2c_{21}^5,\ 2-c_{21}^1-2c_{21}^2+3c_{21}^3+2c_{21}^4-2c_{21}^5,\ -\frac{3+\sqrt{21}}{2},\ 0;\ \ -c_{21}^2-2c_{21}^3-c_{21}^4+c_{21}^5,\ -\frac{3+\sqrt{21}}{2},\ 0;\ \ 1,\ \frac{7+\sqrt{21}}{2};\ \ -\frac{7+\sqrt{21}}{2})$

9. ind $= (6;9)_1^1$: $d_i = (1.0,\ 0.347,\ 1.0,\ 1.532,\ -1.0,\ -1.879)$
$D^2 = 9.0 = 9$
$T = (0,\ \frac{1}{9},\ \frac{2}{3},\ \frac{4}{9},\ \frac{1}{3},\ \frac{7}{9}),$
$S = (1,\ c_9^2,\ 1,\ c_9^1,\ -1,\ c_9^4;\ \ 1,\ c_9^1,\ 1,\ -c_9^4,\ 1;\ \ 1,\ c_9^4,\ -1,\ c_9^2;\ \ 1,\ -c_9^2,\ 1;\ \ 1,\ -c_9^1;\ \ 1)$

10. ind $= (6;13)_1^1$: $d_i = (1.0,\ 1.941,\ 2.770,\ 3.438,\ 3.907,\ 4.148)$
$D^2 = 56.746 = 21 + 15c_{13}^1 + 10c_{13}^2 + 6c_{13}^3 + 3c_{13}^4 + c_{13}^5$
$T = (0,\ \frac{4}{13},\ \frac{2}{13},\ \frac{7}{13},\ \frac{6}{13},\ \frac{12}{13}),$
$S = (1,\ \xi_{13}^2,\ \xi_{13}^3,\ \xi_{13}^4,\ \xi_{13}^5,\ \xi_{13}^6;\ \ -\xi_{13}^4,\ \xi_{13}^6,\ -\xi_{13}^5,\ \xi_{13}^3,\ -1;\ \ \xi_{13}^4,\ 1,\ -\xi_{13}^2,\ -\xi_{13}^5;\ \ \xi_{13}^3,\ -\xi_{13}^6,\ \xi_{13}^2;\ \ -1,\ \xi_{13}^4;\ \ -\xi_{13}^3)$

12. ind $= (6;16)_1^1$: $d_i = (1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.414,\ 1.414)$
$D^2 = 8.0 = 8$
$T = (0,\ \frac{1}{2},\ \frac{1}{4},\ \frac{3}{4},\ \frac{1}{16},\ \frac{5}{16}),$
$S = (1,\ 1,\ 1,\ 1,\ \sqrt{2},\ \sqrt{2};\ \ 1,\ 1,\ 1,\ -\sqrt{2},\ -\sqrt{2};\ \ -1,\ -1,\ \sqrt{2},\ -\sqrt{2};\ \ -1,\ -\sqrt{2},\ \sqrt{2};\ \ 0,\ 0;\ \ 0)$

16. ind $= (6;16)_2^1$: $d_i = (1.0,\ 1.0,\ 1.0,\ 1.0,\ 1.414,\ 1.414)$
$D^2 = 8.0 = 8$
$T = (0,\ \frac{1}{2},\ \frac{1}{4},\ \frac{3}{4},\ \frac{1}{16},\ \frac{13}{16}),$
$S = (1,\ 1,\ 1,\ 1,\ \sqrt{2},\ \sqrt{2};\ \ 1,\ 1,\ 1,\ -\sqrt{2},\ -\sqrt{2};\ \ -1,\ -1,\ -\sqrt{2},\ \sqrt{2};\ \ -1,\ \sqrt{2},\ -\sqrt{2};\ \ 0,\ 0;\ \ 0)$

20. ind $= (6;35)_1^1$: $d_i = (1.0,\ 1.618,\ 1.801,\ 2.246,\ 2.915,\ 3.635)$
$D^2 = 33.632 = 15 + 3c_{35}^1 + 2c_{35}^4 + 6c_{35}^5 + 3c_{35}^6 + 3c_{35}^7 + 2c_{35}^{10} + 2c_{35}^{11}$
$T = (0,\ \frac{2}{5},\ \frac{1}{7},\ \frac{5}{7},\ \frac{19}{35},\ \frac{4}{35}),$
$S = (1,\ \frac{1+\sqrt{5}}{2},\ \xi_7^2,\ \xi_7^3,\ c_{35}^1+c_{35}^6,\ c_{35}^1+c_{35}^4+c_{35}^6+c_{35}^{11};\ \ -1,\ c_{35}^1+c_{35}^6,\ c_{35}^1+c_{35}^4+c_{35}^6+c_{35}^{11},\ -\xi_7^2,\ -\xi_7^3;\ \ -\xi_7^3,\ 1,\ -c_{35}^1-c_{35}^4-c_{35}^6-c_{35}^{11},\ \frac{1+\sqrt{5}}{2};\ \ -\xi_7^2,\ \frac{1+\sqrt{5}}{2},\ -c_{35}^1-c_{35}^6;\ \ \xi_7^3,\ -1;\ \ \xi_7^2)$

