SPACETIME STRUCTURE OF THE VERY EARLY UNIVERSE

by

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ABSTRACT

We investigate two different facets of the spacetime structure of the very early universe.

We determine the stability of finite temperature Kaluza - Klein theories built upon spacetimes whose internal manifolds are toroidally compact by studying the one loop contribution to the effective potential of bosons, fermions, and gravitons. The addition of twisted bosons (usually) or untwisted fermions (always) can stabilise these manifolds at sufficiently low temperatures. In all cases, however the manifolds are unstable if the temperature is above some critical value set by the mass of the matter fields.

We explore the dynamics of spherically (O(3)) symmetric universes consisting of a region of false vacuum separated from an infinite region of true vacuum by a domain wall. Sufficiently massive false vacuum bubbles can inflate to arbitrarily large volumes without moving into the true vacuum. A consideration of the non-Euclidean geometry of the spacetime under study elucidates this pheneomenon.

CONTENTS

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INTRODUCTION

The two papers which comprise the body of this thesis describe rather different facets of the spacetime structure of the early universe. The first discusses Kaluza - Klein theories with matter built upon a spacetime whose macroscopic part is Minkowski space (M^4) and whose internal part is toroidally compact. The second is a study of the dynamics of a spherically (O(3)) symmetric universe consisting of a false vacuum bubble separated from an infinite region of true vacuum by a domain wall.

Kaluza - Klein theories describe gravitation in higher dimensional spacetimes. In these theories spacetime is taken to be the direct product of a four dimensional spacetime with some as yet undetected N dimensional compact space. Observers living in this extended spacetime would interpret certain components of the $4+N$ dimensional metric tensor as gauge fields on the macroscopic four dimensional spacetime. In this manner, Kaluza - Klein theories unify gravitation with gauge theories.

Several authors [1-4] have studied classical gravitation in extended spacetimes and have discussed solutions in which three of the spatial dimensions evolve to become much larger than the rest. We consider one aspect of the finite temperature quantum dynamics of Kaluza - Klein theories whose vacuum manifold is $M^4 \times$ N torus (T^N) (and, for a twist, $M^4 \times$ Klein bottle (K^2)): we study the stability, at the one loop level, of solutions wherein one of the internal dimensions is much smaller than the others. The solutions are stable if the effective action considered as a function of this small dimension has a global minimum.

In calculating the effective action we consider the one loop contributions of bosons, fermions and gravitons. We also carefully describe how to renormalise the $4+N$ dimensional cosmological constant. This is an important consideration in extended spacetimes because the cosmological constant contributes to the effective action a term proportional to the volume of the internal manifold. Hence the effective action considered as a function of the smallest compact dimension picks up a term linear in this dimension. We show that the effective four dimensional cosmological constant measured by observers unaware of the compact dimensions is equal to the equilibrium value of the effective action. The $4+N$ dimensional cosmological constant is then normalised so that the effective four dimensional cosmological constant vanishes. Finally, we determine that the spacetimes $M^4 \times T^N$ are stable below some critical temperature if they contain sufficiently many twisted (antiperiodic) bosons or untwisted (periodic) fermions. The spacetime $M^4 \times K^2$ can be stabilised by the addition of untwisted fermi fields but not by bose fields. The equilibrium value of the small length scale and the critical temperature are set by the mass of the matter fields introduced into the theory. Typically these length scales are taken to be on the order of the Planck length. They can not be much smaller or the associated gauge couplings would be unreasonably large [5].

The Kaluza - Klein mechanism might be relevant to the inflationary cosmologies [6]. One can imagine that the universe is 'initialised' with the length scales of the compact dimensions not equal to their equilibrium values. If the effective action is rather flat at these nonequilibrium values then the internal manifold may take a long time to relax to its equilibrium configuration. Thus, the length scales of the compact dimensions may play the role usually assigned to the Higgs field in the standard inflationary models. As the internal manifold slowly relaxes to its equilibrium configuration, the macroscopic four dimensional spacetime may grow exponentially. For the moment, this sort of inflationary scenerio remains a matter of conjecture.

One of the great strengths of the (standard) inflationary universe scenerios is the wide variety of initial configurations they allow. The only requirement is that some initially hot regions of the universe supercool to temperatures below that of the inflationary phase transition and that some of these regions approach the false vacuum state. In practice, however, most calculations for the inflationary scenerios have been carried out assuming a homogeneous universe. There is good reason to believe that the results of such calculations may be applied to an inhomogeneous universe consisting of large (*i.e.*, bigger than the de Sitter length) bubbles of false vacuum enveloped by a region of true vacuum. Consider an observer living deep within such a large bubble. Because of the existence of the de Sitter horizon, he has no means of knowing that this bubble is not infinite. Therefore, he expects to see inflation. We, on the other hand, may adopt a more global view and consider the pressure forces acting on the boundary which separates the true and false vacua. The false vacuum has negative pressure and the true vacuum zero pressure, so these forces are inward, reflecting the inherent instability of the false vacuum. We are led to the paradoxical conclusion that the false vacuum region inflates without moving out into the true vacuum. Moreover, the continuity of the spacetime manifold, implicit in general relativity, guarantees that the metric on the boundary separating true and false vacua is well defined. Thus, two observers stationed on either side of the boundary must agree on whether or not a small area of this boundary is growing even though they might not agree on whether or not the volume of the false vacuum is growing. It is these paradoxes which we adress in the second part of this thesis.

In order to make the mathematics of our system manageable, we make two major simplifying assumptions which, in our opinion, leave the essential physics of the problem intact. First, we consider a spherically ($O(3)$) symmetric universe consisting of a false vacuum bubble separated from an infinite region of true vacuum by a domain wall. In particular, the solution to the Einstein equations in the true vacuum region is taken to be the Schwarzschild space. Second, we work in the 'thin wall' approximation which assumes that the thickness of the domain wall is small compared to all other length scales in the problem and that the scalar field configuration has dynamically relaxed to its equilibrium form.

The spherically symmetric universes we have described above fall into four classes. The first two classes contain false vacuum bubbles whose mass is less than some critical mass $\bar{M}$. These bubbles ultimately collapse into black holes. The distinction between the two classes is technical and need not be elaborated here. The third class contains false vacuum bubbles whose mass is greater than $\bar{M}$. These are the inflationary bubbles which do indeed grow to arbitrarily large radii without encroaching upon the true vacuum. The resolution of this paradox hinges upon the fact that the manifold we are studying is not Euclidean and in particular upon the fact that the standard Schwarzschild coordinates (T_S, R, Θ, Φ) do not cover the true vacuum region of the manifold. The domain wall, when viewed from the true vacuum, must be thought of as evolving in the maximally extended Schwarzschild manifold; any intuitions gained solely by considering its radius are suspect. The fourth class of universe contains false vacuum bubbles whose mass is again less than $\bar{M}$. They are generalisations of the Coleman - De Luccia bounce [7].

The fact that the maximally extended Schwarzschild manifold must be considered in order to properly understand inflationary scenerios with inhomogeneous initial conditions may shed light upon a number of interesting cosmological questons. Our results may provide a key to solving the cosmological constant problem and may help demonstrate that it is not the case that all information lost to a black hole may, in principle, be recovered as the black hole evaporates. We shall touch upon these issues in the discussion section.

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ON THE STABILITY OF

TOROIDALLY COMPACT KALUZA-KLEIN THEORIES

Things are bad!...The world's collapsing!...

Louis-Ferdinand Celine
<u>Guignol's Band</u>

ON THE STABILITY OF TOROIDALLY COMPACT KALUZA–KLEIN THEORIES ☆

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We study the stability at the one loop level, of finite temperature Kaluza–Klein theories coupled to matter fields. We restrict our attention to space–times containing compact manifolds which are toruses and Klein bottles. If the cosmological constant is chosen so that the effective potential vanishes at its minimum, and if twisted bosons or untwisted fermions are introduced into the theory, then these space–times are stable below a critical temperature of the order of the particle masses. We also discuss some subtleties that arise when Fermi fields are defined on non-simply connected manifolds.

1. Introduction. Kaluza–Klein theories describe gravitation in higher dimensional space–times. In these theories, the vacuum manifold of space–time is taken to be a direct product of four dimensional Minkowski space (M^4) and some as yet undetected N dimensional compact space. Observers on this manifold would interpret certain components of the $4 + N$ dimensional metric tensor as gauge fields on Minkowski space. In this manner, Kaluza–Klein theories unify gravitation with gauge theories.

The compact space would have escaped experimental detection only if its length scales are extremely small; typically they are taken to be no more than a few orders of magnitude greater than the Planck length. On the other hand, these length scales can not be arbitrarily small since each gauge coupling of the theory is inversely proportional to an appropriate root mean square circumference of the compact manifold. Thus, if the length scales are too small, the coupling constants will be unreasonably large [1]. The question arises, can one create stable $4 + N$ dimensional space–times with compact dimensions of such an extraordinary size? The answer, at least for the toroidally compact manifolds $M^4 \times S^1$, $M^4 \times T^N$ and $M^4 \times K^2$ (K^2 symbolizes the Klein bottle) is no, not unless massive matter fields are added to the theory. On $M^4 \times S^1$ and $M^4 \times T^N$, the matter fields so introduced may be either Fermi fields periodic (also called "untwisted") or Bose fields antiperiodic ("twisted") in all the toroidal dimensions; both give the same qualitative results. The space–time $M^4 \times K^2$ can be stabilized by the addition of appropriate Fermi fields to be described in the sequel, but under no circumstances can Bose fields stabilize this manifold.

To explore the stability of these $4 + N$ dimensional space–times, one may study the one loop effective action

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as a function of the length scales of the compact space. Appelquist and Chodos [2] have calculated the zero temperature graviton effective action for $M^4 \times S^1$. They showed that this action decreases monotonically and without bound as the radius of the circle decreases, which seems to imply that the compact dimension tends to shrink to a point. However, the one loop approximation breaks down at scales below the Planck length, and nothing definite can be said about this regime. Rubin and Roth [3] have discovered that at non-zero temperature, there is a temperature dependent unstable equilibrium length for the radius of the circle. The finite temperature graviton effective potential for $M^4 \times T^N$ has been calculated by Appelquist et al. [4], who find that one compact dimension is driven to the Planck length while the others tend to infinity. The same result is obtained for the manifold $M^4 \times K^2$ as we shall demonstrate presently.

We have noted that by adding the appropriate matter fields to the theory, one can cure these instabilities while leaving the compact manifold with an acceptable size. This was first discussed by Rubin and Roth [5] and by Tsokos [6] for the case of $M^4 \times S^1$. We shall extend their techniques to prove the same result for the manifolds $M^4 \times T^N$ and $M^4 \times K^2$; in the latter case we shall take care to define precisely what we mean by twisted and untwisted fields on the Klein bottle.

When studying Fermi fields on non-simply connected manifolds one should bear in mind the following remarks: There can be as many distinct spin connections on a manifold M as the number of elements in the homology group $H_1(M; Z_2)$ [7]. This group is trivial when M is simply connected but need not be if M is not simply connected. If one quantizes fermions by selecting a specific spin structure, the resulting Dirac action is not invariant under arbitrary local proper Lorentz transformations. It is an open question as to whether or not the action need be invariant under such transformations. However, this aesthetic condition may be realized by constructing an action in which all possible spin structures are summed upon. For example, $H_1(M^4 \times T^N; Z_2) = Z_2^N$ so that there are 2^N distinct spin connections on $M^4 \times T^N$. These correspond to the 2^N possible ways to construct fields which may be independently periodic or antiperiodic on each of the N circles. The effective potential induced by these 2^N Fermi fields acts to mitigate the instabilities on $M^4 \times T^N$; with sufficiently many Fermi fields, the instabilities can be cured.

The lagrangian contains a bare $4 + N$ dimensional cosmological constant which induces on the effective potential a term proportional to the volume of the compact space. Some authors [5,6] fix this cosmological constant by insisting that its contribution to the effective potential cancel against a similar term induced by the matter fields. The vacuum so constructed, while stable at zero temperature, is unstable against tunneling at any non-zero temperature. We shall specify the value of the $4 + N$ dimensional cosmological constant by requiring the total (i.e., graviton plus matter) effective potential to vanish at its minimum. This prescription cures the quantum instability of refs. [5,6] for all temperatures below some critical value which is determined by the masses of the matter fields. It also guarantees that the effective four dimensional cosmological constant is zero.

2. Five dimensional Kaluza–Klein model. We begin by considering the simplest possible Kaluza–Klein model, $M^4 \times S^1$. In our analysis we fix the bare cosmological constant by requiring the effective potential to vanish at its minimum; indeed, we shall use the same procedure for all toroidally compact manifolds we study. This prescription differs from that of refs. [5,6] and has important physical consequences.

The euclidian action S for gravity coupled to a twisted scalar field ϕ of mass M is:

$$S = S_g + S_m, \quad S_g = \frac{1}{16\pi G_5}\int d^5x \sqrt{g}(R - \Lambda_0), \quad S_m = \tfrac{1}{2}\int d^5x \sqrt{g}(g^{AB}\partial_A\phi\partial_B\phi + M^2\phi^2). \tag{2.1}$$

At the one loop level the total effective action at finite temperature (Γ_{tot}) can be decomposed into a sum of the contributions from gravitons (Γ_g) and from twisted scalars (Γ_{tb}). These two contributions have been calculated in refs. [3,5]. When the classical field configurations are constants the effective actions are proportional to $\beta\int d^3x$, where β is the inverse temperature, so it is more convenient to work with the effective potentials

$$\tilde{\Gamma}_{tot} = \tilde{\Gamma}_g + \tilde{\Gamma}_{tb}, \quad \tilde{\Gamma}_g = \Gamma_g \Big/ \beta\int d^3x, \quad \tilde{\Gamma}_{tb} = \Gamma_{tb} \Big/ \beta\int d^3x. \tag{2.2}$$

In refs. [5,6] Λ_0 is renormalized so that $\lim_{L_5\to\infty}\Gamma_{tot} = 0$. Then the asymptotic behaviour of $\tilde{\Gamma}_{tot}$ is

$$\tilde{\Gamma}_{tot} = -\frac{15\zeta(5)}{4\pi^2}\frac{1}{L_5^4}, \quad T=0, \quad M^{-1} \ll L_5,$$

$$\tilde{\Gamma}_{tot} = -\frac{15\zeta(5)}{4\pi^2}\frac{L_5}{\beta^5} - \frac{N_{tb}}{4\pi^2}\frac{M^2 L_5 e^{-\beta M}}{\beta^3}, \quad M^{-1}, \beta \ll L_5, \tag{2.3}$$

where N_{tb} is the number of twisted boson degrees of freedom. According to this choice the value of the zero temperature effective potential at its minimum is nonzero. We shall see in the next section that this amounts to having a nonzero cosmological constant which is bad from an observational point of view.

We shall choose Λ_0 differently, so as to avoid this difficulty. We require that $\langle g_{MN}\rangle = \eta_{MN}$ for our vacuum state, where η_{MN} represents the Minkowsky metric on $M^4 \times S^1$. A priori, it is not obvious that such a restriction can be imposed when quantum corrections are taken into account, however in the presence case we shall be able to enforce this condition consistently up to one loop level. Let us consider the vacuum expectation value of the Einstein equations for $g_{MN} = \eta_{MN} + h_{MN}$,

$$\langle L_5 | G_{MN}(\eta) + G^1_{MN} - \Lambda_0 h_{MN} | L_5\rangle = \langle L_5 | \tau_{MN} | L_5\rangle + \Lambda_0 \eta_{MN}, \tag{2.4}$$

where $G_{MN}(\eta)$ is the Einstein tensor evaluated for the metric η, G^1_{MN} is the part of $G_{MN}(g)$ linear in h_{MN}, τ_{MN} is the energy momentum tensor of matter and gravitation, and $|L_5\rangle$ is the ground state of the quantum theory built on $M^4 \times S^1$ where L_5 is the circumference of S^1.

Now $G_{MN}(\eta)$ and $\langle L_5 | G^1_{MN} - \Lambda_0 h_{MN} | L_5\rangle$ is zero because $\langle h_{MN}\rangle$ is zero. Thus the lhs of (2.4) vanishes. For any value of L_5, then, we must find a Λ_0 such that the rhs of (2.4) vanishes. Moreover the 00 component of the rhs of (2.4) is equal to L_5^{-1} times the cosmological constant measured by an observer living in $M^4 \times S^1$ who does not know of the existence of the compact dimension. Thus if we find a Λ_0 and L_5 satisfying (2.4), we guarantee that such an observer will measure a vanishing cosmological constant.

