RECONSTRUCTING CONVEX SETS

by

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ABSTRACT

The problem of reconstructing an unknown set from a given partial description arises in a variety of applications areas such as robotics, tomography, pattern recognition, and nondestructive evaluation. For the most part, these applications areas have remained disjoint. This is unfortunate, because there is often substantial overlap in the underlying mathematical set reconstruction problems. This suggests the need for a unified mathematical theory of set reconstruction.

The main contribution of this thesis is to begin to develop a general theory of set reconstruction. To this end, we develop results for some generally stated reconstruction problems, explore and organize the existing literature on problems related to set reconstruction, and abstract several themes from our work. Our focus is on the important special class of reconstruction problems in which the unknown set is known to be a convex set. This class is selected because convex sets are important in many applications, and the theory of convex sets is well-developed.

First, we study the case where the given partial description consists of three sets $\mathbf{G}_i$, $\mathbf{G}_e$, and $\mathbf{G}_b$ that are known to respectively be subsets of the interior, exterior, and boundary of the unknown convex set **A**. For this problem, the following three sets are characterized: the largest set that can be guaranteed to be contained in the interior of **A**, the largest set that can be guaranteed to be contained in the exterior of **A**, and the smallest set that can be guaranteed to contain the boundary of **A**. We also focus upon the special case where only a finite number of points on the boundary of **A** is given. For this case, conditions that must hold in order to conclude that the set **A** is bounded are given.

Next, we consider the case where the set **A** is known to be a simplex and we are given two sets **F** and **G** that satisfy $\mathbf{F} \subset \mathbf{A} \subset \mathbf{G}$. We show how this problem arises in many signal analysis problems, when we attempt to estimate a collection of unknown positive functions from a given set of functions that are positive combinations of the unknown component functions. For this simplex reconstruction problem, we characterize the smallest set **V** that can be guaranteed to contain all the vertices of the unknown simplex **A**. Key results are given for the special but practically important 3-component case, where **F** and **G** are convex planar sets. For this case, we present several geometric fixed point algorithms that may be used to obtain an approximation to **V**. In addition, we describe the special structure of the boundary of **V** in the case where **F** and **G** are convex polygons.

We then unify the literature on problems related to reconstructions where the unknown set **A** is known to be a sphere, plane, or polytope. First, we discuss the problem of reconstructing an unknown sphere S from two given sets **F** and **G** that satisfy $\mathbf{F} \subset S \subset \mathrm{com}(\mathbf{G})$. Then, we consider the problem of reconstructing an unknown m-dimensional plane **P** from k sets $\mathbf{A}_1, \ldots, \mathbf{A}_k$ that intersect **P**. Finally, we describe some existing algorithms that generate a sequence of boundary points to reconstruct an unknown polytope. We show that this problem is the dual to that of reconstructing a polytope by using a sequence of support planes. The relevant literature on minimum spanning spheres, largest empty spheres, α-hulls, motion planning, and interval linear equations is also discussed.

Finally, the following themes are isolated: penumbras, star-shaped sets, extreme sets (i.e. largest and smallest), iterative algorithms, and duality. Some of these themes recur throughout our work, others (such as duality) have surfaced in the work and bear further study in the development of a general theory.

Several of the mathematical results obtained in our investigation are of interest in their own right. In addition, many of our results may be directly applicable to a variety of areas such as robotics, tomography, pattern recognition, and chemometrics.

Thesis Supervisor: George C. Verghese
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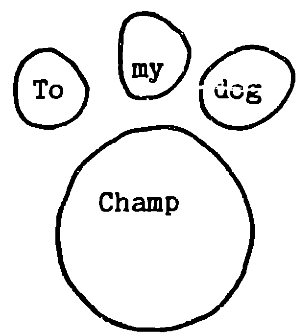

To
my
dog
Champ

CONTENTS

LIST OF SYMBOLS

Most of the mathematical terms that are used in the following list of symbols are defined in Appendix 1.

$B(p,\delta)$	The open ball about the point p with radius δ.
bdy(A)	The boundary of A.
clo(A)	The closure of A.
com(A)	The complement of A.
dim(A)	The dimension of A.
dist(A,B)	The distance between two sets A and B.
ext(A)	The exterior of A.
gcd(i,j)	The greatest common divisor of i and j.
hul(A)	The convex hull of A.
hul(A,B)	The convex hull of the union of the sets A and B.
$\text{hul}(p_1,\ldots,p_m)$	The convex hull of $\{p_1,\ldots,p_m\}$.
iff	If and only if.
int(A)	The interior of A.
ker(A)	The kernel of A.
pen(A,B)	The penumbra of A with respect to B.
R^n	n-dimensional Euclidean space.
[a,b]	$\{ x \mid x \text{ is in } R \text{ and } a \le x \le b \}$, or the line segment joining points a and b.
(a,b)	$\{ x \mid x \text{ is in } R \text{ and } a < x < b \}$, or the open line segment joining points a and b.
[a,b)	$\{ x \mid x \text{ is in } R \text{ and } a \le x < b \}$, or the union of a and the open line segment (a,b).
(a,b]	$\{ x \mid x \text{ is in } R \text{ and } a < x \le b \}$, or the union of b and the open line segment (a,b).

$[f:\alpha]$	$\{ x \mid f(x) = \alpha \}$.
$\cup$	union.
$\cap$	intersection.
$\subset$	contained in.
$\in$	element of.
v^T	the transpose of v.
$\emptyset$	The empty set.
[9]	Reference number 9.
(9)	Equation number 9.
$\lvert p-q \rvert$	The distance between points p and q.
$f(A)>\alpha$	$f(x)>\alpha$, for all x in A.
$\phi(n)$:	The Euler phi-function.

CHAPTER 1. INTRODUCTION

1. Set Reconstruction Problems

There are many practical situations where one is faced with the problem of reconstructing an unknown subset of R^n from a given partial description of the set. We shall refer to any problem of this type as a set reconstruction problem. Set reconstruction problems arise from the areas of robotics, tomography, nondestructive evaluation, and chemometrics (just to name a few). Let us take a closer look at each of these areas in order to illustrate some typical set reconstruction problems.

Set reconstruction problems arise in robotics when we attempt to process tactile or visual information. Refer to Fig.1.

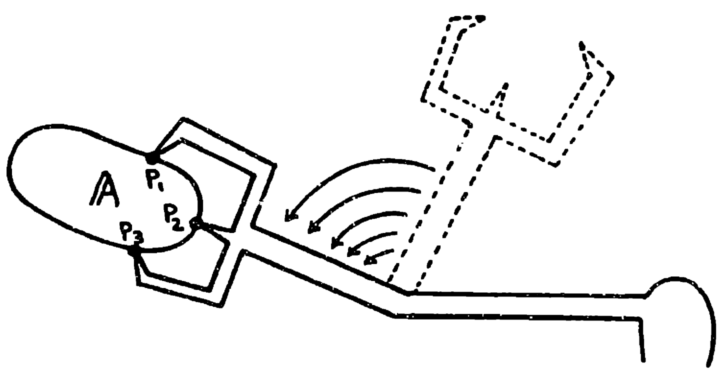


Fig.1a

Fig.1a shows a robot arm that has swept out a subset of R^3 and grasped an unknown object A. In this case, the partial description of A might consist of the following two facts: the points p_1, p_2, and p_3 are on the boundary of A, and the set swept by the robot's arm is contained in the complement of A. Some reconstruction problems of this type were considered in [3], [8], [17], [46], and [49]; see Chapter 4 for a discussion of some of these papers. A more general form of this set reconstruction problem will be considered in Chapter 2.

0	0	0	0	0	0	0
0	1	1	0	1	1	0
1	0	0	1	0	0	1
0	1	1	0	1	1	0
0	0	0	0	0	0	0

Fig.1b

The visual input to a robot often takes the form of a picture that has been discretized and digitized. An example is given in Fig.1b, which shows an overhead view of an unknown object **A** that is resting on a table. The number 0 has been assigned to the grid points that do not lie in **A** and the number 1 has been assigned to the points that lie in **A**. In this case the partial description of the unknown set **A** might consist of the planar grid of points together with a mapping from the grid into the set {0,1}.

A number of set reconstruction problems arise from the area of tomography. In positron-emission tomography, the distribution of an ingested radioisotope must be deduced from a partial description that consists of a collection of lines that pass through an unknown radioactive region **A** in the body [30]. This situation is depicted in Fig.2a. In other types of tomography [55], the partial description consists of a set of projections of the unknown set **A**, see Fig.2b.

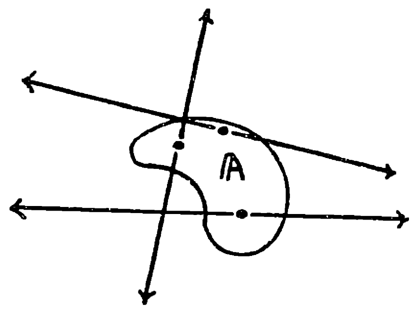


Fig.2a

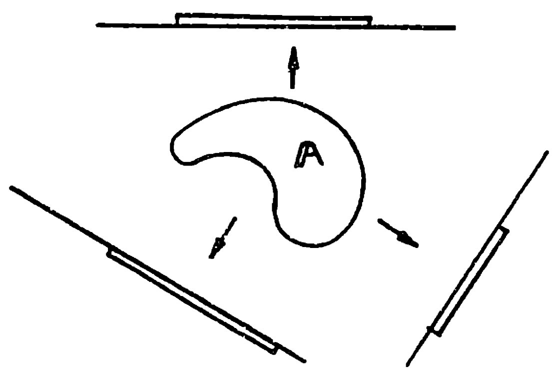


Fig.2b

In the area of nondestructive evaluation, reconstruction problems arise when we attempt to identify a flaw in a piece of material. The flaw could either be a void (an air pocket) or an inclusion (a piece of another material).

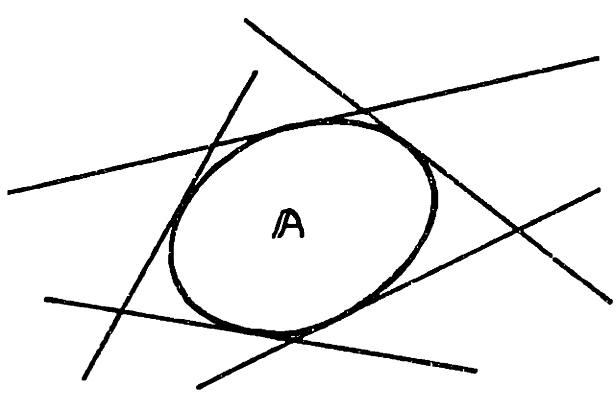


Fig.3

In [22], it was shown how ultrasound measurements of an unknown flaw **A** may be used to obtain a partial description that consists of a collection of support (or tangent) planes for **A**, see Fig.3. The results in [22] were obtained by taking advantage of the fact that many flaws are ellipsoidal (or nearly ellipsoidal).

In reconstruction problems that are derived from robotics, tomography, and nondestructive evaluation, the set that must be reconstructed is an actual physical object. As a result, these problems are typically 3-dimensional in nature. Reconstruction problems in higher dimensions may arise from situations where the set that must be reconstructed is a byproduct of some geometric interpretation of a problem that is not obviously geometric. Some reconstruction problems of this type arise from the area of chemometrics. In Chapter 3, we shall show how a geometric interpretation of the problem of identifying the components of a chemical mixture, by using a collection of absorption spectra of chromatographic fractions, leads us to consider the following set reconstruction problem: estimate an unknown n-simplex **A** from two given sets **F** and **G** that satisfy $\mathbf{F} \subset \mathbf{A} \subset \mathbf{G}$. A 2-dimensional example of this reconstruction problem is illustrated in Fig.4.

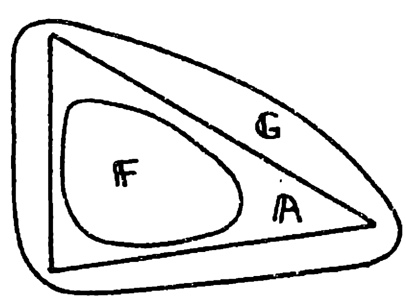


Fig.4

Although set reconstruction problems arise in a wide variety of applications and thus are very important, there is no unified mathematical theory of set reconstruction. By this we mean that there is no collection of general concepts and tools for reconstructing an unknown set from a given partial description. This is not to say that researchers have not considered any reconstruction problems. On the contrary, many authors have considered the particular problems that arise in robotics or tomography. Unfortunately, the literature does not recognize or take advantage of the fact that when we abstract the underlying mathematical problems from the particular set reconstruction problems that arise in these applications, there is considerable overlap.

The main contribution of this thesis is to begin to develop a general theory of set reconstruction. The steps that must be taken to develop a useful theory of set reconstruction are similar to those that have been taken to obtain a useful theory of numbers, probability, groups, linear equations, or any other entity. First, some tractable mathematical problems must be abstracted from the potential applications for the general theory. Next, we must attack these mathematical problems. Then, a collection of recurring themes must be abstracted from our solutions to these problems. If there are no significant recurring themes, then additional tractable problems must be solved. Once a collection of themes has been identified, we can develop results for these. The results and themes that are obtained by this process form the body of the general theory. [It seems as though we have the seeds for the development of a general theory of generating general theories! If we really did understand the process of generating general theories, we might be able to program a computer to generate the general theory of set reconstruction whilst we eat bonbons. But this sounds like another thesis.]

To accomplish our goal, first we develop results for some generally stated reconstruction problems that arise from the application areas that we have listed. Then we explore and organize the existing literature on problems related to set reconstruction. Finally, a collection of recurring themes will be abstracted from our work. A more detailed outline of these tasks is given in the next section.

2. Outline of the Thesis

We have decided to focus upon the important special class of reconstruction problems in which the unknown set is known to be a convex set. We have selected this class of problems because convex sets are important in many applications (e.g. see the reconstruction problems that arise from the areas of nondestructive evaluation and chemometrics that were discussed in Section 1) and the theory of convex sets is well-developed (so we should be able to formulate some tractable problems).

A detailed study of the following two reconstruction problems will be given (note that the terms interior, exterior, boundary, convex, and n-simplex that are used in these problem statements are defined in Appendix 1):

> Reconstruction Problem 1:
>
> Estimate an unknown convex set **A** from three given sets G_i, G_e, and G_b that are contained in the interior, exterior, and boundary of **A**, respectively.

Reconstruction Problem 2:

Estimate an unknown n-simplex **A** from two given sets **F** and **G** that satisfy $\mathbf{F} \subset \mathbf{A} \subset \mathbf{G}$.

Reconstruction Problem 1 will be discussed in Chapter 2. Reconstruction Problem 2, which arises when we attempt to estimate a collection of unknown positive functions from a given set of functions that are positive combinations of the unknown functions, will be considered in Chapter 3. Many interesting results will be given for each of these problems. In order to provide some feeling for the flavor of these results, we shall describe one result for each problem here.

In Chapter 2, we show that if we are given eight points on the boundary of an unknown 3-dimensional convex set **A**, then in some cases it is possible to conclude that the set **A** is bounded. We also show that this conclusion cannot be drawn if we are given less than eight points (i.e. for every set of m points $\{p_1,\ldots,p_m\}$ in $\mathbf{R}^3$, if m is in the interval [1,7] and there is a convex subset **A** of $\mathbf{R}^3$ for which p_i is in bdy(**A**), for all i, then there is an unbounded convex subset B of $\mathbf{R}^3$ for which p_i is in bdy(B), for all i).

In Chapter 3, we shall describe a collection of iterative algorithms that may be used to obtain solutions to Reconstruction Problem 2 for the case where the unknown simplex **A** is a triangle. One of these algorithms is illustrated in Fig.5 for a particular choice of the given sets **F** and **G**. In Chapter 3, we show that this procedure converges to a triangle **T** that has all of its vertices on the boundary of **G** and all of its legs tangent to **F** (if such a triangle exists). A triangle **T** of this type has the property that none of the vertices of the unknown triangle $\mathbb{A}$ can be contained in int($\mathbb{T}$). By exploiting this property, certain possible locations for the unknown triangle $\mathbb{A}$ may be ruled out.

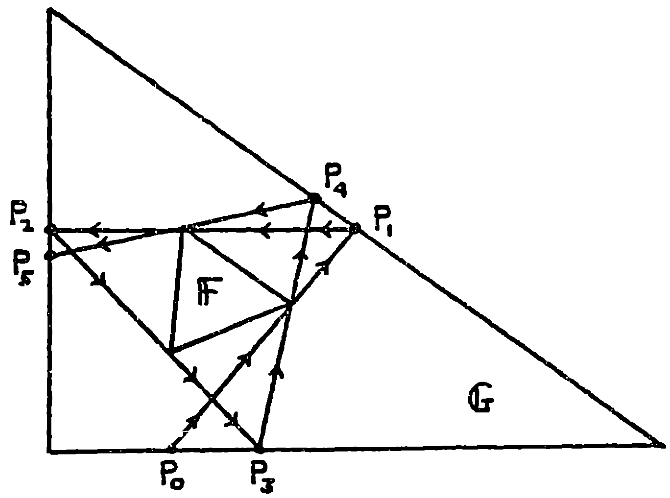


Fig.5

The two results that we have just described focus on the 2 and 3-dimensional cases. Most of the results that are presented in this thesis are only developed for low dimensional cases. The problem of extending these results to higher dimensions is not trivial. However, the low dimensional case is of interest in many applications, and much of the literature in this area is restricted to this.

The approaches that we shall use in solving Reconstruction Problems 1 and 2 will not yield a single estimate of the unknown set $\mathbb{A}$. Rather than focusing upon a <u>single set</u> that satisfies the constraints imposed by the given partial description, we shall explore the structural properties of the <u>class of all sets</u> that satisfy these constraints. This approach has been applied to a variety of other problems, see [60] and [38].

Apart from helping us formulate a general theory of set reconstruction, the results given in Chapters 2 and 3 may be directly applicable to a variety of areas such as robotics, tomography, pattern recognition, and chemometrics. These applications will be cited on a result-by-result basis throughout the thesis.

In Chapter 4, we unify the literature on problems related to set reconstruction where the unknown convex set **A** is known to be a sphere, plane, or polytope. We first discuss the problem of reconstructing an unknown sphere **S** from two given sets **F** and **G** that satisfy $\mathbf{F} \subset \mathbf{S} \subset com(\mathbf{G})$. We then consider the problem of reconstructing an unknown m-dimensional plane **P** from k sets $\mathbf{A}_1,\ldots,\mathbf{A}_k$ that intersect **P**. Finally, we describe some existing algorithms that generate a sequence of boundary points to reconstruct an unknown polytope. We show that this problem is the dual of the problem of reconstructing a polytope by using a sequence of support planes. The relevant literature on minimum spanning spheres, largest empty spheres, α-hulls, motion planning, and interval linear equations will also be discussed.

In his frequently cited book [54], Rockafellar defines the penumbra of a set **A** with respect to a set **B** as the set of all points p of the form

$$p = (1-\lambda)a + \lambda b, \text{ for some } a \text{ in } \mathbf{A}, \text{ some } b \text{ in } \mathbf{B}, \text{ and some } \lambda \leq 0 .$$

Rockafellar refers to the penumbra as "an interesting construction". However, as far as we can tell, he never actually uses it in the book. It turns out that the penumbra is a very important set that surfaces throughout our work. As a result, this is one of the recurring themes that we shall describe. A discussion of some recurring themes may be found in Chapter 5. In addition to penumbras, we shall discuss star-shaped sets, extreme sets, iterative algorithms, and duality. Chapter 5 also contains a summary of the results of the Chapters 2, 3, and 4.

This thesis represents a solid step toward the development of a general theory of set reconstruction. However, the job is far from finished. Chapter 5 contains a discussion of the steps that should be taken in order to continue towards a general theory of set reconstruction.

In order to make this thesis accessible to a wide audience, Appendix 1 has been included to present most of the required background material. In this appendix, we shall review many of the mathematical terms and basic results that are used in the thesis.

The proofs for the results given in Chapters 2 and 3 are given in Appendices 2 and 3, respectively.

Throughout the thesis, we make use of the important notion of a convex hull to form compact statements of our results. Thus, in order to directly apply many of our results, one must equipped with an algorithm for determining the convex hull of a given set. A discussion of some convex hull algorithms is given in [61], [19], [33], and in the forthcoming book [51].

CHAPTER 2. RECONSTRUCTING A CONVEX SET FROM SUBSETS OF ITS INTERIOR, EXTERIOR, AND BOUNDARY

In this chapter, we shall consider the problem of reconstructing as well as possible an unknown convex set **A** from three given sets G_i, G_e, and G_b that are known to be subsets of the interior, exterior, and boundary of **A**, respectively.

Our results for this problem form a foundation for the work described in Chapter 3. In addition, these results are of interest in their own right, for they may be directly applicable to many problems in robotics. These latter applications will be cited throughout the chapter. Proofs of the results of this chapter are given in Appendix 2.

1. Outline of the Chapter

This section contains a description of the particular problems that will be considered in this chapter.

The chapter uses several basic results concerning the interior, relative interior, exterior, boundary, complement, closure, convex hull, and kernel of a set. Most of these basic results are given in Appendix 1, which also contains a dictionary of terms. Additional results, properties, and definitions may be found in [5] and [31].

<u>Notation</u>:

a. If **V** is a subset of R^n, then int(**V**), rint(**V**), ext(**V**), bdy(**V**), com(**V**), clo(**V**), hul(**V**), and ker(**V**) denote the interior, relative interior, exterior, boundary, complement, closure, convex hull, and kernel of the set **V**, respectively. int(V,W) denotes the interior of the union of the sets V and W (the sets ext(V,W) through ker(V,W) are defined similarly).

b. If p is a point in R^n, then $B(p,\delta)$ denotes the open ball about p with radius δ.

c. dist(V,W) denotes the distance between two sets V and W.

d. pen(V,W) denotes the penumbra of V with respect to W.

Now our problem is the following: let C denote the class of all convex subsets A of R^n that satisfy all three of the following:

$$G_i \subset \text{int}(A) \,, \tag{1a}$$

$$G_b \subset \text{bdy}(A) \,, \tag{1b}$$

$$G_e \subset \text{ext}(A) \,, \tag{1c}$$

where G_i, G_b, and G_e are some given subsets of R^n. Fig.1 illustrates the family of sets C for a particular choice of G_i, G_b, and G_e.

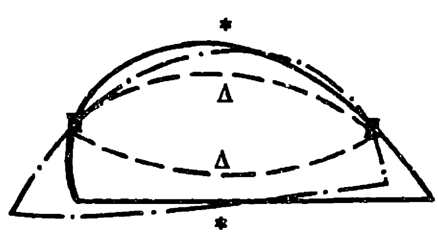


Fig.1

In our figures we shall use the symbols Δ, ■, and * to represent points that are in G_i, G_b, and G_e, respectively. Note that although our figures consist of 2-dimensional configurations of a finite number of points, all our results hold for the case where G_i, G_b, and G_e are arbitrary subsets of R^n.

Let **I**, **B**, **E**, and **O** denote the subsets of R^n given by

$$\mathbf{I} = \{ p \mid p \in \mathrm{int}(\mathbf{A}),\ \forall \mathbf{A} \in \mathbf{C} \} , \tag{2a}$$

$$\mathbf{B} = \{ p \mid p \in \mathrm{bdy}(\mathbf{A}),\ \forall \mathbf{A} \in \mathbf{C} \} , \tag{2b}$$

$$\mathbf{E} = \{ p \mid p \in \mathrm{ext}(\mathbf{A}),\ \forall \mathbf{A} \in \mathbf{C} \} , \tag{2c}$$

$$\mathbf{O} = \mathrm{com}(\mathbf{I} \cup \mathbf{B} \cup \mathbf{E}) . \tag{2d}$$

If **A** is a set in **C**, then **I**, **B**, and **E** represent the largest sets that can be guaranteed to be contained in int(**A**), bdy(**A**), and ext(**A**), respectively.

The sets **I**, **B**, **E**, and **O** could play an important role in many problems in robotics. Suppose that **A** is an unknown convex set and we have obtained the measurements $\mathbf{G}_i$, $\mathbf{G}_b$, and $\mathbf{G}_e$ of the interior, boundary, and exterior of **A**. Since **A** must be contained in the set com(**E**) and **I** must be contained in **A**, the sets **I** and com(**E**) are bounds for the set **A**. If a robot could determine both **I** and com(**E**), then it could use this information to help it decide which object it has selected from a bin of parts containing several different types of convex objects.

In this chapter, we shall discuss several properties of the class **C** and sets **I**, **B**, **E**, and **O**. In Section 2.1, we shall give some necessary and sufficient conditions for **C** to be nonempty. Some structural properties of **C** and a characterization of the smallest set in **C** are given in Section 2.2. In Section 3.1, we shall characterize the sets **I** and com(**E**). In addition, we show that com(**E**) is a star-shaped set, and give conditions under which com(**E**) is bounded. In Section 3.2, we shall characterize the smallest set that contains the boundary of every set in **C**. In Section 3.3, we shall suggest a way by which new characterizations of the sets **I**, **B**, **E**, and **O** can be obtained by using conditions for **C** to be nonempty. Section 4 contains a description of some open problems.

2. Properties of C

Several properties of the class **C** will be discussed in this section. In Section 2.1, we shall develop some necessary and sufficient conditions for **C** to be nonempty. In Section 2.2 we shall show that if $\{A_1, \ldots A_k\}$ is known to be a subset of **C**, then we can also classify some other types of sets with respect to **C**.

2.1 Conditions for C to be Nonempty

Conditions for **C** to be nonempty are important for two reasons. First, they suggest ways by which one could reject the initial hypothesis that a particular unknown set **A** that satisfies (1) is convex. A robot with this ability could determine whether it has selected a ball or a ring from a bin of parts containing these two types of objects.

A second use of conditions for **C** to be nonempty will be discussed in Section 3, where we shall show that such conditions may be used to classify a given point with respect to the sets **I**, **B**, **E**, and **O**.

We shall give our conditions for **C** to be nonempty after a sequence of intermediate lemmas.

Let G denote the set given by

$$G = G_1 \cup G_b \ . \qquad (3)$$

Lemma 1:

If **A** is a set in the class **C**, then hul(**G**) is contained in clo(**A**).

Lemma 1 is illustrated in Fig.2. The closure of the set **A** in Fig.2 contains the convex hull of **G**.

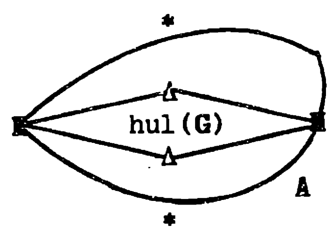


Fig.2

The next lemma shows that if **C** is nonempty, then hul(**G**) satisfies all but one of constraints that define **C**.

Lemma 2:

If **C** is nonempty, then

$$G_b \subset \text{bdy}[\text{hul}(G)] \ , \quad \text{and} \tag{4a}$$

$$G_e \subset \text{ext}[\text{hul}(G)] \ . \tag{4b}$$

From Lemma 2, it is clear that if **C** is nonempty and G_i is contained in int[hul(**G**)], then hul(**G**) must be in **C**. Let **Y** be the set given by

$$Y = \text{bdy}[\text{hul}(G)] \cap G_i . \tag{5}$$

Fig.3 illustrates **Y** for a particular choice of G_i, G_b, and G_e. For this example **Y** equals {a,b}.

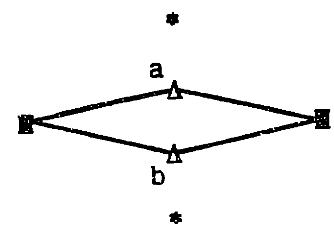


Fig.3

With this definition, we have the following result:

Lemma 3a:

If **C** is nonempty and **Y** is empty, then hul(**G**) is in **C**.

Next, we shall investigate what can be said in the case where **C** is nonempty and **Y** is nonempty. In this case, hul(**G**) cannot be in **C** because G_i is not contained in int[hul(**G**)]. However, there are sets in **C** that are only slightly larger than hul(**G**), as we discuss next.

Let **A** be a set in **C**. G_i must be contained in int(**A**). Thus, for each point p in G_i, there is some positive number r(p) for which the open ball B[p,r(p)] is contained in **A**. Let T(r) be the set given by

$$T(r) = \text{hul}[G, [\bigcup_{y \in Y} B(y, r(y))]] \ . \tag{6}$$

The set T(r) is depicted in Fig.4 for a particular choice of the sets G_i, G_b, G_e, and **A**.

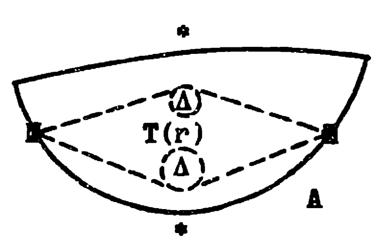


Fig.4

It is easily shown that

$$\text{hul}(\mathbf{G}) \subset \mathbf{T}(r) \subset \text{clo}(\mathbf{A}) \ . \tag{7}$$

We then have the following:

Lemma 4:

The set **T**(r) given by (6) is in **C**.

