Asymptotics, exact results, and analogies in p-adic random matrix theory

by

Roger Van Peski

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Pure Mathematics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2023

Author . . . .
Department of Mathematics
July 25, 2023

Certified by . . . .
Alexei Borodin
Professor
Thesis Supervisor

Accepted by . . . .
Davesh Maulik
Graduate Chair, Department of Mathematics

Asymptotics, exact results, and analogies in p-adic random matrix theory

by

Roger Van Peski

Submitted to the Department of Mathematics
on July 25, 2023, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy in Pure Mathematics

Abstract

This thesis is a compilation of exact results regarding p-adic random matrices and Hall-Littlewood polynomials, and asymptotic results proven using these tools. Many of the results of both types are motivated and guided by analogies to existing results in classical random matrix theory over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, but often exhibit probabilistic behaviors which differ markedly from these known cases. Specifically, we prove the following:

(1) We show exact relations between products and corners of random matrices over $\mathbb{Q}_p$ and Hall-Littlewood processes, which are direct analogues of the classical relations between singular values of real or complex random matrices and type A Heckman-Opdam hypergeometric functions. (2) We prove that the boundary of the Hall-Littlewood t-deformation of the Gelfand-Tsetlin graph is parametrized by infinite integer signatures, extending results of Gorin and Cuenca on boundaries of related deformed Gelfand-Tsetlin graphs. (3) In the special case when $1/t$ is a prime p we combine this with the aforementioned relations between matrix corners and Hall-Littlewood polynomials to recover results of Bufetov-Qiu [BQ17] and Assiotis [Ass22] on infinite p-adic random matrices. (4) Using the above relation between matrix products and Hall-Littlewood polynomials, together with explicit formulas for the latter, we obtain exact product formulas for the joint distribution of the cokernels of products $A_1, A_2A_1, A_3A_2A_1, \ldots$ of independent additive-Haar-distributed matrices A_i over the p-adic integers $\mathbb{Z}_p$. This generalizes the explicit formula for the classical Cohen-Lenstra measure on abelian p-groups. (5) We give an exact sampling algorithm for products of corners of Haar $\mathrm{GL}_N(\mathbb{Z}_p)$-distributed matrices, and show by analyzing it that the singular numbers of such products obey a law of large numbers and their fluctuations converge dynamically to independent Brownian motions. (6) We consider the singular numbers of a certain explicit continuous-time Markov jump process on $\mathrm{GL}_N(\mathbb{Q}_p)$, which we argue gives the closest p-adic analogue of multiplicative Dyson Brownian motion. We do so by explicitly classifying the possible dynamics of singular numbers of processes on $\mathrm{GL}_N(\mathbb{Q}_p)$ satisfying natural properties possessed by Brownian motion on $\mathrm{GL}_N(\mathbb{C})$. Computing the evolution of singular numbers explicitly, we find that the N-tuple of singular numbers in decreasing order evolves as a Poisson jump process on $\mathbb{Z}^N$, with ordering enforced by reflection off the walls of the positive type A Weyl chamber. (7) As N and time go to ∞, we show that this process converges to a stationary limit, with density explicitly expressed in terms of certain intricate exponential

sums. The proof uses new Macdonald process computations, which feature a symmetric function incarnation of the explicit solution to the inverse moment problem for abelian p-groups shown recently by Sawin and Wood [SW22b]. (8) We prove that this reflected Poisson walk is universal, governing dynamical local limits for the singular numbers of p-adic random matrix products at both the bulk and edge, and may thus be viewed as a p-adic analogue of the extended sine and Airy processes. (9) Extrapolating this process to general real $p > 1$, we analyze the limit as $p \to 1$. We prove a law of large numbers, a central limit theorem relating it to stationary solutions of certain SDEs, and a bulk limit to a certain explicit stationary Gaussian process on $\mathbb{R}$. Unlike most previously studied limits of Macdonald processes, the latter exhibits scaling exponents characteristic of the Edwards-Wilkinson universality class in $(1 + 1)$ dimensions, which may be seen as a reflection of locality of interactions between singular numbers which differs markedly from classical random matrix theory.

Thesis Supervisor: Alexei Borodin
Title: Professor

Acknowledgments

My deepest thanks go to my advisor Alexei Mikhailovich Borodin, without whom this thesis and I would both not exist in their present form. He has taught me much both directly and obliquely over these five years, and had my back when I needed it; it has been a great privilege to simultaneously find my own path and learn from a master. When I think of how my outlook on science, life, and how to be a mathematician has changed over my PhD, I see it everywhere informed by his ethos, and this perhaps is the sincerest thanks I can give.

I wish also to thank my good friend Andrew Ahn, with whom I have had the great pleasure of spending many, many hours over the past few years shooting the mathematical breeze, and sometimes working together toward something more concrete. I discussed much of what is in this thesis with him as I came to understand it, and was often helped by his knowledge of related results in complex random matrix theory.

I thank Vadim Gorin for many helpful comments about the work here and encouragement to pursue these directions—one very concrete example, his suggestion to recover results of [Ass22] became Theorem 1.7.3. Some of the ideas in this work began in my undergraduate thesis, and I am very grateful to Ju-Lee Kim for advising that project and for convivial conversations here at MIT since then. Many other professors at MIT enriched my experience by exposing me to some nice things and other viewpoints and aesthetics—I wish in particular to thank Mehran Kardar, Alex Postnikov, and Scott Sheffield for this—and many simply by being friendly and interesting people. I thank Michel Goemans for his good cheer and hard work as department head during my time here.

Outside those mentioned above, I am grateful to Theo Assiotis, Alexander Bufetov, Ivan Corwin, Evgeni Dimitrov, Jason Fulman, Jimmy He, Nathan Kaplan, Mario Kieburg, Hoi Nguyen, Matthew Nicoletti, Grigori Olshanski, Melanie Matchett Wood, and several anonymous referees for helpful comments and conversations about the work here. I thank my collaborators Andrew Ahn, Korina Digalaki, Pei-Ken Hung, Hoi Nguyen, Marianna Russkikh, Ethan Sussman, and Yilin Wang, for joint work which in some cases has led to theorems (not covered in this thesis) and in all cases has taught me something new. I gratefully acknowledge support from an NSF Graduate Research Fellowship under grant #1745302, along with partial funding from the NSF FRG grant DMS-1664619 during some summers.

Last but not least, I thank my friends, Boston-area and otherwise, math and nonmath, for enriching my life in their various ways over the past five years, and my family for their support.

Scholarship has not been cheerful always and everywhere, although it ought to be.

—Hermann Hesse, *The Glass Bead Game*

THIS PAGE INTENTIONALLY LEFT BLANK

Contents

THIS PAGE INTENTIONALLY LEFT BLANK

List of Figures

Chapter 1

Introduction

1.1 Preface

In the 1950s, Eugene Wigner [Wig51, Wig55, Wig57] proposed eigenvalues of large random matrices as a tractable model for the energy levels observed in experiments with heavy nuclei such as uranium. Quantum theory predicted that the behavior of such nuclei was governed by an infinite-dimensional Hermitian operator, its Hamiltonian, the eigenvalues of which corresponded to these energy levels. Such operators were far too complicated to study in practice, and Wigner instead made the bold guess that the eigenvalues of a large $N \times N$ real symmetric or complex Hermitian matrix, with a naive Gaussian distribution that allowed the eigenvalue distribution to be exactly computed, should behave similarly enough to the Hamiltonian's spectrum to make physical predictions. This helped birth the field of random matrix theory[1], which has continued along these lines far past the original application of nuclear physics.

Essentially the same origin story played out independently in the field of arithmetic statistics. Better computers allowed number theorists to compile tables of class groups of quadratic imaginary number fields, allowing Henri Cohen and Hendrik Lenstra Jr. [CL84] to make detailed conjectures regarding the frequency with which certain groups appeared[2]. While these conjectures were made from empirical data, soon af-

[1] An independent origin was the earlier work of Wishart [Wis28] on singular values of random matrices, motivated by statistics.

[2] More precisely, their conjectures concerned the limiting proportion of quadratic imaginary number fields $\mathbb{Q}(\sqrt{-d})$, $1 \leq d \leq D$ squarefree, for which a given finite abelian p-group occurred as the p-Sylow subgroup of the class group $\mathrm{Cl}(\mathbb{Q}(\sqrt{-d}))$ (which is a finite abelian group). The actual computations were

terwards Friedman and Washington [FW87] considered function field analogues and realized that the *Cohen-Lenstra distribution* on abelian p-groups, which was the subject of these conjectures, also appeared in random matrix theory over non-archimedean local fields such as $\mathbb{Q}_p$. Ellenberg and Venkatesh [EV10, Section 4.1] (see also [Woo19]) subsequently gave a random matrix heuristic for the original Cohen-Lenstra conjectures which was remarkably similar to Wigner's. Namely, $\mathrm{Cl}(\mathbb{Q}(\sqrt{-d}))$ is the quotient of an infinite-rank $\mathbb{Z}$-module, the group of fractional ideals, by the full-rank submodule of principal fractional ideals. A natural model is to take N large and consider $\mathrm{coker}(A) := \mathbb{Z}^N/A\mathbb{Z}^N$ where $A \in \mathrm{Mat}_N(\mathbb{Z})$ is a random matrix, producing a random submodule $A\mathbb{Z}^N \subset \mathbb{Z}^N$. Passing to the p-Sylow subgroup of the class group[3], the heuristic predicted that it should be modeled by the $N \to \infty$ limit of $\mathrm{coker}(A') = \mathbb{Z}_p^N/A'\mathbb{Z}_p^N$ for random $A' \in \mathrm{Mat}_N(\mathbb{Z}_p)$, which had been proven in [FW87] to reproduce the Cohen-Lenstra distribution for certain specific choices of distribution on A'. Such results have now been shown to be universal for any generic choices of distribution of matrix entries by Wood and coauthors [Woo19, Woo16, Woo18, NW22b]. Other classes of matrices—symmetric [Woo17, NW22a], alternating [BKL$^+$15], rectangular [Woo18], Hermitian [Lee22]—yield different limiting distributions on the cokernel, many of which appear elsewhere in arithmetic statistics and combinatorics.

The two bodies of work above have not interacted too much. While both feature random matrices, eigenvalues and singular values of random real or complex matrices seem far removed from random groups. Nonetheless, there were some hints at connections between the two. The theory of spherical functions on Lie groups had already proven a useful tool in complex random matrix theory—see e.g. [For10, BG15, GM20] and the references therein—and structural parallels between special functions on real/complex Lie groups and p-adic groups are well-studied (see e.g. [Mac98a, Chapters V and VII]), so from this perspective it was natural to look for a corresponding story for p-adic random matrices. Additionally, Fulman [Ful14] and Fulman-Kaplan [FK19] had previously noted that the Cohen-Lenstra measure on abelian p-groups and certain related measures could be written elegantly in terms of Hall-Littlewood polynomials, which are spherical functions on p-adic groups. This thesis began as an attempt to understand this story,

done by D. Buell, C. P. Schnorr, D. Shanks and H. Williams, according to [CL84].

[3]See [Woo19] for more details on this heuristic.

and once the structural analogies between the complex and p-adic settings were clarified, it became apparent that many interesting probabilistic questions which had been studied extensively on the complex side had never been considered on the p-adic side. At the same time, the aforementioned structural parallels allowed us to bring results and intuition from symmetric function theory, in particular the methods of Macdonald processes introduced by Borodin and Corwin [BC14], to bear on these problems. Let us now describe in more detail these results and their context.

1.2 Exact results and Hall-Littlewood polynomials

For any nonsingular complex matrix $A \in M_{n\times m}(\mathbb{C})$, by singular value decomposition there exist $U \in \mathrm{U}(n), V \in \mathrm{U}(m)$ with $UAV = \mathrm{diag}(e^{-r_1}, e^{-r_2}, \ldots, e^{-r_{\min(m,n)}})$ for some $\infty > r_1 \geq \cdots \geq r_{\min(m,n)}$. Studying the distributions of the singular values of various random matrices A, and their asymptotics, is a classical but still very active line of research.

For any nonsingular p-adic matrix[4] $A \in M_{n\times m}(\mathbb{Q}_p)$, there similarly exist $U \in \mathrm{GL}_n(\mathbb{Z}_p), V \in \mathrm{GL}_m(\mathbb{Z}_p)$ such that $UAV = \mathrm{diag}(p^{\lambda_1}, p^{\lambda_2}, \ldots, p^{\lambda_{\min(m,n)}})$ for some integers $\infty > \lambda_1 \geq \ldots \geq \lambda_{\min(m,n)}$. We refer to the integers λ_i as the *singular numbers* of A and write $\mathrm{SN}(A) = (\lambda_1, \ldots, \lambda_{\min(m,n)}) = \lambda$ in the above case. In the case where $A \in M_{n\times m}(\mathbb{Z}_p)$ so $A : \mathbb{Z}_p^m \to \mathbb{Z}_p^n$ is a linear map, we have

$$\mathrm{coker}(A) := \mathbb{Z}_p^n / A\mathbb{Z}_p^m \cong \bigoplus_{i=1}^{n} \mathbb{Z}/p^{\lambda_i}\mathbb{Z} \tag{1.2.1}$$

where λ_i are the singular numbers[5]. One can study the distribution of $\mathrm{SN}(A)$ for random $A \in M_{n\times m}(\mathbb{Q}_p)$ just as with singular values, and if $A \in M_{n\times m}(\mathbb{Z}_p)$ then this is equivalent to the cokernel studied in the works above. Specifically, one may ask the following.

(Q1) For 'natural' choices of the distribution of A, what is the distribution of $\mathrm{SN}(A)$?

(Q2) Let A_{col} be the matrix given by removing the last column from A. What is the conditional distribution of $\mathrm{SN}(A_{col})$ given $\mathrm{SN}(A)$?

[4]For background on the p-adic numbers and matrix groups over them, see Chapter 2.
[5]Assume $m \geq n$ so the cokernel has no free part.

(Q3) Let $B \in M_{m\times k}(\mathbb{Q}_p)$ be another independent random matrix. Given the distributions of $\mathrm{SN}(A), \mathrm{SN}(B)$, what is the distribution of $\mathrm{SN}(AB)$?

As an example of (Q1), the original work of Friedman-Washington [FW87] explicitly computed the distribution of $\mathrm{coker}(A^{(n)})$ for $A^{(n)} \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_p)$ with iid entries distributed according to the Haar measure on the additive group $\mathbb{Z}_p$. They found the attractive limiting formula that for any integer partition (an infinite sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$ which is eventually 0),

$$\lim_{n\to\infty} \Pr(\mathrm{SN}(A^{(n)}) = (\lambda_1, \ldots, \lambda_n)) = \frac{\prod_{j\geq 1}(1-p^{-j})}{|\mathrm{Aut}(\bigoplus_i \mathbb{Z}/p^{\lambda_i}\mathbb{Z})|}. \tag{1.2.2}$$

The right hand side defines a probability measure on integer partitions, equivalently on abelian p-groups, which appeared in the original conjecture of [CL84] and is hence known as the Cohen-Lenstra measure.

We give complete answers to (Q2), (Q3) and a family of cases of (Q1) including the above case in Theorem 1.2.1 below, when the distribution of A is invariant under left- and right-multiplication by $\mathrm{GL}_n(\mathbb{Z}_p)$ and $\mathrm{GL}_m(\mathbb{Z}_p)$ respectively. These results use relations between p-adic matrices and the classical Hall-Littlewood symmetric polynomials $P_\lambda(x_1, \ldots, x_n; t)$, symmetric polynomials which reduce to Schur polynomials at $t = 0$ and monomial symmetric polynomials at $t = 1$, and play key roles in geometry, representation theory, and algebraic combinatorics. Explicitly they are defined by

$$P_\lambda(x_1, \ldots, x_n; t) := \frac{1}{v_\lambda(t)} \sum_{\sigma \in S_n} \sigma \left(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{1\leq i<j\leq n} \frac{x_i - tx_j}{x_i - x_j} \right), \tag{1.2.3}$$

where σ acts by permuting the variables and $v_\lambda(t)$ is the normalizing constant such that the $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ term has coefficient 1. They form a distinguished basis for the ring $\Lambda_n := \mathbb{C}[x_1, \ldots, x_n]^{S_n}$ of symmetric polynomials in n variables $x_1, \ldots, x_n$, indexed by the set $\mathrm{Sig}_n^+ := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n : \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}$ of nonnegative *integer signatures*, and feature an additional parameter t which we take to be real. Using only the property that they form a basis and some positivity properties below, one can use them to define probability measures, Markov dynamics, and randomized convolution operations on signatures[6].

[6] These operations also make sense when the indices lie in the set $\mathrm{Sig}_n := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n : \lambda_1 \geq$

1. (Probability measures) For real $a_1, \ldots, a_n \geq 0$, and $t \in [0,1)$ one has $P_\lambda(a_1, \ldots, a_n; t) \geq 0$. Hence for any sets $\{a_i\}, \{b_i\}$ of nonnegative reals with all $a_i b_j < 1$ one may define the *Hall-Littlewood measure* on Sig_n^+ via

$$\Pr(\lambda) = \frac{1}{\Pi_{(0,t)}(a_1, \ldots, a_n; b_1, \ldots, b_n)} P_\lambda(a_1, \ldots, a_n; t) Q_\lambda(b_1, \ldots, b_n; t) \tag{1.2.4}$$

where $\Pi_{(0,t)}(a_1, \ldots, a_n; b_1, \ldots, b_n)$ is a normalizing constant and Q_λ is a certain constant multiple of P_λ, see Chapter 2.

2. (Markov dynamics) Because the P_λ form a basis for the vector space of symmetric polynomials in n variables,

$$P_\lambda(x_1, \ldots, x_n; t) = \sum_\mu P_{\lambda/\mu}(x_{k+1}, \ldots, x_n; t) P_\mu(x_1, \ldots, x_k; t) \tag{1.2.5}$$

for some symmetric polynomials $P_{\lambda/\mu} \in \Lambda_{n-k}$, called *skew Hall-Littlewood polynomials*. Substituting positive real numbers a_i for the variables naturally yields Markov dynamics $\mathrm{Sig}_n^+ \to \mathrm{Sig}_k^+$ given by

$$\Pr(\lambda \to \mu) = \frac{P_{\lambda/\mu}(a_{k+1}, \ldots, a_n; t) P_\mu(a_1, \ldots, a_k; t)}{P_\lambda(a_1, \ldots, a_n; t)}.$$

3. (Product convolution) Again using that the P_λ form a basis,

$$P_\lambda(x_1, \ldots, x_n; t) \cdot P_\mu(x_1, \ldots, x_n; t) = \sum_\nu c^\nu_{\lambda,\mu}(0,t) P_\nu(x_1, \ldots, x_n; t)$$

for some structure coefficients $c^\nu_{\lambda,\mu}(0,t)$–these are often called *Littlewood-Richardson coefficients*, particularly in the case $t = 0$ corresponding to the classical Schur polynomials. One may then, given two fixed signatures λ, μ, define their 'convolution' $\lambda \boxtimes_{\mathbf{a}} \mu$ (a random signature) by

$$\Pr(\lambda \boxtimes_{\mathbf{a}} \mu = \nu) = \frac{P_\nu(a_1, \ldots, a_n; t)}{P_\lambda(a_1, \ldots, a_n; t) P_\mu(a_1, \ldots, a_n; t)} c^\nu_{\lambda,\mu}(0,t)$$

for each $\nu \in \mathrm{Sig}_n$. Convolutions of signatures which are themselves random may be

$\cdots \geq \lambda_n\}$ of integer signatures (possibly with some negative parts), but we refer to Chapter 2 for conventions on these.

obtained from this by mixtures.

We may now state the main structural result, which shows that the matrix operations of products and corners mirror the above operations on the level of symmetric functions.

Theorem 1.2.1. *Fix a prime p and let $t = 1/p$.*

1. (Truncated Haar ensemble) *Let $1 \leq n \leq m \leq N$ be integers, and A be the top-left $n \times m$ submatrix of a Haar-distributed element of $\mathrm{GL}_N(\mathbb{Z}_p)$. Then $\mathrm{SN}(A)$ is a random nonnegative signature with distribution given by the Hall-Littlewood measure*
$$\Pr(\mathrm{SN}(A) = \lambda) = \frac{P_\lambda(1, t, \ldots, t^{n-1}; t) Q_\lambda(t^{m-n+1}, \ldots, t^{N-n}; t)}{\Pi_{(0,t)}(1, t, \ldots, t^{n-1}; t^{m-n+1}, \ldots, t^{N-n})}. \tag{1.2.6}$$

2. (Corners process) *Let n, k, N be integers with $1 \leq n \leq N$ and $1 \leq k \leq N - n$, $\lambda \in \mathrm{Sig}_n$, and $A \in M_{n \times N}(\mathbb{Q}_p)$ be random with $\mathrm{SN}(A) = \lambda$ and distribution invariant under $\mathrm{GL}_n(\mathbb{Z}_p) \times \mathrm{GL}_N(\mathbb{Z}_p)$ acting on the right and left. Let $A_{col} \in M_{n \times (N-k)}(\mathbb{Q}_p)$ be the first $N - k$ columns of A. Then $\mathrm{SN}(A_{col})$ is a random element of Sig_n with distribution given by*
$$\Pr(\mathrm{SN}(A_{col}) = \nu) = \frac{Q_{\nu/\lambda}(1, \ldots, t^{-(k-1)}; t) P_\nu(t^{N-n}, \ldots, t^{N-1}; t)}{P_\lambda(t^{N-n}, \ldots, t^{N-1}; t) \Pi_{(0,t)}(1, \ldots, t^{-(k-1)}; t^{N-n}, \ldots, t^{N-1})}. \tag{1.2.7}$$
Now let $1 \leq d \leq n$ and $A_{row} \in M_{(n-d) \times N}$ be the first $n - d$ rows of A. Then $\mathrm{SN}(A_{row})$ is a random element of Sig_{n-d} with distribution
$$\Pr(\mathrm{SN}(A_{row}) = \mu) = P_{\lambda/\mu}(1, \ldots, t^{d-1}; t) \frac{P_\mu(t^d, \ldots, t^{n-1}; t)}{P_\lambda(1, \ldots, t^{n-1}; t)}. \tag{1.2.8}$$

3. (Product process) *Let A, B be random elements of $M_n(\mathbb{Q}_p)$ with fixed singular numbers $\mathrm{SN}(A) = \lambda, \mathrm{SN}(B) = \mu$, invariant under left- and right-multiplication by $\mathrm{GL}_n(\mathbb{Z}_p)$. Then for any $\nu \in \mathrm{Sig}_n$, $\mathrm{SN}(AB)$ has distribution $\lambda \boxtimes_{(1,\ldots,t^{n-1})} \mu$, i.e.*
$$\Pr(\mathrm{SN}(AB) = \nu) = c^\nu_{\lambda,\mu}(0, t) \frac{P_\nu(1, \ldots, t^{n-1}; t)}{P_\lambda(1, \ldots, t^{n-1}; t) P_\mu(1, \ldots, t^{n-1}; t)}. \tag{1.2.9}$$

In the limit $N \to \infty$, Theorem 1.2.1 Part 1 recovers the distribution of singular numbers of matrices with iid additive Haar entries.

Corollary 1.2.2. *Fix a prime p and let $t = 1/p$. Let $1 \leq n \leq m$, and $A \in M_{n\times m}(\mathbb{Z}_p)$ be random with iid entries distributed according to the additive Haar measure on $\mathbb{Z}_p$. Then for any $\lambda \in \mathrm{Sig}_n^+$,*

$$\Pr(\mathrm{SN}(A) = \lambda) = \frac{P_\lambda(1, \ldots, t^{n-1}; t) Q_\lambda(t^{m-n+1}, t^{m-n+2}, \ldots; t)}{\Pi_{(0,t)}(1, \ldots, t^{n-1}; t^{m-n+1}, t^{m-n+2}, \ldots)}$$

The $m = n$ case of Corollary 1.2.2 was the original ensemble studied by Friedman-Washington [FW87], and appeared also in the work of Evans [Eva02]. The rectangular case was considered in work of Wood [Woo19, Thm. 1.3], which studies the $n \to \infty$ asymptotics of $n \times (n+u)$ matrices for fixed u; this work shows that the limit is universal for many choices of the distribution of matrix entries, but does not consider the exact result of Corollary 1.2.2 for finite n. Subcases of the Hall-Littlewood measures we consider also appear in the work of Fulman [Ful02, Ful14] on Jordan blocks of uniformly random elements of $\mathrm{GL}_n(\mathbb{F}_q)$, though the language of Hall-Littlewood measures was not used.

Explicit formulas for the probabilities in Corollary 1.2.2 and Theorem 1.2.1 may be obtained using the explicit formulas for Hall-Littlewood polynomials, Proposition 2.2.15 and Theorem 2.2.16, recovering those given in [FW87] and yielding several new ones. One such consequence is a generalization of the results of [FW87] to cokernels of products of additive Haar matrices. Here $\lambda'_x := \#\{i : \lambda_i \geq x\}$, $(a; t)_n := \prod_{i=0}^{n-1}(1 - a \cdot t^i)$ is the t-Pochhammer symbol,

$$\begin{bmatrix} a \\ b \end{bmatrix}_t := \frac{(t;t)_a}{(t;t)_b (t;t)_{a-b}} \tag{1.2.10}$$

is the t-binomial coefficient, and

$$n(\lambda) := \sum_i (i-1)\lambda_i. \tag{1.2.11}$$

Theorem 1.2.3. *Let $t = 1/p$, fix $n \geq 1$, and let A_i be iid $n \times n$ matrices with iid entries distributed by the additive Haar measure on $\mathbb{Z}_p$. Then the joint distribution of*

$\mathrm{SN}(A_1), \mathrm{SN}(A_2A_1), \ldots$ *is given by*

$$\begin{aligned}
&\Pr(\mathrm{SN}(A_i \cdots A_1) = \lambda(i) \textit{ for all } i = 1, \ldots, k) \\
&= (t;t)_n^k t^{n(\lambda(k))} \prod_{1 \le i \le k} \prod_{x \in \mathbb{Z}} t^{\binom{\lambda(i)'_x - \lambda(i-1)'_x + 1}{2}} \begin{bmatrix} \lambda(i)'_x - \lambda(i-1)'_{x+1} \\ \lambda(i)'_x - \lambda(i)'_{x+1} \end{bmatrix}_t
\end{aligned} \tag{1.2.12}$$

for any k and $\lambda(1), \ldots, \lambda(k) \in \mathrm{Sig}_n^{\geq 0}$, where we take $\lambda(0) = (0, \ldots, 0)$ in (1.2.12).

Note that the product over $x \in \mathbb{Z}$, which may appear uninviting, in fact has only finitely many nontrivial terms. We mention also that in work of Nguyen and the author [NVP22] which does not appear in this thesis, the $n \to \infty$ limit of this distribution was shown to be universal for products of random matrices with iid entries from a generic distribution, and interpreted in terms of automorphisms of nested sequences of abelian p-groups. Currently we are not aware of this distribution appearing elsewhere, but given the various matrix models mentioned in the Preface which have found application elsewhere, it seems any natural enough distribution on p-adic random matrices may model some class of random abelian p-groups appearing in nature.

Remark 1. Though we have chosen to state them in the case of $\mathbb{Q}_p$ because it is most commonly considered in the literature, all of our results and proofs for p-adic matrices are actually valid for matrices over any non-Archimedean local field K with finite residue field, i.e. any algebraic extensions of $\mathbb{Q}_p$ or $\mathbb{F}_q((t))$. Any such field has a ring of integers R which plays the role of $\mathbb{Z}_p$, and a unique maximal ideal $(\omega) \subset R$ generated by a uniformizer ω which plays the role of p. The residue field $R/(\omega)$ is a finite field $\mathbb{F}_q$ for some q. Replacing $\mathbb{Q}_p$ by K, $\mathbb{Z}_p$ by R, p by ω (in the context of matrix entries), and setting $t = 1/q$ in Hall-Littlewood specializations, our results translate mutatis mutandis. This is essentially a consequence of the fact that Part 3 of Theorem 1.2.1 holds in this generality, see [Mac98a, Ch. V] and the discussion in Chapter 3.

1.2.1 Macdonald polynomials and connections to the complex case

Let us say a few words about the proof of Theorem 1.2.1. Part 3 of the theorem is essentially a probabilistic reframing of results on the Hecke ring of the pair $(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{GL}_n(\mathbb{Z}_p))$

in [Mac98a, Ch. V]. To prove Part 1 and Part 2, we use limiting cases of Part 3 corresponding to projection matrices. For example, if $U \in \mathrm{GL}_N(\mathbb{Z}_p)$ is random with Haar distribution, then the corners described in Part 1 are given by the convolution $P_n U P_m$ of projection matrices of rank n and m, which may be treated by a limiting case of the product operation in Part 3. This link also explains the appearance of similar geometric progressions in t in the formulas in Parts 1, 2, 3: those in Parts 1, 2 come from those in Part 3 via this degeneration. We remark that in the complex case, the relation between products of randomly-rotated projection matrices and the so-called truncated unitary/Jacobi ensembles was observed and exploited by Collins [Col05].

To implement this strategy in the p-adic setting, in view of (1.2.9) it is necessary to establish some combinatorial results on asymptotics of the structure coefficients $c^{\nu}_{\lambda,\mu}(0,t)$. We prove some quite general results in this direction in Chapter 3, which are valid for the more general class of *Macdonald polynomials* $P_\lambda(x_1,\ldots,x_n;q,t)$, another family of symmetric polynomials indexed by signatures. These polynomials have two parameters q,t and specialize to Hall-Littlewood polynomials when $q=0$, but the measures, Markov dynamics and convolution operations on signatures defined for Hall-Littlewood polynomials work exactly the same way. We chose to work at the level of Macdonald polynomials partially to highlight the similarities between our results and existing results for complex random matrices. One may define measures, Markov kernels and randomized convolution operations on integer signatures by the formulas (1.2.6), (1.2.8), (1.2.7) and (1.2.9) in Theorem 1.2.1 but with Macdonald polynomials $P_\lambda(x_1,\ldots,x_n;q,t)$ substituted in for Hall-Littlewood polynomials $P_\lambda(x_1,\ldots,x_n;0,t)$, and all probabilities will still be nonnegative provided $q \in [0,1)$. Sending $q \to 0$ recovers the probabilities in Theorem 1.2.1. There is another limit where $t = q^{\beta/2}$ and $q,t \to 1$ while the signatures are also scaled at some rate dependent on q, which yields probability measures, Markov kernels and convolutions on *real* signatures. In these limits, the formulas in Theorem 1.2.1 (with Macdonald polynomials in place of Hall-Littlewood, but no other changes) degenerate when $\beta = 1,2,4$ to formulas for singular values under the corresponding corners and product matrix operations on real, complex and quaternion random matrices. We discuss this limit in more detail in Section 3.3.

1.3 Law of large numbers and central limit theorem

The asymptotic distributions obtained in the previous p-adic random matrix literature look quite different from their counterparts in the world of singular values. For instance, (1.2.2) yields that with probability 1 only an asymptotically finite number of the λ_i are nonzero. The motivating questions in number theory concern group-theoretic properties such as the probability of cyclicity (i.e. probability that $\mathrm{SN}(A) = (k, 0, \dots, 0)$ for some k) or the distribution of ranks (the number of nonzero parts of $\lambda = \mathrm{SN}(A)$).

By contrast, the singular values of an $n \times n$ matrix with iid standard Gaussian entries, usually referred to as the *Ginibre ensemble*, converge with rescaling to the celebrated Marchenko-Pastur law [MP67] (a compactly supported probability distribution on $\mathbb{R}$). As far as we are aware, no continuous probability distributions on $\mathbb{R}$ appeared previously governing limits of singular numbers of random p-adic matrices. Indeed, such limits are in a sense orthogonal to the viewpoint of random abelian p-groups taken in most of the previous literature.

Remark 2. In a probabilistic context it is often helpful to view the additive Haar measure on $\mathbb{Z}_p$ as an analogue of the Gaussian on $\mathbb{R}$ or $\mathbb{C}$. One shared feature is that additive convolution preserves both classes of measures: if X, Y are distributed according to the additive Haar measure on $\mathbb{Z}_p$, $X + Y$ is as well. In fact, random vectors $v \in \mathbb{Z}_p^n$ with iid Haar-distributed entries are invariant under $\mathrm{GL}_n(\mathbb{Z}_p)$ just as Gaussian vectors are invariant under $\mathrm{U}(n)$, and both are characterized up to scaling by this invariance together with independence of entries. Another shared feature is that both the Gaussian and the additive Haar measure are their own Fourier transform. See Tao [Tao08] and Evans [Eva01] for more discussion.

In the next result, we find Gaussian limits in the setting of products of a large number of p-adic matrices of finite size. Given random matrices $A_1, A_2, \ldots \in M_n(\mathbb{Q}_p)$, one may view the $\mathrm{SN}(A_1), \mathrm{SN}(A_2 A_1), \ldots$ as defining a discrete-time Markov chain on the set of weakly decreasing n-tuples of integers. Equivalently, for each i the i^{th} largest singular number evolves as some random walk on $\mathbb{Z}$, with the n such random walks sometimes colliding but never crossing. See Figure 1-1 below.

In the case when A_i are $n \times n$ corners of independent Haar-distributed elements of $\mathrm{GL}_N(\mathbb{Z}_p)$ with $N > n$, we show that the singular numbers of their products satisfy an explicit law of large numbers as $k \to \infty$, and furthermore the fluctuations converge to n independent Brownian motions. In the limit as $N \to \infty$, the entries of such corners become independent and distributed according to the additive Haar measure on $\mathbb{Z}_p$, recovering the matrices studied in the previous literature, so we allow the case '$N = \infty$' below.

Theorem 1.3.1. *Fix $n \geq 1$, and let $N_1, N_2, \ldots \in \mathbb{Z} \cup \{\infty\}$ with $N_j > n$ for all j. For each j, if $N_j < \infty$ let A_j be the top left $n \times n$ corner of a Haar distributed element of $\mathrm{GL}_{N_j}(\mathbb{Z}_p)$, and if $N_j = \infty$ let A_j have iid entries distributed by the additive Haar measure on $\mathbb{Z}_p$. For $k \in \mathbb{N}$ let*

$$(\lambda_1(k), \ldots, \lambda_n(k)) := \mathrm{SN}(A_k \cdots A_1).$$

Then we have a strong law of large numbers

$$\frac{\lambda_i(k)}{\sum_{j=1}^k \sum_{\ell=0}^{N_j-n-1} \frac{p^{-i-\ell}(1-p^{-1})}{(1-p^{-i-\ell-1})(1-p^{-i-\ell})}} \to 1 \text{ a.s. as } k \to \infty.$$

Let

$$\bar{\lambda}_i(k) := \lambda_i(k) - \sum_{j=1}^{k} \sum_{\ell=0}^{N_j-n-1} \frac{p^{-i-\ell}(1-p^{-1})}{(1-p^{-i-\ell-1})(1-p^{-i-\ell})}$$

and define the random function of $f_{\bar{\lambda}_i,k} \in C[0,1]$ as follows: set $f_{\bar{\lambda}_i,k}(0) = 0$ and

$$(f_{\bar{\lambda}_i,k}(1/k), f_{\bar{\lambda}_i,k}(2/k), \ldots, f_{\bar{\lambda}_i,k}(1)) = \frac{1}{\sqrt{\sum_{j=1}^k \sum_{\ell=0}^{N_j-n-1} \frac{p^{-i-\ell}(1-p^{-1})(1-p^{-2i-2\ell-1})}{(1-p^{-i-\ell})^2(1-p^{-i-\ell-1})^2}}}(\bar{\lambda}_i(1), \ldots, \bar{\lambda}_i(k)),$$

then linearly interpolate from these values on each interval $[\ell/k, (\ell+1)/k]$. Then as $k \to \infty$, the n-tuple of random functions $(f_{\bar{\lambda}_1,k}, \ldots, f_{\bar{\lambda}_n,k})$ converges in law in the sup norm topology on $C[0,1]$ to n independent standard Brownian motions.

In particular, we have the central limit theorem that

$$\frac{\bar{\lambda}_i(k)}{\sqrt{\sum_{j=1}^k \sum_{\ell=0}^{N_j-n-1} \frac{p^{-i-\ell}(1-p^{-1})(1-p^{-2i-2\ell-1})}{(1-p^{-i-\ell})^2(1-p^{-i-\ell-1})^2}}} \to \mathcal{N}(0,1)$$

in law for each i.

Remark 3. If all N_j are equal to some N, then the law of large numbers takes the more

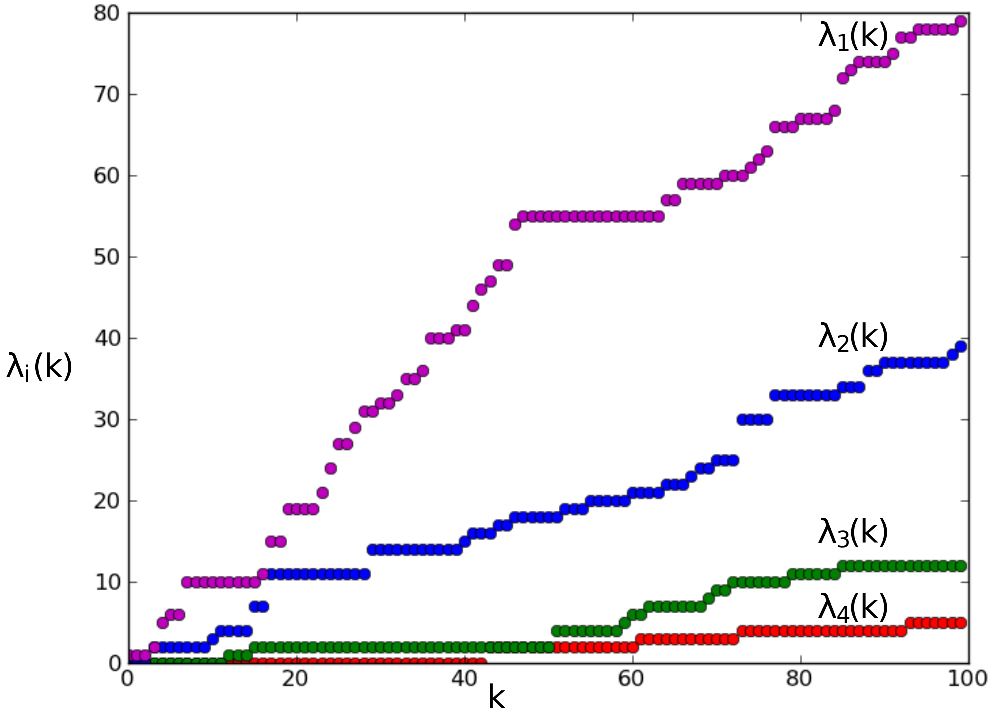


Figure 1-1: A plot of $(\lambda_1(k), \lambda_2(k), \lambda_3(k), \lambda_4(k)) = \mathrm{SN}(A_k \cdots A_1)$ where $A_1, \ldots, A_{100} \in M_4(\mathbb{Z}_2)$ are random matrices with iid entries drawn from the additive Haar measure on $\mathbb{Z}_2$.

standard form

$$\frac{\lambda_i(k)}{k} \to \sum_{\ell=0}^{N-n-1} \frac{p^{-i-\ell}(1-p^{-1})}{(1-p^{-i-\ell-1})(1-p^{-i-\ell})}.$$

When $N = \infty$, evaluating the sum via the q-Gauss identity [Koe98, (3.5)] yields an even more explicit limit:

$$\frac{\lambda_i(k)}{k} \to \frac{1}{p^i - 1}.$$

The matrices A_i above lie in $\mathrm{GL}_n(\mathbb{Q}_p)$ with probability 1, so Theorem 1.3.1 may be viewed as a statement about certain random walks on the group $\mathrm{GL}_n(\mathbb{Q}_p)$. Previous work by Brofferio-Schapira [BS11] takes this perspective of random walks on groups and studies similar random walks from an ergodic theory perspective. They prove a law of large numbers for products of iid random matrices from a quite general class of probability distributions on $\mathrm{GL}_n(\mathbb{Q}_p)$ via a generalization of Oseledets' multiplicative ergodic theorem [Ose68] to matrices over $\mathbb{Q}_p$, due to Raghunathan [Rag79]. The family of probability distributions considered in [BS11] is more general than that of Theorem 1.2.1, but the latter covers cases when the matrices A_i are not identically distributed and shows Gaussian fluctuations, which are not shown in [BS11]. A different family of random walks on $\mathrm{GL}_n(\mathbb{Q}_p)$ were studied by Chhaibi [Chh17]; the perspective in this work is more similar to ours in that it is heavily based on special functions on p-adic groups, though the presentation and problems considered are quite different. Finally, there is a body of work on random walks on Bruhat-Tits buildings which translates to results on random walks on p-adic groups, see for instance Cartwright-Woess [CW04, Section 8], Schapira [Sch09], and especially the survey of Parkinson [Par17]. These works contain central limit theorems, but for quite different quantities and settings than ours.

In the setting of real and complex matrices, the study of asymptotics of singular values of products $A_k A_{k-1} \cdots A_1$ of random matrices as $k \to \infty$ dates back at least as far as the 1960 work of Furstenberg and Kesten [FK60], who showed Gaussian fluctuations for the logarithm of the largest singular value under some assumptions on the A_i. In the case where the A_i are iid square with complex Gaussian entries, Gaussian fluctuations for logarithms of all singular values (not just the largest as in [FK60]) were obtained in the physics literature by Akemann-Burda-Kieburg [ABK14] and in the mathematics literature by Liu-Wang-Wang [LWW23, Thm. 1.1]. The case of products of $n \times n$ corners of unitary matrices, often referred to as the *truncated unitary ensemble*, has also been considered;

the term Jacobi ensemble is also sometimes used for such corners due to the relation with the classical Jacobi orthogonal polynomial ensemble. Dynamical convergence of the logarithm of the largest singular value of $A_k \cdots A_1$ to Brownian motion for products of such unitary corners was suggested, though not directly implied, by the work of Ahn (one should send $\hat{T} \to \infty$ in [Ahn22b, Thm. 1.7]). Together these results strongly suggest that as the number of products goes to infinity, the fluctuations of the n logarithms of singular values should converge to n independent Brownian motions, as holds in the p-adic case by Theorem 1.3.1, and indeed a result of this form was shown by Ahn [Ahn22a] after our result originally appeared.

There is now a large body of both mathematics and physics literature on asymptotics of matrix products in various regimes, often from the perspective of ergodic theory and often motivated by connections to chaotic dynamical systems and disordered systems in statistical physics, neural networks, and other areas. See for example Ahn [Ahn22b, AS22], Akemann-Burda-Kieburg [ABK14, ABK19, ABK20], Akemann-Ipsen [AI15], Akemann-Ipsen-Kieburg [AIK13], Akemann-Kieburg-Wei [AKW13], Crisanti-Paladin-Vulpiani [CPV12], Forrester [For15] and Forrester-Liu [FL16], Gol'dsheid-Margulis [GM89], Gorin-Sun [GS22], Kieburg-Kösters [KK$^+$19], and Liu-Wang-Wang [LWW23]. Such considerations motivate the study of the *Lyapunov exponents*, so named because of the connection with dynamical systems: given random complex matrices $A_1, A_2, \ldots$, the i^{th} Lyapunov exponent is defined as

$$\lim_{k \to \infty} \frac{1}{k} \log(i^{th} \text{ largest singular value of } A_k \cdots A_1).$$

In the limit as the sizes of the A_i grows, the largest Lyapunov exponents converge (with appropriate scaling) to an evenly spaced sequence $0, -1, -2, \ldots$ in several known cases. Note that this statement has no content if one may scale each Lyapunov exponent individually, but it is quite surprising that applying the same additive shift and multiplicative scaling to all of the Lyapunov exponents together produces this evenly spaced sequence. For complex Ginibre matrices this convergence statement is an easy corollary of results of Liu-Wang-Wang[7] [LWW23]. For products of corners of Haar-distributed unitary matrices and Ginibre matrices, the same result holds by work of Ahn and the author [AVP23]. These examples seem to support the notion that such evenly spaced Lyapunov exponents

[7] To obtain $0, -1, -2, \ldots$, take $N \to \infty$ in (1.7) of [LWW23] with scaling and use that the digamma function $\psi(z)$ is asymptotic to $\log(z)$.

are universal, though we are not aware of a precise conjecture in the literature regarding the scope of this class.

In the p-adic case, we are able to prove that at least within the class of truncated $\mathrm{GL}_N(\mathbb{Z}_p)$ matrices–and iid additive Haar matrices, by the $N \to \infty$ limit)–the appropriate analogues of Lyapunov exponents have universal limits. Consider random $A_1, A_2, \ldots \in M_n(\mathbb{Q}_p)$. For appropriate $U, V \in \mathrm{GL}_n(\mathbb{Z}_p)$ such that $U(A_k \cdots A_1)V = \mathrm{diag}(p^{\lambda_1(k)}, \ldots, p^{\lambda_n(k)})$ with $\lambda_1(k) \geq \ldots \geq \lambda_n(k)$, we have that the i^{th} *smallest* part $\lambda_{n-i+1}(k)$ of $\mathrm{SN}(A_k \cdots A_1)$ is the analogue of $-\log(i^{th}$ *largest* singular value) in the complex setting, because p^D is small in the p-adic norm for large D. Hence

$$\lim_{k\to\infty} \frac{\lambda_{n-i+1}(k)}{k}$$

should be regarded as the appropriate analogue of the i^{th} Lyapunov exponent in the p-adic setting. Our next result shows that within the class of products of arbitrary Haar corners, these analogues of Lyapunov exponents converge to values $1, p, p^2, \ldots$ in geometric progression, much like the arithmetic progression $0, -1, -2, \ldots$ in the complex case mentioned previously.

Theorem 1.3.2 (Large-n universality of Lyapunov exponents)**.** *For each $n \in \mathbb{N}$, let $N_1^{(n)}, N_2^{(n)}, \ldots \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ be such that $N_j^{(n)} > n$ and the limiting frequencies*

$$\rho_n(N) := \lim_{k\to\infty} \frac{|\{1 \leq j \leq k : N_j^{(n)} = N\}|}{k}$$

exist for all $N > n$. Let $A_j^{(n)}$ be $n \times n$ corners of independent Haar distributed matrices in $\mathrm{GL}_{N_j^{(n)}}(\mathbb{Z}_p)$ (with the case $N_j^{(n)} = \infty$ treated as in Theorem 1.3.1). Then for each n, the Lyapunov exponents

$$L_i^{(n)} := \lim_{k\to\infty} \frac{\lambda_{n-i+1}(k)}{k}$$

exist almost surely, where $\lambda_{n-i+1}(k)$ is as in Theorem 1.3.1. Furthermore, the Lyapunov exponents have limits

$$\lim_{n\to\infty} \frac{L_i^{(n)}}{p^{-n}(1-c(n))} = p^{i-1}$$

for every i, where $c(n) := \sum_{N>n} \rho_n(N) p^{-(N-n)}$.

We note that Theorem 1.3.2 does not require any relation between the $N_j^{(n)}$ for different

n; one can for example alternate $N_j^{(n)} = n+1$ for n even and $N_j^{(n)} = \infty$ for n odd, and the result still holds. In the next section we see that this geometric progression was in fact a hint of a new universal object.

1.4 p-adic analogues of Dyson Brownian motion, Poisson walks, and local limits

Theorem 1.3.2 shows that in that setting, at least near the 'edge' $j \approx N$ for N large, the j^{th} singular number $\mathrm{SN}(A_\tau^{(N)} \cdots A_1^{(N)})_j$ evolves in discrete time τ about p times faster than the $(j+1)^{th}$ does. It is natural to ask the finer-grained question of what this discrete-time dynamics actually looks like near the edge, and whether there is any hope that an $N \to \infty$ limit exists which might be universal. A related question is the *bulk local limit* of this evolution far away from either edge, e.g. for $j \approx N/2$. Both of these questions differ from the previous section in that the matrix size N is sent to ∞ along with the number of products, rather than fixed as the number of products goes to ∞.

In the bulk, we find the exact distribution of singular numbers at a fixed time, which we now define, though we suggest to skip over the details of the formulas at a first reading. These formulas feature the q-Pochhammer symbols and q-binomial coefficients defined earlier in (1.2.10), and for signatures $\mu \in \mathrm{Sig}_{k-1}, \lambda \in \mathrm{Sig}_k$ we write

$$\mu \prec \lambda \iff \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \ldots \geq \mu_{k-1} \geq \lambda_k$$

$|\mu| = \sum_i \mu_i$, and $\mu - (d[k-1]) = (\mu_1 - d, \ldots, \mu_{k-1} - d)$. They also feature q-Whittaker polynomials $P_\lambda(\cdots; q, 0)$ and Plancherel specializations of Hall-Littlewood polynomials, for which we refer to Chapter 2 for definitions.

Theorem 1.4.1. *For any $k \in \mathbb{Z}_{\geq 1}$ and $\chi \in \mathbb{R}_{>0}$, there is a Sig_k-valued random variable*

$\mathcal{L}_{k,\chi}$ *with law defined by*

$$\begin{aligned}
&\Pr(\mathcal{L}_{k,\chi} = \boldsymbol{L}) \\
&\quad = \frac{(t;t)_\infty^{k-1}}{k!(2\pi\mathbf{i})^k} \prod_{i=1}^{k-1} \frac{t^{\binom{L_i-L_k}{2}}}{(t;t)_{L_i-L_{i+1}}} \int_{\tilde{\Gamma}^k} e^{\chi t^{L_k}(w_1+\ldots+w_k)} \frac{\prod_{1\le i\ne j\le k}(w_i/w_j;t)_\infty}{\prod_{i=1}^k(-w_i^{-1};t)_\infty(-tw_i;t)_\infty} \\
&\times \sum_{j=0}^{L_{k-1}-L_k} t^{\binom{j+1}{2}} \begin{bmatrix} L_{k-1}-L_k \\ j \end{bmatrix}_t P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1},\ldots,w_k^{-1};t,0) \prod_{i=1}^k \frac{dw_i}{w_i} \qquad (1.4.1)
\end{aligned}$$

for any $\boldsymbol{L} = (L_1,\ldots,L_k) \in \mathrm{Sig}_k$*, with contour (see Figure 1-2)*

$$\tilde{\Gamma} := \{x+\mathbf{i} : x \le 0\} \cup \{x-\mathbf{i} : x \le 0\} \cup \{x+\mathbf{i}y : x^2+y^2=1, x>0\} \qquad (1.4.2)$$

in usual counterclockwise orientation. Its density also has a series representation

$$\begin{aligned}
\Pr(\mathcal{L}_{k,\chi} = \boldsymbol{L}) &= \frac{1}{(t;t)_\infty} \sum_{d\le L_k} e^{-\chi t^d} \frac{t^{\sum_{i=1}^k \binom{L_i-d}{2}}}{(t;t)_{L_k-d}\prod_{i=1}^{k-1}(t;t)_{L_i-L_{i+1}}} \\
\times \sum_{\substack{\mu\in\mathrm{Sig}_{k-1}\\ \mu\prec\boldsymbol{L}}} &(-1)^{|\boldsymbol{L}|-|\mu|-d} \prod_{i=1}^{k-1} \begin{bmatrix} L_i-L_{i+1} \\ L_i-\mu_i \end{bmatrix}_t Q_{(\mu-(d[k-1]))'}(\gamma((1-t)t^d\chi),\alpha(1);0,t). \qquad (1.4.3)
\end{aligned}$$

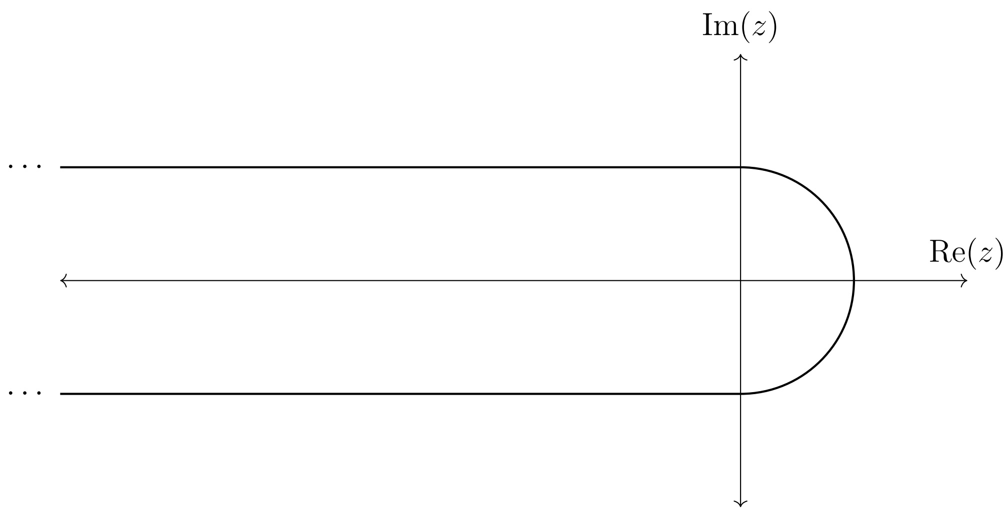


Figure 1-2: The contour $\tilde{\Gamma}$ in $\mathbb{C}$.

When $k = 1$ the above formulas require setting $L_{k-1} = \infty$ above and suitably interpreting the result, see the related Theorem 6.3.1 for the precise statement. The fact that the above probabilities sum to 1 for a given pair k, χ is a nontrivial q-series identity for which we are not aware of a reference. Our proof of this fact requires probabilistic arguments to show that certain prelimit random variables which limit to the above form a tight sequence (Proposition 6.5.1), but it would certainly be interesting to give an algebraic proof. It is however manifest from the formulas that

$$\mathcal{L}_{k,\chi} + (\underbrace{1, \ldots, 1}_{k \text{ times}}) = \mathcal{L}_{k,t\chi} \tag{1.4.4}$$

in distribution. Informally, this tells us that the random variables $\mathcal{L}_{k,\chi}$ for different χ do not look so different from one another, and sending $\chi \to 0$ or $\chi \to \infty$ should not result in any interesting limit behavior because it can be absorbed by translation.

We may now state the bulk limit result. We note that it allows substantial freedom in how the number of matrix products s_N varies with the matrix size N.

Theorem 1.4.2. *Fix p prime and $k \in \mathbb{Z}_{\geq 1}$, and for each $N \in \mathbb{Z}_{\geq 1}$ let $A_i^{(N)}, i \geq 1$ be iid matrices with iid entries distributed by the additive Haar measure on $\mathbb{Z}_p$. Let $(s_N)_{N \geq 1}$ be a sequence of natural numbers such that s_N and $N - \log_{t^{-1}} s_N$ both go to ∞ and as $N \to \infty$. Let $(s_{N_j})_{j \geq 1}$ be any subsequence for which $\log_t s_{N_j}$ converges in $\mathbb{R}/\mathbb{Z}$, and let α be any preimage in $\mathbb{R}$ of this limit. Then*

$$(\mathrm{SN}(A_{s_{N_j}}^{(N_j)} \cdots A_1^{(N_j)})_i' - [\log_{t^{-1}}(s_{N_j}) + \alpha])_{1 \leq i \leq k} \to \mathcal{L}_{k,t^{\alpha+1}/(1-t)} \tag{1.4.5}$$

in distribution as $j \to \infty$, where $[\cdot]$ is the nearest integer function and as always $t = 1/p$.

Note that we did not specify which preimage α to choose, but choosing a different one simply translates the left hand side of (1.4.5) by an integer and multiplies the parameter of $\mathcal{L}_{k,\cdot}$ by an integer power of t on the right hand side, which in light of (1.4.4) leaves (1.4.5) invariant. However, it was necessary to choose a subsequence of the s_N such that $\log_t s_{N_j}$ converges in $\mathbb{R}/\mathbb{Z}$, because $\mathrm{SN}(A_{s_{N_j}}^{(N_j)} \cdots A_1^{(N_j)})_i'$ is an integer.

It is interesting to compare Theorem 1.4.2 to corresponding results in classical random matrix theory. The local limits of eigenvalues or singular values of a single complex matrix, or spacings between them, is a classical topic originally studied as a model for the observed repulsion of energy levels in heavy nuclei. By these local limits we mean the following: the singular values form a random finite (multi-)set of points on $\mathbb{R}$, and as the size N of the matrix goes to infinity, this collection of points becomes larger and larger, and one may speak of a limit to an infinite random collection of points on $\mathbb{R}$ after suitable rescaling. Dyson [Dys62b] computed the limiting spacings between eigenvalues of Hermitian matrices in the bulk (i.e. far away from the largest and smallest eigenvalue) in certain exactly solvable cases, finding them governed by the celebrated sine kernel. If one zooms in near the expected location of the largest singular value, the so-called soft edge, one obtains an infinite collection of points with a rightmost point (corresponding to the largest singular value). This random point configuration is the Airy point process defined by Prähofer and Spohn [PS02], the correlation functions of which were computed earlier by Forrester [For93]. The distribution of this rightmost point, the limit of the largest singular value, is the eponymous distribution studied by Tracy and Widom [TW94].

Many of the works on real/complex matrix products mentioned in the previous section also consider logarithms of singular values of matrix products in joint limits where both the matrix size and number of products go to ∞. In the opposite regime to the one of the previous section, i.e. the case of the large N limit of product of a fixed number of matrices, Liu-Wang-Zhang [LWZ16] showed for Ginibre matrices that the local limits are governed by the sine kernel in the bulk and Airy at the soft edge. In joint limits as the matrix size and number of products grow together, one sees stochastic processes which interpolate between the bulk-sine/edge-Airy statistics of a single matrix, and the independent Gaussian ones of products of fixed-size matrices mentioned in the previous section, see e.g. Akemann-Burda-Kieburg [ABK19, ABK20] and Liu-Wang-Wang [LWW23]. As mentioned, our result Theorem 1.4.2 yields essentially the same limit for many different choices of how the number of products s_N goes to ∞ with the matrix size N, since the only difference is the constant in $\mathcal{L}_{k,\cdot}$, and by the previous discussion this does not meaningfully change the probabilistic behavior, in contrast to the complex case.

These limits of complex matrix products are also shared by a continuous-time prelimit object, the (multiplicative) Dyson Brownian motion. Dyson [Dys62a] observed that for an $N \times N$ Hermitian matrix with above-diagonal entries evolving as independent complex Brownian motions (and real ones on the diagonal), the eigenvalues evolve as N independent Brownian motions conditioned to never intersect, often called Dyson Brownian motion. One may also define a canonical multiplicative Brownian motion $Y^{(N)}(T), T \in \mathbb{R}_{\geq 0}$ on $\mathrm{GL}_N(\mathbb{C})$, for which the logarithms of singular values evolve as N independent Brownian motions with drifts in arithmetic progression, again conditioned never to intersect, a process known as multiplicative Dyson Brownian motion[8]. That the eigenvalues/singular values of the additive and multiplicative Brownian motion have such a simple description is a beautiful fact and seems unexpected given the nontrivial way that eigenvalues and singular values depend on matrix entries; it is related to the fact that these Brownian motions on matrices are coordinate-invariant in a certain sense, so the number of free parameters is much smaller than it appears. We mention also that one may view the usual Hermitian Brownian motion as a tangent space version of the multiplicative one, see Klyachko [Kly00] for a discussion in the case of discrete-time versions.

1.4.1 Invariant stochastic processes on matrices.

For multiplicative Brownian motion $\mathcal{Y}^{(N)}(T)$, it is natural to consider its *multiplicative increments* $\mathcal{Y}^{(N)}(t_i)\mathcal{Y}^{(N)}(t_{i-1})^{-1}$ for a series of times $t_1 < t_2 < \ldots < t_k$, as the value of the multiplicative Brownian motion at a given t_i is the product of the corresponding increments. These increments $\mathcal{Y}^{(N)}(T+s)\mathcal{Y}^{(N)}(T)^{-1}$ satisfy

1. Independence: $\mathcal{Y}^{(N)}(T+s)\mathcal{Y}^{(N)}(T)^{-1}$ is independent of the trajectory $\mathcal{Y}^{(N)}(\tau), 0 \leq \tau \leq T$,
2. Stationarity: $\mathcal{Y}^{(N)}(T+s)\mathcal{Y}^{(N)}(T)^{-1} = \mathcal{Y}^{(N)}(s)\mathcal{Y}^{(N)}(0)^{-1}$ in distribution for any $T \geq 0$, and
3. Isotropy with respect to $\mathrm{U}(N)$: For any $U \in \mathrm{U}(N)$, $\mathcal{Y}^{(N)}(T+s)\mathcal{Y}^{(N)}(T)^{-1} = U\mathcal{Y}^{(N)}(T+s)\mathcal{Y}^{(N)}(T)^{-1}U^{-1}$ in distribution. Equivalently, $\Pr(\mathcal{Y}^{(N)}(T+s) \in S|\mathcal{Y}(T) = x) = \Pr(\mathcal{Y}^{(N)}(T+s) \in US|\mathcal{Y}(T) = Ux)$ for any $U \in \mathrm{U}(N)$, $x \in \mathrm{GL}_N(\mathbb{C})$ and measurable $S \subset \mathrm{GL}_N(\mathbb{C})$.

The first two are familiar from the theory of Brownian motion on $\mathbb{R}$, while a version of the latter with respect to the rotation group $O(N)$ may be seen as soon as one considers Brownian motion on $\mathbb{R}^N$. The fixed-time marginals of any process satisfying the above and starting at the Haar measure must be infinitely-divisible measures invariant with respect to the action of $\mathrm{U}(N)$ on the left and right, which were classified by the generalized Lévy-Khintchine theorem of Gangolli [Gan64] (see also earlier work of Hunt

[8]See the introduction of [AVP23] for short discussion of multiplicative Brownian motion and its relation to matrix products, with references to the literature.

[Hun56]). Later, Gangolli [Gan65] explicitly constructed stochastic processes with these fixed-time marginals, finding them to be mixtures of multiplicative Brownian motion and Poisson jump processes, as with the classical Lévy-Khintchine theorem on $\mathbb{R}$. Of these, only the multiplicative Brownian motion[9] has continuous sample paths. We note that strictly speaking, the uniqueness statement applies only to the infinitely divisible measures which are the single-time marginals of the process, and we are not aware of a uniqueness statement at the process level analogous to the characterization of Brownian motion on $\mathbb{R}$.

One might optimistically hope for a similar classification in the p-adic case, and hope that the 'right' analogue of multiplicative Brownian motion yields an elegant stochastic process on singular numbers similar to the above multiplicative Dyson Brownian motion. Both hopes will turn out to be well-founded.

1.4.2 Classifying invariant processes.

Now we turn to the question of Markov processes on $\mathrm{GL}_N(\mathbb{Q}_p)$ with stationary, independent, $\mathrm{GL}_N(\mathbb{Z}_p)$-isotropic increments. The following definition gives a wide class of processes which are easily seen to have these properties.

Definition 1. Let $N \in \mathbb{Z}_{\geq 1}$, let M be any probability measure on Sig_N, and let $c \in \mathbb{R}_{\geq 0}$. Then we define the process $Y^{(N,M,c)}(\tau), \tau \in \mathbb{R}_{\geq 0}$ on $\mathrm{GL}_N(\mathbb{Q}_p)$ by

$$Y^{(N,M,c)}(\tau) := U_{P(\tau)} \operatorname{diag}(p^{\nu_1^{(P(\tau))}}, \ldots, p^{\nu_N^{(P(\tau))}}) V_{P(\tau)} \cdots U_1 \operatorname{diag}(p^{\nu_1^{(1)}}, \ldots, p^{\nu_N^{(1)}}) V_1 U_0 \tag{1.4.6}$$

where $P(\tau)$ is a Poisson process on $\mathbb{Z}_{\geq 0}$ with rate c, and $\nu^{(i)} \sim M$ and $U_i, V_i \sim M_{Haar}(\mathrm{GL}_N(\mathbb{Z}_p))$ are iid.

Our first result, Theorem 1.4.3, says that at the level of singular numbers Definition 1 is the only example.

[9]This uniqueness actually applies after restricting to the subgroup $\mathrm{SL}_N(\mathbb{C})$. In the case of $\mathrm{GL}_N(\mathbb{C})$ there is an additional multiplicative Brownian motion on $\mathbb{R}_+$ corresponding to the determinant, leading to a two-parameter family of processes with continuous sample paths. See Jones-O'Connell [JO06, p108] for discussion; this work also contains an excellent Lie-theoretic exposition of multiplicative Brownian motions in the cases of common matrix groups and Brownian motions on Weyl chambers in general Lie type.

Theorem 1.4.3. *Let $N \in \mathbb{Z}_{\geq 1}$ and let $X(\tau), \tau \in \mathbb{R}_{\geq 0}$ be a Markov process on $\mathrm{GL}_N(\mathbb{Q}_p)$ started at the identity with stationary, independent, $\mathrm{GL}_N(\mathbb{Z}_p)$-isotropic increments. Then there exists a constant $c \in \mathbb{R}_{\geq 0}$ and a probability measure M_X on Sig_N such that*

$$\mathrm{SN}(X(\tau)) = \mathrm{SN}(Y^{(N,M_X,c)}(\tau)) \qquad \textit{in multi-time distribution.} \tag{1.4.7}$$

We deduce Theorem 1.4.3 from a later result stated as Proposition 6.1.2, which works at the level of the homogeneous space $\mathrm{GL}_N(\mathbb{Q}_p)/\mathrm{GL}_N(\mathbb{Z}_p)$ analogously to [Gan64]. We note that because the dynamics of Definition 1 commutes with the right action of $\mathrm{GL}_N(\mathbb{Z}_p)$, it projects to Markov dynamics on $\mathrm{GL}_N(\mathbb{Q}_p)/\mathrm{GL}_N(\mathbb{Z}_p)$. Even at this level, unlike the homogeneous space $\mathrm{GL}_N(\mathbb{C})/\mathrm{U}(N)$, the space $\mathrm{GL}_N(\mathbb{Q}_p)/\mathrm{GL}_N(\mathbb{Z}_p)$ is countable and naturally carries the discrete topology. Hence one cannot expect an analogue of the continuous multiplicative Brownian motion, but it turns out that the same Poisson jump/matrix product processes appear as in the complex case—the processes $Y^{(N,M,c)}$ provide examples of these.

Given that no process on $\mathrm{GL}_N(\mathbb{Q}_p)/\mathrm{GL}_N(\mathbb{Z}_p)$ with continuous paths exists, one may at least ask for a Poisson jump process with the smallest or simplest jumps. Such a process should have M_X supported on $\delta_{(0[N])}$ and the smallest nontrivial signature $\delta_{(1,0[N-1])}$, where here and below we use the notation $a[k]$ in signatures to denote a repeated k times. Note that one might equally well replace $\delta_{(1,0[N-1])}$ by $\delta_{(0[N-1],-1)}$, but this is related to the previous case by taking inverse matrices and reversing the left and right actions, so there is no loss in our choice. Because the singular numbers of $Y^{(N,M_X,c)}$ do not change at the jumps where $\nu^{(i)} = (0[N])$, as far as the singular numbers are concerned one may take $M_X = \delta_{(1,0[N-1])}$, up to changing the Poisson rate constant c. We see next that the singular numbers of this process have an elegant description.

1.4.3 p-adic Dyson Brownian motion.

Definition 2. For $n \in \mathbb{N} \cup \{\infty\}$ and $\mu \in \mathrm{Sig}_n$ and $t \in (0,1)$, we define the stochastic process $\mathcal{S}^{\mu,n}(\tau) = (\mathcal{S}_1^{\mu,n}(\tau), \ldots, \mathcal{S}_n^{\mu,n}(\tau))$ on Sig_n as follows. For each $1 \leq i \leq n$, $\mathcal{S}_i^{\mu,n}$ has an exponential clock of rate t^i, and when the clock associated to $\mathcal{S}_i^{\mu,n}$ rings, $\mathcal{S}_i^{\mu,n}$ increases by 1 if the resulting n-tuple is still weakly decreasing. If not, then $\mathcal{S}_{i-d}^{\mu,n}$ increases by 1 instead and $\mathcal{S}_i^{\mu,n}$ remains the same, where $d \geq 0$ is the smallest index so that the resulting tuple is weakly decreasing. In the case of trivial initial condition we will often write $\mathcal{S}^{(n)}$ for $\mathcal{S}^{(0[n]),n}$.

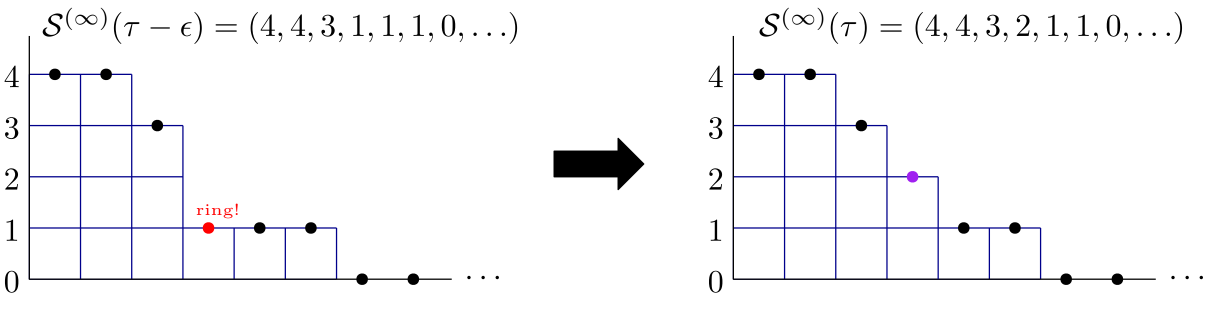


Figure 1-3: An example of the dynamics described in Definition 2. If the clock associated to $\mathcal{S}_4^{(\infty)} = 1$ rings at time τ and the process was previously in state $(4,4,3,1,1,1,0,\dots)$ (i.e. $\mathcal{S}^{(\infty)}(\tau-\epsilon) = (4,4,3,1,1,1,0,\dots)$ for all sufficiently small $\epsilon > 0$), then $\mathcal{S}_4^{(\infty)}$ increases by 1 and so $\mathcal{S}^{(\infty)}(\tau) = (4,4,3,2,1,1,0,\dots)$.

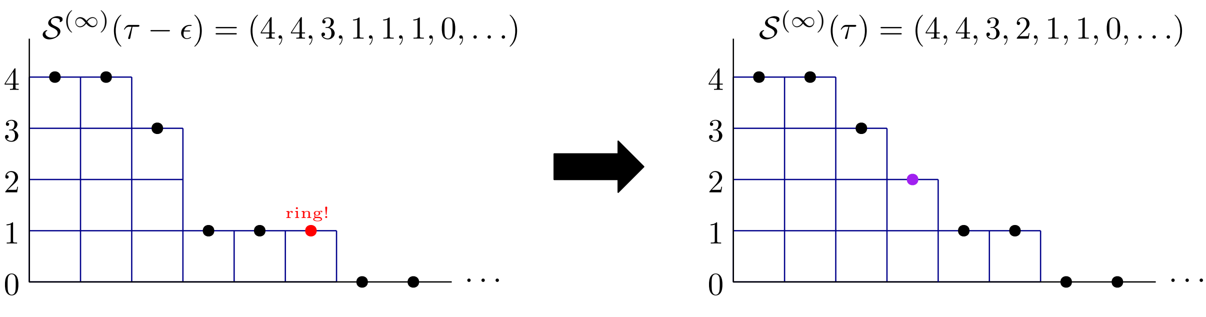


Figure 1-4: An example of the dynamics of Definition 2 in the case where the part associated to the clock that rings—in this case, $\mathcal{S}_6^{(\infty)}$—cannot increase without violating the weakly decreasing condition, so $\mathcal{S}_4^{(\infty)}$ is 'pushed' instead.

Theorem 1.4.4. *Let $N \in \mathbb{Z}_{\geq 1}, c \in \mathbb{R}_{>0}$, and $X^{(N)}(\tau) := Y^{(N,\delta_{(1,0[N-1])},c)}(\tau)$ in the notation of Definition 1. Then*

$$\mathrm{SN}\left(X^{(N)}(\tau)\right) = \mathcal{S}^{(N)}\left(\left(c\frac{1}{t}\frac{1-t}{1-t^N}\right)\tau\right) \tag{1.4.8}$$

in multi-time distribution, where we take the parameter t in $\mathcal{S}^{(N)}$ to be $1/p$.

We find later in Theorem 6.3.1 that this 'p-adic Dyson Brownian motion' $\mathcal{S}^{(n)}(\tau)$ has the same bulk limit as the one for singular numbers additive Haar matrix products given in Theorem 1.4.2. Both are Hall-Littlewood processes, which for $\mathcal{S}^{(n)}(\tau)$ is the statement

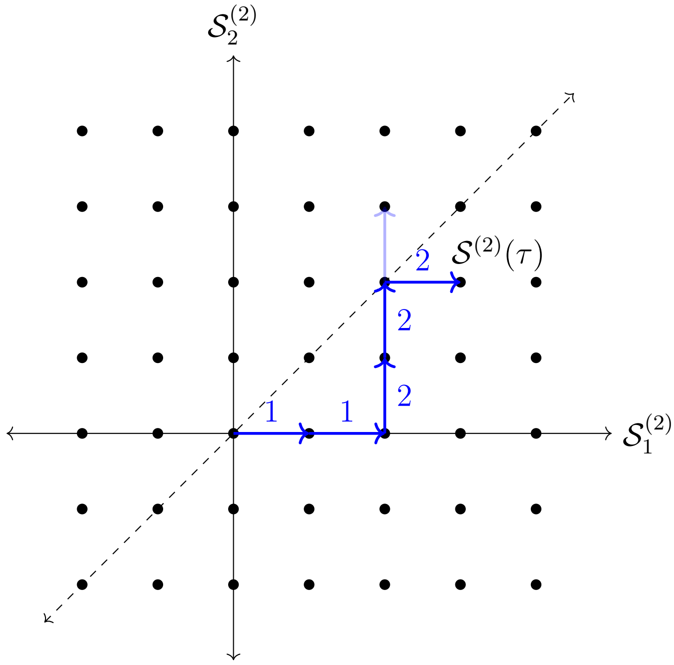


Figure 1-5: The reflection condition of Definition 2 in the case $n = 2$: here $(\mathcal{S}_1^{(2)}(\tau), \mathcal{S}_2^{(2)}(\tau))$ is portrayed as an up-right walk in the $x - y$ plane lying below the line $y = x$, and each jump is labeled by which clock rings. In the final jump, the second clock rings, but due to the reflection condition, $\mathcal{S}_2^{(2)}$ does not increase—the result of such an increase is shown as an opaque arrow—but rather $\mathcal{S}_1^{(2)}$ increases instead.

that it is a continuous-time Markov process on the set $\mathbb{Y}$ of partitions starting at the empty partition, with transition probabilities defined by

$$\begin{aligned}\Pr(\mathcal{S}^{(n)}(\tau_0+\tau)=\nu|\mathcal{S}^{(n)}(\tau_0)=\mu) \\ \propto \left(\lim_{D\to\infty} Q_{\nu/\mu}\left(\underbrace{t\tau/D,\ldots,t\tau/D}_{D \text{ times}};t\right)\right)\frac{P_\nu(1,t,t^2,\ldots,t^{n-1};t)}{P_\mu(1,t,t^2,\ldots,t^{n-1};t)};\end{aligned} \quad (1.4.9)$$

see Corollary 6.2.5 for the full statement for $\mathcal{S}^{(n)}$ and Corollary 3.1.3 for the statement for the Haar matrix product process. This allows us to bring tools from symmetric functions to bear, which are key for Theorem 1.4.2 and Theorem 6.3.1. However, it was not clear how to use the usual tool for Macdonald process asymptotics, namely contour integral formulas for 't-moment' observables coming from difference operators, because such moments do not uniquely determine the distribution. Our proofs nonetheless use results on Macdonald polynomials extensively, but in a nonstandard fashion which relies on a Markovian projection property specific to the Hall-Littlewood case. An interesting feature of the proof is that its starting point is a symmetric function incarnation of an explicit moment inversion formula for abelian p-groups, recently worked out in greater generality by Sawin-Wood [SW22b], which we discuss further in Section 6.3. We mention also that the Hall-Littlewood process corresponding to $\mathcal{S}^{(N)}(\tau)$ appears in earlier work of Borodin [Bor99] and Bufetov-Petrov [BP15] on the related problem of Jordan blocks of random upper-triangular matrices over $\mathbb{F}_q$.

It is natural to ask about the multi-time bulk limits of these processes as well, which we discuss in the next section. In particular, we find that the evolution of singular numbers in the bulk for matrix products matches the Poisson jump rules of $\mathcal{S}^{(n)}(\tau)$, so in some sense $\mathcal{S}^{(n)}(\tau)$ is the most natural prelimit incarnation of our bulk limit. This gives some explanation as to why the limits in Theorem 1.4.2 and Theorem 6.3.1 are the same.

Remark 4. Random walks on Weyl chambers conditioned to never intersect are the subject of an extensive literature with connections to representation theory, total positivity, and other parts of combinatorics, as well as random matrices. See Biane [Bia91, Bia92], Grabiner [Gra99], Baryshnikov [Bar01], Bougerol-Jeulin [BJ02], O'Connell-Yor [OY02], Biane-Bougerol-O'Connell [BBO05], and the references therein. However, we are not aware of corresponding work for reflected (rather than conditioned) random walks on a positive Weyl chamber, and believe it is worth understanding whether the combinatorics is similarly rich. To clarify a potential point of confusion, let us note that reflections across the walls of the Weyl chamber appear across the works which treat conditioned random walks, in analogues of the classical reflection principle for Brownian motion following Gessel-Zeilberger [GZ92]; however, the random walks themselves are not reflected at the boundary, but conditioned to avoid it.

Remark 5. A related body of literature deals with random walks on Bruhat-Tits buildings, of which $\mathrm{SL}_N(\mathbb{Q}_p)/\mathrm{SL}_N(\mathbb{Z}_p)$ is the type $\tilde{A}$ case, see e.g. Parkinson [Par17] and the references therein. These typically treat random walks satisfying a stronger notion of isotropy than ours: theirs in our context would be the assumption that

$$\Pr(X(\tau+s)=y|X(\tau)=x)=\Pr(X(\tau+s)=Uy|X(\tau)=Vx) \tag{1.4.10}$$

for any fixed $U, V \in \mathrm{GL}_N(\mathbb{Z}_p)$, while ours only requires

$$\Pr(X(\tau+s)=y|X(\tau)=x)=\Pr(X(\tau+s)=Uy|X(\tau)=Ux). \tag{1.4.11}$$

It is not hard to show by slight modifications of our arguments that the only processes satisfying the strong isotropy condition (1.4.10) and stationary independent increments are of the form $Y^{(N,M,c)}(\tau)$, and indeed this is remarked in the discrete-time setting in Parkinson [Par07, p381].

However, multiplicative Brownian motion on $\mathrm{GL}_N(\mathbb{C})/\mathrm{U}(N)$ does *not* satisfy the strong isotropy condition (1.4.10); indeed, this condition in continuous time precludes continuous sample paths. This is our reason for taking the weaker condition (1.4.11), which multiplicative Brownian motion does satisfy, even though the resulting constraints on the processes in Proposition 6.1.2 are weaker than one obtains with (1.4.10).

Remark 6. We believe Theorem 1.4.3 and the discussion directly after it, together with Theorem 1.4.4, give a satisfactory answer to the question of what a p-adic multiplicative Dyson Brownian motion. However, let us be clear that we have *not* answered the stronger question

What is the analogue for $\mathrm{GL}_N(\mathbb{Q}_p)$ *of multiplicative Brownian motion?*

We have only shown that the singular numbers of such a process should agree with those of $Y^{(N,\delta_{(1,0[N-1])},c)}$, but we have made no uniqueness statement at the level of a process on $\mathrm{GL}_N(\mathbb{Q}_p)$. The process $Y^{(N,\delta_{(1,0[N-1])},c)}$ is quite natural, but there may certainly be a more natural one. Returning to the previous remark, we expect that such a process may not satisfy the strong isotropy condition (1.4.10), and it seems plausible that a good analogue of multiplicative Brownian motion should not wait at any state for a nonzero amount of time, which $Y^{(N,\delta_{(1,0[N-1])},c)}$ does.

In the simplest case $N = 1$, the above discussion concerns processes in continuous ($\mathbb{R}$-valued) time on the group $\mathbb{Q}_p^\times$, for which existing literature on p-adic Brownian motions such as Albeverio-Karwowski [AK91, AK94] or Evans [Eva89] (which studies more general totally disconnected abelian groups) likely provides a natural route to answering the question above when $N = 1$. However, we are not aware of works concerning stochastic processes on nonabelian p-adic groups.

To prevent confusion for one who wishes to begin reading the primary sources on p-adic Brownian motions, it is worth mentioning that many previous works referring to p-adic Brownian motions such as Evans [Eva93, Eva98] and Bikulov-Volovich [BV97] are instead studying a process where the time parameter lives in $\mathbb{Q}_p$ rather than $\mathbb{R}$, leading to a different object which has no *a priori* relation to our setting.

1.5 Universal limits of the product process at the bulk and edge

We wish to describe the evolution of the singular numbers $\mathrm{SN}\left(A_\tau^{(N)} \cdots A_1^{(N)}\right), \tau \in \mathbb{Z}_{\geq 0}$ in the bulk, i.e. evolution of $\mathrm{SN}\left(A_\tau^{(N)} \cdots A_1^{(N)}\right)_i$ for $i = r_N + O(1)$ with $1 \ll r_N \ll N$. We also wish to describe the evolution at the right edge $i = N - O(1)$. The reason we do not consider the left edge $i = 1 + O(1)$ is that it is nonuniversal; this is not obvious given the discussion so far, but one can see it for the examples of Theorem 1.3.1 later in Section 8.1. Let us first consider what the type of a putative limit object for the bulk and right edge must be. In the bulk, it should be a stochastic process with state space

$$\mathrm{Sig}_{2\infty} := \left\{(\mu_n)_{n\in\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}} : \mu_{n+1} \leq \mu_n \text{ for all } n \in \mathbb{Z}\right\} \tag{1.5.1}$$

because there should be infinitely many smaller and infinitely many larger singular numbers in the limit. For the edge it should be a process on

$$\mathrm{Sig}_{\mathrm{edge}} := \left\{(\mu_n)_{n\in\mathbb{Z}_{\leq 0}} \in \mathbb{Z}^{\mathbb{Z}_{\leq 0}} : \mu_{n+1} \leq \mu_n \text{ for all } n \in \mathbb{Z}_{<0}\right\} \tag{1.5.2}$$

because there is a smallest singular number $\mathrm{SN}\left(A_\tau^{(N)}\cdots A_1^{(N)}\right)_N$. Furthermore, in the cases covered by Theorem 1.3.2, that result suggests that the $(j+1)^{th}$ singular number of $A_\tau^{(N)}\cdots A_1^{(N)}$ moves (in discrete time τ) about t times slower than the j^{th} does. This suggests that a singular number $\mathrm{SN}\left(A_\tau^{(N)}\cdots A_1^{(N)}\right)_j$ in either the bulk or the right edge must change very seldom, so we should expect a continuous-time Poisson-type limit. This is what we will construct.

Theorem 1.5.1. *For each $t \in (0,1)$ there exists a continuous-time Markov process $\mathcal{S}^{(2\infty)}(T) = \left(\mathcal{S}_i^{(2\infty)}(T)\right)_{i\in\mathbb{Z}}, T \in \mathbb{R}_{\geq 0}$ on $\overline{\mathrm{Sig}}_{2\infty}$, which we call the 'reflecting Poisson sea,' enjoying the following properties:*

1. *For any $D \in \mathbb{N}$ and sequence of 'bulk observation points' $r_n, n \geq 1$ with $r_n \to \infty$ and $n - r_n \to \infty$,*

$$\left(\mathcal{S}_{r_n-D}^{(n)}\left(t^{-r_n}T\right), \dots, \mathcal{S}_{r_n+D}^{(n)}\left(t^{-r_n}T\right)\right) \to \left(\mathcal{S}_{-D}^{(2\infty)}(T), \dots, \mathcal{S}_{D}^{(2\infty)}(T)\right) \tag{1.5.3}$$

   in multi-time distribution.

2. *(Shift-stationarity) The process $\left(\mathcal{S}_{i-1}^{(2\infty)}\left(t^{-1}T\right)\right)_{i\in\mathbb{Z}}$ is equal to $\mathcal{S}^{(2\infty)}(T)$ in multi-time distribution.*

3. *(Markovian projections) For any $d \in \mathbb{Z}$, the truncated process $\left(\min\left(d, \mathcal{S}_i^{(2\infty)}(T)\right)\right)_{i\in\mathbb{Z}}$ is also Markov.*

In particular, the bulk limit property in Theorem 1.5.1 shows that $\mathcal{S}^{(2\infty)}$ appears as the dynamical bulk limit of the process $\mathrm{SN}\left(X^{(N)}(\tau)\right)$ where $X^{(N)}(\tau)$ is as in Theorem 1.4.4, and hence has single-time marginals given by Theorem 1.4.1. The shift-stationarity property is visible at the level of these marginals in (1.4.4).

The process $\mathcal{S}^{(2\infty)}(T)$ should be thought of as a two-sided version of $\mathcal{S}^{(\infty)}$ where each part $\mathcal{S}^{(2\infty)}(T)_i$ has an exponential clock of rate t^i and attempts to jump when it rings, subject to the same pushing/reflection rules as in Definition 2. The reason this is a difficult object to make sense of is that the sum of these jump rates is infinite, unlike $\mathcal{S}^{(\infty)}$ where the sum of the jump rates is finite. We give an explicit construction of the reflecting Poisson sea in Section 7.1 by coupling a collection of processes $\mathcal{S}^{(\infty)}$ on the same probability space.

In the bulk limit result we state, we actually require a version of $\mathcal{S}^{(2\infty)}(T)$ with a general initial condition $\mu \in \mathrm{Sig}_{2\infty}$, and which we denote by $\mathcal{S}^{\mu,2\infty}(T)$. The construction is the same and given in the general case in Proposition 7.1.2. Our next result shows that it is the universal object governing local bulk limits of p-adic matrix products, when the singular numbers are started at certain nonzero initial conditions.

Theorem 1.5.2. *Let $\mu \in \mathrm{Sig}_{2\infty}$ be any signature with all parts nonnegative and $\mu_{-n} \to \infty$ as $n \to \infty$. For each $N \in \mathbb{N}$, let $A_i^{(N)}, i \geq 1$ be iid left-$\mathrm{GL}_N(\mathbb{Z}_p)$-invariant random matrices in $\mathrm{Mat}_N(\mathbb{Z}_p)$, and let r_N be a 'bulk observation point,' such that*

(i) The matrix product process is nontrivial: $\Pr\left(A_i^{(N)} \in \mathrm{GL}_N(\mathbb{Z}_p)\right) < 1$ for every N,

(ii) r_N is in the bulk: $r_N \to \infty$ and $N - r_N \to \infty$ as $N \to \infty$, and

(iii) The coranks $X_N := \mathrm{corank}\left(A_i^{(N)} \pmod{p}\right)$ are far away from r_N with high probability: for every $j \in \mathbb{N}$,

$$\Pr\left(X_N > r_N - j \mid X_N > 0\right) \to 0 \qquad \text{as } N \to \infty. \tag{1.5.4}$$

Let $B^{(N)} \in \mathrm{Mat}_N(\mathbb{Z}_p), N \geq 1$ be left-$\mathrm{GL}_N(\mathbb{Z}_p)$-invariant 'initial condition' matrices with fixed singular numbers

$$\mathrm{SN}\left(B^{(N)}\right)_i = \mu_{i-r_N} \tag{1.5.5}$$

for all $1 \leq i \leq N$. Finally, define the matrix product process

$$\Pi^{(N)}(\tau) := \mathrm{SN}\left(A_\tau^{(N)} \cdots A_1^{(N)} B^{(N)}\right), \tau \in \mathbb{Z}_{\geq 0}. \tag{1.5.6}$$

Then for any $D \in \mathbb{N}$ we have convergence

$$\left(\Pi_{r_N-D}^{(N)}\left(\lfloor c_N T \rfloor\right), \ldots, \Pi_{r_N+D}^{(N)}\left(\lfloor c_N T \rfloor\right)\right) \xrightarrow{N\to\infty} \left(\mathcal{S}_{-D}^{(2\infty)}(T), \ldots, \mathcal{S}_{D}^{(2\infty)}(T)\right) \tag{1.5.7}$$

in finite-dimensional distribution, where the constant c_N determining the time scaling is given explicitly by

$$c_N := \frac{t^{-r_N}}{\mathbb{E}\left[\mathbb{1}(X_N \leq r_N)\left(t^{-X_N} - 1\right)\right]} \tag{1.5.8}$$

The hypotheses are very general: (i) and (ii) are clearly required to speak of a bulk limit, and as we discuss further in Chapter 7, having X_N close to r_N with nontrivial probability presents a direct impediment to the existence of a continuous-time limit. We in fact prove a stronger theorem, with a weaker (but more technical to state) version of the hypothesis that X_N is much smaller than r_N with high probability, in Theorem 7.2.1, and explain why this hypothesis is essentially optimal. We note also that the matrices $A_i^{(N)}$ are not required to be nonsingular. It is also quite surprising that the time scaling c_N in (1.5.8) depends only on $A_i^{(N)} \pmod p$.

The condition $\mu_{-n} \to \infty$ as $n \to \infty$ in Theorem 1.5.2 is convenient for the following reason. For such μ, for any d there will be some i_0 with $\mu_{i_0} \geq d$, and so the Markovian truncations $\left(\min\left(d, \mathcal{S}^{\mu,2\infty}(T)_i\right)\right)_{i\in\mathbb{Z}}$ will have all parts $\min\left(d, \mathcal{S}^{\mu,2\infty}(T)_i\right), i \leq i_0$ equal to d for all time, and these may be ignored in the dynamics. This removes the complicated feature of the sum of jump rates diverging and makes $\left(\min\left(d, \mathcal{S}^{\mu,2\infty}(T)_i\right)\right)_{i\in\mathbb{Z}}$ a much simpler object.

We next state the edge version, which is exactly the same except r_N and $\mathrm{Sig}_{2\infty}$ in Theorem 1.5.2 are replaced by N and $\mathrm{Sig}_{\mathrm{edge}}$. The limit object, $\mathcal{S}^{\mu,edge}(T)$, is constructed the same way as $\mathcal{S}^{\mu,2\infty}(T)$, see Definition 53.

Theorem 1.5.3. *Let $\mu \in \mathrm{Sig}_{edge}$ have $\mu_0 \geq 0$ and $\mu_{-n} \to \infty$ as $n \to \infty$. For each $N \in \mathbb{N}$, let $A_i^{(N)}, i \geq 1$ be iid left-$\mathrm{GL}_N(\mathbb{Z}_p)$-invariant random matrices in $\mathrm{Mat}_N(\mathbb{Z}_p)$ such that*

(i) The matrix product process is nontrivial: $\Pr\left(A_i^{(N)} \in \mathrm{GL}_N(\mathbb{Z}_p)\right) < 1$ for every N, and

(ii') The coranks $X_N := \operatorname{corank}\left(A_i^{(N)} \pmod p\right)$ are far away from N with high probability: for every $j \in \mathbb{N}$,

$$\Pr\left(X_N > N - j \mid X_N > 0\right) \to 0 \quad \text{as } N \to \infty. \tag{1.5.9}$$

Let $B^{(N)} \in \mathrm{Mat}_N(\mathbb{Z}_p), N \geq 1$ be left-$\mathrm{GL}_N(\mathbb{Z}_p)$-invariant 'initial condition' matrices with *fixed singular numbers*

$$\mathrm{SN}\left(B^{(N)}\right)_i = \mu_{i-N} \tag{1.5.10}$$

for all $1 \le i \le N$*. Finally, define the matrix product process*

$$\Pi^{(N)}(\tau) := \mathrm{SN}\left(A_\tau^{(N)} \cdots A_1^{(N)} B^{(N)}\right), \tau \in \mathbb{Z}_{\ge 0}. \tag{1.5.11}$$

Then for any $D \in \mathbb{N}$ *we have convergence*

$$\left(\Pi_{N-D}^{(N)}\left(\lfloor c_N T \rfloor\right), \ldots, \Pi_N^{(N)}\left(\lfloor c_N T \rfloor\right)\right) \xrightarrow{N \to \infty} \left(\mathcal{S}_{-D}^{\mu,edge}(T), \ldots, \mathcal{S}_0^{\mu,edge}(T)\right) \tag{1.5.12}$$

in finite-dimensional distribution, where c_N *is as in* (1.5.8) *with* $r_N = N$*.*

In the complex case, recent work of Ahn [Ahn22a] showed that a universality statement for multi-time limits of singular values of complex matrix products from fairly generic $\mathrm{U}(N)$-invariant distributions, finding that these limits match those of multiplicative Dyson Brownian motion. At a structural level this is analogous to the results of this section, but the discrete, Poisson-type limit objects we see in the p-adic setting are quite different.

1.6 Extrapolating to the $p \to 1$ limit

Given that the bulk limit $\mathcal{S}^{(2\infty)}$ of p-adic matrix products depends on p only as a real parameter $t = 1/p \in (0, 1)$, it is natural to wonder if it has any interesting limit behavior as t approaches 0 or 1. For β-ensembles, which extrapolate the classical real, complex and quaternion random matrix ensembles, many works have examined analogous limits as $\beta \to \infty$, e.g. [DE05, EPS14, GM20, GK22]. From the explicit description of our limit, we see that the $t \to 0$ regime is degenerate since the ratios between the jump rates of parts $\mathcal{S}_i^{(n)}$ will diverge so the parts will spread apart and not interact. However, in the limit $t \to 1$ the jump rates all converge to 1, which suggests the parts $\mathcal{S}_i^{(2\infty)}$ will interact more and more with one another and interesting limit behavior may result. Rather than studying the $t \to 1$ limit of $\mathcal{S}^{(2\infty)}$, which is itself a long-time bulk limit of $\mathcal{S}^{(n)}$ or $\mathcal{S}^{(\infty)}$ by Theorem 1.5.1, we study $\mathcal{S}^{(\infty)}$ in a simultaneous limit as $t \to 1$ and time goes to ∞.

An alternative motivation to study this process comes from interacting particle systems. Letting

$$x_k(\tau) = \mathcal{S}^{(\infty)} \left(\left(t^{-1} - 1 \right) \tau \right)'_k - k, \tag{1.6.1}$$

we have that $x_1(\tau) > x_2(\tau) > \dots$ for all time τ, and we view the $x_i(\tau)$ as positions of particles on $\mathbb{Z}$. Each particle $x_i(\tau)$ jumps by 1 to the right according to a Poisson process with rate $t^{x_i(\tau)+i}(1 - t^{\mathrm{gap}_i})$, where we use shorthand $\mathrm{gap}_i = x_{i-1} - x_i - 1$ for the distance between particles. Hence each particle has a base jump rate t^i which is slower for particles further behind the leading particle, but also has a position-dependent slowing t^{x_i} which causes it to slow down as it moves further to the right. We refer to this system as slowed t-TASEP, by analogy with the more commonly studied q-TASEP which we now recall.

Consider a configuration of particles on $\mathbb{Z}$ at some positions $x_1 > x_2 > \cdots$, at most one particle per site, evolving in continuous time. Each particle has an independent Poisson clock and jumps 1 unit to the right whenever it rings. The clock of the i^{th} particle from the right has rate $1 - q^{x_{i-1} - x_i - 1}$, often simply written $1 - q^{\mathrm{gap}}$, where $0 \leq q < 1$ and we take $x_0 := \infty$. This is the well-known *q-TASEP*, introduced in [BC14], which reduces to the usual *totally asymmetric simple exclusion process (TASEP)* when $q = 0$. The asymptotics of q-TASEP and its relatives in various regimes have been the subject of much recent work, for example [Bar15, BCS14, BC15, FV15, OP17, IS19b, IS19a, Vet21]. These asymptotics crucially rely on the exact solvability of the model, which derives from its connection to Macdonald processes [BC14].

An inhomogeneous version of q-TASEP, where the i^{th} particle has jump rate $a_i(1 - q^{\mathrm{gap}})$ for some fixed positive real parameters $a_1, a_2, \dots$, was introduced simultaneously in [BC14]. Such inhomogeneities often yield different asymptotic behaviors: for instance, [Bar15] showed that by tuning the a_i correctly, one may see the Baik-Ben Arous-Peche distributions in the limit, generalizing Tracy-Widom asymptotics established in [FV15].

The position-dependent damping means that slowed t-TASEP behaves quite differently, as is already apparent with the rightmost particle. In q-TASEP, this particle jumps according to a Poisson process with rate 1, hence has asymptotically $(\text{time})^{1/2}$ order Gaussian fluctuations. By contrast, it is immediate from Theorem 6.3.1 later that the particles of slowed t-TASEP have asymptotically finite fluctuations. However, in the above-mentioned regime in which time $\to \infty$ and $t \to 1$ simultaneously, which ameliorates but does not obliterate the position-dependent slowing, one may expect a continuous limit. Our first asymptotic result is that the position of each particle obeys an explicit law of large numbers in this regime.

Theorem 1.6.1. *Let $(x_1(s), x_2(s), \ldots)$, $s \in \mathbb{R}_{\geq 0}$ be the particles of slowed t-TASEP with $t = e^{-\epsilon}$. Then for any $\tau > 0$ and $k \in \mathbb{Z}_{>0}$,*

$$\epsilon \cdot x_k(\tau/\epsilon) \to \log\left(\sum_{j=0}^{k} \frac{\tau^j}{j!}\right) - \log\left(\sum_{j=0}^{k-1} \frac{\tau^j}{j!}\right) \qquad \text{in probability as } \epsilon \to 0^+.$$

In particular, particles become macroscopically far apart in the limit. In simulations with fixed $t \approx 1$, one may observe the first particle 'peeling off' from the bulk while the second particle barely moves at all due to the $1 - t^{\mathrm{gap}}$ component of its jump rate until the gap becomes large. Then the second particle 'peels off', and once it is far away the third begins to move nontrivially, etc.

However, since particles affect those behind them due to this $1 - t^{\mathrm{gap}}$ factor in the jump rates, despite the macroscopic separation they continue to influence one another at the level of fluctuations. In the scaling of time and t of Theorem 1.6.1, we have that the rescaled fluctuations

$$\epsilon^{1/2}\left(x_k(\tau/\epsilon) - \mathbb{E}\left[x_k(\tau/\epsilon)\right]\right)$$

converge to Gaussians $X_\tau^{(k)}$ with nontrivial covariances determined by an $(r+s)$-fold contour integral formula for

$$\mathrm{Cov}(X_\tau^{(1)} + \ldots + X_\tau^{(r)}, X_\tau^{(1)} + \ldots + X_\tau^{(s)}),$$

see Proposition 8.4.1. These limiting covariances still depend on τ, but converge without rescaling as $\tau \to \infty$, a manifestation of the convergence to stationary distribution at fixed t which follows from Theorem 6.3.1.

Theorem 1.6.2. *As $\tau \to \infty$, the random variables $X_\tau^{(i)}$ converge in distribution to the fixed-time marginal of the unique stationary solution $(Z_T^{(1)}, Z_T^{(2)}, \ldots)$ to the system*

$$dZ_T^{(k)} = \left((k-1)Z_T^{(k-1)} - kZ_T^{(k)}\right) dT + dW_T^{(k)} \qquad k = 1, 2, \ldots \tag{1.6.2}$$

where $W_T^{(k)}$ are independent standard Brownian motions. Their covariances furthermore *have the explicit form*

$$\mathrm{Cov}(Z_T^{(r)}, Z_T^{(s)}) = \frac{1}{4\pi^2}\oint_{\Gamma_0}\oint_{\Gamma_{0,w}} \frac{w}{z-w}\frac{r!s!}{z^r w^s} e^{z+w}(1-z/r)(1-w/s)\frac{dz}{z}\frac{dw}{w}$$

with the w-contour enclosing 0 and enclosed by the z-contour.

In addition to reflecting prelimit convergence to stationarity, Theorem 1.6.2 yields a 2-fold rather than $(r+s)$-fold contour integral formula, which allows analysis in the bulk regime $r, s \to \infty$. The proof of this reduction of covariance formulas is by orthogonal polynomial methods inspired by the similar arguments of [BCF18, §5.1], though the interpretation as a stationary solution to a system of SDEs is not present there.

A natural way to study the bulk limit of slowed t-TASEP is to string the $Z_0^{(k)}$, which represent asymptotic fluctuations of particle positions, together into a stochastic process $Y_T, T \in \mathbb{R}_{>0}$ by linear interpolation. Explicitly, set $Y_0 = 0$, $Y_T = Z_0^{(T)}$ when $T \in \mathbb{Z}_{>0}$, and linearly interpolate times between these, see Figure 1-6. The bulk limit is then encoded by the scaling limit of Y_T for large T, which we explicitly compute by taking asymptotics of the covariance formula in Theorem 1.6.2 via steepest descent.

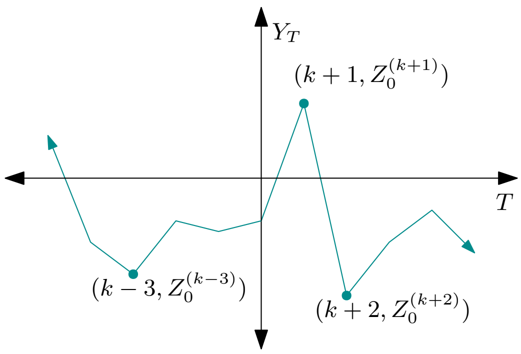


Figure 1-6: The graph of the process Y_T, which is just a piecewise linear interpolation from the (random) points $(T, Z_0^{(T)})$, shown in a window around $T = k$.

Theorem 1.6.3. *The process*

$$R_s^{(T)} := T^{1/4} Y_{T+s\sqrt{T}}$$

converges in finite-dimensional distributions as $T \to \infty$ to the unique stationary Gaussian *process $R_s, s \in \mathbb{R}$ with covariances*

$$\mathrm{Cov}(R_a, R_b) = \int_0^\infty y^2 e^{-y^2-|b-a|y} dy.$$

The scaling exponents in Theorem 1.6.3 are characteristic of the *Edwards-Wilkinson universality class* in $(1+1)$ dimensions (1 spatial dimension plus time), see [Sep10] for other examples of interacting particle systems in this class. The specific integral form of the covariance is somewhat similar to, but not the same as, covariances for solutions to the $(1+1)$-dimensional additive stochastic heat equation, see for example [Hai09, §2.3.2]. We suspect it may arise from some transform of solutions to this or a similar stochastic PDE, but do not have any results in this direction. However, the fact that the limiting fluctuations are described by a 1-dimensional Gaussian process of any kind is surprising given the algebraic origins of slowed t-TASEP, which we discuss next.

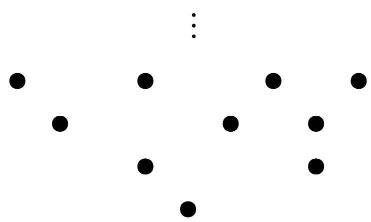
1.6.1 Discussion: Macdonald processes, locality, and dynamics in $(1+1)$ and $(2+1)$ dimensions

The dynamics on partitions described above appear naturally as marginals of dynamics on triangular arrays of integers, or *Gelfand-Tsetlin patterns*, which are simply sequences $\lambda^{(n)} \in \mathbb{Y}_n, n \geq 1$ which *interlace* in the sense that

$$\lambda_1^{(n+1)} \geq \lambda_1^{(n)} \geq \lambda_2^{(n+1)} \geq \ldots \geq \lambda_{n+1}^{(n+1)}.$$

These are often visualized as infinite configurations of particles in the plane by placing a particle at each point $(\lambda_i^{(n)}, n)$ as in Figure 1-7 (middle).

$$\begin{array}{ccccccc} & & & \vdots & & & \\ 0 & & 3 & & 6 & & 8 \\ & 1 & & 5 & & 7 & \\ & & 3 & & 7 & & \\ & & & 4 & & & \end{array}$$



$$\begin{array}{c} \vdots \\ a_4 \\ a_3 \\ a_2 \\ a_1 \end{array}$$

Figure 1-7: The bottom four rows of an infinite Gelfand-Tsetlin pattern, visualized as a sequence of partitions/array of integers (left) and as a particle configuration (middle), with jump rates a_i for the Poisson clocks of each row (right).

Simple continuous-time dynamics on such arrays, such that the n^{th} row evolves by Hall-Littlewood process dynamics for each n, were given in [BBW16, §6]. In such dynamics, the n^{th} row has a Poisson clock of rate a_i for every i, where in our setting $a_i = t^{i-1}$. When a row's clock rings, one of the particles in that row jumps, specifically the leftmost one whose jump would not violate interlacing with the row below. This jump in turn triggers a particle in the row above to jump by 1 according to certain rules, which triggers one in the row above that, etc., in such a way that interlacing is preserved and the triggering of moves is local.

At least in special cases, such dynamics also exist for more general Macdonald processes, given by replacing the Hall-Littlewood polynomials by Macdonald polynomials $P_\lambda(x_1, \ldots, x_n; q, t)$. Such dynamics are still local in that particles' jump rates are only affected by the particles corresponding to adjacent entries of the original Gelfand-Tsetlin pattern, and were studied in depth in [BP16]. For the Schur ($q = t$) and q-Whittaker ($t = 0$) cases with Poisson rates $a_i \equiv 1$, it was shown in [BF14] and [BCF18] that global bulk asymptotics of such arrays are governed by the 2-dimensional Gaussian free field, which exhibits logarithmic correlations. We mention also the related works [BF09, BCT17] dealing with similar asymptotics of so-called $(2+1)$-dimensional growth models (i.e. growth models in two spatial and one time dimension). This is in marked contrast to the correlations $\mathrm{Cov}(R_s, R_{s+d})$ in the bulk limit Theorem 1.6.3, which decay like const $\cdot |d|^{-3}$ for large d and do not diverge for small d.

Why this difference? *The surprising feature of the Hall-Littlewood case is that not only are the dynamics on* 2*-dimensional Gelfand-Tsetlin patterns governed by local interactions, but their projection to a given row of the Gelfand-Tsetlin pattern results in* $(1+1)$*-dimensional dynamics with only local interactions.* After the transform (1.6.1) this becomes the statement, visible in the definition of slowed t-TASEP above, that a particle's jump rate is independent of the other particles except for the one in front of it. Strictly speaking, projecting to a row of the Gelfand-Tsetlin pattern corresponds to the finite n version of (1.4.9), which does not yield the full slowed t-TASEP. However, the $n = \infty$ case of (1.4.9), which corresponds to the full slowed t-TASEP, can interpreted as the projection of Hall-Littlewood process dynamics to a row 'at infinity'. A fuller account is given in Section 4.5, but the basic idea is that with the initial condition where every entry of the Gelfand-Tsetlin pattern is 0, at each time all sufficiently high rows of the Gelfand-Tsetlin pattern will yield the same partition, and the projection of the dynamics to this partition is Markovian and yields the $n = \infty$ case of (1.4.9). For a precise formulation of this statement in terms of the boundary of a branching graph, see Section 4.5.

The locality of these dynamics on rows of the Gelfand-Tsetlin pattern is quite special to the Hall-Littlewood case, and does not hold for the general Macdonald dynamics above. Even in the $q = t$ Schur case, which usually is the simplest case of Macdonald processes, a given row of the continuous-time dynamics evolves as n independent Poisson random walks conditioned in the sense of Doob h-transform not to intersect for all time. It therefore has highly nonlocal interactions, see [BG13]. In light of this, it makes sense that the asymptotics of slowed t-TASEP are characteristic of $(1+1)$-dimensional growth models, while the asymptotics observed in e.g. [BF14, BCF18] are characteristic of $(2+1)$-dimensional models. Thus the apparent dissonance between our results and those discussed above is explained by the unusual locality of interactions of Hall-Littlewood processes[10] with one principal specialization $1, t, \ldots, t^{n-1}$. The nonlocality of interactions in most Macdonald processes is also inherited by limits which pertain to classical random matrix theory, see e.g. Section 3.3 and the references therein for more discussion of these. We have already seen the probabilistic differences between classical and p-adic random matrix theory above, and one may view the different $t \to 1$ asymptotics observed here as a manifestation of these at the level of symmetric functions after extrapolating to generic real $p = t^{-1} \in \mathbb{R}_{>1}$.

[10]Let us clarify a point of potential confusion, which is that ordinary q-TASEP features only local interactions but arises as the projection to the leftmost particles in the array (see Figure 1-7) of the above-mentioned dynamics on q-Whittaker processes. The difference is that this projection to a $(1+1)$-dimensional system with local interactions is special and occurs only at the edge of the array, while in our Hall-Littlewood case projecting to any row yields only local interactions. It should in fact not be difficult to obtain the same asymptotics as we do in the bulk for Hall-Littlewood dynamics on Gelfand-Tsetlin patterns by considering the dynamics of $\mathcal{S}^{(n)}$ with finite n taken to infinity sufficiently fast along with time, which is very different from the Gaussian free field type asymptotics present in e.g. the q-Whittaker case [BCF18].

1.7 Infinite p-adic random matrices and boundaries of branching graphs

In 1976, Voiculescu [Voi76] classified the characters of the infinite unitary group $\mathrm{U}(\infty)$, defined as the inductive limit of the chain $\mathrm{U}(1) \subset \mathrm{U}(2) \subset \ldots$. This was later shown to be equivalent to earlier results by Aissen, Edrei, Schoenberg and Whitney, stated without reference to representation theory. A similar story unfolded for the infinite symmetric group S_∞ [KOO98, VK81], related to the classical Thoma theorem [Tho64]. See [BO12a, §1.1] and the references therein for a more detailed exposition of both.

In later works such as [VK82, OO98] the result for $\mathrm{U}(\infty)$ was recast in terms of classifying the boundary of the so-called *Gelfand-Tsetlin branching graph*, defined combinatorially in terms of Schur polynomials. This led to natural generalizations to other branching graphs defined in terms of degenerations of Macdonald polynomials $P_\lambda(x_1, \ldots, x_n; q, t)$, which feature two parameters q, t and specialize to Hall-Littlewood polynomials when $q = 0$; see [BO12a, Cue18, Gor12, OO98, Ols21]. In special cases these combinatorial results take on additional significance in representation theory and harmonic analysis; the Schur case was already mentioned, and two other special cases of the result of [OO98] for the Jack polynomial case specialize to statements about the infinite symmetric spaces $\mathrm{U}(\infty)/\mathcal{O}(\infty)$ and $\mathrm{U}(2\infty)/\mathrm{Sp}(\infty)$. For the Young graph, the boundary of its natural Hall-Littlewood deformation was conjectured in equivalent form in [Ker92], proved in [Mat19], and used to deduce results on infinite matrices over finite fields in [CO22]. Surprisingly, however, the boundary of the Hall-Littlewood deformation of the Gelfand-Tsetlin graph has not previously been carried out, despite the fact that the appearance of Hall-Littlewood polynomials in harmonic analysis on p-adic groups suggests interpretations beyond the purely combinatorial setting.

Let us describe the setup of the Hall-Littlewood branching graph; we refer to [BO17, Chapter 7] for an expository account of the general formalism of graded graphs and their boundaries. Let $\mathrm{Sig}_n = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n : \lambda_1 \geq \ldots \geq \lambda_n\}$ be the set of *integer signatures of length* n, not necessarily nonnegative. Allowing λ to be an arbitrary signature, (1.2.3) yields a symmetric 'Hall-Littlewood Laurent polynomial' which we also denote P_λ.

Let $\mathscr{G}_t$ be the weighted graph with vertices

$$\bigsqcup_{n\geq 1} \mathrm{Sig}_n$$

and edges between $\lambda \in \mathrm{Sig}_n, \mu \in \mathrm{Sig}_{n+1}$ with weights

$$L_n^{n+1}(\mu, \lambda) := P_{\mu/\lambda}(t^n; t)\frac{P_\lambda(1, \dots, t^{n-1}; t)}{P_\mu(1, \dots, t^n; t)}, \tag{1.7.1}$$

known as *cotransition probabilities.* These cotransition probabilities are stochastic by (1.2.5), so any probability measure on Sig_{n+1} induces another probability measure on Sig_n. A sequence of probability measures $(M_n)_{n\geq 1}$ which is consistent under these maps is called a *coherent system.* As these form a simplex, understanding coherent systems reduces to understanding the extreme points, called the *boundary* of the branching graph. Our first main result is an explicit description of the boundary of $\mathscr{G}_t$. Here we recall the notation used in Theorem 1.2.3.

Theorem 1.7.1. *For any $t \in (0, 1)$, the boundary of $\mathscr{G}_t$ is naturally in bijection with Sig_∞. Under this bijection $\mu \in \mathrm{Sig}_\infty$ corresponds to the coherent system $(M_n^\mu)_{n\geq 1}$ defined explicitly by*

$$M_n^\mu(\lambda) := (t; t)_n \prod_{x\in\mathbb{Z}} t^{(\mu'_x - \lambda'_x)(n-\lambda'_x)} \begin{bmatrix} \mu'_x - \lambda'_{x+1} \\ \lambda'_x - \lambda'_{x+1} \end{bmatrix}_t$$

for $\lambda \in \mathrm{Sig}_n$.

We note that the product over $x \in \mathbb{Z}$ in fact has only finitely many nontrivial terms. The fact that the extreme measures have simple explicit formulas is unusual for results of this type–usually, the measures are characterized implicitly by certain generating functions.

The proof in Section 4.2 follows the general outline of the so-called *Vershik-Kerov ergodic method*, as do those of many related results mentioned above. One of the closest works to our setting is [Gor12], which studies the Schur analogue with edge weights

$$s_{\mu/\lambda}(t^n)\frac{s_\lambda(1, \dots, t^{n-1})}{s_\mu(1, \dots, t^n)}$$

for $t \in (0, 1)$, where $s_\lambda(x)$ is the Schur polynomial. The boundary is shown to be naturally in bijection with Sig_∞ as in our case[11].

The boundary classification results of [Gor12] are generalized in [Cue18] to the Macdonald case with cotransition probabilities

$$P_{\mu/\lambda}(t^n;q,t=q^k)\frac{P_\lambda(1,\dots,t^{n-1};q,t=q^k)}{P_\mu(1,\dots,t^n;q,t=q^k)} \tag{1.7.2}$$

for any $k \in \mathbb{N}$, and the boundary is again identified with Sig_∞; when $k=1$ this reduces to the result of [Gor12]. We do not see how Theorem 1.7.1 could be accessed by the methods of [Cue18] or the newer work [Ols21], which treats the related *Extended Gelfand-Tsetlin graph* with weights coming from Macdonald polynomials with arbitrary $q,t \in (0,1)$. Instead, we rely on explicit expressions, Proposition 4.1.2 and Theorem 2.2.16, for the skew Hall-Littlewood polynomials appearing in (1.7.1). This means that Theorem 1.7.1 gives explicit formulas for the extreme coherent measures, while in previous works they were defined implicitly by certain generating functions.

1.7.1 Ergodic measures on infinite p-adic random matrices

In the special case $t = 1/p$, the purely combinatorial results on Hall-Littlewood polynomials have consequences in p-adic random matrix theory, and we may deduce results of [BQ17, Ass22] from Theorem 1.2.1 and Theorem 1.7.1 above.

We earlier defined singular numbers for nonsingular matrices, but the extension to possibly-singular matrices is straightforward by allowing the singular numbers to be infinite and letting $p^\infty := 0$. For technical reasons[12] we will consider the negatives of the singular numbers, which are parametrized by

$$\underline{\mathrm{Sig}}_n = \{(\lambda_1,\dots,\lambda_n) \in (\mathbb{Z} \cup \{-\infty\})^n : \lambda_1 \geq \dots \geq \lambda_n\}. \tag{1.7.3}$$

For fixed $n \leq m$, the $\mathrm{GL}_n(\mathbb{Z}_p) \times \mathrm{GL}_m(\mathbb{Z}_p)$ bi-invariant measures on $\mathrm{Mat}_{n\times m}(\mathbb{Q}_p)$ are all convex combinations of those parametrized by $\lambda \in \underline{\mathrm{Sig}}_n$ via $U\,\mathrm{diag}_{n\times m}(p^{-\lambda_1},\dots,p^{-\lambda_n})V$ with U,V distributed by the Haar measures on $\mathrm{GL}_n(\mathbb{Z}_p), \mathrm{GL}_m(\mathbb{Z}_p)$ respectively. One may

[11]Our t corresponds to the q^{-1} in the notation [Gor12]. The setting of [Gor12] actually corresponds to $t>1$, and the boundary corresponds to infinite *increasing* tuples of integers, but this statement is equivalent to ours upon interchanging signatures with their negatives–see the comment after Theorem 1.1 in [Gor12].

[12]See Remark 8.

define $\mathrm{GL}_\infty(\mathbb{Z}_p)$ as a direct limit of the system

$$\mathrm{GL}_1(\mathbb{Z}_p) \subset \mathrm{GL}_2(\mathbb{Z}_p) \subset \dots$$

and it is natural to ask for the extension of Smith normal form to infinite matrices, i.e. for the extreme points in the set of $\mathrm{GL}_\infty(\mathbb{Z}_p)$ bi-invariant measures on $\mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p)$. This problem was previously solved in [BQ17], which gave an explicit family of measures in bijection with $\underline{\mathrm{Sig}}_\infty$. We give a new proof that the extreme measures are naturally parametrized by $\underline{\mathrm{Sig}}_\infty$ in Theorem 1.7.2 below.

Theorem 1.7.2. *The set of extreme* $\mathrm{GL}_\infty(\mathbb{Z}_p)\times\mathrm{GL}_\infty(\mathbb{Z}_p)$*-invariant measures on* $\mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p)$ *is naturally in bijection with* $\underline{\mathrm{Sig}}_\infty$*. Under this bijection, the measure* E_μ *corresponding to* $\mu \in \underline{\mathrm{Sig}}_\infty$ *is the unique measure such that its* $n \times m$ *truncations are distributed by the unique* $\mathrm{GL}_n(\mathbb{Z}_p) \times \mathrm{GL}_m(\mathbb{Z}_p)$*-invariant measure on* $\mathrm{Mat}_{n\times m}(\mathbb{Q}_p)$ *with singular numbers distributed according to a certain measure* $M^{\mu}_{m,n}$*, which is defined later in Theorem 4.3.3, in the case* $t = 1/p$*.*

Our proof goes by deducing this parametrization by $\underline{\mathrm{Sig}}_\infty$ from an augmented version of the parametrization by Sig_∞ appearing in Theorem 1.7.1. The key fact which relates the random matrix setting to the purely combinatorial setting is Theorem 1.2.1, which relates the distribution of singular numbers of a p-adic matrix after removing a row or column to the cotransition probabilities (1.7.1).

We note that while Hall-Littlewood polynomials are not mentioned by name in [BQ17], it should be possible to extrapolate many of their Fourier analytic methods to statements about Hall-Littlewood polynomials at general t. Our methods, which are based on explicit formulas for certain skew Hall-Littlewood polynomials, nonetheless differ substantially from those of [BQ17] in a manner which is not merely linguistic. Let us also be clear that while both Theorem 1.7.2 and [BQ17] show that the extreme measures are parametrized by $\mu \in \underline{\mathrm{Sig}}_\infty$, it is not obvious from the descriptions that the measures corresponding to a given $\mu \in \underline{\mathrm{Sig}}_\infty$ under [BQ17] and Theorem 1.7.2 are in fact the same. A separate argument, assuming the result of [BQ17], is required to prove that the two parametrizations by $\underline{\mathrm{Sig}}_\infty$ match, see Proposition 4.3.4. This additionally provides a computation of the distribution of singular numbers of finite corners of matrices drawn from the measures in [BQ17], which is new. We refer to Remark 23 for more detail on the differences between

Theorem 1.7.2 and [BQ17, Theorem 1.3], in particular an explanation of how our results carry over to a general non-Archimedean local field as is done in [BQ17]. We mention also that the other main result of [BQ17] is a classification of the extreme measures on infinite symmetric matrices $\{A \in \mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p) : A^T = A\}$ invariant under $\mathrm{GL}_\infty(\mathbb{Z}_p)$; it would be interesting to have an analogous Hall-Littlewood proof of this result as well, see Remark 24 for further discussion of possible strategy and difficulties.

