Scaling limits of fluctuations of extended-source internal DLA

SCALING LIMITS OF FLUCTUATIONS OF EXTENDED-SOURCE INTERNAL DLA

By

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Abstract. In a previous work, we showed that the 2D, extended-source internal DLA (IDLA) of Levine and Peres is $\delta^{3/5}$-close to its scaling limit, if δ is the lattice size. In this paper, we investigate the scaling limits of the fluctuations themselves. Namely, we show that two naturally defined error functions, which measure the "lateness" of lattice points at one time and at all times, respectively, converge to geometry-dependent Gaussian random fields. We use these results to calculate point-correlation functions associated with the fluctuations of the flow. Along the way, we demonstrate similar $\delta^{3/5}$ bounds on the fluctuations of the related divisible sandpile model of Levine and Peres, and we generalize the results of our previous work to a larger class of extended sources.

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1 Introduction

Internal diffusion-limited aggregation (IDLA) is a lattice growth model, tracking the growth of a random set $A(t) \subset \mathbb{Z}^d$ defined as follows. At each time t, we start a particle at the origin, and we let it undergo a simple random until it first exits the set $A(t-1)$—supposing it exits at the point z_t, we set $A(t) = A(t-1) \cup \{z_t\}$. Intuitively, this process follows the diffusion of particles from an origin-centered source. In fact, it was originally proposed by the chemical physicists Meakin and Deutch [MD86] in order to model such diffusive processes, such as the smoothing of a spherical surface by electrochemical polishing.

We are interested in a generalization of this model to the extended-source case, wherein particles start instead from discretizations of a fixed mass distribution, and the lattice size is allowed to grow arbitrarily small. This generalization was first introduced and studied by Levine and Peres [LP08], although it corresponds to Diaconis and Fulton's earlier notion of a "smash sum" of two sets [DF91].

In both cases, a primary question of study is the overall smoothness of the occupied set $A(t)$. Following the work of Lawler, Bramson, and Griffeath [LBG92], it is well-known that—in the point-source case—these sets closely approximate an origin-centered ball for large t. Several authors have shown strong convergence rates for this process [Law95, AG10]; most recently, Jerison, Levine, and Sheffield proved that the fluctuations away from the disk are at most of order $\log t$ in dimension 2, narrowly improving a $\log^2 t$ result by Asselah and Gaudiliére [JLS12, AG13a]. Independent works by Asselah and Gaudilliére and by Jerison, Levine, and Sheffield proved bounds of order $\sqrt{\log t}$ in higher dimensions [AG13b, JLS13], which have been shown to be tight [AG11]. In the extended-source case, Levine and Peres first showed that the scaling limits of IDLA correspond to solutions of a closely related free boundary problem [LP08]. We recently proved that, if the lattice size is δ, the fluctuations of IDLA away from this expected set are at most of order $\delta^{3/5}$.

The fluctuations can also be studied "on the aggregate", however, which provides interesting insight into the geometry of the problem. Namely, we are interested in studying mean fluctuations over an area of finite volume, as weighted by a test function $u \in C^\infty(\mathbb{R}^d)$. To do this in the point-source case, Jerison et al. [JLS14] introduced natural error functions on the lattice $\delta\mathbb{Z}^d$, which quantify how late or early the IDLA process is in getting to a given point. Specifically, they introduced a fluctuation function E^s and a lateness function L, that capture fluctuations at a single time s and at all times, respectively. They proved that these error functions weakly approach certain Gaussian random fields as the lattice spacing δ

decreases, allowing them to find the scaling limits of fluctuations integrated against a test function u. Eli Sadovnik studied this question more recently for an extended source, in the special case of the single-time fluctuation function and with discrete harmonic test functions [Sad16].

In this paper, we extend the techniques used in [JLS14] and [Sad16] in order to prove more general scaling limits of random error functions in the extended-source case. Our main results, Theorems 3.1 and 3.2, show that the fluctuation function E^s and the lateness function L converge weakly to geometry-dependent Gaussian random fields, allowing for any C^4 test functions. In particular, by choosing highly localized test functions, we will be able to calculate "point-correlation functions", which encode the correlations between fluctuations of IDLA at two different points in space. Furthermore, in Appendix B, we generalize these results and the results of our previous paper to a larger class of extended sources, removing the most restrictive hypothesis of our setting.

It must be noted that, in the point-source case of [JLS14], the functions E^s and L measure fluctuations away from a previously-calculated continuous limit of IDLA; specifically, they measure the difference between $A(t)$ and a smooth sphere. To our knowledge, this is not possible in the general-source case without stronger estimates on the convergence of discrete harmonic functions—as such, our general-source versions of E^s and L compare IDLA to a closely related deterministic process: the divisible sandpile model of Levine and Peres [LP10]. We show in Theorem 2.8 that the divisible sandpile converges at least as quickly as the best known estimates (from [Dar20]) on IDLA. In fact, we believe that it converges faster than IDLA, but the estimate from Theorem 2.8 is sufficient for our purposes.

After briefly reviewing the necessary theory, we introduce our primary results in Section 3. The following sections are spent proving these results; Section 4 proves the scaling limit of the fixed-time fluctuation function, and Section 5 proves that of the lateness function. Finally, we use these results to calculate point correlation functions of IDLA fluctuations in Section 6.

2 Review of lattice growth processes

Here we provide a background on extended-source IDLA and on a related deterministic process, the divisible sandpile growth introduced also by Levine and Peres [LP08]. Many of our specific definitions are taken from our preceding paper, [Dar20]; see that paper for more information.

Following from [Dar20], we restrict attention to IDLA processes started from concentrated mass distributions.

Definition 2.1. Let $D_0 \subset \mathbb{R}^2$ be a compact, connected domain with smooth boundary, and fix $N \in \mathbb{Z}^{\geq 0}$ and $T_1, \ldots, T_N \in \mathbb{R}^{\geq 0}$. For each $i = 1, \ldots, N$ and $s \in [0, T_i]$, suppose $Q_i^s \subset D_0$ satisfies the following properties:

(1) Q_i^s is a compact domain with $\mathrm{Vol}(Q_i^s) = s$.
(2) Q_i^s is bounded away from ∂D_0—that is, $Q_i^s \subset\subset \mathrm{int}(D_0)$. This requirement is lifted in Appendix B.
(3) $Q_i^s \subset Q_i^{s'}$ for $s \leq s' \leq T_i$.
(4) ∂Q_i^s is rectifiable, with arclength bounded independently of s.

Finally, set $T = \sum_k T_k$, and fix increasing functions $s_i : [0, T] \to [0, T_i]$ satisfying $\sum_k s_k(s) = s$ for all $s \in [0, T]$. In this setting, the **concentrated mass distribution** associated to the data $(D_0, \{Q_i^s\}, \{s_i\})$ is the map $\sigma_s : \mathbb{R}^2 \to \mathbb{Z}^{\geq 0}$ defined by

$$\sigma_s = \mathbf{1}_{D_0} + \sum_{i=1}^{N} \mathbf{1}_{Q_i^{s_i(s)}}.$$

In short, a concentrated mass distribution is a collection of increasing compact subsets $Q_i^{s_i}$ of D_0, such that the total mass at any time s is $\mathrm{vol}(D_0) + s$. The functions s_i give the mass of each subset $Q_i^{s_i}$ at the time s.

The second requirement above—that $Q_i^s \subset\subset \mathrm{int}(D_0)$—merits explanation. This is used in [Dar20] to guarantee that the source points never fall too close to the pole of a certain Poisson kernel, and thus to bound the values of this Poisson kernel on each source point. However, we provide a proof in Appendix B that versions of our main theorems (Theorems 3.1 and 3.2) and versions of our fluctuation bounds (Lemma 2.4 and Theorem 2.8) hold even with this requirement lifted.

The analysis of this paper holds in its entirety for infinite mass distributions, where $T = \infty$. For these, we simply require that the finite-time collections $\{Q_i^s\}|_{s \leq T'}$ and $\{\sigma_s\}|_{s \leq T'}$ give rise to concentrated mass distributions for any $T' > 0$. We will assume that $T < \infty$, but we can also imagine that we have simply "cut off" an infinite mass distribution in the manner just described.

Since we are studying processes on discrete lattices, we are primarily interested in the restrictions of these mass functions to the grid $\frac{1}{m}\mathbb{Z}^2$. Write S_s^m for the multiset defined by $\sum_{\frac{1}{m}\mathbb{Z}^2} (\sigma_s - \mathbf{1}_{D_0})$; that is, for any $z \in \frac{1}{m}\mathbb{Z}^2 \cap \bigcup Q_i^{s_i}$, we have that $z \in S_s^m$ with multiplicity $\sigma_s(z) - 1$. We can order S_T^m into a sequence $z_{m,j}$ of source points as follows:

(1) If $z \in S_T^m$ with multiplicity k, let $\tau_i(z) := \inf\{t \mid z \in S_t^m \text{ with multiplicity } i+1\}$ for $i \leq k - 1$.
(2) Given $\{z_{m,1}, \ldots, z_{m,j-1}\}$, choose $z_{m,j} \in S_T^m \setminus \{z_{m,1}, \ldots, z_{m,j-1}\}$ to minimize $\tau_{i(z,j)}(z)$, where $i(z, j)$ is the multiplicity of z in $\{z_{m,1}, \ldots, z_{m,j-1}\}$.

In short, we are simply ordering the particles in S_T^m in the order they appear (accounting for multiplicity) in the sets $\{S_s^m\}$. Given this sequence, we define the discrete densities $\sigma_{m,n} = \mathbf{1}_{\frac{1}{m}\mathbb{Z}^2\cap D_0} + \sum_{i=1}^n \mathbf{1}_{\{z_{m,i}\}}$. It is clear that $\sigma_{m,n}$ differs from its continuum limit σ_{n/m^2} at only $O(m^{-2})$ points, accounting for multiplicity.

The (resolution m) internal DLA (IDLA) associated with the mass distribution is the following process:

Definition 2.2 (Internal DLA)**.** Suppose we have a concentrated mass distribution with initial set D_0 giving rise to the sequences $\{z_{m,t}\}$. The **IDLA** $A_m(t)$ with the mass distribution is as follows. Define the initial set $A_m(0) = \frac{1}{m}\mathbb{Z}^2 \cap D_0$. Then, for each integer $t \geq 1$, start a simple random walk at $z_{m,t}$, and let z_t' be the first point in the walk outside the set $A_m(t-1)$—then $A_m(t) := A_m(t-1) \cup \{z_t'\}$.

Importantly, the law of $A_m(t)$ does not depend on the order of $\{z_{m,1}, \ldots, z_{m,t}\}$, as proven by Diaconis and Fulton [DF91, Lemma 2.2] (see [DF91, Section 3] for the application of this result to our setting).

For each time s, the sets $A_m(m^2 s)$ approach a deterministic limit D_s almost surely, where D_s is the Diaconis–Fulton "smash" sum

$$D_s = D_0 \oplus Q_1^{s_1} \oplus \cdots \oplus Q_N^{s_N}.$$

The smash sum operation is as defined in [LP10]:

Definition 2.3. If $A, B \subset \frac{1}{m}\mathbb{Z}^2$, we define the discrete smash sum $A \oplus B$ as follows. Let $C_0 = A \cup B$, and for each $x_i \in \{x_1, \ldots, x_n\} = A \cap B$, start a simple random walk at x_i and stop it upon exiting C_{i-1}. Let y_i be its final position, and define $C_i = C_{i-1} \cup \{y_i\}$. Then $A \oplus B := C_n$ is a random set.

As proven in [LP10, Theorem 1.3], if we instead take domains $A, B \subset \mathbb{R}^2$, the smash sums $A_m \oplus B_m$ of

$$A_m := \frac{1}{m}\mathbb{Z}^2 \cap A, \quad B_m := \frac{1}{m}\mathbb{Z}^2 \cap B$$

approach a deterministic limit, which we label $A \oplus B$. Figure 1 (taken from [Dar20]) gives an example of this.

The convergence $A_m(m^2 s) \to D_s$ was shown originally by Levine and Peres [LP10]. In [Dar20, Theorem 3.1], we have recently shown the following convergence rate for this scaling limit, in the special case that D_s is a **smooth** flow—that is, that $s \mapsto D_s$ is a smooth isotopy for $s \in [0, T]$.

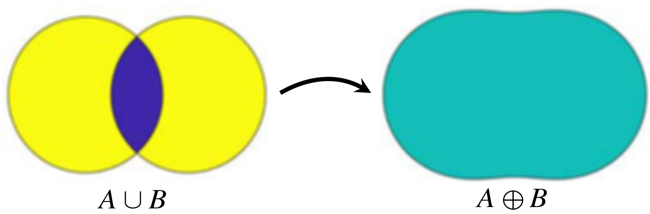

Figure 1: The smash sum $A \oplus B$ is the deterministic limit of an IDLA-type growth process starting from the sets A and B, representing the dispersal of particles in $A \cap B$ (in dark blue above) to the edges of $A \cup B$ (in yellow above).

Lemma 2.4. *Suppose D_s is a smooth flow arising from a concentrated mass distribution. For large enough m, the fluctuation of the associated IDLA $A_m(t)$ is bounded as*

$$\mathbb{P}\left\{(D_s)_{Cm^{-3/5}} \cap \frac{1}{m}\mathbb{Z}^2 \subset A_m(m^2 s) \subset (D_s)^{Cm^{-3/5}} \text{ for all } s \in [0,T]\right\}^c \leq e^{-m^{2/5}}$$

for a constant C depending on the flow, where $(D_s)^\varepsilon$ and $(D_s)_\varepsilon$ denote outer- and inner-ε-neighborhoods of D_s, respectively.

