Problem Solving in Engineering with Multiple Solution Methods

by

Hao Li

B.S., Rice University (2012)
S.M., Massachusetts Institute of Technology (2015)

Submitted to the Department of Mechanical Engineering
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mechanical Engineering

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2023

Authored by: Hao Li
Department of Mechanical Engineering
August 24, 2023

Certified by: Anette E. Hosoi
Pappalardo Professor of Mechanical Engineering
Thesis Supervisor

Accepted by: Nicolas G. Hadjiconstantinou
Professor of Mechanical Engineering
Graduate Officer, Department of Mechanical Engineering

Problem Solving in Engineering with Multiple Solution Methods

by

Hao Li

Submitted to the Department of Mechanical Engineering
on August 24, 2023, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy in Mechanical Engineering

Abstract

One of the key challenges of Engineering Education is developing students' ability to navigate and solve problems that have multiple solution paths. In order to accomplish this, the process of solving these moderately- and ill-structured problems needs to be better understood. We used two approaches to achieve this.

First, we performed problem solving experiments with students (two preliminary studies and a main study). One preliminary study found that expert problem solvers tended to start solving problems with simpler methods compared to novices. The other preliminary study found that students using reasoning and intuition had better outcomes than students who "dived in" to detailed analysis. The main experiment was conducted to illustrate the possibilities of student problem solving activity in a more open-ended way. Here, the subject population consisted of 72 undergraduate and graduate students recruited from the author's institution. The participants were given a problem with a well-defined goal but no well-defined method. After attempting to solve the problem, the participant was given a short questionnaire. The results were coded to extract the method used and the approximate time used for each method. Student performance was compared against school year, the choice of method, and the number of methods used. No significant differences in performance were found between students in different years. However, it was found that students who either 1) used simpler methods (methods with lower solve time) or 2) used more than one method tended to perform better than average, though the results are not statistically significant. Additionally, survey results were analyzed to understand the reasons for students' method choices.

Second, we built a mathematical model to describe the behavior of a problem solver with multiple methods at their disposal. Each solution method was modeled with a fixed solve time, and the problem solver may switch between methods. We start with a basic model with two solution methods, and additional complexities are successively added. Next, we present two versions of the model: using Markov and Poisson processes to describe the method transition behavior. Two optimization problems are presented: one whose objective is to maximize the solve probability given a time limit, and one whose objective is to minimize the average solve time to achieve problem solving success. We give analytic solutions for the solve probability and average solve time for the case with two methods. We also present conditions for which switching methods is beneficial. It was found that whenever there existed sufficiently short methods for solving a problem, using multiple methods (i.e. switching methods) can improve the problem solving outcome. The model and experiment are then matched, and the results are used to develop a framework of strategies for teaching students to solve problems with multiple solution methods.

Thesis supervisor: Anette E. Hosoi
Title: Pappalardo Professor of Mechanical Engineering

Acknowledgments

First, I would like to thank my advisor Peko for her energy and optimism in guiding me through several research projects during my time at MIT. Her patience and dedication were especially important during several difficult periods during the PhD process. Peko's ability to think laterally and combine different subject areas was essential for an interdisciplinary project such as this one.

Next, I would like to thank my committee members Dan and Warren. They have given me encouragement and actionable feedback during our relatively numerous committee meetings and paper feedback cycles.

I would also like to thank the professors and students I worked with during the numerous times I was a TA. They gave me energy and motivation to continue my work. Special thanks to the 200+ research participants who took part in educational studies, without whom this thesis would not have been possible.

Additionally I would like to thank the HML lab members of several generations. They made our workspace a relaxed, friendly area where I could talk openly about research, life, or other interesting things. Shoutout to past and present members of our diverse research group, TeamPeko, which includes IDSS, Course 6, and students in other disciplines, for the cross-pollination of research ideas. Thank you to Susan and Christina, who administered our research group, and allowed me to meet with Peko when it was needed.

Several student groups deserve a mention here. I would like to thank the Frisbros and Fris-broettes who I got to know while playing Ultimate here. They made the early years of my PhD enjoyable and dynamic. I would also like to thank the members of the MIT Rocket Team, as they were important during a slow time in my PhD when I was exploring research topics. I would like to acknowledge the Korean language teachers, classmates, and friends who made learning the language a fruitful experience. And a special shoutout to my living community Sidney-Pacific for providing a welcoming home for my time here.

I would like to thank my close friends Mohammad, Sandro, Arko, and Cody. They kept me sane during the COVID-19 pandemic during our many Zoom chats. Through activities such as games, running, and hiking, we kept each other engaged.

Finally, I would like to thank my family for their support throughout the PhD process.

THIS PAGE INTENTIONALLY LEFT BLANK

Contents

1 Introduction **23**

1.1 Previous Work: Problem Solving 23

1.1.1 Stages of Problem Solving 24

1.1.2 Types of Representations 26

1.1.3 Open and Closed Problems 27

1.1.4 Role of Estimation in Problem Solving 28

1.2 Previous Work: Related Areas 29

1.2.1 Early Stage Design 29

1.2.2 Connections to Theoretical and Conceptual Frameworks . . . 29

2 Preliminary Studies **33**

2.1 Differences between Experts and Novices in Starting a Problem . . . 34

2.1.1 Participant Description and Experimental Method 34

2.1.2 Problems 35

2.1.3 Analysis 38

2.1.4 Results 38

2.1.5 Discussion 41

2.2 Relationship between Starting Approach and Solution Outcome . . . 42

2.2.1 Participant Description and Experimental Method 42

2.2.2 Problem Given to Participants 43

2.2.3 Results and Analysis 44
2.2.4 Discussion 46

3 Main Experiment **47**
3.1 Recruitment of Participants 47
3.2 Procedure 48
3.3 Experimental Results 48
3.3.1 Failure Mode 50
3.3.2 Student Year 51
3.3.3 Type of Method 52
3.3.4 Number of Methods 53
3.4 Data Analysis and Discussion 54
3.4.1 Sankey Diagrams 54
3.4.2 Choice of Starting Method and Ending Method 54
3.4.3 Starting Method and Problem Solving Outcome 58
3.4.4 Patterns of Transitions 58
3.4.5 Survey Results 59

4 Analytic Model **63**
4.1 Assumptions 63
4.2 Model with Two Solution Methods 65
4.3 Additional Elements of the Model 67
4.3.1 Effect of Starting Method 67
4.3.2 Generalization to n Solution Methods 70
4.3.3 Partial Solve Probabilities 72
4.3.4 Improving Transitions between Solution Methods 73
4.4 Matching Analytic Model with Experiment 75
4.4.1 Monte Carlo Sensitivity Analysis 76

4.4.2 Analysis of Students who Used Multiple Methods 76

5 Analytic Solutions and Proofs **79**

5.1 Formulations 80

5.1.1 Discrete (Markov) Formulation of the Primal Problem 80

5.1.2 Discrete (Markov) Formulation of the Dual Problem 80

5.1.3 Continuous (Poisson) Formulation of the Primal Problem . . . 81

5.1.4 Continuous (Poisson) Formulation of the Dual Problem 81

5.2 Optimality Criteria 81

5.2.1 Discrete Primal Problem 81

5.2.2 Discrete Dual Problem 88

5.2.3 Continuous Primal Problem 89

5.2.4 Continuous Dual Problem 93

5.3 Analytic Solutions for P_{solve} and t_{solve} for $n = 2$ Solution Methods . . 94

5.3.1 Discrete Primal Problem $P_{solve}(\alpha)$ 95

5.3.2 Discrete Dual Problem $t_{solve}(\alpha)$ 96

5.3.3 Continuous Primal Problem $P_{solve}(\lambda)$ 96

5.3.4 Continuous Dual Problem $t_{solve}(\lambda)$ 102

5.4 Remarks 109

5.5 Applications 110

6 Conclusions **113**

6.1 Recommendations for Improving Problem Solving Outcome 113

6.1.1 Improving Tools 114

6.1.2 Improving Strategies 115

6.1.3 Order of Deployment 118

6.2 Future Work 121

A Questionnaire Given after the Volume Problem **123**

B Example of P_{solve} Calculation **125**

C Optimality Condition for P_{solve} with $n = 2$ Solution Methods (Markov Formulation of Primal Problem) **129**

D Proof of Equation 5.3 (Optimality Condition for Markov Formulation of Dual Problem) **133**

E Proof of Equation 5.7 (Optimality Condition for Poisson Formulation of Dual Problem) **137**

F Proof of Proposition 5 (Analytic Solution for $P_{solve}(\alpha)$) **141**

G Proof of Proposition 6 (Analytic Solution for $t_{solve}(\alpha)$) **147**

List of Figures

1-1 Research space summary, depicting the areas that are relevant to this work. The category Stage of Problem Solving is adapted from Wankat and Oreovicz [25]. The category Type of Representation is adapted from a previous paper by Li and Hosoi [13]. The category Openness of Problems was posed by Bahar and Maker [4]. The category Accuracy of Solution is derived from work by Linder [15] and Shakerin [20]. . . 24

1-2 The cycle of self-regulated learning. If this cycle is applied to problem solving, the problem solver would proceed through three stages: they plan their solution, implement their plan, and reflect on their performance. The feedback from one attempt can inform their next attempt (possibly using a different method). Image from [1]. 30

2-1 Problem 1 graph 35

2-2 Problem 1 data table 36

2-3 Problem 2 circle and ellipse 36

2-4 Problem 3 robot and model 37

2-5 Problem 3 rigid model relationship 37

2-6 Problem 1 responses by technique 39

2-7 Problem 2 responses by technique 40

2-8 Problem 3 responses by by whether Free Body Diagram (FBD) technique was drawn 40

2-9 Cantilevered beam to support traffic lights 43

3-1 Volume Problem used in the main experiment. Figure reproduced from previous work by Li and Hosoi [14]. 49

3-2 Example of student work. The participant's pen color was switched every two minutes so that work can be identified within time intervals. Figure reproduced from previous work by Li and Hosoi [14]. 50

3-3 Sankey diagram for the Volume Problem, 10 minute section only. Darker transitions represent a larger fraction of participants with the correct answer. The left nodes represent the method used at the start of the 10 minute section, and the right nodes represent the method used at the end of the 10 minute section. There is one intermediate node, corresponding to one participant who used three methods in the 10 minute section. Note that the majority of students did not switch methods. . 55

3-4 Sankey diagram for the Volume Problem, 5 minute and 10 minute sections. Intermediate steps have been omitted for clarity. Darker transitions represent a larger fraction of participants with the correct answer. The left nodes represent the method used at the start of the 5 minute section, and the right nodes represent the method used at the end of the 10 minute section. If a participant only used one method in both the 5 and 10 minute sections, the node would appear once on the left and once on the right. Note that the last method (10 minute section) was a better predictor of problem solving success than the first method (5 minute section). The right nodes for Visual Estimation, Geometric Approximation, CAD, and Monte Carlo tended to attract transitions with higher fraction correctly solved than Integral or Other. There was no clear pattern for transitions originating from the left nodes. 56

4-1 P_{solve} with two solution methods and time limit $t_f = 10$ and $t_2 > t_f$. Note that for $t_1 = 2, 3, 4, 5$ there exists an $\alpha > 0$ for which P_{solve} is maximized. In other words, a strategy of not switching ($\alpha = 0$) is not optimal for maximizing solve probability; some switching is beneficial. Values of t_1 equal to 6 or greater do not benefit from switching at all. If the problem solver is aware that they do not have a simple way to solve the problem (i.e. t_1 is 6 or greater), they should consider improving their skills, i.e., reducing their t_1. Figure reproduced from previous work by Li and Hosoi [14]. 68

4-2 Heatmap showing the dependence of P_{solve} on t_1 and α with $t_f = 10$ and $t_2 > t_f$. Note that as t_1 increases, the maximum P_{solve} decreases, and it occurs at a smaller switching tendency α. This shows the benefit of improving both skills (i.e. reducing the solve time t_1) and strategy (i.e. choosing the correct α for the problem solver's value of t_1). . . . 68

4-3 An example P_{solve} showing the effect of starting method probability. There are two solution methods, the time limit is $t_f = 10$, method 1 has solve time $t_1 = 5$, and method 2 has $t_2 > t_f$. Note that for curves where $\beta_1 \leq 0.5$, a switching strategy is beneficial. However, for curves where $\beta_1 > 0.5$, there is no benefit from switching, and $\alpha = 0$ gives the highest solve probability. In other words, the more likely the problem solver starts on the shorter method, the less need they have for a switching strategy. Figure reproduced from previous work by Li and Hosoi [14]. 69

4-4 (a) An example of P_{solve} with six solution methods, equal starting probabilities, and time limit $t_f = 10$. Four of the solution methods have solve times less than 10 timesteps; their solve times are 2, 4, 4, and 6 timesteps, respectively. The partial solution probabilities p_i are equal to one for all methods. Two methods have solve times longer than 10 timesteps. This is a situation that benefits from a switching strategy, i.e. P_{solve} is maximized from some $\alpha > 0$. (b) An example of P_{solve} where the partial solution probabilities can be between 0 and 1. The result of the analytic model (solid black line) is overlaid with a Monte Carlo simulation (gray line). There are six solution methods with solve times 2, 3, 3, 6, 6, and 6 timesteps. The partial solution probabilities are 0.8, 0.45, 0.8, 0.5, 0.21, and 0, respectively. The starting probabilities are equal, and the time limit is $t_f = 10$ timesteps. Note that methods with lower solve times are associated with higher partial solution probabilities, potentially increasing the benefits of a switching strategy. 73

4-5 Matching model and data, with Monte Carlo sensitivity analysis. Starting method probabilities β_i of the model were set by the fraction of students starting on the corresponding method in the experiment's 10 minute section. Partial solve probabilities p_i of the model were set by the fraction of students successfully using a given method. In the model, transitions to only the first two methods (Visual Estimation and Geometric Approximation) were allowed, unless the first method is "other," in which case transitions to all five other methods were allowed. Monte Carlo simulation regions are also shown, which were generated by overlaying the $P_{solve}(\alpha)$ curves generated from varying the model parameters t_i and p_i. The four regions, corresponding to the different values of t_1 and p_1, are clearly distinct. These regions were generated by overlaying all Monte Carlo P_{solve} curves with the corresponding t_1 and p_1. Here, t_1 and p_1 represent the characteristics of the first method, Visual Estimation; a low solve time t_1 and a high solve probability p_1 led to the largest P_{solve}. 77

5-1 Example of probabilities of sequences for $t_f = 6$ with 2 methods. The right subplot is a zoomed in version of the left subplot, with the $B = 0$ curve omitted. Note that $P'_{\mathcal{S}}(0) \neq 0$ for only $B = 0, 1$ transitions. . . 86

5-2 Pathological case for three solution methods where $P'_{solve}(0) < 0$ but the maximum occurs at $\alpha \approx 0.376$ and $P_{solve} \approx 0.686$ instead of $\alpha = 0$. For reference, $P_{solve} = \frac{2}{3}$ when $\alpha = 0$. The three methods have solve times 2, 10, and 11, while the time limit is $t_f = 10$. The P_{solve}-axis does not start at zero in order to clearly display the maximum. . . . 87

5-3 Visualization of the Primal Problem $P_{solve} = \frac{V_r}{V_0}$ for cases with $B = 2$ and $B = 3$ transitions. For each case, the problem solver started on method 1 (the shorter method). Light shaded areas represent regions where the problem is solved because one inequality is satisfied. Dark shaded areas represent regions where the problem is solved, but two inequalities are satisfied. We are interested in the union of all regions where at least one inequality is satisfied. Therefore, when computing the total P_{solve}, the dark shaded areas must be accounted for using the Principle of Inclusion-Exclusion. 102

5-4 Optimum location for two solution methods for the Poisson Primal formulation as a function of the nondimensionalized solve time of the shorter method, t_1^*. Optimal λ (left) is the switching tendency λ_{opt} that maximizes P_{solve} for $t_f = 10$. P_{solve} improvement (right) is the improvement in solve probability over the no-switch case, $P_{solve}(\lambda_{opt}) - P_{solve}(0)$. 103

5-5 Decision tree for the Poisson Dual Problem with two solution methods. The problem solver will start on either method with probability $P = \frac{1}{2}$. If the problem solver solves the problem with method i, they will do so with probability $P = e^{-\lambda t_i}$. If the problem solver does not solve the problem with method i, they will do so with probability $P = 1 - e^{-\lambda t_i}$, and switch to the other method. The problem solver will continue to switch methods until they solve the problem. 104

5-6 Optimum location for two solution methods for Poison Dual formulation as a function of the ratio of the two methods' solve times, $\frac{t_1}{t_2}$. Optimal λ (left) is the switching tendency λ_{opt} that minimizes t_{solve} for $t_2 = 10$. Fraction t_{solve} reduction (right) is the reduction in solve time over the no-switch case, $\left|\frac{t_{solve}(\lambda_{opt}) - t_{solve}(0)}{t_{solve}(0)}\right|$. 108

6-1 Examples of improving tools. (a) Introducing more low complexity methods will improve probability of solve. The analytic model was computed for three scenarios for a problem with four possible solution methods and a time limit of 10 timesteps. In the first scenario (light gray), the problem solver could solve the problem using one low complexity method; the other solution methods cannot be used to solve within the time limit. In the second scenario (dark gray), the problem solver could solve using two low complexity methods. In the third scenario (black), the problem solver could solve with three low complexity methods. Note that as the number of usable low complexity methods increases, P_{solve} significantly increases for every value of switching tendency $\alpha \in [0, 1)$. The optimal α is similar for the three scenarios. (b) Reducing solve times can improve the solve probability, given that the student chooses the optimal switching tendency. The analytic model was computed for three scenarios for a problem with three solution methods and a time limit of 10 timesteps. Two of these three solution methods can be completed under the time limit. The scenarios show the effect of reducing solve time on the two methods that can be completed within the time limit. In the first scenario (light gray), the solve times of these two "effective" methods were close to the time limit. In the second scenario (dark gray), the solve times of the effective methods were approximately half the time limit. In the third scenario (black), the solve time of one effective method was only one fifth the time limit, while the solve time of the other was half the time limit. As the solve times of the effective methods were reduced, P_{solve} is increased for every switching strategy $\alpha \in (0, 1)$. However, the benefit for switching was significant for only the third scenario. 116

6-2 Examples of improving strategies. (a) Switching methods with optimal tendency can improve P_{solve}. The analytic model was computed for a problem with four solution methods and a time limit $t_f = 10$. Two of these four solution methods can be completed within t_f. Compared to the no switch solve probability $P_{\alpha=0}$, the maximum solve probability P_{max} at the optimal switching tendency α_{opt} is significantly higher. (b) Choosing a better starting method will improve P_{solve}, especially when the switching tendency α is low. The model was computed for three scenarios with $t_f = 10$ and three solution methods (solve times 2, 5, and 12 timesteps). In the first scenario (light gray), the starting probabilities were weighted towards the longest method. In the second scenario (dark gray), the starting probabilities were weighted equally. In the third scenario (black), the starting probabilities were weighted towards the shortest method. Note that as the weight is shifted towards the shorter methods, P_{solve} is improved for every $\alpha \in [0, 1)$. However, the effects are greatest for small α. Also, the optimal α decreases as starting method is improved. (c) Judiciously switching methods can improve P_{solve}. Improvement occurs when the student chooses to switch from more complex to less complex methods. The model was computed for three scenarios for a problem with four solution methods and $t_f = 10$. Two of these four solution methods can be completed within t_f. In the first scenario (light gray), random transitions were used. In the second scenario (dark gray), the problem solver avoided the most complex method when switching. In the third scenario (black), the problem solver only switched from a more complex to a simpler method. As the transition strategy is improved, P_{solve} is increased for every $\alpha \in (0, 1)$. Additionally, the better the transition strategy, the more benefit there is from switching, so the optimal α is increased. 119

6-3 Order of deployment for recommendations to improve problem solving outcomes. Note that tools and strategies are interleaved. The tool of low solve time methods is essential and prerequisite to other strategies and tools; it directly unlocks the strategy of switching and the strategy of starting on a low complexity method. Subsequently, the strategy of switching unlocks the optimization of switching strategy and the optimization of solve time for already known methods. 120

B-1 Graph of the solve probability $P_{solve}(\alpha)$. Note that at $\alpha = 0$, P_{solve} is $\frac{1}{2}$. This corresponds to the no-switch case, where the problem solver stays on the first method chosen. As α is increased, P_{solve} reaches a maximum of $\frac{9}{16}$ at $\alpha = \frac{1}{4}$. Further increasing α will lead to a decrease in P_{solve} until the solve probability reaches zero at $\alpha = 1$. 127

G-1 Decision tree for the Markov Dual Problem with two solution methods. The problem solver will start on either method with probability $P = \frac{1}{2}$. If the problem solver solves the problem with method i, they will do so with probability $P = (1-\alpha)^{t_i - 1}$. If the problem solver does not solve the problem with method i, they will do so with probability $P = 1 - (1-\alpha)^{t_i - 1}$, and switch to the other method. The problem solver will continue to switch methods until they solve the problem. . . 148

THIS PAGE INTENTIONALLY LEFT BLANK

List of Tables

2.1 Problem 1 results 39

2.2 Problem 2 results 39

2.3 Problem 3 results 39

2.4 Characteristics of students' 5 minute submission 45

2.5 p-values of characteristics vs. problem solving outcome 45

3.1 Failure mode analysis of participants. A large majority of students who finished the problem obtained a correct solution; a large majority of students who did not obtain the correct solution did not finish solving the problem. 51

3.2 Volume Problem results by school year. Clear differences between students in different years were not observed. Table reproduced from previous work by Li and Hosoi [14]. 51

3.3 Accuracy and corresponding solve times. Fraction correct is the fraction of participants who obtained a correct solution with a given method. Solve time is the time spent on the method. These methods are in the order of least complex (least average time spent) to most complex (most average time spent). Table reproduced from previous work by Li and Hosoi [14]. 53

3.4 Number of methods used vs. solution outcome. Note that a higher fraction correct was associated with more solution methods. Table reproduced from previous work by Li and Hosoi [14]. 54

3.5 Survey results for participants' confidence in their solution method and answer. Most, but not all, participants responded. Responses were recorded on a five-point Likert scale (1-5). A t-test was used to calculate confidence intervals. Overall, students had a much higher confidence in their method than their answer. Students who obtained the correct answer reported similar confidence in their method as those with incorrect answers. However, students with correct answers reported much higher confidence in their answer than those with incorrect answers. 61

5.1 Possible formulations of the model. Differences between the Discrete and Continuous models, as well as differences between Primal and Dual problems, are shown. 82

5.2 Optimality criteria for the different formulations in the model. The criteria for $n = 2$ solution methods are given. References for $n \geq 2$ solution methods are included. 94

B.1 Sequence probabilities for an example for two solution methods with $t_1 = 2, t_2 = 6$, and $t_f = 4$. 126

Chapter 1

Introduction

One of the key challenges of engineering education is to develop students' ability to solve complex, open-ended problems [5, 16]. According to the literature, one way to improve problem solving skills is to teach students multiple representations or multiple solution methods [9, 26]. However, once students acquire the ability to use multiple solution methods, how should they use it? Given a problem, how should they decide which method to use first? If they get stuck on their first method, should they persevere or switch to a different method? Should instructional time be spent solely on teaching students tools, or would it be beneficial to teach them problem solving strategies as well?

The purpose of this section is to introduce the previous work necessary to understanding problem solving with multiple solution methods. The subject matter is not limited to mechanical engineering and can include other engineering, science, or mathematics fields as well.

1.1 Previous Work: Problem Solving

Previous work has deconstructed problem solving in several ways. These are explained below and summarized in Figure 1-1. These four deconstructions categorize different aspects of problem solving and serve as a foundation for understanding the focus of this work: multiple solution methods.

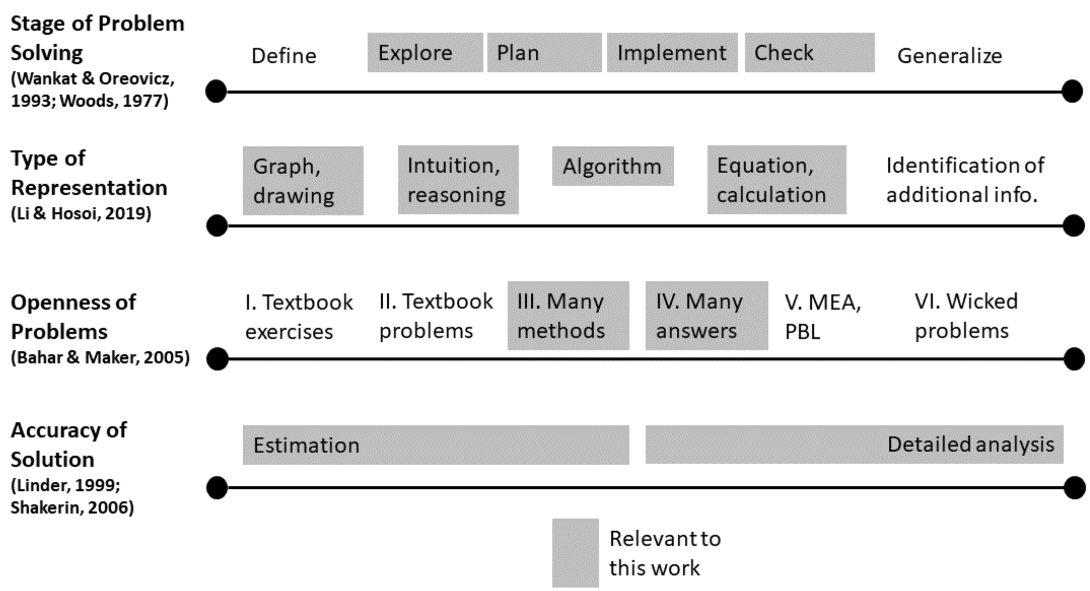


Figure 1-1: Research space summary, depicting the areas that are relevant to this work. The category Stage of Problem Solving is adapted from Wankat and Oreovicz [25]. The category Type of Representation is adapted from a previous paper by Li and Hosoi [13]. The category Openness of Problems was posed by Bahar and Maker [4]. The category Accuracy of Solution is derived from work by Linder [15] and Shakerin [20].

1.1.1 Stages of Problem Solving

Previous work has defined the different stages of problem solving, namely the steps that a problem solver must carry out in order to successfully arrive at a solution. Various formulations of the stages of problem solving were collected by Woods [26]. Among these are Polya's four stages of problem solving: Define, Plan, Carry out the Plan, and Look back [17]. Another formulation, adapted from Wankat and Oreovicz [25], is shown in Figure 1-1. Schoenfeld developed timing diagrams to display students' problem solving stages over time [19], and Kohl applied these diagrams in the context of experiments in physics problem solving [11]. Previous efforts by Li and

Hosoi sought to understand the impact of the early decisions students make in problem solving [12, 13]. In these papers, there was a need to clearly define the start of problem solving. Within the context of the stages of problem solving (as formulated by Polya, Wankat and Oreovicz, Schoenfeld, and Kohl), Li and Hosoi divided student work into "How would you start solving this problem?," which corresponds roughly to the "Define" and "Plan" stages in Polya's breakdown, and "Solve as much of the problem as you can," which corresponds roughly to the "Carry out the Plan," and "Look back" stages. The stage of problem solving can affect what solution method a student may take. The focus of this work is on "Solve as much of the problem as you can," the portion of problem solving in which the student has already decided which method they will attempt. Note that during the defining and exploring stages, it is possible that a student has not yet decided on a solution method. However, during the planning and execution stages, students are likely to know which method they will use. Furthermore, it is possible that a solution method may fail, and the student will need to start over, thereby transitioning from executing the solution to additional exploration and planning. When the student has arrived at an answer, they may decide to double check using a different solution method, thereby briefly repeating the stages of problem solving. These complexities are inherent in the use of multiple solution methods.

1.1.1.1 Start of Problem Solving

A student's ability to start solving a problem is important for several reasons. Training students in problem solving skills helps students gain confidence in their abilities [3, 18]. Ancel found that taking a problem solving course improved the self-efficacy beliefs of nursing students [3]; Psycharis and Kallia found that taking a programming course improved the mathematical self-efficacy of high school students [18]. Furthermore, how a student starts a problem may have significant effect on their ability to fully carry out the solution. A student may solve the wrong problem, spend all their time pursuing the wrong approach, or forget their goal if they do not properly define the problem and formulate a plan [17].

Additionally, the ability to start problems is beneficial to the trust between teachers and students. According to a professor in the author's department, if students cannot start problems, they often believe that they are unfairly treated in a class [8]. A large difference in the expectation of difficulty between students and teachers may call into question a teacher's competence and integrity, necessary components of teacher-student trust according to a review performed by Trahan [23]. Thus, the instructor may need to facilitate this process by getting students "unstuck." One way to do this is to suggest initial approaches such as exploring problem specifications or trying a simple case.

1.1.2 Types of Representations

There is also prior work that defines types of representations, i.e. the various formats that problem solving activity can take. These "formats" are broad categories defined by the qualitative nature of the problem solver's solution steps. For example, Kohl's work on physics problem solving used four types of representations: Forces, Picture, Math, and Written. In this work, Kohl found differences between experts and novices in the frequency of transitions between representations; novices tend to use more representations and switch between them more often, while experts tend to use fewer representations and solve the problem in a more straightforward way [11]. Another example of representation types, shown in Figure 1-1, was used by Li and Hosoi in a previous paper. This set of representations included Graph/drawing, Intuition/reasoning, Algorithm, Equation/calculation, and Identification of additional information [13]. In a separate work, Van Huevelen and Zou used multiple representations to teach problem solving in work-energy processes. Their analysis showed that a significant majority of students found multiple representations useful, especially when solving unfamiliar problems [24]. In this work, we are interested in students' use of multiple solution methods, which can be interpreted as analogous to multiple representations. It is possible to map solution methods to different representations, though the mapping may not be straightforward. One solution method might correspond one-to-one with one type of representation, or one solution method may include multiple representations, or multiple solution methods may correspond to one type of representation. The nature of a solution method can determine which representations are involved. For example, a simple solution method, such as estimation, may only need a few lines of calculations. However, a more complex solution method, such as integration, may involve a figure, several sentences to explain the reasoning, multiple lines of equations, and potentially some computer code as well. The need to manage several representations can increase the time requirements of more complex solution methods, making them difficult to execute within a time limit. When there are constraints to problem solving, it is important for the problem solver to be aware of the tradeoffs inherent in using solution methods with more representational requirements.