Remark 7. In addition to [OO98], another work somewhat similar in spirit to Theorem 1.7.1 and Theorem 1.7.2 is [AN21]. This work finds the boundary of a certain branching graph defined via multivariate Bessel functions–another degeneration of Macdonald polynomials–and related to β-ensembles at general β. In the classical values $\beta = 1, 2, 4$ this recovers results on branching graphs coming from random matrix theory. Results such as ours in terms of Hall-Littlewood polynomials may be regarded as extrapolations of p-adic random matrix theory to arbitrary real $p > 1$ in the same way β-ensembles extrapolate classical random matrix theory to real $\beta > 0$, see also Remark 1.

1.7.2 Ergodic decompositions of p-adic Hua measures

For finite random matrices over $\mathbb{Q}_p$ or $\mathbb{C}$, one wishes to compute the distribution of singular numbers, singular values or eigenvalues of certain distinguished ensembles such as the classical GUE, Wishart and Jacobi ensembles (over $\mathbb{C}$), or the additive Haar measure over $\mathbb{Z}_p$. The infinite-dimensional analogue of this problem is to compute how distinguished measures on infinite matrices decompose into extreme points, which correspond to ergodic measures. Such a decomposition is given by a probability measure on the space of ergodic measures, which in our case corresponds to a probability measure on $\underline{\mathrm{Sig}}_\infty$.

One such distinguished family of measures on p-adic matrices is given by the *p-adic Hua measures* defined in [Ner13], which are analogues of the complex Hua-Pickrell measures[13]. There is a p-adic Hua measure $\mathbb{M}_n^{(s)}$ on $\mathrm{Mat}_{n\times n}(\mathbb{Q}_p)$ for each $n \in \mathbb{Z}_{\geq 1}, s \in \mathbb{R}_{>-1}$, which is defined by an explicit density with respect to the underlying additive Haar measure on $\mathrm{Mat}_{n\times n}(\mathbb{Q}_p)$, see Definition 33. A motivating property of these measures is that they are consistent under taking corners, and hence define a measure $\mathbb{M}_\infty^{(s)}$ on $\mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p)$. The decomposition of this measure into ergodic measures on $\mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p)$

[13]See [BO01], which coined the term for these measures, for an historical discussion of these measures and summary of the contents of the earlier works [Hua63, Pic87].

was computed recently in [Ass22], and we reprove the result using the aforementioned relation between p-adic matrix corners and the Hall-Littlewood branching graph $\mathscr{G}_t$. Below E_μ is as in Theorem 1.7.2, $\mathbb{Y}$ is the set of integer partitions, Q_λ is the dual normalization of the Hall-Littlewood symmetric function, and the normalizing constant $\Pi(1,\ldots;u,\ldots)$ is the *Cauchy kernel*–see Chapter 2 for precise definitions.

Theorem 1.7.3. *Fix a prime p and real parameter $s > -1$, and let $t = 1/p$ and $u = p^{-1-s}$. Then the infinite p-adic Hua measure $\mathbb{M}_\infty^{(s)}$ decomposes into ergodic measures according to*

$$\mathbb{M}_\infty^{(s)} = \sum_{\mu \in \mathbb{Y}} \frac{P_\mu(1,t,\ldots;t)Q_\mu(u,ut,\ldots;t)}{\Pi(1,\ldots;u,\ldots)} E_\mu \tag{1.7.4}$$

where E_μ is as defined in Theorem 1.7.2.

The key ingredient in the original proof of Theorem 1.7.3 given previously in [Ass22] is a certain Markov chain which generates the finite Hua measures $\mathbb{M}_n^{(s)}$, and which was guessed from Markov chains appearing in similar settings [Ful02]. The arguments there did not use Hall-Littlewood polynomials, but the limiting measure on $\underline{\mathrm{Sig}}_\infty$ which describes the ergodic decomposition was observed in [Ass22] to be the Hall-Littlewood measure in (1.7.4), by matching explicit formulas. From our perspective, by contrast, the fact that this measure is a Hall-Littlewood measure is natural and is key to the proof.

1.8 What is in the next chapters, where else is this all written down, and why read this thesis (or not)

In Chapter 2 we give preliminaries on p-adic random matrices and on symmetric functions and Macdonald processes. We then prove Theorem 1.2.1 and discuss the structural analogy with the complex case through Macdonald polynomials in Chapter 3. In Chapter 4 we expand on Section 1.7 and prove the results there using the matrix corners part of Theorem 1.2.1. We then turn attention to matrix products, and Chapters 5 to 7 expand upon and provide proofs for material discussed in Sections 1.3 to 1.5. Finally, in Chapter 8 we consider the $p \to 1$ limit results summarized in Section 1.6.

Much of the material in this thesis has already appeared in a peer-reviewed journal at the time of writing. Specifically, the main results of Chapter 3 and Chapter 5 appeared in [VP21]. The exception is Theorem 1.2.3, which appeared in [VP22a] together with the contents of Chapter 4. The results of Chapter 8 appeared in [VP22b]. Those of Chapter 6 and Chapter 7 have not yet appeared, and at the time of this writing are intended to be posted in separate papers after submission of this thesis; the posted versions will surely have benefitted from additional comments and edits by the time they are finalized. In all cases, the background material—much of which is shared between works—has been combined into Chapter 2.

While the above introduction borrows heavily from those of the published versions, we have also changed it substantially in order to present what we hope is a more unified and panoramic treatment of these results and the links between them, and improve the exposition in many places with the benefit of hindsight. This, in our view, is the main potential value of the present document to a future reader.

Chapter 2

Preliminaries

2.1 p-adic random matrix background

The following is a condensed version of the exposition in [Eva02, Section 2], to which we refer any reader desiring a more detailed introduction to p-adic numbers geared toward a probabilistic viewpoint. Fix a prime p. Any nonzero rational number $r \in \mathbb{Q}^\times$ may be written as $r = p^k(a/b)$ with $k \in \mathbb{Z}$ and a, b coprime to p. Define $|\cdot| : \mathbb{Q} \to \mathbb{R}$ by setting $|r|_p = p^{-k}$ for r as before, and $|0|_p = 0$. Then $|\cdot|_p$ defines a norm on $\mathbb{Q}$ and $d_p(x, y) := |x - y|_p$ defines a metric. We additionally define $\mathrm{val}_p(r) = k$ for r as above and $\mathrm{val}_p(0) = \infty$, so $|r|_p = p^{-\mathrm{val}_p(r)}$. We define the *field of p-adic numbers* $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to this metric, and the *p-adic integers* $\mathbb{Z}_p$ to be the unit ball $\{x \in \mathbb{Q}_p : |x|_p \leq 1\}$. It is not hard to check that $\mathbb{Z}_p$ is a subring of $\mathbb{Q}_p$. We remark that $\mathbb{Z}_p$ may be alternatively defined as the inverse limit of the system $\ldots \to \mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \to \cdots \to \mathbb{Z}/p\mathbb{Z} \to 0$, and that $\mathbb{Z}$ naturally includes into $\mathbb{Z}_p$.

$\mathbb{Q}_p$ is noncompact but is equipped with a left- and right-invariant (additive) Haar measure; this measure is unique if we normalize so that the compact subgroup $\mathbb{Z}_p$ has measure 1. The restriction of this measure to $\mathbb{Z}_p$ is the unique Haar probability measure on $\mathbb{Z}_p$, and is explicitly characterized by the fact that its pushforward under any map $r_n : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ is the uniform probability measure. For concreteness, it is often useful to view elements of $\mathbb{Z}_p$ as 'power series in p' $a_0 + a_1 p + a_2 p^2 + \ldots$, with $a_i \in \{0, \ldots, p-1\}$; clearly these specify a coherent sequence of elements of $\mathbb{Z}/p^n\mathbb{Z}$ for each n. The Haar probability measure then has the alternate explicit description that each a_i is iid uniformly random from $\{0, \ldots, p-1\}$. Additionally, $\mathbb{Q}_p$ is isomorphic to the ring of Laurent series in p, defined in exactly the same way.

Similarly, $\mathrm{GL}_N(\mathbb{Q}_p)$ has a unique left- and right-invariant measure for which the total mass of the compact subgroup $\mathrm{GL}_N(\mathbb{Z}_p)$ is 1. We denote this measure by $\mathbb{M}$. The restriction of $\mathbb{M}$ to $\mathrm{GL}_N(\mathbb{Z}_p)$ pushes forward to $\mathrm{GL}_N(\mathbb{Z}/p^n\mathbb{Z})$; these measures are the uniform measures on the finite groups $\mathrm{GL}_N(\mathbb{Z}/p^n\mathbb{Z})$. This gives an alternative characterization of the measure.

The following standard result is sometimes known as Smith normal form and holds also for more general rings.

Proposition 2.1.1. *Let $n \leq m$. For any $A \in M_{n\times m}(\mathbb{Q}_p)$, there exist $U \in \mathrm{GL}_n(\mathbb{Z}_p), V \in \mathrm{GL}_m(\mathbb{Z}_p)$ such that $UAV = \mathrm{diag}_{n\times m}(p^{\lambda_1}, \ldots, p^{\lambda_n})$ where λ is a weakly decreasing n-tuple of integers when A is nonsingular, when A is singular we formally allow parts of λ to equal ∞ and define $p^\infty = 0$. Furthermore, there is a unique such n-tuple λ.*

Definition 3. We denote by $\overline{\mathrm{Sig}}_n$ the set of *extended signatures* $\lambda = (\lambda_1, \ldots, \lambda_n)$ where $\lambda_i \in \mathbb{Z} \cup \{\infty\}$, $\lambda_i \geq \lambda_{i+1}$ for all i, and we take $\infty > k$ for all $k \in \mathbb{Z}$. We similarly let $\underline{\mathrm{Sig}}_n$ be the set of weakly decreasing n-tuples with entries in $\mathbb{Z} \cup \{-\infty\}$, and we will often abuse terminology and refer to these as extended signatures as well. For any $n \leq m$ and $A \in M_{n\times m}(\mathbb{Q}_p)$, we let $\mathrm{SN}(A) \in \overline{\mathrm{Sig}}_n$ denote the extended signature of Proposition 2.1.1, which we refer to as the *singular numbers* of A. Note the convention that the length of λ is the smaller dimension of A.

Remark 8. The reason for defining both $\overline{\mathrm{Sig}}_n$ and $\underline{\mathrm{Sig}}_n$ is an unfortunate sign convention mismatch (for which, let us not neglect to mention, we are wholly responsible) regarding singular numbers. Two previous works on infinite p-adic random matrices [BQ17, Ass22] used the opposite one, calling $-\mathrm{SN}(A)_i$ (in our notation) the singular numbers, and indeed it is more natural for the problems they consider. However, the one given in Definition 3 is more natural when one is considering matrices over $\mathbb{Z}_p$ because then the singular numbers will be nonnegative rather than nonpositive and the connection to symmetric functions appears more cleanly. Since much of this thesis works over $\mathbb{Z}_p$, we have taken the above convention, so $\mathrm{SN}(A)$ lies in $\overline{\mathrm{Sig}}_n$, but when working with infinite matrices it will be useful to consider the negative singular numbers which lie in $\underline{\mathrm{Sig}}_n$.

Similarly to eigenvalues and singular values, singular numbers have a variational characterization. We first recall the version for singular values, one version of which states that for $A \in \mathrm{Mat}_{n\times m}(\mathbb{C})$ (assume without loss of generality $n \le m$) with singular values $a_1 \ge \ldots \ge a_n$,

$$\prod_{i=1}^{k} a_i = \sup_{\substack{V\subset\mathbb{C}^m:\dim(V)=k \\ W\subset\mathbb{C}^n:\dim(W)=k}} |\det(\mathrm{Proj}_W \circ A|_V)| \tag{2.1.1}$$

where Proj is the orthogonal projection and $A|_V$ is the restriction of the linear operator A to the subspace V. (2.1.1) holds because the right hand side is unchanged by multiplying A by unitary matrices, hence A may be taken to be diagonal with singular values on the diagonal by singular value decomposition, at which point the result is easy to see. For a slightly different version which picks out the k^{th} largest singular value rather than the product of the k largest, see [Ful00, Section 5].

For p-adic matrices, we state the result slightly differently to avoid referring to orthogonal projection, the reason being that unlike $U(n)$, $\mathrm{GL}_n(\mathbb{Z}_p)$ does not preserve a reasonable inner product, only the norm.

Proposition 2.1.2. *Let $1 \le n \le m$ be integers and $A \in \mathrm{Mat}_{n\times m}(\mathbb{Q}_p)$ with $\mathrm{SN}(A) = (\lambda_1, \ldots, \lambda_n)$. Then for any $1 \le k \le n$,*

$$\lambda_n + \ldots + \lambda_{n-k+1} = \inf_{\substack{P:\mathbb{Q}_p^n\to\mathbb{Q}_p^n \text{ rank } k \text{ projection} \\ W\subset\mathbb{Q}_p^m:\dim W=k}} \mathrm{val}_p(\det(PA|_W)) \tag{2.1.2}$$

Proof. If $U_1 \in \mathrm{GL}_n(\mathbb{Z}_p), U_2 \in \mathrm{GL}_m(\mathbb{Z}_p)$, then for any a rank k projection P the matrix $U_1PU_1^{-1}$ is also a rank k projection, and similarly for any W as above U_2W is also a dimension k subspace. Hence

$$\inf_{\substack{P:\mathbb{Q}_p^n\to\mathbb{Q}_p^n \text{ rank } k \text{ projection} \\ W\subset\mathbb{Q}_p^m:\dim W=k}} \mathrm{val}_p(\det(PA|_W)) = \inf_{\substack{P:\mathbb{Q}_p^n\to\mathbb{Q}_p^n \text{ rank } k \text{ projection} \\ W\subset\mathbb{Q}_p^m:\dim W=k}} \mathrm{val}_p(\det(P(U_1AU_2)|_W)). \tag{2.1.3}$$

By Smith normal form we may choose U_1, U_2 so that $U_1AU_2 = \mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_n})$, hence

$$\mathrm{RHS}(2.1.2) = \inf_{\substack{P:\mathbb{Q}_p^n\to\mathbb{Q}_p^n \text{ rank } k \text{ projection} \\ W\subset\mathbb{Q}_p^m:\dim W=k}} \mathrm{val}_p(\det(P\,\mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_n})|_W)). \tag{2.1.4}$$

The infimum on the right hand side is clearly achieved by taking $W = \mathrm{span}(\boldsymbol{e_{n-k+1}}, \ldots, \boldsymbol{e_n})$ (where $\boldsymbol{e_i}$ are the standard basis vectors) and P to be the projection onto $\mathrm{span}(\boldsymbol{e_{n-k+1}}, \ldots, \boldsymbol{e_n})$.

This proves (2.1.2).

$\square$

It turns out that one does not have to work with arbitrary projections and subspaces, but may instead consider only minors of the matrix A. Here by $k \times k$ minor, we mean any $k \times k$ matrix obtained by deleting rows and columns of the original matrix (and do not, as is also standard, mean the determinant of such a matrix). Though this version is slightly cumbersome to prove, it makes it much easier to relate matrix entries to singular numbers, and we expect it to be useful in future work as well as in this one.

Proposition 2.1.3. *With the same setup as Proposition 2.1.2,*

$$\lambda_n + \ldots + \lambda_{n-k+1} = \inf_{A' \; k \times k \; \textit{minor of } A} \mathrm{val}_p(\det(A')). \tag{2.1.5}$$

Proof. Clearly

$$\lambda_n + \ldots + \lambda_{n-k+1} = \inf_{A' \; k \times k \text{ minor of } \mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_n})} \mathrm{val}_p(\det(A')). \tag{2.1.6}$$

Since $U_1 A U_2 = \mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_n})$ for some $U_1 \in \mathrm{GL}_m(\mathbb{Z}_p), U_2 \in \mathrm{GL}_m(\mathbb{Z}_p)$, to show equality of the right hand side of (2.1.6) and (2.1.5) it therefore suffices to show that

$$\inf_{A' \; k \times k \text{ minor of } B} \mathrm{val}_p(\det(A')) = \inf_{A' \; k \times k \text{ minor of } UBV} \mathrm{val}_p(\det(A')) \tag{2.1.7}$$

for any $B = (b_{i,j})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \in \mathrm{Mat}_{n \times m}(\mathbb{Q}_p)$ and $U \in \mathrm{GL}_n(\mathbb{Z}_p), V \in \mathrm{GL}_m(\mathbb{Z}_p)$.

First note that since $\mathrm{GL}_n(\mathbb{Z}_p)$ is generated by the three elementary row operations

(i) elementary transposition matrices,

(ii) unit multiple matrices $\mathrm{diag}(1[i-1], u, 1[n-i])$ for $u \in \mathbb{Z}_p^\times, 1 \leq i \leq n$, and

(iii) matrices $(\mathbb{1}(i=j) + \mathbb{1}(i=x, j=y))_{1 \leq i,j \leq n}$ for some $x \neq y$,

it suffices to prove (2.1.7) when U and V are each one of the above types. This is clear for types (i) and (ii). Suppose that one of U, V is of type (iii), without loss of generality U is of type (iii) and V is the identity. Then

$$UBV = (b_{i,j} + \mathbb{1}(i=x) b_{y,j})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \tag{2.1.8}$$

differs from B only in the x^{th} row. For any two sets of indices $I_x = \{x, i_1, \ldots, i_{k-1}\}$ and $J = \{j_1, \ldots, j_k\}$ which include the row x, let $B_{I_x,J}$ be the corresponding minor. Then

$$\det(UB)_{I_x,J} = \det B_{I_x,J} + \det B_{I_y,J} \tag{2.1.9}$$

so by the ultrametric inequality

$$\mathrm{val}_p(\det(UB)_{I_x,J}) \geq \min(\mathrm{val}_p(\det B_{I_x,J}), \mathrm{val}_p(\det B_{I_y,J})). \tag{2.1.10}$$

The $\leq$ direction of (2.1.7) follows immediately. For the $\geq$ direction, suppose that the strict equality case of (2.1.10) holds. It is a standard fact about $\mathbb{Q}_p$ that if strict inequality in (2.1.10) is achieved, then $\mathrm{val}_p(\det B_{I_x,J}) = \mathrm{val}_p(\det B_{I_y,J})$, so since $(UB)_{I_y,J} = B_{I_y,J}$ we have

$$\mathrm{val}_p(\det(UB)_{I_y,J}) = \mathrm{val}_p(\det B_{I_y,J}) = \mathrm{val}_p(\det B_{I_x,J}). \tag{2.1.11}$$

It follows that any value achieved by the infimum on the left hand side of (2.1.7) must also be achieved by the one on the right hand side, and this proves (2.1.7). □

Remark 9. As mentioned, Proposition 2.1.2 is a straightforward p-adic analogue of the corresponding statement (2.1.1) for complex matrices. However, the analogue of Proposition 2.1.3 for complex matrices, namely that products of singular values are related to an infimum over $k \times k$ minors, is manifestly false. As is apparent from the above proof, specific properties of the p-adic numbers such as ultrametricity are required to make the reduction from an infimum over all subspaces in Proposition 2.1.2 to an infimum only over subspaces generated by subsets of the standard basis (and similarly for projections) in Proposition 2.1.3.

We record a few other simple facts about singular numbers which will be useful.

Corollary 2.1.4. *If $d \leq m$ and $\ell \leq n$ are nonnegative integers, $A \in \mathrm{Mat}_{m\times n}(\mathbb{Q}_p)$, and B is any $d \times \ell$ submatrix of A, then the j^{th} smallest singular numbers satisfy*

$$\sum_{j=1}^{k} \mathrm{SN}(B)_{\min(d,\ell)-j+1} \geq \sum_{j=1}^{k} \mathrm{SN}(A)_{\min(m,n)-j+1} \tag{2.1.12}$$

for any $1 \leq j \leq \min(k, \ell)$.

Proof. By Proposition 2.1.2 both sides of (2.1.12) may be expressed as an infimum, and the left hand side is an infimum over a smaller set. $\square$

Proposition 2.1.5. *Let $n \le m$, $A \in \mathrm{Mat}_{n\times m}(\mathbb{Q}_p)$, and $\kappa \in \mathrm{Sig}_m$. Then*

$$|\operatorname{SN}(\operatorname{diag}(p^{\kappa_1}, \ldots, p^{\kappa_n})A)| = |\operatorname{SN}(A)| + |\kappa|.$$

Proof. In the case $m = n$ this follows immediately since $\det(p^\kappa A) = \det(p^\kappa)\det(A)$. In general, A is equivalent by column operations to a matrix A' with nonzero entries only in the left $m \times m$ submatrix A''; clearly $\operatorname{SN}(A) = \operatorname{SN}(A'')$. Since column operations commute with left-multiplication by p^κ, we have $\operatorname{SN}(p^\kappa A) = \operatorname{SN}(p^\kappa A'')$, so we may appeal to the square case. $\square$

Proposition 2.1.6. *Let $n \le m$, $A \in \mathrm{Mat}_{n\times m}(\mathbb{Q}_p)$, and suppose $B \in \mathrm{Mat}_m(\mathbb{Q}_p)$ has all singular numbers nonnegative (resp. nonpositive). Then $\operatorname{SN}(AB)_i \ge \operatorname{SN}(A)_i$ (resp. $\operatorname{SN}(AB)_i \le \operatorname{SN}(A)_i$) for each singular number $1 \le i \le n$. If $C \in \mathrm{Mat}_n(\mathbb{Q}_p)$ has nonnegative (resp. nonpositive) singular numbers, the same holds with CA replacing AB.*

Proof. Write $B = UDV$ where $D = \operatorname{diag}(p^{\operatorname{SN}(B)})$. Then $\operatorname{SN}(A) = \operatorname{SN}(AU)$ and $\operatorname{SN}(AB) = \operatorname{SN}(AUD)$. Each minor determinant of AUD is a nonnegative (resp. nonpositive) power of p times the corresponding minor determinant of AU, so the desired inequality follows from Proposition 2.1.3. The proof for CA is the same. $\square$

We will often write $\operatorname{diag}_{n\times N}(p^\lambda)$ for $\operatorname{diag}_{n\times N}(p^{\lambda_1}, \ldots, p^{\lambda_n})$, and also omit the dimensions $n \times N$ when they are clear from context. We note also that for any $\lambda \in \mathrm{Sig}_N$, the double coset $\mathrm{GL}_N(\mathbb{Z}_p)\operatorname{diag}(p^\lambda)\mathrm{GL}_N(\mathbb{Z}_p)$ is compact. The restriction of $\mathbb{M}$ to such a double coset, normalized to be a probability measure, is the unique $\mathrm{GL}_N(\mathbb{Z}_p)\times\mathrm{GL}_N(\mathbb{Z}_p)$-invariant probability measure on $\mathrm{GL}_N(\mathbb{Q}_p)$ with singular numbers λ, and all $\mathrm{GL}_N(\mathbb{Z}_p)\times\mathrm{GL}_N(\mathbb{Z}_p)$-probability measures and convex combinations of these for different λ. These measures may be equivalently described as $U\operatorname{diag}(p^{\lambda_1}, \ldots, p^{\lambda_N})V$ where U, V are independently distributed by the Haar probability measure on $\mathrm{GL}_N(\mathbb{Z}_p)$. More generally, if $n \le m$ and $U \in \mathrm{GL}_n(\mathbb{Z}_p), V \in \mathrm{GL}_m(\mathbb{Z}_p)$ are Haar distributed and $\mu \in \overline{\mathrm{Sig}}_n$, then $U\operatorname{diag}_{n\times m}(p^\mu)V$ is invariant under $\mathrm{GL}_n(\mathbb{Z}_p)\times\mathrm{GL}_m(\mathbb{Z}_p)$ acting on the left and right, and is the unique such bi-invariant measure with singular numbers given by μ.

The Haar measure on $\mathrm{GL}_N(\mathbb{Z}_p)$ also has an explicit characterization which will be very useful in Chapter 7.

Proposition 2.1.7. *Let*

$$A \in \mathrm{Mat}_N(\mathbb{Z}_p) \tag{2.1.13}$$

be a random matrix with distribution given as follows: sample its columns $v_N, v_{N-1}, \ldots, v_1$ from right to left, where the conditional distribution of v_i given $v_{i+1}, \ldots, v_N$ is that of a random column vector with additive Haar distribution conditioned on the event

$$v_i \pmod p \notin \mathrm{span}(v_{i+1} \pmod p), \ldots, v_N \pmod p)) \subset \mathbb{F}_p^N, \tag{2.1.14}$$

where in the case $i = N$ we take the span in (2.1.14) to be the 0 subspace. Then A is distributed by the Haar measure on $\mathrm{GL}_N(\mathbb{Z}_p)$.

Proof. Because $\mathrm{GL}_N(\mathbb{Z}_p)$ is compact, it suffices to show the above is a left Haar measure, i.e. for any $B \in \mathrm{GL}_N(\mathbb{Z}_p)$ we must show $BA = A$ in distribution. We show $(v_{N-j}, \ldots, v_N) = (Bv_{N-j}, \ldots, Bv_N)$ in distribution for any j by induction, which suffices. For the base case, recall (see e.g. [Eva01]) that additive Haar measure on $\mathbb{Z}_p^N$ is invariant under $\mathrm{GL}_N(\mathbb{Z}_p)$, and $Bv_N \equiv 0 \pmod p$ if and only if $v_N \equiv 0 \pmod p$, hence $Bv_N = v_N$ in distribution. For the inductive step, we have that $Bv_{N-j+1}, \ldots, Bv_N$ satisfy (2.1.14) with $i = N - j + 1, \ldots, N$ if and only $v_{N-j+1}, \ldots, v_N$ do. Furthermore, for any $(w_{N-j+1}, \ldots, w_N)$ in the support of $\mathrm{Law}(v_{N-j+1}, \ldots, v_N)$, we have

$$\mathrm{Law}(Bv_{N-j} | v_{N-i} = w_{N-i} \text{ for all } 0 \le i < j) = \mathrm{Law}(v_{N-j} | v_{N-i} = Bw_{N-i} \text{ for all } 0 \le i < j). \tag{2.1.15}$$

It follows by the inductive hypothesis that

$$\mathrm{Law}(v_{N-j}, \ldots, v_N) = \mathrm{Law}(Bv_{N-j}, \ldots, Bv_N), \tag{2.1.16}$$

completing the proof. □

2.2 Symmetric function background

In this section we give a review of general symmetric functions, Macdonald (Laurent) polynomials, and measures and Markov dynamics on signatures/partitions arising from these. For a more detailed introduction to symmetric functions see [Mac98a], and for Macdonald measures see [BC14].

Definition 4. We denote by $\mathbb{Y}$ the set of all integer partitions $(\lambda_1, \lambda_2, \ldots)$, i.e. sequences of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ which are eventually 0. We call the integers λ_i the *parts* of λ, set $\lambda'_i = \#\{j : \lambda_j \geq i\}$, and write $m_i(\lambda) = \#\{j : \lambda_j = i\} = \lambda'_i - \lambda'_{i+1}$. We write $\text{len}(\lambda)$ for the number of nonzero parts, and denote the set of partitions of length $\leq n$ by $\mathbb{Y}_n$. We write $\mu \prec \lambda$ or $\lambda \succ \mu$ if $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots$, and refer to this condition as *interlacing*. Finally, we denote the partition with all parts 0 by $\emptyset$.

The above integer partition notation is standard in the symmetric functions literature, e.g. in [Mac98a]. However, as we saw for Smith normal form and related results in the previous section, it is natural to consider weakly decreasing n-tuples of integers for Smith normal form and related results. We thus define notation analogous to Definition 4 for signatures as well, with a few twists.

Definition 5. Sig_n denotes the set of integer signatures of length n, which are weakly decreasing n-tuples of integers. As in Definition 26, $\overline{\text{Sig}}_n$ denotes the set of extended signatures. Given $\lambda = (\lambda_1, \ldots, \lambda_n) \in \overline{\text{Sig}}_n$, we refer to the λ_i as the *parts* of λ. When the length of λ is not clear from context we will denote it by $\text{len}(\lambda)$. Sig_n^+ and $\overline{\text{Sig}}_n^+$ are the sets of (extended) signatures with all parts nonnegative. We set $|\lambda| := \sum_{i=1}^n \lambda_i$ and $m_k(\lambda) = |\{i : \lambda_i = k\}|$. For $\lambda \in \overline{\text{Sig}}_n$ and $\mu \in \overline{\text{Sig}}_{n-1}$, write $\mu \prec_P \lambda$ if $\lambda_i \geq \mu_i$ and $\mu_i \geq \lambda_{i+1}$ for $1 \leq i \leq n-1$. For $\nu \in \overline{\text{Sig}}_n$ write $\nu \prec_Q \lambda$ if $\lambda_i \geq \nu_i$ for $1 \leq i \leq n$ and $\nu_i \geq \lambda_{i+1}$ for $1 \leq i \leq n-1$. We write $c[k]$ for the signature $(c, \ldots, c)$ of length k, and occasionally abuse notation by writing (λ, μ) to refer to the tuple $(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m)$ when $\lambda \in \text{Sig}_n, \mu \in \text{Sig}_m$. We additionally write $-\lambda := (-\lambda_n, \ldots, -\lambda_1) \in \text{Sig}_n$ for any $\lambda \in \text{Sig}_n$. Finally, we denote the empty signature by $()$.

Remark 10. In later sections it will sometimes be cleanest to work with signatures of infinite length as well, but we introduce that notation later on.

We denote by Λ_n the ring $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$ of symmetric polynomials in n variables $x_1, \ldots, x_n$. It is a very classical fact that the power sum symmetric polynomials $p_k = \sum_{i=1}^n x_i^k, k = 1, \ldots, n$, are algebraically independent and algebraically generate Λ_n. An immediate consequence is that Λ_n has a natural basis given by the polynomials

$$p_\lambda := \prod_{i \geq 1} p_{\lambda_i}$$

for $n \geq \lambda_1 \geq \lambda_2 \geq \ldots$ a weakly decreasing sequence of nonnegative integers which is eventually 0 (i.e. an integer partition). Hence given generic real parameters q, t, one may define an inner product on Λ_n by setting

$$\langle p_\lambda, p_\mu \rangle_{(q,t)} = \delta_{\lambda\mu} \prod_{i \geq 1 : \lambda_i > 0} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} \prod_{j \geq 1} j^{m_j(\lambda)} \cdot m_j(\lambda)!$$

The *Macdonald symmetric polynomials* $\{P_\lambda(x_1, \ldots, x_n; q, t)\}_{\lambda \in \mathrm{Sig}_n^+}$ are a distinguished basis for Λ_n characterized by the properties

- They are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_{(q,t)}$.

- They may be written as

$$P_\lambda(x_1, \ldots, x_n; q, t) = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} + (\text{lower-order monomials in the lexicographic order}).$$

It is not *a priori* clear that such polynomials exist, see [Mac98b] for a proof of this, but it is clear that they form a basis for Λ_n. The dual basis $Q_\lambda(\mathbf{x}; q, t)$ is defined by

$$Q_\lambda(x_1, \ldots, x_n; q, t) := \frac{P_\lambda(x_1, \ldots, x_n; q, t)}{\langle P_\lambda, P_\lambda \rangle_{(q,t)}}, \tag{2.2.1}$$

and we note that the normalizing constant $\langle P_\lambda, P_\lambda \rangle_{(q,t)}$ is an explicit rational function of q and t which is computed below in Lemma 2.2.13.

Because the P_λ form a basis for the vector space of symmetric polynomials in n variables, there exist symmetric polynomials $P_{\lambda/\mu}(x_1, \ldots, x_{n-k}; q, t) \in \Lambda_{n-k}$ indexed by

$\lambda \in \mathrm{Sig}_n^+, \mu \in \mathrm{Sig}_k^+$ which are defined by

$$P_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\mu \in \mathrm{Sig}_k^+} P_{\lambda/\mu}(x_{k+1}, \ldots, x_n; q, t) P_\mu(x_1, \ldots, x_k; q, t). \tag{2.2.2}$$

Similarly to the usual Macdonald polynomials, we will sometimes write $P_{\lambda/\mu}$ for $\lambda \in \mathbb{Y}_n, \mu \in \mathbb{Y}_k$ by identifying the partitions with nonnegative signatures.

We define two different versions of the skew Q polynomials. One is the standard one, the other is a slightly nonstandard way where the lengths of both signatures are the same, in contrast to the skew P polynomials. This differs from the classical treatment [Mac98a], and is inspired by the higher spin Hall-Littlewood polynomials introduced in [Bor17]; we say more by way of motivation in Remark 11 below. The standard version is given by (2.2.2) with P replaced by Q everywhere, or equivalently

$$Q_{\lambda/\mu}(x_1, \ldots, x_{n-k}; q, t) = \frac{\langle P_\mu, P_\mu \rangle_{(q,t)}}{\langle P_\lambda, P_\lambda \rangle_{(q,t)}} P_{\lambda/\mu}(x_1, \ldots, x_{n-k}; q, t). \tag{2.2.3}$$

The nonstandard one, which we denote by $\tilde{Q}$, is as follows. For $k, n \geq 1$ arbitrary and $\lambda, \nu \in \mathrm{Sig}_k^+$, define $\tilde{Q}_{\lambda/\nu}(x_1, \ldots, x_n; q, t)$ by

$$Q_{(\lambda, 0[n])}(x_1, \ldots, x_{n+k}; q, t) = \sum_{\nu \in \mathrm{Sig}_k^+} \tilde{Q}_{\lambda/\nu}(x_{k+1}, \ldots, x_{n+k}; q, t) Q_\nu(x_1, \ldots, x_k; q, t). \tag{2.2.4}$$

Note that the subscripts of $Q_{\lambda/\mu}$ are signatures of different length, while for $\tilde{Q}_{\lambda/\nu}$ they have the same length.

By comparing the terms of (2.2.2) (with Q instead of P and n replaced by $n+k$) and (2.2.4) with $\mu = \nu = (0[k])$, and using that $Q_\nu(x_1, \ldots, x_k; q, t), \nu \in \mathrm{Sig}_k^+$ is a basis for Λ_k, we see that

$$\tilde{Q}_{\lambda/(0[k])}(y_1, \ldots, y_n; q, t) = Q_{(\lambda, 0[n])/(0[k])}(y_1, \ldots, y_n; q, t) = Q_\lambda(y_1, \ldots, y_n; q, t) \tag{2.2.5}$$

where the second equality holds since $Q_\lambda(y_1, \ldots, y_n, 0[k]; q, t) = Q_\lambda(y_1, \ldots, y_n; q, t)$ (see Section 2.2.2). Recall the notation $(a; q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geq 0$, with $(a; q)_0 = 1$ and $(a; q)_\infty$ defined in the obvious way.

Definition 6. For $\lambda \in \mathrm{Sig}_n, \mu \in \mathrm{Sig}_{n-1}$ with $\mu \prec_P \lambda$, let

$$\psi_{\lambda/\mu} := \prod_{1 \le i \le j \le n-1} \frac{f(t^{j-i}q^{\mu_i-\mu_j})f(t^{j-i}q^{\lambda_i-\lambda_{j+1}})}{f(t^{j-i}q^{\lambda_i-\mu_j})f(t^{j-i}q^{\mu_i-\lambda_{j+1}})} \tag{2.2.6}$$

where $f(u) := (tu;q)_\infty/(qu;q)_\infty$. For $\nu \in \mathrm{Sig}_n$ with $\nu \prec_Q \lambda$, let

$$\varphi_{\lambda/\nu} := \prod_{1 \le i \le j \le n} \frac{f(t^{j-i}q^{\lambda_i-\lambda_j})}{f(t^{j-i}q^{\lambda_i-\nu_j})} \prod_{1 \le i \le j \le n-1} \frac{f(t^{j-i}q^{\nu_i-\nu_{j+1}})}{f(t^{j-i}q^{\nu_i-\lambda_{j+1}})}. \tag{2.2.7}$$

The following lemma may be easily derived from the corresponding statement for symmetric functions in infinitely many variables [Mac98a, VI.6 Ex. 2(a)].

Lemma 2.2.1. *For $\lambda, \nu \in \mathrm{Sig}_n^+, \mu \in \mathrm{Sig}_{n-k}^+$, we have*

$$P_{\lambda/\mu}(x_1, \ldots, x_k; q, t) = \sum_{\mu=\lambda^{(0)} \prec_P \lambda^{(1)} \prec_P \cdots \prec_P \lambda^{(k)}=\lambda} \prod_{i=0}^{k-1} x_i^{|\lambda^{(i+1)}|-|\lambda^{(i)}|} \psi_{\lambda^{(i+1)}/\lambda^{(i)}} \tag{2.2.8}$$

and

$$\tilde{Q}_{\lambda/\nu}(x_1, \ldots, x_k; q, t) = \sum_{\nu=\lambda^{(0)} \prec_Q \lambda^{(1)} \prec_Q \cdots \prec_Q \lambda^{(k)}=\lambda} \prod_{i=0}^{k-1} x_i^{|\lambda^{(i+1)}|-|\lambda^{(i)}|} \varphi_{\lambda^{(i+1)}/\lambda^{(i)}}. \tag{2.2.9}$$

This suggests using the above formulas to extend the definition of P and $\tilde{Q}$ to arbitrary signatures, possibly with negative parts, which we do now.

Definition 7. For $\mu \in \mathrm{Sig}_n, \lambda \in \mathrm{Sig}_{n+k}$, we define $\mathrm{GT}_P(\lambda/\mu)$ to be the set of sequences of interlacing signatures $\mu = \lambda^{(0)} \prec_P \lambda^{(1)} \prec_P \cdots \prec_P \lambda^{(k)} = \lambda$. We will often write $\mathrm{GT}_P(\lambda)$ for $\mathrm{GT}_P(\lambda/())$.

For $\lambda, \nu \in \mathrm{Sig}_n$, we define $\mathrm{GT}_{Q,k}(\lambda/\nu)$ to be the set of sequences of $k+1$ length n interlacing signatures $\nu = \lambda^{(0)} \prec_Q \lambda^{(1)} \prec_Q \cdots \prec_Q \lambda^{(k)} = \lambda$. We refer to elements of either GT_P or $\mathrm{GT}_{Q,k}$ as Gelfand-Tsetlin patterns, see Section 2.2.

For $T \in \mathrm{GT}_P(\lambda/\mu)$ with $\mathrm{len}(\lambda) = \mathrm{len}(\mu) + k$, set $\psi(T) := \prod_{i=0}^{k-1} \psi_{\lambda^{(i+1)}/\lambda^{(i)}}$. For $T \in \mathrm{GT}_{Q,k}(\lambda/\nu)$, set $\varphi(T) := \prod_{i=0}^{k-1} \varphi_{\lambda^{(i+1)}/\lambda^{(i)}}$. In both cases, let $wt(T) := (|\lambda^{(1)}| - |\lambda^{(0)}|, \ldots, |\lambda^{(k)}| - |\lambda^{(k-1)}|) \in \mathbb{Z}^k$.

In what follows, we often write $\mathbf{x}$ for the collection of variables $x_1, \ldots, x_n$ when n is clear from context, and $\mathbf{a} = (a_1, \ldots, a_n)$ for a collection of n real numbers. For $\mathbf{d} \in \mathbb{Z}^n$

$$
\begin{array}{ccccccccc}
\lambda_N^{(N)} & & \lambda_{N-1}^{(N)} & & \cdots\cdots\cdots & & \lambda_2^{(N)} & & \lambda_1^{(N)} \\
& \lambda_{N-1}^{(N-1)} & & \lambda_{N-2}^{(N-1)} & \cdots & \lambda_2^{(N-1)} & & \lambda_1^{(N-1)} & \\
& & \lambda_{N-2}^{(N-2)} & & & & \lambda_1^{(N-2)} & & \\
& & & \cdots\cdots\cdots\cdots\cdots\cdots & & & & & \\
& & & & \lambda_1^{(1)} & & & &
\end{array}
$$

$$
\begin{array}{cccccc}
 & \nu_N^{(k)} & \nu_{N-1}^{(k)} & \cdots\cdots\cdots & \nu_2^{(k)} & \nu_1^{(k)} \\
 & \nu_N^{(k-1)} & \nu_{N-1}^{(k-1)} & \cdots\cdots\cdots & \nu_2^{(k-1)} & \nu_1^{(k-1)} \\
 & \iddots & \iddots & \cdots\cdots\cdots & \iddots & \iddots \\
\nu_N^{(0)} & \nu_{N-1}^{(0)} & \cdots\cdots\cdots & \nu_2^{(0)} & \nu_1^{(0)} &
\end{array}
$$

Figure 2-1: An element of $\mathrm{GT}_P(\lambda^{(N)})$ (top) and an element of $\mathrm{GT}_{Q,k}(\nu^{(k)}/\nu^{(0)})$ (bottom). Note that $\mathrm{GT}_{Q,k}$ depends on a parameter k specifying the number of rows, while for GT_P the data of the number of rows was already determined by the respective lengths of λ and μ.

we write $\mathbf{x}^{\mathbf{d}} := x_1^{d_1} \cdots x_n^{d_n}$ and similarly for $\mathbf{a}^{\mathbf{d}}$.

Definition 8. For any $n \geq 0, k \geq 1$ and $\lambda \in \mathrm{Sig}_{n+k}, \mu \in \mathrm{Sig}_n$, we let

$$P_{\lambda/\mu}(x_1, \ldots, x_k; q, t) = \sum_{T \in \mathrm{GT}_P(\lambda/\mu)} \psi(T) \mathbf{x}^{wt(T)}$$

For any $\nu, \kappa \in \mathrm{Sig}_n$ we let

$$\tilde{Q}_{\kappa/\nu}(x_1, \ldots, x_k; q, t) = \sum_{T \in \mathrm{GT}_{Q,k}(\kappa/\nu)} \varphi(T) \mathbf{x}^{wt(T)}.$$

Note that these are just the formulas in Lemma 2.2.1, with the only change being that we do not require the signatures to be nonnegative. These combinatorial formulas make some symmetries readily apparent, as noted in [GM20].

Lemma 2.2.2. *Let $n, k \in \mathbb{Z}_{\geq 0}$, $\lambda, \nu \in \mathrm{Sig}_{n+k}, \mu \in \mathrm{Sig}_n$. Then*

$$P_{-\lambda/-\mu}(x_1, \ldots, x_k; q, t) = P_{\lambda/\mu}(x_1^{-1}, \ldots, x_k^{-1}; q, t) \tag{2.2.10}$$

$$\tilde{Q}_{-\lambda/-\nu}(x_1, \ldots, x_r; q, t) = \tilde{Q}_{\lambda/\nu}(x_1^{-1}, \ldots, x_r^{-1}; q, t) \tag{2.2.11}$$

$$P_{(\lambda+(d[n+k]))/(\mu+d[n])}(x_1, \ldots, x_k; q, t) = (x_1 \cdots x_k)^d P_{\lambda/\mu}(x_1, \ldots, x_k; q, t) \tag{2.2.12}$$

$$\tilde{Q}_{(\lambda+(d[n+k]))/(\nu+(d[n+k]))}(x_1, \ldots, x_r; q, t) = \tilde{Q}_{\lambda/\nu}(x_1, \ldots, x_r; q, t). \tag{2.2.13}$$

Proof. First interpret the P and $\tilde{Q}$ polynomials as sums over GT patterns by Definition 8. For (2.2.10) and (2.2.11), note that we have bijections

$$\mathrm{GT}_P(\lambda/\mu) \leftrightarrow \mathrm{GT}_P(-\lambda/-\mu)$$
$$\mathrm{GT}_{Q,r}(\lambda/\nu) \leftrightarrow \mathrm{GT}_{Q,r}(-\lambda/-\nu)$$

which can be directly verified to preserve the branching coefficients ψ defined in (2.2.6) and φ defined in (2.2.7). For (2.2.12) and (2.2.13), one similarly has bijections

$$\mathrm{GT}_P(\lambda/\mu) \leftrightarrow \mathrm{GT}_P((\lambda + (d[n+k]))/(\nu + (d[n]))$$
$$\mathrm{GT}_{Q,r}(\lambda/\nu) \leftrightarrow \mathrm{GT}_{Q,r}((\lambda + (d[n+k]))/(\nu + (d[n+k]))$$

by adding d to each entry of the GT patterns, and these preserve the branching coefficients but change $wt(T)$ in the P case. □

Remark 11. Lemma 2.2.2 provides some motivation for our definition of the modified skew Q functions $\tilde{Q}$. The naive generalization of (2.2.12) to the unmodified Q functions, even the $n = 0$ special case

$$Q_{(\lambda+(d[k]))/()}(x_1, \ldots, x_k; q, t) \neq (x_1 \cdots x_k)^d Q_{\lambda/()}(x_1, \ldots, x_k; q, t), \tag{2.2.14}$$

here taking $\lambda \in \mathrm{Sig}_k^+, d \geq 0$ since we only defined skew Q functions in (2.2.3) for nonnegative signatures. The reason is that

$$\langle P_\lambda, P_\lambda \rangle_{(q,t)} \neq \left\langle P_{\lambda+(d[k])}, P_{\lambda+(d[k])} \right\rangle_{(q,t)}, \tag{2.2.15}$$

as is easy to check from Lemma 2.2.12. Our purpose in defining $\tilde{Q}$ is to repair this translation-invariance while still keeping a version of the Cauchy identity (see below) intact. We note also that Definition 8 applies also to infinite signatures, in particular to partitions, and for $\lambda, \mu \in \mathbb{Y} \subset \mathrm{Sig}_\infty$ the polynomial $\tilde{Q}_{\lambda/\mu}$ is exactly the usual skew Macdonald polynomial $Q_{\lambda/\mu}$, and so we will usually write Q instead of $\tilde{Q}$ when the indices are partitions.

Lemma 2.2.3 (Modified skew Cauchy identity)**.** *Let* $\nu \in \mathrm{Sig}_N, \mu \in \mathrm{Sig}_{N+k}$*, and* $x_1, \ldots, x_k, y_1, \ldots, y_r$ *be indeterminates. Then*

$$\begin{aligned} &\sum_{\kappa \in \mathrm{Sig}_{N+k}} \tilde{Q}_{\kappa/\mu}(y_1, \ldots, y_r; q, t) P_{\kappa/\nu}(x_1, \ldots, x_k; q, t) \\ &= \Pi_{(q,t)}(x_1, \ldots, x_k; y_1, \ldots, y_r) \sum_{\kappa \in \mathrm{Sig}_N} \tilde{Q}_{\nu/\lambda}(y_1, \ldots, y_r; q, t) P_{\mu/\lambda}(x_1, \ldots, x_k; q, t) \end{aligned} \tag{2.2.16}$$

where

$$\Pi_{(q,t)}(x_1, \ldots, x_k; y_1, \ldots, y_r) = \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq r}} \frac{(tx_iy_j; q)_\infty}{(x_iy_j; q)_\infty} \tag{2.2.17}$$

and (2.2.16) *is interpreted as an equality of formal power series in the variables.*

Proof. When $\nu \in \mathrm{Sig}_N^+, \mu \in \mathrm{Sig}_{N+k}^+$, the result follows by specializing the usual Cauchy identity [Mac98a, Ch. VI.7, Ex. 6(a)] to finitely many variables and replacing partitions by nonnegative signatures. The result then follows for general signatures since replacing

ν, μ by $\nu + (d[N]), \mu + (d[N+k])$ in (2.2.16) multiplies both sides by $(x_1 \cdots x_k)^d$ by Lemma 2.2.2. $\square$

We note that the set $\{P_\lambda(\mathbf{x}; q, t) : \lambda \in \mathrm{Sig}_n\}$ forms a basis for the ring of symmetric Laurent polynomials $\Lambda_n[(x_1 \cdots x_n)^{-1}]$. Hence for any $\lambda, \mu \in \mathrm{Sig}_n$ one has

$$P_\lambda(\mathbf{x}; q, t) \cdot P_\mu(\mathbf{x}; q, t) = \sum_{\nu \in \mathrm{Sig}_n} c^{\nu}_{\lambda,\mu}(q, t) P_\nu(\mathbf{x}; q, t) \tag{2.2.18}$$

for some structure coefficients $c^{\nu}_{\lambda,\mu}(q, t)$. By matching degrees it is clear that these coefficients are nonzero only if $|\lambda| + |\mu| = |\nu|$. These multiplicative structure coefficients for the P polynomials are related to the 'comultiplicative' structure constants of the Q polynomials.

Proposition 2.2.4. *Let $m, n \in \mathbb{N}$ and $\lambda, \nu \in \mathrm{Sig}_n$. Then*

$$\tilde{Q}_{\lambda/\nu}(x_1, \ldots, x_m; q, t) = \sum_{\mu \in \mathrm{Sig}_n} c^{\lambda}_{\nu,\mu} Q_\mu(x_1, \ldots, x_m; q, t).$$

When all signatures are nonnegative this follows by specializing the corresponding statement for symmetric functions, see [Mac98a, Ch. VI.7] where Proposition 2.2.4 is taken as the definition of the skew Q polynomials. The case of general signatures follows by shifting arguments as before.

2.2.1 Another scalar product

There is another scalar product on Λ_n, given by an explicit integral formula, which is related to the one $\langle \cdot, \cdot \rangle_{(q,t)}$ above by a certain limit (see [Mac98a, Chapter VI, (9.9)]). The explicit formula will be useful to us for asymptotics in Chapter 6.

Definition 9 ([Mac98a, Chapter VI, (9.10)])**.** For polynomials $f, g \in \Lambda_n$, define

$$\langle f, g \rangle'_{q,t;n} := \frac{1}{n!(2\pi \mathbf{i})^n} \int_{\mathbb{T}^n} f(z_1, \ldots, z_n) \overline{g(z_1, \ldots, z_n)} \prod_{1 \le i \neq j \le n} \frac{(z_i z_j^{-1}; q)_\infty}{(t z_i z_j^{-1}; q)_\infty} \prod_{i=1}^n \frac{dz_i}{z_i}, \tag{2.2.19}$$

where $\mathbb{T}$ denotes the unit circle with usual counterclockwise orientation, and to avoid confusion we clarify that the product is over $\{(i, j) \in \mathbb{Z} : 1 \le i, j \le n, i \neq j\}$.

Proposition 2.2.5. *If $\lambda, \mu \in \mathrm{Sig}_n$ and $\lambda \neq \mu$, then*

$$\langle P_\lambda(\mathbf{z}; q, t), P_\mu(\mathbf{z}; q, t)\rangle'_{q,t;n} = 0. \tag{2.2.20}$$

Proof. Let $D \in \mathbb{Z}$ be such that $\lambda + (D[n])$ and $\mu + (D[n])$ both lie in Sig_n^+. Then

$$\left\langle P_{\lambda+(D[n])}(\mathbf{z}; q, t), P_{\mu+(D[n])}(\mathbf{z}; q, t)\right\rangle'_{q,t;n} = 0 \tag{2.2.21}$$

by [Mac98a, Chapter VI, (9.5)]. However,

$$\begin{aligned} P_{\lambda+(D[n])}(\mathbf{z}; q, t)\overline{P_{\mu+(D[n])}(\mathbf{z})} &= (z_1 \cdots z_n)^D P_\lambda(\mathbf{z}; q, t)\overline{(z_1 \cdots z_n)^D P_\mu(\mathbf{z}; q, t)} \\ &= P_\lambda(\mathbf{z}; q, t)\overline{P_\mu(\mathbf{z}; q, t)} \end{aligned} \tag{2.2.22}$$

for any $z_1, \ldots, z_n \in \mathbb{T}$, so

$$\left\langle P_{\lambda+(D[n])}(\mathbf{z}; q, t), P_{\mu+(D[n])}(\mathbf{z}; q, t)\right\rangle'_{q,t;n} = \langle P_\lambda(\mathbf{z}; q, t), P_\mu(\mathbf{z}; q, t)\rangle'_{q,t;n}, \tag{2.2.23}$$

which completes the proof. □

2.2.2 Symmetric functions

It is often convenient to consider symmetric polynomials in an arbitrarily large or infinite number of variables, which we formalize through the ring of symmetric functions. One has a chain of maps

$$\cdots \to \Lambda_{n+1} \to \Lambda_n \to \Lambda_{n-1} \to \cdots \to 0$$

where the map $\Lambda_{n+1} \to \Lambda_n$ is given by setting x_{n+1} to 0. In fact, writing $\Lambda_n^{(d)}$ for symmetric polynomials in n variables of total degree d, one has

$$\cdots \to \Lambda_{n+1}^{(d)} \to \Lambda_n^{(d)} \to \Lambda_{n-1}^{(d)} \to \cdots \to 0$$

with the same maps. The inverse limit $\Lambda^{(d)}$ of these systems may be viewed as symmetric polynomials of degree d in infinitely many variables. From the ring structure on each Λ_n one gets a natural ring structure on $\Lambda := \bigoplus_{d\geq 0} \Lambda^{(d)}$, and we call this the *ring of symmetric functions*. Because $p_k(x_1, \ldots, x_{n+1}) \mapsto p_k(x_1, \ldots, x_n)$ and $m_\lambda(x_1, \ldots, x_{n+1}) \mapsto$

$m_\lambda(x_1, \ldots, x_n)$ (for $n \geq \text{len}(\lambda)$) under the natural map $\Lambda_{n+1} \to \Lambda_n$, these families of symmetric polynomials define symmetric functions $p_k, m_\lambda \in \Lambda$. An equivalent definition of Λ is $\Lambda := \mathbb{C}[p_1, p_2, \ldots]$ where p_i are indeterminates; under the natural map $\Lambda \to \Lambda_n$ one has $p_i \mapsto p_i(x_1, \ldots, x_n)$.

For any $\lambda, \mu \in \mathbb{Y}$ with $\text{len}(\mu), \text{len}(\lambda) \leq n$, the skew Macdonald polynomials satisfy a consistency property

$$P_{\lambda/\mu}(x_1, \ldots, x_n, 0; q, t) = P_{\lambda/\mu}(x_1, \ldots, x_n; q, t) \tag{2.2.24}$$

where we identify the subscripts on both sides with signatures of length $n+1$ and n respectively. For $\lambda, \nu \in \text{Sig}_k$,

$$\tilde{Q}_{\lambda/\nu}(x_1, \ldots, x_n, 0; q, t) = \tilde{Q}_{\lambda/\nu}(x_1, \ldots, x_n; q, t) \tag{2.2.25}$$

as well. Hence here exist *(skew) Macdonald symmetric functions*, denoted $P_{\lambda/\mu}, \tilde{Q}_{\lambda/\nu}$ as well, such that $P_{\lambda/\mu} \mapsto P_{\lambda/\mu}(\mathbf{x}; q, t)$ under the natural map $\Lambda \to \Lambda_n$ and similarly for $\tilde{Q}$. From Lemma 2.2.3, or from [Mac98a], we obtain the skew Cauchy identity

$$\begin{aligned}\sum_{\kappa \in \mathbb{Y}} & P_{\kappa/\nu}(\mathbf{x}; q, t) Q_{\kappa/\mu}(\mathbf{y}; q, t) \\ &= \exp\left(\sum_{\ell=1}^\infty \frac{1-t^\ell}{1-q^\ell} \frac{1}{\ell} p_\ell(\mathbf{x}) p_\ell(\mathbf{y})\right) \sum_{\lambda \in \mathbb{Y}} Q_{\nu/\lambda}(\mathbf{y}; q, t) P_{\mu/\lambda}(\mathbf{x}; q, t). \end{aligned} \tag{2.2.26}$$

Here $P_{\kappa/\nu}(\mathbf{x}; q, t)$ is an element of Λ, a polynomial in $p_1(\mathbf{x}), p_2(\mathbf{x}), \ldots \in \Lambda$, and summands such as $P_{\kappa/\nu}(\mathbf{x}; q, t) Q_{\kappa/\mu}(\mathbf{y}; q, t)$ are interpreted as elements of a ring $\Lambda \otimes \Lambda$ and both sides interpreted as elements of a completion thereof.

To get a probability measure on $\mathbb{Y}$ from the skew Cauchy identity, we would like homomorphisms $\phi : \Lambda \to \mathbb{C}$ which take P_λ and Q_λ to $\mathbb{R}_{\geq 0}$—here we recall that we take $q, t \in (-1, 1)$. Simply plugging in nonnegative real numbers for the variables in Lemma 2.2.3 works, but does not yield all of them. However, a full classification of such homomorphisms, called *Macdonald nonnegative specializations* of Λ, was conjectured by Kerov [Ker92] and proven by Matveev [Mat19]. We describe them now: they are associated to triples of $\{\alpha_n\}_{n \geq 1}, \{\beta_n\}_{n \geq 1}, \tau$ (the Plancherel parameter) such that $\tau \geq 0$, $0 \leq \alpha_n, \beta_n < 1$ for all $n \geq 1$, and $\sum_n \alpha_n, \sum_n \beta_n < \infty$. These are typically called usual (or alpha) parameters, dual (or beta) parameters, and the Plancherel parameter[1] respectively. Given such a triple, the corresponding specialization is defined by

$$\begin{aligned} p_1 &\mapsto \sum_{n\geq 1} \alpha_n + \frac{1-q}{1-t}\left(\tau + \sum_{n\geq 1}\beta_n\right) \\ p_k &\mapsto \sum_{n\geq 1} \alpha_n^k + (-1)^{k-1}\frac{1-q^k}{1-t^k}\sum_{n\geq 1}\beta_n^k \qquad \text{for all } k \geq 2. \end{aligned} \tag{2.2.27}$$

Note that the above formula defines a specialization for arbitrary tuples of reals α_n, β_n and τ satisfying convergence conditions, but it will not in general be nonnegative.

Definition 10. For the specialization θ defined by the triple $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}, \tau$, we write

$$P_\lambda(\alpha(\alpha_1, \alpha_2, \ldots), \beta(\beta_1, \beta_2, \ldots), \gamma(\tau); q, t) := P_\lambda(\theta; q, t) := \theta(P_\lambda) \tag{2.2.28}$$

and similarly for skew and dual Macdonald polynomials. For any other specialization ϕ defined by parameters $\{\alpha'_n\}_{n\geq 1}, \{\beta'_n\}_{n\geq 1}, \tau'$, we let $\theta \cup \phi$ be the specialization with usual parameters $\{\alpha_n\}_{n\geq 1} \sqcup \{\alpha'_n\}_{n\geq 1}$, dual parameters $\{\beta_n\}_{n\geq 1} \sqcup \{\beta'_n\}_{n\geq 1}$, and Plancherel parameter $\tau + \tau'$. For $k \in \mathbb{N}$ we write

$$\phi[k] = \underbrace{\phi \cup \cdots \cup \phi}_{k \text{ times}}, \tag{2.2.29}$$

similarly to our notation for repeated variables. We will omit the $\alpha(\cdots)$ in notation if all alpha parameters are zero for the given specialization, and similarly for β and γ. Additionally, because the α variables correspond to the variables in the original Macdonald symmetric polynomial, for 'pure alpha' specializations with no β or Plancherel variables we often write $P_\lambda(a_1, a_2, \ldots; q, t)$ in place of $P_\lambda(\alpha(a_1, a_2, \ldots); q, t)$.

[1]The terminology 'Plancherel' comes from the fact that in the case $q = t$ where the Macdonald polynomials reduce to Schur polynomials, it is related to (the poissonization of) the Plancherel measure on irreducible representations of the symmetric group S_N, see e.g. [BO17].

Additionally, we define notation

$$\begin{aligned}\Pi_{q,t}(\alpha(\alpha_1,\ldots),\beta(\beta_1,\ldots),\gamma(\tau);\alpha(\alpha'_1,\ldots),\beta(\beta'_1,\ldots),\gamma(\tau')) &:= \Pi_{q,t}(\theta;\phi) \\ &:= \exp\left(\sum_{\ell=1}^{\infty}\frac{1-t^\ell}{1-q^\ell}\frac{1}{\ell}\theta(p_\ell)\phi(p_\ell)\right).\end{aligned} \tag{2.2.30}$$

We refer to a specialization as

- *pure alpha* if τ and all $\beta_n, n \geq 1$ are 0.
- *pure beta* if τ and all $\alpha_n, n \geq 1$ are 0.
- *Plancherel* if all $\alpha_n, \beta_n, n \geq 1$ are 0.

On Macdonald polynomials these act as follows.

Proposition 2.2.6. *Let $\lambda, \mu \in Y$ and $c_1, \ldots, c_n \in \mathbb{R}_{\geq 0}$. Then*

$$\begin{aligned} P_\lambda(\alpha(c_1,\ldots,c_n);q,t) &= P_\lambda(c_1,\ldots,c_n;q,t) \\ Q_\lambda(\alpha(c_1,\ldots,c_n);q,t) &= Q_\lambda(c_1,\ldots,c_n;q,t) \\ P_\lambda(\beta(c_1,\ldots,c_n);q,t) &= Q_{\lambda'}(c_1,\ldots,c_n;t,q) \\ Q_\lambda(\beta(c_1,\ldots,c_n);q,t) &= P_{\lambda'}(c_1,\ldots,c_n;t,q), \end{aligned} \tag{2.2.31}$$

where in each case the left hand side is a specialized Macdonald symmetric function while the right hand side is a Macdonald polynomial with real numbers plugged in for the variables. Furthermore,

$$P_\lambda(\gamma(\tau);q,t) = \lim_{D\to\infty} P_\lambda\left(\tau\cdot\frac{1-q}{1-t}\frac{1}{D}[D];q,t\right) \tag{2.2.32}$$

and similarly for Q.

The alpha case of (2.2.31), and (2.2.32), are straightforward from (2.2.27). The β case follows from properties of a certain involution on Λ, see [Mac98a, Chapter VI], and explains the terminology 'dual parameter'. The Plancherel parameter is related to the others through limit transitions, of which the α version is below. The convergence statement is standard, but the fact that it is monotonic is useful for later convergence statements and we are not aware of a reference.

Lemma 2.2.7. *For any $\tau \geq 0, q, t \in (-1,1)$ and $\lambda, \mu \in \mathbb{Y}$, the sequence*

$$P_{\lambda/\mu}\left(\tfrac{1-q}{1-t}\tfrac{\tau}{D}[D]; q, t\right), D = 1, 2, \ldots \tag{2.2.33}$$

is nondecreasing and converges to $P_{\lambda/\mu}(\gamma(\tau); q, t)$, and similarly for Q.

Proof. The fact that $\lim_{D\to\infty} P_{\lambda/\mu}\left(\tfrac{1-q}{1-t}\tfrac{\tau}{D}[D]; q, t\right) = P_{\lambda/\mu}(\gamma(\tau); q, t)$ is standard, and follows because (a) clearly $p_k(\tfrac{1-q}{1-t}\tfrac{\tau}{D}[D]) \to p_k(\gamma(\tau))$ for each k, and (b) $P_{\lambda/\mu}$ is a polynomial in the p_k.

Let us now prove the sequence (2.2.33) is nondecreasing. Specializing Lemma 2.2.1 to our case,

$$P_{\lambda/\mu}(\tfrac{1-q}{1-t}\tfrac{\tau}{D}[D]) = \sum_{\mu=\lambda^{(0)}\prec\ldots\prec\lambda^{(D)}=\lambda} \left(\frac{\tau(1-q)}{D(1-t)}\right)^{|\lambda|-|\mu|} \prod_{i=0}^{D-1} \psi_{\lambda^{(i+1)}/\lambda^{(i)}}.$$

There are many distinct sequences $\mu = \lambda^{(0)} \prec \ldots \prec \lambda^{(D)} = \lambda$ for which the sets $\{\lambda^{(i)} : 0 \leq i \leq D\}$ are the same but the multiplicities of the partitions in the sequence are different, and we wish to group these together. Hence we collect terms according to the set of distinct partitions appearing:

$$P_{\lambda/\mu}(\tfrac{1-q}{1-t}\tfrac{\tau}{D}[D]) = \sum_{S\subset\mathbb{Y}} \sum_{\substack{\mu=\lambda^{(0)}\prec\ldots\prec\lambda^{(D)}=\lambda \\ \{\lambda^{(0)},\ldots,\lambda^{(D)}\}=S}} \left(\frac{\tau(1-q)}{(1-t)D}\right)^{|\lambda|-|\mu|} \prod_{i=0}^{D-1} \psi_{\lambda^{(i+1)}/\lambda^{(i)}}. \tag{2.2.34}$$

where clearly only sets S of the form $\{\mu^{(1)}, \ldots, \mu^{(k)}\}$ with $\mu = \mu^{(1)} \prec \ldots \prec \mu^{(k)} = \lambda$ contribute. We now fix such an S, and claim that the term

$$\sum_{\substack{\mu=\lambda^{(0)}\prec\ldots\prec\lambda^{(D)}=\lambda \\ \{\lambda^{(0)},\ldots,\lambda^{(D)}\}=S}} \left(\frac{\tau(1-q)}{(1-t)D}\right)^{|\lambda|-|\mu|} \prod_{i=0}^{D-1} \psi_{\lambda^{(i+1)}/\lambda^{(i)}} \tag{2.2.35}$$

in (2.2.34) is nondecreasing in D. Note first that

$$\prod_{i=0}^{D-1} \psi_{\lambda^{(i+1)}/\lambda^{(i)}}$$

is the same for all terms in (2.2.35), independent of D (provided $D+1 \geq |S|$ so the sum is non-empty), and that it is nonnegative because $q, t \in (-1, 1)$. The number of summands in (2.2.35) is $\binom{D-1}{|S|-1}$. Hence

$$(2.2.35) = \mathbb{1}(D \geq |S|) \prod_{i=1}^{|S|} \psi_{\mu^{(i+1)}/\mu^{(i)}} \left(\frac{\tau(1-q)}{(1-t)D}\right)^{|\lambda|-|\mu|} \binom{D-1}{|S|-1}. \tag{2.2.36}$$

Since the RHS of (2.2.36) is nonnegative, we need only show it is nondecreasing in D when $D \geq |S|$, as otherwise it is 0. The ratio of successive (nonzero) terms is

$$\begin{aligned}
\frac{\left(\frac{\tau}{D+1}\right)^{|\lambda|-|\mu|}\binom{D}{|S|-1}}{\left(\frac{\tau}{D}\right)^{|\lambda|-|\mu|}\binom{D-1}{|S|-1}} &= \left(\frac{D}{D+1}\right)^{|\lambda|-|\mu|} \frac{D}{D-|S|+1} \\
&\geq \left(1 - \frac{1}{D+1}\right)^{|\lambda|-|\mu|} \frac{D}{D-(|\lambda|-|\mu|+1)+1} \\
&\geq \left(1 - \frac{|\lambda|-|\mu|}{D+1}\right) \frac{D}{D-(|\lambda|-|\mu|)} \\
&\geq 1.
\end{aligned}$$

In the first inequality we used that $|S| \leq |\lambda| + 1$, as the sizes of the partitions in S must each differ by at least one. In the second we used the elementary inequality $(1-x)^n \geq 1 - nx$ for $x \in [0, 1], n \geq 1$, which follows by noting equality holds at $x = 0$ and the LHS has larger derivative on the interval. This completes the proof. □

We record one more useful fact about specializations which will be needed in Chapter 6.

Proposition 2.2.8. *Let $u \in \mathbb{R}$ and let θ, ϕ be the specializations $\theta = \alpha(u, ut, \ldots)$ and $\phi = \beta(-u, -uq, \ldots)$ (note these are not in general nonnegative specializations). Then*

$$\theta \cup \phi = 0 \tag{2.2.37}$$

(i.e. $(\theta \cup \phi)(p_i) = 0$ for all $i \geq 1$ and $(\theta \cup \phi)(1) = 1$), and consequently

$$P_{\lambda/\mu}(\theta; q, t) = \mathbb{1}(\lambda = \mu). \tag{2.2.38}$$

Proof. By the explicit formula (2.2.27), $(\theta \cup \phi)(p_i) = 0$ for all $i \geq 1$, and $P_{\lambda/\mu}$ is a polynomial in the p_i with no constant term unless $\lambda = \mu$. □

2.2.3 Probabilistic constructions from symmetric functions

For the remainder of this section assume $q, t \in [0, 1)$. We note that for any nonnegative specializations θ, ϕ with

$$\sum_{\lambda \in \mathbb{Y}} P_\lambda(\theta; q, t) Q_\lambda(\phi; q, t) < \infty, \tag{2.2.39}$$

the specialized Cauchy identity

$$\sum_{\kappa \in \mathbb{Y}} P_{\kappa/\nu}(\theta; q, t) Q_{\kappa/\mu}(\phi; q, t) = \Pi_{q,t}(\theta; \psi) \sum_{\lambda \in \mathbb{Y}} Q_{\nu/\lambda}(\phi; q, t) P_{\mu/\lambda}(\theta; q, t). \tag{2.2.40}$$

holds by applying $\theta \otimes \phi$ to (2.2.26).

Definition 11. The *Macdonald measure* with specializations θ, ϕ satusfying (2.2.39) is the measure on $\mathbb{Y}$ given by

$$\Pr(\lambda) = \frac{P_\lambda(\theta; q, t) Q_\lambda(\phi; q, t)}{\Pi_{(q,t)}(\theta; \phi)}.$$

Slightly more generally, the skew Cauchy identity may be used to define Markov transition dynamics. Note that now the signatures do not have to be nonnegative.

Proposition 2.2.9. *For specializations $\theta = \alpha(a_1, \ldots, a_n), \phi$ satisfying (2.2.39), the formulas*

$$\Pr(\nu \to \lambda) = \frac{P_\lambda(a_1, \ldots, a_n; q, t) \tilde{Q}_{\lambda/\nu}(\phi; q, t)}{P_\nu(a_1, \ldots, a_n; q, t) \Pi_{(q,t)}(a_1, \ldots, a_n; \phi)}$$

and

$$\Pr(\lambda \to \mu) = \frac{P_{\lambda/\mu}(a_{k+1}, \ldots, a_n; q, t) P_\mu(a_1, \ldots, a_k; q, t)}{P_\lambda(a_1, \ldots, a_n; q, t)}.$$

define Markov transition dynamics $\mathrm{Sig}_n \to \mathrm{Sig}_n$ and $\mathrm{Sig}_n \to \mathrm{Sig}_k$ respectively. Here we allow $n = \infty$.

The following product convolution is related to the above operations but more general in a certain sense, as we explain in Chapter 3. For simplicity we give only the definition for pure α specializations.

Definition 12. Let $\mathbf{a} = \alpha(a_1, \ldots, a_n)$ be nonnegative reals. Then given $\lambda, \mu \in \mathrm{Sig}_n$, we define a random signature $\lambda \boxtimes_{\mathbf{a}} \mu$ by

$$\Pr(\lambda \boxtimes_{\mathbf{a}} \mu = \nu) = \frac{P_\nu(\mathbf{a}; q, t)}{P_\lambda(\mathbf{a}; q, t) P_\mu(\mathbf{a}; q, t)} c^\nu_{\lambda,\mu}(q, t).$$

Definition 13. Let θ and $\psi_1, \ldots, \psi_k$ be Macdonald-nonnegative specializations such that each pair θ, ψ_i satisfies (2.2.39). The associated *ascending Macdonald process* is the probability measure on sequences $\lambda^{(1)}, \ldots, \lambda^{(k)}$ given by

$$\Pr(\lambda^{(1)}, \ldots, \lambda^{(k)}) = \frac{Q_{\lambda^{(1)}}(\psi_1) Q_{\lambda^{(2)}/\lambda^{(1)}}(\psi_2) \cdots Q_{\lambda^{(k)}/\lambda^{(k-1)}}(\psi_k) P_{\lambda^{(k)}}(\theta)}{\prod_{i=1}^k \Pi(\psi_i; \theta)}.$$

The $k = 1$ case of Definition 13 is a measure on partitions, referred to as a *Hall-Littlewood measure*. Instead of defining joint distributions all at once as above, one can define Markov transition kernels on $\mathbb{Y}$.

Definition 14. Let θ, ψ be Hall-Littlewood nonnegative specializations satisfying (2.2.39) and λ be such that $P_\lambda(\theta) \neq 0$. The associated *Cauchy Markov kernel* is defined by

$$\Pr(\lambda \to \nu) = Q_{\nu/\lambda}(\psi) \frac{P_\nu(\theta)}{P_\lambda(\theta)\Pi(\psi; \theta)}. \tag{2.2.41}$$

It is clear that the ascending Hall-Littlewood process above is nothing more than the joint distribution of k steps of a Cauchy Markov kernel with specializations ψ_i, θ at the i^{th} step. The product convolution $\boxtimes$ of Definition 12 is related to Macdonald processes as follows.

Proposition 2.2.10. *Let $n \in \mathbb{N} \cup \{\infty\}$ and $\mathbf{b} = \alpha(b_1, \ldots, b_n)$ with $b_i \in \mathbb{R}_{>0}$. Let $\theta^{(i)}, 1 \leq i \leq k$ be arbitrary Macdonald-nonnegative specializations and $\nu^{(i)}$ be distributed by the Macdonald measure with specializations $\theta^{(i)}, \mathbf{b}$ for each $i = 1, \ldots, k$. Then for any fixed $\lambda^{(i)} \in \mathrm{Sig}_n, i = 1, \ldots, k$ we have*

$$\begin{aligned}&\Pr(\nu^{(1)} \boxtimes_{\mathbf{b}} \cdots \boxtimes_{\mathbf{b}} \nu^{(\tau)} = \lambda^{(\tau)} \textit{ for all } \tau = 1, \ldots, k) \\ &\qquad = \frac{\tilde{Q}_{\lambda^{(1)}/(0[n])}(\theta^{(1)}; q, t) \tilde{Q}_{\lambda^{(2)}/\lambda^{(1)}}(\theta^{(2)}; q, t) \cdots \tilde{Q}_{\lambda^{(k)}/\lambda^{(k-1)}}(\theta^{(k)}; q, t) P_{\lambda^{(k)}}(\mathbf{b}; q, t)}{\Pi_{(q,t)}(\mathbf{b}; \theta^{(1)}, \ldots, \theta^{(k)})}.\end{aligned}$$

Proof. A simple algebraic manipulation using Proposition 2.2.4 to absorb the structure coefficients $c^\nu_{\lambda,\mu}$. ☐

2.2.4 Hall-Littlewood and q-Whittaker polynomials.

We collect useful explicit formulas for the Hall-Littlewood special case ($q = 0$) and the q-Whittaker special case ($t = 0$). The former will be the relevant one for almost the entirety of this thesis, but q-Whittaker polynomials play a role in Chapter 6. When not otherwise indicated, everything here may be found in [Mac98a, Ch. III] or simply derived from results there.

Proposition 2.2.11 (Explicit formulas for Hall-Littlewood polynomials). *For $\lambda \in \mathrm{Sig}_n$, let*

$$v_\lambda(t) = \prod_{i \in \mathbb{Z}} \frac{(t;t)_{m_i(\lambda)}}{(1-t)^{m_i(\lambda)}}.$$

When $q = 0$ we have

$$P_\lambda(\mathbf{x};0,t) = \frac{1}{v_\lambda(t)} \sum_{\sigma \in S_n} \sigma \left(\mathbf{x}^\lambda \prod_{1 \le i < j \le n} \frac{x_i - tx_j}{x_i - x_j} \right)$$

where σ acts by permuting the variables.

For going between P and Q polynomials, we compute the proportionality constant

$$b_\lambda(q,t) := \frac{1}{\langle P_\lambda, P_\lambda \rangle_{(q,t)}}. \tag{2.2.42}$$

Lemma 2.2.12 (See [Mac98a, p339, (6.19)]). *The constant $b_\lambda(q,t)$ of (2.2.42) is given explicitly as follows. We associate to λ its Ferrers diagram as in Section 2.2.4. The boxes in the diagram correspond to pairs (i,j) with $1 \le i \le \lambda'_j, 1 \le j \le \lambda_i$. For such a box $s = (i,j)$, we define its* arm-length $a(s)$ *and* leg-length $\ell(s)$ *by the horizontal (resp. vertical) distance from s to the edge of the diagram as in Section 2.2.4, explicitly*

$$a(s) = \lambda_i - j \tag{2.2.43}$$

$$\ell(s) = \lambda'_j - i. \tag{2.2.44}$$

The formula is then

$$b_\lambda(q,t) = \prod_{s \in \lambda} \frac{1 - q^{a(s)} t^{\ell(s)+1}}{1 - q^{a(s)+1} t^{\ell(s)}} \tag{2.2.45}$$

where the product is over boxes s inside the diagram of λ.

$$\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & \\ \cline{1-2} & \\ \cline{1-2} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 3 & 2 & 1 & 0 \\ \hline 1 & 0 \\ \cline{1-2} 1 & 0 \\ \cline{1-2} 0 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 3 & 2 & 0 & 0 \\ \hline 2 & 1 \\ \cline{1-2} 1 & 0 \\ \cline{1-2} 0 \\ \cline{1-1} \end{array} \tag{2.2.46}$$

Figure 2-2: The Ferrers diagram of $\lambda = (4, 2, 2, 1)$ (left), with $a(s)$ and $\ell(s)$ listed for each box s (middle and right respectively).

The Hall-Littlewood and q-Whittaker specializations of (2.2.45) will be useful.

Lemma 2.2.13. *In the q-Whittaker specialization,*

$$b_\lambda(q, 0) = \prod_i \frac{1}{(q; q)_{\lambda_i - \lambda_{i+1}}}. \tag{2.2.47}$$

In the Hall-Littlewood specialization,

$$b_\lambda(0, t) = \prod_{i>0} (t; t)_{m_i(\lambda)} \tag{2.2.48}$$

Proof. Because $\ell(s) \geq 0$, the numerator of (2.2.45) is always 1 when $t = 0$, so

$$b_\lambda(q, 0) = \prod_{\substack{s \in \lambda \\ \ell(s)=0}} \frac{1}{1 - q^{a(s)+1}} = \prod_i \frac{1}{(q; q)_{\lambda_i - \lambda_{i+1}}}. \tag{2.2.49}$$

Similarly, when $q = 0$ the denominator of (2.2.45) is always 1, and

$$b_\lambda(0, t) = \prod_{\substack{s \in \lambda \\ a(s)=0}} (1 - t^{\ell(s)+1}) = \prod_{i>0} (t; t)_{\lambda'_i - \lambda'_{i+1}}. \tag{2.2.50}$$

$\square$

Remark 12. It is easy to see from Lemma 2.2.13 that translation invariance in the sense of (2.2.15) does not hold even in the Hall-Littlewood case when λ has some parts equal to 0, as mentioned in Remark 11.

The explicit forms of the Hall-Littlewood and q-Whittaker special cases of the formulas in Lemma 2.2.1 will also be useful.

Lemma 2.2.14. *Let $\lambda, \nu \in \mathrm{Sig}_n$ and $\mu \in \mathrm{Sig}_{n+1}$ with $\mu \prec_P \lambda$ and $\nu \prec_Q \lambda$. In the Hall-Littlewood case $q = 0$ the formulas of Definition 6 specialize to*

$$\begin{aligned}
\psi_{\lambda/\mu}(0,t) &= \prod_{\substack{i \in \mathbb{Z} \\ m_i(\mu)=m_i(\lambda)+1}} (1 - t^{m_i(\mu)}) \\
\varphi_{\lambda/\nu}(0,t) &= \prod_{\substack{i \in \mathbb{Z} \\ m_i(\lambda)=m_i(\nu)+1}} (1 - t^{m_i(\lambda)}),
\end{aligned} \tag{2.2.51}$$

In the q-Whittaker case $t = 0$ they specialize to

$$\begin{aligned}
\psi_{\lambda/\mu}(q,0) &= \prod_{i=1}^{\mathrm{len}(\mu)} \begin{bmatrix} \lambda_i - \lambda_{i+1} \\ \lambda_i - \mu_i \end{bmatrix}_q \\
\varphi_{\lambda/\mu}(q,0) &= \frac{1}{(q;q)_{\lambda_1 - \mu_1}} \prod_{i=1}^{\mathrm{len}(\lambda)-1} \begin{bmatrix} \mu_i - \mu_{i+1} \\ \mu_i - \lambda_{i+1} \end{bmatrix}_q .
\end{aligned} \tag{2.2.52}$$

As in the Macdonald case, the above branching coefficients are 0 when the conditions $\mu \prec_P \lambda$ (resp. $\nu \prec_Q \lambda$) do not hold.

Proof. Direct computation from Definition 6. □

The pure alpha specialization $\alpha(u, ut, \ldots, ut^{n-1})$, often referred as a *principal specialization*, produces simple factorized expressions for Macdonald polynomials. For brevity we give only the Hall-Littlewood case which is needed later, though we will mention the Macdonald case in Section 3.3. Let

$$n(\lambda) := \sum_{i=1}^{n} (i-1)\lambda_i = \sum_{i \geq 1} \binom{\lambda_i'}{2}. \tag{2.2.53}$$

The following formulas may be easily derived from Proposition 2.2.11 and (2.2.5).

Proposition 2.2.15 (Principal specialization formulas). *For $J, n \geq 1$ and $\lambda \in \mathrm{Sig}_n^+$,*

$$\begin{aligned}
P_\lambda(x, xt, \ldots, xt^{n-1}; 0, t) &= x^{|\lambda|} t^{n(\lambda)} \frac{(t;t)_n}{\prod_{i \in \mathbb{Z}} (t;t)_{m_i(\lambda)}} \\
\tilde{Q}_{\lambda/(0[n])}(x, xt, \ldots, xt^{J-1}; 0, t) &= x^{|\lambda|} t^{n(\lambda)} \frac{(t;t)_J}{(t;t)_{m_0(\lambda)+J-n}} \mathbb{1}(m_0(\lambda) + J - n \geq 0)
\end{aligned}$$

Note that the principal specialization formula for Q differs from the statement in

[Mac98a, Ch. III.2, Ex. 1] due to our conventions on signatures, but it may be derived directly from that statement using (2.2.5) to translate between skew and non-skew Q polynomials. There are also nice explicit formulas for principally specialized Macdonald polynomials, see [Mac98a, Ch. VI], but we will not need these except briefly in Section 3.3.

When the principal specialization is infinite, nice formulas for the principally specialized skew Hall-Littlewood polynomials were shown in [Kir98].

Theorem 2.2.16. *For $\mu, \lambda \in \mathbb{Y}$, we have*

$$P_{\mu/\lambda}(u, ut, \ldots; 0, t) = u^{|\mu|-|\lambda|} t^{n(\mu/\lambda)} \prod_{x \geq 1} \frac{(t^{1+\mu'_x-\lambda'_x}; t)_{m_x(\lambda)}}{(t; t)_{m_x(\mu)}}. \tag{2.2.54}$$

For $\lambda, \nu \in \mathrm{Sig}_n$,

$$Q_{\nu/\lambda}(u, ut, \ldots; 0, t) = u^{|\nu|-|\lambda|} t^{n(\nu/\lambda)} \prod_{x \in \mathbb{Z}} \frac{(t^{1+\nu'_x-\lambda'_x}; t)_{m_x(\lambda)}}{(t; t)_{m_x(\lambda)}} \tag{2.2.55}$$

Analogues of Theorem 2.2.16 for finite principal specializations have to our knowledge not appeared before. We derive them in Section 4.2, where they are necessary to prove Theorem 1.7.1, and recover Theorem 2.2.16 as a limiting case.

In later sections, we will often omit the '$; q, t$' in the arguments of Macdonald or Hall-Littlewood polynomials when they are clear from context.

THIS PAGE INTENTIONALLY LEFT BLANK

Chapter 3

Exact results on Hall-Littlewood polynomials and p-adic random matrices

In this chapter, we prove several exact results relating p-adic random matrices to Hall-Littlewood polynomials. In Section 3.1 we prove Theorem 1.2.1 from the Introduction, conditional on certain symmetric functions results proven in Section 3.2. We then extract a few consequences in p-adic random matrix theory, Corollary 1.2.2 and Theorem 1.2.3. In an optional appendix Section 3.3, we discuss the structural parallels to real/complex/quaternion random matrix theory and indicate how similar results may be proven there.

3.1 Products, corners, and the classical ensembles via Hall-Littlewood polynomials

Theorem 1.2.1. *Fix a prime p and let $t = 1/p$.*

1. (Truncated Haar ensemble) *Let $1 \leq n \leq m \leq N$ be integers, and A be the top-left $n \times m$ submatrix of a Haar-distributed element of* $\mathrm{GL}_N(\mathbb{Z}_p)$*. Then* $\mathrm{SN}(A)$ *is a*

random nonnegative signature with distribution given by the Hall-Littlewood measure

$$\Pr(\mathrm{SN}(A) = \lambda) = \frac{P_\lambda(1, t, \ldots, t^{n-1}; t) Q_\lambda(t^{m-n+1}, \ldots, t^{N-n}; t)}{\Pi_{(0,t)}(1, t, \ldots, t^{n-1}; t^{m-n+1}, \ldots, t^{N-n})}. \tag{1.2.6}$$

2. (Corners process) *Let n, k, N be integers with $1 \leq n \leq N$ and $1 \leq k \leq N - n$, $\lambda \in \mathrm{Sig}_n$, and $A \in M_{n \times N}(\mathbb{Q}_p)$ be random with $\mathrm{SN}(A) = \lambda$ and distribution invariant under $\mathrm{GL}_n(\mathbb{Z}_p) \times \mathrm{GL}_N(\mathbb{Z}_p)$ acting on the right and left. Let $A_{col} \in M_{n \times (N-k)}(\mathbb{Q}_p)$ be the first $N - k$ columns of A. Then $\mathrm{SN}(A_{col})$ is a random element of Sig_n with distribution given by*

$$\Pr(\mathrm{SN}(A_{col}) = \nu) = \frac{Q_{\nu/\lambda}(1, \ldots, t^{-(k-1)}; t) P_\nu(t^{N-n}, \ldots, t^{N-1}; t)}{P_\lambda(t^{N-n}, \ldots, t^{N-1}; t) \Pi_{(0,t)}(1, \ldots, t^{-(k-1)}; t^{N-n}, \ldots, t^{N-1})}. \tag{1.2.7}$$

Now let $1 \leq d \leq n$ and $A_{row} \in M_{(n-d) \times N}$ be the first $n - d$ rows of A. Then $\mathrm{SN}(A_{row})$ is a random element of Sig_{n-d} with distribution

$$\Pr(\mathrm{SN}(A_{row}) = \mu) = P_{\lambda/\mu}(1, \ldots, t^{d-1}; t) \frac{P_\mu(t^d, \ldots, t^{n-1}; t)}{P_\lambda(1, \ldots, t^{n-1}; t)}. \tag{1.2.8}$$

3. (Product process) *Let A, B be random elements of $M_n(\mathbb{Q}_p)$ with fixed singular numbers $\mathrm{SN}(A) = \lambda, \mathrm{SN}(B) = \mu$, invariant under left- and right-multiplication by $\mathrm{GL}_n(\mathbb{Z}_p)$. Then for any $\nu \in \mathrm{Sig}_n$, $\mathrm{SN}(AB)$ has distribution $\lambda \boxtimes_{(1, \ldots, t^{n-1})} \mu$, i.e.*

$$\Pr(\mathrm{SN}(AB) = \nu) = c^\nu_{\lambda,\mu}(0, t) \frac{P_\nu(1, \ldots, t^{n-1}; t)}{P_\lambda(1, \ldots, t^{n-1}; t) P_\mu(1, \ldots, t^{n-1}; t)}. \tag{1.2.9}$$

Let us flesh out the discussion from the Introduction on how Theorem 1.2.1 is proven. For Part 3, the results are essentially already contained in [Mac98a, Ch. V] and must be translated to probabilistic language. Parts 1 and 2 both concern the operation of taking submatrices of a random matrix, which is equivalent to the multiplicative convolution of Part 3 with projection matrices. There is a slight difficulty because Part 3 is a statement about the pair $(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{GL}_n(\mathbb{Z}_p))$ and hence holds only for nonsingular matrices A, B, so one must make a limiting argument with nonsingular matrices which are very close to projection matrices, e.g. $\mathrm{diag}(1[N-k], p^D[k])$ for large D–recall that in the p-adic norm, p^D for large D is very small. Since Part 3 relates matrix products to the structure coefficients $c^\nu_{\lambda,\mu}(0, t)$, to implement the above idea of matrices which limit to projectors we must understand asymptotics of $c^\nu_{\lambda,\mu}(0,t)$ for sequences of λ, μ which approach the singular numbers of projection matrices. We will first state the relevant asymptotic results on Macdonald structure coefficients, Proposition 3.1.1 and Proposition 3.1.2, then prove Theorem 1.2.1 conditional on these. This illustrates why these are the right asymptotic results on structure coefficients for our setting, which may not be apparent from the first glance. In the next subsection we will then develop the machinery to establish Proposition 3.1.1 and Proposition 3.1.2.

Proposition 3.1.1. *Let $q, t \in (-1,1)$ be such that the structure coefficients $c^\nu_{\lambda,\mu}(q,t)$ are all nonnegative*[1]*. Let $n \leq m \leq N$ be integers such that $n \leq N-m$, let $\lambda \in \mathrm{Sig}_n$, and let $a_1 \geq a_2 \geq \ldots \geq a_N > 0$ be real numbers and $\mathbf{a} = (a_1, \ldots, a_N)$. Let M_D^{Cauchy} be the probability measure on Sig_n with distribution defined by taking the last n parts of a random signature $\kappa = (D[N-n], \lambda) \boxtimes_{\mathbf{a}} (D[N-m], 0[m])$, or explicitly,*

$$M_D^{Cauchy}(\nu) = \sum_{\substack{\kappa \in \mathrm{Sig}_N \\ \kappa_{N-n+i} = \nu_i \text{ for all } i=1,\ldots,n}} c^\kappa_{(D[N-n],\lambda),(D[N-m],0[m])}(q,t) \frac{P_\kappa(\mathbf{a})}{P_{(D[N-n],\lambda)}(\mathbf{a}) P_{(D[N-m],0[m])}(\mathbf{a})}.$$

Then for each $\nu \in \mathrm{Sig}_n$,

$$M_D^{Cauchy}(\nu) \to \frac{\tilde{Q}_{\nu/\lambda}(a_1^{-1}, \ldots, a_{N-m}^{-1}) P_\nu(a_{N-n+1}, \ldots, a_N)}{P_\lambda(a_{N-n+1}, \ldots, a_N) \Pi(a_1^{-1}, \ldots, a_{N-m}^{-1}; a_{N-n+1}, \ldots, a_N)} \tag{3.1.1}$$

as $D \to \infty$.

Proposition 3.1.2. *Let $q, t \in (-1,1)$ be such that the structure coefficients $c^\nu_{\lambda,\mu}(q,t)$ are all nonnegative. Let $0 < k \leq n$ be integers, let $\lambda \in \mathrm{Sig}_n$, and let $a_1 \geq a_2 \geq \ldots \geq a_n > 0$ be real numbers and $\mathbf{a} = (a_1, \ldots, a_n)$. Let M_D^{branch} be the probability measure on Sig_{n-k} with distribution defined by taking the last $n-k$ parts of a random signature $\kappa = \lambda \boxtimes_{\mathbf{a}} (D[k], 0[n-k])$, or explicitly,*

$$M_D^{branch}(\mu) = \sum_{\substack{\kappa \in \mathrm{Sig}_n \\ \kappa_{k+i} = \mu_i \text{ for all } i=1,\ldots,n-k}} c^\kappa_{\lambda,(D[k],0[n-k])}(q,t) \frac{P_\kappa(\mathbf{a}; q,t)}{P_\lambda(\mathbf{a}; q,t) P_{(D[k],0[n-k])}(\mathbf{a}; q,t)}.$$

[1] Conjecturally, this is true if $q, t \in [0,1)$ or if $q, t \in (-1, 0]$, see Matveev [Mat19]. For our application we will only need the case $q = 0, t = 1/p \in (0,1)$, for which the nonnegativity follows from the interpretation of the structure coefficients in terms of the Hall algebra [Mac98a, Ch. III], or alternatively from Theorem 1.2.1 Part 3.

Then for each $\mu \in \mathrm{Sig}_{n-k}$,

$$M_D^{branch}(\mu) \to P_{\lambda/\mu}(a_1, \ldots, a_k)\frac{P_\mu(a_{k+1}, \ldots, a_n)}{P_\lambda(\mathbf{a})} \text{ as } D \to \infty.$$

Proof of Theorem 1.2.1, conditional on Proposition 3.1.2 and Proposition 3.1.1. We first prove Part 3. In this proof we will use essentially the notation of [Mac98a, Ch. V] to state the relevant results and then show how ours follow. Let $\mathscr{G}_t = \mathrm{GL}_n(\mathbb{Q}_p)$, $K = \mathrm{GL}_n(\mathbb{Z}_p)$, and $L(\mathscr{G}_t, K)$ denote the algebra of compactly supported functions $f : \mathscr{G}_t \to \mathbb{C}$ which are bi-invariant under K, i.e. $f(k_1 x k_2) = f(x)$ for $x \in \mathscr{G}_t, k_1, k_2 \in K$. Define a convolution operation on $L(\mathscr{G}_t, K)$ by

$$(f * g)(x) = \int_{\mathscr{G}_t} f(xy^{-1})g(y)dy$$

where the integration is with respect to the Haar measure on $\mathscr{G}_t$ normalized such that K has measure 1, mentioned earlier. This multiplication is associative, and may be checked to be commutative as well. By Proposition 2.1.1, each double coset KxK of $K\backslash\mathscr{G}_t/K$ has a unique representative of the form $\mathrm{diag}(p^\lambda)$ for some $\lambda \in \mathrm{Sig}_n$. We abuse notation slightly and write such a double coset as $Kp^\lambda K$. We denote by $\mathbb{1}_\lambda$ the indicator function on such a double coset; clearly $\mathbb{1}_\lambda \in L(\mathscr{G}_t, K)$. We will use the following two results, both of which may be found in the discussion after [Mac98a, Ch. V, (2.7)]:

- The map $\theta : L(\mathscr{G}_t, K) \to \Lambda_n[(x_1 \cdots x_n)^{-1}]$ given by $\theta(\mathbb{1}_\lambda) = t^{n(\lambda)} P_\lambda(x_1, \ldots, x_n; 0, t)$ is a $\mathbb{C}$-algebra isomorphism, where $n(\lambda) := \sum_i (i-1)\lambda_i$. Equivalently,

$$\mathbb{1}_\lambda * \mathbb{1}_\mu = \sum_{\nu \in \mathrm{Sig}_n} t^{n(\lambda)+n(\mu)-n(\nu)} c^\nu_{\lambda,\mu}(0,t)\mathbb{1}_\nu.$$

- The measure of each double coset $Kp^\lambda K$ is

$$\mathbb{M}(Kp^\lambda K) = t^{n(\lambda)-(n-1)|\lambda|} P_\lambda(1, t, \ldots, t^{n-1}; 0, t).$$

It follows directly that

$$\frac{\int_{x \in Kp^\nu K}(\mathbb{1}_\lambda * \mathbb{1}_\mu)(x)dx}{\mathbb{M}(Kp^\lambda K)\mathbb{M}(Kp^\mu K)} = c^\nu_{\lambda,\mu}(0,t)\frac{P_\nu(1, \ldots, t^{n-1}; 0, t)}{P_\lambda(1, \ldots, t^{n-1}; 0, t)P_\mu(1, \ldots, t^{n-1}; 0, t)}, \tag{3.1.2}$$

using the fact that either $(n-1)|\lambda| + (n-1)|\mu| = (n-1)|\nu|$ or the equality is trivial.

But the LHS of (3.1.2), by the definition of the convolution product, is

$$\frac{1}{\mathbb{M}(Kp^{\lambda}K)\mathbb{M}(Kp^{\mu}K)} \int_{x,y \in \mathscr{G}_t} \mathbb{1}_{\nu}(x)\mathbb{1}_{\lambda}(xy^{-1})\mathbb{1}_{\mu}(y)dxdy$$

Setting $A = xy^{-1}, B = y$, this is exactly the conditional probability $\Pr(\mathrm{SN}(AB) = \nu)$ as A, B vary over $Kp^{\lambda}K$ and $Kp^{\mu}K$ respectively (both normalized to have total measure 1), so Part 3 of Theorem 1.2.1 is proven.

Now consider Part 1. The distribution of the top n rows of a Haar-distributed element of $\mathrm{GL}_N(\mathbb{Z}_p)$ is just the unique $\mathrm{GL}_n(\mathbb{Z}_p) \times \mathrm{GL}_N(\mathbb{Z}_p)$-invariant distribution on $M_{n\times N}(\mathbb{Q}_p)$ with singular numbers $\mathrm{SN}(A) = (0[n])$. Hence Part 1 is the special case of (1.2.7) when $\lambda = (0[n])$. Let us deduce (1.2.7) from Proposition 3.1.1.

Fix $\lambda \in \mathrm{Sig}_n$. We wish to compute the distribution of the singular numbers $\mathrm{SN}(U \operatorname{diag}_{n\times N}(p^{\lambda})VP_k)$, where $U \in \mathrm{GL}_n(\mathbb{Z}_p), V \in \mathrm{GL}_N(\mathbb{Z}_p)$ are Haar distributed and $P_k = \operatorname{diag}_{N\times N}(1[N-k], 0[k])$ is a corank-k projector. Consider a fixed, deterministic $V_0 \in \mathrm{GL}_N(\mathbb{Z}_p)$, and let $\nu = \mathrm{SN}(U \operatorname{diag}_{n\times N}(p^{\lambda})V_0P_k) \in \mathrm{Sig}_n$. First note that this is independent of U, and setting

$$A(V_0) := \operatorname{diag}_{N\times N}(p^{\lambda}, 0[N-n])V_0P_k$$

we have $\mathrm{SN}(A(V_0)) = (\infty[N-n], \nu) \in \overline{\mathrm{Sig}}_N$. For $D > \nu_1$ set

$$A_D(V_0) := \operatorname{diag}_{N\times N}(p^{\lambda}, p^D[N-n])V_0 \operatorname{diag}_{N\times N}(1[N-k], p^D[k]).$$

We claim that $\mathrm{SN}(A_D)_{N-n+i} = \nu_i$ for each $i = 1, \ldots, n$.

Let $r_D : M_N(\mathbb{Z}_p) \to M_N(\mathbb{Z}/p^D\mathbb{Z})$ be the obvious map. Proposition 2.1.1 holds also for matrices over $\mathbb{Z}/p^D\mathbb{Z}$, so we may abuse notation and define SN on both $M_N(\mathbb{Z}_p)$ and $M_N(\mathbb{Z}/p^D\mathbb{Z})$ as in Definition 3. Let $\varphi_D : \overline{\mathrm{Sig}}_N \to \overline{\mathrm{Sig}}_N$ be the map defined as follows: for any $\kappa \in \overline{\mathrm{Sig}}_N$, let $j = \max(\{i : \kappa_i \geq D\})$, and define $\varphi_D(\kappa) := (\infty[j], \kappa_{j+1}, \ldots, \kappa_N)$. It is clear that the diagram

$$\begin{array}{ccc} M_N(\mathbb{Z}_p) & \xrightarrow{\mathrm{SN}} & \overline{\mathrm{Sig}}_N \\ \Big\downarrow r_D & & \Big\downarrow \varphi_D \\ M_N(\mathbb{Z}/p^D\mathbb{Z}) & \xrightarrow{\mathrm{SN}} & \overline{\mathrm{Sig}}_N \end{array}$$

commutes, hence $\mathrm{SN}(A_D(V_0))_{N-n+i} = \nu_i$ for each $i = 1, \ldots, n$, because $r_D(A_D(V_0)) = r_D(A(V_0))$ and $\nu_i < D$ for all i. Thus for any fixed ν and $D > \nu_1$, recalling that $V \in \mathrm{GL}_N(\mathbb{Z}_p)$ is Haar distributed, we have

$$\Pr(\mathrm{SN}(A_D(V))_{N-n+i} = \nu_i \text{ for } i = 1, \ldots, n) = \Pr(\mathrm{SN}(A(V))_{N-n+i} = \nu_i \text{ for } i = 1, \ldots, n).$$

This stabilization for $D > \nu_1$ in particular implies that

$$\lim_{D\to\infty} \Pr(\mathrm{SN}(A_D(V))_{N-n+i} = \nu_i \text{ for } i = 1, \ldots, n) = \Pr(\mathrm{SN}(A(V))_{N-n+i} = \nu_i \text{ for } i = 1, \ldots, n).$$

The RHS is what we want to compute. By Part 3 of Theorem 1.2.1, the LHS is equal to

$$\lim_{D\to\infty} \sum_{\substack{\kappa \in \mathrm{Sig}_N \\ \kappa_{N-n+i} = \nu_i \text{ for all } i=1,\ldots,n}} \frac{c^{\kappa}_{(D[N-n],\lambda),(D[k],0[N-k])}(0,t) P_\kappa(1, \ldots, t^{n-1}; 0, t)}{P_{(D[N-n],\lambda)}(1, \ldots, t^{n-1}; 0, t) P_{(D[k],0[N-k])}(1, \ldots, t^{n-1}; 0, t)}.$$

By Proposition 3.1.1 this is equal to

$$\frac{\tilde{Q}_{\nu/\lambda}(1, \ldots, t^{-(k-1)}) P_\nu(t^{N-n}, \ldots, t^{N-1})}{P_\lambda(t^{N-n}, \ldots, t^{N-1}) \Pi(1, \ldots, t^{-(k-1)}; t^{N-n}, \ldots, t^{N-1})},$$

which is (1.2.7). Setting $m = N - k$ and $\lambda = (0[n])$, and dividing all variables in both P specializations by t^{N-n} and multiplying those in the Q specialization by t^{N-n}, yields Theorem 1.2.1 Part 1.

It remains to prove the other case of Part 2, namely (4.3.1). One wishes to compute the distribution of $P_d U \operatorname{diag}_{n\times N}(p^\lambda) V$ where $P_d \in M_n(\mathbb{Z}_p)$ is a corank d projector and U, V are as above. We may ignore V, and by the same argument as before it suffices to consider the matrix $\operatorname{diag}_{n\times n}(0[n-d], p^D[d]) U \operatorname{diag}_{n\times n}(p^\lambda)$ for large D. One then applies Proposition 3.1.2 to yield (4.3.1). $\square$

From Theorem 1.2.1 Part 1 we deduce the following. Recall that the intuition for this statement is that if N is very large compared to m, n, then the entries of an $n \times m$ corner of a Haar distributed element of $\mathrm{GL}_N(\mathbb{Z}_p)$ become asymptotically iid from the additive Haar measure on $\mathbb{Z}_p$. The analogous statement holds in the complex case, namely that an $n \times m$ corner of a Haar distributed element of $U(N)$ becomes a matrix of iid Gaussians as $N \to \infty$ if one rescales appropriately.

Corollary 1.2.2. *Fix a prime p and let $t = 1/p$. Let $1 \leq n \leq m$, and $A \in M_{n \times m}(\mathbb{Z}_p)$ be random with iid entries distributed according to the additive Haar measure on $\mathbb{Z}_p$. Then for any $\lambda \in \mathrm{Sig}_n^+$,*

$$\Pr(\mathrm{SN}(A) = \lambda) = \frac{P_\lambda(1, \ldots, t^{n-1}; t) Q_\lambda(t^{m-n+1}, t^{m-n+2}, \ldots; t)}{\Pi_{(0,t)}(1, \ldots, t^{n-1}; t^{m-n+1}, t^{m-n+2}, \ldots)}$$

Proof. Let $A \in M_{n \times m}(\mathbb{Z}_p)$ be distributed as in Corollary 1.2.2, and $B_N \in M_{n \times m}(\mathbb{Z}_p)$ be an $n \times m$ corner of a Haar distributed element of $\mathrm{GL}_N(\mathbb{Z}_p)$. By Part 1 of Theorem 1.2.1 one has

$$\lim_{N \to \infty} \Pr(\mathrm{SN}(B_N) = \lambda) = \frac{P_\lambda(1, \ldots, t^{n-1}) Q_\lambda(t^{m-n+1}, t^{m-n+2}, \ldots)}{\Pi_{(0,t)}(1, \ldots, t^{n-1}; t^{m-n+1}, t^{m-n+2}, \ldots)},$$

hence it suffices to prove

$$\lim_{N \to \infty} \Pr(\mathrm{SN}(B_N) = \lambda) = \Pr(\mathrm{SN}(A) = \lambda).$$

It is clear that A and B_N are nonsingular with probability 1, so $\mathrm{SN}(B_N)$ and $\mathrm{SN}(A)$ lie in Sig_n^+ (rather than $\overline{\mathrm{Sig}}_N$) with probability 1. Letting $D > \lambda_1$ be an integer, we have by the argument in the proof of Theorem 1.2.1 that if $\mathrm{SN}(E) = \lambda$ then $\mathrm{SN}(r_D(E)) = \lambda$ as well for any fixed nonsingular $E \in M_{n \times m}(\mathbb{Z}_p)$, where r_D is the reduction modulo p^D map. Therefore $\Pr(\mathrm{SN}(B_N) = \lambda) = \Pr(\mathrm{SN}(r_D(B_N)) = \lambda)$ and similarly with B_N replaced by A, so it suffices to prove

$$\lim_{N \to \infty} \Pr(\mathrm{SN}(r_D(B_N) = \lambda) = \Pr(\mathrm{SN}(r_D(A) = \lambda).$$

From the discussion of measures at the beginning of the section it follows that $r_D(A)$ has entries iid uniform over $\mathbb{Z}/p^D\mathbb{Z}$, and $r_D(B_N)$ has the distribution of an $n \times m$ corner of a uniformly random element of $\mathrm{GL}_N(\mathbb{Z}/p^D\mathbb{Z})$. A uniformly random element $(a_{ij})_{1 \leq i,j \leq N} \in \mathrm{GL}_N(\mathbb{Z}/p^D\mathbb{Z})$ may be sampled by first sampling a uniformly random element of $(a'_{ij})_{1 \leq i,j \leq N} \in \mathrm{GL}_N(\mathbb{Z}/p\mathbb{Z})$, then choosing $a_{ij} \in \mathbb{Z}/p^D\mathbb{Z}$ independently, uniform in the congruence class of a'_{ij}. Thus it suffices to show that for any $C \in M_{n \times m}(\mathbb{Z}/p\mathbb{Z})$,

$$\lim_{N \to \infty} \Pr(r_1(B_N) = C) = \Pr(r_1(A) = C).$$

For fixed C,

$$\frac{(p^{N-m}-1)(p^{N-m}-p)\cdots(p^{N-m}-p^{n-1})}{p^{nN}} \leq \Pr(r_1(B_N)=C) \leq \frac{p^{n(N-m)}}{p^{nN}};$$

the lower bound is sharp when C is the zero matrix, and the upper bound is sharp when C is nonsingular. Both bounds come from counting the number of $n \times N$ nonsingular matrices over $\mathbb{Z}/p\mathbb{Z}$ with left $n \times m$ submatrix C, and dividing by $\#M_{n\times N}(\mathbb{Z}/p\mathbb{Z})$. The upper bound is p^{-nm} and the lower bound goes to p^{-nm} as $N \to \infty$. Since $\Pr(r_1(A) = C) = p^{-nm}$ for any C, we are done. $\square$

Note also that Parts 2, 3 of Theorem 1.2.1 (and the limiting case Corollary 1.2.2) together with Proposition 2.2.10 immediately imply that joint distributions of singular numbers of products of p-adic Haar corners are distributed according to Hall-Littlewood processes.

Corollary 3.1.3. *Let $t = 1/p$, fix $n \geq 1$ and let $N_1, N_2, \ldots \in \mathbb{Z} \cup \{\infty\}$ with with $N_i > n$ for all i. For each i, let A_i be the top left $n \times n$ corner of a Haar distributed element of $\mathrm{GL}_{N_i}(\mathbb{Z}_p)$ if $N_i < \infty$, and let A_i have iid entries distributed by the additive Haar measure on $\mathbb{Z}_p$ if $N_i = \infty$. Then for $\lambda^{(1)}, \ldots, \lambda^{(k)} \in \mathrm{Sig}_n^+$,*

$$\begin{aligned}&\Pr(\mathrm{SN}(A_\tau \cdots A_1) = \lambda^{(\tau)} \text{ for all } \tau = 1, \ldots, k) \\ &= \frac{\tilde{Q}_{\lambda^{(1)}/(0[n])}(t, \ldots, t^{N_1-n})\tilde{Q}_{\lambda^{(2)}/\lambda^{(1)}}(t, \ldots, t^{N_2-n}) \cdots \tilde{Q}_{\lambda^{(k)}/\lambda^{(k-1)}}(t, \ldots, t^{N_k-n})P_{\lambda^{(k)}}(1, \ldots, t^{n-1})}{\Pi_{(0,t)}(1, \ldots, t^{n-1}; t, \ldots, t^{N_1-n}, t, \ldots, t^{N_2-n}, \ldots, t, \ldots, t^{N_k-n})}\end{aligned}$$

The analogue of this result in the real/complex/quaternion case is given in [Ahn22b, Thm. 3.12]. Additionally, Theorem 1.2.1 together with Hall-Littlewood combinatorics yields exact formulas for the distribution of singular numbers of a product of additive Haar matrices stated earlier as Theorem 1.2.3, generalizing the explicit formula for the Cohen-Lenstra measure which is the case of one matrix.

Proposition 3.1.4. *For $n \geq 1$ and $\lambda, \nu \in \mathrm{Sig}_n$, we have*

$$\frac{\tilde{Q}_{\nu/\lambda}(u, ut, \ldots)P_\nu(1, \ldots, t^{n-1})}{P_\lambda(1, \ldots, t^{n-1})\Pi(1, \ldots, t^{n-1}; u, ut, \ldots)} = (u;t)_n u^{|\nu|-|\lambda|} t^{n(\nu)-n(\lambda)+n(\nu/\lambda)} \prod_{x \in \mathbb{Z}} \begin{bmatrix} \nu'_x - \lambda'_{x+1} \\ \nu'_x - \nu'_{x+1} \end{bmatrix}_t . \tag{3.1.3}$$

Proof. It follows from the definition in (2.2.17) and telescoping that

$$\frac{1}{\Pi(1,\ldots,t^{n-1};u,ut,\ldots)} = (u;t)_n.$$

Combining Proposition 2.2.15 with Theorem 2.2.16 to evaluate P_ν, P_λ and $\tilde{Q}_{\nu/\lambda}$ respectively yields

$$\frac{\tilde{Q}_{\nu/\lambda}(u,ut,\ldots)P_\nu(1,\ldots,t^{n-1})}{P_\lambda(1,\ldots,t^{n-1})} = u^{|\nu|-|\lambda|}t^{n(\nu/\lambda)+n(\nu)-n(\lambda)}\prod_{x\in\mathbb{Z}}\frac{(t^{1+\nu'_x-\lambda'_x};t)_{m_x(\lambda)}}{(t;t)_{m_x(\nu)}}$$

Noting that

$$\prod_{x\in\mathbb{Z}}\frac{(t^{1+\nu'_x-\lambda'_x};t)_{m_x(\lambda)}}{(t;t)_{m_x(\nu)}} = \prod_{x\in\mathbb{Z}}\begin{bmatrix}\nu'_x-\lambda'_{x+1}\\ \nu'_x-\nu'_{x+1}\end{bmatrix}_t$$

completes the proof. □

Proof of Theorem 1.2.3. Follows immediately by combining Theorem 1.2.1 and Proposition 3.1.4 with $u=t=1/p$. □

3.2 Asymptotics of Macdonald polynomials and structure coefficients

We now develop the machinery to prove Proposition 3.1.1 and Proposition 3.1.2. Both of these statements involve limits of normalized structure coefficients

$$\frac{P_{\nu(D)}(a_1,\ldots,a_n;q,t)}{P_{\lambda(D)}(a_1,\ldots,a_n;q,t)P_{\mu(D)}(a_1,\ldots,a_n;q,t)}c^{\nu(D)}_{\lambda(D),\mu(D)}(q,t)$$

for some signatures $\lambda(D),\mu(D),\nu(D)$, so we must establish asymptotics both on the Macdonald polynomials with real specializations $\mathbf{a}$, and on the structure coefficients $c^\nu_{\lambda,\mu}(q,t)$ themselves. Both come from Theorem 3.2.1 below, which treats the asymptotics of Macdonald polynomials in formal variables $x_1,\ldots,x_N$.

Theorem 3.2.1. *Let $q,t\in(-1,1)$. Fix positive integers k,N, let $r_1,\ldots,r_k$ be positive integers such that $\sum_i r_i = N$, and set $s_i=\sum_{j=1}^i r_j$ with the convention $s_0=0$. Let $L_1>\cdots>L_k$ be integers and $\lambda^{(i)}\in\mathrm{Sig}_{r_i}$ be any signatures, and define the signature*

$\lambda(D) = (L_1D + \lambda_1^{(1)}, \ldots, L_1D + \lambda_{r_1}^{(1)}, \ldots, L_kD + \lambda_1^{(k)}, \ldots, L_kD + \lambda_{r_k}^{(k)}) \in \mathrm{Sig}_N$ *for each* $D \in \mathbb{N}$ *large enough so that this is a valid signature. Then*

$$\frac{P_{\lambda(D)}(x_1, \ldots, x_N; q, t)}{\prod_{i=1}^k (x_{s_{i-1}+1} \cdots x_{s_i})^{L_iD}} \to \prod_{i=1}^k P_{\lambda^{(i)}}(x_{s_{i-1}+1}, \ldots, x_{s_i}; q, t) \prod_{i=1}^{k-1} \Pi_{(q,t)}(x_{s_{i-1}+1}^{-1}, \ldots, x_{s_i}^{-1}; x_{s_i+1}, \ldots, x_N) \tag{3.2.1}$$

as $D \to \infty$ *in the sense that the coefficient of each Laurent monomial* $x_1^{d_1} \cdots x_N^{d_N}$ *on the LHS converges to the corresponding coefficient on the RHS.*

Proof. First, note that in the branching rule (2.2.8) there is exactly on Gelfand-Tsetlin pattern $T_{max} \in \mathrm{GT}_P(\lambda(D)/())$ with weight $\lambda(D)$, namely the one with all entries as large as possible. One can check that $\psi(T_{max}) = 1$, so T_{max} contributes the lexicographically highest-degree monomial $\mathbf{x}^{\lambda(D)}$ of $P_{\lambda(D)}(x_1, \ldots, x_N)$.

Define the signature $\hat{\lambda}(D) := ((L_0 \cdot D)[r_0], \ldots, (L_k \cdot D)[r_k])$, so $\mathbf{x}^{\hat{\lambda}(D)} = \prod_{i=1}^k (x_{s_{i-1}+1} \cdots x_{s_i})^{L_iD}$. The idea of the proof is that for another monomial $\mathbf{x}^{\hat{\lambda}(D)+\mathbf{d}}$, the set of GT patterns of weight $\mathbf{x}^{\hat{\lambda}(D)+\mathbf{d}}$ stabilizes in size for all large D, and furthermore the structure of these GT patterns will be in a sense independent of D.

Fix $\mathbf{d} \in \mathbb{Z}^n$ for the remainder of the proof. For any fixed monomial $\mathbf{x}^{\mathbf{d}}$ and sufficiently large D, all Gelfand-Tsetlin patterns contributing to the coefficient of $\mathbf{x}^{\mathbf{d}}$ in $P_{\lambda(D)}(x_1, \ldots, x_N)/\mathbf{x}^{\hat{\lambda}(D)}$ will be as in Figure 3-1, or in other words, all entries will be close to those of T_{max}.

It is natural to divide each of the 'strips' of entries $\approx L_iD$ into two GT patterns, one triangular of the type in the P branching rule and one rectangular of the type in the Q branching rule, by splitting into the parts above and below row s_i inclusive, see Figure 3-2. Where each strip intersects the s_i^{th} row one has a signature $\kappa^{(i)} + L_iD$, so any $T \in \mathrm{GT}_P(\lambda(D))$ uniquely specifies smaller *constituent GT patterns* $T_i^P \in \mathrm{GT}_P(\kappa^{(i)}/()), T_i^Q \in \mathrm{GT}_{Q,N-s_i}(\lambda^{(i)}/\kappa^{(i)})$ for each $i = 1, \ldots, k$. It is also clear from the picture that any choice of these smaller GT patterns, i.e. choice of $\kappa^{(i)} \in \mathrm{Sig}_{r_i}, i = 1, \ldots, k$ and elements of $\mathrm{GT}_P(\kappa^{(i)}/())$ and $\mathrm{GT}_{Q,s_i}(\lambda^{(i)}/\kappa^{(i)})$ for $i = 1, \ldots, k$, uniquely specifies an element of $\mathrm{GT}(\lambda(D))$ provided D is large enough that the rows are still weakly decreasing.

This motivates the following. For signatures $\kappa^{(1)} \in \mathrm{Sig}_{r_1}, \ldots, \kappa^{(k)} \in \mathrm{Sig}_{r_k}$ and GT patterns $T_i^P \in \mathrm{GT}_P(\kappa^{(i)}/()), T_i^Q \in \mathrm{GT}_{Q,N-s_i}(\lambda^{(i)}/\kappa^{(i)})$ for $i = 1, \ldots, k$, define (for all D large enough that this makes sense) $BP_D(T_1^P, \ldots, T_k^P, T_1^Q, \ldots, T_k^Q) \in \mathrm{GT}_P(\lambda(D)/())$ to

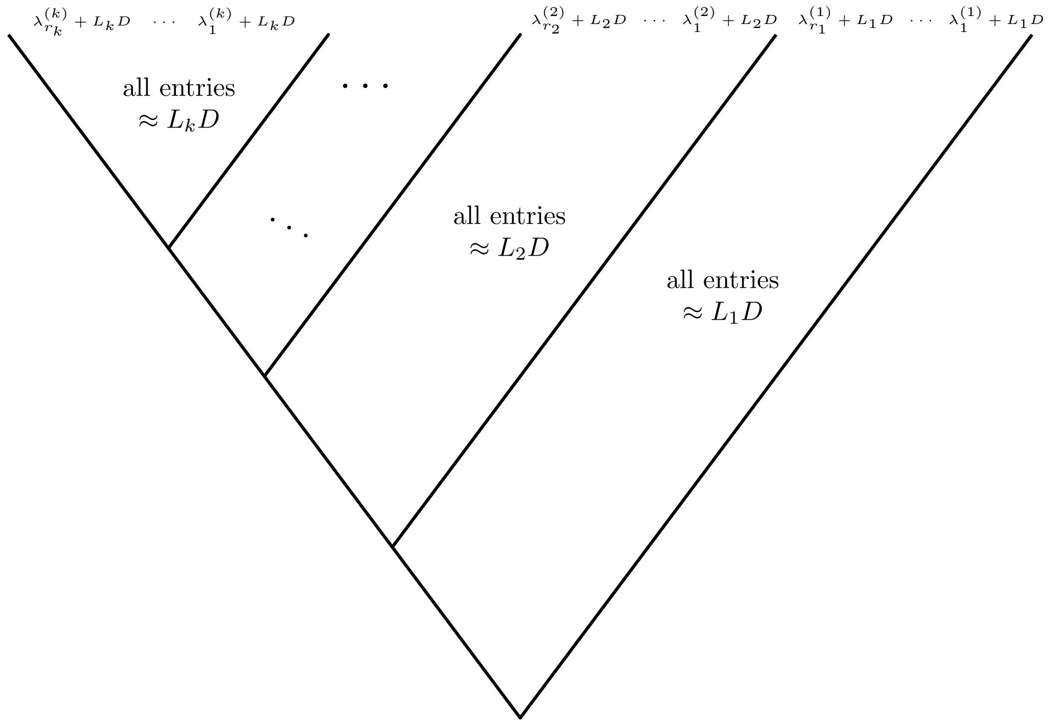


Figure 3-1: The form of a Gelfand-Tsetlin pattern T with $wt(T)$ close to $\lambda(D)$ for large D.