Let **T** denote the class of all sets **T**(r) that satisfy (6) for some map r from **Y** to $(0,\infty)$. From the discussion thus far, we have the following result:

Lemma 3b:

If **A** is a set in **C** and **Y** is nonempty, then there is a set **T**(r) in **T** that is in **C** and contained in clo(**A**).

By combining Lemmas 3a and 3b, we obtain the following necessary and sufficient conditions for the class **C** to be nonempty (note that sufficiency is trivial):

Theorem 1: Conditions for C to be Nonempty:

C is nonempty <u>iff</u>

a. **Y** is empty and hul(**G**) is in the class **C**, or

b. **Y** is nonempty and there is a set **T**(r) in **T** that is in the class **C**.

2.2 The Structure of C

In this section we investigate what may be deduced from the knowledge that some given sets belong to **C**. In particular, we attempt to classify certain additional types of sets with respect to **C**. We shall also state a theorem that gives the smallest set in **C**.

If **A** is a set in **C**, then clo(**A**) is convex, int(**A**) equals int[clo(**A**)], and bdy(**A**) equals bdy[clo(**A**)]. In addition, rint(**A**) is convex and bdy(**A**) equals bdy[rint(**A**)]. From this, we have the following lemma:

Lemma 5:

If **A** is in **C**, then clo(**A**) and rint(**A**) are also in **C**.

The converse of Lemma 5 does not hold. If clo(**A**) is in **C**, we cannot conclude that **A** is also in **C**. The problem lies in the fact that the closure of a nonconvex set may be convex. For example, if **A** is the 1-dimensional set given by the union of the intervals [-1,0) and (0,1], then **A** is not convex. However, clo(**A**) equals the interval [-1,1] which is convex. If rint(**A**) is in **C**, we cannot conclude that **A** is in **C**. Here the problem lies in the fact that the interior of a nonconvex set may be convex. For example, consider the set **A** shown in Fig.5. Since **A** does not contain the open segment that joins the points p and q, **A** is not convex. However, int(**A**) is convex.

Fig.5

Although the converse of Lemma 5 does not hold, we do have the following result:

Lemma 6:

If A is a convex set and clo(A) or rint(A) is in C, then A is also in C.

In the previous subsection, we defined the class of sets T for the case where Y (the intersection of bdy[hul(G)] and G_i) is nonempty. The next lemma gives a structural result for C with respect to this class T.

Lemma 7:

Suppose Y is nonempty. If $T(r_2)$ is a set in T that is in C, then every set $T(r_1)$ in T that satisfies

$$T(r_1) \subset T(r_2) \;, \tag{8}$$

is also in C.

We shall now state a theorem that gives the smallest set in C for the case where C is nonempty. This result could be applied in the area of robotics. A robot that could determine the smallest set in C, might use this ability to help it decide whether it has selected a small ball or a large ball from a bin of parts containing two different size balls.

The collection of all subsets of R^n is an ordered set with respect to the set containment relation $\subset$ (see [24] for the definition of an ordered set). The next theorem characterizes the smallest set in C with respect to this ordering.

Theorem 2: Smallest Set in C:

Suppose C is nonempty and the dimension of hul(G) equals n.

a. If Y is empty, then I is the smallest set in C (i.e. I is in C and I is contained in every set in C).

b. If Y is nonempty, then I is the infimum of C (i.e. I is the largest subset of R^n that is contained in every set in C).

Theorem 2 may be obtained by combining Lemmas 1, 3, 5, and 7 with the following fact: if W is a proper subset of the relative interior of a convex set V, then either W is not convex or clo(W) is a proper subset of clo(V).

Next we shall consider the case where we are given two sets A_1 and A_2 in C. Suppose that A_1 is contained in A_2. This situation is depicted in Fig.6 for a particular choice of C_i, G_b, and G_e.

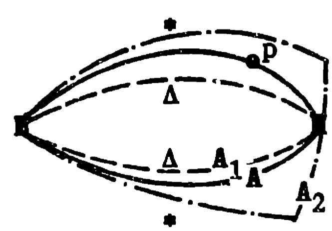


Fig.6

The next lemma shows that all the convex sets between A_1 and A_2 must be in C.

Lemma 8:

If A_1 and A_2 are in C, then every convex set A that satisfies

$$A_1 \subset A \subset A_2 , \tag{9}$$

is also in C.

3. Properties of I, B, E, and O

This section contains a discussion of several properties of the sets I, B, E and O given by (2).

In Section 3.1, we shall characterize the sets I and com(E).

If A is an unknown convex set and we have obtained the measurements G_i, G_b, and G_e of the interior, boundary, and exterior of A, then com(E) is the smallest set that can be guaranteed to contain A. Suppose we also know that A is bounded. In this case, we would hope that the set com(E) is also bounded. This can happen for certain choices of the sets G_i, G_b, and G_e. In Section 3.1, we shall consider the case where we take m boundary point measurements of A. We shall state results for both the 2 and 3-dimensional cases and conjecture a result for the n-dimensional case. Our results imply the following: 8 boundary point measurements of an unknown bounded convex set A in R^3 can provide enough information to determine a bounded set that must contain A. Said another way, in some cases it is possible to conclude that an unknown convex set in R^3 is bounded, from only 8 boundary point measurements. Whether or not this can be done in a particular case depends upon the configuration of the 8 boundary points.

If A is an unknown convex set and we have obtained the measurements G_i, G_b, and G_e of the interior, boundary, and exterior of A, then A must be contained in the set com(E) and I must be contained in A. In Section 3.2, we shall show that any point in com(E) that is not in I could be a boundary point for A.

In Section 2.1, we gave some conditions under which C is nonempty. In Section 3.3, we shall show how conditions of this type may be used to obtain new characterizations of the sets I, B, E, and O.

3.1 Properties of I and com(E)

In this section, we characterize the sets I and com(E) and describe some properties of the set com(E). We shall show that com(E) is a star-shaped set with kernel clo[hul(G)]. Then we shall discuss some conditions under which com(E) is bounded.

First, we observe that I must contain G_i, B must contain G_b, and E must contain G_e. Since the sets in C are convex, it is possible to say much more. Fig.7 depicts some of the implications that the convexity assumption gives us.

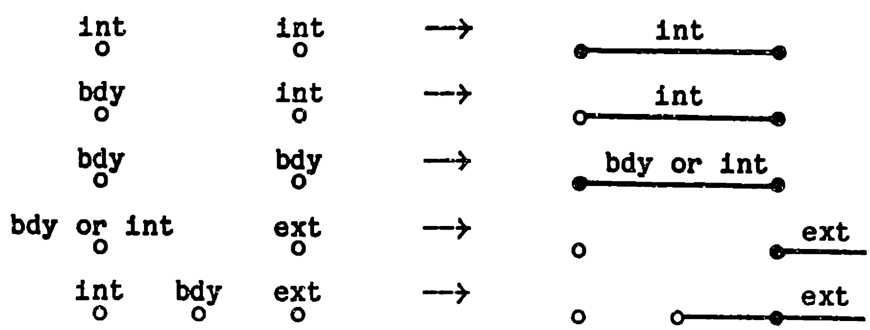


Fig.7

For example, the first statement in Fig.7 is as follows: if two points are in the interior of a convex set, then the line segment that joins them is also contained in the interior of the set, i.e. the interior of a convex set is convex. By using such statements, it is possible to grow the sets I, B, and E from the seeds G_i, G_b, and G_e. An example of how this can be done is given in Fig.8.

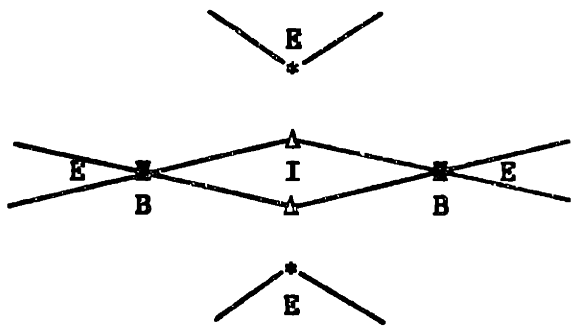


Fig.8

The next two lemmas are used to obtain a characterization of I.

Lemma 9:

Suppose Y is nonempty. If p is a point in ext[hul(G)], then there is a set T(r) in T for which p is in ext[T(r)].

Fig.9 illustrates Lemma 9. If Y is nonempty and p is a point in ext[hul(G)], then we may shrink the open balls about the points in Y that are used to define T(r) to force p into the exterior of T(r).

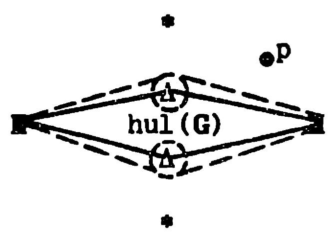


Fig.9

Let Z denote the set of points p that satisfy the following constraints: p is in bdy[hul(G)] and all the support hyperplanes for clo[hul(G)] at p intersect G_i. In Fig.9, Z equals bdy[hul(G)] minus the two points in G_b.

Lemma 10:

Suppose Y is nonempty. A point p in bdy[hul(G)] is in bdy[T(r)] for some set T(r) in T iff p is not in Z.

Lemma 10 is illustrated in Fig.10. There is only one support line of the set clo[hul(G)] at the point p and this line intersects G_i. From Fig.10, we can see that p is in int[T(r)] for all sets T(r) in T. There are support lines for clo[hul(G)] that do not intersect G_i at the two points that are not in Z and these points are in bdy[T(r)] for some of the T(r) in T.

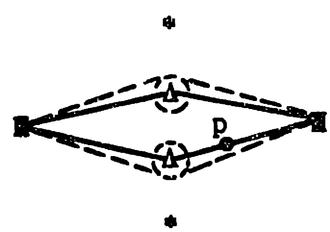


Fig.10

The next two theorems characterize the sets I and com(E).

Theorem 3: Characterization of I:

If C is nonempty, then

$$I = int[hul(G)] \cup Z \ .$$

Theorem 4: Characterization of com(E):

A point p in R^n is in com(E) <u>iff</u>

a. Y is empty and hul(p,G) is in C, or

b. Y is nonempty and hul[p,T(r)] is in C for some set T(r) in T.

Theorem 4 is illustrated in Fig.11. Since the intersection of G_b and the interior of hul[p,T(r)] is nonempty for all sets T(r) in T, hul[p,T(r)] cannot be in C. Thus p must be in E. Fig.11 shows that there is a set T(r) in T for which hul[q,T(r)] is in C. Thus, q must be in com(E).

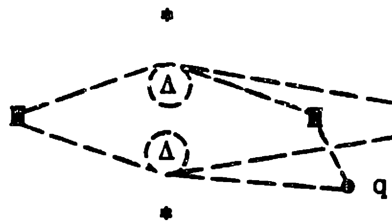


Fig.11

If p is a point in com(E), then there must be a set A in C for which p is in int(A) or bdy(A), see Fig.12. By Lemma 1, we known that clo[hul(G)] must be contained in clo(A). Thus by using the first three statements described in Fig.7, we may conclude that any line segment that joins p with a point q in clo[hul(G)] must be contained in clo(A), and hence in com(E). The next result gives a formal statement of this property.

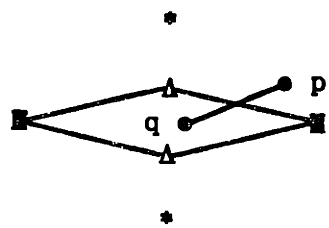


Fig.12

Lemma 11:

$$clo[hul(G)] \subset ker[com(E)] \ .$$

The kernel of com(E) may be larger than the set clo[hul(G)]. For example, consider the case illustrated in Fig.13. Theorem 4 may be used to show that com(E) equals the triangle abc. Thus ker[com(E)] equals abc which properly contains clo[hul(G)].

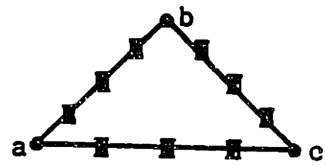


Fig.13

Suppose that we have determined that a point p is in com(E). Then by Lemma 11 we may conclude that all of the points in hul[p,clo[hul(G)]] are also in com(E). An analogous conclusion may be reached for the case where we have determined that a point p is in E. Suppose p is a point in E. Let S_p denote the penumbra of p with respect to the set clo[hul(G)] (S_p equals the set of points of the form $(1-\lambda)p+\lambda q$, for some point q in clo[hul(G)] and some $\lambda \leq 0$, see Fig.14). We may conclude that S_p is contained in E (if a point x in S_p was in com(E) then by Lemma 11, p would also have to be in com(E)). Thus, we have the following:

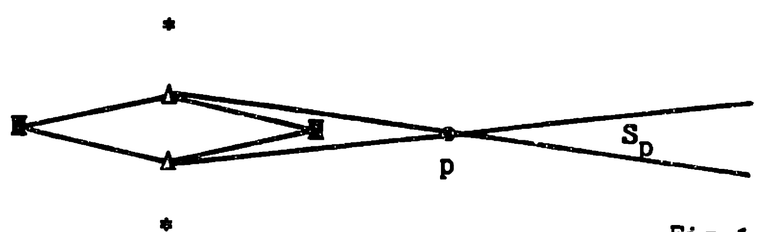


Fig.14

Lemma 12:

If p is in E, then S_p is contained in E.

We close this section with some conditions under which the set com(E) is bounded. We shall focus upon the case where G_i and G_e are empty, and G_b equals a finite set of points. An example of this case is shown in Fig.15. Note that in this example the set com(E) is bounded.

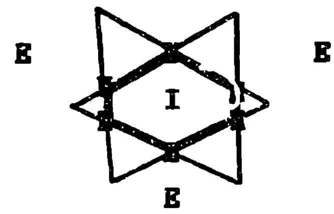


Fig.15

The next result gives some conditions under which com(E) may be bounded. These conditions are stated in terms of the cardinality of G_b.

Result 1:

Suppose G_i and G_e are empty, and G_b consists of m points.

a. In the 2-dimensional case, if $m<5$, then com(E) is unbounded. If $m\geq 5$, then com(E) could be bounded.

b. In the 3-dimensional case, if $m<8$, then com(E) is unbounded. If $m\geq 8$, then com(E) could be bounded.

c. In the n-dimensional case, if $m\geq 2n+2$, then com(E) could be bounded.

We conjecture that in the $(n>3)$-dimensional case, com(E) must be unbounded if m is less than 2n+2.

3.2 The Boundary Points of Sets in C

A characterization of the set of points that are in bdy(A) for some set A in C will be given in this section. We need the following lemma:

Lemma 13:

Let p be a point in R^n and let A_1 and A_2 be sets in C. If p is in both ext(A_1) and int(A_2), then there is a set A in C for which p is in bdy(A).

Lemma 13 is illustrated in Fig.6.

The next theorem follows from Lemma 13.

Theorem 5: The Boundary Points of Sets in C:

Suppose C is nonempty. A point p is in bdy(A) for some set A in C *iff* p is in the union of B and O.

If A is an unknown convex set and we have obtained the measurements G_i, G_b, and G_e of the interior, boundary, and exterior of A, then Theorem 5 tells us that the union of B and O is the smallest set that can be guaranteed to contain the boundary of A.

3.3 An Application of Conditions for C to be Nonempty

We now show how new characterizations for the sets I, B, E, and O may be obtained from conditions for C to be nonempty.

Let p be a point in R^n and let $C_i(p)$, $C_b(p)$, and $C_e(p)$ denote the subclasses of C given by

$$C_i(p) = \{ A \mid A \in C \text{ and } p \in \text{int}(A) \} , \quad (10a)$$

$$C_b(p) = \{ A \mid A \in C \text{ and } p \in \text{bdy}(A) \} , \quad (10b)$$

$$C_e(p) = \{ A \mid A \in C \text{ and } p \in \text{ext}(A) \} . \quad (10c)$$

We then obtain the following theorem from Lemma 13:

Theorem 6: Characterization of I, B, E, and O:

Let p be a point in R^n.

p is in I <u>iff</u> $C_b(p) = C_e(p) = \emptyset$,

p is in B <u>iff</u> $C_i(p) = C_e(p) = \emptyset$,

p is in E <u>iff</u> $C_i(p) = C_b(p) = \emptyset$.

p is in O <u>iff</u> $C_b(p) \neq \emptyset$, and $C_i(p) \neq \emptyset$ or $C_e(p) \neq \emptyset$.

Suppose that we would like to classify a given point with respect to the sets I, B, E, and O. First, we observe that the classes $C_i(p)$, $C_b(p)$, and $C_e(p)$ all have the same form as the class C. For example, $C_i(p)$ is the class of convex sets A that satisfy the constraints

$$p \cup G_i \subset int(A) \ , \tag{11a}$$

$$G_b \subset bdy(A) \ , \tag{11b}$$

$$G_e \subset ext(A) \ , \tag{11c}$$

As a result, any test that determines whether or not C is empty (for example, Theorem 1) can also be used to determine whether or not $C_i(p)$, $C_b(p)$, or $C_e(p)$ is empty. Thus Theorem 6 can be used in conjunction with a test that determines whether or not C is empty, to classify a given point p with respect to I, B, E, and O. In this way, new conditions for C to be nonempty can generate new characterizations for the sets I, B, E, and O.

4. Extensions and Open Problems

In Result 1, we gave some conditions under which the set com(E) may be bounded in the case where G_i and G_e are empty and G_b consists of m points. This result only handles the 2 and 3-dimensional cases. We do not have any similar results for the n-dimensional case. We conjecture that in the (n>3)-dimensional case, com(E) must be unbounded if m is less than 2n+2.

In Theorems 3 and 4 we characterized the sets I and com(E). To complete the story, a characterization of B must be developed. It should be possible to obtain such a characterization by examining the points at which the sets bdy(I) and bdy[com(E)] intersect.

CHAPTER 3. RECONSTRUCTING A SIMPLEX: THE COMPONENT ANALYSIS PROBLEM

The problem of reconstructing as well as possible an unknown simplex S from two given sets F and G that satisfy $F \subset S \subset G$ will be considered in this chapter. Proofs of the results in this chapter may be found in Appendix 3.

1. Introduction

The Component Analysis Problem

Consider the following problem:

> Given r functions $f_1(x),\ldots,f_r(x)$ from R into R that are known to be of the form
>
> $$f_j(x) = w_{1j}c_1(x) + \ldots + w_{mj}c_m(x) \ , \tag{1}$$
>
> for some unknown number m of unknown nonnegative functions $c_1(x),\ldots,c_m(x)$ and some unknown nonnegative weights w_{ij}, estimate the functions $c_1(x),\ldots,c_m(x)$.

We shall refer to this as the <u>component analysis problem</u>.

This problem arises in many practical situations. For example, we must solve a component analysis problem when we attempt to estimate a collection of positive signals emitted from multiple sources by using a set of measurements that have been obtained from an array of sensors; see [57] for some related problems. In this case, at the j^{th} sensor we observe a signal $f_j(x)$ that is of the form (1), where $c_1(x),\ldots,c_m(x)$ are the appropriately delayed unknown source signals and w_{ij} is the attenuation factor from the i^{th} source to the j^{th} sensor.

The component analysis problem also arises when we attempt to identify the components in a collection of r chemical mixtures by using the absorption spectra of the mixtures. The absorption spectrum of a chemical component is a nonnegative function of frequency that measures how well the given component absorbs various frequencies of light, and the absorption spectrum $f_j(x)$ of the j^{th} mixture is of the form (1), where $c_1(x),\ldots,c_m(x)$ are the spectra of unit concentrations of the components of the mixture and the weight w_{ij} is the concentration of the i^{th} component in the j^{th} mixture. One typically observes the spectra of r mixtures of the same m components, where each mixture is formed from different relative concentrations of the components. We then must solve a component analysis problem to identify the component spectra $c_1(x),\ldots,c_m(x)$, see [32], [45].

This chapter focuses upon the 3-component problem (note that a brief discussion of the general case is given in Section 4). This special case is of considerable significance in the chemical analysis problem mentioned above, and yet the only treatment we are aware of is that of [45], which is rather limited.

A Related Simplex Reconstruction Problem

A subset S of R^n is called a k-*simplex* if it may be expressed as the convex hull of a k-dimensional set consisting of k+1 points (note that the *dimension of a subset S* of R^n is given by the dimension of the smallest plane containing it).

Next we show that if the functions $f_1(x),\ldots,f_r(x)$ in (1) are only known for d values of x, then the component analysis problem reduces to the following simplex reconstruction problem (where n will be defined shortly):

> Estimate an unknown n-simplex S from two given n-dimensional convex polytopes F and G that satisfy

$$F \subset S \subset G \ . \tag{2}$$

Suppose that we are given the values of the functions $f_1(x),\ldots,f_r(x)$ at d values $x_1,\ldots,x_d$ of the variable x. Let f_j denote the points in R^d given by

$$f_j = [f_j(x_1), f_j(x_2), \ldots, f_j(x_d)] \ / \ \alpha_j \ , \tag{3a}$$

where

$$\alpha_j = f_j(x_1) + \ldots + f_j(x_d) \ . \tag{3b}$$

Let F and G be the polytopes given by

$$F = \text{hul}(f_1,\ldots,f_r) \ , \tag{4a}$$

$$G = P \cap \{ \ p \mid p \geq 0, \text{ component-wise } \} \ , \tag{4b}$$

where P denotes the minimal-dimension plane that contains F. Let n equal dim(F). The sets F and G are illustrated in Fig.1 for the case where d=4, r=6, and n=2. In this case, the points $f_1,\ldots,f_6$ must lie in the regular tetrahedron with vertices (1,0,0,0), (0,1,0,0), (0,0,1,0), and (0,0,0,1). For the particular case shown in Fig.1, F is a pentagon and G is the quadrilateral abcd which equals the intersection of the 2-dimensional plane that contains F with the regular tetrahedron.

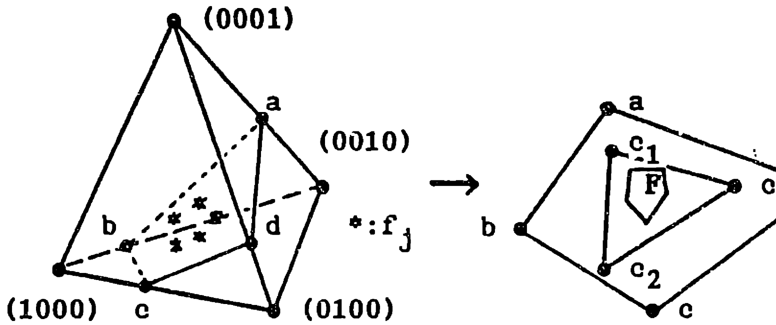


Fig.1

Suppose there is an n-simplex S that satisfies (2). Let $c_1,\ldots,c_{n+1}$ denote the vertices of S. For the example given in Fig.1, S could be any triangle $c_1c_2c_3$ that contains the pentagon F, and is contained in the quadrilateral G.

Since F is contained in S, each point f_j may be written as

$$f_j = w_{1j}c_1 + \ldots + w_{n+1,j}c_{n+1} , \tag{5a}$$

for some $w_{1j},\ldots,w_{n+1,j}$ for which

$$w_{ij} \geq 0, \text{ for all } i, \text{ and} \tag{5b}$$

$$w_{1j} + \ldots + w_{n+1,j} = 1 . \tag{5c}$$

In addition, since S is contained in G, the vertices of S satisfy

$$c_i \geq 0 , \quad \text{component-wise, for all } i . \tag{6}$$

Suppose $c_i(x)$ is a nonnegative function from R into R for which

$$c_i = [c_i(x_1), c_i(x_2), \ldots, c_i(x_d)] . \tag{7}$$

From (3), (5a), (5b), (6), and (7), we see that (1) holds for x equal to $x_1,\ldots,x_d$. Thus, the functions $c_1(x),\ldots,c_{n+1}(x)$ could be the actual components. In this way, every n-simplex S that satisfies (2) gives rise to an (n+1)-tuple $(c_1(x),\ldots,c_{n+1}(x))$ that could represent the actual components that combine to generate the functions

$f_1(x),\ldots,f_r(x)$. In fact, since a positive scaling λ_i of the function $c_i(x)$ may be absorbed by the weights w_{ij}, any (n+1)-tuple of the form $(\lambda_1 c_1(x),\ldots,\lambda_{n+1}c_{n+1}(x))$, for some positive scale factors $\lambda_1,\ldots,\lambda_{n+1}$, could also represent the actual components.

Next we show that, under certain conditions, one of the simplexes that satisfies (2) must correspond to the actual components. Suppose $c_1(x),\ldots,c_m(x)$ are the actual components. Let c_i denote the points in $\mathbf{R}^d$ given by

$$c_i = [c_i(x_1), c_i(x_2), \ldots, c_i(x_d)] \;/\; \beta_i \;, \tag{8a}$$

where

$$\beta_i = c_i(x_1) + \ldots + c_i(x_d) \;, \tag{8b}$$

and let S be the set $\mathrm{hul}(c_1,\ldots,c_m)$. From (1), it can be shown that each point f_j is a convex combination of the points $c_1,\ldots,c_m$. Thus, the set $\{f_1,\ldots,f_r\}$ must be contained in S. Therefore, since S is convex, F must be contained in S. If the number of components m satisfies

$$m = n + 1 \;, \tag{9}$$

then S must be an n-simplex that is contained in the minimal-dimension plane P that contains F (it can be shown that if (9) holds, then the points $c_1,\ldots,c_m$ must be linearly independent, and if $c_1,\ldots,c_m$ are linearly independent and $r \geq m$, then (9) will hold for almost all choices of the weights w_{ij} in (1)). In addition, since $c_1(x),\ldots,c_m(x)$ are nonnegative functions, S must be contained in the n-dimensional polytope G. Thus if m satisfies (9), S must satisfy (2).

We have shown how the component analysis problem may be reduced to the geometric problem of estimating an n-simplex from two n-dimensional convex polytopes F and G that satisfy (2). This geometric interpretation of the component analysis problem gives us a simple representation of the constraints imposed upon the functions $c_1(x),\dots,c_m(x)$ and $f_1(x),\dots,f_r(x)$. The first containment relation in (2) arises from the constraint that each function $f_j(x)$ is a positive combination of the functions $c_1(x),\dots,c_m(x)$. The second containment relation arises from the fact that the functions $c_1(x),\dots,c_m(x)$ are nonnegative.

2. Outline of the Chapter

Let G be a given n-dimensional subset of R^n and let F be a given bounded subset of G. Let C be the set defined by

$$C = \{ S \mid S \text{ is an n-simplex and } F \subset S \subset G \} . \qquad (10)$$

The class C is illustrated in Fig.2 for a particular choice of F and G.

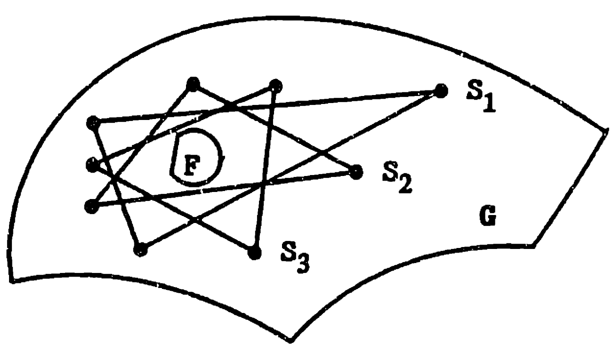


Fig.2

Let V denote the subset of R^n given by

$$V = \{ p \mid p \text{ is a vertex of some simplex in } C \} . \qquad (11)$$

We shall refer to this set as the vertex domain. The vertex domain is illustrated in Fig.3 for a particular choice of the sets F and G. In this figure, the point p is in V and the point q is not in V.

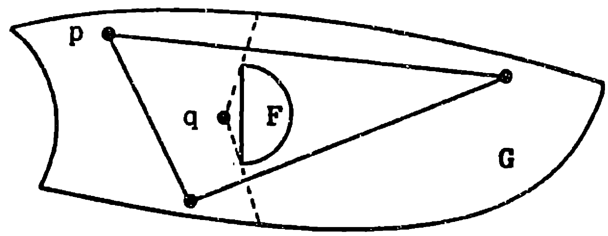


Fig.3

The set V plays an important role in the component analysis problem. Suppose that F and G are given by (4). Any point that does not lie in the vertex domain for the sets F and G cannot correspond to one of the underlying components. Thus, V may be used to obtain bounds that all of the underlying components must satisfy. Later we shall see that in some cases the set V is formed from the union of n+1 disjoint sets $V_1,\ldots,V_{n+1}$ with the property that each simplex S in C must have exactly one vertex in each of the sets $V_1,\ldots,V_{n+1}$. In these cases, we may approximate C as the Cartesian product of the sets $V_1,\ldots,V_{n+1}$ (recall that the Cartesian product of k sets $A_1,\ldots,A_k$ is the set of all k-tuples of the form $(a_1,\ldots,a_k)$, where a_i is in A_i). Thus, in some cases V may be used to obtain a separate bound for each of the underlying components in the component analysis problem.