1.1.3 Open and Closed Problems

The literature has also categorized the degree of openness of problems. These range from completely closed ("well-structured"), where both the problem and the solution method are prescribed, to completely open-ended ("ill-structured"), where not even the problem is known to the instructor [4, 7, 10]. In engineering education, examples of closed problems include textbook exercises, and examples of open-ended problems include Problem-Based Learning (PBL) [16] and Model-Eliciting Activities (MEA) [5]. The continuum of open and closed problems is shown in Figure 1-1. Of special interest to this work are problems with intermediate openness, where there are multiple possible solution methods to achieve a well-defined goal. These types of problems are valuable because the experience learned by solving them can be transferred to new situations [7]. Furthermore, the existence of a well-defined objective makes the analysis of problem solving activity potentially tractable, yet the possibility of multiple methods gives ample opportunities to find optimal pathways.

1.1.4 Role of Estimation in Problem Solving

Previous literature has also analyzed the role of estimation in engineering education. When solving open-ended problems, the problem solver is often faced with a range of approaches: from simple estimation to detailed analysis leading to an exact solution. There is a need to choose a method that is appropriate given the desired precision and time constraints. Previous work has assessed students' performance on one end of the range: simple estimation. Linder found that students often had difficulty making basic estimates [15]. Shakerin noted that engineering classes overwhelmingly emphasized detailed analysis over estimation and suggested several activities to help students improve estimation skills [20]. Smith observed that students were unwilling to make rough estimates before and after performing Finite Element Analysis, often trusting the computer simulations without reservation [21]. Furthermore, these deficiencies in estimation ability were observed from undergraduate seniors [15, 21]. Given the lack of emphasis on estimation in the curriculum and the current state of students' estimation ability, we pose the question: Would a student choose estimation, detailed analysis, or a method of intermediate complexity if given the freedom to choose among methods of different complexity? Understanding this may inform recommendations for using multiple solution methods.

1.2 Previous Work: Related Areas

1.2.1 Early Stage Design

In analogous work in design education, Yang analyzed the role of sketching [27] and prototyping in design outcomes [28]. Sketches and prototypes are two types of representations engineers might use when solving design problems, where the expected result can be a physical product or object instead of a mathematical solution. These two activities also represent the early stages of solving design problems. Because design inherently allows for multiple solution methods, these activities can portray a wide range of approaches. Sketching is used for concept generation [27] and can be linked to the exploration and planning stages of problem solving. Prototyping requires a higher level of effort and involvement and can be used for transitioning to product production [28]; it can be linked to not only the exploration and planning stages but also the implementation stage of problem solving.

It is possible to map the results of design education research to those of problem solving research. For example, Yang found that simpler prototypes were associated with better design outcome [28]. Additionally, since design problems are relatively open, or ill-structured, compared to classroom problems, a better solution can be found by exploring a larger design space. Estimation or simple analysis allows the designer to explore a larger design space, potentially improving design outcome [6]. This work addresses whether simple solution methods, such as estimation, lead to better outcomes on moderately-structured problems (which may not necessarily be design problems).

1.2.2 Connections to Theoretical and Conceptual Frameworks

Existing theoretical and conceptual frameworks can provide a basis for understanding how students solve moderately- or ill-structured problems with multiple solution paths. The framework of self-regulated learning can be applied to problem solving. In self-regulated learning, the problem solver (or learner) first plans, sets goals, and lays out strategies. Then, they implement these strategies. Finally, the problem solver reflects on their performance [1, 29]. For ill-structured problems where the solution path is not immediately obvious, the systematic approach of self-regulated learning can help students navigate the possible difficulties and dead ends. If a solution method does not work out, the problem solver can reflect on this and try a different approach. This process is depicted in Figure 1-2.

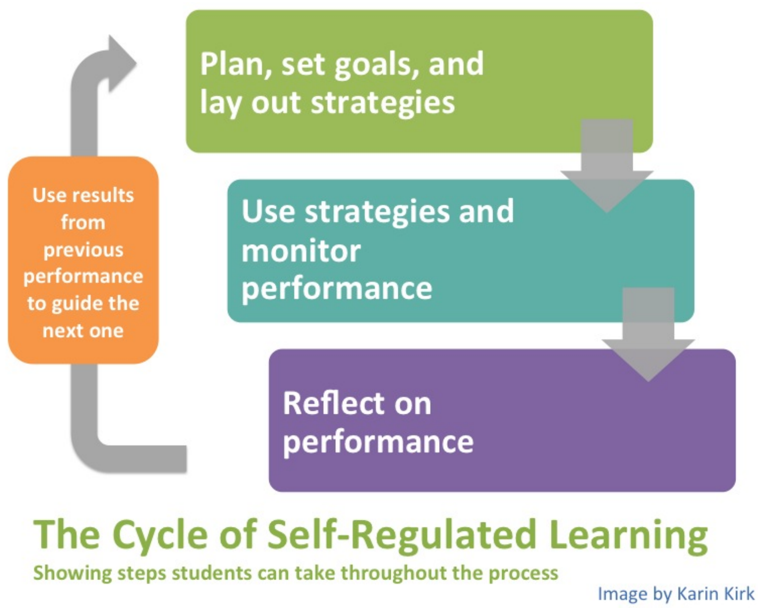


Figure 1-2: The cycle of self-regulated learning. If this cycle is applied to problem solving, the problem solver would proceed through three stages: they plan their solution, implement their plan, and reflect on their performance. The feedback from one attempt can inform their next attempt (possibly using a different method). Image from [1].

The Model of Domain Learning is another conceptual framework that can be applied to problem solving. The goal is to understand how novices build expertise and become experts [2, 22]. In this framework, the learner progress through three stages. In the first stage, Acclimation, the learner has little knowledge of a field, and this knowledge is unstructured. For example, a novice problem solver may use the first approach that comes to mind when solving a problem, never changing approach if their attempt fails. In the second stage, Competence, the learner has begun to understand the key principles of the field and can accomplish basic tasks easily. For example, an intermediate problem solver may attempt to solve a problem using a standard approach, get stuck, and then switch to a simpler approach to obtain an answer. In the third stage, Proficiency, the learner has accumulated large stores of organized knowledge and can efficiently accomplish a wide variety of tasks. For example, an expert problem solver may use a back-of-the-envelope calculation to first estimate the solution to a complex problem before investing time in a more precise method.

THIS PAGE INTENTIONALLY LEFT BLANK

Chapter 2

Preliminary Studies

The purpose of this section is to document two studies that explore different ways to analyze the start of a problem. Both studies feature problems that can be solved in multiple ways. The first study (Section 2.1) investigates how problem solvers start problems. This study seeks to answer the following research question:

RQ1: How do problem solvers of different expertise levels start solving problems?

The second study (Section 2.2) investigates the types of representations students use to solve problems. This study seeks to answer the following research question:

RQ2: Are there initial approaches to starting a problem that are more likely to get the problem solved?

The contents of both studies are based on previous publications by Li and Hosoi [12, 13].

2.1 Differences between Experts and Novices in Starting a Problem

2.1.1 Participant Description and Experimental Method

The subject population consisted of undergraduates, graduate students, and professors in the Mechanical Engineering department of the author's institution. Undergraduates were chosen from the approximately 55 enrollees of Numerical Computation for Mechanical Engineers, a second-year course in Mechanical Engineering. Graduate students were the attendees of a seminar hosted by the Graduate Association of Mechanical Engineers, and professors were chosen by appointment. All participants consented to the study.

The participants in this study were given three problems and asked "what is the first thing you would try to start this problem?" Participants were not required to solve the entire problem. Problems 1 and 2 had a five-minute time limit, and problem 3 had a ten-minute time limit. Undergraduates were given the problems during the recitation period attached to the Numerical Computation class. Graduate students were given the problems during a seminar. Faculty were given the problems by appointment.

Problem 1 provided participants with a table and a graph of data collected from a prosthetic testing context. Participants were then asked to estimate the value of a variable in between data points. Techniques suggested by the participants include reading the graph (least complex), interpolation, and curve fitting (most complex). Responses that did not fit in any of these three categories were classified as "other."

Problem 2 provided participants with equations of a circle and an ellipse and their graphs. Participants were then asked to find the area lying inside both curves. Techniques suggested by the participants included visual estimation (least complex), Monte Carlo, and integration (most complex). Responses that did not fit in any of these three categories were classified as "other."

Problem 3 provided participants with a simple, rigid-body model of a robot leg. The relationship for hip torque and leg angle was also given. Participants were asked to consider a situation in which the leg can deform. They were then asked to find the hip torque at a given angle for this deformable model. Techniques suggested by the participants included drawing a free body diagram (FBD). Responses were separated by whether an FBD was drawn. If a FBD was not drawn, responses were binned by whether progress was made.

2.1.2 Problems

2.1.2.1 Problem 1

A PhD student and her advisor published the graph below (Figure 2-1) of the physiological gait cycle describing position and orientation of an ankle. Let's say that the table below (Figure 2-2) represents the data on the Physiological gait (not the Model). Estimate the y-position of the ankle at 34% of the gait cycle.

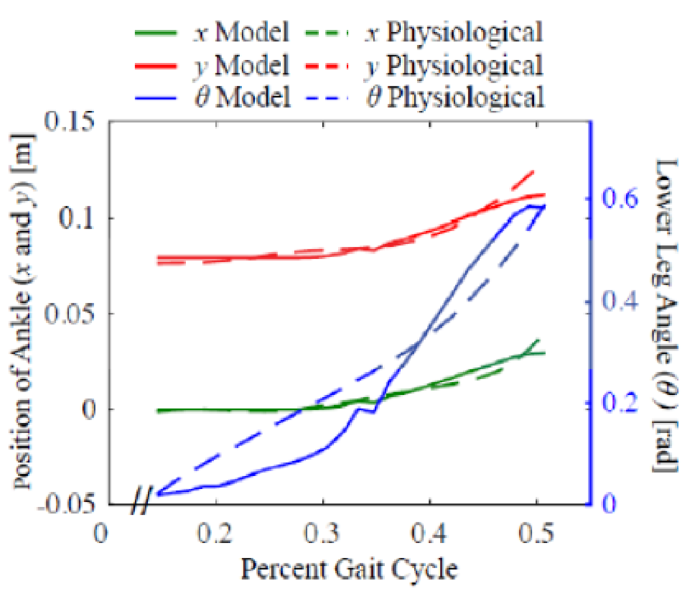


Figure 2-1: Problem 1 graph

% gait cycle	*x (mm)*	*y (m)*	*Θ (rad)*
0.15	0.0	0.080	0.06
0.2	0.0	0.082	0.11
0.3	0.1	0.085	0.21
0.4	4.1	0.089	0.37
0.45	11.0	0.098	0.43
0.5	26.2	0.115	0.59

Figure 2-2: Problem 1 data table

What is the first thing you can try to start the problem? "I don't know" is a valid response.

2.1.2.2 Problem 2

We are interested in the area A that lies inside both the circle of radius 2, centered at (1, 0.5) and the ellipse with equation $\frac{x^2}{25} + \frac{y^2}{4} = 1$. The curves are shown in Figure 2-3.

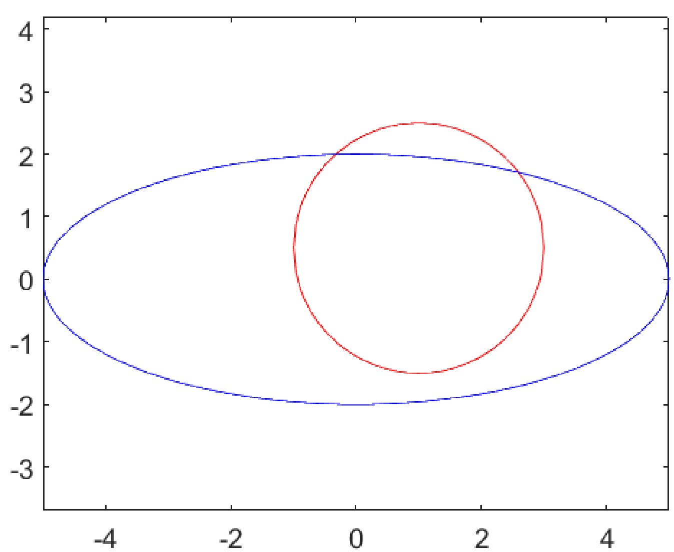


Figure 2-3: Problem 2 circle and ellipse

List all the different things you might try to find this area A. What is the first thing you would try? "I don't know" is a valid answer.

2.1.2.3 Problem 3

A leg for a walking robot is being tested (Figure 2-4a). As a rough first estimate, you model the leg as a mass m attached to one end of a rigid stick with length L (Figure 2-4b). The assembly is standing up at an angle β to the ground because a "hip" torque τ is applied on the end with the mass. The stick does not slide on the ground, but its contact point with the ground can rotate.

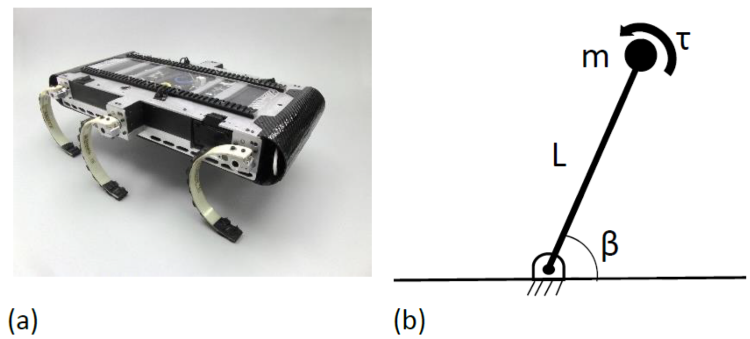


Figure 2-4: Problem 3 robot and model

Modeling the leg as a rigid stick, you find the following relationship between the applied torque τ and the equilibrium angle β (Figure 2-5).

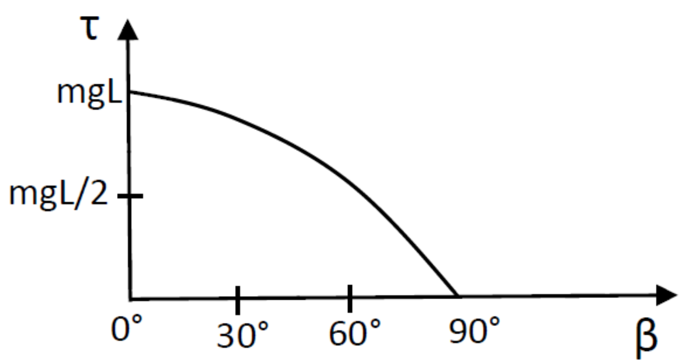


Figure 2-5: Problem 3 rigid model relationship

This is not a bad estimate for rigid legs, but it may be inaccurate if the legs are deformable (as in Figure 2-4a). If, instead of a rigid leg, the leg is a **compression spring** with spring constant k, **what is the first thing you would do to estimate the torque required to hold the leg at** $\beta = 60°$ **and why?** Please consider two cases: (1) $\frac{mg}{kL} \ll 1$ and (2) $\frac{mg}{kL} \approx 1$.

2.1.3 Analysis

The characteristics of how the participants started the problems were studied. Characteristics include (1) techniques used: drawing a free body diagram, finding equations of motion, reading a graph, and suggesting mathematical or computational methods and (2) progress made or result obtained: answer obtained, correct equation of motion stated, or effective technique suggested.

For problems 1 and 2, the technique suggested by the participant was analyzed. If multiple techniques were suggested, the first one written down was used for analysis. If a student wrote “I would try this first...", then this technique was used for analysis.

For problem 3, whether the participant used a FBD was analyzed. If the participant suggested drawing a FBD or drew a FBD, the response was counted. Partially drawn FBDs were also counted. This type of binning was chosen because drawing a FBD is a canonical part of solving statics problems. Participants who did not draw FBDs were further binned into those who made progress and those who did not make progress. Progress was defined as giving a correct torque balance or using the graph for the case $\frac{mg}{kL} \ll 1$.

2.1.4 Results

The results for problems 1, 2, and 3 are tabulated below.

Table 2.1: Problem 1 results

Participants	***Graph***	***Interp.***	***Curve Fit***	***Other***	***Total***
Undergraduates	6	20	18	7	51
Graduate students	8	5	2	1	16
Faculty	4	3	0	3	10

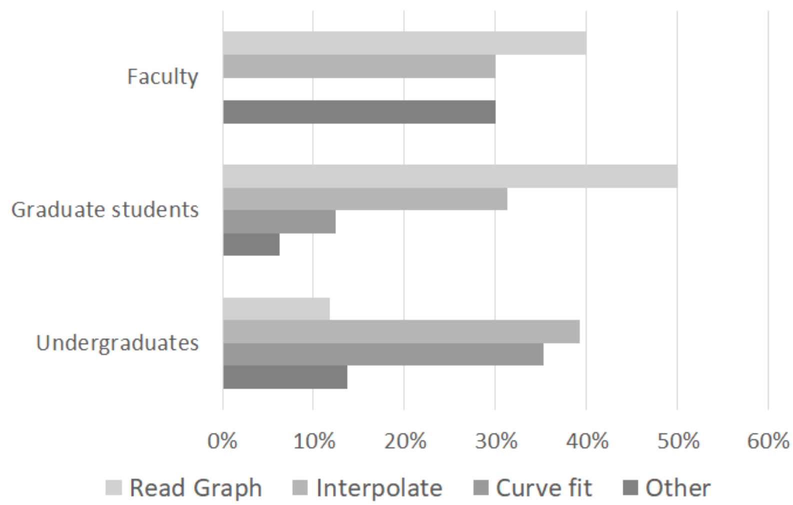


Figure 2-6: Problem 1 responses by technique

Table 2.2: Problem 2 results

Participants	***Visual***	***MC***	***Integral***	***Other***	***Total***
Undergraduates	5	13	16	5	39
Graduate students	0	0	3	1	4
Faculty	2	1	3	4	10

Table 2.3: Problem 3 results

Participants	***FBD***	***No FBD, Progress***	***No FBD, No Progress***	***Total***
Undergraduates	18	3	8	39
Graduate students	12	0	0	12
Faculty	5	4	1	10

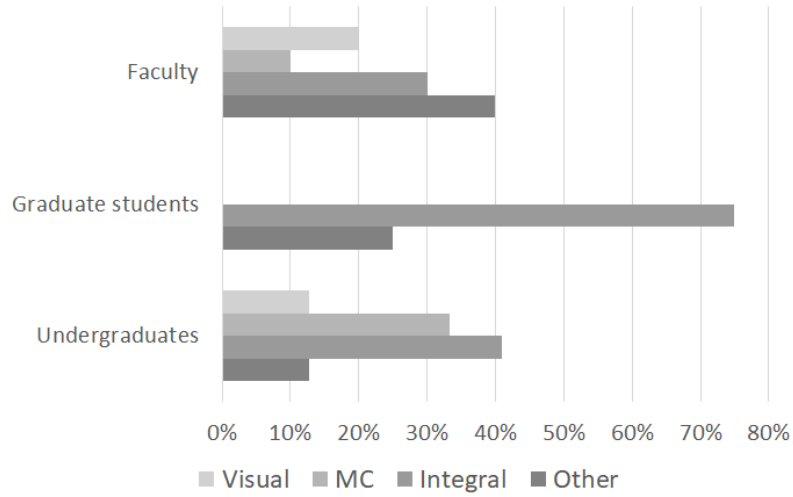


Figure 2-7: Problem 2 responses by technique

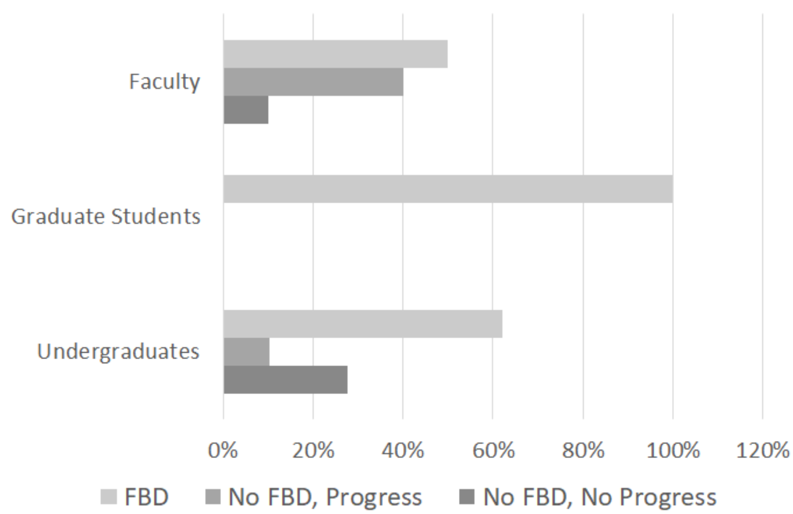


Figure 2-8: Problem 3 responses by by whether Free Body Diagram (FBD) technique was drawn

2.1.5 Discussion

A typical problem solving heuristic is to first try a simple technique to get a rough approximation, followed by a more complicated method if more accuracy is needed.

During the PhD oral qualifying exams at the author's institution, it has been said that students who ignore simple approaches and begin problems with a complex method make the graders uneasy, because it is not clear these students understand the problem [8].

The problem 1 results suggest that graduate students and faculty prefer a "faster" or "simpler" technique as the first step to solving the problem. That is, graduate students and faculty tended to suggest reading the graph instead of interpolating or doing a curve fit. More than 40% of faculty and graduate students chose to read the graph for problem 1, compared to only 12% of undergraduates. All three faculty whose responses fit into the "other" category corroborated the data and the graph as a first step. This cross-checking approach was not used by undergraduates and graduate students.

The problem 2 results show that about 35% of undergraduate students mentioned Monte Carlo (MC), none of the graduate students mentioned MC, and one of the of ten professors mentioned MC. Professors tended to avoid the most complex route as the first step. Of the professors who used an "other" method, one suggested a simple geometric approach and another suggested to cut and weigh the desired shape; both of these are relatively "simple" techniques. Seven of ten professors avoided integration, which is the most complex way to solve this problem. It should be noted that MC is a technique that is emphasized in the Numerical Computation for Mechanical Engineers class. All of the students have had some exposure to MC, and two of the ten professors surveyed were present or past instructors of this class. However, none of the graduate students have had recent experience with MC.

The problem 3 results show that compared to undergraduates, graduate students were more likely to draw a free body diagram. Five of ten faculty drew free body diagrams. Of the five faculty who did not draw a free body diagram, four were still able to make progress by giving correct approaches. Of the eleven undergraduates who did not draw free body diagrams, only three were able to make progress.

It was found that while a novice may know how to apply a relatively complex technique (e.g. doing a curve fit in problem 1), they may not be adept at choosing among multiple approaches of varying complexity. Engineers of greater experience were able to find the “simple” approaches; this ability may reflect their higher level of sophistication in applying engineering reasoning.

2.2 Relationship between Starting Approach and Solution Outcome

2.2.1 Participant Description and Experimental Method

The participants were 69 students recruited from an undergraduate Mechanics and Materials course. The participants were given 15 minutes to solve a problem. They made two submissions: one with the prompt “How would you start solving this problem?” at 5 minutes and another with the prompt “Solve as much of the problem as you can” at 15 minutes. This problem (see Section 2.2.2) was somewhat open-ended and had multiple solution paths. Participants were prompted to explore the problem space, and were discouraged from erasing any ideas or approaches they wrote down. Participants were not required to finish the problem, though it is possible to obtain an answer within 15 minutes. Students did not provide their names on submissions. The type of technique the participant used was analyzed relative to their progress in solving the problem.

2.2.2 Problem Given to Participants

Figure 2-9: Cantilevered beam to support traffic lights

You work at an engineering design company, and your supervisor has given you the following project:

A horizontal cantilevered beam is used to support traffic lights as shown in Figure 2-9. For the horizontal part of the beam, several designs are possible:

1. Circular cross-section with radius 5cm at the fixed end (where it's attached to the vertical pole) tapering to a circular cross section with radius 10cm at the free end

2. Circular cross-section with radius 10cm at the fixed end tapering to a circular cross section with radius 5cm at the free end

3. Circular cross section with radius 7.5cm throughout the beam

4. A different design

You will need to find the best design and justify it with reasoning.

2.2.3 Results and Analysis

Based on previous work, it is expected that students will use one of several approaches to solve each problem. These approaches will depend on the problem given [12]. For the problem given in Section 2.2.2, a correlation will be performed between participants' 5 minute and 15 minute responses. The 5 minute "starting approach" response was analyzed by the presence or absence of one of five characteristics. These five characteristics are: free body diagram or drawing, calculations or analysis, intuition or estimation, identification of additional background, and consideration of alternative designs. The coding of student responses is described in more detail in Table 2.4. These five characteristics, while specifically selected for this problem, are general enough to be applicable to other engineering problems as well. The 15 minute "solution" response was binned by two items: the presence of an answer and whether the answer was the preferred choice. For the problem given in Section 2.2.2, a choice of design 2 or 4, accompanied by the appropriate justification, was considered the preferred choice.

For each of the five problem-starting characteristics described above, the students were divided into two categories: those whose response contained the characteristic and those who did not. It was possible for a student's 5 minute response to contain more than one characteristic. For each of the two solution outcomes described above, the students were divided in two categories of participants: those who were successful and those who were not. For each pair of starting characteristic and solution outcome, the chi-squared test was used to determine whether the relationship was due to random chance. The p-values are given in Table 2.5. A p-value of less than .05 was considered statistically significant.

Table 2.4: Characteristics of students' 5 minute submission

Characteristic present in in 5 minute submission	**Example keywords**
Free body diagram (FBD) or drawing	Draw, diagram, moment and shear diagram
Analysis or calculation	Solve, calculate, find, optimize, minimize, Solidworks, moment and force balance
Intuition or estimation	Approximation, estimation, elimination of choices with reasoning
Identification of add'l info	Cost, building standards, material properties, wind loads
Alternate design	Alternate design, advantages of alternate design

Table 2.5: p-values of characteristics vs. problem solving outcome

Characteristic present in 5 minute submission	**p-value vs. answer is choice 2 or 4**	**p-value vs. answer present**	**correlation direction**
FBD or Drawing	.848	.866	Neutral
Analysis or calculation	**.035**	.056	**Negative**
Intuition or estimation	.078	.142	**Positive**
Identification of add'l info	.600	.260	Slightly positive
Alternate design	.720	.141	Slightly positive

2.2.4 Discussion

The presence of analysis or calculation in the 5 minute response had a negative relationship with outcome in the 15 minute response. We found statically significant $p = .035$ for the outcome indicator of obtaining the preferred answer, and $p = .056$ for the outcome indicator of obtaining an answer at all. Participants preferring analysis or calculation tended to "dive in" to solving the problem analytically, often running out of time.

On the other hand, the presence of intuition or estimation in the 5 minute response had a positive relationship with outcome in the 15 minute response. We found $p = .078$ for the outcome indicator of obtaining the preferred answer, and $p = .142$ for the outcome indicator of obtaining an answer at all. This result is not statistically significant, but it does suggest that students using intuition obtained a result more often than students who did not. Participants using intuition tended to "cut through" the difficulty of detailed analysis and obtain an answer within the time limit. The other three categories had relatively weak relationships with outcome. This data suggests that for relatively open-ended problems, it may be advantageous for students to use their intuition or do an estimation first, and then follow up with detailed analysis.

Chapter 3

Main Experiment

This experiment investigates the methods used by problem solvers and the associated problem solving outcome. Specifically, we seek to answer the following research question:

RQ: How would a student select from a range of low complexity to high complexity methods, and how would this choice affect their problem solving outcome?

Portions of this Chapter are based on a previous work by Li and Hosoi [14].

3.1 Recruitment of Participants

The subject population consisted of 72 undergraduates and graduate students of the author's institution. The objective of recruitment was to maximize the number of participants, so participants were not limited to students in the Mechanical Engineering department; the diversity in students' disciplines may potentially result in a larger variety of solution methods.

Multiple recruitment methods were used. Subjects were recruited from the enrollees of Numerical Computation for Mechanical Engineers (a second-year course in Mechanical Engineering), through announcements made to student organizations, and from flyers posted in the author's institution. Additionally, participants were allowed to refer fellow students via snowball sampling, and some subjects were recruited informally from existing networks.

3.2 Procedure

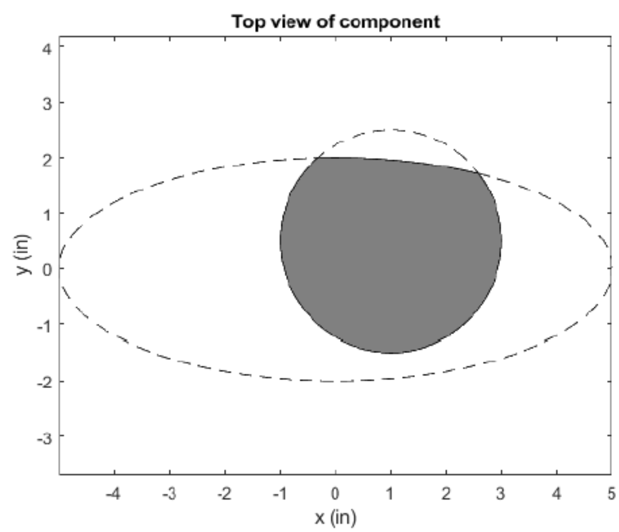
The participants in this study were given the Volume Problem (see Figure 3-1), which consisted of two sections. The first section asked students "How would you start solving this problem?" and was five minutes long. The second section asked students to "Solve as much of the problem as you can" and was ten minutes long. The participants submitted their answers on paper answer sheets. Every two minutes the participant switched pen color so that work can be identified within time intervals. A sample of student work is shown in Figure 3-2. The participant was able to access a computer connected to the internet. There were no restrictions on what tools they could use, but they could not consult other people. After completing the two sections of the Volume Problem, the participant completed a short questionnaire about their problem solving process (see Appendix A).