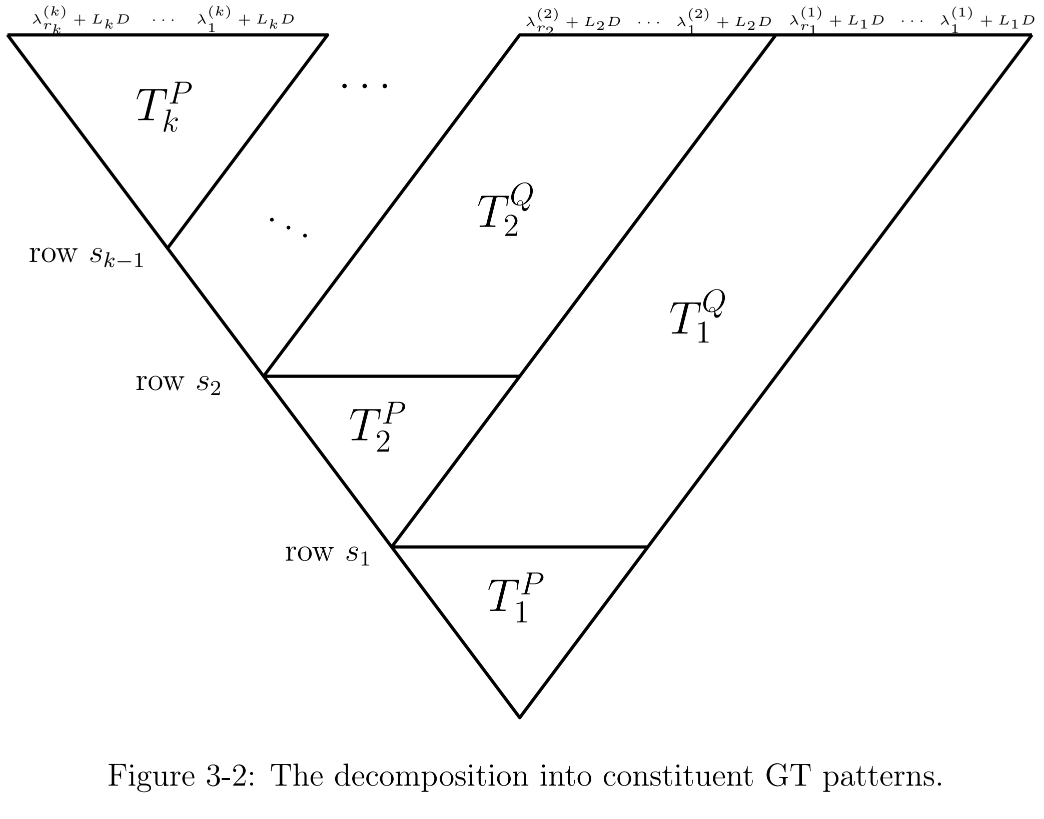


Figure 3-2: The decomposition into constituent GT patterns.

be the GT pattern of top row $\lambda(D)$ which decomposes into $T_1^P, \dots, T_k^P, T_1^Q, \dots, T_k^Q$ as above. For sufficiently large D, all GT patterns contributing to $\mathbf{x}^{\mathbf{d}}$ will be of the form $BP_D(T_1^P, \dots, T_k^P, T_1^Q, \dots, T_k^Q)$ for some $T_1^P, \dots, T_k^Q$, and furthermore only finitely many such patterns will contribute to $\mathbf{x}^{\mathbf{d}}$ (and this finite number does not grow with D). Hence we may focus on describing the GT patterns $BP_D(T_1^P, \dots, T_k^P, T_1^Q, \dots, T_k^Q)$.

Given any $T \in \mathrm{GT}_{Q,s}(\mu/\nu)$ given by $\nu = \lambda^{(1)} \prec_Q \lambda^{(2)} \prec_Q \cdots \prec_Q \lambda^{(s)} = \mu$, define $\tilde{T} \in \mathrm{GT}_{Q,s}(-\nu/-\mu)$ by $-\mu \prec -\lambda^{(s-1)} \prec \cdots \prec -\lambda^{(1)} = -\nu$. Similarly, given $T \in \mathrm{GT}_P(\kappa/())$ defined by $() \prec_P \lambda^{(1)} \prec_P \cdots \prec_P \lambda^{(\mathrm{len}(\kappa))} = \kappa$, let $\tilde{T} \in \mathrm{GT}_P(-\kappa/())$ be the GT pattern with i^{th} row $-\lambda^{(i)}$. We claim that

$$\lim_{D\to\infty} \psi(BP_D(T_1^P, \dots, T_k^P, T_1^Q, \dots, T_k^Q)) = \prod_{i=1}^{k} \psi(\tilde{T}_i^P)\varphi(\tilde{T}_i^Q). \tag{3.2.2}$$

In the GT pattern $BP_D(T_1^P, \dots, T_k^P, T_1^Q, \dots, T_k^Q)$, we may view each entry as coming from one of the constituent GT patterns T_i^P or T_i^Q. As $D \to \infty$, the difference between any two entries in a given row of $BP_D(T_1^P, \dots, T_k^P, T_1^Q, \dots, T_k^Q)$ which come from the same constituent GT pattern remains constant, while the difference between any two entries which come from different constituent GT patterns goes to infinity.

Recall from (2.2.6) that the P branching coefficient $\psi(BP_D(T_1^P, \dots, T_k^P, T_1^Q, \dots, T_k^Q))$ is a product of factors

$$\frac{f(t^{j-i}q^{\mu_i-\mu_j})}{f(t^{j-i}q^{\lambda_i-\mu_j})}, \tag{3.2.3}$$

and

$$\frac{f(t^{j-i}q^{\lambda_i-\lambda_{j+1}})}{f(t^{j-i}q^{\mu_i-\lambda_{j+1}})}. \tag{3.2.4}$$

Notice that when for example μ_i and μ_j come from different constituent GT patterns of the GT pattern $BP_D(T_1^P, \dots, T_k^P, T_1^Q, \dots, T_k^Q)$, $q^{\mu_i-\mu_j} \to 0$ as $D \to \infty$ since $|q| < 1$, and hence $f(t^{j-i}q^{\mu_i-\mu_j}) \to f(0) = 1$. Similarly the other three factors $f(t^{j-i}q^{\lambda_i-\lambda_{j+1}})$, $f(t^{j-i}q^{\lambda_i-\mu_j})$, $f(t^{j-i}q^{\mu_i-\lambda_{j+1}})$ converge to 1. Because the number of these factors in (2.2.6) is finite independent of D, this implies that

$$\lim_{D\to\infty} \psi(BP_D(T_1^P, \dots, T_k^P, T_1^Q, \dots, T_k^Q))$$

is equal the product of those $f(\cdots)^{\pm 1}$ factors corresponding to pairs of entries coming from the same constituent GT pattern.

First let us consider the constituent GT patterns T_i^P. Because $f(t^{j-i}q^{\mu_i-\mu_j})$ depends only on the differences $j-i$ and $\mu_i-\mu_j$ but is independent of overall translation of the indices or the entries (and similarly for the other three f terms), we see that the product of $f(\cdots)^{\pm 1}$ terms in $\psi(BP_D(T_0^P,\ldots,T_k^P,T_1^Q,\ldots,T_k^Q))$ corresponding to entries from a given constituent GT pattern T_i^P is exactly $\psi(T_i^P)$. By the symmetry of (2.2.6) this is equal to $\psi(\tilde{T}_i^P)$, cf. Lemma 2.2.2.

Now consider a constituent GT pattern T_i^Q. It follows from (2.2.7) that the product of factors (3.2.3) and (3.2.4) corresponding to pairs of entries in T_i^Q is exactly $\varphi(\tilde{T}_i^Q)$, proving (3.2.2).

It is an easy check from our decomposition into constituent GT patterns that

$$\frac{\mathbf{x}^{wt(BP_D(T_1^P,\ldots,T_k^P,T_1^Q,\ldots,T_k^Q))}}{\mathbf{x}^{\hat{\lambda}(D)}}=\prod_{i=1}^{k}(x_{s_{i-1}+1}^{wt(T_i^P)_1}\cdots x_{s_i}^{wt(T_i^P)_{r_i}})(x_{s_i+1}^{wt(T_i^Q)_1}\cdots x_N^{wt(T_i^Q)_{N-s_i}}). \quad (3.2.5)$$

Combining (3.2.2), (3.2.5), and Lemma 2.2.2, and summing over the $\kappa^{(i)}$, yields

$$\begin{aligned}
&\lim_{D\to\infty}\frac{P_{\lambda(D)/()}(x_1,\ldots,x_N)}{\mathbf{x}^{\hat{\lambda}(D)}}[\mathbf{x}^{\mathbf{d}}]\\
&=\sum_{\kappa^{(i)}\in\mathrm{Sig}_{r_i},i=1,\ldots,k}\prod_{i=1}^{k}P_{\kappa^{(i)}/()}(x_{s_{i-1}+1},\ldots,x_{s_i})\tilde{Q}_{-\kappa^{(i)}/-\lambda^{(i)}}(x_{s_i+1},\ldots,x_N)[\mathbf{x}^{\mathbf{d}}]\\
&=\prod_{i=1}^{k}P_{-\kappa^{(i)}/()}(x_{s_{i-1}+1}^{-1},\ldots,x_{s_i}^{-1})\tilde{Q}_{-\kappa^{(i)}/-\lambda^{(i)}}(x_{s_i+1},\ldots,x_N)[\mathbf{x}^{\mathbf{d}}],
\end{aligned}$$

where $(\cdot)[\mathbf{x}^{\mathbf{d}}]$ denotes the coefficient of the $\mathbf{x}^{\mathbf{d}}$ term of the Laurent polynomial $(\cdot)$. Finally, applying the Cauchy identity Lemma 2.2.3 to the RHS of the above completes the proof.

□

Remark 13. We have made no attempt to find the most general hypothesis on q and t under which the above result holds, as $q,t\in(-1,1)$ is the only range typically used in probabilistic applications. Some extra complications arise if q,t are such that some of the $f(\cdots)$ factors in the denominator may vanish, or if $|q|\geq 1$–as then the argument that $f(\cdots)$ factors involving pairs of entries from different constituent GT patterns go to 1 no longer holds. However, we believe that one may be able to prove the same result for more general values of q,t with some additional analysis of cancellation between $f(\cdots)$ factors.

Remark 14. For the Hall-Littlewood case $q = 0$ or the Schur case $q = t$, the convergence statement (3.2.2) is actually stabilization to equality for all sufficiently large D, and hence the coefficients of monomials in the statement of Theorem 3.2.1 also stabilize for large D. This is because the branching coefficients $\psi(T)$ are 'local' in these cases, meaning that they may be expressed as products over entries of the GT pattern rather than pairs of entries, so in particular entries of different constituent GT patterns do not interact. In the case $q = t$ this is particular clear, as $\psi(T) = 1$ for any valid GT pattern T.

Furthermore, at $q = t$ this stabilization is monotonic from below, i.e. for each $\mu \in \mathbb{Z}^N$, the coefficient of $\mathbf{x}^\mu$ in the LHS of Theorem 3.2.1 (which is an integer, as follows from the fact that the $\psi(T)$ are all 1) is increasing in D for all D such that $\lambda(D)$ is a signature.

Remark 15. An asymptotic factorization statement somewhat similar to Theorem 3.2.1 was proven for Jack functions in infinitely many variables by Okounkov-Olshanski [OO98, Thm. 4.1], though we do not believe the two are directly related as our polynomials are in only finitely many variables.

We now convert Theorem 3.2.1 into a statement about Macdonald polynomials specialized at real variables.

Definition 15. For any finite subset $S \subset \mathbb{Z}^N$, let $\mathrm{Proj}_S : \mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}] \to \mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$ be the $\mathbb{C}$-linear operator with $\mathrm{Proj}_S \, \mathbf{x}^{\mathbf{d}} = \mathbb{1}(\mathbf{d} \in S)\mathbf{x}^{\mathbf{d}}$.

Proposition 3.2.2. *Let $t, q \in (-1, 1)$, $\mathbf{a} = (a_1, \ldots, a_N)$ with $a_1 > \ldots > a_N > 0$ be real numbers, and $L_i, r_i, s_i, \lambda(D), \hat{\lambda}(D)$ be as in Theorem 3.2.1. Then*

$$\lim_{D \to \infty} \frac{P_{\lambda(D)}(\mathbf{a})}{\mathbf{a}^{\hat{\lambda}(D)}} = \prod_{i=1}^{k} P_{\lambda^{(i)}}(a_{s_{i-1}+1}, \ldots, a_{s_i}) \prod_{i=1}^{k-1} \Pi_{(q,t)}(a_{s_{i-1}+1}^{-1}, \ldots, a_{s_i}^{-1}; a_{s_i+1}, \ldots, a_N) \tag{3.2.6}$$

Proof. Let $R_{q,t}(\mathbf{a})$ be the RHS of (3.2.6). For any $\epsilon > 0$ we must find D_0 so that

$$\left| \frac{P_{\lambda(D)}(\mathbf{a})}{\mathbf{a}^{\hat{\lambda}(D)}} - R_{q,t}(\mathbf{a}) \right| < \epsilon$$

for all $D > D_0$. Below we will abuse notation and write Proj_S acting on a Laurent polynomial in the real numbers $a_1, \ldots, a_N$ to mean 'take the corresponding polynomial in formal variables x_i, apply Proj_S, then specialize $x_i = a_i$ for each i'. Then for any S, we write

$$\left|\frac{P_{\lambda(D)}(\mathbf{a})}{\mathbf{a}^{\hat{\lambda}(D)}} - R_{q,t}(\mathbf{a})\right| \leq \left|\mathrm{Proj}_S\left(\frac{P_{\lambda(D)}(\mathbf{a})}{\mathbf{a}^{\hat{\lambda}(D)}} - R_{q,t}(\mathbf{a})\right)\right| + \left|(\mathrm{Id}-\mathrm{Proj}_S)\frac{P_{\lambda(D)}(\mathbf{a})}{\mathbf{a}^{\hat{\lambda}(D)}}\right| + |(\mathrm{Id}-\mathrm{Proj}_S)R_{q,t}(\mathbf{a})|, \quad (3.2.7)$$

The first and third terms of the RHS are easy to bound. The first term of (3.2.7), is a finite sum of $|S|$ Laurent monomials in $a_1, \ldots, a_N$ with coefficients that converge to 0 as $D \to \infty$ by Theorem 3.2.1. Hence for any S we may choose D_0 so that

$$\left|\mathrm{Proj}_S\left(\frac{P_{\lambda(D)}(\mathbf{a})}{\mathbf{a}^{\hat{\lambda}(D)}} - R_{q,t}(\mathbf{a})\right)\right| < \frac{\epsilon}{3}$$

for all $D > D_0$. For the third term, $R_{q,t}(\mathbf{a})$ is a convergent power series in the variables $a_j/a_i, j > i$, so we may choose S sufficiently large that

$$|(\mathrm{Id}-\mathrm{Proj}_S)R_{q,t}(\mathbf{a})| < \frac{\epsilon}{3}. \quad (3.2.8)$$

The second term is slightly trickier. Recall from Lemma 2.2.1 that $f(u) := (tu;q)_\infty/(qu;q)_\infty$. In particular, since $q, t \in (-1,1)$, $f(u)$ is defined, continuous, and nonzero on $[-1,1]$, so

$$\left|\frac{f(u_1)}{f(u_2)}\right| \leq \frac{\sup_{u\in[-1,1]} f(u)}{\inf_{u\in[-1,1]} f(u)}. \quad (3.2.9)$$

Recall that $\psi(T)$ is a finite product of factors $\frac{f(u_1)}{f(u_2)}$ for u_1, u_2 products of powers of q, t (and in particular lying in $[-1,1]$). Hence there is a constant C depending only on N, which is an appropriate power of the RHS of (3.2.9), such that for any $\kappa \in \mathrm{Sig}_N$ and $T \in \mathrm{GT}_P(\kappa)$ we have

$$|\psi(T)| \leq C. \quad (3.2.10)$$

As in (3.2.8) we may choose S large enough so that

$$|(\mathrm{Id}-\mathrm{Proj}_S)R_{q,q}(\mathbf{a})| < \frac{\epsilon}{3C}. \quad (3.2.11)$$

By Remark 14 on the Schur case, for any $\mathbf{d} \in \mathbb{Z}^d$ one has that $(P_{\lambda(D)}(\mathbf{x};q,q)/\mathbf{x}^{\hat{\lambda}(D)})[\mathbf{x}^{\mathbf{d}}]$ is an increasing sequence (in D) of integers which stabilizes to $R_{q,q}(\mathbf{x})[\mathbf{x}^{\mathbf{d}}]$ for large D. It follows from this and the nonnegativity of the Laurent monomials $\mathbf{a}^{\mathbf{d}}$ that

$$(\mathrm{Id}-\mathrm{Proj}_S)\frac{P_{\lambda(D)}(a_1,\ldots,a_N;q,q)}{\mathbf{a}^{\hat{\lambda}(D)}} \le (\mathrm{Id}-\mathrm{Proj}_S)R_{q,q}(\mathbf{a}) \tag{3.2.12}$$

(we drop absolute values because all terms in the hidden summations on each side are positive). Putting together (3.2.10), (3.2.11) and (3.2.12) yields

$$\begin{aligned}\left|(\mathrm{Id}-\mathrm{Proj}_S)\frac{P_{\lambda(D)}(\mathbf{a};q,t)}{\mathbf{a}^{\hat{\lambda}(D)}}\right| &\le C\left|(\mathrm{Id}-\mathrm{Proj}_S)\frac{P_{\lambda(D)}(\mathbf{a};q,q)}{\mathbf{a}^{\hat{\lambda}(D)}}\right| \\ &\le C|(\mathrm{Id}-\mathrm{Proj}_S)R_{q,q}(\mathbf{a})| \\ &< \epsilon/3.\end{aligned}$$

This handles the second term of (3.2.7), so choosing S large enough that all three $\epsilon/3$ bounds are simultaneously satisfied completes the proof. $\square$

Theorem 3.2.1 may also be used to control asymptotics of the structure coefficients $c^{\nu}_{\lambda,\mu}(q,t)$. We first prove a simple lemma. Define the *dominance order* $\trianglelefteq$ on $\mathbb{Z}^n$ by $\mathbf{v}\trianglelefteq\mathbf{w}$ if $\sum_i v_i = \sum_i w_i$ and

$$\sum_{i=1}^{j} v_i \le \sum_{i=1}^{j} w_i \qquad \text{for } j=1,\ldots,n.$$

Lemma 3.2.3. *Let $\lambda(D),\mu(D),\kappa(D)\in\mathrm{Sig}_N$ be three sequences of signatures of the form in Theorem 3.2.1 (possibly for different L_i,r_i), such that $\hat{\lambda}(D)+\hat{\mu}(D)=\hat{\kappa}(D)$. Let $\tilde{\lambda}\in\mathbb{Z}^N$ be the tuple such that $\lambda(D)=\hat{\lambda}(D)+\tilde{\lambda}(D)$, and define $\tilde{\mu},\tilde{\kappa}$ similarly. Let S denote the (finite) interval $[\tilde{\kappa},\tilde{\lambda}+\tilde{\mu}]$ in $\mathbb{Z}^N$ with respect to the dominance order, or explicitly $S=\{\mathbf{d}\in\mathbb{Z}^N:\tilde{\kappa}\trianglelefteq\mathbf{d}\trianglelefteq\tilde{\lambda}+\tilde{\mu}\}$. Then*

1. *The set $\{\lim_{D\to\infty}\mathrm{Proj}_S\, P_{\hat{\kappa}(D)+\mathbf{d}}(\mathbf{x})/\mathbf{x}^{\hat{\kappa}(D)}:\mathbf{d}\in S\}$ is a basis for $\mathrm{Proj}_S\,\mathbb{C}[x_1^{\pm},\ldots,x_N^{\pm}]$. Here the limits of Laurent polynomials are in the sense of convergence of coefficients of each monomial.*

2. *The coefficient of $\lim_{D\to\infty}\mathrm{Proj}_S\, P_{\kappa(D)}/\mathbf{x}^{\hat{\kappa}(D)}$ in the decomposition of*

$$\lim_{D\to\infty}\mathrm{Proj}_S\,\frac{P_{\lambda(D)}(\mathbf{x})P_{\mu(D)}(\mathbf{x})}{\mathbf{x}^{\hat{\kappa}(D)}}$$

in the above basis is $\lim_{D\to\infty} c^{\kappa(D)}_{\lambda(D),\mu(D)}(q,t)$.

Proof. The first part follows since the Macdonald polynomials are homogeneous and may be written as $P_\lambda(\mathbf{x}) = \mathbf{x}^\lambda +$(lower order terms in the dominance order). The second part is then clear. $\square$

Definition 16. Let $\lambda, \nu \in \mathrm{Sig}_n, \mu \in \mathrm{Sig}_r$. Define $d^\nu_{\lambda,\mu}(q,t)$ by[2]

$$\tilde{Q}_{\nu/\lambda}(x_1, \ldots, x_r) = \sum_{\mu \in \mathrm{Sig}_r} d^\nu_{\lambda,\mu}(r) P_\mu(x_1, \ldots, x_r).$$

Proposition 3.2.4. *Let $n \leq m \leq N$ such that $n \leq N - m$, let $\lambda, \nu \in \mathrm{Sig}_n$, and let $q, t \in (-1, 1)$. Then*

$$\lim_{D\to\infty} c^{(2D[N-m-n],2D-\eta_n,\ldots,2D-\eta_1,D[m-n],\nu_1,\ldots,\nu_n)}_{(D[N-n],\lambda),(D[N-m],0[m])}(q,t) = d^\nu_{\lambda,\eta}(q,t) \tag{3.2.13}$$

and

$$\lim_{D\to\infty} c^{(2D[N-m]+\alpha,D[m-n]+\beta,\nu_1,\ldots,\nu_n)}_{(D[N-n],0[n]),(D[N-m],0[m])}(q,t) = 0 \tag{3.2.14}$$

for all $\alpha \in \mathrm{Sig}_{N-m}, \beta \in \mathrm{Sig}_{m-n}$ not as in (3.2.13).

Proof. Let $\lambda(D) = (D[N-n], \lambda)$, $\mu(D) = (D[N-m], 0[m])$, and $\kappa(D; \alpha, \beta, \nu) = (2D[N-m] + \alpha, D[m-n] + \beta, \nu_1, \ldots, \nu_n)$, which we will write as simply $\kappa(D)$ when the other signatures are clear from context. Fix α, β, ν, and let S be as in Lemma 3.2.3. By Theorem 3.2.1 we have

$$\begin{aligned}
&P_{\lambda(D)}(\mathbf{x})/\mathbf{x}^{\hat{\lambda}(D)} \to P_\lambda(x_{N-n+1}, \ldots, x_N)\Pi(x_1^{-1}, \ldots, x_{N-n}^{-1}; x_{N-n+1}, \ldots, x_N) \\
&P_{\mu(D)}(\mathbf{x})/\mathbf{x}^{\hat{\mu}(D)} \to \Pi(x_1^{-1}, \ldots, x_{N-m}^{-1}; x_{N-m+1}, \ldots, x_N).
\end{aligned}$$

Splitting the latter Cauchy kernel,

$$\begin{aligned}
&\lim_{D\to\infty} \mathrm{Proj}_S \frac{P_{\lambda(D)}(\mathbf{x})P_{\mu(D)}(\mathbf{x})}{\mathbf{x}^{\hat{\lambda}(D)+\hat{\mu}(D)}} \\
&\quad = \mathrm{Proj}_S\, \Pi(x_1^{-1}, \ldots, x_{N-n}^{-1}; x_{N-n+1}, \ldots, x_N)\Pi(x_1^{-1}, \ldots, x_{N-m}^{-1}; x_{N-m+1}, \ldots, x_{N-n}) \\
&\qquad\qquad \cdot \left(\Pi(x_1^{-1}, \ldots, x_{N-m}^{-1}; x_{N-n+1}, \ldots, x_N)P_\lambda(x_{N-n+1}, \ldots, x_N)\right)
\end{aligned}$$

[2]These are related by duality to the structure coefficients $c^\nu_{\lambda,\mu}$, see [Mac98a, Ch. VI], but we will not elaborate on this because we do not need it and due to our conventions with signatures it would be somewhat cumbersome to state.

Applying Lemma 2.2.3, the definition of the coefficients $d^{\nu}_{\lambda,\eta}$, and Lemma 2.2.2, one has

$$\begin{aligned}
&\Pi(x_1^{-1},\dots,x_{N-m}^{-1};x_{N-n+1},\dots,x_N)P_\lambda(x_{N-n+1},\dots,x_N)\\
&=\sum_{\nu\in\mathrm{Sig}_n} P_\nu(x_{N-n+1},\dots,x_N)\tilde{Q}_{\nu/\lambda}(x_1^{-1},\dots,x_{N-m}^{-1})\\
&=\sum_{\substack{\nu\in\mathrm{Sig}_n\\ \eta\in\mathrm{Sig}_{N-m}}} d^{\nu}_{\lambda,\eta}P_\nu(x_{N-n+1},\dots,x_N)P_\eta(x_1^{-1},\dots,x_{N-m}^{-1})\\
&=\sum_{\substack{\nu\in\mathrm{Sig}_n\\ \eta\in\mathrm{Sig}_{N-m}}} d^{\nu}_{\lambda,\eta}P_\nu(x_{N-n+1},\dots,x_N)P_{-\eta}(x_1,\dots,x_{N-m}).
\end{aligned}$$

By straightforward application of Theorem 3.2.1,

$$\begin{aligned}
\lim_{D\to\infty}\mathrm{Proj}_S\frac{P_{\kappa(D)}(\mathbf{x})}{\mathbf{x}^{\hat{\kappa}(D)}} &= \mathrm{Proj}_S\, P_\alpha(x_1,\dots,x_{N-m})P_\beta(x_{N-m+1},\dots,x_{N-n})P_\nu(x_{N-n+1},\dots,x_N)\\
&\cdot\Pi(x_1^{-1},\dots,x_{N-m}^{-1};x_{N-m+1},\dots,x_N)\Pi(x_{N-m+1}^{-1},\dots,x_{N-n}^{-1};x_{N-n+1},\dots,x_N)\\
&= \mathrm{Proj}_S\, P_\alpha(x_1,\dots,x_{N-m})P_\beta(x_{N-m+1},\dots,x_{N-n})P_\nu(x_{N-n+1},\dots,x_N)\\
&\cdot\Pi(x_1^{-1},\dots,x_{N-n}^{-1};x_{N-n+1},\dots,x_N)\Pi(x_1^{-1},\dots,x_{N-m}^{-1};x_{N-m+1},\dots,x_{N-n}) \quad (3.2.15)
\end{aligned}$$

We see that

$$\begin{aligned}
\lim_{D\to\infty}\mathrm{Proj}_S\frac{P_{\lambda(D)}(\mathbf{x})P_{\mu(D)}(\mathbf{x})}{\mathbf{x}^{\hat{\lambda}(D)+\hat{\mu}(D)}} &= \mathrm{Proj}_S\,\Pi(x_1^{-1},\dots,x_{N-n}^{-1};x_{N-n+1},\dots,x_N)\\
\cdot\Pi(x_1^{-1},\dots,x_{N-m}^{-1};x_{N-m+1},\dots,x_{N-n}) &\sum_{\substack{\nu\in\mathrm{Sig}_n\\ \eta\in\mathrm{Sig}_{N-m}}} d^{\nu}_{\lambda,\eta}P_\nu(x_{N-n+1},\dots,x_N)P_{-\eta}(x_1,\dots,x_{N-m})
\end{aligned}$$

is a finite (because S is finite) sum of terms of the form RHS(3.2.15) for those $\kappa(D;\alpha,\beta,\nu)$ for which $\alpha=-\eta,\beta=(0[m-n])$. Furthermore, the coefficients of these terms are $d^{\nu}_{\lambda,\eta}$. Applying Lemma 3.2.3 completes the proof. □

We now combine these results.

Proof of Proposition 3.1.1. Let $\lambda(D),\mu(D),\kappa(D;\alpha,\beta,\nu)$ be as in the previous proof, and $S\subset\mathrm{Sig}_{N-m}$ be finite. Since we have assumed q,t are such that the structure coefficients are nonnegative,

$$M_D^{Cauchy}(\nu) \geq \sum_{\eta \in S} c^{\kappa(D;-\eta,(0[m-n]),\nu)}_{\lambda(D),\mu(D)} \frac{P_{\kappa(D;-\eta,(0[m-n]),\nu)}(\mathbf{a})}{P_{\lambda(D)}(\mathbf{a})P_{\mu(D)}(\mathbf{a})} \tag{3.2.16}$$

by simply taking only finitely many of the terms in the sum used to define M_D^{Cauchy}. Note that $\hat{\kappa} = \hat{\lambda} + \hat{\mu}$, so we may divide numerator by $\mathbf{a}^{\hat{\kappa}}$ and denominator by $\mathbf{a}^{\hat{\lambda}} \cdot \mathbf{a}^{\hat{\mu}}$ to obtain the LHS of Proposition 3.2.2. Applying Proposition 3.2.4 and Proposition 3.2.2 to the structure coefficients and specialized Macdonald polynomials on the RHS of (3.2.16), respectively, we obtain

$$\begin{aligned}\lim_{D\to\infty} M_D^{Cauchy}(\nu) &\geq \sum_{\eta \in S} \lim_{D\to\infty} c^{\kappa(D;-\eta,(0[m-n]),\nu)}_{\lambda(D),\mu(D)} \frac{P_{\kappa(D;-\eta,(0[m-n]),\nu)}(\mathbf{a})/\mathbf{a}^{\hat{\kappa}}}{P_{\lambda(D)}(\mathbf{a})/\mathbf{a}^{\hat{\lambda}} \cdot P_{\mu(D)}(\mathbf{a})/\mathbf{a}^{\hat{\mu}}} \\ &= \sum_{\eta \in S} d^{\nu}_{\lambda,\eta} \frac{P_{-\eta}(a_1,\ldots,a_{N-m})P_\nu(a_{N-n+1},\ldots,a_N)}{P_\lambda(a_{N-n+1},\ldots,a_N)\Pi(a_1^{-1},\ldots,a_{N-m}^{-1};a_{N-n+1},\ldots,a_N)}.\end{aligned}$$

Because the bound holds for any finite S, we may replace S by Sig_{N-m} in the above, and also replace $P_{-\eta}(a_1,\ldots,a_{N-m})$ by $P_\eta(a_1^{-1},\ldots,a_{N-m}^{-1})$ by Lemma 2.2.2. Then the above becomes

$$\begin{aligned}&\sum_{\eta \in \mathrm{Sig}_{N-m}} d^{\nu}_{\lambda,\eta} \frac{P_\eta(a_1^{-1},\ldots,a_{N-m}^{-1})P_\nu(a_{N-n+1},\ldots,a_N)}{P_\lambda(a_{N-n+1},\ldots,a_N)\Pi(a_1^{-1},\ldots,a_{N-m}^{-1};a_{N-n+1},\ldots,a_N)} \\ &\qquad = \frac{\tilde{Q}_{\nu/\lambda}(a_1^{-1},\ldots,a_{N-m}^{-1})P_\nu(a_{N-n+1},\ldots,a_N)}{P_\lambda(a_{N-n+1},\ldots,a_N)\Pi(a_1^{-1},\ldots,a_{N-m}^{-1};a_{N-n+1},\ldots,a_N)}\end{aligned}$$

by definition of the coefficients $d^{\nu}_{\lambda,\eta}$. Because

$$\sum_{\nu \in \mathrm{Sig}_n} \frac{\tilde{Q}_{\nu/\lambda}(a_1^{-1},\ldots,a_{N-m}^{-1})P_\nu(a_{N-n+1},\ldots,a_N)}{P_\lambda(a_{N-n+1},\ldots,a_N)\Pi(a_1^{-1},\ldots,a_{N-m}^{-1};a_{N-n+1},\ldots,a_N)} = 1$$

by Lemma 2.2.3, the inequalities

$$\lim_{D\to\infty} M_D^{Cauchy}(\nu) \geq \frac{\tilde{Q}_{\nu/\lambda}(a_1^{-1},\ldots,a_{N-m}^{-1})P_\nu(a_{N-n+1},\ldots,a_N)}{P_\lambda(a_{N-n+1},\ldots,a_N)\Pi(a_1^{-1},\ldots,a_{N-m}^{-1};a_{N-n+1},\ldots,a_N)}$$

must all be equalities, completing the proof. □

Proof of Proposition 3.1.2. The proof is very similar to that of Proposition 3.1.1, so we will go through the argument but neglect some of the analytic details which are the same as before. Proceeding as in Proposition 3.2.4, we compute the limiting structure coefficients

$$\lim_{D\to\infty} c^{(D[k]+\eta,\mu)}_{\lambda,(D[k],0[n-k])}$$

for $\mu \in \mathrm{Sig}_{N-k}$. Define $\tilde{c}^{\lambda}_{\mu,\eta}$ by

$$P_{\lambda/\mu}(x_1,\ldots,x_k) = \sum_{\eta\in\mathrm{Sig}_k} \tilde{c}^{\lambda}_{\mu,\eta} P_\eta(x_1,\ldots,x_k)$$

(these are the dual Littlewood-Richardson coefficients and are related to the usual $c^{\lambda}_{\mu,\eta}$, see [Mac98a, Ch. VI], though we will not need this). We have

$$\begin{aligned}P_\lambda(\mathbf{x})\frac{P_{\mu(D)}(\mathbf{x})}{\mathbf{x}^{(D[k],0[n-k])}} &\to P_\lambda(\mathbf{x})\Pi(x_1^{-1},\ldots,x_k^{-1};x_{k+1},\ldots,x_n)\\ &= \sum_{\mu\in\mathrm{Sig}_{n-k}} \tilde{c}^{\lambda}_{\mu,\eta}P_\eta(x_1,\ldots,x_)P_\mu(x_{k+1},\ldots,x_n)\Pi(x_1^{-1},\ldots,x_k^{-1};x_{k+1},\ldots,x_n).\end{aligned}$$

Likewise we have

$$\frac{P_{(D[k]-\mu,\mu)}(\mathbf{x})}{\mathbf{x}^{(D[k],0[n-k])}} \to P_\eta(x_1,\ldots,x_k)P_\mu(x_{k+1},\ldots,x_n)\Pi(x_1^{-1},\ldots,x_k^{-1};x_{k+1},\ldots,x_n).$$

Hence by the same argument as before,

$$\lim_{D\to\infty} c^{(D[k]+\eta,\mu)}_{\lambda,(D[k],0[n-k])} = \tilde{c}^{\lambda}_{\mu,\eta}.$$

We thus have, again using positivity of all of the structure coefficients and specialized Macdonald polynomials, that

$$\begin{aligned}\lim_{D\to\infty} M_D^{branch}(\mu) &\geq \sum_{\eta\in\mathrm{Sig}_k} \lim_{D\to\infty} c^{(D[k]+\eta,\mu)}_{\lambda,(D[k],0[n-k])} \frac{P_{(D[k]+\eta,\mu)}(\mathbf{a})/\mathbf{a}^{(D[k],0[N-k])}}{P_\lambda(\mathbf{a})P_{(D[k],0[n-k])}(\mathbf{a})/\mathbf{a}^{(D[k],0[N-k])}}\\ &= \sum_{\eta\in\mathrm{Sig}_k} \tilde{c}^{\lambda}_{\mu,\eta}\frac{P_\eta(a_1,\ldots,a_k)P_\mu(a_{k+1},\ldots,a_n)}{P_\lambda(a_1,\ldots,a_n)}\\ &= \frac{P_{\lambda/\mu}(a_1,\ldots,a_k)P_\mu(a_{k+1},\ldots,a_n)}{P_\lambda(a_1,\ldots,a_n)}.\end{aligned}$$

These sum to 1 by the branching rule, so the inequalities are equalities, completing the proof. □

3.3 Appendix: Relations to the Archimedean case and alternate proof of Proposition 3.1.1

This section is not logically necessary for the rest of this thesis, but is a comment on the relations between Theorem 1.2.1 and results on singular values of corners and products in the real, complex and quaternion cases, through the lens of symmetric function theory, which was alluded to in the Introduction. We will first informally state these results in more detail than in the Introduction in order to highlight the parallel, and give references to more complete treatments. We will then give an alternate proof of Proposition 3.1.1 which is simpler, but valid only under additional assumptions which do *not* cover the Hall-Littlewood case $q = 0$. This was the first proof we found, but we were unable to justify the $q \to 0$ limit and hence resorted to the stronger results proven in the previous section. However, the proof below has the advantage that it survives the limit to the real/complex/quaternion cases which we are about to describe, and hence could be used to adapt the convolution-of-projectors method of Theorem 1.2.1 to prove the analogous result in this setting. At the end of the appendix we will outline how this could be carried out.

Fix a parameter $\beta > 0$ and let $q = e^{-\epsilon}$, $t = q^{\beta/2}$. In all cases below, we assume that the integers n, m, N, k satisfy the same constraints as in Theorem 1.2.1. Below we give an informal statement of the analogue of Theorem 1.2.1 in the real, complex and quaternion setting.

1. Define the random signature $\lambda(\epsilon)$ by

$$\Pr(\lambda(\epsilon) = \lambda) = \frac{P_\lambda(1, t, \ldots, t^{n-1}; q, t)Q_\lambda(t^{m-n+1}, \ldots, t^{N-n}; q, t)}{\Pi_{(q,t)}(1, t, \ldots, t^{n-1}; t^{m-n+1}, \ldots, t^{N-n})}$$

   for any $\lambda \in \mathrm{Sig}_n^+$, with q, t depending on ϵ as above. Then as $\epsilon \to 0$, the random real signature $\epsilon\lambda(\epsilon) = (\epsilon\lambda_1(\epsilon), \ldots, \epsilon\lambda_n(\epsilon))$ converges in distribution to some limiting random real signature $\lambda(0)$. When $\beta = 1, 2, 4$, $\lambda(0)$ has the same distribution as $(-\log(r_n), \ldots, -\log(r_1))$, where $r_1 \geq \cdots \geq r_n$ are the squared singular values of an $n \times m$ corner of a Haar-distributed element of $\mathcal{O}(n), \mathrm{U}(n)$ or $\mathrm{Sp}^*(n)$ respectively. This is due to Forrester-Rains [FR05], see also Borodin-Gorin [BG15, Thm. 2.8].

2. Fix a real signature ℓ of length n and define the nonrandom signature

$$\lambda(\epsilon) := (\lfloor \ell_1/\epsilon \rfloor, \dots, \lfloor \ell_n/\epsilon \rfloor) \in \mathrm{Sig}_n .$$

Define the random signature $\nu(\epsilon)$ by

$$\Pr(\nu(\epsilon) = \nu) = \frac{\tilde{Q}_{\nu/\lambda(\epsilon)}(1, \dots, t^{-(k-1)}; q, t) P_\nu(t^{N-n}, \dots, t^{N-1}; q, t)}{P_\lambda(t^{N-n}, \dots, t^{N-1}; q, t)\Pi_{(q,t)}(1, \dots, t^{-(k-1)}; t^{N-n}, \dots, t^{N-1})}$$

for any $\nu \in \mathrm{Sig}_n$. Then as $\epsilon \to 0$, $\nu(\epsilon)$ converges to a random real signature $\nu(0)$. Suppose $\beta = 1, 2, 4$ and $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively, and $A_{col} \in M_{n \times (N-k)}(\mathbb{F})$ is the first $N - k$ columns of $A \in M_{n \times N}(\mathbb{F})$ with fixed singular values $e^{-\ell} := (e^{-\ell_1}, \dots, e^{-\ell_n})$ and distribution invariant under the orthogonal, unitary or symplectic groups acting on the right and left. Then the distribution of the negative logarithms of the squared singular values of A_{col} is given by $\nu(0)$. The statement for (4.3.1) is exactly analogous. We could not locate these exact statements in the literature but essentially equivalent ones appear in Borodin-Gorin [BG15] and Sun [Sun16] when considering the Jacobi corners process.

3. Fix real signatures r, ℓ of length n and define nonrandom integer signatures $\lambda(\epsilon)$ as above and $\rho(\epsilon)$ similarly with r in place of ℓ. Then as $\epsilon \to 0$, $\epsilon \cdot (\rho(\epsilon) \boxtimes_{(1,\dots,t^{n-1})} \lambda(\epsilon))$ (where we abuse notation and use $\boxtimes$ to refer to the convolution operation with Macdonald polynomials instead of Hall-Littlewood) converges to a random real signature s. When $\beta = 1, 2, 4$, e^{-s} gives the distribution of singular values of AB where A, B are bi-invariant under the orthogonal, unitary or symplectic group and have fixed singular values e^{-r} and $e^{-\ell}$. See Gorin-Marcus [GM20, Prop. 2.2].

More general background on these limits may be found in Ahn [Ahn22b], Borodin-Gorin [BG15], Gorin-Marcus [GM20], and Sun [Sun16].

Remark 16. The explicit formulas for the above distributions are uniform expressions in terms of β, and the distributions for general $\beta \in [0, \infty)$ are referred to as β-*ensembles*. β is then seen as an inverse temperature parameter, and the zero-temperature limit $\beta \to \infty$ has in particular been studied, both because it provides tractable though accurate approximations to $\beta = 1, 2, 4$, and because it exhibits asymptotic behaviors interesting in their own right. In particular, the product convolution and corners operation–the analogues of Theorem 1.2.1 Parts 3 and 2 respectively–become deterministic in this limit and are controlled by certain orthogonal polynomials. See Gorin-Marcus [GM20] and Gorin-Kleptsyn [GK22] for a discussion of the eigenvalue (as opposed to singular value) case, and Borodin-Gorin [BG15, Cor. 5.4] for the deterministic $\beta \to \infty$ limit of Jacobi corners; we are not aware of anywhere the $\beta \to \infty$ limits of general corners and products (the analogues of Parts 2, 3 of Theorem 1.2.1) are worked out explicitly in the literature. In our setting, viewing the measures and operations of Theorem 1.2.1 for arbitrary $t \in (0,1)$ not necessarily a prime power is exactly analogous to this extrapolation to general β.

We observe the exact same freezing to a deterministic operation in the p-adic case of products and corners in the limit $p \to \infty$, i.e. $t \to 0$. It is interesting to note that while the $\beta \to \infty$ limit requires extrapolation away from the usual matrix models, the $t \to 0$ limit does not because one can find arbitrarily large primes. In the corners case, the partition ν in the notation of Theorem 1.2.1 concentrates around λ, and the partition μ concentrates around $(\lambda_{d+1}, \ldots, \lambda_n)$. In the product case, ν concentrates around $(\lambda_1 + \mu_1, \ldots, \lambda_n + \mu_n)$. These facts may be easily verified using the explicit formulas for Hall-Littlewood polynomials in Section 2.2.4, and may also be seen heuristically directly from the matrix models without any formulas.

Below we prove Proposition 3.1.1 under the additional assumptions that $\mathbf{a} = (1, t, \ldots, t^{N-1})$ and $q, t \in (0,1)$. We remark that Proposition 3.1.2 may be proven by similar label-variable duality manipulations under the restricted hypotheses as above; the modifications to the proof below are not difficult.

Proof. For the remainder of the proof, we will denote $\boxtimes_{(1,\ldots,t^{N-1})}$ by $\boxtimes_t$ and use Supp for the support of a measure. Let $\lambda(D) = (D[N-n], \lambda)$ and $\mu(D) = (D[N-m], 0[m])$. Recall that

$$M_D^{Cauchy}(\nu) := \sum_{\substack{\kappa \in \mathrm{Sig}_N \\ \kappa_{N-n+i} = \nu_i \text{ for all } i=1,\ldots,n}} c^{\kappa}_{\lambda(D),\mu(D)}(q,t) \frac{P_\kappa(t^{N-1}, \ldots, 1)}{P_{\lambda(D)}(t^{N-1}, \ldots, 1) P_{\mu(D)}(t^{N-1}, \ldots, 1)}$$

and we wish to show

$$M_D^{Cauchy}(\nu) \to \frac{P_\nu(t^{n-1}, \ldots, 1) \tilde{Q}_{\nu/\lambda}(t^{N-n}, \ldots, t^{m-n+1})}{P_\lambda(t^{n-1}, \ldots, 1) \Pi(t^{n-1}, \ldots, 1; t^{N-n}, \ldots, t^{m-n+1})} \tag{3.3.1}$$

(note that we have written the measure in a different form from Proposition 3.1.1 by using homogeneity to rearrange powers of t).

Denote the limiting measure of (3.3.1) by $\mathcal{M}$. The proof is by a kind of moments method which consists of showing the convergence of expectations of observables

$$\mathbb{E}_{\nu \sim M_D^{Cauchy}}[P_\alpha(q^{\nu_1}t^{n-1}, \ldots, q^{\nu_n})] \to \mathbb{E}_{\nu \sim \mathcal{M}}[P_\alpha(q^{\nu_1}t^{n-1}, \ldots, q^{\nu_n})] \tag{3.3.2}$$

as $D \to \infty$ for each $\alpha \in \mathrm{Sig}_n^+$, followed by an argument that these 'moments' are sufficient to give convergence of measures. We rely on the nontrivial label-variable duality satisfied by these observables, see [Mac98a, Section 6]:

$$\frac{P_\nu(q^{\alpha_1}t^{n-1}, \ldots, q^{\alpha_n})}{P_\nu(t^{n-1}, \ldots, 1)} = \frac{P_\alpha(q^{\nu_1}t^{n-1}, \ldots, q^{\nu_n})}{P_\alpha(t^{n-1}, \ldots, 1)}. \tag{3.3.3}$$

Such a strategy is used to prove similar statements in [GM20, Section 4].

We first show

$$\left|\mathbb{E}_{\nu \sim M_D^{Cauchy}}[P_\alpha(q^{\nu_1}t^{n-1}, \ldots, q^{\nu_n})] - \mathbb{E}_{\kappa \sim \lambda(D) \boxtimes_t \mu(D)}[P_{(\alpha, 0[N-n])}(q^{\kappa_1}t^{N-1}, \ldots, q^{\kappa_N})]\right| \to 0 \tag{3.3.4}$$

as $D \to \infty$. To show (3.3.4) it suffices to show that there exist constants $C(\alpha, D)$ independent of $\kappa \in \mathrm{Supp}(\lambda(D) \boxtimes_t \mu(D))$ such that $C(\alpha, D) \to 0$ as $D \to \infty$ and

$$|P_{(\alpha, 0[N-n])}(q^{\kappa_1}t^{N-1}, \ldots, q^{\kappa_N}) - P_\alpha(q^{\nu_1}t^{n-1}, \ldots, q^{\nu_n})| < C(\alpha, D). \tag{3.3.5}$$

where ν is defined by $\nu_i = \kappa_{N-n+i}$. This suffices because the support $\mathrm{Supp}(\lambda(D) \boxtimes_t \mu(D))$ of this measure contains only κ for which $\kappa \supset \lambda(D)$ by basic properties of the structure coefficients, hence

$$\left|\mathbb{E}_{\nu \sim M_D^{Cauchy}}[P_\alpha(q^{\nu_1}t^{n-1}, \ldots, q^{\nu_n})] - \mathbb{E}_{\kappa \sim \lambda(D) \boxtimes_t \mu(D)}[P_{(\alpha, 0[N-n])}(q^{\kappa_1}t^{N-1}, \ldots, q^{\kappa_N})]\right| < C(\alpha, D)$$

by (3.3.5) and linearity of expectation. So let us prove (3.3.5).

$P_{(\alpha, 0[N-n])}$ is a polynomial in N variables $q^{\kappa_1}t^{N-1}, \ldots, q^{\kappa_N}$, which we split into two collections of variables, the first $N-n$ and the last n. As $D \to \infty$, the first $N-n$ variables go to 0 because $\kappa_i \geq \lambda(D)_i = D$ for $i = 1, \ldots, N-n$, for any $\kappa \in \mathrm{Supp}(\lambda(D) \boxtimes_t \mu(D))$, hence

$$P_{(\alpha,0[N-n])}(q^{\kappa(D)_1}t^{N-1},\dots,q^{\kappa(D)_N}) \to P_{(\alpha,0[N-n])}(0[N-n],t^{n-1}q^{\kappa(D)_{N-n+1}},\dots,q^{\kappa(D)_N})$$
$$= P_\alpha(q^{\nu_1}t^{n-1},\dots,q^{\nu_n})$$

for any sequence $\kappa(D) \in \mathrm{Supp}(\lambda(D) \boxtimes_t \mu(D))$ with last n parts given by ν. The last n variables always lie in a compact interval $[0, q^{\lambda_n}]$ because $\kappa_i \geq \lambda_n$ for $i = N-n+1,\dots,N$ by interlacing, for any $\kappa \in \mathrm{Supp}(\lambda(D) \boxtimes_t \mu(D))$. Hence the above convergence is uniform over ν and κ, i.e. (3.3.5) holds.

Thus to show (3.3.2), it suffices to show

$$\mathbb{E}_{\kappa\sim\lambda(D)\boxtimes_t\mu(D)}[P_{(\alpha,0[N-n])}(q^{\kappa_1}t^{N-1},\dots,q^{\kappa_N})] \to \mathbb{E}_{\nu\sim\mathcal{M}}[P_\alpha(q^{\nu_1}t^{n-1},\dots,q^{\nu_n})]. \tag{3.3.6}$$

Now, using label-variable duality (3.3.3),

$$\begin{aligned}
&\mathbb{E}_{\kappa\sim\lambda(D)\boxtimes_t\mu(D)}[P_{(\alpha,0[N-n])}(q^{\kappa_1}t^{N-1},\dots,q^{\kappa_N})]\\
&= \mathbb{E}_{\kappa\sim\lambda(D)\boxtimes_t\mu(D)}\left[\frac{P_\kappa(q^{\alpha_1}t^{N-1},\dots,1)}{P_\kappa(t^{N-1},\dots,1)}P_{(\alpha,0[N-n])}(t^{N-1},\dots,1)\right]\\
&= \sum_{\kappa\in\mathrm{Sig}_N} c^\kappa_{\lambda(D),\mu(D)}(q,t)\frac{P_\kappa(t^{N-1},\dots,1)P_{(\alpha,0[N-n])}(t^{N-1},\dots,1)}{P_{\lambda(D)}(t^{N-1},\dots,1)P_{\mu(D)}(t^{N-1},\dots,1)}\frac{P_\kappa(q^{\alpha_1}t^{N-1},\dots,1)}{P_\kappa(t^{N-1},\dots,1)}\\
&= \frac{P_{(\alpha,0[N-n])}(t^{N-1},\dots,1)}{P_{\lambda(D)}(t^{N-1},\dots,1)P_{\mu(D)}(t^{N-1},\dots,1)}\sum_{\kappa\in\mathrm{Sig}_N} c^\kappa_{\lambda(D),\mu(D)}(q,t)P_\kappa(q^{\alpha_1}t^{N-1},\dots,1)\\
&= \frac{P_{(\alpha,0[N-n])}(t^{N-1},\dots,1)}{P_{\lambda(D)}(t^{N-1},\dots,1)P_{\mu(D)}(t^{N-1},\dots,1)}P_{\lambda(D)}(q^{\alpha_1}t^{N-1},\dots,1)P_{\mu(D)}(q^{\alpha_1}t^{N-1},\dots,1)\\
&= \frac{P_{(\alpha,D[N-n])}(q^Dt^{N-1},\dots,q^Dt^n,q^{\lambda_1}t^{n-1},\dots,q^{\lambda_n})P_{(\alpha,D[N-n])}(q^Dt^{N-1},\dots,q^Dt^m,t^{m-1},\dots,1)}{P_{(\alpha,D[N-n])}(t^{N-1},\dots,1)}.
\end{aligned}$$

As $D \to \infty$, the above clearly converges to

$$\frac{P_\alpha(q^{\lambda_1}t^{n-1},\dots,q^{\lambda_n})P_\alpha(t^{m-1},\dots,1)}{P_\alpha(t^{N-1},\dots,1)},$$

so we must show

$$\mathbb{E}_{\nu\sim\mathcal{M}}[P_\alpha(q^{\nu_1}t^{n-1},\dots,q^{\nu_n})] = \frac{P_\alpha(q^{\lambda_1}t^{n-1},\dots,q^{\lambda_n})P_\alpha(t^{m-1},\dots,1)}{P_\alpha(t^{N-1},\dots,1)}. \tag{3.3.7}$$

Again using label-variable duality, and the Cauchy identity Lemma 2.2.3, we have

$$\begin{aligned}
&\mathbb{E}_{\nu\sim\mathcal{M}}[P_\alpha(q^{\nu_1}t^{n-1},\dots,q^{\nu_n})]\\
&=\mathbb{E}_{\nu\sim\mathcal{M}}\left[\frac{P_\alpha(t^{n-1},\dots,1)P_\nu(q^{\alpha_1}t^{n-1},\dots,q^{\alpha_n})}{P_\nu(t^{n-1},\dots,1)}\right]\\
&=\frac{P_\alpha(t^{n-1},\dots,1)\sum_{\nu\in\mathrm{Sig}_n}P_\nu(q^{\alpha_1}t^{n-1},\dots,q^{\alpha_n})\tilde{Q}_{\nu/\lambda}(t^{N-n},\dots,t^{m-n+1})}{\Pi(t^{n-1},\dots,1;t^{N-n},\dots,t^{m-n+1})P_\lambda(t^{n-1},\dots,1)}\\
&=\frac{\Pi(q^{\alpha_1}t^{n-1},\dots,q^{\alpha_n};t^{N-n},\dots,t^{m-n+1})}{\Pi(t^{n-1},\dots,1;t^{N-n},\dots,t^{m-n+1})}\frac{P_\alpha(t^{n-1},\dots,1)P_\lambda(q^{\alpha_1}t^{n-1},\dots,q^{\alpha_n})}{P_\lambda(t^{n-1},\dots,1)}\\
&=\frac{\Pi(q^{\alpha_1}t^{n-1},\dots,q^{\alpha_n};t^{N-n},\dots,t^{m-n+1})}{\Pi(t^{n-1},\dots,1;t^{N-n},\dots,t^{m-n+1})}P_\alpha(q^{\lambda_1}t^{n-1},\dots,q^{\lambda_n}).
\end{aligned}$$

Hence (3.3.7) is equivalent to

$$\frac{\Pi(q^{\alpha_1}t^{n-1},\dots,q^{\alpha_n};t^{N-n},\dots,t^{m-n+1})}{\Pi(t^{n-1},\dots,1;t^{N-n},\dots,t^{m-n+1})}=\frac{P_\alpha(t^{m-1},\dots,1)}{P_\alpha(t^{N-1},\dots,1)}. \tag{3.3.8}$$

(3.3.8) follows by applying the explicit formula for principally specialized Macdonald polynomials, [Mac98a, (6.11')], to the numerator and denominator of the RHS, expanding the LHS into infinite products and noting that all but finitely many terms cancel, and comparing the resulting expressions.

We have proven convergence of 'moments', so let us upgrade this to convergence of measures. Consider the compact set

$$\mathrm{U}^n:=\{(u_1,\dots,u_n)\in\mathbb{R}^n:0\le u_1\le\cdots\le u_n\le q^{\lambda_n}\}.$$

Then we have a map $\phi:\mathrm{Sig}_n\to\mathrm{U}^n$ given by $\phi(\nu_1,\dots,\nu_n)=(q^{\nu_1},\dots,q^{\nu_n})$. Also,

$$f_\alpha(u_1,\dots,u_n):=\frac{P_\alpha(u_1t^{n-1},\dots,u_n)}{P_\alpha(t^{n-1},\dots,1)}$$

defines a function on U^n. The subalgebra of $\mathcal{C}(\mathrm{U}^n)$ generated by the functions f_α is just the set of finite linear combinations of f_α because products of Macdonald polynomials may be expanded as linear combinations of Macdonald polynomials. This algebra contains the constant functions ($f_{(0[n])}$ is constant) and separates points, so by the Stone-Weierstrass theorem it is dense in $\mathcal{C}(\mathrm{U}^n)$ with sup norm.

By hypothesis, the structure coefficients are nonnegative and hence M_D^{Cauchy} is indeed a probability measure for each D. To show weak convergence $M_D^{Cauchy} \to \mathcal{M}$, we must show for any $f \in \mathcal{C}(\mathrm{U}^n)$ that $\int_{\mathrm{U}^n} f d\phi_*(M_D^{Cauchy}) \to \int_{\mathrm{U}^n} f d\phi_*(\mathcal{M})$. By the above, there exists a linear combination g of f_αs such that $\sup_{u \in U^n} |f(u) - g(u)| < \epsilon/3$. Since $\mathcal{M}$ and M_D^{Cauchy} are probability measures it follows that $\int_{\mathrm{U}^n} |f - g| d\phi_*(\mathcal{M}) < \epsilon/3$ and similarly with $\mathcal{M}$ replaced by any M_D^{Cauchy}. By (3.3.2), we may choose D such that

$$\left| \int_{\mathrm{U}^n} g d\phi_*(M_D^{Cauchy}) - \int_{\mathrm{U}^n} g d\phi_*(\mathcal{M}) \right| < \epsilon/3.$$

Putting together the three inequalities yields

$$\left| \int_{\mathrm{U}^n} f d\phi_*(M_D^{Cauchy}) - \int_{\mathrm{U}^n} f d\phi_*(\mathcal{M}) \right| < \epsilon,$$

hence $\phi_*(M_D^{Cauchy})$ converges weakly to $\phi_*(\mathcal{M})$. Because both measures are supported on a discrete subset $\phi(\{\nu \in \mathrm{Sig}_n : \nu_n \geq \lambda_n\})$ of U^n, this implies $M_D^{Cauchy}(\nu) = \phi_*(M_D^{Cauchy})(\phi(\nu)) \to \phi_*(\mathcal{M})(\phi(\nu)) = \mathcal{M}(\nu)$ for each $\nu \in \mathrm{Sig}_n$, completing the proof. □

The proofs of Proposition 3.1.1 and Proposition 3.1.2 in Section 3.2 heavily used the discrete structure of the set of integer signatures, and we have no idea how they would be modified to the continuum limit to real signatures described earlier. However, we claim that the above proof could be modified with no substantial changes. Let us briefly outline why this is so.

Definition 17. Let $r = (r_1, \ldots, r_n) \in \mathrm{Sig}_n^{\mathbb{R}}$ have distinct parts, $\theta > 0$ a parameter, and $y_1, \ldots, y_n$ complex variables. Setting $\lambda(\epsilon) = \lfloor \epsilon^{-1}(r_1, \ldots, r_n) \rfloor$, we define the (type A) Heckman-Opdam hypergeometric function

$$\mathcal{F}_r(y_1, \ldots, y_n; \theta) := \lim_{\epsilon \to 0} \epsilon^{\theta \binom{n}{2}} P_\lambda(e^{\epsilon y_1}, \ldots, e^{\epsilon y_n}; q = e^{-\epsilon}, t = e^{-\theta \epsilon})$$

The dual Heckman-Opdam function may be obtained similarly by degenerating Q. Instead of defining the measures appearing in the singular value setting as limits of Macdonald measures, as we did earlier in this Appendix, one may instead first take the limit to Heckman-Opdam functions and then define measures in terms of these. When one takes the limit of (3.3.3) in the above regime, one obtains

$$\frac{\mathcal{F}_r(-\lambda_1-(n-1)\theta,-\lambda_2-(n-2)\theta,\dots,-\lambda_n;\theta)}{\mathcal{F}_r(-(n-1)\theta,\dots,0;\theta)}=\frac{J_\lambda(e^{-r_1},\dots,e^{-r_n};\theta)}{J_\lambda(1,\dots,1;\theta)}$$

where J_λ is the classical Jack polynomial. The same argument used to prove Proposition 3.1.1 above may be used after this limit, with the Macdonald polynomials replaced by Heckman-Opdam functions or Jack polynomials as appropriate given the above, and the sums replaced by integrals. This post-limit version of Proposition 3.1.1 may then be used to implement the convolution-of-projectors strategy we used in Section 3.1 to prove the analogue of Theorem 1.2.1 in the real/complex/quaternion setting. We refer to [GM20] for similar random matrix arguments utilizing label-variable duality and Jack/Heckman-Opdam functions.

Chapter 4

Branching graphs and infinite p-adic matrices

In Section 4.1 we prove formulas for principally specialized skew Hall-Littlewood polynomials. These form the main tool for the classification of $\partial\mathscr{G}_t$ in Section 4.2. In Section 4.3 we prove an augmented boundary result (Theorem 4.3.3) tailored to the p-adic random matrix situation, and use it to prove Theorem 1.7.2 and Theorem 1.7.3. Finally, in Section 4.5 we prove a result about Markov dynamics on $\partial\mathscr{G}_t$ which is motivated by the dynamics in Chapter 6 and Chapter 8.

4.1 Principally specialized skew Hall-Littlewood polynomials

In this section we prove Theorem 2.2.16 stated earlier, as well as extensions when the geometric progression is finite and the formulas are less simple in Proposition 4.1.2. Let us introduce a bare minimum of background on higher spin Hall-Littlewood polynomials $F_{\mu/\lambda}, G_{\nu/\lambda}$, which generalize the usual Hall-Littlewood polynomials P, Q by the addition of an extra parameter s. We omit their definition, which may be found in [Bor17, BP17], as we will only care about the case $s = 0$ when they reduce to slightly renormalized

Hall-Littlewood polynomials. When $s = 0$, for $\lambda, \nu \in \mathrm{Sig}_n^{\geq 0}, \mu \in \mathrm{Sig}_{n+k}^{\geq 0}$ one has

$$F_{\mu/\lambda}(x_1, \ldots, x_k)\big|_{s=0} = \prod_{i \geq 0} \frac{(t;t)_{m_i(\mu)}}{(t;t)_{m_i(\lambda)}} P_{\mu/\lambda}(x_1, \ldots, x_k) \tag{4.1.1}$$

and

$$G_{\nu/\lambda}(x_1, \ldots, x_k)\big|_{s=0} = \tilde{Q}_{\nu/\lambda}(x_1, \ldots, x_k) \tag{4.1.2}$$

by [Bor17, §8.1]. We recall from Chapter 2 that $\tilde{Q}$ is our modified version of the usual skew dual Hall-Littlewood polynomial where both signatures in the subscript are the same length; as the same is true for G, this facilitates comparison between them. Formulas for principally specialized skew F and G functions were shown in [Bor17], though we will state the version given later in [BP17]. We apologize to the reader for giving a formula for an object which we have not actually defined, but will immediately specialize to the Hall-Littlewood case, so we hope no confusion arises. We need the following notation.

Definition 18. The normalized terminating q-hypergeometric function is

$${}_{r+1}\bar{\phi}_r \left(\begin{matrix} t^{-n}; a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix} ; t, z \right) := \sum_{k=0}^{n} z^k \frac{(t^{-n};t)_k}{(t;t)_k} \prod_{i=1}^{r} (a_i;t)_k (b_i t^k; t)_{n-k} \tag{4.1.3}$$

for $n \in \mathbb{Z}_{\geq 0}$ and $|z|, |t| < 1$.

Proposition 4.1.1 ([BP17, Proposition 5.5.1])**.** *Let $J \in \mathbb{Z}_{\geq 1}, \lambda \in \mathrm{Sig}_n^{\geq 0}, \mu \in \mathrm{Sig}_{n+J}^{\geq 0}$. Then*

$$F_{\mu/\lambda}(u, tu, \ldots, t^{J-1}u) = \prod_{x \in \mathbb{Z}_{\geq 0}} w_u^{(J)}(i_1(x), j_1(x); i_2(x), j_2(x)), \tag{4.1.4}$$

where the product is over the unique collection of $n+J$ up-right paths on the semi-infinite horizontal strip of height 1 with paths entering from the bottom at positions $\lambda_i, 1 \leq i \leq n$, J paths entering from the left, and paths exiting from the top at positions $\mu_i, 1 \leq i \leq n+J$, see Figure 4-1. Here $i_1(x), j_1(x), i_2(x), j_2(x)$ are the number of paths on the south, west, north and east edge of the vertex at position x as in Figure 4-2, and the weights in the *product are given by*

$$w_u^{(J)}(i_1,j_1;i_2,j_2) := \delta_{i_1+j_1,i_2+j_2}\frac{(-1)^{i_1+j_2}t^{\frac{1}{2}i_1(i_1+2j_1-1)}s^{j_2-i_1}u^{i_1}(t;t)_{j_1}(us^{-1};t)_{j_1-i_2}}{(t;t)_{i_1}(t;t)_{j_2}(us;t)_{i_1+j_1}} \times {}_4\bar{\phi}_3\left(\begin{matrix}t^{-i_1};t^{-i_2},t^J su,tsu^{-1}\\ s^2,t^{1+j_1-i_2},t^{1+J-i_1-j_1}\end{matrix};t,t\right). \quad (4.1.5)$$

Similarly, for $\lambda,\nu \in \mathrm{Sig}_n^{\geq 0}$, $G_{\nu/\lambda}(u,tu,\ldots,t^{J-1}u)$ is given by the product of the same weights over the unique collection of n up-right paths on the same strip entering from the bottom at positions $\lambda_i, 1 \leq i \leq n$ and exiting from the top at positions $\nu_i, 1 \leq i \leq n$.

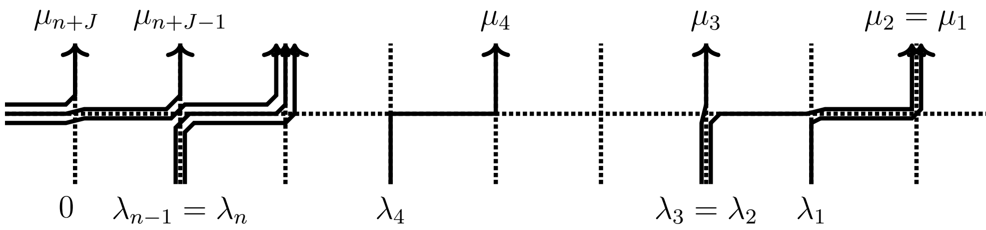


Figure 4-1: The unique path collection corresponding to the function $\mathbb{F}_{\mu/\lambda}(u,qu,\ldots,q^{J-1}u)$ with $J=3, n=6, \lambda=(7,6,6,4,1,1), \mu=(8,8,6,4,2,2,2,1,0)$.

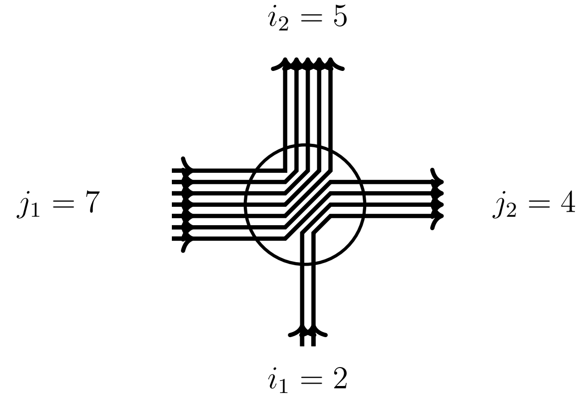


Figure 4-2: Illustration of the notation for edges, in the example $(i_1,j_1;i_2,j_2)=(2,7;5,4)$.

Remark 17. To avoid confusion with [Bor17, BP17], we note that the parameter which we call t for consistency with Hall-Littlewood notation is denoted by q in these references.

We now introduce some notation and specialize Proposition 4.1.1 to the Hall-Littlewood case $s=0$.

Definition 19. For $\lambda, \nu \in \mathrm{Sig}_n$, we define

$$n(\nu/\lambda) := \sum_{1 \le i < j \le n} \max(\nu_j - \lambda_i, 0) = \sum_{x \ge \lambda_n} \binom{\nu'_{x+1} - \lambda'_{x+1}}{2}.$$

We additionally allow the case when $\lambda, \nu \in \mathbb{Y}$; the first formula makes sense with the $\le n$ removed, while for the second we simply replace the sum over $x \ge \lambda_n$ by $x \ge 0$.

Note that $n(\nu/\lambda)$

1. is translation-invariant, $n((\nu + D[n])/(\lambda + D[n])) = n(\nu/\lambda)$, and
2. generalizes the standard definition of $n(\nu)$ in (2.2.53), namely when $\nu \in \mathrm{Sig}_n^{\ge 0}$ then $n(\nu) = n(\nu/(0[n]))$.

One may also view $n(\nu/\lambda)$ as quantifying the failure of ν and λ to interlace; it is 0 when ν, λ interlace, and increases by 1 when a part of λ is moved past a part of ν.

Proposition 4.1.2. *For $J \in \mathbb{Z}_{\ge 1}, \lambda \in \mathrm{Sig}_n^{\ge 0}, \mu \in \mathrm{Sig}_{n+J}^{\ge 0}$,*

$$P_{\mu/\lambda}(u, \ldots, ut^{J-1}) = (t;t)_J u^{|\mu|-|\lambda|} \prod_{x \ge 0} \frac{t^{m_x(\lambda)m_x(\mu) + \binom{\mu'_{x+1} - \lambda'_{x+1}}{2}}}{(t;t)_{m_x(\mu)}} \, {}_2\bar{\phi}_2 \left(\begin{matrix} t^{-m_x(\lambda)}; t^{-m_x(\mu)}, 0 \\ t^{1+\mu'_{x+1}-\lambda'_x}, t^{1+J-\mu'_x+\lambda'_{x+1}} \end{matrix} ; t, t \right). \tag{4.1.6}$$

For $\lambda, \nu \in \mathrm{Sig}_n$,

$$\tilde{Q}_{\nu/\lambda}(u, \ldots, ut^{J-1}) = u^{|\nu|-|\lambda|} t^{n(\nu/\lambda)} \prod_{x \in \mathbb{Z}} \frac{t^{m_x(\lambda)m_x(\nu)}}{(t;t)_{m_x(\lambda)}} \, {}_2\bar{\phi}_2 \left(\begin{matrix} t^{-m_x(\lambda)}; t^{-m_x(\nu)}, 0 \\ t^{1+\nu'_{x+1}-\lambda'_x}, t^{1+J-\nu'_x+\lambda'_{x+1}} \end{matrix} ; t, t \right). \tag{4.1.7}$$

Proof. We begin with (4.1.6). In this case we may apply Proposition 4.1.1 to compute

$$\mathrm{LHS}(4.1.6) = F_{\mu/\lambda}(u, \ldots, ut^{J-1})\big|_{s=0} \prod_{i \ge 0} \frac{(t;t)_{m_i(\lambda)}}{(t;t)_{m_i(\mu)}}. \tag{4.1.8}$$

When $s \to 0$, the factor $s^{j_2 - i_1}(us^{-1};t)_{j_1 - i_2}$ in (4.1.5) converges to $(-u)^{j_1 - i_2} t^{\binom{j_1 - i_2}{2}}$ (using that $j_2 - i_1 = j_1 - i_2$). The sign cancels with the sign in (4.1.5), and the power of u

combines with the u^{i_1} in (4.1.5) to give u^{j_2}, so (4.1.5) becomes

$$w_{t^n}^{(J)}(i_1, j_1; i_2, j_2) = \delta_{i_1+j_1, i_2+j_2} u^{j_2} \frac{t^{\frac{1}{2}i_1(i_1+2j_1-1)+\binom{j_1-i_2}{2}}(t;t)_{j_1}}{(t;t)_{i_1}(t;t)_{j_2}} \, {}_4\bar{\phi}_3 \left(\begin{matrix} t^{-i_1}; t^{-i_2}, 0, 0 \\ 0, t^{1+j_1-i_2}, t^{1+J-i_1-j_1} \end{matrix} ; t, t \right). \tag{4.1.9}$$

In the product (4.1.4) when the weights are specialized to (4.1.9), some of the factors simplify, as

$$\prod_{x \geq 0} \frac{(t;t)_{j_1}}{(t;t)_{i_1}(t;t)_{j_2}} = \frac{(t;t)_J}{\prod_{x \in \mathbb{Z}} (t;t)_{i_1(x)}} \tag{4.1.10}$$

because the $\frac{(t;t)_{j_1(x)}}{(t;t)_{j_2(x)}}$ factor cancels except for a $(t;t)_J$ from the paths incoming from the left. Hence

$$\prod_{x \geq 0} w_u^{(J)}(i_1, j_1; i_2, j_2) = (t;t)_J \prod_{x \geq 0} u^{j_2} \frac{t^{\frac{1}{2}i_1(i_1+2j_1-1)+\binom{j_1-i_2}{2}}}{(t;t)_{m_x(\lambda)}} \, {}_2\bar{\phi}_2 \left(\begin{matrix} t^{-i_1}; t^{-i_2}, 0 \\ t^{1+j_1-i_2}, t^{1+J-i_1-j_1} \end{matrix} ; t, t \right). \tag{4.1.11}$$

Using that $j_2 = i_1 + j_1 - i_2$ simplifies the exponent of t in (4.1.11) to

$$\frac{1}{2} i_1(i_1 + 2j_1 - 1) + \binom{j_1 - i_2}{2} = \binom{j_2}{2} + i_1 i_2.$$

To convert to the form in terms of partitions, we record the following translations between the i's and j's and the usual conjugate partition notation:

$$\begin{aligned} i_1(x) &= \lambda'_x - \lambda'_{x+1} = m_x(\lambda), \\ j_1(x) &= \mu'_x - \lambda'_x, \\ i_2(x) &= \mu'_x - \mu'_{x+1} = m_x(\mu), \\ j_2(x) &= \mu'_{x+1} - \lambda'_{x+1}. \end{aligned} \tag{4.1.12}$$

Translating (4.1.11) into partition notation and multiplying by the $\prod_{i \geq 0} \frac{(t;t)_{m_i(\lambda)}}{(t;t)_{m_i(\mu)}}$ factor of (4.1.8) yields (4.1.6).

To prove (4.1.7) we first note that both sides of (4.1.7) are translation-invariant, so without loss of generality we may take $\lambda, \nu \in \mathrm{Sig}_n^{>0}$. We then likewise appeal to Proposition 4.1.1 and either make the same argument as above or deduce it from (4.1.6) by considering $F_{(\nu, 0[J])/\lambda}(u, \ldots, ut^{J-1})$ for $\lambda, \nu \in \mathrm{Sig}_n^{>0}$ and using the relation between P

and Q polynomials. Since λ, ν are of the same length we have

$$\prod_{x \geq 0} t^{\binom{j_2(x)}{2}} = t^{n(\nu/\lambda)}$$

by (4.1.12). Finally, note that the product can be extended from $x \geq 0$ to $x \in \mathbb{Z}$, which in this translation-invariant setting is more aesthetically appealing. $\square$

Remark 18. While it follows from the branching rule that for nonnegative signatures μ, λ of appropriate lengths,

$$P_{(\mu,0)/(\lambda,0)}(u, \ldots, ut^{J-1}) = P_{\mu/\lambda}(u, \ldots, ut^{J-1}),$$

see (2.2.24), this relation is not readily apparent from (4.1.6). The only term on the RHS of (4.1.6) which a priori might differ after padding λ, μ with zeros is the $x = 0$ term of the product. It may be checked that this term is in fact unchanged by padding with zeros, but this is not immediately obvious from the formula as written.

Taking $J \to \infty$, we recover the theorem of [Kir98] which we stated in Chapter 2.

Proof of Theorem 2.2.16. For $n \geq \mathrm{len}(\lambda), n + J \geq \mathrm{len}(\mu)$, we may identify $\mu, \lambda \in \mathbb{Y}$ with nonnegative signatures $\mu(n + J) \in \mathrm{Sig}^{\geq 0}_{n+J}, \lambda(n) \in \mathrm{Sig}^{\geq 0}_{n}$ given by truncating. Hence to compute

$$P_{\mu/\lambda}(u, ut, \ldots)$$

it suffices to take $J \to \infty$ in (4.1.6). The polynomial $P_{\mu(n+J)/\lambda(n)}$ is independent of n for all n sufficiently large, see (2.2.24), so we will fix n and will abuse notation below and write λ for $\lambda(n)$. We first pull the $1/(t;t)_{m_x(\mu)}$ out of the product, and note that $m_0(\mu(n + J)) \to (t;t)_\infty$ as $J \to \infty$, cancelling the $(t;t)_J$ term of (4.1.6). We write the remaining term inside the product in (4.1.6) as

$$\left(t^{m_x(\lambda) m_x(\mu(n+J))} \, {}_2\bar{\phi}_2 \left(\begin{matrix} t^{-m_x(\lambda)}; t^{-m_x(\mu(n+J))}, 0 \\ t^{1+\mu(n+J)'_{x+1} - \lambda'_x}, t^{1+J-\mu(n+J)'_x + \lambda'_{x+1}} \end{matrix} ; t, t \right) \right). \tag{4.1.13}$$

To show (2.2.54) it suffices to show that for $x > 0$,

$$\lim_{J\to\infty} t^{m_x(\lambda)m_x(\mu(n+J))}\ {}_2\bar{\phi}_2\left(\begin{matrix} t^{-m_x(\lambda)}; t^{-m_x(\mu(n+J))}, 0 \\ t^{1+\mu(n+J)'_{x+1}-\lambda'_x}, t^{1+J-\mu(n+J)'_x+\lambda'_{x+1}} \end{matrix}; t, t\right) = (t^{1+\mu'_x-\lambda'_x}; t)_{m_x(\lambda)}. \tag{4.1.14}$$

and for $x = 0$,

$$\lim_{J\to\infty} t^{m_x(\lambda)m_x(\mu(n+J))}\ {}_2\bar{\phi}_2\left(\begin{matrix} t^{-m_x(\lambda)}; t^{-m_x(\mu(n+J))}, 0 \\ t^{1+\mu(n+J)'_{x+1}-\lambda'_x}, t^{1+J-\mu(n+J)'_x+\lambda'_{x+1}} \end{matrix}; t, t\right) = 1. \tag{4.1.15}$$

We begin with (2.2.54). Then $1 + J - \mu(n+J)'_x + \lambda'_{x+1} \to \infty$ and all other arguments in the q-hypergeometric function remain the same, so the LHS of (4.1.14) is

$${}_2\bar{\phi}_1\left(\begin{matrix} t^{-m_x(\lambda)}; t^{-m_x(\mu)} \\ t^{1+\mu'_{x+1}-\lambda'_x} \end{matrix}; t, t\right) = \sum_{\ell=0}^{m_x(\lambda)} t^{\ell}\frac{(t^{-m_x(\lambda)}; t)_\ell}{(t;t)_\ell}(t^{-m_x(\mu)}; t)_\ell (t^{1+\mu'_{x+1}-\lambda'_x+\ell}; t)_{m_x(\lambda)-\ell}. \tag{4.1.16}$$

To apply known identities, we reexpress the above in terms of the more standard terminating q-hypergeometric series ${}_2\phi_1$ as

$$\begin{aligned}(t^{1+\mu'_{x+1}-\lambda'_x}; t)_{m_x(\lambda)} &\sum_{\ell=0}^{m_x(\lambda)} t^{\ell}\frac{(t^{-m_x(\lambda)}; t)_\ell (t^{-m_x(\mu)}; t)_\ell}{(t;t)_\ell (t^{1+\mu'_{x+1}-\lambda'_x}; t)_\ell} \\ &= (t^{1+\mu'_{x+1}-\lambda'_x}; t)_{m_x(\lambda)}\ {}_2\phi_1\left(\begin{matrix} t^{-m_x(\lambda)}; t^{-m_x(\mu)} \\ t^{1+\mu'_{x+1}-\lambda'_x} \end{matrix}; t, t\right).\end{aligned} \tag{4.1.17}$$

By a special case of the q-Gauss identity, see e.g. [Koe98, Exercise 3.17],

$${}_2\phi_1\left(\begin{matrix} t^{-n}; b \\ c \end{matrix}; t, t\right) = \frac{(c/b; t)_n}{(c;t)_n} b^n. \tag{4.1.18}$$

Applying (4.1.18) with $b = t^{-m_x(\mu)}, c = t^{1+\mu'_{x+1}-\lambda'_x}$ to (4.1.17) yields

$${}_2\bar{\phi}_1\left(\begin{matrix} t^{-m_x(\lambda)}; t^{-m_x(\mu)} \\ t^{1+\mu'_{x+1}-\lambda'_x} \end{matrix}; t, t\right) = (t^{1+\mu'_x-\lambda'_x}; t)_{m_x(\lambda)} t^{-m_x(\lambda)m_x(\mu)}, \tag{4.1.19}$$

which shows (4.1.14).

We now show (4.1.15), so let $x = 0$. Then $\mu_x(n+J)' = n + J$, so the arguments of the q-hypergeometric function in (4.1.14) are independent of J except for $t^{-m_0(\mu(n+J))}$. In the sum

$$t^{m_x(\lambda)m_x(\mu(n+J))} {}_2\bar{\phi}_2 \left(\begin{matrix} t^{-m_x(\lambda)}; t^{-m_x(\mu(n+J))}, 0 \\ t^{1+\mu(n+J)'_{x+1}-\lambda'_x}, t^{1+J-\mu(n+J)'_x+\lambda'_{x+1}} \end{matrix} ; t, t \right) = t^{m_x(\lambda)m_x(\mu(n+J))}$$
$$\times \sum_{k=0}^{m_x(\lambda)} t^k \frac{(t^{-m_x(\lambda)}; t)_k}{(t;t)_k} (t^{-m_x(\mu(n+J))}; t)_k (t^{k+1+\mu(n+J)'_{x+1}-\lambda'_x}; t)_{m_x(\lambda)-k} (t^{k+1-n+\lambda'_{x+1}}; t)_{m_x(\lambda)-k},$$

the dominant term as $J \to \infty$ is the $k = m_x(\lambda)$ term, and its limit when normalized by $t^{m_x(\lambda)m_x(\mu(n+J))}$ is 1. This shows (4.1.15).

The proof of (2.2.55) using (4.1.7) is exactly analogous except that only (4.1.14) is needed because there are only n paths. □

4.2 The t-deformed Gelfand-Tsetlin graph and its boundary

Let $t \in (0, 1)$ for the remainder of the section. In this section we introduce the Hall-Littlewood Gelfand-Tsetlin graph and the notion of its boundary, the set of extreme coherent systems. The main result stated earlier, Theorem 1.7.1, is that the boundary is naturally in bijection with the set Sig_∞ of infinite signatures. We will break it into three parts: Proposition 4.2.2 gives an explicit coherent system of measures $(M_n^\mu)_{n\geq1}$ for each $\mu \in \mathrm{Sig}_\infty$, Proposition 4.2.7 tells that every extreme coherent system must be one of these, and Proposition 4.2.10 tells that each system $(M_n^\mu)_{n\geq1}$ is extreme.

The general structure of the proof of Theorem 1.7.1, via the so-called Vershik-Kerov ergodic method, is similar to e.g. [Ols16, Theorem 6.2] or [Cue18]. A good general reference for (unweighted) graded graphs, with references to research articles, is [BO17, Chapter 7].

4.2.1 Classifying the boundary.

Definition 20. $\mathscr{G}_t$ is the weighted, graded graph with vertices

$$\bigsqcup_{n\geq 1} \mathrm{Sig}_n$$

partitioned into *levels* indexed by $\mathbb{Z}_{\geq 1}$. The only edges of $\mathscr{G}_t$ are between vertices on levels differing by 1. Between every $\lambda \in \mathrm{Sig}_n, \mu \in \mathrm{Sig}_{n+1}$ there is a weighted edge with weight

$$L_n^{n+1}(\mu, \lambda) := P_{\mu/\lambda}(t^n) \frac{P_\lambda(1, \ldots, t^{n-1})}{P_\mu(1, \ldots, t^n)},$$

and these weights are called *cotransition probabilities* or *(stochastic) links*. We use L_n^{n+1} to denote the (infinite) $\mathrm{Sig}_{n+1} \times \mathrm{Sig}_n$ matrix with these weights.

Note L_n^{n+1} is a stochastic matrix by the branching rule. More generally, for $m \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, $1 \leq n < m$, and $\mu \in \mathrm{Sig}_m, \lambda \in \mathrm{Sig}_n$ we let

$$L_n^m(\mu, \lambda) := P_{\mu/\lambda}(t^n, \ldots, t^{m-1}) \frac{P_\lambda(1, \ldots, t^{n-1})}{P_\mu(1, \ldots, t^{m-1})}. \tag{4.2.1}$$

When m is finite one has $L_n^m = L_n^{n+1} L_n^{n+2} \cdots L_{m-1}^m$, where the product is just the usual matrix product.

Remark 19. The cotransition probabilities define (deterministic) maps $\mathcal{M}(\mathrm{Sig}_m) \to \mathcal{M}(\mathrm{Sig}_n)$, where here and below we use $\mathcal{M}$ to denote the space of Borel probability measures, in this case with respect to the discrete topology on the set of signatures.

Remark 20. Lemma 2.2.2 implies translation-invariance

$$L_n^m(\mu, \lambda) = L_n^m(\mu + D[m], \lambda + D[n]) \tag{4.2.2}$$

of the cotransition probabilities.

The cotransition probabilities have explicit formulas courtesy of the results of Section 4.1, which will be useful in the proofs of Lemma 4.2.5 and Proposition 4.2.7 later. For $\lambda \in \mathrm{Sig}_n$, we let

$$\begin{bmatrix} n \\ \lambda \end{bmatrix}_t = \frac{(t;t)_n}{\prod_{i \in \mathbb{Z}} (t;t)_{m_i(\lambda)}}$$

(the t-deformed multinomial coefficient).

Corollary 4.2.1. *For $\mu \in \mathrm{Sig}_{n+J}, \lambda \in \mathrm{Sig}_n$,*

$$L_n^{n+J}(\mu,\lambda) = \frac{1}{\begin{bmatrix} n+J \\ J \end{bmatrix}_t} \prod_{x\in\mathbb{Z}} \frac{t^{(n-\lambda'_x)(\mu'_x-\lambda'_x)+m_x(\lambda)m_x(\mu)}}{(t;t)_{m_x(\lambda)}} \, {}_2\bar{\phi}_2 \left(\begin{matrix} t^{-m_x(\lambda)}; t^{-m_x(\mu)}, 0 \\ t^{1+\mu'_{x+1}-\lambda'_x}, t^{1+J-\mu'_x+\lambda'_{x+1}} \end{matrix} ; t, t \right). \tag{4.2.3}$$

Proof. By the translation-invariance of Remark 20, it suffices to prove the case when μ, λ are nonnegative signatures. We combine the formula of Proposition 4.1.2 for $P_{\mu/\lambda}(t^n, \ldots, t^{n+J-1})$ with the one from Proposition 2.2.15 for the principally specialized non-skew Hall-Littlewood polynomial. By the latter,

$$\frac{P_\lambda(1,\ldots,t^{n-1})}{P_\mu(1,\ldots,t^{n+J-1})} = \frac{(t;t)_n}{(t;t)_{n+J}} t^{n(\lambda)-n(\mu)} \prod_{i\geq 0} \frac{(t;t)_{m_i(\mu)}}{(t;t)_{m_i(\lambda)}}. \tag{4.2.4}$$

Note also that by the definition of $n(\lambda)$,

$$t^{n(\lambda)-n(\mu)} = \prod_{x\geq 0} t^{\binom{\lambda'_{x+1}}{2} - \binom{\mu'_{x+1}}{2}}, \tag{4.2.5}$$

so by the identity

$$\binom{a+b}{2} - \binom{a}{2} - \binom{b}{2} = ab$$

we have

$$t^{n(\lambda)-n(\mu)} \prod_{x\geq 0} t^{\binom{\mu'_{x+1}-\lambda'_{x+1}}{2}} = \prod_{x\geq 0} t^{-\lambda'_{x+1}(\mu'_{x+1}-\lambda'_{x+1})}. \tag{4.2.6}$$

Simplifying the product of (4.1.6) with (4.2.4) by the above manipulations yields

$$L_n^{n+J}(\mu,\lambda) = \frac{1}{\begin{bmatrix} n+J \\ J \end{bmatrix}_t} \prod_{x\geq 0} \frac{t^{(n-\lambda'_{x+1})(\mu'_{x+1}-\lambda'_{x+1})+m_x(\lambda)m_x(\mu)}}{(t;t)_{m_x(\lambda)}} \, {}_2\bar{\phi}_2 \left(\begin{matrix} t^{-m_x(\lambda)}; t^{-m_x(\mu)}, 0 \\ t^{1+\mu'_{x+1}-\lambda'_x}, t^{1+J-\mu'_x+\lambda'_{x+1}} \end{matrix} ; t, t \right).$$

The product may be extended to all $x \in \mathbb{Z}$ since all other terms are 1, at which point it is manifestly translation-invariant, which yields the result for arbitrary signatures. $\square$

Definition 21. A sequence $(M_n)_{n\geq 1}$ of probability measures on $\mathrm{Sig}_1, \mathrm{Sig}_2, \ldots$ is *coherent* if

$$\sum_{\mu \in \mathrm{Sig}_{n+1}} M_{n+1}(\mu) L_n^{n+1}(\mu, \lambda) = M_n(\lambda)$$

for each $n \geq 1$ and $\lambda \in \mathrm{Sig}_n$.

Definition 22. A coherent system of measures $(M_n)_{n\geq 1}$ is *extreme* if there do not exist coherent systems $(M'_n)_{n\geq 1}, (M''_n)_{n\geq 1}$ different from $(M_n)_{n\geq 1}$ and $s \in (0,1)$ such that $M_n = sM'_n + (1-s)M''_n$ for each n. The set of extreme coherent systems of measures on a weighted, graded graph is called its *boundary*, and denoted in our case by $\partial \mathscr{G}_t$.

In the previous section we considered both signatures (of finite length), and integer partitions, which have infinite length but stabilize to 0. To describe points on the boundary $\partial \mathscr{G}_t$ in this section, it turns out that it will be necessary to introduce signatures of infinite length which are not partitions.

Definition 23. We denote the set of infinite signatures by

$$\mathrm{Sig}_\infty := \{(\mu_1, \mu_2, \ldots) \in \mathbb{Z}^\infty : \mu_1 \geq \mu_2 \geq \ldots\}.$$

We refer to the μ_i as *parts* just as with partitions, and define μ'_i and $m_i(\mu)$ the exact same way, though we must allow them to be equal to ∞.

A distinguished subset of Sig_∞ is $\mathbb{Y}$, the set of partitions. Translating by any $D \in \mathbb{Z}$ yields

$$\mathbb{Y} + D = \{\mu \in \mathrm{Sig}_\infty : \mu_i = D \text{ for all but finitely many } i\}.$$

However, Sig_∞ also contains infinite signatures with parts not bounded below, the set of which we denote by

$$\mathrm{Sig}_\infty^{unstable} := \{\mu \in \mathrm{Sig}_\infty : \lim_{i \to \infty} \mu_i = -\infty\}.$$

It is clear that

$$\mathrm{Sig}_\infty = \mathrm{Sig}_\infty^{unstable} \sqcup \bigsqcup_{D \in \mathbb{Z}} (\mathbb{Y} + D)$$

and we will use this decomposition repeatedly in what follows. To treat the unbounded signatures we will approximate by signatures in $\mathbb{Y} + D$, which are no more complicated than partitions, and to this end we introduce the following notation.

Definition 24. For $\lambda \in \mathrm{Sig}_\infty^{unstable}$ and $D \in \mathbb{Z}$, we let

$$\lambda^{(D)} = (\lambda_1, \ldots, \lambda_k, D, D, \ldots) \in \mathbb{Y} + D$$

where k is the largest index such that $\lambda_k > D$.

The first step to proving Theorem 1.7.1 is, for each element of Sig_∞, an explicit formula for a corresponding coherent system of measures on $\mathscr{G}_t$; we will later show that these are exactly the boundary points.

Proposition 4.2.2. *For each $\mu \in \mathrm{Sig}_\infty$, there exists a coherent system of measures $(M_n^\mu)_{n \geq 1}$ on $\mathscr{G}_t$, given explicitly by*

$$M_n^\mu(\lambda) := \begin{bmatrix} n \\ \lambda \end{bmatrix}_t \prod_{x \in \mathbb{Z}} t^{(\mu'_x - \lambda'_x)(n - \lambda'_x)} (t^{1+\mu'_x - \lambda'_x}; t)_{m_x(\lambda)}. \tag{4.2.7}$$

for $\lambda \in \mathrm{Sig}_n$.

Before proving Proposition 4.2.2 we will calculate explicit formulas for the links L_n^m in Proposition 4.1.2, which are a corollary to the formula for principally specialized skew functions in Theorem 2.2.16.

Corollary 4.2.3. *Let $n \geq 1$. If $\lambda \in \mathrm{Sig}_n^{\geq 0}, \mu \in \mathbb{Y}$, then*

$$\frac{P_{\mu/\lambda}(t^n, t^{n+1}, \ldots) P_\lambda(1, \ldots, t^{n-1})}{P_\mu(1, t, \ldots)} = \begin{bmatrix} n \\ \lambda \end{bmatrix}_t \prod_{x \in \mathbb{Z}_{>0}} t^{(\mu'_x - \lambda'_x)(n - \lambda'_x)} (t^{1+\mu'_x - \lambda'_x}; t)_{m_x(\lambda)}. \tag{4.2.8}$$

Furthermore, if instead $\lambda \in \mathrm{Sig}_n, \mu \in \mathrm{Sig}_\infty^{unstable}$, then

$$\frac{P_{(\mu^{(D)} - D[\infty])/(\lambda - D[n])}(t^n, t^{n+1}, \ldots) P_{(\lambda - D[n])}(1, \ldots, t^{n-1})}{P_{(\mu^{(D)} - D[\infty])}(1, t, \ldots)} \tag{4.2.9}$$

increases monotonically as $D \to -\infty$, and stabilizes to

$$\begin{bmatrix} n \\ \lambda \end{bmatrix}_t \prod_{x \in \mathbb{Z}} t^{(\mu'_x - \lambda'_x)(n - \lambda'_x)} (t^{1+\mu'_x - \lambda'_x}; t)_{m_x(\lambda)} \tag{4.2.10}$$

for all $D < \lambda_n$.

Proof. (4.2.8) follows from Theorem 2.2.16 and Proposition 2.2.15 by the same proof as that of Corollary 4.2.1, so let us show the monotonicity and stabilization statement. Substituting (4.2.9) into (4.2.8) and changing variables $x \mapsto x + D$ in the product yields

$$\frac{P_{(\mu^{(D)}-D[\infty])/(\lambda-D[n])}(t^n, t^{n+1}, \ldots)P_{(\lambda-D[n])}(1, \ldots, t^{n-1})}{P_{(\mu^{(D)}-D[\infty])}(1, t, \ldots)} = \begin{bmatrix} n \\ \lambda \end{bmatrix}_t \prod_{x \in \mathbb{Z}_{>D}} t^{(\mu'_x-\lambda'_x)(n-\lambda'_x)}(t^{1+\mu'_x-\lambda'_x}; t)_{m_x(\lambda)}.$$

The factors in the product are all in $[0, 1]$ and are equal to 1 when $x \leq \lambda_n$, and since the product is over $x \in \mathbb{Z}_{>D}$ this completes the proof. $\square$

Remark 21. Given the translation-invariance of the links L_n^m noted in Remark 20, when $\mu \in \mathbb{Y} + D$ it is natural to view the expression

$$\frac{P_{(\mu-D[\infty])/(\lambda-D[n])}(t^n, t^{n+1}, \ldots)P_{(\lambda-D[n])}(1, \ldots, t^{n-1})}{P_{(\mu-D[\infty])}(1, t, \ldots)}$$

as simply

$$\frac{P_{\mu/\lambda}(t^n, t^{n+1}, \ldots)P_\lambda(1, \ldots, t^{n-1})}{P_\mu(1, t, \ldots)},$$

even though in our setup the expressions $P_{\mu/\lambda}(t^n, t^{n+1}, \ldots)$ and $P_\mu(1, t, \ldots)$ are not well-defined when μ is not in $\mathbb{Y}$. Hence in view of Theorem 2.2.16 it is natural to view the coherent systems $(M_n^\mu)_{n\geq 1}$ of Theorem 1.7.1 as being given by links 'at infinity'

$$M_n^\mu(\lambda)\text{“} = \text{”}L_n^\infty(\mu, \lambda) = \frac{P_{\mu/\lambda}(t^n, t^{n+1}, \ldots)P_\lambda(1, \ldots, t^{n-1})}{P_\mu(1, t, \ldots)}$$

for general $\mu \in \mathrm{Sig}_\infty$, though we must take a slightly roundabout path to make rigorous sense of the RHS. Many of the proofs below follow the same pattern of proving a result for $\mu \in \mathbb{Y}$ by usual symmetric functions machinery, appealing to translation-invariance for $\mu \in \mathbb{Y} + D$, and then approximating $\mu \in \mathrm{Sig}_\infty^{unstable}$ by elements $\mu^{(D)} \in \mathbb{Y} + D$ and using Corollary 4.2.3 to apply the monotone convergence theorem. We note also that the formula (4.2.7) is clearly translation-invariant.

Proof of Proposition 4.2.2. We first show M_n^μ is indeed a probability measure. Clearly it is a nonnegative function on Sig_n, but we must show it sums to 1. When $\mu \in \mathbb{Y}$ this is by Corollary 4.2.3 and the definition of skew HL polynomials, and the case $\mu \in \mathbb{Y} + D$

reduces to this one. Hence it remains to show that for $\mu \in \mathrm{Sig}_\infty^{unstable}$,

$$\sum_{\lambda \in \mathrm{Sig}_n} \lim_{D \to -\infty} \frac{P_{(\mu^{(D)}-D[\infty])/(\lambda-D[n])}(t^n, t^{n+1}, \ldots) P_{(\lambda-D[n])}(1, \ldots, t^{n-1})}{P_{(\mu^{(D)}-D[\infty])}(1, t, \ldots)} = 1. \tag{4.2.11}$$

By Corollary 4.2.3, the functions

$$\frac{P_{(\mu^{(D)}-D[\infty])/(\lambda-D[n])}(t^n, t^{n+1}, \ldots) P_{(\lambda-D[n])}(1, \ldots, t^{n-1})}{P_{(\mu^{(D)}-D[\infty])}(1, t, \ldots)} = \begin{bmatrix} n \\ \lambda \end{bmatrix}_t \prod_{x \in \mathbb{Z}_{>D}} t^{(\mu'_x - \lambda'_x)(n-\lambda'_x)} (t^{1+\mu'_x-\lambda'_x}; t)_{m_x(\lambda)}$$

converge to the summand in (4.2.11) from below as $D \to -\infty$. Hence (4.2.11) follows by the monotone convergence theorem.

For $\mu \in \mathbb{Y} + D$ for some D, coherency again follows from the definition of skew functions and the first part of Corollary 4.2.3. For $\mu \in \mathrm{Sig}_\infty^{unstable}$ we must show

$$\begin{aligned} &\sum_{\kappa \in \mathrm{Sig}_{n+1}} \lim_{D \to -\infty} \frac{P_{(\mu^{(D)}-D[\infty])/(\kappa-D[n+1])}(t^{n+1}, \ldots) P_{(\kappa-D[n+1])}(1, \ldots, t^n)}{P_{(\mu^{(D)}-D[\infty])}(1, t, \ldots)} \frac{P_{\kappa/\lambda}(t^n) P_\lambda(1, \ldots, t^{n-1})}{P_\kappa(1, \ldots, t^n)} \\ &\qquad = \lim_{D \to -\infty} \frac{P_{(\mu^{(D)}-D[\infty])/(\lambda-D[n])}(t^n, t^{n+1}, \ldots) P_{(\lambda-D[n])}(1, \ldots, t^{n-1})}{P_{(\mu^{(D)}-D[\infty])}(1, t, \ldots)}. \end{aligned} \tag{4.2.12}$$

Again the monotone convergence theorem allows us to interchange the limit and sum. The result then follows by translation invariance of the links (4.2.2) and the definition of skew HL polynomials. □

It remains to show that the coherent systems identified in Proposition 4.2.2 are extreme and that all extreme coherent systems are of this form. Just from the definition, an arbitrary extreme coherent system is an elusive object. Luckily, the general results of the Vershik-Kerov ergodic method guarantee that extreme coherent systems can be obtained through limits of cotransition probabilities for certain *regular sequences* of signatures, which are much more concrete.

Definition 25. A sequence $(\mu(n))_{n \geq 1}$ with $\mu(n) \in \mathrm{Sig}_n$ is *regular* if for every $k \in \mathbb{Z}_{\geq 1}$ and $\lambda \in \mathrm{Sig}_k$, the limit

$$M_k(\lambda) := \lim_{n \to \infty} L_k^n(\mu(n), \lambda)$$

exists and M_k is a probability measure.

Proposition 4.2.4. *For any extreme coherent system $(M_k)_{k\geq 1} \in \partial G$ there exists a regular sequence $(\mu(n))_{n\geq 1}$ such that*

$$M_k(\cdot) = \lim_{n\to\infty} L_k^n(\mu(n), \cdot).$$

Proof. Follows from [OO98, Theorem 6.1]. ☐

The space of extreme coherent systems obtained from regular sequences as in Proposition 4.2.4 is sometimes referred to as the *Martin boundary*. It naturally includes into the boundary, and Proposition 4.2.4 says that in this setup they are in fact equal.

Lemma 4.2.5. *Let $(\mu(n))_{n\geq 1}$ be a sequence with $\mu(n) \in \mathrm{Sig}_n$, such that*

$$\lim_{n\to\infty} \mu(n)_i =: \mu_i$$

exists and is finite for every i. Then $(\mu(n))_{n\geq 1}$ is regular and the corresponding coherent family is $(M_n^\mu)_{n\geq 1}$, where $\mu = (\mu_1, \mu_2, \ldots) \in \mathrm{Sig}_\infty$.

Proof. Let $(\mu(n))_{n\geq 1}$ satisfy the hypothesis. We must show for arbitrary $k, \lambda \in \mathrm{Sig}_k$ that

$$\lim_{n\to\infty} L_k^n(\mu(n), \lambda) = M_k^\mu(\lambda). \tag{4.2.13}$$

It is easy to see from the explicit formula in Corollary 4.2.1 that $L_k^n(\mu(n), \lambda)$ depends only on the parts of $\mu(n)$ which are $\geq \lambda_k$. For any fixed x, it is easy to see that $\mu(n)'_x \to \mu'_x$. In fact, for all sufficiently large n, it must be true that $\mu(n)'_x = \mu'_x$ for all $x \geq \lambda_k$ such that μ'_x is finite. Hence for all sufficiently large n the product in (4.2.3) only has nontrivial terms when $\lambda_k \leq x \leq \mu_1$, so it suffices to show that each term converges. This follows by the exact same argument as the proof of Theorem 2.2.16, again with two cases based on whether $\mu(n)'_x$ stabilizes or $\mu(n)'_x \to \infty$. ☐

Lemma 4.2.6. *Let $1 \leq k < n$ be integers and $\lambda \in \mathrm{Sig}_k$.*

1. *If $\mu \in \mathrm{Sig}_n$ is such that $\lambda'_x > \mu'_x$ for some x, then $L_k^n(\mu, \lambda) = 0$.*

2. *If $\mu \in \mathrm{Sig}_\infty$ is such that $\lambda'_x > \mu'_x$ for some x, then $M_k^\mu(\lambda) = 0$.*

Proof. If $\mu \in \mathbb{Y}, \lambda \in \mathbb{Y}$ and $\lambda'_x > \mu'_x$ for any x, it follows from the upper-triangularity of the branching rule [Mac98a, Chapter III, (5.5')] that $P_{\lambda/\mu}(t^k, \ldots, t^{n-1}) = 0$, showing (1). Approximating $\mu \in \mathrm{Sig}_\infty$ with $(\mu_1, \ldots, \mu_n) \in \mathrm{Sig}_n$ and invoking Lemma 4.2.5 yields (2). $\square$

Lemma 4.2.6 could also be shown by the explicit formula (4.2.7), but as the above proof shows, it in fact requires only the very basic properties of symmetric functions.

Proposition 4.2.7. *Every extreme coherent system is given by* $(M_n^\mu)_{n \geq 1}$ *for some* $\mu \in \mathrm{Sig}_\infty$.

Proof. Let $(M_n)_{n\geq 1}$ be an extreme coherent system and $(\mu(n))_{n\geq 1}$ be a regular sequence converging to it, the existence of which is guaranteed by Proposition 4.2.4. We wish to find $\mu \in \mathrm{Sig}_\infty$ such that

$$\lim_{n\to\infty} L_k^n(\mu(n), \lambda) = M_k^\mu(\lambda) \tag{4.2.14}$$

for all k and $\lambda \in \mathrm{Sig}_k$, and will construct μ as a limit of the signatures $\mu(n)$.

Our first step is to show the sequence of first parts $(\mu_1(n))_{n\geq 1}$ is bounded above (and hence all other $(\mu_i(n))_{n\geq 1}$ are as well). Suppose for the sake of contradiction that this is not the case. Then there is a subsequence $(\mu_1(n_j))_{j\geq 1}$ of $(\mu_1(n))_{n\geq 1}$ for which $\mu_1(n_j) \to \infty$. We claim that for any k and $\lambda \in \mathrm{Sig}_k$,

$$\lim_{j\to\infty} L_k^{n_j}(\mu^(n_j), \lambda) = 0. \tag{4.2.15}$$

This suffices for the contradiction, as then (4.2.15) holds also with n_j replaced by n by regularity of $(\mu(n))_{n\geq 1}$, therefore the sequence of probability measures $L_k^n(\mu(n), \cdot)$ converges to the zero measure, which contradicts the definition of regular sequence. So let us prove (4.2.15), and to declutter notation let us without loss of generality denote the subsequence by $(\mu(n))_{n\geq 1}$ as well.

We claim there exists a constant C_k such that for all $J \geq 1$ and $\nu \in \mathrm{Sig}_{k+J}$,

$$\left| t^{m_x(\nu) m_x(\lambda)} \, {}_2\bar{\phi}_2 \left(\begin{matrix} t^{-m_x(\lambda)}; t^{-m_x(\nu)}, 0 \\ t^{1+\nu'_{x+1}-\lambda'_x}, t^{1+J-\nu'_x+\lambda'_{x+1}} \end{matrix} ; t, t \right) \right| \leq C_k \tag{4.2.16}$$

For fixed λ, $1 + \nu'_{x+1} - \lambda'_x$ and $1 + J - \nu'_x + \lambda'_{x+1}$ are both bounded below independent of

ν by $1-k$. This gives an upper bound on the factors $(bt^\ell; t)_{m_x(\lambda)-\ell}, 0 \le \ell \le m_x(\lambda)$ where $b \in \{t^{1+\nu'_{x+1}-\lambda'_x}, t^{1+J-\nu'_x+\lambda'_{x+1}}\}$ which appear in the sum expansion (4.1.3) of (4.2.16). The term $t^{m_x(\nu)m_x(\lambda)}(t^{-m_x(\nu)}; t)_\ell$ is likewise bounded above independent of ν. Because $m_x(\lambda)$ and λ'_x can only take finitely many values, the claim follows. Furthermore, the LHS of (4.2.16) is simply 1 whenever $m_x(\lambda) = 0$, which is true for all but finitely many x. Plugging this bound into Corollary 4.2.1 yields

$$L_k^n(\mu(n), \lambda) \le \frac{C_k^k}{\begin{bmatrix} n \\ k \end{bmatrix} \prod_{i \in \mathbb{Z}} (t;t)_{m_i(\lambda)}} \prod_{x \in \mathbb{Z}} t^{(k-\lambda'_x)(\mu(n)'_x - \lambda'_x)}. \tag{4.2.17}$$

For $\lambda_1 < x \le \mu(n)_1$, one has $t^{(k-\lambda'_x)(\mu(n)'_x-\lambda'_x)} \le t^k < 1$, and our claim (4.2.15) follows.

Now, suppose for the sake of contradiction that there exists k for which $(\mu(n)_k)_{n\ge1}$ is not bounded below. Then for any $\lambda \in \mathrm{Sig}_k$, there are infinitely many n for which $\mu(n)_k < \lambda_k$ and consequently $\mu(n)'_x < \lambda'_x = k$ for $x = \lambda_k$. By Lemma 4.2.6, $L_k^n(\mu(n), \lambda) = 0$ for all such n, therefore $L_k^n(\mu(n), \lambda) \to 0$ as $n \to \infty$ since $(\mu(n))_{n\ge1}$ is a regular sequence. This is a contradiction, therefore $(\mu(n)_k)_{n\ge1}$ is bounded below for each k.

Since $(\mu(n)_k)_{n\ge1}$ is bounded above and below for each k, there is a subsequence on which these converge, and by a diagonalization argument there exists a subsequence $(\mu(n_j))_{j\ge1}$ on which $\mu(n_j)_k$ converges for every k. Letting $\mu_i = \lim_{j\to\infty} \mu(n_j)_i$ and $\mu = (\mu_1, \mu_2, \ldots) \in \mathrm{Sig}_\infty$, we have by Lemma 4.2.5 that

$$\lim_{j\to\infty} L_k^{n_j}(\mu(n_j), \lambda) = M_k^\mu(\lambda)$$

for each $\lambda \in \mathrm{Sig}_k$. Since $\lim_{n\to\infty} L_k^n(\mu(n), \lambda)$ exists by the definition of regular sequence, it must also be equal to $M_k^\mu(\lambda)$. This shows (4.2.14), completing the proof. □

For the other direction, Proposition 4.2.10, we will need the basic fact that general coherent systems are convex combinations of extreme ones.

Proposition 4.2.8. *For any coherent system $(M_n)_{n\ge1}$ on $\mathscr{G}_t$, there exists a Borel*[1] *mea-*

[1]The topology on $\partial\mathscr{G}_t$ here is the following. For each n, the set of probability measures on Sig_n inherits a topology from the product topology on $\mathbb{R}^\infty$ by viewing the measures as functions, which gives a topology on the inverse limit $\partial\mathscr{G}_t$.

sure π on $\partial\mathscr{G}_t$ such that

$$M_k = \int_{M' \in \partial\mathscr{G}_t} M'_k \pi(dM')$$

for each k, where M' is shorthand for a coherent system $(M'_n)_{n\geq 1}$.

Proof. Follows from [Ols03, Theorem 9.2]. $\square$

It will also be necessary to put a topology on Sig_∞, namely the one inherited from the product topology on $\mathbb{Z}^\infty$ where $\mathbb{Z}$ is equipped with the cofinite topology. The following lemma shows that these natural choices of topology on Sig_∞ and $\partial\mathscr{G}_t$ are compatible.

Lemma 4.2.9. *The map*

$$\begin{aligned} f : \mathrm{Sig}_\infty &\to \mathcal{M}(\partial\mathscr{G}_t) \\ \mu &\mapsto (M_n^\mu)_{n\geq 1} \end{aligned}$$

is continuous, hence in particular Borel.

Proof. Since Sig_∞ is first-countable, to show f is continuous it suffices to show it preserves limits of sequences. Hence we must show that for any $\mu \in \mathrm{Sig}_\infty$, if $\nu^{(1)}, \nu^{(2)}, \ldots \in \mathrm{Sig}_\infty$ and $\nu_i^{(k)} \to \mu_i$ for all i, then $M_n^{\nu^{(k)}} \to M_n^\mu$ pointwise as functions on Sig_n. This follows straightforwardly from the explicit formula (4.2.7) of Proposition 4.2.2. $\square$

Proposition 4.2.10. *For every $\mu \in \mathrm{Sig}_\infty$, the coherent system $(M_n^\mu)_{n\geq 1}$ is extreme.*

Proof. Fix $\mu \in \mathrm{Sig}_\infty$. By Proposition 4.2.7,there is a Borel measure $\pi \in \mathcal{M}(\partial\mathscr{G}_t)$.

$$M_k^\mu = \int_{M' \in \partial\mathscr{G}_t} M'_k \pi(dM') = \int_{\nu \in \mathrm{Sig}_\infty} M_k^\nu (\iota_* \pi)(d\nu) \tag{4.2.18}$$

where $\iota : \partial\mathscr{G}_t \hookrightarrow \mathrm{Sig}_\infty$ is the inclusion guaranteed by Proposition 4.2.7. Because $f \circ \iota = \mathrm{Id}$ and f is Borel, ι is a Borel isomorphism onto its image, hence $\iota_*\pi$ is a Borel measure in the topology on Sig_∞ above.

We first claim that $\iota_*\pi$ is supported on

$$S_{\leq \mu} := \{\nu \in \mathrm{Sig}_\infty : \nu_i \leq \mu_i \text{ for all } i\}.$$

Suppose not. Since

$$\mathrm{Sig}_\infty \setminus S_{\leq \mu} = \bigcup_{k \geq 1} \{\nu \in \mathrm{Sig}_\infty : \nu_i > \mu_i \text{ for at least one } 1 \leq i \leq k\}$$

and

$$\{\nu \in \mathrm{Sig}_\infty : \nu_i > \mu_i \text{ for at least one } 1 \leq i \leq k\} = \bigcup_{\substack{\lambda \in \mathrm{Sig}_k : \\ \exists i \text{ s.t. } \lambda_i > \mu_i}} \{\nu \in \mathrm{Sig}_\infty : \nu_i = \lambda_i \text{ for all } 1 \leq i \leq k\},$$

if $(\iota_* \pi)(\mathrm{Sig}_\infty \setminus S_{\leq \mu}) > 0$ then there exists k and $\lambda \in \mathrm{Sig}_k$ such that

$$(\iota_* \pi)(\{\nu \in \mathrm{Sig}_\infty : \nu_i = \lambda_i \text{ for all } 1 \leq i \leq k\}) > 0. \tag{4.2.19}$$

Denoting the set in (4.2.19) by $S_k(\lambda) \subset \mathrm{Sig}_\infty$, we have

$$M_k^\mu(\lambda_1, \ldots, \lambda_k) = \int_{\nu \in S_k(\lambda)} M_k^\nu(\lambda_1, \ldots, \lambda_k)(\iota_* \pi)(d\nu) + \int_{\nu \in \mathrm{Sig}_\infty \setminus S_k(\lambda)} M_k^\nu(\lambda_1, \ldots, \lambda_k)(\iota_* \pi)(d\nu). \tag{4.2.20}$$

The LHS is 0 by Lemma 4.2.6. If $\nu \in S_k(\lambda)$, then the only factor in

$$M_k^\nu(\lambda_1, \ldots, \lambda_k) = \begin{bmatrix} k \\ \lambda \end{bmatrix}_t \prod_{x \in \mathbb{Z}_{\geq \lambda_k}} t^{(\nu_x' - \lambda_x')(k - \lambda_x')} (t^{1 + \nu_x' - \lambda_x'}; t)_{m_x(\lambda)}$$

which depends on ν is $(t^{1+\nu'_{\lambda_k} - k}; t)_{m_{\lambda_k}(\lambda)}$, which is clearly bounded below by $(t;t)_\infty$. Hence the RHS of (4.2.20) is bounded below by

$$(\iota_* \pi)(S_k(\lambda))(t;t)_\infty \begin{bmatrix} k \\ \lambda \end{bmatrix}_t > 0,$$

a contradiction. Therefore $\iota_* \pi$ is indeed supported on $S_{\leq \mu}$.

For each $k \geq 1$ we may decompose

$$S_{\leq \mu} = (S_{\leq \mu} \cap S_k(\mu_1, \ldots, \mu_k)) \sqcup (S_{\leq \mu} \cap (S_k(\mu_1, \ldots, \mu_k))^c)$$

into those signatures which agree with μ on the first k coordinates and those which do not, and

$$M_k^\mu(\mu_1, \dots, \mu_k) = \int_{\nu \in S_{\leq \mu} \cap S_k(\mu_1,\dots,\mu_k)} M_k^\nu(\mu_1, \dots, \mu_k)(\iota_* \pi)(d\nu) + \int_{\nu \in S_{\leq \mu} \cap (S_k(\mu_1,\dots,\mu_k)^c)} M_k^\nu(\mu_1, \dots, \mu_k)(\iota_* \pi)(d\nu). \quad (4.2.21)$$

The second integral in (4.2.21) is always 0 by Lemma 4.2.6. If $\nu \in S_{\leq \mu} \cap S_k(\mu_1, \dots, \mu_k)$ then $\nu'_x = \mu'_x$ for $x > \mu_k$ and $\nu'_x \leq \mu'_x$ when $x = \mu_k$. Hence

$$(t^{1+\nu'_x - k}; t)_{m_x(\mu_1,\dots,\mu_k)} \leq (t^{1+\mu'_x - k}; t)_{m_x(\mu_1,\dots,\mu_k)}$$

for all x, and all other factors in (4.2.7) are the same for $M_k^\nu(\mu_1, \dots, \mu_k)$ and $M_k^\mu(\mu_1, \dots, \mu_k)$, therefore

$$M_k^\nu(\mu_1, \dots, \mu_k) \leq M_k^\mu(\mu_1, \dots, \mu_k) \text{ for all } \nu \in S_{\leq \mu} \cap S_k(\mu_1, \dots, \mu_k).$$

Hence (4.2.21) reduces to

$$M_k^\mu(\mu_1, \dots, \mu_k) \leq M_k^\mu(\mu_1, \dots, \mu_k) \cdot (\iota_* \pi)(S_{\leq \mu} \cap S_k(\mu_1, \dots, \mu_k)). \quad (4.2.22)$$

Since $M_k^\mu(\mu_1, \dots, \mu_k) > 0$ by (4.2.7), it follows that

$$(\iota_* \pi)(S_{\leq \mu} \cap S_k(\mu_1, \dots, \mu_k)) = 1.$$

Since this is true for all k and $\bigcap_k (S_{\leq \mu} \cap S_k(\mu_1, \dots, \mu_k)) = \{\mu\}$, it follows that $(\iota_* \pi)(\{\mu\}) = 1$, i.e. $\iota_* \pi$ is the delta mass at μ. Hence $(M_n^\mu)_{n \geq 1}$ is an extreme coherent system, completing the proof. $\square$

4.3 Infinite p-adic random matrices and corners

In this section, we turn to p-adic random matrix theory and prove Theorem 1.7.2 and Theorem 1.7.3. We will first give the basic setup of p-adic random matrices and the key result Proposition 4.3.1 which relates the operations of removing rows and columns to Hall-Littlewood polynomials. In Section 4.3.2 we prove auxiliary boundary results on a slightly more complicated branching graph which extends the one in the previous section, which are tailored to the random matrix corner situation. We then use these to deduce the result Theorem 1.7.2, that extreme bi-invariant measures on $\mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p)$ are parametrized by the set $\underline{\mathrm{Sig}}_\infty$ defined in Definition 26 below, from the parametrization of the boundary of this augmented branching graph by $\underline{\mathrm{Sig}}_\infty$ (Theorem 4.3.3).

4.3.1 p-adic background.

Definition 26. Recall from Definition 3 that for $n \in \mathbb{Z}_{\geq 1}$, we let

$$\underline{\mathrm{Sig}}_n = \{(\lambda_1, \ldots, \lambda_n) \in (\mathbb{Z} \cup \{-\infty\})^n : \lambda_1 \geq \ldots \geq \lambda_n\},$$

where we take $-\infty < a$ for all $a \in \mathbb{Z}$, and refer to elements of $\underline{\mathrm{Sig}}_n$ as *extended signatures*. The definition of $\underline{\mathrm{Sig}}_\infty$ is exactly analogous. For $0 \leq k \leq n$, we denote by $\underline{\mathrm{Sig}}_n^{(k)} \subset \underline{\mathrm{Sig}}_n$ the set of all extended signatures with exactly k integer parts and the rest equal to $-\infty$. For $\lambda \in \underline{\mathrm{Sig}}_n^{(k)}$, we denote by $\lambda^* \in \mathrm{Sig}_k$ the signature given by its integer parts.

Definition 27. For $1 \leq n \leq m < \infty$ and $A \in \mathrm{Mat}_{n\times m}(\mathbb{Q}_p)$, we denote by $\mathrm{ESN}(A) \in \underline{\mathrm{Sig}}_\infty$ the extended signature with first n parts given by $-\mathrm{SN}(A)$, and all others by $-\infty$. We refer to the finite parts of $\mathrm{ESN}(A)$ as the *negative finite singular numbers* of A.

Remark 22. The reason for padding with $-\infty$ is to allow us to treat matrices of different sizes on equal footing, essentially viewing them as corners of a large matrix of low rank. This is why it makes sense to work with negative singular numbers in the infinite matrix context, as otherwise we would have to pad $\mathrm{SN}(A)$ with infinitely many ∞ entries on the left, and we like our infinite tuples to read left to right. Our formalism is somewhat unwieldy but seemed to be the least awkward one for the problem at hand. We note that the negative singular numbers are referred to as the singular numbers in [Ass22, BQ17] and in [VP22a] where the results of this chapter first appeared.