In Section 3, we shall focus upon the case where F and G are compact convex planar sets. This would correspond to the 3-component problem. In Section 3.1 we characterize V and specialize our results to the case where F and G are convex polygons. Section 3.2 describes three interesting iterative procedures that may be used to obtain an approximation to V. We shall discuss the case where F and G are compact convex subsets of $\mathbb{R}^n$ in Section 4. In Section 4.1 we characterize V and in Section 5 we discuss some open questions.

3. The 3-Component Case

3.1 Characterization of V

Let F and G be two compact convex subsets of $\mathbb{R}^2$ that satisfy

$$\mathrm{bdy}(F) \cap \mathrm{bdy}(G) = \emptyset \text{ , and} \tag{12a}$$

$$\mathrm{int}(F) \neq \emptyset \text{ .} \tag{12b}$$

An example of two sets F and G that satisfy these constraints is shown in Fig.4.

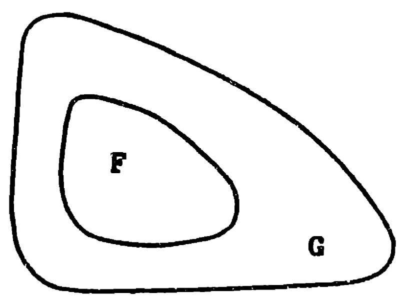


Fig.4

Let X be the subset of $\mathbb{R}^2$ given by

$$X = \{ \, p \mid \text{there are points } q \text{ and } r \text{ in } G \text{ for which the triangle } pqr \text{ contains } F \, \} \text{ .} \tag{13}$$

The set X is illustrated in Fig.5 for a particular choice of the sets F and G. In this figure, the point p_1 is in X, and the point p_2 is in com(X).

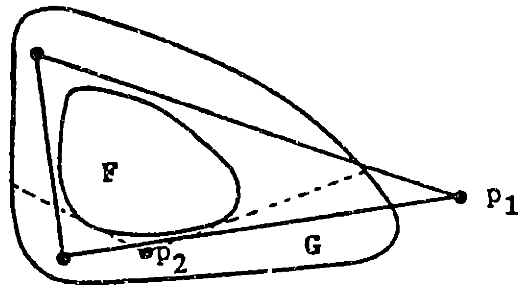


Fig.5

Since **G** is a convex set, the vertex domain **V** equals the intersection of **G** and **X**. Thus, any characterization of the set **X** may be used to characterize **V**.

Let f be the map from ext(**F**) into bdy(**G**) that is illustrated in Fig.6. For any point p in ext(**F**), there are two support lines for **F** that pass through p. To obtain f(p), we move from p along the support line for **F** that passes to the right of **F**. f(p) is taken to be the point in bdy(**G**) at which we exit from the set **G**.

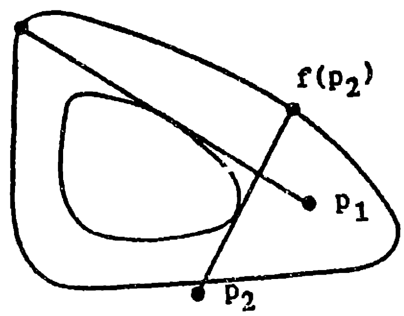


Fig.6

The <u>penumbra</u> of a set **A** with respect to a set **B** is the set of all points p that may be written as $(1-\lambda)a + \lambda b$, for some a in **A**, some b in **B**, and some $\lambda \leq 0$ [54] (some problems that involve penumbras are considered in [68]). We shall denote this set by pen(**A**,**B**). The set pen(**A**,**B**) is illustrated in Fig.7 for a particular choice of the sets **A** and **B**.

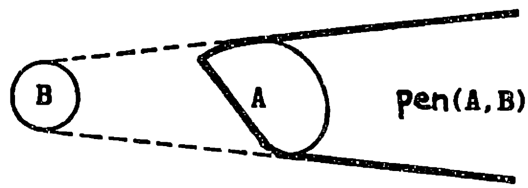


Fig.7

We shall denote the i^{th} iterate of a function g at x by $g^i(x)$. For a point p in the domain of f, let $X_e(p)$ and $X_i(p)$ be defined by

$$X_e(p) = \text{int}\{\text{pen}[F, f(p)] \cap \text{pen}[F, f^2(p)]\} \quad , \tag{14a}$$

$$X_i(p) = \text{int}\{\text{pen}[X_b(p), F]\} \quad , \tag{14b}$$

where $X_b(p)$ denotes the intersection of the rays $f(p)p$ and $f^2(p)f^3(p)$. The sets $X_i(p)$, $X_e(p)$, and $X_b(p)$ are illustrated in Fig.8 for a particular choice of the sets F and G. Fig.8a shows the case where the rays $f(p)p$ and $f^2(p)f^3(p)$ intersect at a point, and Fig.8b shows the case where these rays do not intersect.

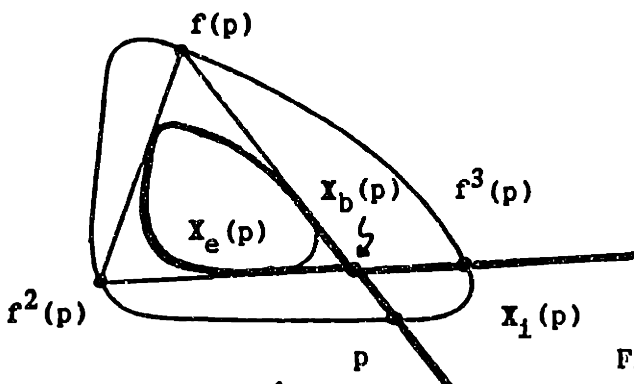


Fig.8a

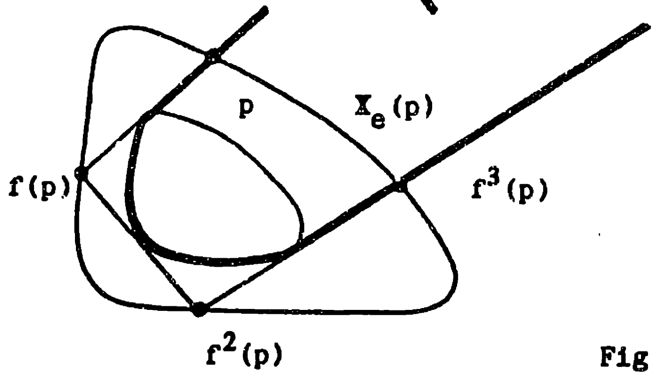


Fig.8b

The next theorem shows that the sets $X_i(p)$, $X_e(p)$, and $X_b(p)$ can be used to characterize the interior, exterior, and boundary of X.

Theorem 1: Characterization of X:

$$\mathrm{int}(X) = \bigcup_{p\in \mathrm{bdy}(G)} X_i(p) \ , \tag{15a}$$

$$\mathrm{ext}(X) = \bigcup_{p\in \mathrm{bdy}(G)} X_e(p) \ , \tag{15b}$$

$$\mathrm{bdy}(X) = \bigcup_{p\in \mathrm{bdy}(G)} X_b(p) \ . \tag{15c}$$

From Theorem 1, we see that the boundary of the set X is generated by rotating a triangle that circumscribes F and has a chord of G for its base, see Fig.9.

Corollary 1 gives the characterization of V that corresponds to the characterization of X given in Theorem 1.

Corollary 1: Characterization of V:

$$V = \Big(\bigcup_{p\in \mathrm{bdy}(G)} [X_i(p) \cup X_b(p)] \Big) \cap G \ . \tag{16}$$

Fig.9 illustrates the vertex domain for the sets F and G of Fig.4.

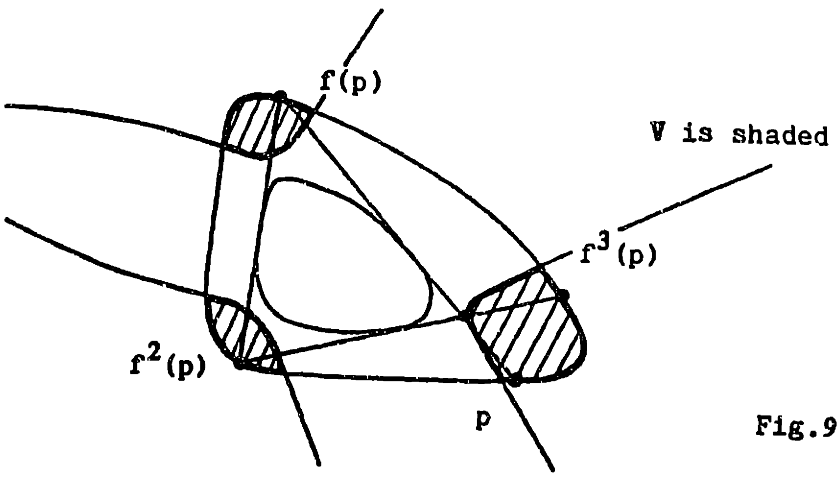


Fig.9

In Fig.9, the intersection of com(V) and G is star-shaped about each point in the interior of F. The next corollary shows that this property is general.

Corollary 2:

$$\mathrm{int}(F) \subset \ker[\mathrm{com}(V) \cap G] \quad . \tag{17}$$

Corollary 1 gives a characterization of V for the case where F and G are compact convex planar sets that satisfy (12).

The Polygonal Case

In Section 1, we showed that a geometric interpretation of the component analysis problem leads us to consider the class C for the case where F and G are convex polytopes. Here, we shall discuss some results for the 2-dimensional version of this problem, which results when three components are present.

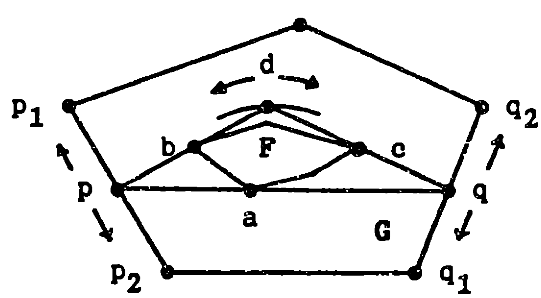


Fig.10

Suppose F and G are convex polygons that satisfy (12). The set bdy(X) is generated by rotating a triangle that circumscribes F and has a chord of G for its base. This procedure is illustrated in Fig.10 for the polygonal case. As the point p moves along the segment p_1p_2, the point q moves along the segment q_1q_2 and the point d traces a portion of the c 've bdy(X). This basic operation is isolated in Fig.11.

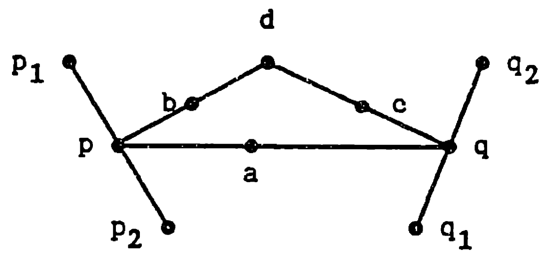


Fig.11

For each line pq that passes through the point a and intersects the lines p_1p_2 and q_1q_2, we obtain a point d. The set of all points obtained in this way will be a conic, i.e. all such points d=(x,y) satisfy

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0 , \tag{18}$$

for some constants A through F. MacLaurin [35] in 1735 observed this fact and used it along with several other similar constructions in an effort to describe curves of various degrees (to the best of our knowledge, no one has generalized Maclaurin's results to higher dimensions).

The conic passes through the following five points: b, c, the intersection of the lines p_1p_2 and q_1q_2, the intersection of the lines ab and q_1q_2, and the intersection of the lines ac and p_1p_2. If we have determined these five points, then the coefficients of (18) may be obtained in the following way: evaluate (18) at the five known points to obtain five linear equations in the six unknowns, normalize F to 1 (provided that the curve does not pass through (0,0), we can take F to be 1), and solve the five linear equations in the five remaining unknowns.

Alternatively, we can obtain a parametric description of the conic. Since d equals the intersection of the lines

$$\lambda p + (1 - \lambda)b \quad \text{and} \tag{19a}$$

$$\gamma q + (1 - \gamma)c \ , \tag{19b}$$

d may be written as

$$d = b + (p - b)\{\ |c\text{-}b\ \ c\text{-}q|\ /\ |p\text{-}b\ \ c\text{-}q|\ \}\ , \tag{20a}$$

where the points in this equation are written as vectors in $\mathbf{R}^2$, with respect to some coordinate system, and $|\cdot|$ denotes the determinant. Similarly, since q equals the intersection of the lines pa and q_1q_2,

$$q = a + (p - a)\{\ |q_2\text{-}a\ \ q_2\text{-}q_1|\ /\ |p\text{-}a\ \ q_2\text{-}q_1|\ \}. \tag{20b}$$

In addition, p can be written as

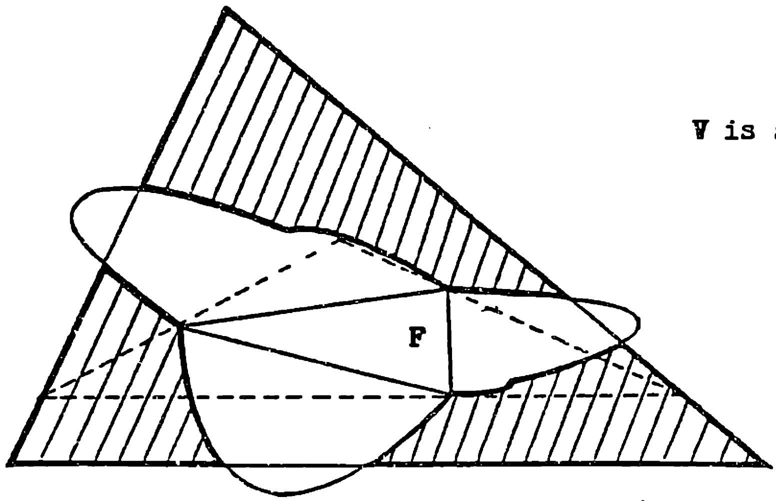
$$p = p_1 + (p_2 - p_1)t\ . \tag{20c}$$

By substituting (20b) and (20c) into (20a), it can be seen that d is of the form

$$d = [\ x(t)\ \ y(t)\ ]\ /\ w(t)\ , \tag{20d}$$

where x(t), y(t), and w(t) are second degree polynomials in t. (20d) is a rational parametrization of the conic in terms of the points p_1, p_2, q_1, q_2, a, b, and c. [Note: This actually proves that the curve is a conic, for when x and y are the ratio of two quadratics in t, (18) becomes a fourth degree polynomial in t (after clearing the denominators) where each of the five coefficients is a linear function of A, B, C, D, E, and F. Thus, there exist values for A, B, C, D, E, and F such that this polynomial is zero for all t.]

Fig.12 illustrates the set **V** for a particular choice of the polygons **F** and **G**. The curve bdy(**V**) is formed from several conic arcs. (20) may be used to parametrize each of these arcs in terms of the vertices of **F** and **G**.



V is shaded

Fig.12

A conic is said to be <u>degenerate</u> if it may be expressed as the union of two lines. Otherwise, it is <u>nondegenerate</u>. The next result gives an upper bound on the number of nondegenerate conics that can be involved in forming the boundary of the vertex domain.

Result 1: The Number of Conic Arcs in bdy(V):

Let f and g be the number of vertices of **F** and **G**, respectively. The curve bdy(**V**) consists of at most 3f + 2g nondegenerate conic arcs.

(3f+2g) is only an upper bound. First, some of the conic arcs that we have counted may not intersect **G**, and thus could not contribute to the boundary of the vertex domain. In other cases, $\mathbf{X}_b(p)$ may remain at a vertex of **F** while p (in Fig.8) varies over an interval of bdy(**X**). For example, this happens when both **F** and **G** are triangles, see Fig.12.

3.2 Iterative Procedures to Approximate V

In this section, we shall describe a set of iterative procedures that may be used to approximate V for the case where F and G are compact convex sets that satisfy (12). First, we give a procedure that converges to the members of a special class of triangles that may be used to approximate V. Then we shall discuss a procedure that converges to the asymptotes of the curve bdy(X) and a procedure that converges to the point at which a given ray that is emitted from F intersects the curve bdy(X).

We shall refer to a triangle that is inscribed in G (i.e. has each of its vertices in bdy(G)) and circumscribes F (i.e. has each of its legs tangent to F) as a circumscribing-inscribed (CI) triangle. Fig.13 illustrates the CI triangles S_1 and S_2 for the particular choice of the sets F and G that was given in Fig.4.

If S is a CI triangle, then the set int(S) may be written as the union of $X_e(p_1)$, $X_e(p_2)$, and $X_e(p_3)$, where p_1, p_2, and p_3 are the vertices of S. From Theorem 1, it follows that the interior of a CI triangle must be contained in com(V). Thus V must be contained in the set G-int($S_1,\ldots,S_m$), where $S_1,\ldots,S_m$ are the CI triangles. For the case shown in Fig.13, this set equals the union of 6 sets $V_1,\ldots,V_6$ that lie along bdy(G). More generally, if there are m CI triangles, then there will be 3m sets of this form. We may approximate V by the union of all sets V_j for which the intersection of int(V_j) and V is nonempty (later we show that a simple test may be performed to determine whether or not such an intersection is nonempty). From Fig.9, we see that V does not intersect V_1, V_3, and V_5. Thus in this case, the resulting approximation to V is the union of the sets V_2, V_4, and V_6. Note that this approximation to V is conservative in the sense that V is contained in this approximation.

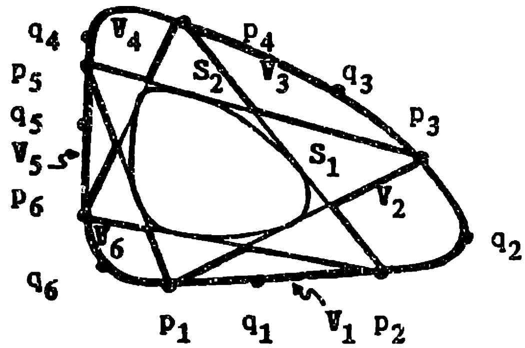


Fig.13

To obtain the approximation described above, we must devise methods to

1. locate CI triangles, and (21a)

2. test for whether $\text{int}(V_j) \cap V = \phi$. (21b)

It can be shown that if a pair of sets F and G satisfying (12) has a CI triangle, then for every point p in the domain of the function f defined in Fig.6, the iterates of f at p will converge to a CI triangle. Said another way, if f has a 3-periodic point, then every point in the domain of f is an asymptotically 3-periodic point of f (a point p is said to be a <u>k-periodic point</u> of a function g if $g^k(p)=p$ and $g^i(p) \neq p$, for all i in the interval $[0,k-1]$ ($g^0(p)$ is defined to be the identity function), and p is said to be an <u>asymptotically k-periodic point</u> of g if the set

$$\{g^{ki}(p), g^{ki+1}(p), \ldots, g^{k(i+1)-1}(p)\} \quad , \tag{22}$$

converges to k different points).

The structural properties of the periodic points of f depend upon the sets F and G. As can be seen from the example in Fig.13, for some choices of F and G the corresponding function f will possess 3-periodic points. Fig.14 illustrates that there are sets F and G for which the corresponding function f has a 4-periodic point but no 3-periodic points, and Fig.15 shows that more than one type of 5-periodic point can be obtained.

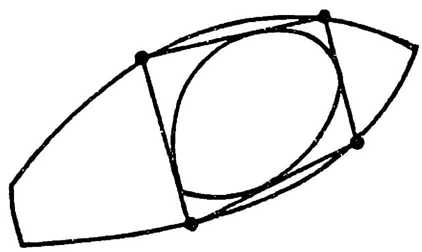


Fig.14

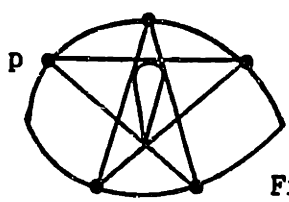


Fig.15a

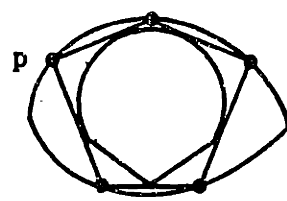


Fig.15b

The word "type" here can be rigorously defined, but corresponds essentially to the kind of polygon that is generated by the iterates of a periodic point (a rigorous definition may be made by using the permutations that were introduced in [10] to analyze billiard ball paths in a planar set). In Fig.15a, the iterates of p form a 5-point star, and in Fig.15b the iterates of p generate a pentagon. It can be shown that for $k \geq 3$ the number of different types of k-periodic points that can be obtained is $\phi(k)/2$, where $\phi(k)$ is the Euler phi-function, which equals the the number of integers r in the interval $[1,k)$ for which $\gcd(r,k)=1$.

The next theorem shows that the function f that is associated with a particular pair of sets F and G can only possess k-periodic points for one value of k, and all k-periodic points must be of the same type.

Theorem 2: The Periodic Points of f:

All of the periodic points of the function f that is associated with a particular pair of compact convex sets F and G that satisfy (12) are of the same period and type.

The next theorem is even stronger than Theorem 2. It states that if f has a k-periodic point, then every point in the domain of f is asymptotically k-periodic.

Theorem 3: The Asymptotically Periodic Points of f:

Suppose the function f that is associated with a particular pair of compact convex sets F and G the satisfy (12) has a k-periodic point.

a. Then, every point in the domain of f is an asymptotically k-periodic point of f.

b. Let p_1p_2 be an open segment of bdy(G) for which p_1 and p_2 are k-periodic points of f and p_1p_2 does not contain any periodic points of f. Let f_r denote the restriction of f to bdy(G) (so that f_r is invertible, though f is not). If p is a point in the segment p_1p_2, then either $\{f_r^{ki}(p)\}$ converges to p_1 and $\{f_r^{-ki}(p)\}$ converges to p_2, or $\{f_r^{ki}(p)\}$ converges to p_2 and $\{f_r^{-ki}(p)\}$ converges to p_1. In addition, both of the sequences converge monotonically with respect to the curve bdy(G).

Theorem 3 is illustrated in Fig.16 for the case where f has a 3-periodic point. In this case $\{f_r^{3i}(p)\}$ converges to p_1 and $\{f_r^{-3i}(p)\}$ converges to p_2.

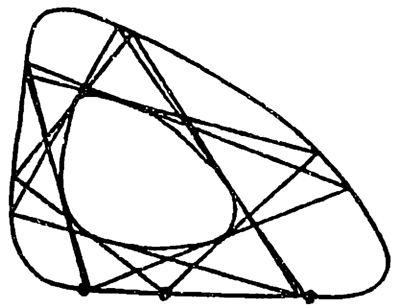


Fig.16

The following procedure, which is based upon the results given in Theorem 3, may be used to locate CI triangles:

Procedure: Find CI Triangles:

1. **A** ←-- bdy(**G**),
2. Select a point p from **A**,
3. Compute $\{f_r^{3i}(p)\}$ and $\{f_r^{-3i}(p)\}$ until convergence is detected and set q and r to the estimated limits of $\{f_r^{3i}(p)\}$ and $\{f_r^{-3i}(p)\}$, respectively,
4. **A** ←-- **A** minus the arcs qpr, $f_r(q)f_r(p)f_r^{-2}(r)$, $f_r^2(q)f_r^2(p)f_r^{-1}(r)$ of bdy(**G**),
5. If **A** is empty (to within the tolerance used in detecting convergence), then stop, else go to 2.

Since some sets **F** and **G** have an infinite number of CI triangles (for example see Fig.17), this procedure will not terminate for all choices of the sets **F** and **G**.

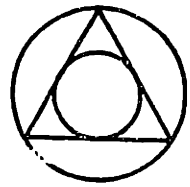


Fig.17

The next result shows that the number of CI triangles is finite in the case where F and G are convex polygons. In this case the procedure will always terminate.

Result 2: The Number of CI Triangles:

A pair of convex polygons F and G that satisfy (12) will only have a finite number of CI triangles.

The function f can also be used to perform the test (21b). Refer to Fig.13. Let $p_1,\dots,p_{3m}$ be the 3-periodic points of f numbered in counterclockwise order around bdy(G), let $S_1,\dots,S_m$ be the CI triangles, and let V_j be the subset of G-int($S_1,\dots,S_m$) that corresponds to the arc p_jp_{j+1} of bdy(G). Let q_j be a point on the open arc p_jp_{j+1}. It can be shown that V intersects int(V_j) if and only if $X_b(q_j)$ is a point that is in G.

The function f may be used to obtain a second type of approximation to V. Let $\{A_j\}$ and $\{B_j\}$ be the sequences of sets that are generated by the recurrence relations

$$A_{j+1} = A_j \cup X_e[f^j(p)] \quad , \tag{23a}$$

$$B_{j+1} = B_j \cup X_i[f^j(p)] \quad , \tag{23b}$$

$$A_1 = B_1 = \emptyset \; . \tag{23c}$$

We m / use A_j and B_j for a j^{th} approximation to the sets com(V)$\cap$G and V, respectively. This alternative approach is illustrated in Fig.18.

Whereas the first approximation method that we discussed only applies to the cases where there are CI triangles, this second method may be used in all cases.

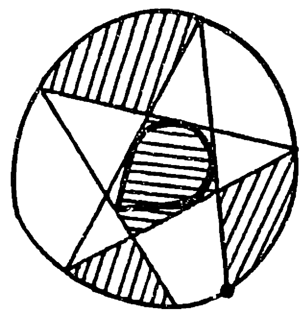


Fig.18

The function f may be used to determine the points at which bdy(X) intersects bdy(G). We conclude this section with a brief discussion of two additional iterative procedures that also may be used to expose important features of X.

In some cases the set com(X) will be unbounded. This happens for the example given in Fig.4. It can be shown that if com(X) is unbounded, then the asymptotes of the curve bdy(X) will be of the same form as the parallel rays s_1 and s_2 in Fig.19a.

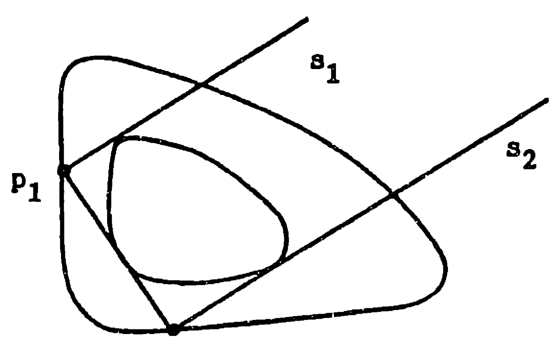


Fig.19a

Let g be the function from bdy(G) into bdy(G) that is defined in Fig.19b (the lines $f(p)f^2(p)$ and g(p)a in Fig.19b are parallel). It can be shown that the iterates of g and g^{-1} can be used to locate the asymptotes of bdy(X).

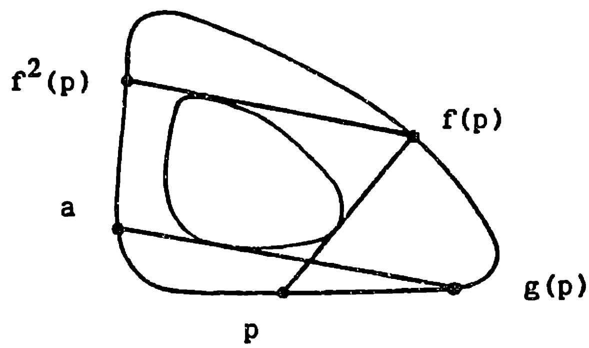


Fig.19b

The approximations to V that were given in this section could be improved in a particular case by determining the point at which a given ray emitted from F intersects bdy(X). Refer to Fig.20a. Let q_1q_2 be a ray that is emitted from F. Suppose the ray q_1q_2 intersects bdy(X) at a point r. From Theorem 1, we know that the point r must equal X(p), for some point p in bdy(G), and the corresponding sets $X_i(p)$ and $X_e(p)$ must be contained in X and com(X), respectively. These two sets may be used to improve our approximation to V.

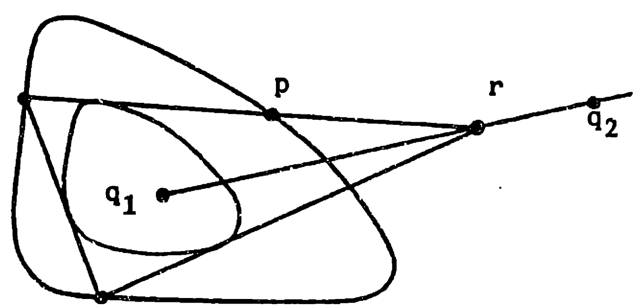


Fig.20a

It can be shown that the iterates of the function h that is defined in Fig.20b converge to the point at which the given ray intersects the curve bdy(X).

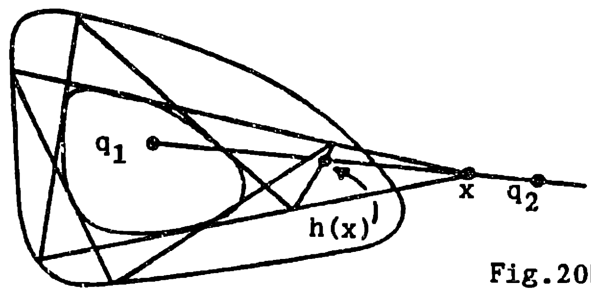


Fig.20b

4. The (n+1)-Component Case

A characterization for **V** in the case where the sets **F** and **G** are compact convex subsets of $\mathbf{R}^n$ will be given in this section. In addition, we shall describe some open problems for this case.