The problem solving task was to "estimate the volume of the component." This task was chosen such that participants are likely to consider both low complexity and high complexity methods. While it is possible to obtain an analytic (exact) solution, most students are unlikely to do so within the time constraints.

3.3 Experimental Results

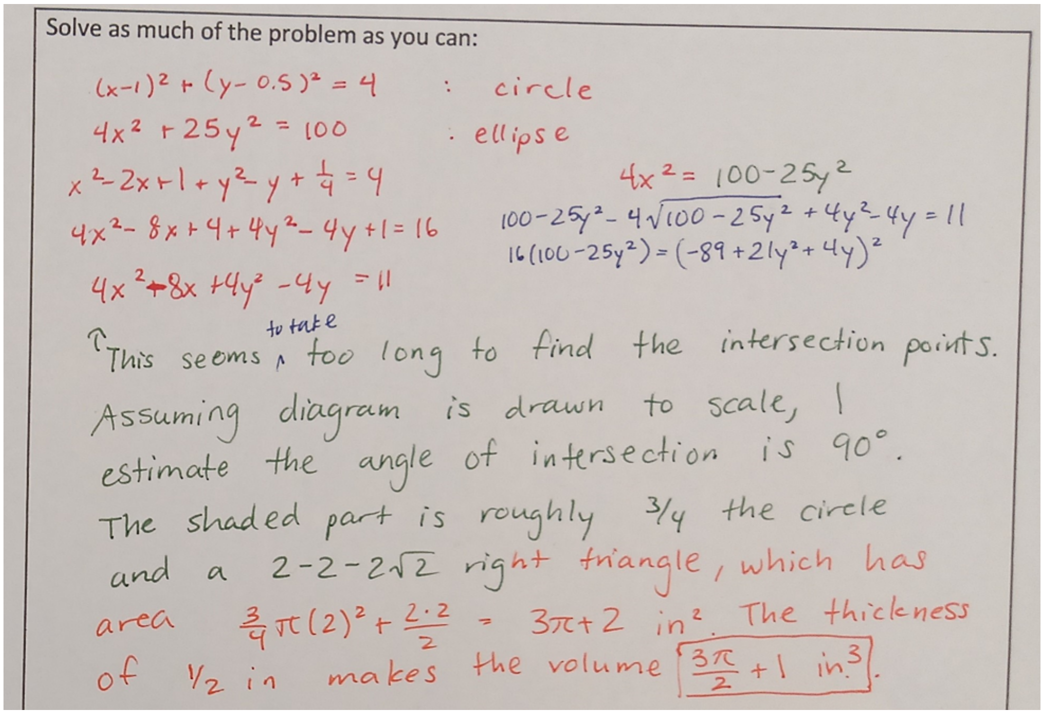
Participants' responses were analyzed for three categories of items: stage of problem solving, type of representation, and solution method. The first two categories, stage of problem solving and type of representation, are based on previous literature (see Figure 1-1). The third category, solution method, is central to this work. Solution methods were extracted from interpreting students' answer sheets and corroborating

A circular steel disc of radius 2 in and thickness ½ in will be turned into a crankshaft component. For the first cut, the center of the disc is placed at the location (x, y) = (1, 0.5)in on a mill. The cutting head will trace a toolpath such that all material outside of the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$ will be removed.



Your task is to estimate the volume of the component after this cut **and justify your answer with reasoning**. You can use any resources you want, except talking to someone else.

Figure 3-1: Volume Problem used in the main experiment. Figure reproduced from previous work by Li and Hosoi [14].


Solve as much of the problem as you can:

$(x-1)^2 + (y-0.5)^2 = 4$: circle

$4x^2 + 25y^2 = 100$: ellipse

$x^2 - 2x + 1 + y^2 - y + \frac{1}{4} = 4$

$4x^2 - 8x + 4 + 4y^2 - 4y + 1 = 16$

$4x^2 - 8x + 4y^2 - 4y = 11$

$4x^2 = 100 - 25y^2$

$100 - 25y^2 - 4\sqrt{100-25y^2} + 4y^2 - 4y = 11$

$16(100-25y^2) = (-89 + 21y^2 + 4y)^2$

This seems to take too long to find the intersection points. Assuming diagram is drawn to scale, I estimate the angle of intersection is 90°. The shaded part is roughly 3/4 the circle and a 2-2-$2\sqrt{2}$ right triangle, which has area $\frac{3}{4}\pi(2)^2 + \frac{2 \cdot 2}{2} = 3\pi + 2$ in^2. The thickness of 1/2 in makes the volume $\boxed{\frac{3\pi}{2} + 1 \text{ in}^3}$.

Figure 3-2: Example of student work. The participant's pen color was switched every two minutes so that work can be identified within time intervals. Figure reproduced from previous work by Li and Hosoi [14].

with questionnaire (Appendix A) responses. Specifically, the results to the following questions were used: "Which method did you pick and why?," "Did you switch methods?," and "If so, why did you switch?"

The Volume Problem instructed participants to estimate a quantity. Any answer within 10% of exact was considered correct. 26 of 72 (36%) participants solved the problem correctly. We first analyze failure mode, and then compare the problem solving outcome against the student's year, the method used, and number of methods used.

3.3.1 Failure Mode

The nature of successful and unsuccessful solutions was analyzed (see Table 3.1). We found that 26 of 29 (90%) of student who obtained an answer got the correct answer. Also, 43 of 46 (93%) of students who did not obtain the correct answer obtained no answer at all. The majority of incorrect solutions was due to the problem solver not finishing, not because they made a mistake. In other words, students who did not solve the problem correctly tended to work until the time limit expired, while students who solved the problem correctly sometimes finished their work before the time limit was reached.

	Obtained answer	**Did not obtain answer**	**Total**
Correct	26	-	26
Incorrect	3	43	46
Total	29	43	72

Table 3.1: Failure mode analysis of participants. A large majority of students who finished the problem obtained a correct solution; a large majority of students who did not obtain the correct solution did not finish solving the problem.

3.3.2 Student Year

The participants' results were analyzed by school year. Students of different years performed somewhat similarly (See Table 3.2). The second year students did not perform as well as average, but the result was within the margin of error. Overall, a strong trend was not observed.

School year	**Number correct**	**Total number**	**Fraction correct (with 95% confidence intervals)**
First year	9	19	0.47 [0.17, 0.68]
Second year	2	11	0.18 [0.05, 0.48]
Third year	6	15	0.40 [0.20, 0.64]
Fourth year	4	11	0.36 [0.15, 0.64]
Graduate student	5	16	0.31 [0.14, 0.56]
Overall	26	72	0.36 [0.26, 0.48]

Table 3.2: Volume Problem results by school year. Clear differences between students in different years were not observed. Table reproduced from previous work by Li and Hosoi [14].

3.3.3 Type of Method

Participants' solution methods were placed into six categories formulated from a broad, high-level view of students' solution attempts. The categories are: Visual Estimation, Geometric Approximation, CAD (e.g. Solidworks or Onshape), Monte Carlo, Integral, and Other. With the exception of "Other," which includes approaches that do not fit in the first five categories, these are in approximate order of implementation complexity from simplest to most complex (see Table 3.3). They are also in approximate order of accuracy from least precise to most precise. It is possible for a student to use more than one method. In this case, the incomplete method (i.e. the one the student switched away from) is not analyzed, while the method that the student "completed" (i.e.. the one the student finished, or ran out of time doing so) is analyzed. Fraction correct is defined as the number of students who correctly solved with a given method, divided by the total number of students who "completed" the given method (as defined in this context).

On average, Visual Estimation was associated with the lowest solve time. Geometric Approximation, CAD, and Monte Carlo were associated with moderate solve time, while Integral was associated with the longest solve time. The fraction of students who correctly solved the problem also depended on which method was used. Integration was associated with the lowest fraction correct. Geometric Approximation and Monte Carlo were associated with a moderate fraction correct, while Visual Estimation and CAD were associated with the highest fraction correct. In general, shorter methods were associated with a higher fraction correct.

Students who correctly solved the problem using a given method tended to use less time on that method than students who did not correctly solve the problem. This is consistent with the observation from Section 3.3.1 that students who did not solve the problem correctly tended to work until the time limit expired, while students who solved the problem correctly sometimes finished their work before the time limit was reached.

Method	**Number students**	**Fraction correct (with 95% CI)**	**Solve time (min)**	**Solve time (correct soln. only, min)**
Visual Estimation	6	0.83 [0.44, 0.97]	3.8	3.4
Geometric Approx.	23	0.39 [0.22, 0.59]	7.8	6.3
CAD	5	0.80 [0.38, 0.96]	7.8	7.3
Monte Carlo	4	0.50 [0.15, 0.85]	8.3	6.5
Integral	28	0.21 [0.10, 0.40]	9.5	9.0

Table 3.3: Accuracy and corresponding solve times. Fraction correct is the fraction of participants who obtained a correct solution with a given method. Solve time is the time spent on the method. These methods are in the order of least complex (least average time spent) to most complex (most average time spent). Table reproduced from previous work by Li and Hosoi [14].

3.3.4 Number of Methods

We also analyzed the number of methods used by the participants (See Table 3.4). It was found that most students used only one method (64); in comparison, few students used two or more solution methods (8). The potential reasons for this are discussed with survey results in Section 3.4.5. Of seven participants who switched methods exactly once, four obtained correct answers, an accuracy of 57% compared to the overall accuracy of 36%. Additionally, one participant switched methods twice and obtained the correct answer. Even though this data is not statistically significant, it may suggest the existence of an optimal number of solution methods for maximizing the probability of correctly solving this problem.

Number methods	Number correct	Total number students	Fraction correct (with 95% CI)
1	21	64	0.33 [0.23, 0.45]
2	4	7	0.57 [0.25, 0.84]
3	1	1	1.00 [0.21, 1.00]

Table 3.4: Number of methods used vs. solution outcome. Note that a higher fraction correct was associated with more solution methods. Table reproduced from previous work by Li and Hosoi [14].

3.4 Data Analysis and Discussion

3.4.1 Sankey Diagrams

Sankey diagrams were used to visualize the participants' method use on the Volume Problem. Figure 3-3 represents the methods used during the 10 minute "solve" section only. Figure 3-4 represents the methods used during both the 5 minute and 10 minute sections, with intermediate methods omitted for clarity. The Sankey diagrams were compiled such that the first method nodes used were aligned on the left, and the last method nodes used were aligned on the right. In Figure 3-3, intermediate method nodes were aligned in the order they appear. If a participant only used one method, the node would appear once on the left and once on the right.

It was found that participants with more transitions tended to solve the problem successfully. However, the number of transitions between solution methods cannot be indefinitely high, suggesting that there is an optimum number of transitions that maximizes the likelihood of solving the problem.

3.4.2 Choice of Starting Method and Ending Method

The participants' choice of starting method (for the 5 minute section) was notable. Of the 72 participants, 29 did not immediately suggest a method. Among the remaining 43 that did, 27 suggested Integral, 8 suggested Geometric Approximation, and 3 sug-

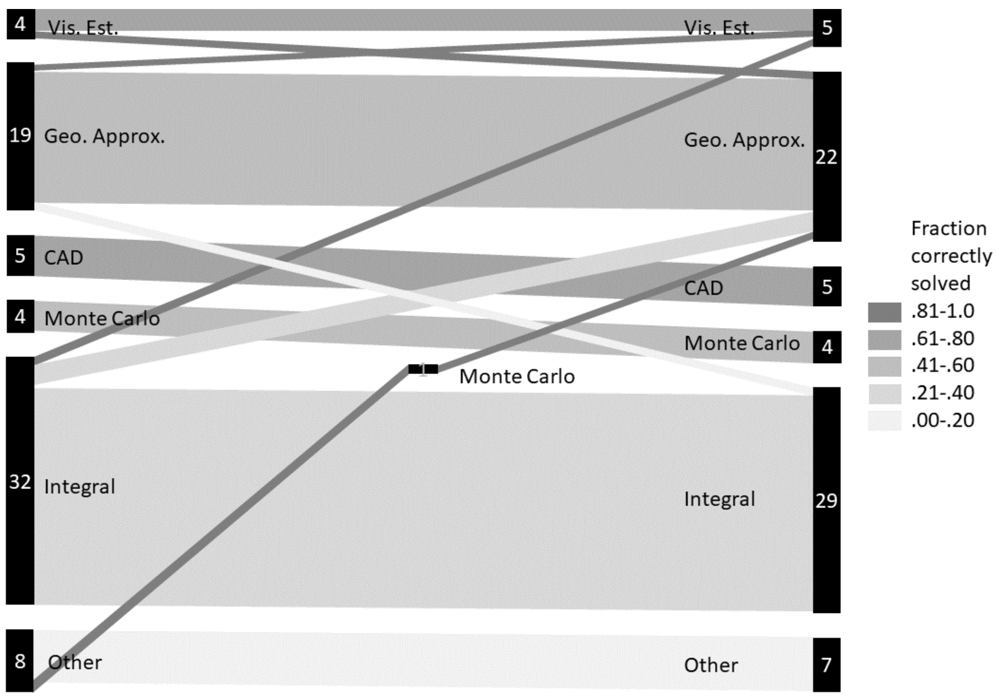


Figure 3-3: Sankey diagram for the Volume Problem, 10 minute section only. Darker transitions represent a larger fraction of participants with the correct answer. The left nodes represent the method used at the start of the 10 minute section, and the right nodes represent the method used at the end of the 10 minute section. There is one intermediate node, corresponding to one participant who used three methods in the 10 minute section. Note that the majority of students did not switch methods.

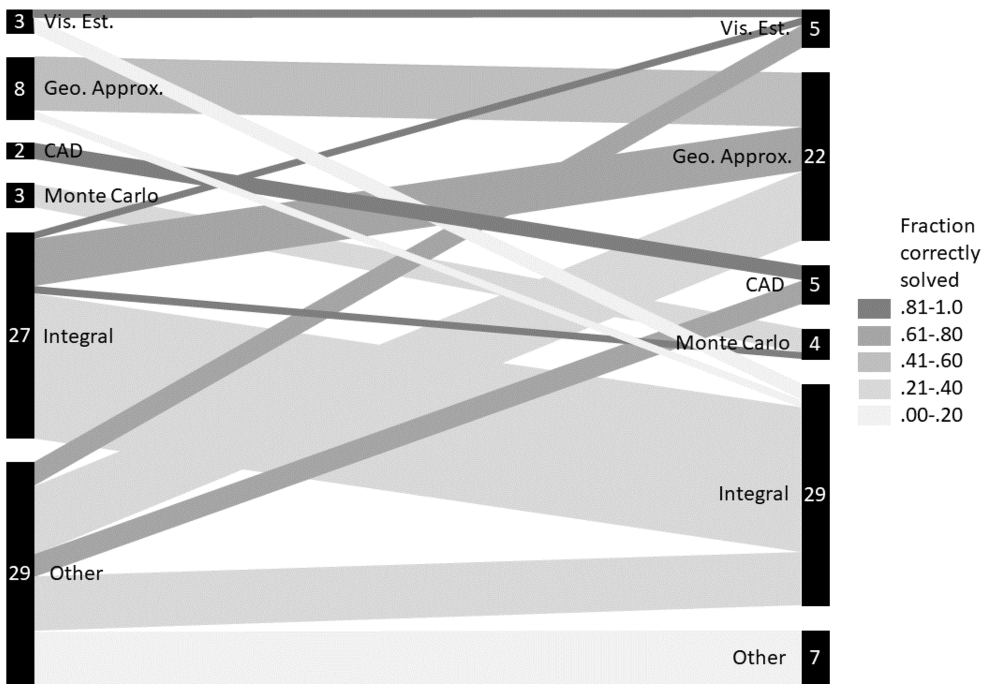


Figure 3-4: Sankey diagram for the Volume Problem, 5 minute and 10 minute sections. Intermediate steps have been omitted for clarity. Darker transitions represent a larger fraction of participants with the correct answer. The left nodes represent the method used at the start of the 5 minute section, and the right nodes represent the method used at the end of the 10 minute section. If a participant only used one method in both the 5 and 10 minute sections, the node would appear once on the left and once on the right. Note that the last method (10 minute section) was a better predictor of problem solving success than the first method (5 minute section). The right nodes for Visual Estimation, Geometric Approximation, CAD, and Monte Carlo tended to attract transitions with higher fraction correctly solved than Integral or Other. There was no clear pattern for transitions originating from the left nodes.

gested Visual Estimation. The tendency to suggest an integral method first indicates a preference for detailed analysis over simple estimation, at least when starting the problem. This pattern could reflect participants' lack of awareness or knowledge of lower complexity methods or absence of a starting strategy. However, this should be considered in the following context: when starting to solve the problem, the student does not have perfect information about which solution methods are available and their relative solve times. Therefore the first method they chose may or may not be the optimal (shortest) one.

There were also differences between the number of participants starting and ending with a given method. Of the 72 participants, only 7 did not choose a method by the end of the 10 minute section. Among the remaining 65, 29 ended with Integral, 22 ended with Geometric Approximation, and 5 ended with Visual Estimation. There was a shift from other methods to Geometric Approximation, especially from the "Other" category. In particular, 8 participants started the 5 minute section with geometric approximation, but 22 participants ended the 10 minute section with geometric approximation. A significant number of participants shifted to this less complex method from a more complex method; Geometric Approximation was taken to be a method with intermediate complexity and accuracy. In contrast to geometric approximation, the number of participants starting with visual estimation, CAD, Monte Carlo, and integral was similar to the number of participants ending with those methods. Most notably, a significant number of participants remained on the most complex method, Integral, throughout the entire time allotted. These patterns reflect mixed results: while some participants switched to a less complex method, others remained on more complex methods throughout the problem solving period. For some participants, there is evidence of switching strategy or knowledge of lower complexity methods. For others, this evidence is absent.

3.4.3 Starting Method and Problem Solving Outcome

Data from the Volume Problem suggests that the last method used in the 10 minute section is a better predictor of successfully solving the problem than the first method used in the 5 minute section. In the 5 minute section, participants would often suggest methods without following through on a full solution. For example, of the 3 participants who started with visual estimation in the 5 minute section, only one obtained the correct answer. However, of the 5 participants who ended with visual estimation in the 10 minute section, 4 obtained the correct answer.

3.4.4 Patterns of Transitions

It is possible to have the following types of transitions between methods:

- Transitions from methods of lower complexity to higher complexity. For example, a participant may finish the problem early using estimation and wishes to confirm the answer with more detailed analysis. Note that in the analytic model in Chapter 4, we assumed (for simplicity) that problem solving activity ceases upon the first correct answer obtained. However, for the experiment, we allow this type of transition to be a possibility.

- Transitions from methods of higher complexity to lower complexity. For example, a participant may fail to solve the problem using detailed analysis and change to estimation in order to obtain an answer within the time limit.

- Transitions between methods of similar complexity. For example, a participant may start the problem on one method, decide that they are more familiar with another method, and switch to that method.

Method transitions within the 5 minute section were not counted because the participant was not expected to solve the problem during that section. Method transitions within the 10 minute section were observed for 8 of 72 participants; of these, six transitioned from higher complexity methods to lower complexity methods, one transitioned from a lower complexity method to a higher complexity method after obtaining an answer with the lower complexity method, and one combined two methods in their solution attempt.

The pattern of method transitions suggests that the participants would use a lower complexity method to "descope" if their higher complexity method failed. This strategy seems to have been beneficial, as the participants using it had higher solve percentage than average. However, the number of participants using transitions was small, so limited conclusions can be drawn.

The "descoping" strategy is consistent with predictions made by the analytic model in Chapter 4; the presence of low solve time methods, in addition to a tendency to start on high solve time methods, presented ample opportunities to switch to the lower solve time method, thereby increasing solve probability.

Note that very few students (11%) used two or more solution methods in the 10 minute section. The small amount of students using estimation in the Volume Problem is consistent with the findings from the literature: that students had difficulty with estimation, and that engineering classes overwhelmingly emphasized detailed analysis over estimation [15, 20]. To remedy this, the instructor can teach estimation as a backup option. This may or may not be the perfect solution (instead, students could deploy estimation as a first-line strategy), but the benefits from its use here on the Volume Problem suggest it is an improvement over the status quo; students are no longer saddled with the inflexibility of using only detailed analyses when solving problems.

3.4.5 Survey Results

In the questionnaire completed after the problem (Appendix A), the participants were asked, "On the previous problem, which method did you pick and why?" They were also asked whether they switched methods and "If so, why did you switch?' The following patterns were observed from the responses:

- Some students decided to solve the problem using equations and integrals, but then tried to estimate or approximate because their initial method was too time consuming. Sometimes the students saw their approximations as "bad" or "crude."
- Some students deliberately chose a method that they felt was easiest
- Some student chose a method that they perceived to be the most accurate
- Some student chose a method based on what they remember (e.g. calculus techniques)
- Some students decided to visualize the problem instead of doing detailed calculations
- Some students spent considerable time searching on the internet for a black-box automatic solver for this problem.

While some students did use lower complexity methods, their overall attitudes toward higher complexity methods and lower complexity methods were not the same; they favored higher complexity methods and attempted to use them whenever possible because of higher perceived accuracy. Estimations and simple approximations were not as highly regarded due to potential inaccuracy or "crudeness." However, at the same time, the students looked for solution techniques that were simple to implement and time-efficient. Some students attempted to find a solution method that is both short and precise, spending considerable time searching on the internet for a black-box automatic solver.

It is possible that some students believed that the experimenter expected fancy, exact, or clever solution methods. Alternately, it is possible that students were not accustomed to using approximations and estimations in the context of homework or exams. Nevertheless, this pattern is quite different from the strategies suggested by the analytic model (in subsequent Chapters 4–6) , namely using a switching strategy or starting on a low complexity method.

Additionally, the participants were asked, "What is your confidence in your solution method" and "What is your confidence in your answer?" Answers were given on a five-point Likert scale. The results are shown in Table 3.5. It was found that students' overall confidence in their answer was significantly lower than their confidence in their method. When comparing students who obtained the correct answer with those that did not, we found that the method confidence was nearly the same. However, the answer confidence was much higher for students who obtained the correct answer.

	N, method confidence	**Average confidence in method (with 95% CI)**	**N, answer confidence**	**Average confidence in answer (with 95% CI)**
Correct	25	3.52 [3.07, 3.97]	25	3.00 [2.48, 3.52]
Incorrect	44	3.41 [3.04, 3.78]	43	1.37 [1.10, 1.65]
Overall	69	3.45 [3.17, 3.73]	68	1.97 [1.65, 2.29]

Table 3.5: Survey results for participants' confidence in their solution method and answer. Most, but not all, participants responded. Responses were recorded on a five-point Likert scale (1-5). A t-test was used to calculate confidence intervals. Overall, students had a much higher confidence in their method than their answer. Students who obtained the correct answer reported similar confidence in their method as those with incorrect answers. However, students with correct answers reported much higher confidence in their answer than those with incorrect answers.

One possible reason is that most students believed that their method was correct even though only some students completed their solution. There was no additional reward for obtaining the correct answer, so some students may have been content to demonstrate a method they believed would work, without worrying too much whether they obtained an answer or not. This could also explain why a large majority of students only used one method in Section 3.3.4. It is possible some students did not understand the time requirements of a more complex method like integration, even though they were confident such a method would work. Thus they did not take the time to consider other (lower complexity) methods as well.

Chapter 4

Analytic Model

The purpose of the model is to build a framework to articulate the interplay of skills and strategies in solving problems with multiple solution methods. The model was motivated by the better performance of students using shorter methods in Section 3.3.3 and potentially better performance of students using multiple solution methods in Section 3.3.4; we sought to more clearly understand these phenomena.

This model shares some similarities with the main experiment presented in Chapter 3. Just as in the experiment, our objective for the model is to maximize the probability of solving the problem within a time limit. Different solution methods are each associated with a solve time and a solve probability, and switching between methods is allowed. We begin with the simplest case (Two Solution Methods) to illustrate the concepts. Next, we successively add additional complexities. Finally, we match the model with the experiment.

Portions of this Chapter are based on a previous work by Li and Hosoi [14].

4.1 Assumptions

This model makes several assumptions about the types of problems and solutions that are relevant to this analysis, namely:

1. The problem has a well-defined solution, and can be solved via more than one solution method. The different solution methods may have different solve times.

2. There is a time limit (e.g. an in-class exam), and this limit is known to the problem solver.

3. From the point of view of the problem solver, there can be uncertainty in awareness of solve time of various solution methods. The student does not necessarily know, before solving the problem, which solution method will take less time. The student also does not necessarily know exactly how long a problem will take. If the student did have perfect information about the relative solve time or difficulty of the various solution methods, it would be obvious which solution path to take. (They would be doing exercises instead of solving problems.)

4. The student may tailor their solution according to what they think that the instructor (or in this case, the experimenter) wants for an acceptable solution. This may influence their problem solving behavior. In this thesis, this effect is assumed to be negligible; in the experiment (Chapter 3), the experimenter attempted to take a neutral approach by defining the goal of problem solving, but not prescribing or recommending a specific method.

Additionally, this model makes the following assumptions about the problem solving process:

1. The time variables in the model are discrete. Key variables, such as the solve time of various solution methods, are multiples of a discrete time interval. This approach can be generalized to the continuous time limit, but, for simplicity, we focus on the discrete formulation in this Chapter.

2. The student may switch methods at any point during their problem solving time. In the simplest case with two solution methods, we assume equal probability of

starting on each method and random switching at discrete time steps. In later sections, these assumptions are relaxed as the student is allowed to bias their starting or switching to certain solution methods.

3. The solve time and solve probability for a given method is fixed regardless of how long the student has already spent working on solving the problem, or what order position the solution attempt in their list of solution attempts. Work done for one method that leads to an unsuccessful outcome does not influence the solve time or solve probability for another method.

4. The student "starts over" when they switch to a new method; they lose all progress on their current method. They must complete the requisite number of consecutive time steps on the new method to successfully solve the problem.

5. The student stops problem solving activity when the goal is achieved (i.e. the problem is solved).

4.2 Model with Two Solution Methods

Consider a problem that has two solution methods (e.g. the student could solve the problem graphically or via integration). Let the time required to solve the problem using these methods be T_1 and T_2. (Note that T_1 and T_2 are continuous quantities.) If we consider time intervals of length Δ, the total number of intervals needed to solve the problem for each method is $t_1 = T_1/\Delta$ and $t_2 = T_2/\Delta$, respectively. Unlike T_1 and T_2, we let t_1 and t_2 be discrete quantities and do not consider noninteger multiples of Δ. The problem solver successfully solves the problem if they spend t_1 consecutive timesteps on method 1 or t_2 consecutive timesteps on method 2. In addition, they are allocated a total of t_f timesteps for problem solving; i.e. t_f is a time limit, and it is not possible to solve the problem if both methods have solve times longer than t_f.

Suppose initially, the problem solver has an equal probability, $\frac{1}{2}$, of selecting either method 1 or method 2. Then at each subsequent timestep, the problem solver will switch to the other method with probability α and remain on the current method with probability $1-\alpha$. Here, α can be considered a *strategy*. For example, a problem solver can choose a strategy of switching more often (higher α), a strategy of switching less often (lower α), or a strategy of staying on an existing method (α=0). Whether a strategy is favorable may depend on the problem solver's *skills*, i.e. their ability to use the *tools* at their disposal to solve the problem within a time constraint; this is reflected in their solve times t_1 and t_2.

We can now ask, what is the best strategy (i.e. value of α) that maximizes the probability of solving the problem. Assume that $t_1, t_2 \geq 2$ to avoid a degenerate case. Without loss of generality, also assume that method 1 has a shorter or equal solve time compared to the method 2; that is, $t_1 \leq t_2$. Then the following cases are possible:

- Case I: $t_1 \leq t_2 \leq t_f$. In this case, the best strategy is to not switch methods, since the problem can be solved with both methods within the time limit. Therefore, $\alpha = 0$ gives the maximum probability of solving the problem, $P_{solve} = 1$. Note that this model assumes that switching methods requires the student to "start over" on the new method; they must stay on the new method for the needed number of consecutive timesteps to solve the problem.

- Case II: $t_1 = t_f < t_2$. In this case, it is impossible to solve the problem with method 2, but it is just possible with method 1. The best strategy is to not switch at all, hence $\alpha = 0$ gives the maximum $P_{solve} = \frac{1}{2}$.

- Case III: $t_f < t_1 \leq t_2$. In this case, it is impossible to solve the problem with either method 1 or method 2. Any value of $\alpha \in [0, 1]$ will give $P_{solve} = 0$.

- Case IV: $t_1 < t_f < t_2$. This is the interesting case in which there may exist an optimal strategy $\alpha > 0$ which maximizes P_{solve}. This case is nontrivial and will be further analyzed below.

It can be shown (see Appendix C for proof) that if $0 < t_1 \leq \lfloor \frac{t_f}{2} \rfloor$ and $t_2 > t_f$, there exists $\alpha > 0$ that maximizes P_{solve}. That is, for these values of t_1, the best outcome is obtained if the problem solver explores both solution methods. (Here, the floor function symbol $\lfloor\ \rfloor$ means to round down any noninteger values.) On the other hand, if $\lfloor \frac{t_f}{2} \rfloor < t_1 < t_f$ and $t_2 > t_f$, $\alpha = 0$ maximizes P_{solve}, so the problem solver is better off if they stick to the original starting method. As t_1 decreases, the optimal switching frequency α increases and the associated solve probability P_{solve} also increases (see Figure 4-1). That is, if the solve time for method 1 is very short (i.e. the student is able to solve the problem quickly), the optimal switching tendency is high; i.e. the problem solver is rewarded for a higher tendency to switch methods. The corresponding probability of solving the problem also increases. If the solve time for method 1 exceeds a critical limit (in this case $\lfloor \frac{t_f}{2} \rfloor$), there is no reward for switching, and it is optimal for the problem solver to stay on the same method they started with. See Appendix B for a numerical example of calculating $P_{solve}(\alpha)$. See Figures 4-1 and 4-2 for a numerical example of the effect of t_1 on solve probability.