Proposition 4.3.1. *Let $n, m \geq 1$ be integers, $\mu \in \underline{\mathrm{Sig}}_\infty$ with $\mathrm{len}(\mu^*) \leq \min(m+1, n)$, let $A \in \mathrm{Mat}_{n\times(m+1)}(\mathbb{Q}_p)$ be distributed by the unique bi-invariant measure with negative singular numbers μ, and let $t = 1/p$. If $A' \in \mathrm{Mat}_{n\times m}$ is the first m columns of A, then*

$\mathrm{ESN}(A')$ *is a random element of* $\underline{\mathrm{Sig}}_\infty$ *with*

$$\Pr(\mathrm{ESN}(A') = \lambda) = \begin{cases} \frac{\tilde{Q}_{-\lambda^*/-\mu^*}(t^{m+1-k})P_{-\lambda^*}(1,\ldots,t^{k-1})}{P_{-\mu^*}(1,\ldots,t^{k-1})\Pi(t^{m+1-k};1,\ldots,t^{k-1})} & \mu,\lambda \in \underline{\mathrm{Sig}}_\infty^{(k)} \text{ for some } 0 \le k \le \min(m,n) \\ P_{\mu^*/\lambda^*}(t^m)\frac{P_{\lambda^*}(1,\ldots,t^{m-1})}{P_{\mu^*}(1,\ldots,t^m)} & \mu \in \underline{\mathrm{Sig}}_\infty^{(m+1)}, \lambda \in \underline{\mathrm{Sig}}_\infty^{(m)} \\ 0 & \text{otherwise} \end{cases} \tag{4.3.1}$$

for any $\lambda \in \underline{\mathrm{Sig}}_\infty$.

Proof. In the case where $\mathrm{len}(\mu^*) = \min(m+1,n)$ so that A is full-rank, the result follows by applying Part 2 of Theorem 1.2.1 (taking care with sign conventions). The non full-rank case $\mathrm{len}(\mu^*) < \min(m+1,n)$ follows from the full-rank case with $m+1 > n$, as in this case the rank of A does not change after removing the $(m+1)^{th}$ column. $\square$

Because $\mathrm{ESN}(A) = \mathrm{ESN}(A^T)$, Proposition 4.3.1 obviously holds for removing rows rather than columns after appropriately relabeling the indices. By relating matrix corners to Hall-Littlewood polynomials, Proposition 4.3.1 provides the key to applying the results on Hall-Littlewood branching graphs to study p-adic random matrices. In the second case of the transition probabilities in (4.3.1), one immediately recognizes the cotransition probabilities of Section 4.2. However, one now has two added features not present in that section: (1) the signatures may have infinite parts, and (2) with matrices one may remove either rows or columns, so there are in fact two (commuting) corner maps. In the next subsection, we augment the branching graph formalism and results of Section 4.2 to handle this more complicated setup. However, let us first introduce the setup of infinite matrices.

Definition 28. $\mathrm{GL}_\infty(\mathbb{Z}_p)$ is the direct limit $\varinjlim \mathrm{GL}_N(\mathbb{Z}_p)$ with respect to inclusions

$$\begin{aligned} \mathrm{GL}_N(\mathbb{Z}_p) &\hookrightarrow \mathrm{GL}_{N+1}(\mathbb{Z}_p) \\ A &\mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Equivalently, $\mathrm{GL}_\infty(\mathbb{Z}_p) = \bigcup_{N\ge1} \mathrm{GL}_N(\mathbb{Z}_p)$ where we identify $\mathrm{GL}_N(\mathbb{Z}_p)$ with the group of infinite matrices for which the top left $N \times N$ corner is an element of $\mathrm{GL}_N(\mathbb{Z}_p)$ and all other entries are 1 on the diagonal and 0 off the diagonal.

The definition

$$\mathrm{Mat}_{n\times m}(\mathbb{Q}_p) := \left\{ Z = (Z_{ij})_{\substack{1\le i\le n\\ 1\le j\le m}} : Z_{ij} \in \mathbb{Q}_p \right\}.$$

still makes sense when n or m is equal to ∞ by replacing $1 \le i \le n$ with $i \in \mathbb{Z}_{\ge 1}$ and similarly for m. When n or m is ∞, $\mathrm{GL}_\infty(\mathbb{Z}_p)$ clearly acts on this space on the appropriate side.

4.3.2 Auxiliary boundary results and proof of Theorem 1.7.2.

In this subsection we prove a similar result to Theorem 1.7.1, Theorem 4.3.2, and deduce an extension to a 'two-dimensional' version of the branching graph $\mathscr{G}_t$ in Theorem 4.3.3.

Definition 29. For each $k \ge 1$, we define a graded graph

$$\mathscr{G}_t^{(k)} = \bigsqcup_{n\ge 1} \mathscr{G}_t^{(k)}(n)$$

with vertex set at each level given by $\mathscr{G}_t^{(k)}(n) = \mathrm{Sig}_k$. Edges are only between adjacent levels, and to each edge from $\nu \in \mathscr{G}_t^{(k)}(n+1)$ to $\lambda \in \mathscr{G}_t^{(k)}(n)$ is associated a cotransition probability

$$\tilde{\Lambda}_n^{n+1}(\nu, \lambda) = \tilde{Q}_{-\lambda/-\nu}(t^n) \frac{P_{-\lambda}(1, t, \ldots, t^{k-1})}{P_{-\nu}(1, t, \ldots, t^{k-1})\Pi(1, t, \ldots, t^{k-1}; t^n)}.$$

We define $\tilde{\Lambda}_n^m = \tilde{\Lambda}_n^{n+1} \cdots \tilde{\Lambda}_{m-1}^m$ for general $1 \le n < m < \infty$ as before.

The next result is a version of Theorem 1.7.1 for this smaller branching graph $\mathscr{G}_t^{(k)}$. Recall the definition of boundary from earlier in this section.

Theorem 4.3.2. *For any $t \in (0, 1)$, the boundary $\partial\mathscr{G}_t^{(k)}$ is naturally in bijection with Sig_k. Under this bijection, $\mu \in \mathrm{Sig}_k$ corresponds to the coherent system $(M_n^\mu)_{n\ge 1}$ defined explicitly by*

$$M_n^\mu(\lambda) = \tilde{Q}_{-\lambda/-\mu}(t^n, t^{n+1}, \ldots) \frac{P_{-\lambda}(1, t, \ldots, t^{k-1})}{P_{-\mu}(1, t, \ldots, t^{k-1})\Pi(1, t, \ldots, t^{k-1}; t^n, t^{n+1}, \ldots)} \tag{4.3.2}$$

for $\lambda \in \mathrm{Sig}_k$.

Note we have simultaneously suppressed the k-dependence in our notation for the measure M_n^μ on Sig_k and abused notation by using the same for measures on $\mathscr{G}_t$ and $\mathscr{G}_t^{(k)}$, but there is no ambiguity if one knows the length of μ. The proof of Theorem 4.3.2 is an easier version of the proof of Theorem 1.7.1, so we simply give a sketch and outline the differences.

Proof. We first prove that every extreme coherent system is of the form (4.3.2) for some $\mu \in \mathrm{Sig}_k$. The analogue of Proposition 4.2.4 similarly follows from the general result [OO98, Theorem 6.1], so there exists a regular sequence $(\mu(n))_{n\geq 1}$ approximating any extreme coherent system. Using the explicit formula (4.1.7) of Proposition 4.1.2, a naive bound as in the proof of Proposition 4.2.7 establishes that $\mu(n)_1$ is bounded above.

The analogue of Lemma 4.2.6, namely that $\tilde{\Lambda}_n^m(\mu,\lambda) = 0$ and $M_n^\mu(\lambda) = 0$ if there exists an x for which $\lambda'_x > \mu'_x$, holds similarly by the branching rule. Using this one obtains that a regular sequence $(\mu(n))_{n\geq 1}$ must have last parts $\mu(n)_k$ bounded below. Together with the upper bound this yields that $(\mu(n))_{n\geq 1}$ has a convergent subsequence, where here convergence simply means that all terms in the subsequence are equal to the same $\mu \in \mathrm{Sig}_k$. It now follows as in the proof of Proposition 4.2.7 that in fact the coherent system approximated by $(\mu(n))_{n\geq 1}$ must be $(M_n^\mu)_{n\geq 1}$ for this μ.

It remains to prove that every coherent system of the form (4.3.2) is in fact extreme. The proof is the same as that of Proposition 4.2.10 using the above analogue of Lemma 4.2.6, except that no measure-theoretic details are necessary because the decomposition of an arbitrary coherent system into extreme ones takes the form of a sum over the countable set Sig_k. □

For applications in the next section it is desirable to in some sense combine $\mathscr{G}_t$ and $\mathscr{G}_t^{(k)}$ by working with extended signatures. We wish to define a doubly-graded graph with cotransition probabilities which generalize the earlier $L_n^{n+1}, \tilde{\Lambda}_n^{n+1}$ and which correspond to the situation of removing rows and columns from a matrix in Proposition 4.3.1.

Definition 30. Define

$$\widetilde{\mathscr{G}}_t = \bigsqcup_{m,n\geq 1} \widetilde{\mathscr{G}}_t(m,n)$$

with $\widetilde{\mathscr{G}}_t(m,n) = \underline{\mathrm{Sig}}_\infty$ for each m,n, and edges from $\widetilde{\mathscr{G}}_t(m+1,n)$ to $\widetilde{\mathscr{G}}_t(m,n)$ with weights

$$L_{m,n}^{m+1,n}(\mu,\lambda) = \begin{cases} \frac{\tilde{Q}_{-\lambda^*/-\mu^*}(t^{m+1-k})P_{-\lambda^*}(1,\dots,t^{k-1})}{P_{-\mu^*}(1,\dots,t^{k-1})\Pi(t^{m+1-k};1,\dots,t^{k-1})} & \mu,\lambda \in \underline{\mathrm{Sig}}_\infty^{(k)} \text{ for some } 0 \le k \le \min(m,n) \\ P_{\mu^*/\lambda^*}(t^m)\frac{P_{\lambda^*}(1,\dots,t^{m-1})}{P_{\mu^*}(1,\dots,t^m)} & \mu \in \underline{\mathrm{Sig}}_\infty^{(m+1)}, \lambda \in \underline{\mathrm{Sig}}_\infty^{(m)} \\ 0 & \text{otherwise} \end{cases} \tag{4.3.3}$$

and edges from $\widetilde{\mathscr{G}}_t(m,n+1)$ to $\widetilde{\mathscr{G}}_t(m,n)$ with weights $L_{m,n}^{m,n+1}(\mu,\lambda) = L_{n,m}^{n+1,m}(\mu,\lambda)$.

It follows immediately from the Cauchy identity Lemma 2.2.3 that

$$L_{m,n}^{m+1,n}L_{m+1,n}^{m+1,n+1} = L_{m,n}^{m,n+1}L_{m,n+1}^{m+1,n+1},$$

so there is no ambiguity in defining coherent systems of probability measures on $\widetilde{\mathscr{G}}_t$.

Theorem 4.3.3. *For $t \in (0,1)$, the boundary $\partial\widetilde{\mathscr{G}}_t$ is in bijection with $\underline{\mathrm{Sig}}_\infty$. The extreme coherent system $(M_{m,n}^\mu)_{m,n\ge1}$ corresponding to $\mu \in \mathrm{Sig}_\infty$ is determined by*

$$M_{m,n}^\mu(\nu) = \sum_{\lambda\in\mathrm{Sig}_n} M_n^\mu(\lambda)M_{m-n+1}^\lambda(\nu^*)$$

for $m \ge n$ and hence for all m,n by coherency. The extreme coherent system corresponding to $\mu \in \mathrm{Sig}_\infty^{(k)}$ is determined by

$$M_{m,n}^\mu(\nu) = \sum_{\lambda\in\mathrm{Sig}_k} M_{n-k+1}^{\mu^*}(\lambda)M_{m-k+1}^\lambda(\nu^*)$$

for $m,n \ge k$ and hence for all m,n by coherency.

Proof. First note that every coherent system on $\widetilde{\mathscr{G}}_t$ is determined by a sequence of coherent systems on the subgraphs with vertex sets

$$\bigsqcup_{m\ge n}\widetilde{\mathscr{G}}_t(m,n) \tag{4.3.4}$$

for $n \ge 1$, which are themselves coherent with one another under the links $L_{m,n}^{m,n+1}$. By the definition of the cotransition probabilities (4.3.3), a coherent system on (4.3.4) must decompose as a convex combination of $n+1$ coherent systems, each one having all measures supported on $\underline{\mathrm{Sig}}_\infty^{(k)}$ for $0 \le k \le n$. Hence extreme coherent systems on (4.3.4) are parametrized by $\underline{\mathrm{Sig}}_k$ by applying Theorem 4.3.2 for each k.

It follows by the above-mentioned commutativity $L_{m,n}^{m+1,n} L_{m+1,n}^{m+1,n+1} = L_{m,n}^{m,n+1} L_{m,n+1}^{m+1,n+1}$ that given a coherent system $(M_m)_{m \ge n}$ on the graph (4.3.4), $(M_m L_{m,n-1}^{m,n})_{m \ge n}$ is a coherent system on

$$\bigsqcup_{m \ge n} \widetilde{\mathscr{G}}_t(m, n-1).$$

Since $L_{m,n-1}^{m,n}$ takes coherent systems to coherent systems, by decomposing these into extreme coherent systems it induces a map $\mathcal{M}(\underline{\mathrm{Sig}}_n) \to \mathcal{M}(\underline{\mathrm{Sig}}_{n-1})$ between spaces of probability measures on the respective boundaries, i.e. a Markov kernel. It follows from the explicit formulas (4.3.2), (4.3.3) and the Cauchy identity (2.2.26) that this Markov map is itself given by $L_{m,n-1}^{m,n}$ on the appropriately restricted domain, after identifying $\underline{\mathrm{Sig}}_n$ and $\underline{\mathrm{Sig}}_{n-1}$ as subsets of $\underline{\mathrm{Sig}}_\infty$ in the obvious way.

Hence $\partial \widetilde{\mathscr{G}}_t$ is in bijection with coherent systems on the graph with vertex set

$$\bigsqcup_{n \ge 1} \underline{\mathrm{Sig}}_n$$

and edges between n^{th} and $(n-1)^{\mathrm{st}}$ level given by $L_{m,n-1}^{m,n}$ for any $m \ge n$ (note the these links are independent of $m \ge n$ by (4.3.3)). The boundary of this graph is classified by $\underline{\mathrm{Sig}}_\infty$ by combining Theorem 1.7.1 (for coherent systems supported on Sig_∞) and Theorem 4.3.2 (for coherent systems supported on $\underline{\mathrm{Sig}}_\infty^{(k)}$), and the explicit coherent systems in the statement follow from the above computations. □

Proof of Theorem 1.7.2. Any $\mathrm{GL}_\infty(\mathbb{Z}_p) \times \mathrm{GL}_\infty(\mathbb{Z}_p)$-invariant measure on $\mathrm{Mat}_{\infty \times \infty}(\mathbb{Q}_p)$ is uniquely determined by its marginals on $m \times n$ truncations for finite m, n, which are each $\mathrm{GL}_m(\mathbb{Z}_p) \times \mathrm{GL}_n(\mathbb{Z}_p)$-invariant. The $\mathrm{GL}_n(\mathbb{Z}_p) \times \mathrm{GL}_m(\mathbb{Z}_p)$-invariant probability measures on $\mathrm{Mat}_{n \times m}(\mathbb{Q}_p)$ are in bijection with probability measures on $\underline{\mathrm{Sig}}_\infty$ supported on signatures with at most $\min(m, n)$ finite parts, via the map ESN. Hence removing a row (resp. column) induces a Markov kernel $\mathcal{M}(\underline{\mathrm{Sig}}_\infty) \to \mathcal{M}(\underline{\mathrm{Sig}}_\infty)$, and by Proposition 4.3.1 this Markov kernel is exactly $L_{m,n-1}^{m,n}$ (resp. $L_{m-1,n}^{m,n}$). Hence Theorem 4.3.3 yields that the set of extreme $\mathrm{GL}_\infty(\mathbb{Z}_p) \times \mathrm{GL}_\infty(\mathbb{Z}_p)$-invariant measures on $\mathrm{Mat}_{\infty \times \infty}(\mathbb{Q}_p)$ is in bijection with $\underline{\mathrm{Sig}}_\infty$. Here the measure E_μ corresponding to μ is determined by the fact that each $m \times n$ corner has negative singular numbers distributed by the measure $M_{m,n}^\mu$ in

Theorem 4.3.3. ☐

We have shown that the extreme bi-invariant measures are parametrized somehow by $\underline{\mathrm{Sig}}_\infty$, but in [BQ17] the measure corresponding to a given $\mu \in \underline{\mathrm{Sig}}_\infty$ is defined quite differently, and it is not at all clear a priori that it is the same as our measure E_μ. Let us describe these measures.

In the finite or infinite setting, there are two natural families of random matrices in $\mathrm{Mat}_{n\times m}(\mathbb{Q}_p)$ which are invariant under the natural action of $\mathrm{GL}_n(\mathbb{Z}_p) \times \mathrm{GL}_m(\mathbb{Z}_p)$:

- (Haar) $p^{-k}Z$, where $k \in \mathbb{Z} \cup \{-\infty\}$ and Z has iid entries distributed by the additive Haar measure on $\mathbb{Z}_p$.
- (Nonsymmetric Wishart-type) $p^{-k}X^TY$, where $X \in \mathbb{Z}_p^n, Y \in \mathbb{Z}_p^m$ have iid additive Haar entries.

One can of course obtain invariant measures by summing the above random matrices, which motivates the following class of measures.

Definition 31. Let $\mu \in \underline{\mathrm{Sig}}_\infty$, and let $\mu_\infty := \lim_{\ell\to\infty} \mu_\ell \in \mathbb{Z} \cup \{-\infty\}$. Let $X_i^{(\ell)}, Y_j^{(\ell)}, Z_{ij}$ be iid and distributed by the additive Haar measure on $\mathbb{Z}_p$ for $i, j, \ell \geq 1$. Then we define the measure $\tilde{E}_\mu$ on $\mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p)$ as the distribution of the random matrix

$$\left(\sum_{\ell:\mu_\ell > \mu_\infty} p^{-\mu_\ell} X_i^{(\ell)} Y_j^{(\ell)} + p^{-\mu_\infty} Z_{ij} \right)_{i,j\geq 1}.$$

It is shown in [BQ17, Theorem 1.3] that the $\tilde{E}_\mu, \mu \in \underline{\mathrm{Sig}}_\infty$ are exactly the extreme $\mathrm{GL}_\infty(\mathbb{Z}_p) \times \mathrm{GL}_\infty(\mathbb{Z}_p)$-invariant measures on $\mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p)$.

Proposition 4.3.4. *For any $\mu \in \underline{\mathrm{Sig}}_\infty$, $\tilde{E}_\mu = E_\mu$.*

Proof. By combining Theorem 1.7.2 with the result [BQ17, Theorem 1.3] that the $\tilde{E}_\mu$ are exactly the extreme measures, we have that $\{\tilde{E}_\mu : \mu \in \underline{\mathrm{Sig}}_\infty\} = \{E_\mu : \mu \in \underline{\mathrm{Sig}}_\infty\}$. Hence for each $\mu \in \underline{\mathrm{Sig}}_\infty$ there exists $\nu \in \underline{\mathrm{Sig}}_\infty$ such that $\tilde{E}_\mu = E_\nu$. Suppose for the sake of contradiction that $\nu \neq \mu$. Let $k \geq 1$ be the smallest index for which $\mu_k \neq \nu_k$, let

$$f : \mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p) \to \underline{\mathrm{Sig}}_k$$

be the map to the first k singular numbers of the top left $k \times k$ corner, and let

$$S_{\leq\mu}^{(k)} := \{\lambda \in \underline{\mathrm{Sig}}_k : \lambda_i \leq \mu_i \text{ for } 1 \leq i \leq k\}.$$

We claim that

$$f_*(\tilde{E}_\mu) \text{ is supported on } S_{\leq\mu}^{(k)} \text{ and } (f_*(\tilde{E}_\mu))(\mu_1, \ldots, \mu_k) > 0, \tag{4.3.5}$$

and

$$f_*(E_\nu) \text{ is supported on } S_{\leq\nu}^{(k)} \text{ and } (f_*(E_\nu))(\nu_1, \ldots, \nu_k) > 0. \tag{4.3.6}$$

The first, (4.3.5), follows straightforwardly from Definition 31, while (4.3.6) follows from Theorem 4.3.3 and Lemma 4.2.6.

If $\mu_k > \nu_k$, then $\mathrm{Supp}(f_*(\tilde{E}_\mu)) \supsetneq \mathrm{Supp}(f_*(E_\nu))$, while if $\mu_k < \nu_k$ then $\mathrm{Supp}(f_*(\tilde{E}_\mu)) \subsetneq \mathrm{Supp}(f_*(E_\nu))$, contradicting the claim $f_*(\tilde{E}_\mu) = f_*(E_\nu)$. Therefore there does not exist k as above, so $\mu = \nu$, completing the proof. □

Combining Proposition 4.3.4 with Theorem 1.7.2 in fact provides a (quite indirect!) computation of the singular numbers of $m \times n$ truncations of the infinite matrices in Definition 31.

Corollary 4.3.5. *The negative singular numbers of an $m \times n$ corner of an infinite matrix with distribution $\tilde{E}_\mu$ are distributed by the measure $M_{m,n}^\mu$ of Theorem 4.3.3.*

It seems possible that the summation which defines the measures $M_{m,n}^\mu$ may be simplified to get more explicit formulas for the above distributions, though we do not address this question here.

Remark 23. There are several comments on the relation between our setup and that of [BQ17] which are worth highlighting:

- We work over $\mathbb{Q}_p$ while [BQ17] works over an arbitrary non-Archimedean local field F. Such a field has a ring of integers $\mathcal{O}_F$ playing the role of $\mathbb{Z}_p$ and a uniformizer ω playing the role of p, and a finite residue field $\mathcal{O}_F/\omega\mathcal{O}_F \cong \mathbb{F}_q$. Our results transfer mutatis mutandis to this setting with $t = 1/q$, as the only needed input Proposition 4.3.1 transfers in view of Remark 1.

- While we simply prove a bijection, a short additonal argument shows that the space of extreme invariant measures on $\mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p)$ is homeomorphic to $\underline{\mathrm{Sig}}_\infty$ with natural topologies on both spaces, see the proof of Theorem 1.3 of [BQ17] for details.

- We have used the language of extreme and ergodic measures interchangeably, but for an explanation of how the extreme measures are exactly the ergodic ones in the conventional sense, for this problem and more general versions, see [BQ17, Section 2.1].

Remark 24. As mentioned in the Introduction, [BQ17] also classify extreme measures on infinite symmetric matrices $\mathrm{Sym}(\mathbb{N}, \mathbb{Q}_p) := \{A \in \mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p) : A^T = A\}$ invariant under the action of $\mathrm{GL}_\infty(\mathbb{Z}_p)$ by $(g, A) \mapsto gAg^T$. The statement is more involved, essentially due to the fact that the $\mathrm{GL}_n(\mathbb{Z}_p)$-orbits on $\mathrm{Sym}(n, \mathbb{Q}_p)$ are parametrized by their singular numbers together with additional data, unlike the $\mathrm{GL}_n(\mathbb{Z}_p) \times \mathrm{GL}_m(\mathbb{Z}_p)$-orbits on $\mathrm{Mat}_{n\times m}(\mathbb{Z}_p)$. To pursue a similar strategy to our proof of Theorem 1.7.2 one would need an analogue of Proposition 4.3.1, i.e. a result giving the distribution of the $\mathrm{GL}_{n-1}(\mathbb{Z}_p)$-orbit of an $(n-1) \times (n-1)$ corner of an $n \times n$ symmetric matrix drawn uniformly from a fixed $\mathrm{GL}_n(\mathbb{Z}_p)$-orbit. Given that the parametrization of these orbits involves more data than the (extended) signature specifying their singular numbers, it is not immediately clear how the answer would be expressed in terms of Hall-Littlewood polynomials.

We do however expect a solution in terms of Hall-Littlewood polynomials to a related problem which is coarser. The problem is to find the distribution of just the singular numbers, rather than $\mathrm{GL}_{n-1}(\mathbb{Z}_p)$-orbits, of an $(n-1) \times (n-1)$ corner of a random element of $\mathrm{Sym}(n, \mathbb{Q}_p)$ with fixed singular numbers and $\mathrm{GL}_n(\mathbb{Z}_p)$-invariant distribution. The existence of such a result is suggested by a known expression for the singular numbers of an $n \times n$ symmetric matrix with iid (apart from the symmetry constraint) entries distributed by the additive Haar measure on $\mathbb{Z}_p$. This distribution was computed in [CKL$^+$15], and shown to be equivalent to a measure coming from one of the so-called Littlewood identities for Hall-Littlewood polynomials in [Ful16]. It seems natural that a solution to this problem could be augmented with the extra data required to parametrize $\mathrm{GL}_n(\mathbb{Z}_p)$-orbits, answering the question of the previous paragraph. We have not attempted to pursue this direction.

4.4 Ergodic decomposition of p-adic Hua measures

We now define a special family of measures on $\mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p)$, the *$p$-adic Hua measures*, introduced in [Ner13]. Their decomposition into the ergodic measures $\tilde{E}_\mu$ of Definition 31 was computed in [Ass22]. We will rederive that result, showing in the process that the p-adic Hua measures have a natural interpretation in terms of measures on partitions derived from Hall-Littlewood polynomials.

Definition 32. For $\lambda \in \mathrm{Sig}_n$, we set

$$\lambda^+ := (\max(\lambda_1, 0), \dots, \max(\lambda_n, 0)) \in \mathrm{Sig}_n^{\geq 0}.$$

Definition 33. The p-adic Hua measure $\mathbb{M}_n^{(s)}$ on $\mathrm{Mat}_{n\times n}(\mathbb{Q}_p)$ is defined by

$$d\mathbb{M}_n^{(s)}(A) = \frac{(p^{-1-s};p^{-1})_n^2}{(p^{-1-s};p^{-1})_{2n}} p^{|\,\mathrm{ESN}(A)^+|(-s-2n)} d\mathbb{M}^{(n)}(A),$$

where $\mathbb{M}^{(n)}$ is the product over all n^2 matrix entries of the additive Haar measure $\mathbb{M}$ on $\mathbb{Q}_p$.

The following computation of the distribution of the singular numbers of $\mathbb{M}_n^{(s)}$ is done in [Ass22, Proposition 3.1], using Definition 33 and results of [Mac98a, Chapter V].

Proposition 4.4.1. *The pushforward of $\mathbb{M}_n^{(s)}$ under $-\mathrm{SN} : \mathrm{Mat}_{n\times n}(\mathbb{Q}_p) \to \underline{\mathrm{Sig}}_n$ (the map SN of Definition 3 composed with $\lambda \mapsto -\lambda$) is supported on Sig_n and given by*

$$\left(-\mathrm{SN}_*(\mathbb{M}_n^{(s)})\right)(\lambda) = \frac{(u;t)_n^2}{(u;t)_{2n}} u^{|\lambda^+|} t^{(2n-1)(|\lambda^+|-|\lambda|)+2n(\lambda)} \frac{(t;t)_n^2}{\prod_{x\in\mathbb{Z}}(t;t)_{m_x(\lambda)}},$$

where as usual $t = 1/p$, and $u = t^{1+s}$.

We may now prove the main result, which we recall. Note that by Proposition 4.3.4 the same result holds with E_μ replaced by $\tilde{E}_\mu$, and it is the latter version which was proven in [Ass22].

Theorem 1.7.3. *Fix a prime p and real parameter $s > -1$, and let $t = 1/p$ and $u = p^{-1-s}$. Then the infinite p-adic Hua measure $\mathbb{M}_\infty^{(s)}$ decomposes into ergodic measures according to*

$$\mathbb{M}_\infty^{(s)} = \sum_{\mu\in\mathbb{Y}} \frac{P_\mu(1,t,\dots;t)Q_\mu(u,ut,\dots;t)}{\Pi(1,\dots;u,\dots)} E_\mu \tag{1.7.4}$$

where E_μ is as defined in Theorem 1.7.2.

Proof. The p-adic Hua measure is uniquely determined by its projections to $n \times n$ corners, and by extremality of the measures E_μ any decomposition into a convex combination of them is unique. Hence it suffices to show that a matrix A, distributed by the measure on $\mathrm{Mat}_{\infty\times\infty}(\mathbb{Q}_p)$ described by RHS(1.7.4), has $n \times n$ corners given by the finite p-adic Hua measure $\mathbb{M}_n^{(s)}$. By Proposition 4.4.1 and Theorem 1.7.2, it suffices to show

$$\sum_{\mu \in \mathbb{Y}} \frac{P_\mu(1,t,\ldots)Q_\mu(u,ut,\ldots)}{\Pi(1,\ldots;u,\ldots)} \sum_{\lambda \in \mathrm{Sig}_n^{\geq 0}} \frac{P_{\mu/\lambda}(t^n,\ldots)P_\lambda(1,\ldots,t^{n-1})}{P_\mu(1,t,\ldots)} \tilde{Q}_{-\nu/-\lambda}(t,t^2,\ldots)$$
$$\times \frac{P_{-\nu}(1,\ldots,t^{n-1})}{P_{-\lambda}(1,\ldots,t^{n-1})\Pi(1,\ldots,t^{n-1};t,\ldots)} = \frac{(u;t)_n^2}{(u;t)_{2n}} u^{|\nu^+|} t^{(2n-1)(|\nu^+|-|\nu|)+2n(\nu)} \frac{(t;t)_n^2}{\prod_{x\in\mathbb{Z}}(t;t)_{m_x(\nu)}} \tag{4.4.1}$$

The proof is a surprisingly long series of applications of the Cauchy identity/branching rule and principal specialization formulas. We first cancel the $P_\mu(1,\ldots)$ factors and apply the Cauchy identity (2.2.26) to the sum over μ to obtain

$$\frac{P_{-\nu}(1,\ldots,t^{n-1})}{\Pi(1,\ldots;u,\ldots)\Pi(1,\ldots,t^{n-1};t,\ldots)}$$
$$\times \sum_{\lambda\in\mathrm{Sig}_n^{\geq 0}} \frac{P_\lambda(1,\ldots,t^{n-1})}{P_{-\lambda}(1,\ldots,t^{n-1})} \tilde{Q}_{-\nu/-\lambda}(t,\ldots)\tilde{Q}_{\lambda/(0[n])}(u,\ldots)\Pi(t^n,\ldots;u,\ldots). \tag{4.4.2}$$

Using that

$$\Pi(1,\ldots,t^{n-1};t,\ldots) = (t;t)_n,$$

and

$$P_{-\nu}(1,\ldots,t^{n-1}) = P_\nu(1,\ldots,t^{-(n-1)}) = t^{-(n-1)|\nu|}P_\nu(1,\ldots,t^{n-1})$$

and similarly for λ, (4.4.2) becomes

$$\frac{(t;t)_n P_\nu(1,\ldots,t^{n-1})t^{(n-1)(|\lambda|-|\nu|)}}{\Pi(1,\ldots,t^{n-1};u,\ldots)} \sum_{\lambda\in\mathrm{Sig}_n^{\geq 0}} \tilde{Q}_{-\nu/-\lambda}(t,\ldots)\tilde{Q}_{\lambda/(0[n])}(u,\ldots). \tag{4.4.3}$$

It follows from the explicit branching rule Lemma 2.2.14 and the principal specialization formula Proposition 2.2.15 for P that

$$\tilde{Q}_{-\nu/-\lambda}(x) = \tilde{Q}_{\lambda/\nu}(x)\frac{t^{-n(\lambda)}P_\lambda(1,\dots,t^{n-1})}{t^{-n(\nu)}P_\nu(1,\dots,t^{n-1})}. \tag{4.4.4}$$

By definition of skew Q functions (4.4.4) immediately extends to

$$\tilde{Q}_{-\nu/-\lambda}(x_1,\dots,x_k) = \tilde{Q}_{\lambda/\nu}(x_1,\dots,x_k)\frac{t^{-n(\lambda)}P_\lambda(1,\dots,t^{n-1})}{t^{-n(\nu)}P_\nu(1,\dots,t^{n-1})}$$

for any k, hence to an equality of symmetric functions and hence specializes to

$$\tilde{Q}_{-\nu/-\lambda}(t^n,\dots) = \tilde{Q}_{\lambda/\nu}(t^n,\dots)\frac{t^{-n(\lambda)}P_\lambda(1,\dots,t^{n-1})}{t^{-n(\nu)}P_\nu(1,\dots,t^{n-1})}. \tag{4.4.5}$$

By first absorbing the $t^{(n-1)(|\lambda|-|\nu|)}$ into $\tilde{Q}_{-\nu/-\lambda}$ in (4.4.3) and then substituting (4.4.5) and simplifying $\tilde{Q}_{\lambda/(0[n])}$ via Proposition 2.2.15, (4.4.3) becomes

$$\begin{aligned}&\frac{(t;t)_n P_\nu(1,\dots,t^{n-1})}{\Pi(1,\dots,t^{n-1};u,\dots)}\sum_{\lambda\in\mathrm{Sig}_n^{\geq 0}}\tilde{Q}_{\lambda/\nu}(t^n,\dots)\frac{t^{-n(\lambda)}P_\lambda(1,\dots,t^{n-1})}{t^{-n(\nu)}P_\nu(1,\dots,t^{n-1})}u^{|\lambda|}t^{n(\lambda)}\\ &=\frac{(t;t)_n t^{n(\nu)}}{\Pi(1,\dots,t^{n-1};u,\dots)}\sum_{\lambda\in\mathrm{Sig}_n^{\geq 0}}\tilde{Q}_{\lambda/\nu}(t^n,\dots)P_\lambda(u,\dots,ut^{n-1}).\end{aligned} \tag{4.4.6}$$

At first glance, the sum on the RHS of (4.4.6) looks like the one in the Cauchy identity (2.2.16), but there is a nontrivial difference: the sum is over only nonnegative signatures. If $\nu \in \mathrm{Sig}_n^{\geq 0}$ itself, this poses no issue and the Cauchy identity applies directly, but in general this is not the case.

Luckily, using the explicit formula in Theorem 2.2.16 we may relate the sum in (4.4.6) to one to which the Cauchy identity applies. By slightly rearranging terms in Theorem 2.2.16, we have that for $\lambda \in \mathrm{Sig}_n^{\geq 0}$,

$$\begin{aligned}\tilde{Q}_{\lambda/\nu}(t^n,\dots) &= \frac{t^{n\cdot(|\lambda|-|\nu|)}}{\prod_{x\in\mathbb{Z}}(t;t)_{m_x(\nu)}}\prod_{x\leq 0}(t^{1+n-\nu'_x};t)_{m_x(\nu)}t^{\binom{n-\nu'_x}{2}}\prod_{x>0}(t^{1+\lambda'_x-\nu'_x};t)_{m_x(\nu)}t^{\binom{\lambda'_x-\nu'_x}{2}}\\ \tilde{Q}_{\lambda/\nu^+}(t^n,\dots) &= \frac{t^{n\cdot(|\lambda|-|\nu^+|)}}{\prod_{x>0}(t;t)_{m_x(\nu)}}\prod_{x>0}(t^{1+\lambda'_x-\nu'_x};t)_{m_x(\nu)}t^{\binom{\lambda'_x-\nu'_x}{2}}\end{aligned} \tag{4.4.7}$$

where ν^+ is the truncation as in Definition 32. Since

$$\prod_{x \le 0} (t^{1+n-\nu'_x}; t)_{m_x(\nu)} = (t;t)_{|\{i:\nu_i \le 0\}|} = (t;t)_{m_0(\nu^+)},$$

(4.4.7) implies that

$$\tilde{Q}_{\lambda/\nu}(t^n, \ldots) = t^{n\cdot(|\nu^+|-|\nu|)+\sum_{x\le 0}\binom{n-\nu'_x}{2}} \frac{(t;t)_{m_0(\nu^+)}}{\prod_{x\le 0}(t;t)_{m_x(\nu)}} \tilde{Q}_{\lambda/\nu^+}(t^n, \ldots).$$

Therefore

$$\begin{aligned}
&\sum_{\lambda \in \mathrm{Sig}_n^{\ge 0}} \tilde{Q}_{\lambda/\nu}(t^n, \ldots) P_\lambda(u, \ldots, ut^{n-1}) \\
&= t^{n\cdot(|\nu^+|-|\nu|)+\sum_{x\le 0}\binom{n-\nu'_x}{2}} \frac{(t;t)_{m_0(\nu^+)}}{\prod_{x\le 0}(t;t)_{m_x(\nu)}} \sum_{\lambda \in \mathrm{Sig}_n^{\ge 0}} \tilde{Q}_{\lambda/\nu^+}(t^n, \ldots) P_\lambda(u, \ldots, ut^{n-1}) \\
&= t^{n\cdot(|\nu^+|-|\nu|)+\sum_{x\le 0}\binom{n-\nu'_x}{2}} \frac{(t;t)_{m_0(\nu^+)}}{\prod_{x\le 0}(t;t)_{m_x(\nu)}} \Pi(t^n, \ldots; u, \ldots, ut^{n-1}) P_{\nu^+}(u, \ldots, ut^{n-1}) \\
&= \frac{(t;t)_n}{(ut^n;t)_n} \frac{u^{|\nu^+|} t^{n\cdot(|\nu^+|-|\nu|)+\sum_{x\le 0}\binom{n-\nu'_x}{2}+n(\nu^+)}}{\prod_{x\in\mathbb{Z}}(t;t)_{m_x(\nu)}}
\end{aligned} \tag{4.4.8}$$

by applying (2.2.26) and Proposition 2.2.15. It is an elementary check from the definitions that

$$n \cdot (|\nu^+| - |\nu|) + \sum_{x \le 0} \binom{n-\nu'_x}{2} + n(\nu^+) = (2n-1)(|\nu^+| - |\nu|) + n(\nu). \tag{4.4.9}$$

Substituting (4.4.9) into (4.4.8) and the result into (4.4.6) yields

$$\begin{aligned}
&(t;t)_n (u;t)_n t^{n(\nu)} \frac{(t;t)_n}{(ut^n;t)_n} \frac{u^{|\nu^+|} t^{(2n-1)(|\nu^+|-|\nu|)+n(\nu)}}{\prod_{x\in\mathbb{Z}}(t;t)_{m_x(\nu)}} \\
&= \frac{(u;t)_n^2}{(u;t)_{2n}} u^{|\nu^+|} t^{(2n-1)(|\nu^+|-|\nu|)+2n(\nu)} \frac{(t;t)_n^2}{\prod_{x\in\mathbb{Z}}(t;t)_{m_x(\nu)}},
\end{aligned} \tag{4.4.10}$$

which is the formula in Proposition 4.4.1, completing the proof. □

In some sense, the interpretation of the measures $M_n^{(s)}$ which we have given here explains their special nature and gives a natural non-historical route to their discovery. Let us suppose that one knew only Proposition 4.3.1 and Theorem 1.7.2, and wished to look for family of measures on $\mathrm{Mat}_{n\times n}(\mathbb{Q}_p)$ which are consistent under taking corners. Any measure on the boundary yields such a family (and vice versa), but only for very nice measures on the boundary do we expect the resulting measure on corners to have any reasonable description. Because the cotransition probabilities feature principal specializations, the natural candidate for this measure on the boundary is a Hall-Littlewood measure with two principal specializations $u_1, u_1t, \ldots$ and $u_2, u_2t, \ldots$. Indeed, the above combinatorics would break down entirely for other Hall-Littlewood measures. This leaves one free parameter because one may divide one specialization and multiply the other by any positive real number without changing the measure, and this free parameter is exactly the one in the p-adic Hua measure.

In another direction we note that, if one did not already know the result of [Ass22], the above considerations could help guess it. Since known natural measures on finite p-adic matrices have singular numbers distributed by Hall-Littlewood measures by Theorem 1.2.1 and Corollary 1.2.2, and the ergodic decomposition of a measure on infinite matrices is the analogue of the distribution of singular numbers of a finite matrix, it is natural to search for the ergodic decomposition within the space of Hall-Littlewood measures. As mentioned above, essentially the only Hall-Littlewood measures with nice explicit densities are those with principal specializations, of finite or infinite length. If one were of finite length, say N, then it is a straightforward consequence of Theorem 1.7.2 that at most N singular numbers of any corner are nonzero, which contradicts Proposition 4.4.1. Hence if the ergodic decomposition is according to a well-behaved (principally specialized) Hall-Littlewood measure, both specializations must be infinite, and this leads exactly to the one-parameter family of Hall-Littlewood measures which do indeed appear.

4.5 Markov dynamics on the boundary

For finite n, one has natural Hall-Littlewood process dynamics on Sig_n, as discussed in Chapter 2. It is natural to ask whether these yield dynamics on the boundary Sig_∞, and whether anything interesting may be said about them. For the q-Gelfand-Tsetlin graph mentioned in the Introduction, the resulting dynamics on Sig_∞ were studied in [BG13], see also the references therein for previously studied instances of this question on branching graphs in which the boundary is continuous rather than discrete. In this section, we show in Proposition 4.5.1 that the Hall-Littlewood process dynamics on the levels of $\mathscr{G}_t$ indeed lift to dynamics on $\partial\mathscr{G}_t$. This is motivated by Chapter 6 and Chapter 8, which study a continuous-time limit of these dynamics corresponding to a Hall-Littlewood process with one Plancherel specialization and one $1, t, \ldots$. We are not presently aware of an interpretation of the latter in terms of infinite p-adic random matrices when $t = 1/p$, as with earlier results in this chapter.

While the fact that the dynamics in Chapter 8 may be viewed as dynamics on $\partial\mathscr{G}_t$ is not technically necessary for their analysis in Chapter 8, it provides an interesting context for the results of Chapter 8. There exist other dynamics which arise in a structurally similar manner for different degenerations of Macdonald polynomials, but nonetheless have quite different asymptotic behavior, as we recall from the discussion in the Section 1.6. Because Proposition 4.5.1 requires branching graph formalism which is orthogonal to Chapter 8 apart from this motivation, we chose to prove it within this chapter.

We now consider Markovian dynamics on the boundary $\partial\mathscr{G}_t$. We will show that the Cauchy dynamics of Definition 14 with fixed principal specialization commute with the cotransition probabilities of $\mathscr{G}_t$ and hence extend to dynamics on the boundary, which are given by essentially the same formula after identifying the boundary with Sig_∞. Skew Q-polynomials generalize easily to infinite signatures: For $\nu, \lambda \in \mathrm{Sig}_\infty$, define

$$Q_{\nu/\lambda}(\alpha) := \begin{cases} \alpha^{\sum_i \nu_i - \lambda_i} \varphi_{\nu/\lambda} & \nu_i \geq \lambda_i \text{ for all } i \text{ and } \sum_{i \geq 1} \nu_i - \lambda_i < \infty \\ 0 & \text{otherwise} \end{cases} \tag{4.5.1}$$

where $\varphi_{\nu/\lambda}$ is extended from Lemma 2.2.14 to infinite signatures in the obvious way. In the case $\nu, \lambda \in \mathbb{Y}$, this agrees with the standard branching rule in Lemma 2.2.14.

Definition 34. For $0 < \alpha < 1$, define

$$\Gamma_\alpha^n(\lambda, \nu) = \tilde{Q}_{\nu/\lambda}(\alpha) \frac{P_\nu(1, \ldots, t^{n-1})}{P_\lambda(1, \ldots, t^{n-1}) \Pi(\alpha; 1, \ldots, t^{n-1})} \tag{4.5.2}$$

for $n \in \mathbb{Z}_{\geq 1}$ and $\lambda, \nu \in \mathrm{Sig}_n$. For $\mu, \kappa \in \mathbb{Y} + D$, define

$$\Gamma_\alpha^\infty(\mu, \kappa) = Q_{(\kappa - D[\infty])/(\mu - D[\infty])}(\alpha) \frac{P_{(\kappa - D[\infty])}(1, \ldots)}{P_{(\mu - D[\infty])}(1, \ldots) \Pi(\alpha; 1, t, \ldots)}. \tag{4.5.3}$$

Finally, for $\mu, \kappa \in \mathrm{Sig}_\infty^{unstable}$, define

$$\Gamma_\alpha^\infty(\mu, \kappa) = \lim_{D \to -\infty} \Gamma_\alpha^\infty(\mu^{(D)}, \kappa^{(D)}). \tag{4.5.4}$$

When $\mu \in \mathbb{Y}$, the dynamics defined by (4.5.3) yields a Hall-Littlewood process with one infinite specialization $1, t, \ldots$. The dynamics studied in Chapter 8 are a continuous-time limit of these by Lemma 2.2.7. We prove Proposition 4.5.1 in the above discrete-time setting to minimize technicalities, though the statement for the limiting continuous-time process is the exactly analogous.

Proposition 4.5.1. *For $n \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, Γ_α^n is a Markov kernel. For $1 \leq n < m < \infty$ it commutes with the links L_n^m in the sense that*

$$\Gamma_\alpha^n L_n^m = L_n^m \Gamma_\alpha^m. \tag{4.5.5}$$

Therefore given any coherent system $(M_n)_{n \geq 1}$ on $\mathscr{G}_t$, the pushforward measures $(M_n \Gamma_\alpha^n)_{n \geq 1}$ also form a coherent system. The induced map on $\partial \mathscr{G}_t$ is given by Γ_α^∞.

Proof. The fact that (4.5.2) and (4.5.3) define Markov kernels follows directly from the Cauchy identity, Lemma 2.2.3 and (2.2.26) respectively. For the infinite case (4.5.4), we must show

$$\sum_{\kappa \in \mathrm{Sig}_\infty} \lim_{D \to -\infty} Q_{(\kappa^{(D)} - D[\infty])/(\mu^{(D)} - D[\infty])}(\alpha) \frac{P_{(\kappa^{(D)} - D[\infty])}(1, \ldots)}{P_{(\mu^{(D)} - D[\infty])}(1, \ldots) \Pi(\alpha; 1, t, \ldots)} = 1. \tag{4.5.6}$$

Note that

$$Q_{(\kappa^{(D)} - D[\infty])/(\mu^{(D)} - D[\infty])}(\alpha) \frac{P_{(\kappa^{(D)} - D[\infty])}(1, \ldots)}{P_{(\mu^{(D)} - D[\infty])}(1, \ldots) \Pi(\alpha; 1, t, \ldots)} \mathbb{1}(\kappa_i = \mu_i \text{ whenever } \kappa_i < D)$$

increases monotonically as $D \to -\infty$ in a trivial way, namely it is either 0 (for D such that the indicator is 0) or its final constant value (when the indicator function is nonzero). Hence we again interchange limit and sum by monotone convergence, obtaining

$$\lim_{D \to -\infty} \sum_{\kappa \in \mathbb{Y} + D} Q_{(\kappa - D[\infty])/(\mu^{(D)} - D[\infty])}(\alpha) \frac{P_{(\kappa - D[\infty])}(1, \ldots)}{P_{(\mu^{(D)} - D[\infty])}(1, \ldots) \Pi(\alpha; 1, t, \ldots)}.$$

This is 1 by the Cauchy identity (2.2.26).

Below we will show (4.5.5), from which it follows that the maps Γ_α^n preserve coherent systems and hence induce a Markov kernel on $\partial\mathscr{G}_t$. To show that this Markov kernel is given by Γ_α^∞ we must show the '$m=\infty$' analogue of (4.5.5), namely for any $\mu \in \mathrm{Sig}_\infty, \nu \in \mathrm{Sig}_n$ one has

$$\sum_{\kappa \in \mathrm{Sig}_\infty} \Gamma_\alpha^\infty(\mu,\kappa) M_n^\kappa(\nu) = \sum_{\lambda \in \mathrm{Sig}_n} M_n^\mu(\lambda)\Gamma_\alpha^n(\lambda,\nu). \tag{4.5.7}$$

We will treat (4.5.5) and (4.5.7) simultaneously, and so introduce the notation $L_m^\infty(\mu,\cdot) := M_n^\mu(\cdot)$. For (4.5.7), if $\mu \in \mathbb{Y} + D$ for some D, then by translation-invariance and the Cauchy identity,

$$\begin{aligned}
\Gamma_\alpha^n L_n^\infty(\mu,\nu) &= \sum_{\lambda \in \mathrm{Sig}_n} L_n^\infty(\mu,\lambda)\Gamma_\alpha^n(\lambda,\nu) \\
&= \sum_{\lambda \in \mathrm{Sig}_n} L_n^\infty(\mu - D[\infty], \lambda - D[n])\Gamma_\alpha^n(\lambda - D[n], \nu - D[n]) \\
&= \sum_{\lambda \in \mathrm{Sig}_n} P_{(\mu - D[\infty])/(\lambda - D[n])}(t^n,\ldots)\frac{P_{(\lambda - D[n])}(1,\ldots,t^{n-1})}{P_{(\mu - D[\infty])}(1,\ldots)} \\
&\times \tilde{Q}_{(\nu - D[n])/(\lambda - D[n])}(\alpha)\frac{P_{(\nu - D[n])}(1,\ldots,t^{n-1})}{P_{(\lambda - D[n])}(1,\ldots,t^{n-1})\Pi(\alpha;1,\ldots,t^{n-1})} \\
&= \frac{P_{(\nu - D[n])}(1,\ldots,t^{n-1})}{(1,\ldots,t^{n-1})\Pi(\alpha;1,\ldots,t^{n-1})}\left(\frac{1}{\Pi(\alpha;t^n,\ldots)}\sum_{\kappa \in \mathbb{Y}} P_{\kappa/(\nu - D[n])}(t^n,\ldots)Q_{\kappa/(\mu - D[\infty])}(\alpha)\right) \\
&= \sum_{\kappa \in \mathbb{Y}} L_n^\infty(\kappa + D[\infty],\nu)\Gamma_\alpha^\infty(\mu,\kappa + D[\infty]) \\
&= L_n^\infty \Gamma_\alpha^\infty(\mu,\nu)
\end{aligned}$$

The proof of (4.5.5) is the same after replacing ∞ with m, without the translation by D issues. The case $\mu \in \mathrm{Sig}_\infty^{unstable}$ of (4.5.7) requires a limiting argument:

$$\begin{aligned}
\Gamma_\alpha^n L_n^\infty(\mu,\nu) = \sum_{\lambda \in \mathrm{Sig}_n} &\tilde{Q}_{\nu/\lambda}(\alpha)\frac{P_\nu(1,\ldots,t^{n-1})}{P_\nu(1,\ldots,t^{n-1})\Pi(\alpha;1,\ldots,t^{n-1})} \\
&\times \lim_{D \to -\infty} P_{(\mu^{(D)} - D[\infty])/(\lambda - D[n])}(t^n,\ldots)\frac{P_{(\lambda - D[n])}(1,\ldots,t^{n-1})}{P_{(\mu^{(D)} - D[\infty])}(1,\ldots)},
\end{aligned}$$

and by Theorem 2.2.16 and monotone convergence this is equal to

$$\lim_{D \to -\infty} \sum_{\lambda \in \mathrm{Sig}_n} \frac{\tilde{Q}_{\nu/\lambda}(\alpha)P_\nu(1,\ldots,t^{n-1})}{P_\nu(1,\ldots,t^{n-1})\Pi(\alpha;1,\ldots,t^{n-1})}P_{(\mu^{(D)} - D[\infty])/(\lambda - D[n])}(t^n,\ldots)\frac{P_{(\lambda - D[n])}(1,\ldots,t^{n-1})}{P_{(\mu^{(D)} - D[\infty])}(1,\ldots)}.$$

Using that $\Gamma^n_\alpha(\lambda,\nu)=\Gamma^n_\alpha(\lambda-D[n],\nu-D[n])$ yields

$$\lim_{D\to-\infty}\frac{P_{(\nu-D[n])}(1,\ldots,t^{n-1})}{P_{(\mu^{(D)}-D[\infty])}(1,\ldots)}\sum_{\lambda\in\mathrm{Sig}_n}\tilde{Q}_{(\nu-D[n])/(\lambda-D[n])}(\alpha)P_{(\mu^{(D)}-D[\infty])/(\lambda-D[n])}(t^n,\ldots).$$

Applying the Cauchy identity (2.2.26) and the fact that

$$\Pi(\alpha;1,\ldots,t^{n-1})\Pi(\alpha;t^n,\ldots)=\Pi(\alpha;1,\ldots),$$

and rearranging, yields

$$\lim_{D\to-\infty}\sum_{\tilde{\kappa}\in\mathbb{Y}}L^\infty_n(\tilde{\kappa},\nu-D[n])\Gamma^\infty_\alpha(\mu^{(D)}-D[\infty],\tilde{\kappa}).$$

Changing variables to $\kappa=\tilde{\kappa}+D[\infty]$ this is

$$\lim_{D\to-\infty}\sum_{\kappa\in\mathbb{Y}+D}L^\infty_n(\kappa-D[\infty],\nu-D[n])\Gamma^\infty_\alpha(\mu^{(D)}-D[\infty],\kappa-D[\infty]). \tag{4.5.8}$$

For each fixed D, there is an obvious bijection between $\mathbb{Y}+D$ and

$$\{\kappa\in\mathrm{Sig}^{unstable}_\infty:\kappa_i=\mu_i\text{ for all }i\text{ such that }\mu_i\le D\},$$

as signatures in either set are determined by their parts which are $>D$. Hence the sum in (4.5.8) is equal to

$$\sum_{\kappa\in\mathrm{Sig}^{unstable}_\infty}L^\infty_n(\kappa^{(D)}-D[\infty],\nu-D[n])\Gamma^\infty_\alpha(\mu^{(D)}-D[\infty],\kappa^{(D)}-D[\infty])I_D(\kappa,\mu), \tag{4.5.9}$$

where

$$I_D(\kappa,\mu):=\mathbb{1}(\kappa_i=\mu_i\text{ for all }i\text{ such that }\mu_i\le D)$$

The summands in (4.5.9), as functions of D, take at most two values, namely 0 (for all $\kappa\ne\mu$, for D positive enough that the indicator function is 0) and $L^\infty_n(\kappa,\nu)\Gamma^\infty_\alpha(\mu,\kappa)$ when the indicator function is nonzero. Hence monotone convergence again applies, yielding

$$\sum_{\kappa\in\mathrm{Sig}_\infty}\lim_{D\to-\infty}L^\infty_n(\kappa^{(D)}-D[\infty],\nu-D[n])\Gamma^\infty_\alpha(\mu^{(D)}-D[\infty],\kappa^{(D)}-D[\infty])I_D(\kappa,\mu).$$

The summand stabilizes to $L_n^\infty(\kappa,\nu)\Gamma_\alpha^\infty(\mu,\kappa)$ (using translation-invariance of L_n^∞), hence the above is equal to $L_n^\infty\Gamma_\alpha^\infty(\mu,\nu)$ as desired. This completes the proof. $\square$

THIS PAGE INTENTIONALLY LEFT BLANK

Chapter 5

Matrix products at fixed size: Limits and Gaussian fluctuations

The plan of this chapter is as follows. In Section 5.1 we state a general law of large numbers and functional central limit theorem for Hall-Littlewood processes, Theorem 5.1.1, and deduce the LLN and CLT for matrix products Theorem 1.3.1 from it. We spend the remainder of the section proving Theorem 5.1.1. In Section 5.2 we introduce the random sampling algorithm for Hall-Littlewood processes with one principal specialization $1, t, \ldots, t^{n-1}$ by a PushTASEP-like particle system. In Section 5.3 we introduce a simpler variant of this particle system which is easier to analyze asymptotically, and show that the two may be coupled with small error. In Section 5.4 we complete the proof by analyzing this particle system. In Section 5.5 we prove the universality of Lyapunov exponents stated earlier as Theorem 1.3.2.

5.1 Asymptotics of products of random matrices

Recall the main result.

Theorem 1.3.1. *Fix $n \geq 1$, and let $N_1, N_2, \ldots \in \mathbb{Z} \cup \{\infty\}$ with $N_j > n$ for all j. For each j, if $N_j < \infty$ let A_j be the top left $n \times n$ corner of a Haar distributed element of $\mathrm{GL}_{N_j}(\mathbb{Z}_p)$, and if $N_j = \infty$ let A_j have iid entries distributed by the additive Haar measure on $\mathbb{Z}_p$. For $k \in \mathbb{N}$ let*

$$(\lambda_1(k), \ldots, \lambda_n(k)) := \mathrm{SN}(A_k \cdots A_1).$$

Then we have a strong law of large numbers

$$\frac{\lambda_i(k)}{\sum_{j=1}^k \sum_{\ell=0}^{N_j-n-1} \frac{p^{-i-\ell}(1-p^{-1})}{(1-p^{-i-\ell-1})(1-p^{-i-\ell})}} \to 1 \text{ a.s. as } k \to \infty.$$

Let

$$\bar{\lambda}_i(k) := \lambda_i(k) - \sum_{j=1}^{k} \sum_{\ell=0}^{N_j-n-1} \frac{p^{-i-\ell}(1-p^{-1})}{(1-p^{-i-\ell-1})(1-p^{-i-\ell})}$$

and define the random function of $f_{\bar{\lambda}_i,k} \in C[0,1]$ as follows: set $f_{\bar{\lambda}_i,k}(0) = 0$ and

$$(f_{\bar{\lambda}_i,k}(1/k), f_{\bar{\lambda}_i,k}(2/k), \ldots, f_{\bar{\lambda}_i,k}(1)) = \frac{1}{\sqrt{\sum_{j=1}^k \sum_{\ell=0}^{N_j-n-1} \frac{p^{-i-\ell}(1-p^{-1})(1-p^{-2i-2\ell-1})}{(1-p^{-i-\ell})^2(1-p^{-i-\ell-1})^2}}}(\bar{\lambda}_i(1), \ldots, \bar{\lambda}_i(k)),$$

then linearly interpolate from these values on each interval $[\ell/k, (\ell+1)/k]$. Then as $k \to \infty$, the n-tuple of random functions $(f_{\bar{\lambda}_1,k}, \ldots, f_{\bar{\lambda}_n,k})$ converges in law in the sup norm topology on $C[0,1]$ to n independent standard Brownian motions.

In view of Corollary 3.1.3, this is a special case of the result below.

Theorem 5.1.1. *Fix the Hall-Littlewood parameter $t \in (0,1)$, and $n \in \mathbb{Z}_{>0}$. Let $x_1, x_2, \ldots \in (\delta, 1-\delta)$ for some $\delta > 0$, and let $\hat{x}_i = (x_i, tx_i, \ldots, t^{m_i-1}x_i)$ be collections of variables in t-geometric progression, possibly infinite, for each i. Let $(\lambda(1), \lambda(2), \ldots)$ be an infinite sequence of random signatures whose marginals are given by a Hall-Littlewood process,*

$$\Pr(\lambda(1) = \lambda^{(1)}, \ldots, \lambda(N) = \lambda^{(N)}) = \frac{\tilde{Q}_{\lambda^{(N)}/\lambda^{(N-1)}}(\hat{x}_N) \cdots \tilde{Q}_{\lambda^{(2)}/\lambda^{(1)}}(\hat{x}_2)\tilde{Q}_{\lambda^{(1)}}(\hat{x}_1)P_{\lambda_N}(1, \ldots, t^{n-1})}{\Pi(1, \ldots, t^{n-1}; \hat{x}_1, \ldots, \hat{x}_N)}.$$

Then we have the following strong law of large numbers. For each $i = 1, \ldots, n$,

$$\frac{\lambda_i(k)}{\sum_{j=1}^k \sum_{\ell=0}^{m_j-1} \frac{t^{i+\ell-1}x_j(1-t)}{(1-t^{i+\ell}x_j)(1-t^{i+\ell-1}x_j)}} \to 1 \text{ a.s. as } k \to \infty. \tag{5.1.1}$$

We also have the following functional central limit theorem. Let

$$\bar{\lambda}_i(k) := \lambda_i(k) - \sum_{j=1}^{k} \sum_{\ell=0}^{m_j-1} \frac{t^{i+\ell-1}x_j(1-t)}{(1-t^{i+\ell}x_j)(1-t^{i+\ell-1}x_j)}.$$

Let $f_{\bar{\lambda}_i,k}$ be the random element of $C[0,1]$ defined as follows: set $f(0)=0$ and

$$(f_{\bar{\lambda}_i,k}(1/k), f_{\bar{\lambda}_i,k}(2/k), \ldots, f_{\bar{\lambda}_i,k}(1)) = \frac{1}{\sqrt{\sum_{j=1}^{k}\sum_{\ell=0}^{m_j-1} \frac{t^{i+\ell-1}x_j(1-t)(1-t^{2i+2\ell-1}x_j^2)}{(1-t^{i+\ell-1}x_j)^2(1-t^{i+\ell}x_j)^2}}}(\bar{\lambda}_i(1), \ldots, \bar{\lambda}_i(k)),$$

then linearly interpolate from these values on each interval $[\ell/k, (\ell+1)/k]$. Then as $k \to \infty$, the n-tuple of random functions $(f_{\bar{\lambda}_1,k}, \ldots, f_{\bar{\lambda}_n,k})$ converges in law in the sup norm topology to n independent standard Brownian motions.

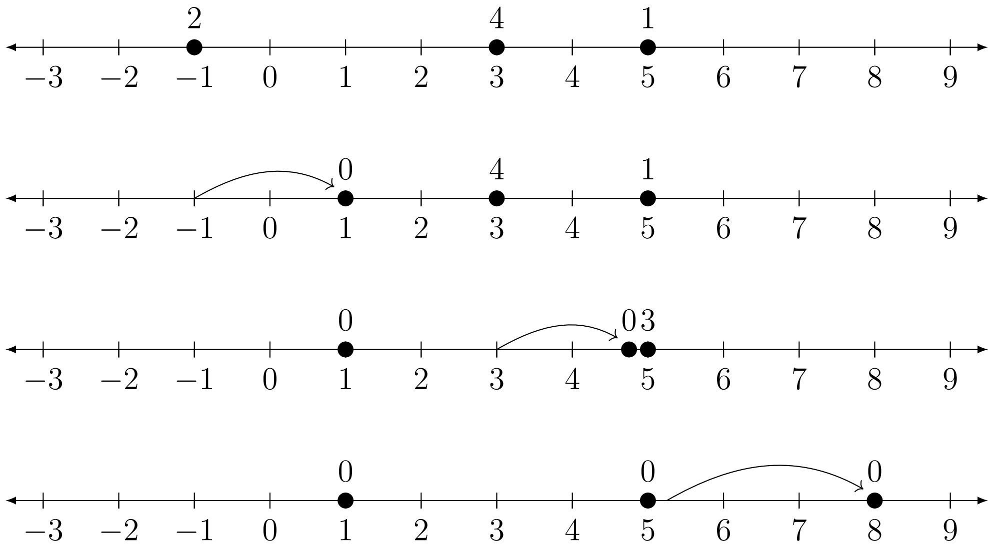
Remark 25. Though we have avoided it for the sake of simplicity, it is possible to define the product process more generally, allowing for nonsquare matrices. In the usual archimedean case this is done in [Ahn22b, Appendix A], and the p-adic case is exactly the same.

5.2 Sampling algorithm for Hall-Littlewood processes with one principal specialization

In what follows, we will identify signatures $\lambda \in \mathrm{Sig}_n$ with configurations of n particles on $\mathbb{Z}$ by placing $m_i(\lambda)$ particles at each position $i \in \mathbb{Z}$. Each particle corresponds to a part of λ, and we will refer to them as the $1^{\text{st}}, \ldots, n^{th}$ particle or 'particle $1, \ldots,$ particle n' to reflect this, even when some are in the same location. In this numbering, particle j will correspond to a particle at position λ_j.

Definition 35. Define the 'insertion map' $\iota : \mathbb{Z}_{\geq 0}^n \times \mathrm{Sig}_n \to \mathrm{Sig}_n$ by defining $\iota(a_1, \ldots, a_n; \lambda)$ as follows. First assign to each particle j an 'impulse' a_j. Particle n then moves to the right until it has either moved a_n steps or encountered particle $n-1$. If it encounters particle $n-1$, then it is 'blocked' by particle $n-1$ and donates the remainder $a_n - (\lambda_{n-1} - \lambda_n)$ of its impulse to particle $n-1$. Particle $n-1$ now has impulse $a_{n-1} + \max(0, a_n - (\lambda_{n-1} - \lambda_n))$, and moves in the same manner, possibly donating some of its impulse to particle $n-2$; all further particle evolve in the same manner.

Example 5.2.1. *To compute $\iota(1,4,2;(5,3,-1)) = (8,5,1)$ the particles jump as above. The numbers above the particles represent their impulses; note that impulse-donation from particle 2 to particle 1 occurs at the third step shown.*



It is obvious from Definition 35 that $\lambda \prec_Q \iota(a_1, \ldots, a_n; \lambda)$ for any $\mathbf{a} \in \mathbb{Z}_{\geq 0}^n$. It is also not hard to check by induction on i that one may equivalently define ι by defining the $(n-i)^{th}$ part

$$\iota(a_1, \ldots, a_n; \lambda)_{n-i} = \min(\lambda_{n-i-1}, \max(\lambda_{n-i}+a_{n-i}, \lambda_{n-i+1}+a_{n-i}+a_{n-i+1}, \ldots, \lambda_n+a_{n-i}+\ldots+a_n)) \tag{5.2.1}$$

for each $i = 0, \ldots, n-1$, where we formally take $\lambda_0 = \infty$ in the edge case $i = n-1$.

We now use the insertion ι with random input $a_1, \ldots, a_n$ to define random signatures, which we will show in Proposition 5.2.2 yields the 'Cauchy' Markov transition dynamics of Proposition 2.2.9. First we define the measures which will be the distributions of the a_i.

Definition 36. Let G_x be the measure on $\mathbb{Z}_{\geq 0}$ which is the distribution of $\max(X-T, 0)$ where $X \sim \mathrm{Geom}(x), T \sim \mathrm{Geom}(t)$. Explicitly,

$$G_x(\ell) = \frac{1-x}{1-tx}(1-t)^{\mathbb{1}(\ell>0)}x^\ell. \tag{5.2.2}$$

Equivalently G_x is defined by the generating function

$$\sum_{\ell \geq 0} G_x(\ell) z^\ell = \frac{1-x}{1-tx}\frac{1-txz}{1-xz} = \frac{\Pi_{(0,t)}(z;x)}{\Pi_{(0,t)}(1;x)}. \tag{5.2.3}$$

Proposition 5.2.2. *For $0 < x < 1$, let $X_1, \ldots, X_n$ be independent with $X_i \sim G_{xt^{i-1}}$. Let $\lambda, \nu \in \mathrm{Sig}_n$ with $\lambda \prec_Q \nu$. Then*

$$\Pr(\iota(X_1, \ldots, X_n; \lambda) = \nu) = \frac{1-x}{1-t^n x} \prod_{j:m_j(\lambda)=m_j(\nu)+1} (1 - t^{m_j(\lambda)}) \prod_{i=1}^{n} (xt^{i-1})^{\nu_i - \lambda_i} \tag{5.2.4}$$

$$= \frac{\tilde{Q}_{\nu/\lambda}(x) P_\nu(1, \ldots, t^{n-1})}{P_\lambda(1, \ldots, t^{n-1}) \Pi_{(0,t)}(x; 1, \ldots, t^{n-1})}. \tag{5.2.5}$$

Proof. We let $\Pr_x(\lambda \to \nu) := \Pr(\iota(X_1, \ldots, X_n; \lambda) = \nu)$. The equality of the RHS of (5.2.4) with (5.2.5) follows by Proposition 2.2.15 while the first requires proof. We will explicitly compute $\Pr_x(\lambda \to \nu)$ from the definition of ι.

Let $\lambda = (a_1[k_1], \ldots, a_r[k_r])$, where the a_i are distinct, k_i are integers ≥ 1 with $\sum_i k_i = n$. To avoid cumbersome notation for edge cases, we formally take $\lambda_0 = a_0 = \infty$ in some formulas below.

It is clear from the definition that $\Pr_x(\lambda \to \nu)$ is nonzero only if $\lambda \prec \nu$. By interlacing, only the rightmost particle in the group of k_i particles at location a_i can exit to the right; the location where it stops is $\iota(X_1, \ldots, X_n; \lambda)_{n-(k_r+\ldots+k_i)+1}$.

Let us define random variables $N_i, 1 \leq i \leq r$, to be the location of the particle that jumps out of the i^{th} clump after its jump (if no particle leaves the clump, then $N_i = a_i$. Explicitly, $N_i = \iota(X_1, \ldots, X_n; \lambda)_{n-(k_r+\ldots+k_i)+1}$, and so

$$\Pr(N_1 = \nu_1, N_2 = \nu_{k_1+1}, \ldots, N_r = \nu_{n-k_r+1}) = \Pr_x(\lambda \to \nu)$$

for any $\nu \succ_Q \lambda$. We will explicitly compute the joint distribution of the N_i, starting with the distribution of N_r.

By Definition 35, N_r has distribution

$$\min(a_{r-1}, a_r + X_{n-k_r+1} + \ldots + X_n).$$

Using the probability generating function (5.2.3), we have that

$$\Pr(X_{n-k_r+1} + \ldots + X_n = \ell) = \left(\prod_{i=n-k_r+1}^{n} \frac{1 - t^{i-1}x}{1 - t^i x} \frac{1 - t^i xz}{1 - t^{i-1}xz} \right) [z^\ell] \tag{5.2.6}$$

$$= \left(\frac{1 - t^{n-k_r}x}{1 - t^n x} \frac{1 - t^n xz}{1 - t^{n-k_r}xz} \right) [z^\ell]. \tag{5.2.7}$$

Expanding this out, we have

$$\Pr(N_r = a_r + \ell) = \begin{cases} \frac{1-t^{n-k_r}x}{1-t^n x} & \ell = 0 \\ \frac{1-t^{n-k_r}x}{1-t^n x}(1 - t^{k_r})(xt^{n-k_r})^\ell & 0 < \ell < a_{r-1} - a_r \\ \frac{1}{1-t^n x}(1 - t^{k_r})(xt^{n-k_r})^\ell & \ell = a_{r-1} - a_r \end{cases} . \tag{5.2.8}$$

Note that this formula still makes sense when $r = 1$, as the last case $\ell = \infty$ has probability 0.

Now let us find the distribution of N_{r-1}. Its distribution, conditional on N_r, depends on whether $N_r < a_{r-1}$ or $N_r = a_{r-1}$.

Case I: $N_r < a_{r-1}$**.**

In this case, we may compute the conditional distribution of N_r exactly as before, obtaining

$$\Pr(N_{r-1} = a_{r-1} + \ell) = \begin{cases} \frac{1-t^{n-k_r-k_{r-1}}x}{1-t^{n-k_r}x} & \ell = 0 \\ \frac{1-t^{n-k_r-k_{r-1}}x}{1-t^{n-k_r}x}(1 - t^{k_{r-1}})(xt^{n-k_r-k_{r-1}})^\ell & 0 < \ell < a_{r-2} - a_{r-1} \\ \frac{1}{1-t^{n-k_r}x}(1 - t^{k_{r-1}})(xt^{n-k_r-k_{r-1}})^\ell & \ell = a_{r-2} - a_{r-1} \end{cases} . \tag{5.2.9}$$

Case II: $N_r = a_{r-1}$**.**

In this case, the computation is different: Because N_r may donate some of its jump, Definition 35 yields that N_{r-1} has distribution

$$\min(a_{r-2}, a_{r-1} + Y + X_{n-k_r-k_{r-1}+1} + \ldots + X_{n-k_r}),$$

where Y comes from the possible jump-donation of N_r and has distribution given by

$$\Pr(Y = \ell) = \Pr((X_{n-k_r+1} + \ldots + X_n) - (a_{r-1} - a_r) = \ell | X_{n-k_r+1} + \ldots + X_n \geq a_{r-1} - a_r). \tag{5.2.10}$$

This looks overly complicated, but let us back up and see what it all means. As we noted before, $X_{n-k_r+1} + \ldots + X_n$ has probability generating function

$$\frac{1 - t^{n-k_r}x}{1 - t^n x} \frac{1 - t^n xz}{1 - t^{n-k_r}xz},$$

hence

$$X_{n-k_r+1} + \ldots + X_n \sim \max(\mathrm{Geom}(t^{n-k_r}x) - \mathrm{Geom}(t^{k_r}), 0).$$

How, in general, would one sample $Z \sim \mathrm{Geom}(x) - \mathrm{Geom}(w)$? A simple way is to take two coin with probability x and w of heads respectively, and keep flipping them until one comes up tails, then see how many additional flips it takes before the other comes up tails–call this (random) number ℓ. If the w-coin came up tails first, then $Z = \ell$; if the x-coin came up tails first, $Z = -\ell$. From this description it is clear that if we condition on $Z \geq 1$, or indeed $Z \geq c$ for any $c \geq 1$, we are conditioning on the event that the w-coin comes up tails first *and* the x-coin comes up heads for at least c additional rounds. It is thus clear that the conditional distribution of Z, given $Z \geq c$, is $c + \mathrm{Geom}(x)$.

Applying this to our above situation, we have that conditioning $X_{n-k_r+1} + \ldots + X_n$ to be above some positive number, it will have a geometric distribution. Specifically,

$$\Pr((X_{n-k_r+1} + \ldots + X_n) - (a_{r-1} - a_r) = \ell | X_{n-k_r+1} + \ldots + X_n \geq a_{r-1} - a_r) = (1 - t^{n-k_r}x)(t^{n-k_r}x)^\ell.$$

Hence by (5.2.10), $Y \sim \mathrm{Geom}(t^{n-k_r}x)$, so Y has probability generating function $\frac{1-t^{n-k_r}x}{1-t^{n-k_r}xz}$. Thus $Y + X_{n-k_r-k_{r-1}+1} + \ldots + X_{n-k_r}$ has probability generating function

$$\frac{1 - t^{n-k_r}x}{1 - t^{n-k_r}xz} \cdot \left(\frac{1 - t^{n-k_r-k_{r-1}}x}{1 - t^{n-k_r}x} \frac{1 - t^{n-k_r}xz}{1 - t^{n-k_r-k_{r-1}}xz} \right) = \frac{1 - t^{n-k_r-k_{r-1}}x}{1 - t^{n-k_r-k_{r-1}}xz},$$

i.e.

$$Y + X_{n-k_r-k_{r-1}+1} + \ldots + X_{n-k_r} \sim \mathrm{Geom}(t^{n-k_r-k_{r-1}}x).$$

Thus at last we have

$$\Pr(N_{r-1} = a_{r-1}+\ell) = \begin{cases} (1 - t^{n-k_r-k_{r-1}}x)(t^{n-k_r-k_{r-1}}x)^\ell & 0 \le \ell < a_{r-2} - a_{r-1} \\ (t^{n-k_r-k_{r-1}}x)^\ell & \ell = a_{r-2} - a_{r-1} \end{cases}, \quad (5.2.11)$$

which concludes the computation of Case II.

A key feature of the distributions computed in (5.2.8), (5.2.9) and (5.2.11) is that the $1 - t^{n-k_r}x$ term in (5.2.8), which appears only in the case $N_r < a_{r-1}$ (Case I), cancels with the $1 - t^{n-k_r}x$ appearing in the computation (5.2.9) for Case I; meanwhile, when $N_r = a_{r-1}$ (Case II), it appears neither in (5.2.8) nor in (5.2.11).

Together, (5.2.8), (5.2.9) and (5.2.11) imply the joint distribution

$$\begin{aligned}
&\Pr(N_r = a_r + \ell_1 \text{ and } N_{r-1} = a_{r-1} + \ell_2) \\
&= \frac{(1 - t^{n-k_r-k_{r-1}}x)^{\mathbb{1}(\ell_2 < a_{r-2}-a_{r-1})}}{1 - t^n x}(1 - t^{k_r})^{\mathbb{1}(\ell_1 > 0)}(1 - t^{k_{r-1}})^{\mathbb{1}(\ell_1 < a_{r-1} - a_r \text{ and } \ell_2 > 0)} \\
&\cdot (t^{n-k_r}x)^{\ell_1}(t^{n-k_r-k_{r-1}}x)^{\ell_2} \\
&= \frac{(1 - t^{n-k_r-k_{r-1}}x)^{\mathbb{1}(\ell_2 < a_{r-2}-a_{r-1})}}{1 - t^n x}(1 - t^{k_r})^{\mathbb{1}(m_{a_r}(\lambda) = m_{a_r}(\nu)+1)}(1 - t^{k_{r-1}})^{\mathbb{1}(m_{a_{r-1}}(\lambda) = m_{a_{r-1}}(\nu)+1)} \\
&\cdot (t^{n-k_r}x)^{\ell_1}(t^{n-k_r-k_{r-1}}x)^{\ell_2}
\end{aligned}$$

But we see that the computation of the distribution of N_{r-1} is exactly the same for any N_i. There is the same division into Case I and Case II depending on whether N_{i+1} achieves its maximum, and the feature that the $1 - t^{n-k_r}x$ terms cancel in both Case I and Case II is also the same. Hence these terms telescope, and we are left with

$$\frac{1 - t^{n-k_r-\ldots-k_1}x}{1 - t^n x} = \frac{1 - x}{1 - t^n x},$$

where the $1 - t^{n-k_r-\ldots-k_1}x$ appears because the last such term does not cancel. Hence continuing the above computation yields

$$\Pr(N_r = \nu_{n-k_r+1}, N_{r-1} = \nu_{n-k_r-k_{r-1}+1}, \ldots, N_1 = \nu_1) \tag{5.2.12}$$

$$= \frac{1-x}{1-t^n x} \prod_{i=1}^{r} (1-t^{k_i})^{\mathbb{1}(m_{a_i}(\lambda)=m_{a_i}(\nu)+1)} \prod_{i=0}^{r-1} (t^{n-k_r-\ldots-k_{r-i}} x)^{\nu_{n-k_r-\ldots-k_{r-i}+1}-a_{r-i}} \tag{5.2.13}$$

$$= \frac{1-x}{1-t^n x} \prod_{j:m_j(\lambda)=m_j(\nu)+1} (1-t^{m_j(\lambda)}) \prod_{i=1}^{n} (xt^{i-1})^{\nu_i-\lambda_i}, \tag{5.2.14}$$

concluding the proof. □

Remark 26. Since the sum over ν of the LHS of (5.2.4) is clearly 1, Proposition 5.2.2 implies that the sum of the RHS of (5.2.4) is 1, which gives a proof of the corresponding case of the skew Hall-Littlewood Cauchy identity (Lemma 2.2.3).

It is very important to note that the random variables X_i above satisfy $\mathbb{E}[X_i] > \mathbb{E}[X_j]$ when $i < j$. This means that the i^{th} particle, which is already ahead of the j^{th} particle, is likely to pull even further ahead if one iterates the above dynamics. Empirically this may be seen in Figure 1-1. This observation is key to the proof of Theorem 5.1.1, as it implies that while there may be some interactions between particles, as one iterates the above dynamics the particles should spread apart and interactions should not contribute to the limit. Hence by Donsker's theorem the rescaled fluctuations of the particles should look like independent Brownian motions.

The rest of this chapter is devoted to making the above heuristic argument precise. We implement it by coupling the interacting particle dynamics of Proposition 5.2.2 to dynamics in which the particles do not interact at all, and showing that the error between the two is small in the limit.

5.3 Coupling to non-interacting particle dynamics

Proposition 5.2.2 gives an explicit sampling algorithm for Hall-Littlewood processes

$$\Pr(\lambda_1, \ldots, \lambda_N) = \frac{\tilde{Q}_{\lambda_N/\lambda_{N-1}}(x_N) \cdots \tilde{Q}_{\lambda_2/\lambda_1}(x_2) \tilde{Q}_{\lambda_1/(0[n])}(x_1) P_{\lambda_N}(1, \ldots, t^{n-1})}{\Pi(1, \ldots, t^{n-1}; x_1, \ldots, x_N)}.$$

Theorem 1.3.1 and Theorem 5.1.1 treat Hall-Littlewood processes as above but with the variables x_i replaced by geometric progressions $\hat{x}_i$ (sometimes infinite), and we must extend our notation slightly to deal with these. We begin by setting up the appropriate probability space on which the random variables X_i of Proposition 5.2.2 can be defined in this more general setting.

Definition 37. A *generalized variable* $\hat{x}$ is a tuple $(x, tx, \ldots, t^{m-1}x)$ or $(x, tx, \ldots)$ in finite or infinite geometric progression with common ratio t. For a generalized variable, define probability spaces

$$\Omega_{\hat{x}} = \begin{cases} (\mathbb{Z}^n_{\geq 0})^m & \hat{x} = (x, \ldots, t^{m-1}x) \\ \{\boldsymbol{\omega} = (\omega^{(1)}, \omega^{(2)}, \ldots) \in (\mathbb{Z}^n_{\geq 0})^\infty : \text{ only finitely many } \omega^{(i)} \text{ nonzero}\} & \hat{x} = (x, tx, \ldots) \end{cases}.$$

Recall the definition of the measure G_x in Definition 36. Now define the measure $G_{\hat{x}}$ on $\Omega_{\hat{x}}$ by

$$G_{\hat{x}} = \begin{cases} G_x \times \cdots \times G_{t^{m-1}x} & \hat{x} = (x, tx, \ldots, t^{m-1}x) \\ G_x \times G_{tx} \times \cdots & \hat{x} = (x, tx, \ldots) \end{cases}.$$

Two things must be justified in this definition. The first is that the infinite product measure $G_x \times G_{tx} \times \cdots$ on $(\mathbb{Z}^n_{\geq 0})^\infty$ makes sense, which follows from the Kolmogorov extension theorem. The second is that this measure is actually supported on the subset $\Omega_{\hat{x}}$, which follows from a standard Borel-Cantelli argument.

Definition 38. We inductively define ι on $(\mathbb{Z}^n_{\geq 0})^m, m > 1$ as follows. For $\omega_i \in \mathbb{Z}^n_{\geq 0}$, set

$$\iota((\omega_1, \ldots, \omega_m); \lambda) := \iota(\omega_m; \iota((\omega_1, \ldots, \omega_{m-1}); \lambda)), \tag{5.3.1}$$

We define $\iota : \Omega_{\hat{x}} \times \mathrm{Sig}_n \to \mathrm{Sig}_n$ as above when $\hat{x}$ is a finite geometric progression, and when $\hat{x}$ is an infinite geometric progression the definition readily extends because $\Omega_{\hat{x}}$ consists of sequences with only finitely many nonzero $\omega_i \in \mathbb{Z}^n_{\geq 0}$. Given a sequence $\hat{x}_1, \hat{x}_2, \ldots$ of generalized variables as in Theorem 5.1.1, we will use the following notations.

- $\Omega := \Omega_{\hat{x}_1} \times \Omega_{\hat{x}_2} \times \cdots$.
- $\boldsymbol{\omega} = (\omega^{(1)}, \omega^{(2)}, \ldots)$ will denote an element of Ω, with each $\omega^{(i)}$ denoting an element of $\Omega_{\hat{x}_i}$.

Definition 39. Define the sequence $\lambda(0), \lambda(1), \dots$ of random signatures on the probability space Ω of Definition 38 by setting $\lambda(0) = (0[n])$ and inductively defining

$$\lambda(k, \boldsymbol{\omega}) := \iota(\omega^{(k)}; \lambda(k-1, \boldsymbol{\omega}))$$

for $\boldsymbol{\omega} \in \Omega$, where ι is defined on $\omega^{(k)} \in \Omega_{\hat{x}_k}$ via (5.3.1). We will usually omit the dependence on the element of the probability space Ω and simply write $\lambda(k)$.

In other words, if $\hat{x}_k = (x_k, \dots, t^{m-1}x_k)$, then $\lambda(k+1)$ comes from $\lambda(k)$ by inserting random arrays as in Proposition 5.2.2 with distributions corresponding to the variables $x_k, tx_k, \dots, t^{m-1}x_k$.

We now define the non-interacting variant of the randomized insertion algorithm of Proposition 5.2.2, where each particle's movement is independent of the others. This is easier to analyze, and it will be shown in Proposition 5.3.1 that the two may be coupled with asymptotically negligible effect on the particles' positions, thus reducing the analysis of the sampling algorithm in Proposition 5.2.2 to something much simpler.

Definition 40. Define the *non-interacting insertion map* $\eta : \mathbb{Z}_{\geq 0}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ by

$$\eta(a_1, \dots, a_n; \mathbf{v}) = (v_1 + a_1, \dots, v_n + a_n),$$

and extend to $\eta : \Omega_{\hat{x}} \times \mathbb{Z}^n \to \mathbb{Z}^n$ as in (5.3.1). Define a random sequence $\mathbf{v}(0), \mathbf{v}(1), \dots$ with $\mathbf{v}(i) \in \mathbb{Z}^n$ on Ω by setting $\mathbf{v}(0) = (0[n])$ and

$$\mathbf{v}(k, \boldsymbol{\omega}) := \eta(\omega^{(k)}, \mathbf{v}(k-1, \boldsymbol{\omega})).$$

Remark 27. Neither the input tuple nor the output tuple of η must be a signature, and if either one happens to be, it does not imply that the other one is.

We now state the result mentioned earlier, that the 'interacting' and 'non-interacting' dynamics $\lambda(k)$ and $\mathbf{v}(k)$ may be coupled together with a negligible difference between them.

Proposition 5.3.1. *Let $\hat{x}_1, \hat{x}_2, \dots$ be a sequence of generalized variables, $\hat{x}_i = (x_i, tx_i, \dots, t^{m_i - 1}x_i)$ (where we allow $m_i = \infty$), such that there exists $\delta > 0$ for which $x_i \in (\delta, 1-\delta)$.*

Then with probability 1 *with respect to the product measure*[1] $G_{\hat{x}_1} \times G_{\hat{x}_2} \times \cdots$ *on* Ω*,*

$$\sup_{k \in \mathbb{Z}_{\geq 0}} |\lambda_i(k) - v_i(k)| \tag{5.3.2}$$

is bounded for every i*.*

Informally, the particles interact when a particle behind jumps to the position of the particle in front. The following lemma shows that in the non-interacting case, such overlaps occur a negligible amount, which will be used in the proof of Proposition 5.3.1 to show that interactions contribute negligibly overall as well.

Lemma 5.3.2. *With the same hypotheses on the* $\hat{x}_i$ *as in Proposition 5.3.1, we have that with probability* 1*, the set*

$$\{k \in \mathbb{Z}_{\geq 0} : v_i(k) \leq v_{i+1}(k+1) + B\}$$

is finite for all $B \in \mathbb{Z}$ *and all* i*.*

The proof of Lemma 5.3.2 will be deferred to Section 5.4.

Proof of Prop. 5.3.1. We construct a sequence $\lambda^{(1)}(k) = \mathbf{v}(k), \lambda^{(2)}(k), \ldots, \lambda^{(n)}(k) = \lambda(k)$ of discrete-time stochastic processes on the space of particle configurations, all defined on $\Omega_{\hat{x}_1} \times \Omega_{\hat{x}_2} \times \cdots$. Informally, $\lambda^{(j)}$ is the process in which the last j particles $\lambda^{(j)}_n, \ldots, \lambda^{(j)}_{n-j}$ interact as in Definition 39, but particles $\lambda^{(j)}_{n-j+1}, \ldots, \lambda^{(j)}_1$ do not interact with any other particles, as in Definition 40. We will then prove by induction on j that with probability 1,

$$\sup_{k \in \mathbb{Z}_{\geq 0}} |\lambda^{(j)}_i(k) - v_i(k)| \tag{5.3.3}$$

is finite for all i. When $j = n$, this will prove (5.3.2).

Now let us be more formal. Let $\mathrm{Sig}^{(j)}_n = \{(v_1, \ldots, v_n) \in \mathbb{Z}^n : v_n \leq \ldots \leq v_{n-j+1}\}$. Following the indexing theme above, we see that $\mathrm{Sig}^{(1)}_n = \mathbb{Z}^n$ and $\mathrm{Sig}^{(n)}_n = \mathrm{Sig}_n$. Once we have defined $\lambda^{(i)}$ it will be true that $\lambda^{(i)}$ takes values in $\mathrm{Sig}^{(i)}_n$.

[1] Defined on $\Omega_{\hat{x}_1} \times \Omega_{\hat{x}_2} \times \cdots$ via the Kolmogorov extension theorem.

Now, define $\eta^{(j)} : \mathbb{Z}^n_{\geq 0} \times \mathrm{Sig}^{(j)}_n \to \mathrm{Sig}^{(j)}_n$ by

$$\eta^{(j)}(a_1, \ldots, a_n; v) = (\eta(a_1, \ldots, a_{n-j}; v_1, \ldots, v_{n-j}), \iota(a_{n-j+1}, \ldots, a_n; v_{n-j+1}, \ldots, v_n)). \tag{5.3.4}$$

In other words, particles $v_{n-j+1}, \ldots, v_n$ try to jump by $a_{n-j+1}, \ldots, a_n$ units respectively, but may donate some of their jumps to the next particle as in the definition of ι, while particles $v_1, \ldots, v_{n-j}$ each jump by $a_1, \ldots, a_{n-j}$ units respectively, independent of the positions of all other particles. It is clear from this description that the image of $\eta^{(j)}$ is indeed $\mathrm{Sig}^{(j)}_n$. It is also clear that $\eta^{(n)} = \iota$, and that

$$\lambda^{(j)}_i(k) = v_i(k) \text{ for } i = 1, \ldots, n-j.$$

When $j = 1$ this means that the first $n-1$ particles do not interact and hence the n^{th} particle has no one to interact with, therefore $\eta^{(1)} = \eta$. Just as in (5.3.1), we extend $\eta^{(j)}$ to a map $\Omega_{\hat{x}} \times \mathrm{Sig}^{(j)}_n \to \mathrm{Sig}^{(j)}_n$ for any generalized variable $\hat{x}$.

Finally, given generalized variables $\hat{x}_1, \hat{x}_2, \ldots$, we define the discrete-time stochastic processes $\lambda^{(j)}(k)$ on the probability space $\Omega := \Omega_{\hat{x}_1} \times \Omega_{\hat{x}_2} \times \cdots$ by setting $\lambda^{(j)}(0) = (0[n])$ and

$$\lambda^{(j)}(k, \boldsymbol{\omega}) = \eta^{(j)}(\omega^{(k)}, \lambda^{(j)}(k-1, \boldsymbol{\omega})) \tag{5.3.5}$$

where $\boldsymbol{\omega} = (\omega^{(1)}, \ldots) \in \Omega$ as in Definition 38. We will usually write the random variable $\lambda^{(j)}(k)$ without the dependence on $\boldsymbol{\omega}$.

We claim that the inequalities

$$\lambda^{(j+1)}_{n-j}(k) \geq \lambda^{(j)}_{n-j}(k) \tag{5.3.6}$$

and

$$\lambda^{(j+1)}_{n-j+i}(k) \leq \lambda^{(j)}_{n-j+i}(k) \text{ for } i = 1, \ldots, j \tag{5.3.7}$$

hold for all k. We prove this by induction on k, the base case $k = 0$ following since $\lambda^{(\ell)}(0) = (0[n])$. Suppose that (5.3.6) and (5.3.7) hold for some k. Since (5.3.5) defines $\lambda^{(j+1)}(k+1)$ and $\lambda^{(j)}(k+1)$ by inserting $\omega^{(k)} \in \Omega_{\hat{x}_k}$, which is a sequence of elements of $\mathbb{Z}^n_{\geq 0}$, it suffices to show that the inequalities (5.3.6) and (5.3.7) remain true after inserting a single element of $\mathbb{Z}^n_{\geq 0}$. To be precise, it suffices to show that for any $\nu \in \mathrm{Sig}^{(j+1)}_n$, $\mu \in \mathrm{Sig}^{(j)}_n$

such that[2]

$$\nu_{n-j} \geq \mu_{n-j} \tag{5.3.8}$$

$$\nu_{n-j+i} \leq \mu_{n-j+i} \text{ for } i = 1, \ldots, j \tag{5.3.9}$$

and $a \in \mathbb{Z}^n_{\geq 0}$, one has

$$\eta^{(j+1)}(a;\nu)_{n-j} \geq \eta^{(j)}(a;\mu)_{n-j} \tag{5.3.10}$$

$$\eta^{(j+1)}(a;\nu)_{n-j+i} \leq \eta^{(j)}(a;\mu)_{n-j+i} \text{ for } i = 1, \ldots, j. \tag{5.3.11}$$

(5.3.10) is clear because $\eta^{(j)}(a;\mu)_{n-j} = \mu_{n-j} + a_{n-j}$ while $\eta^{(j+1)}(a;\nu)_{n-j} \geq \nu_{n-j} + a_{n-j}$ (where the possible $>$ comes from the fact that ν_{n-j} may get pushed by the preceding particle). We now turn to (5.3.11)

Applying (5.2.1) to the ι in (5.3.4), we have

$$\begin{aligned}&\eta^{(j+1)}(a;\nu)_{n-j+i}\\ &= \min(\nu_{n-j+i-1}, \max(\nu_{n-j+i}+a_{n-j+i}, \nu_{n-j+i+1}+a_{n-j+i}+a_{n-j+i+1}, \ldots, \nu_n+a_{n-j+i}+\ldots+a_n))\end{aligned} \tag{5.3.12}$$

for $i = 1, \ldots, j$. Similarly, (5.2.1) implies

$$\eta^{(j)}(a;\mu)_{n-j+i} = \begin{cases} \min(\mu_{n-j+i-1}, \max(\mu_{n-j+i} + a_{n-j+i}, \ldots, \mu_n + a_{n-j+i} + \ldots + a_n)) & i \geq 2 \\ \max(\mu_{n-j+i} + a_{n-j+i}, \ldots, \mu_n + a_{n-j+i} + \ldots + a_n) & i = 1 \end{cases} \tag{5.3.13}$$

Because $\nu_{n-j+i} \leq \mu_{n-j+i}, \ldots, \nu_n \leq \mu_n$, we have

$$\begin{aligned}&\max(\nu_{n-j+i} + a_{n-j+i}, \nu_{n-j+i+1} + a_{n-j+i} + a_{n-j+i+1}, \ldots, \nu_n + a_{n-j+i} + \ldots + a_n))\\ &\quad \leq \max(\mu_{n-j+i} + a_{n-j+i}, \mu_{n-j+i+1} + a_{n-j+i} + a_{n-j+i+1}, \ldots, \mu_n + a_{n-j+i} + \ldots + a_n)).\end{aligned} \tag{5.3.14}$$

[2]Note that (5.3.8), (5.3.9) are the same as (5.3.6) and (5.3.7).

and

$$\nu_{n-j+i-1} \leq \mu_{n-j+i-1} \text{ when } i \geq 2. \tag{5.3.15}$$

Combining the definitions (5.3.12) and (5.3.13) with the inequalities (5.3.14) and (5.3.15) yields the desired

$$\eta^{(j+1)}(a;\nu)_{n-j+i} \leq \eta^{(j)}(a;\mu)_{n-j+i} \text{ for } i = 1, \ldots, j.$$

Thus we have proven (5.3.6) and (5.3.7).

We finally turn to the proof of (5.3.3), by induction on j. The base case $j = 1$ follows because $\lambda^{(1)}(k) = \mathbf{v}(k)$ for all k as noted earlier. Thus we will suppose that (5.3.3) holds for some $j \geq 1$ and verify that it holds for $j + 1$.

We will first show

$$\sup_{k \in \mathbb{Z}_{\geq 0}} |\lambda^{(j+1)}_{n-j}(k) - v_{n-j}(k)| < \infty \tag{5.3.16}$$

almost surely. First note that for k such that $\lambda^{(j+1)}_{n-j+1}(k+1) \leq \lambda^{(j+1)}_{n-j}(k)$,

$$\lambda^{(j+1)}_{n-j}(k+1) - \lambda^{(j+1)}_{n-j}(k) = v_{n-j}(k+1) - v_{n-j}(k)$$

because no pushing occurs. For k such that

$$\lambda^{(j+1)}_{n-j+1}(k+1) > \lambda^{(j+1)}_{n-j}(k) \tag{5.3.17}$$

$\lambda^{(j+1)}_{n-j}$ may receive some push from $\lambda^{(j+1)}_{n-j+1}$, causing it to move further than v_{n-j} does during that round. Hence to show (5.3.16), it suffices to show that the number of k for which (5.3.17) holds is almost surely finite, as then the error $\sup_{k \in \mathbb{Z}_{\geq 0}} |\lambda^{(j+1)}_{n-j}(k) - v_{n-j}(k)|$ is a sum of a finite number of almost surely finite random variables (each one representing the amount by which the $(n-j)^{th}$ particle gets pushed).

By (5.3.6), $\lambda^{(j+1)}_{n-j}(k) \geq \lambda^{(j)}_{n-j}(k)$, and by (5.3.7) $\lambda^{(j+1)}_{n-j+1}(k+1) \leq \lambda^{(j)}_{n-j+1}(k+1)$. Hence for k such that (5.3.17) holds,

$$\lambda^{(j)}_{n-j+1}(k+1) > \lambda^{(j)}_{n-j}(k)$$

also holds, and since $\lambda^{(j)}_{n-j}(k) = v_{n-j}(k)$, we have that

$$v_{n-j+1}(k+1) + (\lambda^{(j)}_{n-j+1}(k+1) - v_{n-j+1}(k+1)) > v_{n-j}(k) \tag{5.3.18}$$

holds as well, so it suffices to show that

$$|\{k : v_{n-j+1}(k+1) + (\lambda^{(j)}_{n-j+1}(k+1) - v_{n-j+1}(k+1)) > v_{n-j}(k)\}| < \infty \text{ a.s.} \tag{5.3.19}$$

By the inductive hypothesis that (5.3.3) holds for j, we have that

$$\sup_{k \in \mathbb{Z}_{\geq 0}} |\lambda^{(j)}_{n-j+1}(k) - v_{n-j+1}(k)| < \infty \tag{5.3.20}$$

almost surely. Since by Lemma 5.3.2,

$$\{k : v_{n-j+1}(k+1) + B > v_{n-j}(k)\}$$

is almost surely finite for all B, it is in particular almost surely finite for the *random*

$$B = \sup_{k \in \mathbb{Z}_{\geq 0}} |\lambda^{(j)}_{n-j+1}(k) - v_{n-j+1}(k)|$$

(the order of quantifiers in Lemma 5.3.2 is important for this conclusion). Since $\lambda^{(j)}_{n-j}(k+1) - v_{n-j}(k+1)$ is almost surely bounded, (5.3.19) follows. This completes the proof of (5.3.16).

Now, since

$$\lambda^{(j+1)}_{n-j}(k) \geq \lambda^{(j)}_{n-j}(k) \tag{5.3.21}$$

$$\lambda^{(j+1)}_{n-j+i}(k) \leq \lambda^{(j)}_{n-j+i}(k) \text{ for } i = 1, \dots, j \tag{5.3.22}$$

$$\lambda^{(j+1)}_{i}(k) = \lambda^{(j)}_{i}(k) \text{ for } i = 1, \dots, n-j-1 \tag{5.3.23}$$

$$\sum_{i=1}^{n} \lambda^{(j+1)}_{i}(k) = \sum_{i=1}^{n} \lambda^{(j)}_{i}(k), \tag{5.3.24}$$

it follows that

$$\sum_{i=1}^{j} (\lambda^{(j)}_{n-j+i}(k) - \lambda^{(j+1)}_{n-j+i}(k)) = \lambda^{(j+1)}_{n-j}(k) - \lambda^{(j)}_{n-j}(k) \tag{5.3.25}$$

(this is a kind of conservation of momentum: the amount that the $(n-j)^{th}$ particle is pushed forward from collisions equals the amount that the particles behind it are pushed backward). We have

$$\sup_k |\lambda_{n-j}^{(j+1)}(k) - \lambda_{n-j}^{(j)}(k)| \leq \sup_k |\lambda_{n-j}^{(j+1)}(k) - v_{n-j}(k)| + \sup_k |\lambda_{n-j}^{(j)}(k) - v_{n-j}(k)| < \infty \text{ a.s.}$$

by applying (5.3.16) to the first term and the inductive hypothesis to the second. Because the summands $\lambda_{n-j+i}^{(j)}(k) - \lambda_{n-j+i}^{(j+1)}(k)$ on the LHS of (5.3.25) are nonnegative, it follows that

$$\sup_k |\lambda_{n-j+i}^{(j)}(k) - \lambda_{n-j+i}^{(j+1)}(k)| < \infty \text{ a.s.} \tag{5.3.26}$$

for $i = 1, \ldots, j$. We thus have

$$\sup_k |\lambda_{n-j+i}^{(j+1)}(k) - v_{n-j+i}(k)| \leq \sup_k |\lambda_{n-j+i}^{(j)}(k) - \lambda_{n-j+i}^{(j+1)}(k)| + \sup_k |\lambda_{n-j+i}^{(j)}(k) - v_{n-j+i}| < \infty \text{ a.s.}$$

by applying (5.3.26) to the first summand and the inductive hypothesis to the second. This establishes (5.3.3) for $j+1$, for $i = n-j, n-j+1, \ldots, n$, and the equation is trivial (the supremum is just 0) when $i = 1, \ldots, n-j-1$. This completes the induction on j, showing that (5.3.3) holds for all i and j. In particular it holds for $j = n$, which proves Proposition 5.3.1.

□

5.4 Analysis of non-interacting particle dynamics $\mathbf{v}(k)$ and proof of Theorem 5.1.1

In the previous subsection, we phrased the relevant Hall-Littlewood process in terms of a particle system $\lambda(k)$ in which particles interact, then coupled it to a system $\mathbf{v}(k)$ where they do not interact. In this subsection we analyze $\mathbf{v}(k)$ to prove our results. We first record facts about the means, variances and fourth moments of jumps of $\mathbf{v}(k)$ in Lemma 5.4.1, some of which are used to give an overdue proof of Lemma 5.3.2, used in the previous subsection. We then apply them and Donsker's theorem to prove an analogue of Theorem 5.1.1 for $\mathbf{v}(k)$, and conclude the desired result for $\lambda(k)$ by our coupling and Proposition 5.3.1

Lemma 5.4.1. *Let $\delta > 0$ and let $\hat{x}_1, \hat{x}_2, \dots$ be generalized variables such that $\delta < x_i < 1-\delta$ for all i as in Proposition 5.3.1. Let $\mathbf{v}(k)$ be as in the previous subsection, $Y_i(k) := v_i(k) - v_i(k-1)$ for $k \geq 0$, and $\bar{Y}_i(k) = Y_i(k) - \mathbb{E}Y_i(k)$. Then*

1. *If $\hat{x}_k = (x_k, \dots, t^{m-1}x_k)$, then*

$$\mathbb{E}Y_i(k) = \sum_{j=0}^{m-1} \frac{x_k t^{j+i-1}(1-t)}{(1-t^{j+i}x_k)(1-t^{j+i-1}x_k)} \tag{5.4.1}$$

   where we allow $m = \infty$. Consequently, there exist constants $b_i, B_i > 0$ such that $b_i < \mathbb{E}Y_i(k) < B_i$ for all k.

2. *For $\hat{x}_k$ as above, we have*

$$\mathbb{E}[\bar{Y}_i(k)^2] = \sum_{j=0}^{m-1} \frac{t^{j+i-1}x_k(1-t)(1-t^{2j+2i-1}x_k^2)}{(1-t^{j+i-1}x_k)^2(1-t^{j+i}x_k)^2}. \tag{5.4.2}$$

   Consequently, there exist constants $c_i, C_i > 0$ such that $c_i < \mathbb{E}[\bar{Y}_i(k)^2] < C_i$ for all k.

3. *There exist constants $D_i > 0$ such that $\mathbb{E}[\bar{Y}_i(k)^4] < D_i$ for all k.*

Definition 41. We let $\mu(t^{i-1}\hat{x}_k)$ denote the RHS of (5.4.1), and $\sigma^2(t^{i-1}\hat{x}_k)$ denote the RHS of (5.4.2), which by Lemma 5.4.1 are the mean and variance of $Y_i(k)$ respectively.

Proof. It follows from the definition of v that

$$Y_i(k) = v_i(k) - v_i(k-1) = \sum_{j=0}^{m-1} Z_{t^{j+(i-1)}x_k}, \tag{5.4.3}$$

where $Z_x \sim G_x$ are independent. We compute

$$\begin{aligned}
\mathbb{E}Z_x &= \frac{d}{dy}|_{y=1} \sum_{\ell \geq 0} \Pr(Z_x = \ell) y^\ell \\
&= \frac{d}{dy}|_{y=1} \sum_{\ell \geq 0} \frac{1-x}{1-tx}(1-t)^{\mathbb{1}(\ell>0)}(xy)^\ell \\
&= \frac{d}{dy}|_{y=1} \frac{1-x}{1-tx}\frac{1-txy}{1-xy} \\
&= \left[\frac{1-x}{1-tx}\frac{-tx(1-xy)+x(1-txy)}{(1-xy)^2}\right]_{y=1} \\
&= \frac{x(1-t)}{(1-tx)(1-x)}.
\end{aligned}$$

Combining with (5.4.3) yields (5.4.1). We have

$$\mathbb{E}Y_i(k) \geq \mathbb{E}Z_{t^{i-1}x_k} = \frac{t^{i-1}x_k(1-t)}{(1-t^i x_k)(1-t^{i-1}x_k)} > t^{i-1}x_k(1-t) \geq t^{i-1}(1-t)\delta,$$

so setting $b_i = t^{i-1}(1-t)\delta$ we have $b_i < \mathbb{E}Y_i(k)$. For the other bound,

$$\begin{aligned}
\mathbb{E}Y_i(k) &\leq \sum_{j=0}^{\infty} \frac{t^{j+i-1}x_k(1-t)}{(1-t^{j+i}x_k)(1-t^{j+i-1}x_k)} \\
&< \frac{1}{(1-t^i(1-\delta))(1-t^{i-1}(1-\delta)))} \sum_{j=0}^{\infty}(1-t)t^{i-1}x_k \cdot t^j \\
&< \frac{t^{i-1}\delta}{(1-t^i(1-\delta))(1-t^{i-1}(1-\delta)))},
\end{aligned}$$

so we may set $B_i = \frac{t^{i-1}\delta}{(1-t^i(1-\delta))(1-t^{i-1}(1-\delta)))}$. This proves Part 1 of the lemma.

For Part 2, we have

$$\bar{Y}_i(k) = \sum_{j=0}^{m-1}(Z_{t^{j+(i-1)}x_k} - \mathbb{E}[Z_{t^{j+(i-1)}x_k}]).$$

Set $\bar{\mathbb{Z}}_x = Z_x - \mathbb{E}Z_x$. Since the $\bar{\mathbb{Z}}$'s above are independent, the variances add. Hence the lower bound $c_i < \mathbb{E}[\bar{Y}_i(k)^2]$ follows because $\mathbb{E}[\bar{\mathbb{Z}}^2_{t^{j+(i-1)}x_k}]$ is bounded below for $x_k \in$

$(\delta, 1-\delta)$. For the upper bound, we compute

$$\begin{aligned}\mathbb{E}Z_x^2 &= \frac{d}{dy}|_{y=1} y \frac{d}{dy} \sum_{\ell \geq 0} \Pr(Z_x = \ell) y^\ell \\ &= \frac{d}{dy}|_{y=1} y \frac{1-x}{1-tx} \frac{-tx(1-xy) + x(1-txy)}{(1-xy)^2} \\ &= \frac{x(1-t)(1+x)}{(1-x)^2(1-tx)},\end{aligned}$$

so

$$\mathbb{E}[\bar{\mathbb{Z}}_x^2] = \mathbb{E}Z_x^2 - (\mathbb{E}Z_x)^2 = \frac{x(1-t)(1-tx^2)}{(1-x)^2(1-tx)^2}, \tag{5.4.4}$$

proving (5.4.2). This is bounded above by $\frac{x}{\delta^2(1-t(1-\delta))^2}$ for $0 < x < 1-\delta$, hence

$$\begin{aligned}\mathbb{E}[\bar{Y}_i(k)^2] &= \sum_{j=0}^{m-1} \mathbb{E}[\bar{\mathbb{Z}}^2_{t^{j+(i-1)}x_k}] && (5.4.5) \\ &\leq \frac{1}{\delta^2(1-t(1-\delta))^2} \frac{t^{i-1}x_k}{1-t} && (5.4.6) \\ &< \frac{1}{\delta^2(1-t(1-\delta))^2} \frac{t^{i-1}(1-\delta)}{1-t} && (5.4.7)\end{aligned}$$

for all $x_k \in (\delta, 1-\delta)$, so we may set C_i to be the final expression. This proves Part 2.

For the fourth moment,

$$\mathbb{E}[\bar{Y}_i(k)^4] = \sum_{j=0}^{m-1} \mathbb{E}[\bar{\mathbb{Z}}^4_{t^{j+(i-1)}x_k}] + \sum_{0 \leq j \neq \ell \leq m-1} \mathbb{E}[\bar{\mathbb{Z}}^2_{t^{j+(i-1)}x_k}]\mathbb{E}[\bar{\mathbb{Z}}^2_{t^{\ell+(i-1)}x_k}]. \tag{5.4.8}$$

We have

$$\mathbb{E}[\bar{\mathbb{Z}}_x^4] = \frac{(1-t)x(x^2+4x+1)}{(1-x)^3(1-tx)}$$

by a similar generating function computation as before, and bounding the sum of these for $x, tx, t^2x, \ldots$ in terms of geometric series as before yields that

$$\sum_{j=0}^{\infty} \mathbb{E}[\bar{\mathbb{Z}}^4_{t^{j+i-1}x}]$$

is bounded uniformly over $x \in (0, 1-\delta)$. Likewise,

$$\begin{aligned}
&\sum_{0 \le j \ne \ell \le m-1} \mathbb{E}[\bar{\mathbb{Z}}^2_{t^{j+(i-1)}x_k}]\mathbb{E}[\bar{\mathbb{Z}}^2_{t^{\ell+(i-1)}x_k}] \\
&\le \left(\sum_{0 \le j \le m-1} \mathbb{E}[\bar{\mathbb{Z}}^2_{t^{j+(i-1)}x_k}] \right) \\
&< \left(\frac{1}{\delta^2(1-t(1-\delta))^2} \frac{t^{i-1}(1-\delta)}{1-t} \right)^2
\end{aligned}$$

by using our previous variance bound at the last step. Hence we have bounded both sums on the RHS of (5.4.8) uniformly in $x_k \in (0, 1-\delta)$, and Part 3 follows. □

We now prove Lemma 5.3.2 as promised.

Proof of Lemma 5.3.2. We first claim that it suffices to show that for any given B,

$$|\{k \in \mathbb{Z}_{\ge 0} : v_i(k) \le v_{i+1}(k+1) + B\}| < \infty \text{ a.s.}$$

(this differs from the statement of Lemma 5.3.2 in order of quantifiers). This is immediate because

$$\begin{aligned}
&\{\boldsymbol{\omega} \in \Omega : |\{k \in \mathbb{Z}_{\ge 0} : v_i(k, \boldsymbol{\omega}) \le v_{i+1}(k+1, \boldsymbol{\omega}) + B'\}| = \infty \text{ for some } B'\} \\
&\qquad = \bigcup_{B \in \mathbb{N}} \{\boldsymbol{\omega} \in \Omega : |\{k \in \mathbb{Z}_{\ge 0} : v_i(k, \boldsymbol{\omega}) \le v_{i+1}(k+1, \boldsymbol{\omega}) + B\}|\} \quad (5.4.9)
\end{aligned}$$

so it suffices to show the sets on the RHS have measure 0. This is what we will now do.

It follows from the formula in Lemma 5.4.1 Part 1 that $\mathbb{E}[Y_{i+1}(k)] \le t\mathbb{E}[Y_i(k)]$, hence

$$\sum_{j=1}^{k} \mathbb{E}Y_i(j) - \sum_{j=1}^{k+1} \mathbb{E}Y_{i+1}(j) \ge (1-t) \sum_{j=1}^{k} \mathbb{E}Y_i(k) - \mathbb{E}Y_{i+1}(k+1) \ge (1-t)b_i \cdot k - B_{i+1} \quad (5.4.10)$$

where b_i, B_{i+1} are the constants in Lemma 5.4.1. For k such that the RHS of (5.4.10) is positive,

$$\begin{aligned}
\Pr(v_i(k) \le v_{i+1}(k+1) + B) &= \Pr\left(\sum_{j=1}^{k} \bar{Y}_i(j) - \sum_{j=1}^{k+1} \bar{Y}_{i+1}(j) \le B + \sum_{j=1}^{k+1} \mathbb{E}Y_{i+1}(j) - \sum_{j=1}^{k} \mathbb{E}Y_i(j)\right) \\
&\le \Pr\left(\sum_{j=1}^{k} \bar{Y}_i(j) - \sum_{j=1}^{k+1} \bar{Y}_{i+1}(j) \le B + (1-t)b_i \cdot k - B_{i+1}\right) \\
&\le \Pr\left(\left|\sum_{j=1}^{k} \bar{Y}_i(j) - \sum_{j=1}^{k+1} \bar{Y}_{i+1}(j)\right| \le B + (1-t)b_i \cdot k - B_{i+1}\right)
\end{aligned}$$

(the last step is the only one using the positivity assumption). By Markov's inequality,

$$\begin{aligned}
&\Pr\left(\left|\sum_{j=1}^{k} \bar{Y}_i(j) - \sum_{j=1}^{k+1} \bar{Y}_{i+1}(j)\right| \le B + (1-t)b_i \cdot k - B_{i+1}\right) \\
&\qquad\qquad \le \frac{\mathbb{E}\left[\left(\sum_{j=1}^{k} \bar{Y}_i(j) - \sum_{j=1}^{k+1} \bar{Y}_{i+1}(j)\right)^4\right]}{(B + (1-t)b_i \cdot k - B_{i+1})^4}. \qquad (5.4.11)
\end{aligned}$$

By Lemma 5.4.1,

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{j=1}^{k} \bar{Y}_i(j) - \sum_{j=1}^{k+1} \bar{Y}_{i+1}(j)\right)^4\right] \\
&= \sum_{j=1}^{k} \mathbb{E}[\bar{Y}_i(j)^4] + \sum_{j=1}^{k+1} \mathbb{E}[\bar{Y}_{i+1}(j)^4] + \sum_{j=1}^{k}\sum_{\ell=1}^{k+1} \mathbb{E}[\bar{Y}_i(j)^2]\mathbb{E}[\bar{Y}_{i+1}(j)^2] \\
&< kD_i + (k+1)D_{i+1} + k(k+1)C_iC_{i+1} \\
&= O(k^2).
\end{aligned}$$

Hence the RHS of (5.4.11) is $O(1/k^2)$. Thus

$$\sum_k \Pr(v_i(k) \le v_{i+1}(k+1) + B) < \infty$$

and so by Borel-Cantelli, $\{k \in \mathbb{Z}_{\ge 0} : v_i(k) \le v_{i+1}(k+1) + B\}$ is almost-surely finite, completing the proof. □

Proof of Theorem 5.1.1. We begin with the first claim (5.1.1), the law of large numbers. By Lemma 5.4.1, $\mu(t^{i-1}\hat{x}_k)$ and $\sigma^2(t^{i-1}\hat{x}_k)$ are the mean and variance, respectively, of

$Y_i(k)$. Since $\sum_{j=1}^k Y_i(j) = v_i(k)$, it suffices to show

$$\frac{\lambda_i(k) - \mathbb{E}v_i(k)}{k} \to 1 \text{ a.s. as } k \to \infty. \tag{5.4.12}$$

By Proposition 5.3.1, $|\lambda_i(k) - v_i(k)|$ is almost-surely bounded as $k \to \infty$. It follows that

$$\frac{v_i(k) - \lambda_i(k)}{k} \to 0 \text{ a.s. as } k \to \infty. \tag{5.4.13}$$

The uniform variance bound in Lemma 5.4.2 Part 2 ensures that the sequence of random variables $Y_i(1), Y_i(2), \dots$ satisfies the hypothesis of Kolmogorov's strong law of large numbers [Shi96, Ch. IV.§3, Thm. 2], hence

$$\frac{\sum_{j=1}^k Y_i(j) - \sum_{j=1}^k \mathbb{E}Y_i(j)}{k} \to 0 \text{ a.s. as } k \to \infty. \tag{5.4.14}$$

We have

$$\frac{\lambda_i(k)}{\sum_{j=1}^k \mu(t^{i-1}\hat{x}_j)} = 1 + \frac{k}{\sum_{j=1}^k \mu(t^{i-1}\hat{x}_j)} \left(\frac{v_i(k) - \mathbb{E}[v_i(k)]}{k} + \frac{\lambda_i(k) - v_i(k)}{k} \right). \tag{5.4.15}$$

By Lemma 5.4.1 Part 1,

$$\left| \frac{k}{\sum_{j=1}^k \mu(t^{i-1}\hat{x}_j)} \right| \le \frac{1}{b_i},$$

so

$$\left| \frac{\lambda_i(k)}{\sum_{j=1}^k \mu(t^{i-1}\hat{x}_j)} - 1 \right| \le \frac{1}{b_i} \left(\frac{v_i(k) - \mathbb{E}[v_i(k)]}{k} + \frac{\lambda_i(k) - v_i(k)}{k} \right).$$

By (5.4.14) and (5.4.13) respectively, the two terms inside the parentheses on the RHS go to 0 almost surely as $k \to \infty$. This proves (5.1.1), the law of large numbers.

To show the second claim of Theorem 5.1.1, namely the convergence of the rescaled λ_i to Brownian motions, we will use the same strategy of first showing convergence for the v_i and then utilizing the coupling. Let

$$\bar{v}_i(k) = v_i(k) - \sum_{j=1}^k \mu(t^{i-1}\hat{x}_j) = \sum_{j=1}^k \bar{Y}_i(j).$$

Define $f_{\bar{v}_i,k}$, a $C[0,1]$-valued random variable on the probability space Ω of Definition 38 by setting $f_{\bar{v}_i,k}(0) = 0$,

$$(f_{\bar{v}_i,k}(1/k), f_{\bar{v}_i,k}(2/k), \ldots, f_{\bar{v}_i,k}(1)) = \frac{1}{\sqrt{\sum_{j=1}^{k} \sigma^2(t^{i-1}\hat{x}_j)}}(\bar{v}_i(1), \ldots, \bar{v}_i(k))$$

and linearly interpolating $f_{\bar{v}_i,k}$ at other values in $[0,1]$. Let the measure $M_{\bar{v}_i,k}$ on $C[0,1]$ be the distribution of $f_{\bar{v}_i,k}$. We claim that as $k \to \infty$, $M_{\bar{v}_i,k}$ converges weakly to the Wiener measure P_W on $C[0,1]$. By Donsker's theorem[3], this convergence holds if the Lindeberg condition is satisfied, and it is well-known (see e.g. [Shi96, Ch. III, §4.I.2]) that the Lindeberg condition is implied by the Lyapunov condition. The latter, in our case, reads that for some $\delta > 0$,

$$\frac{1}{(\sum_{j=1}^{k} \sigma^2(t^{i-1}\hat{x}_j))^{1+\delta/2}} \sum_{j=1}^{k} \mathbb{E}[\bar{Y}_i(j)^{2+\delta}] \to 0 \text{ as } k \to \infty. \tag{5.4.16}$$

We will prove (5.4.16) when $\delta = 2$. Letting c_i, D_i be as in Lemma 5.4.1, we have

$$\left(\sum_{j=1}^{k} \sigma^2(t^{i-1}\hat{x}_j)\right)^2 > k^2 c_i^2$$

and

$$\sum_{j=1}^{k} \mathbb{E}[\bar{Y}_i(j)^4] < kD_i.$$

Hence the expression in (5.4.16) is bounded above by $\frac{D_i}{c_i^2}\frac{1}{k}$, and (5.4.16) follows immediately. This verifies that $M_{\bar{v}_i,k}$ converges weakly to P_W as $k \to \infty$. Because $v_1, \ldots, v_n$ are independent, we also have that the product measure $M_{\bar{v}_i,k} \times \cdots \times M_{\bar{v}_i,k}$ on $(C[0,1])^n$ converges weakly to P_W^n, i.e. $(f_{\bar{v}_1,k}, \ldots, f_{\bar{v}_n,k})$ converges in distribution to n independent Brownian motions.

We wish to show via our coupling that that $(f_{\bar{\lambda}_1,k}, \ldots, f_{\bar{\lambda}_n,k})$ converges in distribution to P_W^n as well. We will use the following basic lemma.

Lemma 5.4.2. *Let S be a metric space with Borel σ-algebra Σ, and P a probability measure on (S, Σ). Let X_n, Y_n be random variables defined on the same probability space and taking values in S, such that $|X_n - Y_n| \to 0$ in probability (where $|\cdot|$ denotes the norm*

[3]Many versions in print require that the increments $\bar{Y}_i(j)$ be identically distributed as well as independent, but a version for random walks with distinct independent increments may be obtained by specializing Donsker's theorem for martingales [Bro71, Thm. 3] to this case.

induced by the metric on S) and such that the distribution μ_{Y_n} of Y_n converges weakly to P. Then μ_{X_n} converges weakly to P as well.