In other words, the fluctuations of the random set $A_m(m^2 s)$ are unlikely to be larger in magnitude than $Cm^{-3/5}$, for some fixed $C > 0$. We will also use this to bound the maximum fluctuations of a closely related, deterministic process—the "divisible sandpile growth" of Levine and Peres [LP10]—defined as follows:

Definition 2.5 (Divisible Sandpile)**.** Suppose we have a concentrated mass distribution with initial set D_0 giving rise to the sequences $\{z_{m,i}\}$. The **divisible sandpile aggregation** associated with our mass distribution is characterized by its **final mass distributions** $\nu_{m,t}$. Let $\nu_{m,0} := \mathbf{1}_{A_m(0)} = \mathbf{1}_{D_0 \cap \frac{1}{m}\mathbb{Z}^2}$, and define $\nu_{m,n}$ inductively as follows.

Given $\nu_{m,n}$, define the intermediate function $\nu^0_{m,n} = \nu_{m,n} + \mathbf{1}_{\{z_{m,n+1}\}}$. At each time step t, choose a point $z = z(t) \in \operatorname{supp} \nu^t_{m,n}$ such that $\nu^t_{m,n}(z) > 1$. Set

$$\nu^{t+1}_{m,n}(z) = 1, \quad \nu^{t+1}_{m,n}(z \pm 1/m) = \nu^t_{m,n}(z \pm 1/m) + \frac{1}{4}(\nu^t_{m,n}(z) - 1),$$
$$\nu^{t+1}_{m,n}(z \pm i/m) = \nu^t_{m,n}(z \pm i/m) + \frac{1}{4}(\nu^t_{m,n}(z) - 1).$$

In other words, we define $\nu^{t+1}_{m,n}$ by taking the "excess mass" at z in $\nu^t_{m,n}$ and splitting it evenly between the neighbors of z. For a large enough (but finite) t', we will have $\nu^{t'}_{m,n} \leq 1$ everywhere, and the above process must stop; then define $\nu_{m,n+1} = \nu^{t'}_{m,n}$.

A primary property of the divisible sandpile is a mean-value property for discrete harmonic functions; we review the definition of the latter below:

Definition 2.6. Suppose $K \subset \frac{1}{m}\mathbb{Z}^2$, and define the interior int(K) to be the set of $z \in K$ with all neighboring lattice points $z' \sim z$ also contained in K.

We say that $h : K \to \mathbb{R}$ is discrete harmonic on K if, for all $z \in \text{int}(K)$, we have

$$\sum_{z' \sim z} (h(z) - h(z')) = 0.$$

The following proposition follows directly from the definition of the divisible sandpile:

Proposition 2.7. *Suppose* $h : \frac{1}{m}\mathbb{Z}^2 \to \mathbb{R}$ *is discrete harmonic on* $\operatorname{supp} \nu_{m,n}$. *Then,*

$$\sum_{z \in \frac{1}{m}\mathbb{Z}^2} h(z)\nu_{m,n}(z) = \sum_{z \in \frac{1}{m}\mathbb{Z}^2 \cap D_0} h(z)\sigma_{m,n}(z).$$

Finally, we find the same $m^{-3/5}$-bound on maximum fluctuations for the divisible sandpile as we do for IDLA; the following theorem is proved in the Appendix:

Theorem 2.8. *Suppose D_τ is a smooth flow arising from a concentrated mass distribution. For large enough m and any time $s \in [0, T]$, the fluctuations of the occupied set* $\operatorname{supp} \nu_{m,m^2 s}$ *are bounded as*

$$(D_s)_{Cm^{-3/5}} \cap \frac{1}{m}\mathbb{Z}^2 \subset \operatorname{supp} \nu_{m,m^2 s} \subset (D_s)^{Cm^{-3/5}}$$

for a constant C depending on the flow.

For both internal DLA and the divisible sandpile, we expect that this $m^{-3/5}$-bound is not optimal for the divisible sandpile. Indeed, in the single-source case, it was shown in [LP08] that the fluctuations of the divisible sandpile are $O(m^{-1})$, and it was shown in [JLS12] that the fluctuations of internal DLA in two dimensions are $O(\log(m)/m)$, and we expect the same to be true in the extended-source case. However, as we apply the techniques used in [Dar20] to the divisible sandpile, we are restricted in that our bound on the L^1 convergence rate of discrete Green's functions to their continuum limit—Lemma 5.2(c) of [Dar20]—is of order $m^{-8/5}$ rather than $m^{-2} \log m$, as was shown in [JLS12] in the specific case of the disk.

The specific form of the limiting shapes D_s only enters into our analysis indirectly. In particular, the critical fact is that both internal DLA and the divisible sandpile converge to the same sets; this allows us to achieve an approximate version of the mean value property—Lemma 2.7—for the IDLA occupied set. By choosing appropriate discrete harmonic functions, we then show that this approximate equality is violated when IDLA fluctuations are too great.

To simplify notation, we will continue to write $\varepsilon_m = Cm^{-3/5}$, as in Lemmas 2.4 and 2.8. We further define

$$F_m^s = (D_s)^{\varepsilon_m} \setminus (D_s)_{\varepsilon_m}.$$

3 Main results

Our two primary results pertain to scaling limits of the fluctuations of $A_m(t)$ away from its deterministic limit. Following [JLS14], we quantify these fluctuations using the following random functions.

First, the (**time** s) **error function** $E_m^s : \frac{1}{m}\mathbb{Z}^2 \to \mathbb{R}$ is defined as

$$E_m^s(x) := m(1_{A_m(m^2 s)}(x) - \nu_{m,m^2 s}(x)).$$

This takes a positive value on "early" points, where the IDLA A_m has reached by time $m^2 s$ but where the expected set—represented here by the divisible sandpile occupied set—has not yet reached. It takes a negative value on "late" points, where the divisible sandpile occupied set has reached but the IDLA has not.

Although E_m^s itself does not converge (in m) to a well-defined random variable, our primary objects of interest are the limits of inner products

$$(E_m^s, u) = m^{-2} \sum_{x \in \frac{1}{m}\mathbb{Z}^2} E_m^s(x) u(x) = m^{-1} \sum_{x \in \frac{1}{m}\mathbb{Z}^d} u(x)(1_{A_m(m^2 s)}(x) - \nu_{m,m^2 s}(x)).$$

We can think of (E_m^s, u) as a snapshot of the discrepancy at the fixed time s, weighted by the function $u \in C^4(\mathbb{R}^2)$.

Through the following theorem, we show that E_m^s converges weakly to a Gaussian random field on the fixed-time curve ∂D_s:

Theorem 3.1. *Suppose $u \in C^4(\mathbb{R}^d)$. The random variables (E_m^s, u) converge in law as $m \to \infty$ to a normal variable of mean 0 and variance*

$$\int_{D_s} |\psi|^2 (1 - \sigma_s),$$

where ψ solves the Dirichlet problem on D_s with boundary values $\psi|_{\partial D_s} \equiv u|_{\partial D_s}$.

Of course, convergence holds for finite-dimensional distributions of E_m^s (for fixed s); as such, we can turn this into a covariance formula using a polarization identity; if $u, v \in C^4(\mathbb{R}^2)$, the variables (E^s_m, u) and (E^s_m, v) form a joint Gaussian random variable with covariance

$$\int_{D_s} \psi\varphi(1-\sigma_s), \tag{1}$$

where ψ and φ are harmonic on D_s with $\psi|\partial D_s \equiv u$ and $\varphi|\partial D_s \equiv v$, respectively.

The requirement that $u \in C^4$ arises when we attempt to compare ψ to a discrete-harmonic approximation ψ_m (see Definition 2.6 for details on discrete harmonic functions). As shown in [Che, Theorem 3.5], the error in this approximation can be bounded by $\|\nabla^4 \psi\|$, which in turn can be bounded by $\|\nabla^4 u\|$ from the maximum principle. This is a wider class of test functions than were allowed in [JLS14], though the same generalization to C^4 can be made in the point-source case.

A more sensitive metric is given by the **lateness function**,

$$L^s_m = \sum_{n=1}^{\lfloor m^2 s \rfloor} \frac{n}{m} 1_{A_{m,n} \setminus A_{m,n-1}} - \sum_{n=1}^{\lfloor m^2 s \rfloor} \frac{n}{m} (\nu_{m,n} - \nu_{m,n-1}).$$

Up to a scaling factor, the first term in this expression is the actual time of arrival at each point. The latter term is an approximation of the expected time of arrival, which we can see as follows.

Suppose $x \in \frac{1}{m}\mathbb{Z}^d$, and $\langle T_{m,x} \rangle$ is the expected time for $A_{m,n}$ to arrive at x. For a brief window around T_x, the quantity $\nu_{m,n}(x)$ lies strictly between 0 and 1—say, when $n \in \{n', n'+1, \ldots, n'+\Delta-1\}$. As $\nu_{m,n}(x)$ is constant before n' and after $n'+\Delta$, only the terms involving $\{n', n'+1, \ldots, n'+\Delta-1\}$ contribute to the sum

$$\sum_{n=1}^{m^2 s} n(\nu_{m,n} - \nu_{m,n-1})(x). \tag{2}$$

Of course, the increments $(\nu_{m,n} - \nu_{m,n-1})(x)$ are non-negative, and

$$\sum_{n=n'}^{n'+\Delta} (\nu_{m,n} - \nu_{m,n-1})(x) = \nu_{m,n'+\Delta}(x) - \nu_{m,n'-1}(x) = 1,$$

so the sum (2) is a weighted average of $\{n', n'+1, \ldots, n'+\Delta\}$. As this interval is tightly centered around $\langle T_{m,x} \rangle$, we expect the overall sum to converge (in m) to $\langle T_{m,x} \rangle$.

Our second result is in the same spirit as Theorem 3.1, showing now that the lateness function converges weakly to a 2D Gaussian random field:

Theorem 3.2. *Suppose $u \in C^4_0(\mathbb{R}^d)$, with* supp $u \subset D_s$. *The random variables (L^s_m, u) converge in law as $m \to \infty$ to a normal variable of mean 0 and variance*

$$2\int_0^s ds' \int_0^{s'} ds'' \int_{D_{s''}} \psi_{s'}\psi_{s''}(1-\sigma_{s''}),$$

where ψ_t solves the Dirichlet problem on D_t with boundary values $\psi_t|_{\partial D_t} \equiv u|_{\partial D_t}$.

As before, this implies convergence along finite-dimensional distributions of L^s_m. In particular, if $u, v \in C_0^4(\mathbb{R}^d)$ with $\operatorname{supp} u, \operatorname{supp} v \subset D_s$, the variables (L^s_m, u) and (L^s_m, v) form a joint Gaussian random variable with covariance

$$\int_0^s ds' \int_0^{s'} ds'' \int_{D_{s''}} (\psi_{s'}\varphi_{s''} + \psi_{s''}\varphi_{s'})(1 - \sigma_{s''}). \tag{3}$$

It bears mentioning that the parameter s can be disposed of, by always taking it large enough that $D_s \supset \operatorname{supp} u$. In particular, the function ψ_s vanishes uniformly for larger s, so increasing the parameter past this point does not change the value of the covariance (3). This gives rise to the perhaps-more-natural field L defined by

$$(L, u) = \lim_{s\to\infty} \left(\lim_m (L^s_m, u) \right),$$

which always satisfies the covariance formula (3) without the boundary terms introduced in Lemma 5.1.

After proving these results in the following sections, we will turn to an interesting application of Theorem 3.2. Namely, we will use Equations (1) and (3) to compute point-correlation functions, which encode the correlations between fluctuations at two different points. In some sense, point-correlation functions will be local versions of the above results.

4 Proof of Theorem 3.1

In our analysis below, we will make use of the grids

$$\mathcal{G}_m := \left\{ (x, y) \in \frac{1}{m}\mathbb{R}^2 \;\middle|\; x \in \frac{1}{m}\mathbb{Z} \text{ or } y \in \frac{1}{m}\mathbb{Z} \right\}.$$

In particular, we will use the following notion of a grid harmonic function on $\mathcal{G}_m$:

Definition 4.1. A continuous function $\phi : U \subset \mathcal{G}_m \to \mathbb{R}$ is **grid harmonic** if

$$\Delta_h\phi(z) := \frac{m^2}{4}(\phi(z + 1/m) + \phi(z - 1/m) + \phi(z + i/m) + \phi(z - i/m)) - m^2\phi(z) = 0$$

on the nodes $z \in U \cap \frac{1}{m}\mathbb{Z}^2$, and ϕ is linear on each edge of $\mathcal{G}_m$.

Finally, we will let $\mathscr{F}_{m,t}$ be the filtration generated by $A_m(t)$.

Proof of Theorem 3.1, Step 1. We will first relate (E^s_m, u) to a family of martingales and show that the difference converges in law to zero.

Let $\varepsilon_m = Cm^{-3/5}$, as in Lemmas 2.4 and 2.8, and let ψ_m be harmonic on $(D_s)^{2\varepsilon_m}$ with boundary values $u|_{\partial(D_s)^{2\varepsilon_m}}$. Let $\psi_{(m)}$ solve the corresponding grid Dirichlet problem on $\mathcal{G}_m \cap (D_s)^{2\varepsilon_m}$. That is, $\psi_{(m)}$ is grid-harmonic in $\mathcal{G}_m \cap (D_s)^{2\varepsilon_m}$, and

$$\psi_{(m)}|_{\partial(D_s)^{2\varepsilon_m}} \equiv \psi_m|_{\partial(D_s)^{2\varepsilon_m}} \equiv u|_{\partial(D_s)^{2\varepsilon_m}}.$$

Since $\psi_{(m)} = \psi_m$ on the boundary, standard estimates (for instance, see [Che, Theorem 3.5]) give

$$\|\psi_{(m)} - \psi_m\|_\infty \le C_1/m^2, \tag{4}$$

where $C_1 \sim \|\nabla^4 u\|$. Next, define the martingales

$$\begin{aligned} M_m(t) &:= m^{-1}\Bigg(\sum_{x\in A_m(t\wedge\tau^*)\setminus D_0} \psi_{(m)}(x) - \sum_{i=1}^{t\wedge\tau^*}\psi_{(m)}(z_{m,i})\Bigg) \\ &= m^{-1}\sum_{A_m(t\wedge\tau^*)} \psi_{(m)}\cdot(1-\sigma_{m,t\wedge\tau^*}), \end{aligned}$$

where τ^* is the first time that $A_m(\tau^*) \not\subset (D_s)^{\varepsilon_m}$. Note that $A_m(\tau^*) \subset (D_s)^{2\varepsilon_m}$, so the function $\psi_{(m)}$ is defined on all of $A_m(\tau^*)$.