4.3 Additional Elements of the Model

4.3.1 Effect of Starting Method

Now that we have illustrated the basic interplay between strategies (as reflected in α) and skills (as reflected in t_1 and t_2), we can consider the impact of other parameters. One way to extend the model is to allow the probability of starting on each method to be unequal. Let β_i be the probability of starting on method i. In Section 4.2, we set $\beta_1 = \beta_2 = \frac{1}{2}$, but here, we can freely vary the β_i, subject to the constraint that the sum of all the starting probabilities, $\Sigma_i \beta_i$, is 1.

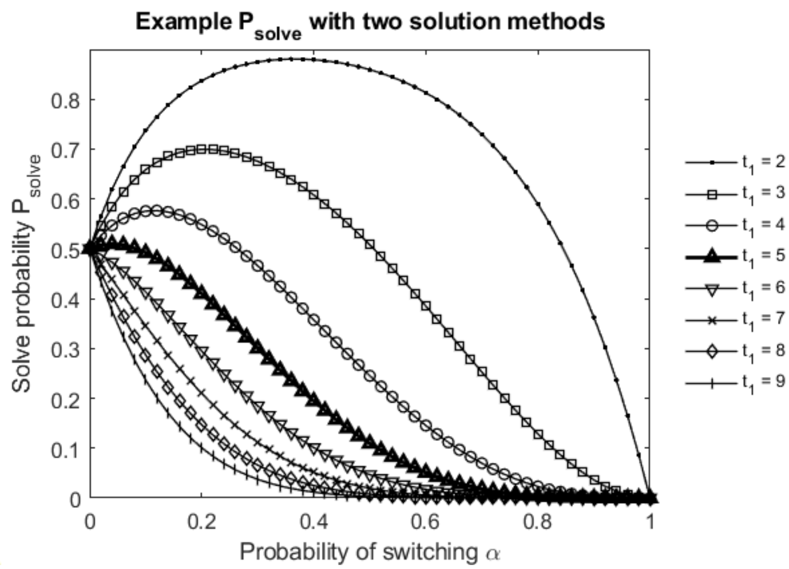


Figure 4-1: P_{solve} with two solution methods and time limit $t_f = 10$ and $t_2 > t_f$. Note that for $t_1 = 2, 3, 4, 5$ there exists an $\alpha > 0$ for which P_{solve} is maximized. In other words, a strategy of not switching ($\alpha = 0$) is not optimal for maximizing solve probability; some switching is beneficial. Values of t_1 equal to 6 or greater do not benefit from switching at all. If the problem solver is aware that they do not have a simple way to solve the problem (i.e. t_1 is 6 or greater), they should consider improving their skills, i.e., reducing their t_1. Figure reproduced from previous work by Li and Hosoi [14].

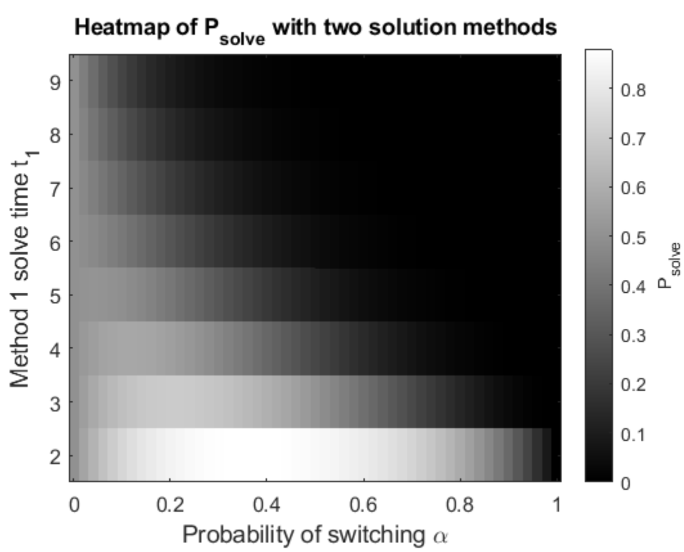


Figure 4-2: Heatmap showing the dependence of P_{solve} on t_1 and α with $t_f = 10$ and $t_2 > t_f$. Note that as t_1 increases, the maximum P_{solve} decreases, and it occurs at a smaller switching tendency α. This shows the benefit of improving both skills (i.e. reducing the solve time t_1) and strategy (i.e. choosing the correct α for the problem solver's value of t_1).

From the point of view of the problem solver, the ability to more reliably identify a more efficient solution method can have significant benefits for increasing P_{solve}. For example, consider using this strategy for the case with two solution methods. If the probability of starting on the shorter method, β_1, is increased, the maximum achievable P_{solve} is correspondingly increased for every value of α (see Figure 4-3). Furthermore, increasing β_1 decreases the α required to achieve this maximum. In other words, if a problem solver is more proficient at selecting shorter methods, they have less need for a switching strategy. In fact, if β_1 is sufficiently high, there is no longer any need for any switching, so $\alpha = 0$ would result in the maximum P_{solve}. See Figure 4-3 for an example of this situation. Conversely, students who tend to start with the higher solve-time method (i.e. smaller β_1) are less likely to find a solution in the allotted time.

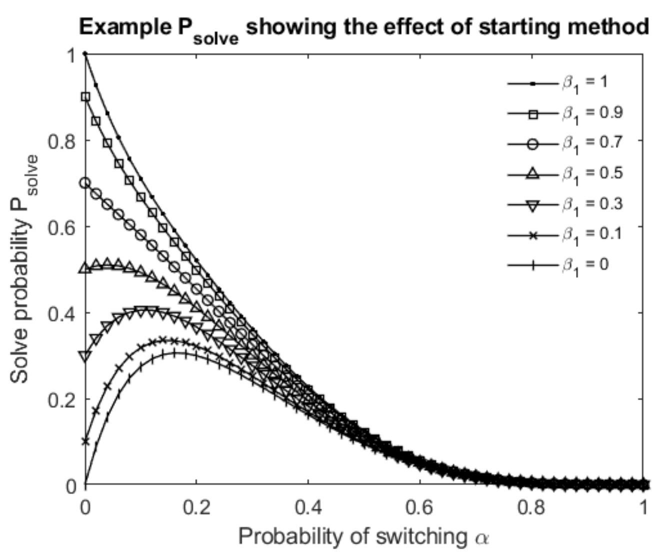


Figure 4-3: An example P_{solve} showing the effect of starting method probability. There are two solution methods, the time limit is $t_f = 10$, method 1 has solve time $t_1 = 5$, and method 2 has $t_2 > t_f$. Note that for curves where $\beta_1 \leq 0.5$, a switching strategy is beneficial. However, for curves where $\beta_1 > 0.5$, there is no benefit from switching, and $\alpha = 0$ gives the highest solve probability. In other words, the more likely the problem solver starts on the shorter method, the less need they have for a switching strategy. Figure reproduced from previous work by Li and Hosoi [14].

4.3.2 Generalization to n Solution Methods

In the previous two sections, we developed a two solution method model to demonstrate the basic interplay between strategy and skills. The next step is to generalize this model to three or more solution methods. Consider $n \geq 3$ methods with solve times $2 \leq t_1 \leq t_2 \leq \cdots \leq t_n$. At each timestep, let the total probability of switching remain α. However, we will have to modify the probability of switching to a given method. There are $n-1$ other methods, so we set the probability of switching to each of the other methods to be $\frac{\alpha}{n-1}$. Note that the probabilities of switching to each of the other methods are equal, and they sum up to α. Thus the probability of remaining on the current method is $1-\alpha$.

Just as in the basic model with two solution methods, several cases are possible:

- Case I: it is possible to solve the problem with all of the methods within the time limit, i.e.

  $$t_1 \leq t_2 \leq \cdots \leq t_n \leq t_f$$

  If this occurs, then not switching methods ($\alpha = 0$) results in a solve probability of one, so there is no need for a switching strategy. Any positive value of α will result in the possibility of a sequence of solve methods that does not solve the problem. (For example, it is possible that the problem solver switches at each time step among several solution methods, never accumulating two or more consecutive time steps on any method, and thereby never solving the problem.)

- Case II: It is not possible to solve the problem with any of the methods within the time limit, i.e.

  $$t_f < t_1 \leq t_2 \leq \cdots \leq t_n$$

  If this occurs, then there is no way the problem solver can solve the problem regardless of switching strategy. Thus the solve probability is zero.

- Case III: Neither I or II is true. It is possible to solve the problem within the time limit for at least one method, but not for all of the methods. This is the nontrivial case and will be analyzed further.

The presence of additional solution methods makes analysis more difficult, but the goal remains the same: to increase the solve probability P_{solve} as much as possible, and to determine the conditions for which a switching strategy $\alpha > 0$ can maximize the solve probability. There are several characteristics of the solve times that affect the P_{solve} curve.

The first characteristic is the number of methods with solve times less than the time limit t_f. This must be at least one and at most $n-1$ in accordance with Case III above. Because the problem solver can complete these methods within the time limit, these are the solution methods that the problem solver can execute *effectively*, i.e. they are the handiest tools in their analytical toolbox. For a fixed number of total methods n, the more methods that fall into this category, the more likely they will start on an effective method. This will increase their no-switch ($\alpha = 0$) solve probability because an effective method will have a solve time less than t_f. A problem solver may also start on a solution method they cannot execute within the time limit. In this case, the solution method will not lead to a solution unless the problem solver switches sufficiently quickly to an effective method.

The second characteristic is the solve times of the effective solution methods. These solve times can be small compared to t_f, representing short methods that are simple for the problem solver to execute. Alternately, these solve times can be as large, or nearly as large as t_f, representing longer methods that are more complex but still feasible for the problem solver to execute within the time limit. This characteristic represents a combination of how well the problem solver knows the method (their skill in using their tools) and the method's intrinsic complexity. The magnitude of these solve times do not affect the problem solver's no-switch ($\alpha = 0$) solve probability, but they can determine whether a switching strategy $\alpha > 0$ can improve P_{solve}. As a rule of thumb, the lower these solve times, the more benefit there is to implementing a switching strategy. Figure 4-4a represents an example of a situation that benefits from a switching strategy.

4.3.3 Partial Solve Probabilities

In the previous sections, each solution method is modeled such that after the problem solver spends a fixed number of timesteps on the method, the probability of a correct solution is one. However, in real world situations, it is possible for a problem solver to arrive at an incorrect solution due to, for example, a conceptual or calculation mistake. In other words, the solve probability for a given method can be a positive number less than one.

Therefore, it is natural to extend the model such that probability of solving the problem with each method can be any number in the interval $[0, 1]$. We define the *partial solution probability* $p_i \in [0, 1]$ as the probability of solving with method i after t_i timesteps. Therefore, after t_i timesteps, the problem is successfully solved with probability p_i and not successfully solved with probability $1 - p_i$. For clarity of modeling, we reset the counter for the consecutive timesteps if the problem is not successfully solved. In other words, if the problem solver is unable to solve with method i after t_i timesteps, they will start over at the next time step with a method that is determined by their switching strategy. It is possible for multiple solve attempts to occur within the time limit. For example, a problem solver can attempt to solve using one method, complete the required number of timesteps for that method, fail, and then attempt to solve with another method (provided the solve attempts fit within the time constraint t_f).

The partial solution probability p_i represents a problem solver's ability to correctly use a tool in their toolbox. Higher partial solution probabilities will increase a problem solver's no-switch ($\alpha = 0$) overall solve probability P_{solve}, as each effective method will make a greater contribution. Higher partial solution probability can also change the optimal switching strategy α. As a rule of thumb, the higher the partial solution probability, the more benefit there is to implement a switching strategy; this can be especially true if the higher partial solution probability is associated with low solve time methods. Figure 4-4b represents an example of a situation involving partial solution probabilities that benefits from a switching strategy.

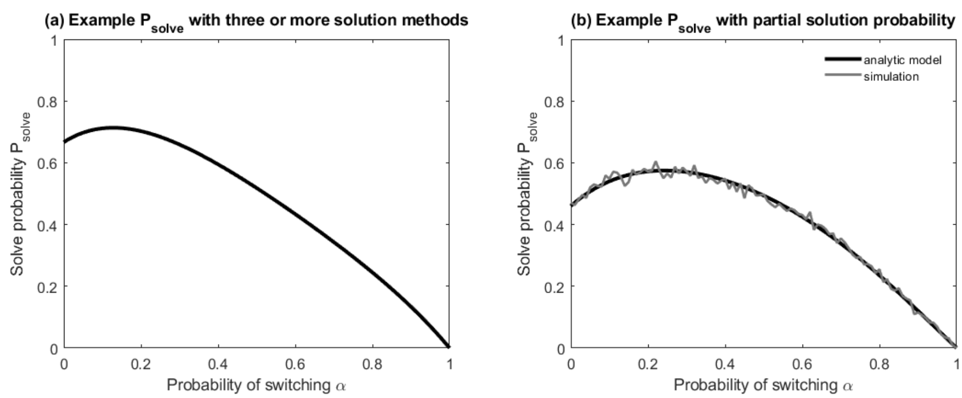


Figure 4-4: (a) An example of P_{solve} with six solution methods, equal starting probabilities, and time limit $t_f = 10$. Four of the solution methods have solve times less than 10 timesteps; their solve times are 2, 4, 4, and 6 timesteps, respectively. The partial solution probabilities p_i are equal to one for all methods. Two methods have solve times longer than 10 timesteps. This is a situation that benefits from a switching strategy, i.e. P_{solve} is maximized from some $\alpha > 0$. (b) An example of P_{solve} where the partial solution probabilities can be between 0 and 1. The result of the analytic model (solid black line) is overlaid with a Monte Carlo simulation (gray line). There are six solution methods with solve times 2, 3, 3, 6, 6, and 6 timesteps. The partial solution probabilities are 0.8, 0.45, 0.8, 0.5, 0.21, and 0, respectively. The starting probabilities are equal, and the time limit is $t_f = 10$ timesteps. Note that methods with lower solve times are associated with higher partial solution probabilities, potentially increasing the benefits of a switching strategy.

4.3.4 Improving Transitions between Solution Methods

Up to this point, the model has assumed a switching strategy where the probability of switching from any method to any other method is equal. However, this may not be the optimal strategy; it is possible to improve P_{solve} by allowing some transitions but not others. For example, the following guidelines can improve the switching strategy. These are rules of thumb; they are true for many, but not necessarily all situations:

- It is favorable for a problem solver to switch from a complex method (high solve time) to a simple method (low solve time) but not vice versa. Once the problem solver arrives at a low solve time method, it is favorable for them to either stay on the method, or switch to another low solve time method. This increases the probability that a problem solver spends a sufficient number of timesteps on a favorable method, thereby increasing P_{solve}.

- It is prudent to avoid switching from one high solve time method (e.g. solve time more than half the time limit) to another high solve time method. The second high solve time method is unlikely to be completed in time. If the problem solver failed to solve using a low solve time method, it may not be favorable to try a more complicated, higher solve time method. Instead, it may be favorable to attempt another low solve time method.

When implementing any improved switching strategy in our model, we set the total switching probability at each timestep to α for simplicity. However, this α is divided only among other methods that are favorable for switching according the heuristics above. In this improved strategy, if the problem solver is working on a simple, low solve time method, they can only switch to other, comparably simple methods. If the problem solver is working on a complex, high solve time method, they can switch to any simpler method. This strategy represents an optimization of the switching strategy outlined in the two solution method model. It requires the problem solver to be aware of solution methods they can implement in a short period of time and with high accuracy. If a problem solver can implement this strategy, they will be able to significantly improve the effectiveness of the tools in their toolbox.

4.4 Matching Analytic Model with Experiment

When comparing the model with the data, we make the following assumption: in the application of the model, the five minute section was not counted in the solve time for model parameter fitting. So, for the problem solving time period corresponding with the model fit, the student had already seen and read the problem, and started brainstorming for the solution. We assume that the student had possibly already read the problem and done some defining and exploring before starting the execution of the solution. In other words, the model does not account for reading, defining, exploring, and planning.

The model was matched with the experimental data from the Volume Problem. The fitting parameters were such that the model's P_{solve} at $\alpha = 0$ was matched with the fraction correct of the study participants who did not switch. Starting probabilities β_i were set by the fraction of students starting on method i in the 10 minute section. Additionally, partial solve probabilities p_i were set by the fraction correct for each method. Furthermore, the model assumed that students only transitioned to visual estimation or geometric approximation (if they are currently on a method that is not "other"). This is consistent with the majority of the transitions observed in the data (Figure 3-3). The resulting P_{solve} vs α curve is shown in Figure 4-5. This curve was overlaid with three data points representing the 64 participants who did not switch methods ($\alpha = 0$), the 72 total participants ($\alpha = .014$) , and the 7 participants who switched methods once ($\alpha = .111$).

The confidence intervals were calculated assuming a Bernoulli distribution for the solve probabilities P_{solve}. The model's P_{solve} prediction is within the confidence intervals for the participants who did not switch and for the total participants. Since the number of students who switched once was low compared to the total number of students, the confidence interval for this data point is wide. However, the model is still within the 95% confidence interval.

4.4.1 Monte Carlo Sensitivity Analysis

Additionally, a Monte Carlo sensitivity analysis was performed to better understand the range of possible P_{solve} curves. Parameters for the model were first estimated from the measured data and then varied. The 95% confidence intervals for solve times t_i and solve probabilities p_i were first calculated from data of the 72 participants who worked the Volume Problem. Then, for the Monte Carlo sensitivity analysis, each value of t_i and p_i was randomly chosen to be either the top or bottom of its respective 95% confidence interval in order to map out the boundaries of our data. Next, the P_{solve} vs α curve was calculated. The results are shown in Figure 4-5.

Note that the Monte Carlo curves coalesced into four "bundles," or groups of similar curves. These bundles correspond with the t_1 and p_1 of the solution method with the lowest solve time and highest solve accuracy, Visual Estimation. The parameters p_1 and t_1 can each take the value of either the top or bottom of its 95% confidence interval, for a total of four combinations (see Figure 4-5 legend). The top bundle (corresponding to the highest P_{solve}) resulted from the solve time t_1 at the bottom of its interval and p_1 at the top of its interval. The existence of these bundles shows the large effect of changing the characteristics of the solution method with the shortest solve time (and in this case, the highest solve probability).

4.4.2 Analysis of Students who Used Multiple Methods

Participants who switched methods on the Volume Problem performed better than the model would have suggested (see Figure 4-5). While no definitive conclusions can be reached due to limited data, it may be illuminating to analyze trends. Of seven participants who switched methods exactly once, four obtained correct answers, an accuracy of 57% compared to the overall accuracy of 36%. Additionally, one participant switched methods twice and obtained the correct answer. Therefore, there were a total of five participants who obtained the correct answer and switched

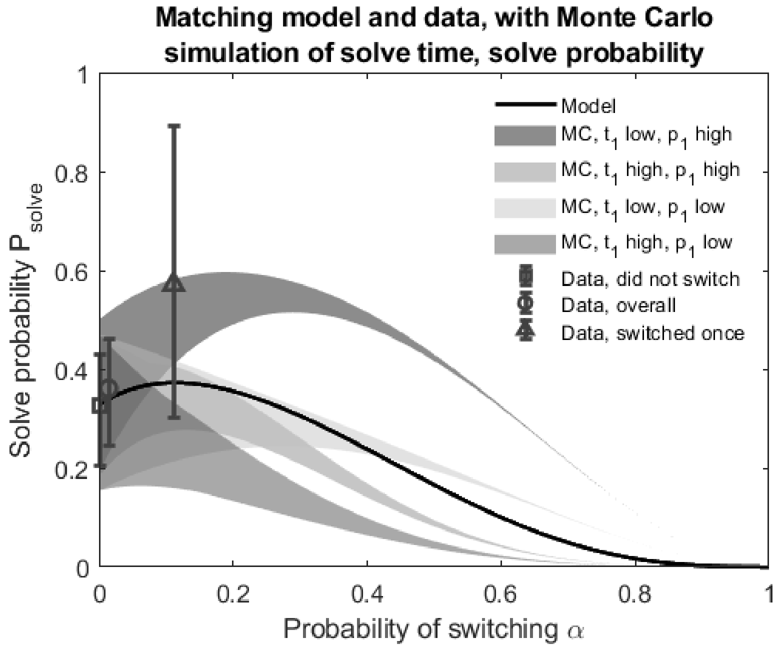


Figure 4-5: Matching model and data, with Monte Carlo sensitivity analysis. Starting method probabilities β_i of the model were set by the fraction of students starting on the corresponding method in the experiment's 10 minute section. Partial solve probabilities p_i of the model were set by the fraction of students successfully using a given method. In the model, transitions to only the first two methods (Visual Estimation and Geometric Approximation) were allowed, unless the first method is "other," in which case transitions to all five other methods were allowed. Monte Carlo simulation regions are also shown, which were generated by overlaying the $P_{solve}(\alpha)$ curves generated from varying the model parameters t_i and p_i. The four regions, corresponding to the different values of t_1 and p_1, are clearly distinct. These regions were generated by overlaying all Monte Carlo P_{solve} curves with the corresponding t_1 and p_1. Here, t_1 and p_1 represent the characteristics of the first method, Visual Estimation; a low solve time t_1 and a high solve probability p_1 led to the largest P_{solve}.

methods. These five participants used various methods, but they obtained the correct answer only through two methods: Visual Estimation and Geometric Approximation.

Three of these five participants reached the correct answer through Visual Estimation; their average solve time was 4 minutes, compared to 3.4 minutes for all students who obtained the correct answers with Visual Estimation. The other two reached the correct answer through Geometric Approximation; their average solve time was 2 minutes, compared to 6.3 minutes for all students who obtained correct answers with Geometric Approximation. These five students had an average solve time of 3.2 minutes, but the average solve time of all students who obtained the correct answer (through any method) was 6.5 minutes, approximately two times greater. This difference in solve times may imply that students who switched methods may be more adept at using strategy; they were not necessarily better at using a specific tool (such as Visual Estimation or Geometric Approximation). Instead, they were able to reduce the solve times through switching to a simpler method. In fact, all five of these participants spent five minutes or less on the method they used to successfully solve the problem. Given that the time limit is 10 minutes, this provides some corroboration for a key prediction of the model: that it would be beneficial to switch if there were methods with solve time of $\frac{t_f}{2}$ or less.

Chapter 5

Analytic Solutions and Proofs

In Chapter 4, we modeled the problem solving process with a time limit and discrete timesteps. In this Chapter, we extend the model to consider additional cases: one with no time limit and one with continuous time. For the cases in this Chapter, we consider $n \geq 2$ solution methods, but do not consider unequal starting probabilities, partial solve probabilities, or unequal method transition probabilities.

First, we present the various formulations. A summary of the formulations is given in Table 5.1. Next, for each formulation, we derive the criteria necessary for which switching methods leads to the optimal outcome. A summary of the optimality criteria is given in Table 5.2. Finally, we derive analytic solutions for the solve probability P_{solve} and the solve time t_{solve} as a function of the method switching tendency (α or λ) for $n = 2$ solution methods. These analytic solutions are then used to calculate the location of the optima for P_{solve} and t_{solve} for the case of continuous time. The purpose of this Chapter is to derive the mathematical facts, which are interesting problems in themselves, and to apply these results.

5.1 Formulations

5.1.1 Discrete (Markov) Formulation of the Primal Problem

This formulation is the basis for the model presented in Chapter 4. Recall that we consider a problem that has n solution methods. Let the time required to solve the problem using method i be discrete quantity t_i. The problem solver successfully solves the problem if they spend t_i consecutive timesteps on method i for any $i \in \{1, 2, \ldots, n\}$. In addition, the problem solver is constrained by a time limit t_f. P_{solve} is the overall probability of solving the problem. The problem solver will start on each method i with probability $\frac{1}{n}$. At each subsequent timestep, the problem solver will switch to another method with probability α and remain on the current method with probability $1 - \alpha$. For simplicity, we set the probability of switching to each of the $n - 1$ other methods to be $\frac{\alpha}{n-1}$.

Define a *Sequence* $\mathcal{S}$ as a progression of methods used by the problem solver. An example of a sequence with ten elements is $\{1, 3, 3, 2, 2, 2, 1, 1, 1, 1\}$. In this work, a sequence can also be represented as a block of digits (e.g. 1332221111) for simplicity. For the Primal Problem, a sequence of methods will be t_f elements long, with each element representing the method used at each timestep.

5.1.2 Discrete (Markov) Formulation of the Dual Problem

The goal of the Dual Problem is to minimize the average solve time, t_{solve}, required to achieve $P_{solve} = 1$. In this problem, we retain the solution methods i with solve times t_i, $i = \{1, 2, \ldots, n\}$. Additionally, we set the probability of starting on each method to be $\frac{1}{n}$. However, there is no time limit. The problem solver will attempt to solve the problem until they succeed. A sequence can be indefinitely long for the Dual Problem, since there is no time limit.

5.1.3 Continuous (Poisson) Formulation of the Primal Problem

We modify the Markov version to obtain the Poisson formulation. To construct the Poisson model, we take the discretization size $\Delta t \to 0$. As before, let t_i represent the time required to solve the problem using method i for $i = \{1, 2, \ldots, n\}$ and let t_f represent the time limit. However, t_i and t_f are now continuous quantities that are not defined in terms of a discrete timestep size. As before, the probability of starting on each method is $\frac{1}{n}$. However, method switching is now modeled as Poisson process. The switching events arrive at a rate $\lambda \geq 0$ with units of arrivals per unit time (instead of arrivals per timestep α in the Markov model). Additionally, the arrivals are independent. In each switching event, a transition to each of the $n-1$ other methods remains equally likely.

5.1.4 Continuous (Poisson) Formulation of the Dual Problem

We use the Poisson formulation with continuous time and retain the solution methods i with solve times t_i, $i = \{1, 2, \ldots, n\}$. We set the probability of starting on each method to be $\frac{1}{n}$ and do not impose a time limit. The problem solver switches methods at a rate $\lambda \geq 0$, and the goal is to minimize the average solve time $t_{solve}(\lambda)$.

5.2 Optimality Criteria

5.2.1 Discrete Primal Problem

Proposition 1. *The existence of an optimum for P_{solve} occurs when*

$$m(1+t_f) - \sum_{i=1}^{m} t_i - \frac{1}{n-1} \sum_{\substack{i \neq j \\ 1 \leq i \leq n \\ 1 \leq j \leq m}} \max(t_j, t_f - t_i + 1) > 0, \tag{5.1}$$

	Discrete	**Continuous**
Primal	Time $t \in \{0, 1, 2, \dots\}$ Method solve times t_i, $i = 1, 2, \dots, n$ Time limit t_f Switching tendency α per timestep Find $\max P_{solve}(\alpha)$ such that $t_{solve} \leq t_f$	Time $t \in [0, \infty)$ Method solve times t_i, $i = 1, 2, \dots, n$ Time limit t_f Switching tendency λ per unit time Find $\max P_{solve}(\lambda)$ such that $t_{solve} \leq t_f$
Dual	Time $t \in \{0, 1, 2, \dots\}$ Method solve times t_i, $i = 1, 2, \dots, n$ No time limit Switching tendency α per timestep Find $\min t_{solve}(\alpha)$ such that $P_{solve} = 1$	Time $t \in [0, \infty)$ Method solve times t_i, $i = 1, 2, \dots, n$ No time limit Switching tendency λ per unit time Find $\min t_{solve}(\lambda)$ such that $P_{solve} = 1$

Table 5.1: Possible formulations of the model. Differences between the Discrete and Continuous models, as well as differences between Primal and Dual problems, are shown.

where solution methods $1, 2, \dots, m$ have solve times less than or equal to t_f, and solution methods $m+1, m+2, \dots, n$ have solve times greater than t_f.

Proof. Consider a problem with n solution methods, of which methods $1, 2, \dots, m$ have solve time not exceeding t_f. A sequence of t_f timesteps may now have numbers $1, 2, \dots, n$ denoting the method used at the timestep. For example, 1222144324 is a possible sequence for $n = 4$ and $t_f = 10$.

The solve probability can be expressed as

$$P_{solve}(\alpha) = \sum_{\text{all } \mathcal{S}} \delta_{\mathcal{S}} P_{\mathcal{S}}(\alpha),$$

where

$$\delta_{\mathcal{S}} = \begin{cases} 1 & \text{Sequence } \mathcal{S} \text{ solves problem} \\ 0 & \text{Otherwise.} \end{cases}$$

The probability of each sequence occurring is

$$P_{\mathcal{S}}(\alpha) = \frac{1}{n} \left(\frac{\alpha}{n-1} \right)^B (1-\alpha)^{t_f - 1 - B},$$

where B is the number of transitions, $\frac{1}{n}$ is the probability of starting on a given method, $\frac{\alpha}{n-1}$ is the probability of switching to a given method, and $1 - \alpha$ is the probability of staying on the same method.

We compute

$$P'_{\mathcal{S}}(\alpha) = \frac{1}{n}\left(\frac{B}{n-1}\left(\frac{\alpha}{n-1}\right)^{B-1}(1-\alpha)^{t_f-1-B} - \left(\frac{\alpha}{n-1}\right)^{B}(t_f-1-B)(1-\alpha)^{t_f-2-B}\right).$$

For $B=0$, we have

$$P'_{\mathcal{S}}(\alpha) = -\frac{1}{n}(t_f-1)(1-\alpha)^{t_f-2},$$

so

$$P'_{\mathcal{S}}(0) = -\frac{1}{n}(t_f-1).$$

If $B=0$, there is only one method used. The problem is solvable with the m solution methods that have solve times not exceeding t_f. The total $P'_{solve}(0)$ contribution is $-\frac{m}{n}(t_f-1)$.