Proof. We must show that for all $f \in \mathcal{C}_b(S)$, $\mathbb{E}[f(X_n)] \to \int_S f dP$. By hypothesis, $\mathbb{E}[f(Y_n)] \to \int_S f dP$, so it suffices to show $\mathbb{E}[f(X_n) - f(Y_n)] \to 0$. By the convergence in probability hypothesis, $\Pr(|X_n - Y_n| > \delta) \to 0$ for any $\delta > 0$.

f is bounded, so let B be such that $f \leq B$. We have

$$\begin{aligned}
\mathbb{E}[|f(X_n) - f(Y_n)|] &= \mathbb{E}[\mathbb{1}_{|X_n - Y_n| > \delta}|f(X_n) - f(Y_n)|] + \mathbb{E}[\mathbb{1}_{|X_n - Y_n| \leq \delta}|f(X_n) - f(Y_n)|] \\
&\leq 2B \Pr(|X_n - Y_n| > \delta) + \mathbb{E}\left[\sup_{x : |x - Y_n| \leq \delta} |f(x) - f(Y_n)|\right].
\end{aligned}$$

Since $g_\delta(y) := \sup_{x \in \bar{B}_\delta(y)} |f(x) - f(y)|$ is a continuous, bounded function of y, the weak convergence hypothesis yields $\mathbb{E}[g_\delta(Y_n)] \to \int_S g_\delta dP$ as $n \to \infty$. Because f is continuous, $\limsup_{\delta \to 0} g_\delta(y) = 0$ for all y, and since g_δ is uniformly bounded by $2B$ which is integrable on (S, Σ, P), we have by reverse Fatou's lemma that

$$\limsup_{\delta \to 0} \int_S g_\delta dP \leq \int_S \limsup_{\delta \to 0} g_\delta dP = 0.$$

By the above, for any $\epsilon > 0$, we may first choose δ so that $|\int_S g_\delta dP| < \epsilon/3$, then for all large enough n we have $|\mathbb{E}[g_\delta(Y_n)] - \int_S g_\delta dP < \epsilon/3$ and $2B \Pr(|X_n - Y_n| > \delta) < \epsilon/3$, yielding that $\mathbb{E}[|f(X_n) - f(Y_n)|] < \epsilon$. Hence

$$\mathbb{E}[f(X_n) - f(Y_n)] \to 0$$

as $n \to \infty$, completing the proof. $\square$

When $S = C[0,1]^n$ with metric

$$d((f_1, \ldots, f_n), (g_1, \ldots, g_n)) = \sup_{1 \leq i \leq n} \sup_{x \in [0,1]} |f_i(x) - g_i(x)|,$$

and

$$X_k = (f_{\bar{\lambda}_1,k}, \ldots, f_{\bar{\lambda}_n,k}) \tag{5.4.17}$$

$$Y_k = (f_{\bar{v}_1,k}, \ldots, f_{\bar{v}_n,k}) \tag{5.4.18}$$

we have from above that Y_k converges in distribution to P_W^n. To conclude from Lemma 5.4.2 that X_k converges in distribution to P_W^n, it suffices to show that $d(X_k, Y_k) \to 0$ in probability. So we must show that for any $\delta, \epsilon > 0$, $\Pr(d(X_k, Y_k) > \delta) < \epsilon$ for all sufficiently large k. Proposition 5.3.1, together with the fact that $\lambda_i(k) - \bar{\lambda}_i(k) = \mathbb{E}[v_i(k)] = v_i(k) - \bar{v}_i(k)$, yields that

$$B(\boldsymbol{\omega}) := \sup_{1 \le i \le n} \sup_{k} |\bar{\lambda}_i(k, \boldsymbol{\omega}) - \bar{v}_i(k, \boldsymbol{\omega})|$$

is an almost-surely finite random variable on Ω. Hence there exists D such that

$$\Pr(B(\boldsymbol{\omega}) > D) < \epsilon. \tag{5.4.19}$$

By the lower bound c_i in Lemma 5.4.1 Part 2, $\sum_{j=1}^{\infty} \sigma^2(t^{i-1}\hat{x}_j)$ diverges, hence there exists K such that for all $k > K$,

$$\frac{D}{\sum_{j=1}^{k} \sigma^2(t^{i-1}\hat{x}_j)} < \delta \tag{5.4.20}$$

for $i = 1, \ldots, n$.

We therefore have that for $k > K$ and $\boldsymbol{\omega}$ such that $B(\boldsymbol{\omega}) \le D$,

$$\sup_{1 \le i \le n} \sup_{0 \le \ell \le k} \frac{1}{\sum_{j=1}^{k} \sigma^2(t^{i-1}\hat{x}_j)} |\bar{\lambda}_i(\ell, \boldsymbol{\omega}) - \bar{v}_i(\ell, \boldsymbol{\omega})| < \delta. \tag{5.4.21}$$

Because $\sup_{x \in [0,1]} |f - g| = \sup_{x = 0, 1/k, \ldots, 1} |f - g|$ if f, g are piecewise linear on each interval $[\frac{\ell}{k}, \frac{\ell+1}{k}]$, we have

$$\sup_{0 \le \ell \le k} \frac{1}{\sum_{j=1}^{k} \sigma^2(t^{i-1}\hat{x}_j)} |\bar{\lambda}_i(\ell, \boldsymbol{\omega}) - \bar{v}_i(\ell, \boldsymbol{\omega})| = \sup_{x \in [0,1]} |f_{\bar{v}_i,k}(x) - f_{\bar{\lambda}_i,k}(x)|.$$

Thus (5.4.21) implies that for $\boldsymbol{\omega}$ such that $B(\boldsymbol{\omega}) \le D$ and $k > K$,

$$d(X_k(\boldsymbol{\omega}), Y_k(\boldsymbol{\omega})) = \sup_{1 \le i \le n} \sup_{x \in [0,1]} |f_{\bar{v}_i,k}(x) - f_{\bar{\lambda}_i,k}(x)| < \delta.$$

Together with (5.4.19) this implies that

$$\Pr(d(X_k, Y_k) > \delta) < \epsilon$$

for all $k > K$. Since δ, ϵ were arbitrary, this is exactly the statement that $d(X_k, Y_k) \to 0$ in probability. This completes the proof of Theorem 5.1.1. $\square$

Remark 28. It is worth noting that fact that λ_i jumps further than λ_j in expectation for $i < j$ comes from the fact that $t < 1$. Hence our technique would no longer hold if one were to take a simultaneous limit $t \to 1$ as well, because the hopping particles would not outpace the ones behind them and hence interactions between them could contribute nontrivially in the limit. Such $t \to 1$ limits of Hall-Littlewood processes have been studied by Dimitrov [Dim18] and Corwin-Dimitrov [CD18], though we do not see how their results would apply directly to our specific case. We note also that the connection to p-adic random matrices is lost in this regime.

5.5 Lyapunov exponents

Given the law of large numbers in Theorem 1.3.1 and the formulas in Lemma 5.4.1, the proof of Theorem 1.3.2 is quite easy. First recall the statement.

Theorem 1.3.2 (Large-n universality of Lyapunov exponents)**.** *For each $n \in \mathbb{N}$, let $N_1^{(n)}, N_2^{(n)}, \ldots \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ be such that $N_j^{(n)} > n$ and the limiting frequencies*

$$\rho_n(N) := \lim_{k\to\infty} \frac{|\{1 \leq j \leq k : N_j^{(n)} = N\}|}{k}$$

exist for all $N > n$. Let $A_j^{(n)}$ be $n \times n$ corners of independent Haar distributed matrices in $\mathrm{GL}_{N_j^{(n)}}(\mathbb{Z}_p)$ (with the case $N_j^{(n)} = \infty$ treated as in Theorem 1.3.1). Then for each n, the Lyapunov exponents

$$L_i^{(n)} := \lim_{k\to\infty} \frac{\lambda_{n-i+1}(k)}{k}$$

exist almost surely, where $\lambda_{n-i+1}(k)$ is as in Theorem 1.3.1. Furthermore, the Lyapunov exponents have limits

$$\lim_{n\to\infty} \frac{L_i^{(n)}}{p^{-n}(1-c(n))} = p^{i-1}$$

for every i, where $c(n) := \sum_{N>n} \rho_n(N) p^{-(N-n)}$.

Proof. For existence of Lyapunov exponents, we have

$$L_i^{(n)} := \lim_{k\to\infty} \frac{\lambda_{n-i+1}(k)}{k} = \lim_{k\to\infty} \frac{\frac{\lambda_{n-i+1}(k)}{\sum_{j=1}^k \mu(t^{n-i+1},\ldots,t^{N_j^{(n)}-i})}}{\frac{k}{\sum_{j=1}^k \mu(t^{n-i+1},\ldots,t^{N_j^{(n)}-i})}}.$$

The limit of the numerator exists almost surely by Theorem 1.3.1. For the denominator we have

$$\frac{\sum_{j=1}^k \mu(t^{n-i+1},\ldots,t^{N_j^{(n)}-i})}{k} = \sum_{N>n} \frac{|\{1 \le j \le k : N_j^{(n)} = N\}|}{k} \mu(t^{n-i+1},\ldots,t^{N_j^{(n)}-i}),$$

hence

$$\lim_{k\to\infty} \frac{\sum_{j=1}^k \mu(t^{n-i+1},\ldots,t^{N_j^{(n)}-i})}{k} = \sum_{N>n} \rho_n(N) \mu(t^{n-i+1},\ldots,t^{N_j^{(n)}-i}),$$

(this uses the fact that $\mu(t^{n-i+1},\ldots,t^{N-i})$ is bounded as a function of N). Therefore $L_i^{(n)}$ exists almost surely.

Recall that for $Z_x \sim G_x$,

$$\mathbb{E} Z_x = \frac{x(1-t)}{(1-tx)(1-x)}.$$

It is an elementary check that there exist constants $B(i)$ depending only on t and i such that for any $x < 1$ and $n \ge i$,

$$\left|\mathbb{E} Z_{t^{n-i}x} - (1-t)xt^{n-i}\right| < B(i)t^{2n}x. \tag{5.5.1}$$

Let

$$\tilde{L}_i^{(n)} := \lim_{k\to\infty} \frac{\sum_{j=1}^k \sum_{\ell=1}^{N_j^{(n)}-n} (1-t)t^{n-i} \cdot t^\ell}{k}.$$

Then

$$\tilde{L}_i^{(n)} = \frac{\sum_{j=1}^k t^{n-i+1}(1-t^{N_j^{(n)}-n})}{k} = t^{n-i+1}(1-c(n)). \tag{5.5.2}$$

But also by (5.5.1),

$$\left|\sum_{j=1}^{k}\mu(t^{n-i+1},\ldots,t^{N_j^{(n)}-i})-\sum_{j=1}^{k}\sum_{\ell=1}^{N_j^{(n)}-n}(1-t)t^{n-i}\cdot t^{\ell}\right|<\sum_{j=1}^{k}\sum_{\ell=1}^{N_j^{(n)}-n}B(i)t^{2n}t^{\ell}<kB(i)t^{2n}\frac{t}{1-t},$$

hence

$$\left|L_i^{(n)}-\tilde{L}_i^{(n)}\right|<B(i)t^{2n}\frac{t}{1-t}.$$

Since $c(n)\leq t<1$, $\frac{B(i)t^{2n}\frac{t}{1-t}}{t^n(1-c(n))}=o(1)$ as $n\to\infty$, so (5.5.2) implies

$$\lim_{n\to\infty}\frac{L_i^{(n)}}{t^n(1-c(n))}=\lim_{n\to\infty}\frac{\tilde{L}_i^{(n)}}{t^n(1-c(n))}+\frac{L_i^{(n)}-\tilde{L}_i^{(n)}}{t^n(1-c(n))}=t^{1-i}.$$

□

Remark 29. It is worth noting that if one instead considers

$$\lim_{k\to\infty}\frac{1}{k}\log(\lambda_i(k))$$

(which in our analogy corresponds to the i^{th} smallest singular value), then the $n\to\infty$ limits are not universal and indeed the limits may not exist for some choices of the $N_j^{(n)}$. If the $N_j^{(n)}$ are all the same for any fixed n, then one has

$$\lim_{k\to\infty}\frac{1}{k}\log(\lambda_i(k))=\mathbb{E}[Y_i(k)]$$

and this clearly depends on the choice of $N_j^{(n)}$.

THIS PAGE INTENTIONALLY LEFT BLANK

Chapter 6

Local bulk limits of p-adic Dyson Brownian motion

6.1 Classifying isotropic processes

In this section we prove Theorem 1.4.3, by deducing it from a result (Proposition 6.1.2) which translates the constraints of isotropy and stationarity of processes on $\mathrm{GL}_N(\mathbb{Q}_p)/\mathrm{GL}_N(\mathbb{Z}_p)$ into a usable form. For expository purposes, we first prove a version in discrete time (Proposition 6.1.1) which makes the basic ideas of Proposition 6.1.2 slightly more apparent.

Definition 42. A stochastic process $X(\tau), \tau \in \mathbb{Z}_{\geq 0}$ on a group G has *independent increments* if for any $s, \tau \in \mathbb{Z}_{\geq 0}$, $X(\tau+s)X(\tau)^{-1}$ is independent of the trajectory of $X(y)$ up to time τ. It has *stationary increments* if

$$\mathrm{Law}(X(\tau+s)X(\tau)^{-1}) = \mathrm{Law}(X(s)X(0)^{-1}) \tag{6.1.1}$$

for all such s, τ. For a subgroup $K \leq G$, we further say $X(\tau)$ has *K-isotropic increments* if

$$\mathrm{Law}(X(\tau+s)X(\tau)^{-1}) = \mathrm{Law}(kX(\tau+s)X(\tau)^{-1}k^{-1}) \tag{6.1.2}$$

for any $k \in K, s, \tau \geq 0$. We use the same terminology for continuous-time processes with $\mathbb{Z}_{\geq 0}$ replaced everywhere by $\mathbb{R}_{\geq 0}$.

We now state and prove the discrete-time result, which follows directly from the definitions.

Proposition 6.1.1. *Let $X(\tau), \tau \in \mathbb{Z}_{\geq 0}$ be a discrete-time stochastic process on $\mathrm{GL}_N(\mathbb{Q}_p)$ started at the identity, with stationary, independent, $\mathrm{GL}_N(\mathbb{Z}_p)$-isotropic increments, and set $M_X^{(d)} := \mathrm{Law}(\mathrm{SN}(X(1)X(0)^{-1}))$. Then there exists a distribution on triples (U, V, ν), such that the marginal distribution of each pair (U, ν) and (V, ν) is $M_{Haar}(\mathrm{GL}_N(\mathbb{Z}_p)) \times M_X^{(d)}$, for which*

$$\mathrm{Law}(X(\tau), \tau \in \mathbb{Z}_{\geq 0}) = \mathrm{Law}(U_\tau \operatorname{diag}(p^{\nu_1^{(\tau)}}, \ldots, p^{\nu_N^{(\tau)}})V_\tau \cdots U_1 \operatorname{diag}(p^{\nu_1^{(1)}}, \ldots, p^{\nu_N^{(1)}})V_1, \tau \in \mathbb{Z}_{\geq 0}) \tag{6.1.3}$$

where $(U_i, V_i, \nu^{(i)})$ are iid copies of (U, V, ν).

Remark 30. We note that while the pairs (U, ν) and (V, ν) are each distributed by product measures, the pair (U, V) need not be. For example, one may have $U = V^{-1} \sim M_{Haar}(\mathrm{GL}_N(\mathbb{Z}_p))$. Of course, U and V can also be independent Haar matrices.

Proof of Proposition 6.1.1. Consider the increment

$$X(\tau + 1) = (X(\tau + 1)X(\tau)^{-1})X(\tau) \tag{6.1.4}$$

corresponding to the time step $\tau \to \tau + 1$. By the independent increments property $X(\tau+1)X(\tau)^{-1}$ is independent of $X(0), \ldots, X(\tau)$ and distributed as $X(1)X(0)^{-1}$. Hence $X(\tau+1)X(\tau)^{-1} = WDV$ for $W, V \in \mathrm{GL}_N(\mathbb{Z}_p)$ and $D = \operatorname{diag}(p^{\nu(\tau+1)})$ with $\nu(\tau+1) \sim M_X^{(d)}$ by definition of $M_X^{(d)}$ and stationary increments, and all of these are independent of $X(0), \ldots, X(\tau)$. By isotropy,

$$WDV = \tilde{U}WDV\tilde{U}^{-1} \tag{6.1.5}$$

in distribution for any fixed $\tilde{U} \in \mathrm{GL}_N(\mathbb{Z}_p)$. Hence by averaging, (6.1.5) also holds when $\tilde{U}$ is random with Haar distribution independent of W, D, V and $X(0), \ldots, X(\tau)$. Because $\tilde{U}$ is Haar-distributed independent of D and of W and V, $\tilde{U}W$ and $V\tilde{U}^{-1}$ are each Haar-distributed independent of D. Defining $U_{\tau+1} = \tilde{U}W$ and $V_{\tau+1} = V\tilde{U}^{-1}$, we thus have that the increments are of the form in the right hand side of (6.1.3), which completes the proof.

□

We now explicitly define the measure and Poisson rate constant claimed to exist in Theorem 1.4.3. We will work on the homogeneous space $\mathrm{GL}_N(\mathbb{Q}_p)/\mathrm{GL}_N(\mathbb{Z}_p)$, since it is discrete and so all processes on it are Poisson jump processes.

Definition 43. For any $X \in \mathrm{GL}_N(\mathbb{Q}_p)$, we denote by $[X]$ the corresponding coset in $\mathrm{GL}_N(\mathbb{Q}_p)/\mathrm{GL}_N(\mathbb{Z}_p)$.

Definition 44. Given a stochastic process $X(\tau), \tau \in \mathbb{R}_{\geq 0}$ on $\mathrm{GL}_N(\mathbb{Q}_p)$ satisfying the conditions of Theorem 1.4.3, we define

$$\begin{aligned} \tau' &= \inf\{\tau > 0 : [X(\tau)] \neq [X(0)]\} \\ M_X &= \mathrm{Law}(\mathrm{SN}(X(\tau'))) \\ c &= \mathbb{E}[\tau']. \end{aligned} \tag{6.1.6}$$

Proposition 6.1.2. *Let $N \in \mathbb{Z}_{\geq 1}$ and let $X(\tau), \tau \in \mathbb{R}_{\geq 0}$ be a Markov process on $\mathrm{GL}_N(\mathbb{Q}_p)$ started at the identity with stationary, independent, $\mathrm{GL}_N(\mathbb{Z}_p)$-isotropic increments. Then there exists a distribution on triples (U, V, ν), such that the marginal distribution of each pair (U, ν) and (V, ν) is $M_{Haar}(\mathrm{GL}_N(\mathbb{Z}_p)) \times M_X$, for which*

$$\mathrm{Law}([X(\tau)], \tau \in \mathbb{R}_{\geq 0}) = \mathrm{Law}([U_{P(\tau)} \operatorname{diag}(p^{\nu_1^{(P(\tau))}}, \ldots, p^{\nu_N^{(P(\tau))}}) V_{P(\tau)} \cdots U_1 \operatorname{diag}(p^{\nu_1^{(1)}}, \ldots, p^{\nu_N^{(1)}}) V_1], \tau \in \mathbb{R}_{\geq 0} \tag{6.1.7}$$

where $(U_i, V_i, \nu^{(i)})$ are iid copies of (U, V, ν).

Proof. First note that the dynamics of $X(\tau)$ commutes with right-multiplication by $\mathrm{GL}_N(\mathbb{Z}_p)$ (in fact, by $\mathrm{GL}_N(\mathbb{Q}_p)$), so $[X(\tau)]$ is Markov. Define the $\mathbb{Z}_{\geq 0}$-valued process

$$N_X(\tau) = |\{0 < s \leq \tau : [X(\tau)] \neq \lim_{\epsilon \to 0^+} [X(\tau - \epsilon)]\}|, \tag{6.1.8}$$

i.e. the number of times $[X(\tau)]$ has changed value up to time τ. By the Markov property and stationary increments, $N_X(\tau)$ is a Poisson process $P(\tau)$ with rate c as defined in Definition 44. Let $t_1 < t_2 < \ldots$ be the (random) times realizing $\mathrm{SN}(X(\tau)) \neq \lim_{\epsilon \to 0^+} \mathrm{SN}(X(\tau - \epsilon))$ and $t_0 = 0$, so $X(\tau) = X(t_{N_X(\tau)})$ and this is equal in distribution to $X(t_{P(\tau)})$. By Kolmogorov's extension theorem it suffices to show

$$\mathrm{Law}([X(t_{P(s)})], 0 \le s \le \tau) = \mathrm{Law}([U_{P(s)} \operatorname{diag}(p^{\nu(P(s))})V_{P(s)} \cdots U_1 \operatorname{diag}(p^{\nu^{(1)}})V_1, 0 \le s \le \tau) \tag{6.1.9}$$

for any fixed $\tau \in \mathbb{R}_{\ge 0}$. It further suffices to show the equality of conditional laws

$$\begin{aligned}&\mathrm{Law}([X(t_{P(s)})], 0 \le s \le \tau | P(s), 0 \le s \le \tau)\\ &\quad = \mathrm{Law}([U_{P(s)} \operatorname{diag}(p^{\nu(P(s))})V_{P(s)} \cdots U_1 \operatorname{diag}(p^{\nu^{(1)}})V_1], 0 \le s \le \tau | P(s), 0 \le s \le \tau),\end{aligned} \tag{6.1.10}$$

as then one may average over the distribution of $P(s), 0 \le s \le \tau$ to obtain (6.1.9). By the strong Markov property, $X(t_{i+1})X(t_i)^{-1}$ is independent of $X(\tau), 0 \le \tau \le t_i$ and independent of $t_0, \ldots, t_i$, hence it suffices to show

$$\mathrm{Law}([X(t_i)], 0 \le i \le n) = \mathrm{Law}([U_i \operatorname{diag}(p^{\nu^{(i)}})V_i \cdots U_1 \operatorname{diag}(p^{\nu^{(1)}})V_1], 0 \le i \le n) \tag{6.1.11}$$

for all $n \in \mathbb{Z}_{\ge 1}$. Since each increment $X(t_{i+1})X(t_i)^{-1}$ is distributed as $X(t_1)X(0)^{-1}$ by independent increments, (6.1.11) is exactly the discrete case Proposition 6.1.1 and we are done. □

Proof of Theorem 1.4.3. Since Smith normal form is independent of right-multiplication by $\mathrm{GL}_N(\mathbb{Z}_p)$, we may write $\mathrm{SN}([X])$ for $[X] \in \mathrm{GL}_N(\mathbb{Q}_p)/\mathrm{GL}_N(\mathbb{Z}_p)$ with no ambiguity. By Proposition 6.1.2,

$$\mathrm{SN}(X(\tau)) = \mathrm{SN}([X(\tau)]) = \mathrm{SN}([\tilde{U}_{P(\tau)} \operatorname{diag}(p^{\tilde{\nu}_1^{(P(\tau))}}, \ldots, p^{\tilde{\nu}_N^{(P(\tau))}})\tilde{V}_{P(\tau)} \cdots \tilde{U}_1 \operatorname{diag}(p^{\tilde{\nu}_1^{(1)}}, \ldots, p^{\tilde{\nu}_N^{(1)}})\tilde{V}_1]) \tag{6.1.12}$$

in multi-time distribution, where $\tilde{U}_i, \tilde{V}_i, \tilde{\nu}^{(i)}$ correspond to the $U_i, V_i, \nu^{(i)}$ in Proposition 6.1.2. We write them with the tildes to distinguish them from

$$Y^{(N,M_X,c)}(\tau) = U_{P(\tau)} \operatorname{diag}(p^{\nu_1^{(P(\tau))}}, \ldots, p^{\nu_N^{(P(\tau))}})V_{P(\tau)} \cdots U_1 \operatorname{diag}(p^{\nu_1^{(1)}}, \ldots, p^{\nu_N^{(1)}})V_1U_0, \tag{6.1.13}$$

for which U_i and V_i are independent as we recall from Definition 1. As in the proof of

Proposition 6.1.2 we are reduced to showing that

$$\begin{aligned}\mathrm{Law}(\mathrm{SN}(U_i \operatorname{diag}(p^{\nu^{(i)}})V_i \cdots U_1 \operatorname{diag}(p^{\nu^{(1)}})V_1), 0 \le i \le n) \\ = \mathrm{Law}(\mathrm{SN}(\tilde{U}_i \operatorname{diag}(p^{\tilde{\nu}^{(i)}})\tilde{V}_i \cdots \tilde{U}_1 \operatorname{diag}(p^{\tilde{\nu}^{(1)}})\tilde{V}_1), 0 \le i \le n),\end{aligned} \tag{6.1.14}$$

which we do by induction. The base case is trivial, so assume it holds for some n. Then by inductive hypothesis,

$$\begin{aligned}&\mathrm{Law}(\mathrm{SN}(\tilde{U}_i \operatorname{diag}(p^{\tilde{\nu}^{(i)}})\tilde{V}_i \cdots \tilde{U}_1 \operatorname{diag}(p^{\tilde{\nu}^{(1)}})\tilde{V}_1), 0 \le i \le n+1) \\ &= \mathrm{Law}(\mathrm{SN}(U_1 \operatorname{diag}(p^{\nu^{(1)}})V_1, \ldots, \mathrm{SN}(U_n \operatorname{diag}(p^{\nu^{(n)}})V_n \cdots U_1 \operatorname{diag}(p^{\nu^{(1)}})V_1), \\ &\mathrm{SN}(\operatorname{diag}(p^{\tilde{\nu}^{(n+1)}})\tilde{V}_{n+1}U_n \operatorname{diag}(p^{\nu^{(n)}})V_n \cdots U_1 \operatorname{diag}(p^{\nu^{(1)}})V_1)),\end{aligned} \tag{6.1.15}$$

where we have removed the $\tilde{U}_{n+1}$ on the left since it does not affect the singular numbers. Because $(\tilde{\nu}^{(n+1)}, \tilde{V}_{n+1}) \sim M_X \times M_{Haar}(\mathrm{GL}_N(\mathbb{Z}_p))$ and $(\nu^{(n+1)}, V_{n+1}) \sim M_X \times M_{Haar}(\mathrm{GL}_N(\mathbb{Z}_p))$ as well,

$$\mathrm{RHS}(6.1.15) = \mathrm{Law}(\mathrm{SN}(U_i \operatorname{diag}(p^{\nu^{(i)}})V_i \cdots U_1 \operatorname{diag}(p^{\nu^{(1)}})V_1), 0 \le i \le n+1) \tag{6.1.16}$$

(adding back in the U_{n+1} which does not affect the singular numbers). This completes the induction to show (6.1.14) and hence the proof.

□

Remark 31. The above results and proofs apply mutatis mutandis with the groups $\mathrm{GL}_N(\mathbb{Z}_p) \le \mathrm{GL}_N(\mathbb{Q}_p)$ replaced by any groups $K \le G$ with K compact and G/K discrete, and Sig_N replaced by $K \backslash G/K$.

We now turn attention to those processes with $M_X = \delta_{(1,0[N-1])}$. Theorem 1.4.3 implies that the evolution of singular numbers of such a process are determined by a rate parameter which may be absorbed by time change. We refer to the discussion directly after Theorem 1.4.3 for why these are natural analogues of multiplicative Brownian motion.

Definition 45. For any $N \in \mathbb{Z}_{\ge 1}$, we define a continuous-time stochastic process $X^{(N)}(\tau)$

on $\mathrm{GL}_N(\mathbb{Q}_p)$ by

$$X^{(N)}(\tau) = Y^{(N,\delta_{(1,0[N-1])},1)}(\tau) = U_{P(\tau)} \operatorname{diag}(p, 1[N-1])V_{P(\tau)} \cdots U_1 \operatorname{diag}(p, 1[N-1])V_1 U_0, \tag{6.1.17}$$

where $P(\tau)$ is a rate 1 Poisson process and $U_i, V_i, i \in \mathbb{Z}_{\geq 1}$ are independent matrices distributed by the Haar measure on $\mathrm{GL}_N(\mathbb{Z}_p)$.

6.2 p-adic Dyson Brownian motion and reflected Poisson walks

Much of this section consists of computing and comparing Markov generators. We wish to go from equalities of generators to equalities of stochastic processes, for which we use the following standard result (see [Fel15] or [BO12b, Section 4.1], which also give stronger ones than we need):

Proposition 6.2.1. *Let $Y_\tau, \tau \in \mathbb{R}_{\geq 0}$ be a Markov process on a countable state space $\mathcal{X}$ with well-defined generator Q satisfying*

$$\sup_{a \in \mathcal{X}} |Q(a,a)| < \infty. \tag{6.2.1}$$

Then the law of $Y_\tau, \tau \in \mathbb{R}_{\geq 0}$ is uniquely determined by Q.

We now give the generator for $\mathcal{S}$, which in light of Proposition 6.2.1 serves as an alternative and more formal definition.

Definition 46. Let $n \in \mathbb{N} \cup \{\infty\}$, $\mu \in \mathrm{Sig}_n$ and $t \in (0,1)$. We define the stochastic process $\mathcal{S}^{\mu,n}(\tau) = (\mathcal{S}_1^{\mu,n}(\tau), \ldots, \mathcal{S}_n^{\mu,n}(\tau))$ on Sig_n as the Markov process with initial condition μ and generator given by[1]

$$B_{\mathcal{S}}(\kappa,\nu) := \begin{cases} -t\frac{1-t^n}{1-t} & \kappa = \nu \\ t^\ell + \ldots + t^{\ell + m_{\kappa_\ell}(\kappa) - 1} = t^\ell \frac{1-t^{m_{\kappa_\ell}(\kappa)}}{1-t} & \kappa \prec \nu \text{ and } (\nu_i)_{1 \leq i \leq n} = (\kappa_i + \mathbb{1}(i=\ell))_{1 \leq i \leq n} \\ 0 & \text{otherwise} \end{cases} \tag{6.2.2}$$

[1]We suppress n and t in the notation for the generator, but of course it depends on both.

As before, when $n = \infty$ we take $t^n = 0$ in the above formulas, and may have $m_{\kappa_\ell}(\kappa) = \infty$.

To see that the above generator corresponds to the informal description of the dynamics in the Introduction, note that by those rules the transition rate from κ to ν as above is given by the sum of jump rates of $\kappa_\ell, \ldots, \kappa_{\ell+m_{\kappa_\ell}(\kappa)-1}$, which are $t^\ell, \ldots, t^{\ell+m_{\kappa_\ell}(\kappa)-1}$. We next see that $\mathcal{S}^{(n)}$ corresponds to a Hall-Littlewood process, for which we use the following notation.

Definition 47. For any $n \in \mathbb{N} \cup \{\infty\}$, we denote by $\lambda^{(n)}(\tau), \tau \geq 0$ the stochastic process on $\mathbb{Y}_n$ in continuous time τ with finite-dimensional marginals given by the Hall-Littlewood process

$$\begin{aligned}\Pr(\lambda^{(n)}(\tau_i) = \lambda(i) \text{ for all } i = 1, \ldots, k)\\ = \frac{\left(\prod_{j=1}^k Q_{\lambda(j)/\lambda(j-1)}(\gamma(\tau_j - \tau_{j-1}); 0, t)\right) P_{\lambda(k)}(1, t, \ldots, t^{n-1}; 0, t)}{\exp\left(\frac{\tau_k(1-t^n)}{1-t}\right)}\end{aligned} \tag{6.2.3}$$

for each sequence of times $0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_k$ and $\lambda(1), \ldots, \lambda(k) \in \mathbb{Y}_n$, where in the product we take the convention $\tau_0 = 0$ and $\lambda(0)$ is the zero partition. More generally, for $\mu \in \mathrm{Sig}_n^+$ we denote by $\lambda^{(n,\mu)}(\tau)$ the same process started at initial condition μ, i.e. with marginals defined by $1/P_\mu(1, \ldots, t^{n-1}; 0, t)$ times the expression in (6.2.3) where we instead take $\lambda(0) = \mu$.

Lemma 6.2.2. *For any $n \in \mathbb{N} \cup \{\infty\}$, $\lambda^{(n)}(\tau)$—and more generally $\lambda^{(n,\mu)}$—has a Markov generator given by*

$$\begin{aligned}B_{HL}(\kappa, \nu) &:= \begin{cases} -\frac{1-t^n}{1-t} & \kappa = \nu \\ \frac{\varphi_{\nu/\kappa}(0,t)}{1-t} \frac{P_\nu(1,t,\ldots,t^{n-1};0,t)}{P_\kappa(1,t,\ldots,t^{n-1};0,t)} & \kappa \prec \nu \text{ and } |\nu| = |\kappa| + 1 \\ 0 & \text{otherwise} \end{cases} &(6.2.4)\\ &= \frac{1}{t} B_{\mathcal{S}}(\kappa, \nu) &(6.2.5)\end{aligned}$$

for any $\kappa, \nu \in \mathrm{Sig}_n$.

Finally, we compute the generator of the process on $\mathrm{GL}_N(\mathbb{Q}_p)/\mathrm{GL}_N(\mathbb{Z}_p)$ described in the previous section.

Lemma 6.2.3. *Let $N \in \mathbb{Z}_{\geq 1}, c \in \mathbb{R}_{>0}$, and $X(\tau) = Y^{(N,\delta_{(1,0[N-1])},c)}$ in the notation of Definition 1. Then the stochastic proces* $\mathrm{SN}(X(\tau))$ *has Markov generator* $c\frac{1-t}{1-t^N}B_{HL}$.

Having stated the results, the rest of the section consists of doing the computations.

Proof of Lemma 6.2.2. We will first prove (6.2.4). Recall

$$B_{HL}(\kappa, \nu) := \frac{d}{d\tau}\bigg|_{\tau=0} \Pr(\lambda^{(n)}(T+\tau) = \nu | \lambda^{(n)}(T) = \kappa) \tag{6.2.6}$$

(this is of course independent of T by the Markov property). By the equivalence of the Hall-Littlewood process with the Cauchy dynamics of Definition 14 and the explicit formula (2.2.30) for the Cauchy kernel, we have

$$\Pr(\lambda^{(n)}(T+\tau) = \nu | \lambda^{(n)}(T) = \kappa) = \frac{Q_{\nu/\kappa}(\gamma(\tau);t)}{\exp\left(\tau\frac{1-t^n}{1-t}\right)} \frac{P_\nu(1,\ldots,t^{n-1};t)}{P_\kappa(1,\ldots,t^{n-1};t)},$$

and only the term

$$\frac{Q_{\nu/\kappa}(\gamma(\tau);t)}{\exp\left(\tau\frac{1-t^n}{1-t}\right)}$$

depends on τ. When $\nu = \kappa$, $Q_{\nu/\kappa} = 1$, so

$$B_{HL}(\kappa, \kappa) = \frac{d}{d\tau}\bigg|_{\tau=0} \Pr(\lambda^{(n)}(T+\tau) = \kappa | \lambda^{(n)}(T) = \kappa) = -\frac{1-t^n}{1-t}. \tag{6.2.7}$$

In general, $Q_{\nu/\kappa}$ (viewed as an element of the ring of symmetric functions) is a polynomial in the $p_k, k \geq 1$ which is homogeneous of degree $|\nu| - |\kappa|$ if we define each p_k to have degree k. Under the Plancherel specialization, all $p_k, k \geq 2$ are sent to 0, hence $Q_{\nu/\kappa}(\gamma(\tau);t) = O(\tau^{|\nu|-|\kappa|})$ as $\tau \to 0$. It follows that

$$B_{HL}(\kappa, \nu) = 0 \text{ if } |\nu| > |\kappa| + 1. \tag{6.2.8}$$

When $|\nu| = |\kappa| + 1$, it follows from Lemma 2.2.14 that $Q_{\nu/\kappa} = \varphi_{\nu/\kappa} p_1$. Hence

$$Q_{\nu/\kappa}(\gamma(\tau);t) = \varphi_{\nu/\kappa}\frac{\tau}{1-t}$$

and

$$\left.\frac{d}{d\tau}\right|_{\tau=0} \Pr(\lambda^{(n)}(T+\tau)=\nu|\lambda^{(n)}(T)=\kappa) = \frac{\varphi_{\nu/\kappa}}{1-t}\frac{P_\nu(1,\ldots,t^{n-1};t)}{P_\kappa(1,\ldots,t^{n-1};t)}, \tag{6.2.9}$$

Combining (6.2.7), (6.2.8) and (6.2.9) yields (6.2.4), so it remains to prove (6.2.5).

The latter equality (6.2.5) is immediate except for the case where $\kappa \prec \nu$ and all parts in κ and ν are the same except for one part which differs by 1. In this case there are some integers $k, a \geq 0, b \geq 1$ such that $\kappa = (\ldots,(k+1)[a],k[b],\ldots)$ and $\nu = (\ldots,(k+1)[a+1],k[b-1],\ldots)$ where we use $x[c]$ to denote x repeated c times in the partition. Let ℓ be the smallest integer so that $\kappa_\ell = k$. To compute (6.2.9) we specialize Lemma 2.2.14 to obtain

$$\varphi_{\nu/\kappa} = 1-t^{a+1},$$

and plugging this and Proposition 2.2.15 into (6.2.9) yields that in our situation

$$B_{HL}(\kappa,\nu) = \frac{t^{\ell-1}(1-t^b)}{1-t} = t^{\ell-1}+\ldots+t^{(\ell+b-1)-1}, \tag{6.2.10}$$

proving (6.2.5). □

Lemma 6.2.4. *Let $\lambda,\mu \in \mathrm{Sig}_N$ and let U be a Haar-distributed element of $\mathrm{GL}_N(\mathbb{Z}_p)$. Then*

$$\Pr(\mathrm{SN}(\mathrm{diag}(p^\mu)U\,\mathrm{diag}(p^\lambda))=\nu) = c^\nu_{\lambda,\mu}(0,t)\frac{P_\nu(1,\ldots,t^{N-1};0,t)}{P_\mu(1,\ldots,t^{N-1};0,t)P_\lambda(1,\ldots,t^{N-1};0,t)}. \tag{6.2.11}$$

Proof. This is essentially Part 3 of Theorem 1.2.1, though let us remark on slight differences in setup. That result was stated for two matrices $A,B \in \mathrm{GL}_N(\mathbb{Q}_p)$ with fixed singular numbers λ,μ respectively, and distribution invariant under left- and right-multiplication by $\mathrm{GL}_N(\mathbb{Z}_p)$. Such matrices are given by $U\,\mathrm{diag}(p^\lambda)V$ and $U'\,\mathrm{diag}(p^\mu)V'$ for U,V,U',V' independent Haar-distributed elements of $\mathrm{GL}_N(\mathbb{Z}_p)$. Hence

$$\mathrm{SN}(AB) = \mathrm{SN}(U\,\mathrm{diag}(p^\lambda)VU'\,\mathrm{diag}(p^\mu)V') = \mathrm{SN}(\mathrm{diag}(p^\lambda)VU'\,\mathrm{diag}(p^\mu)), \tag{6.2.12}$$

and VU' has Haar distribution. Hence

$$\Pr(\mathrm{SN}(\operatorname{diag}(p^{\mu})U\operatorname{diag}(p^{\lambda})) = \nu) = \Pr(\mathrm{SN}(AB) = \nu), \tag{6.2.13}$$

and (6.2.11) now follows by Part 3 of Theorem 1.2.1. □

Proof of Lemma 6.2.3. By the definition of $X(\tau)$,

$$\begin{aligned}&\Pr(\mathrm{SN}(X(\tau+\epsilon)) = \nu | \mathrm{SN}(X(\tau)) = \mu)\\ &\quad = \mathbb{1}(\nu = \mu)\cdot(1-c\epsilon) + c\epsilon\Pr(\mathrm{SN}(\operatorname{diag}(p^{\mu})U\operatorname{diag}(p,1,\ldots,1)) = \nu) + O(\epsilon^2)\end{aligned} \tag{6.2.14}$$

as $\epsilon \to 0$. Hence the generator of the process $\mathrm{SN}(X(\tau))$ on Sig_N is given by

$$B_{SN}(\mu,\nu) := -c\mathbb{1}(\mu = \nu) + c\Pr(\mathrm{SN}(\operatorname{diag}(p^{\mu})U\operatorname{diag}(p,1,\ldots,1)) = \nu) \tag{6.2.15}$$

for any $\mu, \nu \in \mathrm{Sig}_N$.

By Lemma 6.2.4,

$$\begin{aligned}&\Pr(\mathrm{SN}(\operatorname{diag}(p^{\mu})U\operatorname{diag}(p,1,\ldots,1)) = \nu)\\ &\quad = c^{\nu}_{\mu,(1,0[N-1])}(0,t)\frac{P_{\nu}(1,\ldots,t^{N-1};0,t)}{P_{\mu}(1,\ldots,t^{N-1};0,t)P_{(1,0[N-1])}(1,\ldots,t^{N-1};0,t)}.\end{aligned} \tag{6.2.16}$$

By Proposition 2.2.4 and (2.2.51), when $\nu \succ \mu$ and $|\nu| - |\mu| = 1$ we have

$$\varphi_{\nu/\mu}(0,t) = Q_{\nu/\mu}(1) = c^{\nu}_{\mu,(1,0[N-1])}(0,t)Q_{(1,0[N-1])}(1) = c^{\nu}_{\mu,(1,0[N-1])}(0,t)(1-t). \tag{6.2.17}$$

Additionally,

$$P_{(1,0[N-1])}(1,\ldots,t^{N-1};0,t) = \frac{1-t^N}{1-t} \tag{6.2.18}$$

by Proposition 2.2.15 (one may also simply use that this Hall-Littlewood polynomial is the elementary symmetric polynomial e_1). Substituting (6.2.18) and (6.2.17) into (6.2.16) yields

$$\mathrm{RHS}(6.2.16) = \left(c\frac{1-t}{1-t^N}\right)\frac{\varphi_{\nu/\mu}}{1-t}\frac{P_{\nu}(1,\ldots,t^{N-1};0,t)}{P_{\mu}(1,\ldots,t^{N-1};0,t)}. \tag{6.2.19}$$

Combining with (6.2.15) and Lemma 6.2.2 yields that

$$B_{SN}(\mu,\nu) = \left(c\frac{1-t}{1-t^N}\right) B_{HL}(\mu,\nu) \tag{6.2.20}$$

for all $\mu,\nu \in \mathrm{Sig}_N^+$, completing the proof. □

Proof of Theorem 1.4.4. Follows from the equality of generators of Lemma 6.2.3, together with Proposition 6.2.1. □

The following corollary also follows from the equality of generators given in Lemma 6.2.2, together with Proposition 6.2.1.

Corollary 6.2.5. *Let $n \in \mathbb{N}$ and $\mu \in \mathrm{Sig}_n$, and let $\lambda^{(n,\mu)}, \mathcal{S}^{\mu,n}$ be as in Definition 47 and Definition 46 respectively. Then*

$$\lambda^{(n,\mu)}(\tau) = \mathcal{S}^{\mu,n}(\tau/t) \tag{6.2.21}$$

in multi-time distribution.

6.3 The stationary law

In this section we compute explicit contour integral formulas, given in Theorem 6.3.1, for the limiting distribution of conjugate parts of the half-infinite Poisson walk $\mathcal{S}^{\nu,\infty}(\tau)$, which will be used to show the corresponding formula for the reflecting Poisson sea in Theorem 1.4.1. These formulas are valid for suitably small initial conditions ν, which will also be useful in upcoming random matrix coupling arguments. Because our methods come from Macdonald processes, we will state things in terms of the Hall-Littlewood process $\lambda^{(\infty,\nu)}(\tau)$ of Definition 47, but this is the same as $\mathcal{S}^{\nu,\infty}(\tau)$ up to time change by Corollary 6.2.5.

Let us briefly outline the proof before giving details. We define a family of observables $f_\mu, \mu \in \mathbb{Y}_k$ of a random partition κ by

$$f_\mu(\kappa) = \frac{P_{\kappa/\mu'}(1,t,\ldots;0,t)}{P_\kappa(1,t,\ldots;0,t)}. \tag{6.3.1}$$

These have nice expectations when κ is distributed by the Hall-Littlewood measure we are interested in, but also may be explicitly inverted to yield a different form, given in Lemma 6.3.2, for the prelimit probability we wish to take asymptotics of. Though this expression may look *a priori* more complicated, it is suitable for converting to contour integrals (Lemma 6.3.3), and hence for asymptotic analysis. We remark that in the case when $t = 1/p$, this computation of the probabilities from our observables is in fact equivalent to that of determining the distribution of a random abelian p-group from its 'moments', as done in [Woo19, Woo17] and later made more computationally explicit in [SW22a, SW22b]; we will give the details of the translation between these two settings in an upcoming publication.

Theorem 6.3.1. *Let $\lambda(\tau) = \lambda^{(\infty,\emptyset)}(\tau)$ as defined in Definition 47, and fix $k \in \mathbb{Z}_{\geq 2}$ and $\alpha \in \mathbb{R}$. Then for any integers $L_1 \geq \ldots \geq L_k$,*

$$\begin{aligned}\lim_{\substack{\tau\to\infty \\ \log_{t^{-1}}(\tau)+\alpha\in\mathbb{Z}}} & \Pr(\lambda_i'(\tau) - \log_{t^{-1}}(\tau) = L_i + \alpha \text{ for all } 1 \leq i \leq k) \\ &= \frac{(t;t)_\infty^{k-1}}{k!(2\pi \mathbf{i})^k} \prod_{i=1}^{k-1} \frac{t^{\binom{L_i - L_k}{2}}}{(t;t)_{L_i - L_{i+1}}} \int_{\tilde{\Gamma}^k} e^{\frac{t^{L_k+\alpha}}{1-t}(w_1+\ldots+w_k)} \frac{\prod_{1\leq i\neq j\leq k}(w_i/w_j;t)_\infty}{\prod_{i=1}^k(-w_i^{-1};t)_\infty(-tw_i;t)_\infty} \\ &\times \sum_{j=0}^{L_{k-1}-L_k} t^{\binom{j+1}{2}} \begin{bmatrix} L_{k-1}-L_k \\ j \end{bmatrix}_t P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1},\ldots,w_k^{-1};t,0) \prod_{i=1}^k \frac{dw_i}{w_i} \qquad (6.3.2)\end{aligned}$$

with contour

$$\tilde{\Gamma} := \{x + \mathbf{i} : x \leq 0\} \cup \{x - \mathbf{i} : x \leq 0\} \cup \{x + \mathbf{i}y : x^2 + y^2 = 1, x > 0\} \qquad (6.3.3)$$

in usual counterclockwise orientation. When $k = 1$, the equality (6.3.2) holds with the sum over j replaced by

$$\sum_{j=0}^{\infty} t^{\binom{j+1}{2}} \frac{1}{(t;t)_j} P_{(j)}(w_1^{-1};t,0). \qquad (6.3.4)$$

Furthermore, if $\nu(\tau), \tau \in t^{\mathbb{Z}+\alpha}$ is a sequence of partitions such that

$$\nu_1'(\tau) \leq \log_{t^{-1}} \tau - 2\left(\frac{k}{\log t^{-1}}\right)^2 (\log\log\tau)^2 \qquad (6.3.5)$$

for all sufficiently large τ, then the same result holds with $\lambda(\tau) = \lambda^{(\infty,\nu(\tau))}(\tau)$.

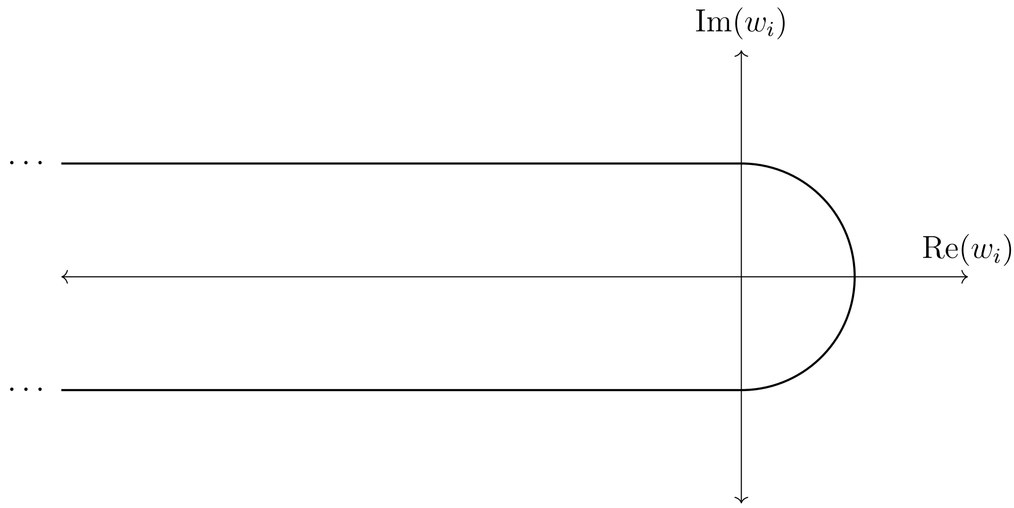


Figure 6-1: The contour $\tilde{\Gamma}$ in $\mathbb{C}$.

Remark 32. The sum in (6.3.4) is equal to $(-tw_1^{-1};t)_\infty$ by the q-binomial theorem. One should view this sum as what one naively obtains in the general $k \geq 2$ form (6.3.2) by taking $k = 1$ and substituting $L_{k-1} = \infty$ and

$$\begin{bmatrix} \infty \\ j \end{bmatrix}_t := \frac{1}{(t;t)_j}. \tag{6.3.6}$$

Remark 33. The reason for the parameter α is that for general τ, $\lambda_i'(\tau) - \log_{t^{-1}}(\tau)$ will not be an integer but rather lie on some shift of the integer lattice, and it is necessary to consider a sequence of τ where $\lambda_i'(\tau) - \log_{t^{-1}}(\tau)$ all lie on the same shift $\mathbb{Z} + \alpha$ of the integer lattice in order to have any hope of a $T \to \infty$ limiting distribution.

Remark 34. The restriction (6.3.5) ensures that the initial condition ν is sufficiently far from the observation point at $\approx \log_{t^{-1}} \tau$ that there is time for the process to relax to its stationary distribution. We believe that Theorem 6.3.1 continues to hold under the weaker condition that $\log_{t^{-1}} \tau - \nu_1'(\tau) \to \infty$ at any rate, and it is easy to see that this condition is necessary to obtain a limit distribution supported on $\mathbb{Z}$. The bound $(\log \log \tau)^2$ is nonetheless quite good, and arises as a technical condition in certain error bounds on contour integrals in the proof. We discuss in more detail why it is technically necessary for our arguments in Remark 36.

It is also clear that the formula on the right hand side of (6.3.2) is invariant under replacing $(L_1, \ldots, L_k) \mapsto (L_1 + 1, \ldots, L_k + 1)$ and $\alpha \mapsto \alpha - 1$, which it should be since the left hand side obviously has this invariance.

Lemma 6.3.2. *Fix $\nu \in \mathbb{Y}$ and set $\lambda(\tau) = \lambda^{(\infty,\nu)}(\tau)$ as defined in Definition 47. Then for any $k \in \mathbb{Z}_{\geq 1}$ and $\eta \in \mathbb{Y}_k$,*

$$\begin{aligned} &\Pr((\lambda'_1(\tau), \ldots, \lambda'_k(\tau)) = \eta) \\ &= P_{\eta'}(1,t,\ldots;0,t) \sum_{\mu \in \mathbb{Y}_k} \sum_{\kappa \in \mathbb{Y}_k} \frac{P_{\nu/\kappa}(1,t,\ldots;0,t)}{P_\nu(1,t,\ldots;0,t)} Q_{\mu'/\kappa}(\gamma(\tau);0,t) P_{\mu'/\eta'}(\beta(-1);0,t). \end{aligned} \tag{6.3.7}$$

At first glance it might seem that we could simplify the expression in (6.3.7) still further by applying the skew Cauchy identity to

$$\sum_{\mu \in \mathbb{Y}_k} Q_{\mu'/\kappa}(\gamma(\tau);0,t) P_{\mu'/\eta'}(\beta(-1);0,t) \tag{6.3.8}$$

to obtain a finite sum. However, this is slightly false: the Cauchy identity would only apply if the sum were over $\mu \in \mathbb{Y}$ rather than $\mathbb{Y}_k$. The fact that our sum is written over $\mathbb{Y}_k$ rather than $\mathbb{Y}$ is not purely cosmetic: if η' has length k then there will be $\mu \in \mathbb{Y}_{k+1}$ with $\mu' \succ \kappa, \mu' \succ \eta'$ (for κ of length $\leq k$) and consequently nonzero values of $Q_{\mu'/\kappa}(\gamma(\tau);0,t) P_{\mu'/\eta'}(\beta(-1);0,t)$ for such $\mu \in \mathbb{Y} \setminus \mathbb{Y}_k$. Hence our sum is different from the one appearing the Cauchy identity, and in our view this fact is largely responsible for the fact that both the computations and the final formula in this section do not bear much resemblance to the previously studied asymptotics of Macdonald processes of which we are aware. Understanding the sum (6.3.8) was a key difficulty in the computations, as it is not dominated by one or a small collection of terms. While $P_{\mu'/\eta'}(\beta(-1);0,t)$ is explicit by the branching rule Lemma 2.2.14, the branching rule yields a much more complicated sum formula for $Q_{\mu'/\kappa}(\gamma(\tau);0,t)$, the number of terms of which grows superexponentially in $|\mu|$ with no clear way to separate into a main term and subleading terms as $T \to \infty$. It turns out, however, that after reexpressing $Q_{\mu'/\kappa}(\gamma(\tau);0,t)$ using the torus scalar product (Definition 9) there are surprising simplifications, yielding an expression which is finally suitable to asymptotics and is given in the next lemma.

Lemma 6.3.3. *Keep the notation of Lemma 6.3.2, and let $\tilde{\nu} = (\max(\nu_1, k), \max(\nu_2, k), \ldots) =$*

$(\nu_1', \ldots, \nu_k', 0, \ldots)'$. *Then*

$$\Pr((\lambda_1'(\tau), \ldots, \lambda_k'(\tau)) = \eta) = \frac{(t;t)_\infty^{k-1}}{k!(2\pi \mathbf{i})^k \prod_{i=1}^{k-1} (t;t)_{\eta_i - \eta_{i+1}}} \int_{c\mathbb{T}^k} e^{\frac{\tau}{1-t}(z_1 + \ldots + z_k)} t^{\sum_{i=1}^k \binom{\eta_i}{2}}$$
$$\times \frac{P_{\tilde{\nu}}(\beta(z_1, \ldots, z_k), 1, t, \ldots; 0, t)}{P_{\tilde{\nu}}(1, t, \ldots; 0, t)} \prod_{1 \le i \neq j \le k} (z_i/z_j; t)_\infty$$
$$\times \sum_{j=0}^{\eta_{k-1} - \eta_k} t^{j(\eta_k + 1)} t^{\binom{j}{2}} \begin{bmatrix} \eta_{k-1} - \eta_k \\ j \end{bmatrix}_t \frac{P_{\eta + j\boldsymbol{e_k}}(z_1^{-1}, \ldots, z_k^{-1}; t, 0)}{\prod_{i=1}^k (-z_i^{-1}; t)_\infty} \prod_{i=1}^k \frac{dz_i}{z_i}, \tag{6.3.9}$$

where $\mathbb{T}$ *denotes the unit circle with counterclockwise orientation, and* $c \in \mathbb{R}_{>1}$ *is arbitrary.*

The rest of the section consists of proofs of the above statements.

Proof of Lemma 6.3.2. By the explicit formulas Proposition 2.2.15 and Theorem 2.2.16, for any $\kappa \in \mathbb{Y}, \mu \in \mathbb{Y}_k$ we have

$$\frac{P_{\kappa/\mu'}(1, t, \ldots; 0, t)}{P_\kappa(1, t, \ldots; 0, t)} = \prod_{i=1}^k t^{\binom{\kappa_i' - \mu_i}{2} - \binom{\kappa_i'}{2}} (t^{1 + \kappa_i' - \mu_i}; t)_{\mu_i - \mu_{i+1}}, \tag{6.3.10}$$

By the skew Cauchy identity (2.2.40) we also have

$$\begin{aligned} \mathbb{E}\left[\frac{P_{\lambda(\tau)/\mu'}(1, t, \ldots; 0, t)}{P_{\lambda(\tau)}(1, t, \ldots; 0, t)}\right] &= \frac{\sum_{\lambda \in \mathbb{Y}} Q_{\lambda/\nu}(\gamma(\tau); 0, t) P_{\lambda/\mu}(1, t, \ldots; 0, t)}{\Pi_{0,t}(1, t, \ldots; \gamma(\tau)) P_\nu(1, t, \ldots; 0, t)} \\ &= \sum_{\kappa \in \mathbb{Y}_k} \frac{P_{\nu/\kappa}(1, t, \ldots; 0, t)}{P_\nu(1, t, \ldots; 0, t)} Q_{\mu'/\kappa}(\gamma(\tau); 0, t). \end{aligned} \tag{6.3.11}$$

However, an important property of the observable (6.3.10) is that it depends on the first k parts of κ' but not on the others. In other words, setting $\tilde{\lambda} = (\lambda_1', \ldots, \lambda_k')' \in \mathbb{Y}$, we have

$$\frac{P_{\lambda/\mu'}(1, t, \ldots; 0, t)}{P_\lambda(1, t, \ldots; 0, t)} = \frac{P_{\tilde{\lambda}/\mu'}(1, t, \ldots; 0, t)}{P_{\tilde{\lambda}}(1, t, \ldots; 0, t)}. \tag{6.3.12}$$

Hence by the branching rule Lemma 2.2.14 and Proposition 2.2.8,

$$\begin{aligned}\sum_{\mu\in\mathbb{Y}_k} \frac{P_{\lambda/\mu'}(1,t,\ldots;0,t)}{P_\lambda(1,t,\ldots;0,t)} P_{\mu'/\eta'}(\beta(-1);0,t) &= \sum_{\mu\in\mathbb{Y}_k} \frac{P_{\tilde{\lambda}/\mu'}(1,t,\ldots;0,t)}{P_{\tilde{\lambda}}(1,t,\ldots;0,t)} P_{\mu'/\eta'}(\beta(-1);0,t) \\ &= \frac{P_{\tilde{\lambda}/\eta'}(\beta(-1),1,t,\ldots;0,t)}{P_{\tilde{\lambda}}(1,t,\ldots;0,t)} \\ &= \frac{\mathbb{1}(\lambda_1'(\tau),\ldots,\lambda_k'(\tau))=\eta)}{P_{\eta'}(1,t,\ldots;0,t)},\end{aligned} \tag{6.3.13}$$

where we are using the fact that because $\tilde{\lambda}' \in \mathbb{Y}_k$, each $\mu' \prec \tilde{\lambda}'$ appearing in the branching rule automatically satisfies $\mu_1' \leq \tilde{\lambda}_1'$ so $\mu \in \mathbb{Y}_k$.

Hence

$$\mathrm{LHS}(6.3.7) = P_{\eta'}(1,t,\ldots;0,t)\mathbb{E}\left[\sum_{\mu\in\mathbb{Y}_k} \frac{P_{\lambda(\tau)/\mu'}(1,t,\ldots;0,t)}{P_{\lambda(\tau)}(1,t,\ldots;0,t)} P_{\mu'/\eta'}(\beta(-1);0,t)\right], \tag{6.3.14}$$

and to apply (6.3.11) we wish to commute the expectation and the sum, so we must check the hypothesis of Fubini's theorem. For $t \in (0,1)$, $P_{\lambda(\tau)/\mu'}(1,t,\ldots;0,t) \geq 0$ and $P_{\mu'/\eta'}(\beta(1);0,t) \geq 0$ since these are Hall-Littlewood nonnegative specializations, so since $P_{\mu'/\eta'}(\beta(-1);0,t) = (-1)^{|\mu|-|\eta|} P_{\mu'/\eta'}(\beta(1);0,t)$ by homogeneity we have

$$\begin{aligned}&\mathbb{E}\left[\sum_{\mu\in\mathbb{Y}_k} \left| \frac{P_{\lambda(\tau)/\mu'}(1,t,\ldots;0,t)}{P_{\lambda(\tau)}(1,t,\ldots;0,t)} P_{\mu'/\eta'}(\beta(-1);0,t)\right|\right] \\ &= \mathbb{E}\left[\sum_{\mu\in\mathbb{Y}_k} \frac{P_{\lambda(\tau)/\mu'}(1,t,\ldots;0,t)}{P_{\lambda(\tau)}(1,t,\ldots;0,t)} P_{\mu'/\eta'}(\beta(1);0,t)\right] \\ &\leq \mathbb{E}\left[\sum_{\mu\in\mathbb{Y}} \frac{P_{\lambda(\tau)/\mu'}(1,t,\ldots;0,t)}{P_{\lambda(\tau)}(1,t,\ldots;0,t)} P_{\mu'/\eta'}(\beta(1);0,t)\right] \\ &= \mathbb{E}\left[\frac{P_{\lambda(\tau)/\eta'}(\beta(1),1,t,\ldots;0,t)}{P_{\lambda(\tau)}(1,t,\ldots;0,t)}\right] \\ &= \frac{1}{\Pi_{0,t}(1,t,\ldots;\gamma(\tau))} \sum_{\lambda\in\mathbb{Y}} P_{\lambda/\eta'}(\beta(1),1,t,\ldots;0,t) Q_{\lambda/\nu}(\gamma(\tau);0,t) \\ &= \frac{\Pi_{0,t}(\beta(1),1,t,\ldots;\gamma(\tau))}{\Pi_{0,t}(1,t,\ldots;\gamma(\tau))} \sum_{\kappa\in\mathbb{Y}} P_{\nu/\kappa}(\beta(1),1,t,\ldots;0,t) Q_{\eta'/\kappa}(\gamma(\tau);0,t),\end{aligned} \tag{6.3.15}$$

which is finite since the last sum has finitely many nonzero terms. Hence Fubini's theorem applies, yielding

$$\begin{aligned}\mathrm{RHS}(6.3.14) &= P_{\eta'}(1,t,\ldots;0,t)\sum_{\mu\in\mathbb{Y}_k} P_{\mu'/\eta'}(\beta(-1);0,t)\mathbb{E}\left[\frac{P_{\lambda(\tau)/\mu'}(1,t,\ldots;0,t)}{P_{\lambda(\tau)}(1,t,\ldots;0,t)}\right]\\ &= P_{\eta'}(1,t,\ldots;0,t)\sum_{\mu\in\mathbb{Y}_k}\sum_{\kappa\in\mathbb{Y}_k}\frac{P_{\nu/\kappa}(1,t,\ldots;0,t)}{P_{\nu}(1,t,\ldots;0,t)}Q_{\mu'/\kappa}(\gamma(\tau);0,t)P_{\mu'/\eta'}(\beta(-1);0,t)\end{aligned} \tag{6.3.16}$$

(the two sums commute because the sum over κ is actually over $\kappa\subset\nu$ and hence finite).

□

In what follows, the next lemma will often be useful for bounding Macdonald polynomials of complex arguments.

Lemma 6.3.4. *Let $\lambda\in\mathrm{Sig}_n$, $q,t\in(-1,1)$, and $z_1,\ldots,z_n\in\mathbb{C}$ (assume they are nonzero if $\lambda\notin\mathrm{Sig}_n^+$). Then*

$$|P_\lambda(z_1,\ldots,z_n;q,t)|\le P_\lambda(|z_1|,\ldots,|z_n|;q,t). \tag{6.3.17}$$

Proof. Follows by expanding P_λ via the branching rule Definition 8, noting that the coefficient of each monomial is nonnegative since $q,t\in(-1,1)$, and applying the triangle inequality. □

Proof of Lemma 6.3.3. Our starting point is Lemma 6.3.2, which yields

$$\begin{aligned}&\Pr((\lambda_1'(\tau),\ldots,\lambda_k'(\tau))=\eta')\\ &= P_{\eta'}(1,t,\ldots;0,t)\sum_{\mu,\kappa\in\mathbb{Y}_k}\frac{P_{\nu/\kappa}(1,t,\ldots;0,t)}{P_{\nu}(1,t,\ldots;0,t)}Q_{\mu'/\kappa}(\gamma(\tau);0,t)P_{\mu'/\eta'}(\beta(-1);0,t).\end{aligned} \tag{6.3.18}$$

The only place ν appears above is

$$\frac{P_{\nu/\kappa}(1,t,\ldots;0,t)}{P_{\nu}(1,t,\ldots;0,t)}, \tag{6.3.19}$$

which is independent of $\nu_{k+1}',\nu_{k+2}',\ldots$ by (6.3.10) (with ν,κ substituted for κ,μ'). This independence also follows from the fact that the projection of the stochastic process $\lambda(\tau)$ to $(\lambda_1'(\tau),\ldots,\lambda_k'(\tau))$ is Markovian. Hence the right hand side of (6.3.18) is the same upon replacing ν by $\tilde{\nu}$, and to keep notation sanitary we use ν below and simply assume without loss of generality that $\nu'_{k+1} = 0$.

We now reexpress $Q_{\mu'/\kappa}(\gamma(\tau); 0, t)$ using the torus scalar product. The specific case of the skew Cauchy identity with specializations $\gamma(\tau), \beta(z_1, \ldots, z_k)$ is

$$\sum_{\mu \in \mathbb{Y}} Q_\mu(z_1, \ldots, z_k; t, 0) Q_{\mu'/\kappa}(\gamma(\tau); 0, t) = \Pi_{0,t}(\gamma(\tau); \beta(z_1, \ldots, z_k)) Q_{\kappa'}(z_1, \ldots, z_k; t, 0)$$
$$= e^{\frac{\tau}{1-t}(z_1 + \ldots + z_k)} Q_{\kappa'}(z_1, \ldots, z_k; t, 0). \tag{6.3.20}$$

Since the polynomials $P_\lambda(z_1, \ldots, z_k; t, 0)$ are orthogonal with respect to $\langle \cdot, \cdot \rangle'_{t,0;k}$ and the Q_λ are proportional to them by (2.2.1), (6.3.20) together with the defining orthogonality property of Macdonald polynomials yields

$$Q_{\mu'/\kappa}(\gamma(\tau); 0, t) = \frac{\left\langle e^{\frac{\tau}{1-t}(z_1 + \ldots + z_k)} Q_{\kappa'}(z_1, \ldots, z_k; t, 0), P_\mu(z_1, \ldots, z_k; t, 0) \right\rangle'_{t,0;k}}{\langle Q_\mu(z_1, \ldots, z_k; t, 0), P_\mu(z_1, \ldots, z_k; t, 0) \rangle'_{t,0;k}}. \tag{6.3.21}$$

By the definition of the proportionality constants $b_\lambda(q, t)$,

$$\langle Q_\mu(z_1, \ldots, z_k; t, 0), P_\mu(z_1, \ldots, z_k; t, 0) \rangle'_{t,0;k}$$
$$= b_\mu(t, 0) \langle P_\mu(z_1, \ldots, z_k; t, 0), P_\mu(z_1, \ldots, z_k; t, 0) \rangle'_{t,0;k}. \tag{6.3.22}$$

By Lemma 2.2.13,

$$b_\mu(t, 0) = \prod_{i=1}^{k} \frac{1}{(t; t)_{\mu_i - \mu_{i+1}}}. \tag{6.3.23}$$

By substituting $(t, 0)$ for (q, t) in [BC14, (2.8)], we have

$$\langle P_\mu(z_1, \ldots, z_k; t, 0), P_\mu(z_1, \ldots, z_k; t, 0) \rangle'_{t,0;k} = \prod_{i=1}^{k-1} \frac{(t; t)_{\mu_i - \mu_{i+1}}}{(t; t)_\infty}. \tag{6.3.24}$$

Putting these together, the denominator in (6.3.21) is

$$\langle Q_\mu(z_1, \ldots, z_k; t, 0), P_\mu(z_1, \ldots, z_k; t, 0) \rangle'_{t,0;k} = \frac{1}{(t; t)_{\mu_k} (t; t)_\infty^{k-1}}. \tag{6.3.25}$$

Expressing the $Q_{\mu'/\kappa}(\gamma(\tau); 0, t)$ in (6.3.18) via (6.3.21) and substituting the definition of the torus scalar product for the numerator and (6.3.25) for the denominator in (6.3.21), we obtain

$$\begin{aligned}
&\Pr((\lambda_1'(\tau), \ldots, \lambda_k'(\tau)) = \eta) \\
&\qquad = \frac{(t;t)_\infty^{k-1}}{k!(2\pi\mathbf{i})^k} P_{\eta'}(1, t, \ldots; 0, t) \sum_{\mu,\kappa \in \mathbb{Y}_k} \frac{P_{\nu/\kappa}(1, t, \ldots; 0, t)}{P_\nu(1, t, \ldots; 0, t)} (t;t)_{\mu_k} Q_{\mu/\eta}(-1; t, 0) \\
&\qquad\times \int_{\mathbb{T}^k} e^{\frac{\tau}{1-t}(z_1+\ldots+z_k)} Q_{\kappa'}(z_1, \ldots, z_k; t, 0) P_\mu(\bar{z}_1, \ldots, \bar{z}_k; t, 0) \prod_{1 \le i \neq j \le k} (z_i/z_j; t)_\infty \prod_{i=1}^k \frac{dz_i}{z_i}.
\end{aligned} \tag{6.3.26}$$

We wish to commute the sum and integral in (6.3.26), so we must check that Fubini's theorem applies. We first use the fact that $\bar{z} = z^{-1}$ on $\mathbb{T}$ to rewrite the integrand as function analytic away from 0 and ∞ and then expand the contours to $c\mathbb{T}$ to obtain

$$\begin{aligned}
&\int_{\mathbb{T}^k} e^{\frac{\tau}{1-t}(z_1+\ldots+z_k)} P_\mu(\bar{z}_1, \ldots, \bar{z}_k; t, 0) \prod_{1 \le i \neq j \le k} (z_i/z_j; t)_\infty \prod_{i=1}^k \frac{dz_i}{z_i} \\
&\qquad = \int_{c\mathbb{T}^k} e^{\frac{\tau}{1-t}(z_1+\ldots+z_k)} P_\mu(z_1^{-1}, \ldots, z_k^{-1}; t, 0) \prod_{1 \le i \neq j \le k} (z_i/z_j; t)_\infty \prod_{i=1}^k \frac{dz_i}{z_i}
\end{aligned} \tag{6.3.27}$$

where $c > 1$ may be arbitrary. Now

$$\begin{aligned}
&\sum_{\mu,\kappa \in \mathbb{Y}_k} (t;t)_{\mu_k} P_{\nu/\kappa}(1, t, \ldots; 0, t) |Q_{\mu/\eta}(-1; t, 0)| \\
&\times \left| \int_{c\mathbb{T}^k} \prod_{1 \le i \neq j \le k} (z_i/z_j; t)_\infty e^{\frac{\tau}{1-t}(z_1+\ldots+z_k)} Q_{\kappa'}(z_1, \ldots, z_k; t, 0) P_\mu(z_1^{-1}, \ldots, z_k^{-1}; t, 0) \prod_{i=1}^k \frac{dz_i}{z_i} \right| \\
&\le \sum_{\mu,\kappa \in \mathbb{Y}_k} (t;t)_{\mu_k} P_{\nu/\kappa}(1, t, \ldots; 0, t) Q_{\mu/\eta}(1; t, 0) \left((2\pi)^k Q_{\kappa'}(c[k]; t, 0) e^{Tkc} P_\mu(c^{-1}[k]; t, 0) (-1; t)_\infty^{k^2-k} \right) \\
&\le (2\pi)^k e^{Tkc} (-1; t)_\infty^{k^2-k} \left(\sum_{\mu \in \mathbb{Y}} Q_{\mu/\eta}(1; t, 0) P_\mu(c^{-1}[k]; t, 0) \right) \left(\sum_{\kappa \in \mathbb{Y}} P_{\nu/\kappa}(1, t, \ldots; 0, t) P_\kappa(\beta(c[k]); 0, t) \right) \\
&= (2\pi)^k (-1; t)_\infty^{k^2-k} e^{Tkc} \Pi_{t,0}(1; c^{-1}[k]) P_\nu(\beta(c[k]), 1, t, \ldots; 0, t) < \infty.
\end{aligned} \tag{6.3.28}$$

Here we have applied Lemma 6.3.4 and trivial bounds to the integrand, then used the branching rule and Cauchy identity; note it is important that we have expanded the contours, as the sum in the Cauchy identity would diverge if the variables of P_μ were 1 rather than $c^{-1} < 1$. By (6.3.28), Fubini's theorem applies to (6.3.26) (note that we must apply multiple times as we additionally split the sum over μ, κ into two sums below), hence

$$\mathrm{RHS}(6.3.26) = \frac{(t;t)_\infty^{k-1}}{k!(2\pi\mathbf{i})^k} P_{\eta'}(1,t,\ldots;0,t) \int_{c\mathbb{T}^k} \left(\sum_{\kappa \in \mathbb{Y}_k} \frac{P_{\nu/\kappa}(1,t,\ldots;0,t)}{P_\nu(1,t,\ldots;0,t)} Q_{\kappa'}(z_1,\ldots,z_k;t,0) \right)$$
$$\times \left(\sum_{\mu \in \mathbb{Y}_k} (t;t)_{\mu_k} Q_{\mu/\eta}(-1;t,0) P_\mu(z_1^{-1},\ldots,z_k^{-1};t,0) \right) e^{\frac{\tau}{1-t}(z_1+\ldots+z_k)} \prod_{1 \le i \ne j \le k} (z_i/z_j;t)_\infty \prod_{i=1}^k \frac{dz_i}{z_i} \tag{6.3.29}$$

Since $\mathrm{len}(\nu) \le k$, by the branching rule

$$\sum_{\kappa \in \mathbb{Y}_k} \frac{P_{\nu/\kappa}(1,t,\ldots;0,t)}{P_\nu(1,t,\ldots;0,t)} Q_{\kappa'}(z_1,\ldots,z_k;t,0) = \frac{P_\nu(\beta(z_1,\ldots,z_k),1,t,\ldots;0,t)}{P_\nu(1,t,\ldots;0,t)}. \tag{6.3.30}$$

By Proposition 2.2.15 and the definition of $\eta = \eta(\tau)$ in terms of the L_i,

$$P_{\eta'}(1,t,\ldots;0,t) = \frac{t^{\sum_{i=1}^k \binom{\eta_i}{2}}}{(t;t)_{\eta_k} \prod_{i=1}^{k-1} (t;t)_{L_i - L_{i+1}}}. \tag{6.3.31}$$

Let us bring the $(t;t)_{\eta_k}^{-1}$ factor inside the sum in (6.3.29) and evaluate the resulting sum

$$\sum_{\mu \in \mathbb{Y}_k} \frac{(t;t)_{\mu_k}}{(t;t)_{\eta_k}} Q_{\mu/\eta}(-1;t,0) P_\mu(z_1^{-1},\ldots,z_k^{-1};t,0). \tag{6.3.32}$$

By the q-binomial theorem,

$$\frac{(t;t)_{\mu_k}}{(t;t)_{\eta_k}} = (1-t^{\eta_k+1})\cdots(1-t^{\mu_k}) = \sum_{j=0}^{\mu_k-\eta_k} (-t^{\eta_k+1})^j t^{\binom{j}{2}} \begin{bmatrix} \mu_k - \eta_k \\ j \end{bmatrix}_t, \tag{6.3.33}$$

and we note that we can replace the sum by one over all $j \ge 0$ since the q-binomial coefficient will be 0 when $j > \mu_k - \eta_k$. We will use the identity[2]

$$\begin{bmatrix} \mu_k - \eta_k \\ j \end{bmatrix}_t Q_{\mu/\eta}(-1;t,0) = \begin{bmatrix} \eta_{k-1} - \eta_k \\ j \end{bmatrix}_t (-1)^j Q_{\mu/(\eta + j\boldsymbol{e_k})}(-1;t,0) \tag{6.3.34}$$

to simplify (6.3.32), but first we prove (6.3.34). As before, in the case $k = 1$ we interpret

[2]We observed (6.3.34) through explicit examples and are not aware of any broader context for it in symmetric function theory, though this would certainly be interesting if it exists.

(6.3.34) by taking $\eta_{k-1} = \infty$ and

$$\begin{bmatrix} \infty \\ j \end{bmatrix}_t = \frac{1}{(t;t)_j} \tag{6.3.35}$$

to obtain

$$\begin{bmatrix} \mu_1 - \eta_1 \\ j \end{bmatrix}_t Q_{(\mu_1)/(\eta_1)}(-1;t,0) = \frac{(-1)^j}{(t;t)_j} Q_{(\mu_1)/(\eta_1+j)}(-1;t,0), \tag{6.3.36}$$

which follows immediately from the branching rule Lemma 2.2.14, so we will prove the $k \geq 2$ case. By the explicit branching rule Lemma 2.2.14,

$$Q_{\mu/\eta}(-1;t,0) = (-1)^{|\mu/\eta|} \frac{1}{(t;t)_{\mu_1-\eta_1}} \begin{bmatrix} \eta_1 - \eta_2 \\ \eta_1 - \mu_2 \end{bmatrix}_t \cdots \begin{bmatrix} \eta_{k-2} - \eta_{k-1} \\ \eta_{k-2} - \mu_{k-1} \end{bmatrix}_t \begin{bmatrix} \eta_{k-1} - \eta_k \\ \eta_{k-1} - \mu_k \end{bmatrix}_t \tag{6.3.37}$$

while for any j such that $\eta + j\boldsymbol{e_k} = (\eta_1, \ldots, \eta_{k-1}, \eta_k + j) \prec \mu$ we have

$$Q_{\mu/(\eta+j\boldsymbol{e_k})}(-1;t,0) = (-1)^{|\mu/\eta|-j} \frac{1}{(t;t)_{\mu_1-\eta_1}} \begin{bmatrix} \eta_1 - \eta_2 \\ \eta_1 - \mu_2 \end{bmatrix}_t \cdots \begin{bmatrix} \eta_{k-2} - \eta_{k-1} \\ \eta_{k-2} - \mu_{k-1} \end{bmatrix}_t \begin{bmatrix} \eta_{k-1} - \eta_k - j \\ \eta_{k-1} - \mu_k \end{bmatrix}_t . \tag{6.3.38}$$

By writing out the q-factorials on both sides and cancelling a pair it is elementary to check that

$$\begin{bmatrix} \mu_k - \eta_k \\ j \end{bmatrix}_t \begin{bmatrix} \eta_{k-1} - \eta_k \\ \eta_{k-1} - \mu_k \end{bmatrix}_t = \begin{bmatrix} \eta_{k-1} - \eta_k - j \\ \eta_{k-1} - \mu_k \end{bmatrix}_t \begin{bmatrix} \eta_{k-1} - \eta_k \\ j \end{bmatrix}_t . \tag{6.3.39}$$

Now (6.3.34) follows by combining (6.3.37), (6.3.38) and (6.3.39).

By (6.3.34) (or (6.3.36), if $k = 1$) and the Cauchy identity,

$$
\begin{aligned}
(6.3.32) &= \sum_{j=0}^{\infty} (-t^{\eta_k+1})^j t^{\binom{j}{2}} \sum_{\mu \in \mathbb{Y}_k} \begin{bmatrix} \mu_k - \eta_k \\ j \end{bmatrix}_t Q_{\mu/\eta}(-1;t,0) P_\mu(z_1^{-1},\ldots,z_k^{-1};t,0) \\
&= \sum_{j=0}^{\infty} (-t^{\eta_k+1})^j t^{\binom{j}{2}} \begin{bmatrix} \eta_{k-1} - \eta_k \\ j \end{bmatrix}_t (-1)^j \sum_{\mu \in \mathbb{Y}_k} Q_{\mu/(\eta+j\boldsymbol{e_k})}(-1;t,0) P_\mu(z_1^{-1},\ldots,z_k^{-1};t,0) \\
&= \sum_{j=0}^{\eta_{k-1}-\eta_k} t^{j(\eta_k+1)} t^{\binom{j}{2}} \begin{bmatrix} \eta_{k-1} - \eta_k \\ j \end{bmatrix}_t P_{\eta+j\boldsymbol{e_k}}(z_1^{-1},\ldots,z_k^{-1};t,0) \Pi_{t,0}(-1;z_1^{-1},\ldots,z_k^{-1}).
\end{aligned}
\tag{6.3.40}
$$

Substituting (6.3.30), (6.3.31) and (6.3.40) into (6.3.29) and replacing $\Pi_{t,0}$ by its explicit product formula (2.2.30) yields

$$
\begin{aligned}
\mathrm{RHS}(6.3.29) &= \frac{(t;t)_\infty^{k-1}}{k!(2\pi\mathbf{i})^k \prod_{i=1}^{k-1}(t;t)_{\eta_i-\eta_{i+1}}} \int_{c\mathbb{T}^k} e^{\frac{\tau}{1-t}(z_1+\ldots+z_k)} t^{\sum_{i=1}^k \binom{\eta_i}{2}} \prod_{1\le i\ne j\le k} (z_i/z_j;t)_\infty \\
&\times \frac{P_\nu(\beta(z_1,\ldots,z_k),1,t,\ldots;0,t)}{P_\nu(1,t,\ldots;0,t)} \sum_{j=0}^{\eta_{k-1}-\eta_k} t^{j(\eta_k+1)} t^{\binom{j}{2}} \begin{bmatrix} \eta_{k-1} - \eta_k \\ j \end{bmatrix}_t \frac{P_{\eta+j\boldsymbol{e_k}}(z_1^{-1},\ldots,z_k^{-1};t,0)}{\prod_{i=1}^k (-z_i^{-1};t)_\infty} \prod_{i=1}^k \frac{dz_i}{z_i},
\end{aligned}
\tag{6.3.41}
$$

where if $k = 1$ we interpret as in the theorem statement. This completes the proof. $\square$

In the below proof we will assume the same modification as before to the sum over j in the $k = 1$ case without comment.

Proof of Theorem 6.3.1. Write $\eta(\tau) = (L_1 + \log_{t^{-1}}(\tau) + \alpha, \ldots, L_k + \log_{t^{-1}}(\tau) + \alpha)$. Then

$$
\eta_i - \eta_j = L_i - L_j \tag{6.3.42}
$$

for each i, j, so by Lemma 6.3.3

$$\Pr(\lambda_i'(\tau) = L_i + \log_{t^{-1}}(\tau) + \alpha \text{ for all } 1 \le i \le k) = \frac{(t;t)_\infty^{k-1}}{k!(2\pi\mathbf{i})^k \prod_{i=1}^{k-1}(t;t)_{L_i - L_{i+1}}}$$
$$\times \int_{c\mathbb{T}^k} e^{\frac{\tau}{1-t}(z_1+\ldots+z_k)} t^{\sum_{i=1}^k \binom{\eta_i(\tau)}{2}} \prod_{1 \le i \ne j \le k} (z_i/z_j;t)_\infty \frac{P_{\nu(\tau)}(\beta(z_1,\ldots,z_k),1,t,\ldots;0,t)}{P_{\nu(\tau)}(1,t,\ldots;0,t)}$$
$$\times \sum_{j=0}^{L_{k-1}-L_k} t^{j(\eta_k(\tau)+1)} t^{\binom{j}{2}} \begin{bmatrix} L_{k-1} - L_k \\ j \end{bmatrix}_t \frac{P_{\eta(\tau)+j\boldsymbol{e_k}}(z_1^{-1},\ldots,z_k^{-1};t,0)}{\prod_{i=1}^k (-z_i^{-1};t)_\infty} \prod_{i=1}^k \frac{dz_i}{z_i}. \tag{6.3.43}$$

(Technically the above requires that τ is large enough so $\eta(\tau) \in \mathbb{Y}_k$, which is true as long as $L_k + \log_{t^{-1}}(\tau) + \alpha \ge 0$). We wish to take a limit as $\tau \to \infty$ of the above expression, so to remove the τ-dependence inside the exponential we make a change of variables to $w_i = t^{-\eta_k(\tau)} z_i = \tau t^{-L_k - \alpha} z_i$. For later convenience we also set

$$g_\tau(w_1,\ldots,w_k) := \frac{P_{\nu(\tau)}(\beta(t^{\eta_k(\tau)} w_1,\ldots,t^{\eta_k(\tau)} w_k),1,t,\ldots;0,t)}{P_{\nu(\tau)}(1,t,\ldots;0,t)}. \tag{6.3.44}$$

This yields

$$\mathrm{RHS}(6.3.43) = \frac{(t;t)_\infty^{k-1}}{k!(2\pi\mathbf{i})^k \prod_{i=1}^{k-1}(t;t)_{L_i - L_{i+1}}} \int_{ct^{-\eta_k(\tau)}\mathbb{T}^k} e^{\frac{t^{L_k+\alpha}}{1-t}(w_1+\ldots+w_k)} g_\tau(w_1,\ldots,w_k)$$
$$\times \sum_{j=0}^{L_{k-1}-L_k} t^{j(\eta_k(\tau)+1)+\binom{j}{2}} \begin{bmatrix} L_{k-1} - L_k \\ j \end{bmatrix}_t (t^{-\eta_k(\tau)})^{|\eta(\tau)|+j} P_{\eta(\tau)+j\boldsymbol{e_k}}(w_1^{-1},\ldots,w_k^{-1};t,0)$$
$$\times \frac{\prod_{1 \le i \ne j \le k}(w_i/w_j;t)_\infty \prod_{i=1}^k t^{\binom{\eta_i(\tau)}{2}}}{\prod_{i=1}^k (-t^{-\eta_k(\tau)} w_i^{-1};t)_\infty} \prod_{i=1}^k \frac{dw_i}{w_i}, \tag{6.3.45}$$

where we have used that $P_{\eta+j\boldsymbol{e_k}}$ is homogeneous of degree $|\eta|+j$.

By the elementary identity

$$\binom{a+b}{2} = \binom{a}{2} + \binom{b}{2} + ab \tag{6.3.46}$$

and (6.3.42) we have

$$\binom{\eta_i}{2} = \binom{\eta_k}{2} + \binom{L_i - L_k}{2} + (\eta_i - \eta_k)\eta_k. \tag{6.3.47}$$

Additionally, by Lemma 2.2.2 and (6.3.42),

$$P_{\eta+j\boldsymbol{e_k}}(w_1^{-1},\ldots,w_k^{-1};t,0)=(w_1\cdots w_k)^{-\eta_k}P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1},\ldots,w_k^{-1};t,0). \tag{6.3.48}$$

Substituting (6.3.47) for $1\le i\le k$ and (6.3.48) into (6.3.45) and similarly simplifying the t exponent in the κ term yields

$$\begin{aligned}\mathrm{RHS}(6.3.45)&=\frac{(t;t)_\infty^{k-1}}{k!(2\pi\mathbf{i})^k}\prod_{i=1}^{k-1}\frac{t^{\binom{L_i-L_k}{2}}}{(t;t)_{L_i-L_{i+1}}}\int_{ct^{-\eta_k(\tau)}\mathbb{T}^k}e^{\frac{t^{L_k+\alpha}}{1-t}(w_1+\ldots+w_k)}\prod_{1\le i\neq j\le k}(w_i/w_j;t)_\infty\\&\times\prod_{i=1}^k\frac{w_i^{-\eta_k(\tau)}t^{\binom{\eta_k(\tau)}{2}+(\eta_i(\tau)-\eta_k(\tau))\eta_k(\tau)}t^{-\eta_k(\tau)\eta_i(\tau)}}{(-t^{-\eta_k}w_i^{-1};t)_\infty}g_\tau(w_1,\ldots,w_k)\\&\times\sum_{j=0}^{L_{k-1}-L_k}t^{j+\binom{j}{2}}\begin{bmatrix}L_{k-1}-L_k\\ j\end{bmatrix}_t P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1},\ldots,w_k^{-1};t,0)\prod_{i=1}^k\frac{dw_i}{w_i}.\end{aligned} \tag{6.3.49}$$

Noting that

$$\frac{w_i^{-\eta_k(\tau)}t^{\binom{\eta_k(\tau)}{2}+(\eta_i(\tau)-\eta_k(\tau))\eta_k(\tau)}t^{-\eta_k(\tau)\eta_i(\tau)}}{(-t^{-\eta_k(\tau)}w_i^{-1};t)_\infty}=\frac{1}{(-w_i^{-1};t)_\infty(-tw_i;t)_{\eta_k(\tau)}} \tag{6.3.50}$$

and shifting contours yields

$$\begin{aligned}\mathrm{RHS}(6.3.49)&=\frac{(t;t)_\infty^{k-1}}{k!(2\pi\mathbf{i})^k}\prod_{i=1}^{k-1}\frac{t^{\binom{L_i-L_k}{2}}}{(t;t)_{L_i-L_{i+1}}}\int_{\Gamma(\tau)^k}e^{\frac{t^{L_k+\alpha}}{1-t}(w_1+\ldots+w_k)}\frac{\prod_{1\le i\neq j\le k}(w_i/w_j;t)_\infty}{\prod_{i=1}^k(-w_i^{-1};t)_\infty(-tw_i;t)_{\eta_k(\tau)}}\\&\times g_\tau(w_1,\ldots,w_k)\sum_{j=0}^{L_{k-1}-L_k}t^{\binom{j+1}{2}}\begin{bmatrix}L_{k-1}-L_k\\ j\end{bmatrix}_t P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1},\ldots,w_k^{-1};t,0)\prod_{i=1}^k\frac{dw_i}{w_i}\end{aligned} \tag{6.3.51}$$

where

$$\begin{aligned}\Gamma(\tau):=&\{x+\mathbf{i}y:x^2+y^2=1,x>0\}\cup\{x+\mathbf{i}:-t^{-\eta_k(\tau)-1/2}<x\le 0\}\\&\cup\{x-\mathbf{i}:-t^{-\eta_k(\tau)-1/2}<x\le 0\}\cup\{-t^{-\eta_k(\tau)-1/2}+\mathbf{i}y:-1\le y\le 1\},\end{aligned} \tag{6.3.52}$$

see Figure 6-2. For the asymptotics, we will decompose the integration contour into a main term contour $\Gamma_1(\tau)$ and error term contour $\Gamma_2(\tau)$. First define

$$\xi(\tau) := \left(\frac{2+\frac{1}{k}}{\log t^{-1}}\right) \log\log\tau. \tag{6.3.53}$$

Here we have chosen the constant in front of $\log\log\tau$ in (6.3.53) somewhat arbitrarily so that as $\tau \to \infty$ the limits

$$2\left(\frac{k}{\log t^{-1}}\right)^2 (\log\log\tau)^2 - \frac{k^2-k}{2}\xi(\tau)^2 - \text{const}\cdot\xi(\tau) \to \infty \tag{6.3.54}$$

and

$$\xi(\tau) - \left(\frac{2}{\log t^{-1}}\right)\log\log\tau \to \infty \tag{6.3.55}$$

hold, as these are needed to control certain error terms in the proof below.

$$\begin{aligned}
\Gamma(\tau) &= \Gamma_1(\tau) \cup \Gamma_2(\tau) \\
\Gamma_1(\tau) &= \{x+\mathbf{i} : -t^{-\xi(\tau)} < x \le 0\} \cup \{x-\mathbf{i} : -t^{-\xi(\tau)} < x \le 0\} \cup \{x+\mathbf{i}y : x^2+y^2=1, x>0\} \\
\Gamma_2(\tau) &= \{-t^{-\eta_k(\tau)-1/2} + \mathbf{i}y : -1 \le y \le 1\} \cup \{x+\mathbf{i} : -t^{-\eta_k(\tau)-1/2} < x \le -t^{-\xi(\tau)}\} \\
&\quad \cup \{x-\mathbf{i} : -t^{-\eta_k(\tau)-1/2} < x \le -t^{-\xi(\tau)}\}.
\end{aligned} \tag{6.3.56}$$

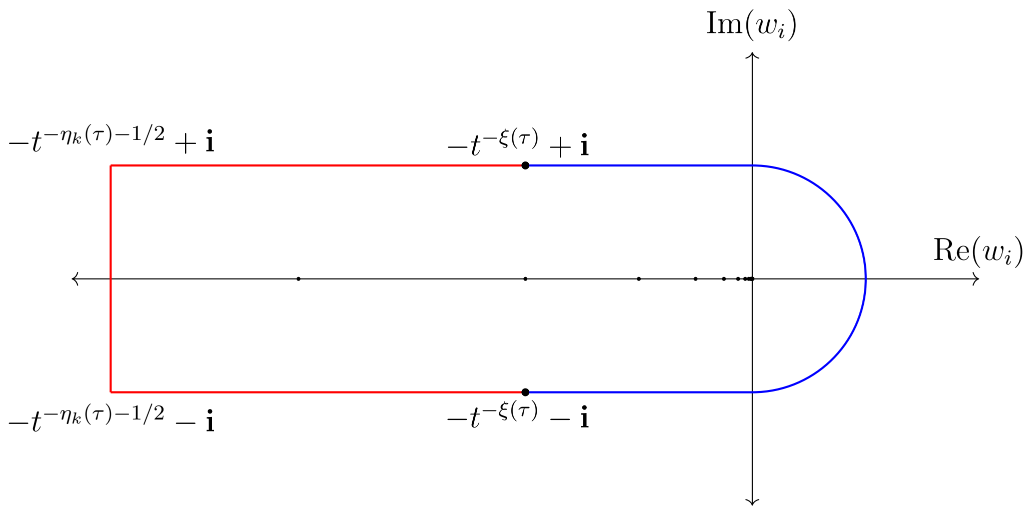


Figure 6-2: The contour $\Gamma(\tau)$ decomposed as in (6.3.56), with $\Gamma_1(\tau)$ in blue and $\Gamma_2(\tau)$ in red, and the poles of the integrand at $w_i = -t^{\mathbb{Z}}$ shown.

We further define another error term contour

$$\Gamma_3(\tau) = \{x + \mathbf{i} : x \le -t^{-\xi(\tau)}\} \cup \{x - \mathbf{i} : x \le -t^{-\xi(\tau)}\}, \tag{6.3.57}$$

so that

$$\Gamma_1(\tau)\cup\Gamma_3(\tau) = \{x+\mathbf{i} : x \le 0\}\cup\{x-\mathbf{i} : x \le 0\}\cup\{x+\mathbf{i}y : x^2+y^2 = 1, x > 0\} = \tilde{\Gamma} \tag{6.3.58}$$

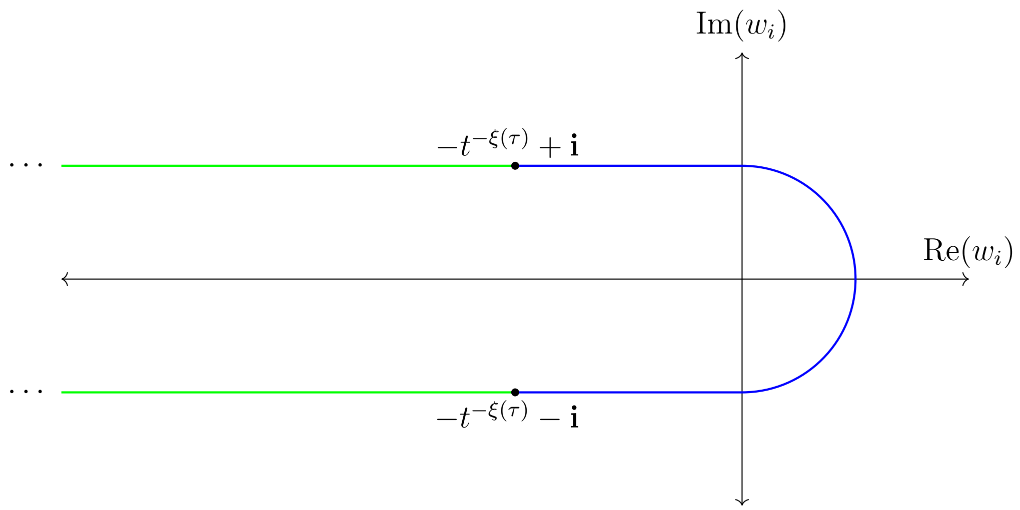


Figure 6-3: The contour $\tilde{\Gamma}$ decomposed as in (6.3.58), with $\Gamma_1(\tau)$ in blue and $\Gamma_3(\tau)$ in green.

is independent of τ. To complete the proof, we must show that

$$\lim_{\substack{\tau\to\infty \\ \log_t(\tau)\equiv\alpha}} \mathrm{RHS}(6.3.51) = \frac{(t;t)_\infty^{k-1}}{k!(2\pi\mathbf{i})^k} \prod_{i=1}^{k-1} \frac{t^{\binom{L_i-L_k}{2}}}{(t;t)_{L_i-L_{i+1}}} \int_{\tilde{\Gamma}^k} e^{\frac{t^{L_k+\alpha}}{1-t}(w_1+\ldots+w_k)} \frac{\prod_{1\le i\ne j\le k}(w_i/w_j;t)_\infty}{\prod_{i=1}^k(-w_i^{-1};t)_\infty(-tw_i;t)_\infty}$$
$$\times \sum_{j=0}^{L_{k-1}-L_k} t^{\binom{j+1}{2}} \begin{bmatrix} L_{k-1}-L_k \\ j \end{bmatrix}_t P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1},\ldots,w_k^{-1};t,0) \prod_{i=1}^k \frac{dw_i}{w_i}. \tag{6.3.59}$$

To compress notation we abbreviate the τ-independent part of the integrand as

$$f(w_1,\ldots,w_k) := \frac{(t;t)_\infty^{k-1}}{k!} \prod_{i=1}^{k-1} \frac{t^{\binom{L_i-L_k}{2}}}{(t;t)_{L_i-L_{i+1}}} e^{\frac{t^{L_k+\alpha}}{1-t}(w_1+\ldots+w_k)} \frac{\prod_{1\le i\ne j\le k}(w_i/w_j;t)_\infty}{\prod_{i=1}^k w_i(-w_i^{-1};t)_\infty}$$
$$\times \sum_{j=0}^{L_{k-1}-L_k} t^{\binom{j+1}{2}} \begin{bmatrix} L_{k-1}-L_k \\ j \end{bmatrix}_t P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1},\ldots,w_k^{-1};t,0). \tag{6.3.60}$$

With this notation, we may rewrite the equality (6.3.59) which we want to show as

$$\lim_{\substack{\tau\to\infty\\ \log_t(\tau)\equiv\alpha}} \frac{1}{(2\pi\mathbf{i})^k}\int_{\Gamma_1(\tau)^k} f(w_1,\ldots,w_k)\left(\frac{g_\tau(w_1,\ldots,w_k)}{\prod_{i=1}^k(-tw_i;t)_{\eta_k(\tau)}} - \frac{1}{\prod_{i=1}^k(-tw_i;t)_\infty}\right)\prod_{i=1}^k dw_i \tag{6.3.61}$$

$$+\lim_{\substack{\tau\to\infty\\ \log_t(\tau)\equiv\alpha}} \frac{1}{(2\pi\mathbf{i})^k}\int_{\Gamma(\tau)^k\setminus\Gamma_1(\tau)^k} \frac{f(w_1,\ldots,w_k)g_\tau(w_1,\ldots,w_k)}{\prod_{i=1}^k(-tw_i;t)_{\eta_k(\tau)}}\prod_{i=1}^k dw_i \tag{6.3.62}$$

$$-\lim_{\substack{\tau\to\infty\\ \log_t(\tau)\equiv\alpha}} \frac{1}{(2\pi\mathbf{i})^k}\int_{\tilde{\Gamma}^k\setminus\Gamma_1(\tau)^k} \frac{f(w_1,\ldots,w_k)}{\prod_{i=1}^k(-tw_i;t)_\infty}\prod_{i=1}^k dw_i = 0. \tag{6.3.63}$$

We will show each of the three lines (6.3.61), (6.3.62) and (6.3.63) above are 0 separately. For this, we first state several needed asymptotics for the functions f and g_τ, the proofs of which we defer to later in the section.

Lemma 6.3.5. *In the notation of the above proof, for any $w_1,\ldots,w_k$ in $\tilde{\Gamma}$ or in $\Gamma(\tau)$ for τ sufficiently large,*

$$|f(w_1,\ldots,w_k)| \le C\prod_{i=1}^k e^{c_1\operatorname{Re}(w_i)+\frac{k-1}{2}(\log t^{-1})\lfloor\log_t|w_i|\rfloor^2+c_2\lfloor\log_t|w_i|\rfloor} \tag{6.3.64}$$

for some positive constants C, c_1, c_2 independent of τ.

Lemma 6.3.6. *In the notation of the above proof,*

$$g_\tau(w_1,\ldots,w_k) = 1 + O(t^{\eta_k(\tau)-\nu_1'(\tau)-\xi(\tau)}) \tag{6.3.65}$$

as $\tau\to\infty$, with implied constant uniform over $w_1,\ldots,w_k\in\Gamma_1(\tau)$.

Outside of $\Gamma_1(\tau)$ we content ourselves with a cruder bound on g_τ:

Lemma 6.3.7. *In the notation of the above proof,*

$$|g_\tau(w_1,\ldots,w_k)| = O(t^{-(k+1)(\nu_1'(\tau))^2}) \tag{6.3.66}$$

as $\tau\to\infty$, with implied constant uniform over $w_1,\ldots,w_k$ in $\Gamma(\tau)$.

To prove (6.3.61) is 0, we claim that

$$\frac{g_\tau(w_1,\ldots,w_k)}{\prod_{i=1}^k(-tw_i;t)_{\eta_k(\tau)}} - \frac{1}{\prod_{i=1}^k(-tw_i;t)_\infty} = \frac{g_\tau(w_1,\ldots,w_k)\prod_{i=1}^k(-t^{\eta_k(\tau)+1}w_i;t)_\infty - 1}{\prod_{i=1}^k(-tw_i;t)_\infty}$$
$$= O(t^{\eta_k(\tau)-\nu_1'(\tau)-\xi(\tau)}) \tag{6.3.67}$$

if $w_i \in \Gamma_1(\tau)$ for all i. For such w_i we have $|t^{\eta_k(\tau)+1}w_i| \leq t^{\eta_k(\tau)-\xi(\tau)}$ (for τ large enough so $|-t^{-\xi(\tau)} \pm i| \leq t^{-\xi(\tau)-1}$). Hence $(-t^{\eta_k(\tau)+1}w_i;t)_\infty = 1 + O(t^{\eta_k(\tau)-\xi(\tau)})$, and combining with Lemma 6.3.6 (which has the bigger error term that dominates) yields that

$$g_\tau(w_1,\ldots,w_k)\prod_{i=1}^k(-t^{\eta_k(\tau)+1}w_i;t)_\infty = 1 + O(t^{\eta_k(\tau)-\nu_1'(\tau)-\xi(\tau)}). \tag{6.3.68}$$

Furthermore, $|(-tw_i;t)_\infty|$ is bounded away from 0 for $w_i \in \tilde{\Gamma}$ and $\Gamma_1(\tau) \subset \tilde{\Gamma}$ for each τ, and combining with (6.3.68) shows (6.3.67).

Since $|w_i| \leq t^{-\xi(\tau)-1}$ holds for all $w_i \in \Gamma_1(\tau)$ (for large τ as above),

$$e^{\frac{k-1}{2}(\log t^{-1})\lfloor \log_t |w_i| \rfloor^2} = O(t^{-\frac{k-1}{2}(\xi(\tau)+1)^2}) \qquad \text{uniformly over } w_i \in \Gamma_1(\tau). \tag{6.3.69}$$

The other terms in the exponent of (6.3.64) are dominated by the $\frac{k-1}{2}(\log t^{-1})\lfloor \log_t |w_i| \rfloor^2$ term (using that $\mathrm{Re}(w_i) \leq 1$ on $\Gamma_1(\tau)$), so

$$|f(w_1,\ldots,w_k)| = O(t^{-k\frac{k-1}{2}(\xi(\tau)+1)^2}) \qquad \text{uniformly over } w_1,\ldots,w_k \in \Gamma_1(\tau). \tag{6.3.70}$$

Multiplying the bounds (6.3.68) and (6.3.70) by the length of the contour $\Gamma_1(\tau)$ which is $O(t^{-\xi(\tau)})$, we find that the first line (6.3.61) is bounded by

$$O(t^{\eta_k(\tau)-\nu_1'(\tau)-\xi(\tau)}) \cdot O(t^{-k\frac{k-1}{2}(\xi(\tau)+1)^2}) \cdot O(t^{-\xi(\tau)}) \tag{6.3.71}$$

as $\tau \to \infty$, and this is $o(1)$ by (6.3.5) and (6.3.54). This shows the vanishing of the first line (6.3.61).

For the other two integrals (6.3.62) and (6.3.63), note that $|(-tw_i;t)_\infty|$ and $|(-tw_i;t)_{\eta_k(\tau)}|$ are both bounded away from 0 uniformly over $w_i \in \tilde{\Gamma} \cup \Gamma(\tau)$; here it is important that the vertical part of $\Gamma(\tau)$ has real part not in $-t^{\mathbb{Z}}$ or else the former product would vanish, which is why we chose it to have real part $-t^{-\eta_k(\tau)-1/2} \in -t^{\mathbb{Z}+1/2}$. To prove that the limits in (6.3.62) and (6.3.63) are 0 it thus suffices to show

$$\lim_{\substack{\tau\to\infty \\ \log_t(\tau)\equiv\alpha}} \int_{\Gamma(\tau)^k\setminus\Gamma_1(\tau)^k} |f(w_1,\ldots,w_k)|\cdot|g_\tau(w_1,\ldots,w_k)|\prod_{i=1}^k dw_i = 0 \tag{6.3.72}$$

and

$$\lim_{\substack{\tau\to\infty \\ \log_t(\tau)\equiv\alpha}} \int_{\tilde{\Gamma}^k\setminus\Gamma_1(\tau)^k} |f(w_1,\ldots,w_k)|\prod_{i=1}^k dw_i = 0. \tag{6.3.73}$$

We first show (6.3.72). Note that

$$\Gamma(\tau)^k\setminus\Gamma_1(\tau)^k = \bigcup_{i=1}^k \Gamma(\tau)^{i-1}\times\Gamma_2(\tau)\times\Gamma(\tau)^{k-i} \tag{6.3.74}$$

(not a disjoint union). Hence by symmetry of the integrand it suffices to show

$$\lim_{\substack{\tau\to\infty \\ \log_t(\tau)\equiv\alpha}} \int_{\Gamma(\tau)} \left(\int_{\Gamma_2(\tau)^{k-1}} |g_\tau(w_1,\ldots,w_k)|\cdot|f(w_1,\ldots,w_k)|\prod_{i=1}^{k-1} dw_i\right) dw_k = 0. \tag{6.3.75}$$

The bound (6.3.64) on $|f|$ factors, so combining Lemma 6.3.5 and Lemma 6.3.7 yields

$$\begin{aligned} \mathrm{LHS}(6.3.75) \leq \lim_{\substack{\tau\to\infty \\ \log_t(\tau)\equiv\alpha}} &\left(\int_{\Gamma_2(\tau)} Ce^{c_1\operatorname{Re}(w)+\frac{k-1}{2}(\log t^{-1})\lfloor\log_t|w|\rfloor^2+c_2\lfloor\log_t|w|\rfloor} dw\right) \\ \times &\left(\int_{\Gamma(\tau)} Ce^{c_1\operatorname{Re}(w)+\frac{k-1}{2}(\log t^{-1})\lfloor\log_t|w|\rfloor^2+c_2\lfloor\log_t|w|\rfloor} dw\right)^{k-1} \times O(t^{-(k+1)\nu_1'(\tau)^2}). \end{aligned} \tag{6.3.76}$$

It is easy to see that

$$\left|\int_{\Gamma(\tau)} Ce^{c_1\operatorname{Re}(w)+\frac{k-1}{2}(\log t^{-1})\lfloor\log_t|w|\rfloor^2+c_2\lfloor\log_t|w|\rfloor} dw\right| < \mathrm{const} \tag{6.3.77}$$

independent of τ, since the $\operatorname{Re}(w)$ term is negative and dominates because $\operatorname{Re}(w)\approx-|w|$ on the contour. Hence by (6.3.76) it suffices to show

$$\lim_{\substack{\tau\to\infty \\ \log_t(\tau)\equiv\alpha}} \int_{\Gamma_2(\tau)} Ce^{c_1\operatorname{Re}(w)+\frac{k-1}{2}(\log t^{-1})\lfloor\log_t|w|\rfloor^2+c_2\lfloor\log_t|w|\rfloor} dw \times O(t^{-(k+1)\nu_1'(\tau)^2}) = 0. \tag{6.3.78}$$

Since

$$\frac{k-1}{2}(\log t^{-1})\lfloor \log_t |w| \rfloor^2 + c_2 \lfloor \log_t |w| \rfloor = o(\mathrm{Re}(w)) \qquad \text{as } w \to \infty \text{ along } \bigcup_\tau \Gamma_2(\tau), \tag{6.3.79}$$

and

$$\inf_{w \in \Gamma_2(\tau)} |w| \to \infty \qquad \text{as } T \to \infty, \tag{6.3.80}$$

there is a constant $0 < c_1' < c_1$ for which

$$Ce^{c_1 \mathrm{Re}(w) + \frac{k-1}{2}(\log t^{-1})\lfloor \log_t |w| \rfloor^2 + c_2 \lfloor \log_t |w| \rfloor} \leq e^{c_1' \mathrm{Re}(w)} \tag{6.3.81}$$

on $\Gamma_2(\tau)$ for all sufficiently large τ. We may therefore bound

$$\left| \int_{\Gamma_2(\tau)} Ce^{c_1 \mathrm{Re}(w) + \frac{k-1}{2}(\log t^{-1})\lfloor \log_t |w| \rfloor^2 + c_2 \lfloor \log_t |w| \rfloor} dw \right| \leq \int_{\Gamma_3(\tau)} e^{c_1' \mathrm{Re}(w)} dw + o(1) \tag{6.3.82}$$

where the $o(1)$ corresponds to the vertical part of $\Gamma_2(\tau)$. Explicitly,

$$\mathrm{RHS}(6.3.82) = 2 \int_{-\infty}^{-t^{-\xi(\tau)}} e^{c_1' x} dx = \frac{2}{c_1'} e^{-t^{-\xi(\tau)}}. \tag{6.3.83}$$

We have thus shown that the expression inside the limit of (6.3.78) is $O(e^{-t^{-\xi(\tau)}}) \times O(t^{-(k+1)\nu_1'(\tau)^2})$. By the naive bound $\nu_1'(\tau) \leq \log_{t^{-1}} \tau$ (for large enough τ, by (6.3.5)), we may rewrite this as

$$O(e^{-t^{-\xi(\tau)}}) \times O(t^{-(k+1)\nu_1'(\tau)^2}) = O(\exp(-e^{(\log t^{-1})\xi(\tau)} + (k-1)(\log t^{-1}) e^{-2 \log \log \tau})), \tag{6.3.84}$$

which is $o(1)$ by (6.3.55). This shows (6.3.72), so the limit (6.3.62) is indeed 0. The case of (6.3.73) is almost exactly the same, except that (a) the analogue of (6.3.76) yields directly to treating a single integral over $\Gamma_3(\tau)$ rather than having to reduce to this as in (6.3.82), and (b) there is no g_τ so the final bound is in fact better requires only that $\xi(\tau) \to \infty$ at any rate rather than the growth rate (6.3.55) to finish. This shows (6.3.59) and completes the proof.

□

Proof of Lemma 6.3.5. If $k \geq 2$ then the sum over j in (6.3.60) is a polynomial in

$w_1^{-1}, \ldots, w_k^{-1}$, hence if $|w_i| \geq 1$ for all i we have

$$\left| \sum_{j=0}^{L_{k-1}-L_k} t^{\binom{j+1}{2}} \begin{bmatrix} L_{k-1} - L_k \\ j \end{bmatrix}_t P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1}, \ldots, w_k^{-1}; t, 0) \right| < \text{const} \tag{6.3.85}$$

independent of τ. If $k = 1$, then the sum is

$$\sum_{j=0}^{\infty} \frac{t^{\binom{j+1}{2}} (w_1^{-1})^j}{(t;t)_j} = (-tw_1^{-1}; t)_\infty \tag{6.3.86}$$

by the q-binomial theorem, and this too is clearly bounded by a constant over all $|w_1| \geq 1$.

For the products in the denominator of (6.3.60),

$$\left| \frac{1}{(-w_i^{-1}; t)_\infty} \right| \leq \left| \frac{1}{1 + w_i^{-1}} \right| \frac{1}{(t;t)_\infty} \tag{6.3.87}$$

since $|w_i|^{-1} \leq 1$, and since out contours do not include -1 the above is bounded by a constant. Similarly, $|1/w_i|$ is clearly bounded above by a constant along our contours. We now treat nonconstant terms. Since $|w_j| \geq 1$,

$$|(w_i/w_j; t)_\infty| \leq (-|w_i|; t)_\infty \leq (-t^{\lfloor \log_t |w_i| \rfloor}; t)_\infty. \tag{6.3.88}$$

By writing

$$(-t^{-b}; t)_\infty = (-1; t)_\infty t^{-b}(t^b + 1) t^{-b+1}(t^{b-1} + 1) \cdots t^{-1}(t + 1) \leq t^{-\binom{b+1}{2}} (-1; t)_\infty^2, \tag{6.3.89}$$

we therefore obtain

$$\left| \prod_{1 \leq i \neq j \leq k} (w_i/w_j; t)_\infty \right| \leq \text{const} \cdot \prod_{i=1}^{k} e^{(-\log t)\left(\frac{k-1}{2} \lfloor \log_t |w_i| \rfloor^2 + \frac{3(k-1)}{2} \lfloor \log_t |w_i| \rfloor\right)}. \tag{6.3.90}$$

Combining (6.3.90) with the previous constant bounds and the trivial bound

$$\left| e^{\frac{t^{L_k+\alpha}}{1-t} w} \right| = e^{c_1 \operatorname{Re}(w)} \tag{6.3.91}$$

yields (6.3.64). $\square$

Proof of Lemma 6.3.6. First note that

$$
\begin{aligned}
g_\tau(w_1, \ldots, w_k) &= \sum_{\kappa \subset \nu(\tau)} \frac{P_{\nu(\tau)/\kappa}(1, t, \ldots; 0, t)}{P_{\nu(\tau)}(1, t, \ldots; 0, t)} t^{\eta_k(\tau)|\kappa|} Q_{\kappa'}(w_1, \ldots, w_k; t, 0) \\
&= \sum_{\kappa \subset \nu(\tau)} Q_{\kappa'}(w_1, \ldots, w_k; t, 0) \prod_{i=1}^{k} t^{\binom{\nu_i'(\tau)-\kappa_i'}{2} - \binom{\nu_i'(\tau)}{2} + \eta_k(\tau)\kappa_i'} (t^{1+\nu_i'(\tau)-\kappa_i'}; t)_{m_i(\kappa)},
\end{aligned}
\tag{6.3.92}
$$

where we have used the branching rule, the fact that $Q_{\kappa'}$ is homogeneous of degree $|\kappa|$, and the explicit formula (6.3.10). We wish to bound

$$
\begin{aligned}
&|g_\tau(w_1, \ldots, w_k) - 1| \\
&= \left| \sum_{\substack{\kappa \subset \nu(\tau) \\ \kappa \neq \emptyset}} Q_{\kappa'}(w_1, \ldots, w_k; t, 0) \prod_{i=1}^{k} t^{\binom{\nu_i'(\tau)-\kappa_i'}{2} - \binom{\nu_i'(\tau)}{2} + \eta_k(\tau)\kappa_i'} (t^{1+\nu_i'(\tau)-\kappa_i'}; t)_{m_i(\kappa)} \right| \\
&\leq \sum_{\substack{\kappa \subset \nu(\tau) \\ \kappa \neq \emptyset}} Q_{\kappa'}(|w_1|, \ldots, |w_k|; t, 0) \prod_{i=1}^{k} t^{\binom{\nu_i'(\tau)-\kappa_i'}{2} - \binom{\nu_i'(\tau)}{2} + \eta_k(\tau)\kappa_i'} (t^{1+\nu_i'(\tau)-\kappa_i'}; t)_{m_i(\kappa)} \\
&\leq \sum_{\kappa \subset \nu(\tau)} Q_{\kappa'}(|w_1|, \ldots, |w_k|; t, 0) \prod_{i=1}^{k} t^{\binom{\nu_i'(\tau)-\kappa_i'}{2} - \binom{\nu_i'(\tau)}{2} + \eta_k(\tau)\kappa_i'} - 1,
\end{aligned}
\tag{6.3.93}
$$

where in the last bound we used that $(t^{1+\nu_i'(\tau)-\kappa_i'}; t)_{m_i(\kappa)} \in [0, 1]$. We rewrite the exponential as

$$
t^{\binom{\nu_i'(\tau)-\kappa_i'}{2} - \binom{\nu_i'(\tau)}{2} + \eta_k(\tau)\kappa_i'} = t^{\kappa_i'(\kappa_i'/2 + 1/2 - \nu_i'(\tau) + \eta_k(\tau))} \leq t^{\kappa_i'(\kappa_i'/2 + 1/2 - \nu_1'(\tau) + \eta_k(\tau))}. \tag{6.3.94}
$$

Then

$$
\prod_{i=1}^{k} t^{\kappa_i'(\kappa_i'/2 + 1/2 - \nu_1'(\tau) + \eta_k(\tau))} = t^{n(\kappa) + (\eta_k(\tau) - \nu_1'(\tau))|\kappa|} = \left(t^{\eta_k(\tau) - \nu_1'(\tau)}\right)^{|\kappa|} Q_\kappa(1, t, \ldots; 0, t). \tag{6.3.95}
$$

Substituting (6.3.94) and (6.3.95) into (6.3.93) yields

$$
\begin{aligned}
|g_\tau(w_1,\ldots,w_k)-1| &\le \sum_{\kappa\subset\nu(\tau)} \left(t^{\eta_k(\tau)-\nu_1'(\tau)}\right)^{|\kappa|} Q_{\kappa'}(|w_1|,\ldots,|w_k|;t,0)Q_\kappa(1,t,\ldots;0,t)-1 \\
&\le \sum_{\kappa\in\mathbb{Y}} \left(t^{\eta_k(\tau)-\nu_1'(\tau)}\right)^{|\kappa|} Q_{\kappa'}(|w_1|,\ldots,|w_k|;t,0)Q_\kappa(1,t,\ldots;0,t)-1 \\
&= \Pi_{0,t}(1,t,\ldots;\beta(t^{\eta_k(\tau)-\nu_1'(\tau)}|w_1|,\ldots,t^{\eta_k(\tau)-\nu_1'(\tau)}|w_k|))-1 \\
&= \prod_{i=1}^k(-t^{\eta_k(\tau)-\nu_1'(\tau)}|w_i|;t)_\infty - 1
\end{aligned}
\tag{6.3.96}
$$

by the Cauchy identity. In our setting, $|w_i|\le t^{-\xi(\tau)-1}$ since $w_i\in\Gamma_1(\tau)$ for all i, so by the q-binomial theorem

$$
\prod_{i=1}^k(-t^{\eta_k(\tau)-\nu_1'(\tau)}|w_i|;t)_\infty - 1 \le \prod_{i=1}^k\left(\sum_{j\ge0}\frac{\left(t^{\eta_k(\tau)-\nu_1'(\tau)}|w_i|\right)^j t^{\binom{j}{2}}}{(t;t)_j}\right)-1 = O(t^{\eta_k(\tau)-\nu_1'(\tau)-\xi(\tau)-1}),
\tag{6.3.97}
$$

completing the proof. $\square$

Proof of Lemma 6.3.7. By the same manipulations as in the first part of the proof of Lemma 6.3.6,

$$
|g_\tau(w_1,\ldots,w_k)| \le \sum_{\kappa\subset\nu(\tau)} Q_{\kappa'}(t^{\eta_k(\tau)}|w_1|,\ldots,t^{\eta_k(\tau)}|w_k|;t,0)\prod_{i=1}^k t^{\kappa_i'(\kappa_i'/2+1/2-\nu_i'(\tau))}. \tag{6.3.98}
$$

We bound this as

$$
\mathrm{RHS}(6.3.98) \le \#\{\kappa\in\mathbb{Y}:\kappa\subset\nu(\tau)\}\times\sup_{\kappa\subset\nu(\tau)} Q_{\kappa'}(t^{\eta_k(\tau)}|w_1|,\ldots,t^{\eta_k(\tau)}|w_k|;t,0)\times\sup_{\kappa\subset\nu(\tau)}\prod_{i=1}^k t^{\kappa_i'(\kappa_i'/2+1/2-\nu_i'(\tau))}
\tag{6.3.99}
$$

Since $\kappa_i'\le\nu_i'(\tau)$ for each $1\le i\le k$ and $\kappa_k'=0$, a naive bound gives

$$
\#\{\kappa\in\mathbb{Y}:\kappa\subset\nu(\tau)\}\le(\nu_1'(\tau)+1)\cdots(\nu_k'(\tau)+1). \tag{6.3.100}
$$

By the branching rule,

$$
\begin{aligned}
&Q_{\kappa'}(t^{\eta_k(\tau)}|w_1|,\dots,t^{\eta_k(\tau)}|w_k|;t,0)\\
&=\sum_{\emptyset=\rho^{(0)}\prec\rho^{(1)}\prec\dots\prec\rho^{(k)}=\kappa'}\prod_{i=1}^{k}\left(t^{\eta_k(\tau)}|w_i|\right)^{|\rho^{(i)}/\rho^{(i-1)}|}\frac{1}{(t;t)_{\rho_1^{(i)}-\rho_1^{(i-1)}}}\prod_{j=1}^{i}\begin{bmatrix}\rho^{(i-1)})_j-\rho^{(i-1)}_{j+1}\\ \rho^{(i-1)})_j-\rho^{(i)}_{j+1}\end{bmatrix}_t\\
&\le \mathrm{const}^{|\kappa|}\cdot\sum_{\emptyset=\rho^{(0)}\prec\rho^{(1)}\prec\dots\prec\rho^{(k)}=\kappa'}(t;t)_\infty^{-2k^2}
\end{aligned}
\tag{6.3.101}
$$

where we have used that $t^{\eta_k(\tau)}|w_i|\le\mathrm{const}$ on $\Gamma(\tau)$, and bounded the branching coefficients by $(t;t)_\infty^{-2k^2}$. Similarly to the proof of Lemma 6.3.6, we bound the number of Gelfand-Tsetlin patterns $\emptyset=\rho^{(0)}\prec\rho^{(1)}\prec\dots\prec\rho^{(k)}=\kappa'$ by

$$\prod_{\ell=1}^{k-1}\#\{(\zeta_1,\dots,\zeta_\ell):0\le\zeta_i\le\kappa_1'\text{ for all }1\le i\le\ell\}=(\kappa_1'+1)^{\binom{k}{2}},\tag{6.3.102}$$

so (6.3.101) becomes

$$Q_{\kappa'}(|w_1|,\dots,|w_k|;t,0)\le\mathrm{const}^{|\kappa|}(t;t)_\infty^{-2k^2}(\kappa_1'+1)^{\binom{k}{2}}.\tag{6.3.103}$$

We now bound the power of t in (6.3.99) as

$$\prod_{i=1}^{k}t^{\kappa_i'(\kappa_i'/2+1/2-\nu_i'(\tau))}\le t^{\frac{k}{8}-\sum_{i=1}^{k}(\nu_i')^2}\le\mathrm{const}\cdot t^{-k(\nu_1')^2}.\tag{6.3.104}$$

Combining (6.3.98), (6.3.103), (6.3.104) and the fact that $\kappa_1'\le\nu_1'$ and so $|\kappa|\le k\nu_1'$, we have

$$|g_\tau(w_1,\dots,w_k)|\le\mathrm{const}'\cdot\mathrm{const}^{k\nu_1'}\cdot(\nu_1'+1)^{\binom{k}{2}}\cdot t^{-k(\nu_1')^2}=O(t^{-(k+1)(\nu_1'(\tau))^2}),\tag{6.3.105}$$

completing the proof. □

Remark 35. We initially proved Theorem 6.3.1 and its auxiliary lemmas in the case of trivial initial condition $\nu(\tau)\equiv\emptyset$ only. It was quite unexpected that the addition of an initial condition merely produces a simple multiplicative factor

$$\frac{P_\nu(\beta(z_1,\ldots,z_k),1,t,\ldots;0,t)}{P_\nu(1,t,\ldots;0,t)}$$

in the integrand of Lemma 6.3.3, which may be treated asymptotically as above.