Finally we consider the generating functional (partition function at zero T)

$$Z(J=0) = \exp\left(-\int d^3x\, dt\, \bar{L}_5 \langle \bar{L}_5 | \tau_{00} + \Lambda_0 | \bar{L}_5\rangle\right) = \exp\left(-\int d^3x\, dt\, \tilde{\Gamma}_{tot}\right), \tag{2.5}$$

where $\bar{L}_5$ is the circumference of the compact dimension for which $\Gamma_{tot}(L_5)$ is a minimum and we have chosen the external source $J = 0$ corresponding to our vacuum state $|\bar{L}_5\rangle$. This implies

$$\tilde{\Gamma}_{tot}(\bar{L}_5) = \langle \bar{L}_5 | \tau_{00} + \Lambda_0 | \bar{L}_5\rangle \bar{L}_5, \tag{2.6}$$

which is the effective four dimensional cosmologial constant when L_5 assumes its equilibrium value $\bar{L}_5$. Thus our recipe for fixing Λ_0 is the following: choose Λ_0 so that the value of the effective potential at its minimum is zero.

The choice of the cosmological constant affects the stability of the vacuum. The vacuum constructed in refs. [5, 6] has a negative absolute minimum at zero temperature. At all finite temperatures, this becomes only a local minimum; the vacuum is unstable against tunneling, see fig. 1. Now, the term induced on the effective potential by the cosmological constant is proportional to L_5. Thus the difference between our effective potential and that of refs. [5,6] is a term linear in L_5 with positive slope. Inspection of the asymptotic form of the total effective potential, eq. (2.3), reveals that the vacuum we have constructed is stable against tunneling for all temperatures below some critical value determined by the masses of the matter fields in the theory, see fig. 2.

3. Toroidally compact Kaluza–Klein theories. Appelquist et al. [4] noted that the space–time $M^4 \times S^1 \times S^1$ is unstable when only gravitons contribute to the effective action [5]. Their expression for the one loop effective potential is

$$\tilde{\Gamma}_g = \Gamma_g \Big/ \beta \int d^3x = -(9/\pi^3) L_1 L_2 \sum{}' (m_1^2 L_1^2 + m_2^2 L_2^2)^{-3}, \tag{3.1}$$

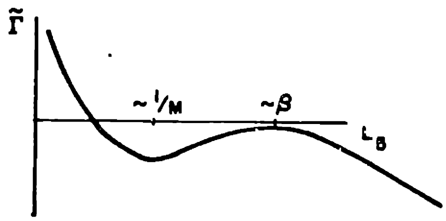

Fig. 1. Finite temperature total effective potential with Λ_0 renormalized as in refs. [5,6].

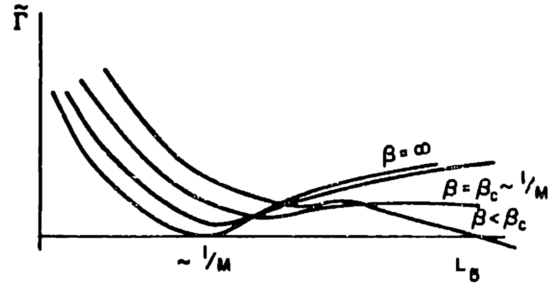

Fig. 2. Finite temperature total effective potential with Λ_0 renormalized as prescribed in this paper.

where $\int d^3x$ is the uncompactified spatial volume; L_1, L_2 are the circumferences of the toroidal dimensions, and the prime on the sum means exclude the term with $m_1 = m_2 = 0$.

When L_2 is much less than L_1

$$\tilde{\Gamma}_g = -(18/\pi^3)\zeta(6) L_1/L_2^5. \tag{3.2}$$

So for fixed L_1, L_2 tends to shrink to zero, and for fixed L_2, L_1 increases indefinitely.

We now show that this instability can be cured by the addition of massive scalar fields antiperiodic in both toroidal dimensions. When considering the manifold $M^4 \times T^N$, we shall reserve the word "twisted" for fields antiperiodic in all toroidal dimensions.

The contribution of a free massive scalar field to the effective action is

$$\tilde{\Gamma}_{tb} = \tfrac{1}{2} \ln \mathrm{Det}\,(-\Box + M^2) \Big/ \beta \int d^3x\ , \tag{3.3}$$

which is to be evaluated for twisted scalar fields at finite temperature. We use the zeta function regularization [10]. Then, in terms of the zeta function of $(\Box + M^2)$

$$\tilde{\Gamma}_{tb} = -\tfrac{1}{2}\zeta'(0) \Big/ \beta \int d^3x\ . \tag{3.4}$$

The zeta function

$$\frac{\zeta(s)}{\beta \int d^3x} = \frac{\mu^{2s}}{\beta} \sum_{p,m,n=-\infty}^{\infty} \int d^3k\, \{M^2 + k^2 + (2\pi p/\beta)^2 + [2\pi(m+\tfrac{1}{2})/L_1]^2 + [2\pi(n+\tfrac{1}{2})/L_2]^2\}^{-s}, \tag{3.5}$$

where μ is an arbitrary mass put in to keep $\zeta(s)$ dimensionless, is well defined for sufficiently large s and may be defined on the entire real axis by analytic continuation. Integrating over k yields

$$\frac{\zeta(s)}{\beta \int d^3x} = \frac{\Gamma(s-\tfrac{3}{2})\mu^{2s}}{8\pi^{3/2}\beta\Gamma(s)} \sum_{p,m,n=-\infty}^{\infty} \{(2\pi p/\beta)^2 + [2\pi(m+\tfrac{1}{2})/L_1]^2 + [2\pi(n+\tfrac{1}{2})/L_2]^2 + M^2\}^{-s+3/2}\ , \tag{3.6}$$

which, after rearranging terms, becomes

$$\begin{aligned}\frac{\zeta(s)}{\beta \int d^3x} = \frac{\Gamma(s-\tfrac{3}{2})\mu^{2s}}{8\pi^{3/2}\beta\Gamma(s)} \sum_{p,m,n=-\infty}^{\infty} &\{[(2\pi p/\beta)^2 + (\pi m/L_1)^2 + (\pi n/L_2)^2 + M^2]^{-s+3/2} \\ &- [(2\pi p/\beta)^2 + (\pi m/L_1)^2 + (2\pi n/L_2)^2 + M^2]^{-s+2/3} - [(2\pi p/\beta)^2 + (2\pi m/L_1)^2 + (\pi n/L_2)^2 + M^2]^{-s+3/2} \\ &+ [(2\pi p/\beta)^2 + (2\pi m/L_1)^2 + (2\pi n/L_2)^2 + M^2]^{-s+3/2}\}\ .\end{aligned} \tag{3.7}$$

The sums above converge for sufficiently large s, and may be analytically continued to functions which are regular at $s = 0$ via the relationship

$$\pi^{-\nu/2}\Gamma(\nu/2) \sum_{n_1 \dots n_p = -\infty}^{\infty} (M^2 + (n_1/a_1)^2 + \dots + (n_p/a_p)^2)^{-\nu/2}$$

$$= a_1 \dots a_p M^{p-\nu} \pi^{(p-\nu)/2} \left(\Gamma((\nu - p)/2) + 2 \sum_{n_1 \dots n_p}{}' \frac{K_{(p-\nu)/2}(2\pi M[(a_1 n_1)^2 + \dots + (a_p n_p)^2]^{1/2})}{\{\pi M[(a_1 n_1)^2 + \dots + (a_p n_p)^2]^{1/2}\}^{(p-2)/2}} \right). \quad (3.8)$$

The prime on the sum means omit the term with $n_1 = n_2 \dots = n_p = 0$, and K is a modified Bessel function. The derivative of $\zeta(s)$ in (3.7) may now be taken at $s = 0$ with the result

$$\frac{\Gamma_{\rm tb}}{\beta \int d^3x} = -\frac{1}{2} \frac{\zeta'(0)}{\text{Vol.}\,\beta}$$

$$= -\frac{M^6 L_1 L_2}{2^6 \pi^3} \left(-\tfrac{11}{36} - \tfrac{1}{3} \ln(\mu/M) + \sum_{p,m,n=-\infty}^{\infty}{}' [4K_3(2z)/z^3 - 2K_3(2y)/y^3 - 2K_3(2t)/t^3 + K_3(2s)/s^3] \right), \quad (3.9)$$

where

$$z = \tfrac{1}{2} M [(\beta p)^2 + (2L_1 m)^2 + (2L_2 n)^2]^{1/2}, \quad y = \tfrac{1}{2} M [(\beta p)^2 + (2L_1 m)^2 + (L_2 n)^2]^{1/2},$$

$$t = \tfrac{1}{2} M[(\beta p)^2 + (L_1 m)^2 + (2L_2 n)^2]^{1/2}, \quad s = \tfrac{1}{2} M[(\beta p)^2 + (L_1 m)^2 + (L_2 n)^2]^{1/2}.$$

In order to ascertain the stability of these manifolds for the case when one compact dimension is much smaller than the others, we consider the regime L_2 much less than M^{-1}, L_1, β. Since $K_3(2x)/x^3$ behaves as x^{-6} as x tends to 0, the sums in eq. (3.9) are dominated by those terms with $m = p = 0$. Therefore we conclude

$$\tilde{\Gamma}_{\rm tb} \to N_{\rm tb} \tfrac{31}{16} \pi^{-3} \zeta(6) \beta L_1 / L_2^5 \qquad (L_2 \to 0), \quad (3.10)$$

where $N_{\rm tb}$ is the number of twisted bosons in the theory and $\zeta(s)$ is the Riemann zeta function. The total one loop effective potential is

$$\tilde{\Gamma}_{\rm net} = -18\pi^{-3}\zeta(6)\beta L_1/L_2^5 + N_{\rm tb} \tfrac{31}{16} \pi^{-3} \zeta(6) \beta L_1/L_2^5 . \quad (3.11)$$

If the number of twisted bosons is greater than 9, the instability of $M^4 \times S^1 \times S^1$ is cured.

Let us briefly consider the behaviour of a field T' antiperiodic in the dimension with circumference L_1 and periodic in the dimension with circumference L_2. If L_1 is much less than L_2, M^{-1}, β, the effective potential is positive and proportional to L_2/L_1^5 just as in the twisted field case, however, if L_2 is much less than L_1, M^{-1}, β, the effective potential is negative and proportional to L_1/L_2^5. The effective potential induced by T' has no minimum and such fields do not stabilize $M^4 \times T^N$. Likewise, a field T'' antiperiodic in the dimension with circumference L_2 and periodic in the other dimension can not stabilize $M^4 \times T^N$.

The matter effective potential contains a term proportional to the volume of the torus: $M^3[-\frac{11}{36} - \frac{1}{3}\ln(\mu/M)] \times L_1 L_2$ as well as a similar term induced by the graviton. These two terms can be combined to give a contribution of the form $\Lambda_i L_1 L_2/16\pi G$. The bare cosmological constant also contributes a term of this form to the effective potential, so we may view Λ_i as an induced cosmological constant.

Let us study the zero temperature limit of the theory and temporarily choose $\Lambda_0 = -\Lambda_i$. Consider the family of curves in L_1, L_2 space defined by $L_1 = L, L_2 = \alpha L$, where α is non-zero. For any given α, we view Γ as a function of L and discover

$$\tilde{\Gamma}(L) = 0^- \quad (L \to \infty), \qquad \tilde{\Gamma}(L) = +\infty \quad (L \to 0). \quad (3.12)$$

This implies that $\tilde{\Gamma}(L_1, L_2)$ has a (possibly degenerate) absolute minimum; and that the effective four dimensional cosmological constant is not zero. Moreover, the vacuum constructed by setting $\Lambda_0 = -\Lambda_i$ is unstable against tunneling at any non-zero temperature. We suggest therefore, that Λ_0 should be fixed by requiring that the zero temperature effective potential vanishes at its minimum. This ensures that the effective four dimensional cosmological constant is zero and cures the vacuum instability against tunneling for all temperatures below some critical value of order M. These considerations parallel those worked out in detail for $M^4 \times S^1$.

In order to generalize our analysis to the space-time $M^4 \times T^N$ we must consider the 2^N Bessel functions which arise in the analogue of eq. (3.9). We must determine what coefficients they will inherit from eq. (3.8) and what contribution they will make to the effective potential in the regime L_N much less than $M^{-1}, L_1, ... L_{N-1}, \beta$. The combinatoric formula

$$\sum_{k=0}^{N} (-1)^k 2^{N-k}\, {}^N C_k = 1 \qquad (3.13)$$

greatly simplifies the result:

$$\tilde{\Gamma}_{tb} \to (N_{tb} L_1 ... L_{N-1}/L_N^{3+N}) [\Gamma(N/2+2)/\pi^{N/2+2}] [1 - (\tfrac{1}{2})^{N+3}]\, \zeta(4+N) \qquad (L_N \to 0)\,. \qquad (3.14)$$

The total one loop effective potential is

$$\tilde{\Gamma}_{tot} \to (L_1 ... L_{N-1}/L_N^{3+N}) [\Gamma(N/2+2)\zeta(4+N)/\pi^{N/2+2}] [-\tfrac{1}{2}(N+4)(N+1) + N_{tb}(1-(\tfrac{1}{2})^{N+3})]. \qquad (3.15)$$

So the space–time is stabilized if the $N_{tb} > 2^{N+3}(N+4)(N+1)/(2^{N+3}-1)$.

The total one loop effective potential contains a term of the form $(\Lambda_0 + \Lambda_i) L_1 ... L_N / 16\pi G$ and we fix Λ_0 by requiring that $\tilde{\Gamma}_{net}$ vanishes at its minimum. Again, the vacuum so constructed is stable for all temperatures below some critical value of order M, and the effective four dimensional cosmological constant is zero.

At zero temperature the one loop effective potential for fermions is the same as that for bosons except for an overall minus sign and a dimension dependent degeneracy factor. Thus it is untwisted fermions which tend to stabilize the toroidally compact space–times. Recall that in order to have a quantum theory invariant under arbitrary proper local Lorentz transformations, one must sum over 2^N Fermi fields on $M^4 \times T^N$. These fields taken together tend to stabilize this space–time.

The Klein bottle, K^2, is a locally flat manifold. It may be viewed as a plane with the points (y_1, y_2) and $(y_1 + ml_1, (-1)^m y_2 + nl_2)$ identified for all integral m and n. In analogy with the simple toroidal case, we define a function, F, of the Klein bottle co-ordinates to be untwisted if

$$F(y_1 + mL_1, (-1)^m y_2 + nL_2) = F(y_1, y_2)\,, \qquad (3.16)$$

twisted if

$$F(y_1 + mL_1, (-1)^m y_2 + nL_2) = (-1)^{m+n} F(y_1, y_2)\,, \qquad (3.17)$$

T' if

$$F(y_1 + mL_1, (-1)^m y_2 + nL_2) = (-1)^m F(y_1, y_2)\,, \qquad (3.18)$$

and T'' *if*

$$F(y_1 + mL_1, (-1)^m y_2 + nL_2) = (-1)^n F(y_1, y_2)\,, \qquad (3.19)$$

for all integers m and n. We find that effective potentials calculated on a Klein bottle with circumferences L_1 and L_2 can be expressed in terms of effective potentials calculated on a torus with circumferences $2L_1$ and L_2. Our results are:

$$\Gamma^K_{ut}(L_1, L_2) = \Gamma^K_{T''}(L_1, L_2) = \tfrac{1}{2}\Gamma^{torus}_{ut}(2L_1, L_2), \quad \Gamma^K_{tb}(L_1, L_2) = \Gamma^K_{T'}(L_1, L_2) = \tfrac{1}{2}\Gamma^{torus}_{T'}(2L_1, L_2)\,, \qquad (3.20)$$

where Γ_{ut} = the effective potential on the Klein bottle with circumference L_1 and L_2 induced by an untwisted field, etc. Eq. (3.20) is obtained for all fields we consider: gravitons, scalars and spinors. Note that the graviton field is single valued and has positive signature; either condition guarantees that it be untwisted. Thus eq. (3.20) has the following consequences: The manifold $M^4 \times K^2$ is unstable in the absence of matter. Moreover, bosonic matter can not stabilize this space–time but massive fermionic untwisted or T'' fields can.

To illustrate our result we consider the effective potential induced by untwisted scalar fields on $M^4 \times K^2$. As in the toroidal cases, the effective action is eq. (3.3)

$$\Gamma = \tfrac{1}{2} \ln \mathrm{Det}(-\Box + M^2) . \tag{3.21}$$

The normal modes for untwisted fields on the Klein bottle are

$$e^{imy_1 2\pi/L_1} \cos(ny_2 2\pi/L_2), \quad e^{i(m+1/2)y_1 2\pi/L_1} \sin(ny_2 2\pi/L_2) . \tag{3.22}$$

These are not the same as those obtained in the toroidal case, and herein lies the only difference between the torus and Klein bottle calculations. Eq. (3.5) becomes:

$$\frac{\zeta(s)}{\beta \int d^3x} = \frac{\mu^{2s}}{\beta} \sum_{m,n,p=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} [\tfrac{1}{2}(k^2 + 4\pi^2p^2/\beta^2 + 4\pi^2m^2/L_1^2 + 4\pi^2n^2/L_2^2)^{-s}$$
$$+ \tfrac{1}{2}(k^2 + 4\pi^2p^2/\beta^2 + 4\pi^2(m+\tfrac{1}{2})^2/L_1^2 + 4\pi^2n^2/L_2^2)^{-s}]$$
$$= \frac{\mu^{2s}}{\beta} \sum_{m,n,p=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \tfrac{1}{2} [k^2 + 4\pi^2p^2/\beta^2 + 4\pi^2m^2/(2L_1)^2 + 4\pi^2n^2/L_2^2]^{-s} , \tag{3.23}$$

which is indeed the same result as is obtained for untwisted scalar fields on a toroidal manifold with circumferences $2L_1$ and L_2.