For $B=1$, we have

$$P'_{\mathcal{S}}(\alpha) = \frac{1}{n}\left(\frac{1}{n-1}(1-\alpha)^{t_f-2} - \frac{\alpha}{n-1}(t_f-2)(1-\alpha)^{t_f-3}\right),$$

so

$$P'_{\mathcal{S}}(0) = \frac{1}{n(n-1)}.$$

There are two cases for $B=1$:

(i) The problem is solved with the first method encountered. The first method must be one of the m methods with solve time t_f or less. The number of timesteps spent on the first method (call this method i) must be in $\{t_i, t_{i+1}, \ldots, t_f-1\}$, for a total of $(t_f-1)-t_i+1 = t_f-t_i$ possibilities. The second method can be any other method, for a total of $n-1$ possibilities. Therefore, total number of possible method pairs is $(n-1)\sum_{i=1}^{m}(t_f - t_i)$. The total contribution to $P'_{solve}(0)$ is

$$\frac{1}{n(n-1)} \cdot (n-1)\sum_{i=1}^{m}(t_f - t_i) = \frac{1}{n}\sum_{i=1}^{m}(t_f - t_i).$$

(ii) The problem is solved with the second method encountered. Let t_i and t_j denote the solve times of the first and second methods used, respectively. The problem solver can spend at most $t_i - 1$ timesteps on the first method for $1 \leq i \leq m$, since they do not solve the problem with the first method. This constraint causes the problem solver to spend at least $t_f - t_i + 1$ timesteps on the second method. Additionally, the problem solver must spend at least t_j timesteps on the second method. Combining these constraints, we find that the timesteps spent on the second method can range from $\max(t_j, t_f - t_i + 1)$ to $t_f - 1$, for a total of

$$(t_f - 1) - \max(t_j, t_f - t_i + 1) + 1 = t_f - \max(t_j, t_f - t_i + 1)$$

combinations. Summing over all possibilities of i and j, we obtain

$$\sum_{\substack{i \neq j \\ 1 \leq i \leq n \\ 1 \leq j \leq m}} (t_f - \max(t_j, t_f - t_i + 1)),$$

for a total $P'_{solve}(0)$ contribution of

$$\frac{1}{n(n-1)} \sum_{\substack{i \neq j \\ 1 \leq i \leq n \\ 1 \leq j \leq m}} (t_f - \max(t_j, t_f - t_i + 1)).$$

For $B \geq 2$, we can show that $Q'(0) = 0$ in a similar way to the $n = 2$ case, so there are no contributions to $P'_{solve}(0)$. For an example, see Figure 5-1.

Adding the $P'_{solve}(0)$ contributions together and setting this sum to be positive, we obtain

$$-\frac{m}{n}(t_f-1)+\frac{1}{n}\sum_{i=1}^{m}(t_f-t_i)+\frac{1}{n(n-1)}\sum_{\substack{i\neq j\\1\leq i\leq n\\1\leq j\leq m}}(t_f-\max(t_j,t_f-t_i+1))>0.$$

Combining terms, we obtain

$$-\frac{m}{n}(t_f-1)+\frac{1}{n}\sum_{i=1}^{m}t_f-\frac{1}{n}\sum_{i=1}^{m}t_i+\frac{1}{n(n-1)}\sum_{\substack{i\neq j\\1\leq i\leq n\\1\leq j\leq m}}t_f-\frac{1}{n(n-1)}\sum_{\substack{i\neq j\\1\leq i\leq n\\1\leq j\leq m}}\max(t_j,t_f-t_i+1)>0$$

$$\Leftrightarrow -\frac{m}{n}t_f+\frac{m}{n}+\frac{m}{n}t_f-\frac{1}{n}\sum_{i=1}^{m}t_i+\frac{m}{n}t_f-\frac{1}{n(n-1)}\sum_{\substack{i\neq j\\1\leq i\leq n\\1\leq j\leq m}}\max(t_j,t_f-t_i+1)>0,$$

which is equivalent to

$$m(1+t_f)-\sum_{i=1}^{m}t_i-\frac{1}{n-1}\sum_{\substack{i\neq j\\1\leq i\leq n\\1\leq j\leq m}}\max(t_j,t_f-t_i+1)>0,$$

as desired.

□

5.2.1.1 Remarks

For $n=2$ solution methods, we can apply Equation 5.1 with $n=2$ and $m=1$ to obtain

$$t_1<\frac{t_f+1}{2}. \tag{5.2}$$

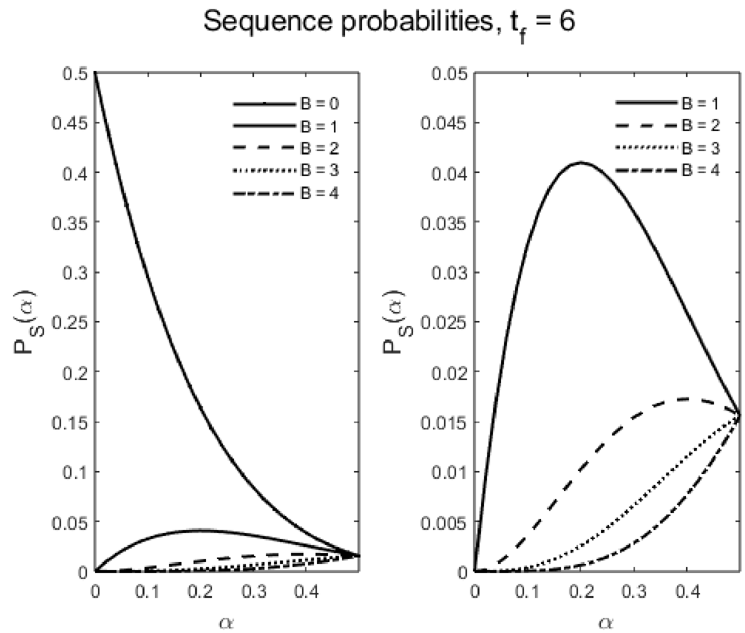


Figure 5-1: Example of probabilities of sequences for $t_f = 6$ with 2 methods. The right subplot is a zoomed in version of the left subplot, with the $B = 0$ curve omitted. Note that $P'_{\mathcal{S}}(0) \neq 0$ for only $B = 0, 1$ transitions.

For integer t_1 and t_f, this is equivalent to $t_1 < \lfloor \frac{t_f}{2} \rfloor$, which agrees with the result in Section 4.2. In the time-limited problem solving scenario, a maximum P_{solve} exists for a switching tendency $\alpha > 0$ if and only if a method with solve time $\lfloor \frac{t_f}{2} \rfloor$ exists. Furthermore, as the solve time of the shorter method is reduced, the corresponding maximum P_{solve} is increased, and this maximum occurs at a higher α (see Figure 4-1). In other words, reducing the solve time of the shorter solution method both improves the solve probability and justifies a higher switching tendency.

Qualitatively, Equation (5.1) means that if there are sufficiently many solution methods of sufficiently low solve time, switching methods with tendency $\alpha > 0$ will improve P_{solve}. However, this optimality condition is a sufficient, but not necessary, condition. There may exist combinations of solve times t_i that do not satisfy (5.1) but still produce a P_{solve} maximum at $\alpha > 0$. If the reader is interested, they can explore a case with $n = 3$, $t_f = 10$, and $t_1, t_2, t_3 = 2, 10, 11$ in Figure 5-2.

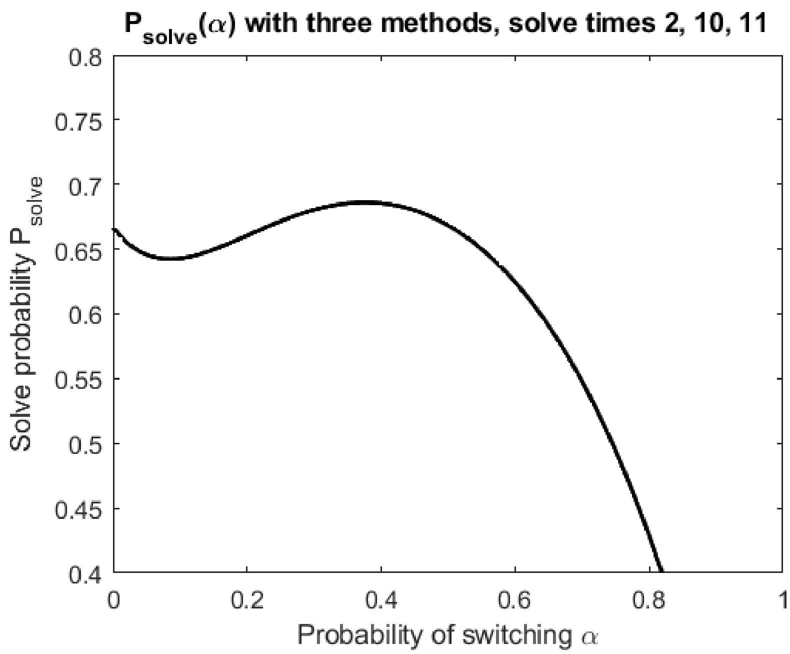


Figure 5-2: Pathological case for three solution methods where $P'_{solve}(0) < 0$ but the maximum occurs at $\alpha \approx 0.376$ and $P_{solve} \approx 0.686$ instead of $\alpha = 0$. For reference, $P_{solve} = \frac{2}{3}$ when $\alpha = 0$. The three methods have solve times 2, 10, and 11, while the time limit is $t_f = 10$. The P_{solve}-axis does not start at zero in order to clearly display the maximum.

5.2.2 Discrete Dual Problem

Proposition 2. *The existence of t_{solve} minimum for $\alpha > 0$ occurs when*

$$\sum_{i=1}^{n}(t_i^2 + t_i) - \frac{4}{n-1}\sum_{i<j} t_i t_j > 0, \tag{5.3}$$

where $t_1 \leq t_2 \leq \cdots \leq t_n$ are the solve times of the n methods.

The proof of Equation (5.3) uses a similar approach as the proof for Equation (5.1). Instead of setting $P'_{solve}(0) > 0$, we let $t'_{solve}(0) < 0$. The full details are given in Appendix D.

5.2.2.1 Remarks

For $n = 2$, the existence of t_{solve} minimum for $\alpha > 0$ occurs when

$$t_2 > \frac{4t_1 - 1 + \sqrt{12t_1^2 - 12t_1 + 1}}{2}, \tag{5.4}$$

where $t_1 \leq t_2$ are the solve times of the two methods. This result can be derived from Equation (5.3). Note that we can write (5.4) as

$$\frac{t_2}{t_1} > \frac{4t_1 - 1 + \sqrt{12t_1^2 - 12t_1 + 1}}{2t_1}$$

and take $t_1 \to \infty$ to obtain

$$\frac{t_2}{t_1} > 2 + \sqrt{3}.$$

This represents the continuous limit of the discrete model as the timestep size Δt approaches zero and the number of timesteps t_1, t_2 approaches infinity.

Compared to the Primal Problem, this $n = 2$ result can be interpreted as more conservative; in order for switching methods to be helpful, the shorter method can have a solve time of at most $\frac{1}{2+\sqrt{3}} \approx 0.268$ times the longer method. In the Primal

Problem with two methods, switching was beneficial when the shorter method is approximately $\frac{1}{2}$ the time limit or less (and the longer method is greater than the time limit). A larger disparity in solve times is needed for switching to be helpful in the time-unlimited scenario of the Dual Problem.

5.2.3 Continuous Primal Problem

Proposition 3. *Let $t_i^* = \frac{t_i}{t_f}$ represent the normalized solve time of method i. Then a maximum P_{solve} exists for $\lambda > 0$ when*

$$m(n-m) - (2n-m-1)\sum_{i=1}^{m} t_i^* + \sum_{\substack{i \neq j \\ 1 \leq i,j \leq m}} \min\left(t_i^*, 1 - t_j^*\right) > 0, \tag{5.5}$$

where solution methods $1, 2, \ldots, m$ have solve times less than or equal to t_f, and solution methods $m+1, m+2, \ldots, n$ have solve times greater than t_f.

Proof. Let $P_{solve|B}$ be the solve probability given B transitions, and let P_B be the probability of B transitions. Using conditional probability and the properties of Poisson distributions, we can write

$$P_{solve}(\lambda) = \sum_{B=0}^{\infty} P_B P_{solve|B}$$

$$= \sum_{B=0}^{\infty} \frac{(\lambda t_f)^B e^{-\lambda t_f}}{B!} P_{solve|B}.$$

Then we compute

$$P'_{solve}(\lambda) = -t_f e^{-\lambda t_f} P_{solve|0} + \sum_{B=1}^{\infty} \frac{B t_f (\lambda t_f)^{B-1} e^{-\lambda t_f} - t_f (\lambda t_f)^B e^{-\lambda t_f}}{B!} P_{solve|B},$$

so

$$P'_{solve}(0) = -t_f P_{solve|0} + t_f P_{solve|1}.$$

For $B = 0$ transitions, the problem is only solved when the problem solver starts on one of the m methods with solve time t_f or less. Thus,

$$P_{solve|0} = \frac{m}{n}.$$

For $B = 1$ transition, the problem solver uses two distinct methods. There are a total of $n(n-1)$ ways to choose these two methods. There are three cases for the two methods chosen:

(i) The two methods chosen have solve time greater than t_f. There are a total of $(n - m)(n - m - 1)$ method combinations. Here, the probability of solving is zero.

(ii) One method chosen has solve time t_f or less, and one method chosen has solve time greater than t_f. There are a total of $2n(n - m)$ method combinations. The probability of solving if the shorter method chosen is $i, 1 \leq i \leq m$, is $1 - t_i^*$, as the method transition occurs after t_i^* amount of time has passed. The probability of solving over all candidate methods in this case will be the average of the $1 - t_i^*$ values:

$$\frac{1}{m}\sum_{i=1}^{m}(1 - t_i^*) = 1 - \frac{1}{m}\sum_{i=1}^{m} t_i^*.$$

This is true regardless of whether the problem is solved with the first or second method chosen.

(iii) Both methods chosen have solve times t_f or less. There are a total of $m(m-1)$ ways to choose the two methods. Let i be the first method chosen and j be the second method chosen. Then:

(a) If the problem is solved with the first method chosen, the probability of solve is $1 - t_i^*$.

(b) If the problem is not solved with the first method but with the second, the transition must land in the interval $(0, t_i^*)$ and it must be true that the remaining time must be sufficient. If $t_i^* + t_j^* \leq 1$, both methods fit within the time limit and the problem is solved regardless of where the transition lands in $(0, t_i^*)$. Else, the transition must land within a subinterval $(0, t_j^*)$, which has the probability $\frac{1-t_j^*}{t_i^*}$ relative to the transition landing in $(0, t_i^*)$. Combining these two possibilities, the probability of solve (solving with second method) is

$$t_i^* \min\left(1, \frac{1 - {t_j}^*}{t_i^*}\right) = \min(t_i^*, 1 - t_j^*).$$

The probability of solve for Case (iii) will be the mean solve probability over all possible choices of i, j:

$$\frac{1}{m(m-1)} \sum_{\substack{i \neq j \\ 1 \leq i,j \leq m}} \left(1 - t_i^* + \min(t_i^*, 1 - t_j^*)\right)$$

$$= \frac{1}{m(m-1)} \left(m(m-1) - (m-1) \sum_{i=1}^{m} t_i^* + \sum_{\substack{i \neq j \\ 1 \leq i,j \leq m}} \min(t_i^*, 1 - t_j^*) \right)$$

$$= 1 - \frac{1}{m} \sum_{i=1}^{m} t_i^* + \frac{1}{m(m-1)} \sum_{\substack{i \neq j \\ 1 \leq i,j \leq m}} \min(t_i^*, 1 - t_j^*).$$

Then we can compute

$$P_{solve|1} = \frac{(n-m)(n-m-1)}{n(n-1)} \cdot 0 + \frac{2m(n-m)}{n(n-1)} \left(1 - \frac{1}{m} \sum_{i=1}^{m} t_i^*\right) +$$

$$+\frac{m(m-1)}{n(n-1)}\left(1-\frac{1}{m}\sum_{i=1}^{m}t_i^*+\frac{1}{m(m-1)}\sum_{\substack{i\neq j\\1\leq i,j\leq m}}\min(t_i^*,1-t_j^*)\right)$$

$$=\frac{2mn-m^2-m}{n(n-1)}-\frac{2n-m-1}{n(n-1)}\sum_{i=1}^{m}t_i^*+\frac{1}{n(n-1)}\sum_{\substack{i\neq j\\1\leq i,j\leq m}}\min(t_i^*,1-t_j^*).$$

We then set $P'_{solve}(0)>0$ to obtain

$$-t_f\frac{m}{n}+t_f\left(\frac{2mn-m^2-m}{n(n-1)}-\frac{2n-m-1}{n(n-1)}\sum_{i=1}^{m}t_i^*+\frac{1}{n(n-1)}\sum_{\substack{i\neq j\\1\leq i,j\leq m}}\min(t_i^*,1-t_j^*)\right)>0$$

$$\Leftrightarrow\frac{m(n-m)}{n(n-1)}-\frac{2n-m-1}{n(n-1)}\sum_{i=1}^{m}t_i^*+\frac{1}{n(n-1)}\sum_{\substack{i\neq j\\1\leq i,j\leq m}}\min(t_i^*,1-t_j^*)>0$$

$$\Leftrightarrow m(n-m)-(2n-m-1)\sum_{i=1}^{m}t_i^*+\sum_{\substack{i\neq j\\1\leq i,j\leq m}}\min(t_i^*,1-t_j^*)>0,$$

as desired.

$\square$

5.2.3.1 Remarks

We can derive the optimality condition for $n=2$. Just as in the Markov formulation of the Primal Problem, we assume that $t_1\leq t_f$ and $t_2>t_f$, so $n=2$ and $m=1$. From Equation (5.5), we have

$$1-2t_1^*>0$$

$$\Leftrightarrow t_1^* < \frac{1}{2}. \tag{5.6}$$

Note that Equation (4.5) for general $n \geq 2$ is a necessary condition. For $n = 2$, we can show that Equation (5.6) is also a sufficient condition. To do this, suppose $t_1 \geq \frac{t_f}{2}$ and consider the space for which the problem is solved. Thus there is an interval with length of at least $\frac{t_f}{2}$ spent on method 1. There is an equally sized space (of equal probability) defined by taking the complement (i.e., wherever there is method 1, replace with method 2, and vice versa). This second space cannot solve the problem because the longest interval ($\geq \frac{t_f}{2}$) will be spent on method 2, which has solve time greater than t_f, and there there is no other interval of length greater than $\frac{t_f}{2}$. (There is one sub-case where $t_1 = \frac{t_f}{2}$ and the method switching arrival occurs at $t = \frac{t_f}{2}$ where the complement also can solve the problem. However, the probability of this occurring is zero since arrivals occur in continuous time).

Because the two spaces have equal probability, the probability of solve (associated with the original space) cannot exceed $\frac{1}{2}$, so there is no benefit from switching. Thus, switching can improve P_{solve} only when $t_1 < \frac{t_f}{2}$. This is similar to the corresponding Markov $n = 2$ result in Equation (5.2).

5.2.4 Continuous Dual Problem

Proposition 4. *The existence of t_{solve} minimum for $\lambda > 0$ occurs when*

$$\sum_{i=1}^{n} t_i^2 - \frac{4}{n-1} \sum_{i<j} t_i t_j > 0, \tag{5.7}$$

where $t_1 \leq t_2 \leq \cdots \leq t_n$ are the solve times of the n methods.

The proof of Equation (5.7) uses a similar approach as the proofs for previous optimality conditions in Equations (5.1), (5.3), and (5.5). The full details are given in Appendix E.

5.2.4.1 Remarks

Applying Equation (5.7) to the $n = 2$ case, we obtain

$$t_1^2 + t_2^2 - 4t_1t_2 > 0.$$

Assuming $t_2 \geq t_1$, this is equivalent to:

$$t_2 > (2 + \sqrt{3})t_1. \tag{5.8}$$

Note that this is the continuous limit of the corresponding Markov Dual $n = 2$ optimality criterion in Equation (5.4). All optimality criteria are summarized in Table 5.2.

	Discrete	Continuous
Primal	$n = 2$: Let $t_1 \leq t_f < t_2$. Then $\max P_{solve}$ occurs at $\alpha > 0$ when $t_1 \leq \left\lfloor \frac{t_f}{2} \right\rfloor$. $n \geq 2$: See Proposition 5.1.	$n = 2$: Let $t_1 \leq t_f < t_2$. Then $\max P_{solve}$ occurs at $\lambda > 0$ when $t_1 < \frac{t_f}{2}$. $n \geq 2$: See Proposition 5.5.
Dual	$n = 2$: Let $t_1 \leq t_2$. Then $\min t_{solve}$ occurs at $\alpha > 0$ when $t_2 > \frac{4t_1 - 1 + \sqrt{12t_1^2 - 12t_1 + 1}}{2}$. $n \geq 2$: See Proposition 5.3.	$n = 2$: Let $t_1 \leq t_2$. Then $\min t_{solve}$ occurs at $\lambda > 0$ when $t_2 > (2 + \sqrt{3})t_1$. $n \geq 2$: See Proposition 5.7.

Table 5.2: Optimality criteria for the different formulations in the model. The criteria for $n = 2$ solution methods are given. References for $n \geq 2$ solution methods are included.

5.3 Analytic Solutions for P_{solve} and t_{solve} for $n = 2$ Solution Methods

We give the analytic solutions for P_{solve} and t_{solve} for $n = 2$ below. The results for the discrete model are proved in Appendices F and G. The results for the continuous model are proved below.

5.3.1 Discrete Primal Problem $P_{solve}(\alpha)$

Proposition 5. *In a Markov process, the probability of exactly B transitions occurring within t_f timesteps is*

$$\binom{t_f - 1}{B}\alpha^B(1-\alpha)^{t_f-1-B}.$$

Then the probability of solving the problem is

$$P_{solve}(\alpha) = \sum_{B=0}^{\infty}\binom{t_f - 1}{B}\alpha^B(1-\alpha)^{t_f-1-B}P_{solve|B}(t_1), \tag{5.9}$$

where $P_{solve|B}(t_1)$ is the probability of solving given a fixed B transitions. We have

$$P_{solve|B} = \begin{cases} \frac{1}{\binom{t_f-1}{B}}p\left(B, \frac{B+1}{2}\right) & \textit{if } B \textit{ is odd} \\ \frac{1}{2\binom{t_f-1}{B}}\left(p\left(B, \frac{B}{2}\right) + p\left(B, \frac{B}{2}+1\right)\right) & \textit{if } B \textit{ is even,} \end{cases} \tag{5.10}$$

where

$$p(B,C) = \begin{cases} \binom{C}{1}\binom{t_r(1)}{B} & \textit{if } t_b(1) < t_1 \le t_b(0) \\ \binom{C}{1}\binom{t_r(1)}{B} - \binom{C}{2}\binom{t_r(2)}{B} & \textit{if } t_b(2) < t_1 \le t_b(1) \\ \vdots & \\ \binom{C}{1}\binom{t_r(1)}{B} - \binom{C}{2}\binom{t_r(2)}{B} + \cdots + (-1)^{k+1}\binom{C}{k}\binom{t_r(k)}{B} & \textit{if } 1 < t_1 \le t_b(k-1), \end{cases} \tag{5.11}$$

$t_r(j) = t_f - jt_1 + (j-1)$, $t_b(j) = \frac{t_f-(B-j)}{j+1}$, *and* $k = \min\left(\left\lfloor \frac{t_f-B-1}{t_1-1}\right\rfloor, C\right)$.

The proof of Proposition 5 is given in Appendix F.

5.3.2 Discrete Dual Problem $t_{solve}(\alpha)$

Proposition 6. *Let* $r_1 = 1 - (1-\alpha)^{t_1 - 1}$ *and* $r_2 = 1 - (1-\alpha)^{t_2 - 1}$ *represent the probabilities of not solving the problem before the next transition for method 1 and method 2, respectively. Let*

$$\tilde{t_1} = \begin{cases} \frac{1-(1+\alpha(t_1-1))(1-\alpha)^{t_1-1}}{\alpha(1-(1-\alpha)^{t_1-1})} & \text{if } \alpha > 0 \\ \frac{t_1}{2} & \text{if } \alpha = 0 \end{cases}$$

and

$$\tilde{t_2} = \begin{cases} \frac{1-(1+\alpha(t_2-1))(1-\alpha)^{t_2-1}}{\alpha(1-(1-\alpha)^{t_2-1})} & \text{if } \alpha > 0 \\ \frac{t_2}{2} & \text{if } \alpha = 0 \end{cases}$$

represent the expected time spent on a problem before the next transition, given that the solution is not successful, for method 1 and method 2, respectively. Then the solve time is

$$t_{solve} = \frac{t_1(1-r_1) + t_2(1-r_2) + (t_2 + \tilde{t_1})r_1(1-r_2) + (t_1 + \tilde{t_2})r_2(1-r_1) + 2r_1r_2(\tilde{t_1} + \tilde{t_2})}{2(1-r_1r_2)},$$

where we have suppressed the dependence on α *for* r_1, r_2, $\tilde{t_1}$, $\tilde{t_2}$, *and* t_{solve} *for compactness of notation.*

The proof of Proposition 6 is given in Appendix G.

5.3.3 Continuous Primal Problem $P_{solve}(\lambda)$

Proposition 7. *Let* $t_1^* = \frac{t_1}{t_f}$ *be the solve time for the first method normalized by the time limit. In a Poisson process, the probability of exactly* B *transitions occurring within a time interval* t_f *is*

$$\frac{(\lambda t_f)^B e^{\lambda t_f}}{B!}.$$

Then the probability of solving the problem is

$$P_{solve}(\lambda) = \sum_{B=0}^{\infty} \frac{(\lambda t_f)^B e^{-\lambda t_f}}{B!} P_{solve|B}(t_1^*), \tag{5.12}$$

where $P_{solve|B}(t_1^*)$ *is the probability of solving given a fixed* B *transitions. Note that this is a function of the nondimensionalized solve time for the first method. We have*

$$P_{solve|B} = \begin{cases} p\left(B, \frac{B+1}{2}\right) & \textit{if } B \textit{ is odd} \\ \frac{1}{2}\left(p\left(B, \frac{B}{2}\right) + p\left(B, \frac{B}{2}+1\right)\right) & \textit{if } B \textit{ is even,} \end{cases} \tag{5.13}$$

where

$$p(B,C) = \begin{cases} \binom{C}{1}(1-t_1^*)^B & \textit{if } t_1^* \geq \frac{1}{2} \\ \binom{C}{1}(1-t_1^*)^B - \binom{C}{2}(1-2t_1^*)^B & \textit{if } \frac{1}{3} \leq t_1^* < \frac{1}{2} \\ \binom{C}{1}(1-t_1^*)^B - \binom{C}{2}(1-2t_1^*)^B + \binom{C}{3}(1-3t_1^*)^B & \textit{if } \frac{1}{4} \leq t_1^* < \frac{1}{3} \\ \vdots & \\ \binom{C}{1}(1-t_1^*)^B - \binom{C}{2}(1-2t_1^*)^B + \cdots + \binom{C}{C}(-1)^{C+1}(1-Ct_1^*)^B & \textit{if } t_1^* < \frac{1}{C}. \end{cases} \tag{5.14}$$

Proof. We can begin by proving the case for which B is odd. Let

$$t^{(1)}, t^{(2)}, \ldots, t^{(B)} \in [0,1]$$

be the times at which the transitions arrive. These times are already normalized by the time limit t_f. Then WLOG assume

$$0 \leq t^{(1)} \leq t^{(2)} \leq \cdots \leq t^{(B)} \leq 1.$$

The volume of this pyramid in B-dimensional space is $\frac{1}{B!}$ because there are $B!$ ways to order the arrivals. This can also be seen by evaluating the nested integral

$$\begin{aligned}
V_0 &= \int_0^1 \int_0^{t^{(B)}} \cdots \int_0^{t^{(3)}} \int_0^{t^{(2)}} dt^{(1)} dt^{(2)} \cdots dt^{(B-1)} dt^{(B)} \\
&= \int_0^1 \int_0^{t^{(B)}} \cdots \int_0^{t^{(3)}} t^{(2)} dt^{(2)} \cdots dt^{(B-1)} dt^{(B)} \\
&\vdots \\
&= \int_0^1 \cdots \int_0^{t^{(j+1)}} \frac{\left(t^{(j)}\right)^{j-1}}{(j-1)!} dt^{(j-1)} \cdots dt^{(B)} \\
&\vdots \\
&= \int_0^1 \frac{\left(t^{(B)}\right)^{B-1}}{(B-1)!} dt^{(B)} \\
&= \frac{1}{B!}.
\end{aligned}$$

In order for the problem solver to solve the problem, additional constraints are needed. Consider the case where the problem solver starts on method 1. For odd B, the problem solver is on method 1 in the disjoint intervals

$$[0, t^{(1)}], [t^{(2)}, t^{(3)}], \ldots, [t^{(B-1)}, t^{(B)}].$$

At least one of these intervals must have length at least t_1^* for the problem solver to solve the problem. Therefore at least one of the following $C = \frac{B+1}{2}$ constraints must be satisfied:

$$
\begin{aligned}
t^{(1)} &\geq t_1^* \\
t^{(3)} - t^{(2)} &\geq t_1^* \\
t^{(5)} - t^{(4)} &\geq t_1^* \\
&\vdots \\
t^{(B)} - t^{(B-1)} &\geq t_1^*.
\end{aligned}
$$

These constraints can be rewritten as:

$$
\begin{aligned}
&t^{(1)} \geq t_1^* \\
&t^{(2)} \leq t^{(3)} - t_1^* \\
&t^{(4)} \leq t^{(5)} - t_1^* \\
&\vdots \\
t^{(B-1)} &\leq t^{(B)} - t_1^*.
\end{aligned}
$$

If $t_1^* > \frac{1}{2}$, exactly one of the above constraints can be satisfied in the solution region. Otherwise two disjoint intervals within $[0, 1]$ will have a combined length greater than one, which is impossible. If $\frac{1}{3} < t_1^* \leq \frac{1}{2}$, one or two constraints can be satisfied (otherwise, three disjoint intervals within $[0, 1]$ will have combined length greater than one). In general, if $\frac{1}{r+1} < t_1^* \leq \frac{1}{r}$, it is possible for $k = 1, 2, \ldots, r$ constraints to be satisfied. This is repeated until $0 \leq t_1^* \leq \frac{1}{C}$. where it is possible for $k = 1, 2, \ldots, C$ constraints to be satisfied. If k of the C constraints are satisfied, where $1 \leq k \leq C$, then the volume of the enclosed region is

$$V_1 = \overbrace{\int_0^1 \int_0^{t^{(B)}-(\cdot)} \cdots \int_0^{t^{(4)}-(\cdot)} \int_0^{t^{(3)}} \int_{(\cdot)}^{t^{(2)}}}^{\text{Replace } k \text{ of the } (\cdot) \text{ with } t_1^*, \text{ the remainder with } 0} dt^{(1)} dt^{(2)} dt^{(3)} \cdots dt^{(B-1)} dt^{(B)}.$$

At the jth (where j is odd) evaluation of the nested integral starting from the innermost integral, we obtain

$$V_1 = \int_0^1 \cdots \underbrace{\int_0^{t^{(j+1)}-(\cdot)} \frac{(t^{(j)} - lt_1^*)^{j-1}}{(j-1)!} dt^{(j)}}_{I} \cdots dt^{(B)},$$

where l is the accumulated number of $(\cdot)$'s that equal t_1^*. For the immediate integral corresponding to the jth evaluation (denoted I),

$$I = \begin{cases} \frac{(t^{(j)} - lt_1^*)^j}{j!} & \text{if } (\cdot) = 0 \\ \frac{(t^{(j)} - (l+1)t_1^*)^j}{j!} & \text{if } (\cdot) = t_1^*. \end{cases}$$

Each time $(\cdot) = t_1^*$, the coefficient of the t_1^* term in the integrand increases by 1. Thus, repeating this process, we obtain

$$V_1 = \frac{(1 - kt_1^*)^B}{B!}.$$

For each value of r, the union of all volumes for $k = 1, 2, \ldots, r$ is found using the Principle of Inclusion-Exclusion. A visualization of the interplay between constraints for $k = 2$ and $k = 3$ is shown in Figure 5-3. There are $\binom{C}{k}$ ways to choose k constraints from C constraints, so we can compute for a fixed r

$$V_r = \frac{1}{B!}\left(\binom{C}{1}(1-t_1^*)^B - \binom{C}{2}(1-2t_1^*)^B + \cdots + \binom{C}{r}(-1)^{(r+1)}(1-rt_1^*)^B\right).$$

Thus the corresponding solve probability is

$$\frac{V_r}{V_0} = \left(\binom{C}{1}(1-t_1^*)^B - \binom{C}{2}(1-2t_1^*)^B + \cdots + \binom{C}{r}(-1)^{(r+1)}(1-rt_1^*)^B\right).$$

By symmetry, this solve probability is the same when the problem solver starts on method 2. Combining the required cases, we obtain Equations 5.13 and 5.14 for odd B. For even B, a similar approach is used; the number of constraints is $C = \frac{B}{2} + 1$ when starting on method 1, and $C = \frac{B}{2}$ when starting on method 2. The process can be repeated to obtain Equations 5.13 and 5.14 for even B.