Remark 36. Our hypothesis $\log_{t^{-1}}\tau-\nu_1'(\tau)\geq 2\left(\frac{k}{\log t^{-1}}\right)^2\log\log\tau$ in Theorem 6.3.1, which we believe is slightly suboptimal as mentioned in Remark 34, comes from the need for existence of a function $\xi(\tau)$ to split the contours as above. The requirement (6.3.55) forces $\xi(\tau)$ to be large, while the requirement (6.3.54) forces $\log_{t^{-1}}\tau-\nu_1'(\tau)$ to be larger than $\xi(\tau)^2$, so improving the bounds in either case could lead to improved versions of the technical hypothesis (6.3.5).

The requirement (6.3.55) in the proof above essentially comes from the need to dominate the error term bounded in Lemma 6.3.7. This error term came from the main term in the proof of Lemma 6.3.7, which comes from bounding

$$\frac{P_{\nu(\tau)/\kappa}(1,t,\ldots;0,t)}{P_{\nu(\tau)}(1,t,\ldots;0,t)}, \tag{6.3.106}$$

and our bound on this term is essentially sharp up to unimportant constants for the case $\kappa_i'\approx\nu_i'/2$, $\nu_k'\approx\nu_1'$. This is why it is not clear to us at the moment how the $\log\log\tau$ growth of $\log_{t^{-1}}\tau-\nu_1'(\tau)$ can be improved beyond improving the constant in front of $\log\log\tau$. It seems likely to us that this can be done by manipulating expressions differently before bounding to take advantage of more cancellations, but the bound we establish suffices for our application in upcoming work so we have not tried hard to do so.

Remark 37. It seems possible that a more involved version of the above manipulations could yield an explicit joint distribution of $\lambda_1'(\tau),\ldots,\lambda_k'(\tau)$ where the initial condition $\nu(\tau)$ have parts $\nu_i'(\tau)$ which grow like $\log_{t^{-1}}(\tau)+c_i$. Such a distribution would in particular be different from the one above, since the fact that $\lambda_i'(\tau)\geq\nu_i'(\tau)$ would make the conjugate parts $\lambda_i'(\tau)$ bounded below in the above regime.

6.4 Residue expansions

The probability distribution in Theorem 6.3.1, which is expressed there by a contour integral, may be residue-expanded to yield formulas for the same limiting probability in terms of certain infinite series in $e^{-t^d}, d \le L_k$, which lead to the series formulas for $\mathcal{S}^{(2\infty)}$ given in the Introduction.

Proposition 6.4.1. *Let $\lambda(\tau)$ be as in Theorem 6.3.1, with or without the initial condition. Then for any $\boldsymbol{L} = (L_1, \ldots, L_k) \in \mathrm{Sig}_k$, we have*

$$\begin{aligned}
\lim_{\substack{\tau\to\infty \\ \log_{t^{-1}}(\tau)+\alpha\in\mathbb{Z}}} & \Pr((\lambda_i'(\tau) - \log_{t^{-1}}(\tau) - \alpha)_{1\le i\le k} = \boldsymbol{L}) \\
&= \frac{1}{(t;t)_\infty} \sum_{d \le L_k} e^{\frac{t^{d+\alpha}}{1-t}} \frac{t^{\sum_{i=1}^k \binom{L_i-d}{2}}}{(t;t)_{L_k-d} \prod_{i=1}^{k-1}(t;t)_{L_i-L_{i+1}}} \\
\times \sum_{\substack{\mu\in\mathrm{Sig}_{k-1} \\ \mu\prec\boldsymbol{L}}} & (-1)^{|\boldsymbol{L}|-|\mu|-d} \prod_{i=1}^{k-1} \begin{bmatrix} L_i - L_{i+1} \\ L_i - \mu_i \end{bmatrix}_t Q_{(\mu-(d[k-1]))'}(\gamma(t^{d+\alpha}), \alpha(1); 0, t). \quad (6.4.1)
\end{aligned}$$

Remark 38. By the branching and principal specialization formulas (Proposition 2.2.15 and Lemma 2.2.14), the formula (6.4.1) may also be written as

$$\frac{1}{(t;t)_\infty} \sum_{d\le L_k} e^{\frac{t^{d+\alpha}}{1-t}} P_{\boldsymbol{L}(d)'}(1,t,\ldots;0,t) \sum_{\substack{\mu\in\mathrm{Sig}_{k-1} \\ \mu\prec\boldsymbol{L}}} P_{\boldsymbol{L}(d)/\mu(d)}(-1;t,0) Q_{\mu(d)'}(\gamma(t^{d+\alpha}), \alpha(1); 0, t) \tag{6.4.2}$$

where $\boldsymbol{L}(d) := (L_1 - d, \ldots, L_k - d), \mu(d) := (\mu_1 - d, \ldots, \mu_k - d)$. The fact that the final answer has such a simple expression in terms of symmetric functions seems in no way justified by the complicated intermediate manipulations we have taken, and it would be very interesting to find a simpler proof of Proposition 6.4.1 which explains this. We remark also that at first glance it might appear that the branching rule (for general specialized Macdonald symmetric functions) would simplify the sum over μ in (6.4.2). The issue is that the sum is over $\mu \in \mathrm{Sig}_{k-1}$, which is a smaller index set, c.f. (6.3.8) and the discussion after for a similar sum in an earlier prelimit expression which appears to be responsible for the above.

The integrand in our previous contour integral formula (6.3.2) has poles at $w_i = -t^x, x \in \mathbb{Z}, 1 \le i \le k$, all of which lie within $\tilde{\Gamma}$. To derive Proposition 6.4.1 we wish to residue expand at these poles to obtain the right hand side of (6.4.1), but because $\tilde{\Gamma}$ is not a closed contour and furthermore the integrand is not meromorphic in a neighborhood of 0, justifying this takes some care. To this end we state the following lemmas. Lemma 6.4.2 is the main algebraic step of computing the residues which arise from shifting contours. Lemma 6.4.3 shows that the contour integral appearing as an error term in Lemma 6.4.2 is indeed negligible, and hence should be thought of as the statement that the integral in (6.3.2) is indeed equal to its naive residue expansion. Recall the function f from (6.3.60).

Lemma 6.4.2. *Fix $k \in \mathbb{Z}_{\geq 1}$, and let $h \in \mathbb{Z}_{\geq 0}$ and Γ be a simple closed contour with interior containing $\{-t^x : x \in \mathbb{Z}, x \geq -h\}$. Then for any $\boldsymbol{L} = (L_1, \ldots, L_k) \in \mathrm{Sig}_k$ and any integer $n \geq L_k$,*

$$\frac{1}{(2\pi \mathbf{i})^k} \int_{\Gamma^k} \frac{f(w_1, \ldots, w_k)}{\prod_{i=1}^k (-tw_i; t)_\infty} \prod_{i=1}^k dw_i = \frac{1}{(t;t)_\infty} \sum_{d=L_k-h}^{L_k} \frac{e^{\frac{t^{d+\alpha}}{1-t}} t^{\sum_{i=1}^k \binom{L_i-d}{2}}}{(t;t)_{L_k-d} \prod_{i=1}^{k-1} (t;t)_{L_i-L_{i+1}}}$$
$$\times \sum_{\substack{\mu \in \mathrm{Sig}_{k-1} \\ \mu \prec \boldsymbol{L}}} (-1)^{|\boldsymbol{L}|-|\mu|-d} \prod_{i=1}^{k-1} \begin{bmatrix} L_i - L_{i+1} \\ L_i - \mu_i \end{bmatrix}_t Q_{(\mu-(d[k-1]))'}(\gamma(t^{d+\alpha}), \alpha(1); 0, t)$$
$$+ \frac{1}{(2\pi \mathbf{i})^k} \int_{(t^{n+1/2}\mathbb{T})^k} \frac{f(w_1, \ldots, w_k)}{\prod_{i=1}^k (-tw_i; t)_\infty} \prod_{i=1}^k dw_i \quad (6.4.3)$$

where $f(w_1, \ldots, w_k)$ is as in (6.3.60)*. In particular, the right hand side is independent of $n \geq L_k$.*

Lemma 6.4.3. *Fix $k \in \mathbb{Z}_{\geq 1}$ and $\boldsymbol{L} = (L_1, \ldots, L_k) \in \mathrm{Sig}_k$. Then for any $n \geq L_k$,*

$$\frac{(t;t)_\infty^{k-1}}{k!(2\pi \mathbf{i})^k} \prod_{i=1}^{k-1} \frac{t^{\binom{L_i-L_k}{2}}}{(t;t)_{L_i-L_{i+1}}} \int_{(t^{n+1/2}\mathbb{T})^k} e^{\frac{t^{L_k+\alpha}}{1-t}(w_1+\ldots+w_k)} \frac{\prod_{1 \leq i \neq j \leq k} (w_i/w_j; t)_\infty}{\prod_{i=1}^k (-w_i^{-1}; t)_\infty (-tw_i; t)_\infty}$$
$$\times \sum_{j=0}^{L_{k-1}-L_k} t^{\binom{j+1}{2}} \begin{bmatrix} L_{k-1} - L_k \\ j \end{bmatrix}_t P_{(L_1-L_k, \ldots, L_{k-1}-L_k, j)}(w_1^{-1}, \ldots, w_k^{-1}; t, 0) \prod_{i=1}^k \frac{dw_i}{w_i} = 0 \quad (6.4.4)$$

with the sum over j interpreted as in Theorem 6.3.1 when $k = 1$.

We also record a certain computation used several times below in the following lemma.

Lemma 6.4.4. *For any $w \in \mathbb{C}^\times$ and $n \in \mathbb{Z}$,*

$$(-t^{-n}w^{-1}; t)_\infty (-t^{n+1}w; t)_\infty = w^{-n} t^{-\binom{n+1}{2}} (-w^{-1}; t)_\infty (-tw; t)_\infty. \quad (6.4.5)$$

Proof. A simple direct computation. □

Remark 39. Since

$$(-w^{-1};t)_\infty(-tw;t)_\infty = \frac{\theta_3(t^{1/2}w;t)}{(t;t)_\infty} \tag{6.4.6}$$

may be written in terms of the Jacobi theta function

$$\theta_3(z;t) := (t;t)_\infty \prod_{n\in\mathbb{Z}_{\geq 0}} (1+t^{n+1/2}z)(1+t^{n+1/2}/z), \tag{6.4.7}$$

the above computation is in fact equivalent to the standard transformation law

$$\theta_3(tz;t) = t^{-1/2}z^{-1}\theta_3(z;t). \tag{6.4.8}$$

It is worth mentioning that Jacobi theta functions have appeared in the related context of periodic Schur processes introduced in [Bor07], used further in e.g. [ARVP22], [BB19], [IMS21b, IMS21a, IMS22], and their above transformation law has been useful there. We are not aware of any closer connection with the present work, however. See e.g. [EMOT81, Chapter 13] for more background on theta functions, though the notation there differs from that of [Bor07] which we use above.

Proof of Lemma 6.4.2. The integrand on the left hand side of (6.4.3) is meromorphic away from 0 and ∞, and for $w_1,\ldots,w_{k-1}$ fixed it has poles at $w_k = -t^m, m \in \mathbb{Z}$. Of these, $-t^h, -t^{h+1}, \ldots$ lie in the interior of Γ. Hence by deforming the w_k contour to $t^{n+1/2}\mathbb{T}$ we obtain

$$\mathrm{LHS}(6.4.3) = \frac{1}{(2\pi\mathbf{i})^{k-1}} \int_{\Gamma^{k-1}} \left(\sum_{m=-h}^{n} \mathrm{Res}_{w_k=-t^m} \frac{f(w_1,\ldots,w_k)}{\prod_{i=1}^k(-tw_i;t)_\infty} \right) \prod_{i=1}^{k-1} dw_i \tag{6.4.9}$$

$$+ \frac{1}{(2\pi\mathbf{i})^{k}} \int_{\Gamma^{k-1}\times t^{n+1/2}\mathbb{T}} \frac{f(w_1,\ldots,w_k)}{\prod_{i=1}^k(-tw_i;t)_\infty} \prod_{i=1}^{k} dw_i. \tag{6.4.10}$$

When $w_k = -t^m$, the factor

$$\prod_{1\leq i\neq j\leq k} (w_i/w_j;t)_\infty$$

in the numerator of f has zeros at all $w_i \in -t^{\mathbb{Z}}, 1 \leq i \leq k-1$. Hence

$$\mathrm{Res}_{w_k=-t^m} \frac{f(w_1,\ldots,w_k)}{\prod_{i=1}^k(-tw_i;t)_\infty}$$

is in fact holomorphic away from 0, so the contours of the $(k-1)$-fold integral in (6.4.9) may be deformed to any simple closed contours containing 0 without any additional residue terms, in particular they may all be deformed to $\mathbb{T}$. For the k-fold integral in (6.4.10) we similarly deform the w_{k-1}-contour to $t^{n+1/2}\mathbb{T}$, yielding a term identical (by symmetry of the variables $w_1, \dots, w_k$) to the $(k-1)$-fold integral of (6.4.9) except that one of the integrals is over $t^{n+1/2}\mathbb{T}$. However, since we may deform to any contours around 0 this makes no difference, and so by pushing each of the k contours in (6.4.3) to $t^{n+1/2}\mathbb{T}$ and using symmetry of the variables to equate the k sums of residues (and commuting the finite sum with the integral) we have

$$\begin{aligned}\mathrm{LHS}(6.4.3) = k\sum_{m=-h}^{n} \frac{1}{(2\pi\mathbf{i})^{k-1}} \int_{\mathbb{T}^{k-1}} \mathrm{Res}_{w_k=-t^m} \frac{f(w_1,\dots,w_k)}{\prod_{i=1}^{k}(-tw_i;t)_\infty} \prod_{i=1}^{k-1} dw_i \\ + \frac{1}{(2\pi\mathbf{i})^{k}} \int_{(t^{n+1/2}\mathbb{T})^k} \frac{f(w_1,\dots,w_k)}{\prod_{i=1}^{k}(-tw_i;t)_\infty} \prod_{i=1}^{k} dw_i.\end{aligned} \tag{6.4.11}$$

Hence to show (6.4.3) we must show

$$\begin{aligned}\frac{1}{(2\pi\mathbf{i})^{k-1}} \int_{\mathbb{T}^{k-1}} \left(\sum_{m=-h}^{n} \mathrm{Res}_{w_k=-t^m} \frac{kf(w_1,\dots,w_k)}{\prod_{i=1}^{k}(-tw_i;t)_\infty} \right) \prod_{i=1}^{k-1} dw_i = \frac{1}{(t;t)_\infty} \sum_{d=L_k-h}^{L_k} \frac{e^{\frac{t^{d+\alpha}}{1-t}} t^{\sum_{i=1}^{k}\binom{L_i-d}{2}}}{\prod_{i=1}^{k-1}(t;t)_{L_i-L_{i+1}}} \\ \times \frac{1}{(t;t)_{L_k-d}} \sum_{\substack{\mu\in\mathrm{Sig}_{k-1}\\ \mu\prec\boldsymbol{L}}} (-1)^{|\boldsymbol{L}|-|\mu|-d} \prod_{i=1}^{k-1} \begin{bmatrix} L_i-L_{i+1} \\ L_i-\mu_i \end{bmatrix}_t Q_{(\mu-(d[k-1]))'}(\gamma(t^{d+\alpha}),\alpha(1);0,t)\end{aligned} \tag{6.4.12}$$

(for $k \geq 2$) and

$$\sum_{m=-h}^{n} \mathrm{Res}_{w_1=-t^m} \frac{f(w_1)}{(-tw_1;t)_\infty} = \frac{1}{(t;t)_\infty} \sum_{d=L_1-h}^{L_1} e^{\frac{t^{d+\alpha}}{1-t}} \frac{t^{\binom{L_1-d}{2}}}{(t;t)_{L_1-d}} (-1)^{L_1-d} \tag{6.4.13}$$

(for $k=1$). We begin with (6.4.13), where

$$\begin{aligned}\frac{f(w_1)}{(-tw_1;t)_\infty} &= e^{\frac{t^{L_1+\alpha}}{1-t}w_1} \frac{1}{w_1(-w_1^{-1};t)_\infty(-tw_1;t)_\infty} \sum_{j=0}^{\infty} \frac{t^{\binom{j+1}{2}}}{(t;t)_j} P_{(j)}(w_1^{-1};t,0) \\ &= e^{\frac{t^{L_1+\alpha}}{1-t}w_1} \frac{1}{w_1(-w_1^{-1};t)_\infty(-tw_1;t)_\infty} (-tw_1^{-1};t)_\infty \\ &= e^{\frac{t^{L_1+\alpha}}{1-t}w_1} \frac{1}{(-w_1;t)_\infty}\end{aligned} \tag{6.4.14}$$

by the q-binomial theorem, since $P_{(j)}(w_1^{-1};t,0) = w_1^{-j}$ by the branching rule. This function has simple poles at $w_1 = -t^{-m}, m \in \mathbb{Z}_{\geq 0}$, of which $-t^{-h}, -t^{-h+1}, \ldots, -1$ are contained in our contour. The pole at $w_1 = -t^{-m}$ contributes

$$e^{\frac{-t^{L_1+\alpha+m}}{1-t}} \frac{1}{(1-t^{-m})\cdots(1-t^{-1})(t^m)(t;t)_\infty} = \frac{1}{(t;t)_\infty} e^{\frac{-t^{L_1+\alpha+m}}{1-t}} \frac{t^{\binom{m}{2}}}{(t;t)_m} \tag{6.4.15}$$

Summing (6.4.15) over $0 \leq m \leq h$ and making the change of variables $d = L_1 + m$ yields (6.4.13) and completes the $k = 1$ case.

We now show the $k \geq 2$ case, (6.4.12). It is not hard to check similarly to above (one may use Lemma 6.4.4 to simplify the computation, though this is not necessary) that

$$\operatorname{Res}_{w_k=-t^m} \frac{1}{w_k(-w_k^{-1};t)_\infty(-tw_k;t)_\infty} = \frac{(-1)^m}{(t;t)_\infty^2 t^{-\binom{m+1}{2}}}, \tag{6.4.16}$$

where we let $\binom{m+1}{2} = (m^2+m)/2$ even when m is negative. Lemma 6.4.4 also implies that

$$\frac{(t^{-m}w_i;t)_\infty(t^m w_i^{-1};t)_\infty}{w_i(-w_i^{-1};t)_\infty(-tw_i;t)_\infty} = (1+t^{-m}w_i)w_i^m t^{-\binom{m+1}{2}}. \tag{6.4.17}$$

Using (6.4.16), (6.4.17) and the explicit formula (6.3.60) for f we compute

$$\begin{aligned}
&\operatorname{Res}_{w_k=-t^m} \frac{kf(w_1,\ldots,w_k)}{\prod_{i=1}^k(-tw_i;t)_\infty} = \frac{(t;t)^{k-1}}{(k-1)!} \prod_{i=1}^{k-1} \frac{t^{\binom{L_i-L_k}{2}}}{(t;t)_{L_i-L_{i+1}}} e^{\frac{t^{L_k+\alpha}}{1-t}(w_1+\ldots+w_{k-1})} \\
&\times \prod_{1\leq i\neq j\leq k-1} (w_i/w_j;t)_\infty \left(\sum_{j=0}^{L_{k-1}-L_k} t^{\binom{j+1}{2}} \begin{bmatrix} L_{k-1}-L_k \\ j \end{bmatrix}_t P_{\tilde{L}+j\mathbf{e}_k}(w_1^{-1},\ldots,w_{k-1}^{-1},-t^{-m};t,0) \right) \\
&\times \frac{\prod_{i=1}^{k-1}(1+t^m w_i^{-1})w_i^m t^{-\binom{m+1}{2}}}{(-1)^m t^{-\binom{m+1}{2}}(t;t)_\infty^2} e^{\frac{t^{L_k+\alpha+m}}{1-t}},
\end{aligned} \tag{6.4.18}$$

where $\tilde{L} = (L_1 - L_k, \dots, L_{k-1} - L_k, 0)$ as before. By the branching rule,

$$\begin{aligned}
&\sum_{j=0}^{L_{k-1}-L_k} t^{\binom{j+1}{2}} \begin{bmatrix} L_{k-1} - L_k \\ j \end{bmatrix}_t P_{\tilde{L}+j\boldsymbol{e}_k}(w_1^{-1}, \dots, w_{k-1}^{-1}, -t^{-m}; t, 0) \\
&= \sum_{j=0}^{L_{k-1}-L_k} t^{\binom{j+1}{2}} \begin{bmatrix} L_{k-1} - L_k \\ j \end{bmatrix}_t \sum_{\substack{\mu \in \mathrm{Sig}_{k-1} \\ \mu \prec \tilde{L}+j\boldsymbol{e}_k}} (-t^{-m})^{|\tilde{L}|+j-|\mu|} \begin{bmatrix} L_1 - L_2 \\ L_1 - L_k - \mu_1 \end{bmatrix}_t \cdots \begin{bmatrix} L_{k-2} - L_{k-1} \\ L_{k-2} - L_k - \mu_{k-2} \end{bmatrix}_t \\
&\times \begin{bmatrix} L_{k-1} - L_k - j \\ L_{k-1} - L_k - \mu_{k-1} \end{bmatrix}_t P_\mu(w_1^{-1}, \dots, w_{k-1}^{-1}; t, 0) \\
&= \sum_{\substack{\mu \in \mathrm{Sig}_{k-1} \\ \mu \prec \tilde{L}}} (-t^{-m})^{|\tilde{L}|-|\mu|} \begin{bmatrix} L_1 - L_2 \\ L_1 - L_k - \mu_1 \end{bmatrix}_t \cdots \begin{bmatrix} L_{k-2} - L_{k-1} \\ L_{k-2} - L_k - \mu_{k-2} \end{bmatrix}_t P_\mu(w_1^{-1}, \dots, w_{k-1}^{-1}; t, 0) \\
&\times \sum_{j=\mu_{k-1}}^{L_{k-1}-L_k} \begin{bmatrix} L_{k-1} - L_k - j \\ L_{k-1} - L_k - \mu_{k-1} \end{bmatrix}_t \begin{bmatrix} L_{k-1} - L_k \\ j \end{bmatrix}_t t^{\binom{j+1}{2}} (-t^{-m})^j \\
&= \sum_{\substack{\mu \in \mathrm{Sig}_{k-1} \\ \mu \prec \tilde{L}}} (-t^{-m})^{|\tilde{L}|-|\mu|} \begin{bmatrix} L_1 - L_2 \\ L_1 - L_k - \mu_1 \end{bmatrix}_t \cdots \begin{bmatrix} L_{k-2} - L_{k-1} \\ L_{k-2} - L_k - \mu_{k-2} \end{bmatrix}_t P_\mu(w_1^{-1}, \dots, w_{k-1}^{-1}; t, 0) \\
&\times \begin{bmatrix} L_{k-1} - L_k \\ L_{k-1} - L_k - \mu_{k-1} \end{bmatrix} (t^{-m+1}; t)_{\mu_{k-1}},
\end{aligned} \tag{6.4.19}$$

where in the last equality we used the identity

$$\begin{bmatrix} L_{k-1} - L_k - j \\ L_{k-1} - L_k - \mu_{k-1} \end{bmatrix}_t \begin{bmatrix} L_{k-1} - L_k \\ j \end{bmatrix}_t = \begin{bmatrix} \mu_{k-1} \\ j \end{bmatrix}_t \begin{bmatrix} L_{k-1} - L_k \\ L_{k-1} - L_k - \mu_{k-1} \end{bmatrix}_t \tag{6.4.20}$$

(which is the same as (6.3.39)), and then applied the q-binomial theorem. By (6.4.18), (6.4.19), and the fact that $w_i^{-1} = \bar{w}_i$ on $\mathbb{T}$,

$$\frac{1}{(2\pi\mathbf{i})^{k-1}} \int_{\mathbb{T}^{k-1}} \mathrm{Res}_{w_k=-t^m} \frac{kf(w_1,\ldots,w_k)}{\prod_{i=1}^k (-tw_i;t)_\infty} \prod_{i=1}^{k-1} dw_i = (t;t)_\infty^{k-3} \prod_{i=1}^{k-1} \frac{t^{\binom{L_i-L_k}{2}}}{(t;t)_{L_i-L_{i+1}}} (-1)^m t^{\binom{m+1}{2}} e^{\frac{t^{L_k+\alpha+m}}{1-t}}$$
$$\times \sum_{\substack{\mu\in\mathrm{Sig}_{k-1}\\ \mu\prec\tilde{L}}} (-t^{-m})^{|\tilde{L}|-|\mu|} (t^{-m+1};t)_{\mu_{k-1}} \prod_{i=1}^{k-1} \begin{bmatrix} L_i - L_{i+1} \\ L_i - L_k - \mu_i \end{bmatrix} \frac{1}{(k-1)!(2\pi\mathbf{i})^{k-1}} \int_{\mathbb{T}^{k-1}} e^{\frac{t^{L_k+\alpha}}{1-t}(w_1+\ldots+w_{k-1})}$$
$$\times \prod_{1\le i\ne j\le k-1} (w_i/w_j;t)_\infty P_\mu(\bar{w}_1,\ldots,\bar{w}_{k-1};t,0) \prod_{i=1}^{k-1} (1+t^{-m}w_i) w_i^{m-1} t^{-\binom{m}{2}} dw_i. \tag{6.4.21}$$

We recognize a factor in the above integrand as

$$\begin{aligned}\prod_{i=1}^{k-1} e^{\frac{t^{L_k+\alpha}}{1-t}w_i}(1+t^{-m}w_i) &= \Pi_{0,t}(\gamma(t^{L_k+\alpha}),\alpha(t^{-m});\beta(w_1,\ldots,w_{k-1})) \\ &= \sum_{\lambda\in\mathbb{Y}} Q_\lambda(\gamma(t^{L_k+\alpha}),\alpha(t^{-m});0,t) Q_{\lambda'}(w_1,\ldots,w_{k-1};t,0),\end{aligned} \tag{6.4.22}$$

by (2.2.40). Note also that since $w_i \in \mathbb{T}$, by Lemma 2.2.2

$$(w_1^m \cdots w_{k-1}^m) P_\mu(\bar{w}_1,\ldots,\bar{w}_{k-1};t,0) = P_{\mu-(m[k-1])}(\bar{w}_1,\ldots,\bar{w}_{k-1};t,0). \tag{6.4.23}$$

If $\mu_{k-1} - m < 0$, then the above is not a polynomial but a Laurent polynomial, and by orthogonality of the q-Whittaker Laurent polynomials (Proposition 2.2.5) the integral in (6.4.21) is 0. Otherwise, if $\mu_{k-1} - m \ge 0$, orthogonality still implies that only the term $\lambda' = \mu - (m[k-1])$ of the sum (6.4.22) contributes to the integral in (6.4.21). Hence we obtain

$$\frac{t^{-(k-1)\binom{m}{2}}}{(k-1)!(2\pi\mathbf{i})^{k-1}} \int_{\mathbb{T}^{k-1}} \Pi_{0,t}(\gamma(t^{L_k+\alpha}),\alpha(t^{-m});\beta(w_1,\ldots,w_{k-1})) P_{\mu-(m[k-1])}(\bar{w}_1,\ldots,\bar{w}_{k-1};t,0)$$
$$\times \prod_{1\le i\ne j\le k-1} (w_i/w_j;t)_\infty \prod_{i=1}^{k-1} \frac{dw_i}{w_i} = t^{-(k-1)\binom{m}{2}} Q_{(\mu-(m[k-1]))'}(\gamma(t^{L_k+\alpha}),\alpha(t^{-m});0,t)$$
$$\times \mathbb{1}(\mu_{k-1}-m \ge 0) \left\langle Q_{\mu-(m[k-1])}(w_1,\ldots,w_{k-1};t,0), P_{\mu-(m[k-1])}(w_1,\ldots,w_{k-1};t,0)\right\rangle'_{t,0;k-1}. \tag{6.4.24}$$

By (6.3.25),

$$\left\langle Q_{\mu-(m[k-1])}(w_1,\ldots,w_{k-1};t,0), P_{\mu-(m[k-1])}(w_1,\ldots,w_{k-1};t,0)\right\rangle'_{t,0;k-1} = \frac{1}{(t;t)_{\mu_{k-1}-m}(t;t)_\infty^{k-2}}. \tag{6.4.25}$$

Combining (6.4.21), (6.4.24) and (6.4.25) and writing

$$\mathbb{1}(\mu_{k-1}-m\geq 0)\frac{(t^{-m+1};t)_{\mu_{k-1}}}{(t;t)_{\mu_{k-1}-m}} = \mathbb{1}(m\leq 0)\frac{1}{(t;t)_{-m}} \tag{6.4.26}$$

yields

$$\begin{aligned}
&\frac{1}{(2\pi\mathbf{i})^{k-1}}\int_{\mathbb{T}^{k-1}}\operatorname{Res}_{w_k=-t^m}\frac{kf(w_1,\ldots,w_k)}{\prod_{i=1}^k(-tw_i;t)_\infty}\prod_{i=1}^{k-1}dw_i = \frac{\mathbb{1}(m\leq 0)}{(t;t)_\infty}\left(\prod_{i=1}^{k-1}\frac{t^{\binom{L_i-L_k}{2}}}{(t;t)_{L_i-L_{i+1}}}\right)\\
&\times\frac{(-1)^m t^{-(k-2)\binom{m}{2}+m}}{(t;t)_{-m}}e^{\frac{t^{L_k+\alpha+m}}{1-t}}\sum_{\substack{\mu\in\mathrm{Sig}_{k-1}\\ \mu\prec\tilde{L}}}(-t^{-m})^{|\tilde{L}|-|\mu|}\prod_{i=1}^{k-1}\begin{bmatrix}L_i-L_{i+1}\\ L_i-L_k-\mu_i\end{bmatrix}_t\\
&\times Q_{(\mu-(m[k-1]))'}(\gamma(t^{L_k+\alpha}),\alpha(t^{-m});0,t).
\end{aligned} \tag{6.4.27}$$

Set $\boldsymbol{L}=(L_1,\ldots,L_k)=\tilde{L}+(L_k[k])$ and relabel $\mu\mapsto\mu+(L_k[k-1])$ so that

$$\begin{aligned}
&\sum_{\substack{\mu\in\mathrm{Sig}_{k-1}\\ \mu\prec\tilde{L}}}(-t^{-m})^{|\tilde{L}|-|\mu|}\prod_{i=1}^{k-1}\begin{bmatrix}L_i-L_{i+1}\\ L_i-L_k-\mu_i\end{bmatrix}_t Q_{(\mu-(m[k-1]))'}(\gamma(t^{L_k+\alpha}),\alpha(t^{-m});0,t) =\\
&\sum_{\substack{\mu\in\mathrm{Sig}_{k-1}\\ \mu\prec\boldsymbol{L}}}(-t^{-m})^{|\boldsymbol{L}|-|\mu|-L_k}\prod_{i=1}^{k-1}\begin{bmatrix}L_i-L_{i+1}\\ L_i-\mu_i\end{bmatrix}_t t^{-m(|\mu|-(k-1)(L_k+m))}Q_{\mu-((L_k+m)[k-1])'}(\gamma(t^{L_k+\alpha+m}),\alpha(1);0,t),
\end{aligned} \tag{6.4.28}$$

where we have also used homogeneity of Q to scale the specializations. Setting $d=L_k+m$, the power of t in (6.4.27) (together with the one in (6.4.28)) is

$$-(k-2)\binom{m}{2}+m-m(|\boldsymbol{L}|-|\mu|-L_k)-m(|\mu|-(k-1)(L_k+m)) = \sum_{i=1}^k\binom{L_i-d}{2} \tag{6.4.29}$$

by a tedious computation using (6.3.46). The sign in (6.4.27) is $(-1)^{|\boldsymbol{L}|-|\mu|-d}$. Putting this together we have

$$\frac{1}{(2\pi\mathbf{i})^{k-1}}\int_{\mathbb{T}^{k-1}}\mathrm{Res}_{w_k=-t^m}\frac{kf(w_1,\ldots,w_k)}{\prod_{i=1}^k(-tw_i;t)_\infty}\prod_{i=1}^{k-1}dw_i = e^{\frac{t^{d+\alpha}}{1-t}}\frac{t^{\sum_{i=1}^k\binom{L_i-d}{2}}}{(t;t)_\infty(t;t)_{L_k-d}\prod_{i=1}^{k-1}(t;t)_{L_i-L_{i+1}}}$$
$$\times\sum_{\substack{\mu\in\mathrm{Sig}_{k-1}\\ \mu\prec\boldsymbol{L}}}(-1)^{|\boldsymbol{L}|-|\mu|-d}\prod_{i=1}^{k-1}\begin{bmatrix}L_i-L_{i+1}\\ L_i-\mu_i\end{bmatrix}_t Q_{(\mu-(d[k-1]))'}(\gamma(t^{d+\alpha}),\alpha(1);0,t). \tag{6.4.30}$$

This shows (6.4.12) and hence completes the proof. $\square$

Proof of Lemma 6.4.3. By Lemma 6.4.2, the integral which we wish to compute is independent of $n\geq L_k$. Hence it suffices to show

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{Z}}}\frac{(t;t)_\infty^{k-1}}{k!(2\pi\mathbf{i})^k}\prod_{i=1}^{k-1}\frac{t^{\binom{L_i-L_k}{2}}}{(t;t)_{L_i-L_{i+1}}}\int_{(t^{n+1/2}\mathbb{T})^k}e^{\frac{t^{L_k+\alpha}}{1-t}(w_1+\ldots+w_k)}\frac{\prod_{1\leq i\neq j\leq k}(w_i/w_j;t)_\infty}{\prod_{i=1}^k(-w_i^{-1};t)_\infty(-tw_i;t)_\infty}$$
$$\times\sum_{j=0}^{L_{k-1}-L_k}t^{\binom{j+1}{2}}\begin{bmatrix}L_{k-1}-L_k\\ j\end{bmatrix}_t P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1},\ldots,w_k^{-1};t,0)\prod_{i=1}^k\frac{dw_i}{w_i}=0. \tag{6.4.31}$$

To simplify expressions we will show (6.4.31) by showing

$$\lim_{\substack{n\to\infty\\ n\in\mathbb{Z}}}\int_{(t^{n+1/2}\mathbb{T})^k}e^{\frac{t^{L_k+\alpha}}{1-t}(w_1+\ldots+w_k)}\frac{\prod_{1\leq i\neq j\leq k}(w_i/w_j;t)_\infty}{\prod_{i=1}^k(-w_i^{-1};t)_\infty(-tw_i;t)_\infty}$$
$$\times P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1},\ldots,w_k^{-1};t,0)\prod_{i=1}^k\frac{dw_i}{w_i}=0 \tag{6.4.32}$$

for each j. Letting $w_i=t^nu_i$ we have

$$\mathrm{LHS}(6.4.32)=\lim_{\substack{n\to\infty\\ n\in\mathbb{Z}}}\int_{(t^{1/2}\mathbb{T})^k}e^{\frac{t^{L_k+\alpha+n}}{1-t}(u_1+\ldots+u_k)}\frac{\prod_{1\leq i\neq j\leq k}(u_i/u_j;t)_\infty}{\prod_{i=1}^k(-t^{-n}u_i^{-1};t)_\infty(-t^{n+1}u_i;t)_\infty}$$
$$\times t^{-n(|\boldsymbol{L}|+j-kL_k)}P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(u_1^{-1},\ldots,u_k^{-1};t,0)\prod_{i=1}^k\frac{du_i}{u_i} \tag{6.4.33}$$

where we have used homogeneity of P. By Lemma 6.4.4,

$$(-t^{-n}u_i^{-1};t)_\infty(-t^{n+1}u_i;t)_\infty = u_i^{-n}t^{-\binom{n+1}{2}}(-u_i^{-1};t)_\infty(-tu_i;t)_\infty, \tag{6.4.34}$$

so

$$\mathrm{RHS}(6.4.33) = \lim_{\substack{n\to\infty \\ n\in\mathbb{Z}}} \int_{(t^{1/2}\mathbb{T})^k} \prod_{i=1}^k \left(u_i^n t^{\binom{n+1}{2}-n(|\boldsymbol{L}|+j-kL_k)}\right) e^{\frac{t^{L_k+\alpha+n}}{1-t}(u_1+\ldots+u_k)} \tag{6.4.35}$$

$$\times \frac{\prod_{1\le i\neq j\le k}(u_i/u_j;t)_\infty}{\prod_{i=1}^k(-u_i^{-1};t)_\infty(-tu_i;t)_\infty} P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(u_1^{-1},\ldots,u_k^{-1};t,0)\prod_{i=1}^k \frac{du_i}{u_i}. \tag{6.4.36}$$

The part of the integrand on the second line (6.4.36) is bounded on $\mathbb{T}^k$ and is independent of n, while the part of the integrand on the first line (6.4.35) goes to 0 in n uniformly over $\mathbb{T}^k$, showing (6.4.32) and hence completing the proof. □

Proof of Proposition 6.4.1. The idea is to take a family of simple closed contours which approach the contour $\tilde{\Gamma}$ defined in Theorem 6.3.1 and also encircle more and more of the poles $-t^x$. The contours $\Gamma(\tau)$ defined in (6.3.56) work, though we write them as $\Gamma(B), B \in \mathbb{Z}$ to avoid a clash of notation with the τ in Proposition 6.4.1. By Lemma 6.4.2 together with Lemma 6.4.3,

$$\frac{1}{(2\pi\mathbf{i})^k}\int_{\Gamma(B)^k} \frac{f(w_1,\ldots,w_k)}{\prod_{i=1}^k(-tw_i;t)_\infty}\prod_{i=1}^k dw_i = \sum_{d=L_k-\eta_k(B)}^{L_k} e^{\frac{t^{d+\alpha}}{1-t}} \frac{t^{\sum_{i=1}^k \binom{L_i-d}{2}}}{(t;t)_\infty(t;t)_{L_k-d}\prod_{i=1}^{k-1}(t;t)_{L_i-L_{i+1}}}$$

$$\times \sum_{\substack{\mu\in\mathrm{Sig}_{k-1} \\ \mu\prec\boldsymbol{L}}} (-1)^{|\boldsymbol{L}|-|\mu|-d}\prod_{i=1}^{k-1}\begin{bmatrix} L_i - L_{i+1} \\ L_i-\mu_i\end{bmatrix}_t Q_{(\mu-(d[k-1]))'}(\gamma(t^{d+\alpha}),\alpha(1);0,t)+0. \tag{6.4.37}$$

It follows immediately that

$$\lim_{\substack{B\to\infty \\ B\in\mathbb{Z}}} \frac{1}{(2\pi\mathbf{i})^k}\int_{\Gamma(B)^k}\frac{f(w_1,\ldots,w_k)}{\prod_{i=1}^k(-tw_i;t)_\infty}\prod_{i=1}^k dw_i = \mathrm{RHS}(6.4.1). \tag{6.4.38}$$

The limit

$$\lim_{\substack{B\to\infty\\ B\in\mathbb{Z}}} \frac{1}{(2\pi\mathbf{i})^k}\int_{\Gamma(B)^k} \frac{f(w_1,\ldots,w_k)}{\prod_{i=1}^k(-tw_i;t)_\infty}\prod_{i=1}^k dw_i$$
$$= \frac{(t;t)_\infty^{k-1}}{k!(2\pi\mathbf{i})^k}\prod_{i=1}^{k-1}\frac{t^{\binom{L_i-L_k}{2}}}{(t;t)_{L_i-L_{i+1}}}\int_{\tilde{\Gamma}^k} e^{\frac{t^{L_k+\alpha}}{1-t}(w_1+\ldots+w_k)}\frac{\prod_{1\le i\neq j\le k}(w_i/w_j;t)_\infty}{\prod_{i=1}^k(-w_i^{-1};t)_\infty(-tw_i;t)_\infty}$$
$$\times\sum_{j=0}^{L_{k-1}-L_k} t^{\binom{j+1}{2}}\begin{bmatrix}L_{k-1}-L_k\\ j\end{bmatrix}_t P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1},\ldots,w_k^{-1};t,0)\prod_{i=1}^k\frac{dw_i}{w_i} \tag{6.4.39}$$

follows by the estimate Lemma 6.3.5 exactly as with (6.3.72) in the proof of Theorem 6.3.1. Combining (6.4.38) with (6.4.39) completes the proof. □

6.5 Tightness and the limiting random variable

In the Introduction, we stated that the limiting formulas on the right hand side of (6.3.2) and (6.4.1) define a Sig_k-valued random variable, but Theorem 6.3.1 and Proposition 6.4.1 do not *a priori* imply this because mass may escape to $\pm\infty$. In this section we show that there is no escape of mass and the formulas indeed define a random variable.

Proposition 6.5.1. *In the notation of Theorem 6.3.1, the sequence of* Sig_k*-valued random variables*

$$(\lambda_i'(\tau)-\log_{t^{-1}}(\tau)-\alpha)_{1\le i\le k}, \tau\in t^{-\mathbb{N}+\alpha} \tag{6.5.1}$$

is tight.

Proof. We must show that for every $\epsilon>0$, there exists $D=D(\epsilon)$ such that

$$\Pr(-D\le\lambda_k'(\tau)-\log_{t^{-1}}(\tau)-\alpha\le\lambda_1'(\tau)-\log_{t^{-1}}(\tau)-\alpha\le D)>1-\epsilon \tag{6.5.2}$$

for all $\tau\in t^{-\mathbb{N}+\alpha}$. For the upper bound in (6.5.2), first note that λ_1' is Markov, and if $\lambda_1'(\tau)=x$ at some τ then the waiting time before λ_1' jumps to $x+1$ follows an exponential distribution with rate $t^x/(1-t)$. This is because λ_1' will increase as soon as one of the clocks $x+1,x+2,\ldots$ rings, and these have rates $t^x,t^{x+1},\ldots$. Hence for $D\in\mathbb{N}$ we have

$$\Pr(\lambda_1'(\tau)>D)=\Pr(\sum_{i=0}^D E_i\le\tau) \tag{6.5.3}$$

where $E_i \sim \mathrm{Exp}(t^i/(1-t))$. Clearly

$$\Pr(\sum_{i=0}^{D} E_i \leq \tau) \leq \Pr(E_D \leq \tau) = 1 - e^{-\frac{t^D}{1-t}\tau}. \tag{6.5.4}$$

Hence

$$\Pr(\lambda_1'(\tau) - \log_{t^{-1}}(\tau) - \alpha \leq D) \geq e^{-\frac{t^{\log_{t^{-1}}(\tau)+\alpha+D}}{1-t}\tau} = e^{-\frac{t^{\alpha+D}}{1-t}}. \tag{6.5.5}$$

Now for the lower bound of (6.5.2). Suppose that $\lambda_k'(\tau) = x$ at some time τ, and consider the waiting time until λ_k' jumps to $x+1$. If clock $x+1$ rings k times then λ_k' will jump, even in the unfavorable case $\lambda_k'(\tau) = \ldots = \lambda_1'(\tau) = x$; note that there are many other ways the clocks can ring to cause λ_k' to jump, we are just choosing this one for the lower bound. So the waiting time until λ_k' jumps to $x+1$ is upper-bounded by a sum of k independent $\mathrm{Exp}(t^x)$ random variables. Denoting this sum of k random variables by $E_{k,x}$, it follows as before that

$$\Pr(\lambda_k'(\tau) \leq H) \leq \Pr(\sum_{i=0}^{H-1} E_{k,i} > \tau) \tag{6.5.6}$$

where in contrast to (6.5.3) we have an inequality rather than an equality because we have only bounded the waiting time rather than giving it exactly. By Markov's inequality,

$$\Pr(\sum_{i=0}^{H-1} E_{k,i} > \tau) \leq \frac{\mathbb{E}[\sum_{i=0}^{H-1} E_{k,i}]}{\tau} = \frac{kt^{-H+1}}{\tau}\frac{1-t^H}{1-t} \leq \frac{kt^{-H+1}}{(1-t)\tau}. \tag{6.5.7}$$

Hence

$$\Pr(\lambda_k'(\tau) - \log_{t^{-1}}(\tau) - \alpha \geq -D) \geq 1 - \frac{kt^{D+2-\log_{t^{-1}}(\tau)-\alpha}}{(1-t)\tau} = 1 - t^D\frac{kt^{2-\alpha}}{1-t}. \tag{6.5.8}$$

Since the bounds (6.5.5) and (6.5.8) are both independent of τ and both go to 1 as $D \to \infty$, together they show (6.5.2). □

Our motivation for tightness was to show that the limit formulas of Theorem 6.3.1 and Proposition 6.4.1 actually define a random variable, which was stated as Theorem 1.4.1 in the Introduction.

Proof of Theorem 1.4.1. For any $\chi \in \mathbb{R}_{>0}$, taking $\alpha = \log_t((1-t)\chi)$ in Theorem 6.3.1

and combining Proposition 6.5.1 with Prokhorov's theorem shows that the formula (1.4.1) defines a valid random variable. By Theorem 6.3.1 and Proposition 6.4.1, the right hand sides of (1.4.3) and (1.4.1) are equal, completing the proof. □

From now on we state limit results in terms of the random variables $\mathcal{L}_{k,\chi}$ rather than limit probabilities, which substantially declutters notation.

6.6 Examples of residue formula for $\mathcal{L}_{k,\chi}$

In this section we compute more explicitly the infinite series formula in Theorem 1.4.1 for $k = 1$, $k = 2$, and an example case of $k = 3$.

Corollary 6.6.1. *In the notation of Theorem 1.4.1, for any $\chi \in \mathbb{R}_{>0}$ the $\mathbb{Z}$-valued random variable $\mathcal{L}_{1,\chi}$ is defined by*

$$\Pr(\mathcal{L}_{1,\chi} = L) = \frac{1}{(t;t)_\infty} \sum_{m \geq 0} e^{-\chi t^{L-m}} \frac{(-1)^m t^{\binom{m}{2}}}{(t;t)_m} \tag{6.6.1}$$

for all $L \in \mathbb{Z}$.

Proof. Follows immediately from Theorem 1.4.1, by noting that the sum over $\mu \in \mathrm{Sig}_{k-1}$ has only one term $\mu = ()$ and is equal to $(-1)^{L-d}$, and changing variables to $m = L - d$. □

In the case $k = 2$, Theorem 1.4.1 has the following reduction.

Corollary 6.6.2. *In the notation of Theorem 1.4.1, for any $L \in \mathbb{Z}$ and $x \in \mathbb{Z}_{\geq 0}$,*

$$\Pr(\mathcal{L}_{2,\chi} = (L+x, L)) = \frac{t^{\binom{x}{2}}}{(t;t)_\infty} \sum_{m \geq 0} e^{-\chi t^{L-m}} (-1)^m t^{m^2 + (x-1)m}$$
$$\times \sum_{i=0}^{x} \frac{(-1)^{x-i}}{(t;t)_{x-i}} \begin{bmatrix} m+i \\ i \end{bmatrix}_t \left(\frac{(t^{L-m}\chi)^{i+m}}{(i+m)!} + \frac{(t^{L-m}\chi)^{i+m-1} \mathbb{1}(i+m \geq 1)}{(i+m-1)!} \right) \tag{6.6.2}$$

Proof. Follows by substituting the branching rule definition Lemma 2.2.1 of $Q_{(\mu-(d))'}$ into the formula in Theorem 1.4.1 and making the change of variables $m = L - d$. Since $k - 1 = 1$, the Q polynomials appearing are all of the form $Q_{(1[m+i])}(\gamma(t^{L-m}(1-t)\chi), \alpha(1); 0, t)$ for $0 \leq i \leq x$, which may be simply expanded by the branching rule to yield the above. □

Example 6.6.3. *In the notation of Theorem 1.4.1,*

$$
\begin{aligned}
&\Pr(\mathcal{L}_{3,\chi} = (L+2, L, L)) \\
&= \frac{1}{(t;t)_\infty (t;t)_2} \left((e^{-t^L \chi} t) \left((t;t)_2 \frac{(t^L\chi)^2}{2!} - t(1-t^2)\frac{t^L\chi}{1!} + t^2 \right) \right. \\
&- \frac{e^{-t^{L-1}\chi} t^3}{1-t} \left((1-t)^4(1+t)(3+2t+t^2)\frac{(t^{L-1}\chi)^4}{4!} + (1-t)^3(1+t)t^3(1-2t-t^2-t^3)\frac{(t^{L-1}\chi)^3}{3!} \right. \\
&\left. +(1-t)^2 t(-1+t+t^2)\frac{(t^{L-1}\chi)^2}{2!} - (1-t)^2 t \frac{t^{L-1}\chi}{1!} + (1-t) \right) \\
&+ \frac{e^{-t^{L-2}\chi} t^8}{(t;t)_2} \left((1-t)^6(1+t)^2(9+13t+12t^2+7t^3+3t^4+t^5)\frac{(t^{L-2}\chi)^6}{6!} \right. \\
&+ (1-t)^5(1+t)^2(4-2t-4t^2-6t^3-4t^4-2t^5-t^6)\frac{(t^{L-2}\chi)^5}{5!} \\
&+ (1-t)^4(1-t)t(-3+3t^2+2t^3+t^4)\frac{(t^{L-2}\chi)^4}{4!} \\
&\left. +(1-t)^4(1+t)t(-2-t)\frac{(t^L\chi)^3}{3!} + (1-t)^3(1-t)\frac{(t^{L-2}\chi)^2}{2!} \right) \\
&\left. - \frac{e^{-t^{L-3}\chi} t^{16}}{(t;t)_3}(\dots) + \frac{e^{-t^{L-4}\chi} t^{27}}{(t;t)_4}(\dots) + \dots \right),
\end{aligned}
\tag{6.6.3}
$$

where we have only computed the first three terms in the series.

We now show the computation. By Theorem 1.4.1 with a change of variables $L - d =$

m,

$$
\begin{aligned}
LHS(6.6.3) &= \frac{1}{(t;t)_\infty} \sum_{d \le L} e^{-\chi t^d} P_{(L-d+2,L-d,L-d)'}(1,t,\ldots;0,t) \\
&\times \sum_{\substack{\mu \in \mathrm{Sig}_{k-1} \\ \mu \prec (L+2,L,L)}} P_{(L-d+2,L-d,L-d)/(\mu_1-d,\mu_2-d)}(-1;t,0) Q_{(\mu_1-d,\mu_2-d)'}(\gamma(t^d(1-t)\chi),\alpha(1);0,t) \\
&= \frac{1}{(t;t)_\infty} \sum_{m \ge 0} e^{-\chi t^{L-m}} \frac{t^{\binom{m+2}{2}+2\binom{m}{2}}}{(t;t)_2 (t;t)_0 (t;t)_m} \left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}_t (-1)^m Q_{(m+2,m)'}(\cdots;0,t) \right. \\
&+ \left. \begin{bmatrix} 2 \\ 1 \end{bmatrix}_t (-1)^{m+1} Q_{(m+1,m)'}(\cdots;0,t) + \begin{bmatrix} 2 \\ 2 \end{bmatrix}_t (-1)^{m+2} Q_{(m,m)'}(\cdots;0,t) \right) \\
&= \frac{1}{(t;t)_\infty (t;t)_2} \sum_{m \ge 0} e^{-t^{L-m}\chi} \frac{t^{\frac{3}{2}m^2+\frac{1}{2}m+1}}{(t;t)_m} (-1)^m \\
&\times \left(Q_{(m+2,m)'}(\cdots;0,t) - (1+t) Q_{(m+1,m)'}(\cdots;0,t) + Q_{(m,m)'}(\cdots;0,t) \right),
\end{aligned} \tag{6.6.4}
$$

where we have used Lemma 2.2.14 and Proposition 2.2.15 and written $\cdots$ for the specialization $\gamma(t^{L-m}(1-t)\chi), \alpha(1)$. It remains to compute the three Hall-Littlewood polynomials in the last line of (6.6.4), and since there is not a closed form we compute the first few terms $m = 0, 1$.

For any $\lambda \in \mathbb{Y}$ the branching rule yields

$$
Q_\lambda(\gamma(g),\alpha(a);q,t) = \sum_{c=0}^{\lambda_1} \phi_{(c)}(q,t) a^c \sum_{B \in SYT(\lambda/(c))} \frac{\phi_B(q,t) g^{|\lambda|-c}}{(|\lambda|-c)!}. \tag{6.6.5}
$$

Here $SYT(\lambda/(c))$ is the set of standard Young tableaux corresponding to this skew shape, and for a tableau B identified with a sequence of partitions $(c) = \lambda^{(0)} \prec \ldots \prec \lambda^{(|\lambda|-c)} = \lambda$, we use the shorthand

$$
\phi_B(q,t) := \prod_{i=1}^{|\lambda|-c} \phi_{\lambda^{(i)}/\lambda^{(i-1)}}(q,t) \tag{6.6.6}
$$

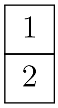
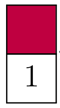
where $\phi_{\lambda^{(i)}/\lambda^{(i-1)}}$ is as in Definition 6. It follows from Lemma 2.2.14 that

$$
\phi_{(c)}(0,t) = (1-t^c), \tag{6.6.7}
$$

so it remains to compute the sum in (6.6.5). For each Hall-Littlewood polynomial appear-

ing in (6.6.4) *with* $m = 0, 1, 2$ *we will enumerate the pairs* (c, B) *with* $B \in SYT(\lambda/(c))$ *and give their coefficients.*

m=0: *We compute the coefficient of* $e^{-t^L\chi}$ *in* (6.6.3)*. Trivially* $Q_{(0,0)'}(\cdots; 0, t) = 1$*. For* $Q_{(1,0)'}(\cdots; 0, t)$ *we may either take* $c = 0$ *or* $c = 1$*: in each case there is one tableau, and the cases contribute* $(1-t)(t^L\chi)^1/1!$ *and* $1 - t$ *respectively. For* $Q_{(2,0)'}(\cdots; 0, t)$ *we again may either take* $c = 0$ *or* $c = 1$*, and in both cases have one skew tableau*[3]*,*

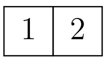
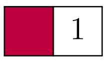
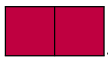
 *and* , (6.6.8)

yielding coefficients $(t;t)_2(t^L\chi)^2/2!$ *and* $(t;t)_2(t^L\chi)/1!$ *respectively in* (6.6.5)*. Summing these yields the desired coefficient.*

m=1: *We obtain*

$$Q_{(1,1)'}(\cdots; 0, t) = Q_{(2)}(\cdots; 0, t) = (1-t)^2\frac{(t^{L-1}\chi)^2}{2!} + (1-t)^2\frac{(t^{L-1}\chi)^1}{1!} + (1-t)\frac{(t^{L-1}\chi)^0}{0!} \tag{6.6.9}$$

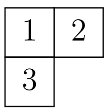
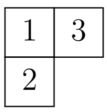
with the three summands coming from the three skew tableaux

 *and*  *and* . (6.6.10)

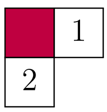
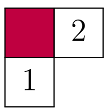
Similarly

$$Q_{(2,1)'}(\cdots; 0, t) = (1-t)^3(2+t)\frac{(t^{L-1}\chi)^3}{3!} + (1-t)^3(2+t)\frac{(t^{L-1}\chi)^2}{2!} + (1-t)^2\frac{(t^{L-1}\chi)^1}{1!} \tag{6.6.11}$$

with tableaux

 *and*  (6.6.12)

contributing $(1-t)^3$ *and* $(1-t)^2(1-t^2)$ *respectively to the degree 3 term,*

 *and*  (6.6.13)

[3]Here and elsewhere we denote the missing boxes, referred to as (c) above, by colored boxes.

also contributing $(1-t)^3$ *and* $(1-t)^2(1-t^2)$ *respectively to the degree* 2 *term, and*

$$\begin{ytableau} *(purple) & *(purple) \\ 1 \end{ytableau} \tag{6.6.14}$$

contributing $(1-t)^2$ *to the degree* 0*. Finally,*

$$Q_{(3,1)'}(\cdots;0,t) = Q_{(2,1,1)}(\cdots;0,t) = (1-t)^2(1-t^2)\frac{(t^{L-1}\chi)^2}{2!}$$
$$+\left((1-t)^3(1-t^2)+(1-t)^2(1-t^2)^2+(1-t)^2(1-t^2)(1-t^3)\right)\left(\frac{(t^{L-1}\chi)^3}{3!}+\frac{(t^{L-1}\chi)^4}{4!}\right). \tag{6.6.15}$$

m=2: *Here the three relevant tableaux for the degree-*4 *term are*

$$\begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau}, \qquad \begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau}, \qquad \begin{ytableau} 1 & 4 \\ 2 \\ 3 \end{ytableau} \tag{6.6.16}$$

in the same order as their coefficients in (6.6.15)*. The skew tableaux for the degree* 3 *term are the same as for* $m=1$*, but with the* $\boxed{1}$ *box replaced by* $\blacksquare$ *and the other indices shifted down by* 1 *to yield a skew standard Young tableau, and their coefficients are exactly the same. Only one tableau contributes to the degree* 2 *term, namely*

$$\begin{ytableau} *(purple) & *(purple) \\ 1 \\ 2 \end{ytableau}. \tag{6.6.17}$$

Summing the above and simplifying yields the coefficient of the $e^{-t^{L-2}\chi}$ *term in* (6.6.3).

One may, either by hand or by computer, generate the coefficient of any given remaining $e^{-t^{L-m}\chi}$ *term for* $m>2$*, but the number of tableaux grows with* m *and there is no closed form of which we are aware. However, because* $e^{-t^{L-m}\chi}$ *shrinks very fast as* m *increases (particularly if* t *is not close to* 1*), it is in fact quite easy to compute good approximations of this probability for any given values of* t,χ,L *by computing a small number of terms.*

6.7 The case of pure α specializations

For concreteness we did the computations in Section 6.3 for the Plancherel process $\lambda(\tau)$, but for matrix products it is desirable to compute a similar limit distribution of Hall-Littlewood alpha Cauchy dynamics

$$\Pr(\lambda \to \nu) = Q_{\nu/\lambda}(\alpha_1, \alpha_2, \ldots; 0, t) \frac{P_\nu(1, t, \ldots; 0, t)}{P_\lambda(1, t, \ldots; 0, t)\Pi_{0,t}(\alpha_1, \ldots; 1, t, \ldots)}, \tag{6.7.1}$$

see Proposition 6.8.1 below. Luckily our computations from Section 6.3 generalize straightforwardly. We use the power sum symmetric polynomial

$$p_1(\alpha_1, \alpha_2, \ldots) = \sum_{i \geq 1} \alpha_i \tag{6.7.2}$$

in that result to highlight the similarity with Theorem 6.3.1, where the Plancherel-specialized p_1 also appears in the formulas.

Proposition 6.7.1. *Let $k \in \mathbb{Z}_{\geq 1}$ and $\alpha \in \mathbb{R}$, and let ϕ be a pure alpha Hall-Littlewood nonnegative specialization determined by α parameters $\alpha_1 \geq \alpha_2 \geq \ldots$ with $0 < \alpha_1 < 1$. Let $\tilde{\lambda}(s), s \in \mathbb{Z}_{\geq 0}$ be distributed by the Hall-Littlewood measure*

$$\Pr(\tilde{\lambda}(s) = \lambda) = \frac{Q_\lambda(\alpha_1[s], \alpha_2[s], \ldots; 0, t)P_\lambda(1, t, \ldots; 0, t)}{\Pi_{0,t}(\alpha_1[s], \alpha_2[s], \ldots; 1, t, \ldots)}. \tag{6.7.3}$$

Let $(s_n)_{n \in \mathbb{N}}$ be any sequence with $s_n \xrightarrow{n \to \infty} \infty$ such that $\log_t s_n$ converges in $\mathbb{R}/\mathbb{Z}$, and let α be a lift of this limit to $\mathbb{R}$. Then

$$(\tilde{\lambda}_i'(s_n) - [\log_{t^{-1}}(s_n) + \alpha])_{1 \leq i \leq k} \to \mathcal{L}_{k, t^\alpha p_1(\alpha_1, \alpha_2, \ldots)} \tag{6.7.4}$$

where $[\cdot]$ is the nearest integer function.

Proof. Since Sig_k is a discrete set, it suffices to show convergence of probabilities

$$\lim_{n\to\infty} \Pr(\tilde{\lambda}'_i(s_n) - [\log_{t^{-1}}(s_n) + \alpha] = L_i \text{ for all } 1 \le i \le k)$$
$$= \frac{(t;t)_\infty^{k-1}}{k!(2\pi\mathbf{i})^k} \prod_{i=1}^{k-1} \frac{t^{\binom{L_i-L_k}{2}}}{(t;t)_{L_i-L_{i+1}}} \int_{\tilde{\Gamma}^k} e^{p_1(\alpha_1,\alpha_2,\ldots)t^{L_k+\alpha}(w_1+\ldots+w_k)} \frac{\prod_{1\le i\neq j\le k}(w_i/w_j;t)_\infty}{\prod_{i=1}^k(-w_i^{-1};t)_\infty(-tw_i;t)_\infty}$$
$$\times \sum_{j=0}^{L_{k-1}-L_k} t^{\binom{j+1}{2}} \begin{bmatrix} L_{k-1}-L_k \\ j \end{bmatrix}_t P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1},\ldots,w_k^{-1};t,0) \prod_{i=1}^k \frac{dw_i}{w_i}, \quad (6.7.5)$$

where if $k=1$ we interpret the sum on the last line as in Theorem 6.3.1.

Lemma 6.3.2 holds with $\lambda(\tau)$ replaced by $\tilde{\lambda}(s)$ on the left hand side and $\gamma(\tau)$ replaced by $\phi[s]$ (i.e. ϕ repeated s times, recall Definition 10) on the right hand side, by the exact same proof. Similarly, Lemma 6.3.3 holds with $\lambda(\tau)$ replaced by $\tilde{\lambda}(s)$ on the left hand side and

$$e^{\frac{\tau}{1-t}(z_1+\ldots+z_k)} = \Pi_{0,t}(\gamma(\tau);\beta(z_1,\ldots,z_k)) \quad (6.7.6)$$

on the right hand side replaced by

$$\Pi_{0,t}(\phi[s];\beta(z_1,\ldots,z_k)) = \prod_{\substack{1\le i\le k\\ j\ge 1}} (1+\alpha_j z_i). \quad (6.7.7)$$

We now explain how the asymptotic analysis used to prove Theorem 6.3.1 carries over. Let

$$a = \sum_{j\ge 1} \alpha_j, \quad (6.7.8)$$

and

$$\eta(s) = (L_i + [\log_{t^{-1}}(s) + \alpha])_{1\le i\le k} \quad (6.7.9)$$

and make the change of variables $w_i = t^{-\eta_k(s)} z_i = st^{-\alpha-L_k} z_i \cdot (1+o(1))$, where $o(1)$ comes from $(\log_{t^{-1}}(s)+\alpha) - [\log_{t^{-1}}(s)+\alpha]$. Fix a constant $\delta > 0$ with $\alpha_1 < (1+\delta)^{-1} < 1$. Then the same manipulations as in the proof of Theorem 6.3.1 to consolidate the powers of t

after the change of variables give

$$\Pr((\tilde{\lambda}(s)'_i)_{1\le i\le k} = \eta(s)) = \frac{(t;t)_\infty^{k-1}}{k!(2\pi\mathbf{i})^k}\prod_{i=1}^{k-1}\frac{t^{\binom{L_i-L_k}{2}}}{(t;t)_{L_i-L_{i+1}}}\int_{\Gamma(s)^k}\prod_{\substack{1\le i\le k\\ j\ge 1}}(1+t^{\alpha+L_k}\alpha_j w_i/s(1+o(1)))^s$$
$$\times \frac{\prod_{1\le i\ne j\le k}(w_i/w_j;t)_\infty}{\prod_{i=1}^k(-w_i^{-1};t)_\infty(-tw_i;t)_{\eta_k(s)}}\sum_{j=0}^{L_{k-1}-L_k} t^{\binom{j+1}{2}}\begin{bmatrix} L_{k-1}-L_k \\ j\end{bmatrix}_t$$
$$\times P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1},\ldots,w_k^{-1};t,0)\prod_{i=1}^k\frac{dw_i}{w_i}, \tag{6.7.10}$$

where the contour $\Gamma'(s)$ is given in Figure 6-4, and is similar to the one $\Gamma(\tau)$ from the proof of Theorem 6.3.1 with $\tau = s$ but lies slightly to the right of it in general.

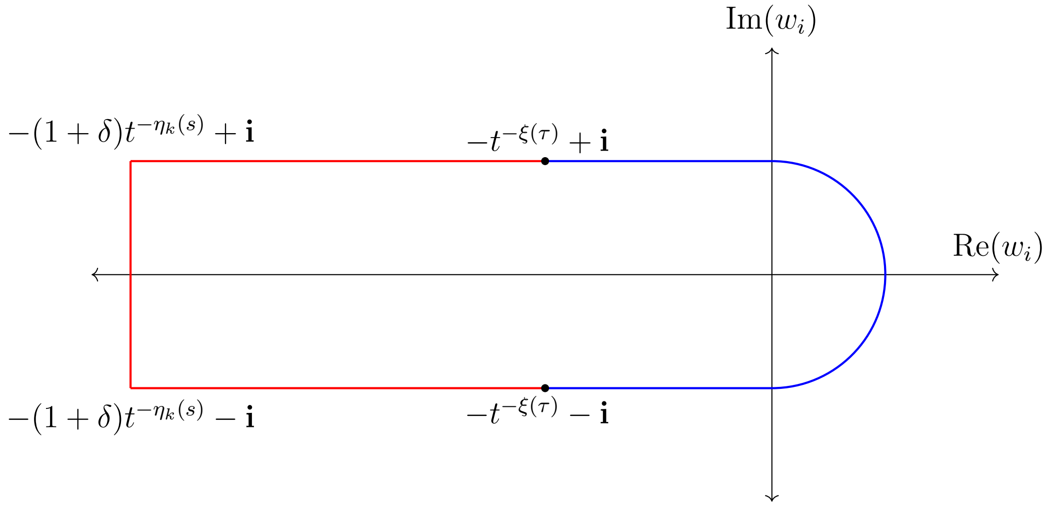


Figure 6-4: The contour $\Gamma'(s)$ with the analogous decomposition to (6.3.56) into $\Gamma_1(s)$ in blue and $\Gamma_2'(s)$ in red.

The reason for the slightly different contour from earlier is that now

$$\mathrm{Re}(t^{\alpha+L_k}\alpha_j w_i/s_n(1+o(1))) \ge -(1+\delta)\alpha_j(1+o(1)) > -1 \tag{6.7.11}$$

for all $w_i \in \Gamma'(s_n)$ and n large enough that the $o(1)$ term is sufficiently small. This implies that

$$\left|1+\frac{t^{\alpha+L_k}\alpha_j}{s_n}w_i(1+o(1))\right| = \sqrt{\left(1+\frac{t^{\alpha+L_k}\alpha_j(1+o(1))}{s_n}\operatorname{Re}(w_i)\right)^2+\left(\frac{t^{\alpha+L_k}\alpha_j(1+o(1))}{s_n}\operatorname{Im}(w_i)\right)^2}$$
$$\leq \left|1+\frac{t^{\alpha+L_k}\alpha_j(1+o(1))}{s_n}\operatorname{Re}(w_i)\right| + \left|\frac{t^{\alpha+L_k}\alpha_j(1+o(1))}{s_n}\operatorname{Im}(w_i)\right|$$
$$\leq 1+\frac{t^{\alpha+L_k}\alpha_j(1+o(1))}{s_n}(\operatorname{Re}(w_i)+1), \tag{6.7.12}$$

where the last line follows by (6.7.11) (to remove the absolute value on the first factor) and the bound $\operatorname{Im}(w_i) \leq 1$.

To simplify notation we express the integrand in terms of

$$\tilde{f}(w_1,\ldots,w_k) := \frac{(t;t)_\infty^{k-1}}{k!}\prod_{i=1}^{k-1}\frac{t^{\binom{L_i-L_k}{2}}}{(t;t)_{L_i-L_{i+1}}}\frac{\prod_{1\leq i\neq j\leq k}(w_i/w_j;t)_\infty}{\prod_{i=1}^k w_i(-w_i^{-1};t)_\infty}$$
$$\times\sum_{j=0}^{L_{k-1}-L_k} t^{\binom{j+1}{2}}\begin{bmatrix} L_{k-1}-L_k \\ j\end{bmatrix}_t P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1},\ldots,w_k^{-1};t,0), \tag{6.7.13}$$

which is the function f of (6.3.60) without the exponential factor. Then similarly to (6.3.61), (6.3.62) and (6.3.63), we must show

$$\lim_{n\to\infty}\frac{1}{(2\pi\mathbf{i})^k}\int_{\Gamma_1(s_n)^k}\frac{\tilde{f}(w_1,\ldots,w_k)g_{s_n}(w_1,\ldots,w_k)\prod_{\substack{1\leq i\leq k\\ j\geq 1}}(1+t^{\alpha+L_k}\alpha_j w_i/s_n(1+o(1)))^{s_n}}{\prod_{i=1}^k(-tw_i;t)_{\eta_k(s_n)}} \tag{6.7.14}$$

$$-\frac{\tilde{f}(w_1,\ldots,w_k)e^{t^{\alpha+L_k}\sum_{i=1}^k w_i}}{\prod_{i=1}^k(-tw_i;t)_\infty}\prod_{i=1}^k dw_i \tag{6.7.15}$$

$$+\lim_{n\to\infty}\frac{1}{(2\pi\mathbf{i})^k}\int_{\Gamma'(s_n)^k\setminus\Gamma_1(s_n)^k}\frac{\tilde{f}(w_1,\ldots,w_k)\prod_{\substack{1\leq i\leq k\\ j\geq 1}}\left(1+\frac{w_i\alpha_j t^{\alpha+L_k}}{s_n(1+o(1))}\right)^{s_n}g_{s_n}(w_1,\ldots,w_k)}{\prod_{i=1}^k(-tw_i;t)_{\eta_k(s_n)}}\prod_{i=1}^k dw_i \tag{6.7.16}$$

$$-\lim_{n\to\infty}\frac{1}{(2\pi\mathbf{i})^k}\int_{\tilde{\Gamma}^k\setminus\Gamma_1(s_n)^k}\frac{\tilde{f}(w_1,\ldots,w_k)e^{t^{\alpha+L_k}\sum_{i=1}^k w_i}}{\prod_{i=1}^k(-tw_i;t)_\infty}\prod_{i=1}^k dw_i = 0, \tag{6.7.17}$$

where g_s is as in (6.3.44). We will show each line is 0 separately. The third line (6.7.17) is exactly the same as (6.3.63) and has been shown in the proof of Theorem 6.3.1. For the first line (6.7.14), we have

$$\prod_{\substack{1\leq i\leq k\\ j\geq 1}} (1+t^{\alpha+L_k}\alpha_j w_i/s_n(1+o(1)))^{s_n} = e^{at^{\alpha+L_k}\sum_{i=1}^k w_i + O(w_i^2/s_n)} \tag{6.7.18}$$
$$= e^{at^{\alpha+L_k}\sum_{i=1}^k w_i}(1+O(w_i^2/s_n))$$

as $n\to\infty$. Combining with the estimate Lemma 6.3.6 and using that w_i^2/s is $o(1)$ uniformly over $w_i\in\Gamma_1(s)$, (6.7.14) follows exactly as earlier with (6.3.61).

For the second line, we need an analogue of Lemma 6.3.5. By (6.7.12) and the elementary inequality

$$\left(1+\frac{x}{n}\right)^n \leq e^x, \tag{6.7.19}$$

we have

$$\prod_{j\geq 1}(1+t^{\alpha+L_k}\alpha_j w_i/s(1+o(1)))^s \leq \prod_{j\geq 1} e^{t^{\alpha+L_k}\alpha_j(\mathrm{Re}(w_i)+1)} = e^{at^{\alpha+L_k}(\mathrm{Re}(w_i)+1)}, \tag{6.7.20}$$

for all large enough s so that the $(1+o(1))$ factor may be neglected. Hence

$$\left|\tilde{f}(w_1,\ldots,w_k)\prod_{\substack{1\leq i\leq k\\ j\geq 1}}(1+t^{\alpha+L_k}\alpha_j w_i/s(1+o(1)))^s\right|$$
$$\leq C\prod_{1\leq i\leq k} e^{at^{\alpha+L_k}(\mathrm{Re}(w_i)+1)+\frac{k-1}{2}(\log t^{-1})\lfloor \log_t|w_i|\rfloor^2 + c_2\lfloor \log_t|w_i|\rfloor}, \tag{6.7.21}$$

where we have bounded $\tilde{f}$ as in the proof of Lemma 6.3.5. The rest of the proof of the vanishing of (6.7.16) is the same as for (6.3.62), with (6.7.21) in place of Lemma 6.3.5. $\square$

Remark 40. One may try to carry through the above argument with dual β parameters in the specialization, but the Cauchy kernel

$$\Pi_{0,t}(\beta(b_1,b_2,\ldots);\beta(z_1,\ldots,z_k)) = \Pi_{t,0}(b_1,\ldots;z_1,\ldots,z_k) = \frac{1}{\prod_{\substack{1\leq i\leq k\\ j\geq 1}}(b_j z_i;t)_\infty} \tag{6.7.22}$$

creates extra poles in the integrand, in contrast to the alpha and Plancherel cases treated in Theorem 6.3.1 and Proposition 6.7.1. These probably can be dealt with, but we did not attempt to carry this out since we only need the alpha and Plancherel cases for our random matrix results.

6.8 From $\mathcal{S}^{(\infty)}$ to matrix bulk limits

In this section we deduce for the limiting distribution of singular numbers in the bulk for the processes $X(\tau)$ of Section 6.1 and for products of additive Haar matrices. We begin with the first, which is slightly easier.

Proposition 6.8.1. *Let $c \in \mathbb{R}_{>0}$, and for each $N \in \mathbb{Z}_{\geq 1}$ let $X(\tau)^{(N)}, \tau \in \mathbb{R}_{\geq 0}$ be the stochastic process $X_\tau^{(N,c)}$ of Definition 45. Fix $\alpha \in \mathbb{R}$, and let $\tau_N \in t^{\alpha+\mathbb{Z}}, N \geq 1$ be a sequence of real numbers such that*

1. *$\tau_N \to \infty$ as $N \to \infty$, and*

2. *$N - \log_{t^{-1}} \tau_N \to \infty$.*

Then for any $k \in \mathbb{Z}_{\geq 1}$,

$$(\mathrm{SN}(X(\tau)^{(N)})'_i - \log_{t^{-1}} \tau_N - \alpha)_{1 \leq i \leq k} \to \mathcal{L}_{k,ct^\alpha} \tag{6.8.1}$$

in distribution.

Proof. By Proposition 6.5.1, it suffices to show that for any $k \in \mathbb{Z}_{\geq 1}$ and integers $L_1 \geq \ldots \geq L_k$,

$$\begin{aligned}
&\lim_{N\to\infty} \Pr((\mathrm{SN}(X(\tau)^{(N)})'_i - \log_{t^{-1}} \tau_N - \alpha)_{1 \leq i \leq k} = (L_1, \ldots, L_k)) \\
&\quad = \frac{(t;t)_\infty^{k-1}}{k!(2\pi \mathbf{i})^k} \prod_{i=1}^{k-1} \frac{t^{\binom{L_i - L_k}{2}}}{(t;t)_{L_i - L_{i+1}}} \int_{\tilde{\Gamma}^k} e^{ct^{L_k+\alpha}(w_1+\ldots+w_k)} \frac{\prod_{1 \leq i \neq j \leq k}(w_i/w_j;t)_\infty}{\prod_{i=1}^k (-w_i^{-1};t)_\infty (-tw_i;t)_\infty} \\
&\quad \times \sum_{j=0}^{L_{k-1}-L_k} t^{\binom{j+1}{2}} \begin{bmatrix} L_{k-1} - L_k \\ j \end{bmatrix}_t P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1}, \ldots, w_k^{-1}; t, 0) \prod_{i=1}^k \frac{dw_i}{w_i},
\end{aligned} \tag{6.8.2}$$

where if $k = 1$ we interpret the sum on the last line as in Theorem 6.3.1. By Theorem 1.4.4 and Corollary 6.2.5,

$$\mathrm{SN}(X(\tau)^{(N)}) = \lambda^{(N)}\left(c\frac{1-t}{1-t^N}\tau\right) \tag{6.8.3}$$

in (multi-time) distribution, where $\lambda^{(N)}$ is as in Definition 47. We claim that it suffices to show

$$\begin{aligned}
&\lim_{N\to\infty} \Pr((\lambda^{(N)}(\tau_N)'_i - \log_{t^{-1}} \tau_N - \alpha)_{1\leq i\leq k} = (L_1, \ldots, L_k)) \\
&\quad = \frac{(t;t)_\infty^{k-1}}{k!(2\pi\mathbf{i})^k} \prod_{i=1}^{k-1} \frac{t^{\binom{L_i-L_k}{2}}}{(t;t)_{L_i-L_{i+1}}} \int_{\tilde{\Gamma}^k} e^{\frac{t^{L_k+\alpha}}{1-t}(w_1+\ldots+w_k)} \frac{\prod_{1\leq i\neq j\leq k}(w_i/w_j;t)_\infty}{\prod_{i=1}^k(-w_i^{-1};t)_\infty(-tw_i;t)_\infty} \\
&\quad \times \sum_{j=0}^{L_{k-1}-L_k} t^{\binom{j+1}{2}} \begin{bmatrix} L_{k-1}-L_k \\ j \end{bmatrix}_t P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1},\ldots,w_k^{-1};t,0) \prod_{i=1}^k \frac{dw_i}{w_i}. \quad (6.8.4)
\end{aligned}$$

First note that replacing τ_N by $c(1-t)\tau_N$ and α by $\alpha - \log_{t^{-1}}(c(1-t))$ in (6.8.4) yields (6.8.2). Furthermore, since $N - \log_{t^{-1}} \tau_N \to \infty$ as $N \to \infty$, we have that

$$c\frac{1-t}{1-t^N}\tau_N = c(1-t)\tau_N + o(1), \tag{6.8.5}$$

and $\lambda^{(N)}$ is a Poisson jump process with the exit rate from any state bounded above, hence if $(\lambda^{(N)}(c(1-t)\tau_N)'_i - \log_{t^{-1}}(\tau_N) - \alpha)_{1\leq i\leq k}$ has a limiting distribution then $(\lambda^{(N)}(c(1-t)\tau_N/(1-t^N))'_i - \log_{t^{-1}}(\tau_N) - \alpha)_{1\leq i\leq k}$ must have the same limiting distribution. Thus it suffices to show (6.8.4), and the remainder of the proof consists of doing so by arguing that the Hall-Littlewood processes $\lambda^{(N)(\tau)}$ and $\lambda(\tau)$ (the latter of which was analyzed in Theorem 6.3.1) are not so different on the timescale we consider.

Define stopping times

$$\begin{aligned}
T_N^{(N)} &:= \inf(\{\tau \in \mathbb{R}_{\geq 0} : \lambda^{(N)}(\tau)'_1 = N\}) \\
T_N &:= \inf(\{\tau \in \mathbb{R}_{\geq 0} : \lambda(\tau)'_1 = N\}) \\
\Xi_N &:= \inf_{j\geq N+1}(\text{time at which clock } j \text{ rings for } \lambda(\tau)).
\end{aligned} \tag{6.8.6}$$

Conditionally on the event that clocks $N+1, N+2, \ldots$ do not ring on a given time interval, both $\lambda_i^{(N)}(\tau), 1 \leq i \leq N$ and $\lambda_i(\tau), 1 \leq i \leq N$ have the same local dynamics controlled by N Poisson clocks on that interval, by Corollary 6.2.5. Taking the time interval $[0, \tau_N]$, since $\min(\tau_N, T_N^{(N)})$ and $\min(\tau_N, T_N)$ are measurable with respect to the σ-algebras generated by $\lambda^{(N)}([0, \tau_N])$ and $\lambda([0, \tau_N])$ respectively, this implies that

$$\mathrm{Law}(\min(\tau_N, T_N^{(N)})) = \mathrm{Law}(\min(\tau_N, T_N) | \Xi_N > \tau_N) \tag{6.8.7}$$

and

$$\mathrm{Law}((\lambda^{(N)}(\tau_N)'_1, \ldots, \lambda^{(N)}(\tau_N)'_k)|T_N^{(N)} > \tau_N) = \mathrm{Law}((\lambda(\tau_N)'_1, \ldots, \lambda(\tau_N)'_k)|T_N > \tau_N \text{ and } \Xi_N > \tau_N). \tag{6.8.8}$$

The explicit description of our dynamics implies the distributional equality

$$\mathrm{Law}(T_N) = \mathrm{Law}\left(\sum_{i=0}^{N-1} Y_{t^i/(1-t)}\right) \tag{6.8.9}$$

where Y_r is an exponential distribution with rate r. Because $N - \log_{t^{-1}} \tau_N \to \infty$,

$$\mathbb{E}[Y_{t^{N-1}/(1-t)}] = (1-t)t^{1-N} \gg \tau_N, \tag{6.8.10}$$

and the fluctuations of $Y_{t^{N-1}/(1-t)}$ are of lower order than its mean, hence

$$\lim_{N\to\infty} \Pr(T_N > \tau_N) = 1. \tag{6.8.11}$$

Furthermore, since the first time one of the clocks $N+1, N+2, \ldots$ rings follows an exponential distribution with rate $t^{N+1}/(1-t)$, the hypothesis $N - \log_{t^{-1}} \tau_N \to \infty$ is exactly what is needed to guarantee that the probability that any of clocks $N+1, N+2, \ldots$ rings on the interval we are concerned with is asymptotically negligible, i.e.

$$\lim_{N\to\infty} \Pr(\Xi_N \leq \tau_N) = 0. \tag{6.8.12}$$

From (6.8.11) and (6.8.12) it follows that

$$\lim_{N\to\infty} \Pr(T_N > \tau_N \text{ and } \Xi_N > \tau_N) = 1. \tag{6.8.13}$$

By (6.8.7) follows by (6.8.12) and (6.8.13),

$$\begin{aligned}
\lim_{N\to\infty} \Pr(T_N^{(N)} > \tau_N) &= \lim_{N\to\infty} \Pr(T_N > \tau_N | \Xi_N > \tau_N) \\
&= \lim_{N\to\infty} \frac{\Pr(T_N > \tau_N \text{ and } \Xi_N > \tau_N)}{\Pr(\Xi_N > \tau_N)} \\
&= 1.
\end{aligned} \tag{6.8.14}$$

From (6.8.14) it follows that

$$\begin{aligned}\mathrm{LHS}(6.8.4) &= \lim_{N\to\infty} \Pr((\lambda^{(N)}(\tau_N)'_i - \log_{t^{-1}}(\tau_N))_{1\le i\le k} = (L_i+\alpha)_{1\le i\le k} | T_N^{(N)} > \tau_N) \cdot \Pr(T_N^{(N)} > \tau_N) \\ &+ \lim_{N\to\infty} \Pr((\lambda^{(N)}(\tau_N)'_i - \log_{t^{-1}}(\tau_N))_{1\le i\le k} = (L_i+\alpha)_{1\le i\le k} | T_N^{(N)} \le \tau_N) \cdot \Pr(T_N^{(N)} \le \tau_N) \\ &= \lim_{N\to\infty} \Pr((\lambda^{(N)}(\tau_N)'_i - \log_{t^{-1}}(\tau_N))_{1\le i\le k} = (L_i+\alpha)_{1\le i\le k} | T_N^{(N)} > \tau_N).\end{aligned} \tag{6.8.15}$$

By (6.8.7),

$$\begin{aligned}\mathrm{RHS}(6.8.15) &= \lim_{N\to\infty} \Pr((\lambda(\tau_N)'_i - \log_{t^{-1}}(\tau_N))_{1\le i\le k} = (L_i+\alpha)_{1\le i\le k} | T_N > \tau_N \text{ and } \Xi_N > \tau_N) \\ &= \lim_{N\to\infty} \frac{1}{\Pr(T_N > \tau_N \text{ and } \Xi_N > \tau_N)} \left(\Pr((\lambda(\tau_N)'_i - \log_{t^{-1}}(\tau_N))_{1\le i\le k} = (L_i+\alpha)_{1\le i\le k}\right) \\ &- \Pr((\lambda(\tau_N)'_i - \log_{t^{-1}}(\tau_N))_{1\le i\le k} = (L_i+\alpha)_{1\le i\le k} \text{ and } (T_N \le \tau_N \text{ or } \Xi_N \le \tau_N)\big).\end{aligned} \tag{6.8.16}$$

Since $\Pr(T_N > \tau_N \text{ and } \Xi_N > \tau_N) = 1 - o(1)$ by (6.8.13), we have

$$\mathrm{RHS}(6.8.16) = \lim_{N\to\infty} \Pr((\lambda(\tau_N)'_i - \log_{t^{-1}}(\tau_N))_{1\le i\le k} = (L_i+\alpha)_{1\le i\le k}). \tag{6.8.17}$$

By Theorem 6.3.1, the above is equal to the right hand side of (6.8.2), and this completes the proof. □

For matrix products, the constraint that the number of matrix products is an integer rather than a real number forces us to make a slightly messier statement (recall Theorem 1.4.2) than with continuous-time processes, but it is essentially the same, and the argument likewise goes by matching to a Hall-Littlewood process with one infinite principal specialization (though in this case the other specialization is alpha rather than Plancherel).

Proof of Theorem 1.4.2. To control subscripts we abuse notation we write N for N_j below, so all limits should be interpreted as along our subsequence $(N_j)_{j\ge1}$. By Proposition 6.5.1 it suffices to show

$$\lim_{N\to\infty} \Pr((\mathrm{SN}(A_{s_N}^{(N)}\cdots A_1^{(N)})_i' - [\log_{t^{-1}}(s)+\alpha])_{1\le i\le k} = (L_1,\ldots,L_k))$$
$$= \frac{(t;t)_\infty^{k-1}}{k!(2\pi\mathbf{i})^k}\prod_{i=1}^{k-1}\frac{t^{\binom{L_i-L_k}{2}}}{(t;t)_{L_i-L_{i+1}}}\int_{\tilde\Gamma^k} e^{\frac{t^{L_k+\alpha+1}}{1-t}(w_1+\ldots+w_k)}\frac{\prod_{1\le i\neq j\le k}(w_i/w_j;t)_\infty}{\prod_{i=1}^k(-w_i^{-1};t)_\infty(-tw_i;t)_\infty}$$
$$\times\sum_{j=0}^{L_{k-1}-L_k} t^{\binom{j+1}{2}}\begin{bmatrix}L_{k-1}-L_k\\ j\end{bmatrix}_t P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1},\ldots,w_k^{-1};t,0)\prod_{i=1}^k\frac{dw_i}{w_i}, \quad (6.8.18)$$

where if $k=1$ we interpret the sum on the last line as in Theorem 6.3.1. For any $s\in\mathbb{N}$, let $\tilde\lambda(s)$ be a Hall-Littlewood measure with one specialization $1,t,\ldots$ and one $\alpha(t,t^2,\ldots)[s]$. Then Proposition 6.7.1 applies with $a=t+t^2+\ldots=t/(1-t)$, yielding

$$\lim_{N\to\infty}\Pr(\tilde\lambda_i'(s_N)-[\log_{t^{-1}}(s_N)+\alpha]=L_i \text{ for all } 1\le i\le k)$$
$$= \frac{(t;t)_\infty^{k-1}}{k!(2\pi\mathbf{i})^k}\prod_{i=1}^{k-1}\frac{t^{\binom{L_i-L_k}{2}}}{(t;t)_{L_i-L_{i+1}}}\int_{\tilde\Gamma^k} e^{\frac{t^{L_k+\alpha+1}}{1-t}(w_1+\ldots+w_k)}\frac{\prod_{1\le i\neq j\le k}(w_i/w_j;t)_\infty}{\prod_{i=1}^k(-w_i^{-1};t)_\infty(-tw_i;t)_\infty}$$
$$\times\sum_{j=0}^{L_{k-1}-L_k} t^{\binom{j+1}{2}}\begin{bmatrix}L_{k-1}-L_k\\ j\end{bmatrix}_t P_{(L_1-L_k,\ldots,L_{k-1}-L_k,j)}(w_1^{-1},\ldots,w_k^{-1};t,0)\prod_{i=1}^k\frac{dw_i}{w_i}. \quad (6.8.19)$$

Let $\tilde\lambda^(N)(s), s\in\mathbb{Z}_{\ge0}$ be a Hall-Littlewood process with transition probabilities

$$\Pr(\tilde\lambda^{(N)}(s+1)=\nu|\tilde\lambda^{(N)}(s)=\kappa)=Q_{\nu/\kappa}(t,t^2,\ldots;0,t)\frac{P_\nu(1,\ldots,t^{N-1};0,t)}{\Pi_{0,t}(t,t^2,\ldots;1,\ldots,t^{N-1})P_\kappa(1,\ldots,t^{N-1};0,t)} \quad (6.8.20)$$

(and initial condition $\emptyset\in\mathbb{Y}$). By Proposition 5.2.2, both $\tilde\lambda$ and $\tilde\lambda^{(N)}$ have a sampling algorithm[4] for which we briefly recall the important points. First, the random step $\tilde\lambda(s)\mapsto\tilde\lambda(s+1)$ involves an infinite number of substeps, indexed by the alpha variables $t,t^2,\ldots$, of which with probability 1 only finitely many are nontrivial. Second, each such substep involves sampling random variables $X_1,\ldots,X_N$ (for $\tilde\lambda^{(N)}$) or $X_1,X_2,\ldots$ (for $\tilde\lambda$) and applying an 'insertion map'

$$(\text{next state})=\iota_N(X_1,\ldots,X_N;(\text{initial state})) \quad (6.8.21)$$

[4]It was stated for simplicity in the case where the fixed principal specialization is finite, but holds for an infinite principal specialization—and hence $\tilde\lambda$—as well.

(for $\tilde{\lambda}^{(N)}$) or

$$(\text{next state}) = \iota_\infty(X_1, X_2, \ldots; (\text{initial state})) \tag{6.8.22}$$

(for $\tilde{\lambda}$). These insertion maps have the property that for any $\kappa \in \mathbb{Y}_N$, if $0 = X_{N+1} = X_{N+2} = \ldots$, then

$$\iota_N(X_1, \ldots, X_N; \kappa) = \iota_\infty(X_1, X_2, \ldots; \kappa). \tag{6.8.23}$$

Now define stopping times

$$\begin{aligned}
T_N^{(N)} &:= \min(\{s \in \mathbb{Z}_{\geq 0} : \tilde{\lambda}^{(N)}(s)'_1 = N\}) \\
T_N &:= \min(\{s \in \mathbb{Z}_{\geq 0} : \tilde{\lambda}(s)'_1 = N\}) \\
\Xi_N &:= \min\{s \in \mathbb{Z}_{\geq 0} : \text{at some substep of } \tilde{\lambda}(s) \mapsto \tilde{\lambda}(s+1), \max_{j \geq N+1} X_j \geq 1\}
\end{aligned} \tag{6.8.24}$$

The rest of the proof proceeds exactly as for Proposition 6.8.1 by showing that the variables $X_{N+1}, X_{N+2}, \ldots$ will all be 0 with high probability on $[0, s_N]$, hence $\tilde{\lambda}'_i(s_N) = \tilde{\lambda}^{(N)}(s_N)'_i$ by the properties of the sampling algorithm outlined above, and then using (6.8.19) in place of (6.8.4). $\square$

Remark 41. The results of Chapter 5 show that the singular numbers of $N \times N$ products of corners of Haar-distributed elements of $\mathrm{GL}_D(\mathbb{Z}_p)$, $D > N$ also form a Hall-Littlewood process, and Theorem 1.4.2 carries over mutatis mutandis to that setting with no changes other than the parameter of $\mathcal{L}_{k,\cdot}$.

THIS PAGE INTENTIONALLY LEFT BLANK

Chapter 7

Universal local limits for p-adic matrix products

7.1 Constructing the limit process

In this section we construct the bulk and edge limit processes mentioned in the Introduction, by coupling together many copies of the process $\mathcal{S}^{\mu,\infty}(T)$ discussed in the previous section. We will give a uniform construction with general initial condition which includes both the bulk and edge cases, and to set up this formalism we define an extended version of earlier signature notation. Throughout this section, $t \in (0,1)$ is a fixed real parameter.

Definition 48. Let $\bar{\mathbb{Z}} = \mathbb{Z} \cup \{\pm\infty\}$. We define

$$\mathrm{Sig}_n := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\}$$

and

$$\widehat{\mathrm{Sig}}_n := \{(\lambda_1, \ldots, \lambda_n) \in \bar{\mathbb{Z}}^n : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\},$$

where we take $\infty > x > -\infty$ for any $x \in \mathbb{Z}$. Furthermore we define the infinite versions

$$\mathrm{Sig}_\infty := \{(\mu_n)_{n \in \mathbb{Z}_{\geq 1}} \in \mathbb{Z}^{\mathbb{Z}_{\geq 1}} : \mu_{n+1} \leq \mu_n \text{ for all } n \in \mathbb{Z}_{\geq 1}\}$$

and

$$\widehat{\mathrm{Sig}}_\infty := \{(\mu_n)_{n \in \mathbb{Z}} \in \bar{\mathbb{Z}}^{\mathbb{Z}_{\geq 1}} : \mu_{n+1} \leq \mu_n \text{ for all } n \in \mathbb{Z}_{\geq 1}\},$$

and the bi-infinite versions

$$\mathrm{Sig}_{2\infty} := \{(\mu_n)_{n\in\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}} : \mu_{n+1} \le \mu_n \text{ for all } n \in \mathbb{Z}\}$$

and

$$\widehat{\mathrm{Sig}}_{2\infty} := \{(\mu_n)_{n\in\mathbb{Z}} \in \bar{\mathbb{Z}}^{\mathbb{Z}} : \mu_{n+1} \le \mu_n \text{ for all } n \in \mathbb{Z}\}.$$

For $x \in \bar{\mathbb{Z}}$, we write $(x[2\infty]) = (x)_{n\in\mathbb{Z}}$. For any finite interval $I \subset \mathbb{Z}$, define

$$\begin{aligned}\pi_I : \widehat{\mathrm{Sig}}_{2\infty} &\to \widehat{\mathrm{Sig}}_{|I|}\\ \mu &\mapsto (\mu_i)_{i\in I}\end{aligned}$$

and for a half-infinite interval $I = [a,\infty)$ define $\pi_I : \widehat{\mathrm{Sig}}_{2\infty} \to \widehat{\mathrm{Sig}}_{\infty}$ in the same way. We refer to the elements λ_n, μ_n above as *parts*, as is standard terminology with integer partitions. Finally, we use $\mathrm{Sig}_n^+, \widehat{\mathrm{Sig}}_n^+, \mathrm{Sig}_{2\infty}^+, \widehat{\mathrm{Sig}}_{2\infty}^+$ to denote the subsets where all parts are either ≥ 0 or equal to $-\infty$.

Definition 49. Given $\mu = (\mu_n)_{n\in\mathbb{Z}} \in \widehat{\mathrm{Sig}}_{2\infty}$, we define $\mu' = (\mu'_n)_{n\in\mathbb{Z}} \in \widehat{\mathrm{Sig}}_{2\infty}$ by

$$\mu'_i = \begin{cases} \text{the unique index } j \text{ such that } \mu_j \ge i, \mu_{j+1} < i & \lim_{n\to\infty} \mu_{-n} \ge i > \lim_{n\to\infty} \mu_n \\ -\infty & i > \lim_{n\to\infty} \mu_{-n} \\ \infty & i \le \lim_{n\to\infty} \mu_n \end{cases}.$$

Though in Theorem 1.5.1 we stated that the limit process $\mathcal{S}^{\mu,2\infty}$ is a bulk limit of the processes $\mathcal{S}^{\nu,n}$ with n particles, for the construction it turns out to be much better to work with the 'half-infinite' version $\mathcal{S}^{\mu,\infty}(\tau)$ defined previously in Definition 2. To couple many such processes together, it is helpful to define notation for certain shifted versions.

Definition 50. For $\mu \in \widehat{\mathrm{Sig}}_{\infty}$ and $t \in (0,1)$, we define the stochastic process

$$\tilde{\mathcal{S}}^{\mu,n}(T) = (\tilde{\mathcal{S}}^{\mu,n}_{-n}(T), \tilde{\mathcal{S}}^{\mu,n}_{-n+1}(T), \ldots) = (\mathcal{S}^{\mu,\infty}_1(t^{-n-1}T), \mathcal{S}^{\mu,\infty}_2(t^{-n-1}T), \ldots). \tag{7.1.1}$$

Strictly speaking, the description in Definition 50 only make sense if finitely many Poisson clocks ring on any interval. This is simple to show, and we do so in Lemma 7.1.1 once we have set up the relevant probability space. We also emphasize that $\tilde{\mathcal{S}}^{\mu,n}(T)$ is merely a notational shift of $\mathcal{S}^{\mu,\infty}(T)$ as defined in Definition 2, where we make the indices start at $-n$ rather than 1, and speed up time by a factor of t^{-n-1} so that $\tilde{\mathcal{S}}_1^{\mu,n}(T)$ has jump rate t, similarly to $\mathcal{S}_1^{\mu,n}(T)$ and $\mathcal{S}_1^{\mu,\infty}(T)$.

Definition 51. Define the probability space

$$\Omega := \prod_{i \in \mathbb{Z}} \mathbb{R}_{\geq 0}^{\mathbb{N}}$$

with the obvious product σ-algebra. Define the probability measure

$$\mathrm{Poiss} := \prod_{i \in \mathbb{Z}} \mathrm{Poiss}_{t^i} \in \mathcal{M}(\Omega)$$

where $\mathrm{Poiss}_r \in \mathcal{M}(\mathbb{R}_{\geq 0}^{\mathbb{N}})$ is the product over the $\mathbb{N}$ factors of the distributions of rate-r exponential variables.

Clearly Poiss_r may be identified with the law of a rate r Poisson jump process on time $T \geq 0$ by viewing each $\mathbb{R}_{\geq 0}$ factor as specifying the waiting time between adjacent jumps (or in the case of the first factor, the waiting time between time $T = 0$ and the first jump). Heuristically, $\mathcal{S}^{\mu,2\infty}(T)$ is defined by giving each $\mathcal{S}_i^{\mu,2\infty}(T)$ an independent exponential clock with rate t^i, and having $\mathcal{S}_i^{\mu,2\infty}(T)$ jump when its clock rings; here, Ω is exactly the space of possible sequences of ring times of all of the $\mathbb{Z}$-many clocks, and the measure Poiss is exactly the desired Poisson measure on the ring times. The main difficulty consists in making sense of this when $\lim_{n \to -\infty} \mu_n$ is finite, i.e. when infinitely many particles with rates in increasing geometric progression are all located at a single point and so infinitely many of their clocks ring on any time interval.

However, we first make formal the above claim that with probability 1 only finitely many clocks with indices belonging to any half-infinite interval $[i, \infty)$ ring on a given time interval, which was necessary for Definition 50 to make sense. First define notation

$$\begin{aligned} \mathrm{jumps} : \mathbb{R}_{\geq 0} \times \left(\mathbb{R}_{\geq 0}^{\mathbb{N}}\right) &\to \mathbb{Z}_{\geq 0} \\ (T, (a_1, a_2, \ldots)) &\mapsto \sup\left(\left\{ n \geq 0 : \sum_{i=1}^{n} a_i \leq T \right\}\right) \end{aligned} \tag{7.1.2}$$

i.e. $\mathrm{jumps}(T, \cdot)$ tells how many times the clock parametrized by the element of $\mathbb{R}_{\geq 0}^{\mathbb{N}}$ has rung by time T.

Definition 52. Denote

$$\widetilde{\Omega} := \{\omega \in \Omega : \sum_{j=i}^{\infty} \mathrm{jumps}(T, \pi_j(\omega)) < \infty \text{ holds for every } T \geq 0 \text{ and } i \in \mathbb{Z}\}.$$

Lemma 7.1.1. *The set $\widetilde{\Omega} \subset \Omega$ has full measure.*

Proof. It is an elementary computation with exponential random variables that for any T and i,

$$\sum_{j=i}^{\infty} \mathrm{jumps}(T, \pi_j(\omega)) < \infty \tag{7.1.3}$$

with probability 1. Hence the set of $\omega \in \Omega$ for which (7.1.3) holds for all $T' \in [0, T]$ is full measure, and the complement $\Omega \setminus \widetilde{\Omega}$ is therefore a union over $i \in \mathbb{Z}, T \in \mathbb{N}$ of measure 0 sets. It therefore has measure 0, so $\widetilde{\Omega}$ has full measure. $\square$

We may couple the processes $\tilde{\mathcal{S}}^{\pi_{[-n,\infty)}(\mu),n}(T)$ on the probability space $\widetilde{\Omega}$ as follows. Simply note that any sequence of clock ring times for $\tilde{\mathcal{S}}_{-n}^{\pi_{[-n,\infty)}(\mu),n}, \tilde{\mathcal{S}}_{-n+1}^{\pi_{[-n,\infty)}(\mu),n}, \ldots$, viewed as an element of $\prod_{i=-n}^{\infty} \mathbb{R}_{\geq 0}^{\mathbb{N}}$, determines $(\tilde{\mathcal{S}}_{-n}^{\pi_{[-n,\infty)}(\mu),n}(T), \tilde{\mathcal{S}}_{-n+1}^{\pi_{[-n,\infty)}(\mu),n}(T), \ldots)$ for all $T \geq 0$ by the jump rules of Definition 2. The random variable $\tilde{\mathcal{S}}^{\pi_{[-n,\infty)}(\mu),n}(T)$ is then a function on this probability space,

$$\tilde{\mathcal{S}}^{\pi_{[-n,\infty)}(\mu),n}(T) : \prod_{i=-n}^{\infty} \mathbb{R}_{\geq 0}^{\mathbb{N}} \to \widehat{\mathrm{Sig}}_{\infty}$$

for any $T \geq 0$. Therefore

$$\prod_{n \geq 1} \tilde{\mathcal{S}}^{\pi_{[-n,\infty)}(\mu),n}(T) \circ \mathrm{Proj}_{[-n,\infty)} : \widetilde{\Omega} \to \prod_{n \geq 1} \widehat{\mathrm{Sig}}_{\infty}$$

defines a coupling of all random variables $\{\tilde{\mathcal{S}}^{\pi_{[-n,\infty)}(\mu),n}(T) : n \geq 1\}$ on $\widetilde{\Omega}$, where $\mathrm{Proj}_{[-n,\infty)}$ denotes projection onto coordinates $-n, -n+1, \ldots$. For each $\omega \in \widetilde{\Omega}$ we denote by $(\mathcal{S}_i^{\pi_{[-n,\infty)}(\mu),n}(T))(\omega) \in \bar{\mathbb{Z}}$ the corresponding coordinate of $\mathcal{S}^{\pi_{[-n,\infty)}(\mu),n}(T)$ under ω. Finally, we may define the desired object.

Definition 53. For any $\mu \in \widehat{\mathrm{Sig}}_{2\infty}$, we define the continuous-time stochastic process

$\mathcal{S}_T^{\mu,2\infty}, T \geq 0$ on $\widehat{\mathrm{Sig}}_{2\infty}$ by setting

$$\begin{aligned} \mathcal{S}^{\mu,2\infty}(T) : \widetilde{\Omega} &\to \widehat{\mathrm{Sig}}_{2\infty} \\ \omega &\mapsto \Big(\lim_{n\to\infty} (\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)(\mu),n}}(T))(\omega) \Big)_{i\in\mathbb{Z}} \end{aligned} \tag{7.1.4}$$

for each $T \geq 0$.

We note that the limit must be taken along $n \in \mathbb{Z}_{\geq -i}$, as $\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)(\mu),n}}(T))(\omega)$ is only well-defined if $n \geq -i$.

Proposition 7.1.2. *For any $T \geq 0$ and $\omega \in \widetilde{\Omega}$, the limit* (7.1.4) *exists and defines a $\widehat{\mathrm{Sig}}_{2\infty}$-valued random variable*[1]*. Furthermore, the resulting stochastic process in $T \geq 0$ is Markov.*

We first establish a preparatory lemma. This

Lemma 7.1.3. *For every $n \in \mathbb{Z}_{\geq 1}, \omega \in \widetilde{\Omega}, T \in \mathbb{R}_{\geq 0}, i \in \mathbb{Z}_{\geq -n}$, the inequality*

$$(\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}(T))(\omega) \geq (\tilde{\mathcal{S}}_i^{\pi_{[-n-1,\infty)}(\mu),n+1}(T))(\omega) \tag{7.1.5}$$

holds.

Lemma 7.1.3 is a purely deterministic/combinatorial fact, and the idea behind it is that $\tilde{\mathcal{S}}_T^{\cdot,n+1}$ has an extra particle in front compared to $\tilde{\mathcal{S}}_T^{\cdot,n}$, which may block the others but will never bring them further ahead. It holds for the half-infinite processes $\tilde{\mathcal{S}}$ but not for the finite n approximations $\mathcal{S}^{\pi_{[-n,n]}(\mu),n}$, as these do not account for pushing by higher-indexed particles. This is the main reason we use the former process rather than the latter in our construction.

Proof of Lemma 7.1.3. Since $\omega \in \widetilde{\Omega}$, the clocks $-n-1, -n, -n+1, \ldots$ only ring a finite number of times in any interval $[0, T]$. Additionally, the lemma clearly holds at time $T = 0$. Hence it suffices to show that if (7.1.5) is true for each i before a given clock rings, then it is also true for each i after that clock rings, for then we may induct on the (finite, by above) number of rings. Let $T \geq 0$ be such that (7.1.5) holds at time T, and under the event ω exactly one clock rings on the interval $[T, T+\epsilon]$.

[1] i.e. it is measurable in the σ-algebra on $\widehat{\mathrm{Sig}}_{2\infty} \subset \bar{\mathbb{Z}}^{\mathbb{Z}}$ inherited from the product σ-algebra, where each $\bar{\mathbb{Z}}$ factor has the discrete σ-algebra

If the strict inequality case of (7.1.5) holds for a given i before the clock rings (i.e. at time T), then clearly (7.1.5) still holds after at time $T+\epsilon$ because the $\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}$ can change by at most 1 when any clock rings. So it remains to consider the case where the equality case

$$(\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}(T))(\omega) = (\tilde{\mathcal{S}}_i^{\pi_{[-n-1,\infty)}(\mu),n+1}(T))(\omega), \tag{7.1.6}$$

of (7.1.5) holds for some index i at time T, and the $(n+1)^{th}$ approximation has a jump at the same index,

$$(\tilde{\mathcal{S}}_i^{\pi_{[-n-1,\infty)}(\mu),n+1}(T+\epsilon))(\omega) = (\tilde{\mathcal{S}}_i^{\pi_{[-n-1,\infty)}(\mu),n+1}(T))(\omega) + 1. \tag{7.1.7}$$

To show that (7.1.5) continues to hold at time $T+\epsilon$, we must show that this jump occurs at the same location for the n^{th} approximation,

$$(\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}(T+\epsilon))(\omega) = (\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}(T))(\omega) + 1 \tag{7.1.8}$$

The clock that rings to induce the jump (7.1.7) must be the j^{th} clock, for some $j \geq i$ for which $(\tilde{\mathcal{S}}_j^{\pi_{[-n-1,\infty)}(\mu),n+1}(T))(\omega) = (\tilde{\mathcal{S}}_i^{\pi_{[-n-1,\infty)}(\mu),n+1}(T))(\omega)$, by the definition of our dynamics. Since (7.1.5) held before the jump, we have

$$\begin{aligned}
(\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}(T))(\omega) &\geq (\tilde{\mathcal{S}}_j^{\pi_{[-n,\infty)}(\mu),n}(T))(\omega) \\
&\geq (\tilde{\mathcal{S}}_j^{\pi_{[-n-1,\infty)}(\mu),n+1}(T))(\omega) \\
&= (\tilde{\mathcal{S}}_i^{\pi_{[-n-1,\infty)}(\mu),n+1}(T))(\omega) \\
&= (\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}(T))(\omega)
\end{aligned} \tag{7.1.9}$$

(using (7.1.6)), so all above inequalities must be equalities. It follows that the particle of $\tilde{\mathcal{S}}_T^{\pi_{[-n,\infty)},n}$ which jumps on $[T, T+\epsilon]$ began at position $(\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}(T))(\omega)$ rather than some other one. Hence one of the following must be true: (a) (7.1.8) holds, or (b) $i > -n$ and $(\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}(T))(\omega) = (\tilde{\mathcal{S}}_{i-1}^{\pi_{[-n,\infty)}(\mu),n}(T))(\omega)$ (for then $\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}$ is blocked by $\tilde{\mathcal{S}}_{i-1}^{\pi_{[-n,\infty)}(\mu),n}$).

Suppose for the sake of contradiction that (b) holds. Then since (7.1.5) holds for $i-1$

at time T by inductive hypothesis,

$$
\begin{aligned}
(\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}(T))(\omega) &= (\tilde{\mathcal{S}}_{i-1}^{\pi_{[-n,\infty)}(\mu),n}(T))(\omega) \\
&\geq (\tilde{\mathcal{S}}_{i-1}^{\pi_{[-n-1,\infty)}(\mu),n+1}(T))(\omega) \\
&\geq (\tilde{\mathcal{S}}_i^{\pi_{[-n-1,\infty)}(\mu),n+1}(T))(\omega) \\
&= (\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}(T))(\omega),
\end{aligned}
\tag{7.1.10}
$$

so again all inequalities must be equalities and

$$
(\tilde{\mathcal{S}}_{i-1}^{\pi_{[-n-1,\infty)}(\mu),n+1}(T))(\omega) = (\tilde{\mathcal{S}}_i^{\pi_{[-n-1,\infty)}(\mu),n+1}(T))(\omega). \tag{7.1.11}
$$

Since only one jump occurs on the interval $[T, T+\epsilon]$, (7.1.7) and (7.1.11) imply that

$$
(\tilde{\mathcal{S}}_i^{\pi_{[-n-1,\infty)}(\mu),n+1}(T+\epsilon))(\omega) = (\tilde{\mathcal{S}}_{i-1}^{\pi_{[-n-1,\infty)}(\mu),n+1}(T+\epsilon))(\omega) + 1, \tag{7.1.12}
$$

which violates the weakly decreasing order. Hence (b) cannot hold, so (7.1.8) holds, which completes the proof. □

Proof of Proposition 7.1.2. We show that for any $\omega \in \widetilde{\Omega}, i \in \mathbb{Z}, T \in \mathbb{R}_{\geq 0}$, the limit

$$
\lim_{n\to\infty} (\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)(\mu),n}}(T))(\omega) \tag{7.1.13}
$$

exists.

The sequence $((\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}(T))(\omega))_{n\geq -i}$ is bounded below by $(\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}(0))(\omega)$ (which is independent of $n \geq -i$), because coordinates of $\tilde{\mathcal{S}}_T^{\cdot;n}$ are nondecreasing in time. Since $((\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}(T))(\omega))_{n\geq -i}$ is also decreasing in n by Lemma 7.1.3, it is immediate that the limit (7.1.13) exists. Hence $\mathcal{S}^{\mu,2\infty}(T)$ is well-defined. Furthermore, each coordinate $\mathcal{S}_i^{\mu,2\infty}(T)$ is a limit of measurable functions $\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}(T) : \widetilde{\Omega} \to \bar{\mathbb{Z}}$ and hence measurable, so $\mathcal{S}^{\mu,2\infty}(T)$ is measurable with respect to the product σ-algebra on $\bar{\mathbb{Z}}^{\mathbb{Z}}$.

We now show $\mathcal{S}^{\mu,2\infty}(T)$ is Markov, which holds by the following facts:

- For any fixed $T \geq 0$, $\mathcal{S}^{\mu,2\infty}(T)$ is determined by $(\tilde{\mathcal{S}}^{\pi_{[-n,\infty)}(\mu),n})_{n\geq 1}(T)$ by the above.
- For $s \geq 0$ and for each $n \geq 1$, $\tilde{\mathcal{S}}^{\pi_{[-n,\infty)}(\mu),n}(T+s)$ is determined by $\tilde{\mathcal{S}}^{\pi_{[-n,\infty)}(\mu),n}(T)$ together with the complete data of which clocks ring when on the interval $[T, T+s]$,

by definition.

- The complete data of which clocks ring when on the interval $[T, T+s]$ is independent of everything earlier, by the memoryless property of exponential distributions.

This completes the proof. $\square$

We now prove that our construction satisfies the property stated in the introduction as Theorem 1.5.1, that it is the bulk limit of the processes $\mathcal{S}^{\nu,n}$. We in fact prove a slightly more general statement which allows arbitrary initial conditions and gives almost-sure convergence, from which Theorem 1.5.1 follows by taking $\mu = (0[2\infty])$.

Proposition 7.1.4. *For any $\mu \in \widehat{\mathrm{Sig}}_{2\infty}$, there exists a stochastic process $\mathcal{S}^{\mu,2\infty}(T), T \geq 0$, with $\mathcal{S}^{\mu,2\infty}(0) = \mu$, which is a bulk limit of the processes $\mathcal{S}^{\nu,n}$ above in the following sense. The processes $\mathcal{S}^{(\mu_{1-r_n},\ldots,\mu_{n-r_n}),n}(T), n \geq 1$ may be coupled on $\widetilde{\Omega}$ such that for any $D \in \mathbb{N}$, $T_1 \in \mathbb{R}_{\geq 0}$ and sequence of 'bulk observation points' $r_n, n \geq 1$ with $r_n \to \infty$ and $n - r_n \to \infty$,*

$$(\mathcal{S}_{r_n-D}^{(\mu_{1-r_n},\ldots,\mu_{n-r_n}),n}(t^{-r_n}T), \ldots, \mathcal{S}_{r_n+D}^{(\mu_{1-r_n},\ldots,\mu_{n-r_n}),n}(t^{-r_n}T)) \to (\mathcal{S}_{-D}^{\mu,2\infty}(T), \ldots, \mathcal{S}_{D}^{\mu,2\infty}(T)) \tag{7.1.14}$$

almost surely for all $0 \leq T \leq T_1$.

Proof. We couple $\mathcal{S}^{(\mu_{1-r_n},\ldots,\mu_{n-r_n}),n}(T), n \geq 1$ on $\widetilde{\Omega}$ in the obvious way, namely by defining

$$\mathcal{S}^{(\mu_{1-r_n},\ldots,\mu_{n-r_n}),n}(T) : \pi_{[1-r_n,n-r_n]}(\widetilde{\Omega}) \to \widehat{\mathrm{Sig}}_n \tag{7.1.15}$$

by identifying the n coordinates of $\pi_{[1-r_n,n-r_n]}(\widetilde{\Omega})$ with the clock times of the n particles of $\mathcal{S}^{(\mu_{1-r_n},\ldots,\mu_{n-r_n}),n}$. Similarly, we have the coupling

$$\tilde{\mathcal{S}}^{(\mu_{1-r_n},\mu_{2-r_n},\ldots),r_n-1}(T) : \pi_{[1-r_n,\infty)}(\widetilde{\Omega}) : \widehat{\mathrm{Sig}}_\infty\,. \tag{7.1.16}$$

For each $\omega \in \widetilde{\Omega}$, there exists an index j_0 such that clocks $j_0, j_0+1, \ldots$ do not ring on the interval $[0, T_1]$. Hence as long as $n - r_n \geq j_0$,

$$\mathcal{S}^{(\mu_{1-r_n},\ldots,\mu_{n-r_n}),n}(T)(\omega) = \pi_{[1,n]}\left(\tilde{\mathcal{S}}^{(\mu_{1-r_n},\mu_{2-r_n},\ldots),r_n-1}(T)(\omega)\right) \tag{7.1.17}$$

for any $T \in [0, T_1]$. Because $n - r_n \to \infty$, this is true for all sufficiently large n. Since $r_n \to \infty$,

$$\lim_{n\to\infty} \tilde{\mathcal{S}}_i^{(\mu_{1-r_n},\mu_{2-r_n},\ldots),r_n-1}(T) = \lim_{n\to\infty} \tilde{\mathcal{S}}_i^{(\mu_{-n},\mu_{-n+1},\ldots),n}(T) = \mathcal{S}^{2\infty,\mu}(T). \tag{7.1.18}$$

Combining (7.1.18) with (7.1.17) completes the proof. □

Definition 54. One may identify the set Sig_{edge} of (1.5.2) with

$$\{\nu \in \widehat{\mathrm{Sig}}_{2\infty} : \nu_i \in \mathbb{Z} \text{ for } i \leq 0 \text{ and } \nu_1 = \nu_2 = \ldots = -\infty\}. \tag{7.1.19}$$

For any $\mu \in \mathrm{Sig}_{edge}$, letting $\hat{\mu} \in \widehat{\mathrm{Sig}}_{2\infty}$ be its image under the above map, we define

$$\mathcal{S}^{\mu,edge}(T) = \mathcal{S}^{\hat{\mu},2\infty}(T). \tag{7.1.20}$$

Some properties of $\mathcal{S}^{\mu,2\infty}(T)$ will be useful later.