4.1 Characterization of V

Let C_p be the set of n-simplexes given by

$$C_p = \{ S \mid S \text{ is an n-simplex with vertices } p, p_1, \ldots, p_n \text{ for which } F \subset S \text{ and } p_1, \ldots, p_n \text{ are in } G \} . \tag{24}$$

Let **X** be the subset of $\mathbf{R}^n$ given by

$$X = \{ p \mid C_p \neq \emptyset \} . \tag{25}$$

The set **X** given by (25) is the n-dimensional generalization of the 2-dimensional set **X** defined in Section 3.1. Since **V** equals the intersection of **X** and **G**, any characterization of the set **X** may be used to characterize **V**.

Theorem 1 gives a characterization of **X** for the 2-dimensional case. In this case, a point p in the set com(**F**) is in bdy(**X**) if and only if C_p consists of a single triangle that circumscribes **F** and has a chord of **G** for its base. We shall show that the set B given by

$$B = \{\, p \mid C_p \neq \emptyset \text{ and if } S \text{ is a simplex in } C_p, \text{ then } F \text{ intersects every face of } S \text{ and } F \text{ intersects the relative interior of at least one face of } S \text{ that contains } p \,\}, \quad (26)$$

is the n-dimensional generalization of this set (the relative interior of a subset A of R^n is the interior of A with respect to the minimal-dimension plane that contains A).

The next lemma describes the ways in which a ray that is emitted from int(F) may intersect X.

Lemma 1: The Structure of X with respect to Rays Emitted from int(F):

Let q be a point in int(F), let p be a point in bdy(F), and let $r(\lambda)$ be the point $(1-\lambda)q + \lambda p$. Either

a. $r(\lambda) \in com(X)$, for all $\lambda \geq 0$, or

b. $r(\lambda) \in com(X)$, for $0 \leq \lambda < \lambda_0$,
$\in B$, for $\lambda = \lambda_0 \geq 1$,
$\in X$, for $\lambda \geq \lambda_0$, or

c. $r(\lambda) \in com(X)$, for $0 \leq \lambda < 1$,
$\in X$, for $\lambda \geq 1$.

Fig.21 illustrates the three cases given in Lemma 1.

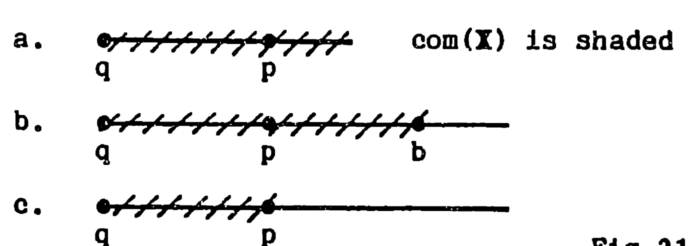


Fig.21

The next lemma shows that the sets X and com(X) are wide about rays that are emitted from the set int(F).

Lemma 2: The Directional Wideness of X and com(X):

Let p be a point in R^n.

a. $p \in X \longrightarrow pen(p,S) \subset X$, for all S in C_p.

b. $p \in com(X) \dashrightarrow hul[p,int(F)] \subset com(X)$.

The next theorem, which follows from Lemmas 1 and 2, gives a characterization of the set X.

Theorem 4: Characterization of X:

$$bdy(X) = B \cup [X \cap bdy(F)] \; , \tag{27a}$$

$$int(X) = \bigcup_{p \in bdy(X)} int[pen(p,F)] \; . \tag{27b}$$

5. Some Open Problems

This section contains a discussion of several open questions. In Section 5.1 we list some results for the 3-component case that have not been generalized to the (n+1)-component case, and in Section 5.2, we describe some problems associated with the iterative procedures that were presented in Section 3.2.

5.1 The (n+1)-Component Case

There are several open questions for the case where F and G are n-dimensional compact convex sets. In this section, we give a brief description of some of these questions.

Fig.22 shows how Lemma 2 may be applied to obtain subsets of X and com(X) in each of the cases described in Lemma 1. If we had an algorithm that could determine whether a given ray that is emitted from int(F) satisfies a, b, or c in Lemma 1 (and also find λ_0 in case b), then we could apply this algorithm to several such rays in order to obtain an approximation of X. In Section 3.2, we saw that in the 2-dimensional case, the function h given in Fig.20b may be used to determine the point at which a given ray that is emitted from F intersects the curve bdy(X). Perhaps there is an n-dimensional generalization of this function.

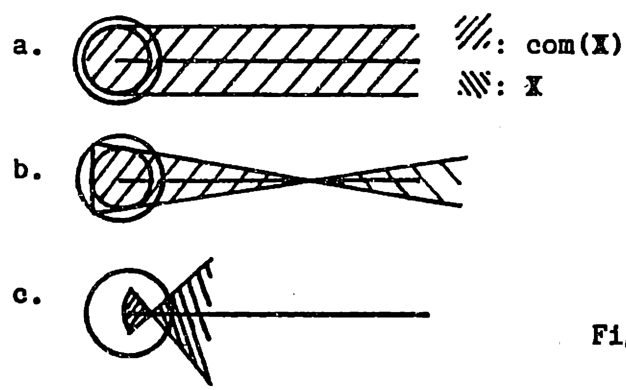


Fig.22

Theorem 1 shows that in the 2-dimensional case the set bdy(X) may be expressed as a mapping of bdy(G). In Section 3.1, we used this fact to show that in the case where F and G are polygons, the set bdy(X) is formed from a collection of conic arcs. We have not been able to generalize these results to the n-dimensional case.

5.2 Iterative Processes in Geometry

In Section 3.2, we described a set of geometric fixed point algorithms that may be used to approximate V for the case where F and G are compact convex sets that satisfy (12). In this section, we shall describe another similar geometric procedure that also appears to converge to its fixed points. Some open questions will be raised regarding the iterative algorithms that we have presented.

Consider the following triangle reconstruction problem:

> Estimate an unknown equilateral triangle T from a given triangle abc and line d that satisfy the following constraints:
>
> 1. each leg of abc contains a vertex of T in its interior,
> 2. d is parallel to one of the legs of T.

An iterative procedure that converges (in many cases) to the unknown equilateral triangle T is illustrated in Fig.23. First we select a point p_0 from the triangle abc. From this initial point, we generate the sequences p_1, p_2, p_3, ..., and p_{-1}, p_{-2}, p_{-3},.... To obtain p_1, we move away from p_0 and parallel to the line d until we intersect abc. From this point we make a 60 degree left turn to obtain p_2. We continue in this way to generate p_i, for $i>2$. If at some point we cannot make a 60 degree left turn, (for example, p_3 in Fig.23) we make a 120 degree right turn. The negative portion of the sequence is obtained from p_1 by reversing these steps. In particular, we move from p_1 to p_0 and then make 60 degree right turns when possible, and 120 degree left turns when a 60 degree right turn cannot be made.

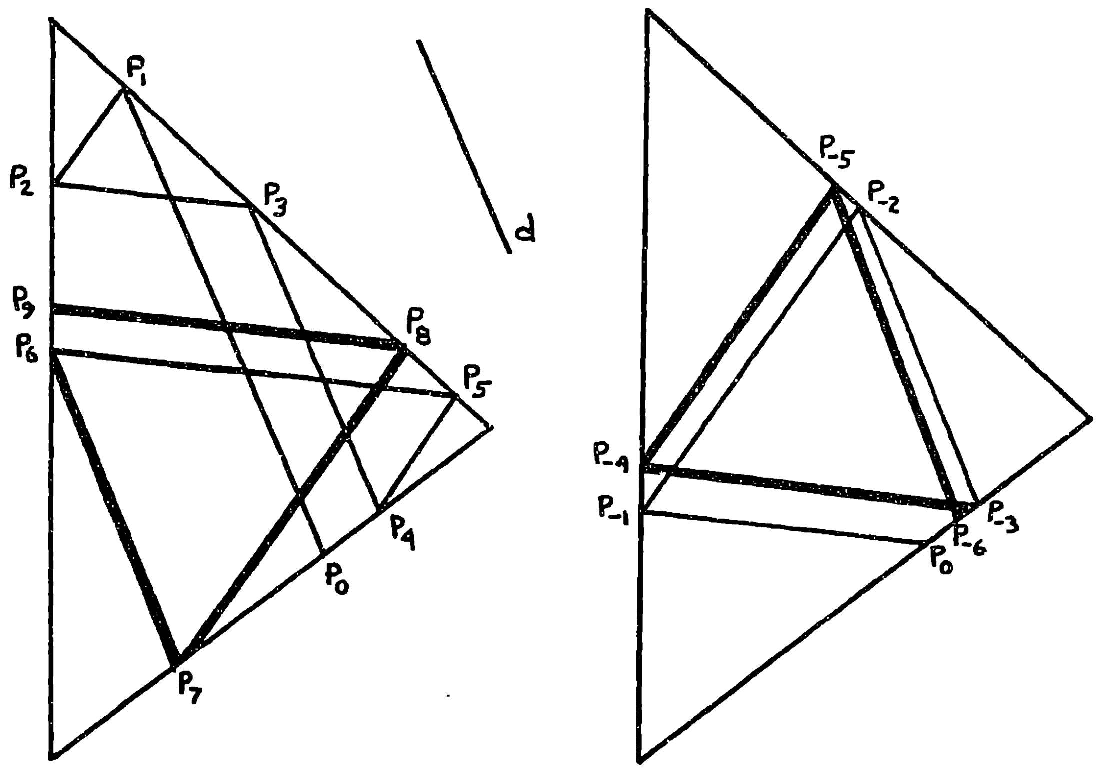


Fig.23

We have implemented this algorithm using the language LOGO on an Apple II Plus personal computer and have studied its behavior for several triangles abc and lines d. The following conjecture characterizes the convergence properties of the algorithm and is based upon our observations:

> Conjecture 1:
>
> In general, both the positive and negative portions of the sequence p_i converge to a triangle that could be the unknown triangle **T**.

Finally, we note that the algorithm does not always converge to an equilateral triangle. Fig.24 shows another fixed path for this procedure.

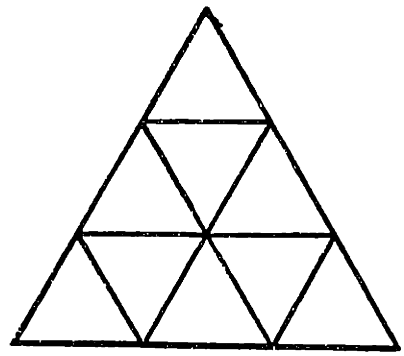


Fig.24

This algorithm and the three algorithms described in Section 3.2 all have a similar structure. Each procedure yields a path within a given convex set that bounces off the boundary of the set according to some set of rules. The fixed paths of these procedures are stable. Is this a consequence of a more general result or is this the bud of a new result on iterative procedures defined on convex sets?

We shall conclude this section with a list of some related papers that may help us formulate general results. In [65], an iterative algorithm is given to determine the point at which a given ray enters a given convex set. The papers [9], [6], and [62] study the convergence properties of various sequences of polygons. [9] considers the sequence P_3, P_4,....., where P_3 is an equilateral triangle, and P_k is the largest regular k-gon that is contained in the polygon P_{k-1}. In [6], three different sequences are examined. One of these sequences is illustrated in Fig.25.

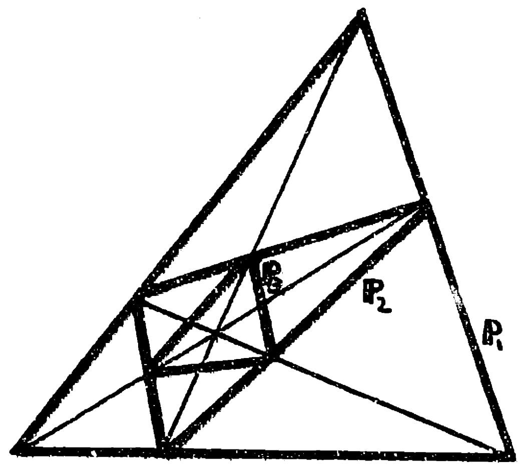


Fig.25

The sequence considered in [62] is shown in Fig.26. The recent paper [28], which describes a new algorithm for the linear programming problem, may also contain some useful ideas.

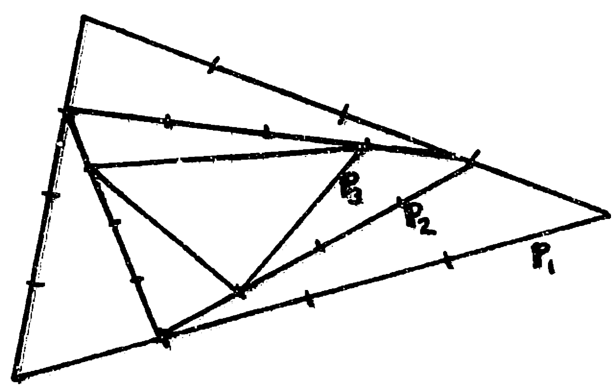


Fig.26

A number of recent articles have studied the trajectory of a billiard ball bouncing in a convex set. [56] discusses the motion of a billiard ball in an annulus bounded by two non-concentric circles. [16] focuses upon the case where the convex set is a polygon or polyhedron. Billiard paths in planar convex sets of constant width are considered in [63].

CHAPTER 4. RECONSTRUCTING SPHERES, PLANES, AND POLYTOPES: A SURVEY

Originally, the results given in Chapters 2 and 3 were developed because we were interested in the following two problems:

1. estimate the shape of an unknown 3-dimensional object from a given collection of boundary point measurements obtained by a robot hand that has touched the unknown object,

2. identify the components of a chemical mixture by using a collection of absorption spectra of chromatographic fractions.

Later, we recognized that the mathematical formulation of each of these seemingly unrelated problems had the following structure: estimate an unknown subset A of R^n from a given partial description of A. We decided to call problems of this form set reconstruction problems.

More recently, we have explored the literature in a variety of fields to identify additional set reconstruction problems that other authors have studied in the past. This search has proven to be quite fruitful. In this chapter, we shall discuss the material uncovered in this search.

In Chapter 3, we investigated a simplex reconstruction problem. The simplex is an important convex set because it arises in many applications. There are many other special types of convex sets that are equally important. In this chapter, we shall discuss some reconstruction problems for the following convex sets: spheres, planes, and polytopes.

In Section 1, we shall discuss the problem of reconstructing an unknown sphere S from two given sets F and G that satisfy F S com(G). We shall pull together the existing literature on minimum spanning spheres, largest empty spheres, α-hulls, and motion planning in our discussion of this problem. In Section 2, we consider the problem of reconstructing an unknown m-dimensional plane P from k sets $A_1,\dots,A_k$ that intersect P. We shall focus upon the case where the k sets are interval polytopes and we shall discuss the related literature on interval linear equations.

In Section 3, we consider some interactive polytope reconstruction problems (the concept of interactive problems will be described in Section 3). First, we describe some existing algorithms that generate a sequence of boundary points to reconstruct an unknown polytope. Then, we shall show that the problem of reconstructing a polytope by using a sequence of support planes is a dual of the boundary points problem, and we shall show how algorithms for one problem may be used to obtain algorithms for the other problem. In Section 4, we shall summarize some of the open problems that are identified in Sections 1 through 3.

1. Sphere Reconstruction Problems

Consider the following sphere reconstruction problem:

Estimate an unknown sphere S in R^n from two given sets F and G that satisfy

$$F \subset S \,, \qquad (1a)$$

$$S \subset \mathrm{com}(G) \,. \qquad (1b)$$

Let C be the class of all spheres in R^n that satisfy (1). The class C is illustrated in Fig.1 for a particular choice of the sets F and G.

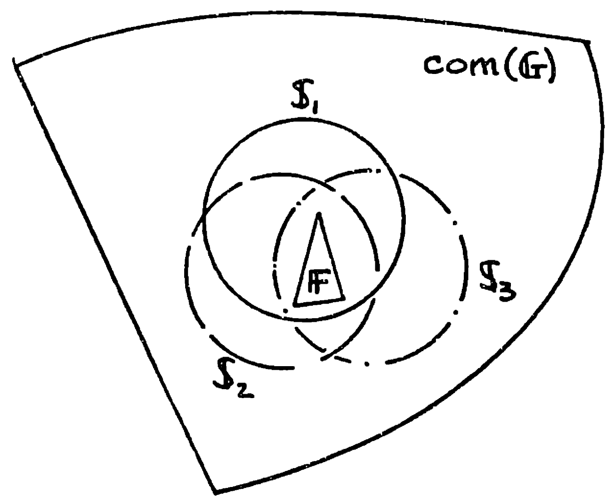


Fig.1

The Largest and Smallest Spheres in C

Many algorithms have been devised to obtain either the largest or smallest sphere in C (or a subset of C) when the sets F and G are of a special form. For example, in [14] and [19] algorithms were given to determine the smallest sphere in C for the case where G is empty and F consists of a finite set of points, see Fig.2.

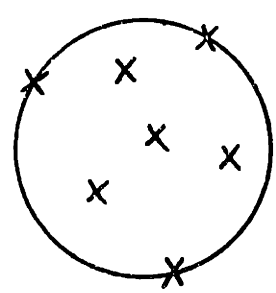


Fig.2

In [11] and [72], it was shown how the largest sphere in **C** may be found in the case where **F** is empty and **G** is the complement of a polytope, see Fig.3.

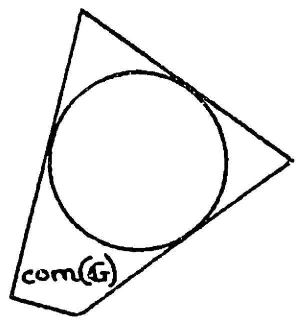


Fig.3

The relationship between these two problems was dicussed in [12]. Toussaint [69] considered the case where **F** is empty and **G** consists of a finite set of points in R^2. For this case, he gave an algorithm to locate the largest sphere in **C** that is centered in some given convex polygon, see Fig.4. (Note that some authors have considered similar scaling problems for ellipsoids, see [50] and [26].)

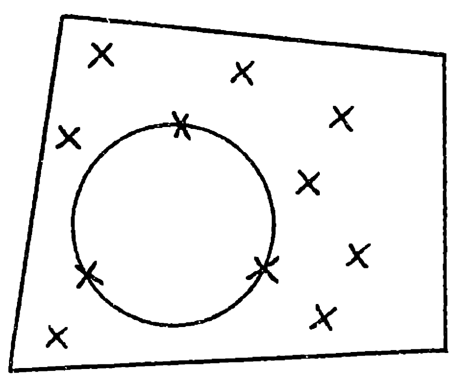


Fig.4

To the best of our knowledge, no algorithms have been developed to find the largest and smallest spheres in **C** for the case where both **F** and **G** are nonempty. Fig.5 shows that in some cases we can decouple these problems, i.e. the smallest sphere in **C** equals the smallest sphere that contains F and the largest sphere in C equals the largest sphere that is contained in com(G). However, in general these problems cannot be decoupled. For example, in Fig.6 the largest sphere in com(G) does not contain the set F, and the smallest sphere that contains F is not contained in com(G). Thus, these extreme spheres are not in C.

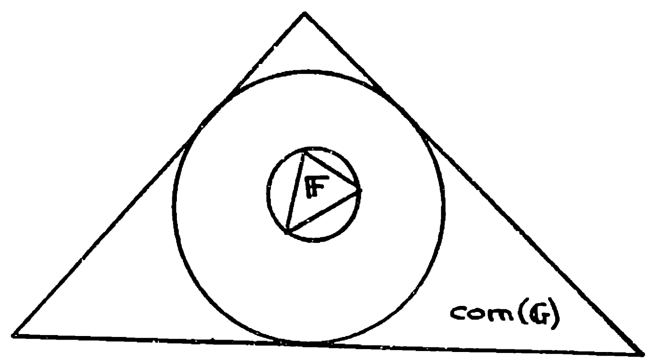


Fig.5

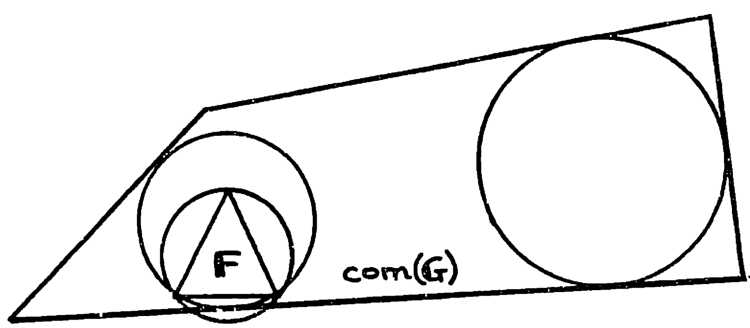


Fig.6

Since these problems may not be decoupled, the algorithms that have been given for the cases in which either F or G is empty may not be applied directly to the problem where both F and G are nonempty. However, it might be possible to modify the approaches that were taken to derive these algorithms to obtain algorithms for the general case.

Estimating the Unknown Sphere

Let **A** and **B** denote the following subsets of $\mathbf{R}^n$:

$$\mathbf{A} = \{\ p \mid p \text{ is in } S, \text{ for all } S \text{ in } \mathbf{C}\ \}\ , \tag{2a}$$

$$\mathbf{B} = \{\ p \mid p \text{ is in com}(S), \text{ for all } S \text{ in } \mathbf{C}\ \}\ . \tag{2b}$$

In Section 2.3.1, we showed how it is possible to grow the sets **I**, **B**, and **E** from the seeds $\mathbf{G}_i$, $\mathbf{G}_b$, and $\mathbf{G}_e$, by using the properties of convex sets given in Fig.2.7; see Fig.2.8. Analogously, we may grow the sets **A** and **B** from the seeds **F** and **G** by using the properties of spheres. For example, for the case shown in Fig.7 the point p must be in **A** and the point q must be in **B**. It should be possible to characterize the sets **A** and **B**, and to develop algorithms to determine whether a given point is in **A**, **B**, or com(**A**,**B**).

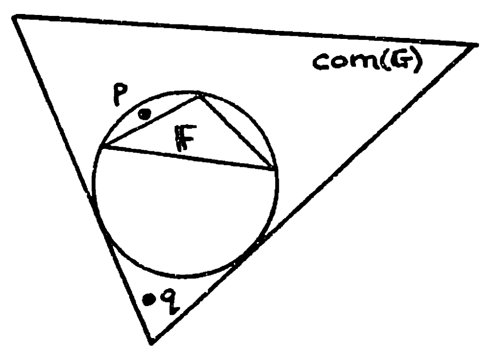


Fig.7

Suppose **G** is empty and we know that the radius of the unknown sphere is r. In this case, the set **A** is given by the intersection of all the spheres of radius r that contain the set **F**. The set **A** is shown in Fig.8 for a particular choice of **F** and r. In [13], the intersection of all spheres of radius 1/α that contain a given set was defined to be the α-hull of the set. In that paper, algorithms were given to compute the α-hull of a given finite subset of $\mathbb{R}^2$.

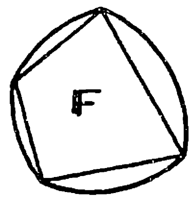


Fig.8

Next, suppose instead that F is empty and that we know the radius of the unknown sphere is r. Suppose G equals the union of a finite set of points and the complement of a polytope, see Fig.9. For the particular choice of G shown in Fig.9, the shaded region denotes the possible locations for the center of the unknown circle. In [64] an algorithm was given to determine the area of the union of a finite collection of circles. This algorithm could be used to determine the ratio of the shaded region in Fig.9 to the bounding polygon. This ratio could be used as an indication of the quality of our estimate of the location of the center of the unknown circle.

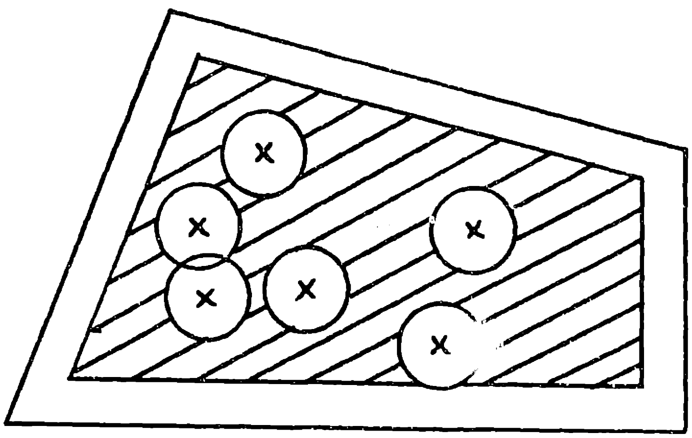


Fig.9

In [75] and [42] the problem of planning the motion of discs in the plane was considered. Perhaps some of the algorithms that have been given for this related problem could be used for sphere reconstruction problems as well.

2. Plane Reconstruction Problems

Suppose that we are given noisy samples $s_1,\ldots,s_k$ of an unknown linear function $f(x)=mx+b$ at k values $x_1,\ldots,x_k$ of the independent variable x. We could use the method of least squares to estimate the function f(x). The estimate obtained in this way, call it $f_e(x)=m_e x+b_e$, would be chosen so as to minimize the sum

$$[s_1-f_e(x_1)]^2 + \ldots + [s_k-f_e(x_k)]^2 \quad (3)$$

If it is known that the noisy samples satisfy the inequalities

$$f(x_i) - a_i \leq s_i \leq f(x_i) + b_i , \quad (4)$$

for some given scalars a_i and b_i (see Fig.10), then the least squares method may not be appropriate.

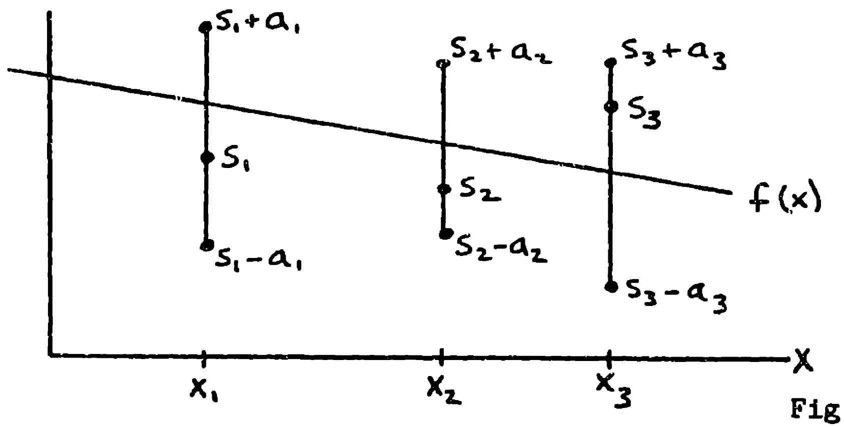


Fig.10

(It is easy to construct examples for which the estimate $f_e(x)$ obtained from the least squares method does not pass through the intervals given by (4).)

In this case, we might try to determine the class **C** of all linear functions g(x) that satisfy the inequalities

$$s_i - b_i \leq g(i) \leq s_i + a_i , \tag{5}$$

i.e. that cut all of the intervals illustrated in Fig.10. From (4) we know that the unknown function f(x) must be in **C**. The class **C** could serve as our estimate of f(x).

The problem of determining the class **C** was considered in [48]. Let $\mathbf{C}_j$ be the class of all linear functions g(x) that satisfy the inequalities (5) for i=1,...,j. In [48], an algorithm was given to compute the class $\mathbf{C}_j$ by using $\mathbf{C}_{j-1}$.

Consider the following more general problem:

> Estimate an unknown m-dimensional plane **P** in $\mathbf{R}^n$ from a collection of k sets $\mathbf{A}_1,\ldots,\mathbf{A}_k$ that satisfy
>
> $$\mathbf{P} \cap \mathbf{A}_i \neq \emptyset . \tag{6}$$

(The following related problem was considered in [12]: for a given set **S**, find the smallest value for α for which there exists a t such that t + α**S** intersects each set in a given collection of sets.)

Example 1:

Refer to Fig.11. Suppose we would like to estimate an unknown plane P from k given points $a_1,\ldots,a_k$ that are known to be of the form

$$a_i = p_i + e_i \ , \qquad (7)$$

where p_i is a point in the unknown plane P, and the coordinates of the point e_i are all less than $\delta/2$.