□

5.3.3.1 Location of Optimum

Using the analytic solution derived in Section 5.3.3, we find the optimal switching tendency λ that maximizes P_{solve} as a function of t_1^*, the nondimensional solve time of the shorter method (see Figure 5-4). For $t_1^* < \frac{1}{2}$, the optimal switching tendency is positive; λ_{opt} increases for decreasing t_1^*. Additionally, we find the improvement in P_{solve} (compared to the no-switch case) associated with the optimal λ.

As a reference value, we find that to obtain a P_{solve} improvement of 10%, we need a t_1^* of 0.32, with a corresponding optimum λ of 0.143 for $t_f = 10$ (1.43 switches per t_f period). To obtain a P_{solve} improvement of 40%, we need a t_1^* of 0.127, with an optimum λ of 0.52 (5.2 switches per t_f period). In order to achieve large improvements in P_{solve}, there must exist a very short method (compared to the time limit), and the

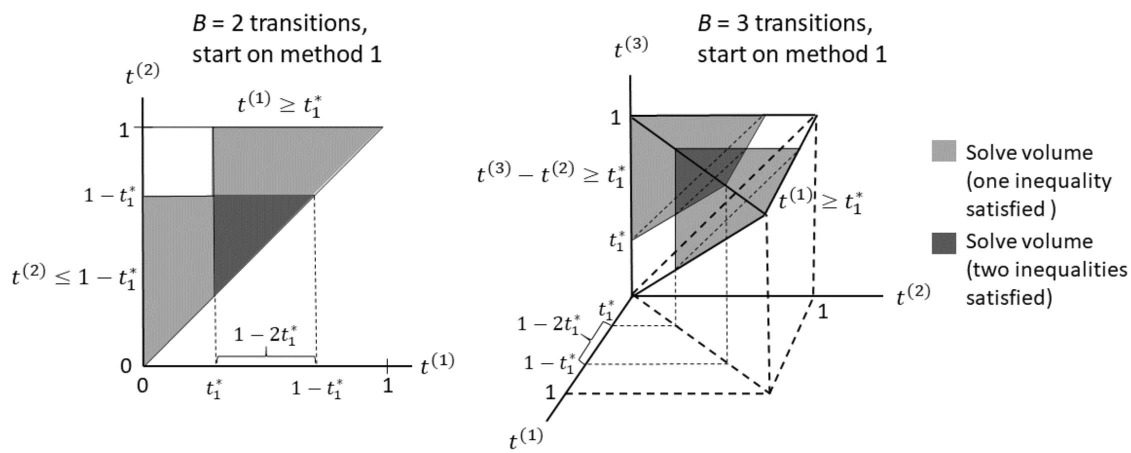


Figure 5-3: Visualization of the Primal Problem $P_{solve} = \frac{V_r}{V_0}$ for cases with $B = 2$ and $B = 3$ transitions. For each case, the problem solver started on method 1 (the shorter method). Light shaded areas represent regions where the problem is solved because one inequality is satisfied. Dark shaded areas represent regions where the problem is solved, but two inequalities are satisfied. We are interested in the union of all regions where at least one inequality is satisfied. Therefore, when computing the total P_{solve}, the dark shaded areas must be accounted for using the Principle of Inclusion-Exclusion.

problem solver must be willing to switch methods with high tendency λ in order to encounter this method.

5.3.4 Continuous Dual Problem $t_{solve}(\lambda)$

Proposition 8. *Let $r_1 = 1 - e^{-\lambda t_1}$ and $r_2 = 1 - e^{-\lambda t_2}$ represent the probabilities of not solving the problem before the next transition for method 1 and method 2, respectively. Let*

$$\tilde{t_1} = \begin{cases} \frac{1}{\lambda} - \frac{t_1 e^{-\lambda t_1}}{1 - e^{-\lambda t_1}} & \textit{if } \lambda > 0 \\ \frac{t_1}{2} & \textit{if } \lambda = 0 \end{cases}$$

and

$$\tilde{t_2} = \begin{cases} \frac{1}{\lambda} - \frac{t_2 e^{-\lambda t_2}}{1 - e^{-\lambda t_2}} & \textit{if } \lambda > 0 \\ \frac{t_2}{2} & \textit{if } \lambda = 0 \end{cases}$$

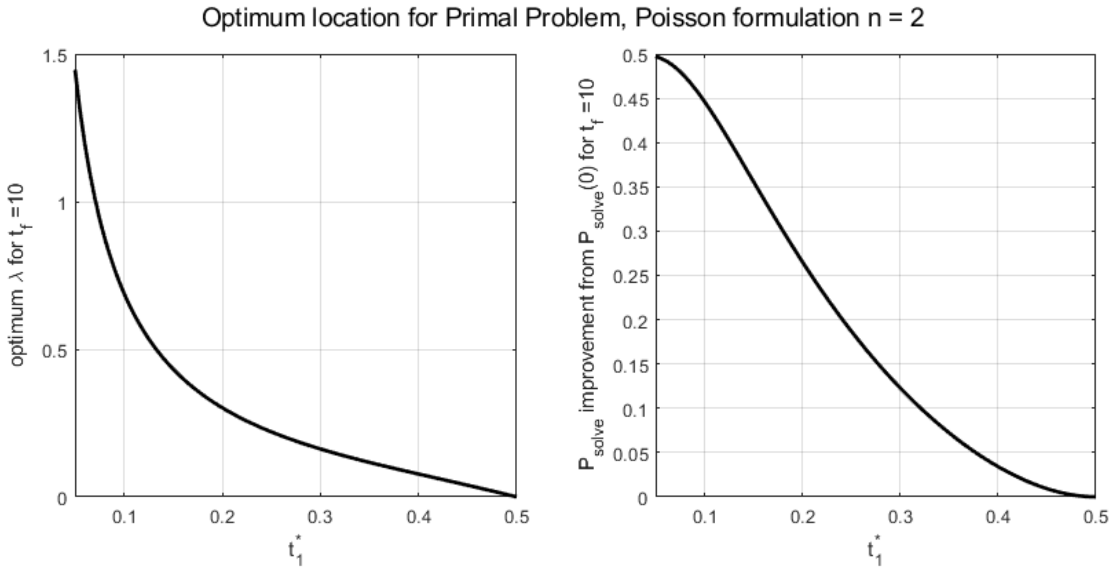


Figure 5-4: Optimum location for two solution methods for the Poisson Primal formulation as a function of the nondimensionalized solve time of the shorter method, t_1^*. Optimal λ (left) is the switching tendency λ_{opt} that maximizes P_{solve} for $t_f = 10$. P_{solve} improvement (right) is the improvement in solve probability over the no-switch case, $P_{solve}(\lambda_{opt}) - P_{solve}(0)$.

represent the expected time spent on a problem before the next transition, given that the solution is not successful, for method 1 and method 2, respectively. Then the solve time is

$$t_{solve} = \frac{t_1(1-r_1) + t_2(1-r_2) + (t_2+\tilde{t_1})r_1(1-r_2) + (t_1+\tilde{t_2})r_2(1-r_1) + 2r_1r_2(\tilde{t_1}+\tilde{t_2})}{2(1-r_1r_2)},$$

where we have suppressed the dependence on λ for r_1, r_2, $\tilde{t_1}$, $\tilde{t_2}$, and t_{solve} for compactness of notation.

Proof. For two solution methods, the problem solver will progress through a decision tree as in Figure 5-5. The following outcomes are possible:

(i) The problem solver starts on method 1, switches methods an even number of times, and eventually solves with method 1.

(ii) The problem solver starts on method 1, switches methods an odd number of times, and eventually solves with method 2.

(iii) The problem solver starts on method 2, switches methods an even number of times, and eventually solves with method 2.

(iv) The problem solver starts on method 2, switches methods an odd number of times, and eventually solves with method 1.

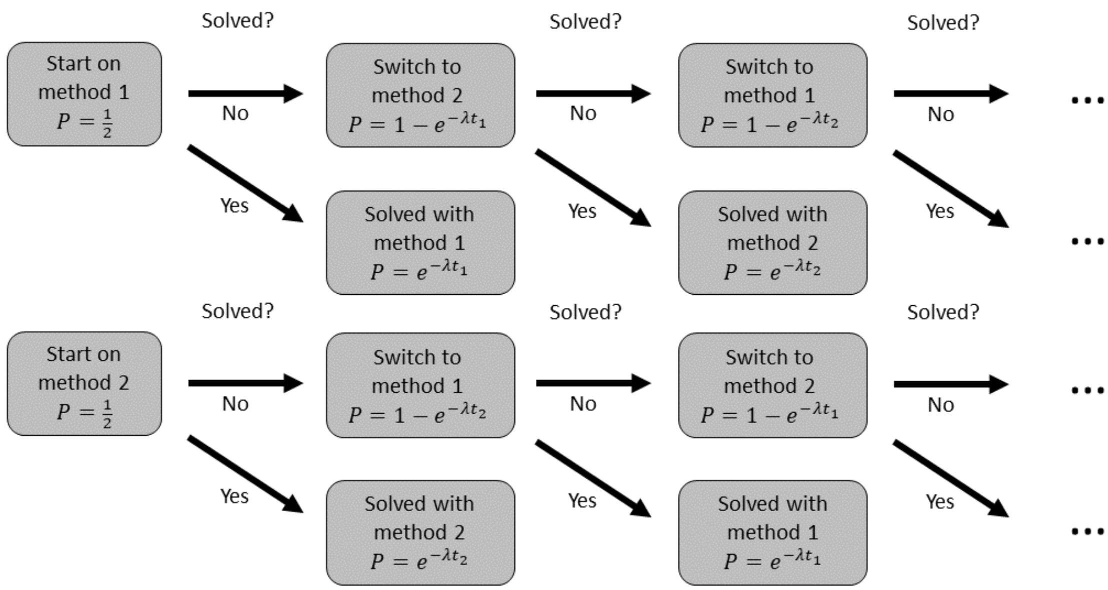


Figure 5-5: Decision tree for the Poisson Dual Problem with two solution methods. The problem solver will start on either method with probability $P = \frac{1}{2}$. If the problem solver solves the problem with method i, they will do so with probability $P = e^{-\lambda t_i}$. If the problem solver does not solve the problem with method i, they will do so with probability $P = 1 - e^{-\lambda t_i}$, and switch to the other method. The problem solver will continue to switch methods until they solve the problem.

Consider Case (i) first. The probability that the problem solver starts on method 1 and solves the problem with zero switching corresponds to the first Poisson arrival occurring in the interval $[t_1, \infty)$, that is, after the problem is solved with method 1. The distribution of Poisson arrival times is the exponential distribution, so the cumulative distribution function $F_T(t) = 1 - e^{-\lambda t}$ can be used to calculate this probability. This probability is equal to $1 - F_T(t_1) = e^{-\lambda t_1}$. The corresponding solve time is t_1.

The probability that the problem solver starts on method 1, switches two times before solving, and then solves with method 1 corresponds to the first Poisson arrival occurring in the interval $[0, t_1)$, the second arrival occurring in the interval $[0, t_2)$ *after the first arrival*, and the third arrival occurring in the interval $[t_1, \infty)$ *after the second arrival.* The probability is

$$F_T(t_1)F_T(t_2)(1 - F_T(t_1)) = (1 - e^{-\lambda t_1})(1 - e^{-\lambda t_2})e^{-\lambda t_1}.$$

The mean time spent on method i, given the problem is not solved, is the mean value of t with exponential probability distribution function $f_T(t) = \lambda e^{-\lambda t}$ over the interval $[0, t_i)$. Call this $\tilde{t_i}$:

$$\begin{aligned}
\tilde{t_i} &\equiv \frac{\int_0^{t_i} t\lambda e^{-\lambda t} dt}{\int_0^{t_i} \lambda e^{-\lambda t} dt} \\
&= \frac{\frac{1}{\lambda}\left(1 - e^{-t_i\lambda})(\lambda t_i + 1\right)}{1 - e^{-\lambda t_i}} \\
&= \frac{1}{\lambda} - \frac{t_i e^{-\lambda t_i}}{1 - e^{-\lambda t_i}}.
\end{aligned}$$

This is not defined at $\lambda = 0$, but

$$\lim_{\lambda \to 0} \left(\frac{1}{\lambda} - \frac{t_i e^{-\lambda t_i}}{1 - e^{-\lambda t_i}} \right) = \frac{t_i}{2}.$$

Therefore the mean solve time given that the problem solver starts on method 1, switches two times, and solves with method 1 is:

$$\frac{1}{\lambda} - \frac{t_1 e^{-\lambda t_1}}{1 - e^{-\lambda t_1}} + \frac{1}{\lambda} - \frac{t_2 e^{-\lambda t_2}}{1 - e^{-\lambda t_2}} + t_1 = \tilde{t_1} + \tilde{t_2} + t_1.$$

In general, the probability of starting on method 1, switching $2k$ times, and solving with method 1 is

$$\left((1 - e^{-\lambda t_1})(1 - e^{-\lambda t_2})\right)^k e^{-\lambda t_1}.$$

The corresponding solve time is

$$k(\tilde{t_1} + \tilde{t_2}) + t_1.$$

Next, consider Case (ii). We can see the probability of starting on method 1, switching $2k - 1$ times, and solving with method 2 is

$$(1 - e^{-\lambda t_1})^k (1 - e^{-\lambda t_2})^{k-1} e^{-\lambda t_2}.$$

The corresponding solve time is

$$k\tilde{t_1} + (k-1)\tilde{t_2} + t_2.$$

For ease of notation, set $r_1 = 1 - e^{-\lambda t_1}$ and $r_2 = 1 - e^{-\lambda t_2}$. The average $t_{solve}(\lambda)$, given that the problem solver starts on method 1 (Cases (i) and (ii)) is

$$t_{solve,1} = \sum_{k=0}^{\infty} \left((k(\tilde{t_1} + \tilde{t_2}) + t_1)(r_1 r_2)^k (1 - r_1) + ((k+1)\tilde{t_1} + k\tilde{t_2} + t_2) r_1^{k+1} r_2^k (1 - r_2) \right).$$

Note that this is a sum of geometric and arithmetic-geometric series. Then we can evaluate

$$t_{solve,1} = \sum_{k=0}^{\infty} t_1 (r_1 r_2)^k (1 - r_1) + \sum_{k=0}^{\infty} k(\tilde{t_1} + \tilde{t_2})(r_1 r_2)^k (1 - r_1) +$$

$$\sum_{k=0}^{\infty} (t_2 + \tilde{t_1}) r_1^{k+1} r_2^k (1 - r_2) + \sum_{k=0}^{\infty} k(\tilde{t_1} + \tilde{t_2}) r_1^{k+1} r_2^k (1 - r_2)$$

$$= \frac{t_1(1-r_1)}{1-r_1r_2} + \frac{(\tilde{t_1}+\tilde{t_2})r_1r_2(1-r_1)}{(1-r_1r_2)^2} + \frac{(t_2+\tilde{t_1})r_1(1-r_2)}{1-r_1r_2} + \frac{(\tilde{t_1}+\tilde{t_2})r_1^2r_2(1-r_2)}{(1-r_1r_2)^2}$$

$$= \frac{t_1(1-r_1)+(t_2+\tilde{t_1})r_1(1-r_2)}{1-r_1r_2} + \frac{(\tilde{t_1}+\tilde{t_2})(r_1r_2(1-r_1)+r_1^2r_2(1-r_2))}{(1-r_1r_2)^2}.$$

By symmetry, the average $t_{solve}(\lambda)$, given that the problem solver starts on method 2 (Cases (iii) and (iv)) is

$$t_{solve,2} = \frac{t_2(1-r_2)+(t_1+\tilde{t_2})r_2(1-r_1)}{1-r_1r_2} + \frac{(\tilde{t_1}+\tilde{t_2})(r_1r_2(1-r_2)+r_2^2r_1(1-r_1))}{(1-r_1r_2)^2}.$$

The overall t_{solve} is

$$t_{solve} = \frac{1}{2}(t_{solve,1}+t_{solve,2})$$

$$= \frac{t_1(1-r_1)+t_2(1-r_2)+(t_2+\tilde{t_1})r_1(1-r_2)+(t_1+\tilde{t_2})r_2(1-r_1)}{2(1-r_1r_2)} +$$

$$\frac{(\tilde{t_1}+\tilde{t_2})\left(r_1r_2(2-(r_1+r_2))+r_1^2r_2(1-r_2)+r_2^2r_1(1-r_1)\right)}{2(1-r_1r_2)^2}$$

$$= \frac{t_1(1-r_1)+t_2(1-r_2)+(t_2+\tilde{t_1})r_1(1-r_2)+(t_1+\tilde{t_2})r_2(1-r_1)+2r_1r_2(\tilde{t_1}+\tilde{t_2})}{2(1-r_1r_2)},$$

as desired.

$\square$

5.3.4.1 Location of Optimum for $n = 2$

We use the analytic solution derived in Section 5.3.4 to find the optimal switching tendency λ that minimizes t_{solve} as a function of $\frac{t_1}{t_2}$, the ratio of the solve times of the longer and shorter methods (see Figure 5-6). For $\frac{t_1}{t_2} < \frac{1}{2+\sqrt{3}}$, the optimal switching tendency is positive; λ_{opt} increases for decreasing $\frac{t_1}{t_2}$. Additionally, we find the fraction reduction in t_{solve} (compared to the no-switch case) associated with the optimal λ.

As a reference value, we find that to obtain a t_{solve} reduction of 10%, we need a $\frac{t_1}{t_2}$ of 0.156, with a corresponding optimum λ of 0.195 for $t_f = 10$ (1.95 switches per time period of length t_2). To obtain a t_{solve} reduction of 50%, we need a $\frac{t_1}{t_2}$ of 0.075, with a optimum λ of 0.76 (7.6 switches per time period of length t_2). In order to achieve large reductions in t_{solve}, one method must be very short compared to the other, and the problem solver must be willing to switch methods with high tendency λ in order to encounter this shorter method.

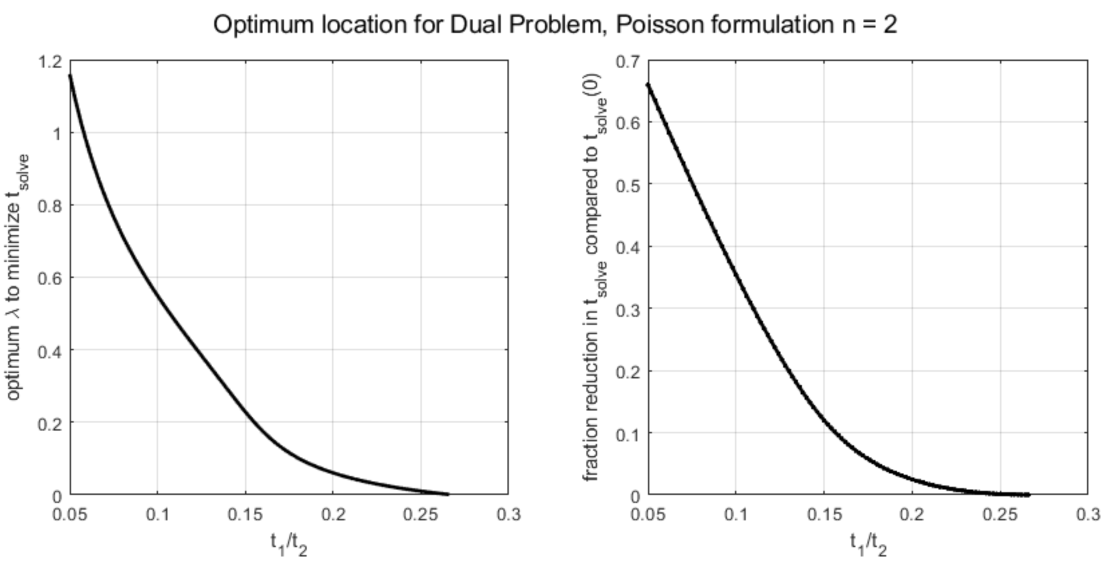


Figure 5-6: Optimum location for two solution methods for Poison Dual formulation as a function of the ratio of the two methods' solve times, $\frac{t_1}{t_2}$. Optimal λ (left) is the switching tendency λ_{opt} that minimizes t_{solve} for $t_2 = 10$. Fraction t_{solve} reduction (right) is the reduction in solve time over the no-switch case, $\left|\frac{t_{solve}(\lambda_{opt}) - t_{solve}(0)}{t_{solve}(0)}\right|$.

5.4 Remarks

In this Chapter, we extended the analytic model formulated in Chapter 4. For both Primal and Dual Problems and both Markov and Poisson formulations, we derived conditions for which the problem solving outcome can be optimized by using the correct switching tendency. While the problem solving implications of the full formulas may not be immediately obvious, the cases with two solution methods ($n = 2$) represent a good approximation of the general trends. In the Primal Problem, if there is a method with a solve time of half the time limit ($\frac{t_f}{2}$) or less, switching methods with optimal tendency will maximize the solve probability P_{solve}. In the Dual Problem, if the ratio of the solve times of the short method and the longer method is $\frac{1}{2+\sqrt{3}} \approx 0.268$ or less, then switching methods with optimal tendency will minimize the average solve time t_{solve}. Increasing the number of solution methods will increase the intricacy of the optimality space for the switching tendency (α or λ). However, general trends remain. The existence of sufficiently short methods compared to t_{solve} for the Primal Problem, or the existence of methods of sufficiently different solve time for the Dual Problem will lead to a nonzero switching tendency optimizing the problem solving outcome.

Additionally, we derived analytic solutions for $n = 2$ Primal and Dual Problems for both the Markov and Poisson formulations. The Poisson Primal Problem solution $P_{solve}(\lambda)$ given in Proposition 7 has a solution space with a regular geometric structure; it is the union of pyramid volumes inside a multi-dimensional unit cube (see Figure 5-3). The Poisson Dual Problem solution $t_{solve}(\lambda)$ given in Proposition 8 can be represented as a series of outcomes given in a flowchart in Figure 5-5. These analytic solutions are a way to visually represent the structure of Poisson formulation solutions. They also allow for the computation of the optimal switching tendency, as well as the extrema of P_{solve} (Figure 5-4) and t_{solve} (Figure 5-6).

5.5 Applications

Note that in the $n = 2$ time-limited case, the threshold for switching ($t_1 < \frac{t_f}{2}$) is less strict than the time unlimited case ($t_1 < \frac{t_2}{2+\sqrt{3}}$). With a time limit, there are more situations where switching is beneficial. An example of a situation that has a restrictive time limit is an exam. In an exam with appropriate difficulty, the time limit and the method solve times are approximately the same order of magnitude, i.e., $t_f \sim t_i$. If the student is able to find a short method of length $\frac{t_f}{2}$ or less, then a switching strategy will help them improve their solve probability.

On the other hand, there are situations that do not have a restrictive time limit. One example is a problem set for an undergraduate class. The time limit is on the order of one week, but the time required to finish the problem set is usually on the order of 3-6 hours. Since $t_f \gg t_i$, we can approximate this as a time unlimited situation. Yet the student still seeks to minimize their solve time while still producing a satisfactory result, so they have time for sleep, socializing, etc.

Another example is research for a PhD Thesis. The completion time is on the order of 5-7+ years and can often slip. There can be some exploration and project changes because the student does not know exactly which project is a good fit at the beginning. Yet the student still seeks to minimize their completion time in order to advance in their career in a timely manner.

A third example is an engineering design project in industry. The time limit is set at the beginning, but can sometimes be extended to allow the project to be completed to specification. Additionally, unforeseen circumstances can cause a change in the techniques or methods used. Yet the project managers seek to minimize the completion time in order to save money.

In these situations, the problem solver should be persistent to minimize their solve time. Switching methods too many times without completing the task will lead to a large amount of time consumed. Only when there exists methods or solution paths that are very short compared to the alternative (by a ratio of $\frac{1}{2+\sqrt{3}} \approx 0.268$ or less) does it make sense to contemplate switching.

THIS PAGE INTENTIONALLY LEFT BLANK

Chapter 6

Conclusions

6.1 Recommendations for Improving Problem Solving Outcome

The framework presented herein suggests the following recommendations for improving problem solving outcomes:

- Learn low solve time methods. Previous studies suggest that methods such as estimation and approximation are underemphasized in the curriculum [15, 20]. This is consistent with our data, in which students gravitate towards more complex methods. Another approach is to reuse and adapt existing solutions for similar problems.

- Choose optimal α. Encourage students to make an effort to solve a problem, but if the solution method fails, encourage them to make the decision to switch methods.

- Improve starting method choice. Recommend students to choose less complex methods when they start solving problems, especially if there are strict time limits.

- Improve choice of subsequent method. When students switch methods, they should move toward less time-consuming or more familiar methods.
- Shorten solve times. Practice methods that students already know to reduce solve time.

According to the modeling assumptions in Chapter 4.1, these recommendations should be applied to problems with multiple ways to solve. Each solution method considered should be sufficient to solve the problem.

These five recommendations fall into two categories: improving tools and improving strategy. They are described in more detail below.

6.1.1 Improving Tools

The priority for improving tools is to have students learn low solve time methods. One way to implement this is to instruct students to use estimation or approximation techniques in addition to traditional tools used for detailed analysis, thereby increasing the number of low complexity methods at the students' disposal. Another way is to reuse a complete solution for an existing problem, as this can save time compared to constructing a solution from scratch.

The experimental results suggest that students who use lower complexity methods are more likely to correctly solve the problem within the time limit. Additionally, the model suggests adding low complexity methods will improve P_{solve} regardless of whether students switch or not. In Figure 6-1a, the analytic model was computed for three scenarios for a problem with four possible solution methods and a time limit of 10 timesteps. In the first scenario, the problem solver could solve the problem using one low complexity method; the other solution methods cannot be used to solve within the time limit. In the second scenario, the problem solver could solve using two low complexity methods. In the third scenario, the problem solver could solve with three low complexity methods. It was found that as the number of usable low complexity methods increases, P_{solve} significantly increases for every value of switching tendency $\alpha \in [0, 1)$. The optimal α is similar for the three scenarios.