Definition 55. For any $d \in \mathbb{Z}$ we define $F_d : \widehat{\mathrm{Sig}}_{2\infty} \to \widehat{\mathrm{Sig}}_{2\infty}$ by

$$F_d((\mu_n)_{n\in\mathbb{Z}}) = (\min(\mu_n, d))_{n\in\mathbb{Z}}.$$

We define F_d on $\widehat{\mathrm{Sig}}_\infty$ and $\widehat{\mathrm{Sig}}_n$ in exactly the same way.

Proposition 7.1.5. *For any $d \in \mathbb{Z}$ and $\mu \in \widehat{\mathrm{Sig}}_\infty$, $F_d(\mathcal{S}^{\mu,2\infty}(T))$ is a Markov process.*

Proof. It is clear from Definition 2 and Definition 50 that $F_d(\tilde{\mathcal{S}}^{\nu,n}(T))$ is a Markov process for any $\nu \in \widehat{\mathrm{Sig}}_\infty$. Clearly $F_d(\mathcal{S}^{\mu,2\infty}(T))$ is a limit of $F_d(\tilde{\mathcal{S}}^{\pi_{[-n,\infty]}(\mu),n}(T))$, by the same proof as Proposition 7.1.2, and the Markov property is inherited by the limit as in that proof. □

Note that if μ has a part $\mu_i \geq d$, then only the parts $F_d(\mathcal{S}^{\mu,2\infty})_j, j > i$ can evolve, leading to a much simpler process because the sum of their jump rates is finite. The following result, which we have stated in terms of Markov generators because we will need this later, says informally that if μ has a part $\geq d$, then $F_d(\mathcal{S}^{\mu,2\infty})$ evolves by the same reflecting Poisson dynamics as the prelimit process. This will be extremely useful for random matrix results, as for such μ we may check convergence to $F_d(\mathcal{S}^{\mu,2\infty})$ by taking asymptotics of generators/transition matrices.

Proposition 7.1.6. *Let $d \in \mathbb{N}$ and let $\mu \in \widehat{\mathrm{Sig}}_{2\infty}$ be such that*

$$i_0 := \mu'_d + 1 = \inf(\{i \in \mathbb{Z} : \mu_i < d\}) > -\infty, \tag{7.1.21}$$

and let

$$N := \begin{cases} \infty & \mu_i > -\infty \text{ for all } i \\ \max(\{i : \mu_i > -\infty\}) & \text{else} \end{cases} \tag{7.1.22}$$

Let $Q : F_d(\widehat{\mathrm{Sig}}_{2\infty}) \to F_d(\widehat{\mathrm{Sig}}_{2\infty})$ be the matrix defined by

$$Q(\nu, \kappa) = \begin{cases} -\frac{t^{\nu'_d+1} - t^{N+1}}{1-t} & \kappa = \nu \\ \frac{t^i(1-t^{m_{\nu_i}(\nu)})}{1-t} & \text{there exists } i \in \mathbb{Z} \text{ such that } \kappa_j = \nu_j + \mathbb{1}(j = i) \text{ for all } j \in \mathbb{Z} \\ 0 & \text{else} \end{cases} \tag{7.1.23}$$

for all $\nu, \kappa \in F_d(\widehat{\mathrm{Sig}}_{2\infty})$. Then the matrix exponential $e^{TQ} : F_d(\widehat{\mathrm{Sig}}_{2\infty}) \to F_d(\widehat{\mathrm{Sig}}_{2\infty})$ is well-defined, and

$$\Pr(F_d(\mathcal{S}^{\mu,2\infty}(T + T_0)) = \kappa | F_d(\mathcal{S}^{\mu,2\infty}(T_0)) = \nu) = (e^{TQ})(\nu, \kappa). \tag{7.1.24}$$

Proof. Applying F_d to Definition 53,

$$F_d(\mathcal{S}^{\mu,2\infty}(T)) = \lim_{n\to\infty} F_d(\tilde{\mathcal{S}}^{\pi_{[-n,\infty)}(\mu),n}(T)). \tag{7.1.25}$$

For all $n > -i_0$, it is clear from the definition of $\tilde{\mathcal{S}}$ that the (multi-time joint) law of $F_d(\tilde{\mathcal{S}}^{\pi_{[-n,\infty)}(\mu),n}(T))$ is independent of n. Note that for any ν as above all entries $F_d(\tilde{\mathcal{S}}^{\pi_{[-n,\infty)}(\nu),n}(T))_j, j \leq \nu'_d$ never change because they are already equal to d. Additionally, if N is finite the entries $F_d(\mathcal{S}^{\mu,2\infty}(T))_j, j > N$ do not change because they are equal to $-\infty$. Meanwhile, all entries $F_d(\tilde{\mathcal{S}}^{\pi_{[-n,\infty)}(\mu),n}(T))_j, i_0 \leq j \leq N$ jump according to Poisson clocks of rate t^j as before, until they reach d, at which point they jump no longer. It follows that $F_d(\tilde{\mathcal{S}}^{\pi_{[-n,\infty)}(\mu),n}(T))$ has Markov generator given by (7.1.23), after identifying the state space with $F_d(\widehat{\mathrm{Sig}}_{2\infty})$ by padding with entries d on the left. Hence by (7.1.25), $F_d(\mathcal{S}^{\mu,2\infty}(T))$ also has Markov generator Q. $\square$

Definition 56. Define the *forward shift map*

$$s : \widehat{\mathrm{Sig}}_{2\infty} \to \widehat{\mathrm{Sig}}_{2\infty}$$
$$(\mu_n)_{n\in\mathbb{Z}} \mapsto (\mu_{n+1})_{n\in\mathbb{Z}}$$

Because the i^{th} coordinate $\mu_i(T)$ of $\mathcal{S}^{\mu,2\infty}(T)$ behaves as a Poisson jump process with rate t^i (neglecting interactions with the other coordinates), the i^{th} coordinate of $s(\mathcal{S}^{\mu,2\infty}(T))$ has rate $t^{i+1} = t \cdot t^i$, i.e. s has the effect of slowing down each jump rate by a factor of t. Heuristically this justifies the following.

Proposition 7.1.7. *If $a \in \mathbb{Z}$ and $\mu = (a)_{n\in\mathbb{Z}}$, then*

$$s(\mathcal{S}^{\mu,2\infty}(t^{-1} \cdot T)) = \mathcal{S}_{\mu,2\infty}(T) \tag{7.1.26}$$

in distribution as stochastic processes.

Proof. Define a map

$$\xi : \Omega \to \Omega$$
$$((a_{n,i})_{i\in\mathbb{N}})_{n\in\mathbb{Z}} \mapsto ((t \cdot a_{n+1,i})_{i\in\mathbb{N}})_{n\in\mathbb{Z}}$$

The map ξ scales the waiting times $a_{n,i}$ by t and shifts which coordinate μ_n they correspond to. Since these waiting times are exponential variables with rates in geometric progression with common ratio t under the measure Poiss $\in \mathcal{M}(\Omega)$ defined in the proof of Proposition 7.1.2, it follows that

$$\xi_*(\mathrm{Poiss}) = \mathrm{Poiss}\,. \tag{7.1.27}$$

It is also immediate from the definition of ξ that for any $T \geq 0$ and $\omega \in \widetilde{\Omega}$,

$$(\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}(T))(\omega) = (\tilde{\mathcal{S}}_{i-1}^{\pi_{[-n-1,\infty)}(\mu),n+1}(t^{-1}T))(\xi(\omega)). \tag{7.1.28}$$

Hence clearly

$$\lim_{n\to\infty}(\tilde{\mathcal{S}}_i^{\pi_{[-n,\infty)}(\mu),n}(T))(\omega) = \lim_{n\to\infty}(\tilde{\mathcal{S}}_{i-1}^{\pi_{[-n-1,\infty)}(\mu),n+1}(t^{-1}T))(\xi(\omega)), \tag{7.1.29}$$

and in view of the construction in Definition 53 this implies (7.1.26). □

For completeness, we record how the results proven above yield what was stated in the Introduction.

Proof of Theorem 1.5.1. In Proposition 7.1.2 we show that $\mathcal{S}^{2\infty,\mu}$ is a well-defined Markov process, and the properties (1), (2) and (3) stated in Theorem 1.5.1 follow from Proposition 7.1.4, Proposition 7.1.7 and Proposition 7.1.5 respectively. □

7.1.1 Convergence of measures on $\widehat{\mathrm{Sig}}_{2\infty}$

Having constructed the putative universal object $\mathcal{S}^{\mu,2\infty}$ and shown some basic properties, we now set up what is needed to prove convergence to it. To speak of weak convergence of $\widehat{\mathrm{Sig}}_{2\infty}$-valued random variables, we must at minimum equip $\widehat{\mathrm{Sig}}_{2\infty}$ with a topology. The space $\bar{\mathbb{Z}}$ has a natural topology with open sets generated by finite subsets of $\mathbb{Z}$ together with intervals[2] $[-\infty, n]$ and $[n, \infty]$ for each $n \in \mathbb{Z}$. For concreteness later we note that the closed sets in this topology are those which, if the contain arbitrarily large positive (resp. negative) finite integers, also contain ∞ (resp. $-\infty$).

We now give $\widehat{\mathrm{Sig}}_{2\infty}$ the topology it inherits from the product topology on $\bar{\mathbb{Z}}^{\mathbb{Z}}$, where each $\bar{\mathbb{Z}}$ factor has the topology above. Equivalently, this is the topology of pointwise convergence on $\bar{\mathbb{Z}}^{\mathbb{Z}}$. When we speak of measures on $\widehat{\mathrm{Sig}}_{2\infty}$, we will always mean measures with respect to the Borel σ-algebra determined by this topology. Note that this is just the product σ-algebra of the discrete σ-algebras on each $\bar{\mathbb{Z}}$ factor, which is the one we took earlier in Proposition 7.1.2.

The space $\bar{\mathbb{Z}}$ is second-countable and separable, hence metrizable by Urysohn's theorem, hence the product $\bar{\mathbb{Z}}^{\mathbb{Z}}$ (and therefore $\widehat{\mathrm{Sig}}_{2\infty}$) is metrizable as well. This makes the following two definitions of weak convergence equivalent by the portmanteau theorem.

Definition 57. A sequence of probability measures $(M_n)_{n\geq 1}$ on $\widehat{\mathrm{Sig}}_{2\infty}$ converges weakly to M if, for every $S \subset \widehat{\mathrm{Sig}}_{2\infty}$ which is a continuity set of M (i.e. $M(\partial S) = 0$), $M_n(S) \to M(S)$ as $n \to \infty$. Equivalently, for every continuous $f : \widehat{\mathrm{Sig}}_{2\infty} \to \mathbb{R}$,

$$\int_{\bar{\mathbb{Z}}^{\mathbb{Z}}} f dM_n \to \int_{\bar{\mathbb{Z}}^{\mathbb{Z}}} f dM.$$

[2]In both cases, the interval includes the infinite endpoint, as indicated by the square braces.

We reduce weak convergence to a more checkable, combinatorial condition, which is what we will actually show. For $I \subset \mathbb{Z}$ let $\pi_I : \bar{\mathbb{Z}}^{\mathbb{Z}} \to \bar{\mathbb{Z}}^I$ be the projection.

Lemma 7.1.8. *A sequence of probability measures $(M_n)_{n\geq 1}$ on $\widehat{\mathrm{Sig}}_{2\infty}$ converges weakly to a probability measure M if, for every finite $I \subset \mathbb{Z}$ and $d \in \mathbb{Z}$, the convergence of pushforward measures*

$$((\pi_I)_*(M_n))(\{b\}) \to ((\pi_I)_*(M))(\{b\})$$

holds for every $b \in \bar{\mathbb{Z}}^I$. The same statement holds for measures on $\widehat{\mathrm{Sig}}^k_{2\infty}$ and finite sets $I \subset \mathbb{Z}^k$.

Proof of Lemma 7.1.8. We prove the $k = 1$ case, as the general case is exactly analogous. Let

$$\mathcal{U} = \{\pi_J^{-1}(A) : J \subset \mathbb{Z} \text{ finite, and } A \subset \pi_J(\bar{\mathbb{Z}}^{\mathbb{Z}}) \text{ a product of singleton sets}\} \cup \{\emptyset\}. \tag{7.1.30}$$

We note that (i) $\mathcal{U}$ is closed under finite intersections, and (ii) every open set in $\bar{\mathbb{Z}}^{\mathbb{Z}}$ is a countable union of elements of $\mathcal{U}$, which follows since $\bar{\mathbb{Z}}$ is countable. By [Bil68, Theorem 2.2], the two properties (i), (ii) imply that for weak convergence $M_n \to M$, it suffices to check that

$$M_n(U) \to M(U) \tag{7.1.31}$$

for every $U \in \mathcal{U}$; this captures the intuitive notion that $\mathcal{U}$ is a 'large enough' collection of sets to determine weak convergence. For a set $\pi_J^{-1}(A)$ as in (7.1.30), by definition

$$((\pi_J)_*(M_n))(A) = M_n(\pi_J^{-1}(A))$$

and similarly for M, therefore our hypothesis implies (7.1.31), which completes the proof.

□

We note that the converse of Lemma 7.1.8 is not true. For instance, the Dirac delta measure at $(D)_{n\in\mathbb{Z}}$ converges weakly as $D \to \infty$ to the Dirac delta measure at $(\infty)_{n\in\mathbb{Z}}$, but the set $\pi_0^{-1}(\{\infty\})$ (which is *not* a continuity set of the latter measure) has probability 0 under the former measures and probability 1 under the latter measure.

7.2 Main theorem statement and comments

We wish to talk about random finite-length partitions—singular numbers of the matrix product process—converging to random elements of $\widehat{\mathrm{Sig}}_{2\infty}$, so it is convenient to define an embedding of $\mathbb{Y}_N$ into $\widehat{\mathrm{Sig}}_{2\infty}$.

Definition 58. For $\lambda \in \mathbb{Y}_N$ define

$$\iota_n(\lambda) = \begin{cases} \infty & n \leq 0 \\ \lambda_n & 1 \leq n \leq N \\ -\infty & n > N \end{cases},$$

and let

$$\begin{aligned} \iota : &\mathbb{Y}_N \hookrightarrow \widehat{\mathrm{Sig}}_{2\infty} \\ &(\lambda_1, \ldots, \lambda_N) \mapsto (\iota_n(\lambda))_{n \in \mathbb{Z}}. \end{aligned}$$

We are now able to state the main dynamical result, which in the bulk case we will later augment to include the single-time marginal as well. It applies to both the bulk and edge: the sequence $(r_N)_{N \geq 1}$, which represents 'observation points' of the matrix product process, may be taken such that $0 \ll r_N \ll N$ for a bulk limit, or $r_N = N - k$ for fixed k for an edge limit.

Theorem 7.2.1. *For each $N \in \mathbb{N}$, let $A_i^{(N)}, i \geq 1$ be an iid sequence of $\mathrm{GL}_N(\mathbb{Z}_p)$-bi-invariant random matrices in $\mathrm{Mat}_N(\mathbb{Z}_p)$, and let r_N be a 'bulk observation point' such that r_N and the random variable*

$$X_N := \mathrm{corank}(A_i^{(N)} \pmod p) \tag{7.2.1}$$

satisfy

(i) $r_N \to \infty$ *as* $N \to \infty$,

(ii) $\Pr(X_N = 0) < 1$ *for every* N, *and*

(iii) X_N *is far away from* r_N *with high probability, in the sense that for every* $j \in \mathbb{N}$,

$$\Pr(X_N > r_N - j | X_N > 0) \to 0 \qquad \text{as } N \to \infty. \tag{7.2.2}$$

Let $\mu \in \widehat{\mathrm{Sig}}_{2\infty}^{+}$ *be such that* $\lim_{n\to\infty} \mu_{-n} = \infty$, *and let* $B^{(N)} \in \mathrm{Mat}_N(\mathbb{Z}_p), N \geq 1$ *be any fixed matrices with singular numbers around* r_N *matching* μ, *i.e. for every* $i \in \mathbb{Z}$

$$(s^{r_N} \circ \iota(\mathrm{SN}(B^{(N)})))_i = \mu_i$$

for all sufficiently large N. *Define the prelimit matrix product process* $\Pi^{(N)}(\tau) = \mathrm{SN}(A_\tau^{(N)} \cdots A_1^{(N)} B^{(N)})$ *for* $\tau \in \mathbb{Z}_{\geq 0}$, *and the shifted version on* $\widehat{\mathrm{Sig}}_{2\infty}$

$$\Lambda^{(N)}(T) := s^{r_N} \circ \iota(\Pi^{(N)}(\lfloor c_N T \rfloor)), T \in \mathbb{R}_{\geq 0}$$

with time change given by

$$c_N := \frac{t^{-r_N}}{\mathbb{E}[t^{-\operatorname{len}(\mathrm{SN}(A_i^{(N)}))} - 1]} \qquad N = 1, 2, \ldots \tag{7.2.3}$$

Then we have convergence

$$\Lambda^{(N)}(T) \xrightarrow{N\to\infty} \mathcal{S}^{\mu,2\infty}(T) \tag{7.2.4}$$

in finite-dimensional distribution.

Many remarks on Theorem 7.2.1 are in order. First of all, the hypothesis (7.2.1) is not the same as what was given in the Introduction. The latter was in fact a stronger hypothesis, as we show now.

Proposition 7.2.2. *Let* $(r_N)_{N\in\mathbb{N}}$ *be a sequence with* $r_N \to \infty$ *and* $r_N \leq N$. *For each* N *let* X_N *be a random variable taking values in* $[[N]]$, *such that for every* $j \in \mathbb{Z}$ *we have*

$$\Pr(X_N \geq r_N + j) = o(\Pr(X_N \geq 1)) \qquad \text{as } N \to \infty. \tag{7.2.5}$$

Then

$$\Pr(X_N \geq r_N + j) = o(\mathbb{E}[\mathbb{1}(X_N \leq r_N)(t^{r_N - X_N} - t^{r_N})]) \tag{7.2.6}$$

for every $j \in \mathbb{Z}$.

Proof. Since $(t^{r_N - X_N} - t^{r_N}) = 0$ when $X_N = 0$,

$$\mathbb{1}(X_N \leq r_N)(t^{r_N - X_N} - t^{r_N}) \leq \mathbb{1}(1 \leq X_N \leq r_N) \leq \mathbb{1}(X_N \geq 1), \tag{7.2.7}$$

hence

$$\mathbb{E}[\mathbb{1}(X_N \leq r_N)(t^{r_N - X_N} - t^{r_N})] \leq \Pr(X_N \geq 1). \tag{7.2.8}$$

□

Proof of Theorem 1.5.2. By Proposition 7.2.2, the hypothesis in Theorem 1.5.2 implies the one in Theorem 7.2.1, and the convergence (7.2.4) clearly implies the version in Theorem 1.5.2. □

Proof of Theorem 1.5.3. Exactly as for Theorem 1.5.2, taking $r_N = N$ in Theorem 7.2.1 and using the natural inclusion $\mathrm{Sig}_{edge} \hookrightarrow \widehat{\mathrm{Sig}}_{2\infty}$ taking $(\mu_i)_{i \in \mathbb{Z}_{\leq 0}}$ to $(\ldots, \mu_{-1}, \mu_0, -\infty, -\infty, \ldots)$. □

One might also wonder where the definition of c_N came from; why $\mathbb{1}(X_N \leq r_N)$ rather than, say, $\mathbb{1}(X_N \leq r_N - 1)$? We show that this is simply a matter of convenience and our hypothesis guarantee that any cutoff near r_N will give the same result.

Proposition 7.2.3. *Suppose r_N and X_N are such that for every $j \in \mathbb{Z}$,*

$$\Pr(X_N \geq r_N + j) = o(\mathbb{E}[\mathbb{1}(X_N \leq r_N)(t^{r_N - X_N} - t^{r_N})]) \qquad \text{as } N \to \infty. \tag{7.2.9}$$

Then for every $j \in \mathbb{Z}$,

$$\mathbb{E}[\mathbb{1}(X_N \leq r_N + j)(t^{r_N - X_N} - t^{r_N})] = (1 + o(1))\mathbb{E}[\mathbb{1}(X_N \leq r_N)(t^{r_N - X_N} - t^{r_N})]. \tag{7.2.10}$$

Proof. We will prove the case $j > 0$, as the case $j < 0$ is the same after replacing r_N by $r_N - j$. It suffices to show

$$\mathbb{E}[\mathbb{1}(r_N < X_N \leq r_N + j)(t^{r_N - X_N} - t^{r_N})] = o(1)\mathbb{E}[\mathbb{1}(X_N \leq r_N)(t^{r_N - X_N} - t^{r_N})]. \tag{7.2.11}$$

Since

$$\mathbb{E}[\mathbb{1}(r_N < X_N \le r_N + j)(t^{r_N - X_N} - t^{r_N})] \le t^{-j}\Pr(r_N < X_N \le r_N + j) \le t^{-j}\Pr(X_N > r_N), \tag{7.2.12}$$

which is $o(\mathbb{E}[\mathbb{1}(X_N \le r_N)(t^{r_N - X_N} - t^{r_N})])$ by (7.2.9), (7.2.11) follows. □

7.3 Reducing Theorem 7.2.1 to Markov generator asymptotics

Our goal is to understand the asymptotic dynamics of singular numbers $\Pi^{(N)}(\tau) = \mathrm{SN}(A_\tau \cdots A_1)$ under matrix products $A_1, A_2, \ldots \in \mathrm{Mat}_N(\mathbb{Z}_p)$ in an 'observation window' around some r_N, i.e. $\Pi_i^{(N)}(\tau)$ where $i = r_N + const$. It is helpful to view the $\Pi_i^{(N)}(\tau)$ as a collection of particles on $\mathbb{Z}$, which may inhabit the same location, and which 'jump' in discrete time τ by $\Pi_i^{(N)}(\tau+1) - \Pi_i^{(N)}(\tau)$ at each 'time step' $\tau \mapsto \tau+1$. To establish a continuous-time Poisson-type limit of this evolution, we show the following:

1. With probability $1 - O(p^{-r_N})$, none of the singular numbers $\Pi_i^{(N)}(\tau), i \approx r_N$ change under the time step $\tau \mapsto \tau + 1$ (and in fact, we see this is true for all $i \ge r_N$ as well).

2. For each $i \approx r_N$, we show the probability $\Pi_i^{(N)}(\tau)$ jumps at a given time step is $cp^{-i} + O(p^{-2r_N})$ for c independent of i which we explicitly compute, in the case when $\Pi_i^{(N)}(\tau)$ is not equal to any other part of $\lambda(\tau)$, and otherwise is given by a slightly different formula since multiple parts may push one another. This leads to the jump rates of the continuous-time process seen in Theorem 7.2.1.

3. We show that the probability that more than one jump occurs among $i \approx r_N$ is $O(p^{-2r_N})$ and hence may be discounted.

This section contains the abstract nonsense portion of the proof of Theorem 7.2.1. We first state three lemmas about random matrices, which correspond to (1), (2) and (3) of the above sketch and contain all of the needed hard computations, and then show how they imply Theorem 7.2.1. The proofs of the lemmas themselves will be deferred to Section 7.5.

Definition 59. Let $d \in \mathbb{N}$ and let A be a random element of $\mathrm{Mat}_N(\mathbb{Z}_p)$ with law invariant under $\mathrm{GL}_N(\mathbb{Z}_p)$ on the right. Then we define the Markov transition matrix on pairs $\kappa, \nu \in F_d(\mathrm{Sig}_N^+)$ by

$$P_{A,d}(\nu, \kappa) := \Pr(F_d(\mathrm{SN}(A^{(N)}B)) = \kappa), \tag{7.3.1}$$

where B is any matrix with $\mathrm{SN}(B) = \nu$.

Lemma 7.3.1. *Let $(r_N)_{N\in\mathbb{N}}$ be a sequence with $r_N \le N$ and $r_N \to \infty$, and for each $N \in \mathbb{N}$ let $A^{(N)}$ be a $\mathrm{GL}_N(\mathbb{Z}_p)$-right-invariant random matrix in $\mathrm{Mat}_N(\mathbb{Z}_p)$ with $\Pr(A^{(N)} \in \mathrm{GL}_N(\mathbb{Z}_p)) < 1$ and*

$$\Pr(\mathrm{corank}(A^{(N)} \pmod p) \ge r_N - j) = o(c_N^{-1}) \qquad \text{for all } j \ge 0 \tag{7.3.2}$$

where

$$c_N^{-1} := \mathbb{E}[\mathbb{1}(\mathrm{corank}(A^{(N)} \pmod p) \le r_N)(t^{r_N - \mathrm{corank}(A^{(N)} \pmod p)} - t^{r_N})]. \tag{7.3.3}$$

Fix $d \in \mathbb{N}$ and let $P_{A^{(N)},d}$ be as in Definition 59. Then as $N \to \infty$,

$$P_{A^{(N)},d}(\nu^{(N)}, \nu^{(N)}) = 1 - \frac{t^{(\nu^{(N)})'_k+1} - t^{N-r_N+1}}{1-t} c_N^{-1} + o(c_N^{-1}). \tag{7.3.4}$$

Furthermore, for any $L \in \mathbb{Z}$ the implied constant is uniform over all $\nu^{(N)} \in F_d(\mathrm{Sig}_N^+)$ with $(\nu^N)'_k \ge L + r_N$.

Remark 42. Note that we do not require the asymptotic in Lemma 7.3.1 and below to be uniform over choices of the distribution of $A^{(N)}$ or the sequence $(r_N)_{N\in\mathbb{N}}$ which we assume to be fixed. Also, we will not need the uniformity of implied constants for Theorem 7.2.1, but we will need it for upcoming results, so we prove it here.

Lemma 7.3.2. *Assume the same setup as Lemma 7.3.1. Then for any sequence of pairs $\nu^{(N)}, \kappa^{(N)} \in F_d(\mathrm{Sig}_N^+)$ with $\nu^{(N)} \prec \kappa^{(N)}$ and $|\kappa^{(N)}/\nu^{(N)}| = 1$,*

$$P_{A^{(N)},d}(\nu^{(N)}, \kappa^{(N)}) = t^{j(N)} \frac{1 - t^{m_{\nu_j}(\nu)}}{1-t} c_N^{-1} + o(c_N^{-1}) \tag{7.3.5}$$

where $j = j(N)$ is the unique index such that $\kappa_j^{(N)} = \nu_j^{(N)} + 1$, and implied constant in (7.3.5) *is uniform over all such sequences of pairs $\nu^{(N)}, \kappa^{(N)}$ with $(\nu^{(N)})'_k \ge L + r_N$.*

Lemma 7.3.3. *Assume the same setup as Lemma 7.3.1. Then*

$$P_{A^{(N)},d}(\nu^{(N)}, \{\kappa \in F_d(\widehat{\mathrm{Sig}}_N) : \kappa \supset \nu^{(N)} \text{ and } |\kappa/\nu^{(N)}| \geq 2\}) = o(c_N^{-1}) \tag{7.3.6}$$

uniformly over all sequences $\nu^{(N)} \in \mathrm{Sig}_N^+, N \geq 1$ *with* $(\nu^{(N)})'_k \geq L$.

Now we show Theorem 7.2.1 conditional on the above lemmas. As a technical convenience, we work with slightly different prelimit processes.

Definition 60. In the setting of Theorem 7.2.1, define the (shifted) discrete-time singular number process $\tilde{\Pi}^{(N)}(\tau) = (\tilde{\Pi}^{(N)}(\tau)_i)_{i\in\mathbb{Z}}, \tau \in \mathbb{Z}_{\geq 0}$ on $\widehat{\mathrm{Sig}}_{2\infty}$ by

$$\tilde{\Pi}^{(N)}(\tau) = \begin{cases} \infty & i < 1 - r_N \\ \Pi^{(N)}(\tau)_{i+r_N} & 1 - r_N \leq i \leq N - r_N \\ \mu_i & i > N - r_N \end{cases}. \tag{7.3.7}$$

Define the continuous-time version by

$$\tilde{\Lambda}^{(N)}(T) = (\tilde{\Lambda}_i^{(N)}(T))_{i\in\mathbb{Z}} = \tilde{\Pi}^{(N)}(\lfloor c_N T \rfloor). \tag{7.3.8}$$

In other words, $\tilde{\Pi}^{(N)}$ agrees with $\Pi^{(N)}$ on all coordinates $i \leq N - r_N$, and all later coordinates are the same as those of μ and do not change as time T increases. The process $\tilde{\Lambda}^{(N)}(T)$ has the advantage that $F_d(\tilde{\Lambda}^{(N)}(0)) = F_d(\mu)$ whenever N is large enough so that $\mathrm{SN}(B^{(N)})$ has at least one part $\geq d$ (of course, the fact that this is true for large N requires the hypothesis $\lim_{n\to\infty} \mu_{-n} = \infty$ of Theorem 7.2.1), hence $F_d(\tilde{\Lambda}^{(N)}(T))$ and $F_d(\mathcal{S}^{\mu,2\infty}(T))$ have the same initial condition.

Lemma 7.3.4. *To prove Theorem 7.2.1, it suffices to prove that under the same hypotheses,*

$$F_d(\tilde{\Lambda}^{(N)}(T)) \xrightarrow{N\to\infty} F_d(\mathcal{S}^{\mu,2\infty}(T))$$

in finite-dimensional distribution for any $d \in \mathbb{N}$.

Proof. To show Theorem 7.2.1 it suffices to show that for any sequence of times $0 \leq T_1 <$

$\ldots < T_k$, we have convergence of random vectors

$$(\Lambda^{(N)}(T_1), \ldots, \Lambda^{(N)}(T_k)) \to (\mathcal{S}^{\mu,2\infty}(T_1), \ldots, \mathcal{S}^{\mu,2\infty}(T_k))$$

in distribution. By Lemma 7.1.8 it suffices to check convergence of measures on sets $\prod_{i=1}^k \pi_{I_i}^{-1}(\{b_i\})$ with $b_i = (b_{i,1}, \ldots, b_{i,|I_i|}) \in \bar{\mathbb{Z}}^{I_i}$. For any projection π_J to finitely many coordinates, $\pi_J(\Lambda^{(N)}) = \pi_J(\tilde{\Lambda}^{(N)})$ for all sufficiently large N in terms of J. Hence it suffices to show

$$(\tilde{\Lambda}^{(N)}(T_1), \ldots, \tilde{\Lambda}^{(N)}(T_k)) \to (\mathcal{S}^{\mu,2\infty}(T_1), \ldots, \mathcal{S}^{\mu,2\infty}(T_k))$$

in distribution. Letting d be some integer satisfying

$$d > \sup_{1 \le i \le k} \sup_{1 \le j \le |I_i|} b_{i,j},$$

it therefore suffices to check $F_d(\tilde{\Lambda}^{(N)}(T))$ converges in joint distribution at $T_1, \ldots, T_k$ to $F_d(\mathcal{S}^{\mu,2\infty}(T))$. $\square$

We now wish to prove the desired convergence in finite-dimensional distribution by analyzing the transition matrix and generator, respectively, of the discrete-time process $F_d(\tilde{\Pi}^{(N)}(\tau)), \tau = 0, 1, \ldots$ and the continuous-time process $F_d(\mathcal{S}^{\mu,2\infty}(T))$. For these considerations it is natural to consider a restricted state space, $\Sigma(d, \mu)$, which we now define.

Definition 61. Define a partial order $\subset$ on $\widehat{\mathrm{Sig}}_{2\infty}$ by

$$\nu \subset \kappa \quad \iff \quad \nu_i \le \kappa_i \quad \text{for all } i. \tag{7.3.9}$$

For $\nu \subset \kappa$, we define

$$|\kappa/\nu| = \sum_{i \in \mathbb{Z}} \kappa_i - \nu_i \in \mathbb{Z}_{\ge 0} \cup \{\infty\}, \tag{7.3.10}$$

where in the sum we take the convention that $\infty - \infty = (-\infty) - (-\infty) = 0$ and $\infty - n = n - (-\infty) = \infty$ for all $n \in \mathbb{Z}$. Finally, we set

$$\Sigma(d, \mu) := \{\nu \in F_d(\widehat{\mathrm{Sig}}_{2\infty}) : \nu \supset \mu, |\nu/\mu| < \infty\}.$$

Lemma 7.3.5. *For $\mu \in \widehat{\mathrm{Sig}}_{2\infty}$ with $\lim_{n\to\infty} \mu_{-n} = \infty$ and any $d \in \mathbb{N}$, the Markov process $F_d(\mathcal{S}^{\mu,2\infty}(T))$ remains on $\Sigma(d,\mu)$ for all time with probability 1, and its transition matrix Q (restricted to $\Sigma(d,\mu)$ is upper-triangular with respect to the partial order $\subset$ of Definition 61.*

Proof. By Proposition 7.1.6, the sum of transition rates of $F_d(\mathcal{S}^{\mu,2\infty})$ out of any state is bounded above by the sum of transition rates out of state μ, which is $(t^{i_0} - t^{N+1})/(1-t)$ and hence finite. Therefore with probability 1, $F_d(\mathcal{S}^{\mu,2\infty}(T))$ stays on $\Sigma(d,\mu)$. Upper-triangularity follows from the explicit definition in Proposition 7.1.6. $\square$

Lemma 7.3.6. *In the setting of Theorem 7.2.1, for any $d \in \mathbb{N}$, $F_d(\tilde{\Pi}^{(N)}(\tau))$ is a Markov chain. Furthermore, the set $\widehat{\mathrm{Sig}}_{2\infty} \setminus \Sigma(d,\mu)$ is absorbing for this process, so it projects to a Markov process on $\Sigma(d,\mu) \cup \{\aleph\}$ by identifying all states in $\widehat{\mathrm{Sig}}_{2\infty} \setminus \Sigma(d,\mu)$ with $\aleph$. Finally, the transition matrix $\tilde{P}_N$ of this Markov process is upper-triangular with respect to the partial order $\subset$ of Definition 61.*

Proof. The diagonal entries of the Smith normal form of any $\tilde{A} \in \mathrm{Mat}_N(\mathbb{Z}/p^d\mathbb{Z})$ will lie in $\{1, p, \ldots, p^{d-1}, 0\}$, and so we define $\mathrm{SN}(\tilde{A}) \in \mathrm{Sig}_N$ to have all parts in $\{0, 1, \ldots, d\}$, where all 0 entries in the diagonal of the Smith normal form correspond to parts d. It is then clear that for any $A \in \mathrm{Mat}_N(\mathbb{Z}_p)$,

$$F_d(\mathrm{SN}(A)) = \mathrm{SN}(A \pmod{p^d}). \tag{7.3.11}$$

Since $A^{(N)}_{\lfloor c_N T \rfloor} \cdots A^{(N)}_1 B^{(N)} \pmod{p^d}$ is a product of independent matrices over $\mathbb{Z}/p^d\mathbb{Z}$, $F_d(\tilde{\Pi}^{(N)}(\tau))$

$$s^{r_N} \circ \iota(\mathrm{SN}(A^{(N)}_\tau \cdots A^{(N)}_1 B^{(N)} \pmod{p^d})) \tag{7.3.12}$$

is a Markov process. Because $\tilde{\Pi}^{(N)}(\tau)$ is the same as above except on coordinates $i > N - r_N$, which do not evolve in time under either process, $\tilde{\Pi}^{(N)}(\tau)$ is also a Markov process. Clearly $F_d(\tilde{\Pi}^{(N)}(\tau))$ has upper-triangular transition matrix with respect to $\supset$, since multiplying matrices over $\mathbb{Z}_p$ can only increase their singular numbers. Hence if it ever leaves $\Sigma(d,\mu)$, it will not return, so it projects to a Markov process on $\Sigma(d,\mu) \cup \{\aleph\}$. $\square$

Note that if $d > \lim_{n\to\infty} \mu_{-n}$, then $F_d(\tilde{\Pi}^{(N)}(\tau))_i = d > \mu_i$ for all sufficiently large negative i, hence in fact $F_d(\tilde{\Pi}^{(N)}(\tau))$ lives on $\widehat{\mathrm{Sig}}_{2\infty} \setminus \Sigma(d,\mu)$. When $d \leq \lim_{n\to\infty} \mu_{-n}$, however, it is not hard to see that $F_d(\tilde{\Pi}^{(N)}(\tau))$ will remain on $\Sigma(d,\mu)$ with probability 1, though we find this fact as a consequence of later statements rather than explicitly deriving it. Finally, we may prove the desired result.

Proof of Theorem 7.2.1, assuming Lemma 7.3.1, Lemma 7.3.2, and Lemma 7.3.3. By Lemma 7.3.4, it suffices to show for any sequence of times $0 \leq T_1 < \ldots < T_k$ that

$$(F_d(\tilde{\Lambda}^{(N)}(T_i))_{1\leq i\leq k} \to (F_d(\mathcal{S}^{\mu,2\infty}(T_i)))_{1\leq i\leq k} \tag{7.3.13}$$

weakly as $N \to \infty$. It follows from Proposition 7.1.6 that

$$\begin{aligned} &\Pr(\mathcal{S}^{\mu,2\infty}(T_i) = \nu^{(i)} \text{ for all } 1 \leq i \leq k) \\ &\quad = (e^{T_1 Q})(F_d(\mu),\nu^{(1)})(e^{(T_2-T_1)Q})(\nu^{(1)},\nu^{(2)})\cdots(e^{(T_k-T_{k-1})Q})(\nu^{(k-1)},\nu^{(k)}) \end{aligned} \tag{7.3.14}$$

when all $\nu^{(i)}$ lie in $\Sigma(d,\mu)$, and (7.3.14) is 0 otherwise. Hence to show (7.3.13), by (7.3.14) we must show

$$\tilde{P}_N^{\lfloor c_N T_i\rfloor - \lfloor c_N T_{i-1}\rfloor}(\nu^{(i-1)},\nu^{(i)}) \xrightarrow{N\to\infty} (e^{(T_i-T_{i-1})Q})(\nu^{(i-1)},\nu^{(i)}). \tag{7.3.15}$$

Let $\tilde{P}_{N,i}$ and Q_i be the restrictions of $\tilde{P}_N$ and Q to the finite poset interval $[\nu^{(i-1)},\nu^{(i)}] \subset \Sigma(d,\mu)$. Then by upper-triangularity (see Lemma 7.3.6 and Lemma 7.3.5 respectively),

$$\tilde{P}_{N,i}^{\lfloor c_N T_i\rfloor - \lfloor c_N T_{i-1}\rfloor}(\nu^{(i-1)},\nu^{(i)}) = P_N^{\lfloor c_N T_i\rfloor - \lfloor c_N T_{i-1}\rfloor}(\nu^{(i-1)},\nu^{(i)}) \tag{7.3.16}$$

$$(e^{(T_i-T_{i-1})Q_i})(\nu^{(i-1)},\nu^{(i)}) = (e^{(T_i-T_{i-1})Q})(\nu^{(i-1)},\nu^{(i)}). \tag{7.3.17}$$

This implies that in order to show (7.3.15), it suffices to show

$$\tilde{P}_{N,i}^{\lfloor c_N T_i\rfloor - \lfloor c_N T_{i-1}\rfloor}(\nu^{(i-1)},\nu^{(i)}) \xrightarrow{N\to\infty} (e^{(T_i-T_{i-1})Q_i})(\nu^{(i-1)},\nu^{(i)}). \tag{7.3.18}$$

The latter is an equality of finite matrices, and because they are finite it suffices to show

$$\tilde{P}_{N,i} = I + c_N^{-1} Q_i + o(c_N^{-1}). \tag{7.3.19}$$

For any $\eta, \kappa \in [\nu^{(i-1)}, \nu^{(i)}] \subset \Sigma(d, \mu)$, we have

$$\tilde{P}_{N,i}(\eta, \kappa) = \tilde{P}_N(\eta, \kappa) = P_{A_i^{(N)},d}((\eta_i)_{1-r_N \le i \le N-r_N}, (\kappa_i)_{1-r_N \le i \le N-r_N}) \tag{7.3.20}$$

and $Q_i(\eta, \kappa) = Q(\eta, \kappa)$. We recall from Proposition 7.1.6 that

$$Q(\eta, \kappa) = \begin{cases} -\frac{t^{\eta_d'+1} - t^{N+1}}{1-t} & \kappa = \eta \\ \frac{t^i(1-t^{m_{\eta_i}(\eta)})}{1-t} & \text{there exists } i \in \mathbb{Z} \text{ such that } \kappa_j = \eta_j + \mathbb{1}(j = i) \text{ for all } j \in \mathbb{Z} \cdot \\ 0 & \text{else} \end{cases} \tag{7.3.21}$$

The asymptotics of Lemma 7.3.1, Lemma 7.3.2, and Lemma 7.3.3 for (7.3.20), which correspond to the three cases of (7.3.21), yield (7.3.19) in these three cases and hence complete the proof. □

7.4 Nonasymptotic linear-algebraic bounds

The purpose of this section is to prove three nonasymptotic statements about random matrix products, Lemma 7.4.1, Lemma 7.4.2, and Lemma 7.4.3. In the next section we will use these to prove Lemma 7.3.1, Lemma 7.3.2, and Lemma 7.3.3 respectively.

For the remainder of this section, we fix the following notation: Let $N \in \mathbb{Z}_{\ge 1}$ and $\mu, \lambda \in \mathbb{Y}_N$ be fixed partitions, let $A = (a_{ij})_{1 \le i,j \le N}$ be a Haar-distributed element of $\mathrm{GL}_N(\mathbb{Z}_p)$, and let $\nu = \mathrm{SN}(\mathrm{diag}(p^\lambda) A \, \mathrm{diag}(p^\mu))$ (a random partition). We write $\mathrm{col}_j(A) = (a_{ij})_{1 \le i \le N} \in \mathbb{Z}_p^N$ and similarly for other matrices.

Lemma 7.4.1. *In the setting of this section, for any $1 \le r \le N$*

$$\Pr(\nu_j = \mu_j \textit{ for all } j \ge r) \ge \prod_{j=r}^{N} \frac{1 - t^{j - \mathrm{len}(\lambda)}}{1 - t^j} = \frac{(t;t)_{N-\mathrm{len}(\lambda)}(t;t)_{r-1}}{(t;t)_{r-1-\mathrm{len}(\lambda)}(t;t)_N} \tag{7.4.1}$$

with equality if $\mu_{r-1} > \mu_r$.

We remark that the right hand side of (7.4.1) is 0 when $r \le \mathrm{len}(\lambda)$. If $\mu_{r-1} = \mu_r$ the statement becomes trivial, but if $\mu_{r-1} > \mu_r$ it is a useful statement.

Lemma 7.4.2. *If* $\mathrm{len}(\lambda)+1 \le r \le N$ *is such that* $\mu_{r-1} > \mu_r$ *and* $m_{\mu_r}(\mu) = m$*, then*

$$(1-t^{r-\mathrm{len}(\lambda)})C(r,N,m,\lambda) \le \Pr(\nu_r = \mu_r + 1 \text{ and } \nu_j = \mu_j \text{ for all } j > r) \le C(r,N,m,\lambda) \tag{7.4.2}$$

where

$$C(r,N,m,\lambda) := (t^{r-\mathrm{len}(\lambda)} - t^r)\frac{1-t^m}{1-t}\frac{(t;t)_{r-1}(t;t)_{N-\mathrm{len}(\lambda)}}{(t;t)_N(t;t)_{r-\mathrm{len}(\lambda)}}$$

Lemma 7.4.3. *For any* $\mathrm{len}(\lambda)+1 \le r \le N$,

$$\Pr\left(\sum_{j=r}^{N} \nu_j - \mu_j \ge 2\right) \le 1 - \frac{(t;t)_{r-1}(t;t)_{N-\mathrm{len}(\lambda)}}{(t;t)_N(t;t)_{r-\mathrm{len}(\lambda)}}\left(1 - t^{r-\mathrm{len}(\lambda)} + t^{r-\mathrm{len}(\lambda)}(1-t^{N-r+1})\frac{1-t^{\mathrm{len}(\lambda)}}{1-t}\right).$$

Proving Lemma 7.4.1, Lemma 7.4.2, and Lemma 7.4.3 requires many auxiliary steps, which we begin proving. The following fact will be useful in proving Lemma 7.4.1 and later.

Lemma 7.4.4. *In the setting of this section, let*

$$v_j = (a_{i,j})_{\mathrm{len}(\lambda) < i \le N} \pmod p \in \mathbb{F}_p^{N-\mathrm{len}(\lambda)}.$$

Then the following implication and partial converse hold:

1. *If the set* $\{v_j : r \le j \le N\}$ *is linearly independent, then* $\nu_j = \mu_j$ *for all* $j \ge r$.

2. *Suppose that additionally* $\mu_{r-1} > \mu_r$*. If* $\nu_j = \mu_j$ *for all* $j \ge r$*, then* $\{v_j : r \le j \le N\}$ *is linearly independent.*

Proof. Let

$$A' := \mathrm{diag}(p^\lambda)A\,\mathrm{diag}(p^\mu) = (p^{\lambda_i+\mu_j}a_{i,j})_{1 \le i,j \le N}. \tag{7.4.3}$$

First, suppose that $\{v_j : r \le j \le N\}$ is a linearly independent set. Then the $\mathrm{col}_N(A')$ has an entry $p^{\mu_N}a_{i,N}$ with valuation μ_N, equivalently $a_{i,N} \in \mathbb{Z}_p^\times$, and all other entries of A' have valuation at least μ_N. Hence by row and column operations we may cancel all entries in the same row and column as $p^{\mu_N}a_{i,N}$ and multiply its row by $a_{i,N}^{-1} \in \mathbb{Z}_p$, obtaining a matrix

$$\begin{pmatrix} p^{\lambda_1+\mu_1}\tilde{a}_{1,1} & \cdots & p^{\lambda_1+\mu_{N-1}}\tilde{a}_{1,N-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ p^{\lambda_{N-1}+\mu_1}\tilde{a}_{N-1,1} & \cdots & p^{\lambda_{N-1}+\mu_{N-1}}\tilde{a}_{N-1,N-1} & 0 \\ 0 & \cdots & 0 & p^{\mu_N} \end{pmatrix} \tag{7.4.4}$$

for some $\tilde{a}_{i,j} \in \mathbb{Z}_p$.

By the linear independence assumption we may then find an entry $p^{\mu_{N-1}}a_{i,N-1}$ in the $(N-1)^{\text{st}}$ column of the matrix in (7.4.4), and cancel again, etc. Continuing, we obtain a matrix

$$\begin{pmatrix} p^{\lambda_1+\mu_1}\hat{a}_{1,1} & \cdots & p^{\lambda_1+\mu_{r-1}}\hat{a}_{1,r-1} & \\ \vdots & \ddots & \vdots & \mathbf{0}_{(N-r+1)\times r} \\ p^{\lambda_{r-1}+\mu_1}\hat{a}_{r-1,1} & \cdots & p^{\lambda_{r-1}+\mu_{r-1}}\hat{a}_{r-1,r-1} & \\ & \mathbf{0}_{r\times(N-r+1)} & & \mathrm{diag}(p^{\mu_r},\ldots,p^{\mu_N}) \end{pmatrix}, \tag{7.4.5}$$

for some $\hat{a}_{i,j} \in \mathbb{Z}_p$, with the same singular numbers as A'. Its top left $(r-1)\times(r-1)$ submatrix $\hat{A} = (p^{\lambda_i+\mu_j}\hat{a}_{i,j})_{1\le i,j\le r-1}$ lies in $p^{\mu_{r-1}}\,\mathrm{Mat}_{(r-1)\times(r-1)}(\mathbb{Z}_p)$, so all parts of $\mathrm{SN}(\hat{A})$ are at least μ_{r-1}, hence

$$\mathrm{SN}(A') = (\mathrm{SN}(\hat{A}), \mu_r, \mu_{r+1}, \ldots, \mu_N).$$

Now for the reverse implication, let us assume that $\mu_{r-1} > \mu_r$ and suppose that the set $\{(a_{i,j})_{\mathrm{len}(\lambda)<i\le N} : r \le j \le N\}$ is not linearly independent modulo p. Let k' be the largest index for which $\{v_{k'}, \ldots, v_N\}$ is linearly dependent, and $k \le k'$ the largest index such that additionally $\mu_k < \mu_{k-1}$. By the assumption $\mu_{r-1} > \mu_r$ it follows that $k \ge r$. We claim $\nu_k > \mu_k$. By definition of k', there must exist a relation

$$c_{k'}v_{k'} + \ldots + c_N v_N = 0$$

with $c_{k'} \neq 0$ in $\mathbb{F}_p^{N-\mathrm{len}(\lambda)}$, so without loss of generality take $c_{k'} = 1$. Letting C_j be a lift of c_j to $\mathbb{Z}^p$, we therefore have that

$$\mathrm{val}_p\left(\mathrm{col}_{k'}(A') + C_{k'+1}\,\mathrm{col}_{k'+1}(A') + \dots C_N\,\mathrm{col}_N(A')\right) \geq \mu_{k'} + 1.$$

The matrix A'', obtained from A' via column operations replacing $\mathrm{col}_{k'}(A')$ by $\mathrm{col}_{k'}(A') + C_{k'+1}\,\mathrm{col}_{k'+1}(A') + \dots C_N\,\mathrm{col}_N(A')$, thus has the same singular numbers as A' and furthermore has $\mathrm{val}_p(\mathrm{col}_j(A'')) \geq \mu_k + 1$ for $j = 1, \dots, k-1, k'$. It follows immediately that ν has at least k parts $\geq \mu_k + 1$, and since $\nu_j \geq \mu_j$ for all j this implies $\nu_k \geq \mu_k + 1$. This proves the reverse implication. □

The forward direction of Lemma 7.4.4 is a corollary of the following inequality, though we are not sure how one would establish the backward direction through the considerations used in the proof below.

Lemma 7.4.5. *Let* $\lambda, \mu \in \mathbb{Y}_N$, $1 \leq r \leq N$ *with* $\mathrm{len}(\lambda) < r$, *and* $k \geq 0$. *Then for any* $B = (b_{ij})_{1 \leq i,j \leq N} \in \mathrm{GL}_N(\mathbb{Z}_p)$,

$$\left|\mathrm{SN}\left((b_{ij})_{\substack{\mathrm{len}(\lambda) < i \leq N \\ r \leq j \leq N}}\right)\right| \leq \sum_{j=r}^{N} \mathrm{SN}(p^\lambda B p^\mu)_j - \mu_j. \tag{7.4.6}$$

Proof of Lemma 7.4.5. Let

$$B' = (b_{ij})_{\substack{\mathrm{len}(\lambda) < i \leq N \\ r \leq j \leq N}}.$$

Since $r > \mathrm{len}(\lambda)$, by Corollary 2.1.4

$$\sum_{j=r}^{N} \mathrm{SN}(p^\lambda B p^\mu)_j \leq \sum_{j=1}^{N-r+1} \mathrm{SN}(B' p^{(\mu_r, \dots, \mu_N)})_j = |\,\mathrm{SN}(B' p^{(\mu_r, \dots, \mu_N)})|. \tag{7.4.7}$$

By Proposition 2.1.5,

$$|\,\mathrm{SN}(B' p^{(\mu_r, \dots, \mu_N)})| = \sum_{j=r}^{N} \mu_j + |\,\mathrm{SN}(B')|. \tag{7.4.8}$$

Combining (7.4.7) with (7.4.8) and subtracting $\sum_{j=r}^{N} \mu_j$ from both sides yields (7.4.6). □

Proof of Lemma 7.4.1. In light of Lemma 7.4.4, we must show (in the notation of that result) that

$$\Pr(\{v_j : r \le j \le N\} \text{ is linearly independent}) = \prod_{j=r}^{N} \frac{p^j - p^{\mathrm{len}(\lambda)}}{p^j - 1}. \tag{7.4.9}$$

When $r \le \mathrm{len}(\lambda)$ this reduces easily to $0 = 0$, so suppose $r > \mathrm{len}(\lambda)$. The columns $\mathrm{col}_r(A), \ldots, \mathrm{col}_N(A)$ are chosen independently from the Haar measure, conditioned to be linearly independent modulo p. This implies that the columns $\mathrm{col}_j(A) \pmod p$ are chosen from the uniform measure on $\mathbb{F}_p^N$, conditionally on being linearly independent. Therefore

$$\Pr(\{v_j : r \le j \le N\} \text{ is linearly independent}) = \frac{\#S_2}{\#S_1} \tag{7.4.10}$$

where

$$S_1 := \{B = (b_{i,j}) \in \mathrm{Mat}_{N \times (N-r+1)}(\mathbb{F}_p) : B \text{ is full rank}\}$$
$$S_2 := \{B = (b_{i,j}) \in \mathrm{Mat}_{N \times (N-r+1)}(\mathbb{F}_p) : (b_{i,j}\mathbb{1}_{i > \mathrm{len}(\lambda)})_{\substack{1 \le i \le N \\ 1 \le j \le N-r+1}} \text{ is full rank}\} \subset S_1.$$

Computing the number of possible first columns, then second columns, etc. of B yields

$$\#S_1 = (p^N - 1) \cdots (p^N - p^{N-r}) \tag{7.4.11}$$

Since the condition

$$(b_{i,j}\mathbb{1}_{i > \mathrm{len}(\lambda)})_{\substack{1 \le i \le N \\ 1 \le j \le N-r+1}} \text{ is full rank}$$

is independent of the upper submatrix $(b_{i,j})_{\substack{1 \le i \le \mathrm{len}(\lambda) \\ 1 \le j \le N-r+1}}$, counting the number of possible first, second, etc. columns we have

$$\#S_2 = (p^N - p^{\mathrm{len}(\lambda)}) \cdots (p^N - p^{\mathrm{len}(\lambda)+N-r}) \tag{7.4.12}$$

Computing the RHS of (7.4.10) via (7.4.11) and (7.4.12) yields (7.4.9) and hence completes the proof. □

For the proofs of Lemma 7.4.2 and Lemma 7.4.3 we will use the following auxiliary computations over $\mathbb{F}_p$.

Lemma 7.4.6. *For $0 \leq r \leq k \leq n$,*

$$\#\{B \in \mathrm{Mat}_{n\times k}(\mathbb{F}_q) : \mathrm{rank}(B) = r\} = q^{rn+rk-r^2} \frac{(q^{-1};q^{-1})_n (q^{-1};q^{-1})_k}{(q^{-1};q^{-1})_r (q^{-1};q^{-1})_{n-r} (q^{-1};q^{-1})_{k-r}}. \tag{7.4.13}$$

Proof. The group $\mathrm{GL}_n(\mathbb{F}_q) \times \mathrm{GL}_k(\mathbb{F}_q)$ acts on $\mathrm{Mat}_{n\times k}(\mathbb{F}_q)$ by $(x,y)\cdot B = xBy^{-1}$, and by Smith normal form orbits are parametrized by their coranks. Letting $B_r \in \mathrm{Mat}_{n\times k}(\mathbb{F}_q)$ be the matrix with ii^{th} entry 1 for $1 \leq i \leq r$ and all other entries 0, we therefore have

$$\mathrm{LHS}(7.4.13) = \frac{\#\,\mathrm{GL}_n(\mathbb{F}_q) \times \mathrm{GL}_k(\mathbb{F}_q)}{\#\,\mathrm{Stab}(B_r)}. \tag{7.4.14}$$

Explicitly,

$$\#\,\mathrm{Stab}(B_r) = \left\{ \left(\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}, \begin{pmatrix} X & 0 \\ P & Q \end{pmatrix} \right) : \substack{X \in \mathrm{GL}_r(\mathbb{F}_q), Z \in \mathrm{GL}_{n-r}(\mathbb{F}_q), Q \in \mathrm{GL}_{k-r}(\mathbb{F}_q) \\ P \in \mathrm{Mat}_{(k-r)\times r}(\mathbb{F}_q), Y \in \mathrm{Mat}_{r \times (n-r)}(\mathbb{F}_q)} \right\},$$

therefore

$$\#\,\mathrm{Stab}(B_r) = (q^r - 1)\cdots(q^r - q^{r-1}) q^{r(n-r)} (q^{n-r}-1)\cdots(q^{n-r} - q^{n-r-1}) q^{r(k-r)} (q^{k-r}-1)\cdots(q^{k-r}-q^{k-r-1}).$$

Combining this with

$$\#\,\mathrm{GL}_n(\mathbb{F}_q) \times \mathrm{GL}_k(\mathbb{F}_q) = (q^n - 1)\cdots(q^n - q^{n-1})(q^k - 1)\cdots(q^k - q^{k-1})$$

and (7.4.14) yields (7.4.13). □

Lemma 7.4.7. *Let d and $n \geq k$ be three nonnegative integers, let $B \in \mathrm{Mat}_{(n+d)\times k}(\mathbb{F}_q)$ be a uniformly random full-rank matrix, and let $B' \in \mathrm{Mat}_{n\times k}(\mathbb{F}_q)$ be its lower $n \times k$ submatrix. Then for any $0 \leq r \leq k$,*

$$\Pr(\mathrm{rank}(B') = r) = q^{-(n-r)(k-r)} \frac{\begin{bmatrix} d \\ k-r \end{bmatrix}_{q^{-1}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^{-1}}}{\begin{bmatrix} n+d \\ k \end{bmatrix}_{q^{-1}}}. \tag{7.4.15}$$

Proof. We first compute

$$\#\{B \in \mathrm{Mat}_{(n+d)\times k}(\mathbb{F}_q) : \mathrm{rank}(B) = k, \mathrm{rank}(B') = r\}$$

where B' is the truncated matrix as in the statement. The number of possible B' is computed in Lemma 7.4.6, so for each B' we must count the number of $d \times k$ matrices B'' such that

$$\begin{pmatrix} B'' \\ B' \end{pmatrix} \in \mathrm{Mat}_{(n+d)\times k}(\mathbb{F}_q)$$

is full rank. By change of basis, the number of such B'' is the same for any B' of rank r, so without loss of generality take $B' = B_r \in \mathrm{Mat}_{n\times k}(\mathbb{F}_q)$, the matrix with ii^{th} entry 1 for $1 \leq i \leq r$ and all other entries 0. Then the first r columns of B'' may be anything, and the last $k - r$ columns must be linearly independent, so there are

$$q^{dr}(q^d - 1)\cdots(q^d - q^{k-r-1}) \tag{7.4.16}$$

possibilities for B'. The result now follows by combining (7.4.16) with Lemma 7.4.6, dividing by the number of full rank $(n + d) \times k$ matrices, and cancelling terms. □

Remark 43. We note that (7.4.15) is a q-analogue of the probability that a uniformly random k-element subset $S \subset A \sqcup B$ has $\#S \cap B = r$, when $\#A = d$ and $\#B = n$.

Lemma 7.4.8. *Let $A \in \mathrm{GL}_N(\mathbb{Z}_p)$ be distributed by the Haar measure and A' be an $n \times m$ submatrix with $n \leq m \leq N$. Then*

$$\Pr(\mathrm{SN}(A') = (1, 0, \ldots, 0)) = t^{m-n+1}\frac{(t;t)_{N-m}(t;t)_m(t;t)_n(t;t)_{N-n}}{(t;t)_1(t;t)_{N-m-1}(t;t)_{n-1}(t;t)_{m-n+1}(t;t)_N}.$$

Proof. Follows immediately by combining Theorem 1.3(1) and Proposition 2.9 of [VP21]. □

We note that Lemma 7.4.8 can also be established by a (longer) direct proof not going through the general results of [VP21].

Proof of Lemma 7.4.2. First write

$$\begin{aligned}
&\Pr(\nu_r = \mu_r + 1 \text{ and } \nu_j = \mu_j \text{ for all } j > r) \\
&= \Pr(\nu_r = \mu_r + 1 \text{ and } \nu_j = \mu_j \text{ for all } r < j < r+m | \nu_j = \mu_j \text{ for all } j \geq r+m) \\
&\times \Pr(\nu_j = \mu_j \text{ for all } j \geq r+m).
\end{aligned} \tag{7.4.17}$$

The second term on the RHS is

$$\Pr(\nu_j = \mu_j \text{ for all } j \geq r+m) = \prod_{j=r+m}^{N} \frac{1 - t^{j-\mathrm{len}(\lambda)}}{1-t^j} \tag{7.4.18}$$

by Lemma 7.4.1, so it suffices to compute the first term on the RHS of (7.4.17). By Lemma 7.4.4,

$$W(A) := (a_{i,j})_{\substack{\mathrm{len}(\lambda)+1 \leq i \leq N \\ r+m \leq j \leq N}} \tag{7.4.19}$$

is full rank modulo p if and only if $\nu_j = \mu_j$ for all $j \geq r+m$. Therefore

$$\begin{aligned}
&\Pr(\nu_r = \mu_r + 1 \text{ and } \nu_j = \mu_j \text{ for all } r < j < r+m | \nu_j = \mu_j \text{ for all } j \geq r+m) \\
&= \Pr(\nu_r = \mu_r + 1 \text{ and } \nu_j = \mu_j \text{ for all } r < j < r+m | W(A) \text{ is full rank modulo } p).
\end{aligned} \tag{7.4.20}$$

We claim that

$$\mathrm{RHS}(7.4.20) = \Pr(\nu_r = \mu_r + 1 \text{ and } \nu_j = \mu_j \text{ for all } r < j < r+m | W(A) = \tilde{I}) \tag{7.4.21}$$

where

$$\tilde{I} = \begin{pmatrix} \mathbf{0}_{(r+m-\mathrm{len}(\lambda)-1)\times(N-(r+m)+1)} \\ I_{N-(r+m)+1} \end{pmatrix}. \tag{7.4.22}$$

First note that any matrix $H \in \mathrm{Mat}_{(N-\mathrm{len}(\lambda))\times(N-(r+m)+1)}(\mathbb{Z}_p)$ which is full-rank modulo p is in the same $\mathrm{GL}_{N-\mathrm{len}(\lambda)}(\mathbb{Z}_p)$-orbit as $\tilde{I}$ (here we use that $\mathrm{len}(\lambda) < r$ and simply apply the necessary row operations to H). This, together with the explicit description of the Haar measure in Proposition 2.1.7, implies that

$$\mathrm{Law}(A | W(A) \text{ is full rank modulo } p) = \mathrm{Law}(BA | W(A) = \tilde{I}) \tag{7.4.23}$$

where

$$B = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & \tilde{B} \end{pmatrix} \in \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathrm{GL}_{N-\mathrm{len}(\lambda)}(\mathbb{Z}_p) \end{pmatrix} \tag{7.4.24}$$

and $\tilde{B}$ is Haar-distributed independent of A, because B mixes $\tilde{I}$ to a matrix distributed by the additive Haar measure conditioned on being full rank. By (7.4.23),

$$\begin{aligned} \mathrm{Law}(\mathrm{SN}(p^\lambda A p^\mu) | W(A) \text{ is full rank modulo } p) &= \mathrm{Law}(\mathrm{SN}(p^\lambda BAp^\mu) | W(A) = \tilde{I}) \\ &= \mathrm{Law}(\mathrm{SN}(Bp^\lambda Ap^\mu) | W(A) = \tilde{I}) \\ &= \mathrm{Law}(\mathrm{SN}(p^\lambda Ap^\mu) | W(A) = \tilde{I}), \end{aligned} \tag{7.4.25}$$

which shows (7.4.21).

For convenience define $\tilde{A} = (\tilde{a}_{i,j})_{1 \le i,j \le N}$ to be a random element of $\mathrm{GL}_N(\mathbb{Z}_p)$ distributed by the Haar measure conditioned on $W(\tilde{A}) = \tilde{I}$, so that

$$\mathrm{RHS}(7.4.21) = \Pr(\mathrm{SN}(p^\lambda \tilde{A} p^\mu)_i = \mu_i + \mathbb{1}(i = r) \text{ for all } r \le i \le N). \tag{7.4.26}$$

For any deterministic matrix $V = (v_{i,j})_{1 \le i,j \le N}$ with $W(V) = \tilde{I}$, first note that $\mathrm{SN}(p^\lambda V p^\mu)_i = \nu_i$ for all $r + m \le i \le N$ by Lemma 7.4.4 as before. We make the following additional claims, which will be used for our upper and lower bounds:

(i) If

$$\mathrm{SN}\left((v_{i,j})_{\substack{\mathrm{len}(\lambda)+1 \le i \le r+m-1 \\ r \le j \le r+m-1}} \right) = (1, 0, \ldots, 0) \tag{7.4.27}$$

then

$$\mathrm{SN}(p^\lambda V p^\mu)_i = \mu_i + \mathbb{1}(i = r) \text{ for all } r \le i \le r + m - 1. \tag{7.4.28}$$

(ii) If (7.4.28) holds, then

$$(v_{i,j})_{\substack{\mathrm{len}(\lambda)+1 \le i \le r+m-1 \\ r \le j \le r+m-1}} \quad (\mathrm{mod}\ p) \text{ has corank } 1. \tag{7.4.29}$$

Let us first show (i), so suppose (7.4.27) holds.

$$p^\lambda V p^\mu = \begin{pmatrix} p^\lambda M^{(1)} & p^\lambda M^{(2)} & p^\lambda M^{(3)} \\ M^{(4)} & M^{(5)} & 0 \\ M^{(6)} & M^{(7)} & \mathrm{diag}(p^{\mu_{r+m}}, \ldots, p^{\mu_N}) \end{pmatrix}, \tag{7.4.30}$$

where $M^{(i)}$ are the appropriate submatrices of Vp^{μ} and $\mathrm{SN}(M^{(5)}) = (1, 0[m-1])$. Here the blocks of (7.4.30) in first, second and third column have widths $r-1$, m and $N-(r+m)+1$ respectively, and the blocks of the first, second, and third rows have heights $\mathrm{len}(\lambda)$, $r+m-\mathrm{len}(\lambda)-1$, and $N-(r+m)+1$ respectively. Hence by further column operations which subtract units times powers $p^{(\mu_i-\mu_\ell)}, 1 \leq i \leq r+m-1, r+m \leq \ell \leq N$ times the ℓ^{th} column from the i^{th} column, we obtain

$$\begin{pmatrix} p^{\lambda}\,\tilde{\mathrm{M}}^{(1)} & p^{\lambda}\,\tilde{\mathrm{M}}^{(2)} & p^{\lambda}M^{(3)} \\ M^{(4)} & M^{(5)} & 0 \\ 0 & 0 & \mathrm{diag}(p^{\mu_{r+m}}, \ldots, p^{\mu_N}) \end{pmatrix} \tag{7.4.31}$$

and because of the $p^{(\mu_i-\mu_\ell)}$ powers the matrices $\tilde{\mathrm{M}}^{(1)}, \tilde{\mathrm{M}}^{(2)}$ still lie in $\mathrm{Mat}_{\mathrm{len}(\lambda)\times(r-1)}(\mathbb{Z}_p)\,\mathrm{diag}(p^{\mu_1}, \ldots, p^{\mu_{r-1}})$ and $\mathrm{Mat}_{\mathrm{len}(\lambda)\times m}(\mathbb{Z}_p)\,\mathrm{diag}(p^{\mu_r}, \ldots, p^{\mu_{r+m-1}})$ respectively. We clearly may further cancel to obtain

$$\begin{pmatrix} p^{\lambda}\,\tilde{\mathrm{M}}^{(1)} & p^{\lambda}\,\tilde{\mathrm{M}}^{(2)} & 0 \\ M^{(4)} & M^{(5)} & 0 \\ 0 & 0 & \mathrm{diag}(p^{\mu_{r+m}}, \ldots, p^{\mu_N}) \end{pmatrix}. \tag{7.4.32}$$

Note that since $\mu_r = \ldots = \mu_{r+m-1}$,

$$M^{(5)} = p^{\mu_r}\tilde{V}, \tag{7.4.33}$$

where $\tilde{V} = (v_{i,j})_{\substack{\mathrm{len}(\lambda)+1\leq i\leq r+m-1 \\ r\leq j\leq r+m-1}}$ is the matrix for which we have assumed $\mathrm{SN}(\tilde{V}) = (1, 0[m-1])$. Hence

$$\mathrm{SN}(M^{(5)}) = (\mu_r + 1, \mu_r[m-1]). \tag{7.4.34}$$

By Corollary 2.1.4 and (7.4.34),

$$\sum_{\ell=1}^{m} \mathrm{SN}\left((p^{\lambda_i+\mu_j}v_{i,j})_{\substack{1\leq i\leq r+m-1 \\ r\leq j\leq r+m-1}}\right)_\ell \leq \sum_{\ell=1}^{m} \mathrm{SN}(M^{(5)})_\ell = m\mu_r + 1. \tag{7.4.35}$$

Since the matrix $(p^{\lambda_i}v_{i,j})_{\substack{1\leq i\leq r+m-1 \\ r\leq j\leq r+m-1}}$ is not full rank modulo p by (7.4.27),

$$\left|\mathrm{SN}\left((p^{\lambda_i}v_{i,j})_{\substack{1\leq i\leq r+m-1 \\ r\leq j\leq r+m-1}}\right)\right| \geq 1, \tag{7.4.36}$$

hence by Proposition 2.1.5 we have

$$\mathrm{LHS}(7.4.35) \geq m\mu_r + 1, \tag{7.4.37}$$

so in fact

$$\mathrm{LHS}(7.4.35) = m\mu_r + 1. \tag{7.4.38}$$

Since every entry of $(p^{\lambda_i+\mu_j} v_{i,j})_{\substack{1 \leq i \leq r+m-1 \\ r \leq j \leq r+m-1}}$ is divisible by p^{μ_r}, each singular number is at least μ_r, and combining this with (7.4.38) yields

$$\mathrm{SN}\left((p^{\lambda_i+\mu_j} v_{i,j})_{\substack{1 \leq i \leq r+m-1 \\ r \leq j \leq r+m-1}}\right) = (\mu_r + 1, \mu_r[m-1]). \tag{7.4.39}$$

Since all entries in columns 1 through $r-1$ of (7.4.32) are divisible by $p^{\mu_{r-1}}$, (7.4.39) together with the equivalence of $p^\lambda V p^\mu$ with (7.4.32) imply (7.4.28). This shows (i).

Now we show (ii), so suppose V is such that (7.4.28) holds. If $(v_{i,j})_{\substack{\mathrm{len}(\lambda)+1 \leq i \leq r+m-1 \\ r \leq j \leq r+m-1}}$ $\pmod p$ were full-rank, then $(v_{i,j})_{\substack{\mathrm{len}(\lambda)+1 \leq i \leq N \\ r \leq j \leq N}}$ $\pmod p$ would be full rank since $W(V) = \tilde{I}$ is full-rank, and by Lemma 7.4.4 this would contradict the fact that $\mathrm{SN}(p^\lambda V p^\mu)_r = \mu_r + 1$. Hence $(v_{i,j})_{\substack{\mathrm{len}(\lambda)+1 \leq i \leq r+m-1 \\ r \leq j \leq r+m-1}}$ $\pmod p$ has corank $k \geq 1$, and it similarly follows that

$$\mathrm{corank}\left((v_{i,j})_{\substack{\mathrm{len}(\lambda)+1 \leq i \leq N \\ r \leq j \leq N}} \pmod p\right) = k \tag{7.4.40}$$

as well. Hence

$$\mathrm{SN}\left((v_{i,j})_{\substack{\mathrm{len}(\lambda)+1 \leq i \leq N \\ r \leq j \leq N}}\right)_{r+i} \geq 1 \qquad \text{for } 0 \leq i \leq k-1. \tag{7.4.41}$$

By Lemma 7.4.5, (7.4.41) implies that

$$\sum_{i=r}^{N} \mathrm{SN}(p^\lambda V p^\mu)_i - \mu_i \geq k, \tag{7.4.42}$$

which contradicts (7.4.28) unless $k = 1$. Therefore $k = 1$, proving (ii).

Using (i) and (ii) for the lower and upper bounds respectively, we have

$$\begin{aligned}
&\Pr\left(\mathrm{SN}\left((\tilde{a}_{i,j})_{\substack{\mathrm{len}(\lambda)<i<r+m\\ r\le j<r+m}}\right)=(1,0[m-1])\right)\\
&\le \Pr(\nu_r=\mu_r+1 \text{ and } \nu_j=\mu_j \text{ for all } r<j<r+m|W(A)=\tilde{I})\\
&\le \Pr\left(\mathrm{corank}\left((\tilde{a}_{i,j})_{\substack{\mathrm{len}(\lambda)<i<r+m\\ r\le j<r+m}} \pmod p\right)=1\right).
\end{aligned}\tag{7.4.43}$$

By applying Lemma 7.4.7 with $r=m-1, n=r+m-\mathrm{len}(\lambda)-1, d=\mathrm{len}(\lambda), k=m$, we obtain

$$\begin{aligned}
\mathrm{RHS}(7.4.43)&=t^{r-\mathrm{len}(\lambda)}\frac{\begin{bmatrix}\mathrm{len}(\lambda)\\ 1\end{bmatrix}_t\begin{bmatrix}r+m-\mathrm{len}(\lambda)-1\\ m-1\end{bmatrix}_t}{\begin{bmatrix}r+m-1\\ m\end{bmatrix}_t}\\
&=t^{r-\mathrm{len}(\lambda)}\frac{(1-t^{\mathrm{len}(\lambda)})(1-t^m)}{1-t}\frac{(t;t)_{r+m-\mathrm{len}(\lambda)-1}(t;t)_{r-1}}{(t;t)_{r-\mathrm{len}(\lambda)}(t;t)_{r+m-1}}
\end{aligned}\tag{7.4.44}$$

By applying Lemma 7.4.8 with $r+m-1, r+m-1-\mathrm{len}(\lambda), m$ substituted for N,m,n respectively,

$$\mathrm{LHS}(7.4.43)=(t^{r-\mathrm{len}(\lambda)}-t^r)\frac{1-t^m}{1-t}\frac{(t;t)_{r-1}(t;t)_{r+m-\mathrm{len}(\lambda)-1}}{(t;t)_{r+m-1}(t;t)_{r-\mathrm{len}(\lambda)-1}}.\tag{7.4.45}$$

Hence

$$\begin{aligned}
&(t^{r-\mathrm{len}(\lambda)}-t^r)\frac{1-t^m}{1-t}\frac{(t;t)_{r-1}(t;t)_{r+m-\mathrm{len}(\lambda)-1}}{(t;t)_{r+m-1}(t;t)_{r-\mathrm{len}(\lambda)-1}}\\
&\le \Pr(\nu_r=\mu_r+1 \text{ and } \nu_j=\mu_j \text{ for all } r<j<r+m|W(A)=\tilde{I})\\
&\le t^{r-\mathrm{len}(\lambda)}\frac{(1-t^{\mathrm{len}(\lambda)})(1-t^m)}{1-t}\frac{(t;t)_{r+m-\mathrm{len}(\lambda)-1}(t;t)_{r-1}}{(t;t)_{r-\mathrm{len}(\lambda)}(t;t)_{r+m-1}}.
\end{aligned}\tag{7.4.46}$$

Combining the reduction (7.4.17) with the computation of (7.4.18) and the bound on the conditional probability coming from combining (7.4.20), (7.4.21), and (7.4.46) completes the proof. $\square$

Proof of Lemma 7.4.3. For any matrix $B=(b_{i,j})_{1\le i,j\le N}\in \mathrm{GL}_N(\mathbb{Z}_p)$, we define $B'=$

$\left((b_{i,j})_{\substack{\text{len}(\lambda)+1 \le i \le N \\ r \le j \le N}}\right)$ as before. By Lemma 7.4.5,

$$\Pr\left(\sum_{j=r}^{N} \nu_j - \mu_j \ge 2\right) \le 1 - \Pr(|\operatorname{SN}(A')| \le 1). \tag{7.4.47}$$

Since $|\operatorname{SN}(A')| = 0$ if and only if $A' \pmod p$ is full rank, Lemma 7.4.7 yields that

$$\Pr(|\operatorname{SN}(A')| = 0) = \frac{(t;t)_{r-1}(t;t)_{N-\text{len}(\lambda)}}{(t;t)_N (t;t)_{r-1-\text{len}(\lambda)}}. \tag{7.4.48}$$

By Lemma 7.4.8,

$$\Pr(|\operatorname{SN}(A')| = 1) = \frac{(t;t)_{r-1}(t;t)_{N-\text{len}(\lambda)}}{(t;t)_N (t;t)_{r-\text{len}(\lambda)}} \left(t^{r-\text{len}(\lambda)}(1 - t^{N-r+1})\frac{1 - t^{\text{len}(\lambda)}}{1-t}\right). \tag{7.4.49}$$

Combining (7.4.47) with (7.4.48) and (7.4.49) completes the proof. □

7.5 Asymptotics of matrix product transition probabilities

In this section, we use the nonasymptotic bounds of the previous section to establish asymptotics for the matrix product process stated earlier as Lemma 7.3.1, Lemma 7.3.2, and Lemma 7.3.3. The technical work of this section essentially amounts to computing the relevant terms of bounds which were left as prelimit explicit formulas in the previous section, with the additional complication of randomizing those bounds over the singular numbers of one of the matrices; we also phrase everything in terms of truncated signatures $F_d(\nu)$, which was not done in the previous section.

Definition 62. In the proofs of Lemma 7.3.1, Lemma 7.3.2 and Lemma 7.3.3, we write $o_{unif}(\cdot)$ to indicate any quantity which is $o(\cdot)$ as $N \to \infty$ with constants uniform over all $\nu^{(N)} \in F_d(\operatorname{Sig}_N^+)$ with $(\nu^N)'_k \ge L + r_N$.

Proof of Lemma 7.3.1. To simplify notation, let

$$j_0 = j_0(N) := (\nu^{(N)})'_k + 1 = \min\{i : \nu^{(N)}_{r_N+i} < d\}. \tag{7.5.1}$$

By hypothesis, $j_0 \geq L$. By the equality case of Lemma 7.4.1, we have

$$\begin{aligned}
&\Pr(F_d(\mathrm{SN}(A^{(N)}\operatorname{diag}(p^{\nu^{(N)}}))) = F_d(\nu^{(N)}) | \, \mathrm{SN}(A^{(N)}) = \ell) \\
&= \Pr(\mathrm{SN}(A^{(N)}\operatorname{diag}(p^{\nu^{(N)}}))_i = \nu_i^{(N)} \text{ for all } j_0 + r_N \leq i \leq N | \, \mathrm{SN}(A^{(N)}) = \ell) \\
&= \begin{cases} \prod_{j=j_0+r_N}^{N} \frac{1-t^{j-\ell}}{1-t^j} & 0 \leq \ell < j_0 + r_N \\ 0 & j_0 + r_N \leq \ell \end{cases}.
\end{aligned} \tag{7.5.2}$$

For notational convenience, here and in the rest of the proof we define the random variable $X_N := \operatorname{len}(\mathrm{SN}(A^{(N)}))$. Taking an expectation over X_N in (7.5.2) yields

$$\Pr(F_d(\mathrm{SN}(A^{(N)}U\operatorname{diag}(p^{\nu^{(N)}}))) = F_d(\nu^{(N)})) = \mathbb{E}\left[\mathbb{1}(X_N < j_0 + r_N) \prod_{j=j_0+r_N}^{N} \frac{1-t^{j-X_N}}{1-t^j}\right]. \tag{7.5.3}$$

Note that (7.5.3) depends on $\nu^{(N)}$ only through j_0, so to establish uniform asymptotics over $\nu^{(N)}$ we simply need them to be uniform over j_0. To show Lemma 7.3.1, we therefore must show

$$\mathbb{E}\left[\mathbb{1}(X_N < j_0 + r_N) \prod_{j=j_0+r_N}^{N} \frac{1-t^{j-X_N}}{1-t^j}\right] = 1 - \frac{t^{j_0} - t^{N-r_N+1}}{1-t} c_N^{-1} + o_{unif}(c_N^{-1}) \tag{7.5.4}$$

(recall the notation o_{unif} from Definition 62 and the definition $c_N := (\mathbb{E}[\mathbb{1}(X_N \leq r_N)t^{r_N - X_N} - t^{r_N}])^{-1})$. Since

$$\Pr(X_N \geq j_0 + r_N) \leq \Pr(X_N \geq L + r_N) = o_{unif}(c_N^{-1}) \tag{7.5.5}$$

by hypothesis, we may write

$$\frac{\mathbb{E}[\mathbb{1}(X_N < j_0 + r_N)(1-t^{j_0+r_N})\cdots(1-t^N)]}{(1-t^{j_0+r_N})\cdots(1-t^N)} = 1 + o_{unif}(c_N^{-1}), \tag{7.5.6}$$

and using this we rearrange (7.5.4) to obtain that it is equivalent to show

$$\begin{aligned}
&\frac{\mathbb{E}[\mathbb{1}(X_N < j_0 + r_N)\left((1-t^{j_0+r_N})\cdots(1-t^N) - (1-t^{j_0+r_N-X_N})\cdots(1-t^{N-X_N})\right)]}{(1-t^{j_0+r_N})\cdots(1-t^N)} \\
&\qquad = \frac{t^{j_0} - t^{N-r_N+1}}{1-t} c_N^{-1} + o_{unif}(c_N^{-1}).
\end{aligned} \tag{7.5.7}$$

This is what we will show.

We write

$$\begin{aligned}&\mathbb{E}[\mathbb{1}(X_N < j_0 + r_N)((1-t^{j_0+r_N})\cdots(1-t^N) - (1-t^{j_0+r_N-X_N})\cdots(1-t^{N-X_N}))] \\ &= \mathbb{E}\left[\mathbb{1}(X_N < j_0 + r_N) \sum_{j=0}^{N-r_N-j_0+1} (-1)^j \begin{bmatrix} N-r_N-j_0+1 \\ j \end{bmatrix}_t t^{\binom{j}{2}+j(j_0+r_N)}(1-t^{-jX_N})\right]\end{aligned} \tag{7.5.8}$$

by expanding both factors inside the expectation via the q-binomial theorem and consolidating term-by-term. Note that the $j=0$ term of (7.5.8) is 0. Since the summands satisfy

$$\left|(-1)^j \begin{bmatrix} N-r_N-j_0+1 \\ j \end{bmatrix}_t t^{\binom{j}{2}+j(j_0+r_N)}(1-t^{-jX_N})\right| \leq \begin{bmatrix} N-r_N-j_0+1 \\ j \end{bmatrix}_t t^{\binom{j}{2}+j(j_0+r_N)} \tag{7.5.9}$$

because $X_N \geq 0$, and the right hand side of (7.5.9) is integrable, Fubini's theorem implies

$$\mathrm{RHS}(7.5.8) = \sum_{j=1}^{N-r_N-j_0+1} (-1)^{j+1} \begin{bmatrix} N-r_N-j_0+1 \\ j \end{bmatrix}_t t^{\binom{j}{2}+j(j_0+r_N)} \mathbb{E}[\mathbb{1}(X_N < j_0+r_N)(1-t^{-jX_N})]. \tag{7.5.10}$$

The contribution of the $j=1$ term of (7.5.10) to (7.5.7) is

$$\frac{1}{(1-t^{j_0+r_N})\cdots(1-t^N)} \frac{t^{j_0}-t^{N-r_N+1}}{1-t} c_N^{-1} = c_N^{-1}\left(\frac{t^{j_0}-t^{N-r_N+1}}{1-t} + o_{unif}(1)\right) \tag{7.5.11}$$

where we use that $r_N \to \infty$ so $(1-t^{j_0+r_N})\cdots(1-t^N) \to 1$. The asymptotic (7.5.11) is uniform over $\nu^{(N)}$ satisfying our hypotheses, since it depends on $\nu^{(N)}$ only through j_0, and is uniform over $j_0 \geq L$.

Hence to prove (7.5.7) it now suffices to show

$$\sum_{j=2}^{N-r_N-j_0+1} (-1)^j \begin{bmatrix} N-r_N-j_0+1 \\ j \end{bmatrix}_t t^{\binom{j}{2}+j(j_0+r_N)} \mathbb{E}[\mathbb{1}(X_N < j_0+r_N)(1-t^{-jX_N})] = o_{unif}(c_N^{-1}), \tag{7.5.12}$$

where we have used the fact that $(1-t^{j_0+r_N})\cdots(1-t^N) = 1 + o_{unif}(1)$ uniformly over $j_0 \geq L$ to remove the denominator of (7.5.7). We rewrite the asymptotic (7.5.12) which we wish to show as

$$-\sum_{j=2}^{N-r_N-j_0+1}(-1)^j \begin{bmatrix} N-r_N-j_0+1 \\ j \end{bmatrix}_t t^{\binom{j}{2}+jj_0}\frac{\mathbb{E}[\mathbb{1}(X_N<j_0+r_N)(t^{j(r_N-X_N)}-t^{jr_N})]}{\mathbb{E}[\mathbb{1}(X_N\le r_N)(t^{r_N-X_N}-t^{r_N})]}=o_{unif}(1). \tag{7.5.13}$$

To show (7.5.13), it suffices to show that for all $\delta>0$,

$$t^{jj_0}\frac{\mathbb{E}[\mathbb{1}(X_N<j_0+r_N)(t^{j(r_N-X_N)}-t^{jr_N})]}{\mathbb{E}[\mathbb{1}(X_N\le r_N)(t^{r_N-X_N}-t^{r_N})]}<\delta \qquad \text{for all } j\ge 2 \tag{7.5.14}$$

for all N sufficiently large independent of j. Since both numerator and denominator in (7.5.14) are 0 when $X_N=0$, by clearing factors of $\Pr(X_N>0)$ we have

$$\begin{aligned} t^{jj_0}\frac{\mathbb{E}[\mathbb{1}(X_N<j_0+r_N)(t^{j(r_N-X_N)}-t^{jr_N})]}{\mathbb{E}[\mathbb{1}(X_N\le r_N)(t^{r_N-X_N}-t^{r_N})]} &= t^{jj_0}\frac{\mathbb{E}[\mathbb{1}(X_N<j_0+r_N)(t^{j(r_N-X_N)}-t^{jr_N})|X_N>0]}{\mathbb{E}[\mathbb{1}(X_N\le r_N)(t^{r_N-X_N}-t^{r_N})|X_N>0]} \\ &\le t^{jj_0}\frac{\mathbb{E}[\mathbb{1}(\tilde{X}_N<j_0+r_N)t^{j(r_N-\tilde{X}_N)}]}{\mathbb{E}[\mathbb{1}(\tilde{X}_N\le r_N)(t^{r_N-\tilde{X}_N}-t^{r_N})]}, \end{aligned} \tag{7.5.15}$$

where to simplify notation we let $\tilde{X}_N$ be a random variable with

$$\mathrm{Law}(\tilde{X}_N)=\mathrm{Law}(X_N|X_N>0). \tag{7.5.16}$$

For any $b\ge 0$,

$$\begin{aligned} \mathrm{RHS}(7.5.15) \le t^{jj_0}&\frac{\Pr(r_N+j_0-b<\tilde{X}_N<r_N+j_0)t^{-jj_0}}{\mathbb{E}[\mathbb{1}(\tilde{X}_N\le r_N)(t^{r_N-\tilde{X}_N}-t^{r_N})]} \\ &+t^{jj_0}\frac{\mathbb{E}[\mathbb{1}(\tilde{X}_N\le r_N+j_0-b)t^{j(r_N-\tilde{X}_N)}]}{\mathbb{E}[\mathbb{1}(\tilde{X}_N\le r_N)(t^{r_N-\tilde{X}_N}-t^{r_N})]}. \end{aligned} \tag{7.5.17}$$

The first term in (7.5.17) is

$$\frac{\Pr(r_N+j_0-b<\tilde{X}_N<r_N+j_0)}{\mathbb{E}[\mathbb{1}(\tilde{X}_N\le r_N)(t^{r_N-\tilde{X}_N}-t^{r_N})]}\le\frac{\Pr(\tilde{X}_N>r_N+L-b)}{\mathbb{E}[\mathbb{1}(\tilde{X}_N\le r_N)(t^{r_N-\tilde{X}_N}-t^{r_N})]}, \tag{7.5.18}$$

which by the hypothesis (7.2.1) is $o_{unif}(1)$ (it is uniform over j_0, since j_0 does not appear).

Note next that for any j and any function $f : \mathbb{R} \to \mathbb{R}$ with $f([1, r_N + j_0 - b]) \subset [0,1]$,

$$\mathbb{E}[\mathbb{1}(\tilde{X}_N \le r_N + j_0 - b) f(\tilde{X}_N) t^{j(r_N + j_0 - \tilde{X}_N)}] \\ \le t^{(j-1)b} \mathbb{E}[\mathbb{1}(\tilde{X}_N \le r_N + j_0 - b) t^{r_N + j_0 - \tilde{X}_N}] \quad (7.5.19)$$

because all nonzero terms come from values of $\tilde{X}_N$ with $r_N + j_0 - \tilde{X}_N \ge b$, and so $t^{j(r_N + j_0 - \tilde{X}_N)} \le t^{(j-1)b} \cdot t^{r_N + j_0 - \tilde{X}_N}$ with probability 1.

Applying (7.5.19) to the numerator and a trivial bound to the denominator of the second term of (7.5.17) yields

$$\frac{\mathbb{E}[\mathbb{1}(\tilde{X}_N \le r_N + j_0 - b) t^{j(r_N + j_0 - \tilde{X}_N)}]}{\mathbb{E}[\mathbb{1}(\tilde{X}_N \le r_N)(t^{r_N - \tilde{X}_N} - t^{r_N})]} \le \frac{t^{(j-1)b} \mathbb{E}[\mathbb{1}(\tilde{X}_N \le r_N + j_0 - b) t^{r_N + j_0 - \tilde{X}_N}]}{(1-t)\mathbb{E}[\mathbb{1}(\tilde{X}_N \le r_N) t^{r_N - \tilde{X}_N}]} \\ \le \frac{t^{(j-1)b}}{1-t} t^{j_0}. \quad (7.5.20)$$

Hence for any $\delta > 0$, by choosing b so that $\frac{t^{(j-1)b}}{1-t} t^L < \delta/2$, we have that (7.5.14) holds for all N large enough that the left hand side of (7.5.18) is $< \delta/2$. As we had previously reduced to (7.5.14), this completes the proof. $\square$

We will prove Lemma 7.3.3 before Lemma 7.3.2 since the former is needed for the latter.

Proof of Lemma 7.3.3. Let $\tilde{X}_N$ be a random variable with $\mathrm{Law}(\tilde{X}_N) = \mathrm{Law}(X_N | X_N > 0)$ as before, so

$$\mathbb{E}[\mathbb{1}(\tilde{X}_N \le r_N)(t^{r_N - \tilde{X}_N} - t^{r_N})] = \frac{c_N^{-1}}{\Pr(X_N > 0)} \quad (7.5.21)$$

(recalling that $X_N \ge 0$ always, and $X_N > 0$ with positive probability by hypothesis). Using the same notation $j_0 = j_0(N)$ defined in (7.5.1), for the proof it suffices to show

$$\Pr\left(\sum_{i=j_0+r_N}^{N} \mathrm{SN}(A^{(N)} \operatorname{diag}(p^{\nu^{(N)}}))_i - \nu_i^{(N)} \ge 2 \,\middle|\, X_N > 0 \right) = o_{unif}(\mathbb{E}[\mathbb{1}(\tilde{X}_N \le r_N)(t^{r_N - \tilde{X}_N} - t^{r_N})]). \quad (7.5.22)$$

For each $0 < x < r_N + j_0$, by applying Lemma 7.4.3 with $r = j_0 + r_N$, $\lambda = \mathrm{SN}(A^{(N)})$, and

$\mu = \nu^{(N)}$ in the notation of that result, we have

$$\begin{aligned}\Pr&\left(\sum_{i=j_0+r_N}^{N} \mathrm{SN}(A^{(N)}\,\mathrm{diag}(p^{\nu^{(N)}}))_i - \nu_i^{(N)} \geq 2 \,\middle|\, X_N = x\right)\\ &\leq \left(1 - \frac{(t;t)_{j_0+r_N-1}(t;t)_{N-x}}{(t;t)_N(t;t)_{j_0+r_N-x}}\left(1 - t^{j_0+r_N-x} + t^{j_0+r_N-x}(1-t^{N-j_0+r_N+1})\frac{1-t^x}{1-t}\right)\right)\end{aligned} \tag{7.5.23}$$

Fix an integer $b \geq 1$ independent of N, and let N be large enough so that $r_N + j_0 - b > 0$ holds (this holds for all sufficiently large N since $r_N \to \infty$ and $j_0(N) \geq L$). Then taking a (conditional, given $X_N > 0$) expectation of (7.5.23) when $0 < x \leq r_N + j_0 - b$ and naively bounding when $x > r_N + j_0 - b$ yields

$$\begin{aligned}\Pr&\left(\sum_{i=j_0+r_N}^{N} \mathrm{SN}(A^{(N)}\,\mathrm{diag}(p^{\nu^{(N)}}))_i - \nu_i^{(N)} \geq 2 \,\middle|\, X_N > 0\right)\\ &\leq \Pr(\tilde{X}_N > r_N + j_0 - b) + \mathbb{E}\left[\mathbb{1}(\tilde{X}_N \leq r_N + j_0 - b)\left(1 - \frac{(t;t)_{j_0+r_N-1}(t;t)_{N-\tilde{X}_N}}{(t;t)_N(t;t)_{j_0+r_N-\tilde{X}_N}}\right.\right.\\ &\times \left.\left.\left(1 - t^{j_0+r_N-\tilde{X}_N} + t^{j_0+r_N-\tilde{X}_N}(1-t^{N-j_0+r_N+1})\frac{1-t^{\tilde{X}_N}}{1-t}\right)\right)\right].\end{aligned} \tag{7.5.24}$$

We first rewrite the expression inside the expectation on the left of (7.5.24) (without the indicator function) as

$$\begin{aligned}1 &- \frac{(t;t)_{j_0+r_N-1}(t;t)_{N-\tilde{X}_N}}{(t;t)_N(t;t)_{j_0+r_N-\tilde{X}_N}}\left(1 - t^{j_0+r_N-\tilde{X}_N} + t^{j_0+r_N-\tilde{X}_N}(1-t^{N-j_0+r_N+1})\frac{1-t^{\tilde{X}_N}}{1-t}\right)\\ &= \frac{(t;t)_{j_0+r_N-1}}{(t;t)_{j_0+r_N-\tilde{X}_N}}\left(\frac{1-t^{j_0+r_N}}{\prod_{i=0}^{\tilde{X}_N-1}(1-t^i\cdot t^{j_0+r_N-\tilde{X}_N+1})}\right.\\ &\left.-\frac{1}{\prod_{i=0}^{\tilde{X}_N-1}(1-t^i\cdot t^{N-\tilde{X}_N+1})}\left(1-t^{j_0+r_N-\tilde{X}_N} + (t^{j_0+r_N-\tilde{X}_N} - t^{N-\tilde{X}_N+1})\frac{1-t^{\tilde{X}_N}}{1-t}\right)\right)\\ &= \frac{(t;t)_{j_0+r_N-1}}{(t;t)_{j_0+r_N-\tilde{X}_N}}\sum_{\ell=0}^{\infty}\begin{bmatrix}\tilde{X}_N-1+\ell\\ \ell\end{bmatrix}_t\left(t^{\ell(j_0+r_N-\tilde{X}_N+1)}(1-t^{j_0+r_N})\right.\\ &\left.-t^{\ell(N-\tilde{X}_N+1)}\left(1-t^{j_0+r_N-\tilde{X}_N} + (t^{j_0+r_N-\tilde{X}_N} - t^{N-\tilde{X}_N+1})\frac{1-t^{\tilde{X}_N}}{1-t}\right)\right)\end{aligned} \tag{7.5.25}$$

where in the second equality, we have expanded both of the $1/(\prod \cdots)$ terms into infinite sums by the q-binomial theorem (here it is important that $\tilde{X}_N \in \mathbb{Z}_{\geq 1}$) and then combined the sums. We further split the sum to write

$$\mathrm{RHS}(7.5.25) = S_1 + S_2 + S_3, \tag{7.5.26}$$

where we define

$$S_1 = \frac{(t;t)_{j_0+r_N-1}}{(t;t)_{j_0+r_N-\tilde{X}_N}} \left(1 - t^{j_0+r_N} - 1 + t^{j_0+r_N-\tilde{X}_N} - (t^{j_0+r_N-\tilde{X}_N} - t^{N-\tilde{X}_N+1})\frac{1-t^{\tilde{X}_N}}{1-t} \right. \\ \left. + \begin{bmatrix} \tilde{X}_N \\ 1 \end{bmatrix}_t (t^{j_0+r_N-\tilde{X}_N+1} - t^{N-\tilde{X}_N+1}) \right) = 0 \tag{7.5.27}$$

(the $\ell = 0$ term in (7.5.25) together with a part of the $\ell = 1$ term chosen so that they exactly cancel),

$$S_2 = \left(-t^{2(j_0+r_N)-\tilde{X}_N+1} - t^{N-\tilde{X}_N+1}\left(-t^{j_0+r_N-\tilde{X}_N} + (t^{j_0+r_N-\tilde{X}_N} - t^{N-\tilde{X}_N+1})\frac{1-t^{\tilde{X}_N}}{1-t}\right)\right) \\ \times \frac{(t;t)_{j_0+r_N-1}}{(t;t)_{j_0+r_N-\tilde{X}_N}} \begin{bmatrix} \tilde{X}_N \\ 1 \end{bmatrix}_t \tag{7.5.28}$$

(the rest of the $\ell = 1$ term), and

$$S_3 = \frac{(t;t)_{j_0+r_N-1}}{(t;t)_{j_0+r_N-\tilde{X}_N}} \sum_{\ell=2}^{\infty} \begin{bmatrix} \tilde{X}_N - 1 + \ell \\ \ell \end{bmatrix}_t \left(t^{\ell(j_0+r_N-\tilde{X}_N)}(1 - t^{j_0+r_N}) \right. \\ \left. - t^{\ell(N-\tilde{X}_N+1)}\left(1 - t^{j_0+r_N-\tilde{X}_N} + t^{j_0+r_N-\tilde{X}_N}(1 - t^{N-j_0-r_N+1})\frac{1-t^{\tilde{X}_N}}{1-t}\right)\right) \tag{7.5.29}$$

(the rest of the sum, i.e. the $\ell \geq 2$ terms). We have observed that $S_1 = 0$, and now argue that S_2 and S_3 are small asymptotically. The t-Pochhammer prefactor

$$\frac{(t;t)_{j_0+r_N-1}}{(t;t)_{j_0+r_N-\tilde{X}_N}}$$

lies in $[0,1]$, and the summands making up S_3 satisfy the bound

$$\left| t^{\ell(j_0+r_N-\tilde{X}_N)}(1-t^{j_0+r_N}) - t^{\ell(N-\tilde{X}_N+1)}\left(1-t^{j_0+r_N-\tilde{X}_N} + (t^{j_0+r_N-\tilde{X}_N} - t^{N-\tilde{X}_N+1})\frac{1-t^{\tilde{X}_N}}{1-t}\right)\right| \times \begin{bmatrix} \tilde{X}_N-1+\ell \\ \ell \end{bmatrix}_t \leq 3 \begin{bmatrix} \tilde{X}_N-1+\ell \\ \ell \end{bmatrix}_t t^{\ell(j_0+r_N-\tilde{X}_N)}, \quad (7.5.30)$$

hence

$$\begin{aligned} |S_3| &\leq 3\sum_{\ell=0}^{\infty} \begin{bmatrix} \tilde{X}_N-1+\ell \\ \ell \end{bmatrix}_t t^{\ell(j_0+r_N-\tilde{X}_N)} \\ &= 3\frac{1}{\prod_{i=0}^{\tilde{X}_N-1} 1-t^i \cdot t^{j_0+r_N-\tilde{X}_N}} \\ &\leq \frac{3}{(t;t)_\infty} \end{aligned} \quad (7.5.31)$$

for all $\tilde{X}_N < j_0 + r_N$ (using that $N+1 \geq j_0+r_N$). Similarly to (7.5.30), we may split S_2 into three terms with a power of t at least $2(j_0+r_N-\tilde{X}_N)$, yielding

$$|S_2| \leq \frac{3}{1-t}\begin{bmatrix} \tilde{X}_N \\ 1 \end{bmatrix}_t t^{2(j_0+r_N-\tilde{X}_N)}. \quad (7.5.32)$$

Below we use shorthand

$$\mathbb{1}_b := \mathbb{1}(\tilde{X}_N \leq r_N + j_0 - b) \quad (7.5.33)$$

to minimize equation overflow. Multiplying (7.5.25) by $\mathbb{1}_b$, taking an expectation, and applying Fubini's theorem (the hypotheses of which we checked in (7.5.31)) to pull it inside the sum yields

$$\begin{aligned} &\mathbb{E}\left[\mathbb{1}_b \times \left(1 - \frac{(t;t)_{j_0+r_N-1}(t;t)_{N-\tilde{X}_N}}{(t;t)_N(t;t)_{j_0+r_N-\tilde{X}_N}}\left(1-t^{j_0+r_N} + t^{j_0+r_N-\tilde{X}_N}(1-t^{N-j_0+r_N+1})\frac{1-t^{\tilde{X}_N}}{1-t}\right)\right)\right] \\ &= \mathbb{E}[\mathbb{1}_b S_2] + \sum_{\ell=2}^{\infty}\mathbb{E}\left[\mathbb{1}_b\frac{(t;t)_{j_0+r_N-1}}{(t;t)_{j_0+r_N-\tilde{X}_N}}\begin{bmatrix} \tilde{X}_N-1+\ell \\ \ell \end{bmatrix}_t \left(t^{\ell(j_0+r_N-\tilde{X}_N)}(1-t^{j_0+r_N})\right.\right. \\ &\left.\left. - t^{\ell(N-\tilde{X}_N+1)}\left(1-t^{j_0+r_N-\tilde{X}_N} + (t^{j_0+r_N-\tilde{X}_N} - t^{N-\tilde{X}_N+1})\frac{1-t^{\tilde{X}_N}}{1-t}\right)\right)\right], \end{aligned} \quad (7.5.34)$$

where we have also used that $S_1 = 0$ to throw away those corresponding terms of the sum. To argue that the remaining terms are small, we first note that by the first bound in (7.5.31), the naive bound

$$\begin{bmatrix} \tilde{X}_N - 1 + \ell \\ \ell \end{bmatrix}_t \le \frac{1}{(t;t)_\infty}, \tag{7.5.35}$$

the bound (7.5.32) on S_2, the nonnegativity of the arguments of all expectations, we have

$$\begin{aligned} |\mathrm{RHS}(7.5.34)| \le \frac{3}{(1-t)(t;t)_\infty} \mathbb{E}[\mathbb{1}_b t^{2(r_N + j_0 - \tilde{X}_N)}] \\ + \frac{3}{(1-t)(t;t)_\infty} \sum_{\ell=2}^{\infty} \mathbb{E}[\mathbb{1}_b t^{\ell(j_0 + r_N - \tilde{X}_N)}]. \end{aligned} \tag{7.5.36}$$

Applying (7.5.19) and collecting terms yields

$$\begin{aligned} \mathrm{RHS}(7.5.36) &\le \mathbb{E}[\mathbb{1}_b t^{r_N + j_0 - \tilde{X}_N}] \frac{3}{(1-t)(t;t)_\infty} \left(t^b + \sum_{\ell=2}^{\infty} t^{(\ell-1)b} \right) \\ &= C' t^b \mathbb{E}[\mathbb{1}_b t^{r_N + j_0 - \tilde{X}_N}] \end{aligned} \tag{7.5.37}$$

for an explicit constant C' independent of b and N. If $j_0 - b > 0$ then (recalling the shorthand $\mathbb{1}_b$ from (7.5.33)) we have

$$\begin{aligned} \mathrm{RHS}(7.5.37) &\le C' t^b \left(t^L \mathbb{E}[\mathbb{1}(\tilde{X}_N \le r_N) t^{r_N - \tilde{X}_N}] + \mathbb{E}[\mathbb{1}(r_N < \tilde{X}_N \le r_N + j_0 - b) t^{r_N + j_0 - \tilde{X}_N}] \right) \\ &\le C' t^b \left(t^L \mathbb{E}[\mathbb{1}(\tilde{X}_N \le r_N) t^{r_N - \tilde{X}_N}] + t^b \Pr(\tilde{X}_N > r_N) \right), \end{aligned} \tag{7.5.38}$$

while if $j_0 - b \le 0$ then

$$\mathrm{RHS}(7.5.37) \le t^L \mathbb{E}[\mathbb{1}(\tilde{X}_N \le r_N) t^{r_N - \tilde{X}_N}], \tag{7.5.39}$$

so the bound (7.5.38) actually holds independent of b and $j_0 \ge L$. Since $\tilde{X}_N \ge 1$,

$$t^{r_N - \tilde{X}_N} \le \frac{1}{1-t} (t^{r_N - \tilde{X}_N} - t^{r_N}), \tag{7.5.40}$$

and combining with (7.5.38) yields

$$\mathrm{RHS}(7.5.37) \leq \frac{C' t^L}{1-t} t^b \mathbb{E}[\mathbb{1}(\tilde{X}_N \leq r_N)(t^{r_N - \tilde{X}_N} - t^{r_N})] + C' t^{2b} \Pr(\tilde{X}_N > r_N). \tag{7.5.41}$$

Substituting (7.5.41) into (7.5.24) and multiplying through by $\Pr(X_N > 0)$ to convert the $\tilde{X}_N$ back to X_N yields

$$\begin{aligned}&\Pr\left(\sum_{i=j_0+r_N}^{N} \mathrm{SN}(A^{(N)} \operatorname{diag}(p^{\nu^{(N)}}))_i - \nu_i^{(N)} \geq 2\right) \\ &\leq \Pr(X_N > r_N + j_0 - b) + \frac{C' t^L}{1-t} t^b c_N^{-1} + C' t^{2b} \Pr(\tilde{X}_N > r_N).\end{aligned} \tag{7.5.42}$$

To show the right hand side is small, we let b depend on N as follows. Since

$$\Pr(X_N > r_N + j_0 - b) \leq \Pr(X_N > r_N + L - b) = o_{unif}(c_N^{-1}) \tag{7.5.43}$$

for any fixed b, by a diagonalization argument there exists a slowly growing sequence $b = b(N)$ not depending on j_0 such that

$$\Pr(X_N > r_N + j_0 - b(N)) = o_{unif}(c_N^{-1}). \tag{7.5.44}$$

Since (7.5.42) holds for any $b > 0$, it holds with b replaced by $b(N)$. Then the first term on the right hand side is $o_{unif}(c_N^{-1})$ by (7.5.44), the second term is $o_{unif}(c_N^{-1})$ because $b(N) \to \infty$, and the third term is $o_{unif}(c_N^{-1})$ as well by hypothesis. Hence

$$\Pr\left(\sum_{i=j_0+r_N}^{N} \mathrm{SN}(A^{(N)} \operatorname{diag}(p^{\nu^{(N)}}))_i - \nu_i^{(N)} \geq 2\right) = o_{unif}(c_N^{-1}), \tag{7.5.45}$$

so we are done. □

Proof of Lemma 7.3.2. First, since $|\kappa^{(N)}/\nu^{(N)}| = 1$ we may rewrite

$$\begin{aligned}&\mathrm{LHS}(7.3.5) = \Pr\left(\mathrm{SN}(A^{(N)} \operatorname{diag}(p^{\nu^{(N)}}))_i = \kappa_i^{(N)} \text{ for all } i \geq j + r_N\right) - \\ &\Pr\left(\mathrm{SN}(A^{(N)} \operatorname{diag}(p^{\nu^{(N)}}))_i = \kappa_i^{(N)} \text{ for all } i \geq j + r_N, \text{ and } |F_d(\mathrm{SN}(A^{(N)} \operatorname{diag}(p^{\nu^{(N)}})))/\kappa^{(N)}| \geq 2\right)\end{aligned} \tag{7.5.46}$$

By trivially bounding the second term in (7.5.46) by

$$\Pr(|F_d(\mathrm{SN}(A^{(N)}\operatorname{diag}(p^{\nu^{(N)}})))/\kappa^{(N)}| \geq 2) \tag{7.5.47}$$

and applying Lemma 7.3.3, (7.5.46) yields

$$\mathrm{LHS}(7.3.5) = \Pr\left(\mathrm{SN}(A^{(N)}\operatorname{diag}(p^{\nu^{(N)}}))_i = \kappa_i^{(N)} \text{ for all } i \geq j + r_N\right) + o_{unif}(c_N^{-1}) \tag{7.5.48}$$

uniformly over $\nu^{(N)}$ as in the statement. For any integer $b \geq \max(0, L)$, we may therefore write

$$\begin{aligned}
&\mathrm{LHS}(7.3.5) = o_{unif}(c_N^{-1}) \\
&+ \Pr\left(\mathrm{SN}(A^{(N)}\operatorname{diag}(p^{\nu^{(N)}}))_i = \kappa_i^{(N)} \text{ for all } i \geq j + r_N \text{ and } X_N > r_N + L - b\right) \\
&+ \Pr\left(\mathrm{SN}(A^{(N)}\operatorname{diag}(p^{\nu^{(N)}}))_i = \kappa_i^{(N)} \text{ for all } i \geq j + r_N \text{ and } X_N \leq r_N + L - b\right)
\end{aligned} \tag{7.5.49}$$

For the first summand in (7.5.49) a naive bound gives

$$\begin{aligned}
\Pr&\left(\mathrm{SN}(A^{(N)}\operatorname{diag}(p^{\nu^{(N)}}))_i = \kappa_i^{(N)} \text{ for all } i \geq j + r_N \text{ and } X_N > r_N + L - b\right) \\
&\leq \Pr(X_N > r_N + L - b).
\end{aligned} \tag{7.5.50}$$

Substituting this and applying Lemma 7.4.2 (with $r = r_N + L, \operatorname{len}(\lambda) = X_N$) to the second summand in (7.5.49), we obtain upper and lower bounds

$$\begin{aligned}
&\mathbb{E}\left[(1 - t^{j+r_N-X_N}) \cdot \mathbb{1}(X_N \leq r_N + L - b) t^j \frac{1-t^m}{1-t}(t^{r_N - X_N} - t^{r_N}) \frac{(t;t)_{j+r_N}(t;t)_{N-X_N}}{(t;t)_N (t;t)_{j+r_N-X_N}}\right] \\
&\qquad + \Pr(X_N > r_N + L - b) + o_{unif}(c_N^{-1}) \\
&\leq \mathrm{LHS}(7.3.5) \\
&\leq \mathbb{E}\left[\mathbb{1}(X_N \leq r_N + L - b) t^j \frac{1-t^m}{1-t}(t^{r_N - X_N} - t^{r_N}) \frac{(t;t)_{j+r_N}(t;t)_{N-X_N}}{(t;t)_N (t;t)_{j+r_N-X_N}}\right] \\
&\qquad + \Pr(X_N > r_N + L - b) + o_{unif}(c_N^{-1})
\end{aligned} \tag{7.5.51}$$

for any $b > 0$ (this condition is required since Lemma 7.4.2 only applies when $X_N < j + r_N$). We will show both bounds have the same asymptotic to obtain the asymptotic for (7.3.5). The difference between the two bounds in (7.5.51) is

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}(X_N \le r_N + L - b)\frac{1-t^m}{1-t}t^{2(j+r_N-X_N)}(1-t^{X_N})\frac{(t;t)_{j+r_N}(t;t)_{N-X_N}}{(t;t)_N(t;t)_{j+r_N-X_N}}\right] \\
&\le \frac{1}{(1-t)(t;t)_\infty}\mathbb{E}[\mathbb{1}(X_N \le r_N + L - b)t^{2(j+r_N-X_N)}(1-t^{X_N})] \\
&\le \frac{1}{(1-t)(t;t)_\infty}t^{j+r_N-(r_N+L-b)}\mathbb{E}[\mathbb{1}(X_N \le r_N + L - b)t^{j+r_N-X_N}(1-t^{X_N})] \\
&\le \frac{1}{(1-t)(t;t)_\infty}t^{b+2j-L}\mathbb{E}[\mathbb{1}(X_N \le r_N)(t^{r_N-X_N} - t^{r_N})] \\
&\le \frac{1}{(1-t)(t;t)_\infty}t^{b+L}c_N^{-1}
\end{aligned} \tag{7.5.52}$$

where we used (7.5.19) in the second bound, and the fact that $b \ge L$ and $j \ge L$ in the penultimate and last bounds respectively. Plugging (7.5.52) into (7.5.51) yields

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}(X_N \le r_N + L - b)t^j\frac{1-t^m}{1-t}(t^{r_N-X_N} - t^{r_N})\frac{(t;t)_{j+r_N}(t;t)_{N-X_N}}{(t;t)_N(t;t)_{j+r_N-X_N}}\right] \\
&- \frac{1}{(1-t)(t;t)_\infty}t^{b+L}c_N^{-1} + \Pr(X_N > r_N + L - b) \\
&\le \Pr\left(\mathrm{SN}(A^{(N)}\operatorname{diag}(p^{\nu^{(N)}}))_i = \kappa_i^{(N)} \text{ for all } i \ge j\right) \\
&\le \mathbb{E}\left[\mathbb{1}(X_N \le r_N + L - b)t^j\frac{1-t^m}{1-t}(t^{r_N-X_N} - t^{r_N})\frac{(t;t)_{j+r_N}(t;t)_{N-X_N}}{(t;t)_N(t;t)_{j+r_N-X_N}}\right] \\
&\qquad + \Pr(X_N > r_N + L - b).
\end{aligned} \tag{7.5.53}$$

We now wish to show that the $\mathbb{E}[\cdots]$ in the lower and upper bounds is uniformly asymptotic to $c_N^{-1}t^j(1-t^m)/(1-t)$. Note that the q-Pochhammer quotient in (7.5.53) is

$$\frac{(t;t)_{j+r_N}(t;t)_{N-X_N}}{(t;t)_N(t;t)_{j+r_N-X_N}} = \prod_{i=1}^{X_N}\frac{1-t^{j+r_N-X_N+i}}{1-t^{N-X_N+i}}. \tag{7.5.54}$$

For $X_N \le r_N + L$, the above is ≤ 1 since $r_N + L \le j + r_N \le N$, and furthermore it is decreasing function of $X_N \in [[r_N + L]]$. Hence since $b \ge 0$ we have

$$\begin{aligned}
0 &\le \mathbb{1}(X_N \le r_N + L - b)\left(1 - \prod_{i=1}^{X_N}\frac{1-t^{j+r_N-X_N+i}}{1-t^{N-X_N+i}}\right) \\
&\le \mathbb{1}(X_N \le r_N + L - b)\left(1 - \prod_{i=1}^{r_N+j-b}\frac{1-t^{b+i}}{1-t^{b+(N-r_N-j)+i}}\right).
\end{aligned} \tag{7.5.55}$$

A naive bound gives

$$1-\prod_{i=1}^{r_N+j-b}\frac{1-t^{b+i}}{1-t^{b+(N-r_N-j)+i}}\leq 1-\prod_{i=1}^{r_N+j-b}(1-t^{b+i})\leq 1-(t^b;t)_\infty\leq Ct^b. \tag{7.5.56}$$

for some constant C and all large b. Hence

$$0\leq \mathbb{1}(X_N\leq r_N+L-b)\left(1-\frac{(t;t)_{j+r_N}(t;t)_{N-X_N}}{(t;t)_N(t;t)_{j+r_N-X_N}}\right)\leq Ct^b \tag{7.5.57}$$

Plugging in the formula for c_N^{-1} and (7.5.57) yields the bound

$$\begin{aligned}
&\left|\mathbb{E}\left[\mathbb{1}(X_N\leq r_N+L-b)t^j\frac{1-t^m}{1-t}(t^{r_N-X_N}-t^{r_N})\frac{(t;t)_{j+r_N}(t;t)_{N-X_N}}{(t;t)_N(t;t)_{j+r_N-X_N}}-t^j\frac{1-t^m}{1-t}c_N^{-1}\right]\right|\\
\leq&\frac{1-t^m}{1-t}t^j\left|\mathbb{E}\left[\mathbb{1}(X_N\leq r_N+L-b)\left(1-\frac{(t;t)_{j+r_N}(t;t)_{N-X_N}}{(t;t)_N(t;t)_{j+r_N-X_N}}\right)(t^{r_N-X_N}-t^{r_N})\right]\right|\\
&+\frac{1-t^m}{1-t}t^j\left|\mathbb{E}\left[\mathbb{1}(r_N+L-b<X_N\leq r_N)(t^{r_N-X_N}-t^{r_N})\right]\right|\\
\leq&Ct^b\frac{1-t^m}{1-t}t^j\mathbb{E}\left[\mathbb{1}(X_N\leq r_N+L-b)(t^{r_N-X_N}-t^{r_N})\right]\\
&+\frac{1-t^m}{1-t}t^j\mathbb{E}\left[\mathbb{1}(r_N+L-b<X_N\leq r_N)(t^{r_N-X_N}-t^{r_N})\right]
\end{aligned} \tag{7.5.58}$$

using that $b\geq L$ (otherwise the bounds in the last indicator function would be reversed). Since $r_N+L-b\leq r_N$ and the argument of the expectation is nonnegative,

$$\mathbb{E}\left[\mathbb{1}(X_N\leq r_N+L-b)(t^{r_N-X_N}-t^{r_N})\right]\leq c_N^{-1}. \tag{7.5.59}$$

Furthermore,

$$\mathbb{E}\left[\mathbb{1}(r_N+L-b<X_N\leq r_N)(t^{r_N-X_N}-t^{r_N})\right]\leq(1-t^{r_N})\Pr(X_N>r_N+L-b). \tag{7.5.60}$$

We thus obtain

$$\mathrm{RHS}(7.5.58)\leq C\frac{t^{b+j}}{1-t}c_N^{-1}+\frac{t^j}{1-t}\Pr(X_N>r_N+L-b). \tag{7.5.61}$$

Finally, we let b depend on N as follows. Since

$$\Pr(X_N>r_N+L-b)=o_{unif}(c_N^{-1}) \tag{7.5.62}$$

for all b, by a diagonalization argument there exists a slowly growing sequence $b = b(N)$ such that

$$\Pr(X_N > r_N + L - b(N)) = o_{unif}(c_N^{-1}) \tag{7.5.63}$$

(the uniformity over j is obvious here but we keep the o_{unif} notation anyway). Substituting (7.5.58), (7.5.61), and (7.5.63) to simplify the upper and lower bounds in our original inequality (7.5.53) thus yields that the inequalities

$$\begin{aligned} &O_{unif}(t^{b(N)}c_N^{-1}) + o_{unif}(c_N^{-1}) + \frac{1-t^m}{1-t}t^j c_N^{-1} \\ &\leq \Pr\left(\mathrm{SN}(A^{(N)}\operatorname{diag}(p^{\nu^{(N)}}))_i = \kappa_i^{(N)} \text{ for all } i \geq j\right) \\ &\leq O_{unif}(t^{b(N)}c_N^{-1}) + o_{unif}(c_N^{-1}) + \frac{1-t^m}{1-t}t^j c_N^{-1}, \end{aligned} \tag{7.5.64}$$

with implied constants which are uniform over all $j \geq L$, hold for all N sufficiently large that $r_N + L - b(N) \geq 0$. Since $b(N) \to \infty$, this shows that

$$\Pr\left(\mathrm{SN}(A^{(N)}\operatorname{diag}(p^{\nu^{(N)}}))_i = \kappa_i^{(N)}\right) = \frac{1-t^m}{1-t}t^j c_N^{-1} + o_{unif}(c_N^{-1}). \tag{7.5.65}$$

□

Chapter 8

The $p \to 1$ limit

In Section 8.1 we introduce discrete-time Markovian dynamics on the boundary, and prove that their continuous-time Poisson limit is equivalent to slowed t-TASEP. In Section 8.2 we state a contour integral formula for observables of this process. We use these in Section 8.3 to prove the law of large numbers Theorem 1.6.1 as $t \to 1$. In Section 8.4 we show Gaussian fluctuations, and the long-time simplification of covariances Proposition 8.4.2 which is half of Theorem 1.6.2. The probabilistic justification of this additional limit via the SDEs in Theorem 1.6.2 is shown in Section 8.5. In Section 8.6 we prove the bulk limit to the Gaussian process given in Theorem 1.6.3.