Consider the event $\mathcal{E}$ that $\operatorname{supp} E_m^s \subset F_m^s$; by Lemmas 2.4 and 2.8, this event occurs with probability $1 - e^{-m^{2/5}} \nearrow 1$. In this case, $\tau^* \ge m^2 s$, so—since $\psi_{(m)}$ is discrete harmonic—we have $M_m(m^2 s) = (E_m^s, \psi_{(m)})$. To relate this to (E_m^s, u), we first want to bound $\sup_{F_m^s} |u - \psi_m|$. Suppose $x \in F_m^s$ achieves this supremum, and choose $x' \in \partial(D_s)^{2\varepsilon_m}$ such that $|x - x'| \le 4\varepsilon_m$. Now, $\partial_i\psi_m$ solves the Laplace equation with boundary values $\partial_i u|_{\partial(D_s)^{2\varepsilon_m}}$, so by the maximum principle,

$$|\partial_i\psi_m| \le \sup\nolimits_{\partial D^{2\varepsilon_m}} |\partial_i u| \le \sup\nolimits_{(\bigcup_m (D_s)^{2\varepsilon_m})} |\nabla u|.$$

In particular, $|\nabla\psi_m| \le C_1 = C_1(u)$ for all m, choosing a larger C_1 if necessary. Without loss of generality, we can take $C_1 \ge \sup_{(\bigcup_m D^{\varepsilon_m})} |\nabla u|$. This implies that

$$\begin{aligned} \sup\nolimits_{F_m^s} |u - \psi_m| &= |u(x) - \psi_m(x)| \\ &\le |u(x) - u(x')| + |u(x') - \psi_m(x')| + |\psi_m(x') - \psi_m(x)| \\ &= |u(x) - u(x')| + |\psi_m(x') - \psi_m(x)| \\ &\le 8\varepsilon_m C_1. \end{aligned}$$

By (4), this means $\sup_{F_m^s} |u - \psi_{(m)}| = O(\varepsilon_m)$. Thus, we find

$$\begin{aligned} |(E_m^s, u) - (E_m^s, \psi_{(m)})| &\le m\cdot \operatorname{Vol}(F_m^s)\sup\nolimits_{F_m^s} |u - \psi_{(m)}| \\ &= O(m\varepsilon_m^2) = O(m^{-1/5}), \end{aligned} \tag{5}$$

which converges to zero.

Now, for any $\delta > 0$, we can choose $m_0 > 0$ such that

$$|(E_m^s, u) - M_m(m^2 s)| = |(E_m^s, u) - (E_m^s, \psi_{(m)})| \le \delta$$

on event $\mathcal{E}$ for any $m > m_0$. The probability of $\mathcal{E}^c$ tends to zero, so we know that $(E_m^s, u) - M_m(m^2 s)$ converges in probability (and thus in law) to zero. □

Step 2. Now, we will show that the family of random variables $M_m(m^2 s)$ converges in law to a zero-mean normal variable with variance $\int_{D_s} |\psi|^2(1-\sigma_s)$.

For this step, we will follow the style of proof in [Sad16]. Define

$$X_{m,t} = M_m(t) - M_m(t-1) = \begin{cases} m^{-1}(\psi_{(m)}(A_m(t) \setminus A_m(t-1)) - \psi_{(m)}(z_{m,t})), & t \le \tau^*, \\ 0, & t > \tau^*. \end{cases}$$

This is a mean-zero martingale difference array adapted to $\mathscr{F}_{m,t}$. The martingale central limit theorem stated in [HH80, Theorem 3.2] thus states that

$$M_m(m^2 s) = \sum_{t \le m^2 s} X_{m,t}$$

converges in law to a normal variable of mean 0 and variance $\int_{D_s} |\psi|^2(1-\sigma_s)$, so long as the following three conditions hold:

(1) $\mathbb{E}[\max_t |X_{m,t}|^2]$ is bounded in m. This also implies that the array is square-integrable, which is one of the hypotheses of the theorem.

(2) $\max_t |X_{m,t}| \to 0$ in probability as $m \to \infty$.

(3) $\sum_{t \le m^2 s} |X_{m,t}|^2 \to \int_{D_s} |\psi|^2(1-\sigma_s)$ in probability as $m \to \infty$.

As in [Sad16], we will handle the first two conditions by showing that

$$\mathbb{E}[\max_t |X_{m,t}|^a] \to 0 \quad \text{for } a \ge 1.$$

This is clear from the following estimate:

$$\begin{aligned} |X_{m,t}|^a &= m^{-a} |\psi_{(m)}(A_m(t) \setminus A_m(t-1)) - \psi_{(m)}(z_{m,t})|^a \\ &\le 2^a m^{-a} \sup |\psi_{(m)}|^a \\ &\le 2^a m^{-a} \sup\nolimits_{(\bigcup_m D^{\varepsilon_m})} |u|^a, \end{aligned}$$

from the maximum principle.

For the final condition, define the random variables

$$S_m(t) = \sum_{i=1}^{t} |X_{m,i}|^2, \quad Z_m(t) = m^{-2} \sum_{x \in A_m(t \wedge \tau^*) \setminus D_0} |\psi_{(m)}(x)|^2 - m^{-2} \sum_{i=1}^{t \wedge \tau^*} |\psi_{(m)}(z_{m,i})|^2,$$

$$N_m(t) = S_m(t) - Z_m(t).$$

Our goal is to show that $N_m(m^2 s) \to 0$ in probability, and thus that $S_m(t)$ can be well-approximated by the simpler variable $Z_m(t)$.

For this, first note that N_m satisfies the martingale property; we only need show this for time intervals before τ^*, as N_m remains constant thereafter. For $t \le \tau^*$,

$$\begin{aligned}
&\mathbb{E}[N_m(t) - N_m(t-1)|\mathscr{F}_{m,t-1}] \\
&\quad = \mathbb{E}[|X_{m,i}|^2 - m^{-2}(\psi_{(m)}(A_m(t) \setminus A_m(t-1))^2 - \psi_{(m)}(z_{m,t})^2)|\mathscr{F}_{m,t-1}] \\
&\quad = \mathbb{E}[m^{-2}(\psi_{(m)}(A_m(t) \setminus A_m(t-1)) - \psi_{(m)}(z_{m,t}))^2 \\
&\qquad - m^{-2}(\psi_{(m)}(A_m(t) \setminus A_m(t-1))^2 - \psi_{(m)}(z_{m,t})^2)|\mathscr{F}_{m,t-1}] \\
&\quad = \mathbb{E}[2m^{-2}\psi_{(m)}(z_{m,t})^2 - 2m^{-2}\psi_{(m)}(A_m(t) \setminus A_m(t-1))\psi_{(m)}(z_{m,t})|\mathscr{F}_{m,t-1}] \\
&\quad = 2m^{-2}\psi_{(m)}(z_{m,t})\mathbb{E}[\psi_{(m)}(z_{m,t}) - \psi_{(m)}(A_m(t) \setminus A_m(t-1))|\mathscr{F}_{m,t-1}] \\
&\quad = 0.
\end{aligned}$$

Since $S_m(0) = Z_m(0) = 0$, we have $N_m(0) = 0$ and thus

$$\mathbb{E}[N_m(m^2 s)^2] = \mathbb{E}[(N_m(m^2 s) - N_m(1))^2] = \sum_{t=1}^{\lfloor m^2 s \rfloor} \mathbb{E}[(N_m(t) - N_m(t-1))^2]$$

from the martingale property. Again taking $t \le \tau^*$, we estimate

$$\begin{aligned}
&\mathbb{E}[(N_m(t) - N_m(t-1))^2] \\
&\qquad \le 2\mathbb{E}[(S_m(t) - S_m(t-1))^2] + 2\mathbb{E}[(Z_m(t) - Z_m(t-1))^2] \\
&\qquad \le 2\mathbb{E}[|X_{m,t}|^4] + 2m^{-4}\mathbb{E}[(|\psi_{(m)}(A_m(t) \setminus A_m(t-1))|^2 + |\psi_{(m)}(z_{m,t})|^2)^2] \\
&\qquad \le 8m^{-4} \sup |\psi_{(m)}|^4 \\
&\qquad \le C_1 m^{-4},
\end{aligned}$$

where $C_1 = C_1(u)$. This implies

$$\mathbb{E}[N_m(m^2 s)^2] = \sum_{t=1}^{\lfloor m^2 s \rfloor} \mathbb{E}[(N_m(t) - N_m(t-1))^2] \le m^2 s \cdot C_1 m^{-4} = O(m^{-2}).$$

Thus, $N_m(m^2 s) \to 0$ in the L^2 norm, and thus also in probability.

Finally, we show that $Z_m(m^2 s) \to \int_{D_s} |\psi|^2(1 - \sigma_s)$ in probability. From the above argument, this would imply that $S_m(m^2 s) \to \int_{D_s} |\psi|^2(1 - \sigma_s)$ in probability, which is exactly the third condition of the martingale central limit theorem.

For this purpose, note that, on event $\mathcal{E}$ (where $\tau^* \ge m^2 s$),

$$\begin{aligned}
Z_m(m^2 s) &= m^{-2} \sum_{A_m(m^2 s) \setminus D_0} |\psi_{(m)}|^2 - m^{-2} \sum_{i=1}^{\lfloor m^2 s \rfloor} |\psi_{(m)}(z_{m,i})|^2 \\
&= m^{-2} \sum_{x \in A_m(m^2 s)} |\psi_{(m)}|^2(1 - \sigma_{m,m^2 s}).
\end{aligned}$$

On Event $\mathcal{E}$, we know that $A_m(m^2 s)$ differs from $D_s \cap \frac{1}{m}\mathbb{Z}^2$ by at most $O(m^2\varepsilon_m)$ points; in this case,

$$\left| Z_m(m^2 s) - m^{-2} \sum_{D_s \cap \frac{1}{m}\mathbb{Z}^2} |\psi_{(m)}|^2 (1-\sigma_s) \right| = O(\varepsilon_m),$$

as $|\psi_{(m)}(x)|^2$ is uniformly bounded (as we saw above) in terms of u and $\sum |\sigma_s - \sigma_{m,m^2 s}| = O(1)$. In turn,

$$\begin{aligned}
&\left| m^{-2} \sum_{D_s \cap \frac{1}{m}\mathbb{Z}^2} |\psi_{(m)}|^2 (1-\sigma_s) - m^{-2} \sum_{D_s \cap \frac{1}{m}\mathbb{Z}^2} |\psi_m|^2 (1-\sigma_s) \right| \\
&\qquad = \left| m^{-2} \sum_{D_s \cap \frac{1}{m}\mathbb{Z}^2} (|\psi_{(m)}|^2 - |\psi_m|^2)(1-\sigma_s) \right| \\
&\qquad = \left| m^{-2} \sum_{D_s \cap \frac{1}{m}\mathbb{Z}^2} (\psi_{(m)} - \psi_m)(\psi_{(m)} + \psi_m)(1-\sigma_s) \right| \\
&\qquad = O(m^{-2}),
\end{aligned}$$

using (4) in the final step. Now, we compare

$$m^{-2} \sum_{D_s \cap \frac{1}{m}\mathbb{Z}^2} |\psi_m|^2 (1-\sigma_s) \quad \text{with} \quad m^{-2} \sum_{D_s \cap \frac{1}{m}\mathbb{Z}^2} |\psi|^2 (1-\sigma_s),$$

where ψ solves the Laplace equation on D_s with $\psi|\partial D_s \equiv u|\partial D_s$. For this, suppose that $x \in \partial D_s$ maximizes $|\psi - \psi_m|$, and take $x' \in \partial(D_s)^{2\varepsilon_m}$ such that $|x - x'| \le 4\varepsilon_m$. As in Step 1, choose C_1 such that $|\nabla u|, |\nabla \psi_m| \le C_1$. Then we find

$$\begin{aligned}
\sup\nolimits_{\partial D} |\psi - \psi_m| &= |\psi(x) - \psi_m(x)| \\
&= |u(x) - \psi_m(x)| \\
&\le |u(x) - u(x')| + |u(x') - \psi_m(x')| + |\psi_m(x') - \psi_m(x)| \\
&= |u(x) - u(x')| + |\psi_m(x') - \psi_m(x)| \\
&\le 8\varepsilon_m C_1.
\end{aligned}$$

Of course, $\psi - \psi_m$ is harmonic in D_s, so the maximum principle implies

$$\sup\nolimits_{D_s} |\psi - \psi_m| \le 8\varepsilon_m C_1.$$

Arguing as before, we find that

$$\left| m^{-2} \sum_{D_s \cap \frac{1}{m}\mathbb{Z}^2} |\psi_m|^2 (1-\sigma_s) - m^{-2} \sum_{D_s \cap \frac{1}{m}\mathbb{Z}^2} |\psi|^2 (1-\sigma_s) \right| = O(\varepsilon_m).$$

Putting these inequalities together shows that

$$\left| Z_m(m^2 s) - m^{-2} \sum_{D_s \cap \frac{1}{m}\mathbb{Z}^2} |\psi|^2 (1-\sigma_s) \right| = O(\varepsilon_n)$$

on Event $\mathcal{E}$. Since $P(\mathcal{E}) \nearrow 1$, this implies that the above difference converges in probability to zero. Finally, $m^{-2} \sum_{D_s \cap \frac{1}{m}\mathbb{Z}^2} |\psi|^2(1-\sigma_s)$ converges to $\int_{D_s} |\psi|^2(1-\sigma_s)$, so the theorem is proved. □

5 Proof of Theorem 3.2

We prove a slight generalization of this result, in the case that $\operatorname{supp} u$ is not necessarily contained in D_s:

Lemma 5.1. *Suppose $u \in C^4(\mathbb{R}^2)$. The random variables (L_m^s, u) converge in law to a normal variable of mean 0 and variance*

$$s^2 \int_{D_s} |\psi_s|^2 (1-\sigma_s) + 2 \int_0^s ds' \int_0^{s'} ds'' \int_{D_{s'}} \psi_{s'} \psi_{s''} (1-\sigma_{s''}) - 2s \int_0^s ds' \int_{D_{s'}} \psi_s \psi_{s'} (1-\sigma_{s'}), \tag{6}$$

where ψ_t solves the Dirichlet problem on D_t with boundary values $\psi|_{\partial D_t} \equiv u|_{\partial D_t}$.