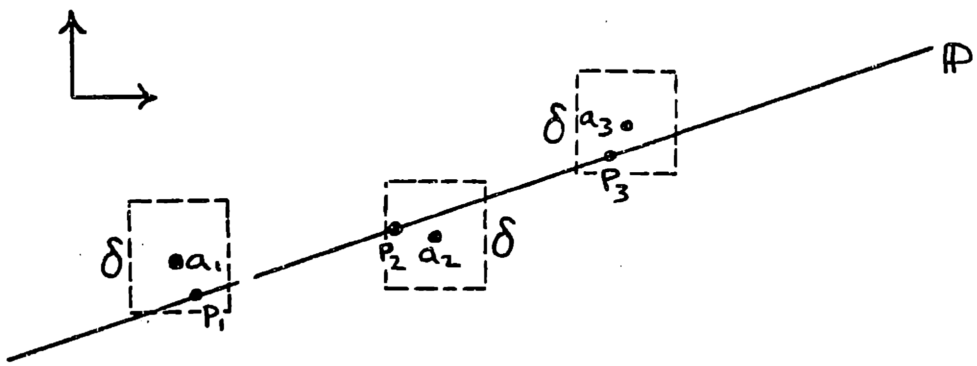


Fig.11

Let A_i be the n-dimensional cube with edges of length δ that is oriented along the coordinate axis and centered at the point a_i. Since p_i is in P and a_i satisfies (7), the plane P must intersect the cube A_i. Thus, we are faced with a special case of the plane reconstruction problem stated above. ∎

Example 2:

Refer to Fig.12. Suppose we would like to estimate an unknown plane P from k given points $a_1,\ldots,a_k$ that are known to be of the form (7), where p_i is a point in P and e_i is a vector with a magnitude r_i that is bounded by some given constant r.

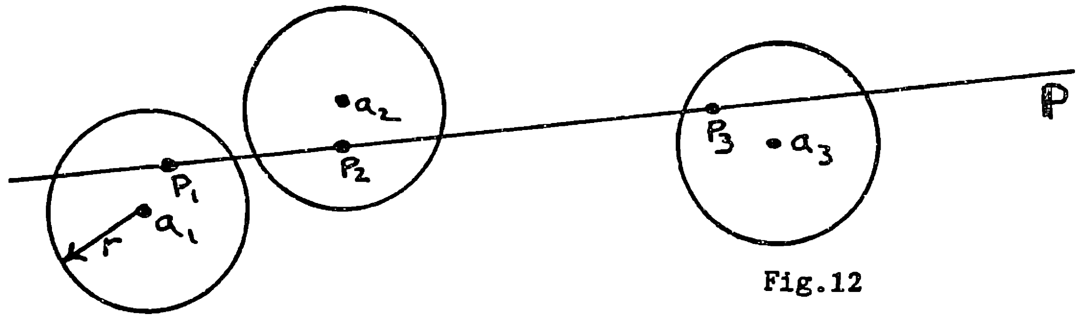


Fig.12

Let A_i be the sphere about the point a_i with radius r. Since p_i is in P and a_i satisfies (7), the plane P must intersect the sphere A_i. Again we are faced with a special case of the plane reconstruction problem stated above. ∎

For the remainder of this section, we shall focus upon the case where each of the sets A_i is an n-dimensional cube.

Suppose that we know that the plane P is actually an (n-1)-dimensional linear subspace, i.e. P is a hyperplane that contains the origin. Let p be the unit normal to the plane P. The problem of estimating P is equivalent to the problem of estimating its unit normal p. A vector x could be a multiple of the unit normal p if and only if x satisfies

$$Ax = 0 \, , \tag{8}$$

for some (kxn) matrix A that has the following form: the i^{th} row of A is a point in the cube A_i. Fig.13 shows the range of possible normals for a particular choice of the cubes $A_1,\ldots,A_k$.

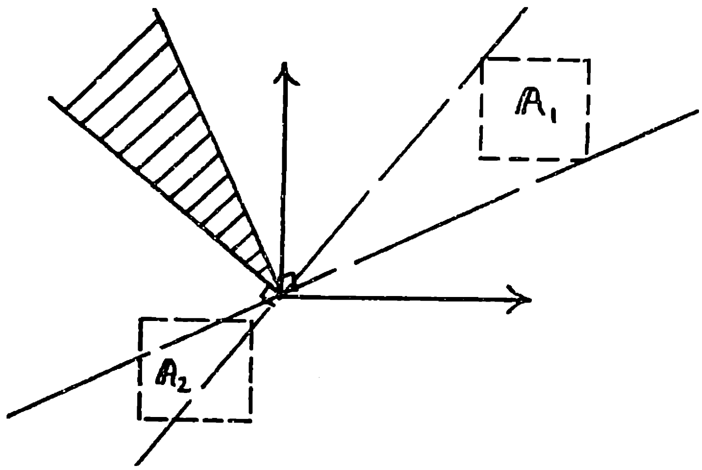


Fig.13

Each element in the matrix A in (8) can range over an interval. Thus, the problem of estimating an unknown (n-1)-dimensional linear subspace from k given cubes that intersect the subspace is equivalent to the problem of solving a system of k linear interval equations.

Linear Interval Equations

During the past 20 years, several results have been developed for more general systems of linear interval equations of the form

$$Ax = b , \tag{9}$$

where A is a (kxn) matrix and each of the elements of A and b is known to lie in some given interval but is otherwise unknown (note that some results have also been given for the problem where b ranges over some general given set, see [39]).

Let $\mathbf{X}$ be the set of vectors x that satisfy (9) for some A and b that satisfy the given interval constraints. In [43] it was shown that $\mathbf{X}$ equals the union of at most 2^n convex polyhedra, each having at most 2^n vertices (except in degenerate cases). The paper [52] focuses upon the case of a single linear interval equation, i.e. the case where the matrix A in (9) is (1xn). Ratschek and Sauer give necessary and sufficient conditions for the existence and uniqueness of solutions, necessary and sufficient conditions for the convexity of the solution set $\mathbf{X}$, and an explicit characterization of $\mathbf{X}$ for the case when it is convex. The set $\mathbf{X}$ has also been studied in [53] and [2].

In general, the set $\mathbf{X}$ is difficult to represent. Thus, many authors have developed methods to approximate $\mathbf{X}$ by a simpler set. In [41], [18], and [20] methods were given to obtain small interval polytopes that contain the set $\mathbf{X}$, and in [2] it was shown how large interval polytopes that are contained in $\mathbf{X}$ may be determined.

3. Interactive Polytope Reconstruction Problems

Up to this point, we have focused upon reconstruction problems in which the unknown set must be estimated from some <u>fixed</u> information. In this section, we shall introduce another important class of reconstruction problems. We shall describe some problems in which the current knowledge of the unknown set is used to formulate a query, and the current estimate of the set is updated by using the answer to this query. Problems of this type will be refered to as <u>interactive</u> reconstruction problems. In this section we shall discuss some interactive problems in which the unknown set is a <u>polytope</u>.

Probing the Boundary of a Polytope

Let P be an unknown bounded polytope in R^n. Suppose that the origin is contained in int(P).

For a point x in $\mathbb{R}^n - 0$, let probe(x) denote the positive multiple of x that lies on the boundary of $\mathbb{P}$.

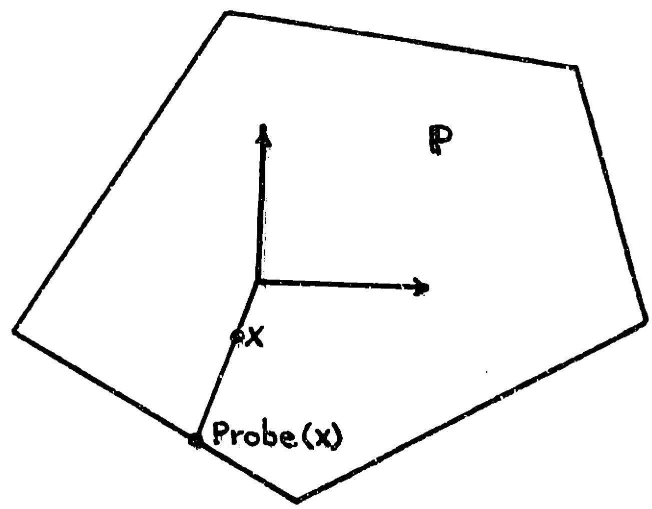


Fig.14

The function probe(x) is illustrated in Fig.14 for a particular choice of $\mathbb{P}$ and x.

Consider the following problem:

> Reconstruct an unknown bounded polytope P by selecting a sequence of points $x_1, \ldots, x_k$ and evaluating the function probe(·) at each of these points.

This problem arises in the area of robotics. A robot can evaluate the function probe(x) for some unknown object P by touching P along the vector x.

The unknown polytope P may always be determined by evaluating probe(·) a finite number of times. An example of how this can be accomplished is shown in Fig.15.

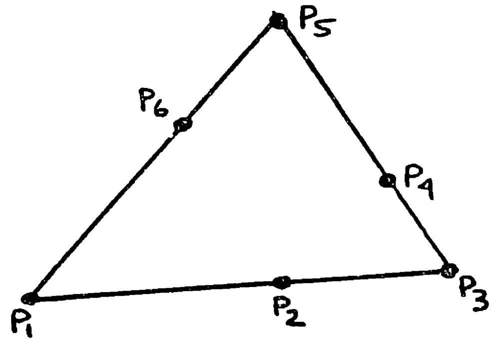


Fig.15

In this example, P is a convex polygon and we have evaluated the function probe(•) at six points to obtain the points $p_1,\ldots,p_6$ on the boundary of P. Since P is convex, it must contain the triangle $p_1p_3p_5$. In addition, the intersection of P and com($p_1p_3p_5$) must be empty (if there was a point p in this intersection, then for some i, p_i would be in int[hul(p,$p_1p_3p_5$) and this would contradict the fact that p_i is on the boundary of P). Thus, from the points $p_1,\ldots,p_6$ we may conclude that P must be the triangle $p_1p_3p_5$. (Note that objects known to be spheres or known to be hyperplanes may also be determined by evaluating the function probe(•) a finite number of times.)

We shall now summarize a simple algorithm that was given in [8] to obtain the desired sequence of points $x_1,\ldots,x_k$ for the case where P is a convex polygon. Refer to Fig.16. Let x_1, x_2, x_3, and x_4 be the points (1,0), (0,1), (-1,0), and (0,-1), respectively. We shall use p_i to denote the point probe(x_i). In Fig.16, Δ, o, and * will be used to denote x_i, p_i, and the origin, respectively. Each step in the remainder of the algorithm uses four points p_i to obtain a new point x_i.

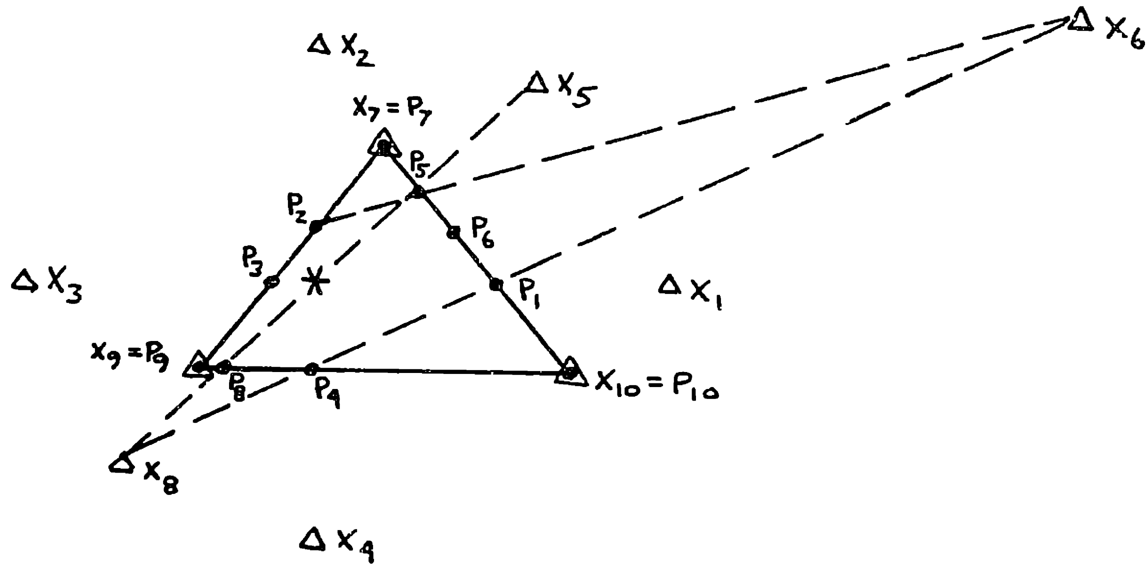


Fig.16

The first step uses the initial points $p_1,\ldots,p_4$ to obtain x_5. Let p be the intersection of the lines p_1p_4 and p_2p_3. If p is on the side of the line p_1p_2 that contains p_3 and p_4, then x_5 is taken to be -p. Otherwise, x_5 is taken to be p. (Note that in [8], x_5 is taken to be the midpoint of the line segment p_1p_2 when p is on the side of the line p_1p_2 that contains p_3 and p_4. We have modified the original algorithm in order to insure that it is a dual of an algorithm that we shall describe in the next section on support planes.) For the case shown in Fig.16, x_5 is taken to be -p. Next, p_1, p_5, p_2, and p_4 assume the roles of $p_1,\ldots,p_4$. These four points are used to obtain x_6. Since p_6 lies on the segment p_1p_5, we known that this segment must lie on an edge of the unknown polygon P. Next, the points p_5, p_2, p_3, and p_6 are used to obtain the point x_7. Since p_7 equals x_7, we may conclude that p_7 is a vertex of the polygon P. Next, p_3, p_4, p_1, and p_2 are used to obtain x_8. Then, p_3, p_8, p_4, and p_2 are used to obtain x_9. Since p_9 equals x_9, we may conclude that p_9 is a vertex of the polygon P. Finally, p_4,

p_1, p_6, and p_8 are used to obtain x_{10}. Since p_{10} equals x_{10}, we may conclude that p_{10} is a vertex of the polygon P. Since p_{10} lies on the line p_1p_6, we may conclude that the unknown polygon is the triangle $p_7p_9p_{10}$.

For the example described above, 10 points were needed to determine the unknown polygon. It can be shown that 3n+1 is an upper bound on the number of points that are needed to identify a convex n-gon by using this algorithm. Sometimes the algorithm only uses 3n points. For the example in Fig.16, an edge of P was identified before a vertex. If instead, we had identified a vertex before an edge (this would have been the case if P had been rotated by π) only 9 points would have been required.

In [8], an improved version of this algorithm is given. This modified algorithm determines an unknown n-gon by using 3n points. In addition, Cole and Yap show that every algorithm that determines the shape of an n-gon requires at least 3n points.

The reports [8] and [3] also discuss the probe function that yields, for a given x and y, the smallest positive number λ for which $y + \lambda x$ is on the boundary of P. Bernstein focuses upon some problems in which the unknown convex polygon P is a member of a given set of polygons. He considers the cases:

1. P contains the origin within its interior,
2. P contains a given disk about the origin, and
3. P contains the origin within its interior and one of the edges of P lies on a given line,

For cases 1, 2, and 3 he gives an algorithm to determine the unknown n-gon in 2n+3, 2n+2, and 2n-1 probes, respectively. He also shows that each of the algorithms may be shortened by one step in the case where it is known that the polygon P has n sides. (Note that we have omitted some of the technical assumptions that are made in [8] and [3] to handle cases where the line $y + \lambda x$ contains an edge of P or only intersects a single vertex of P. In addition to these assumptions, Bernstein places a condition on the set of polygons he considers. However, as he points out, this condition is satisfied for the important special case where the set of polygons is finite.)

Finally, we note that in [17] a similar problem was considered. Gaston and Lozano-Perez considered the problem of identifying an unknown polygon from a set of boundary points and bounds for the surface normals at these points.

Support Planes: A Dual Problem

Let P be an unknown bounded polytope in R^n. Suppose that the origin is contained in int(P).

For a hyperplane r in $R^n - 0$, let support(r) denote the positive multiple of r that is a support plane of P (a multiple of a set is defined point-wise).

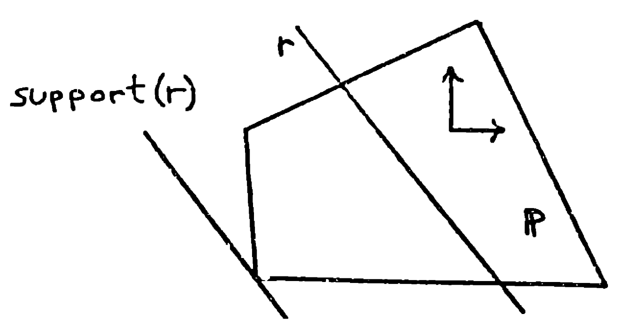


Fig.17

The function support(r) is illustrated in Fig.17 for a particular choice of P and r.

Consider the owing problem:

> Reconstruct an unknown bounded polytope P by selecting a sequence of hyperplanes $r_1,\dots,r_k$ and evaluating the function support(·) at each of these planes.

This problem may be obtained from the one discussed in the previous section by replacing probe(·) by support(·).

Algorithms for this problem could be used in robotics. For example, to identify an u nown polygon standing on one of its edges (this problem is described in [59]), a robot could determine support lines by using LED sensors across two opposing fingers (the IBM RS-1 has this type of gripper).

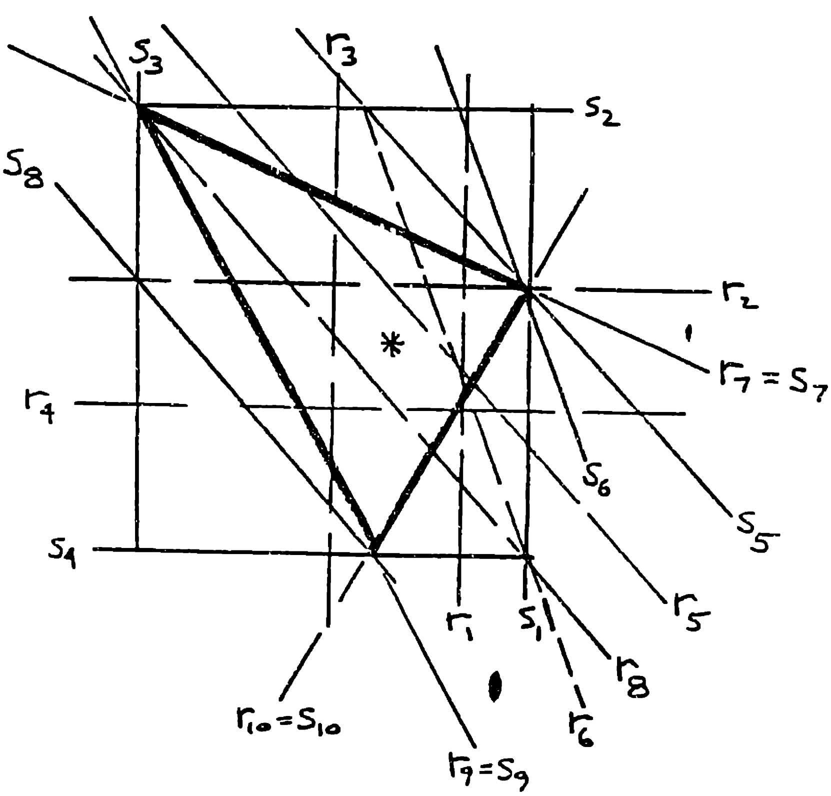


Fig.18

We shall now describe an algorithm that determines the required sequence of hyperplanes $r_1,\ldots,r_K$ for the case where the unknown set P is a convex polygon. Refer to Fig.18. Let * denote the origin of R^2 and let c_1 and c_2 denote the coordinate axes of R^2. Let r_1, r_2, r_3, and r_4 be the lines $c_1=1$, $c_2=1$, $c_1=-1$, and $c_2=-1$, respectively. We shall use s_i to denote the line support(r_i). Each step in the remainder of the algorithm uses four lines s_i to obtain a new line r_i. The first step uses the initial lines $s_1,\ldots,s_4$ to obtain r_5. Let a_1 be the intersection of the lines s_1 and s_4, and let a_2 be the intersection of the lines s_2 and s_3. If the origin and the intersection of the lines s_1 and s_2 are on the same side of the line a_1a_2 then r_5 is taken to be the negative of the line a_1a_2. Otherwise, r_5 is taken to be the line a_1a_2. For the case shown in Fig.18, r_5 is the negative of the line a_1a_2. Next, s_1, s_5, s_2, and s_4 assume the roles of $s_1,\ldots,s_4$. These four lines are used to obtain r_6. Since s_6 passes through the intersection of the lines s_1 and s_5, we known that this point of intersection must be a vertex of the unknown polygon P. Next, the lines s_5, s_2, s_3, and s_6 are used to obtain the line r_7. Since s_7 equals r_7, we may conclude that s_7 contains an edge of the polygon P. Next, s_3, s_4, s_1, and s_2 are used to obtain r_8. Then, s_3, s_8, s_4, and s_2 are used to obtain r_9. Since s_9 equals r_9, we may conclude that s_9 contains an edge of the polygon P. Finally, s_4, s_1, s_6, and s_8 are used to obtain r_{10}. Since s_{10} equals r_{10}, we may conclude that s_{10} contains an edge of the polygon P. Since s_{10} passes through the intersection of the lines s_1 and s_6, we may conclude that the unknown polygon is the triangle that is given by the intersection of the half-planes given by s_7, s_9, and s_{10}.

At this point we observe that a certain duality exists between the the algorithms given in Fig.16 and Fig.18. One algorithm may be obtained from the other by exchanging the roles of points and lines. This notion of duality may be formalized by using the concept of dual polytopes.

For each given polytope P, there is a corresponding dual polytope D that is given by the <u>polar set</u> of P (the polar set of a given set A is the set of all points y that satisfy $y^T x \leq 1$, for all points x in A). The dual polytope D is illustrated in Fig.19 for a particular choice of the polytope P.

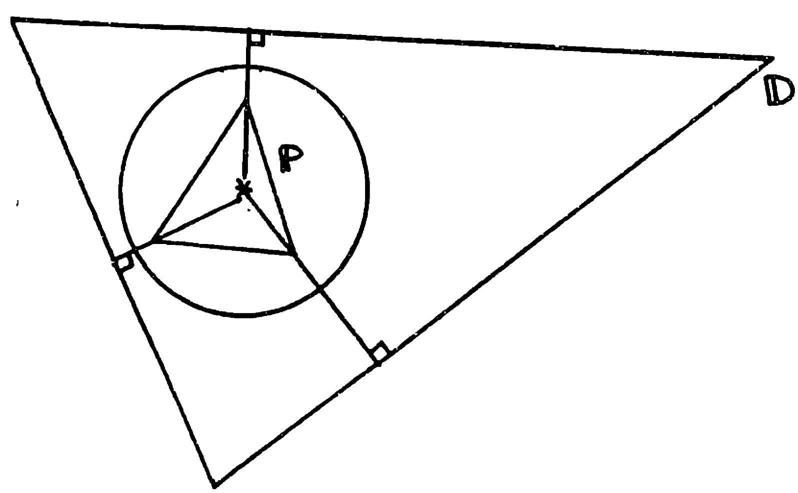


Fig.19

The boundary of D may be generated by replacing each vertex of P by its polar plane and each face of P by its pole. (The polar plane of a point p in R^n-0 is the set of points y for which $y^T p=1$. The pole of a hyperplane S in R^n-0 is the point p for which $p^T x=1$, for all points x in S. Some polar planes are shown with their corresponding poles in Fig.20.)

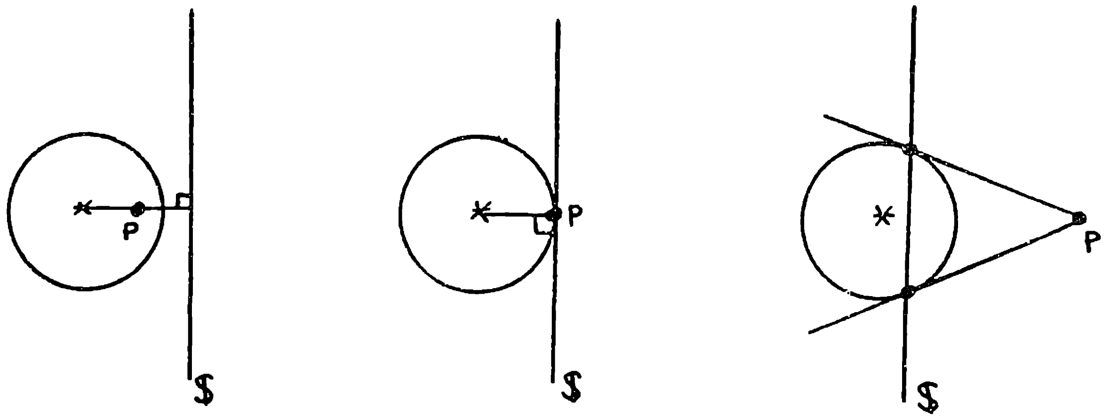


Fig.20

In Fig.21, we have superimposed the algorithms of Fig.16 and Fig.18. In order to clearly show the duality between these algorithms, we chose to have the support plane algorithm search for the dual of the triangle the was used to illustrate the boundary point algorithm.

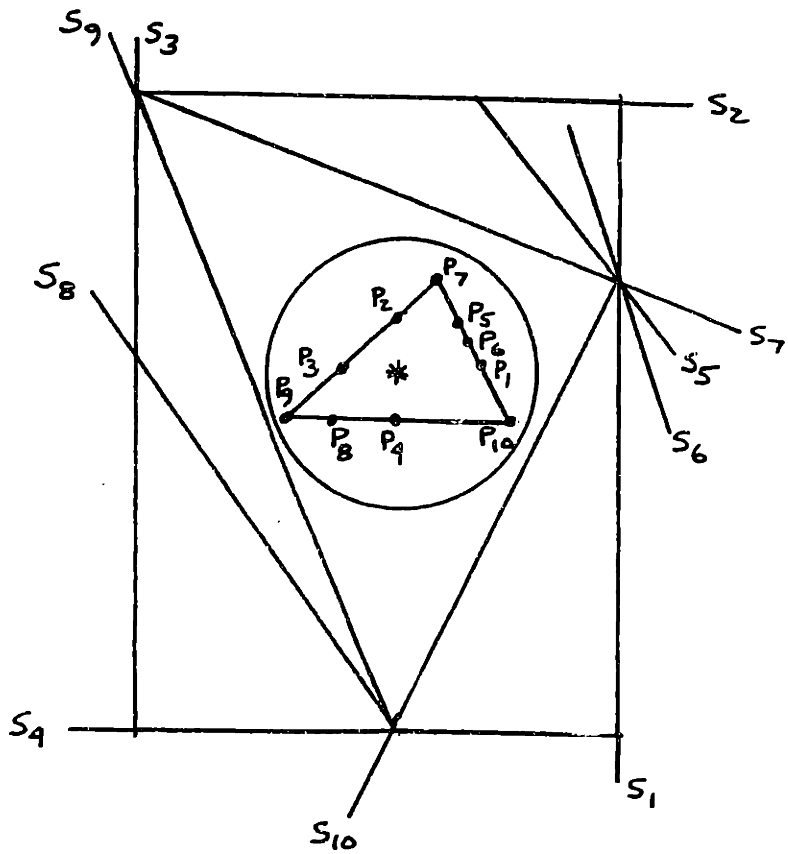


Fig.21

From Fig.21 we see that for each i, the line s_i is the polar line of the point p_i. It is also true that for each i, the line r_i is the polar line of the point x_i.

The next result follows from duality.

Result 1:

If an algorithm A reconstructs an unknown bounded polytope P in R^n by generating a sequence of k points $x_1,\ldots,x_k$ and evaluating the function probe(·) at each of these points, there is a dual algorithm A^* that reconstructs the polar set of P (or dual of P) by generating a sequence of k hyperplanes $r_1,\ldots,r_k$ (that are the polar planes of $x_1,\ldots,x_k$, respectively) and evaluating the function support(·) at each of these planes. The converse also holds.

Result 1 lets us transfer complexity results between these two problems. For example, since 3n+1 is an upper bound on the number of points that are needed to identify a convex n-gon by using the algorithm in Fig.16, 3n+1 is an upper bound on the number of planes that are needed to identify a convex n-gon by using the algorithm in Fig.18.

By using the ideas in this section, efficient dual algorithms may be derived from existing algorithms. For example, the dual of an algorithm in [8] that determines an unknown n-gon by using only 3n points may be used to determine an n-gon by using 3n support planes.

In this section we have discussed the problem of reconstructing a polytope from sequences of boundary points <u>or</u> support planes. We are not aware of any reconstruction algorithms that use <u>both</u> boundary points and support planes.

4. Some Open Problems

A list of the open problems that were described in Sections 1 through 3 is given below.

1. Develop an algorithm to find the largest and smallest spheres in the set of all spheres S that satisfy $A \subset S \subset B$, for two given (nonempty) subsets A and B of R^n.

2. Characterize the sets A and B given by (2), and develop algorithms to determine whether a given point is in A, B, or com(A,B).

3. Characterize the set of all m-dimensional planes that intersect a given collection of spheres (or interval polytopes) in R^n.

4. Develop an algorithm to reconstruct an unknown polytope from sequences of boundary points <u>and</u> support planes.

CHAPTER 5. CONCLUSION

The main goal of this thesis is to begin to develop a general theory of set reconstruction. Thus far, we have considered some particular reconstruction problems (Chapters 2 and 3) and surveyed some of the related literature (Chapter 4). In this chapter, we shall summarize the problems and results that were discussed in Chapters 2 through 4. A number of recurring themes in these results will be described. These themes form the foundation for a general theory of set reconstruction. In addition, we shall outline some directions for future research.