8.1 Between the slowed t-TASEP and Hall-Littlewood processes

In this section we formally define slowed t-TASEP, and show in Theorem 8.1.1 that it is equivalent (in the case of packed initial condition) to a Hall-Littlewood process with one Plancherel specialization and one principal specialization $1, t, \ldots$.

Definition 63. Let

$$\mathbb{X} := \{(x_1, x_2, \ldots) \in \mathbb{Z}^{\mathbb{N}} : x_1 > x_2 > \cdots\}$$

be the space of particle configurations on $\mathbb{Z}$, where the x_i is the position of the i^{th} particle from the right, and

$$\mathbb{X}_0 := \{(x_1, x_2, \dots) \in \mathbb{X} : x_i = -i \text{ for all sufficiently large } i\}.$$

We denote particle configurations $(x_1, x_2, \dots)$ by $\mathbf{x}$, and if $x_{k-1} > x_k + 1$ we write $\mathbf{x}^k := (x_1, \dots, x_{k-1}, x_k + 1, x_{k+1}, \dots)$.

Definition 64. *Slowed t-TASEP with initial condition* $\mathbf{x}_0 \in \mathbb{X}$ is the continuous-time stochastic process $\mathbf{x}_t(\tau) = (x_1(\tau), x_2(\tau), \dots)$ on $\mathbb{X}$ in which $\mathbf{x}_t(0) = \mathbf{x}_0$ and the particles at positions $x_k, k \geq 1$ each have independent Poisson clocks with rates $t^{x_k+k}(1 - t^{x_{k-1}-x_k-1})$, and jump to the right by 1 when they ring. Equivalently, it is defined by the Markov generator

$$\left.\frac{d}{d\tau}\right|_{\tau=0} \Pr(\mathbf{x}_t(T+\tau) = \mathbf{y} | \mathbf{x}_t(T) = \mathbf{x}) = \begin{cases} t^{x_k+k}(1 - t^{x_{k-1}-x_k-1}) & \mathbf{y} = \mathbf{x}^k \text{ for some } k \in \mathbb{Z}_{>0} \\ -1 & \mathbf{y} = \mathbf{x} \\ 0 & \text{otherwise} \end{cases}. \tag{8.1.1}$$

We refer to the initial condition $(-1, -2, \dots)$ as *packed*.

Recall the notation of the Plancherel/principal Hall-Littlewood process $\lambda^{(\infty)}(\tau)$ from Definition 47. For this section we write $\lambda(\tau) = \lambda^{(\infty)}(\tau)$ to pare down notation.

Theorem 8.1.1. *In the notations of Definition 47 and Definition 64,*

$$\mathbf{x}_t(\tau) = (\lambda'_k((1-t)\tau) - k)_{k \geq 1}$$

in (multi-time) distribution, where $\mathbf{x}_t(\tau)$ is a slowed t-TASEP with packed initial condition and parameter t.

Proof. Follows by comparing the generator of slowed t-TASEP in Definition 64 with that of $\lambda(\tau)$, computed in Lemma 6.2.2, and appealing to Proposition 6.2.1. □

8.2 A contour integral formula for t-moments

In this section we we prove contour integral formulas for certain t-moment observables of this particle system, which will be the main tool in subsequent asymptotic results. We again take $t \in (0,1)$, and denote the Weyl denominator/Vandermonde determinant by

$$\Delta(z_1, \dots, z_n) := \prod_{1 \le i < j \le n} (z_i - z_j).$$

Proposition 8.2.1. *Let $\lambda(\tau)$ be distributed as a Hall-Littlewood measure with specializations $1, t, \dots$ and $\gamma(\tau)$, as defined in Chapter 2. Then for any positive integers $r_1, \dots, r_M$,*

$$\begin{aligned}
&\mathbb{E}\left[t^{-\sum_{m=1}^{M}\sum_{j=1}^{r_m}\lambda'_j(\tau)}\right] \\
&= \left(\prod_{m=1}^{M}\frac{(-1)^{\binom{r_m}{2}}}{r_m!(2\pi i)^{r_m}}\right)\oint\cdots\oint\prod_{m=1}^{M}\left(\Delta(z_{1,m},\dots,z_{r_m,m})^2\prod_{s=1}^{r_m}\frac{e^{\tau z_{s,m}}(1+t^{-1}z_{s,m}^{-1})}{z_{s,m}^{r_m}}\right) \\
&\times \prod_{1\le\alpha<\beta\le M}\prod_{\substack{1\le i\le r_\alpha\\1\le j\le r_\beta}}\frac{1-z_{j,\beta}/z_{i,\alpha}}{1-t^{-1}z_{j,\beta}/z_{i,\alpha}}dz_{1,1}\cdots dz_{r_M,M}.
\end{aligned} \tag{8.2.1}$$

with all contours encircling 0 *and satisfying*

$$\begin{aligned}
|z_{j,\beta}| &< t|z_{i,\alpha}| \text{ for all } 1 \le \alpha < \beta \le M, 1 \le i \le r_\alpha, 1 \le j \le r_\beta \\
|z_{s,\alpha}| &< t^{-1} \text{ for all } 1 \le \alpha \le M, 1 \le s \le r_\alpha.
\end{aligned}$$

Proof. First consider a partition $\mu(D,\tau)$ distributed as a Hall-Littlewood process with alpha specializations $1, t, \dots$ and $\left(\frac{\tau}{1-t}\frac{1}{D}\right)[D]$, where again $[D]$ denotes D copies of the same specialization. We recall that this latter specialization is an approximation to the Plancherel specialization $\gamma(\tau)$ and converges to it as $D \to \infty$ by Lemma 2.2.7.

Specializing [BM18, Thm. 2.12] to our case[1] yields

$$\mathbb{E}\left[t^{-\sum_{m=1}^M \sum_{j=1}^{r_m} \mu(D,\tau)_j'}\right] = \prod_{m=1}^M \frac{(-1)^{\binom{r_m}{2}}}{r_m!(2\pi i)^{r_m}} \oint \cdots \oint$$
$$\prod_{m=1}^M \left(\frac{\prod_{1 \le i < j \le r_m} (z_{i,m} - z_{j,m})^2}{(z_{1,m} \cdots z_{r_m,m})^{r_m}} \prod_{s=1}^{r_m} \left(\prod_{j=1}^D \frac{1 + z_{s,m} \frac{\tau}{1-t} \frac{1}{D}}{1 + t z_{s,m} \frac{\tau}{1-t}\frac{1}{D}} \right) \left(\prod_{i \ge 1} \frac{1 + z_{s,m}^{-1} t^{i-2}}{1 + z_{s,m}^{-1} t^{i-1}} \right) \right) \tag{8.2.2}$$
$$\cdot \prod_{1 \le \alpha < \beta \le M} \prod_{\substack{1 \le i \le r_\alpha \\ 1 \le j \le r_\beta}} \frac{1 - z_{j,\beta}/z_{i,\alpha}}{1 - t^{-1} z_{j,\beta}/z_{i,\alpha}} dz_{1,1} \cdots dz_{r_M,M}$$

with all contours encircling 0 and

$$\begin{aligned} |z_{i,\alpha}| < t|z_{j,\beta}| \qquad & \text{for all} \qquad 1 \le \alpha < \beta \le M, 1 \le i \le r_\alpha, 1 \le j \le r_\beta \\ \frac{\tau}{1-t}\frac{1}{D} < |z_{s,\alpha}| < t^{-1} \qquad & \text{for all} \qquad 1 \le \alpha \le M, 1 \le s \le r_\alpha \end{aligned}$$

provided such contours exist. We note that for any fixed $t \in (0,1)$ and $\tau \ge 0$, such contours exist for all D sufficiently large. Picking a choice of contours, we have

$$\lim_{D \to \infty} \mathrm{RHS}(8.2.2) = \prod_{m=1}^M \frac{(-1)^{\binom{r_m}{2}}}{r_m!(2\pi i)^{r_m}} \oint \cdots \oint \prod_{m=1}^M \left(\Delta(z_{1,m}, \ldots, z_{r_m,m})^2 \prod_{s=1}^{r_m} \frac{(1 + t^{-1} z_{s,m}^{-1}) e^{\tau z_{s,m}}}{z_{s,m}^{r_m}} \right)$$
$$\cdot \prod_{1 \le \alpha < \beta \le M} \prod_{\substack{1 \le i \le r_\alpha \\ 1 \le j \le r_\beta}} \frac{1 - z_{j,\beta}/z_{i,\alpha}}{1 - t^{-1} z_{j,\beta}/z_{i,\alpha}} dz_{1,1} \cdots dz_{r_M,M} \tag{8.2.3}$$

where the limit commutes with the integral because the integrand remains bounded as $D \to \infty$ and the contours are compact.

It now suffices to show convergence of the left hand side of (8.2.2) to that of (8.2.1), i.e. we must show

$$\begin{aligned} &\lim_{D \to \infty} \frac{1}{\Pi\left(1, t, \ldots; \frac{\tau}{1-t}\frac{1}{D}[D]\right)} \sum_{\lambda \in \mathbb{Y}} Q_\lambda(1, t, \ldots) P_\lambda\left(\frac{\tau}{1-t}\frac{1}{D}[D]\right) t^{-\sum_{m=1}^M \sum_{j=1}^{r_m} \lambda_j'} \\ &= \frac{1}{\Pi(1, t, \ldots; \gamma(\tau))} \sum_{\lambda \in \mathbb{Y}} Q_\lambda(1, t, \ldots) P_\lambda(\gamma(\tau)) t^{-\sum_{m=1}^M \sum_{j=1}^{r_m} \lambda_j'} \end{aligned} \tag{8.2.4}$$

---

[1] In the notation of [BM18, Thm. 2.12], we are taking $N = M$, $X_1 = \left(\frac{\tau}{1-t}\frac{1}{D}\right)[D]$, $X_2 = \cdots = X_M = 0$ and $Y_1 = \cdots = Y_{M-1} = 0, Y_M = 1, t, \ldots$. Then the product over $1 \le \alpha \le \beta \le M$ in the fourth line of (2.22) of [BM18] only gives nontrivial terms when $\alpha = 1$ or $\beta = M$.

It follows simply from the definition by (2.2.30) that

$$\lim_{D\to\infty} \frac{1}{\Pi\left(1,t,\ldots;\frac{\tau}{1-t}\frac{1}{D}[D]\right)} = \frac{1}{\Pi(1,t,\ldots;\gamma(\tau))},$$

so it suffices to show that the sum on the LHS of (8.2.4) converges to the one on the RHS. We may write the sum on the LHS (resp. RHS) as an integral of the function $f_D(\lambda) = Q_\lambda(1,t,\ldots)P_\lambda(\frac{\tau}{1-t}\frac{1}{D}[D])$ (resp. $f(\lambda) = Q_\lambda(1,t,\ldots)P_\lambda(\gamma(\tau)))$ with respect to the measure on the discrete set $\mathbb{Y}$ determined by $\mathrm{meas}(\{\lambda\}) = t^{-\sum_{m=1}^{M}\sum_{j=1}^{r_m}\lambda'_j}$. By Lemma 2.2.7, $f_D(\lambda)$ converges monotonically from below to $f(\lambda)$, hence the monotone convergence theorem yields the desired convergence of sums, and (8.2.4) follows.

The limit $D \to \infty$ also makes the contour condition $\frac{\tau}{1-t}\frac{1}{D} < |z_{s,\alpha}|$ automatic, so combining (8.2.3) and (8.2.4) completes the proof. □

8.3 Law of large numbers

In this section we establish the law of large numbers for particle positions as $t \to 1$, recalled below.

Theorem 1.6.1. *Let $(x_1(s), x_2(s), \ldots), s \in \mathbb{R}_{\geq 0}$ be the particles of slowed t-TASEP with $t = e^{-\epsilon}$. Then for any $\tau > 0$ and $k \in \mathbb{Z}_{>0}$,*

$$\epsilon \cdot x_k(\tau/\epsilon) \to \log\left(\sum_{j=0}^{k}\frac{\tau^j}{j!}\right) - \log\left(\sum_{j=0}^{k-1}\frac{\tau^j}{j!}\right) \qquad \text{in probability as } \epsilon \to 0^+.$$

We begin with a straightforward heuristic derivation by taking a continuum limit of jump rates to obtain an ODE for particle positions, then give a rigorous proof using the observables in Proposition 8.2.1.

First, we wish to see the scaling of $t = e^{-\epsilon}$, space and time such that both the particle positions and jump rates converge to nontrivial limits. The first particle $x_1(\tau)$ jumps as a rate-1 Poisson process as $t \to 1$, so for it to converge to a nontrivial limit, we should wait a long time (take time to be $\approx \tau/\epsilon$) and rescale space by ϵ, i.e. we should consider $\epsilon x_1(\tau/\epsilon)$. For the first particle, the law of large numbers guarantees concentration, though arguing for the others would be slightly more involved; however, let us suppose

$$\epsilon x_k(\tau/\epsilon) \to c_k(\tau)$$

for some functions $c_1, c_2, \ldots$.

Then we have convergence of jump rates

$$t^{x_k(\tau/\epsilon)+k}(1 - t^{x_{k-1}(\tau/\epsilon)-x_k(\tau/\epsilon)-1}) \to e^{-c_k(\tau)}(1 - e^{c_{k-1}(\tau)-c_k(\tau)}) = e^{-c_k(\tau)} - e^{-c_{k-1}(\tau)}$$

where when $k = 1$ we take $e^{-c_{k-1}(\tau)} = 0$. Because we are scaling time as ϵ^{-1} and then rescaling space by ϵ, by concentration of Poisson variables we should have

$$\lim_{\epsilon \to 0^+} (\text{jump rate of } x_k(\tau/\epsilon)) = \frac{dc_k(\tau)}{d\tau}.$$

Hence the functions $c_k(\tau)$ should satisfy

$$\frac{dc_k(\tau)}{d\tau} = e^{-c_k(\tau)} - e^{-c_{k-1}(\tau)} \qquad \text{for all } k \geq 1, \tag{8.3.1}$$

where when $k = 1$ we take $c_0 = \infty$ so the second term on the RHS is not present. It is easy to verify by inspection that the limits given in Theorem 1.6.1, namely

$$c_k(\tau) := \log\left(\sum_{j=0}^{k} \frac{\tau^j}{j!}\right) - \log\left(\sum_{j=0}^{k-1} \frac{\tau^j}{j!}\right), \tag{8.3.2}$$

satisfy (8.3.1), and furthermore have initial conditions $c_k(0) = 0$ as they should. This concludes the heuristic derivation of Theorem 1.6.1, and we move on to the proof.

Proof of Theorem 1.6.1. We first claim that it suffices to show the same limit as in Theorem 1.6.1 for the slightly different quantity $\epsilon x_k\left(\frac{\tau}{1-e^{-\epsilon}}\right)$. Assuming this result and setting $\tilde{\epsilon} = -\log(1-\epsilon)$ so that $\epsilon = 1 - e^{-\tilde{\epsilon}}$, we have

$$\epsilon x_k(\tau/\epsilon) = \left(1 - e^{-\tilde{\epsilon}}\right) x_k\left(\frac{\tau}{1-e^{-\tilde{\epsilon}}}\right) = \left(\tilde{\epsilon} + O(\tilde{\epsilon}^2)\right) x_k\left(\frac{\tau}{1-e^{-\tilde{\epsilon}}}\right),$$

hence convergence in probability for $\epsilon x_k\left(\frac{\tau}{1-e^{-\epsilon}}\right)$ implies the same for $\epsilon x_k(\tau/\epsilon)$. Since

$$x_k\left(\frac{\tau}{1-t}\right) = \lambda_k(\tau)' - k \qquad \text{in distribution}$$

by Theorem 8.1.1, it suffices to show that

$$\epsilon\lambda_k(\tau) \to \log\left(\sum_{j=0}^{k}\frac{\tau^j}{j!}\right) - \log\left(\sum_{j=0}^{k-1}\frac{\tau^j}{j!}\right) \qquad \text{in probability as } \epsilon \to 0^+. \tag{8.3.3}$$

It therefore suffices to show the convergence of Laplace transforms

$$e^{\epsilon\sum_{j=1}^{r}\lambda_j(\tau)} \to \sum_{j=0}^{r}\frac{\tau^j}{j!} \qquad \text{in probability as } \epsilon \to 0^+ \tag{8.3.4}$$

for each r, as then $\epsilon\sum_{j=1}^{r}\lambda_j(\tau)$ converges in probability, and taking differences yields (8.3.3). Let $Y_r(t) = e^{\epsilon\sum_{j=1}^{r}\lambda_j(\tau)}$ (recall $t = e^{-\epsilon}$). By Chebyshev's inequality, to show (8.3.4) it suffices to show

$$\mathbb{E}[Y_r(t)] \to \sum_{j=0}^{r}\frac{\tau^j}{j!} \tag{8.3.5}$$

and

$$\mathrm{Var}(Y_r(t)) \to 0 \tag{8.3.6}$$

as $\epsilon \to 0^+$.

By Proposition 8.2.1,

$$\mathbb{E}[Y_r(t)] = \frac{(-1)^{\binom{r}{2}}}{r!(2\pi\mathbf{i})^r}\oint\cdots\oint\prod_{1\le i<j\le r}(z_i-z_j)^2\prod_{s=1}^{r}\frac{1+t^{-1}z_s^{-1}}{z_s^{r-1}}e^{\tau z_s}\frac{dz_s}{z_s}, \tag{8.3.7}$$

where all contours are circles around the origin of radius ≤ 1, and

$$\begin{aligned}\mathbb{E}[Y_r(t)^2] =& \frac{1}{(r!)^2(2\pi\mathbf{i})^{2r}}\oint\cdots\oint\prod_{1\le i,j\le r}\frac{1-z_{j,2}/z_{i,1}}{1-t^{-1}z_{j,2}/z_{i,1}}\\ &\cdot\prod_{\ell=1}^{2}\left(\prod_{1\le i<j\le r}(z_{i,\ell}-z_{j,\ell})^2\prod_{s=1}^{r}\frac{1+t^{-1}z_{s,\ell}^{-1}}{z_{s,\ell}^{r-1}}e^{\tau z_{s,\ell}}\frac{dz_{s,\ell}}{z_{s,\ell}}\right)\end{aligned} \tag{8.3.8}$$

where we take the $z_{s,1}$ and $z_{s,2}$ contours to be circles of radii 1 and R respectively, for some $0 < R < 1$ fixed independent of ϵ (for all ϵ sufficiently small that such contours satisfy the conditions in Proposition 8.2.1). It follows by combining two instances of (8.3.7) that

$$\mathbb{E}[Y_r(t)]^2 = \frac{1}{(r!)^2(2\pi\mathbf{i})^{2r}} \oint \cdots \oint \prod_{\ell=1}^{2} \left(\prod_{1 \le i < j \le r} (z_{i,\ell} - z_{j,\ell})^2 \prod_{s=1}^{r} \frac{1 + t^{-1} z_{s,\ell}^{-1}}{z_{s,\ell}^{r-1}} e^{\tau z_{s,\ell}} \frac{dz_{s,\ell}}{z_{s,\ell}} \right)$$

with the same contours as in (8.3.8), and combining with (8.3.8) yields

$$\begin{aligned} \mathrm{Var}(Y_r(t)) =& \frac{1}{(r!)^2(2\pi\mathbf{i})^{2r}} \oint \cdots \oint \left(\prod_{1 \le i,j \le r} \frac{1 - z_{j,2}/z_{i,1}}{1 - t^{-1} z_{j,2}/z_{i,1}} - 1 \right) \\ & \cdot \prod_{\ell=1}^{2} \left(\prod_{1 \le i < j \le r} (z_{i,\ell} - z_{j,\ell})^2 \prod_{s=1}^{r} \frac{1 + t^{-1} z_{s,\ell}^{-1}}{z_{s,\ell}^{r-1}} e^{\tau z_{s,\ell}} \frac{dz_{s,\ell}}{z_{s,\ell}} \right). \end{aligned} \tag{8.3.9}$$

The contours in (8.3.9) are compact and independent of t, and due to the

$$\prod_{1 \le i,j \le r} \frac{1 - z_{j,2}/z_{i,1}}{1 - t^{-1} z_{j,2}/z_{i,1}} - 1$$

term the integrand converges to 0 as $t \to 1$, which yields (8.3.6). It remains to show (8.3.5). Taking all contours in (8.3.7) to be the unit circle so $z_i^{-1} = \bar{z}_i$, by rewriting the Weyl denominator

$$\prod_{1 \le i < j \le r} z_i - z_j = (-1)^{\binom{r}{2}} \left(\prod_{1 \le i < j \le r} \bar{z}_i - \bar{z}_j \right) \prod_{s=1}^{r} z_s^{r-1}$$

we have

$$\mathbb{E}[Y_r(t)] = \frac{1}{r!(2\pi\mathbf{i})^r} \oint \cdots \oint \prod_{1 \le i < j \le r} |z_i - z_j|^2 \prod_{s=1}^{r} (1 + t^{-1} \bar{z}_s) e^{\tau z_s} \frac{dz_s}{z_s}. \tag{8.3.10}$$

It is a classical fact, which follows from the Weyl character formula, Weyl integration formula and character orthogonality (or from the generalization to Macdonald polynomials in [Mac98a, Chapter VI.9]), that the Schur polynomials $s_\lambda(z_1, \ldots, z_r) = P_\lambda(z_1, \ldots, z_r; t = 0)$ are orthonormal with respect to the inner product

$$\langle f, g \rangle = \frac{1}{r!(2\pi\mathbf{i})^r} \oint \cdots \oint \prod_{1 \le i < j \le r} |z_i - z_j|^2 f(z_1, \ldots, z_r) \overline{g(z_1, \ldots, z_r)} \prod_{s=1}^{r} \frac{dz_s}{z_s}$$

where the integrals are over the unit circle in $\mathbb{C}$. Hence to compute (8.3.10) it suffices to expand $e^{\tau \sum_{s=1}^{r} z_s}$ and $\prod_{s=1}^{r}(1+t^{-1}\overline{z_s})$ in terms of the Schur polynomials. We have

$$\prod_{s=1}^{r}(1+t^{-1}\bar{z}_s) = \sum_{k=0}^{r} t^{-k}\overline{e_k(z_1,\dots,z_r)} \tag{8.3.11}$$

where $e_k(z_1,\dots,z_r) = s_{(1[k])}(z_1,\dots,z_r)$ is the elementary symmetric polynomial, and

$$e^{\tau \sum_{s=1}^{r} z_s} = \sum_{j\geq 0} \frac{\tau^j}{j!} e_1(z_1,\dots,z_r)^j. \tag{8.3.12}$$

It follows from the classical Pieri rule for Schur functions, see for example [Mac98a], that for $1 \leq j \leq r$

$$e_1(z_1,\dots,z_r)^j = s_{(1[j])}(z_1,\dots,z_r) + \dots \tag{8.3.13}$$

when expanded in the basis of Schur functions, where the other terms on the RHS of (8.3.13) are Schur functions s_λ where $|\lambda| = j$ and $\lambda \neq (1[j])$. We therefore have

$$\langle e_k, e_1^j \rangle = \delta_{j,k}. \tag{8.3.14}$$

Combining (8.3.11), (8.3.12) and (8.3.14) yields that

$$\mathrm{RHS}(8.3.10) = \sum_{j=0}^{r} \frac{\tau^j}{j!} t^{-j}.$$

Sending $t \to 1$ and tracing back the chain of equalities, this shows (8.3.5) and hence completes the proof. $\square$

8.4 Gaussian fluctuations

In this section we move on from the law of large numbers to study the fluctuations of the particle positions $x_k(\tau)$. Proposition 8.4.1 uses general machinery of [BG15] to show Gaussian fluctuations for the particle positions $x_k\left(\frac{\tau}{1-t}\right)$ and gives a formula for their covariance, but the number of contour integrals in the formula grows with the particle index, making it intractable asymptotically. Taking a further $\tau \to \infty$ limit, this covariance converges (without rescaling) to an expression which can be simplified to a double contour integral with the aid of orthogonal polynomial techniques similar to those used in [BCF18, §5.1].

Definition 65. Letting x_i be the position of the i^{th} particle of slowed t-TASEP with parameter $t = e^{-\epsilon}$, we define $X_\tau^{(i,\epsilon)}$ by

$$x_i\left(\frac{\tau}{1-t}\right) = \mathbb{E}\left[x_i\left(\frac{\tau}{1-t}\right)\right] + \epsilon^{-1/2} X_\tau^{(i,\epsilon)}.$$

Proposition 8.4.1. *For any $n \in \mathbb{Z}_{\geq 1}$, the random vector $(X_\tau^{(1,\epsilon)}, \ldots, X_\tau^{(n,\epsilon)})$ converges in distribution as $\epsilon \to 0^+$ to a mean 0 Gaussian random vector $(X_\tau^{(1)}, \ldots, X_\tau^{(n)})$. The covariances of these Gaussian random vectors are determined by the formula*

$$\begin{aligned}&\mathrm{Cov}(X_\tau^{(1)} + \ldots + X_\tau^{(r)}; X_\tau^{(1)} + \ldots + X_\tau^{(s)}) \\ &= \frac{\oint dz_{2,1} \cdots \oint dz_{2,s} \oint dz_{1,1} \cdots \oint dz_{1,r} \sum\limits_{\substack{1\leq i \leq r \\ 1 \leq j \leq s}} \frac{z_{2,j}}{z_{1,i} - z_{2,j}} F_\tau(z_{2,1}, \ldots, z_{2,s}) F_\tau(z_{1,1}, \ldots, z_{1,r})}{\left(\oint dz_{2,1} \cdots \oint dz_{2,s} F_\tau(z_{2,1}, \ldots, z_{2,s})\right)\left(\oint dz_{1,1} \cdots \oint dz_{1,r} F_\tau(z_{1,1}, \ldots, z_{1,r})\right)},\end{aligned} \tag{8.4.1}$$

for all $r \geq s \geq 1$, where

$$F_\tau(z_1, \ldots, z_k) = \Delta(z_1, \ldots, z_k)^2 \prod_{i=1}^k \frac{e^{\tau z_i}(1 + z_i)}{z_i^{k+1}} \tag{8.4.2}$$

and the contours are all positively oriented, encircle 0 and satisfy $|z_{2,j}| < |z_{1,i}|$ for all $1 \leq i \leq r, 1 \leq j \leq s$.

Remark 44. We note that the integrand in the formula for covariances (8.4.1) is not symmetric in r, s, and in fact the formula is *not* valid if $r < s$. The same is true of the simplified formula (8.4.11) which will be derived from it below in Proposition 8.4.2.

Proof of Proposition 8.4.1. Since $x_i\left(\frac{\tau}{1-t}\right) = \lambda_i'(\tau) - i$,

$$X_\tau^{(i,\epsilon)} = \epsilon^{1/2}\left(\lambda_i'(\tau) - \mathbb{E}\left[\lambda_i'(\tau)\right]\right)$$

Clearly it suffices to show that the family of random variables $(X_\tau^{(1,\epsilon)} + \ldots + X_\tau^{(r,\epsilon)})_{r \geq 1}$ converge jointly to the Gaussian family $(X_\tau^{(1)} + \ldots + X_\tau^{(r)})_{r\geq 1}$. We will first show that another family of random variables

$$V_r(\epsilon, \tau) := t^{-(\lambda'_1(\tau)+\ldots+\lambda'_r(\tau))}, r \geq 1$$

converges jointly to the Gaussian family $(X_\tau^{(1)} + \ldots + X_\tau^{(r)})_{r\geq 1}$ after appropriate scaling, and then argue this suffices.

Proposition 8.2.1 gives us contour integral formulas for all joint moments of these random variables, so it is a matter of analyzing these integral formulas. We will use the general Gaussianity lemma given as Lemma 4.2 of [BG15], which has a self-contained presentation in Section 4.3 of the same paper.

Let

$$Cr_\epsilon(r_\alpha, \bar{\mu}z_\alpha; r_\beta, \bar{\mu}z_\beta) = \prod_{\substack{1\leq i\leq r_\alpha \\ 1\leq j\leq r_\beta}} \frac{1 - z_{j,\beta}/z_{i,\alpha}}{1 - t^{-1}z_{j,\beta}/z_{i,\alpha}} \tag{8.4.3}$$

$$\mathfrak{F}_\epsilon(r_m, \bar{\mu}z_m) = \frac{(-1)^{\binom{r_m}{2}}}{r_m!(2\pi i)^{r_m}} \Delta(z_{1,m}, \ldots, z_{r_m,m})^2 \prod_{s=1}^{r_m} \frac{1 + t^{-1}z_{s,m}^{-1}}{z_{s,m}^{r_m}} e^{\tau z_{s,m}} \tag{8.4.4}$$

where $\bar{\mu}z_\alpha$ is shorthand for the tuple of variables $z_{1,\alpha}, \ldots, z_{r_\alpha,\alpha}$, so that Proposition 8.2.1 reads

$$\mathbb{E}[V_{r_1}(\epsilon, \tau) \cdots V_{r_m}(\epsilon, \tau)] = \oint \cdots \oint \prod_{1\leq\alpha<\beta\leq k} Cr_\epsilon(r_\alpha, \bar{\mu}z_\alpha; r_\beta, \bar{\mu}z_\beta) \prod_{s=1}^{k} \mathfrak{F}_\epsilon(r_m, \bar{\mu}z_m) d\bar{\mu}z_m. \tag{8.4.5}$$

We have

$$Cr_\epsilon(r_\alpha, \bar{\mu}z_\alpha; r_\beta, \bar{\mu}z_\beta) = 1 + \epsilon\mathfrak{Cr}_\epsilon(r_\alpha, \bar{\mu}z_\alpha; r_\beta, \bar{\mu}z_\beta), \tag{8.4.6}$$

and uniform convergence $\mathfrak{Cr}_\epsilon \to \mathfrak{Cr}$ and $\mathfrak{F}_\epsilon \to \mathfrak{F}$ along the contours of interest, where

$$\mathfrak{Cr}(r_\alpha, \bar{\mu}z_\alpha; r_\beta, \bar{\mu}z_\beta) = \sum_{\substack{1\leq i\leq r_\alpha \\ 1\leq j\leq r_\beta}} \frac{z_{j,\beta}}{z_{i,\alpha} - z_{j,\beta}} \tag{8.4.7}$$

$$\mathfrak{F}(r_\alpha, \bar{\mu}z_\alpha) = \frac{(-1)^{\binom{r_m}{2}}}{r_m!(2\pi i)^{r_m}} \Delta(z_{1,\alpha}, \ldots, z_{r_\alpha,\alpha})^2 \prod_{s=1}^{r_\alpha} \frac{e^{\tau z_{s,\alpha}}(1 + z_{s,\alpha})}{z_{s,\alpha}^{r_\alpha+1}}. \tag{8.4.8}$$

By [BG15, Lemma 4.2][2], this implies that the random variables

$$Y_r(\epsilon,\tau) := \frac{V_r(\epsilon,\tau) - \mathbb{E}[V_r(\epsilon,\tau)]}{\sqrt{\epsilon}} \tag{8.4.9}$$

converge jointly to the mean 0, jointly Gaussian family $(Y_r(\tau))_{r\geq 1}$ having covariance

$$\begin{aligned}
&\mathrm{Cov}(Y_r(\tau);Y_s(\tau)) \\
&= \frac{\oint dz_{2,1}\cdots\oint dz_{2,s}\oint dz_{1,1}\cdots\oint dz_{1,r}\mathfrak{Cr}(r,\bar{\mu}z_1;s,\bar{\mu}z_2)\mathfrak{F}(r,\bar{\mu}z_1)\mathfrak{F}(s,\bar{\mu}z_2)}{\left(\oint dz_{2,1}\cdots\oint dz_{2,s}\mathfrak{F}(s,\bar{\mu}z_2)\right)\left(\oint dz_{1,1}\cdots\oint dz_{1,r}\mathfrak{F}(r,\bar{\mu}z_1)\right)}.
\end{aligned} \tag{8.4.10}$$

After cancelling the $\frac{(-1)^{\binom{r_m}{2}}}{r_m!(2\pi i)^{r_m}}$ terms in the numerator and denominator this is exactly the RHS of (8.4.1), hence we indeed have that $(Y_r(\epsilon,\tau))_{r\geq 1}$ converges jointly to $(X_\tau^{(1)}+\ldots+X_\tau^{(r)})_{r\geq 1}$.

It remains to show that $(X_\tau^{(1,\epsilon)}+\ldots+X_\tau^{(r,\epsilon)})_{r\geq 1}$ also converges jointly to $(X_\tau^{(1)}+\ldots+X_\tau^{(r)})_{r\geq 1}$. At a heuristic level this makes perfect sense by Taylor expanding the exponential in

$$V_r(\epsilon,\tau) = e^{\epsilon\left(\mathbb{E}[\lambda_1'(\tau)+\ldots+\lambda_r'(\tau)]+\epsilon^{1/2}\left(X_\tau^{(1,\epsilon)}+\ldots+X_\tau^{(r,\epsilon)}\right)\right)}$$

in (8.4.9), as the leading-order nonconstant term is const $\cdot\,(X_\tau^{(1,\epsilon)}+\ldots+X_\tau^{(r,\epsilon)})$ and the others are small in ϵ. To make this rigorous one uses (joint) tightness in ϵ of the random variables $Y_r(\epsilon,\tau)$ to show joint tightness of the random variables $X_\tau^{(1,\epsilon)}+\ldots+X_\tau^{(r,\epsilon)}$, which are related by a simple transformation, and then argues using Prokhorov's theorem, the convergence of $Y_r(\epsilon,\tau)$ and the previous Taylor expansion. The details are given in the proof of Proposition 4.1 in [BCF18], where our $Y_r(\epsilon,\tau)$ corresponds to their Y_r^ϵ, the analogue of the Gaussian convergence for $Y_r(\epsilon,\tau)$ is Lemma 4.4 of [BCF18], and with these two substitutions the proof carries over mutatis mutandis in our setting. □

We note that for a family of random variables with an only slightly different integral formula for covariances, the analogue of the above Gaussian convergence argument is written in a self-contained manner in the proof of [BCF18, Proposition 4.1]. For a reader wishing to understand all the details of the proof, this might be easier to read than the proof in [BG15] of the general Gaussianity lemma used in our condensed version.

[2] In the notation of [BG15] one should take $\epsilon = L^{-1}$ and $\gamma = 1$.

Returning to our setting, the formula in Proposition 8.4.1 simplifies greatly upon taking another limit $\tau \to \infty$. As was argued in the introduction, and will be fleshed out in the next section, this limit reflects convergence to stationarity of the original particle system (with an additional time change).

Proposition 8.4.2. *As $\tau \to \infty$, the random variables $X_\tau^{(i)}$ converge in distribution to a Gaussian random vector $(\zeta_i)_{i \geq 1}$, with covariances given by*

$$\mathrm{Cov}(\zeta_r; \zeta_s) = \frac{1}{4\pi^2} \oint_{\Gamma_0} \oint_{\Gamma_{0,w}} \frac{w}{z-w} \frac{r!s!}{z^r w^s} e^{z+w} (1 - z/r)(1 - w/s) \frac{dz}{z} \frac{dw}{w} \tag{8.4.11}$$

for each $r \geq s \geq 1$.

Proof. First note that by symmetry of F_τ, we may replace

$$\sum_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} \frac{z_{j,2}}{z_{i,1} - z_{j,2}}$$

by

$$rs \frac{z_{1,2}}{z_{1,1} - z_{1,2}}$$

in (8.4.1). Now, changing variables to $z_i = \tau z_{i,1}, w_j = \tau z_{j,2}$ and cancelling the factors of τ that appear, (8.4.1) becomes

$$\begin{aligned} &\mathrm{Cov}(X_\tau^{(1)} + \ldots + X_\tau^{(r)}; X_\tau^{(1)} + \ldots + X_\tau^{(s)}) \\ &= rs \frac{\oint \cdots \oint \frac{w_1}{z_1 - w_1} \Delta(\bar{\mu} z)^2 \prod_{i=1}^r \frac{e^{z_i}(1 + z_i/\tau)}{z_i^{r+1}} \Delta(\bar{\mu} w)^2 \prod_{j=1}^s \frac{e^{w_j}(1 + w_j/\tau)}{w_j^{s+1}} d\bar{\mu} z d\bar{\mu} w}{\left(\oint \cdots \oint \Delta(\bar{\mu} z)^2 \prod_{i=1}^r \frac{e^{z_i}(1 + z_i/\tau)}{z_i^{r+1}} d\bar{\mu} z \right) \left(\oint \cdots \oint \Delta(\bar{\mu} w)^2 \prod_{j=1}^s \frac{e^{w_j}(1 + w_j/\tau)}{w_j^{s+1}} d\bar{\mu} w \right)}. \end{aligned} \tag{8.4.12}$$

Since

$$\Delta(z_1, \ldots, z_r)^2 \prod_{i=1}^r \frac{e^{z_i}(1 + z_i/\tau)}{z_i^{r+1}} \to \Delta(z_1, \ldots, z_r)^2 \prod_{i=1}^r \frac{e^{z_i}}{z_i^{r+1}}$$

uniformly on the contours of integration, the RHS of (8.4.12) converges as $\tau \to \infty$ to the same expression with the $(1 + z_i/\tau)$ and $(1 + w_j/\tau)$ factors removed. In particular, because the $X_\tau^{(i)}$ are Gaussian, this implies convergence in joint distributions $X_\tau^{(i)} \to \zeta_i$, where the ζ_i form Gaussian random vectors with covariances given by

$$\begin{aligned}&\mathrm{Cov}(\zeta_1 + \ldots + \zeta_r; \zeta_1 + \ldots + \zeta_s) \\ &= rs \frac{\oint \cdots \oint \frac{w_1}{z_1 - w_1} \Delta(z_1, \ldots, z_r)^2 \prod_{i=1}^{r} \frac{e^{z_i}}{z_i^{r+1}} \Delta(w_1, \ldots, w_s)^2 \prod_{j=1}^{s} \frac{e^{w_j}}{w_j^{s+1}} d\bar{\mu} z d\bar{\mu} w}{\left(\oint \cdots \oint \Delta(z_1, \ldots, z_r)^2 \prod_{i=1}^{r} \frac{e^{z_i}}{z_i^{r+1}} d\bar{\mu} z \right) \left(\oint \cdots \oint \Delta(w_1, \ldots, w_s)^2 \prod_{j=1}^{s} \frac{e^{w_j}}{w_j^{s+1}} d\bar{\mu} w \right)}.\end{aligned} \tag{8.4.13}$$

Rewriting the above as

$$\mathrm{Cov}(\zeta_1 + \ldots + \zeta_r; \zeta_1 + \ldots + \zeta_s) = \frac{1}{(2\pi i)^2} \oint \oint \frac{w_1}{z_1 - w_1} \rho_r(z_1) \rho_s(w_1) dz_1 dw_1 \tag{8.4.14}$$

where

$$\rho_r(z_1) = \frac{r \frac{1}{(2\pi i)^{r-1}} \oint \cdots \oint \Delta(z_1, \ldots, z_r)^2 \prod_{i=1}^{r} \frac{e^{z_i}}{z_i^{r+1}} dz_2 \cdots dz_r}{\frac{1}{(2\pi i)^r} \oint \cdots \oint \Delta(z_1, \ldots, z_r)^2 \prod_{i=1}^{r} \frac{e^{z_i}}{z_i^{r+1}} dz_1 dz_2 \cdots dz_r}, \tag{8.4.15}$$

we recognize $\rho_r(z)$ as the 1-point correlation function of the orthogonal polynomial ensemble on the contour Γ_0 with weight $\Delta(z_1, \ldots, z_r)^2 \prod_{i=1}^{r} \frac{e^{z_i}}{z_i^{r+1}}$.

Let p_k^n be the (monic) orthogonal polynomial of degree k with respect to the inner product

$$\langle f, g \rangle_n = \frac{1}{2\pi i} \oint f(z) g(z) \frac{e^z}{z^n} dz. \tag{8.4.16}$$

Then by the classical theory of orthogonal polynomials, see e.g. [Dei99], one has

$$\rho_r(z) = \frac{e^z}{z^{r+1}} \sum_{k=0}^{r-1} \frac{p_k^{r+1}(z)^2}{\langle p_k^{r+1}, p_k^{r+1} \rangle_{r+1}}. \tag{8.4.17}$$

This reduces the computation of (8.4.14) to understanding the orthogonal polynomials p_k^{r+1}. For the observation above that $\rho_r(z)$ is a 1-point correlation function we followed a similar argument in [BCF18], and in fact our orthogonal polynomial ensemble is a special case of the one in that paper. They prove[3] the following explicit formulas by relating the p_k^n to the classical Laguerre polynomials, for which similar explicit formulas are classically

[3]To be specific, one must specialize $T = 1$ in the notation of [BCF18, Lemma 5.3] to arrive at Lemma 8.4.3.

known.

Lemma 8.4.3 ([BCF18, Lemma 5.3])**.** *Let p_k^n be as above. Then*

$$p_k^n(z) = \frac{k!}{(n-k-1)!}\sum_{\ell=0}^{k}\frac{(n-1-\ell)!}{(k-\ell)!\ell!}(-z)^\ell = \frac{1}{(n-1-k)!}\int_0^\infty (y-z)^k y^{n-1-k}e^{-y}dy. \tag{8.4.18}$$

Furthermore

$$\langle p_k^n, p_k^n\rangle_n = (-1)^k\frac{k!}{(n-k-1)!}. \tag{8.4.19}$$

We rewrite (8.4.17) as

$$\rho_r(z) = \left(\frac{e^z}{z^{r+1}}\sum_{k=0}^{r}\frac{p_k^{r+1}(z)^2}{\left\langle p_k^{r+1}, p_k^{r+1}\right\rangle_{r+1}}\right) - \frac{e^z}{z^{r+1}}\frac{p_r^{r+1}(z)^2}{\left\langle p_r^{r+1}, p_r^{r+1}\right\rangle_{r+1}} \tag{8.4.20}$$

and treat the two terms on the RHS separately. We first treat the sum on the RHS of (8.4.20), which will end up not contributing at all. By Lemma 8.4.3,

$$\begin{aligned}
\sum_{k=0}^{r}\frac{p_k^{r+1}(z)^2}{\left\langle p_k^{r+1}, p_k^{r+1}\right\rangle_{r+1}} &= \sum_{k=0}^{r}\frac{(-1)^k}{(r-k)!k!}\int_0^\infty (y-z)^k y^{r-k}e^{-y}dy\int_0^\infty (x-z)^k x^{r-k}e^{-y}dx \\
&= \frac{1}{r!}\int_0^\infty\int_0^\infty\sum_{k=0}^{r}\binom{r}{k}(-(y-z)(x-z))^k(xy)^{r-k}e^{-(x+y)}dxdy \\
&= \frac{1}{r!}\int_0^\infty\int_0^\infty\left(z(x+y)-z^2\right)^r e^{-(x+y)}dxdy \\
&= \frac{z^r}{r!}\int_0^\infty\int_0^u (u-z)^r e^{-u}dudy \\
&= \frac{z^r}{r!}\cdot\left(e^{-z}((r+1)!+r!z)+O(z^{r+1})\right).
\end{aligned}$$

Hence

$$\frac{e^z}{z^{r+1}}\sum_{k=0}^{r}\frac{p_k^{r+1}(z)^2}{\left\langle p_k^{r+1}, p_k^{r+1}\right\rangle_{r+1}} = \frac{r+1}{z}+1+O(z^r). \tag{8.4.21}$$

By (8.4.18),

$$p_r^{r+1}(z) = r!\sum_{\ell=0}^{r}\frac{(-z)^\ell}{\ell!} = r!\left(e^{-z}-f_{r+1}(z)\right), \tag{8.4.22}$$

where $f_{r+1}(z) := \sum_{\ell=r+1}^\infty\frac{(-z)^\ell}{\ell!}$ is just the sum of terms of degree $\geq r+1$ in the Taylor series for e^{-z}. Hence

$$\left(e^{-z} - f_{r+1}(z)\right)^2 = e^{-2z} - 2e^{-z} f_{r+1}(z) + O(z^{2r+2})$$
$$= e^{-z}\left(\sum_{\ell=0}^{\infty}(-1)^{\mathbb{1}(\ell\geq r+1)}\frac{(-z)^\ell}{\ell!} + O(z^{2r+2})\right), \quad (8.4.23)$$

so

$$\frac{e^z}{z^{r+1}}\frac{p_r^{r+1}(z)^2}{\langle p_r^{r+1}, p_r^{r+1}\rangle_{r+1}} = \frac{r!}{(-z)^{r+1}}\sum_{\ell=0}^{\infty}(-1)^{\mathbb{1}(\ell\geq r+1)}\frac{(-z)^\ell}{\ell!} + O(z^{r+1}). \quad (8.4.24)$$

Substituting (8.4.21) and (8.4.24) into (8.4.20) yields

$$\rho_r(z) = \frac{r+1}{z} + 1 - \frac{r!}{(-z)^{r+1}}\sum_{\ell=0}^{\infty}(-1)^{\mathbb{1}(\ell\geq r+1)}\frac{(-z)^\ell}{\ell!} + O(z^r). \quad (8.4.25)$$

Recall that

$$\operatorname{Cov}(\zeta_1 + \ldots + \zeta_r; \zeta_1 + \ldots + \zeta_s) = \frac{1}{(2\pi i)^2}\oint_{\Gamma_0}\oint_{\Gamma_{0,w}}\frac{w}{z-w}\rho_r(z)\rho_s(w)dzdw. \quad (8.4.26)$$

Since $|w| < |z|$ in the region of integration we may expand $\frac{w}{z-w} = \sum_{n=1}^{\infty}\left(\frac{w}{z}\right)^n$ in the integrand, and then interpret the integral as the $\frac{1}{zw}$ term of the resulting Laurent series expansion for the integrand (this may be justified by applying the residue theorem first to the z integral, then the w integral). Since n is positive in the terms $\left(\frac{w}{z}\right)^n$, this yields that only the terms of $\rho_s(w)$ of degree ≤ -2 in w contribute, and only the terms of $\rho_r(z)$ of degree ≥ 0 contribute. The terms of degree ≤ -2 in $\rho_s(w)$ match those of $\frac{s!}{(-w)^{s+1}}e^{-w}$, so we may substitute this for $\rho_s(w)$ in (8.4.26) without changing the integral. Because all terms in the Laurent expansion for $\rho_s(w)$ have degree $\geq -(s+1)$, we additionally have that only the terms of $\rho_r(z)$ of degree $\leq s-1$ contribute. Because $r \geq s$, we may thus ignore the $O(z^r)$ terms in (8.4.25). The terms of degree $0 \leq d \leq s-1$ in the Laurent expansion for $\rho_r(z)$ in (8.4.25) are the same as those in the Laurent series expansion of $-\frac{r!}{(-z)^{r+1}}e^{-z} + 1$. Therefore

$$\operatorname{Cov}(\zeta_1 + \ldots + \zeta_r; \zeta_1 + \ldots + \zeta_s) = \frac{1}{(2\pi i)^2}\oint_{\Gamma_0}\oint_{\Gamma_{0,w}}\frac{w}{z-w}\frac{s!e^{-w}}{(-w)^{s+1}}\left(-\frac{r!}{(-z)^{r+1}}e^{-z} + 1\right)dzdw. \quad (8.4.27)$$

Denoting the RHS above by $C(r,s)$, we have $\operatorname{Cov}(\zeta_r;\zeta_s) = C(r,s) - C(r-1,s) -$

$C(r,s-1)+C(r-1,s-1)$. Writing

$$C(r,s)=\frac{1}{(2\pi i)^2}\oint_{\Gamma_0}\oint_{\Gamma_{0,w}}\frac{w}{z-w}\frac{s!}{(-w)^{s+1}}e^{-w}\left(-\frac{r!}{(-z)^{r+1}}e^{-z}\right)dzdw \\ +\frac{1}{(2\pi i)^2}\oint_{\Gamma_0}\oint_{\Gamma_{0,w}}\frac{w}{z-w}\frac{s!}{(-w)^{s+1}}e^{-w}dzdw, \quad (8.4.28)$$

we see that the second integral on the RHS is independent of r, hence its contribution cancels in $C(r,s)-C(r-1,s)-C(r,s-1)+C(r-1,s-1)$. Thus

$$C(r,s)-C(r-1,s)-C(r,s-1)+C(r-1,s-1)=-\frac{1}{(2\pi i)^2}\oint_{\Gamma_0}\oint_{\Gamma_{0,w}}\frac{w}{z-w}e^{-z-w} \\ \left(\frac{r!s!}{(-z)^{r+1}(-w)^{s+1}}-\frac{(r-1)!s!}{(-z)^{r}(-w)^{s+1}}-\frac{r!(s-1)!}{(-z)^{r+1}(-w)^{s}}+\frac{(r-1)!(s-1)!}{(-z)^{r}(-w)^{s}}\right)dzdw \\ =\frac{1}{4\pi^2}\oint_{\Gamma_0}\oint_{\Gamma_{0,w}}\frac{w}{z-w}\frac{r!s!}{(-z)^{r+1}(-w)^{s+1}}e^{-z-w}(1+z/r)(1+w/s)dzdw. \quad (8.4.29)$$

Changing variables to $-z,-w$ yields (8.4.11), completing the proof. $\square$

8.5 Long-time SDEs and stationarity of fluctuations

In the limit $t=e^{-\epsilon}$, time $=\tau/(1-t)$, we previously derived Gaussian fluctuations $X_\tau^{(k)}$ for the particle positions, with explicit covariances which simplify in the large-time limit $\tau\to\infty$. In this section, we consider the probabilistic meaning of this limit. For particle systems such as ours, one may rigorously show convergence of the multi-time fluctuations to the solution of a system of SDEs as in [BCT17, Theorem 1], but we will instead give a (simpler and more intuitive) formal derivation of such a system of SDEs which closely follows that of [BCF18, Proposition 4.6]. After making a time change

$$Z_T^{(k)}:=X_{e^T-1}^{(k)},$$

this yields a system of SDEs for the $Z_T^{(k)}$ with time-dependent coefficients. As $T\to\infty$ these coefficients converge to nontrivial limits, yielding the system of SDEs

$$dZ_T^{(k)}=\left((k-1)Z_T^{(k-1)}-kZ_T^{(k)}\right)dT+dW_T^{(k)} \qquad k=1,2,\dots \qquad (8.5.1)$$

Though the derivation of the SDEs satisfied by the $X_\tau^{(k)}$ consisted of formal algebraic manipulations and is certainly not a rigorous analytic proof, we will check rigorously in Proposition 8.5.2 using contour integral formulas that the unique Gaussian stationary distribution of the system (8.5.1) is in fact the single-time limit of the fluctuations derived in the previous section.

We now proceed to the heuristic derivation of SDEs for $X_\tau^{(k)}, k = 1, 2, \ldots$. Because each particle jumps according to an independent Poisson clock with rate depending only on its position and that of the particle in front, the fluctuations should satisfy an SDE of the form

$$dX_\tau^{(k)} = f(\tau, X_\tau^{(k-1)}, X_\tau^{(k)})d\tau + g(\tau, X_\tau^{(k-1)}, X_\tau^{(k)})dB_\tau^{(k)},$$

and it remains to compute the drift and diffusion coefficients. To find the drift term $f(\tau, X_\tau^{(k-1)}, X_\tau^{(k)})$, we take expectations of both sides to eliminate the diffusion part. Hence we must compute the $\epsilon \to 0$ limit of the $O(d\tau)$ term in

$$\mathbb{E}\left[X_{\tau+d\tau}^{(k,\epsilon)} - X_\tau^{(k,\epsilon)}\right] = -\epsilon^{-1/2}(c_k(\tau + d\tau) - c_k(\tau)) + \epsilon^{1/2}\mathbb{E}\left[\lambda_k'(\tau + d\tau) - \lambda_k'(\tau)\right], \quad (8.5.2)$$

where $c_k(\tau)$ is the limit of $\epsilon x_k(\tau/\epsilon)$ given explicitly in (8.3.2). The jump rate of $\lambda_k'(\tau)$ is approximately constant on the interval $d\tau$, and equal to

$$\frac{1}{1-t}\left(t^{\lambda_k'(\tau)} - t^{\lambda_{k-1}'(\tau)}\right) \sim \epsilon^{-1}\left(e^{-(c_k(\tau)+\epsilon^{1/2}X_\tau^{(k,\epsilon)})} - e^{-(c_{k-1}(\tau)+\epsilon^{1/2}X_\tau^{(k-1,\epsilon)})}\right) \qquad \text{as } \epsilon \to 0 \quad (8.5.3)$$

where to obtain the RHS we use that $1 - t \approx \epsilon$. Therefore

$$\begin{aligned}\epsilon^{1/2}\mathbb{E}\left[\lambda_k'(\tau + d\tau) - \lambda_k'(\tau)\right] &\sim \epsilon^{-1/2}d\tau\left(e^{-(c_k(\tau)+\epsilon^{1/2}X_\tau^{(k,\epsilon)})} - e^{-(c_{k-1}(\tau)+\epsilon^{1/2}X_\tau^{(k-1,\epsilon)})}\right) \\ &\sim \epsilon^{-1/2}d\tau\left(e^{-c_k(\tau)} - e^{-c_{k-1}(\tau)}\right) - \left(e^{-c_k(\tau)}X_\tau^{(k,\epsilon)} - e^{-c_{k-1}(\tau)}X_\tau^{(k-1,\epsilon)}\right)\end{aligned} \quad (8.5.4)$$

as $\epsilon \to 0$. The other term on the RHS of (8.5.2) is

$$\begin{aligned}-\epsilon^{-1/2}(c_k(\tau + d\tau) - c_k(\tau)) &= -\epsilon^{-1/2}c_k'(\tau)d\tau + O(d\tau^2) \\ &= -\epsilon^{-1/2}\left(e^{-c_k(\tau)} - e^{-c_{k-1}(\tau)}\right)d\tau + O(d\tau^2)\end{aligned} \quad (8.5.5)$$

by the differential equation (8.3.1). Combining (8.5.4) and (8.5.5) yields a term which converges as $\epsilon \to 0$, hence the drift coefficient is

$$f(\tau, X_\tau^{(k-1)}, X_\tau^{(k)}) = \lim_{\epsilon \to 0} \mathrm{RHS}(8.5.2) = -\left(e^{-c_k(\tau)} X_\tau^{(k)} - e^{-c_{k-1}(\tau)} X_\tau^{(k-1)}\right). \tag{8.5.6}$$

We now compute the diffusion coefficient, which is the $O(d\tau)$ term in

$$\mathrm{Var}(X_{\tau+d\tau}^{(k,\epsilon)} - X_\tau^{(k,\epsilon)}) = \epsilon \, \mathrm{Var}(\lambda'_k(\tau + d\tau) - \lambda'_k(\tau)). \tag{8.5.7}$$

We approximate the jump rate to be constant as before, so that

$$\lambda'_k(\tau + d\tau) - \lambda'_k(\tau)$$

is a Poisson random variable with parameter equal to the time step $d\tau$ times its jump rate approximated earlier in (8.5.3). Since variance of $\mathrm{Pois}(r)$ is r,

$$\mathrm{RHS}(8.5.7) = \left(e^{-c_k(\tau)} - e^{-c_{k-1}(\tau)}\right) d\tau + o(1). \tag{8.5.8}$$

Hence

$$g(\tau, X_\tau^{(k-1)}, X_\tau^{(k)}) = \sqrt{e^{-c_k(\tau)} - e^{-c_{k-1}(\tau)}}. \tag{8.5.9}$$

Combining (8.5.6) with (8.5.9), we have derived (again, at a heuristic level) that the $\epsilon \to 0$ limits $X_\tau^{(k)}$ satisfy the system

$$dX_\tau^{(k)} = -\left(e^{-c_k(\tau)} X_\tau^{(k)} - e^{-c_{k-1}(\tau)} X_\tau^{(k-1)}\right) d\tau + \sqrt{e^{-c_k(\tau)} - e^{-c_{k-1}(\tau)}} dB_\tau^{(k)} \qquad k = 1, 2, \ldots \tag{8.5.10}$$

where as before we take $c_0(\tau) \equiv \infty$ identically in the case $k = 1$.

Exponentiating the explicit formula (8.3.2) for $c_k(\tau)$ yields

$$e^{-c_k(\tau)} = \frac{1 + \ldots + \frac{\tau^{k-1}}{(k-1)!}}{1 + \ldots + \frac{\tau^k}{k!}}. \tag{8.5.11}$$

Naively taking the $\tau \to \infty$ limit of the diffusion coefficient in (8.5.10) yields 0, which reflects the fact that particles' jump rates go to 0 as their positions go to ∞ due to the position-dependent slowing. However, the prelimit system also suggests a natural time change to obtain time-independent diffusion rates. The jump rate of λ'_1 is $t^{\lambda'_1}$, so to make this jump rate independent of time one must speed up time by a factor of $t^{-\lambda_1'}$—which, note, depends on the random position of λ_1'. More precisely, if s is the time variable in the original particle system, then letting $h(s)$ be the piecewise-linear random function with $h'(s) = t^{-\lambda_1'(h(s))}$, one has that $\lambda_1'(h(s))$ jumps according to a rate 1 Poisson process. Hence its position is a Poisson random variable with mean s. Since it concentrates around its mean at large s, we have $h'(s) \approx t^{-s}$ and hence

$$h(s) \approx \frac{t^{-s}}{-\log t}$$

for large s. This suggests that the random time change by $t^{-\lambda_1'}$ can be approximated at large times by a deterministic exponential time change, so we make an exponential time change $\tau = e^T$ in the limit SDEs (8.5.10). For notational convenience let us instead shift slightly and take $\tau = e^T - 1$ so that T begins at 0. Setting $Z_T^{(k)} := X_{e^T-1}^{(k)}$ in (8.5.10), one has $d\tau = e^T dT$ and $dB_T^{(k)} = \sqrt{e^T} dW_T^{(k)}$ for $W_T^{(k)}$ independent standard Brownian motions, yielding

$$dZ_T^{(k)} = -\left(e^{-c_k(e^T-1)} Z_T^{(k)} - e^{-c_{k-1}(e^T-1)} Z_T^{(k-1)}\right) e^T dT + \sqrt{e^T \left(e^{-c_k(e^T-1)} - e^{-c_{k-1}(e^T-1)}\right)} dW_T^{(k)} \tag{8.5.12}$$

Plugging in (8.5.11) we obtain

$$dZ_T^{(k)} = \left(-kZ_T^{(k)} + (k-1)Z_T^{(k-1)} + o(1)\right) dT + (1 + o(1)) dW_T^{(k)} \qquad k = 1, 2, \dots$$

which converges to (8.5.1). This mirrors the convergence of the covariances of particle fluctuations without rescaling as $\tau \to \infty$, shown in Proposition 8.4.2, and the main result of this section is that the SDEs (8.5.1) indeed have a stationary solution with the exact covariances of Proposition 8.4.2.

We note also for concreteness that the SDE for $Z_T^{(1)}$ is exactly that of an Ornstein-Uhlenbeck process, and the mean-reversion reflects the fact that the jump rate of λ_1' is smaller when it is further ahead and larger when it is further behind. The dependence of the drift term on $Z_T^{(k)}, Z_T^{(k-1)}$ likewise reflects the prelimit dependence of a particle's jump rate on its own position and that of the particle in front.

Let us now proceed rigorously. We first check that it makes sense to speak of *the* solution to (8.5.1).

Lemma 8.5.1. *Strong existence and uniqueness hold for the system of SDEs*

$$dZ_T^{(k)} = \left((k-1)Z_T^{(k-1)} - kZ_T^{(k)}\right) dT + dW_T^{(k)} \qquad T \geq 0, k = 1, 2, \ldots$$

stated earlier as (8.5.1).

Proof. Note that for each $n \geq 1$, the coefficients in the SDEs (8.5.1) for $(Z_T^{(k)})_{1\leq k\leq n}$ depend only on $(Z_T^{(k)})_{1\leq k\leq n}$, i.e. $(Z_T^{(1)}, \ldots, Z_T^{(n)})$ satisfies an SDE

$$dZ_T^{(k)} = \left((k-1)Z_T^{(k-1)} - kZ_T^{(k)}\right) dT + dW_T^{(k)} \qquad k = 1, \ldots, n. \tag{8.5.13}$$

in $\mathbb{R}^n$ driven by noise $(W_T^{(1)}, \ldots, W_T^{(n)})$. We claim it suffices to prove strong existence and uniqueness of (8.5.13) for each n, which we recall means that given $(Z_0^{(k)})_{1\leq k\leq n}$ and a fixed Brownian motion $(W_T^{(k)})_{1\leq k\leq n}$, there is a process solving (8.5.1) which is unique up to almost-everywhere equivalence. The claim holds because the resulting n-indexed family of solutions is clearly consistent under forgetting the last coordinate $Z_T^{(n)}$, hence the consistent n-indexed family defines a solution $(Z_T^{(k)})_{k\geq 1}$ to the infinite system (8.5.1).

We now argue for fixed n by applying off-the-shelf existence and uniqueness theorems. To aid in matching notation, let

$$\begin{aligned}
b_k(T, \bar{\mu}x) &= (k-1)x_{k-1} - kx_k \\
\bar{\mu}b(T, \bar{\mu}x) &= (b_1(T, \bar{\mu}x), \ldots, b_n(T, \bar{\mu}x)) \\
\sigma_{ij}(T, \bar{\mu}x) &= \mathbb{1}(i = j)
\end{aligned}$$

for $T \geq 0, \bar{\mu}x \in \mathbb{R}^n$, so that (8.5.1) takes the form

$$dZ_T^{(k)} = b_k(T, (Z_T^{(1)}, \ldots, Z_T^{(n)}))dT + \sum_{\ell=1}^{n} \sigma_{k\ell}(T, (Z_T^{(1)}, \ldots, Z_T^{(n)}))dW_T^{(\ell)}.$$

For strong uniqueness, by [KS14, Chapter 5.2, Theorem 2.5] it suffices[4] to show the Lipschitz property that there exists K for which

$$||\bar{\mu}b(T, \bar{\mu}x) - \bar{\mu}b(T, \bar{\mu}y)|| + ||\sigma(T, \bar{\mu}x) - \sigma(T, \bar{\mu}y)|| \leq K||\bar{\mu}x - \bar{\mu}y||. \tag{8.5.14}$$

[4]We here state stronger and easier-to-state hypotheses than in [KS14, Chapter 5.2, Theorem 2.5], which suffice for our purposes.

Here the norm is the standard Euclidean one, viewing σ as a vector in $\mathbb{R}^{n^2}$. For strong existence, by [KS14, Chapter 5.2, Theorem 2.9] it suffices to show (8.5.14) in addition to

$$||\bar{\mu}b(T,\bar{\mu}x)||^2 + ||\sigma(T,\bar{\mu}x)||^2 \leq K^2(1+||\bar{\mu}x||^2). \tag{8.5.15}$$

A crude bound shows

$$||\bar{\mu}b(T,\bar{\mu}x)||^2 \leq 4n^3||x||^2.$$

Take $K^2 = 4n^3$. Since σ is constant and $\bar{\mu}b(T,\bar{\mu}x)$ is linear in $\bar{\mu}x$, (8.5.14) holds. Since $||\sigma(T,\bar{\mu}x)||^2 = n$, (8.5.15) holds as well, completing the proof. □

We now find that the explicit Gaussian vector derived in Proposition 8.4.2 describes the stationary distribution of the above system of SDEs.

Proposition 8.5.2. *Let $(Z_T^{(1)}, Z_T^{(2)}, \ldots)$ be the vector-valued stochastic process satisfying the system of SDEs*

$$dZ_T^{(k)} = \left((k-1)Z_T^{(k-1)} - kZ_T^{(k)}\right)dT + dW_T^{(k)} \tag{8.5.16}$$

where $W_T^{(k)}$ are independent standard Brownian motions, with initial distribution $(Z_0^{(k)})_{k\geq 1}$ given by a Gaussian vector with covariances

$$\mathrm{Cov}(Z_0^{(r)}, Z_0^{(s)}) = \frac{1}{4\pi^2}\oint_{\Gamma_0}\oint_{\Gamma_{0,w}} \frac{w}{z-w}\frac{r!s!}{z^r w^s}e^{z+w}(1-z/r)(1-w/s)\frac{dz}{z}\frac{dw}{w}.$$

Then $(Z_T^{(k)})_{k\geq 1}$ is stationary, i.e.

$$(Z_{T_0}^{(k)})_{k\geq 1} = (Z_0^{(k)})_{k\geq 1} \tag{8.5.17}$$

in distribution, for any fixed time $T_0 > 0$.

Remark 45. A natural further question is whether the finite-τ SDEs (8.5.10) admit a Gaussian solution with fixed-time covariances given by our finite-τ formula in Proposition 8.4.1. This seems more difficult to address without the large-time simplification of Proposition 8.4.2, and we have not attempted to pursue it in this work.

To prepare for the proof, we first give two computational lemmas, which will be proven at the end of the section.

Definition 66. For $r, s \in \mathbb{Z}_{\geq 1}$, let

$$D(r,s) := \frac{1}{4\pi^2} \oint_{\Gamma_0} \oint_{\Gamma_{0,w}} \frac{w}{z-w} \frac{r!s!}{z^r w^s} e^{z+w} (1 - z/r)(1 - w/s) \frac{dz}{z} \frac{dw}{w}. \tag{8.5.18}$$

By Proposition 8.4.2, when $r \geq s$ one has $D(r,s) = \mathrm{Cov}(\zeta_r, \zeta_s)$, but as noted in Remark 44 this is not true when $r < s$. This will be important in computations below.

Lemma 8.5.3. *For any $r \geq s \geq 1$,*

$$(r-1)D(r-1,s) + (s-1)D(r,s-1) - (r+s)D(r,s) = 0. \tag{8.5.19}$$

Lemma 8.5.4. *For any $r \geq 2$,*

$$D(r-1,r) - D(r,r-1) = \frac{1}{r-1}. \tag{8.5.20}$$

Proof of Proposition 8.5.2. It suffices to show

$$(Z_{T_0}^{(k)})_{1 \leq k \leq n} = (Z_0^{(k)})_{1 \leq k \leq n} \qquad \text{in distribution} \tag{8.5.21}$$

for each $n \geq 1$ and $T_0 > 0$. First note that the solution $(Z_T^{(k)})_{1 \leq k \leq n}$ is a Gaussian process, so its distribution at time T_0 is determined by its covariance matrix, i.e. it suffices to check

$$\mathrm{Cov}\left(Z_{T_0}^{(r)}, Z_{T_0}^{(s)}\right) = \mathrm{Cov}\left(Z_0^{(r)}, Z_0^{(s)}\right) \tag{8.5.22}$$

for each $1 \leq s \leq r$. Let

$$A_{r,s}(T) := \mathrm{Cov}\left(Z_T^{(r)}, Z_T^{(s)}\right)$$

for $r, s \geq 1$. It follows by applying Itô's lemma that

$$\frac{d}{dT} A_{r,s}(T) = \mathbb{1}(r = s) + (r-1)A_{r-1,s}(T) + (s-1)A_{r,s-1}(T) - (r+s)A_{r,s}(T) \tag{8.5.23}$$

(this computation can be done for quite general systems of SDEs, see [BCT17, (4.3)]). Hence to check (8.5.22), it suffices to check that the RHS of (8.5.23) is 0 when the constant solution

$$A_{r,s}(T) = D(r,s)$$

is plugged in. When $r > s$, this follows directly from Lemma 8.5.3. When $r = s$, since

$$A_{r-1,r} = A_{r,r-1} = D(r, r-1)$$

we have

$$\text{RHS}(8.5.23) = 1+(r-1)D(r-1,r)+(r-1)D(r,r-1)-2rD(r,r)+(r-1)(D(r,r-1)-D(r-1,r))$$

which is 0 by Lemma 8.5.3 and Lemma 8.5.4. This completes the proof. □

Proof of Lemma 8.5.3. We obtain that

$$\frac{1}{(2\pi i)^2}\oint_{\gamma_0}\oint_{\gamma_{0,w}} \frac{w}{z-w} e^{z+w} z^{-a} w^{-b} \frac{dz}{z}\frac{dw}{w} = \frac{1}{(2\pi i)^2}\oint_{\gamma_0}\oint_{\gamma_{0,w}} \frac{w}{z-w} \frac{(z+w)^{a+b}}{(a+b)!} z^{-a} w^{-b} \frac{dz}{z}\frac{dw}{w} \tag{8.5.24}$$

for $a, b \geq 0$, by expanding

$$\frac{w}{z-w} = \left(\frac{w}{z}\right) + \left(\frac{w}{z}\right)^2 + \dots$$

(using that $|w| < |z|$ along the contours) and taking the residue expansion of both sides. Using (8.5.24) to convert the integral in (8.5.18) to one with integrand of the form

$$\frac{w}{z-w}(\text{Laurent polynomial in } z, w),$$

and combining the three integrals in (8.5.19) into a single double contour integral, it is easily verified (by a computer) that the integrand is 0. □

Proof of Lemma 8.5.4. One has

$$\begin{aligned} D(r-1,r) - D(r,r-1) = \frac{1}{4\pi^2}\oint_{\gamma_0}\oint_{\gamma_{0,w}} \frac{w}{z-w}(r-1)!(r-2)!e^{z+w} \\ \cdot \left(\frac{1}{z^{r-1}w^r}(r-1-z)(r-w) + \frac{1}{z^r w^{r-1}}(r-z)(r-1-w)\right)\frac{dz}{z}\frac{dw}{w}. \end{aligned}$$

Since

$$\left(\frac{1}{z^{r-1}w^r}(r-1-z)(r-w) + \frac{1}{z^r w^{r-1}}(r-z)(r-1-w)\right) = \frac{1}{z^r w^r}(w-z)((r-z)(r-w)-r)$$

which cancels the $\frac{1}{z-w}$ in the integrand, the only poles in the integrand are at $w = 0$ and $z = 0$. The result (8.5.20) now follows by taking this residue. $\square$

Proof of Theorem 1.6.2. Uniqueness of the stationary solution to (1.6.2) was shown in Lemma 8.5.1. In Proposition 8.4.2 we showed that the $X_\tau^{(i)}$ converge to a jointly Gaussian vector with the explicit covariances given in Theorem 1.6.2, which is the second half of the theorem. In Proposition 8.5.2 we showed that this jointly Gaussian vector also describes the unique stationary solution to (1.6.2), which accounts for the first half. $\square$

8.6 Bulk fluctuations

In this section, we gather the random variables ζ_i into a single stochastic process, and compute its covariance in Theorem 1.6.3 by analysis of the contour integral from Proposition 8.4.2.

Definition 67. Let $Y_T, T \in \mathbb{R}^+$ be the stochastic process for which $Y_0 = 0$, $Y_n = \zeta_n$ for all $n \in \mathbb{Z}_{\geq 1}$ with ζ_n as in Proposition 8.4.2, and

$$Y_{n+\alpha} = (1-\alpha)Y_n + \alpha Y_{n+1}$$

for $n \in \mathbb{Z}_{\geq 1}, \alpha \in (0,1)$.

Finally, we recall the main result.

Theorem 1.6.3. *The process*

$$R_s^{(T)} := T^{1/4} Y_{T+s\sqrt{T}}$$

converges in finite-dimensional distributions as $T \to \infty$ to the unique stationary Gaussian process $R_s, s \in \mathbb{R}$ with covariances

$$\mathrm{Cov}(R_a, R_b) = \int_0^\infty y^2 e^{-y^2 - |b-a|y} dy.$$

Proof. Before getting to the main computation, we must take care of some technical details. Firstly, we are justified in speaking of the unique Gaussian process with covariances as in the theorem statement, because a Gaussian process is determined by its (jointly Gaussian) finite-dimensional distributions, and these Gaussian vectors are determined by their covariances. $R_T^{(k)}$ is a Gaussian process; when $k + T\sqrt{k} \in \mathbb{Z}$, $R_T^{(k)} = k^{1/4}\zeta_{k+T\sqrt{k}}$ is Gaussian, and for other values of T $R_T^{(k)}$ is a convex combination of Gaussians and hence also Gaussian. Hence to show convergence of finite-dimensional distributions to R_T, it suffices to show convergence of pairwise covariances of $R_T^{(k)}$ to those of R_T, i.e. we must show

$$\mathrm{Cov}(R_a^{(k)}, R_b^{(k)}) \to \int_0^\infty y^2 e^{-y^2-|b-a|y} dy \text{ as } k \to \infty, \tag{8.6.1}$$

where without loss of generality $a \geq b$.

Since $R_a^{(k)}$ is in general a convex combination $p(a,k)\zeta_{k+\lfloor a\sqrt{k}\rfloor} + (1-p(a,k))\zeta_{k+\lceil a\sqrt{k}\rceil})$ with some $p(a,k) \in [0,1]$, and similarly for $R_b^{(k)}$, to show (8.6.1) it suffices to show

$$k^{1/2}\,\mathrm{Cov}(\zeta_{k+\lfloor a\sqrt{k}\rfloor}, \zeta_{k+\lfloor b\sqrt{k}\rfloor}) \to \int_0^\infty y^2 e^{-y^2-|b-a|y} dy \text{ as } k \to \infty, \tag{8.6.2}$$

along with the same convergence where one or both floor functions are replaced by ceiling functions. We will show (8.6.2) by steepest-descent analysis of the integral formula for covariances (8.4.11), and the versions with one or both floor functions replaced by ceilings are exactly analogous.

Let $r = k + \lfloor a\sqrt{k}\rfloor, s = k + \lfloor b\sqrt{k}\rfloor$.

First change variables in (8.4.11) to $\tilde{z} = z/r, \tilde{w} = w/s$ to obtain

$$\sqrt{k}\,\mathrm{Cov}(\zeta_r, \zeta_s) = \frac{1}{4\pi^2}\oint_{\Gamma_0}\oint_{\Gamma_{0,\frac{s}{r}\tilde{w}}} \sqrt{k}\frac{s\tilde{w}}{r\tilde{z} - s\tilde{w}}\frac{r!s!}{r^r s^s \tilde{z}^r \tilde{w}^s} e^{r\tilde{z}+s\tilde{w}}(1-\tilde{z})(1-\tilde{w})\frac{d\tilde{z}}{\tilde{z}}\frac{d\tilde{w}}{\tilde{w}} \tag{8.6.3}$$

Using Stirling's approximation $n! = \sqrt{2\pi n}(n/e)^n e^{o(1)}$, the above equals

$$\frac{1}{4\pi^2}\oint_{\Gamma_0}\oint_{\Gamma_{0,\frac{s}{r}\tilde{w}}} \sqrt{k}(2\pi\sqrt{rs})\frac{s\tilde{w}}{r\tilde{z} - s\tilde{w}}(1-\tilde{z})(1-\tilde{w})e^{rF(\tilde{z})+sF(\tilde{w})+o(1)}\frac{d\tilde{z}}{\tilde{z}}\frac{d\tilde{w}}{\tilde{w}}, \tag{8.6.4}$$

where here and henceforth $F(z) = z - \log z - 1$. It is easy to check that $F(z)$ has a unique critical point at $z = 1$ which is second-order, and our steepest descent will consist of zooming in on this critical point.

For the contours Γ_0 and $\Gamma_{0,\frac{s}{r}\tilde{w}}$ above, we will use the (counterclockwise-oriented) contours $C_{\tilde{z}} = C_{\tilde{z}}(k)$ and $C_{\tilde{w}} = C_{\tilde{w}}(k)$ which are pictured in Figure 8-1 and which we now describe. First fix any δ with $1/3 < \delta < 1/2$. Let

$$C_{\tilde{z}} = \{\min(x,1) + iy : x^2 + y^2 = 1 + k^{-2\delta}\} \tag{8.6.5}$$

$$C_{\tilde{w}} = \{\min(x, 1 - k^{-1/2}) + iy : x^2 + y^2 = (1 - k^{-1/2})^2 + k^{-2\delta}\}. \tag{8.6.6}$$

Each contour has two parts, one a subset of a circle and one a vertical line; call the circular parts $C''_{\tilde{z}}, C''_{\tilde{w}}$ and the vertical parts $C'_{\tilde{z}}, C'_{\tilde{w}}$.

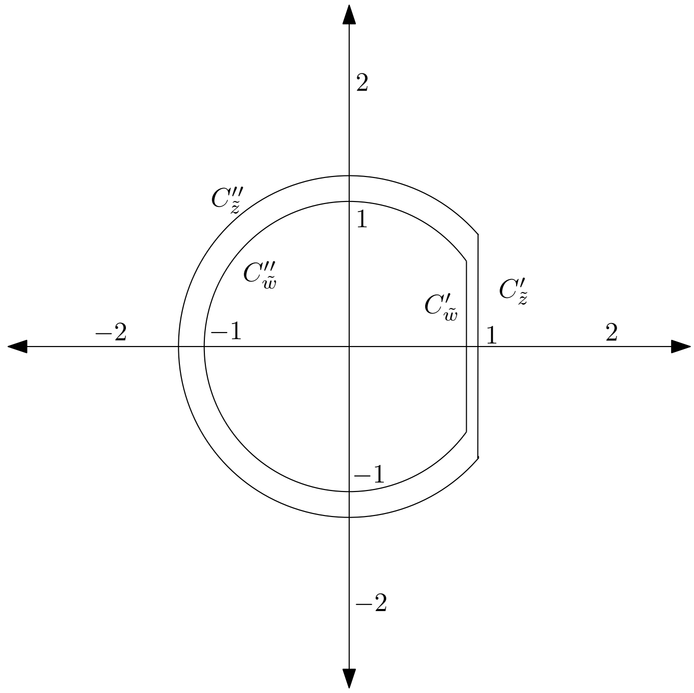


Figure 8-1: The contours $C_{\tilde{z}}$ and $C_{\tilde{w}}$ (figure not to scale).

We first claim that only the integral with $(\tilde{w}, \tilde{z}) \in C'_{\tilde{w}} \times C'_{\tilde{z}}$ contributes asymptotically, i.e.

$$\oint\oint_{(C_{\tilde{w}} \times C_{\tilde{z}}) \setminus (C'_{\tilde{w}} \times C'_{\tilde{z}})} \sqrt{k}(2\pi\sqrt{rs}) \frac{s\tilde{w}}{r\tilde{z} - s\tilde{w}} (1-\tilde{z})(1-\tilde{w}) e^{rF(\tilde{z}) + sF(\tilde{w}) + o(1)} \frac{d\tilde{z}}{\tilde{z}} \frac{d\tilde{w}}{\tilde{w}} \to 0 \tag{8.6.7}$$

as $k \to \infty$.

It is clear from the definition of the contours that the distance between them is const $\cdot k^{-1/2} + o(k^{-1/2})$. Since $r \geq s$, we therefore have

$$\frac{1}{r\tilde{z} - s\tilde{w}} \leq \text{const} \cdot k^{1/2}/s. \tag{8.6.8}$$

Hence

$$\left|\sqrt{k}(2\pi\sqrt{rs})\frac{s\tilde{w}}{r\tilde{z}-s\tilde{w}}\frac{(1-\tilde{z})(1-\tilde{w})}{\tilde{z}\tilde{w}}\right| \leq \text{const}\cdot k^2 \tag{8.6.9}$$

for some constant, for all large enough k. We have

$$\sup_{\tilde{z}\in C''_{\tilde{z}}} \mathrm{Re}(F(\tilde{z})) \leq -\log|\tilde{z}| = -\log\sqrt{1+k^{-2\delta}} = -\frac{1}{2}k^{-2\delta}+o(k^{-2\delta})$$

and similarly

$$\sup_{\tilde{w}\in C''_{\tilde{w}}} \mathrm{Re}\,F(\tilde{w}) \leq -\frac{1}{2}k^{-2\delta}+o(k^{-2\delta}).$$

Hence for such $\tilde{z}\in C''_{\tilde{z}}$ and $\tilde{w}\in C''_{\tilde{w}}$, we have $\mathrm{Re}(rF(\tilde{z})), \mathrm{Re}(sF(\tilde{w})) \leq -\frac{1}{2}k^{1-2\delta}+o(k^{1-2\delta})$. On the vertical segments $C'_{\tilde{z}}$ and $C'_{\tilde{w}}$, $\mathrm{Re}\,F$ is maximized at the unique real value, so $\sup_{\tilde{z}\in C'_{\tilde{z}}}\mathrm{Re}\,F(\tilde{z})=0$ and

$$\sup_{\tilde{w}\in C'_{\tilde{w}}} \mathrm{Re}\,F(\tilde{w}) = -k^{-1/2}-\log(1-k^{-1/2}) = O(k^{-1}). \tag{8.6.10}$$

Thus if $\tilde{z}\in C_{\tilde{z}}, \tilde{w}\in C_{\tilde{w}}$ and at least one of $\tilde{z}\in C''_{\tilde{z}}$ or $\tilde{w}\in C''_{\tilde{w}}$ holds,

$$e^{rF(\tilde{z})+sF(\tilde{w})} \leq e^{-\frac{1}{2}k^{1-2\delta}+o(k^{1-2\delta})}. \tag{8.6.11}$$

It follows that the integrand in (8.6.7) is bounded by

$$\text{const}\cdot k^2 e^{-\frac{1}{2}k^{1-2\delta}+o(k^{1-2\delta})} \tag{8.6.12}$$

uniformly in k over the domain of integration (which, recall, also depends on k). Since the lengths of the k-dependent contours $C_{\tilde{z}}, C_{\tilde{w}}$ are bounded over all k, and the above bound converges to 0 since $1-2\delta>0$, we have established (8.6.7).

Now we consider the remaining part of the integral,

$$\frac{1}{2\pi}\oint_{C'_{\tilde{w}}}\oint_{C'_{\tilde{z}}} \sqrt{krs}\frac{s\tilde{w}}{r\tilde{z}-s\tilde{w}}(1-\tilde{z})(1-\tilde{w})e^{rF(\tilde{z})+sF(\tilde{w})+o(1)}\frac{d\tilde{z}}{\tilde{z}}\frac{d\tilde{w}}{\tilde{w}} \tag{8.6.13}$$

We have $F'(\tilde{z}) = 1-1/\tilde{z}$ and $F''(\tilde{z}) = 1/\tilde{z}^2$, so Taylor expanding about 1 we have $F(\tilde{z}) = (\tilde{z}-1)^2/2+O((\tilde{z}-1)^3)$. Since $C'_{\tilde{z}} = \{1+iy : y\in(-k^{-\delta},k^{-\delta})\}$, for $\tilde{z}\in C'_{\tilde{z}}$ one has $|\tilde{z}-1|^3<k^{-3\delta}$, and similarly $|\tilde{w}-1|^3<k^{-3\delta}$ for $\tilde{w}\in C'_{\tilde{w}}$. Because $\delta>1/3$, we have

$|F(\tilde{z}) - (\tilde{z}-1)^2/2| = o(1)$ as $k \to \infty$ uniformly over $C'_{\tilde{z}}$, and similarly for $\tilde{w}$.

Let us change variables in (8.6.13) to u, v, defined by $\tilde{z} = 1 + ik^{-1/2}u$ and $\tilde{w} = 1 - k^{-1/2} + ik^{-1/2}v$. The condition that $z \in C'_{\tilde{z}}, w \in C'_{\tilde{w}}$ translates to $-k^{1/2-\delta} < u, v < k^{1/2-\delta}$, and by the previous paragraph

$$\begin{aligned} F(\tilde{z}) &= -\frac{1}{2k}u^2 + o_u(1) \\ F(\tilde{w}) &= -\frac{1}{2k}(v+i)^2 + o_v(1) \end{aligned}$$

where the error terms depend on u (resp. v) but are bounded uniformly on the domain of integration. Thus we may write (8.6.13) as

$$\begin{aligned} &\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}(|u|,|v| < k^{1/2-\delta})\sqrt{krs}\frac{s(1+k^{-1/2}(-1+iv))}{(\sqrt{k} + \lfloor a\sqrt{k}\rfloor - \lfloor b\sqrt{k}\rfloor) + \sqrt{k}iu - \sqrt{k}iv + o(\sqrt{k})} \\ &\quad \times \left(-i\frac{u}{\sqrt{k}}\right)\left(\frac{1-iv}{\sqrt{k}}\right) e^{-u^2/2-(v+i)^2/2+o_{u,v}(1)}\frac{ik^{-1/2}du}{1+iu/\sqrt{k}}\frac{ik^{-1/2}dv}{1+k^{-1/2}(-1+iv)} \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}(|u|,|v| < k^{1/2-\delta})\frac{s\sqrt{krs}}{k^{5/2}}\frac{1+k^{-1/2}(-1+iv)}{1+a-b+iu-iv+o(\sqrt{k})} \\ &\quad \times e^{-u^2/2-(v+i)^2/2+o_{u,v}(1)}\frac{u(i+v)}{(1+k^{-1/2}(-1+iv))(1+iu/\sqrt{k})}dudv \end{aligned} \tag{8.6.14}$$

Recalling that r and s are $k + o(k)$ and $u, v = o(k^{1/2})$ in the domain of integration, we see that the integrand in (8.6.14) converges to

$$\frac{1}{1+(a-b)+iu-iv}u(v+i)e^{-u^2/2-(v+i)^2/2}dudv \tag{8.6.15}$$

as $k \to \infty$, and furthermore that there exists a constant C such that it is dominated by the integrable function

$$C\mathbb{1}\left(|u|,|v| < k^{1/2-\delta}\right)\frac{1}{c+(a-b)+iu-iv}u(v+ic)e^{-u^2/2-(v+i)^2/2} \tag{8.6.16}$$

for all $u, v \in \mathbb{R}$ and all large enough k. Hence by dominated convergence, (8.6.14) converges to

$$\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{1+(a-b)+iu-iv}u(v+i)e^{-u^2/2-(v+i)^2/2}dudv. \tag{8.6.17}$$

Note that the convergence above would be exactly the same if $\lfloor a\sqrt{k}\rfloor$ and/or $\lfloor b\sqrt{k}\rfloor$ had been replaced by ceiling functions, as mentioned earlier.

Using the identity

$$\frac{1}{\alpha} = \int_0^\infty e^{-y\alpha} dy \tag{8.6.18}$$

if $\mathrm{Re}(\alpha) > 0$, since $1 + a - b \geq 1$ the above integral is equal to

$$\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\int_0^\infty e^{-y(1+a-b+iu-iv)}dy\right) u(v+i)e^{-u^2/2-(v+i)^2/2}dudv \tag{8.6.19}$$

$$= \frac{1}{2\pi}\int_0^\infty e^{-y^2-y(a-b)}\left(\int_{\mathbb{R}} e^{-\frac{1}{2}(u^2+2iyu-y^2)}du\right)\left(\int_{\mathbb{R}} e^{-\frac{1}{2}((v+i)^2-2iy(v+i)-y^2)}dv\right)dy. \tag{8.6.20}$$

Since

$$\int_{\mathbb{R}} e^{-\frac{1}{2}(u^2+2iyu-y^2)}du = -\sqrt{2\pi}iy$$

and

$$\int_{\mathbb{R}} e^{-\frac{1}{2}((v+i)^2-2iy(v+i)-y^2)}dv = \sqrt{2\pi}iy$$

(for the latter we must shift contours from $\mathbb{R}$ to $\mathbb{R} - i$ before evaluating the Gaussian integral), we have that (8.6.19) is equal to

$$\int_0^\infty y^2 e^{-y^2-y(a-b)}dy, \tag{8.6.21}$$

completing the proof. □

Bibliography

[Ahn22a] Andrew Ahn. Extremal singular values of random matrix products and Brownian motion on $\mathrm{GL}_n(\mathbb{C})$. *arXiv preprint arXiv:2201.11809*, 2022. (Cited on pages 28 and 45.)

[Ahn22b] Andrew Ahn. Fluctuations of β-Jacobi product processes. *Probability Theory and Related Fields*, 183(1-2):57–123, 2022. (Cited on pages 28, 94, 108, and 157.)

[ARVP22] Andrew Ahn, Marianna Russkikh, and Roger Van Peski. Lozenge tilings and the Gaussian free field on a cylinder. *Communications in Mathematical Physics*, 396(3):1221–1275, 2022. (Cited on page 222.)

[AS22] Andrew Ahn and Eugene Strahov. Product matrix processes with symplectic and orthogonal invariance via symmetric functions. *International Mathematics Research Notices*, 2022(14):10767–10821, 2022. (Cited on page 28.)

[AVP23] Andrew Ahn and Roger Van Peski. Lyapunov exponents for truncated unitary and Ginibre matrices. *Ann. Inst. Henri Poincaré Probab. Stat.*, 59(2):1029–1039, 2023. (Cited on pages 28 and 34.)

[ABK14] Gernot Akemann, Zdzislaw Burda, and Mario Kieburg. Universal distribution of Lyapunov exponents for products of Ginibre matrices. *Journal of Physics A: Mathematical and Theoretical*, 47(39):395202, 2014. (Cited on pages 27 and 28.)

[ABK19] Gernot Akemann, Zdzislaw Burda, and Mario Kieburg. From integrable to chaotic systems: Universal local statistics of Lyapunov exponents. *EPL (Europhysics Letters)*, 126(4):40001, 2019. (Cited on pages 28 and 33.)

[ABK20] Gernot Akemann, Zdzislaw Burda, and Mario Kieburg. Universality of local spectral statistics of products of random matrices. *arXiv preprint arXiv:2008.11470*, 2020. (Cited on pages 28 and 33.)

[AI15] Gernot Akemann and Jesper R. Ipsen. Recent exact and asymptotic results for products of independent random matrices. *Acta Physica Polonica B*, 46(9), 2015. (Cited on page 28.)

[AIK13] Gernot Akemann, Jesper R Ipsen, and Mario Kieburg. Products of rectangular random matrices: singular values and progressive scattering. *Physical Review E*, 88(5):052118, 2013. (Cited on page 28.)

[AKW13] Gernot Akemann, Mario Kieburg, and Lu Wei. Singular value correlation functions for products of Wishart random matrices. *Journal of Physics A: Mathematical and Theoretical*, 46(27):275205, 2013. (Cited on page 28.)

[AK91] Sergio Albeverio and Witold Karwowski. Diffusion on p-adic numbers. *Gaussian random fields*, pages 86–99, 1991. (Cited on page 41.)

[AK94] Sergio Albeverio and Witold Karwowski. A random walk on p-adics—the generator and its spectrum. *Stochastic processes and their Applications*, 53(1):1–22, 1994. (Cited on page 41.)

[Ass22] Theodoros Assiotis. Infinite p-adic random matrices and ergodic decomposition of p-adic Hua measures. *Transactions of the American Mathematical Society*, 375(3):1745–1766, 2022. (Cited on pages 3, 5, 54, 57, 60, 135, 144, and 148.)

[AN21] Theodoros Assiotis and Joseph Najnudel. The boundary of the orbital beta process. *Moscow Mathematical Journal*, 21(4):659–694, 2021. (Cited on page 56.)

[Bar15] Guillaume Barraquand. A phase transition for q-TASEP with a few slower particles. *Stochastic Processes and their Applications*, 125(7):2674–2699, 2015. (Cited on page 46.)

[Bar01] Yu Baryshnikov. GUEs and queues. *Probability Theory and Related Fields*, 119:256–274, 2001. (Cited on page 39.)

[BB19] Dan Betea and Jérémie Bouttier. The periodic Schur process and free fermions at finite temperature. *Mathematical Physics, Analysis and Geometry*, 22(1):3, 2019. (Cited on page 222.)

[BKL+15] Manjul Bhargava, Daniel M Kane, Hendrik W Lenstra, Bjorn Poonen, and Eric Rains. Modeling the distribution of ranks, Selmer groups, and Shafarevich–Tate groups of elliptic curves. *Cambridge Journal of Mathematics*, 3(3):275–321, 2015. (Cited on page 16.)

[Bia91] Philippe Biane. Quantum random walk on the dual of SU(n). *Probability theory and related fields*, 89(1):117–129, 1991. (Cited on page 39.)

[Bia92] Philippe Biane. Minuscule weights and random walks on lattices. *Quantum probability & related topics*, pages 51–65, 1992. (Cited on page 39.)

[BBO05] Philippe Biane, Philippe Bougerol, and Neil O'Connell. Littelmann paths and Brownian paths. *Duke Mathematical Journal*, 130:127–167, 2005. (Cited on page 39.)

[BV97] A Kh Bikulov and Igor Vasil'evich Volovich. p-adic Brownian motion. *Izvestiya: Mathematics*, 61(3):537, 1997. (Cited on page 41.)

[Bil68] Patrick Billingsley. *Convergence of probability measures*. John Wiley & Sons, Inc., New York-London-Sydney, 1968. (Cited on page 261.)

[Bor99] Alexei Borodin. The law of large numbers and the central limit theorem for the Jordan normal form of large triangular matrices over a finite field. *Journal of Mathematical Sciences*, 96(5):3455–3471, 1999. (Cited on page 39.)

[Bor07] Alexei Borodin. Periodic Schur process and cylindric partitions. *Duke Math. J.*, 140(3):391–468, 2007. (Cited on page 222.)

[Bor17] Alexei Borodin. On a family of symmetric rational functions. *Advances in Mathematics*, 306:973–1018, 2017. (Cited on pages 68, 115, 116, and 117.)

[BBW16] Alexei Borodin, Alexey Bufetov, and Michael Wheeler. Between the stochastic six vertex model and Hall-Littlewood processes. *arXiv preprint arXiv:1611.09486*, 2016. (Cited on page 50.)

[BC14] Alexei Borodin and Ivan Corwin. Macdonald processes. *Probability Theory and Related Fields*, 158(1-2):225–400, 2014. (Cited on pages 17, 46, 66, and 202.)

[BC15] Alexei Borodin and Ivan Corwin. Discrete time q-TASEPs. *International Mathematics Research Notices*, 2015(2):499–537, 2015. (Cited on page 46.)

[BCF18] Alexei Borodin, Ivan Corwin, and Patrik L Ferrari. Anisotropic (2+1)D growth and Gaussian limits of q-Whittaker processes. *Probability Theory and Related Fields*, 172(1-2):245–321, 2018. (Cited on pages 48, 50, 51, 306, 308, 310, 311, and 313.)

[BCS14] Alexei Borodin, Ivan Corwin, and Tomohiro Sasamoto. From duality to determinants for q-TASEP and ASEP. *The Annals of Probability*, 42(6):2314–2382, 2014. (Cited on page 46.)

[BCT17] Alexei Borodin, Ivan Corwin, and Fabio Lucio Toninelli. Stochastic heat equation limit of a (2+ 1) D growth model. *Communications in Mathematical Physics*, 350(3):957–984, 2017. (Cited on pages 50, 313, and 319.)

[BF09] Alexei Borodin and Patrik L Ferrari. Anisotropic KPZ growth in $2+1$ dimensions: fluctuations and covariance structure. *Journal of Statistical Mechanics: Theory and Experiment*, 2009(02):P02009, 2009. (Cited on page 50.)

[BF14] Alexei Borodin and Patrik L Ferrari. Anisotropic growth of random surfaces in 2+ 1 dimensions. *Communications in Mathematical Physics*, 325(2):603–684, 2014. (Cited on pages 50 and 51.)

[BG13] Alexei Borodin and Vadim Gorin. Markov processes of infinitely many nonintersecting random walks. *Probability Theory and Related Fields*, 155(3-4):935–997, 2013. (Cited on pages 51 and 148.)

[BG15] Alexei Borodin and Vadim Gorin. General β-Jacobi Corners Process and the Gaussian Free Field. *Communications on Pure and Applied Mathematics*, 68(10):1774–1844, 2015. (Cited on pages 16, 107, 108, 109, 305, 307, and 308.)

[BO01] Alexei Borodin and Grigori Olshanski. Infinite random matrices and ergodic measures. *Communications in Mathematical Physics*, 223(1):87–123, 2001. (Cited on page 56.)

[BO12a] Alexei Borodin and Grigori Olshanski. The boundary of the Gelfand–Tsetlin graph: A new approach. *Advances in Mathematics*, 230(4-6):1738–1779, 2012. (Cited on page 52.)

[BO12b] Alexei Borodin and Grigori Olshanski. Markov processes on the path space of the Gelfand–Tsetlin graph and on its boundary. *Journal of Functional Analysis*, 263(1):248–303, 2012. (Cited on page 190.)

[BO17] Alexei Borodin and Grigori Olshanski. *Representations of the infinite symmetric group*, volume 160. Cambridge University Press, 2017. (Cited on pages 52, 76, and 122.)

[BP16] Alexei Borodin and Leonid Petrov. Nearest neighbor Markov dynamics on Macdonald processes. *Advances in Mathematics*, 300:71–155, 2016. (Cited on page 50.)

[BP17] Alexei Borodin and Leonid Petrov. Integrable probability: stochastic vertex models and symmetric functions. *Stochastic Processes and Random Matrices: Lecture Notes of the Les Houches Summer School: Volume 104, 6th-31st July 2015*, 104:26, 2017. (Cited on pages 115, 116, and 117.)

[BJ02] Philippe Bougerol and Thierry Jeulin. Paths in Weyl chambers and random matrices. *Probability theory and related fields*, 124(4):517–543, 2002. (Cited on page 39.)

[BS11] Sara Brofferio and Bruno Schapira. Poisson boundary of $GL_d(\mathbb{Q}_p)$. *Israel Journal of Mathematics*, 185(1):125, 2011. (Cited on page 27.)

[Bro71] Bruce M Brown. Martingale central limit theorems. *The Annals of Mathematical Statistics*, 42(1):59–66, 1971. (Cited on page 178.)

[BQ17] Alexander I Bufetov and Yanqi Qiu. Ergodic measures on spaces of infinite matrices over non-archimedean locally compact fields. *Compositio Mathematica*, 153(12):2482–2533, 2017. (Cited on pages 3, 54, 55, 56, 60, 135, 141, 142, and 143.)

[BM18] Alexey Bufetov and Konstantin Matveev. Hall–Littlewood RSK field. *Selecta Mathematica*, 24(5):4839–4884, 2018. (Cited on page 300.)

[BP15] Alexey Bufetov and Leonid Petrov. Law of large numbers for infinite random matrices over a finite field. *Selecta Mathematica*, 21(4):1271–1338, 2015. (Cited on page 39.)

[CW04] Donald I Cartwright and Wolfgang Woess. Isotropic random walks in a building of type. *Mathematische Zeitschrift*, 247(1):101–135, 2004. (Cited on page 27.)

[Chh17] Reda Chhaibi. Non-Archimedean Whittaker functions as characters: A probabilistic approach to the Shintani–Casselman–Shalika formula. *International Mathematics Research Notices*, 2017(7):2100–2138, 2017. (Cited on page 27.)

[CKL+15] Julien Clancy, Nathan Kaplan, Timothy Leake, Sam Payne, and Melanie Matchett Wood. On a Cohen–Lenstra heuristic for Jacobians of random graphs. *Journal of Algebraic Combinatorics*, 42(3):701–723, 2015. (Cited on page 143.)

[CL84] Henri Cohen and Hendrik W Lenstra. Heuristics on class groups of number fields. In *Number Theory Noordwijkerhout 1983*, pages 33–62. Springer, 1984. (Cited on pages 15, 16, and 18.)

[Col05] Benoît Collins. Product of random projections, Jacobi ensembles and universality problems arising from free probability. *Probability theory and related fields*, 133(3):315–344, 2005. (Cited on page 23.)

[CD18] Ivan Corwin and Evgeni Dimitrov. Transversal fluctuations of the ASEP, stochastic six vertex model, and Hall-Littlewood Gibbsian line ensembles. *Communications in Mathematical Physics*, 363(2):435–501, 2018. (Cited on page 181.)

[CPV12] Andrea Crisanti, Giovanni Paladin, and Angelo Vulpiani. *Products of random matrices: in Statistical Physics*, volume 104. Springer Science & Business Media, 2012. (Cited on page 28.)

[Cue18] Cesar Cuenca. Asymptotic formulas for Macdonald polynomials and the boundary of the (q, t)-Gelfand-Tsetlin graph. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 14:001, 2018. (Cited on pages 52, 54, and 122.)

[CO22] Cesar Cuenca and Grigori Olshanski. Infinite-dimensional groups over finite fields and Hall-Littlewood symmetric functions. *Advances in Mathematics*, 395:108087, 2022. (Cited on page 52.)

[Dei99] Percy Deift. *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, volume 3. American Mathematical Soc., 1999. (Cited on page 310.)

[Dim18] Evgeni Dimitrov. KPZ and Airy limits of Hall–Littlewood random plane partitions. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 54, pages 640–693. Institut Henri Poincaré, 2018. (Cited on page 181.)

[DE05] Ioana Dumitriu and Alan Edelman. Eigenvalues of Hermite and Laguerre ensembles: large beta asymptotics. *Ann. Inst. H. Poincaré Probab. Statist.*, 41(6):1083–1099, 2005. (Cited on page 45.)

[Dys62a] Freeman J Dyson. A Brownian-motion model for the eigenvalues of a random matrix. *Journal of Mathematical Physics*, 3(6):1191–1198, 1962. (Cited on page 33.)

[Dys62b] Freeman J Dyson. Statistical theory of the energy levels of complex systems I, II and III. *Journal of Mathematical Physics*, 3(1):140–156, 157–165, 166–175, 1962. (Cited on page 33.)

[EPS14] Alan Edelman, Per-Olof Persson, and Brian D Sutton. Low-temperature random matrix theory at the soft edge. *Journal of Mathematical Physics*, 55(6):063302, 2014. (Cited on page 45.)

[EV10] Jordan S Ellenberg and Akshay Venkatesh. Statistics of number fields and function fields. In *Proceedings of the International Congress of Mathematicians 2010 (ICM 2010) (In 4 Volumes) Vol. I: Plenary Lectures and Ceremonies Vols. II–IV: Invited Lectures*, pages 383–402. World Scientific, 2010. (Cited on page 16.)

[EMOT81] Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi. *Higher transcendental functions. Vol. II.* Robert E. Krieger Publishing Co., Inc., Melbourne, Fla., 1981. Based on notes left by Harry Bateman, Reprint of the 1953 original. (Cited on page 222.)

[Eva89] Steven N Evans. Local properties of Lévy processes on a totally disconnected group. *Journal of Theoretical Probability*, 2:209–259, 1989. (Cited on page 41.)

[Eva93] Steven N Evans. Local field Brownian motion. *Journal of Theoretical Probability*, 6:817–850, 1993. (Cited on page 41.)

[Eva98] Steven N Evans. Local fields, Gaussian measures, and Brownian motions. *arXiv preprint math/9803046*, 1998. (Cited on page 41.)

[Eva01] Steven N Evans. Local fields, Gaussian measures, and Brownian motions. *Topics in probability and Lie groups: boundary theory*, 28:11–50, 2001. (Cited on pages 24 and 65.)

[Eva02] Steven N Evans. Elementary divisors and determinants of random matrices over a local field. *Stochastic processes and their applications*, 102(1):89–102, 2002. (Cited on pages 21 and 59.)

[Fel15] William Feller. On the integro-differential equations of purely discontinuous Markoff processes. In *Selected Papers I*, pages 539–566. Springer, 2015. (Cited on page 190.)

[FV15] Patrik L Ferrari and Bálint Vető. Tracy–Widom asymptotics for q-TASEP. In *Annales de l'IHP Probabilités et statistiques*, volume 51, pages 1465–1485, 2015. (Cited on page 46.)

[For93] Peter J Forrester. The spectrum edge of random matrix ensembles. *Nuclear Physics B*, 402(3):709–728, 1993. (Cited on page 33.)

[For10] Peter J Forrester. *Log-gases and random matrices (LMS-34)*. Princeton University Press, 2010. (Cited on page 16.)

[For15] Peter J Forrester. Asymptotics of finite system Lyapunov exponents for some random matrix ensembles. *Journal of Physics A: Mathematical and Theoretical*, 48(21):215205, 2015. (Cited on page 28.)

[FL16] Peter J Forrester and Dang-Zheng Liu. Singular values for products of complex Ginibre matrices with a source: hard edge limit and phase transition. *Communications in Mathematical Physics*, 344(1):333–368, 2016. (Cited on page 28.)

[FR05] Peter J Forrester and Eric M Rains. Interpretations of some parameter dependent generalizations of classical matrix ensembles. *Probability theory and related fields*, 131(1):1–61, 2005. (Cited on page 107.)

[FW87] Eduardo Friedman and Lawrence C Washington. On the distribution of divisor class groups of curves over a finite field. *Théorie des Nombres/Number Theory Laval*, 1987. (Cited on pages 16, 18, and 21.)

[Ful02] Jason Fulman. Random matrix theory over finite fields. *Bulletin of the American Mathematical Society*, 39(1):51–85, 2002. (Cited on pages 21 and 57.)

[Ful14] Jason Fulman. Cohen-Lenstra heuristics and random matrix theory over finite fields. *Journal of Group Theory*, 17(4):619–648, 2014. (Cited on pages 16 and 21.)

[Ful16] Jason Fulman. Hall-Littlewood polynomials and Cohen-Lenstra heuristics for Jacobians of random graphs. *Annals of Combinatorics*, 20(1):115–124, 2016. (Cited on page 143.)

[FK19] Jason Fulman and Nathan Kaplan. Random partitions and cohen–lenstra heuristics. *Annals of Combinatorics*, 23:295–315, 2019. (Cited on page 16.)

[Ful00] William Fulton. Eigenvalues, invariant factors, highest weights, and Schubert calculus. *Bulletin of the American Mathematical Society*, 37(3):209–249, 2000. (Cited on page 61.)

[FK60] Harry Furstenberg and Harry Kesten. Products of random matrices. *The Annals of Mathematical Statistics*, 31(2):457–469, 1960. (Cited on page 27.)

[Gan64] Ramesh Gangolli. Isotropic infinitely divisible measures on symmetric spaces. *Acta Mathematica*, 111(1):213–246, 1964. (Cited on pages 34 and 36.)

[Gan65] Ramesh Gangolli. Sample functions of certain differential processes on symmetric spaces. *Pacific Journal of Mathematics*, 15(2):477–496, 1965. (Cited on page 35.)

[GZ92] Ira M Gessel and Doron Zeilberger. Random walk in a Weyl chamber. *Proceedings of the American Mathematical Society*, 115(1):27–31, 1992. (Cited on page 40.)

[GM89] I Ya Gol'dsheid and Grigorii Aleksandrovich Margulis. Lyapunov indices of a product of random matrices. *RuMaS*, 44(5):11–71, 1989. (Cited on page 28.)

[Gor12] Vadim Gorin. The q-Gelfand–Tsetlin graph, Gibbs measures and q-Toeplitz matrices. *Advances in Mathematics*, 229(1):201–266, 2012. (Cited on pages 52, 53, and 54.)

[GK22] Vadim Gorin and Victor Kleptsyn. Universal objects of the infinite beta random matrix theory. *Journal of the European Mathematical Society*, 2022. 53 pages. v2: misprints corrected, more references added. (Cited on pages 45 and 109.)

[GM20] Vadim Gorin and Adam W Marcus. Crystallization of random matrix orbits. *International Mathematics Research Notices*, 2020(3):883–913, 2020. (Cited on pages 16, 45, 71, 108, 109, 110, and 114.)

[GS22] Vadim Gorin and Yi Sun. Gaussian fluctuations for products of random matrices. *American Journal of Mathematics*, 144(2):287–393, 2022. (Cited on page 28.)

[Gra99] David J Grabiner. Brownian motion in a Weyl chamber, non-colliding particles, and random matrices. *Annales de l'IHP Probabilités et statistiques*, 35(2):177–204, 1999. (Cited on page 39.)

[Hai09] Martin Hairer. An introduction to stochastic PDEs. *arXiv preprint arXiv:0907.4178*, 2009. (Cited on page 49.)

[Hua63] Luogeng Hua. *Harmonic analysis of functions of several complex variables in the classical domains*. Number 6. American Mathematical Soc., 1963. (Cited on page 56.)

[Hun56] Gilbert Agnew Hunt. Semi-groups of measures on Lie groups. *Transactions of the American Mathematical Society*, 81(2):264–293, 1956. (Cited on page 35.)

[IMS21a] Takashi Imamura, Matteo Mucciconi, and Tomohiro Sasamoto. Identity between restricted Cauchy sums for the q-Whittaker and skew Schur polynomials. *arXiv preprint arXiv:2106.11913*, 2021. (Cited on page 222.)

[IMS21b] Takashi Imamura, Matteo Mucciconi, and Tomohiro Sasamoto. Skew RSK dynamics: Greene invariants, affine crystals and applications to q-Whittaker polynomials. *arXiv preprint arXiv:2106.11922*, 2021. (Cited on page 222.)

[IMS22] Takashi Imamura, Matteo Mucciconi, and Tomohiro Sasamoto. Solvable models in the KPZ class: approach through periodic and free boundary Schur measures. *arXiv preprint arXiv:2204.08420*, 2022. (Cited on page 222.)

[IS19a] Takashi Imamura and Tomohiro Sasamoto. Fluctuations for stationary q-TASEP. *Probability Theory and Related Fields*, 174(1):647–730, 2019. (Cited on page 46.)

[IS19b] Takeshi Imamura and Tomohiro Sasamoto. The q-TASEP with a random initial condition. *Theoretical and Mathematical Physics*, 198(1):69–88, 2019. (Cited on page 46.)

[JO06] Liza Jones and Neil O'Connell. Weyl chambers, symmetric spaces and number variance saturation. *ALEA Lat. Am. J. Probab. Math. Stat*, 2:91–118, 2006. (Cited on page 35.)

[KS14] Ioannis Karatzas and Steven Shreve. *Brownian motion and stochastic calculus*, volume 113. springer, 2014. (Cited on pages 317 and 318.)

[KOO98] Sergei Kerov, Andrei Okounkov, and Grigori Olshanski. The boundary of the Young graph with Jack edge multiplicities. *International Mathematics Research Notices*, 1998(4):173–199, 1998. (Cited on page 52.)

[Ker92] SV Kerov. Generalized Hall-Littlewood symmetric functions and orthogonal polynomials. *Representation theory and dynamical systems Advances in Soviet Math*, 9:67–94, 1992. (Cited on pages 52 and 75.)

[KK+19] Mario Kieburg, Holger Kösters, et al. Products of random matrices from polynomial ensembles. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 55, pages 98–126. Institut Henri Poincaré, 2019. (Cited on page 28.)

[Kir98] Anatol N Kirillov. New combinatorial formula for modified Hall-Littlewood polynomials. *arXiv preprint math/9803006*, 1998. (Cited on pages 85 and 120.)

[Kly00] Alexander A Klyachko. Random walks on symmetric spaces and inequalities for matrix spectra. *Linear Algebra and its Applications*, 319(1-3):37–59, 2000. (Cited on page 34.)

[Koe98] Wolfram Koepf. Hypergeometric summation. *Vieweg, Braunschweig/Wiesbaden*, 5(6), 1998. (Cited on pages 27 and 121.)

[Lee22] Jungin Lee. Universality of the cokernels of random p-adic hermitian matrices. *arXiv preprint arXiv:2205.09368*, 2022. (Cited on page 16.)

[LWW23] Dang-Zheng Liu, Dong Wang, and Yanhui Wang. Lyapunov exponent, universality and phase transition for products of random matrices. *Communications in Mathematical Physics*, 399(3):1811–1855, 2023. (Cited on pages 27, 28, and 33.)

[LWZ16] Dang-Zheng Liu, Dong Wang, and Lun Zhang. Bulk and soft-edge universality for singular values of products of Ginibre random matrices. *Annales de L'Institut Henri Poincare Section (B) Probability and Statistics*, 52(4):1734–1762, 2016. (Cited on page 33.)

[Mac98a] Ian Grant Macdonald. *Symmetric functions and Hall polynomials*. Oxford university press, 1998. (Cited on pages 16, 22, 23, 66, 68, 69, 72, 73, 74, 75, 77, 82, 85, 88, 89, 90, 103, 106, 110, 112, 130, 144, 304, and 305.)

[Mac98b] Ian Grant Macdonald. *Symmetric functions and orthogonal polynomials*, volume 12. American Mathematical Soc., 1998. (Cited on page 67.)

[MP67] Vladimir Alexandrovich Marchenko and Leonid Andreevich Pastur. Distribution of eigenvalues for some sets of random matrices. *Matematicheskii Sbornik*, 114(4):507–536, 1967. (Cited on page 24.)

[Mat19] Konstantin Matveev. Macdonald-positive specializations of the algebra of symmetric functions: Proof of the Kerov conjecture. *Annals of Mathematics*, 189(1):277–316, 2019. (Cited on pages 52, 75, and 89.)

[Ner13] Yu A Neretin. Hua measures on the space of p-adic matrices and inverse limits of Grassmannians. *Izvestiya. Mathematics*, 77(5):941–953, 2013. (Cited on pages 56 and 144.)

[NVP22] Hoi H Nguyen and Roger Van Peski. Universality for cokernels of random matrix products. *arXiv preprint arXiv:2209.14957*, 2022. (Cited on page 22.)

[NW22a] Hoi H Nguyen and Melanie Matchett Wood. Local and global universality of random matrix cokernels. *arXiv preprint arXiv:2210.08526*, 2022. (Cited on page 16.)

[NW22b] Hoi H Nguyen and Melanie Matchett Wood. Random integral matrices: universality of surjectivity and the cokernel. *Inventiones mathematicae*, 228(1):1–76, 2022. (Cited on page 16.)

[OY02] Neil O'Connell and Marc Yor. A representation for non-colliding random walks. *Electronic Communications in Probability [electronic only]*, 7:1–12, 2002. (Cited on page 39.)

[OO98] Andrei Okounkov and Grigori Olshanski. Asymptotics of Jack polynomials as the number of variables goes to infinity. *International Mathematics Research Notices*, 1998(13):641–682, 1998. (Cited on pages 52, 56, 100, 129, and 138.)

[Ols16] GI Olshanski. Extended gelfand–tsetlin graph, its q-boundary, and q-B-splines. *Functional Analysis and Its Applications*, 50(2):107–130, 2016. (Cited on page 122.)

[Ols03] Grigori Olshanski. The problem of harmonic analysis on the infinite-dimensional unitary group. *Journal of Functional Analysis*, 205(2):464–524, 2003. (Cited on page 132.)

[Ols21] Grigori Olshanski. Macdonald polynomials and extended Gelfand–Tsetlin graph. *Selecta Mathematica*, 27(3):1–61, 2021. (Cited on pages 52 and 54.)

[OP17] Daniel Orr and Leonid Petrov. Stochastic higher spin six vertex model and q-TASEPs. *Advances in Mathematics*, 317:473–525, 2017. (Cited on page 46.)

[Ose68] Valery Iustinovich Oseledets. A multiplicative ergodic theorem. Characteristic Ljapunov exponents of dynamical systems. *Trudy Moskovskogo Matematicheskogo Obshchestva*, 19:179–210, 1968. (Cited on page 27.)

[Par07] James Parkinson. Isotropic random walks on affine buildings. *Annales de l'Institut Fourier*, 57(2):379–419, 2007. (Cited on page 40.)

[Par17] James Parkinson. Buildings, groups of Lie type, and random walks. *Groups, Graphs and Random Walks*, 436:391, 2017. (Cited on pages 27 and 40.)

[Pic87] Doug Pickrell. Measures on infinite dimensional Grassmann manifolds. *Journal of Functional Analysis*, 70(2):323–356, 1987. (Cited on page 56.)

[PS02] Michael Prähofer and Herbert Spohn. Scale invariance of the PNG droplet and the Airy process. *Journal of statistical physics*, 108:1071–1106, 2002. (Cited on page 33.)

[Rag79] Madabusi S Raghunathan. A proof of Oseledec's multiplicative ergodic theorem. *Israel Journal of Mathematics*, 32(4):356–362, 1979. (Cited on page 27.)

[SW22a] Will Sawin and Melanie Matchett Wood. Finite quotients of 3-manifold groups. *arXiv preprint arXiv:2203.01140*, 2022. (Cited on page 196.)

[SW22b] Will Sawin and Melanie Matchett Wood. The moment problem for random objects in a category. *arXiv preprint arXiv:2210.06279*, 2022. (Cited on pages 4, 39, and 196.)

[Sch09] Bruno Schapira. Random walk on a building of type $\tilde{A}_r$ and Brownian motion of the Weyl chamber. In *Annales de l'IHP Probabilités et statistiques*, volume 45, pages 289–301, 2009. (Cited on page 27.)

[Sep10] Timo Seppäläinen. Current fluctuations for stochastic particle systems with drift in one spatial dimension. *ENSAIOS MATEMÁTICOS*, 18:1–81, 2010. (Cited on page 49.)

[Shi96] A. N. Shiryaev. *Probability, 2nd edition.*, volume 95 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1996. Translated from the Russian by R. P. Boas. (Cited on pages 177 and 178.)

[Sun16] Yi Sun. Matrix models for multilevel Heckman-Opdam and multivariate Bessel measures. *arXiv preprint arXiv:1609.09096*, 2016. (Cited on page 108.)

[Tao08] Terence Tao. Tate's proof of the functional equation. https://terrytao.wordpress.com/2008/07/27/tates-proof-of-the-functional-equation/, 2008. (Cited on page 24.)

[Tho64] Elmar Thoma. Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe. *Mathematische Zeitschrift*, 85(1):40–61, 1964. (Cited on page 52.)

[TW94] Craig A Tracy and Harold Widom. Level-spacing distributions and the Airy kernel. *Communications in Mathematical Physics*, 159:151–174, 1994. (Cited on page 33.)

[VP21] Roger Van Peski. Limits and fluctuations of p-adic random matrix products. *Selecta Mathematica*, 27(5):1–71, 2021. (Cited on pages 58 and 277.)

[VP22a] Roger Van Peski. Hall-Littlewood polynomials, boundaries, and p-adic random matrices. *International Mathematics Research Notices*, 2022. (Cited on pages 58 and 135.)

[VP22b] Roger Van Peski. q-TASEP with position-dependent slowing. *Electronic Journal of Probability*, 27:1–35, 2022. (Cited on page 58.)

[VK81] Anatolii Moiseevich Vershik and Sergei Vasil'evich Kerov. Asymptotic theory of characters of the symmetric group. *Funktsional'nyi Analiz i ego Prilozheniya*, 15(4):15–27, 1981. (Cited on page 52.)

[VK82] Anatolii Moiseevich Vershik and Sergei Vasil'evich Kerov. Characters and factor representations of the infinite unitary group. In *Doklady Akademii Nauk*, volume 267, pages 272–276. Russian Academy of Sciences, 1982. (Cited on page 52.)

[Vet21] Bálint Vető. Asymptotic fluctuations of geometric q-TASEP, geometric q-PushTASEP and q-PushASEP. *arXiv preprint arXiv:2106.14013*, 2021. (Cited on page 46.)

[Voi76] Dan-Virgil Voiculescu. Représentations factorielles de type II_1 de $U(\infty)$. *J. Math. Pures Appl*, 55(1):1–20, 1976. (Cited on page 52.)

[Wig51] Eugene P Wigner. On the statistical distribution of the widths and spacings of nuclear resonance levels. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 47, pages 790–798. Cambridge University Press, 1951. (Cited on page 15.)

[Wig55] Eugene P Wigner. Characteristic vectors of bordered matrices with infinite dimensions. *Annals of Mathematics*, pages 548–564, 1955. (Cited on page 15.)

[Wig57] Eugene Paul Wigner. *Statistical properties of real symmetric matrices with many dimensions*. Princeton University, 1957. (Cited on page 15.)

[Wis28] John Wishart. The generalised product moment distribution in samples from a normal multivariate population. *Biometrika*, pages 32–52, 1928. (Cited on page 15.)

[Woo16] Melanie Matchett Wood. Asymptotics for number fields and class groups. In *Directions in Number Theory*, pages 291–339. Springer, 2016. (Cited on page 16.)

[Woo17] Melanie Matchett Wood. The distribution of sandpile groups of random graphs. *Journal of the American Mathematical Society*, 30(4):915–958, 2017. (Cited on pages 16 and 196.)

[Woo18] Melanie Matchett Wood. Cohen-Lenstra heuristics and local conditions. *Research in Number Theory*, 4(4):41, 2018. (Cited on page 16.)

[Woo19] Melanie Matchett Wood. Random integral matrices and the cohen-lenstra heuristics. *American Journal of Mathematics*, 141(2):383–398, 2019. (Cited on pages 16, 21, and 196.)