Remark. In the case of interest, with $\operatorname{supp} u \subset D_s$, we have that $\psi_s \equiv 0$ and thus that the above variance becomes

$$2 \int_0^s ds' \int_0^{s'} ds'' \int_{D_{s'}} \psi_{s'} \psi_{s''} (1-\sigma_{s''}).$$

Step 1. We first want to replace (L_m^s, u) with a suitable martingale. Let ψ_m^t solve the Dirichlet problem for u on $(D_t)^{2\varepsilon_m}$, with $\varepsilon_m = Cm^{-3/5}$. Let $\psi_{(m)}^t$ solve the corresponding grid Dirichlet problem on $\mathcal{G}_m \cap (D_t)^{2\varepsilon_m}$. As in the proof of Theorem 3.1, this means that

$$\|\psi_{(m)}^t - \psi_m^t\|_\infty \le C_1/m^2, \tag{7}$$

where $C_1 \sim \|\nabla^4 u\|$. Now define the martingale

$$M_m(t) = sm^{-1} \sum_{j=1}^{t\wedge\tau^*} (\psi_{(m)}^s (A_m(j) \setminus A_m(j-1)) - \psi_{(m)}^s(z_{m,j})) - m^{-3} \sum_{\ell=1}^{\lfloor m^2 s \rfloor} \sum_{j=1}^{\ell\wedge\tau_\ell\wedge t} (\psi_{(m)}^{\ell/m^2} (A_m(j) \setminus A_m(j-1)) - \psi_{(m)}^{\ell/m^2}(z_{m,j})),$$

where τ_ℓ is the first time that $A_m(j)$ exits $(D_{\ell/m^2})^{\varepsilon_m}$, and $\tau^* := \tau_{m^2 s}$ is the first time that it exits $(D_s)^{\varepsilon_m}$.

Consider the event $\mathcal{E}$, in which

$$\frac{1}{m}\mathbb{Z}^2 \cap (D_{\ell/m^2})_{\varepsilon_m} \subset A_m(\ell) \subset (D_{\ell/m^2})^{\varepsilon_m}$$

for all $\ell \leq m^2 s$. By Lemma 2.4, this occurs with probability $1 - e^{-m^{2/5}} \nearrow 1$. On this event, $\tau_\ell \geq \ell$ for all ℓ, and

$$\begin{aligned} M_m(m^2 s) =& sm^{-1} \sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} \psi^s_{(m)} \cdot (\mathbf{1}_{A_m(m^2 s)} - \sigma_{m,m^2 s}) \\ &- m^{-3} \sum_{\ell=1}^{\lfloor m^2 s \rfloor} \sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} \psi^{\ell/m^2}_{(m)} \cdot (\mathbf{1}_{A_m(\ell)} - \sigma_{m,\ell}) \\ =& sm^{-1} \sum\nolimits_{\frac{1}{m}\mathbb{Z}^d} \psi^s_{(m)} \cdot (\mathbf{1}_{A_m(m^2 s)} - \nu_{m,m^2 s}) \\ &- m^{-3} \sum_{\ell=1}^{\lfloor m^2 s \rfloor} \sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} \psi^{\ell/m^2}_{(m)} \cdot (\mathbf{1}_{A_{m,\ell}} - \nu_{m,\ell}). \end{aligned}$$

Of course, on event $\mathcal{E}$, the function $\mathbf{1}_{A_m(\ell)} - \nu_{m,\ell}$ is supported on

$$F_m^{\ell/m^2} = (D_{\ell/m^2})^{\varepsilon_m} \setminus (D_{\ell/m^2})_{\varepsilon_m};$$

this set has volume $O(\varepsilon_m)$, and $\sup_{F_m^{\ell/m^2}} |u - \psi^{\ell/m^2}_{(m)}| = O(\varepsilon_m)$ as in the previous proof. Then we have

$$\begin{aligned} M_m(m^2 s) &= sm^{-1} \sum_{\frac{1}{m}\mathbb{Z}^2} u \cdot (\mathbf{1}_{A_m(m^2 s)} - \nu_{m,m^2 s}) \\ &\quad - m^{-3} \sum_{\ell=1}^{\lfloor m^2 s \rfloor} \sum_{\frac{1}{m}\mathbb{Z}^2} u \cdot (\mathbf{1}_{A_m(\ell)} - \nu_{m,\ell}) + O(m\varepsilon_m^2) \\ &= m^{-3} \sum_{\ell=1}^{\lfloor m^2 s \rfloor} \sum_{\frac{1}{m}\mathbb{Z}^2} \ell u \cdot (\mathbf{1}_{A_m(\ell)} - \mathbf{1}_{A_m(\ell-1)}) \\ &\quad - m^{-3} \sum_{\ell=1}^{\lfloor m^2 s \rfloor} \sum_{\frac{1}{m}\mathbb{Z}^2} \ell u \cdot (\nu_{m,\ell} - \nu_{m,\ell-1}) + O(m^{-1/5}) \\ &= (L_m^s, u) + O(m^{-1/5}). \end{aligned} \tag{8}$$

Thus, $M_m(m^2 s) - (L_m^s, u)$ converges to zero in probability.

Step 2. Note that the martingale intervals $X_{m,t} = M_m(t) - M_m(t-1)$ take the following form:

$$\begin{aligned} X_{m,t} =& sm^{-1}\mathbf{1}_{\{t\le\tau^*\}} \cdot (\psi^s_{(m)}(A_m(t)\setminus A_m(t-1)) - \psi^s_{(m)}(z_{m,t})) \\ &- m^{-3} \sum_{\substack{t\le\ell\le m^2 s\\ \text{s.t. } \tau_\ell\ge t}} (\psi^{\ell/m^2}_{(m)}(A_m(t)\setminus A_m(t-1)) - \psi^{\ell/m^2}_{(m)}(z_{m,t})). \end{aligned}$$

Now we need to show that $M_m(m^2 s)$ approaches the appropriate normal distribution. We will again make use of the martingale central limit theorem [HH80, Theorem 3.2]—namely, our result is proved if we can show the following three conditions:

(1) $\mathbb{E}[\max_t |X_{m,t}|^2]$ is bounded in m.
(2) $\max_t |X_{m,t}| \to 0$ in probability as $m\to\infty$.
(3) $\sum_t |X_{m,t}|^2$ converges to the expression in (6) in probability as $m\to\infty$.

The first and second conditions follow from the following calculation, that $\mathbb{E}[\max_t |X_{m,t}|^a] \to 0$ for $a \ge 1$.

$$\begin{aligned} |X_{m,t}|^a &\le 2^{a-1}s^a m^{-a} |\psi^s_{(m)}(A_m(t)\setminus A_m(t-1)) - \psi^s_{(m)}(z_{m,t})|^a \\ &\quad + 2^{a-1}s^a m^{-a} \sup_{\ell\ge t} |\psi^{\ell/m^2}_{(m)}(A_m(t)\setminus A_m(t-1)) - \psi^{\ell/m^2}_{(m)}(z_{m,t})|^a \\ &\le 2^{a+1}s^a m^{-a}\sup|\psi_{(m)}|^a \\ &= O(m^{-a}), \end{aligned}$$

which proves the first two conditions. For the final condition, we again define auxiliary variables

$$\begin{aligned} Z_m(t) =& s^2m^{-2}\sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} |\psi^s_{(m)}|^2(\mathbf{1}_{A_m(t\wedge\tau^*)} - \sigma_{m,t\wedge\tau^*}) \\ &+ m^{-6}\sum_{1\le j,k\le m^2 s}\sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} \psi^{k/m^2}_{(m)}\psi^{j/m^2}_{(m)}(\mathbf{1}_{A_m(j\wedge\tau_j\wedge k\wedge\tau_k\wedge t)} - \sigma_{m,j\wedge\tau_j\wedge k\wedge\tau_k\wedge t}) \\ &- 2sm^{-4}\sum_{1\le j\le m^2 s}\sum\nolimits_{\frac{1}{m}\mathbb{Z}^2}\psi^s_{(m)}\psi^{j/m^2}_{(m)}(\mathbf{1}_{A_m(j\wedge\tau_j\wedge t)} - \sigma_{m,j\wedge\tau_j\wedge t}), \\ S_m(t) =& \sum_{j=1}^t |X_{m,j}|^2, \\ N_m(t) =& S_m(t) - Z_m(t). \end{aligned}$$

As before, N_m satisfies the martingale property. To see this, we first factor the intervals of Z_m as follows; below, write $a_{m,t} := A_m(t)\setminus A_m(t-1)$ for the t^{th} point joined to our IDLA.

$$
\begin{aligned}
&Z_m(t) - Z_m(t-1) \\
&\quad = s^2 m^{-2} \mathbf{1}_{\{\tau^* \geq t\}} \cdot \left(\psi^s_{(m)}(a_{m,t})^2 - \psi^s_{(m)}(z_{m,t})^2\right) \\
&\qquad + m^{-6} \sum_{\substack{t \leq j,k \leq m^2 s \\ \text{s.t. } \tau_j, \tau_k \geq t}} \left(\psi^{k/m^2}_{(m)}(a_{m,t}) \psi^{j/m^2}_{(m)}(a_{m,t}) - \psi^{k/m^2}_{(m)}(z_{m,t}) \psi^{j/m^2}_{(m)}(z_{m,t})\right) \\
&\qquad - 2sm^{-4} \sum_{\substack{t \leq j \leq m^2 s \\ \text{s.t. } \tau_j \geq t}} \left(\psi^s_{(m)}(a_{m,t}) \psi^{j/m^2}_{(m)}(a_{m,t}) - \psi^s_{(m)}(z_{m,t}) \psi^{j/m^2}_{(m)}(z_{m,t})\right) \\
&\quad = \left[sm^{-1} \mathbf{1}_{\{\tau^* \geq t\}} \cdot \psi^s_{(m)}(a_{m,t}) - m^{-3} \sum_{\substack{t \leq j \leq m^2 s \\ \text{s.t. } \tau_j \geq t}} \psi^{j/m^2}_{(m)}(a_{m,t}) \right]^2 \\
&\qquad - \left[sm^{-1} \mathbf{1}_{\{\tau^* \geq t\}} \cdot \psi^s_{(m)}(z_{m,t}) - m^{-3} \sum_{\substack{t \leq j \leq m^2 s \\ \text{s.t. } \tau_j \geq t}} \psi^{j/m^2}_{(m)}(z_{m,t}) \right]^2 .
\end{aligned}
$$

We thus see that the only remaining terms of

$$
N_m(t) - N_m(t-1) = |X_{m,t}|^2 - (Z_m(t) - Z_m(t-1))
$$

are the following cross-terms, from which the martingale property follows:

$$
\begin{aligned}
&\mathbb{E}[N_m(t) - N_m(t-1) | \mathscr{F}_{m,t-1}] \\
&\quad = \mathbb{E}[|X_{m,i}|^2 - (Z_m(t) - Z_m(t-1)) | \mathscr{F}_{m,t-1}] \\
&\quad = \mathbb{E}\Bigg[2 \left[sm^{-1} \mathbf{1}_{\{\tau^* \geq t\}} \cdot \psi^s_{(m)}(z_{m,t}) - m^{-3} \sum_{\substack{t \leq j \leq m^2 s \\ \text{s.t. } \tau_j \geq t}} \psi^{j/m^2}_{(m)}(z_{m,t}) \right]^2 \\
&\qquad - 2 \left[sm^{-1} \mathbf{1}_{\{\tau^* \geq t\}} \cdot \psi^s_{(m)}(a_{m,t}) - m^{-3} \sum_{\substack{t \leq j \leq m^2 s \\ \text{s.t. } \tau_j \geq t}} \psi^{j/m^2}_{(m)}(a_{m,t}) \right] \\
&\qquad\quad \times \left[sm^{-1} \mathbf{1}_{\{\tau^* \geq t\}} \cdot \psi^s_{(m)}(z_{m,t}) - m^{-3} \sum_{\substack{t \leq j \leq m^2 s \\ \text{s.t. } \tau_j \geq t}} \psi^{j/m^2}_{(m)}(z_{m,t}) \right] \Bigg| \mathscr{F}_{m,t-1} \Bigg] \\
&\quad \propto \mathbb{E}\Bigg[\left[sm^{-1} \mathbf{1}_{\{\tau^* \geq t\}} \cdot \psi^s_{(m)}(z_{m,t}) - m^{-3} \sum_{\substack{t \leq j \leq m^2 s \\ \text{s.t. } \tau_j \geq t}} \psi^{j/m^2}_{(m)}(z_{m,t}) \right] \\
&\qquad - \left[sm^{-1} \mathbf{1}_{\{\tau^* \geq t\}} \cdot \psi^s_{(m)}(a_{m,t}) - m^{-3} \sum_{\substack{t \leq j \leq m^2 s \\ \text{s.t. } \tau_j \geq t}} \psi^{j/m^2}_{(m)}(a_{m,t}) \right] \Bigg| \mathscr{F}_{m,t-1} \Bigg] \\
&\quad = 0,
\end{aligned}
$$

using the fact that the last variable is a linear combination of martingale intervals adapted to $\mathscr{F}_{m,t}$. As in the proof of Theorem 3.1, we can use this martingale property to show that—since $(S_m(t) - S_m(t-1))^2$ and $(Z_m(t) - Z_m(t-1))^2$ are of order m^{-4}—we have $\mathbb{E}[N_m(m^2 s)^2] = O(m^{-2})$ and thus know that $N_m(m^2 s) \to 0$ in probability.