4. Conclusions. In this paper we have studied the stability at the one loop level of toroidally compact Kaluza–Klein theories. On simple toroidal manifolds, twisted bosons and untwisted fermions cure the instabilities caused by pure gravitons, below a certain temperature provided the theory is renormalized to have zero four-dimensional cosmological constant. The topologically non-trivial Klein bottle manifold can be stabilized only with untwisted fermions.

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DISCUSSION

At the outset of this paper we claimed that the Kaluza - Klein mechanism provides a means for unifying gravitation with gauge theories. One considers a theory of pure gravity in an enlarged spacetime which is the direct product of a four dimensional spacetime and a compact N dimensional 'internal' manifold $\mathcal{M}$. He then integrates out the internal manifold coordinates and discovers the remaining four dimensional effective Lagrangian to be that of a Yang - Mills gauge theory. Moreover, gauge transformations on the Yang - Mills fields may be realised as coordinate transformations on the enlarged spacetime.

Let us study the Kaluza - Klein mechanism in some detail by first considering the simplest possible internal manifold $\mathcal{M} = S^1$. This five dimensional theory unifies gravitation with the abelian gauge group U(1). We shall then describe how to generalise the formalism so as to achieve the unification of gravitation with nonabelian Yang - Mills theories.

Consider a theory of pure gravitation in five dimensions with the metric

$$\bar{g}_{MN} = \begin{pmatrix} g_{\mu\nu}(x) + \tilde{g}s^2 A_\mu(x)A_\nu(x) & -s\tilde{g}A_\mu(x) \\ -s\tilde{g}A_\nu(x) & \tilde{g} \end{pmatrix} \tag{5.1}$$

where x denotes the four dimensional spacetime coordinates, s is a scale factor to be set presently and we have chosen a parametrisation such that $\mathrm{Det}(\bar{g}_{MN}) = \mathrm{Det}(g_{\mu\nu})\mathrm{Det}(\tilde{g})$.

The action is

$$S = \int d^5x \bar{\mathcal{L}} \tag{5.2}$$

with the five dimensional Lagrangian

$$\bar{\mathcal{L}} = \frac{-1}{16\pi \bar{G}} \sqrt{\bar{g}}\,(\bar{R} - \bar{\Lambda}) \tag{5.3}$$

where $\bar{G}$ ($\bar{\Lambda}$) is the five dimensional gravitational (cosmological) constant, $\bar{R}$ is the five dimensional Ricci scalar built with the metric $\bar{g}_{MN}$ and $\bar{g}$ symbolises $\mathrm{Det}(\bar{g}_{MN})$. The effective four dimensional Lagrangian is

$$\mathcal{L} = \int dx^5 \bar{\mathcal{L}} \qquad i.e.,\ S = \int d^4x \mathcal{L} \tag{5.4}$$

which is the Lagrangian used by observers living in the extended spacetime who are unaware of the compact dimensions. A straightforward calculation yields

$$\mathcal{L}(x) = \frac{-1}{16\pi\tilde{G}} \int dx^5 \sqrt{\tilde{g}}\sqrt{g}\big({}^{(4)}R(x) - \overline{\Lambda} \\ + \frac{1}{4} \cdot \tilde{g}s^2 F_{\mu\nu}F^{\mu\nu}\big) \tag{5.5}$$

where ${}^{(4)}R$ is the four dimensional Ricci scalar built with the metric $g_{\mu\nu}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Thus we may identify the fields $g_{\mu\nu}(x)$ and $A_\mu(x)$, originally parameters of the five dimensional metric, as the four dimensional metric and gauge fields, respectively.

Inspection of equation (5.5) reveals that the four dimensional gravitational constant is given by

$$\frac{1}{16\pi G} = \frac{1}{16\pi\tilde{G}} \int dx^5 \sqrt{\tilde{g}} \tag{5.6}$$

and that the four dimensional cosmological constant is

$$\Lambda = \overline{\Lambda}\,. \tag{5.7}$$

Notice that in the classical theory it is the inverse gravitational constant which scales with the volume of the internal manifold. This is in contrast with the quantum theory in which case, at the one loop level, the higher dimensional gravitational constant is a fixed parameter of the Lagrangian, and it is the effective action which scales with the volume of the compact manifold. The cannonical normalisation of the field strength term in equation (5.5) to $-\frac{1}{4}F^2$ sets the scale s:

$$\frac{1}{16\pi\tilde{G}} \int dx^5 \sqrt{\tilde{g}}\,\tilde{g}s^2 = 1\,. \tag{5.8}$$

Next we consider the coordinate transformation $x^M \rightarrow x'^M$ given by

$$(x^\mu, x^5) = \big(x'^\mu, x'^5 - s\lambda(x'^\mu)\big)\,. \tag{5.9}$$

Under this transformation,

$$g_{\mu 5} \rightarrow g'_{\mu 5} = g_{AB}\frac{\partial x^A}{\partial x'^\mu}\frac{\partial x^B}{\partial x'^5} \\ = g_{\sigma 5}\delta^\sigma_\mu - s g_{55}\partial_\mu \lambda\,. \tag{5.10}$$

With the given parametrisation for the components of the metric g_{MN}, equation (5.10) reduces to

$$-sA_\mu\tilde{g} \to -sA_\mu\tilde{g} - s\tilde{g}\partial_\mu\lambda \tag{5.11}$$

that is,

$$A_\mu \to A_\mu + \partial_\mu\lambda \tag{5.12}$$

which justifies the claim that U(1) gauge transformations may be realised as coordinate transformations on a five dimensional spacetime with internal manifold $\mathcal{M} = S^1$.

Let us now generalise the formalism just described so as to be able to unify gravitation with a nonabelian Yang - Mills theory with gauge group $\mathcal{G}$. We require that there exists an N dimensional internal manifold $\mathcal{M}$ on which the group $\mathcal{G}$ acts as an isometry. That is, we represent a group element near the identity as $(1+\theta^a T^a)$ and demand that the group action of this element on the point $p \in \mathcal{M}$ with coordinates y^n be

$$(1+\theta^a T^a)y^n = y^n + \theta^a\xi^{an}(y) \tag{5.13}$$

with $\xi^{an}\partial_n \equiv \xi^a$ a Killing vector. Since the generators T^a satisfy the structure equations

$$[T^a, T^b] = -f^{abc}T^c, \tag{5.14}$$

so must the Killing vectors:

$$[\xi^a, \xi^b] = -f^{abc}\xi^c. \tag{5.15}$$

Here the brackets symbolise the usual Lie bracket operation. In component notation equation (5.15) may be written

$$\xi^{na}\partial_n\xi^{mb} - \xi^{nb}\partial_n\xi^{ma} = -f^{abc}\xi^{mc}. \tag{5.16}$$

The $4+N$ dimensional metric analogous to that given in equation (5.1) is

$$g_{MN} = \begin{pmatrix} g_{\mu\nu}(x) + \tilde{g}(y)\xi^n_a(y)\xi^m_b(y)A^a_\mu(x)A^b_\nu(x) & -\tilde{g}_{mn}(y)\xi^m_a A^a_\mu \\ -\tilde{g}_{mn}(y)\xi^n_a(y)A^a_\nu(x) & \tilde{g}_{mn}(y) \end{pmatrix}, \tag{5.17}$$

and the four dimensional effective Lagrangian is given by

$$\mathcal{L} = \frac{-1}{16\pi\tilde{G}} \int d^N y \sqrt{\tilde{g}}\, \sqrt{g}\, \big({}^{(4)}R(x) + {}^{(N)}\tilde{R}(y) - \Lambda + \frac{1}{4}\tilde{g}_{mn}\xi^n_a \xi^m_b F^a_{\mu\nu} F^{b\mu\nu} \big) \tag{5.18}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc} A^b_\mu A^c_\nu$. In order that the field strength term be cannonically normalised, one must choose the Killing vectors such that

$$\frac{1}{16\pi\tilde{G}} \int d^N y \sqrt{\tilde{g}}\, \tilde{g}_{mn} \xi^n_a \xi^n_b = \delta_{ab}\,. \tag{5.19}$$

It is straightforward to check that the infinitesimal coordinate transformaiton

$$(x^\mu, y^m) \rightarrow (x'^\mu, y'^m) = (x^\mu, y^m - \epsilon^a(x)\xi^{am}) \tag{5.20}$$

realises the gauge transformation

$$A^a_\mu(x) \rightarrow A^a_\mu(x) + D_\mu \epsilon^a(x) \tag{5.21}$$

with D_μ denoting the usual covariant derivative.

In our work we have discussed the quantum dynamics of Kaluza - Klein theories with the internal manifold $\mathcal{M}$ toroidally compact. Such theories allow only for the unification of gravitation with abelian Yang - Mills theories. Therefore, one might wish to explore the quantum dynamics of Kaluza - Klein theories with more complicated internal manifolds. The lowest (= 7) dimensional internal manifolds which admit isometric $SU(3) \times SU(2) \times U(1)$ group actions have been catalogued by Witten [1]. These manifolds are sufficiently complex however, that an analysis along the lines of our work seems untenable. As a modest step in this direction one may wish to explore extended spacetimes whose internal manifolds are spheres. Candelas and Weinberg [2] have studied gravitation in extended spacetimes wherein the energy momentum tensor arises from the one loop quantum fluctuations of light matter fields. If there are sufficiently many matter fields, then this matter contribution to the effective potential will dominate the one loop graviton contribution. Thus, Candelas and Weinberg treat the $4+N$ dimensional metric as a background field. They discover solutions of the Einstein equations whose form is $M^4 \times S^N$, where M^4 denotes Minkowski space, and they compute the quantum effective potential as a function of the spheres radius for odd N. This in turn allows them to fix the radius of the N sphere relative to the Planck length. Gilbert and McClain [3] have studied the stability of these manifolds, again with gravity treated as a background field.

If one ignores quantum gravity, then he must include something like 10^4 matter fields if the internal sphere is to be sufficiently small that the associated gauge couplings are not unreasonably large. For this reason Chodos *et. al.* have been motivated to study the one loop graviton contribution to the effective potential for Kaluza - Klein theories built upon the background geometry $M^4 \times S^N$ [4]. Their work is in progress and preliminary results indicate that $M^4 \times S^N$ is a stable solution to the Einstien equations, at the one loop level, for some but not all odd values of N.

In our work we observed that toroidally compact Kaluza - Klein manifolds are unstable for all temperatures above some critical value set by the Planck scale. Candelas and Weinberg have argued [2] that the same result obtains no matter what the internal manifold. Thus, in the context of Kaluza - Klein theories, the very early universe must undergo a 'dimensional phase transition' at a temperature no higher than something of the order of the Planck temperature. An intriguing possibility is that the dimensional phase transition might occur somewhat later, when the temperature is below that at which some grand unified gauge group is broken down to $\mathcal{H} = SU(3) \times SU(2) \times U(1)$ [5]. In the absense of a dimensional phase transition *viz.*, if the universe is and always was four dimensional, then associated to the nontrivial elements of the homotopy groups $\Pi^1(\mathcal{G}/\mathcal{H})$, $\Pi^2(\mathcal{G}/\mathcal{H})$ and $\Pi^3(\mathcal{G}/\mathcal{H})$ are domain walls, strings and magnetic monopoles, respectively. If, at the time of the gauge group phase transition the universe has $3+N$ spatial dimensions, then there are additional topological singularities associated to nontrivial elements of the homotopy groups $\Pi^{3+n}(\mathcal{G}/\mathcal{H}) \quad (1 \leq n \leq N)$. It would be interesting to study how such topological singularities are manifest in the effective four dimensional world in which we live today.

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THE DYNAMICS OF FALSE VACUUM BUBBLES

I could be bounded in a nutshell and count
myself king of infinite space...

William Shakespeare
Hamlet, Act II, Scene II

THE DYNAMICS OF FALSE VACUUM BUBBLES

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ABSTRACT

The possibility of localized inflation is investigated by calculating the dynamics of a spherically symmetric region of false vacuum which is separated by a domain wall from an infinite region of true vacuum. If the total mass of the system exceeds a critical value, then the false vacuum region will undergo inflation. An observer in the exterior true vacuum region will describe the system as a black hole, while an observer in the interior will describe a closed universe which completely disconnects from the original spacetime. Under these circumstances, it seems clear that information is irretrievably lost to the external spacetime. We suggest that this mechanism is also likely to lead to an instability of Minkowski space: a region of space might undergo a quantum fluctuation into the false vacuum state, evolving into an isolated closed universe; the black hole which remains in the original space would disappear by quantum evaporation.

I. INTRODUCTION

An intriguing feature of the inflationary universe model[1-4] is the wide range of initial conditions which the model allows. One can imagine an initial spacetime manifold which is not at all homogeneous. The spacetime could be hot in some regions, cold in other regions, expanding in some regions, contracting in other regions, etc. One could argue that the regions which were both hot and expanding would cool down to the temperature of the inflationary phase transition. For an appropriate underlying particle theory, these regions would then undergo extreme supercooling, approaching the false vacuum state. The unusual properties of the energy-momentum tensor for this state would then lead to the phenomenon of inflation, causing these regions to expand by many orders of magnitude to become much larger than the observed universe. We would then be living today deep inside one of these inflated regions. We could not be living in one of the regions which did not inflate, because these regions would have remained microscopic in size and would have no chance of producing life.

While the description given above seems plausible, the mathematical details have never been worked out. Most calculations for inflationary models have been carried out under the simplifying assumption of homogeneity, even though one assumes that initial homogeneity is not a necessary condition. There have also been calculations[5-10] which have used perturbation theory to study the mass density inhomogeneities caused by quantum effects, but these calculations rely on a homogeneous zero order approximation. Thus, the consequences of large inhomogeneities in the initial conditions need to be elucidated.

The mathematics of inhomogeneous spacetimes can be very complex, so we will content ourselves to study only the simplest possible example. We will study the dynamics of a spherically ($O(3)$) symmetric universe that consists of a finite region of false vacuum separated by a domain wall from an infinite region of true vacuum. Although this system is highly simplified, it nonetheless raises two significant paradoxes.

The first paradox concerns the behavior of the volume of the false vacuum region. If this region is sufficiently large, then an observer who makes measurements deep within the region would unambiguously expect to see inflation. However, an observer who makes measurements of the domain wall would have a different point of view. He would note that the false vacuum region has negative pressure and is surrounded by the zero pressure true vacuum. The pressure forces are therefore inward, reflecting the inherent instability of the false vacuum. Our assumption of spherical symmetry implies that the metric in the true vacuum region has the usual Schwarzschild form, so gravitational effects are not expected to cause the false vacuum region to expand into the true vacuum region. Thus, the second observer does not expect to see inflation.

In fact, these two points of view are not contradictory. The key to reconciling them is an understanding of the non-Euclidean geometry of the spacetime manifold. We will discover that inflation does take place, for a sufficiently large region of false vacuum, but that the inflating false vacuum region does not move out into the true vacuum region.

The second paradox is concerned with the time evolution of the domain wall's radius of curvature. Consider two observers, one of which is just inside the false vacuum region, and the second of which is just on the other side of the domain wall. Since the two observers can be arbitrarily close to each other, there is no difficulty in defining what it means for them to observe something simultaneously. If both observers were to simultaneously measure the radius of curvature of the wall, then the continuity of the manifold, implicit in general relativity, guarantees that they would measure the same value. Thus, while the two observers can disagree about whether the false vacuum volume seems to be growing or not, they must agree on whether or not the radius of curvature of the domain wall is growing. Again, an understanding of the non-Euclidean geometry is the key to resolving this paradox. In particular, the resolution will hinge on the fact that the standard Schwarzschild coordinates fail to cover the entire manifold.

Although the problem which we solve is very simplified, we believe that it contains the essential physics of more complicated inhomogeneous spacetimes. The paradoxes discussed above will exist whenever an inflating region is surrounded by a noninflating region, and the qualitative behavior of the system will be determined by the manner in which these paradoxes are resolved.