5.1 Summary of Results

We considered the following set reconstruction problem in Chapter 2:

> Estimate an unknown convex subset **A** of R^n from three given sets G_i, G_e, and G_b that are known to be subsets of the interior, exterior, and boundary of **A**, respectively.

We defined **C** to be the collection of all convex sets S that satisfy the constraints $G_i \subset int(S)$, $G_e \subset ext(S)$, and $G_b \subset bdy(S)$. Some necessary and sufficient conditions for **C** to be nonempty were given and some structural properties of **C** were identified.

We let **I** denote the set of points p that are contained in the interior of every set in **C**. The sets **B** and **E** were defined similarly for the boundary and exterior of sets in **C**, respectively, and **O** was defined to be the complement of the union of the sets I, B, and E. Several properties of the sets I, B, E, and O were discussed. We characterized the sets I and com(E). In addition, we showed that com(E) is a star-shaped set, and gave conditions under which com(E) is bounded. The smallest set that contains the boundary of every set in C was shown to be the union of B and O. Finally, we suggested a way by which new characterizations of the sets I, B, E, and O can be obtained by using conditions for C to be nonempty.

In Chapter 3 we focused upon a particular simplex reconstruction problem. This was derived from the problem of estimating a collection of unknown positive functions from a given set of functions that are positive combinations of the unknown functions, and took the following form:

> Estimate an unknown n-simplex S from two given sets F and G that satisfy $F \subset S \subset G$.

Again, we defined the class C to be the collection of all sets that satisfy the constraints imposed by the given partial description of the unknown set, i.e. C was taken to be the collection of n-simplexes that satisfy $F \subset S \subset G$. We defined V to be the union of all the vertices of the n-simplexes in C.

We gave many results for the case where F and G are compact convex planar sets. For this case, we characterized V and specialized our results to the case where F and G are convex polygons. We also described three novel iterative procedures that may be used to obtain an approximation to V. V was also characterized for the more general case where F and G are compact convex subsets of R^n.

We considered some sphere, plane, and polytope reconstruction problems in Chapter 4. First, we discussed the problem of reconstructing an unknown sphere S from two given sets F and G that satisfy $F \subset S \subset com(G)$. Then, we considered the problem of reconstructing an unknown m-dimensional plane P from k sets $A_1,\dots,A_k$ that intersect P. Finally, we considered some interactive polytope reconstruction problems. We showed that the problem of reconstructing a polytope by using sequences of support planes is a dual of the problem of reconstructing a polytope by using a sequence of boundary points. We also showed how algorithms for one problem may be used to obtain algorithms for the other problem. In Chapter 4, numerous connections were made with the existing literature on problems involving spheres, planes, and polytopes.

5.2 Recurring Themes

This section contains a discussion of some topics that have surfaced repeatedly in our work.

Penumbras

The penumbra of a set A with respect to a set B is given by

$$\{ x \mid x = (1-\lambda)a + \lambda b, \text{ for some } a \text{ in } A, \ b \text{ in } B, \text{ and } \lambda \leq 0 \} .$$

We denote this set by pen(A,B). Fig.1 illustrates the set pen(A,B) for a particular choice of the sets A and B.

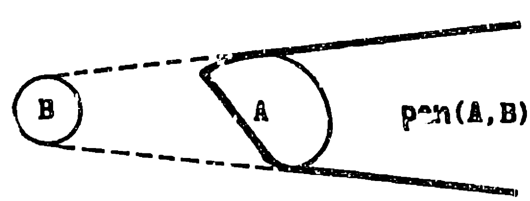


Fig.1

Suppose that we are given three points a, b, and c on the boundary of an unknown convex subset **A** of $\mathbb{R}^2$ and we would like to determine the largest set E that can be guaranteed to be contained in the exterior of **A** (this is a special case of the reconstruction problem considered in Chapter 2). By using Theorem 3 of Chapter 2, it can be shown that E may be expressed as the union of the three penumbras pen[a,int(hul(b,c))], pen[b,int(hul(a,c))], and pen[c,int(hul(a,b))], see Fig.2.

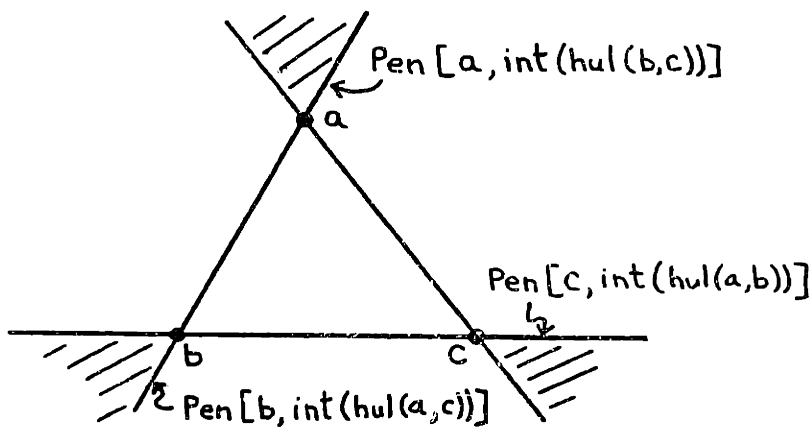


Fig.2

Penumbras also play a role in the reconstruction problem that was considered in Chapter 3. Let **F** and **G** denote compact convex subsets of $\mathbb{R}^2$. Suppose abc is a triangle that contains **F** and is contained in **G**, see Fig.3. It is easy to see that, any triangle of the form abp, where p is a point in pen[c,hul(a,b)], must contain **F**. This is a special case of Lemma 2a of Chapter 3.

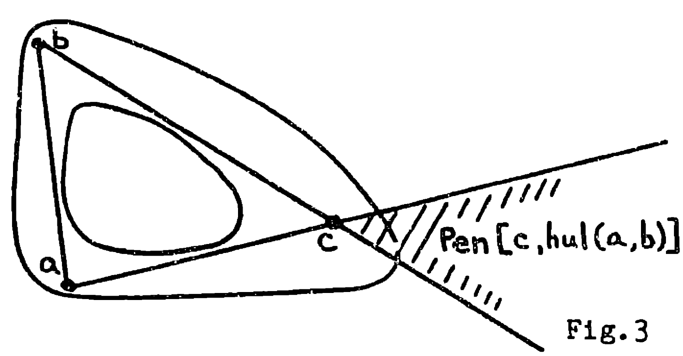


Fig.3

To the best of our knowledge, there are no efficient algorithms for determining the set pen(A,B) (even for the simplest case where A and B are finite subsets of R^2).

Star-Shaped Sets

A subset A of R^n is star-shaped relative to a point x in A if the line segment that joins x with any point in A is contained in A. Star-shaped sets played a role in both Chapters 2 and 3.

Refer to Fig.4. Suppose that we are given six points $p_1,\ldots,p_6$ on the boundary of an unknown convex subset A of R^2. By using Theorem 4 of Chapter 2, it can be shown that the smallest set that can be guaranteed to contain the set A is the star-shaped set shown in Fig.4. This is a special case of Lemma 11 of Chapter 2.

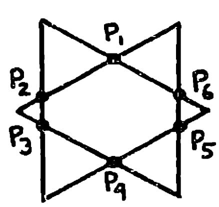


Fig.4

In Chapter 3, we showed that set of points that cannot be a vertex of a triangle between two compact convex planar sets F and G is star-shaped about the set int(F) (see Fig.9 and Corollary 2 of Chapter 3).

Extreme Sets

Suppose that we are given a partial description of an unknown subset A of R^n. In order to bound the size of A we can determine the largest and smallest sets in R^n that satisfy the given partial description. Several different extreme sets of this type were discussed in this thesis.

In Chapter 2, we considered the problem of reconstructing an unknown convex set A from three given sets G_i, G_e, and G_b that are known to be subsets of the interior, exterior, and boundary of A, respectively. In Theorem 2 of that chapter we characterized the smallest subset of R^n that satisfies this partial description. We showed that for some cases, there is no smallest set. For these cases, we determined the infimum of the collection of sets that satisfy the given partial description.

A variety of extreme problems for spheres that have been considered by other authors were listed in our survey in Chapter 4, see Figs.2, 3, and 4.

Iterative Algorithms

Several iterative geometric procedures were introduced in Chapter 3. We described three algorithms that may be used to obtain an approximation of the vertex domain (see Figs.16, 19b, and 20b). A procedure with a similar structure that converges to an equilateral triangle was also discussed. In Section 5.2 of Chapter 3, we identified a number of topics for future research and listed a collection of related papers that should be carefully investigated.

Duality

In Section 3 of Chapter 4, we described some existing algorithms that generate a sequence of boundary points to reconstruct an unknown polytope. We showed that the problem of reconstructing a polytope by using a sequence of support planes is a dual of the boundary points problem. This fact may be exploited to obtain algorithms for one problem from algorithms for the other problem.

Although this is the only instance in which the concept of duality was needed in our work, we feel that duality might be a useful idea in other situations as well. Several authors have already pointed out how duality may be used in the analysis and development of algorithms for a variety of geometric problems, see [33], [51], [7], and [4].

5.3 Future Work

In Sections 2.4, 3.5, and 4.4, we described several specific open problems that are directly related to the particular reconstruction problems that have been considered in this thesis. In this section, we shall describe some general directions for future research.

This thesis represents a first step toward the development of a general theory of set reconstruction. Where do we go from here? There are two important steps that should be taken next. The recurring themes that were discussed in the previous section should be studied, and in parallel with this effort, we should attempt to pull together the related work of other authors.

There is a large body of published work on various specific set reconstruction problems. For example, several authors have considered the problem of reconstructing an unknown set **A** from a set of projections of **A**, see [27], [37], [55], and [71]. This is a fundamental problem in tomography. The problem of reconstructing a set from slices is another important type of problem that arises from tomography (and also microscopy), see [67], [34]. A third type of problem that arises in positron-emission tomography is the problem of reconstructing an unknown set **A** from a collection of lines that intersect **A** [30].

Other authors, within the context of computer vision, have attacked the problem of reconstructing an unknown set A from a set of silhouettes of A [58], [73].

The relatively new mathematical discipline called stereology deals with problems that involve some descriptions such as sections, projections, silhouettes, or intersections with test sets. The main problem of stereology is to determine the characteristic geometric properties (e.g. volume) of an unknown set from a given partial description. A fine survey of this related growing field is given in [74]. This article contains several references that are relavant to the study of set reconstruction.

A list of some of the different types of unknown sets that other authors have considered is given below (A denotes the unknown set).

1. A is a point that moves along a piecewise linear random path [25], [36], and [21].
2. A has a random lifetime [40].
3. A has a hole or inclusion [22].
4. A changes shape with time [23], [1], [44].
5. A rotates and translates with time [23].
6. A is disconnected [23].

It should be possible to expand our list of recurring themes by inspecting the problems and solutions that are contained in the papers cited in this section.

APPENDIX 1. MATHEMATICAL TERMS AND BASIC RESULTS

In this appendix, we shall define several terms and state many basic results that are used throughout the body of the thesis. Most of the material in this appendix has been taken from [31], [5], and [54].

1. Mathematical Terms

asymptotically k-periodic point. A point p is said to be an asymptotically k-periodic point of a function f if the set

$$\{ f^{ki}(p),\ f^{ki+1}(p),\ \ldots,\ f^{k(i+1)-1}(p) \} ,$$

converges to k different points.

bound. A hyperplane $H=[f:\alpha]$ is said to bound the set S if either $f(S)\geq\alpha$ or $f(S)\leq\alpha$.

boundary. The boundary of a subset S of R^n is the set of all points p for which every open ball $B(p,\delta)$ with $\delta>0$, contains at least one point of S and at least one point of com(S). The boundary of a set S is denoted by bdy(S).

bounded. A subset S of R^n is bounded if S is contained in some open ball $B(p,\delta)$.

closed. A subset S of R^n is closed if com(S) is open.

closed half-space. For a given hyperplane $H=[f:\alpha]$, the sets

$$\{ x \mid f(x) \geq \alpha \} \quad \text{and} \quad \{ x \mid f(x) \leq \alpha \}$$

are called the closed half-spaces determined by H.

closure. The closure of a subset S of R^n is the union of S and bdy(S). The closure of S is denoted by clo(S).

compact. A subset S of R^n is compact if S is closed and bounded.

complement. The complement of a subset S of R^n is the set of all points in R^n that are not in S. The complement of S is denoted by com(S).

convex. A subset S of R^n is convex if the line segment joining each pair of points in S is contained in S.

convex combination. A point y in R^n is called a convex combination of the points $x_1, x_2, \ldots, x_k$ in R^n if y may be written as

$$y = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_k x_k ,$$

where

$$\lambda_1 + \lambda_2 + \ldots + \lambda_k = 1 ,$$

$$\lambda_j \geq 0 .$$

convex hull. The convex hull of a subset S of R^n is the intersection of all the convex sets that contain S. The convex hull of a set S is denoted by hul(S).

dimension of a plane. The dimension of a plane is the dimension of its corresponding parallel linear subspace [66].

dimension of a set. The dimension of a subset S of R^n is the dimension of the smallest-dimension plane containing it, and is denoted by dim(S).

distance between sets. If A and B are two nonempty subsets of R^n, then the distance between them is given by

$$\text{dist}(A,B) = \inf\{ \text{ all } |p-q| \text{ for } p \text{ in } A \text{ and } q \text{ in } B \} .$$

dual polytope. The dual of a polytope P equals the polar set of P.

Euclidean distance. The Euclidean distance between two points p and q in R^n is given by the positive square root of

$$(p_1-q_1)^2 + (p_2-q_2)^2 + \ldots (p_n-q_n)^2 ,$$

where p_i and q_i denote the coordinates of the points p and q. The Euclidean distance between p and q is denoted by $|p-q|$.

Euclidean space (n-dimensional). The inner product space [66] given by the collection of all ordered n-tuples of real numbers (for n=1,2,...) together with the operations of addition defined by

$$(a_1,\ldots,a_n) + (b_1,\ldots,b_n) = (a_1+b_1,\ldots,a_n+b_n) ,$$

and scalar multiplication defined by

$$\lambda(a_1,\ldots,a_n) = (\lambda a_1,\ldots,\lambda a_n) ,$$

and the inner product [66] $\langle a,b\rangle$ of $a=(a_1,\ldots,a_n)$ and $b=(b_1,\ldots,b_n)$ defined by

$$\langle a,b\rangle = a_1b_1 + \ldots + a_nb_n ,$$

is called n-dimensional Euclidean space, and is denoted by R^n.

Euler phi-function. The Euler phi-function, denoted by $\phi(n)$, is defined for positive integers n to be the number of integers in the range [1,n] that are relatively prime to n [15].

exterior. The exterior of a subset S of R^n is the set of all points p for which there is an open ball $B(p,\delta)$ with $\delta>0$ that is contained in the set com(S). The exterior of a set S is denoted by ext(S).

functional. A function from R^n to R is called a functional.

hyperplane. An (n-1)-dimensional plane is called a hyperplane. For every hyperplane H, there is a nonidentically zero linear functional f and a real constant α such that $H=[f:\alpha]$.

interior. The interior of a subset S of R^n is the set of all points p for which there is an open ball $B(p,\delta)$ with $\delta>0$ that is contained in S. The interior of a set S is denoted by int(S).

interval hull. The interval hull of a subset S of R^n is the smallest interval polytope that contains S.

interval polytope. A polytope P of R^n is said to be an interval polytope if the linear functional that is associated with each closed half-space used to obtain P only depends on one coordinate of R^n.

iterate (i^{th} iterate). The i^{th} iterate of a function f at x equals f($(i-1)^{th}$ iterate of f at x). The $f^0(x)$ is defined to be the identity function.

kernel. The kernel of a subset S of R^n is the set of all points p in S such that the line segment joining p with any point in S is contained in S.

line. A 1-dimensional plane is called a line.

line segment. The line segment joining two points x and y is the set of all points of the form $\alpha x + (1-\alpha)y$ where α is in the interval [0,1]. The line segment joining x and y is denoted by [x,y].

linear function. A function f from R^n to R^m is said to be linear if

$$f(x+y) = f(x) + f(y) \text{ , and}$$

$$f(\lambda x) = \lambda f(x) \text{ , for scalar } \lambda \text{ .}$$

linear functional. A linear function from R^n to R is called a linear functional. If f is a linear functional, then $[f:\alpha]$ denotes the set of all points x for which $f(x) = \alpha$.

open. A subset S of R^n is open if S equals int(S).

open ball. An open ball $B(p,\delta)$ in R^n is the set of all points q in R^n that satisfy $|p-q| < \delta$.

open half-space. For a given hyperplane $H=[f:\alpha]$, the sets

$$\{ x \mid f(x) > \alpha \} \quad \text{and} \quad \{ x \mid f(x) < \alpha \}$$

are called the open half-spaces determined by H.

open line-segment. The open line segment joining two points x and y is the set of all points of the form $\alpha x + (1-\alpha)y$ where α is in the interval (0,1). The open line segment joining x and y is denoted by (x,y).

parallel planes. Two planes are parallel if one is a translate of the other.

penumbra. The penumbra of the set A with respect to the set B is given by

$$\{ x \mid x = (1-\lambda)a + \lambda b, \text{ for some } a \text{ in } A, b \text{ in } B, \text{ and } \lambda \leq 0 \} .$$

We shall denote this set by pen(A,B).

periodic point (k-periodic point). A point p is said to be a k-periodic point of a function f if $f^k(p)=p$ and $f^i(p) \neq p$, for all i in the interval [0,k-1].

plane (k-dimensional). A translate of a k-dimensional linear subspace of R^n is called a k-dimensional plane [66].

polar plane. The polar plane of a given point p in R^n-0 is the set of points y for which $y^Tp=1$.

polar set. The polar set of a given set **A** is the set of all points y that satisfy $y^T x \leq 1$, for all points x in **A**.

pole. The pole of a hyperplane **P** in $R^n - 0$ is the point p for which $p^T x = 1$, for all points x in **P**.

polytope. A polytope is the intersection of a finite number of closed half-spaces.

relative interior. The relative interior of a set S is the interior of S relative to the minimal-dimension plane that contains S. The relative interior of a set S is denoted by rint(S).

simplex (k-simplex). A k-simplex is the convex hull of a set S consisting of k+1 points for which dim(S) = k.

sphere. A sphere in R^n with center p and radius δ is the set of all points q in R^n that satisfy $|p-q| \leq \delta$.

star-shaped. A subset S of R^n is star-shaped relative to a point x in S if the line segment that joins x with any point in S is contained in S.

support hyperplane. A hyperplane H is said to support a set S at a point p in S if p is in H and if H bounds S.

translate. A subset **B** of R^n is a translate of a set **A** if there is a point x such that every point in **B** can be written as $x + y$ where y is a point in **A**.

2. Basic Results

⟨1⟩ a. $A \subset B \rightarrow \text{int}(A) \subset \text{int}(B)$.

b. $\text{int}(A) \cup \text{bdy}(A) \cup \text{ext}(A) = R^n$,

$\text{int}(A) \cap \text{bdy}(A) = \phi$,

$\text{int}(A) \cap \text{ext}(A) = \phi$,

$\text{bdy}(A) \cap \text{ext}(A) = \phi$.

c. $a \in A \rightarrow a \in \text{int}(A)$ or $a \in \text{bdy}(A)$.

d. $A \subset B \rightarrow \text{ext}(B) \subset \text{ext}(A)$.

e. $\text{ext}(A) = \text{ext}[\text{clo}(A)]$.

f. $\text{int}[\text{int}(A)] = \text{int}(A)$.

g. $\text{int}(A) \subset B \rightarrow \text{int}(A) \subset \text{int}(B)$.

h. clo(A) is closed.

i. $A \subset B$, B closed $\rightarrow \text{clo}(A) \subset B$.

j. $\text{clo}(A) = \text{int}(A) \cup \text{bdy}(A)$.

k. $A \subset B \rightarrow \text{clo}(A) \subset \text{clo}(B)$.

l. $\text{int}(A) \subset A$.

m. p is a point $\rightarrow \text{clo}(p) = p$.

n. If two of the statements

$$\text{int}(A) = \text{int}(B) ,$$

$$\text{bdy}(A) = \text{bdy}(B) ,$$

$$\text{ext}(A) = \text{ext}(B) ,$$

are true, then all three are true.

o. A, B closed $\longrightarrow$ $A \cup B$ closed .

⟨2⟩ C convex $\longrightarrow$

a. int(C) convex ,

b. clo(C) convex ,

c. $\text{clo}[\text{int}(C)] = \text{clo}(C)$,

d. $\text{int}[\text{clo}(C)] = \text{int}(C)$,

e. $\text{bdy}[\text{clo}(C)] = \text{bdy}(C)$.

⟨3⟩ a. $A \subset B$, B convex $\rightarrow$ $\text{hul}(A) \subset B$,

b. $\text{hul}(A) \subset \text{hul}(A,B)$.

c. The intersection of any collection of convex sets is convex.

d. hul(A) is convex .

e. $A \subset \text{hul}(A)$.

f. $A \subset \text{clo}(B)$, B convex $\longrightarrow$ $\text{int}(A) \subset \text{int}(B)$.

g. $\text{hul}(A) = \text{hul}[\text{hul}(A)]$.

h. $\text{hul}(A,B) = \text{hul}[A,\text{hul}(B)]$.

⟨4⟩ Caratheodory's Theorem. Let S be a nonempty subset of R^n. Every point p in hul(S) may be expressed as a convex combination of n+1 or fewer points of S.

⟨5⟩ A subset S of R^n is a hyperplane <u>iff</u> there is a nonidentically zero linear functional f and a real constant α such that $S=[f:\alpha]$.

⟨6⟩ If b is a boundary point of a closed convex subset S of R^n, then there is at least one support hyperplane for S at b.

⟨7⟩ For any points p and q in R^n and real number λ

a. $|p+q| \leq |p| + |q|$ (triangle inequality) ,

b. $|\lambda p| = |\lambda||p|$ (scaling property) .

⟨8⟩ The following set version of the triangle inequality holds:

$$\text{dist}(a,B) \geq \text{dist}[c,B] - |c-a| .$$

⟨9⟩ A set S is convex <u>iff</u> every convex combination of points of S lies in S.

⟨10⟩ The dimension of an open ball in R^n is n.

⟨11⟩ If C is a convex set, then

a. $a,b \in \text{int}(C) \longrightarrow [a,b] \subset \text{int}(C)$,

b. $a \in \text{int}(C),\ b \in \text{bdy}(C) \longrightarrow [a,b) \subset \text{int}(C)$,

c. $a,b \in \text{bdy}(C) \longrightarrow (a,b) \subset \text{int}(C)$, or $(a,b) \subset \text{bdy}(C)$.

⟨12⟩ Let $H=[f:\alpha]$ be a hyperplane in R^n and let p be a point in H. For each $\delta>0$, there are two points p_1 and p_2 in $B(p,\delta)$ that satisfy $f(p_1)>\alpha$ and $f(p_2)<\alpha$, i.e. $B(p,\delta)$ for $\delta>0$, always intersects both open half-spaces determined by H.

⟨13⟩ If $H=[f:\alpha]$ is a support hyperplane for a convex subset S of R^n, then $H=[f':\alpha']$ for some linear functional f′ for which $f'(S)\geq\alpha'$.

⟨14⟩ Planes are closed convex sets.

⟨15⟩ If A is a closed subset of R^n, and p is in com(A), then dist(p,A)>0.

⟨16⟩ Let $H=[f:\alpha]$ be a hyperplane in R^n and let C be a convex subset of R^n. If there are points p_1 and p_2 in C for which $f(p_1)<\alpha$ and $f(p_2)>\alpha$, then H intersects C.

⟨17⟩ For all $\delta>0$ and all points p in R^n, the open ball $B(p,\delta)$ in R^n is

a. open, and

b. convex.

⟨18⟩ Open and closed half-spaces are

a. open and closed, respectively, and

b. convex.

⟨19⟩ Let C be a convex subset of R^n and let p be a point R^n.

a. $hul(p,C) = \{ x \mid x = \lambda p + (1-\lambda)c$, for some c in C and some λ in $[0,1] \}$.

b. $p \notin int(C) \longrightarrow p \in bdy[hul(p,C)]$.

c. $q \in int[hul(p,C)] \longrightarrow (bdy[hul(p,C)] \cap bdy[hul(q,C)]) \subset bdy(C)$.

d. $p \in clo(C) \longrightarrow int(C) = int[hul(p,C)]$,

$bdy(C) = bdy[hul(p,C)]$,

$ext(C) = ext[hul(p,C)]$.

e. $q \in int[hul(p,C)] \longrightarrow q = \lambda p + (1-\lambda)c$, for some λ in (0,1) and some c in C.

<20> For any subset S in R^n, hul(S) consists precisely of all convex combinations of elements of S.

<21> For any convex subset C of R^n, if [a,b] is in bdy(C), then any support hyperplane for C at a point in (a,b) contains the line ab.

<22> Let S be a subset of R^n.

a. $dim(S) < n \longrightarrow int(S) = \emptyset$.

b. $dim(S) = n$, S convex $\longrightarrow int(S) \neq \emptyset$.

<23> If S is a subset of R^n and H is a support hyperplane for S, then

$$H \cap int(S) = \emptyset \ .$$

<25> Same as <3c>.

<26> $A_1 \subset A \subset A_2$, $p \in [bdy(A_1) \cap bdy(A_2)] \longrightarrow p \in bdy(A)$.

APPENDIX 2. PROOFS FOR CHAPTER 2

This appendix contains the proofs for the results given in Chapter 2. For convenience, we shall also restate those results here. The results (and their proofs) that were proved within the text of Chapter 2 will not be discussed here. We shall use (x) to denote equation number x from Chapter 2. Additional equations will be denoted by (*x).

Lemma 1:

If **A** is a set in the class **C**, then hul(**G**) is contained in clo(**A**).

Proof of Lemma 1:

Let **A** be a set in the class **C**. $\mathbf{G}_i$ must be contained in int(**A**), and $\mathbf{G}_b$ must be contained in bdy(**A**). Thus **G** must be contained in clo(**A**), which must be convex. Hence hul(**G**) must be contained in clo(**A**). ∎

Lemma 2:

If **C** is nonempty, then

$$\mathbf{G}_b \subset \text{bdy}[\text{hul}(\mathbf{G})] \quad , \quad \text{and} \tag{4a}$$

$$\mathbf{G}_e \subset \text{ext}[\text{hul}(\mathbf{G})] \quad . \tag{4b}$$

Proof of Lemma 2:

Suppose **C** is nonempty. Let **A** be a set in **C**.

Let b be a point in $\mathbf{G}_b$. Suppose b is in int[hul(**G**)]. By Lemma 1, hul(**G**) must be contained in clo(**A**). Thus, the set int[hul(**G**)] must be contained in int[clo(**A**)]. Since **A** is convex, int[hul(**G**)] must be contained in int(**A**). So the point b must be in int(**A**). But since b is in $\mathbf{G}_b$, this contradicts the fact that **A** is in **C**. Thus, it must be the case that the intersection of G_b and int[hul(G)] is empty. Thus, since $\mathbf{G}_b$ is contained in hul(**G**), $\mathbf{G}_b$ must be contained in bdy[hul(**G**)].

$\mathbf{G}_e$ must be contained in ext(**A**). Thus, $\mathbf{G}_e$ must be contained in ext[clo(**A**)]. By Lemma 1, hul(**G**) must be contained in clo(**A**). Thus, ext[clo(**A**)] must be contained in ext[hul(**G**)]. So, $\mathbf{G}_e$ must be contained in ext[hul(**G**)]. ∎

Lemma 4:

The set **T**(r) given by (6) is in **C**.

Proof of Lemma 4:

Since **A** is in **C**, $\mathbf{G}_e$ must be contained in ext(**A**), and therefore in ext[**T**(r)], by (7).

Since **A** is in **C**, $\mathbf{G}_b$ must be contained in bdy[clo(**A**)]. Thus, since $\mathbf{G}_b$ is contained in bdy[hul(**G**)] and **T**(r) satisfies (7), $\mathbf{G}_b$ must be contained in bdy[**T**(r)].

Let p be a point in $\mathbf{G}_i$. The point p must be in hul(**G**). So, p must either be in int[hul(**G**)] or bdy[hul(**G**)]. Suppose p is in int[hul(**G**)]. Since T(r) satisfies (7), p must be in int[**T**(r)]. Now suppose p is contained in bdy[hul(**G**)]. Since p is also in $\mathbf{G}_i$, p must be in **Y**. Thus, **B**(p,r(p)) is contained in **T**(r). So p must be contained in int[**T**(r)]. Thus, $\mathbf{G}_i$ is contained in int[**T**(r)].

Thus, since **T**(r) is convex, **T**(r) must be in **C**. ∎

Lemma 7:

Suppose Y is nonempty. If $T(r_2)$ is a set in T that is in C, then every set $T(r_1)$ in T that satisfies

$$T(r_1) \subset T(r_2) \quad , \tag{8}$$

is also in C.