Finally, on event $\mathcal{E}$, we estimate $Z_m(m^2 s)$ as follows:

$$\begin{aligned}
Z_m(m^2 s) &= s^2 m^{-2} \sum\nolimits_{\frac{1}{m}\mathbb{Z}^d} |\psi^s_{(m)}|^2 (\mathbf{1}_{A_m(m^2 s)} - \sigma_{m,m^2 s}) \\
&\quad + m^{-6} \sum_{1 \le j,k \le m^2 s} \sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} \psi^{k/m^2}_{(m)} \psi^{j/m^2}_{(m)} (\mathbf{1}_{A_m(j\wedge k)} - \sigma_{m,j\wedge k}) \\
&\quad - 2sm^{-4} \sum_{1 \le j \le m^2 s} \sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} \psi^s_{(m)} \psi^{j/m^d}_{(m)} (\mathbf{1}_{A_m(j)} - \sigma_{m,j}) \\
&= s^2 m^{-2} \sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} |\psi^s_{(m)}|^2 (\mathbf{1}_{A_m(m^2 s)} - \sigma_s) \\
&\quad + 2m^{-6} \sum_{1 \le j \le k \le m^2 s} \sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} \psi^{k/m^2}_{(m)} \psi^{j/m^2}_{(m)} (\mathbf{1}_{A_m(j)} - \sigma_{j/m^2}) \\
&\quad - 2sm^{-4} \sum_{1 \le j \le m^2 s} \sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} \psi^s_{(m)} \psi^{j/m^2}_{(m)} (\mathbf{1}_{A_m(j)} - \sigma_{j/m^2}) + O(m^{-2}).
\end{aligned}$$

Now, from (7), we can continue with the substitutions $\psi^\tau_{(m)} \to \psi^\tau_m$:

$$\begin{aligned}
Z_m(m^2 s) &= s^2 m^{-2} \sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} |\psi^s_m|^2 (\mathbf{1}_{A_m(m^2 s)} - \sigma_s) \\
&\quad + 2m^{-6} \sum_{1 \le j \le k \le m^2 s} \sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} \psi^{k/m^2}_m \psi^{j/m^2}_m (\mathbf{1}_{A_m(j)} - \sigma_{j/m^2}) \\
&\quad - 2sm^{-4} \sum_{1 \le j \le m^2 s} \sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} \psi^s_m \psi^{j/m^2}_m (\mathbf{1}_{A_m(j)} - \sigma_{j/m^2}) + O(m^{-2}).
\end{aligned}$$

On event $\mathcal{E}$, the sets $A_m(j)$ and D_{j/m^2} differ by at most $O(m^2 \varepsilon_m)$ points on the lattice $\frac{1}{m}\mathbb{Z}^2$, so we can replace $\mathbf{1}_{A_m(j)} \to \mathbf{1}_{D_{j/m^2}}$ with only an additional $O(\varepsilon_m)$ error:

$$\begin{aligned}
Z_m(m^2 s) &= s^2 m^{-2} \sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} |\psi^s_m|^2 (\mathbf{1}_{D_s} - \sigma_s) \\
&\quad + 2m^{-6} \sum_{1 \le j \le k \le m^2 s} \sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} \psi^{k/m^2}_m \psi^{j/m^2}_m (\mathbf{1}_{D_{j/m^2}} - \sigma_{j/m^2}) \\
&\quad - 2sm^{-4} \sum_{1 \le j \le m^2 s} \sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} \psi^s_m \psi^{j/m^2}_m (\mathbf{1}_{D_{j/m^2}} - \sigma_{j/m^2}) + O(\varepsilon_m).
\end{aligned}$$

Finally, since the derivatives of ψ^τ_m are uniformly bounded in both m and τ, we can swap these sums with the appropriate integrals with an error of $O(m^{-1})$ (which we wrap into the existing $O(\varepsilon_m)$ term):

$$\begin{aligned}
Z_m(m^2 s) &= s^2 \int_{D_s} |\psi_m^s|^2 (1-\sigma_s) + 2m^{-4} \sum_{1\le j\le k\le m^2 s} \int_{D_{j/m^2}} \psi_m^{k/m^2} \psi_m^{j/m^2} (1-\sigma_{j/m^2}) \\
&\quad - 2sm^{-2} \sum_{1\le j\le m^2 s} \int_{D_{j/m^2}} \psi_m^s \psi_m^{j/m^2} (1-\sigma_{j/m^2}) + O(\varepsilon_m) \\
&= s^2 \int_{D_s} |\psi_m^s|^2 (1-\sigma_s) + 2\int_0^s ds' \int_0^{s'} ds'' \int_{D_{s''}} \psi_m^{s'} \psi_m^{s''} (1-\sigma_{s''}) \\
&\quad - 2s \int_0^s ds' \int_{D_{s'}} \psi_m^s \psi_m^{s'} (1-\sigma_{s'}) + O(\varepsilon_m) \\
&= s^2 \int_{D_s} |\psi_s|^2 (1-\sigma_s) + 2\int_0^s ds' \int_0^{s'} ds'' \int_{D_{s''}} \psi_{s'} \psi_{s''} (1-\sigma_{s''}) \\
&\quad - 2s \int_0^s ds' \int_{D_{s'}} \psi_s \psi_{s'} (1-\sigma_{s'}) + O(\varepsilon_m).
\end{aligned}$$

□

6 Point correlation functions

In this section, we will compute **point-correlation** functions for extended-source IDLA. In short, we want to find a local version of Equation (3), which would tell us the correlation between IDLA fluctuations at two specific points, $p, q \in \operatorname{int}(D_T) \setminus D_0$. We phrase this problem in terms of limits of smooth bump functions, which we already know how to handle from our main results. Fix $\varepsilon > 0$, and let η_p^ε and η_q^ε be smooth functions satisfying

$$\text{(9)} \qquad \operatorname{supp} \eta_p^\varepsilon \subset B_\varepsilon(p), \quad \operatorname{supp} \eta_q^\varepsilon \subset B_\varepsilon(q), \quad \int \eta_p^\varepsilon = \int \eta_q^\varepsilon = 1.$$

Without loss of generality, we will assume that $B_\varepsilon(p), B_\varepsilon(q) \subset D_T$, and we will write $L_m := L_m^T$ for the lateness function at time T. From Theorem 3.2, we know that $(L_m, \eta_p^\varepsilon)$ (resp., $(L_m, \eta_q^\varepsilon)$) tends to a Gaussian variable $L(\eta_p^\varepsilon)$ (resp., $L(\eta_q^\varepsilon)$) in m.

Our primary result is the following:

Theorem 6.1. *Suppose $p, q \in \operatorname{int}(D_T) \setminus D_0$, and s_p and s_q satisfy $p \in \partial D_{s_p}$, $q \in \partial D_{s_q}$. For $\varepsilon > 0$, further suppose that η_q^ε and η_p^ε are smooth functions satisfying* (9)*. The covariance between*

$$L(\eta_q^\varepsilon) = \lim_{m\to\infty} (L_m, \eta_q^\varepsilon) \quad \textit{and} \quad L(\eta_p^\varepsilon) = \lim_{m\to\infty} (L_m, \eta_p^\varepsilon)$$

satisfies

$$g(p,q) := \lim_{\varepsilon\to 0} \mathbb{E}[L(\eta_q^\varepsilon) L(\eta_p^\varepsilon)] = \frac{1}{\upsilon_p \upsilon_q} \int_{D_{s_*}} F_p F_q (1-\sigma_{s_*}),$$

where $s_* = \min(s_p, s_q)$, v_p and v_q are the velocities of the flow $s \mapsto D_s$ at p and q (at times s_p and s_q, respectively), and F_p and F_q are the Poisson kernels of D_{s_p} and D_{s_q} at p and q, respectively.

For completeness' sake, we first recall the notion of a Poisson kernel:

Definition 6.2. Suppose D is a smoothly bounded domain, and $p \in \partial D$. Then the **Poisson kernel** $F_{p,D}$ of D at p is the harmonic function on int(D) satisfying

$$F_p(x) = \frac{\partial}{\partial \mathbf{n}} G_D(x, \xi)|_{\xi = p}, \tag{10}$$

where $G_D(x, y)$ is the Green's function for the domain D, and $\partial/\partial\mathbf{n}$ is the inward normal derivative with respect to the second variable.

Importantly, if $f : C^0(\partial D)$, then the function

$$\phi(x) = \int_{\partial D} d\xi \; F_\xi(x) \tag{11}$$

is harmonic on D and satisfies $\phi|_{\partial D} \equiv f$.

We will use these functions in the following context. If $p \in \mathrm{int}(D_T) \setminus D_0$, there is a unique $s_p > 0$ such that $p \in \partial D_{s_p}$. We write F_p for the Poisson kernel of D_{s_p} at p.

Proof of Theorem 6.1. We first deal with the case that $s_p \neq s_q$, so that p and q are hit at different times by the flow $s \mapsto D_s$. Without loss of generality, suppose $s_p > s_q$, and suppose ε is small enough that

$$\inf\{s \mid B_\varepsilon(p) \cap \partial D_s \neq \emptyset\} > \sup\{s \mid B_\varepsilon(q) \cap \partial D_s \neq \emptyset\}. \tag{12}$$

From (3), we know that

$$\mathbb{E}[L(\eta_q^\varepsilon)L(\eta_p^\varepsilon)] = \int_0^T ds' \int_0^{s'} ds'' \int_{D_{s''}} \psi_{s'}^\varepsilon \varphi_{s''}^\varepsilon (1 - \sigma_{s''}),$$

where ψ_s^ε and φ_s^ε are harmonic functions on D_s satisfying $\psi_s^\varepsilon|\partial D_s \equiv \eta_p^\varepsilon$ and $\varphi_s^\varepsilon|\partial D_s \equiv \eta_q^\varepsilon$. In the above formula, we removed the $\psi_{s''}^\varepsilon \varphi_{s'}^\varepsilon$ term that appears in (3); these terms must all vanish, from (12). Next, note that the remaining terms can only be nonzero when s' (resp., s'') lies in a thin (i.e., $O(\varepsilon)$) band around s_p (resp, s_q). Define

$$s_- := \inf\{s \mid B_\varepsilon(q) \cap \partial D_s \neq \emptyset\}$$

to be the smallest value of s such that φ_s^ε is nonzero. From our above discussion, we can write

$$\mathbb{E}[L(\eta_q^\varepsilon)L(\eta_p^\varepsilon)] = \int_0^T ds' \int_0^{s'} ds'' \int_{D_{s_-}} \psi_{s'}^\varepsilon \varphi_{s''}^\varepsilon (1 - \sigma_{s_-}) + O(\varepsilon),$$

also using the continuity of $s \mapsto \int \sigma_s$. Note that the third integral is now always taken over the same set. Now, introduce coordinates (s, θ) near p such that $(s, \cdot) \in D_s$ and such that $\theta|\partial D_s$ measures the (signed) arclength from $(s, 0)$ along ∂D_s. Introduce similar coordinates (s, α) near q. Without loss of generality, we assume $p = (s_p, 0)$. From (11), we can rewrite

$$\psi_s^\varepsilon(z) = \int d\theta\ \eta_p^\varepsilon(s, \theta) F_{(s,\theta)}(z), \quad \varphi_s^\varepsilon(z) = \int d\alpha\ \eta_q^\varepsilon(s, \alpha) F_{(s,\alpha)}(z),$$

which gives the following formula for the covariance:

$$\begin{aligned}&\mathbb{E}[L(\eta_q^\varepsilon)L(\eta_p^\varepsilon)]\\ &= \int_0^T ds' \int d\theta\ \eta_p^\varepsilon(s', \theta) \int_0^{s'} ds'' \int d\alpha\ \eta_q^\varepsilon(s'', \alpha) \int_{D_{s_-}} F_{(s',\theta)} F_{(s'',\alpha)}(1 - \sigma_{s_-}) + O(\varepsilon).\end{aligned}$$

Now, η_p^ε is supported on an ε-ball around p, so we only have to consider $F_{(s,\theta)}$ for $(s, \theta) \in B_\varepsilon(p)$. For these points, we find that[1] $|F_{(s,\theta)}(z) - F_p(z)| \le \frac{\varepsilon C}{|z-p|^2}$, and thus

$$\begin{aligned}\int_{D_{s_p}} |F_{(s,\theta)} - F_p| &= \int_{B_{\varepsilon^{1/3}}(p) \cap D_{s_p}} |F_{(s,\theta)} - F_p| + \int_{D_{s_p} \setminus B_{\varepsilon^{1/3}}(p)} |F_{(s,\theta)} - F_p|\\ &\le \int_{D_{s_p} \setminus (D_{s_p})_{\varepsilon^{1/3}}} (|F_{(s,\theta)}| + |F_p|) + \varepsilon^{1/3} C \operatorname{vol}(D_{s_p}) = O(\varepsilon^{1/3}).\end{aligned}$$

Repeating the same argument for $F_{(s,\alpha)}$ and F_q (and using the fact that F_p and F_q are bounded near the pole of the other), we find that

$$\begin{aligned}&\mathbb{E}[L(\eta_q^\varepsilon)L(\eta_p^\varepsilon)]\\ &= \int_0^T ds' \int d\theta\ \eta_p^\varepsilon(s', \theta) \int_0^{s'} ds'' \int d\alpha\ \eta_q^\varepsilon(s'', \alpha) \int_{D_{s_-}} F_p F_q (1 - \sigma_{s_-}) + O(\varepsilon^{1/3}).\end{aligned}$$

Finally, we convert from the coordinates (s, θ) and (s, α) back to standard Euclidean coordinates. For this, note that (s, θ) and (s, α) are orthogonal coordinate systems, and that θ and α are unit-speed parametrized, by definition. Thus, the only contributions to $|d(s, \theta)/d(x, y)|$ and $|d(s, \alpha)/d(x, y)|$ are the scaling factors in the s-direction. These are exactly the (inverse) velocities $v(s, \theta)^{-1}$ and $v(s, \alpha)^{-1}$ of the flow $s \mapsto D_s$, and we find that

$$\int ds' \int d\theta\ \eta_p^\varepsilon(s', \theta) = \int dA\ v^{-1} \eta_p^\varepsilon = v_p^{-1} \int dA\ \eta_p^\varepsilon + O(\varepsilon) = v_p^{-1} + O(\varepsilon),$$

[1] For instance, we can find this estimate by first comparing $F_{(s,\theta)}$ and F_p to nearby Green's functions using (10), and then comparing the Green's functions to one another by bounding their gradients above as $|\nabla_x G_D(x, y)| \le C|x - y|^{-1}$.

and similarly $\int ds'' \int d\alpha\ \eta_q^\varepsilon(s'', \alpha) = {v_q}^{-1} + O(\varepsilon)$. Putting these ingredients together, we get

$$\mathbb{E}[L(\eta_q^\varepsilon)L(\eta_p^\varepsilon)] = \frac{1}{v_p v_q} \int_{D_{s_-}} F_p F_q (1 - \sigma_{s_-}) + O(\varepsilon^{1/3})$$
$$= \frac{1}{v_p v_q} \int_{D_{s_*}} F_p F_q (1 - \sigma_{s_*}) + O(\varepsilon^{1/3}),$$

wrapping the $O(\varepsilon)$ error term from switching s_- to s_* into the existing $O(\varepsilon^{1/3})$ error.