In order to make the calculations tractable in closed form, we will make one further assumption in addition to that of spherical symmetry. The domain wall which separates the false vacuum region from the true vacuum region is in reality a dynamical object which can be described properly only by specifying the scalar field as a function of position. We will work, however, in the 'thin-wall' approximation which assumes that the thickness of the wall is small compared to all other length scales in the problem and that the scalar field configuration has dynamically relaxed to its equilibrium form. Thus, the energy-momentum tensor for the wall is determined completely once the position of the wall is known.

The dynamics of the universe are completely specified once we have solved the Einstein equations in the true and false vacuum regions and once we have determined the evolution of the domain wall. The solution in the true vacuum region is guaranteed by Birkhoff's theorem to be a Schwarzschild metric, with the parameter M signifying the mass of the system as detected from asymptotically large distances. In the false vacuum region there is similarly a one-parameter class of spherically symmetric solutions. However, since the false vacuum region comprises the interior of our configuration, we will consider only solutions that are regular at $r = 0$. (Note that for spherically symmetric configurations $r = 0$ can be defined in a coordinate-invariant way as the locus of points which are invariant under rotations.) It is easily shown that this additional requirement singles out the de Sitter space solution. The dynamics of the domain wall is specified by the requirements that the Einstein equations hold at the wall and that the tangential components of the metric remain continuous as the wall is crossed.

The mathematical formalism which we use follows the work of previous authors. The collapse of domain walls separating two regions of true vacuum has been studied by Ipser and Sikivie.[11] Berezin *et. al.*[12] have investigated the collapse of domain walls separating regions of true or false vacua with arbitrary nonnegative energy densities. Aurilia *et. al.*[13] have also considered the evolution of a domain wall separating regions of true and false vacua and have recognized the existence of inflationary solutions. They do not, however, discuss the connection of these solutions to the interesting geometry of the spacetime manifold. This causes them to interpret their massless solution as a candidate for a universe 'created out of nothing' as described by Vilenkin[14]. We show that this massless solution should properly be interpreted as the Coleman - De Luccia bounce.[15]

In the next section we will review the Gauss-Codazzi formalism, which allows one to parameterize four dimensional spacetime by a one-parameter family of three dimensional hypersurfaces, and which expresses four dimensional geometric quantities in terms of three dimensional geometric quantities related to these hypersurfaces. The Einstein equations in this $3+1$ dimensional language yield junction conditions which determine the dynamics of the domain wall, given the wall's energy-momentum tensor. In section III we derive the form of the energy-momentum tensor for a domain wall, and in section IV we use this, along with the solutions to the Einstein equations in the true and false vacuum regions, to determine the equations of motion for the wall. In the fifth section we discuss the solutions of these equations of motion. We end with a summary which discusses some of the implications of these results.

II. JUNCTION CONDITIONS

In this section we will use the Einstein field equations to derive the equations which govern the evolution of the domain wall. These equations are called junction conditions because they describe the discontinuity, or junction, between the true and false vacuum regions.

The four dimensional Einstein equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \tag{2.1}$$

where the metric has one negative eigenvalue, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, and $T_{\mu\nu}$ is the matter energy-momentum tensor.

For the system under consideration,

$$T_{\mu\nu} = \begin{cases} -\rho_0 g_{\mu\nu} & \text{(in false vacuum region)} \\ 0 & \text{(in true vacuum region)}, \end{cases} \tag{2.2}$$

and (in the thin wall approximation) $T_{\mu\nu}$ has a delta function singularity on the domain wall. Here ρ_0 denotes the energy density of the false vacuum.

To describe the behavior of the domain wall, it is simplest to introduce a Gaussian normal coordinate system in the neighborhood of the wall. Denoting the $2+1$ dimensional spacetime hypersurface swept out by the domain wall as Σ, we begin by introducing a coordinate system on Σ. For definiteness, two of the coordinates can be taken to be the angular variables θ and ϕ that are always well-defined, up to an over-all rotation, for a spherically symmetric configuration. For the third coordinate, one can use the proper time variable τ that would be measured by an observer moving along with the domain wall. Next, consider all the geodesics which are orthogonal to Σ. Choose a neighborhood N about Σ so that any point $p \in N$ lies on one, and only one geodesic. The first three coordinates of p are then determined by the coordinates of the intersection of this geodesic with Σ. Since Σ is orientable, we may regard one side of Σ as being the 'positive direction.' For definiteness, we take the true vacuum side as positive. The fourth coordinate $\equiv \eta$ of any $p \in N$ is then taken as the proper distance in the positive direction from Σ to p along the geodesic passing through p. Thus, the full set of coordinates is given by (η, x^i), where $x^i \equiv (r, \theta, \phi)$, and i runs from 1 to 3.

In these coordinates the metric obeys the following simplifying conditions:

$$g^{\eta\eta} = g_{\eta\eta} = 1$$
$$g^{\eta i} = g_{\eta i} = 0. \tag{2.3}$$

Furthermore, one can define a unit vector field $\xi^\mu(x)$ which is normal to each of the $\eta = constant$ hypersurfaces and pointing from the de Sitter to the Schwarzschild spacetime. In the Gaussian normal coordinates, this vector field is given simply by

$$\xi^\mu(x) = \xi_\mu(x) = (1,0,0,0). \tag{2.4}$$

The extrinsic curvature corresponding to each $\eta = constant$ hypersurface is a three dimensional tensor whose components are defined by

$$K_{ij} = \xi_{i;j}, \tag{2.5}$$

Here the semicolon represents the four dimensional covariant derivative with respect to whatever index follows it, but the indices are restricted to the range of 1 to 3. In the Gaussian normal coordinates, the extrinsic curvature acquires the simple form

$$K_{ij} = -\Gamma^\eta_{ij} = \frac{1}{2}\partial_\eta g_{ij}. \tag{2.6}$$

One can easily see that K_{ij} is a symmetric tensor.

The Gauss-Codazzi formalism[16-17] is a method of viewing four dimensional spacetime as being sliced up into three dimensional hypersurfaces. At any point, the four dimensional tensors $R_{\mu\nu\sigma\tau}$, $R_{\mu\nu}$ and R may be expressed in terms of the corresponding three dimensional tensors and the extrinsic curvature of the hypersurface passing through the given point. One begins by noting that the only nonzero components of the affine connection are given by

$$\Gamma^k_{ij} = {}^{(3)}\Gamma^k_{ij},$$
$$\Gamma^\eta_{ij} = -K_{ij}, \tag{2.7}$$
$$\Gamma^i_{\eta j} = K^i{}_j,$$

where the superscript (3) denotes three dimensional geometric quantities. It can then be shown that the Einstein equations become

$$G^{\eta}{}_{\eta} \equiv -\frac{1}{2}{}^{(3)}R + \frac{1}{2}\{(TrK)^2 - Tr(K^2)\} = 8\pi G T^{\eta}{}_{\eta} \tag{2.8a}$$

$$G^{\eta}{}_{i} \equiv K_i{}^m{}_{|m} - (TrK)_{|i} = 8\pi G T^{\eta}{}_{i} \tag{2.8b}$$

$$\begin{aligned} G^i{}_j \equiv {}^{(3)}G^i{}_j - \{K^i{}_j - \delta^i{}_j TrK\}_{,\eta} - (TrK)K^i{}_j \\ + \frac{1}{2}\delta^i{}_j\{TrK^2 + (TrK)^2\} = 8\pi G T^i{}_j \end{aligned} \tag{2.8c}$$

where a comma denotes an ordinary derivative and a subscript bar (|) denotes the three dimensional covariant derivative.

The energy-momentum tensor $T^{\mu\nu}$ is expected to have a delta function singularity at the domain wall, so one can define the surface stress-energy tensor $S^{\mu\nu}$ by writing

$$T^{\mu\nu}(x) = S^{\mu\nu}(x^i)\,\delta(\eta) + (\text{regular terms}), \tag{2.9a}$$

equivalently,

$$S^{\mu\nu}(x^i) = \lim \epsilon \rightarrow 0 \int_{-\epsilon}^{\epsilon} d\eta T^{\mu\nu}(x)\,. \tag{2.9b}$$

In the next section we will discuss the form of $S^{\mu\nu}$ and will show that energy-momentum conservation implies $S^{\eta\eta} = S^{\eta i} = 0$.

When the energy-momentum tensor of Eq. (2.9) is inserted into the field equations (2.8), one sees that (2.8a) and (2.8b) are satisfied automatically provided that they are satisfied for $\eta \neq 0$ and that g_{ij} is continuous at $\eta = 0$ (so that K_{ij} does not acquire a delta function singularity). Eq. (2.8c) then leads to the junction condition

$$\gamma^i{}_j - \delta^i{}_j Tr\gamma = -8\pi G S^i{}_j\,, \tag{2.10}$$

where

$$\gamma_{ij} = \lim_{\epsilon \rightarrow 0}\{K_{ij}(\eta = +\epsilon) - K_{ij}(\eta = -\epsilon)\}\,. \tag{2.11}$$

By taking the trace of Eq. (2.10) we obtain $Tr\,\gamma = 4\pi G Tr\, S$, which can be substituted back into Eq. (2.10) to give

$$\gamma^i{}_j = -8\pi G \left[S^i{}_j - \frac{1}{2}\delta^i{}_j \, Tr\, S \right]. \tag{2.12}$$

A discussion of the meaning of this equation will be postponed until the properties of $S^{\mu\nu}$ are analyzed in the next section.

III. SURFACE STRESS-ENERGY OF A DOMAIN WALL

In this section we will use symmetry arguments and energy-momentum conservation to determine the form of the surface stress-energy defined by Eq. (2.9). A more detailed derivation will be given in Appendix A. For convenience we will use the Gaussian normal coordinate system described in the previous section.

The thin wall approximation assumes that the thickness of the domain wall is much less than any other length scale in the problem. On scales much larger than the thickness, the energy-momentum tensor of the wall can be accurately approximated by an expression proportional to a delta-function on the wall, $\delta(\eta)$, as in Eq. (2.9). Implicit in this description is the assumption that the domain wall has settled into an equilibrium configuration— otherwise it would radiate energy as it approached its equilibrium form, and the energy-momentum distribution would not remain confined to a thin wall.

Using Eq. (2.7), one can easily write down the equations for energy-momentum conservation in Gaussian normal coordinates:

$$T^{i\nu}{}_{;\nu} = T^{ij}{}_{|j} + T^{i\eta}{}_{,\eta} + 2K^{i}{}_{j}T^{j\eta} + (Tr\,K)T^{i\eta} = 0 \tag{3.1a}$$

$$T^{\eta\nu}{}_{;\nu} = T^{\eta i}{}_{|i} + T^{\eta\eta}{}_{,\eta} - K_{ij}T^{ij} + (Tr\,K)T^{\eta\eta} = 0. \tag{3.1b}$$

For the case of interest, $T^{\mu\nu}$ can be written as

$$T^{\mu\nu}(x) = S^{\mu\nu}(x^{i})\delta(\eta) - \rho_0\theta(-\eta)g^{\mu\nu}. \tag{3.2}$$

Combining (3.2) with (3.1a), one finds

$$T^{i\nu}{}_{;\nu} = \left[S^{ij}{}_{|j} + 2K^{i}{}_{j}S^{j\eta} + (Tr\,K)S^{i\eta}\right]\,\delta(\eta) + S^{i\eta}\,\delta'(\eta) = 0\,, \tag{3.3}$$

where the prime $(')$ denotes differentiation with respect to η. Note that Eq. (3.3) appears to contain an ambiguity, since K_{ij} must be evaluated at $\eta = 0$ where it is discontinuous. The problem arises because we are computing the gravitational force on a sheet of mass, a situation which is completely analogous to the elementary problem of evaluating the electrostatic force on a sheet of charge. However, by setting the coefficient of $\delta'(\eta)$ in Eq. (3.3) to zero, one learns that

$$S^{i\eta} = 0, \tag{3.4}$$

and the ambiguity disappears. The vanishing of the term in $\delta(\eta)$ then implies that

$$S^{ij}{}_{|j} = 0. \tag{3.5}$$

From (3.1b) one finds

$$T^{\eta\nu}{}_{;\nu} = [\rho_0 - \overline{K}_{ij}S^{ij} + (Tr\,K)S^{\eta\eta}]\,\delta(\eta) + S^{\eta\eta}\,\delta'(\eta) = 0, \tag{3.6}$$

where

$$\overline{K}_{ij} = \lim_{\epsilon\to 0}\frac{1}{2}\left[K_{ij}(\eta = +\epsilon) + K_{ij}(\eta = -\epsilon)\right]. \tag{3.7}$$

In this case the ambiguity does not disappear, but it will be shown in Appendix B that it can be resolved exactly as in the electrostatic example, with the result which is shown above. One can than deduce that

$$S^{\eta\eta} = 0 \tag{3.8}$$

and that

$$\overline{K}_{ij}S^{ij} = \rho_0. \tag{3.9}$$

Combining the orthogonality conditions (3.4) and (3.8) with rotational invariance, one concludes that $S^{\mu\nu}$ can be written as

$$S^{\mu\nu} = \sigma(\tau)U^{\mu}U^{\nu} - \varsigma(\tau)[h^{\mu\nu} + U^{\mu}U^{\nu}], \tag{3.10}$$

where

$$h^{\mu\nu} = g^{\mu\nu} - \xi^{\mu}\xi^{\nu} \tag{3.11}$$

is the metric projected into the hypersurface of the wall, and

$$U^{\mu} = (0,1,0,0) \tag{3.12}$$

is the four-velocity of the domain wall. Here σ is the surface energy density of the domain wall, and ς is the surface tension. Rotational invariance also implies that the metric on the domain wall can be written as

$$ds^2 = -d\tau^2 + r^2(\tau)\,d\Omega^2, \tag{3.13}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2$. Eq. (3.5) then reduces to

$$\dot{\sigma} = -2(\sigma - \varsigma)\frac{\dot{r}}{r}, \tag{3.14}$$

where the dot denotes a derivative with respect to τ. By introducing the area $A = 4\pi r^2$ of the sphere, the above equation can be rewritten as

$$\frac{d}{d\tau}(A\sigma) = \varsigma \frac{dA}{d\tau}, \tag{3.15}$$

a formula which is easily identified as the conservation of energy.

The values of σ and ς are further restricted by the underlying dynamics of the scalar field which comprises the domain wall, which has an energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu}\left(\frac{1}{2}\partial_\sigma \phi \partial_\tau \phi g^{\sigma\tau} + V(\phi)\right). \tag{3.16}$$

Note that the thin wall approximation assumes that any variation of ϕ along the wall occurs only on length scales much larger than the wall thickness, and so $\partial_\mu \phi \propto \xi_\mu$ to a high degree of accuracy. Thus $T_{\mu\nu}$ can only have terms proportional to $\xi_\mu \xi_\nu$ or to $g_{\mu\nu}$, and it follows that $\varsigma = \sigma$. It then follows from (3.14) that $\dot{\sigma} = 0$, and so finally

$$S^{\mu\nu}(x^i) = -\sigma h^{\mu\nu}(x^i). \tag{3.17}$$

Before closing this section we would like to discuss the intuitive meaning of Eqs. (2.12) and (3.9). Note that

$$\begin{aligned} K_{\tau\tau} &= \xi_{\tau;\tau} \\ &= U^\mu U^\nu \xi_{\mu;\nu} \\ &= -\xi_\mu U^\nu U^\mu{}_{;\nu} \\ &= -\xi_\mu \frac{DU^\mu}{D\tau}, \end{aligned} \tag{3.18}$$

where $DU^\mu/D\tau$ is the covariant acceleration of the wall, and thus $K_{\tau\tau}$ is its component in the normal direction. Thus, the discontinuity of $K_{\tau\tau}$ (proportional to $(\sigma+2\varsigma)$) implied by (2.12) represents a discontinuity in the acceleration of locally inertial frames. One has also

$$K_{\theta\theta} = \frac{K_{\phi\phi}}{\sin^2\theta} = \frac{1}{2}\partial_\eta r^2 , \tag{3.19}$$

so the discontinuity in the angular components of the extrinsic curvature (proportional to σ) measures the discontinuity of geometric distortion.

Finally, Eq. (3.9) can be written as

$$\sigma \overline{\xi_\mu \frac{DU^\mu}{D\tau}} = -\frac{\varsigma}{r^2}\overline{\partial_\eta r^2} - \rho_0 , \tag{3.20}$$

where the overbar means that the indicated quantity is to be averaged over the values it has on either side of the discontinuity at $\eta = 0$. The above equation is easily identified as the equation of motion for a spherical membrane with surface tension ς and a constant pressure difference ρ_0 pointing inward.

IV. EQUATIONS OF MOTION FOR A DOMAIN WALL

The most general O(3) symmetric solution to the Einstein equations for a region of spacetime with vanishing cosmological constant and matter energy momentum tensor is given by Birkhoff's theorem as

$$-d\tau^2 = ds^2 = -(1 - \frac{2GM}{R})dT_S^2 + (1 - \frac{2GM}{R})^{-1}dR^2 + R^2 d\Omega^2 \tag{4.1}$$

where M is an as yet undetermined parameter. Equation 4.1 presents the Schwarzschild line element, which describes the true vacuum side of the domain wall. We will let the lower case letter r and t_S denote the values of the corresponding Schwarzschild coordinates at the domain wall.