Proof of Lemma 7:

Suppose Y is nonempty. Let $T(r_2)$ be a set in T that is in C, and let $T(r_1)$ be a set in T that is contained in $T(r_2)$. $T(r_1)$ is convex and G_i is contained in $\text{int}[T(r_1)]$. G_e must be contained in $\text{ext}[T(r_2)]$, and the intersection of $\text{int}[T(r_2)]$ and G_b must be empty. Thus, by (8), G_e must be contained in $\text{ext}[T(r_1)]$, and the intersection of $\text{int}[T(r_1)]$ and G_b must be empty. Thus since G_b is contained in $T(r_1)$, G_b must be contained in $\text{bdy}[T(r_1)]$. Thus, $T(r_1)$ must be in C. ∎

Theorem 2: Smallest Set in C:

Suppose C is nonempty and the dimension of hul(G) equals n.

a. If Y is empty, then I is the smallest set in C (i.e. I is in C and I is contained in every set in C).

b. If Y is nonempty, then I is the infimum of C (i.e. I is the largest subset of R^n that is contained in every set in C).

Proof of Theorem 2:

Suppose C is nonempty and the dimension of hul(G) equals n.

Suppose Y is empty. By Lemma 3a, hul(G) must be in C. Thus by Lemma 5, rint[hul(G)] must be in C. Since the dimension of hul(G) equals n, rint[hul(G)] equals int[hul(G)]. Thus int[hul(G)] is in C.

From Theorem 3, since **Y** is empty, I equals int[hul(**G**)]. Thus I is in **C**. Also, by definition I is contained in every set in **C**.

Next suppose **Y** is nonempty. From Theorem 3, I equals the union of int[hul(**G**)] and **Z**. Let p be a point in com(I). We shall show that there is a set in **C** that does not contain p. First, suppose p is in ext[hul(**G**)]. By Theorem 1b, there is a set $T(r_c)$ in **T** that is in **C**. By Lemma 9, there is a set $T(r_e)$ in **T** for which p is in ext[$T(r_e)$]. Let r be the map from **Y** to $(0,\infty)$ that equals $\min[r_e(y),r_c(y)]$, for all y in **Y**. T(r) must be contained in both $T(r_e)$ and $T(r_c)$. Thus, by Lemma 7, T(r) must be in **C**. Also, p must be in ext[T(r)]. Thus p is not contained in every set in **C**.

Now suppose p is a in bdy[hul(**G**)] but not in **Z**. By Lemma 10, there is a set $T(r_b)$ in **T** for which p is in bdy[$T(r_b)$]. By Theorem 1, there is a set $T(r_c)$ in **T** that is in **C**. Let r be the map from **Y** to $(0,\infty)$ that equals $\min[r_b(y),r_c(y)]$, for all y in **Y**. T(r) must be contained in both $T(r_b)$ and $T(r_c)$. By Lemma 7, T(r) must be in **C**. Also, p must be in bdy[T(r)]. By Lemma 5, rint[T(r)] must be in **C**. Since the dimension of hul(**G**) equals n, the dimension of T(r) equals n. Thus rint[T(r)] equals int[T(r)]. Hence, int[T(r)] is in **C**. Since p is not in int[T(r)], p is not contained in every set in **C**. ∎

Lemma 8:

If A_1 and A_2 are in **C**, then every convex set A that satisfies

$$A_1 \subset A \subset A_2 , \qquad (9)$$

is also in **C**.

Proof of Lemma 8:

Let A_1 and A_2 be sets in C and let A be a set that satisfies (9). G_i must be contained in $int(A_1)$. Thus, by (9), G_i must be contained in int(A). G_e must be contained in $ext(A_2)$. Thus, by (9) G_e must be contained in ext(A). G_b must be contained in $bdy(A_1)$ and $bdy(A_2)$. Thus, by (9), G_b must be contained in bdy(A). Thus if A is convex, then A must be in C. ∎

Lemma 9:

Suppose Y is nonempty. If p is a point in ext[hul(G)], then there is a set T(r) in T for which p is in ext[T(r)].

Proof of Lemma 9:

Let p be a point ext[hul(G)]. Suppose Y is nonempty. hul(G) must be nonempty. Let d be the distance between hul(G) and the point p. d must be nonzero and finite. Let r be the map from Y to $(0,\infty)$ for which r(y) equals d/2, for all y in Y.

First we shall show that for all points t in T(r), dist[t,hul(G)] is less than d/2. Let t be a point in T(r). t may be written as

$$t = \lambda_1 q_1 + \lambda_2 q_2 + \dots + \lambda_{n+1} q_{n+1} \ , \qquad (*1a)$$

for some scalars $\lambda_1,\dots,\lambda_{n+1}$ for which

$$0 \leq \lambda_j \leq 1 \ , \quad \text{for all } j \ , \quad \text{and} \qquad (*1b)$$

$$\lambda_1 + \lambda_2 + \dots + \lambda_{n+1} = 1 \ , \qquad (*1c)$$

and some points $q_1,\dots,q_{n+1}$ that are in G or B(y,r(y)), for some y in Y. Since $dist[q_j,hul(G)]$ is less than d/2 for all j, there are points q'_j in hul(G) for which $dist(q_j,q'_j)$ is less than d/2. Let t' be the point given by

$$t' = \lambda_1 q_1' + \lambda_2 q_2' + \ldots + \lambda_{n+1} q_{n+1}' \ .$$

Since t′ is a convex combination of points in the convex set hul(G), t′ must be in hul(G). By the repeated application of the triangle inequality and the scaling property for the distance function, we obtain

$$\text{dist}(t,t') \leq |\lambda_1|\text{dist}(q_1,q_1') + \ldots + |\lambda_{n+1}|\text{dist}(q_{n+1},q_{n+1}') \ .$$

Hence, since B_j must be nonzero for some j, and $\text{dist}(q_j,q_j')$ is less than d/2 for all j,

$$\text{dist}(t-t') < |\lambda_1|d/2 + \ldots + |\lambda_{n+1}|d/2 \ . \qquad (*2)$$

Thus by combining (*1b), (*1c), and (*2), we see that the distance between t and t′ must be less than d/2. Therefore, since t′ is in hul(G), dist[t,hul(G)] must be less than d/2.

From a set version of the triangle inequality we have that

$$\text{dist}(p,t) \geq \text{dist}[p,\text{hul}(G)] - \text{dist}[t,\text{hul}(G)] \ .$$

Thus, dist(p,t) is greater than d/2, for all t in T(r). Therefore, B(p,d/2) must be contained in com[T(r)]. Hence, p must be in ext[**T**(r)]. ▩

Lemma 10:

Suppose **Y** is nonempty. A point p in bdy[hul(G)] is in bdy[T(r)], for some set T(r) in **T** <u>iff</u> p is not in **Z**.

Proof of Lemma 10:

Suppose Y is nonempty. Let p be a point in bdy[hul(G)]. Suppose p is in bdy[T(r)], for some set T(r) in T. Let H be a support hyperplane for clo[T(r)] at p. Since clo[hul(G)] is contained in clo[T(r)], and p is in clo[hul(G)], H must be a support hyperplane for clo[hul(G)] at p. Since G_i is contained in int[T(r)], the intersection of H and G_i must be empty.

Now, suppose that there is a support hyperplane H for clo[hul(G)] at p that does not intersect G_i. For each point y in Y, dist[y,H] must be greater than zero. Let r be the map from Y to $(0,\infty)$ that equals dist[y,H], for all y in Y. T(r) is the convex hull of a set that is contained one of the closed half-spaces that is defined by H. Thus since closed half-spaces are convex, T(r) must be contained in a closed half-space defined by H. Since p is in bdy[hul(G)], p must be in clo[T(r)]. Thus since p is in H, p must be in bdy[T(r)]. ∎

Theorem 3: Characterization of I:

If C is nonempty, then

$$I = \text{int}[\text{hul}(G)] \cup Z$$

Proof of Theorem 3:

Suppose C is nonempty. Let A be a set in C. By Lemma 1, hul(G) must be contained in clo(A). Thus since A is convex, int[hul(G)] must be contained in int(A). Thus, int[hul(G)] must be contained in I.

Suppose p is a point in ext[hul(G)]. Say Y is empty. By Lemma 3a, hul(G) must be in C. So p cannot be in I. Now suppose Y is nonempty. By Theorem 1b, there is a set $T(r_C)$ in T that is in C. By Lemma 9, there is a set $T(r_e)$ in **T** for which p is in ext[$T(r_e)$]. Let r be the map from **Y** to (0,∞) that equals min[$r_e(y),r_c(y)$], for all y in **Y**. T(r) must be contained in both $T(r_e)$ and $T(r_c)$. Thus, by Lemma 7, T(r) must be in **C**. Also, p must be in ext[T(r)]. Thus p cannot be in **I**.

Now suppose p is a in bdy[hul(**G**)]. If **Y** is empty, then **Z** must be empty. Thus, p cannot be in **Z**. By Lemma 3a, hul(**G**) must be in **C**. Thus p cannot be in **I**. Suppose **Y** is nonempty.

If p is not in **I**, then there must be a set **A** in **C** for which p is not in int(**A**). By Lemma 3b, there is a set T(r) in **T** that is contained in clo(**A**). Since **A** is convex, int[T(r)] must be contained in int(**A**). Thus, p cannot be in int[T(r)]. Since hul(**G**) is contained in T(r), and p is in bdy[hul(**G**)], p must be in clo[T(r)]. Thus, p must be in bdy[T(r)]. By Lemma 10, p cannot be in **Z**.

By Lemma 10, if p is not in **Z**, then there is a set $T(r_b)$ in **T** for which p is in bdy[$T(r_b)$]. By Theorem 1, there is a set $T(r_c)$ in **T** that is in **C**. Let r be the map from **Y** to (0,∞) that equals min[$r_b(y),r_c(y)$], for all y in **Y**. T(r) must be contained in both $T(r_b)$ and $T(r_c)$. By Lemma 7, T(r) must be in **C**. Also, p cannot be in int[T(r)]. Thus, p cannot be in **I**. ∎

Theorem 4: Characterization of com(E):

A point p in R^n is in com(**E**) <u>iff</u>

a. **Y** is empty and hul(p,**G**) is in **C**, or

b. **Y** is nonempty and hul[p,T(r)] is in **C** for some set T(r) in **T**.

Proof of Theorem 4:

Let p be a point in R^n. If Y is empty and hul(p,G) is in C, then since p must be in hul(p,G), p cannot be in E. Similarly, if there is a set T(r) in T for which hul[p,T(r)] is in the class C, then p cannot be in E.

Suppose p is not in E. In this case, there must be a set A in C for which p is in clo(A). Suppose Y is empty. Since G is contained in clo(A), hul(p,G) must be contained in clo(A). Thus,

$$\text{hul}(G) \subset \text{hul}(p,G) \subset \text{clo}(A) \ .$$

By Lemmas 3a and 5, hul(G) and clo(A) must be in C. Thus, by Lemma 8, hul(p,G) must also be in C.

Now suppose Y is nonempty. By Lemma 3b, there is a set T(r) in T that is in C and contained in clo(A). Thus,

$$T(r) \subset \text{hul}[p,T(r)] \subset \text{clo}(A) \ .$$

Thus, by Lemma 8, hul[p,T(r)] must be in the class C. ∎

Result 1:

Suppose C is nonempty, G_i and G_e are empty, and G_b is nonempty and consists of m points.

a. In the 2-dimensional case, if $m<5$, then com(E) is unbounded. If $m\geq5$, then com(E) could be bounded.

b. In the 3-dimensional case, if $m<8$, then com(E) is unbounded. If $m\geq8$, then com(E) could be bounded.

c. In the n-dimensional case, if $m\geq2n+2$, then com(E) could be bounded.

Proof of Result 1:

Suppose C is nonempty, G_i and G_e are empty, and G_b is nonempty and consists of m points.

Suppose G_b is a subset of R^2. In particular, suppose G_b is given by

$$G_b = \{ (\cos k(2\pi/m), \sin k(2\pi/m)) \mid k \text{ is an integer in } [0,m-1] \} .$$

It can easily be shown that if $m \geq 5$, then com(E) will be bounded. [Note that it is not true that com(E) will be bounded for all choices of G_b for which $m \geq 5$. For example, consider the case where G_b is contained in the intersection of the unit circle and the first quadrant.]

Next we consider the case where G_b is a subset of R^n. Let S be a simplex in R^n. Suppose $m \geq 2n+2$. Suppose G_b contains the (n+1) vertices of S and the center of each of the (n+1) faces of S (the center of a face of a simplex is the average of the vertices on that face). It can easily be shown that com(E) must be S. [Note that it is not true that com(E) will be bounded for all choices of G_b for which $m \geq 2n+2$. For example, consider the case where G_b is contained in the intersection of the unit sphere and the first orthant of R^n.]

We would like to show that for certain values of m, the set com(E) will be unbounded. We shall make use of the fact that com(E) is unbounded when there is an unbounded set in C (recall that every set in C is contained in com(E)).

Let P be the polyhedron hul(G_b). If P is contained in an (n-1)-dimensional plane A, then A is an unbounded set in C. For the remainder of the proof we shall assume that P is an n-dimensional set. Since C is nonempty, G_b must be contained in bdy(P) (see Lemma 3a).

Suppose G_b is a subset of R^3 and we can show that G_b is contained in the union of three sets S_1, S_2, and S_3, where S_i is either a vertex, edge, or face of P. Let T_i be a closed half-space for which

$$P \subset T_i ,$$

$$S_i \subset bdy(T_i) .$$

Let A equal the intersection of T_1, T_2, and T_3. A must be unbounded (the intersection of n closed half-spaces in R^n is unbounded when the intersection is nonempty). Since G_b is contained in bdy(A) and A is convex, A must be in C. Thus, if we can find a list $L=(S_1,S_2,S_3)$, where S_i is either a vertex, edge, or face of P, and G_b is contained in the union of S_1, S_2, and S_3, then we may conclude that com(E) is unbounded. [Note that in the two dimensional case, if we can find a list $L=(S_1,S_2)$, where S_i is either a vertex or an edge of P, and G_b is contained in the union of S_1 and S_2, then we may conclude that com(E) is unbounded.]

First we show that in the 2-dimensional case, if $m<5$, then com(E) is unbounded. Suppose $m<5$ and G_b is a subset of R^2. Let v be the number of vertices of P. Say v=3. Let F be a face of P that contains all the points in G_b that are not vertices of P. Let p be the vertex of P that is not contained in F. Take L=(F,p). Say v=4. Let F_1 and F_2 be two opposite sides of the quadrilateral P. Take $L=(F_1,F_2)$.

Next, we show that in the 3-dimensional case, if $m<8$, then com(E) is unbounded. Suppose $m<8$ and G_b is a subset of R^3. Let v, e, and f be the number of vertices, edges, and faces of P. Say v=4. In this case, P has four faces F_1, F_2, F_3, and F_4. The faces of P may be labaled so that the intersection of rint(F_4) and G_b is empty. Take $L=(F_1,F_2,F_3)$.

Suppose v=5. Let F_1 and F_2 be faces of P for which the union of F_1 and F_2 contains all the points in G_b that are not vertices of P. The union of F_1 and F_2 must contain at least four vertices of P. If this union does not contain all five vertices of P, then let v_5 be the fifth vertex. Otherwise, let v_5 be any vertex of P. Take $L=(F_1,F_2,v_5)$.

Suppose v=6. Let F_1 be a face of P that contains all the points in G_b that are not vertices of P. First, we consider the case where all the faces of P are triangular. Let v_1, v_2, and v_3 be the vertices of P on F_1, and let v_4, v_5, and v_6 be the three remaining vertices of P. Either v_4v_5, v_5v_6, or v_4v_6 is an edge of P. [By Euler's theorem on polyhedra [24] v-e+f=2. Since all the faces of P are triangular, 3f=2e. By combining these relations with the fact that v=6, we find that e=12. If v_4v_5, v_5v_6, and v_4v_6 are not edges of P, then in order for P to have 12 edges, all the other segments that join two vertices of P must be edges. The resulting graph that consists of the vertices and edges of P is nonplanar. But this contadicts the fact that the graph of a polyhedron is always planar. Thus either v_4v_5, v_5v_6, or v_4v_6 is an edge of P.] Without loss of generality, we may assume that v_4v_5 is an edge of P. Take $L=(F_1,v_4v_5,v_6)$.

Next, suppose that v=6 and all the faces of P are triangular except for one which is a quadrilateral. Let F_2 be the four sided face. Let v_1, v_2, v_3 and v_4 be the vertices of P on F_2, and let v_5 and v_6 the other vertices of P. v_5v_6 must be an edge of P. [In this case, $2e=4+3f_3$, where f_3 is the number of triangular faces of P. By combining this with Euler's theorem and v=6, we find that e=11. If v_5v_6 is not an edge of P, then in order for P to have 11 edges, all but one of the segments that join two vertices of P must be edges. It can be shown that this is a nonplanar graph.] Take $L=(F_1,F_2,v_4v_5)$.

If P has two quadrilateral faces F_2 and F_3, then take $L=(F_1,F_2,F_3)$.

Suppose P has a pentagonal face F_2. Let v_6 be the vertex of P that is not in F_2. Take $L=(F_1,F_2,v_6)$.

Next, we consider the case where v=7. Suppose all the faces of P are triangular. Let F be one of the faces of P. Let v_1, v_2, and v_3 be the vertices of P on F_1, and let v_4, v_5, v_6, and v_7 be the four remaining vertices of P. If the graph for P is not given by Fig.*1,

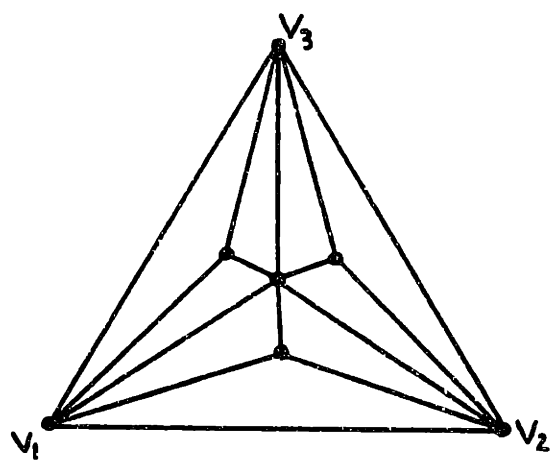


Fig.*1

then either v_4v_5 and v_6v_7, or v_4v_6 and v_5v_7, or v_4v_7 and v_5v_6 are edges of P. If the graph of P is given by Fig.*1, then take L=(face $v_1v_4v_5$, v_3v_7, v_2v_6). If the graph of P is not given by Fig.*1, then we may assume without loss of generality that v_4v_5 and v_6v_7 are edges of P. In this case take $L=(F,v_4v_5,v_6v_7)$.

[Let D be the set of edges of P. Let $D_1=\{v_4v_5,v_6v_7\}$, $D_2=\{v_4v_6,v_5v_7\}$, and $D_3=\{v_4v_7,v_5v_6\}$. We shall show that if D_1, D_2, and D_3 are not subsets of D, then the graph of P is given by Fig.*1.

Since all the faces of P are triangular, $3f=2e$. By combining this with Euler's theorem and $v=7$, we find that $e=15$. Since the triangle $v_1v_2v_3$ is a face of P, the graph of P may be drawn within a triangle labeled $v_1v_2v_3$.

Case 1: v_5v_7 and v_6v_7 are in D, and D_3 and D are disjoint. In order to have 15 edges, we must have two vertices in the set $\{v_4,v_5,v_6,v_7\}$ connected to each of the vertices in the set $\{v_1,v_2,v_3\}$. But this leads to a nonplanar graph in the triangle $v_1v_2v_3$. Thus case 1 cannot be true.

Case 2: v_4v_5, v_4v_6, and v_5v_6 are in D. In this case v_7 must be connected to each of the vertices in the set $\{v_1,v_2,v_3\}$ (v_7 must have three edges). Without loss of generality, we may assume that the triangle $v_4v_5v_6$ is contained in the triangle $v_1v_2v_7$. v_3v_4, v_3v_5, and v_3v_6 cannot be edges. Thus in order to have 15 edges, each vertex in the set $\{v_4,v_5,v_6\}$ must be connected to each of the vertices in the set $\{v_1,v_2\}$. But this leads to a nonplanar graph. Thus case 2 cannot be true.

Case 3: v_4v_5, v_4v_6, and v_4v_7 are in D.

Case a: v_3v_4 is not in D. In this case v_1v_4 and v_2v_4 must be in D (otherwise two of the vertices in the set $\{v_5,v_6,v_7\}$ would have to be connected to each of the vertices in the set $\{v_1,v_2,v_3\}$, and this is a nonplanar graph). It can be shown that in order to obtain 15 edges, we must generate a nonplanar graph.

Case b: v_1v_4, v_2v_4, v_3v_4 are in D. In this case, it can be shown that the graph of P is given by Fig.*1.]

Next, suppose that v=7 and all the faces of P are triangular except for one which is a quadrilateral. Let F be the four sided face. Let v_1, v_2, v_3 and v_4 be the vertices of P on F, and let v_5, v_6, and v_7 be the other vertices of P. Either v_5v_6, v_6v_7, or v_5v_7 must be an edge of P. Without loss of generality, we may assume that v_5v_6 is an edge. Take $L=(F,v_5v_6,v_7)$. [2e=4+3f_3, where f_3 equals the number of triangular faces of P. By combining Euler's theorem with v=7, we find that e=14. Since F is a face of P, we can draw the graph for P within a quadrilateral labeled $v_1v_2v_3v_4$. Suppose v_5v_6, v_6v_7, and v_5v_7 are not edges of P. In order to get 14 edges, one of the vertices in the set $\{v_5,v_6,v_7\}$ must be connected to each of the vertices in the set $\{v_1,v_2,v_3,v_4\}$. Then one of the vertices in the set $\{v_6,v_7\}$ must be connected to three of the vertices in the set $\{v_1,v_2,v_3,v_4\}$. This is a nonplanar graph.]

Suppose v=7 and P has two quadrilateral faces F_1 and F_2. Let v_7 be the vertex of P that is not in the union of F_1 and F_2 (if all the vetices of P are in the union of F_1 and F_2, then let v_7 be any vertex of P). Take $L=(F_1,F_2,v_7)$.

If P has a pentagonal or hexagonal face F then take L=(F, the vertices of P that are not in F).

[Note that most of this proof consists of a case-by-case analysis of the different triangulations of the plane. It might be possible to obtain a more efficient proof by using the ideas in [70] and [29].] ∎

Lemma 13:

Let p be a point in R^n and let A_1 and A_2 be sets in C. If p is in both ext(A_1) and int(A_2), then there is a set A in C for which p is in bdy(A).

Proof of Lemma 13:

Let p be a point in R^n and let A_e and A_i be sets in C. Suppose p is in both ext(A_e) and int(A_i).

By Lemma 5, clo(A_e) and clo(A_i) must be in C.

First, we shall consider the case where Y is empty. We shall show that hul[p,hul(G)] is in C and the point p is in bdy[hul[p,hul(G)]]. By Lemma 3a, hul(G) must be in C. By Lemma 1, hul(G) must be contained in clo(A_i). Thus since p is in clo(A_i), and clo(A_i) is convex, hul[p,hul(G)] must be contained in clc(A_i). Thus, we have

$$\text{hul}(G) \subset \text{hul}[p,\text{hul}(G)] \subset \text{clo}(A_i) \quad ,$$

where hul(G) and clo(A_i) are in C. By Lemma 8, it follows that hul[p,hul(G)] is in C.

Suppose p is in int[hul(G)]. By Lemma 1, hul(G) must be contained in clo(A_e). Thus, int[hul(G)] must be contained in int[clo(A_e)]. Hence, since A_e is convex, int[hul(G)] must be contained in int(A_e). Thus, p must be in int(A_e). But this contradicts the fact that p is in ext(A_e). Thus, it must be the case that p is not in int[hul(G)]. Since hul(G) is convex, it follows that p is in bdy[hul[p,hul(G)]].

Next, we shall consider the case where Y is nonempty. We shall show that there is a set T(r) in T for which hul[p,T(r)] is in C, and p is in bdy[hul[p,T(r)]]. By Lemma 3b, there are two sets T(r_e) and T(r_i) in T that are in C, and contained in clo(A_e) and clo(A_i), respectively.

Let r be the map from Y to $(0,\infty)$ that equals $\min[r_e(y), r_i(y)]$, for all y in Y. Since $r(y) \leq r_i(y)$ for all y in Y, T(r) is contained in $T(r_i)$. By Lemma 7, it follows that T(r) must also be in C. Also T(r) must be contained in $clo(A_i)$. Thus since p is in $clo(A_i)$, and $clo(A_i)$ is convex, hul[p,T(r)] must be contained in $clo(A_i)$. Thus, we have

$$T(r) \subset hul[p,T(r)] \subset clo(A_i) \;,$$

where T(r) and $clo(A_i)$ are in C. By Lemma 8, it follows that hul[p,T(r)] is in C.

Suppose p is in int[T(r)]. Since $r(y) \leq r_e(y)$, for all y in Y, T(r) is contained in $T(r_e)$. Thus T(r) must be contained in $clo(A_e)$. Hence, int[T(r)] is contained in $int[clo(A_e)]$. Thus, since A_e is convex, int[T(r)] must be contained in $int(A_e)$. Hence, p is in $int(A_e)$. But this contradicts the fact that p is in $ext(A_e)$. Thus it must be the case that p is not in int[T(r)]. Since T(r) is convex, it follows that p is in bdy[hul[p,T(r)]]. ∎

APPENDIX 3. PROOFS FOR CHAPTER 3

This appendix contains the proofs for the results given in Chapter 3. For convenience, we shall also restate those results here. The results (and their proofs) that were proved within the text of Chapter 3 will not be discussed here. We shall use (x) and Fig.x to denote equation number x and figure number x from Chapter 3, respectively. Additional equations and figures will be denoted by (*x) and Fig.*x, respectively.

Theorem 1: Characterization of $\mathbf{X}$:

$$\mathrm{int}(\mathbf{X}) = \bigcup_{p \in \mathrm{bdy}(G)} \mathbf{X}_i(p) , \tag{15a}$$

$$\mathrm{ext}(\mathbf{X}) = \bigcup_{p \in \mathrm{bdy}(G)} \mathbf{X}_e(p) , \tag{15b}$$

$$\mathrm{bdy}(\mathbf{X}) = \bigcup_{p \in \mathrm{bdy}(G)} \mathbf{X}_b(p) . \tag{15c}$$

Proof of Theorem 1:

Let ξ be the invertible map from bdy(G) into $[0,2\pi)$ that is given by

$\xi(p)$ = the angle ω of the support line $f(p)f^2(p)$ of the set F.

[The angle of a support line s for a convex planar set A is the angle of a normal to s, pointing away from A.]

For a given angle ω in the interval $[0,2\pi)$, let $\mathbf{X}_i(\omega)$, $\mathbf{X}_e(\omega)$, and $\mathbf{X}_b(\omega)$ denote the sets $\mathbf{X}_i(\xi^{-1}(\omega))$, $\mathbf{X}_e(\xi^{-1}(\omega))$, and $\mathbf{X}_b(\xi^{-1}(\omega))$. The following three equations are equivalent to Theorem 1:

$$\text{int}(X) = \cup X_i(\omega)$$

$$\text{bdy}(X) = \cup X_b(\omega)$$

$$\text{ext}(X) = \cup X_e(\omega)$$

where each union is taken over all ω in the interval $[0,2\pi)$. We shall now prove this equivalent form of Theorem 1.

Part 1:

In the first part of the proof, we shall show that the sets $X_i(\omega)$, $X_b(\omega)$, and $X_e(\omega)$ are contained in the sets int(X), bdy(X), and ext(X), respectively, for each ω.

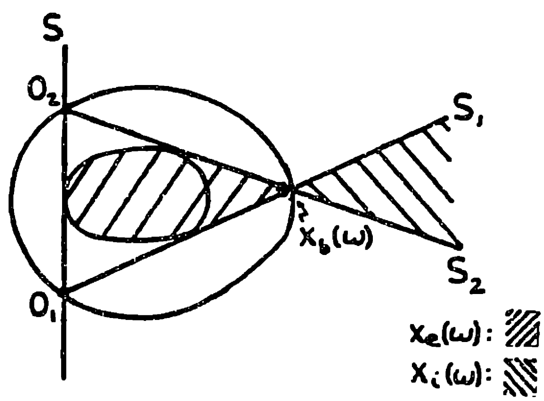


Fig.*1

a.1) The set int(X) contains $X_i(\omega)$, for all ω. Refer to Fig.*1. Suppose P is in $X_i(\omega)$ for some ω. Clearly, the triangle PO_1O_2 contains F. Thus P must be a point in X. Since $X_i(\omega)$ is an open set, this implies that every point in $X_i(\omega)$ must be in the interior of X.

c.1) The set ext(X) contains $X_e(\omega)$, for all ω. Suppose P is a point in $X_e(\omega)$ for some ω. If P is a point in the int(F), then it must be in the exterior of X. Thus, suppose P is not in int(F). We shall show that there can be no points P_1 and P_2, in G, such that the triangle PP_1P_2

contains **F**.