Another approach is to improve the solve time of methods that can already be completed under the time limit. To implement this, students can simply practice techniques they already know to become more proficient. The experimental results show that a few students who completed the Volume Problem made mistakes that led to an incorrect answer. Finishing the problem faster can lead to more time for double checking. Additionally, the model suggests this does not improve P_{solve} if the student remains on the first method selected ($\alpha = 0$), but can increase P_{solve} if the student switches methods. In Figure 6-1b, the analytic model was computed for three scenarios for a problem with three solution methods and a time limit of 10 timesteps. Two of these three solution methods can be completed under the time limit. The scenarios show the effect of reducing solve time on the two methods that can be completed within the time limit. In the first scenario, the solve times of these two "effective" methods were close to the time limit. In the second scenario, the solve times of the effective methods were approximately half the time limit. In the third scenario, the solve time of one effective method was only one fifth the time limit, while the solve time of the other was half the time limit. As the solve times of the effective methods were reduced, P_{solve} is increased for every switching strategy $\alpha \in (0, 1)$. Note that this approach should be implemented only if students already know low solve time methods, and are able to switch between methods (See Figure 6-3). The strategy of switching methods is discussed in more detail below.

6.1.2 Improving Strategies

The most fundamental strategy is to switch methods with the right frequency, that is, selecting an optimal α. To implement this, students can be taught that it is okay

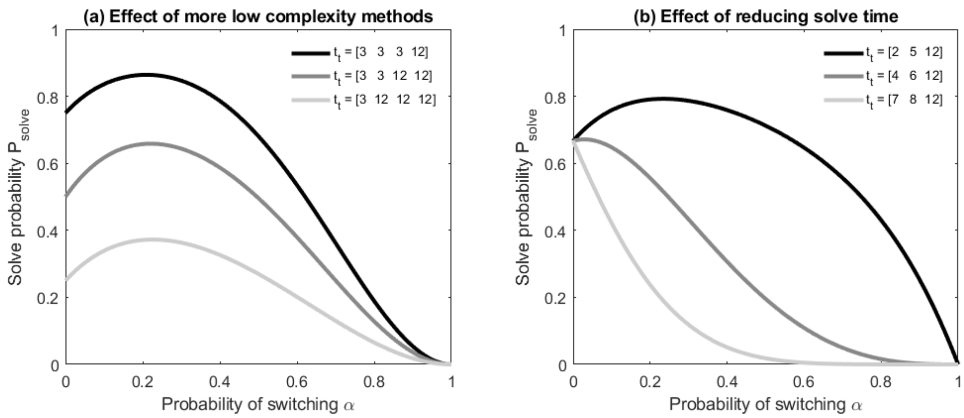


Figure 6-1: Examples of improving tools. (a) Introducing more low complexity methods will improve probability of solve. The analytic model was computed for three scenarios for a problem with four possible solution methods and a time limit of 10 timesteps. In the first scenario (light gray), the problem solver could solve the problem using one low complexity method; the other solution methods cannot be used to solve within the time limit. In the second scenario (dark gray), the problem solver could solve using two low complexity methods. In the third scenario (black), the problem solver could solve with three low complexity methods. Note that as the number of usable low complexity methods increases, P_{solve} significantly increases for every value of switching tendency $\alpha \in [0, 1)$. The optimal α is similar for the three scenarios. (b) Reducing solve times can improve the solve probability, given that the student chooses the optimal switching tendency. The analytic model was computed for three scenarios for a problem with three solution methods and a time limit of 10 timesteps. Two of these three solution methods can be completed under the time limit. The scenarios show the effect of reducing solve time on the two methods that can be completed within the time limit. In the first scenario (light gray), the solve times of these two "effective" methods were close to the time limit. In the second scenario (dark gray), the solve times of the effective methods were approximately half the time limit. In the third scenario (black), the solve time of one effective method was only one fifth the time limit, while the solve time of the other was half the time limit. As the solve times of the effective methods were reduced, P_{solve} is increased for every switching strategy $\alpha \in (0, 1)$. However, the benefit for switching was significant for only the third scenario.

to switch their approach, if they see their current method as unlikely to succeed. This strategy works by allowing the student to find a method that they are able to complete within the time limit, and can only improve P_{solve} if the student knows a sufficient amount of low solve time methods. By default, the model switches from method to method at random, without regard to method choice. However, it is still possible to improve P_{solve} if there are enough low complexity (low solve time) methods. In Figure 6-2a, the analytic model was computed for a problem with four solution methods and a time limit of 10 timesteps. Two of these four solution methods can be completed under the time limit. Compared to the no switch solve probability $P_{\alpha=0}$, the maximum solve probability P_{max} at the optimal switching tendency α_{opt} is significantly higher. Additionally, this strategy is consistent with experimental results, which suggest that students who switch methods are more likely to correctly solve the problem.

Another strategy is to start on less complex, lower solve time methods. It is a strategy consistent with how experts start problems, according to previous work by Li and Hosoi [12]. This strategy can be implemented by emphasizing to the student that they should first try a simple approach, and only proceed to a more detailed method if more accuracy is needed. This strategy requires the student to already know low solve time methods. According to the experimental results, students who started on simpler methods were more likely to solve the problem correctly (See Figure 3-3). Additionally, the model suggests that a better starting method improves P_{solve} regardless of whether the student switches methods or not. It has the biggest effect at low α; the better the starting method, the less need there is to switch methods. In Figure 6-2b, the analytic model was computed for three scenarios with a time limit of 10 timesteps and three solution methods (solve times 2, 5, and 12 timesteps). In the first scenario, the starting probabilities were weighted towards the method with longest solve time. In the second scenario, the starting probabilities were weighted equally. In the third scenario, the starting probabilities were weighted towards the method with the shortest solve time. As the weight is shifted towards the shorter solve time methods, P_{solve} is improved for every value of switching tendency $\alpha \in [0, 1)$.

A third strategy is to improve how you switch. To implement this strategy, students can be taught to switch to a less complex method than their current method, instead of switching methods haphazardly. Note that this strategy does not depend on the previous strategy and can be implemented independently (See Figure 6-3). This type of switching should be applied when there are simple approaches that are sufficiently precise to solve the problem to the desired resolution. By default, the model allows switching from method to method at random. Implementing a better switching strategy will increase both P_{solve} and the corresponding optimal α as well. This strategy cuts down on undesirable switching from less complex to more complex methods, so it also increases the benefit from switching, thereby increasing optimal α. In Figure 6-2c, the analytic model was computed for three scenarios for a problem with four solution methods and a time limit of 10 timesteps. Two of these four solution methods can be completed under the time limit. In the first scenario, random transitions were used. In the second scenario, the problem solver avoided the most complex method when transitioning. In the third scenario, the problem solver only transitioned from a more complex to a simpler method. As the transition strategy is improved, P_{solve} is increased for every switching strategy $\alpha \in (0, 1)$. Additionally, this strategy is consistent with the experimental results, where six of eight students who switched did so from more complex to less complex methods (See Figure 3-3). This subgroup of students performed better than average (four of six students obtained the correct answer), though limited conclusion can be drawn due to small sample size.

6.1.3 Order of Deployment

Our framework suggests that the recommendations for teaching tools and strategies should be deployed in a certain order for maximum effect. This is because some

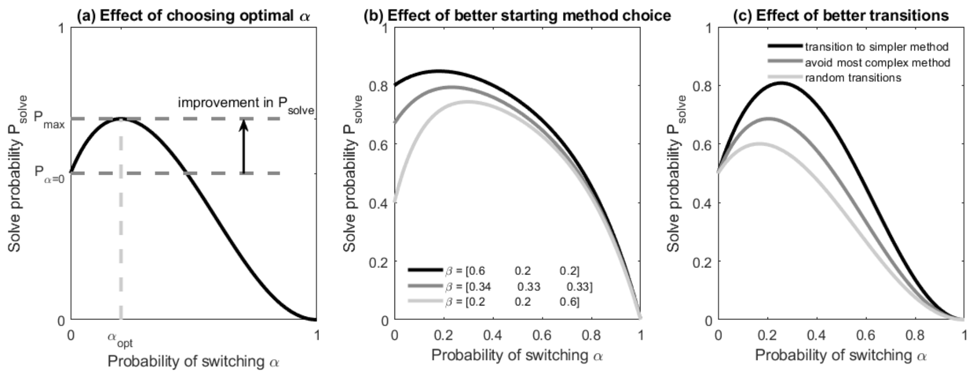


Figure 6-2: Examples of improving strategies. (a) Switching methods with optimal tendency can improve P_{solve}. The analytic model was computed for a problem with four solution methods and a time limit $t_f = 10$. Two of these four solution methods can be completed within t_f. Compared to the no switch solve probability $P_{\alpha=0}$, the maximum solve probability P_{max} at the optimal switching tendency α_{opt} is significantly higher. (b) Choosing a better starting method will improve P_{solve}, especially when the switching tendency α is low. The model was computed for three scenarios with $t_f = 10$ and three solution methods (solve times 2, 5, and 12 timesteps). In the first scenario (light gray), the starting probabilities were weighted towards the longest method. In the second scenario (dark gray), the starting probabilities were weighted equally. In the third scenario (black), the starting probabilities were weighted towards the shortest method. Note that as the weight is shifted towards the shorter methods, P_{solve} is improved for every $\alpha \in [0, 1)$. However, the effects are greatest for small α. Also, the optimal α decreases as starting method is improved. (c) Judiciously switching methods can improve P_{solve}. Improvement occurs when the student chooses to switch from more complex to less complex methods. The model was computed for three scenarios for a problem with four solution methods and $t_f = 10$. Two of these four solution methods can be completed within t_f. In the first scenario (light gray), random transitions were used. In the second scenario (dark gray), the problem solver avoided the most complex method when switching. In the third scenario (black), the problem solver only switched from a more complex to a simpler method. As the transition strategy is improved, P_{solve} is increased for every $\alpha \in (0, 1)$. Additionally, the better the transition strategy, the more benefit there is from switching, so the optimal α is increased.

tools or strategies are prerequisites for others. The recommended deployment order is shown in Figure 6-3.

First, low solve time methods, such as estimation and approximation, should be taught to students. Once they have low solve time methods in their toolbox, students can increase their solve probability by switching methods with optimal tendency α_{opt}. Without low solve time methods, our analytic model shows that it is unlikely students will be able to improve problem solving outcomes with switching. In parallel to switching, they can improve their solve probability by starting with lower solve time methods. Once students can improve P_{solve} via switching, they can optimize their switching strategy by moving to simpler methods instead of complex ones. In parallel, they can further improve the solve time of the methods they already know.

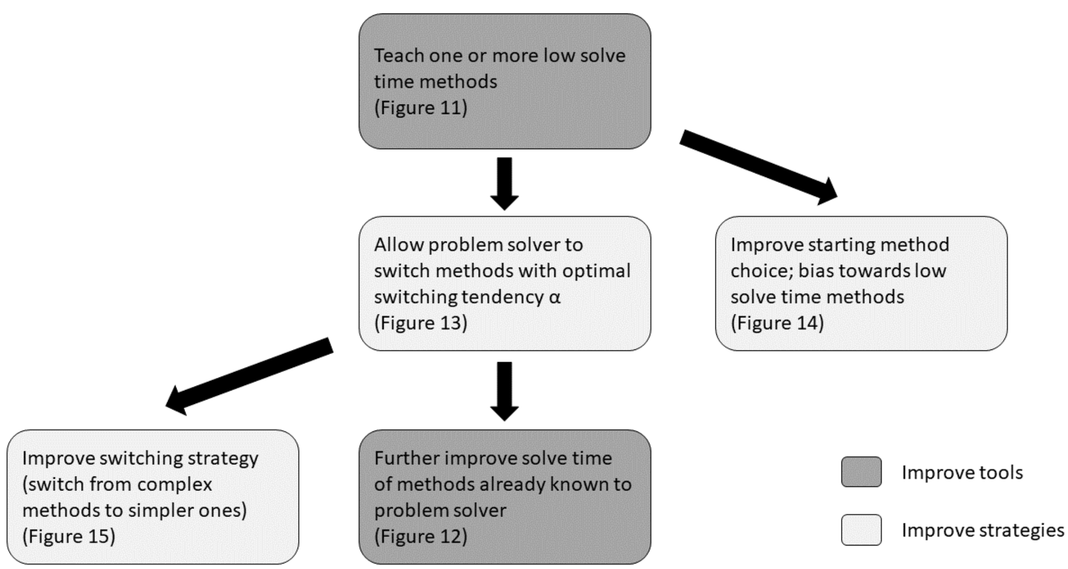


Figure 6-3: Order of deployment for recommendations to improve problem solving outcomes. Note that tools and strategies are interleaved. The tool of low solve time methods is essential and prerequisite to other strategies and tools; it directly unlocks the strategy of switching and the strategy of starting on a low complexity method. Subsequently, the strategy of switching unlocks the optimization of switching strategy and the optimization of solve time for already known methods.

6.2 Future Work

One potential area of future work is to extend the model. For example, the model can be extended to answer the question "does attempting a simpler method first prepare you for the more complex ones?" For example, a student starting off on a simpler method may find it easier to move onto a more complex method than one starting with a complex method, possibly due to more familiarity with the problem. More generally, it is possible for a student's experience on their first method to influence their performance or outcome on subsequent methods.

Previous work framed estimation as a standalone skill currently missing from the curriculum [15, 20]. However, the findings of this work suggest that estimation, as a low complexity solution method, is a useful tool that can supplement higher complexity solution methods in problem solving. The small time requirements and high probability of solve associated with lower complexity methods suggest that there may be concrete situations in the standard curriculum in which these can be used. Another potential area of future work would be to determine where in the curriculum it would make sense to deploy these lower complexity methods. Would it be beneficial to include estimation or simple approximations in Fluid Dynamics, Statics, or Controls? Or when teaching Numerical Computation or Design? If case studies can demonstrate effectiveness, it would go a long way in changing the "detailed analysis only" tendency of today's engineering curriculum.

There could be a range of ways to include low complexity methods in the curriculum. An example of curriculum integration would be to ask a student to solve a problem two ways on an assignment or examination. For an assignment, this could potentially be helpful for increasing student awareness of multiple solution paths. For an examination, this could allow the instructor to better assess students' understanding.

There are additional ways that low complexity methods can be used to enhance students' in-class problem solving. For example, in a statics class, where there are relatively "standard" ways to solve problems, it may be helpful to introduce students to dimensional analysis and units-checking. Such techniques can help students to verify their solutions and reduce their need to ask the instructor, "is my answer right?"

When introducing the idea of low solve time methods to students, they may object that they are not allowed to simply use any method they choose in class. In lower level, theory-based classes (such as statics, dynamics, fluids, and thermodynamics), there is a legitimate need to learn problem solving tools. This is consistent with the flowchart in Figure 6-3. However, nothing prevents the student from making an initial estimate to guide their full solution, or double checking the units in their answer. And when the student proceeds to upper-level design classes, they are expected to apply the theory to their design projects. In this case, they will have opportunities to consider different solution methods and apply the problem solving strategies described in this work.

Another area that would benefit from additional work is student attitudes towards the use of estimations and approximations. Results from this work suggest that students have negative attitudes towards these less complex techniques. To what extent is this true and why? A better understanding of this would help the educator community better convey to students the value of lower complexity techniques in problem solving.

Appendix A

Questionnaire Given after the Volume Problem

1. On the previous problem, which method did you pick and why?

2. Was your method successful? (circle one)

   (a) Yes

   (b) No

3. Did you switch methods? (circle one)

   (a) Yes

   (b) No

4. If so, why did you switch?

5. How frustrated were you? (circle a number)

   not at all frustrated 1 2 3 4 5 extremely frustrated

6. How difficult did you find the problem? (circle a number)

   not at all difficult 1 2 3 4 5 extremely difficult

7. On the previous problem, which best describes your experience? (circle one):

   (a) I do not have an effective solution approach, even after trying to solve the problem

   (b) I now have an effective solution approach, but only after trying to solve the problem

   (c) I knew of an effective solution approach immediately after reading the problem

8. What is your confidence in your solution method? (circle a number)

   no confidence 1 2 3 4 5 complete confidence

9. What is your confidence in your answer? (circle a number)

   no confidence 1 2 3 4 5 complete confidence

Appendix B

Example of P_{solve} Calculation

The following example shows how the total solve probability P_{solve} is calculated as function of switching probability α. This section is from a previous publication by Li and Hosoi [14].

Consider a problem with time limit $t_f = 4$ and two solution methods with solve times $t_1 = 2$ and $t_2 = 6$. Let the probability of starting on each method be $\frac{1}{2}$. Because the problem solver can work on either method 1 or method 2 on each of the four time steps, the following $2^4 = 16$ sequences of methods are possible:

$$1111, 1112, 1121, 1122, 1211, 1212, 1221, 1222,$$

$$2111, 2112, 2121, 2122, 2211, 2212, 2221, 2222$$

Let B be the number of method transitions in the sequence (e.g. going from method 1 to 2 or vice versa). For a sequence length of t_f, there are $t_f - 1$ opportunities between timesteps where the problem solver can switch methods or remain on the same method. For each sequence, the probability of the sequence occurring is the product of $\frac{1}{2}$ (the probability of starting on the either method), α^B (associated with the B method transitions), and $(1-\alpha)^{t_f-1-B}$ (associated with the $t_f - 1 - B$ instances of staying on the same method):

$$P_S = \frac{1}{2}\alpha^B(1-\alpha)^{t_f-1-B}.$$

A sequence solves the problem if there are $t_1 = 2$ consecutive ones in the sequence. Method 2 cannot be used to solve the problem because the solve time is greater than the time limit. We can then summarize sequence probabilities and whether the sequence solves the problem (see Table B.1).

Sequence	**Solves problem?**	**Sequence probability**
1111	Yes	$(1-\alpha)^3/2$
1112	Yes	$\alpha(1-\alpha)^2/2$
1121	Yes	$\alpha^2(1-\alpha)/2$
1122	Yes	$\alpha(1-\alpha)^2/2$
1211	Yes	$\alpha^2(1-\alpha)/2$
1212	No	$\alpha^3/2$
1221	No	$\alpha^2(1-\alpha)/2$
1222	No	$\alpha(1-\alpha)^2/2$
2111	Yes	$\alpha(1-\alpha)^2/2$
2112	Yes	$\alpha^2(1-\alpha)/2$
2121	No	$\alpha^3/2$
2122	No	$\alpha^2(1-\alpha)/2$
2211	Yes	$\alpha(1-\alpha)^2/2$
2212	No	$\alpha^2(1-\alpha)/2$
2221	No	$\alpha(1-\alpha)^2/2$
2222	No	$(1-\alpha)^3/2$

Table B.1: Sequence probabilities for an example for two solution methods with $t_1 = 2, t_2 = 6$, and $t_f = 4$.

For all sequences that solve the problem, we add up their probabilities to obtain the total solve probability. We ignore sequences that don't solve the problem. Thus

$$\begin{aligned} P_{solve}(\alpha) &= \frac{1}{2}(1-\alpha)^3 + 4\cdot\frac{1}{2}\alpha(1-\alpha)^2 + 3\cdot\frac{1}{2}\alpha^2(1-\alpha) \\ &= \frac{1}{2}(1+\alpha-2\alpha^2). \end{aligned}$$

Checking the graph of $P_{solve}(\alpha)$ (see Figure B-1), we see that the maximum solve probability occurs at where $P'_{solve}(\alpha) = 0$, or $\alpha = \frac{1}{4}, P_{solve} = \frac{9}{16}$.

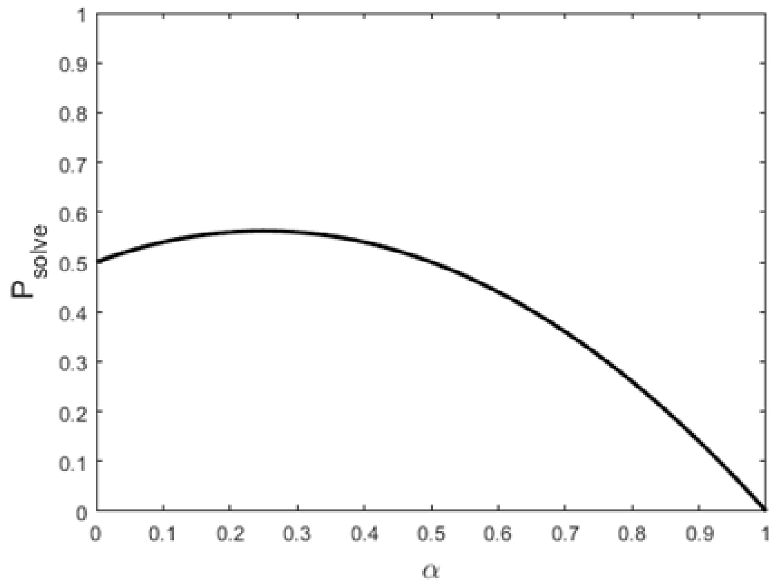


Figure B-1: Graph of the solve probability $P_{solve}(\alpha)$. Note that at $\alpha = 0$, P_{solve} is $\frac{1}{2}$. This corresponds to the no-switch case, where the problem solver stays on the first method chosen. As α is increased, P_{solve} reaches a maximum of $\frac{9}{16}$ at $\alpha = \frac{1}{4}$. Further increasing α will lead to a decrease in P_{solve} until the solve probability reaches zero at $\alpha = 1$.

THIS PAGE INTENTIONALLY LEFT BLANK

Appendix C

Optimality Condition for P_{solve} with $n = 2$ Solution Methods (Markov Formulation of Primal Problem)

In Section 4.2, it was claimed that if $0 < t_1 \leq \lfloor \frac{t_f}{2} \rfloor$, there exists $\alpha > 0$ that maximizes P_{solve}. On the other hand, if $\lfloor \frac{t_f}{2} \rfloor < t_1 < t_f$, $\alpha = 0$ maximizes P_{solve}. We will show this result here.

First, consider the case where $0 < t_1 \leq \lfloor \frac{t_f}{2} \rfloor$. Let us start by defining a *sequence* as a string of t_f digits representing the method used by the problem solver on each of the t_f timesteps. For the simple model in Section 4.2, the digits can only be 1 or 2, representing methods 1 and 2. Let B represent the number of transitions between methods. For example, the sequence 12211222 has eight timesteps and three transitions, so $t_f = 8$ and $B = 3$.

The probability of obtaining a given sequence with B transitions is the product of the starting method probability (equal to $\frac{1}{2}$), the probability of transitions at B timesteps (equal to α^B), and the probability of no transitions at the remaining timesteps (equal to $(1-\alpha)^{t_f-1-B}$). This product is $\frac{1}{2}\alpha^B(1-\alpha)^{t_f-1-B}$.

If this sequence does not solve the problem, it contributes zero to P_{solve}. On the other hand, if it solves the problem, it contributes a term

$$\frac{1}{2}\alpha^B(1-\alpha)^{t_f-1-B} \equiv Q(\alpha)$$

to P_{solve}. We can write P_{solve} informally as the sum of these $Q(\alpha)$ terms:

$$P_{solve}(\alpha) = \Sigma_{\text{sequence solves problem}} Q(\alpha)$$

We evaluate

$$\frac{dQ}{d\alpha} = \frac{1}{2}\left(B\alpha^{B-1}(1-\alpha)^{t_f-1-B} - \alpha^B(t_f-1-B)(1-\alpha)^{t_f-2-B}\right)$$

Note that $Q'(0) = 0$ except when $B = 0$ or 1. If $B = 0$,

$$Q'(0) = \frac{1}{2}\left(-\alpha^0(t_f-1)(1-0)^{t_f-2}\right) = -\frac{1}{2}(t_f-1)$$

If $B = 1$,

$$Q'(\alpha) = \frac{1}{2}\left(\alpha^0(1-\alpha)^{t_f-2} - \alpha^1(t_f-2)(1-\alpha)^{t_f-3}\right)$$

Evaluating this at $\alpha = 0$ gives $Q'(0) = \frac{1}{2}$.

The number of $B = 0$ contributions is always 1, as there exist one sequence of t_f 1s, which will always solve the problem. There also exists a sequence of t_f 2s, but this will never solve the problem. The number of $B = 1$ contributions can be calculated in a straightforward manner. There are two forms of sequences with one transition:

$$\underbrace{11...1}_{k \text{ ones}}2...2 \qquad \text{or} \qquad 2...2\underbrace{11..1}_{k \text{ ones}}$$

In order for the problem to be solved, there needs to be t_1 consecutive ones, so k can take the values $t_1, t_1+1, ..., t_f-1$. There are a total of $2(t_f-1-t_1+1) = 2(t_f-t_1)$

possible ways for the problem to be solved. Thus we can compile

$$P'_{solve}(0) = -\frac{1}{2}(t_f - 1) + \frac{1}{2} \cdot 2(t_f - t_1)$$

Our goal is to find the condition for which $P'_{solve}(0) > 0$. If this is shown, then $\alpha = 0$ does not maximize $P_{solve}(\alpha)$ on the interval [0, 1], so there must exist a maximum in (0, 1]. So we rearrange

$$-\frac{1}{2}(t_f - 1) + \frac{1}{2} \cdot 2(t_f - t_1) > 0$$

$$\Leftrightarrow t_1 < \frac{t_f + 1}{2}$$

Since t_1 and t_f are integers, this is equivalent to $t_1 \leq \lfloor \frac{t_f}{2} \rfloor$, as desired.

Next, consider the case where $\lfloor \frac{t_f}{2} \rfloor < t_1 < t_f$. We analyze the number of times a sequence solves the problem for a given number of transitions B. Note that crucially, if $B > t_f - t_1$, there are no instances for which the sequence solves the problem. This is because there can be at most $t_f - 1$ transitions, and solving the problem requires t_1 consecutive timesteps on method 1, thereby removing $t_1 - 1$ of the $t_f - 1$ transition slots. Thus the maximum number of transitions is $(t_f - 1) - (t_1 - 1) = t_f - t_1$. This can be illustrated via the following example: if $t_f = 6$ and $t_1 = 4$, a sequence that solves the problem, such as 211112, can have at most two transitions, else there cannot be four consecutive timesteps on method 1.

Note that if $t_1 > \lfloor \frac{t_f}{2} \rfloor$, we have $t_1 > \frac{t_f}{2}$ because t_1 and t_f are integers. For each sequence that solves the problem, there is one that does not. We see this by writing the sequence ...2 $\underbrace{11...1}_{\text{at least } t_1}$ 2.... If we swap 1s and 2s, then the sequence ...1 $\underbrace{22...2}_{\text{at least } t_1}$ 1... cannot solve the problem because there are $t_f - t_1 < \frac{t_f}{2}$ remaining timesteps to place method 1. This means that, for a fixed t_1 and B, the number of sequence that solve the problem can be at most half the total number of sequences.

We can then bound the number of times that the term $\frac{1}{2}\alpha^B(1-\alpha)^{t_f-1-B}$ appears in P_{solve} for each B. Note that the total number of such terms can be expressed as a binomial coefficient where B method transition locations are chosen from $t_f - 1$ possible transition opportunities. Multiply this by 2 to account for the first method, which can be method 1 or method 2. Multiply again by $\frac{1}{2}$ to incorporate the result of the previous paragraph to obtain $\frac{1}{2} \cdot 2\binom{t_f-1}{B} = \binom{t_f-1}{B}$. Thus the following bound applies:

$$P_{solve}(\alpha) \leq \sum_{i=0}^{\lfloor \frac{t_f-1}{2} \rfloor} \frac{1}{2}\binom{t_f-1}{i}\alpha^i(1-\alpha)^{t_f-1-i} \equiv X$$

However, we know that

$$Y \equiv \sum_{i=0}^{t_f-1} \frac{1}{2}\binom{t_f-1}{i}\alpha^i(1-\alpha)^{t_f-1-i} = \frac{1}{2}(\alpha+1-\alpha)^{t_f-1} = \frac{1}{2}$$

by the binomial theorem. Taking the difference $Y - X$ gives

$$Y - X = \sum_{i=\lfloor \frac{t_f-1}{2} \rfloor+1}^{t_f-1} \frac{1}{2}\binom{t_f-1}{i}\alpha^i(1-\alpha)^{t_f-1-i}$$

Note that for $\alpha > 0$, $Y - X > 0$, so

$$P_{solve}(\alpha) \leq X < Y = \frac{1}{2}$$

If $\alpha = 0$, then the problem solver stays on the starting method. There is $\frac{1}{2}$ chance of starting on method 1, so $P_{solve}(0) = \frac{1}{2}$. Thus $P_{solve}(\alpha)$ is maximized at $\alpha = 0$, as desired.

Appendix D

Proof of Equation 5.3 (Optimality Condition for Markov Formulation of Dual Problem)

Proof. The average solve time can be expressed as

$$t_{solve}(\alpha) = \sum_{\text{all } \mathcal{S}} \delta_{\mathcal{S}} t_{\mathcal{S}}(\alpha),$$

where

$$\delta_{\mathcal{S}} = \begin{cases} 1 & \text{Sequence } \mathcal{S} \text{ solves problem} \\ 0 & \text{Otherwise,} \end{cases}$$

and $t_{\mathcal{S}}(\alpha)$ is the average solve time of the sequence $\mathcal{S}$ as a function of the switching tendency α.

We can write

$$t_{\mathcal{S}}(\alpha) = \frac{t}{n} \left(\frac{\alpha}{n-1} \right)^{B} (1-\alpha)^{t_{max}-1-B},$$

where we set the placeholder variable $t_{max} \gg t_n$ to represent the maximum time analyzed. t_{max} is much larger than all the solve times and approaches infinity in the Dual Problem. Here, t is the solve time required to solve the problem using sequence $\mathcal{S}$, B is the number of method transitions in $\mathcal{S}$, and n is the number of methods.

Differentiating with respect to α, we obtain

$$t'_{\mathcal{S}}(\alpha) = \frac{t}{n}\left(\frac{B}{n-1}\left(\frac{\alpha}{n-1}\right)^{B-1}(1-\alpha)^{t_{max}-1-B} - \left(\frac{\alpha}{n-1}\right)^{B}(t_{max}-1-B)(1-\alpha)^{t_{max}-2-B}\right).$$

We would like to impose the condition

$$t'_{solve}(0) = \sum_{\text{all } \mathcal{S}} \delta_S t'_{\mathcal{S}}(0) < 0$$

to obtain a minimum t_{solve} for $\alpha > 0$. Note that $t'_{\mathcal{S}}(\alpha) = 0$ unless $B = 0, 1$.