The standard Schwarzschild coordinates displayed in Eq. (4.1) are not in one to one correspondence with the points of the Schwarzschild manifold— indeed each coordinate (T_S, R, Θ, Φ) is associated with two points of this spacetime. It is possible to construct coordinate systems which do not have this pathology. An example is the Kruskal-Szekeris coordinate system defined by

$$U = \left(\frac{R}{2GM} - 1\right)^{\frac{1}{2}} \exp\left(\frac{R}{4GM}\right) \cosh\left(\frac{T_S}{4GM}\right) \tag{4.2a}$$

$$V = \left(\frac{R}{2GM} - 1\right)^{\frac{1}{2}} \exp\left(\frac{R}{4GM}\right) \sinh\left(\frac{T_S}{4GM}\right) \tag{4.2b}$$

which defines what we shall call region I;

$$U = -\left(1 - \frac{R}{2GM}\right)^{\frac{1}{2}} \exp\left(\frac{R}{4GM}\right) \sinh\left(\frac{T_S}{4GM}\right) \tag{4.2c}$$

$$V = -\left(1 - \frac{R}{2GM}\right)^{\frac{1}{2}} \exp\left(\frac{R}{4GM}\right) \cosh\left(\frac{T_S}{4GM}\right) \tag{4.2d}$$

defining region II;

$$U = -\left(\frac{R}{2GM} - 1\right)^{\frac{1}{2}} \exp\left(\frac{R}{4GM}\right) \cosh\left(\frac{T_S}{4GM}\right) \tag{4.2e}$$

$$V = -\left(\frac{R}{2GM} - 1\right)^{\frac{1}{2}} \exp\left(\frac{R}{4GM}\right) \sinh\left(\frac{T_S}{4GM}\right) \tag{4.2f}$$

defining region III;

$$U = \left(1 - \frac{R}{2GM}\right)^{\frac{1}{2}} \exp\left(\frac{R}{4GM}\right) \sinh\left(\frac{T_S}{4GM}\right) \tag{4.2g}$$

$$V = \left(1 - \frac{R}{2GM}\right)^{\frac{1}{2}} \exp\left(\frac{R}{4GM}\right) \cosh\left(\frac{T_S}{4GM}\right) \tag{4.2h}$$

which defines region IV. The values of the coordinates U and V at the domain wall will be called u and v respectively.

In the Kruskal-Szekeres coordinate system the line element (4.1) is written

$$-d\tau^2 = ds^2 = \frac{32M^3}{R}\exp\left(\frac{-R}{2M}\right)(-dV^2 + dU^2) + R^2 d\Omega^2\,, \tag{4.3}$$

where R is a function of U and V given by

$$\left(\frac{R}{2M} - 1\right)\exp\left(\frac{R}{2M}\right) = U^2 - V^2\,. \tag{4.4}$$

The manifold contains a region of Schwarzschild space, a region of de Sitter space, and a boundary which separates them. Since the Schwarzschild spacetime has the symmetry $U \rightarrow -U$, we can always choose the region of Schwarzschild space so that, for a given spacelike slice, arbitrarily large radii are contained in region I as defined by equations (4.2a) and (4.2b). We will discover the domain wall trajectories to fall into several classes, representative elements of which are illustrated in Figs. 3,4,6, and 7. In each figure the arrow points into the region of Schwarzschild space.

The value of the coordinate R at the domain wall, $R = r(\tau)$, has a meaning which is invariant under all coordinate transformations which leave the Schwarzschild spacetime and bounding domain wall spherically symmetric; $r^2(\tau)$ is the proper area on the domain wall subtended by a solid angle $d\Omega$, divided by $d\Omega$. One calls $r(\tau)$ the proper circumferential radius of the domain wall.

The line element for the de Sitter spacetime may be expressed in a Robertson-Walker, $k = 1$ coordinate system as

$$-d\tau^2 = ds^2 = -dT_D^2 + \frac{\cosh^2(\chi T_D)}{\chi^2}(d\Psi^2 + \sin^2\Psi d\Omega^2) \tag{4.5}$$

where $\chi = \sqrt{\frac{8\pi}{3}G\rho_0}$ is called the inverse de Sitter length. Let $\psi(\tau)$ be the value of the coordinate Ψ at the domain wall, and likewise define $t_D(\tau)$. Then the proper circumferential radius is

$$r(\tau) = \frac{\cosh(\chi t_D(\tau))}{\chi}\sin\psi(\tau)\,. \tag{4.6}$$

Our goal is to use the junction condition (2.12) to derive a dynamical equation for the proper circumferential radius of the domain wall. The Schwarzschild and de Sitter line elements along with the behaviour of $r(\tau)$ specify completely the system parameterized by the mass M.

By combining (2.12) with the expression (3.17) for the surface stress-energy $S^{\mu\nu}$, one finds

$$\gamma^i{}_j = -4\pi\sigma G\delta^i{}_j . \tag{4.7}$$

Because the domain wall is spherically symmetric, the four velocity of any point on the wall, as seen by a Schwarzschild observer assumes the form

$$U^\mu_S = (\dot{v}, \dot{u}, 0, 0) \tag{4.8a}$$

in Kruskal-Szekeres coordinates, with a dot signifying differentiation with respect to proper time, and

$$U^\mu_S = (\dot{t}_S, \dot{r}, 0, 0) \tag{4.8b}$$

in the standard Schwarzschild coordinates. We shall choose the flow of proper time so that future directed world lines satisfy $\dot{V} > 0$.

The Schwarzschild normal ξ^μ_S is a smooth vector field defined on the domain wall, of unit length, orthogonal to U^μ_S, and pointing from the de Sitter to the Schwarzschild spacetime. Our convention that, for a given spacelike slice, arbitrarily large radii are contained in region I, along with the choice $\dot{v} > 0$ guarantees that

$$\xi^\mu_S = (\dot{u}, \dot{v}, 0, 0) \tag{4.9}$$

in Kruskal-Szekeres coordinates. Transforming to the standard Schwarzschild coordinates one finds

$$\xi^\mu_S = (A^{-1}\dot{r}, (A + \dot{r}^2)^{\frac{1}{2}} \text{sign}(\dot{t}_S \cdot A), 0, 0) \tag{4.10}$$

where $A = (1 - \frac{2GM}{r})$ and sign $(\dot{t}_S \cdot A) = +1$ or -1 according to whether $(\dot{t}_S \cdot A)$ is positive or negative. Notice that since T_S is not a timelike coordinate everywhere on the Schwarzschild manifold, and since we chose the proper time flow so that $\dot{V}$ was positive, there is no guarantee that $\dot{t}_S > 0$, even if the domain wall is beyond the bubble's Schwarzschild radius *i.e.*, even if $A > 0$.

The four velocity of any point on the domain wall as seen by a de Sitter observer has components

$$U^{\mu}_{D} = (\dot{t}_D, \dot{\psi}, 0, 0) \tag{4.11}$$

and the de Sitter normal is determined to within a sign to be

$$\xi^{\mu}_{D} = \pm(\frac{\cosh \chi t_D}{\chi}\dot{\psi}, \frac{\chi \dot{t}_D}{\cosh \chi t_D}, 0, 0) \tag{4.12}$$

We are free to use the isometries

$$T_D \rightarrow T'_D = -T_D$$

and

$$\Psi \rightarrow \Psi' = \pi - \Psi$$

so as to ensure that (dropping primes as necessary) $\dot{t}_D$ is positive and that the normal has components

$$\xi^{\mu}_{D} = +(\frac{\cosh \chi t_D}{\chi}\dot{\psi}, \frac{\chi \dot{t}_D}{\cosh \chi t_D}, 0, 0) \tag{4.13}$$

Notice that with these conventions ξ_{ψ} is positive. Since the normal points from de Sitter to Schwarzschild space, this means that by convention we have chosen the north pole, $\psi = 0$, to always be a part of the de Sitter manifold.

It is now just a matter of some algebra to calculate the components of the extrinsic curvature as seen by Schwarzschild and de Sitter observers, and to plug these values into Eq. (4.7) in order to assertain the dynamics of the bubble wall. The calculation is facilitated by working in Gaussian normal coordinates, so that K_{ij} is given simply by Eq. (2.6).

Inspection of Eq. (4.7) reveals that only two of the components are linearly independent; the off-diagonal components vanish identically and the diagonal angular components are equal.

We may evaluate the angular components of Eq. (4.7) with the help of Eq. (3.19). In the Schwarzschild region

$$\begin{aligned} K_{\theta\theta} &= \frac{1}{2}\partial_\eta r^2 \\ &= \frac{1}{2}\xi^\mu \partial_\mu r^2 = r\xi^r \\ &= r(A+\dot{r}^2)^{\frac{1}{2}}\text{sign}(\dot{t}_S \cdot A) \equiv r\beta_S \end{aligned} \tag{4.14}$$

where, in the penultimate step we have used Eq. (4.10).

In the de Sitter region

$$\begin{aligned} K_{\theta\theta} &= \frac{1}{2}\partial_\eta r^2 \\ &= \frac{1}{2}\xi^\mu \partial_\mu r^2 \\ &= \frac{1}{2}\xi^\mu \partial_\mu (\frac{1}{\chi^2} \cdot \cosh^2 \chi t_D \sin^2 \psi) \\ &= r(\dot{t}_D \cos\psi + \frac{\dot{\psi}}{2\chi}\sinh(2\chi t_D)\sin\psi) \equiv r\beta_D \end{aligned} \tag{4.15}$$

where, we have used Eq. (4.13).

In terms of the proper circumferential radius

$$K_{\theta\theta} = \pm r\sqrt{1-\chi^2 r^2 + \dot{r}^2} \tag{4.16}$$

in the de Sitter region.

The angular components of Eq. (4.7) may now be written

$$\beta_D - \beta_S = 4\pi\sigma G r\,, \tag{4.17}$$

with β_D and β_S defined in Eqs. (4.15) and (4.14) respectively.

With the help of Eq. (3.18) one can calculate the $\tau\tau$ component of our equations of motion (4.7). The result is simply the proper time derivative of Eq. (4.17). Indeed, it is not surprising that there should be a functional relationship between the two linearly independent components of (4.7); such a relationship is guaranteed by the fact that the Einstein equations imply conservation of $T_{\mu\nu}$, but the $T_{\mu\nu}$ used in this calculation is manifestly conserved. One can easily check that Eq. (3.20), derived directly from conservation of $T_{\mu\nu}$, is equivalent to the $\tau\tau$ component of Eq. (4.7).

Eq. (4.17) allows one to express the mass M of the bubble wall in terms of r and $\dot{r}$ as

$$\begin{aligned} M = \frac{\chi^2 r^3}{2G} &+ 4\pi\sigma r^2(1-\chi^2 r^2+\dot{r}^2)^{\frac{1}{2}}\,\mathrm{sign}\beta_D \\ &- 8\pi^2 G\sigma^2 r^3. \end{aligned} \tag{4.18}$$

It is instructive to consider the limiting case $\chi^2 r^2,\ \dot{r}^2 \ll 1$, $\mathrm{sign}\beta_D = +1$. Then

$$\begin{aligned} M \approx \frac{\chi^2 r^3}{2G} &+ 4\pi\sigma r^2(1+\dot{r}^2)^{\frac{1}{2}} \\ &- 2\pi\sigma\chi^2 r^4 - 8\pi^2 G\sigma^2 r^3. \end{aligned} \tag{4.19}$$

We recognize the four terms of Eq. (4.19) in order as the volume energy of the bubble, the surface energy of the bubble, with lowest order relativistic correction, the Newtonian surface-volume binding energy, and the Newtonian surface-surface binding energy. Curiously, there is no volume-volume interaction term.

V. DISCUSSION OF THE SOLUTIONS

In this section we will describe the solutions to the equations of motion for $r(\tau)$ by presenting the possible trajectories of the domain wall in $(\dot{r}(\tau), r(\tau))$ space. Our understanding will be guided by several 'landmarks' on the $(\dot{r}(\tau), r(\tau))$ plot, namely:

1) the locus of points for which $\beta_D = 0$,

2) the locus of points for which $\beta_S = 0$,

3) the locus of points for which $r =$ the Schwarzschild radius of the bubble $= R_{SCH}$.

We will discover the trajectories of the bubble wall to fall into four main classes— the black hole solutions discussed by Ipser and Sikivie[11] and Berezin *et. al.*,[13] the wormhole solutions studied by Berezin *et. al.*, bubbles which inflate to arbitrarily large proper circumferential radius, and bounce solutions, including the massless Coleman-De Luccia[15] bounce.

The locus of points for which $\beta_D = 0$ is given by

$$\beta_D = \sqrt{(1 - \chi^2 r^2 + \dot{r}^2)} = 0. \tag{5.1}$$

Eq. (5.1) gives a kinematically determined lower bound for $|\dot{r}|$ given $r \geq \frac{1}{\chi}$; since β_D must be real, one has $|\dot{r}| \geq (\chi^2 r^2 - 1)^{\frac{1}{2}}$. We will find it convenient to view the trajectories of the bubble wall as lying on one of two sheets, according to whether β_D is positive or negative.

The locus of points for which $\beta_S = 0$ is determined by

$$\dot{r}^2 = -1 + (\chi^2 + (4\pi\sigma G)^2) r^2, \tag{5.2}$$

but note that Eq. (5.2) corresponds to $\beta_S = 0$ only on the sheet with sign$\beta_D = +1$. If sign$\beta_D = -1$, Eq. (5.2) corresponds to $\beta_S = -8\pi\sigma G r$. Eq. (4.14) indicates that the sign of β_S is related to the sign of $\dot{t}_S$, a relation upon which we will comment shortly. For now we note that if the domain wall is within the Schwarzschild radius of the bubble, then t_S is not the timelike coordinate, hence $\dot{t}_S$ may assume any value whatsoever, including zero. Thus it is of interest to consider the locus of points for which the domain wall is located precisely at the bubble's Schwarzschild radius. This locus is

$$\dot{r} = \text{sign}\beta_S \frac{1 - \left[\chi^2 + (4\pi\sigma G)^2\right] r^2}{8\pi\sigma G r}. \tag{5.3}$$

Figure 1 displays the $(\dot{r}, r)$ space with the landmarks given by Eqs. (5.1)-(5.3) filled in. We consider only $\dot{r} \geq 0$ because the equations of motion are symmetric under $(\ddot{r}, \dot{r}, r) \rightarrow (\ddot{r}, -\dot{r}, r)$, that is, $\tau \rightarrow -\tau$.

In discussing the equations of motion we introduce the simplifying notation

$$4\pi\sigma G \equiv \kappa, \tag{5.4a}$$

$$\chi^2 - \kappa^2 \equiv \hat{\chi}^2, \tag{5.4b}$$

$$\chi^2 + \kappa^2 \equiv \check{\chi}^2. \tag{5.4c}$$

Since $\frac{\kappa}{\chi} \sim \frac{\text{grand unified scale}}{\text{Planck scale}} \ll 1$, we will assume $\hat{\chi}^2$ is positive. We write the equations of motion in the form

$$\beta_D - \beta_S = \kappa r, \tag{5.5a}$$

which implies

$$2GM = \hat{\chi}^2 r^3 + 2\kappa r^2 \beta_D, \tag{5.5b}$$

the time derivative of which may be put into the form

$$\kappa r \ddot{r} = \frac{-GM\beta_D}{r^2} - \kappa\beta_D\beta_S - \chi^2 r \beta_S. \tag{5.6}$$

Consider the possibility of trajectories which satisfy $r_{max} \leq 1/\check{\chi}$. Since $\beta_S = 0$ only if $\dot{r}^2 = -1 + \check{\chi}^2 r^2$, such trajectories would have a constant sign for β_S. It turns out that the only consistant choice is sign$\beta_S =$ $+1$. Then, from Eq. (5.5a), sign$\beta_D = +1$ and Eq. (5.6) shows that $\ddot{r} < 0$ at all times. A Schwarzschild observer who initializes proper time according to $\tau(r_{max}) = 0$ would see the domain wall emerge from a white hole (*i.e.*, cross the bubble's Schwarzschild radius from within) at a finite (negative) proper time, expand to a maximum radius, then contract, crossing the Schwarzschild radius from without, and hence becoming invisible, at some finite (positive) proper time. Indeed, since $\dot{r} \rightarrow r^{-2}$ as $r \rightarrow 0$ via Eq. (5.5b), the proper lifetime of the bubble is finite. These trajectories are the black hole solutions discussed by Ipser and Sikivie[11] and Berezin *et. al.*[12] A representative element of this family is labeled '1' in Fig. 2. In Fig. 3 we have plotted this trajectory in the Kruskal-Szekeres (U, V) coordinates, which, unlike the standard (R, T_S) coordinates are in 1-1 correspondence with the points on the Schwarzschild manifold.[16] The arrow in Fig. 3 indicates the side of the domain wall in which the Schwarzschild geometry lies.