Refer to Fig.*2. Let T, T_1, and T_2 be points where s, s_1, and s_2 are tangent to **F**, respectively. Since **F** is convex, it must contain the triangle TT_1T_2. So, any triangle that contains **F**, must also contain the triangle TT_1T_2. Consider the disjoint regions A and B defined in

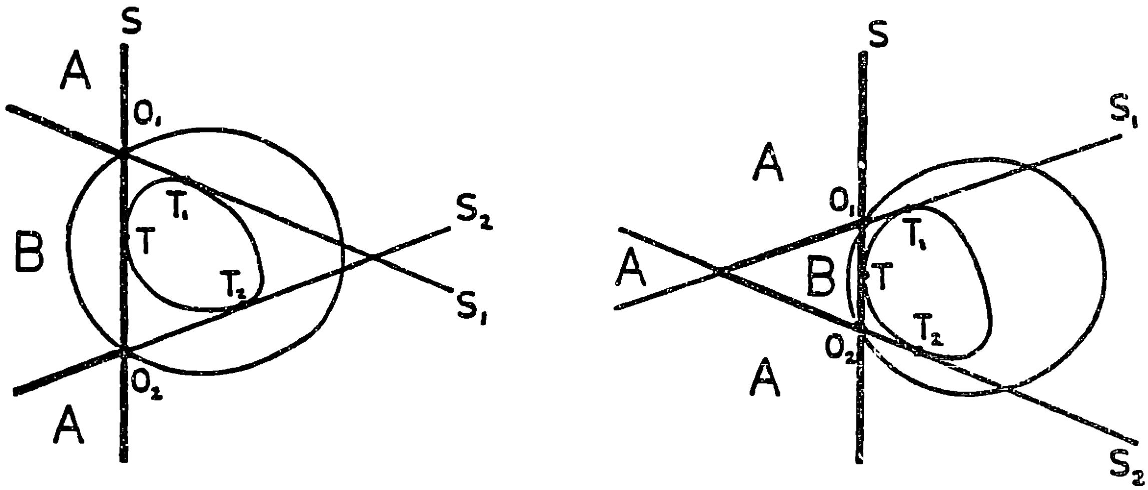


Fig.*2a

Fig.*2b

Fig.*2. (Except for the points O_1 and O_2, region A contains its boundary. Region B contains the segment O_1O_2.) Region A cannot contain any points from **G**. Hence, P_1 or P_2 cannot be placed in region A. If P_1 or P_2 is placed in region B, then the triangle PP_1P_2 will not contain the triangle TT_1T_2. Hence, P, P_1, and P_2 must all lie on the same side of the supporting line s. As a result, the triangle PP_1P_2 cannot contain the triangle TT_1T_2, and thus cannot contain **F**. So, for any point P in $\mathbf{X}_e(\omega)$, there are no triangles PP_1P_2 with P_1 and P_2 in **G**, that contain the oval **F**. In other words, $\mathbf{X}_e(\omega)$ is contained in the complement of **X** for each ω. Since $\mathbf{X}_e(\omega)$ is an open set, this implies that every point in $\mathbf{X}_e(\omega)$ must be in the exterior of **X**.

b.1) The set bdy(X) contains $X_b(\omega)$, for all ω. Refer to Fig.*2a. For each ω, $X_i(\omega)$ is contained in X and $X_e(\omega)$ is contained in the complement of X. Thus, it is clear that $X_b(\omega)$ must be a boundary point of X, for all ω.

Part 2:

In this part of the proof, we shall show that the sets int(X), bdy(X), and ext(X) are contained in the sets $\cup X_i(\omega)$, $\cup X_b(\omega)$, and $\cup X_e(\omega)$, respectively. To simplify the proof, first we shall classify the points in R^2. For each class, we shall show that membership in the class implies membership in one of the sets: $X_i(\omega)$, $X_b(\omega)$, or $X_e(\omega)$, for some ω, which, by the results in Part 1, implies membership in the sets int(X), bdy(X), and ext(X), respectively.

Let P by a point in R^2.

Class 1: P is a point in Class 1 if it is either a point in the interior of F or a point on the boundary of F and there is a unique support line for F at P. Every point in Class 1 is contained in $X_e(\omega)$, for some ω. [This is obviously true when P is a point in the interior of F. Thus, suppose P is a boundary point for F. In this case, we may use the following construction: Refer to Fig.*3. Let s be the support line for F that is parallel to the support line at P. Let ω be the angle of the support line s, let T be a point where s is tangent to F, and let O_1 and O_2 be the points where s intersects the boundary of G. Let s_1 (s_2) be the support line for F, distinct from s, passing through O_1 (O_2). No support line for F can intersect the open segment PT. Since there is only one support line for F at the point P, s_1 and s_2 cannot pass through P. Thus, P must be contained in the set $X_e(\omega)$.] Thus, by c.1 from Part 1, every point in Class 1 must be contained in the set ext(X).

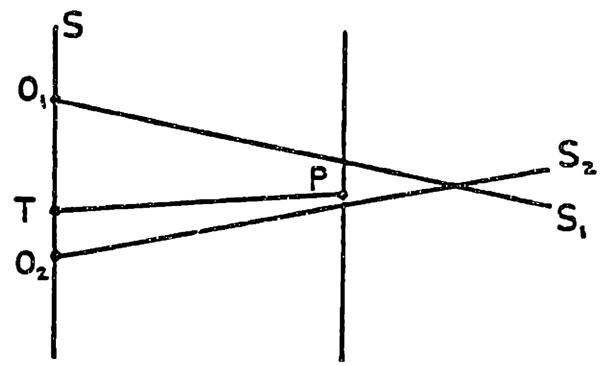


Fig.*3

Class 2: P is a point in Class 2 if it is either a point in the complement of F or a point on the boundary of F and there is more than one support line for F at P. Class 2 must be subdivided because some of the points in this class may be in $X_e(\omega)$, for some ω, while other points may be in either $X_b(\omega)$, for some ω, or $X_i(\omega)$, for some ω. We shall use the following construction to subdivide Class 2: Refer to Fig.*4. Let r_1 and r_2 be the two support lines

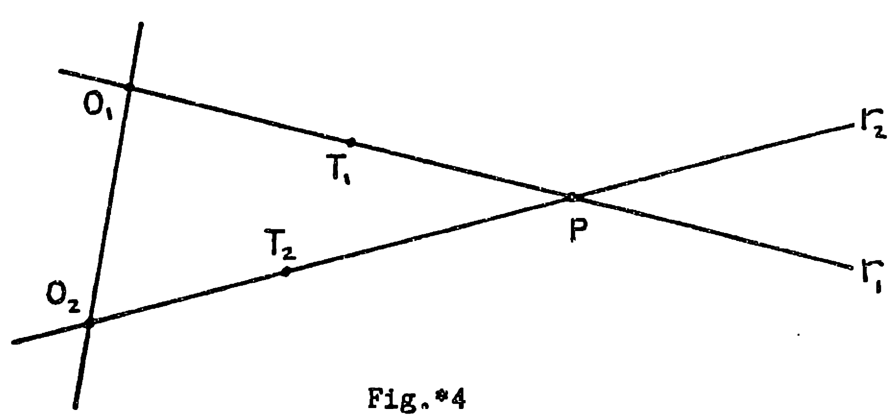


Fig.*4

for F passing through P. (In the case when P is a boundary point of F, let r_1 and r_2 be the support lines for F, at the point P, that are also tangent to F at another point.) Let T_1 and T_2 be points where r_1 and r_2 are tangent to F, respectively. r_1 and r_2 must intersect the boundary of G at points O_1 and O_2 as shown in Fig.*4.

Class 2.1: P is a point in Class 2.1 if the line O_1O_2 does not intersect F. Every point in Class 2.1 is contained in $X_1(\omega)$, for some ω. [This can be shown by completing the construction in Fig.*4 in the following way: Refer to Fig.*5a. Let s be the support line for F that is parallel to the line O_1O_2 and that separates O_1O_2 from F. Let ω be the angle of the support line s. Let O_3 and O_4 be the points where s intersects the boundary of G. O_3 and O_4 cannot lie on the segment A_1A_2 defined in Fig.*5a.

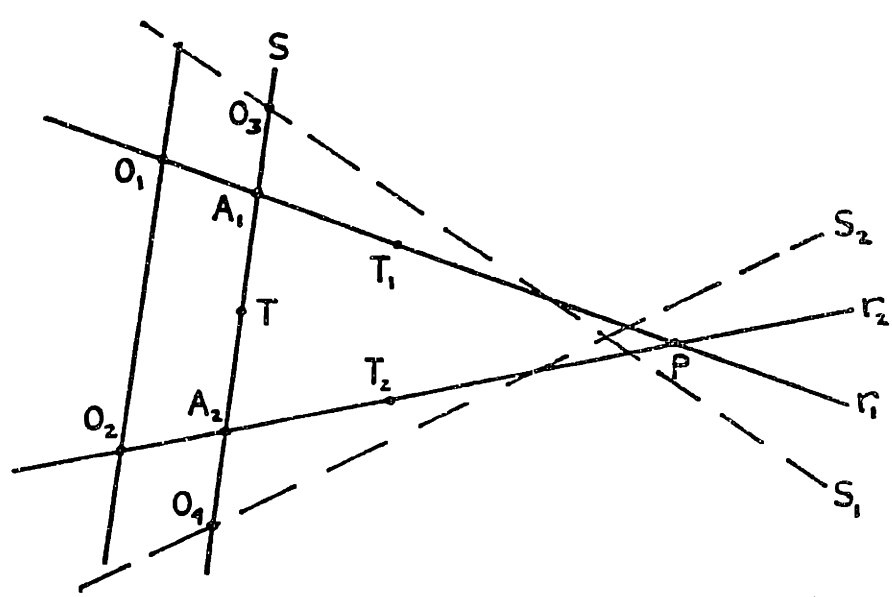


Fig.*5a

Thus, O_3 and O_4 must be as shown in Fig.*5a. Let s_1 (s_2) be the support line for F, distinct from s, passing through O_3 (O_4). By considering the possible angles for s_1 and s_2, we see that P must be a point in $X_i(\omega)$.] Thus, by a.1 from Part 1, every point in Class 2.1 must be contained in the set int(X).

Class 2.2: P is a point in Class 2.2 if the line O_1O_2 is a support line for F. Every point in Class 2.2 is equal to $X_b(\omega)$, for some ω. Thus, by b.1 from Part 1, every point in Class 2.2 must be contained in the set bdy(X).

Class 2.3: P is a point in Class 2.3 if the line O_1O_2 intersects the interior of F. Every point in Class 2.3 is contained in $X_c(\omega)$, for some ω. [This can be shown by completing the construction in Fig.*4 in the following way: Refer to Fig.*5b.

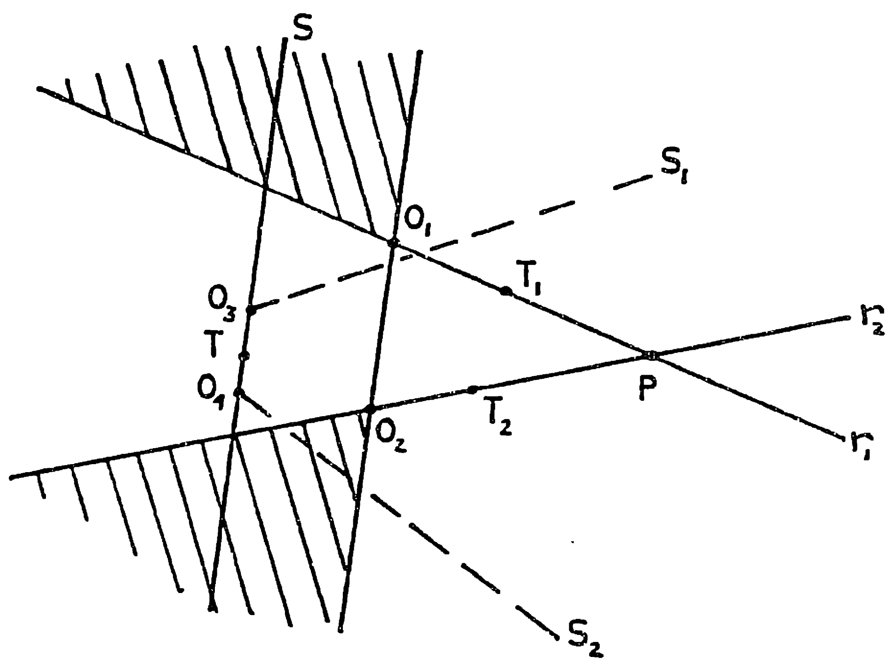


Fig.*5b

Let s be the support line for F that is parallel to the line O_1O_2 and is on the side of O_1O_2 that does not contain P. Let T be a point where the support line s is tangent to F and let ω be the angle of s. Let O_3 and O_4 be the points where s intersects the boundary of G. G cannot have any points in the shaded regions in Fig.*5b. Thus, O_3 and O_4 must be as shown in Fig.*5b. Let s_1 (s_2) be the support line for F, distinct from s, passing through O_3 (O_4). By considering the possible angles for s_1 and s_2, we see

that P must be a point in $X_e(\omega)$.] Thus, by c.1 from Part 1, every point in Class 2.3 must be contained in the set ext(X).

It is clear that any point P in R^2, must be a point in one of these classes. We shall now use this classification of the points in R^2 to complete the proof.

a.2) If P is a point in the set int(X), then it must be contained in $X_i(\omega)$, for some ω. If P is a point in the interior of X, then it cannot be a point in Classes 1, 2.2, or 2.3 because membership in these classes, implies membership in the sets ext(X) or bdy(X). Thus, P must be in Class 2.1, in which case, P must be contained in $X_i(\omega)$, for some ω.

b.2) If P is a point in the set bdy(X), then P must equal $X_b(\omega)$, for some ω. If P is a point on the boundary of X, then P cannot be a point in Classes 1, 2.1, or 2.3 because membership in these classes, implies membership in the sets ext(X) or int(X). Thus, P must be in Class 2.2, in which case, P must equal $X_b(\omega)$, for some ω.

c.2) If P is a point in the set ext(X), then it must be contained in $X_e(\omega)$, for some ω. If P is a point in the exterior of X, then it cannot be in the Classes 2.1 or 2.2 because membership in these classes, implies membership in the sets bdy(X) or int(X). Thus, P must be in Classes 1 or 2.3, in which case, P must be contained in $X_e(\omega)$, for some ω. ∎

Result 1: The Number of Conic Arcs in bdy(V):

Let f and g be the number of vertices of **F** and **G**, respectively. The curve bdy(V) consists of at most 3f + 2g nondegenerate conic arcs.

Proof of Result 1:

Extend the edges of **F** until they intersect bdy(**G**). Mark these points of intersection. As we rotate the triangle specified in Theorem 1 around **F** to obtain bdy(**V**), the conic changes if and only if one of the endpoints of the leg of the triangle that is constrained to be a chord of **G** meets one of the marked points or one of the vertices of **G**. This happens twice for each marked point; once for each end of the rotating chord. That is, 4f + 2g of these events occur. But we have over counted by f, because on f occasions the chord meets two of the marked points simultaneously. Thus, there can be at most 3f + 2g conic arcs. ∎

We shall use the following definitions and results in our proofs of Theorems 1 and 2:

Definition:

Let A be a compact convex subset of R^2. Let a and b be points in bdy(A) and let d be a point in int(A). Refer to Fig.*6. We define the $arc_A ab$ to be the set of boundary points of A obtained as ray r sweeps from da to db in a counterclockwise way.

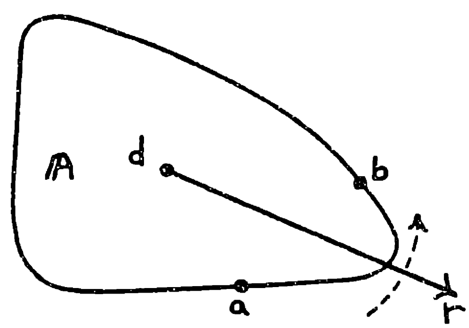


Fig.*6

We shall denote $arc_G ab$ by arc(ab).

Properties of f:

1. f is a continuous function.
2. Let a, b, and c be three distinct points in bdy(G). b is in arc(ac) <u>iff</u> f(b) is in arc(f(a)f(c)).

Theorem 2: The Periodic Points of f:

All of the periodic points of the function f that is associated with a particular pair of compact convex sets F and G that satisfy (12) are of the same period and type.

Proof of Theorem 2:

Let x be an n-periodic point. An example is given in Fig.*7 to illustrate the definitions that we shall introduce in this proof.

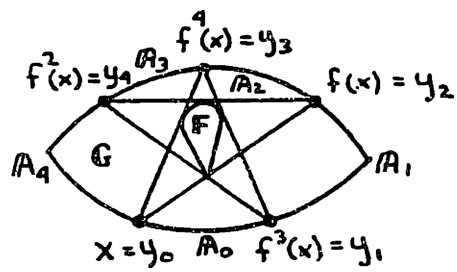


Fig.#7

Starting with x and going counterclockwise around bdy(G), label the points x, f(x), $f^2(x)$,..., $f^{n-1}(x)$ as y_0, y_1, y_2,..., y_{n-1}. It can be shown that

$$f(y_j) = y_{(j+r) \bmod n} ,$$

for some fixed interger r in [1,n/2] for which gcd(r,n)=1 (for the example shown in Fig.#7, n=5 and r=2). For i=0,1,...,n-1, let A_i be the set given by

$$A_i = \mathrm{arc}(y_i y_{(i+1) \bmod n}) - \{y_i, y_{(i+1) \bmod n}\} .$$

Let z be a point in A_j for some j in [0,n-1]. We shall show that z cannot be an m-periodic point for m<n. From Property 2, we have that

$$f(A_j) = A_{(j+r) \bmod n} .$$

It follows that

$$f^i(A_j) = A_{(j+ir) \bmod n} .$$

Thus since z is in A_j, $f^i(z)$ must be in $A_{(j+ir) \bmod n}$. Since gcd(r,n)=1, the numbers (j)mod n, (j+r)mod n, (j+2r)mod n,..., and (j+(n-1)r)mod n, must be distinct. Thus the sets $A_{(j) \bmod n}$, $A_{(j+r) \bmod n}$,...,

$A_{(j+(n-1)r) \bmod n}$ must be disjoint. Thus $f^i(z) \neq z$, for all i in [1,n-1].

Suppose z is an n-periodic point of f. Since $f^i(z)$ must be in $A_{(j+ir) \bmod n}$, we may conclude that the n-periodic points x and z are of the same type. ∎

Theorem 3: The Asymptotically Periodic Points of f:

Suppose the function f that is associated with a particular pair of compact convex sets **F** and **G** the satisfy (12) has a k-periodic point.

a. Then, every point in the domain of f is an asymptotically k-periodic point of f.

b. Let p_1p_2 be an open segment of bdy(**G**) for which p_1 and p_2 are k-periodic points of f and p_1p_2 does not contain any periodic points of f. Let f_r denote the restriction of f to bdy(**G**) (so that f_r is invertible, though f is not). If p is a point in the segment p_1p_2, then either $\{f_r^{ki}(p)\}$ converges to p_1 and $\{f_r^{-ki}(p)\}$ converges to p_2, or $\{f_r^{ki}(p)\}$ converges to p_2 and $\{f_r^{-ki}(p)\}$ converges to p_1. In addition, both of the sequences converge monotonically with respect to the curve bdy(**G**).

Proof of Theorem 3:

Let p_1 and p_2 be points in bdy(**G**) that are k-periodic points of f. Suppose the set $arc(p_1p_2) - \{p_1,p_2\}$ does not contain any periodic points of f. In this proof, we shall use f to denote the function f_r defined in Theorem 3.

By using Property 2, it can be shown that

If $f^k(p)$ is in arc(pp_2), then (*1a)

$f^{k(i+1)}(p)$ is in arc($f^{ki}(p)p_2$) - $\{f^{ki}(p),p_2\}$, for $i \geq 0$,

$f^{k(i-1)}(p)$ is in arc($p_1f^{ki}(p)$) - $\{p_1,f^{ki}(p)\}$, for $i \leq 0$,

If $f^k(p)$ is in arc(p_1p), then (*1b)

$f^{k(i+1)}(p)$ is in arc($p_1f^{ki}(p)$) - $\{p_1,f^{ki}(p)\}$, for $i \geq 0$,

$f^{k(i-1)}(p)$ is in arc($f^{ki}(p)p_2$) - $\{f^{ki}(p),p_2\}$, for $i \leq 0$.

Let b be a point in the set arc(p_2p_1) - $\{p_1,p_2\}$. Let t be a point in int(G). Let the angle of a point in bdy(G) be the angle (in radians) between the rays tb and ty measured counterclockwise from the ray tb.

Let α_i be given by

$$\alpha_i = \text{the angle of } f^{ki}(p).$$

From (*1) we have,

If $f^k(p)$ is in arc(pp_2), then

$0 < \text{angle}(p_1) < \ldots < \alpha_{-1} < \alpha_0 < \alpha_1 < \ldots < \text{angle}(p_2) < 2\pi$,

If $f^k(p)$ is in arc(p_1p), then

$0 < \text{angle}(p_1) < \ldots < \alpha_1 < \alpha_0 < \alpha_{-1} < \ldots < \text{angle}(p_2) < 2\pi$.

It can be shown that $\inf\{\alpha_i\}=p_1$ and $\sup\{\alpha_i\}=p_2$. [Suppose $\inf\{\alpha_i\}=\beta$, for some β for which angle(p_1)<β<angle(p_2). The sequence $\{\alpha_i\}$ must converge to some angle δ for which α<δ<angle(p_2). Thus the sequence $\{f^i(p)\}$ must converge to a point c of bdy(G) with angle δ (it can be shown that if the angles of a sequence of boundary points converge to δ, then the boundary points converge to the point in bdy(G) with angle δ). But this implies that c is a k-periodic point of f

(asymptotically periodic points of a continuous function converge to periodic points, and f is continuous). But since c is in $arc(p_1p_2) - \{p_1,p_2\}$, we have a contradiction. Thus $\inf\{\alpha_i\}=p_1$. Similarly, we can show that $\sup\{\alpha_i\}=p_2$.]

Either

$$\{\alpha_i\} \text{ converges to angle}(p_2), \text{ for } i>0, \text{ and to angle}(p_1), \text{ for } i<0, \text{ or}$$

$$\{\alpha_i\} \text{ converges to angle}(p_1), \text{ for } i>0, \text{ and to angle}(p_2), \text{ for } i<0.$$

It follows that either

$$\{f^{ki}(p)\} \text{ converges to } p_2, \text{ for } i>0, \text{ and to } p_1, \text{ for } i<0, \text{ or}$$

$$\{f^{ki}(p)\} \text{ converges to } p_2, \text{ for } i>0, \text{ and to } p_1, \text{ for } i<0.$$ ∎

Result 2: The Number of CI Triangles:

A pair of convex polygons F and G that satisfy (12) will only have a finite number of CI triangles.

Proof of Result 2:

Let N be the number of points in the intersection of the boundary of **X** and the boundary of **G**. The number of CI triangles equals N/3. Thus, it is sufficient to show that N is finite. By using an argument that is similar to that which was used in the proof of Result 1, it can be shown that the boundary of **X** is formed from at most 3f + 2g nondegenerate conic arcs, where f and g are the number of vertices of the polygons **F** and **G**, respectively. Since none of these conics is a line, and the boundary of **G** is the union of g line segments, each conic can intersect the boundary of **O** at most 2(g) times. Thus, N must be finite. ▣

Lemma 1: The Structure of X with respect to Rays Emitted from int(F):

Let q be a point in int(**F**), let p be a point in bdy(**F**), and let $r(\lambda)$ be the point $(1-\lambda)q + \lambda p$. Either

a. $r(\lambda) \in com(X)$, for all $\lambda \geq 0$, or

b. $r(\lambda) \dot{\in} com(X)$, for $0 \leq \lambda < \lambda_0$,
$\in B$, for $\lambda = \lambda_0 \geq 1$,
$\in X$, for $\lambda \geq \lambda_0$, or

c. $r(\lambda) \dot{\in} com(X)$, for $0 \leq \lambda < 1$,
$\in X$, for $\lambda \geq 1$.

Proof of Lemma 1:

Suppose (a) and (b) are not true. Then it must be the case that $r(1)$ is in com(**X**), and $r(\lambda_1)$ is in **X** for some $\lambda_1 > 1$.

It can be shown that if $r(\lambda)$ is in com(**X**), then $r(\lambda+\varepsilon)$ is in com(X) for some $\varepsilon > 0$. [Let $c=r(\lambda_c)$ be a point in com(**X**). Let K be the set given by

$$\mathbf{K} = \{ (p_1,\ldots,p_n) \mid p_i \text{ is in } \mathbf{K} \text{ for all } i \} .$$

K is a compact subset of R^{n^2}. Let v be the mapping from K to R that is given by

$$v(p_1,\ldots,p_n) = vol[\ hul(c,p_1,\ldots,p_n) \cap \mathbf{F}\ ] ,$$

where vol[A] denotes the volume of a set A. Since v is a continuous function on the compact set K, v attains a maximum value, call it v_{max}. Since c is not in X, $v_{max} < vol(F)$. For $\lambda \geq \lambda_c$, let $\Delta(\lambda)$ be the function given by

$$\Delta(\lambda) = \max_{K} [\ vol[\ hul(r(\lambda),p_1,\ldots,p_n)\ ] - vol[\ hul(c,p_1,\ldots,p_n)\ ]\ ] \ .$$

The function Δ is continuous. In addition, $\Delta(\lambda_c)=0$. Thus there is an $\varepsilon>0$ such that

$$\Delta(\lambda_c+\varepsilon) + v_{max} < vol(F) \ .$$

For this value of ε

$$vol[\ hul(r(\lambda_c+\varepsilon),p_1,\ldots,p_n) \cap F\ ] < vol(F) \ .$$

Thus, $r(\lambda_c+\varepsilon)$ is in com(X).]

It can also be shown that the intersection of X-B and $r(\lambda)$ is an open subset of $\{r(\lambda)\}$. [Suppose $c=r(\lambda_c)$ is a point in X-B. Since c is in X there are points $p_1,\ldots,p_n$ in G for which the set $hul(c,p_1,\ldots,p_n)$ contains F. From the fact that c is not in B, it can be shown that there is an $\varepsilon>0$ for which $hul(r(\lambda-\varepsilon),p_1,\ldots,p_n)$ contains F.]

It follows that $\{r(\lambda)\}$ cannot be written as the union of points in com(X) and X-B. So $\{r(\lambda)\}$ must intersect B. It can shown that this intersection can only contain one point, say $r(\lambda_0)$. It can also be shown that $r(\lambda)$ must be in com(X), for $0 \leq \lambda < \lambda_0$, and $r(\lambda)$ must be in X, for $\lambda \geq \lambda_0$. ∎

Lemma 2: The Directional Wideness of X and com(X):

Let p be a point in R_n.

a. $p \in X \longrightarrow pen(p,S) \subset X$, for all S in C_p.

b. $p \in com(X) \dashrightarrow hul[p,int(F)] \subset com(X)$.

Proof of Lemma 2:

Suppose p is a point in X and S is a simplex in C_p. Let the vertices of $p,p_1,\ldots,p_n$ be the vertices of S. Let q be a point in pen(p,S). It can be shown that $hul(q,p_1,\ldots,p_n)$ contains S which in turn contains F. Thus q must be in X.

Suppose p is in com(X). Let q be a point in hul[p,int(F)]. Suppose q is in X. Let S be a simplex in C_q. Since p is in pen(q,S) and q is in X, p must be in X (Lemma 2a). But this contradicts the fact that p is in com(X). Thus q must be in com(X). ∎

Theorem 4: Characterization of X:

$$bdy(X) = B \cup [X \cap bdy(F)] \ , \tag{27a}$$

$$int(X) = \bigcup_{p \in bdy(X)} int[pen(p,F)] \ . \tag{27b}$$

Proof of Theorem 4:

Since int(F) is contained in com(X), the intersection of X and bdy(F) must be contained in bdy(X). If c is a point in B, then construct a ray $\{r(\lambda)\}$ through c, as in Lemma 1. From Lemma 1, we see that c must be in bdy(X).

By combining Lemmas 1 and 2, it can be shown that if a point c in bdy(X) is not in the intersection of X and bdy(F), then c must be in B.

Suppose p is in bdy(X). p is either in B or the intersection of X and bdy(F). In each case, p must be in X. Thus, pen(p,F) must be contained in X. So int[pen(p,F)] must be contained in int(X).

Let c be a point in int(X). c cannot be in B or bdy(F). Construct a ray {r(λ)} through c, as in Lemma 1. From Lemma 1, there must be a point d in {r(λ)} that is either in B or in the intersection of X and bdy(F). c must be in int[pen(p,F)]. ∎

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