24. ind $= (6;56)_1^1$: $d_i = (1.0,\ 1.0,\ 1.801,\ 1.801,\ 2.246,\ 2.246)$
$D^2 = 18.591 = 12 + 6c_7^1 + 2c_7^2$
$T = (0,\ \frac{1}{4},\ \frac{1}{7},\ \frac{11}{28},\ \frac{5}{7},\ \frac{27}{28}),$
$S = (1,\ 1,\ \xi_7^2,\ \xi_7^2,\ \xi_7^3,\ \xi_7^3;\ \ -1,\ \xi_7^2,\ -\xi_7^2,\ \xi_7^3,\ -\xi_7^3;\ \ -\xi_7^3,\ -\xi_7^3,\ 1,\ 1;\ \ \xi_7^3,\ 1,\ -1;\ \ -\xi_7^2,\ -\xi_7^2;\ \ \xi_7^2)$

28. ind $= (6;80)_1^1$: $d_i = (1.0,\ 1.0,\ 1.414,\ 1.618,\ 1.618,\ 2.288)$
$D^2 = 14.472 = 10 + 2\sqrt{5}$
$T = (0,\ \frac{1}{2},\ \frac{1}{16},\ \frac{2}{5},\ \frac{9}{10},\ \frac{37}{80}),$
$S = (1,\ 1,\ \sqrt{2},\ \frac{1+\sqrt{5}}{2},\ \frac{1+\sqrt{5}}{2},\ c_{40}^3+c_{40}^5-c_{40}^7;\ \ 1,\ -\sqrt{2},\ \frac{1+\sqrt{5}}{2},\ \frac{1+\sqrt{5}}{2},\ -c_{40}^3-c_{40}^5+c_{40}^7;\ \ 0,\ c_{40}^3+c_{40}^5-c_{40}^7,\ -c_{40}^3-c_{40}^5+c_{40}^7,\ 0;\ \ -1,\ -1,\ -\sqrt{2};\ \ -1,\ \sqrt{2};\ \ 0)$

36. ind $= (6;80)^{1}_{\frac{1}{2}}$: $d_i = (1.0,\ 1.0,\ 1.414,\ 1.618,\ 1.618,\ 2.288)$

$D^2 = 14.472 = 10 + 2\sqrt{5}$

$T = (0, \frac{1}{2}, \frac{3}{16}, \frac{2}{5}, \frac{9}{10}, \frac{47}{80})$,

$S = (1,\ 1,\ \sqrt{2},\ \frac{1+\sqrt{5}}{2},\ \frac{1+\sqrt{5}}{2},\ c^3_{40}+c^5_{40}-c^7_{40};\ 1,\ -\sqrt{2},\ \frac{1+\sqrt{5}}{2},\ \frac{1+\sqrt{5}}{2},\ -c^3_{40}-c^5_{40}+c^7_{40};\ 0,\ c^3_{40}+c^5_{40}-c^7_{40},\ -c^3_{40}-c^5_{40}+c^7_{40},\ 0;\ \ -1,\ -1,\ -\sqrt{2};\ \ -1,\ \sqrt{2};\ \ 0)$

The above list include all rank-6 modular data with non-integral D^2 and coming from resolved $\mathrm{SL}_2(\mathbb{Z})$ representations (as well as some with D^2 integral, as we filter using conditions that imply $D^2 \in \mathbb{Z}$, but not conversely). The list misses two known modular data with non-integral $D^2 = 74.617$, whose topological spins are $s_i = (0, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{3}, \frac{2}{3})$ or $s_i = (0, \frac{8}{9}, \frac{8}{9}, \frac{8}{9}, \frac{1}{3}, \frac{2}{3})$. From those s_i's, we find that they must come from the unresolved GT orbit $(4,1,1;9,1,1)$. In the main text of this paper, we showed that the unresolved $\mathrm{SL}_2(\mathbb{Z})$ representations can only produce such modular data (and its conjugations by Galois action and signed diagonal matrices). The unresolved cases are handled in the main text of the paper, which leads to a complete classification of all rank-6 modular data.

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