Of all the black hole trajectories described above, the one with $r_{max} = 1/\check{\chi}$ has the greatest speed, $|\dot{r}|$, given r. What, then, is the fate of a domain wall with given r and speed greater than that attained by this extremal black hole trajectory, and which reaches some maximum radius $> 1/\check{\chi}$. A representative trajectory of this type is labeled '2' in Fig. 2. Inspection of this figure reveals that all such trajectories cross the line $\beta_S = 0$, moreover, the domain wall is always within the bubble's Schwarzschild radius at the time of crossing. Further, one can check that

$$\frac{d}{d\tau}\beta_S \,|_{\beta_S=0} < 0, \tag{5.7}$$

so that when the line $\beta_S = 0$ is crossed, the sign of $\dot{t}_S$ changes. The behavior of such a domain wall trajectory in Kruskal-Szekeres coordinates is displayed in Fig. 4, in which the dots represent the points at which $\beta_S = 0$. In other words, the dots represent the points at which the domain wall is tangent to a line of constant t_S. The most striking feature of Fig. 4 is that the domain wall enters the region of the Schwarzschild manifold (labeled III) causally disconnected from the region (labeled I) which contains arbitrarily large radii The Schwarzschild geometry thus develops a wormhole as the domain wall evolves, as was observed by Berezin *et. al.* In describing the history of the domain wall from the Schwarzschild point of view, we must appreciate that there are two qualitatively different Schwarzschild observers who may view the domain wall from beyond the Schwarzschild radius of the bubble; they reside in regions I and III. The observer in region I may receive information about the bubble wall history until, at the latest, such time as the domain wall enters the causally disconnected region III. From this (proper) time on, the wall is invisible to the region I observer. A Schwarzschild observer in region III will be crushed as the domain wall contracts to zero radius. There is an *a priori* possibility which, when we examine this trajectory from the point of view of a de Sitter observer, we will see is not realized. It is that as $r \to 0$, $\psi \to 2\pi$ so that the de Sitter region has finite spatial volume even though it has vanishing proper circumferential radius. In this fanciful scenario, a Schwarzschild observer in region III could cross the domain wall into de Sitter space and survive.

Suppose a bubble wall is stationary, $\dot{r} = 0$, at some maximum radius r_{max}. Evidently, as r_{max} increases, the radius at which the wall trajectory crosses the $r = R_{SCH}$ line increases. That is, as r_{max} increases, M increases. A plot of $2GM$ versus r at fixed $\dot{r} = 0$ and for $\beta_D > 0$ is illustrated in Fig. 5. Bubbles which ultimately collapse cannot attain a radius greater than some maximum $M \equiv \bar{M}$, nor can they achieve a radius greater than $r \equiv \bar{r} \leq \frac{1}{\chi}$ with equality if, and only if, $\kappa = 0$. Bubbles with mass greater than $\bar{M}$ have no maximum radius; they inflate to arbitrarily large r. A typical such inflationary trajectory is labeled '3' in Fig. 2, and is plotted in Kruskal-Szekeres coordinates in Fig. 6. Note that these inflationary trajectories enter into the sign$\beta_D = -1$ sheet, and that they do so by approaching the $\beta_D = 0$ line tangentially. Like the wormhole trajectories of Fig. 4, the inflationary trajectories may be viewed by two different types of Schwarzschild observers who are beyond the bubble's Schwarzschild radius. An observer in region I would see a wall history similar to that of the wormhole trajectories: he may receive information from the wall until such time as (at the latest) the domain wall enters into region III of the Schwarzschild manifold. A region I observer cannot see an inflationary bubble expand to arbitrarily large radius. On the other hand, region III observers can view an inflationary domain wall receeding to arbitrarily large radius.

Let us estimate the mass $\bar{M}$ which separates inflating solutions from those that collapse by noting that if $\chi \gg \sigma$ then

$$\bar{M}(\sigma) \approx \bar{M}(\sigma = 0) = \frac{1}{2G\chi} = \frac{4}{3}\pi \rho_0 / \chi^3 \tag{5.8}$$

where $\rho_0 = \frac{3\chi^2}{8\pi G}$ is the energy density of a false vacuum region. Thus, $\bar{M}$ is approximately the volume energy of a false vacuum bubble whose radius is the de Sitter length. The energy density ρ_0 is determined by the grand unification scale and should be of order $(10^{14}\text{ GeV})^4$ which implies that $\bar{M} \sim 10^{28}$ GeV ~ 10 kg.

We have observed that it is not possible for domain walls which ultimately collapse to attain radii greater than $\bar{r}$ defined in Fig. 5. However, there do exist solutions to the equations of motion for r which do not collapse, and which have extremal radii, r_{ext}, greater than $\bar{r}$. Let us begin our study of these solutions by considering their mass as a function of r_{ext}. If the domain wall is on the sign$\beta_D = +1$ sheet when r_{ext} is reached, then Fig. 5 shows that $2GM$ decreases monotonically from $\bar{M}$ to $\hat{\chi}^2\chi^{-3}$ as r_{ext} increases from $\bar{r}$ to $\frac{1}{\chi}$. If the domain wall is on the sign$\beta_D = -1$ sheet when r_{ext} is attained, then M decreases monotonically as r_{ext} decreases, and $M = 0$ for some positive r_{ext}. Since

$$\ddot{r} = \frac{-\beta_D}{2\kappa r^2} \cdot \frac{\partial}{\partial r} 2GM \,|_{\dot{r}}, \tag{5.9}$$

it follows that these extremal radii are, in fact, minima. Moreover, since $M < \bar{M}$ for all 'bounce' solutions, the domain walls of these bounces must always be beyond the Schwarzschild radius of the bubble. Typical domain wall trajectories for bounce solutions are labeled '4' and '5' on Fig. 2; note that the bounce labeled '5' lives entirely on the sheet with sign$\beta_D = -1$. The corresponding trajectories look almost identical when displayed in Kruskal-Szekeres coordinates. They appear as in Fig. 7. From the point of view of a Schwarzschild observer in region I, the bounce solution is completely invisible, that is, it would be seen as a black hole created by a point mass, M, located at the origin. A Schwarzschild observer in region III would view the domain wall at a radius greater than that at which he is located, and, like the observer in region I, would measure a vanishing energy momentum tensor $T_{\mu\nu}$. Therefore, he too would interpret M as the mass of a point source located at the origin. Since all Schwarzschild observers interpret M to be the mass of a point source, we may require, on physical grounds, that M be nonnegative.

The bounce solution with $M = 0$ that we have exhibited was first discovered by Coleman and De Luccia.[15] In the context of the original inflationary cosmology, one interprets these $M = 0$ bubbles as nucleating at $\tau = 0$ in a medium of 'false vacuum' , that is, of de Sitter space. At any given time, these $M = 0$ bubbles describe a spatial region of 'true vacuum' (flat space) surrounded by false vacuum. Indeed, the behavior of all our bounce solutions is qualitatively similar to that of the Coleman - De Luccia bounce. Conservation of energy though, prohibits the nucleation of any bubbles with nonzero mass.[15] In Appendix C we will demonstrate the equivalence between the description of the $M = 0$ bounce solution given by Coleman and De Luccia, and by ourselves.

We have described various domain wall trajectories with an emphasis on the behavior of the proper circumferential radius $r(\tau)$. We would now like to briefly review these trajectories which ultimately collapse and focus on the Robertson - Walker angular coordinate $\psi(\tau)$. The relationship between the two is

$$r(\tau) = \frac{1}{\chi} \sin\psi(\tau) \cosh \chi t_D(\tau) . \tag{5.10}$$

Recall that the parameter β_D is given by

$$\beta_D = \dot{t}_D \cos\psi + \frac{\dot{\psi}}{2\chi} \sinh(2\chi t_D) \sin\psi, \tag{5.11}$$

which is dominated by the first term if $\chi r \leq 1$. Thus in this regime the signs of β_D and $\cos\psi$ are identical. For the black hole and wormhole trajectories, $\chi r < 1$ and $\mathrm{sign}\beta_D = +1$. Therefore, $r = 0$ must correspond to $\psi = 0$, which justifies our earlier claim that an observer in region III of the Schwarzschild manifold watching a wormhole trajectory collapse cannot escape to safety by crossing the domain wall into the de Sitter region. Note that, in general, the point $\Psi = 0$ on the de Sitter manifold is not singular, as may be seen by inspecting the de Sitter metric. In the present case, however, the entire de Sitter manifold collapses to a point— a singular occurence indeed.

While we can determine that the collapsing domain wall trajectories terminate with $\psi = 0$, we cannot make an unambiguous statement about the asymptotic behavior of ψ for the inflationary and bounce trajectories. The reason is that the evolution of ψ depends on how the coordinate time t_D is initialized. There are two natural ways one can initialize the time: one can require that inflationary trajectories commence at $t_D = 0$ and that bubble trajectories are invariant under $t_D \rightarrow -t_D$, or one can require that the angular coordinate ψ approach $\frac{\pi}{2}$ aysmptotically. These two conditions are generically incompatible, an exception, however, is the $M = 0$ Coleman - De Luccia bounce, as is shown in Appendix C.

We have plotted, in Figs. 3, 4, 6, and 7 several domain wall trajectories in Kruskal-Szekeres coordinates. Our spacetime consists of a region of Schwarzschild space lying (by convention) to the right of these trajectories and a region of de Sitter space lying to the left. In Fig. 8 we exhibit our spacetime for the case of the inflationary trajectory illustrated in Fig. 6. Our coordinates correspond to the usual Kruskal-Szekeres coordinates in the Schwarzschild region, and to some appropriate extension thereof in the de Sitter region. In both regions, null trajectories are oriented at 45° to the vertical and timelike worldlines are generally vertical. Several constant V (spacelike) hypersurfaces are indicated in Fig. 8. By fixing the Θ coordinate at some constant value we generate a two dimensional spacelike surface which may be embedded[16] in R^3. We illustrate these embedded surfaces in Fig. 9. Note that it is the two dimensional surfaces of our three dimensional representations which correspond to the constant V and Θ surfaces of our spacetime. At large V our spacetime consists of two disjoint universes only one of which contains any region of finite, nonzero energy density. It is this region which inflates and ultimately produces the Friedmann-Robertson-Walker universe observed today.

VI. DISCUSSION AND CONCLUSIONS

In this paper we have analyzed the behavior of a spherically symmetric region of false vacuum which is separated from an infinite region of true vacuum by a domain wall. Several issues of cosmological interest may be viewed in a new light because of these results.

First there are the implications for the early universe. While some theories[14,18] predict an early universe which is very homogeneous, other approaches[19] describe an early universe which is highly chaotic. An important feature of the inflationary mechanism is that it allows for the possibility of such highly chaotic initial conditions. In a chaotic theory it is plausible that a region which inflates might be surrounded by regions which do not inflate, and one would like to understand the evolution of such a system. If our idealized spherically symmetric problem is indicative, then one would expect that the inflating region would appear from the 'outside' (*i.e.*, from region I of the Schwarzschild manifold) as a black hole, and essentially all of the inflation would take place in the causally disconnected region III. The inflating region would then disconnect completely from the manifold which spawned it, forming an isolated closed universe. We have illustrated several snapshots of our evolving system in Fig. 9, where we have used the Kruskal-Szekeres coordinate V as the time variable. The isolated closed universe seen in the final snapshot consists of two regions, as may be seen in Fig. 8, hypersurface D. The first consists of false vacuum which soon decays into thermal radiation, producing a huge region which behaves as a standard Friedmann-Robertson-Walker universe. The other region is an essentially empty Minkowski space, with a Schwarzschild black hole in the center. The black hole evaporates by emitting Hawking radiation. The evaporation rate depends inversely on the cube of the black hole mass, so that a sufficiently light black hole can evaporate very quickly.

Since the detachment of an isolated closed universe is a significant feature of the solution, it is important to consider the extent to which this description depends on the choice of spacetime slicing. Let us for the moment ignore the possibility of black hole evaporation. The future singularity of Schwarzschild space lies on a spacelike hypersurface, and it is therefore possible to choose equal-time slices which approach the singularity without ever reaching it— in such a coordinate system, the manifold would remain connected at all times. Nonetheless it remains true that in all coordinate systems there exists a time T^* such that for all times greater than T^* the de Sitter space region of the spacetime is causally disconnected from the Minkowski space region, that is, from region I. Upon decay of the false vacuum, the resulting Friedmann - Robertson - Walker space is likewise causually disconnected from the Minkowski space. If we admit the possibility of black hole evaporation then we can make a stronger statement. Let us work at times greater then T^* and define p to be the spacetime event which is the terminus of the black hole and which is causally disconnected from the de Sitter space region. Let S denote the set of points which may be reached from p by future directed nonspacelike curves. Not only is S causally disconnected from the de Sitter region, there are no spacelike curves connecting any point in S with any point in the de Sitter region; S is topologically disconnected from the de Sitter region. The region S is also topologically disconnected from the Friedmann - Robertson - Walker region created when the false vacuum decays. Thus we conclude that the detachment of the inflationary region is a meaningful statement.

The question of whether or not it is possible in principle[20] to produce an inflationary universe in the laboratory remains unanswered, but the issues appear to be clearly defined. The exact solutions for inflationary bubbles described in Section V all begin with an initial singularity, a feature which has to be avoided if one is to produce such an object in the laboratory. Thus, one must imagine trying to produce the same final state from a different initial state. It seems to us that an arbitrarily low initial mass density is one criterion that an acceptable 'laboratory' state must satisfy. The extraordinary mass densities involved in the inflationary solutions should be developed by concentrating low density matter from a much larger region. The difficulty stems from the fact that the standard picture of the gravitational collapse of ordinary matter produces the situation shown in Fig. 10, which is quite different from the inflationary solution shown in Fig. 8. In particular, the standard picture of gravitational collapse does not lead to the full future singularity as seen in the inflationary solution.

The challenge, then, is to find a way to set up as an initial condition the configuration corresponding to a nonsingular spacelike hypersurface of the exact solution, such as the hypersurfaces marked B or C in

Fig. 8. Note that there is no topological barrier to constructing such an initial condition, since Fig. 9 shows clearly that these hypersurfaces are topologically equivalent to Euclidean three-space. However, it is still not clear whether or not it is possible to set up such a configuration without an initial singularity. We hope to investigate this topic more fully in the future.[21]

Our results suggest the possibility that an inflationary universe could be created by quantum mechanical tunneling from a Minkowski space. To see how this might occur, note that Minkowski space is not an eigenstate of the energy density operator. Although the total energy of the Minkowski space is zero, the mean energy density in any given region is constantly fluctuating between positive and negative values, with the average at zero. Thus it is conceivable that a local region could fluctuate into a high energy false vacuum state, producing a situation similar to that shown as hypersurface B or C in Fig. 9. The region could then evolve temporarily according to the classical evolution shown in Fig. 9, resulting in a closed inflationary universe which disconnects from the original Minkowski space. The Minkowski space is then left with a virtual hole, which then disappears by Hawking evaporation. For a black hole with a mass of order $M \approx 10^{28}$ GeV as expected for this kind of process, the time scale for Hawking evaporation is given by $M^3/M_P^4 \approx 10^{-14}$ sec. Thus, the net result is an initial Minkowski space which tunnels to become a final Minkowski space plus a closed inflationary universe. It seems clear that no conservation laws are violated in this hypothetical process. The possibility of such tunneling remains for now a matter of speculation, but perhaps further work can clarify the situation.

If an inflationary universe can be created by tunneling from Minkowski space, then the process may be key step in a solution to the cosmological constant problem. Abbott[22] has recently proposed a model which, given some assumptions about the underlying particle physics, explains how the universe could evolve into a huge region of very nearly Minkowskian spacetime. The idea of remaining for a long time in a Minkowski space seems to be an attractive feature for any scheme which solves the cosmological constant problem by dynamical relaxation, since it is hard to see how the delicate cancellations required to fix the cosmological constant could be the result of processes which take place at high energy. However, in order to make such a scenario workable, one must have a mechanism for producing an acceptable universe from a Minkowski space. The tunneling process described above provides such a possibility.

Finally, the results obtained here are apparently relevant to the question of whether or not information is irretrievably lost in a black hole. At the classical level it is clear that the loss of information is irreversible, but some authors[23] have argued that this information might be returned to the external spacetime during the process of black hole evaporation. In the case of false vacuum bubbles, however, it seems clear that the detached inflationary universe will not disappear when the black hole in the Minkowski space evaporates. Thus, at least in this one example, one is led to believe that a repository for information exists outside the Minkowski space, and it is hard to believe that information is not lost.