Now, assume that $s_p = s_q$, and suppose ε is small enough that $B_\varepsilon(p)$ and $B_\varepsilon(q)$ are disjoint. Let

$$s_- := \inf\{s \mid (B_\varepsilon(q) \cup B_\varepsilon(p)) \cap \partial D_s \neq \emptyset\},$$

so that, as before,

$$\mathbb{E}[L(\eta_q^\varepsilon)L(\eta_p^\varepsilon)] = \int_0^T ds' \int_0^{s'} ds'' \int_{D_{s_-}} (\psi_{s'}^\varepsilon \varphi_{s''}^\varepsilon + \psi_{s''}^\varepsilon \varphi_{s'}^\varepsilon)(1 - \sigma_{s_-}) + O(\varepsilon).$$

We can split these terms as follows:

$$\int_0^T ds' \int_0^{s'} ds'' \int_{D_{s_-}} (\psi_{s'}^\varepsilon \varphi_{s''}^\varepsilon + \psi_{s''}^\varepsilon \varphi_{s'}^\varepsilon)(1 - \sigma_{s_-})$$
$$= \int_0^T ds' \int_0^{s'} ds'' \int_{D_{s_-}} \psi_{s'}^\varepsilon \varphi_{s''}^\varepsilon (1 - \sigma_{s_-}) + \int_0^T ds' \int_0^{s'} ds'' \int_{D_{s_-}} \psi_{s''}^\varepsilon \varphi_{s'}^\varepsilon (1 - \sigma_{s_-})$$
$$= \int_0^T ds' \int_0^{s'} ds'' \int_{D_{s_-}} \psi_{s'}^\varepsilon \varphi_{s''}^\varepsilon (1 - \sigma_{s_-}) + \int_0^T ds'' \int_{s''}^{T} ds' \int_{D_{s_-}} \psi_{s''}^\varepsilon \varphi_{s'}^\varepsilon (1 - \sigma_{s_-})$$
$$= \int_0^T ds' \int_0^{T} ds'' \int_{D_{s_-}} \psi_{s'}^\varepsilon \varphi_{s''}^\varepsilon (1 - \sigma_{s_-}).$$

At this point, we can follow the same logic as in the first case, and the theorem follows. □

We can apply this formula concretely to the case of a radially-expanding disk. Suppose that D_0 is the unit disk, and set $Q_0^s = B_{\sqrt{s/\pi}}$ for $s \in [0, 1)$. In this setting, we can imagine our source as a collection of outwardly moving rings of radius $0 \leq r < 1$, as shown in Figure 2. From symmetry considerations, it is clear that $D_s = B_{\sqrt{1+s/\pi}}$ are outwardly expanding disks. Suppose p and q lie in the plane, on origin-centered circles of radii $1 < r_q \leq r_p < \sqrt{1 + 1/\pi}$ and at polar angles θ_p, θ_q. The functions F_p and F_q take the following forms:

$$F_p(re^{i\theta}) = \sum_{n \in \mathbb{Z}} (r/r_p)^{|n|} e^{in(\theta - \theta_p)}, \quad F_q(re^{i\theta}) = \sum_{n \in \mathbb{Z}} (r/r_q)^{|n|} e^{in(\theta - \theta_q)}.$$

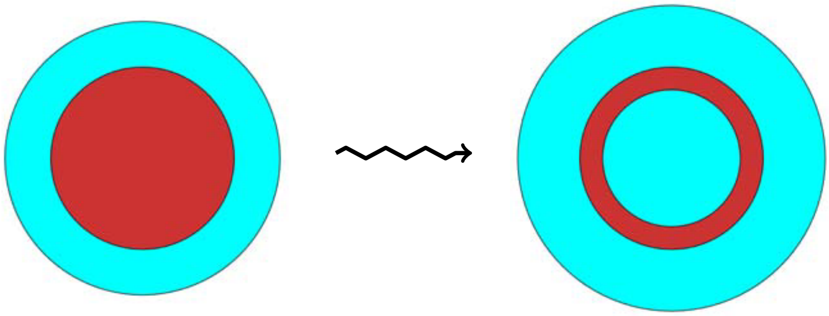

Figure 2: An illustration of the flow $s \mapsto D_s$ in the case of a radially-expanding disk. Here, D_0 is the unit disk (cyan, left), and our single source set Q_0^T is a smaller disk within it (red, left). As time passes, source points move from the center of Q_0^s to the outer boundary of D_s.

Then, from Theorem 6.1, we can calculate

$$\begin{aligned} g(p,q) &= \frac{1}{v_p v_q}\int_0^{r_q} rdr \int_0^{2\pi} d\theta\, F_p F_q (1-\sigma_{\pi(r_q^2-1)}) \\ &= (2\pi)^2 r_p r_q \int_0^{r_q} dr \int_0^{2\pi} d\theta\, rF_p F_q (1-\sigma_{\pi(r_q^2-1)}) \\ &= (2\pi)^2 r_p r_q \int_0^{r_q} dr \int_0^{2\pi} d\theta\, r \sum_{j,k\in\mathbb{Z}} \frac{r^{|j|+|k|}}{r_p^{|j|} r_q^{|k|}} e^{i(j+k)\theta - ij\theta_p - ik\theta_q}(1-\sigma_{\pi(r_q^2-1)}). \end{aligned}$$

Only the terms with $j+k=0$ survive when integrating θ:

$$g(p,q) = (2\pi)^3 r_p r_q \int_0^{r_q} dr \sum_{j\in\mathbb{Z}} \frac{r^{2|j|+1}}{r_p^{|j|} r_q^{|j|}} e^{ij(\theta_p-\theta_q)}(1-\sigma_{\pi(r_q^2-1)}).$$

Now, we break this into two integrals using $\sigma_{\pi(r_q^2-1)} = \mathbf{1}_{\{r\le\sqrt{r_q^2-1}\}} + \mathbf{1}_{\{r\le 1\}}$:

$$\begin{aligned} g(p,q) &= (2\pi)^3 r_p r_q \int_1^{r_q} dr \sum_{j\in\mathbb{Z}} \frac{r^{2|j|+1}}{r_p^{|j|} r_q^{|j|}} e^{ij(\theta_p-\theta_q)} \\ &\qquad - (2\pi)^3 r_p r_q \int_0^{\sqrt{r_q^2-1}} dr \sum_{j\in\mathbb{Z}} \frac{r^{2|j|+1}}{r_p^{|j|} r_q^{|j|}} e^{ij(\theta_p-\theta_q)} \\ &= (2\pi)^3 r_p r_q \sum_{j\in\mathbb{Z}} \frac{e^{ij(\theta_p-\theta_q)}}{r_p^{|j|} r_q^{|j|}} \left(\int_1^{r_q} r^{2|j|+1} dr - \int_0^{\sqrt{r_q^2-1}} r^{2|j|+1} dr \right) \\ &= (2\pi)^3 r_p r_q \sum_{j\in\mathbb{Z}} \frac{1}{2|j|+2} \frac{e^{ij(\theta_p-\theta_q)}}{r_p^{|j|} r_q^{|j|}} (r_q^{2|j|+2} - 1 - (r_q^2-1)^{|j|+1}). \end{aligned}$$

We can discard the negative frequency modes by rewriting this sum as twice the real part of its positive frequency modes:

$$
\begin{aligned}
&g(p,q)\\
&= (2\pi)^3 (r_p r_q)^2 \operatorname{Re}\left(\sum_{j=0}^{\infty} \frac{1}{j+1} \frac{e^{ij(\theta_p-\theta_q)}}{r_p^{j+1} r_q^{j+1}} (r_q^{2j+2} - 1 - (r_q^2 - 1)^{j+1}) \right)\\
&= (2\pi)^3 (r_p r_q)^2\\
&\quad \times \operatorname{Re}\left(e^{-i(\theta_p-\theta_q)} \sum_{j=0}^{\infty} \frac{1}{j+1} e^{i(j+1)(\theta_p-\theta_q)} \left(\frac{r_q^{j+1}}{r_p^{j+1}} - \frac{1}{r_p^{j+1} r_q^{j+1}} - \frac{(r_q^2-1)^{j+1}}{r_p^{j+1} r_q^{j+1}} \right) \right)\\
&= -(2\pi)^3 (r_p r_q)^2 \operatorname{Re}[e^{-i(\theta_p-\theta_q)} (\operatorname{Log}(1 - e^{i(\theta_p-\theta_q)} r_q/r_p) - \operatorname{Log}(1 - e^{i(\theta_p-\theta_q)} 1/r_p r_q)\\
&\qquad\qquad - \operatorname{Log}(1 - e^{i(\theta_p-\theta_q)} (r_q^2 - 1)/r_p r_q))]\\
&= -(2\pi)^3 |pq| \operatorname{Re}[\overline{p} q (\operatorname{Log}(1 - \overline{q}/\overline{p}) - \operatorname{Log}(1 - 1/\overline{p} q) - \operatorname{Log}(1 - (|q|^2 - 1)/\overline{p} q))],
\end{aligned}
$$

where Log denotes the principle value of the logarithm, and we view p and q as complex numbers. This function is plotted in Figure 3.

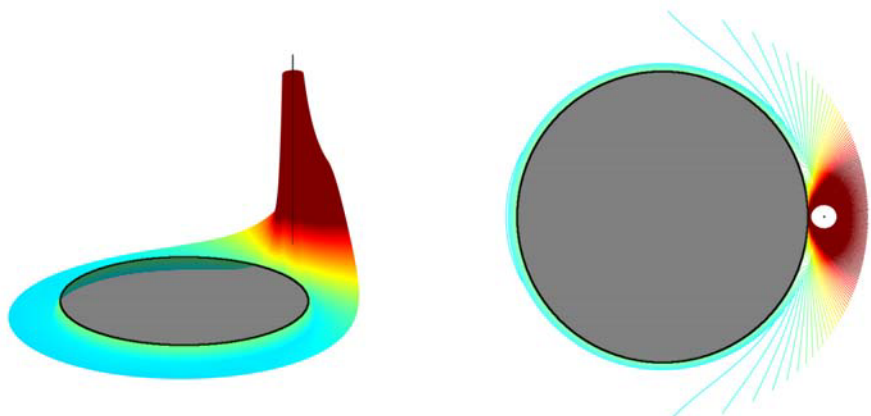

Figure 3: A plot and contour map of the function $g(p,q)$, where $q = (\frac{1}{2} + \frac{\sqrt{5}}{4}, 0)$ is fixed. In each image, the set D_0—the unit disk—is shaded, and q is marked as a black point [resp., line] alongside. Note the logarithmic singularity at $p = q$, and that the function vanishes for p along the unit circle (the inner boundary of the domain).

The function $g(p,q)$ has several key properties, which we can see in Figure 3. For one, $g(p,q)$ is positive if and only if θ_p and θ_q are nearby. This confirms the geometric intuition that, for instance, an early point leads to other nearby early points, but that it prevents distant early points (by using up particle mass itself). Secondly, $g(p, q)$ vanishes as either p or q approaches the unit disk, likely reflecting the fact that points nearer to D_0 are "more deterministic"—i.e., that the variance of their lateness decreases to 0 as they get closer to D_0. It is easy to check from Theorem 6.1 that this property holds true for other flows, as well, with the unit disk replaced by D_0 in general.

Next, note the logarithmic singularity present at $p = q$. This is to be expected, in analogy to the (free space) Green's function $G(x, y) = \log|x - y|$. Indeed, just as we can view the Green's function as giving an inner product

$$(u, v)_{-1} := \int dxdy\, u(x)G(x, y)v(y) = \int dy\, (\nabla^{-2}u)(y)v(y),$$

we can view the point-correlation function g as the kernel of the inner product defined in Equation (3):

$$(u, v)_g := \int dxdy\, u(x)g(x, y)v(y) = \int_0^T ds' \int_0^{s'} ds'' \int_{D_{s''}} (\psi_{s'}\varphi_{s''} + \psi_{s''}\varphi_{s'})(1 - \sigma_{s''}),$$

where, as before, ψ_s and φ_s are the solutions of the Dirichlet problem on D_s for u and v, respectively.

7 Directions for further research

One interesting extension of this work would be to extend these results to higher dimensions. In dimension d, the appropriate scaling factor for the fluctuation functions E_m^s and L_m^s would be $m^{d/2}$ (just as it is $m = m^{2/2}$ here). In general, then, the error found in (5) and (8) would come out to be $O(m^{d/2}\varepsilon_m^2)$. For this to decrease, we then need the bound $\varepsilon_m = o(m^{-d/4})$ on the maximum fluctuations. This is likely possible to achieve if $d = 3$, but clearly impossible for $d \geq 4$.