If $B = 0$,

$$t'_{\mathcal{S}}(\alpha) = -\frac{t}{n}(t_{max}-1)(1-\alpha)^{t_{max}-2},$$

so

$$t'_{\mathcal{S}}(0) = -\frac{t}{n}(t_{max}-1).$$

There are n sequences for which this is true: the problem solver stays on method i for $i = 1, 2, \ldots, n$ without switching. If the problem solver stays on method i, the time required to solve is $t = t_i$. Thus, the total $B = 0$ contribution to $t'_{solve}(0)$ is

$$-\frac{1}{n}(t_{max}-1)\sum_{i=1}^{n} t_i.$$

If $B = 1$,

$$t'_{\mathcal{S}}(\alpha) = \frac{t}{n}\left(\frac{(1-\alpha)^{t_{max}-2}}{n-1} - \left(\frac{\alpha}{n-1}\right)(t_{max}-2)(1-\alpha)^{t_{max}-3}\right),$$

so

$$t'_{\mathcal{S}}(0) = \frac{t}{n(n-1)}.$$

Here, there is one method transition. Let this transition be between method i and j. Then the sequence of methods is $\mathcal{S} = \{\underbrace{i, \ldots i}_{k\ i'\text{s}}, \underbrace{j, \ldots, j}_{t_{max}-k\ j'\text{s}}\}$. There are two cases:

(i) The problem is solved with method i (the first method). For this case, the time required to solve is $t = t_i$. The number of steps spent on the first method, k, can take the values $t_i, t_i + 1, \ldots, t_{max} - 1$ for a total of $(t_{max} - 1) - t_i + 1 = t_{max} - t_i$ possibilities. For each first method, there are $n - 1$ choices for the second method. Thus the $t'_{solve}(0)$ contribution for this case is

$$\sum_{i=1}^{n} \left(\frac{t_i}{n(n-1)}(t_{max} - t_i)(n-1) \right) = \frac{1}{n} \sum_{i=1}^{n} t_i(t_{max} - t_i).$$

(ii) The problem is not solved with the first method i but is solved with the second method j. For each choice of i and j, the problem solver spends $1, 2, \ldots t_i - 1$ timesteps on method i, for a total of $t_i - 1$ possibilities The possible solve times are $t = 1 + t_j, 2 + t_j, \ldots, t_i - 1 + t_j$, for an average of $\frac{t_i}{2} + t_j$. Thus the $t'_{solve}(0)$ contribution for this case is

$$\sum_{i \neq j} \left(\frac{\frac{t_i}{2} + t_j}{n(n-1)}(t_i - 1) \right) = \frac{1}{n(n-1)} \sum_{i \neq j} \left(\frac{t_i}{2} + t_j \right)(t_i - 1).$$

Adding the terms of $t'_{solve}(0)$ together and setting this sum to be negative, we obtain

$$t'_{solve}(0) = -\frac{1}{n}(t_{max} - 1) \sum_{i=1}^{n} t_i + \frac{1}{n} \sum_{i=1}^{n} t_i(t_{max} - t_i) + \frac{1}{n(n-1)} \sum_{i \neq j} \left(\frac{t_i}{2} + t_j \right)(t_i - 1) < 0$$

$$\Leftrightarrow \sum_{i=1}^{n} t_i - \sum_{i=1}^{n} t_i^2 + \frac{1}{n-1}\left(\sum_{i\neq j} \frac{t_i^2}{2} - \sum_{i\neq j} \frac{t_i}{2} + \sum_{i\neq j} t_i t_j - \sum_{i\neq j} t_j\right) < 0$$

$$\Leftrightarrow \sum_{i=1}^{n} t_i - \sum_{i=1}^{n} t_i^2 + \frac{1}{n-1}\left((n-1)\sum_{i=1}^{n} \frac{t_i^2}{2} - (n-1)\sum_{i=1}^{n} \frac{t_i}{2} + 2\sum_{i<j} t_i t_j - (n-1)\sum_{i=1}^{n} t_i\right) < 0$$

$$\Leftrightarrow \sum_{i=1}^{n}(t_i^2 + t_i) - \frac{4}{n-1}\sum_{i<j} t_i t_j > 0,$$

as desired. $\square$

Appendix E

Proof of Equation 5.7 (Optimality Condition for Poisson Formulation of Dual Problem)

Proof. We can write the mean solve time as

$$t_{solve}(\lambda) = \sum_{B=0}^{\infty} P_B t_{solve|B}(\lambda),$$

where $t_{solve|B}(\lambda)$ is the mean solve time given that B transitions occurred, and P_B is the probability of B transitions.

For $B = 0$, if the problem solver starts on method i, they solve the problem if the first transition lands in the interval $[t_i, \infty)$. This occurs with probability $e^{-\lambda t_1}$. The solve time using method i is t_i. We take the mean over all the methods to obtain

$$P_0 t_{solve|0}(\lambda) = \frac{1}{n} \sum_{i=0}^{\infty} t_i e^{-\lambda t_i}.$$

Then we can differentiate both sides with respect to λ:

$$P_0 t'_{solve|0}(\lambda) = -\frac{1}{n}\sum_{i=0}^{\infty} t_i^2 e^{-\lambda t_i}.$$

Thus

$$P_0 t'_{solve|0}(0) = -\frac{1}{n}\sum_{i=0}^{\infty} t_i^2.$$

For $B = 1$, the problem solver starts on method i, switches to method j before they are able to solve, and then solves with method j. This occurs with probability $(1 - e^{-\lambda t_i})e^{-\lambda t_j}$. The expected solve time is sum of the average time spent on method i,

$$\frac{\int_0^{t_i} t\lambda e^{-\lambda t} dt}{\int_0^{t_i} \lambda e^{-\lambda t} dt} = \frac{1}{\lambda} - \frac{t_i e^{-\lambda t_i}}{1 - e^{-\lambda t_i}},$$

and the time spent on method j, t_j. We take the product to find the contribution to t_{solve}:

$$\left(\frac{1}{\lambda} - \frac{t_i e^{-\lambda t_i}}{1 - e^{-\lambda t_i}} + t_j\right)(1 - e^{-\lambda t_i})e^{-\lambda t_j}.$$

Taking the mean over all choices of i, j, we obtain

$$P_1 t_{solve|1}(\lambda) = \frac{1}{n(n-1)} \sum_{\substack{i \neq j \\ 1 \leq i,j \leq n}} \left(\frac{1}{\lambda} - \frac{t_i e^{-\lambda t_i}}{1 - e^{-\lambda t_i}} + t_j\right)(1 - e^{-\lambda t_i})e^{-\lambda t_j}.$$

It can be found that

$$P_1 t'_{solve|1}(0) = \frac{1}{n(n-1)} \sum_{\substack{i \neq j \\ 1 \leq i,j \leq n}} t_i \left(\frac{t_i}{2} + t_j\right).$$

For $B = 2$, the problem solver starts on method i, switches to method j before they are able to solve, then switches to method k before they are able to solve, and then solves with method k. Using a similar approach as the $B = 1$, we find the t_{solve} contribution to be

$$\left(\frac{1}{\lambda} - \frac{t_i e^{-\lambda t_i}}{1-e^{-\lambda t_i}} + \frac{1}{\lambda} - \frac{t_i e^{-\lambda t_j}}{1-e^{-\lambda t_j}} + t_k\right)(1-e^{-\lambda t_i})(1-e^{-\lambda t_j})e^{-\lambda t_k}.$$

It can be shown that differentiating the above expression with respect to λ and evaluating at $\lambda = 0$ gives zero. Thus we conclude

$$P_2 t'_{solve|2}(\lambda) = 0.$$

For $B \geq 3$, we can use a similar approach as $B = 2$ to show that

$$P_B t'_{solve|B}(\lambda) = 0.$$

We can then set $t'_{solve}(0) < 0$ to obtain the optimality condition:

$$t'_{solve}(0) = \sum_{B=0}^{\infty} P_B t'_{solve|B}(0)$$

$$= -\frac{1}{n}\sum_{i=0}^{\infty} t_i^2 + \frac{1}{n(n-1)} \sum_{\substack{i \neq j \\ 1 \leq i,j \leq n}} t_i \left(\frac{t_i}{2} + t_j\right) < 0$$

$$\Leftrightarrow -\frac{1}{n}\sum_{i=0}^{\infty} t_i^2 + \frac{1}{2n}\sum_{i=1}^{n} t_i^2 + \frac{1}{n(n-1)} \sum_{\substack{i \neq j \\ 1 \leq i,j \leq n}} t_i t_j < 0$$

$$\Leftrightarrow \sum_{i=1}^{n} t_i^2 - \frac{4}{n-1}\sum_{i<j} t_i t_j > 0,$$

as desired.

$\square$

THIS PAGE INTENTIONALLY LEFT BLANK

Appendix F

Proof of Proposition 5 (Analytic Solution for $P_{solve}(\alpha)$)

Proof. We can begin by proving the case for which B is odd and the problem solver starts on method 1. Let $t^{(1)}, t^{(2)}, \ldots, t^{(B)}$ be the timesteps immediately preceding the method transitions. A transition cannot occur immediately after the zeroth timestep or after the last timestep. Then WLOG assume $0 < t^{(1)} < t^{(2)} < \cdots < t^{(B)} < t_f$. The number of possible sets of arrival times $\{t^{(1)}, t^{(2)}, \ldots, t^{(B)}\}$ is $\binom{t_f-1}{B}$ because there are $t_f - 1$ possible timesteps after which a transition can occur.

In order for the problem solver to solve the problem, additional constraints are needed. Consider the case where the problem solver starts on method 1. For odd B, the problem solver is on method 1 in the following disjoint subsets of consecutive timesteps:

$$\{1, 2, \ldots, t^{(1)}\}, \{t^{(2)} + 1, \ldots, t^{(3)}\}, \{t^{(4)} + 1, \ldots, t^{(5)}\}, \ldots, \{t^{(B-1)} + 1, \ldots, t^{(B)}\}.$$

At least one of these sets must have at least t_1 elements for the problem solver to solve the problem. Therefore at least one of the following $C = \frac{B+1}{2}$ constraints must be satisfied:

$$
\begin{aligned}
t^{(1)} &\geq t_1 \\
t^{(3)} - t^{(2)} &\geq t_1 \\
t^{(5)} - t^{(4)} &\geq t_1 \\
&\vdots \\
t^{(B)} - t^{(B-1)} &\geq t_1.
\end{aligned}
$$

Suppose that exactly j of these C constraints are satisfied and fix the choice of constraints. Consider the differences

$$
\begin{aligned}
&t_f - t^{(B)} \\
&t^{(B)} - t^{(B-1)} \\
&t^{(B-1)} - t^{(B-2)} \\
&\vdots \\
&t^{(3)} - t^{(2)} \\
&t^{(2)} - t^{(1)} \\
&t^{(1)} - 0.
\end{aligned}
$$

j of these differences must be at least t_1, and the remaining $B + 1 - j$ differences must be at least 1. However the sum of these differences is exactly t_f, so it must also be true that $t_f \geq jt_1 + B + 1 - j$. Thus we have

$$t_1 \leq \frac{t_f - (B + 1 - j)}{j}.$$

If there are no constraints, method transitions can occur at any location in $\{1, 2, \dots, t_f - 1\}$. However, imposing one constraint removes $t_1 - 1$ locations from the set of possible locations. Thus, imposing j constraints removes $j(t_1 - 1)$ possible transition locations. Thus the number of transition locations remaining is $(t_f - 1) - j(t_1 - 1) = t_f - jt_1 + (j - 1)$, so the the number of possible sets $\{t^{(1)}, t^{(2)}, \dots, t^{(B)}\}$ for which the problem is solved with these j constraints is

$$\binom{t_f - jt_1 + (j-1)}{B}.$$

Additionally, there are $\binom{C}{j}$ ways to choose these constraints. In order to calculate solve probability, we must find which interval t_1 belongs to. If

$$\frac{t_f - (B - j)}{j + 1} < t_1 \leq \frac{t_f - (B + 1 - j)}{j},$$

then there can be at most j constraints. Some solve regions are counted multiple times, so we must use the Principle of Inclusion-Exclusion up to j terms to obtain the solve probability:

$$\frac{1}{\binom{t_f-1}{B}} \left(\binom{C}{1}\binom{t_f - t_1}{B} - \binom{C}{2}\binom{t_f - 2t_1 + 1}{B} + \cdots + (-1)^{j+1}\binom{C}{j}\binom{t_f - jt_1 + (j-1)}{B} \right).$$

Also, we need to impose limits on j. In order for the binomial coefficients to be computed, so we need $t_f - jt_1 + (j - 1) \geq B$ and $j \leq C$. Thus we have

$$j \leq \min\left(\left\lfloor \frac{t_f - B - 1}{t_1 - 1} \right\rfloor, C \right).$$

Thus the solve probability for this case (fixed, odd B; start on method 1) is

$$P_{solve|B\text{ transitions}} = \frac{1}{\binom{t_f-1}{B}} p\left(B, \frac{B+1}{2}\right), \tag{F.1}$$

where

$$p(B,C) = \begin{cases} \binom{C}{1}\binom{t_r(1)}{B} & \text{if } t_b(1) < t_1 \le t_b(0) \\ \binom{C}{1}\binom{t_r(1)}{B} - \binom{C}{2}\binom{t_r(2)}{B} & \text{if } t_b(2) < t_1 \le t_b(1) \\ \vdots & \\ \binom{C}{1}\binom{t_r(1)}{B} - \binom{C}{2}\binom{t_r(2)}{B} + \cdots + (-1)^{k+1}\binom{C}{k}\binom{t_r(k)}{B} & \text{if } 1 < t_1 \le t_b(k-1), \end{cases} \tag{F.2}$$

$t_r(j) = t_f - jt_1 + (j-1)$, $t_b(j) = \frac{t_f-(B-j)}{j+1}$, and $k = \min\left(\left\lfloor \frac{t_f-B-1}{t_1-1} \right\rfloor, C\right)$.

By symmetry, the solve probability for the case (fixed, odd B; start on method 2) is the same as the case (fixed, odd B; start on method 1). Thus, whenever B is odd, Equation F.1 applies.

For the case where B is even, we make minor modifications. If the problem solver starts on method 1, the constraints are

$$\begin{aligned} t^{(1)} &\ge t_1 \\ t^{(3)} - t^{(2)} &\ge t_1 \\ t^{(5)} - t^{(4)} &\ge t_1 \\ &\vdots \\ t_f - t^{(B)} &\ge t_1, \end{aligned}$$

and the number of constraints is $C = \frac{B}{2} + 1$. If we repeat the steps above for this case, we find that the expression in Equation F.2 is the same. Similarly, if the problem solver starts on method 2, the constraints are

$$
\begin{aligned}
t^{(2)} - t^{(1)} &\geq t_1 \\
t^{(4)} - t^{(3)} &\geq t_1 \\
&\vdots \\
t^{(B)} - t^{(B-1)} &\geq t_1,
\end{aligned}
$$

and the number of constraints is $C = \frac{B}{2}$. If we repeat the steps above for this case, we find that the expression in Equation F.2 is again the same. Thus for even B, we have

$$
P_{solve|B} = \frac{1}{2\binom{t_f - 1}{B}} \left(p\left(B, \frac{B}{2}\right) + p\left(B, \frac{B}{2} + 1\right) \right), \tag{F.3}
$$

where $p(B, C)$ is defined in Equation F.2, as desired.

$\square$

THIS PAGE INTENTIONALLY LEFT BLANK

Appendix G

Proof of Proposition 6 (Analytic Solution for $t_{solve}(\alpha)$)

Proof. For two solution methods, the problem solver will progress through a decision tree as in Figure 5-5. The following outcomes are possible:

(i) The problem solver starts on method 1, switches methods an even number of times, and eventually solves with method 1.

(ii) The problem solver starts on method 1, switches methods an odd number of times, and eventually solves with method 2.

(iii) The problem solver starts on method 2, switches methods an even number of times, and eventually solves with method 2.

(iv) The problem solver starts on method 2, switches methods an odd number of times, and eventually solves with method 1.

Consider Case (i) first. The probability that the problem solver starts on method 1 and solves the problem with zero switching corresponds to not switching for the first $t_1 - 1$ timesteps. This has probability $(1-\alpha)^{t_1-1}$. The corresponding solve time is t_1.

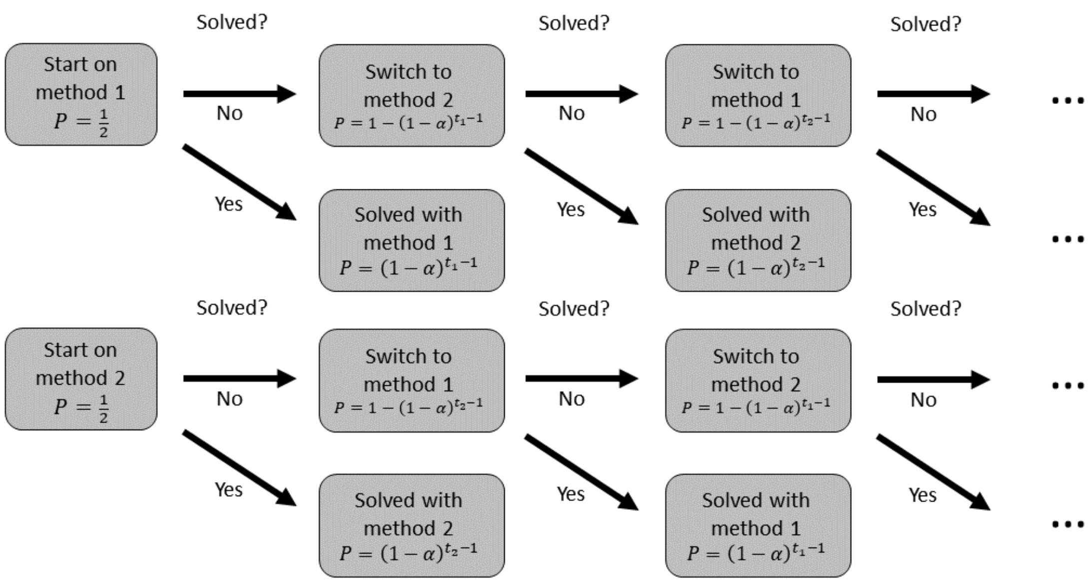


Figure G-1: Decision tree for the Markov Dual Problem with two solution methods. The problem solver will start on either method with probability $P = \frac{1}{2}$. If the problem solver solves the problem with method i, they will do so with probability $P = (1 - \alpha)^{t_i - 1}$. If the problem solver does not solve the problem with method i, they will do so with probability $P = 1 - (1 - \alpha)^{t_i - 1}$, and switch to the other method. The problem solver will continue to switch methods until they solve the problem.

The probability that the problem solver starts on method 1, switches two times before solving, and then solves with method 1 corresponds to switching during the first $t_1 - 1$ timesteps, switching during the next $t_2 - 1$ timesteps *after the first switch*, and not switching for the next $t_1 - 1$ timesteps *after the second switch.* The probability is

$$(1 - (1 - \alpha)^{t_1 - 1})(1 - (1 - \alpha)^{t_2 - 1})(1 - \alpha)^{t_1 - 1}.$$

The mean time spent on method i, given the problem is not solved, is the mean value of the number of timesteps j restricted to $j = 1, 2, \ldots, t_i - 1$. Each value of j has corresponding probability $(1 - \alpha)^{j-1}$. Call this mean time $\tilde{t_i}$:

$$\begin{aligned}
\tilde{t_i} &\equiv \frac{\displaystyle\sum_{j=1}^{t_i - 1} j\alpha(1 - \alpha)^{j-1}}{\displaystyle\sum_{j=1}^{t_i - 1} \alpha(1 - \alpha)^{j-1}} \\
&= \frac{\displaystyle\sum_{j=1}^{t_i - 1} j(1 - \alpha)^{j-1}}{\displaystyle\sum_{j=1}^{t_i - 1} (1 - \alpha)^{j-1}} \\
&= \frac{\dfrac{1 - (1 + \alpha(t_i - 1))(1 - \alpha)^{t_i - 1}}{\alpha^2}}{\dfrac{1 - (1 - \alpha)^{t_i - 1}}{\alpha}} \\
&= \frac{1 - (1 + \alpha(t_i - 1))(1 - \alpha)^{t_i - 1}}{\alpha(1 - (1 - \alpha)^{t_i - 1})}.
\end{aligned}$$

For $\alpha = 0$,

$$\tilde{t_i} \equiv \frac{\displaystyle\sum_{j=1}^{t_i - 1} j}{\displaystyle\sum_{j=0}^{t_i - 1} 1} = \frac{\frac{(t_i - 1)t_i}{2}}{t_i - 1} = \frac{t_i}{2}.$$

Therefore the mean solve time given that the problem solver starts on method 1, switches two times, and solves with method 1 is:

$$\tilde{t_1} + \tilde{t_2} + t_1.$$

In general, the probability of starting on method 1, switching $2k$ times, and solving with method 1 is

$$\left((1-(1-\alpha)^{t_1-1})(1-(1-\alpha)^{t_2-1})\right)^k e^{-\lambda t_1}.$$

The corresponding solve time is

$$k(\tilde{t_1} + \tilde{t_2}) + t_1.$$

Next, consider Case (ii). We can see the probability of starting on method 1, switching $2k-1$ times, and solving with method 2 is

$$(1-(1-\alpha)^{t_1-1})^k(1-(1-\alpha)^{t_2-1})^{k-1}(1-\alpha)^{t_2-1}.$$

The corresponding solve time is

$$k\tilde{t_1} + (k-1)\tilde{t_2} + t_2.$$

For ease of notation, set $r_1 = 1-(1-\alpha)^{t_1-1}$ and $r_2 = 1-(1-\alpha)^{t_2-1}$. The average $t_{solve}(\alpha)$, given that the problem solver starts on method 1 (Cases 1 and 2) is

$$t_{solve,1} = \sum_{k=0}^{\infty} \left((k(\tilde{t_1}+\tilde{t_2})+t_1)(r_1r_2)^k(1-r_1) + ((k+1)\tilde{t_1}+k\tilde{t_2}+t_2)r_1^{k+1}r_2^k(1-r_2)\right).$$

Note that this is a sum of geometric and arithmetic-geometric series. Then we can evaluate

$$t_{solve,1} = \sum_{k=0}^{\infty} t_1(r_1r_2)^k(1-r_1) + \sum_{k=0}^{\infty} k(\tilde{t_1}+\tilde{t_2})(r_1r_2)^k(1-r_1) +$$

$$\sum_{k=0}^{\infty}(t_2+\tilde{t_1})r_1^{k+1}r_2^k(1-r_2) + \sum_{k=0}^{\infty} k(\tilde{t_1}+\tilde{t_2})r_1^{k+1}r_2^k(1-r_2)$$

$$= \frac{t_1(1-r_1)}{1-r_1r_2} + \frac{(\tilde{t_1}+\tilde{t_2})r_1r_2(1-r_1)}{(1-r_1r_2)^2} + \frac{(t_2+\tilde{t_1})r_1(1-r_2)}{1-r_1r_2} + \frac{(\tilde{t_1}+\tilde{t_2})r_1^2r_2(1-r_2)}{(1-r_1r_2)^2}$$

$$= \frac{t_1(1-r_1)+(t_2+\tilde{t_1})r_1(1-r_2)}{1-r_1r_2} + \frac{(\tilde{t_1}+\tilde{t_2})(r_1r_2(1-r_1)+r_1^2r_2(1-r_2))}{(1-r_1r_2)^2}.$$

By symmetry, the average $t_{solve}(\alpha)$, given that the problem solver starts on method 2 (Cases (iii) and (iv)) is

$$t_{solve,2} = \frac{t_2(1-r_2)+(t_1+\tilde{t_2})r_2(1-r_1)}{1-r_1r_2} + \frac{(\tilde{t_1}+\tilde{t_2})(r_1r_2(1-r_2)+r_2^2r_1(1-r_1))}{(1-r_1r_2)^2}.$$

The overall t_{solve} is

$$t_{solve} = \frac{1}{2}(t_{solve,1}+t_{solve,2})$$

$$= \frac{t_1(1-r_1)+t_2(1-r_2)+(t_2+\tilde{t_1})r_1(1-r_2)+(t_1+\tilde{t_2})r_2(1-r_1)}{2(1-r_1r_2)} +$$

$$\frac{(\tilde{t_1}+\tilde{t_2})\left(r_1r_2(2-(r_1+r_2))+r_1^2r_2(1-r_2)+r_2^2r_1(1-r_1)\right)}{2(1-r_1r_2)^2}$$

$$= \frac{t_1(1-r_1) + t_2(1-r_2) + (t_2+\tilde{t_1})r_1(1-r_2) + (t_1+\tilde{t_2})r_2(1-r_1) + 2r_1r_2(\tilde{t_1}+\tilde{t_2})}{2(1-r_1r_2)},$$

as desired. $\square$

Bibliography

[1] What is self-regulated learning?, 2017. Accessed on July 3, 2023. URL: `https://serc.carleton.edu/sage2yc/self_regulated/what.html`.

[2] Patricia A Alexander. Mapping the multidimensional nature of domain learning: The interplay of cognitive, motivational, and strategic forces. *Advances in motivation and achievement*, 10:213–250, 1997.

[3] Gulsum Ancel. Problem-solving training: Effects on the problem-solving skills and self-efficacy of nursing students. *Eurasian Journal of Educational Research*, 16(64), 2016.

[4] Abdulkadir Bahar and C June Maker. Cognitive backgrounds of problem solving: A comparison of open-ended vs. closed mathematics problems. *Eurasia Journal of Mathematics, Science and Technology Education*, 11(6):1531–1546, 2015.

[5] Heidi A Diefes-Dux, Tamara Moore, Judith Zawojewski, PK Imbrie, and Deborah Follman. A framework for posing open-ended engineering problems: Model-eliciting activities. In *34th Annual Frontiers in Education, 2004. FIE 2004.*, pages F1A–3. IEEE, 2004.

[6] Derek Dunn-Rankin. Evaluating design alternatives the role of simple engineering analysis and estimation. In *2001 Annual Conference*, Albuquerque, New Mexico, June 2001. ASEE Conferences.

[7] Rob Foshay and Jamie Kirkley. Principles for teaching problem solving. Technical Report PLATO-TP-4, PLATO Learning, Bloomington, MN, 1998.

[8] Daniel Frey. Personal Interview, April 20, 2018.

[9] Cornelia S Große. Mathematics learning with multiple solution methods: effects of types of solutions and learners' activity. *Instructional Science*, 42(5):715–745, 2014.

[10] David H Jonassen. Instructional design models for well-structured and ill-structured problem-solving learning outcomes. *Educational technology research and development*, 45(1):65–94, 1997.

[11] Patrick B Kohl and Noah D Finkelstein. Patterns of multiple representation use by experts and novices during physics problem solving. *Physical review special topics-Physics education research*, 4(1):010111, 2008.

[12] Hao Li and Anette E Hosoi. Starting problems in mechanical engineering. In *2018 IEEE Frontiers in Education Conference (FIE)*, pages 1–5. IEEE, 2018. ©2018 IEEE. Reprinted with permission.

[13] Hao Li and Anette E Hosoi. Work in progress: Initial approaches for starting open-ended problems in mechanical engineering. In *2019 ASEE Annual Conference & Exposition*, Tampa, Florida, June 2019. ASEE Conferences. ©2019 ASEE. Reprinted with permission.

[14] Hao Li and Anette E Hosoi. An approach to understanding problem solving using multiple solution methods. In *2023 ASEE Annual Conference & Exposition*, Baltimore, Maryland, June 2023. ASEE Conferences. ©2023 ASEE. Sections, figures, and tables reprinted with permission.

[15] Benjamin M Linder. *Understanding estimation and its relation to engineering education*. PhD thesis, Massachusetts Institute of Technology, 1999.

[16] Seren Mabley, Esther Ventura-Medina, and Anthony Anderson. 'I'm lost'–a qualitative analysis of student teams' strategies during their first experience in problem-based learning. *European Journal of Engineering Education*, 45(3):329–348, 2020.

[17] George Polya. *How to solve it: A new aspect of mathematical method*, volume 85. Princeton university press, 2004.

[18] Sarantos Psycharis and Maria Kallia. The effects of computer programming on high school students' reasoning skills and mathematical self-efficacy and problem solving. *Instructional science*, 45(5):583–602, 2017.

[19] Alan H. Schoenfeld. *Mathematical problem solving*. Academic Press, Orlando, FL, 1985.

[20] Said Shakerin. The art of estimation. *International Journal of Engineering Education*, 22(2):273, 2006.

[21] David R Smith, John P D'Angelo, Robert A DaSilva, and James A Sherwood. An experiment in improving student estimation skills. In *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, volume 80333, page V003T03A025. American Society of Mechanical Engineers, 1998.

[22] Marilla D Svinicki. A guidebook on conceptual frameworks for research in engineering education. *Rigorous Research in Engineering Education*, 7(13):1–53, 2010.

[23] MayBelle N Trahan. *A Conceptual and Measurement Model of Student-Teacher Classroom Social Capital: An Exploration of Relationships.* Education Resources Information Center, 2013.

[24] Alan Van Heuvelen and Xueli Zou. Multiple representations of work–energy processes. *American Journal of Physics*, 69(2):184–194, 2001.

[25] Phillip C Wankat and Frank S Oreovicz. *Teaching Engineering.* McGraw Hill, 1993.

[26] Donald R Woods. On teaching problem solving part II: the challenges. *Chemical Engineering Education*, 11(3):140–144, 1977.

[27] Maria C Yang. Concept generation and sketching: Correlations with design outcome. In *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, volume 37017, pages 829–834, 2003.

[28] Maria C Yang. A study of prototypes, design activity, and design outcome. *Design Studies*, 26(6):649–669, 2005.

[29] Barry J Zimmerman. Becoming a self-regulated learner: An overview. *Theory into practice*, 41(2):64–70, 2002.