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APPENDIX A: SURFACE STRESS-ENERGY OF A DOMAIN WALL

We have described a domain wall as a world sheet which separates two regions of spacetime with differing energy densities. This is an idealization; in reality the energy density changes smoothly over some region of nonzero thickness ϵ. Such a configuration may be realized by a scalar field ϕ, which changes dramatically over this region. We employ the 'thin wall' approximation which is valid if the thickness of this region is rather less than the other length scales of the problem. In particular, we assume that this thickness is much smaller than the length scales over which ϕ varies appreciably in the directions transverse to the wall. We may give this a physical interpretation by noting that the four velocity U^μ is tangent to the domain wall. Thus our approximation is valid only if $| \xi^\mu \partial_\mu \phi | \gg | U^\mu \partial_\mu \phi |$, in other words, only if to leading order in ϵ, the domain wall is static in its rest frame. This more realistic conception of the structure of the domain wall obliges us to modify slightly our definition of the surface stress-energy: we do not regard the integral in the definition

$$S^\alpha{}_\beta = \int_{-\epsilon}^{\epsilon} d\eta\, T^\alpha{}_\beta \tag{A.1}$$

as being over a region of infinitesimal thickness, rather, it is an integral over a region of nonzero but small thickness— the region we have been calling the domain wall. In order to analyze the surface stress-energy of a domain wall it is convenient to erect a Gaussian normal coordinate system in a neighborhood N about a three dimensional hypersurface contained within the domain wall. We assume that this neighborhood is large enough to encompass the entire domain wall. The energy-momentum tensor of the scalar field is

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu}\left(\frac{1}{2}\partial_\sigma \phi \partial_\tau \phi g^{\sigma\tau} + V(\phi)\right), \tag{A.2}$$

where V is the potential energy. The action receives a contribution from the neighborhood N of

$$I = \int_N d\eta\, d^3x \sqrt{{}^{(3)}g} \cdot \left(\frac{1}{2}\partial_\mu \phi \partial_\nu \phi g^{\mu\nu} - V(\phi)\right). \tag{A.3}$$

We may perform variations whose support is solely in the neighborhood N in order to obtain there the equations of motion

$$\left[\sqrt{{}^{(3)}g} \cdot \left(g^{\mu\nu}\partial_\mu\partial_\nu\phi - V'(\phi)\right)\right] + g^{\mu\nu}\left(\partial_\nu \sqrt{{}^{(3)}g}\right) \cdot \partial_\mu \phi + \left(\partial_\nu g^{\mu\nu}\right)\sqrt{{}^{(3)}g} \cdot \partial_\mu \phi = 0\,. \tag{A.4}$$

We now multiply Eq. (A.4) by $\partial_\eta \phi$, and integrate across the domain wall with respect to a proper distance normal to the wall from the edge of the domain wall with coordinate $\eta = -\epsilon$ to some point in the interior of the domain wall with coordinate $\eta = \eta(\phi)$. In so doing we note that the tangential components of ϕ and of the four metric are continuous across the wall, and so then is $\sqrt{^{(3)}g}$. Moreover, in Gaussian normal coordinates $g^{\eta\eta} = +1$. As a consequence we may ignore the last term in Eq. (A.4) and we may treat $\sqrt{^{(3)}g}$ as a constant when we perform our integration to obtain

$$\frac{1}{2}g_{\eta\eta}(\partial_\eta \phi)^2 = V(\phi) + \left(\frac{1}{2}g_{\eta\eta}(\partial_\eta\phi)^2 - V(\phi)\right)\big|_{-\epsilon} - 1/\sqrt{^{(3)}g} \cdot \int_{-\epsilon}^{\eta(\phi)} d\eta g^{\mu\nu}(\partial_\nu \sqrt{^{(3)}g})\partial_\mu \phi \partial_\eta \phi\,. \qquad \text{(A.5)}$$

Finally, we require that the surface stress-energy tensor defined by Eq. (A.1) be independent of ϵ for sufficiently small ϵ. Let us consider the orders of ϵ of the various quantities appearing in Eq. (A.5). Because *ipso facto* nothing dramatic happens at the domain wall we have $\partial_\eta \phi \mid_{-\epsilon} = O(1)$ and $V(\phi) \mid_{-\epsilon} = O(1)$. Whatever is the order of $\partial_\eta \phi$, the integrated term in Eq. (A.5) is smaller than the $\frac{1}{2}g_{\eta\eta}(\partial_\eta\phi)^2$ term by a factor of ϵ and so may be ignored. We conclude that if the surface stress-energy is to be of order unity, then $\partial_\eta \phi = O(\frac{1}{\sqrt{\epsilon}})$ and $V(\phi) = O(\frac{1}{\epsilon})$ within the domain wall.

Inserting the form for $\partial_\eta \phi$ given by Eq. (A.5) into the equation for the energy-momentum tensor, (A.2), one discovers that in the wall

$$T_{i\eta} = O(\frac{1}{\sqrt{\epsilon}})\,, \qquad \text{(A.6a)}$$

$$T_{\eta\eta} = O(1) \qquad \text{(A.6b)}$$

and

$$T_{ij} = -g_{ij} \cdot 2V(\phi) + O(1) = -^{(3)}g_{ij} \cdot 2V(\phi) + O(1) \qquad \text{(A.6c)}$$

The surface stress-energy tensor is then

$$S_{i\eta} = O(\sqrt{\epsilon})\,, \qquad \text{(A.7a)}$$

$$S_{\eta\eta} = O(\epsilon) \qquad \text{(A.7b)}$$

and

$$S_{ij} = -\sigma g_{ij} + O(\epsilon) \tag{A.7c}$$

where

$$\sigma = \int_{-\epsilon}^{\epsilon} d\eta \, 2V(\phi) \, .$$

For timelike domain walls such as we are studying, $g_{\eta\eta} = +1$ and Eq. (A.5) shows that $\sigma > 0$. Note that in the limit $\epsilon \rightarrow 0$ only S_{ij} is nonzero, so the surface stress-energy may be interpreted as a three dimensional tensor. We saw in section III that covariant energy-momentum conservation implies σ is constant as the domain wall evolves.

APPENDIX B: THE FORCE ON A THIN WALL

In Section III we applied energy-momentum conservation to the singular expression for $T^{\mu\nu}$ which is applicable to the thin wall approximation, and we found an ambiguous expression in Eq. (3.6). As was pointed out in the text, the problem arose because we were computing the gravitational force on a sheet of mass, a situation which is completely analogous to the elementary problem of evaluating the electrostatic force on a sheet of charge. In this appendix we will show that the ambiguity can be resolved by methods very similar to those used in the electrostatics case, with a result which is also very similar.

The problem arises when one evaluates the integral

$$I \equiv \int_{-\epsilon}^{\epsilon} d\eta\, K^i{}_j T^j{}_i \tag{B.1}$$

which appears when one tries to extract the consequences of Eq. (3.1b) for the behavior of the wall. When evaluated naively with the delta-function expression (3.2) for $T^{\mu\nu}$, the expression is ambiguous because, according to Eq. (2.12), $K^i{}_j$ is discontinuous at the bubble wall ($\eta = 0$). Eq. (B.1) is analogous to the expression $\int d\eta\, E_\eta \rho$ which expresses the normal component of the electrostatic force on a charged sheet.

To evaluate (B.1), one needs information about how $K^i{}_j$ varies as one crosses the domain wall, and this information is contained in Eq. (2.8c). Only the singular terms are important at the wall, so

$$\partial_\eta \left\{ K^i{}_j - \delta^i{}_j \, Tr\, K \right\} = -8\pi G T^i{}_j (\text{singular}) \,, \tag{B.2}$$

which is analogous to $\partial_\eta E_\eta = 4\pi\rho$ in electrostatics.

Thus

$$I = -\frac{1}{8\pi} \int_{-\epsilon}^{\epsilon} d\eta\, K^i{}_j \partial_\eta \left\{ K^i{}_j - \delta^i{}_j \, Tr\, K \right\} . \tag{B.3}$$

Straightforward manipulations then lead to the result

$$I = \overline{K}^i{}_j \int_{-\epsilon}^{\epsilon} d\eta\, T^j{}_i \,, \tag{B.4}$$

where $\overline{K}^i{}_j$ is defined by Eq. (3.7).

APPENDIX C: THE COLEMAN-DE LUCCIA BOUNCE

In the Coleman - De Luccia approach, de Sitter space is embedded in five dimensional Minkowski space, that is, the de Sitter line element is written

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + dz^2 \tag{C.1}$$

where

$$0 \le \mu, \nu \le 3,$$

$$\eta_{\mu\nu} = \text{diag}(-1,1,1,1,),$$

$$\eta_{\mu\nu} x^\mu x^\nu + z^2 = \frac{1}{\chi^2}.$$

Eliminating the coordinate z from the line element one obtains

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \frac{\chi^2 (dx^\mu x^\nu \eta_{\mu\nu})^2}{1 - \chi^2 \eta_{\mu\nu} x^\mu x^\nu}. \tag{C.2}$$

Coleman and De Luccia define

$$\rho^2 \equiv \eta_{\mu\nu} x^\mu x^\nu \tag{C.3}$$

and discover

$$\rho^2 = \frac{2\kappa}{\check{\chi}^2} \tag{C.4}$$

for the domain wall of a bounce solution. The domain wall has a constant value of ρ so that

$$d(\rho^2) = 0 = d(\eta_{\mu\nu} x^\mu x^\nu) = 2(dx^\mu) x^\nu \eta_{\mu\nu} \tag{C.5}$$

and

$$ds^2 \,|_{\text{wall}} = \eta_{\mu\nu} dx^\mu dx^\nu \,|_{\text{wall}} . \tag{C.6}$$

Thus the de Sitter spacetime joins smoothly to the flat spacetime. Moreover, one may identify

$$r^2 = x^i x^j \eta_{ij} \tag{C.7}$$

implying

$$\rho^2 = r^2 - (x^o)^2. \tag{C.8}$$

The Coleman-De Luccia solution has parametric representation

$$r = \frac{2\kappa}{\check{\chi}^2}\cosh\left(\frac{\check{\chi}^2}{2\kappa}\tau\right), \qquad x^o = \frac{2\kappa}{\check{\chi}^2}\sinh\left(\frac{\check{\chi}^2}{2\kappa}\tau\right). \tag{C.9}$$

By virtue of Eq. (C.6) one may identify τ with the proper time, *i.e.*, $\dot{t}^2 = 1 + \dot{r}^2$ and $\tau = 0$ when $x^o = 0$. In our presentation we have

$$0 = 2GM = \check{\chi}^2 r^3 - 2\kappa r^2\sqrt{1 - \chi^2 r^2 + \dot{r}^2} \tag{C.10}$$

(recall that $\mathrm{sign}\beta_D = -1$ for the $M = 0$ bounce), which likewise has the solution

$$r = \frac{2\kappa}{\check{\chi}^2}\cosh\left(\frac{\check{\chi}^2}{2\kappa}\tau\right), \qquad t_D = \frac{1}{\chi}\mathrm{arcsinh}\left\{\left(\frac{2\chi\kappa}{\check{\chi}^2}\right)\cdot\sinh\left(\frac{\check{\chi}^2}{2\kappa}\tau\right)\right\}. \tag{C.11}$$

After identifying t_D with $\frac{1}{\chi}\mathrm{arcsinh}\chi x^o$, one may compare Eqs. (C.9) and (C.11) and confirm that the $M = 0$ bounces discovered by Coleman and De Luccia and rederived by us are identical.

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20. This question is academic, since the energies that are be required are totally inaccessible. We found in Section V that the minimum mass necessary for what we called an inflationary trajectory is about 10^{28} GeV $\approx$ 10 kg, and we presume that an energy of this order would be necessary to produce an inflationary universe under any circumstances.

21. We thank Bill Press for very stimulating and useful discussions on these issues.

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Fig. 1

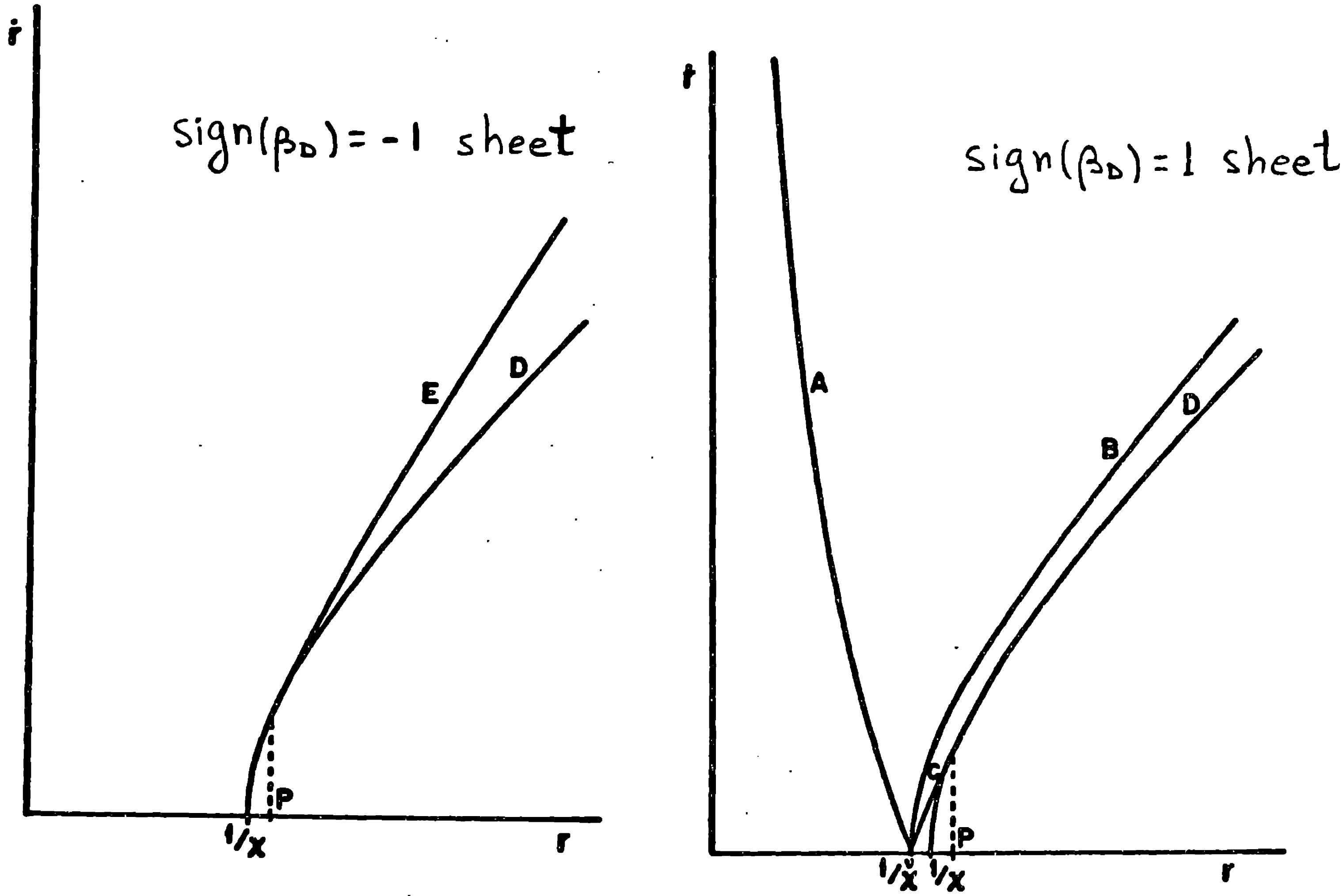
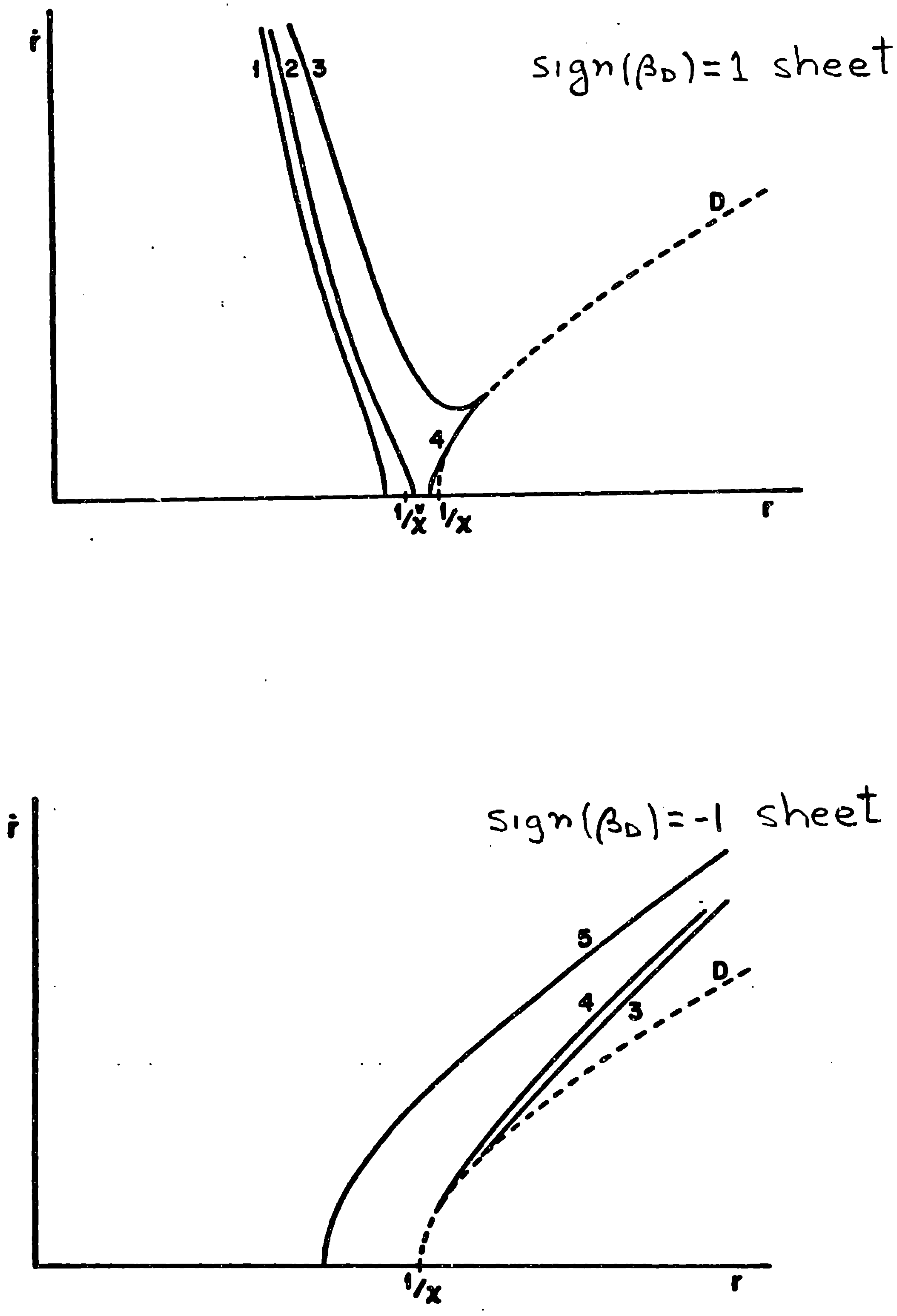
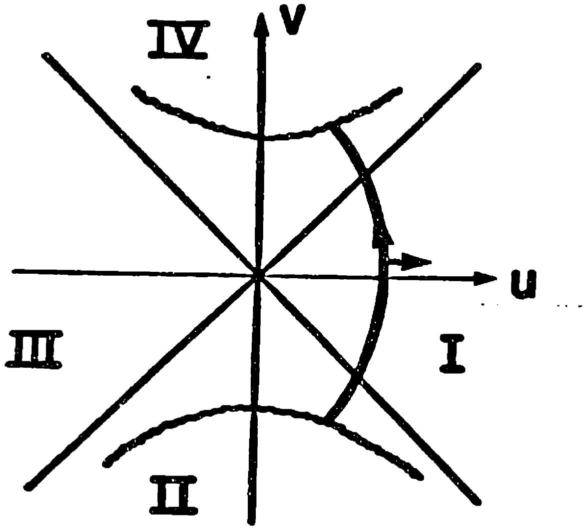