However, there are weaker results that remain possible for $d \geq 4$. For one, if we require the test function u to be harmonic, then we could achieve $\|u - \psi_{(m)}\|_\infty = O(m^{-2})$ on the domain of interest, rather than our existing $\|u - \psi_{(m)}\|_\infty = O(\varepsilon_m)$. In this case, the requirement on ε_m becomes $\varepsilon_m = o(m^{2-d/2})$, which now appears possible for dimensions 4 and 5.

Another important direction of research would be to generalize the sorts of possible sources for IDLA. For instance, it would be interesting to see if corresponding scaling limits hold if, instead of starting from a concentrated mass distribution, we were to start points evenly from a submanifold of D_0. Starting from the boundary of D_0, for example, may provide a good substitute for starting particles evenly across D_0 itself. In chemical applications, this adjusted setting could model a solid particle source of a particular shape.

Fortunately, the methods used in this paper translate fairly straightforwardly to other settings. The greatest obstacle to generalizing our results is finding an analogue to Lemma 2.4, which was the primary result of the preceding paper [Dar20]. Indeed, if it could be shown that the fluctuations of IDLA from a particular source satisfy a similar $O(m^{-1/2-\varepsilon})$ bound (for any $\varepsilon > 0$), the remainder of our argument could likely be repeated.

A Appendix: maximum fluctuations of the divisible sandpile

We will use the capital $N_m(t)$ to denote the fully occupied set

$$N_m(t) := \{\nu_{m,t} = 1\} \subset \frac{1}{m}\mathbb{Z}^2.$$

We will also use the notation of [Dar20]—in particular, for any $\zeta \in \frac{1}{m}\mathbb{Z}^2 \setminus D_0$, we will write $\tau = \tau(\zeta)$ for the time at which $\zeta \in \partial D_\tau$, and we will use H_ζ and Ω_ζ exactly as in that paper. We will not give more details on these objects here.

Now, we say that a point $z \in \frac{1}{m}\mathbb{Z}^2$ is ε-early at time t if $z \in N_m(t)$, but $z \notin (D_{t/m^2})^\varepsilon$. Similarly, z is ε-late at time t if $z \in (D_{t/m^2})_\varepsilon$, but $z \notin N_m(t)$.

Finally, we will define a stopped version of $\nu_{m,t}$, as follows:

Definition A.1 (Stopped Sandpile)**.** Given $\nu_{\zeta,n}$, define the intermediate function $\nu^0_{\zeta,n} = \nu_{\zeta,n} + \mathbf{1}_{\{z_{m,n+1}\}}$. At each time step t, choose a point $z = z(t) \in \operatorname{supp} \nu^t_{\zeta,n} \setminus \partial D_\tau$ such that $\nu^t_{\zeta,n}(z) > 1$. Let $W^t_{m,n}(s)$ be a Brownian motion started from z on the grid

$$\mathcal{G}_m := \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; x \in \frac{1}{m}\mathbb{Z} \text{ or } y \in \frac{1}{m}\mathbb{Z}\right\},$$

as defined in Definition 4.3 of [Dar20]. Define the stopping time

$$\tau^* := \inf\left\{s \;\middle|\; W^t_{m,n}(s) \in \left(\frac{1}{m}\mathbb{Z}^2 \setminus N_m(n)\right) \cup \partial D_\tau\right\},$$

and set

$$\nu^{t+1}_{\zeta,n}(z') = \nu^t_{\zeta,n}(z') + (\nu^t_{\zeta,n}(z) - 1) \cdot (\mathbb{P}[W^t_{m,n}(\tau^*) = z] - \delta_{z,z'}).$$

For a large enough t', we have that $\nu^{t'}_{\zeta,n}(z) \le 1$ everywhere in $\operatorname{supp} \nu^{t'}_{m,n} \setminus \partial D_\tau$; then we define $\nu_{\zeta,n+1} = \nu^{t'}_{\zeta,n}$.

In parallel with the original divisible sandpile model, we define $\nu_{\zeta,n+1}$ by taking the excess mass at z in $\nu_{\zeta,n}$ and splitting it around the edge of $\operatorname{supp}\nu_{\zeta,n}$ according to a discrete harmonic measure. New in this case, however, is that we stop mass before it exits the domain D_τ.

Note that this satisfies the same key equality as the original harmonic measure; namely, for any grid harmonic (see [Dar20]) H defined in $\frac{1}{m}\mathbb{Z}^2 \cap D_\tau$,

$$\sum H \cdot (\nu_{\zeta,t} - \sigma_{\zeta,t}) = 0.$$

Lemma A.2 (No Thin Tentacles)**.** *There is an absolute constant $b > 0$ such that for all $z \in N_m(t) \subset \frac{1}{m}\mathbb{Z}^2$ with $d(z, D_0) \geq r$,*

$$\#(N_m(t) \cap B(z, r)) > bm^2r^2.$$

Proof. For this, we define the intermediate processes $\nu^\ell_{m,t}$, for each integer $\ell \geq 1$:

(1) Define the initial set $\nu^\ell_m(0) = \nu_m(0) = \mathbf{1}_{\frac{1}{m}\mathbb{Z}^2 \cap D_0}$.

(2) For each $i \in \frac{1}{\ell}\mathbb{Z}_{>0}$, start a random walk at $z_{m,\lceil i \rceil}$, and let z'_i be the first point in the walk at which $\nu^\ell_{m,i-\ell^{-1}}(z'_i) < 1$. Let $\nu^\ell_{m,i} = \nu^\ell_{m,i-\ell^{-1}} + \ell^{-1}\mathbf{1}_{\{z'_i\}}$.

Now, $\nu^0_{m,t}$ is simply an IDLA, by definition, and $\nu^\ell_{m,t} \to \nu_{m,t}$ in law (pointwise) in ℓ. We can now lift the proof of Lemma 2 of [JLS12] (Lemma 3.2 of [Dar20]) verbatim,[2] to show that

$$\mathbb{P}[\nu^\ell_{m,t}(z) > 0, \#(\{\nu^\ell_{m,t} = 1\} \cap B(z, r)) \leq bm^2r^2] \leq C_0 e^{-c_0 mr}$$

for constants c_0, C_0 independent of ℓ. In particular, this probability is uniformly bounded below 1 in ℓ; since $\nu^\ell_{m,t}$ converges in law to the deterministic function $\nu_{m,t}$, we see that

$$\lim_{\ell\to\infty} \mathbb{P}[\nu^\ell_{m,t}(z) > 0, \#(\{\nu^\ell_{m,t} = 1\} \cap B(z, r)) \leq bm^2r^2] = 0,$$

from which the lemma follows. □

The next theorem is a restatement of Theorem 2.8, and a stronger version of Theorem 3.9 of [LP08]:

Theorem A.3 (Theorem 2.8)**.** *There is a constant $C > 0$ dependent on the flow such that, for large enough m and any $s \in [0, T]$,*

$$\frac{1}{m}\mathbb{Z}^d \cap (D_s)_{Cm^{-3/5}} \subset N_m(m^2 s) \subset (D_s)^{Cm^{-3/5}}.$$

[2]The only significant difference in proving the new result is that the total number of "trials", as well as the total number of required "failures" (in the language of [JLS12]), is scaled up by a factor of ℓ.

Lemma A.4. *There are constants $C, \alpha > 0$ dependent only on the flow such that, for large enough m, $s \in [0, T]$, $a \geq Cm^{2/5}$, and $\ell \leq \alpha a$, an a/m-early point in $N_m(t)$ by time $m^2 s$ implies a different, ℓ/m-late point at time $m^2 s$.*

Proof. Suppose $z \in N_m(t) \setminus N_m(t-1)$ is the first a/m-early point in N_m—that is, $z \notin (D_{t/m^2})^{a/m}$, but $N_m(t-1) \subset (D_{(t-1)/m^2})^{a/m}$. Further assume that there are no ℓ/m-late points by the time t, or equivalently that $(D_{t/m^2})_{\ell/m} \subset N_m(t)$.

Since z is adjacent to $N_m(t-1)$, we know that

$$N_m(t) \subset (D_{t/m^2})^{(a+1)/m}.$$

Let $\zeta = \zeta(z, t)$ be the nearest point to z in the annulus

$$\frac{1}{m}\mathbb{Z}^2 \cap (D_{t/m^2})^{V(4a+2C)/m\upsilon+2/m} \setminus (D_{t/m^2})^{V(4a+2C)/m\upsilon},$$

where C will be specified later, and $\upsilon, V > 0$ are as in Lemma 3.5 of [Dar20]. Let $\tau > 0$ be such that $\zeta \in \partial D_\tau$, and note that

$$d_H(D_{t/m^2}, D_\tau) \geq d(D_{t/m^2}, \zeta) \geq V(4a + 2C)/m\upsilon.$$

By Lemma 3.5 of [Dar20], this implies

$$\begin{aligned} d(N_m(t), D_\tau^c) &\geq d((D_{t/m^2})^{(a+1)/m}, D_\tau^c) \\ &\geq d(D_{t/m^2}, D_\tau^c) - (a+1)/m \\ &\geq \frac{\upsilon}{V} d_H(D_{t/m^2}, D_\tau) - (a+1)/m \\ &\geq (3a + 2C - 1)/m > 1/m. \end{aligned}$$

From Lemma 4.2(a) of [Dar20], this means $\operatorname{supp}(\nu_{m,t}) \subset \Omega_\zeta$, and thus we can replace $\nu_{m,t}$ with $\nu_{\zeta,t}$. As in [Dar20], we can show that if $\#(N_m(t) \cap B(z, a/m)) > ba^2$ and no points are ℓ/m-late by time t, then

$$\sum_{z' \in \frac{1}{m}\mathbb{Z}^2} (\nu_{\zeta,t}(z') - \sigma_{m,t}(z')) H_\zeta(z') \geq \upsilon ba/12V > 0.$$

Both of the listed assumptions are true; we know $\#(N_m(t) \cap B(z, a/m)) > ba^2$ by Lemma A.2, and we have assumed that no points are ℓ/m-late by time t. However, $\sum(\nu_{\zeta,t} - \sigma_{m,t})H = 0$ for any H harmonic on $\operatorname{supp}(\nu_{\zeta,t})$, so this is a contradiction. □

Lemma A.5. *There is a constant $C > 0$ dependent only on the flow such that, for large enough m, $s \in [0, T]$, and $\ell \geq Cm^{2/5}$, there can be no ℓ/m-late point at time $m^2 s$.*

Proof. Without loss of generality, let $a = \ell^2/Cm^{2/5} \geq \ell$. Fix an integer $t \leq m^2 s$, and suppose that $\zeta \in \frac{1}{m}\mathbb{Z}^2 \cap ((D_{t/m^2})_{\ell/m} \setminus D_0)$ is ℓ/m-late by time t. Then $d(\zeta, \partial D_{t/m^2}) \geq \ell/m$, so by Lemma 3.5 of [Dar20],

$$\begin{aligned} t - m^2\tau &\geq 2m^2\sqrt{1+\tau}\left(\sqrt{1+t/m^2} - \sqrt{1+\tau}\right) \\ &\geq \frac{2m^2}{V}\sqrt{1+\tau}\cdot d_H(D_{t/m^2}, D_\tau) \geq \frac{2m^2}{V}\sqrt{1+\tau}\cdot d(\zeta, \partial D_{t/m^2}) \geq \frac{2m\ell}{V}. \end{aligned}$$

Since ζ is ℓ/m-late at time t, we know that $\zeta \notin N_m(t)$, so $\nu_{\zeta,t}(\zeta) < 1$. As in [Dar20], the sum

$$\tilde{M}_\zeta(t) := \sum_{z'} (\nu_{\zeta,t}(z') - \sigma_{m,t}(z'))H_\zeta(z')$$

is maximized if the interior of $\tilde{\Omega}_\zeta \cap \frac{1}{m}\mathbb{Z}^2$ is fully occupied by $N_m(t)$. We can then show, exactly as in [Dar20], that

$$\tilde{M}_\zeta(t) < 1 - c\ell$$

for some $c > 0$; the new constant term comes from the $\nu_{\zeta,t}(\zeta)H_\zeta(\zeta) < 1$ contribution. So long as C is large enough, this is still negative; of course, we know that $\tilde{M}_\zeta(t) = 0$, so this is a contradiction. □

Proof of Theorem 2.8. We can work with the set $N_m(t)$ instead of $\operatorname{supp} \nu_{m,t}$; indeed, the latter only differs from the former within one unit of the boundary. By Lemma A.5, we only need to show that no $O(m^{-3/5})$-early point can exist. Suppose a point is $\alpha^{-1}Cm^{-3/5}$-early at a time $t \leq m^2 s$. By Lemma A.4, this implies that another point is $Cm^{-3/5}$-late by the same time; this contradicts Lemma A.5, and we retrieve our result. □

B Appendix: sources on the boundary of D_0

The most restrictive hypothesis in our definition of concentrated mass distributions, as outlined in Definition 2.1, is that the sets Q_i^s are bounded away from ∂D_0. For instance, this prohibits the situation depicted in Figure 1, where we might take $D_0 := A \cup B$ (or, rather, a smooth equivalent thereof) and $Q_1 := A \cap B$. In this appendix, we introduce and prove an alternate version of Lemma 2.4 necessary to lift this hypothesis, and we discuss necessary modifications of the proofs of Theorems 3.1 and 3.2 to accommodate this setting.

Definition B.1. Suppose the data $(D_0, \{Q_i^s\}, \{s_i\}, \sigma_s)$ satisfies all of the hypotheses of Definition 2.1, except that Q_i^s need not be bounded away from ∂D_0. In this case, we say that $\sigma_s : \mathbb{R}^2 \to \mathbb{Z}^{\geq 0}$ is a **connected mass distribution**.

The internal DLA associated with σ_s can be defined as with a concentrated mass distribution, and as follows from [LP10], its scaling limit is the smooth smash sum

$$D_s = D_0 \oplus Q_1^{s_1} \oplus \cdots \oplus Q_N^{s_N}.$$

Our main results on connected mass distributions are the following three theorems, which respectively bound the fluctuations of IDLA, bound the fluctuations of the divisible sandpile, and show convergence of normalized fluctuations to the Gaussian fields introduced in this paper.