Fig. 2





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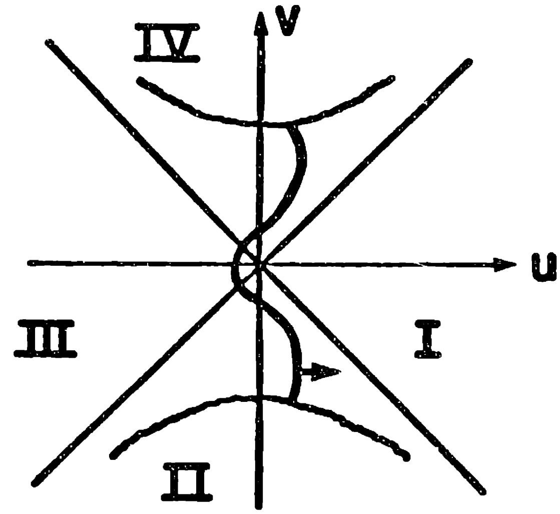


Fig 4

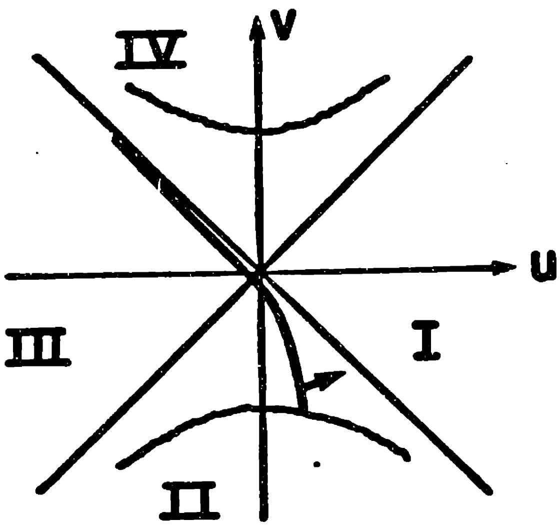
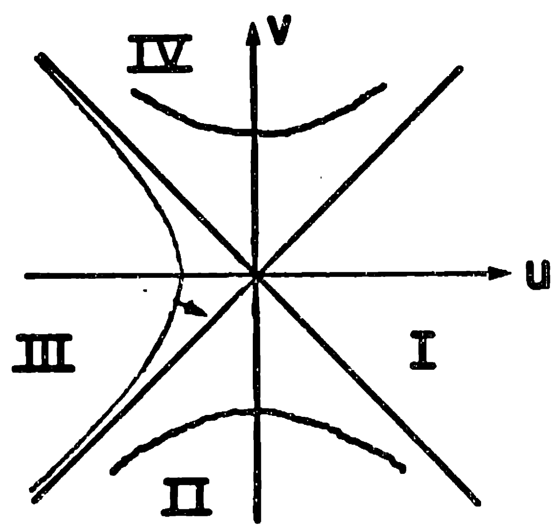


Fig 6



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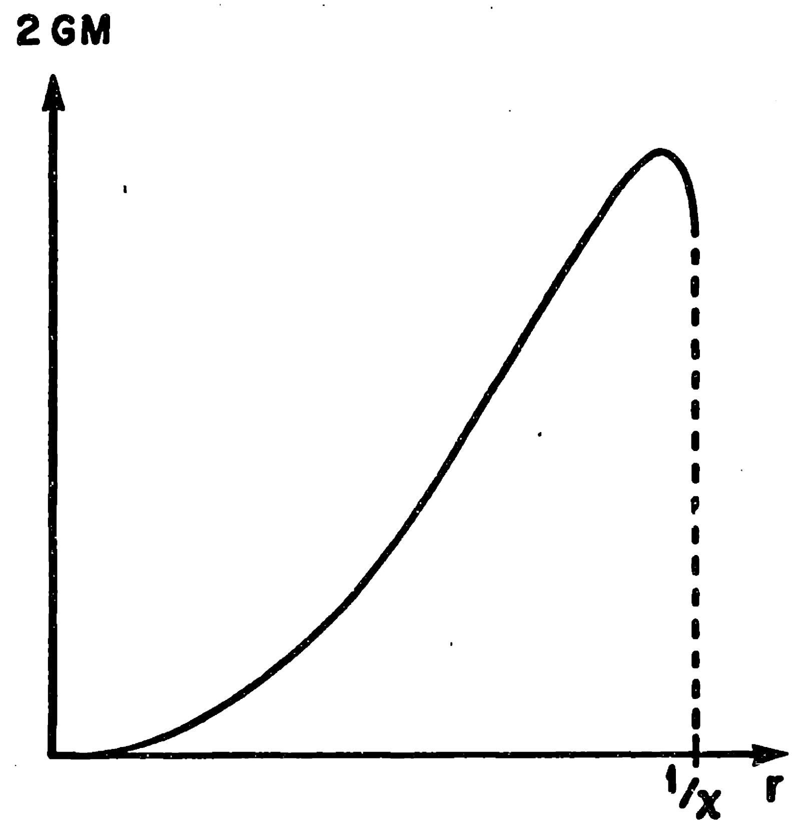
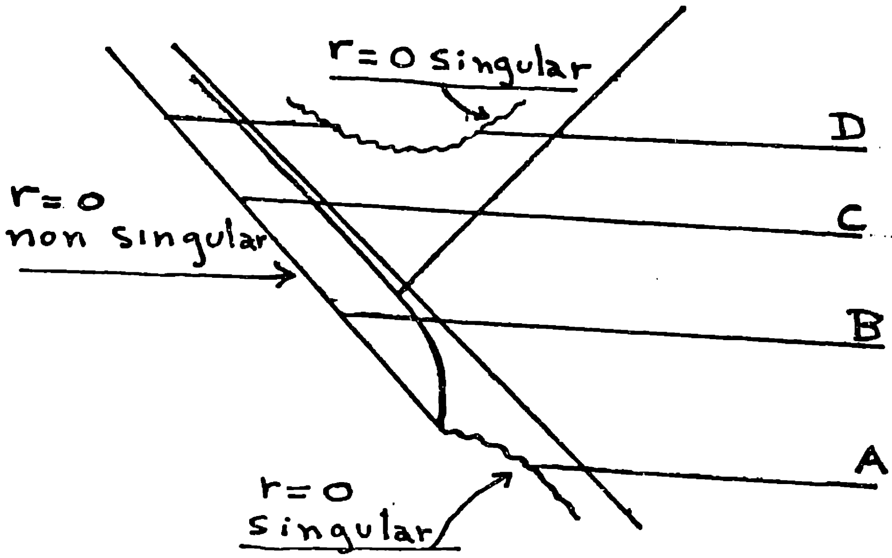


Fig. 5



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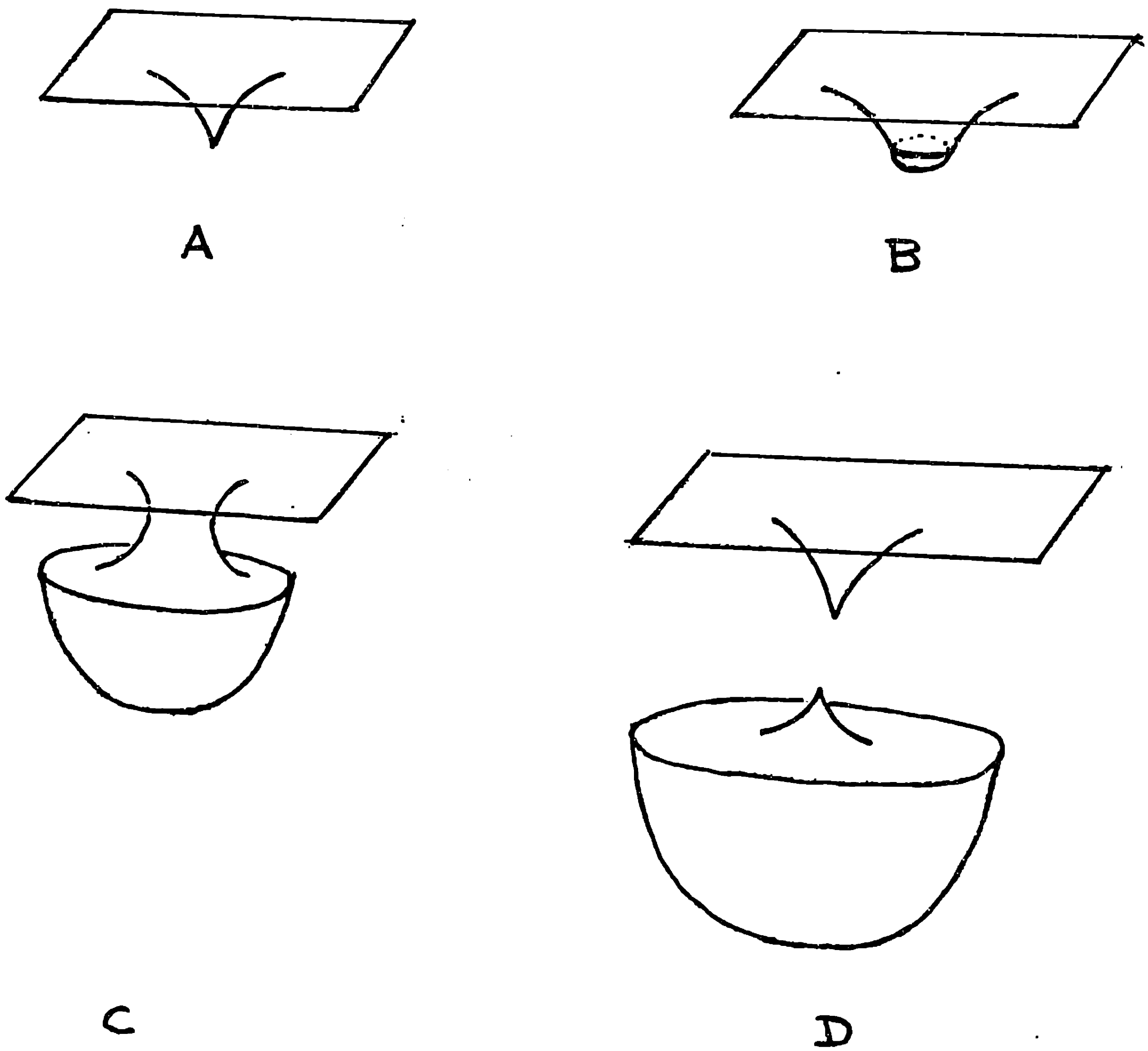

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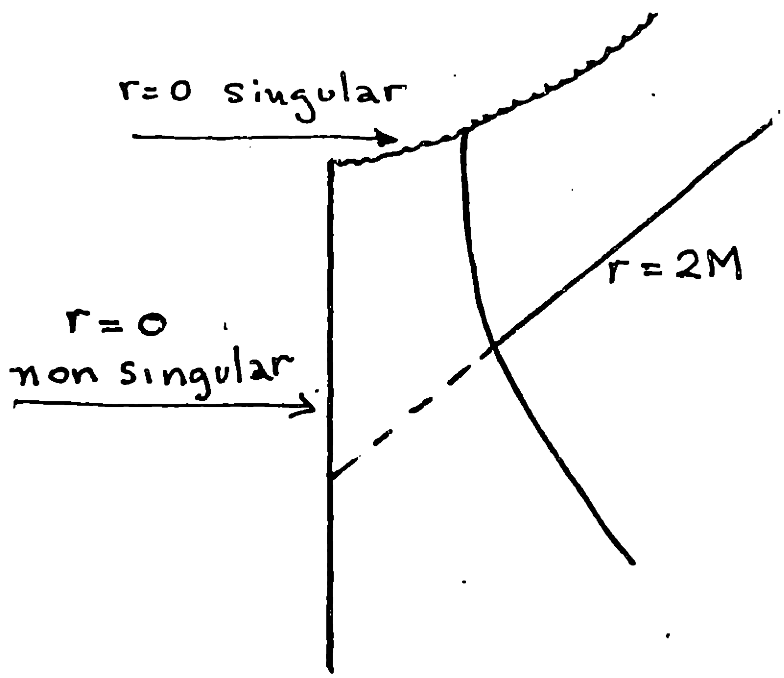

Fig. 10

FIGURE CAPTIONS

Figure 1: The $(\dot{r}, r)$ space with 'landmarks' given by equations 5.1-3 filled in.

A, C, and E denote loci of points for which $r = r_{SCH}$,

B denotes the locus of points for which $\beta_S = 0$,

D denotes the locus of points for which $\beta_D = 0$.

Figure 2: Representative domain wall trajectories plotted in $(\dot{r}, r)$ space.

Figure 3: Representative black hole trajectory plotted in Kruskal-Szekeres coordinates.

Figure 4: Representative trajectory of an (ultimately collapsing) domain wall which enters the region of the Schwarzschild manifold (III) causally disconnected from the region (I) into which the black hole trajectories are conventionally defined to enter. The dots represent points at which $\dot{t}_S = 0$.

Figure 5: A plot of $2GM$ vs. r at fixed $\dot{r} = 0$ and for $\beta_D > 0$.

Figure 6: Representative inflationary trajectory plotted in Kruskal-Szekeres coordinates. The dot represents the point at which $\dot{t}_S = 0$.

Figure 7: Representative bounce trajectory plotted in Kruskal-Szekeres coordinates.

Figure 8: The entire spacetime for the case of an inflationary bubble. Several constant V hypersurfaces are indicated.

Figure 9: Three dimensional representations of the hypersurfaces indicated in Figure 8.

Figure 10: Spacetime diagram illustrating the collapse of ordinary matter.