Theorem B.2. *Fix $\varepsilon > 0$. Suppose D_s is a smooth flow arising from a connected mass distribution. For large enough m, the fluctuation of the associated IDLA $A_m(t)$ is bounded as*

$$\mathbb{P}\Big\{(D_s)_{Cm^{-3/5}} \cap \frac{1}{m}\mathbb{Z}^2 \subset A_m(m^2 s) \subset (D_s)^{Cm^{-3/5}} \text{ for all } s \in [\varepsilon, T]\Big\}^c \le e^{-m^{2/5}}$$

for a constant $C = C(\varepsilon) > 0$, where $(D_s)^\delta$ and $(D_s)_\delta$ denote outer- and inner-δ-neighborhoods of D_s, respectively.

Theorem B.3. *Fix $\varepsilon > 0$. Suppose D_τ is a smooth flow arising from a connected mass distribution. For large enough m and any time $s \in [\varepsilon, T]$, the fluctuations of the occupied set* $\operatorname{supp} \nu_{m,m^2 s}$ *are bounded as*

$$(D_s)_{Cm^{-3/5}} \cap \frac{1}{m}\mathbb{Z}^2 \subset \operatorname{supp} \nu_{m,m^2 s} \subset (D_s)^{Cm^{-3/5}}$$

for a constant $C = C(\varepsilon) > 0$.

Theorem B.4. *Theorem 3.1 holds as stated when σ_s is only a connected mass distribution. Secondly, fix $\varepsilon > 0$ and let $L_m^{(\varepsilon,s)}$ be the ε-**delayed lateness function***

$$L_m^{(\varepsilon,s)} = \sum_{n=m^2\varepsilon}^{m^2 s} \frac{n}{m} 1_{A_{m,n} \setminus A_{m,n-1}} - \sum_{n=m^2\varepsilon}^{m^2 s} \frac{n}{m} (\nu_{m,n} - \nu_{m,n-1}).$$

Suppose $u \in C_0^4(\mathbb{R}^d)$ with $\operatorname{supp} u \subset D_s$. *The random variables $(L_m^{(\varepsilon,s)}, u)$ converge in law as $m \to \infty$ to a normal variable $(L^{(\varepsilon,s)}, u)$ of mean 0 and variance*

$$\begin{aligned} &\varepsilon^2 \int_{D_\varepsilon} |\psi_\varepsilon|^2 (1-\sigma_\varepsilon) + 2 \int_\varepsilon^s ds' \int_\varepsilon^{s'} ds'' \int_{D_{s'}} \psi_{s'} \psi_{s''} (1 - \sigma_{s''}) \\ &\quad + 2s \int_\varepsilon^s ds' \int_{D_\varepsilon} \psi_{s'} \psi_\varepsilon (1-\sigma_\varepsilon) \\ &= 2 \int_0^s ds' \int_0^{s'} ds'' \int_{D_{s'}} \psi_{s'} \psi_{s''} (1 - \sigma_{s''}) + O(\varepsilon). \end{aligned} \tag{13}$$

In particular, the limit (in law) $\lim_{\varepsilon \to 0} \lim_{m \to \infty} (L_m^{(\varepsilon,s)}, u)$ is a normal variable of mean zero and variance as given in Theorem 3.2.

Remark. Removing the assumption that $\operatorname{supp} u \subset D_s$ gives an expression akin to that of Lemma 6; we omit this general case for the sake of clarity.

In the above theorems, we see that the trade-off of allowing sources on the boundary of D_0 comes in the form of requiring the system to evolve a short time before our estimates hold. Although we expect that similar results hold when $\varepsilon = 0$, different methods would be necessary for their proof.

Below, suppose that σ_s is a connected mass distribution with data $(D_0, \{Q_i^s\}, \{s_i\})$. We continue with the notation of [Dar20].

Our proof of Theorem B.2 will be presented as a modification of the proof of [Dar20, Thm. 3.1]. As discussed in our preceding paper (in the discussion following [Dar20, Def. 2.1]), the lifted hypothesis is necessary only for the second statement of [Dar20, Lemma 4.1(c)] and for [Dar20, Lemma 5.2(b)], which is used to employ the constant R_1 in subsequent lemmas. In its place, we introduce the following:

Lemma B.5. *Fix $\varepsilon > 0$. There is a constant $R_1' = R_1'(\varepsilon) > 0$ such that, for any $\zeta \in \frac{1}{m}\mathbb{Z}^2 \cap (D_T \setminus D_\varepsilon)$, we have*

$$|H_\zeta(z_{m,i}) - F_\zeta(z_{m,i})| \leq C_1 m^{-2} (R_1')^{-2}$$

for any source points $z_{m,i}$, and where H_ζ, F_ζ, and C_1 are as in [Dar20]. Taking R_1' slightly smaller if necessary, for any $\zeta \in \frac{1}{m}\mathbb{Z}^2 \cap (D_T \setminus D_\varepsilon)$, we also have

$$\left| G_{D_\tau}(\zeta', z_{m,i}) - \frac{c_\zeta'}{m} J_\zeta(z_{m,i}) \right| \leq C_2 m^{-2} (R_1')^{-2}$$

for any source points $z_{m,i}$, where G_{D_τ}, J_ζ, and C_2 are as in [Dar20].

Proof. The two statements follow directly from [Dar20, Lemma 4.1(c)] and [Dar20, Lemma 5.2(b)], respectively, keeping in mind that $d(D_0, D_\varepsilon^c) = O(\varepsilon)$ from [Dar20, Lemma 3.5]. □

Similarly, we replace [Dar20, Lemma 4.5] with the following:

Lemma B.6. *Fix $\varepsilon > 0$, and suppose D_s is a smooth flow arising from a connected mass distribution. For*

$$m \geq \max(3a + C_2, 2C_2 / \inf_\zeta R_1'),$$

all $s \in [\varepsilon, T]$, and $\zeta \notin (D_s)^{(4a+2C_2)/m}$, we have

$$\mathbb{E}[e^{S_\zeta(m^2 s)} \mathbf{1}_{\mathcal{E}_{(a+1)/m}(m^2 s)^c}] \leq m^K,$$

where $K = K(\varepsilon)$.

Proof. Applying Lemma B.5 in place of [Dar20, Lemma 4.1(c)] to retrieve the bound

$$-\frac{1}{2mR_0} - \frac{1}{mR_1' - C_2} \leq H_\zeta(z) - H_\zeta(z_{m,i}),$$

and defining $R_2' = \min(R_0/2, R_1'/4)$ in place of R_2, the proof follows verbatim. □

This implies the following modification of [Dar20, Lemma 4.6]:

Lemma B.7. *Fix $\varepsilon > 0$, suppose D_s is smooth, and fix $a \geq 2C_2 + 2$, $\ell \leq a$, and $s \in [\varepsilon, T]$. For*

$$m \geq \max(3a + C_2, 5a/\inf_\zeta R_1')$$

and $\zeta \in \frac{1}{m}\mathbb{Z}^2 \cap ((D_s)_{\ell/m} \setminus D_\varepsilon)$, we have

$$\mathbb{E}[e^{S_\zeta(m^2 s)} \mathbf{1}_{\mathcal{E}_{(a+1)/m}(m^2 s)^c}] \leq m^K e^{K'a},$$

where K is as in Lemma B.6 and $K' = K'(\varepsilon) > 0$.

In turn, replacing R_1 by R_1' and R_2 by R_2' everywhere in the proof of [Dar20, Lemma 4.7] gives

Lemma B.8. *Fix $\varepsilon > 0$. For large enough m, $s \in [\varepsilon, T]$, $3a + C_2 \geq a \geq C_3 m^{2/5}$, and $\ell \leq \alpha a$, we have*

$$\mathbb{P}(\mathcal{E}_{a/m}[m^2 s] \cap \mathcal{L}_{\ell/m}[m^2 s]^c) \leq e^{-2m^{2/5}}.$$

Making the same substitutions in the setting of [Dar20, Section 5] gives a parallel to our second estimate:

Lemma B.9. *Fix $\varepsilon > 0$. There is a constant $C_4 = C_4(\varepsilon) > 0$ such that, for large enough m, if $s \in [\varepsilon, T]$, $\ell \geq C_4 m^{2/5}$, and $a \leq \ell^2/C_4 m^{2/5}$, then*

$$\mathbb{P}(\mathcal{L}_{\ell/m}[m^2 s] \cap \mathcal{E}_{a/m}[m^2 s]^c) \leq e^{-2m^{2/5}}.$$

Finally, putting these together as in [Dar20, Section 6], but considering only $s \geq \varepsilon$, gives Theorem B.2. To prove Theorem B.3, we can take the analysis of Appendix A nearly verbatim; the only step where our lifted hypothesis becomes relevant is at the end of the proof of Lemma A.5, where we show that $\tilde{M}_\zeta(t) < 1 - c\ell$. This makes use of the estimate [Dar20, Lemma 5.2(b)], which can now be safely replaced with Lemma B.5 at the cost of requiring that $s \geq \varepsilon$.

Proof of Theorem B.2. The first statement of Theorem B.2—that Theorem 3.1 holds in our generalized setting—follows from simply letting $\varepsilon < s$ and replacing our references to Lemmas 2.4 and 2.8 by Lemmas B.2 and B.3. Indeed, in our proof of Theorem 3.1, we only make use of the fact that no point is too early or too late at the fixed time s.

For the second statement, we adapt the proof of Lemma 6 by introducing the martingale

$$M_m^{\varepsilon}(t) = sm^{-1}\sum_{j=0}^{t\wedge\tau^*}(\psi_{(m)}^{s}(A_m(j)\setminus A_m(j-1)) - \psi_{(m)}^{s}(z_{m,j})) - \varepsilon m^{-1}\sum_{j=0}^{t\wedge m^2\varepsilon\wedge\tau^*}(\psi_{(m)}^{\varepsilon}(A_m(j)\setminus A_m(j-1)) - \psi_{(m)}^{\varepsilon}(z_{m,j})) - m^{-3}\sum_{\ell=m^2\varepsilon}^{m^2 s}\sum_{j=0}^{\ell\wedge\tau_\ell\wedge t}(\psi_{(m)}^{\ell/m^2}(A_m(j)\setminus A_m(j-1)) - \psi_{(m)}^{\ell/m^2}(z_{m,j})).$$

We replace the event $\mathcal{E}$ with the event $\mathcal{E}^{\varepsilon}$, in which

$$\frac{1}{m}\mathbb{Z}^2 \cap (D_{\ell/m^2})_{\varepsilon_m} \subset A_m(\ell) \subset (D_{\ell/m^2})^{\varepsilon_m}$$

for all $m^2\varepsilon \le \ell \le m^2 s$; this event occurs with probability $1 - e^{-m^{2/5}}$ by Theorem B.2, and in this event, $\tau_\ell \ge \ell$ for all such ℓ. Thus, as in the case of Lemma 6, we find that

$$\begin{aligned}
&M_m^{\varepsilon}(m^2 s)\\
&= sm^{-1}\sum\nolimits_{\frac{1}{m}\mathbb{Z}^d}\psi_{(m)}^{s}\cdot(\mathbf{1}_{A_m(m^2s)} - \nu_{m,m^2s}) - \varepsilon m^{-1}\sum\nolimits_{\frac{1}{m}\mathbb{Z}^d}\psi_{(m)}^{\varepsilon}\cdot(\mathbf{1}_{A_m(m^2\varepsilon)} - \nu_{m,m^2\varepsilon})\\
&\quad - m^{-3}\sum_{\ell=m^2\varepsilon}^{m^2 s}\sum\nolimits_{\frac{1}{m}\mathbb{Z}^2}\psi_{(m)}^{\ell/m^2}\cdot(\mathbf{1}_{A_{m,\ell}} - \nu_{m,\ell})\\
&= sm^{-1}\sum\nolimits_{\frac{1}{m}\mathbb{Z}^d} u\cdot(\mathbf{1}_{A_m(m^2s)} - \nu_{m,m^2s}) - \varepsilon m^{-1}\sum\nolimits_{\frac{1}{m}\mathbb{Z}^d} u\cdot(\mathbf{1}_{A_m(m^2\varepsilon)} - \nu_{m,m^2\varepsilon})\\
&\quad - m^{-3}\sum_{\ell=m^2\varepsilon}^{m^2 s}\sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} u\cdot(\mathbf{1}_{A_{m,\ell}} - \nu_{m,\ell}) + O(m^{-1/5})\\
&= m^{-3}\sum_{\ell=m^2\varepsilon}^{m^2 s}\sum\nolimits_{\frac{1}{m}\mathbb{Z}^d} u\cdot(\mathbf{1}_{A_m(\ell)} - \mathbf{1}_{A_m(\ell-1)})\\
&\quad - m^{-3}\sum_{\ell=m^2\varepsilon}^{m^2 s}\sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} u\cdot(\nu_{m,\ell} - \nu_{m,\ell-1}) + O(m^{-1/5})\\
&= (L_m^{(\varepsilon,s)}, u) + O(m^{-1/5}),
\end{aligned}$$

using Lemma B.3 in place of Lemma 2.8 to bound $|u - \psi_{(m)}^{\ell/m^2}|$. Now, since $\operatorname{supp} u \subset D_s$, we know that $\psi^s \equiv 0$, and we can reduce

$$M_m^\varepsilon(t) = -\varepsilon m^{-1} \sum_{j=0}^{t \wedge m^2\varepsilon} (\psi_{(m)}^\varepsilon(A_m(j) \setminus A_m(j-1)) - \psi_{(m)}^\varepsilon(z_{m,j})) \\ - m^{-3} \sum_{\ell=m^2\varepsilon}^{m^2 s} \sum_{j=0}^{\ell \wedge t} (\psi_{(m)}^{\ell/m^2}(A_m(j) \setminus A_m(j-1)) - \psi_{(m)}^{\ell/m^2}(z_{m,j})).$$

Finally, following the logic of Step 2 of the proof of Lemma 6 shows that the quadratic variation of $M_m^\varepsilon(m^2 s)$ converges to the expression (13), which completes the proof. □

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